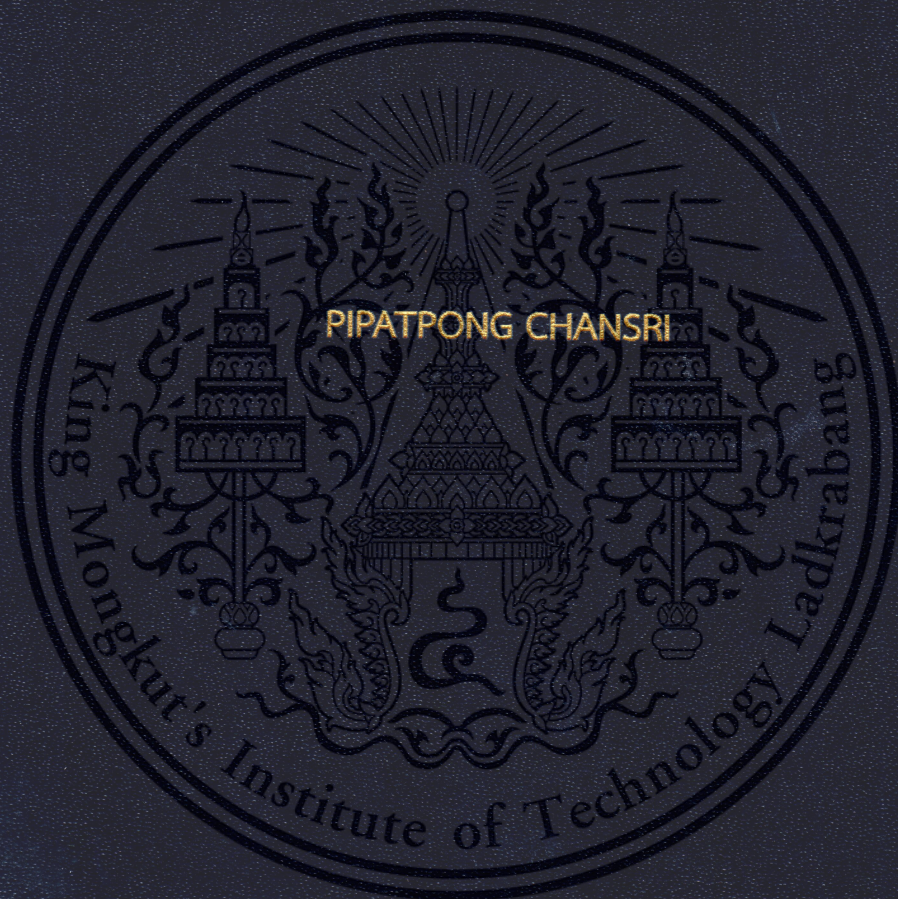


# GEOMETRY ON THE SLOPE OF A MOUNTAIN



A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE  
DEGREE OF DOCTOR OF PHILOSOPHY IN APPLIED MATHEMATICS  
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE  
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

2020

KMITL-2020-SC-D-001-030

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### บทคัดย่อ

ในงานวิจัยนี้เราศึกษาการมีอยู่จริงของการวัดเชิงความชันบนพื้นผิวของการหมุนรอบ นอกจากนี้เราได้แสดงวิธีการสร้างการวัดแบบพินสเตอร์ในสองมิติและสามมิติ โดยที่อินดิเคทริกซ์ของการวัดเหล่านั้นเป็นเส้นโค้งพีแดลหรือพื้นผิวพีแดลของเส้นโค้งหรือพื้นผิวที่กำหนดตามลำดับ

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### Abstract

We study the existence of globally defined slope metrics on surfaces of revolution. We obtain necessary and sufficient conditions for any surface of revolution to admit a strongly convex slope metric. We also show how to construct new Finsler metrics, in two and three dimensions, whose indicatrices are pedal curves or pedal surfaces of some other curves or surfaces.

**Keywords :** slope metrics, pedal, surfaces of revolution, Finsler metrics, indicatrices

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# Table of Contents

	Page
Abstract in Thai .....	i
Abstract in English.....	ii
Acknowledgements .....	iii
Table of Contents .....	iv
List of Figures.....	vi
<b>Chapter 1. Introduction</b> .....	<b>1</b>
1.1 Research Motivation .....	1
1.2 Objectives of the study .....	3
1.3 Scopes of the study.....	3
1.4 Benefits of the study .....	3
1.5 Research Methodology.....	3
<b>Chapter 2. Preliminaries</b> .....	<b>4</b>
2.1 Elementary Riemannian geometry.....	4
2.1.1 Riemannian surface and tangent bundle.....	4
2.1.2 Surface of revolution.....	4
2.1.3 Examples of surfaces of revolution .....	5
2.2 The curvature of a plane curve.....	6
2.3 The convexity of a plane curve.....	8
2.4 Finsler metric.....	8
2.5 Slope metrics.....	9
2.6 The pedal curve of a plane algebraic curve .....	13
2.7 The pedal surface.....	16
<b>Chapter 3. The slope metric on a surface of revolution</b> .....	<b>20</b>
3.1 Main Result.....	21
3.2 Examples.....	22
3.2.1 Two-sheeted hyperboloid .....	22
3.2.2 Cone.....	23
3.2.3 Ellipsoid .....	23
3.2.4 One-sheeted hyperboloid .....	23
3.2.5 The elliptic paraboloid .....	24
3.2.6 The Gauss function.....	25
<b>Chapter 4. Pedals and Finsler metrics</b> .....	<b>28</b>
4.1 Some properties of the pedal on a plane curve.....	28
4.2 Examples of pedals and their corresponding Finsler metrics.....	31
4.2.1 The slope metric whose indicatrix is a limaçon.....	31

4.2.2 The pedal curve of an ellipse.....	32
4.2.3 The pedal surface of a 2-sphere.....	34
4.2.4 The Pedal surface of an ellipsoid.....	37
<b>Chapter 5. Conclusions and Suggestions.....</b>	<b>40</b>
5.1 Conclusions.....	40
5.2 Suggestions.....	40
References.....	40
Appendix.....	43
Appendix A.....	44
Appendix B.....	56
Author Biography.....	72



# List of Figures

Figure	Page
1.1 Zermelo Navigation with velocity $V$ under constant wind $W$ .....	1
1.2 Picture show that the person walk in direction on the inclined plane of angle $\varepsilon$ to the horizontal sea level.....	2
2.1 surface of revolution.....	5
2.2 Cylinder.....	5
2.3 Cone.....	6
2.4 Hyperboloid.....	6
2.5 Picture show that the person walk in direction on the inclined plane of angle $\varepsilon$ to the horizontal sea level.....	11
2.6 The pedal curve of $(C)$ with respect to $P(x_0, y_0)$ .....	14
2.7 The pedal curve of a plane curve with pedal point at the origin (left) and at an arbitrary point $P(x_0, y_0)$ (right).....	15
2.8 The pedal surface of $(S)$ with respect to $P(x_0, y_0, z_0)$ .....	17
2.9 The pedal surface of a surface with pedal point at the origin (left) and at an arbitrary point $P(x_0, y_0, z_0)$ (right).....	17
3.1 Hyperboloid of two sheeted .....	22
3.2 Cone.....	23
3.3 Ellipsoid.....	23
3.4 Hyperboloid of one sheeted .....	24
3.5 The elliptic paraboloid (a).....	24
3.6 The elliptic paraboloid (b).....	25
3.7 The elliptic paraboloid (c).....	25
3.8 The Gauss function (a).....	26
3.9 The Gauss function (b).....	26
4.1 A limaçon curve for $a = 1, b = 3$ . (left), and the non-convex case for $a = 1, b = 1$ (right). .....	32
4.2 The convex curve in (4.7) for $a = 10, b = 9, k = 2$ (left), and the non- convex case for $a = 10, b = 6, k = 2$ (right). .....	33
4.3 The pedal surface of sphere for $k = 1, r = \frac{1}{3}$ .....	36
4.4 The pedal surface of sphere for $k = 1, r = 1$ . This surface is not convex.	36
4.5 The pedal surface of an ellipsoid for $a = 2, c = \sqrt{6}$ .....	38
4.6 The pedal surface of an ellipsoid for for $k = 1/3, a = b = 2, c = 4$ . This surface is not convex.....	38

# Chapter 1

## Introduction

### 1.1 Research Motivation

Let  $M$  be an  $n$ -dimensional smooth manifolds. Denote by  $T_x M$  the tangent space at  $x$  in  $M$ , and  $TM := \cup_{x \in M} T_x M$  the tangent bundle of  $M$ . Riemannian manifold is  $n$ -dimensional smooth manifolds  $M$  equipped with an inner product on the tangent space  $T_x M$  at each point  $x$ . In more general, a Finsler manifolds is a smooth manifolds where each tangent space is equipped with a Minkowski norm, that is, a norm that is not necessarily induced by an inner product. The main difference is that the metric itself and all Finsler geometric quantities depend not only on the point  $x \in M$  of the manifold, but also on the direction  $y \in T_x M$  the tangent space at  $x$  in  $M$ , where  $(x, y)$  are the canonical coordinates of the tangent bundle  $TM$ . This directional dependence reveals many hidden geometrical features that are usually obscured by the quadratic form in the  $y$ -variable of a Riemannian metric. On the other hand, most of the geometrical properties of Finsler spaces are highly nonlinear, this is the case with the non-linear connection or the parallel displacement, making most of the traditional Riemannian methods unapplicable.

It is well-known that one of the most important problems in differential geometry and calculus of variations is the time minimizing travel between two points on a Riemannian or Finsler manifold. The problem of finding these time minimizing paths goes back to Caratheodory ([5]) and Finsler himself.

An important insight in to the problem is due to Shen ([13]) who related the Zermelo's navigation problem to the geometry of Randers metrics. Indeed, it is now clear that the time minimizing travel paths on a Riemannian manifold  $(M, v)$  under the influence a mild wind  $W \in TM$ ,  $\|W\|_v < 1$ , are exactly the geodesics of a Randers metric  $F = \alpha + \beta$  uniquely determined by the navigation data  $(v, W)$  (see [4] for details). Moreover, a singular solution of the Zermelo's navigation problem can be found in the

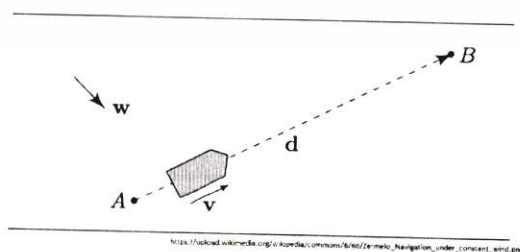


Figure 1.1: Zermelo Navigation with velocity  $V$  under constant wind  $W$

case  $\|W\|_v = 1$ , namely the geodesics of a Kropina metric ([16]). The Randers metrics

$F = \alpha + \beta$  and the Kropina metrics  $F = \frac{\alpha^2}{\beta}$  belong to a larger class of Finsler metrics called  $(\alpha, \beta)$ -metrics since they are obtained by deformations of a Riemannian metric by means of a linear 1-form  $\beta = b_i(x)y^i$  on  $TM$ . The common characteristic is that they are obtained by rigid translation of a Riemannian unit sphere by a vector field  $W$ . The local and global geometries of these Finslerian metrics have been extensively studied ([12]).

Another interesting but much less studied problem is the Matsumoto's slope metric  $F = \frac{\alpha^2}{\alpha - \beta}$ .

Based on a letter of P. Finsler (1969), M. Matsumoto considered the following problem:

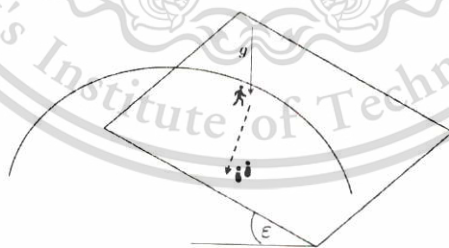
Suppose a person walking on a horizontal plane with velocity  $c$ , while the gravitational force is acting perpendicularly on this plane. The person is almost ignorant of the action of this force. Imagine the person walks now with same velocity on the inclined plane of angle  $\epsilon$  to the horizontal sea level. Under the influence of gravitational forces, what is the trajectory the person should walk in the center to reach a given destination in the shortest time?

Based on this, he has formulated the following **Slope principle** ([9],[10]).

With respect to the time measure, a plane  $(\pi)$  with an angle  $\epsilon$  inclination can be regarded as a Minkowski plane. The indicatrix curve of the corresponding Minkowski metric is a limaçon, contained in this plane, given by

$$r = c + a \cos \theta,$$

in the polar coordinates  $(r, \theta)$  of  $(\pi)$ , whose pole is the origin  $O$  of  $(\pi)$  and the polar axis is the most steepest downhill direction, where  $a = \frac{g}{2} \sin \epsilon$ , and  $g$  is the acceleration constant.



**Figure 1.2:** Picture show that the person walk in direction on the inclined plane of angle  $\epsilon$  to the horizontal sea level

From calculus of variations it follows that for a hiker walking the slope of a mountain under the influence of gravity, the most efficient time minimizing paths are not the Riemannian geodesics, but the geodesics of the slope metric  $F = \frac{\alpha^2}{\alpha - \beta}$ .

More recently, it was shown that the fire fronts evolution can be modeled by Finsler metrics of slope type and their generalizations (see [8]). In this setting the

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geodesics behaviour and the cut locus have real interpretations and concrete applications for the firefighters activity as well as preventing of wild fires. All these applications show that slope metrics deserve a more detailed study making in this way the motivation of the present paper.

## 1.2 Objectives of the study

- 1) We show that there are some examples of surfaces admitting globally defined slope metrics.
- 2) We describe in some detail the geometry of the surface of revolution endowed with a slope metric.
- 3) We study the geometry of Finsler surface by using the geometry of algebraic curve.

## 1.3 Scopes of the study

- 1) We suppose in this work that the mountain is represent by some surface of revolution.
- 2) We specialize to Riemannian surfaces of revolution admitting globally defined slope metrics.
- 3) We construct explicitly the slope metric on a surface of revolution and show that there are some such surfaces admitting globally defined strongly convex slope metrics.
- 4) We turn to study of geodesics of slope metrics on a surface of revolution.

## 1.4 Benefits of the study

- 1) To develop the theorem of geometry on the slope of the mountain in terms of finlser metrics.
- 2) To find new Finsler metric and slope metric in high dimensional.

## 1.5 Research Methodology

- 1) Study Finsler geometry and Matsumoto metric.
- 2) Study the form of surfaces of revolution that admit a slope metrics.
- 3) Study the pedal curve of algebraic curve.
- 4) Summarize a results and writing the theorem.

# Chapter 2

## Preliminaries

### 2.1 Elementary Riemannian geometry

#### 2.1.1 Riemannian surface and tangent bundle

**Definition 2.1.** (see e.g. [11]) A **surface** in  $\mathbb{R}^3$  is a subset  $M$  of  $\mathbb{R}^3$  such that each point  $p$  of  $M$  there exists a one-to-one mapping  $X : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ , where image contain a neighbourhood of  $p$  in  $M$ .

**Definition 2.2.** A **Riemannian metric** on a surface  $M$  is defined by

$$ds^2 = g_{ij} dx^i dx^j$$

where  $x^i$  is the local coordinate of  $M$ .

**Definition 2.3.** The pair  $(M, h)$  is called a **Riemannian surface** whenever  $M$  is a surface in  $\mathbb{R}^3$  and  $h$  is a Riemannian metric on  $M$ .

**Definition 2.4.** (see e.g. [15]) The **tangent space** over a point  $x$  on  $M$  is a vector space with the basis  $(\frac{\partial}{\partial x^i})_x, i = 1, \dots, n$  such that for every tangent vector  $y \in T_x M$ , we can write it in the form of the linear combination

$$y = \sum_{i=1}^n y^i(x) \left( \frac{\partial}{\partial x^i} \right) = y^i \frac{\partial}{\partial x^i}.$$

**Definition 2.5.** The set of all tangent vectors on a surface  $M$  is called the **tangent bundle** of  $M$  and denoted by

$$TM = \{(x, y) \mid x \in M, y \in T_x M\} \rightarrow M.$$

#### 2.1.2 Surface of revolution

We will show one example of Riemannian surface which is a surface of revolution. The surface of revolution  $M \rightarrow \mathbb{R}^3$  can be parametrized as  $M \rightarrow \mathbb{R}^3$  can be parametrization as

$$(u, v) \mapsto (x = m(u) \cos v, y = m(u) \sin v, z = u) \tag{2.1}$$

where  $u \in (0, \infty), v \in [0, 2\pi]$ . Here  $m : (0, \infty) \rightarrow (0, \infty)$  is a smooth function such that  $m'(0) = 1$  (see [14] for details). It is known that a curve  $u = u(t), v = v_0$ : constant is called a **meridian**, and  $u = u_0$ : constant,  $v = v(t)$  is called a **parallel**.

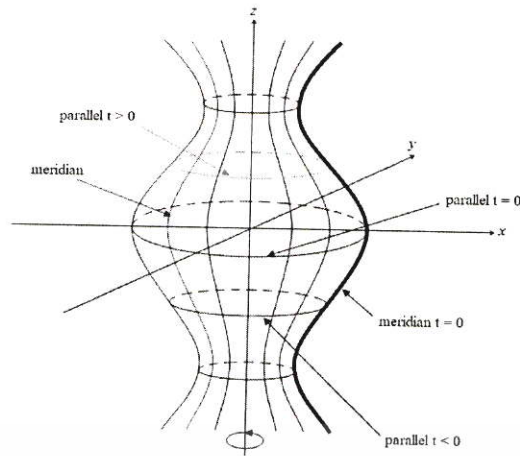


Figure 2.1: surface of revolution

The induced Riemannian metric is

$$(a_{ij}) = \begin{pmatrix} 1 + (m')^2(u) & 0 \\ 0 & m^2(u) \end{pmatrix} \quad (2.2)$$

### 2.1.3 Examples of surfaces of revolution

1. **Cylinder.** The surface revolution by rotating line  $(r, 0, t)$  around  $z$ -axis parametrized as

$$(r \cos \theta, r \sin \theta, t).$$

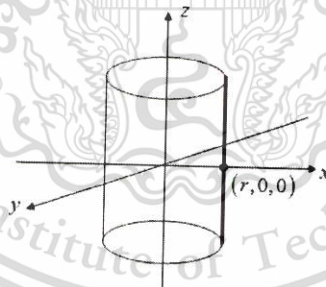


Figure 2.2: Cylinder.

2. **Cone.** The surface revolution by rotating line  $(t, 0, t)$  around  $z$ -axis parametrized as

$$(t \cos \theta, t \sin \theta, t).$$

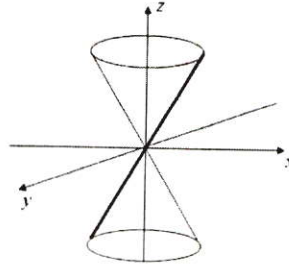


Figure 2.3: Cone.

3. **Hyperboloid** . The surface revolution by rotating line  $(t, 0, t^2)$  around  $x$ -axis parametrized as

$$(t, t^2 \cos \theta, t^2 \sin \theta).$$

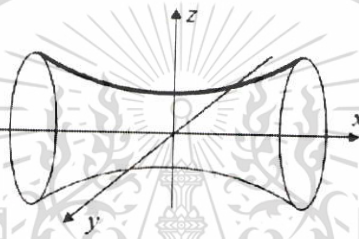


Figure 2.4: Hyperboloid.

## 2.2 The curvature of a plane curve

Let  $c : [a, b] \rightarrow \mathbb{R}^2$ ,  $t \mapsto c(t) = (x(t), y(t))$ , be a plane curve endowed with the arbitrary parameter  $t$ . Observe that

$$\left\| \frac{dc(t)}{dt} \right\| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$$

where we denote by  $\dot{x}(t) = \frac{dx}{dt}$ . The unit tangent vector is defined by

$$T(t) = \frac{dc/dt}{\|dc/dt\|}.$$

By computation in components we obtain

$$\begin{aligned} T(t) &= \frac{(\dot{x}(t), \dot{y}(t))}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \\ \dot{T}(t) &= \frac{[\dot{y}(t)(\ddot{x}(t)\dot{y}(t) - \dot{x}(t)\ddot{y}(t)), \dot{x}(t)(\ddot{y}(t)\dot{x}(t) - \ddot{x}(t)\dot{y}(t))]}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{3/2}} \\ \|\dot{T}(t)\| &= \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\dot{x}(t)^2 + \dot{y}(t)^2}. \end{aligned} \tag{2.3}$$

The definition for the **unit normal vector** is that  $N$  is orthonormal to  $T$ , that is we have the formulas

$$\begin{aligned} T(t) \cdot T(t) &= 1 \\ T(t) \cdot N(t) &= 0 \\ N(t) \cdot N(t) &= 1. \end{aligned} \tag{2.4}$$

Since  $T(t) = \frac{(\dot{x}(t), \dot{y}(t))}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}$  it follows that

$$N(t) = \frac{(-\dot{y}(t), \dot{x}(t))}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}.$$

Hence, from the first formula,  $\dot{T}$  and  $T$  are orthogonal, and since  $T$  and  $N$  are orthonormal, it follows that  $\dot{T}$  must be multiple of  $N$ , that is, there exists some function  $k(t)$  such that

$$\dot{T}(t) = k(t)N(t).$$

In the same way, from the third formula it follows that  $\dot{N}$  and  $N$  are orthogonal, and since  $T$  and  $N$  are orthonormal, it follows that  $\dot{N}$  must be multiple of  $T$ , that is, there exists some function  $\tilde{k}(t)$  such that

$$\dot{N}(t) = \tilde{k}(t)T(t).$$

By substituting these formulas in the second formula, we have

$$\dot{T}(t) \cdot N(t) + T(t) \cdot \dot{N}(t) = kN \cdot N + \tilde{k}T \cdot T = k + \tilde{k} = 0,$$

therefore  $\tilde{k} = -k$ , and the moving equations of the orthonormal frame  $\{T(t), N(t)\}$  are

$$\begin{aligned} \frac{dT(t)}{dt} &= k(t)\|\dot{c}(t)\|N(t) \\ \frac{dN(t)}{dt} &= -k(t)\|\dot{c}(t)\|T(t), \end{aligned} \tag{2.5}$$

where

$$k(t) := \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{[\dot{x}(t)^2 + \dot{y}(t)^2]^{3/2}},$$

or, in vectorial representation

$$k(t) = \frac{\|dT/dt\|}{\|dc/dt\|} = \frac{\dot{c} \cdot \ddot{c}}{\|\dot{c}\|^3}.$$

Hence  $k(t)$  is called the **curvature** of a curve  $c$ .

**Remark 2.1.** Observe that if the curve  $c$  is arclength parameter with an arclength parameter  $s$ , that is

$$|\dot{c}(s)| = \left\| \frac{dc}{ds} \right\| = 1,$$

for every  $s \in [a, b]$ . Then we can write

$$\begin{cases} x'y'' - x''y' = k \\ y'y'' + x'x'' = 0 \end{cases} \tag{2.6}$$

where we denote by  $x' = x'(s) = \frac{dx}{ds}$ . By solving this algebraic system we get

$$x'' = -ky', \quad y'' = kx'.$$

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## 2.3 The convexity of a plane curve

**Definition 2.6.** A **convex curve** is a curve in the Euclidean plane which lies completely on one side of each and every one of its tangent lines.

**Definition 2.7.** A **strictly convex curve** is a convex curve that does not contain any line segments. Equivalently, a strictly convex curve is a curve that intersects any line in at most two points.

We recall that, for an arclength parameterized curve  $c(s)$ , parametrized counterclockwise, the original meaning of the curvature  $k(s)$  is the variation of the angle  $\varphi$  measured counterclockwise of the tangent line to the curve with the  $x$ -axis, that is

$$k(s) = \frac{d\varphi}{ds}. \quad (2.7)$$

Indeed, a computation to show that

$$k = \frac{dT}{ds} \cdot N = \frac{d\varphi}{ds}.$$

In this case, it follows that a curve  $c$  is convex if it is counterclockwise parameterized and the angle  $\varphi$  is increasing along the curve (i.e.  $k > 0$ ). Observe that the curve is counterclockwise parameterized if and only if  $c(s) \times \dot{c}(s) > 0$ , that is  $xy' - yx' > 0$ . Hence, the conditions for the curve to be strongly convex are

$$c(s) \times \dot{c}(s) > 0, \quad k > 0. \quad (2.8)$$

These conditions do not depend on the parameterization, so in general, for an arbitrary parameterized curve  $c(t)$  the criterion for strong convexity, in vectorial representation, is

$$c(s) \times \dot{c}(s) > 0, \quad \dot{c} \times \ddot{c} > 0. \quad (2.9)$$

On the other hand, if the curve is parameterized clockwise, then the convexity of it means  $k < 0$ , hence in this case, the strong convexity criterion is

$$c(s) \times \dot{c}(s) < 0, \quad \dot{c} \times \ddot{c} < 0. \quad (2.10)$$

Observe that we can unify this conditions in

$$\frac{\dot{c} \times \ddot{c}}{c \times \dot{c}} > 0. \quad (2.11)$$

If we write this formula in coordinates we have

$$\frac{\dot{c} \times \ddot{c}}{c \times \dot{c}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{x\dot{y} - y\dot{x}} > 0. \quad (2.12)$$

## 2.4 Finsler metric

We introduce the definition of Finslerian norm. Let  $(M, F)$  be a smooth  $n$ -dimensional differentiable manifold (see, e.g., [15] for details of manifold) endowed with a Finslerian norm  $F : TM \rightarrow [0, \infty)$  with properties

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1.  $F$  is positive and differentiable on  $\widetilde{TM}$ ,
2.  $F$  is 1-positive homogeneous, that is  $F(x, \lambda y) = \lambda F(x, y)$  for any  $\lambda > 0$  and for all  $(x, y) \in \widetilde{TM}$ ,
3. Hessian matrix  $g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$  is positive definite on  $\widetilde{TM}$ .

Then  $(M, F)$  is called a **Finsler manifold**,  $F$  is called the **Finsler metric**.

**Definition 2.8.** For  $\dim M = 2$ ,  $(M, F)$  is called Finsler surface. We denote the **indicatrix** (unit tangent bundle) of  $F$  by

$$\mathcal{I}_F = \{y \in TM : F(y) = 1\} \subset TM.$$

## 2.5 Slope metrics

In this section we will show that slope metric can be obtained as a movement on a Riemannian surface under the influence of the gravity attraction force. Let us consider a surface  $S$  embedded in the Euclidean space  $\mathbb{R}^3$  parametrized by

$$S \rightarrow \mathbb{R}^3, (x, y) \mapsto (x, y, z = f(x, y)), \quad (2.13)$$

that is  $S$  is the graph of smooth function  $z = f(x, y)$ . The tangent plane  $\pi_p = T_p S$  at point  $p = (x, y, f(x, y)) \in S$  is spanned by the vector fields

$$\partial_x := (1, 0, f_x), \quad \partial_y := (0, 1, f_y),$$

where  $f_x$  and  $f_y$  are the partial derivative of  $f$  with respect to  $x$  and  $y$ , respectively. It follows that the induced Riemannian metric from  $\mathbb{R}^3$  on the surface  $S$  is

$$a_{ij} = \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix}.$$

It is customary to denote this Riemannian metric by  $\alpha$ . If  $Y \in T_p M$  is an arbitrary tangent vector to  $S$ , i.e.  $Y = u \cdot \partial_x + v \cdot \partial_y$ , then the Riemannian length of  $Y$  is given by

$$|Y|_a^2 := a(Y, Y) = u^2 + v^2 + (u \cdot f_x + v \cdot f_y)^2.$$

We will construct the slope metric on the surface  $M$  by considering

- the plane  $x, y$  to be the sea level;
- the  $z \geq 0$  coordinate to be the altitude above the sea level;
- the surface  $M : z = f(x, y)$  to be the mountain.

For a given point  $p \in S$ , we consider the tangent plane  $\pi_p = \langle \partial_x, \partial_y \rangle$  and construct an  $\alpha$ -orthogonal basis  $e_1, e_2$  in  $\pi_p$  by choosing  $e_1$  to point on the steepest downhill

direction of  $\pi_p$ . The normal vector on  $S$  at a point  $p$  is  $\vec{n}_p = (f_x, f_y, -1)$ , and that the tangent plane at  $S$  through  $p = (x_0, y_0, z_0 = f(x_0, y_0))$  is

$$\pi_p : f_x(x - x_0) + f_y(y - y_0) - (z - z_0) = 0.$$

Since we have chosen the plane  $xy$  to be the sea level, the gravitational force  $\vec{F}$  acts perpendicular on the plane  $xy$  and it is given by

$$e_3 = (0, 0, -1).$$

In order to get the steepest downhill direction on  $\pi_p$  from  $p$  we compute the projection of  $e_3|_p$  on  $\pi_p$ . that is

$$u_p = \vec{n}_p \times (e_3|_p \times \vec{n}_p) = -(f_x, f_y, f_x^2 + f_y^2) = -f_x \cdot \partial_x - f_y \cdot \partial_y.$$

Remark that  $-(f_x, f_y)$  is the projection of  $e_3$  on  $\pi_p$  regarded as vector in  $\pi_p$ . It follows that its Riemannian length is

$$\begin{aligned} |u|_a^2 &= a_{11} \cdot f_x^2 + 2a_{12} \cdot f_x f_y + a_{22} \cdot f_y^2 \\ &= (1 + f_x^2) \cdot f_x^2 + 2f_x^2 f_y^2 + (1 + f_y^2) \cdot f_y^2 \\ &= (1 + f_x^2 + f_y^2)(f_x^2 + f_y^2), \end{aligned}$$

and we obtain the  $\alpha$ -unit vector

$$\begin{aligned} e_1 &:= \frac{1}{\sqrt{(1 + f_x^2 + f_y^2)(f_x^2 + f_y^2)}} (f_x, f_y, f_x^2 + f_y^2) \\ &= \frac{1}{\sqrt{(1 + f_x^2 + f_y^2)(f_x^2 + f_y^2)}} (f_x \partial_x + f_y \partial_y). \end{aligned}$$

The orthogonal vector on  $u$  is, clearly

$$u_p^\perp = (-f_y, f_x, 0) = -f_y \partial_x + f_x \partial_y.$$

The Riemannian length of  $u_p^\perp$  is

$$\begin{aligned} |u_p^\perp|_a^2 &= a_{11} \cdot f_y^2 - 2a_{12} \cdot f_x f_y + a_{22} \cdot f_x^2 \\ &= (1 + f_x^2) \cdot f_y^2 - 2f_x^2 f_y^2 + (1 + f_y^2) f_x^2 \\ &= f_x^2 f_y^2 + f_y^2 - 2f_x^2 f_y^2 + f_x^2 + f_x^2 f_y^2 = f_x^2 + f_y^2, \end{aligned}$$

and hence we obtain the unit vector

$$\begin{aligned} e_2 &= \frac{1}{\sqrt{f_x^2 + f_y^2}} (-f_y, f_x, 0) \\ &= \frac{1}{\sqrt{f_x^2 + f_y^2}} (-f_y \partial_x + f_x \partial_y). \end{aligned}$$

Clearly  $e_1, e_2$  is an  $\alpha$ -orthonormal basis in  $\pi_p$ . An arbitrary tangent vector  $v \in \pi_p$  will be written:

$$v = X e_1 + Y e_2 = \dot{x} \partial_x + \dot{y} \partial_y, \quad (2.14)$$

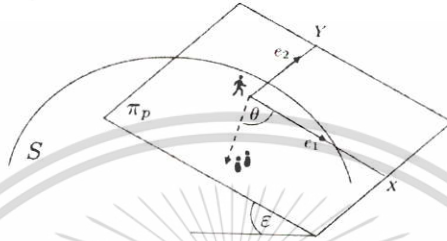
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where relation between the coordinates  $(X, Y)$  in the basis  $\{e_1, e_2\}$  and the canonical coordinates  $(\dot{x}, \dot{y})$  read

$$\begin{aligned}\dot{x} &= -\frac{f_x}{\sqrt{(1+f_x^2+f_y^2)(f_x^2+f_y^2)}} \cdot X - \frac{f_y}{\sqrt{f_x^2+f_y^2}} \cdot Y, \\ \dot{y} &= -\frac{f_y}{\sqrt{(1+f_x^2+f_y^2)(f_x^2+f_y^2)}} \cdot X + \frac{f_x}{\sqrt{f_x^2+f_y^2}} \cdot Y.\end{aligned}\quad (2.15)$$

The equality of the 3<sup>rd</sup> component is now an identity.



**Figure 2.5:** Picture show that the person walk in direction on the inclined plane of angle  $\varepsilon$  to the horizontal sea level

In the tangent plane  $\pi_p$  with origin  $p = (0, 0)$  and  $(X, Y)$  in the basis  $\{e_1, e_2\}$ , the slope principle tells us that the time minimizing trajectory of a hiker on the plane  $\pi_p$  is given by walking in the direction given by the limaçon

$$r = v + a \cdot \cos \theta, \quad (2.16)$$

where  $(r, \theta)$  are the polar coordinates of the  $XY$  plane,  $v$  is the velocity of the hiker on the flat plane  $xy$ ,  $a = \frac{g}{2} \cdot \sin \varepsilon$  is constant for fixed  $p \in S$ . Taking into account the parametrization

$$\begin{aligned}X(t) &= (v + a \cos t) \cdot \cos t, \\ Y(t) &= (v + a \cos t) \cdot \sin t,\end{aligned}\quad (2.17)$$

we obtain the implicit equation of the limaçon

$$X^2 + Y^2 = v\sqrt{X^2 + Y^2} + a \cdot X, \quad (2.18)$$

in the coordinates  $(X, Y)$  of the adapted Riemannian frame  $e_1, e_2$ . By Okubo's method we obtain the Minkowski metric

$$F(X, Y) = \frac{X^2 + Y^2}{v\sqrt{X^2 + Y^2} + a \cdot X}.$$

We rewrite this in the canonical coordinates  $(\dot{x}, \dot{y})$  of  $\pi_p$ . From (2.15) a computation shows

$$\begin{pmatrix} X \\ Y \end{pmatrix} = -\sqrt{1+f_x^2+f_y^2} \begin{pmatrix} \frac{f_x}{\sqrt{f_x^2+f_y^2}} \dot{x} + \frac{f_y}{\sqrt{f_x^2+f_y^2}} \dot{y} \\ \frac{f_y}{\sqrt{(1+f_x^2+f_y^2)(f_x^2+f_y^2)}} \dot{x} - \frac{f_x}{\sqrt{(1+f_x^2+f_y^2)(f_x^2+f_y^2)}} \dot{y} \end{pmatrix}.$$

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and hence we obtain

$$\begin{aligned} X^2 + Y^2 &= (1 + f_x^2)\dot{x}^2 + (1 + f_y^2)\dot{y}^2 + 2f_x f_y \dot{x}\dot{y} \\ &= \dot{x}^2 + \dot{y}^2 + (f_x \dot{x} + f_y \dot{y})^2. \end{aligned}$$

The limaçon equation (2.18) now reads in canonical coordinates

$$\dot{x}^2 + \dot{y}^2 + (f_x \dot{x} + f_y \dot{y})^2 = v \sqrt{\dot{x}^2 + \dot{y}^2 + (f_x \dot{x} + f_y \dot{y})^2} - a \sqrt{\frac{1 + f_x^2 + f_y^2}{f_x^2 + f_y^2}} (f_x \dot{x} + f_y \dot{y}). \quad (2.19)$$

We recall from the slope principle that  $a = \frac{g}{2} \sin \varepsilon$ , where  $\varepsilon$  is the angle of plane  $\pi_p$  and  $xy$ . Since  $\pi_p$  is smoothly move on  $S$ , the angle  $\varepsilon$  and therefore  $a$  will depend on the coordinates  $(x, y, f(x, y))$  of  $p \in S$ .

From the construction of the Riemannian orthonormal frame  $\{e_1, e_2\}$  on  $S$  it is clear that we can write

$$\langle e_1, (0, 0, -1) \rangle = |e_1| |(0, 0, -1)| \cos\left(\frac{\pi}{2} - \varepsilon\right),$$

that is

$$\sin \varepsilon = \frac{\sqrt{f_x^2 + f_y^2}}{\sqrt{1 + f_x^2 + f_y^2}}.$$

Then

$$a(x, y) = \frac{g}{2} \sin \varepsilon = \frac{g}{2} \frac{\sqrt{f_x^2 + f_y^2}}{\sqrt{1 + f_x^2 + f_y^2}},$$

that is

$$a(x, y) \frac{\sqrt{1 + f_x^2 + f_y^2}}{\sqrt{f_x^2 + f_y^2}} = \frac{g}{2}, \quad \text{i.e. } X = -\frac{g}{2a(x, y)} (f_x \dot{x} + f_y \dot{y}).$$

Using this formula, the indicatrix equation (2.19) reads

$$\dot{x}^2 + \dot{y}^2 + (f_x \dot{x} + f_y \dot{y})^2 = v \cdot \sqrt{\dot{x}^2 + \dot{y}^2 + (f_x \dot{x} + f_y \dot{y})^2} - \frac{g}{2} \cdot (f_x \dot{x} + f_y \dot{y}), \quad (2.20)$$

or, equivalently

$$\alpha^2 = v \cdot \alpha - \frac{g}{2} \cdot \beta,$$

where

$$\begin{cases} \alpha &= \sqrt{(1 + f_x^2)\dot{x}^2 + 2f_x f_y \dot{x}\dot{y} + (1 + f_y^2)\dot{y}^2} \\ \beta &= f_x \dot{x} + f_y \dot{y}. \end{cases} \quad (2.21)$$

Okubo's method gives

$$\begin{aligned} X &= -\frac{g}{2a} \cdot \beta, \\ F(x, y, \dot{x}, \dot{y}) &= \frac{\alpha^2}{v\alpha - \frac{g}{2}\beta}, \end{aligned}$$

where  $v$  is the speed of the hiker on the ground and  $g$  is the gravitational acceleration constant.

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We recall that the limaçon  $r = v + a \cos \theta$  is convex if and only if  $v > 2a$ , that is in our case

$$\frac{1}{2} > \frac{a}{v} = \frac{g}{2v} \frac{\sqrt{f_x^2 + f_y^2}}{\sqrt{1 + f_x^2 + f_y^2}} \Leftrightarrow \frac{v}{g} > \frac{\sqrt{f_x^2 + f_y^2}}{\sqrt{1 + f_x^2 + f_y^2}},$$

i.e.

$$\frac{1 + f_x^2 + f_y^2}{f_x^2 + f_y^2} = 1 + \frac{1}{f_x^2 + f_y^2} > \frac{g^2}{v^2},$$

that is

$$\frac{1}{f_x^2 + f_y^2} > \frac{g^2 - v^2}{v^2}.$$

This is the necessary and sufficient condition for  $F$  to be a positive defined Finsler metric.

**Remark 2.2.** It is customary to normalize the formula for  $F$ . We can further assume  $v = \frac{g}{2}$ , i.e. the speed of the hiker on flat ground is exactly half of gravitational acceleration constant, i.e.  $v = \frac{9.8}{2} = 4.9$ . In this case we obtain the Finsler metric

$$F = \frac{\alpha^2}{\alpha - \beta} \quad \text{with } \alpha \text{ and } \beta \text{ given in (2.21).} \quad (2.22)$$

The strongly convexity condition of the limaçon reads

$$f_x^2 + f_y^2 < \frac{\left(\frac{g}{2}\right)^2}{g^2 - \frac{g^2}{4}} = \frac{1}{3}, \quad \text{i.e. } f_x^2 + f_y^2 < \frac{1}{3}. \quad (2.23)$$

The metric  $\frac{\alpha^2}{\alpha - \beta}$  will be called the **slope metric**. The normalized slope metric  $F$  can be extended eventually to a more general setting by simply by taking  $\alpha$  to be an arbitrary Riemannian metric on an  $n$ -dimensional manifold  $M$  and  $\beta = b_i dx^i$  to be any 1-form on  $M$ . This type of Finsler metric is usually called a **Matsumoto metric**.

## 2.6 The pedal curve of a plane algebraic curve

Let us recall that  $(C)$  be an algebraic curve in plane given by  $f(x, y) = 0$ , if  $f(x, y)$  is a real polynomial in the variables  $x, y$ , i.e.

$$f(x, y) = \sum_{ij} a_{ij} x^i y^j,$$

where the sum is finite and  $a_{ij}$  are real numbers. If we write  $(C)$  in parametric form

$$(C) : \begin{cases} x = x(t) \\ y = y(t), \end{cases} \quad (2.24)$$

then, at regular values of the parameter  $t$ , the tangent line to  $(C)$  is

$$(\ell) : \dot{y}(t) \cdot x - \dot{x}(t) \cdot y + \{\dot{x}(t) \cdot y(t) - x(t) \cdot \dot{y}(t)\} = 0, \quad (2.25)$$

where dots represent the derivative of a function of one variable with respect to  $t$ .

**Definition 2.9.** Let  $(C)$  be a regular curve as in (2.24) and let  $P(x_0, y_0)$  be a fixed point, called the **pedal point**. Then the **pedal curve** of the curve  $(C)$  with respect to  $P$  is the parametrized curve obtained by associating to the parameter  $t$  the orthogonal projection  $p(t)$  of  $P$  onto the tangent line at  $t$  (see Figure 2.6, and [7] for details on algebraic curves).

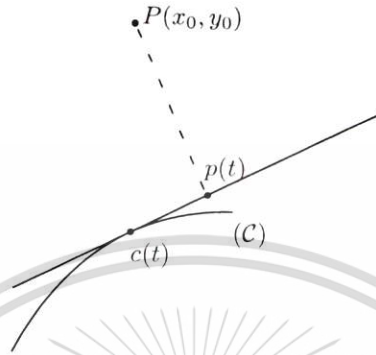


Figure 2.6: The pedal curve of  $(C)$  with respect to  $P(x_0, y_0)$ .

The pedal curve are curves considered important in geometrical optics and kinematics. For the curve  $(C)$  in the parametrization (2.24), the unit tangent vector and the unit normal vectors are

$$\begin{aligned} T(t) &= \frac{dc/dt}{|dc/dt|} = \frac{\langle \dot{x}(t), \dot{y}(t) \rangle}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}, \\ N(t) &= \frac{\langle -\dot{y}(t), \dot{x}(t) \rangle}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}, \end{aligned} \quad (2.26)$$

respectively. Here  $|\cdot|$  is the usual Euclidean norm in  $\mathbb{R}^2$ . It is known that the moving equation of the Frenet frame  $(T(t), N(t))$  along  $(C)$  are given by

$$\begin{cases} \frac{dT}{dt} = |c'(t)| \cdot k_c \cdot N(t) \\ \frac{dN}{dt} = -|c'(t)| \cdot k_c \cdot T(t), \end{cases}$$

where  $|c'(t)| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$  is the speed of  $(C)$  and

$$k_c := \frac{\langle c''(t), N(t) \rangle}{|c'(t)|^2}$$

is the curvature of the curve  $(C)$ . Here  $\langle \cdot, \cdot \rangle$  is the usual inner product of the Euclidean plane. The tangent line  $(\ell)$  to  $(C)$  at a point  $(x(t), y(t)) \in C$  is given by (2.25) and the orthogonal line on  $(\ell)$  through a point  $P(x_0, y_0)$  is given by

$$(\ell)^\perp : \quad y - y_0 = -\frac{\dot{x}(t)}{\dot{y}(t)}(x - x_0).$$

It follows

**Theorem 2.3.** [7] Let  $(C)$  be an algebraic plane curve in parametric form (2.24) and  $P(x_0, y_0)$  an arbitrary point in  $\mathbb{R}^2$ ,  $P \notin (C)$ . Then the pedal curve of  $(C)$  with respect to the point  $P$  is given by

$$p(t) = \langle c, N \rangle \cdot N + \langle r_0, T \rangle \cdot T, \tag{2.27}$$

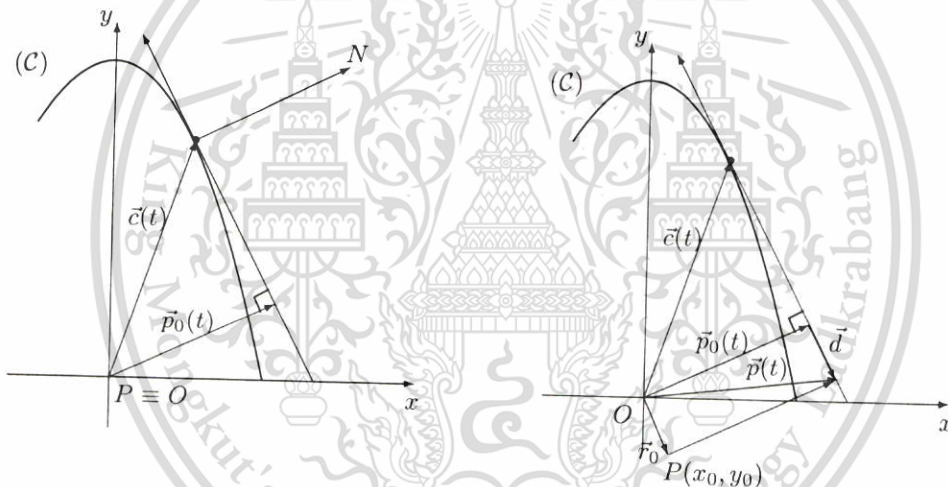
where we denote by  $r_0$  the position vectors of the point  $P(x_0, y_0)$ .

**Proof.** Recall that if  $\vec{a}$  and  $\vec{b}$  are two vector in  $\mathbb{R}^2$  with the some origin  $O$ , then the projection of  $\vec{b}$  on  $\vec{a}$  is given by

$$\vec{b}_1 = pr_{\vec{a}}\vec{b} = \langle \vec{b}, \frac{1}{\|\vec{a}\|} \cdot \vec{a} \rangle \cdot \vec{a}. \tag{2.28}$$

We consider first the special case where  $P \equiv O$  is the origin of  $\mathbb{R}^2$ . In this case, it is trivial to observe that the position vector of the pedal curve point  $\vec{p}_o(t)$  is just the projection of the vector  $\vec{c}(t)$  on the normal direction of  $\vec{N}$ , hence we see that

$$p_o = \langle c, N \rangle \cdot N. \tag{2.29}$$



**Figure 2.7:** The pedal curve of a plane curve with pedal point at the origin (left) and at an arbitrary point  $P(x_0, y_0)$  (right).

Let us consider now the general case when  $P$  is not origin, and  $P$  is not on the curve  $(C)$  either. Observe that in this case

$$\vec{p}(t) = \vec{p}_o(t) + \vec{d}, \tag{2.30}$$

where  $\vec{d}$  is the vector with origin in the foot of the projection of origin on the tangent line, and tip at the point on the pedal curve. We can see that  $\vec{d}$  is actually the projection of  $\vec{r}_0$  on the tangent direction, i.e.

$$\vec{d} = \langle \vec{r}_0, T \rangle \cdot T, \tag{2.31}$$

and hence using (2.28), (2.29) in (2.30) we obtain the desired conclusion.

□

**Remark 2.4.** Observe that  $\vec{r}_0$  can be written as the sum of projection on  $T$  and projection on  $N$ , i.e.

$$\vec{r}_0 = \langle r_0, N \rangle \cdot N + \langle r_0, T \rangle \cdot T.$$

Hence formula (2.27) can be written as

$$\begin{aligned} p(t) &= \langle c, N \rangle + r_0 - \langle r_0, N \rangle \cdot N \\ &= \langle c - r_0, N \rangle \cdot N + r_0. \end{aligned}$$

## 2.7 The pedal surface

We are going to extend our considerations from curves of surfaces. Instead to the curve ( $\mathcal{C}$ ), we are going to consider a smooth surface  $\mathcal{S} \hookrightarrow \mathbb{R}^3$  embedded in  $\mathbb{R}^3$  with parametric equations

$$(\mathcal{S}) : \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v). \end{cases} \quad (2.32)$$

Observe that, at any regular vector  $(u, v)$  of the parameters, the tangent plane of  $\mathcal{S}$  at  $(x(u, v), y(u, v), z(u, v)) \in \mathcal{S}$  is

$$(\pi) : \frac{\partial(y, z)}{\partial(u, v)}(x - x(u, v)) + \frac{\partial(z, x)}{\partial(u, v)}(y - y(u, v)) + \frac{\partial(x, y)}{\partial(u, v)}(z - z(u, v)) = 0,$$

where

$$\frac{\partial(y, z)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix}$$

and so on. The normal vector to  $(\pi)$  at a point  $(u, v)$  is given by

$$(\pi^\perp) : \frac{x}{\frac{\partial(y, z)}{\partial(u, v)}} = \frac{y}{\frac{\partial(z, x)}{\partial(u, v)}} = \frac{z}{\frac{\partial(x, y)}{\partial(u, v)}}.$$

Let  $(\mathcal{S})$  be a regular surface parametrized on in (2.32) and let  $P(x_0, y_0, z_0)$  be a fixed point, the **pedal point**. Then the **pedal surface** of the surface  $(\mathcal{S})$  with respect to the point  $P$  is the parametrized surface obtained by associating to the parameter  $(u, v)$  the orthogonal projection  $p(u, v)$  of  $P$  onto the tangent plane  $(\pi)$  at  $\mathcal{S}(u, v)$ , see Figure 2.8.

The tangent plane  $(\pi)$  is generated by the vectors

$$\mathcal{S}_u = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \quad \text{and} \quad \mathcal{S}_v = \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix},$$

while the unit normal vector to  $(\mathcal{S})$  is

$$N = \frac{\mathcal{S}_u \times \mathcal{S}_v}{\|\mathcal{S}_u \times \mathcal{S}_v\|} = \frac{1}{\|\mathcal{S}_u \times \mathcal{S}_v\|} \left( \frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)} \right)^t.$$

A straightforward computation gives

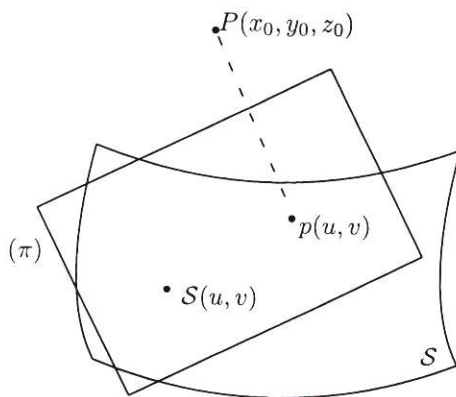


Figure 2.8: The pedal surface of (S) with respect to  $P(x_0, y_0, z_0)$ .

**Theorem 2.5.** [7] Let  $(\sigma)$  be a smooth surface embedded in  $\mathbb{R}^3$  and  $P(x_0, y_0, z_0)$  an arbitrary point in  $\mathbb{R}^3$ ,  $P \notin \sigma$ . The pedal surface of the surface (S) with respect to the point P is given by

$$p(u, v) = \langle S - r_0, N \rangle \cdot N + r_0. \tag{2.33}$$

**Proof.** The proof is similar to the proof of Theorem 2.3. Indeed, consider first the case when  $P \equiv O$  is the origin of  $\mathbb{R}^3$ .

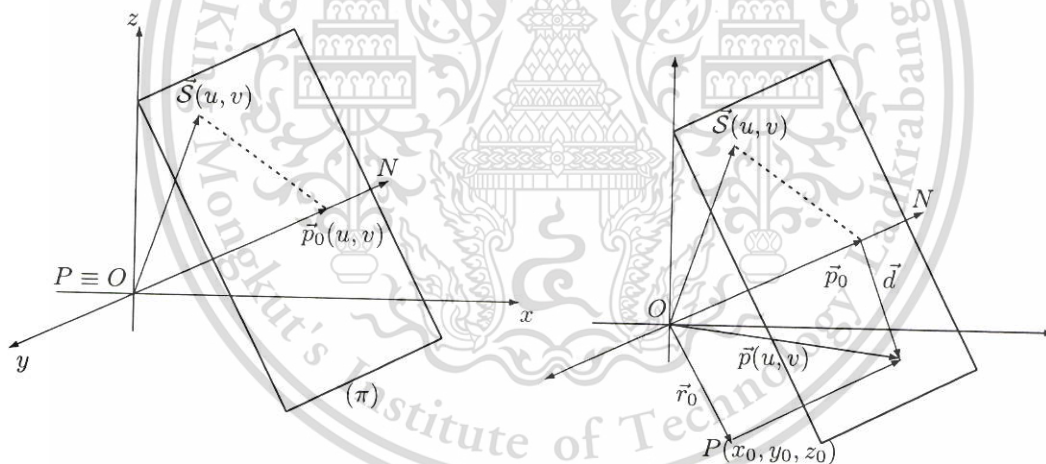


Figure 2.9: The pedal surface of a surface with pedal point at the origin (left) and at an arbitrary point  $P(x_0, y_0, z_0)$  (right).

Since  $\vec{p}_0$  is in the normal direction, it is perpendicular on  $(\pi)$  and hence on any line in plane. It follows that  $\vec{p}_0(u, v)$  is given just the projection of  $\vec{v}_0$  on the normal direction i.e.

$$\vec{p}_0(u, v) = \langle S(u, v), N(u, v) \rangle \cdot N(u, v).$$

We consider the general case when  $P(x_0, y_0, z_0)$  is not origin, and not on the surface (S). Observe again that

$$\vec{p}(u, v) = \vec{p}_0 + \vec{d}(u, v),$$

where  $\vec{d}$  is the same in proof of theorem 2.3, In this case, we can see that  $\vec{d}$  is the projection of  $\vec{r}_0$  on  $(\pi)$ . If we decompose  $\vec{r}_0$  by the component  $\vec{d} \subset \pi$  and its component in the normal direction, it is clear that the latter is given by the projection of  $\vec{r}$  on the normal direction, i.e.

$$\langle \vec{r}, N \rangle \cdot N,$$

hence

$$\vec{d} = \vec{r}_0 - \langle \vec{r}_0, N \rangle \cdot N.$$

Finally, we get

$$\vec{p} = \vec{p}_0 + \vec{d} = \langle \mathcal{S}, N \rangle \cdot N + r_0 - \langle r_0, N \rangle, \text{ that is the fomula in the theorem.}$$

□

Observe that

$$p_u = \langle \mathcal{S} - r_0, N_u \rangle \cdot N + \langle \mathcal{S} - r_0, N \rangle \cdot N_u,$$

$$p_v = \langle \mathcal{S} - r_0, N_v \rangle \cdot N + \langle \mathcal{S} - r_0, N \rangle \cdot N_v.$$

By denoting  $\Phi = \mathcal{S} - r_0$ , hence the first fundamental form of the pedal surface of  $(\mathcal{S})$  reads

$$(I) = \begin{vmatrix} E & F \\ F & G \end{vmatrix} = \begin{vmatrix} \langle p_u, p_u \rangle & \langle p_u, p_v \rangle \\ \langle p_u, p_v \rangle & \langle p_v, p_v \rangle \end{vmatrix} \\ = \langle \Phi, N \rangle^2 \left\{ \begin{vmatrix} \langle \Phi, N_v \rangle & |N_u|^2 \\ \langle \Phi, N_u \rangle & \langle \Phi, N_v \rangle \end{vmatrix} + \langle \Phi, N_u \rangle \begin{vmatrix} \langle \Phi, N_u \rangle & \langle \Phi, N_v \rangle \\ \langle N_u, N_v \rangle & |N_v|^2 \end{vmatrix} \right. \\ \left. + \langle \Phi, N_v \rangle \begin{vmatrix} |N_u|^2 & \langle N_u, N_v \rangle \\ \langle N_u, N_v \rangle & |N_v|^2 \end{vmatrix} \right\}.$$

Since

$$\begin{vmatrix} |N_u|^2 & \langle N_u, N_v \rangle \\ \langle N_u, N_v \rangle & |N_v|^2 \end{vmatrix} = |N_v|^2 |N_u|^2 - |N_v|^2 |N_u|^2 \cos^2 \theta_{N_u, N_v} > 0$$

and

$$\begin{aligned} & \langle \Phi, N_v \rangle \begin{vmatrix} |N_u|^2 & \langle N_u, N_v \rangle \\ \langle \Phi, N_u \rangle & \langle \Phi, N_v \rangle \end{vmatrix} + \langle \Phi, N_u \rangle \begin{vmatrix} \langle \Phi, N_u \rangle & \langle \Phi, N_v \rangle \\ \langle N_u, N_v \rangle & |N_v|^2 \end{vmatrix} \\ & = \langle \langle \Phi, N_v \rangle N_u - \langle \Phi, N_u \rangle N_v, \langle \Phi, N_v \rangle N_u - \langle \Phi, N_u \rangle N_v \rangle \\ & = |\langle \Phi, N_v \rangle N_u - \langle \Phi, N_u \rangle N_v|^2 \\ & > 0, \end{aligned}$$

it results that the first fundamental form is positive. Likewise,

$$P_{uu}(u, v) = \langle \mathcal{S}_u, N_u \rangle N + \langle \Phi, N_{uu} \rangle N + \langle \Phi, N_u \rangle N_u + \langle \Phi, N_u \rangle N_u + \langle \Phi, N \rangle N_{uu},$$

$$P_{uv}(u, v) = \langle \mathcal{S}_v, N_u \rangle N + \langle \Phi, N_{uv} \rangle N + \langle \Phi, N_u \rangle N_v + \langle \Phi, N_v \rangle N_u + \langle \Phi, N \rangle N_{uv},$$

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$$P_{vv}(u, v) = \langle S_v, N_v \rangle N + \langle \Phi, N_{vv} \rangle N + \langle \Phi, N_v \rangle N_v + \langle \Phi, N_v \rangle N_v + \langle \Phi, N \rangle N_{vv}.$$

Hence to the second fundamental form of the pedal surface of  $(\sigma)$  is

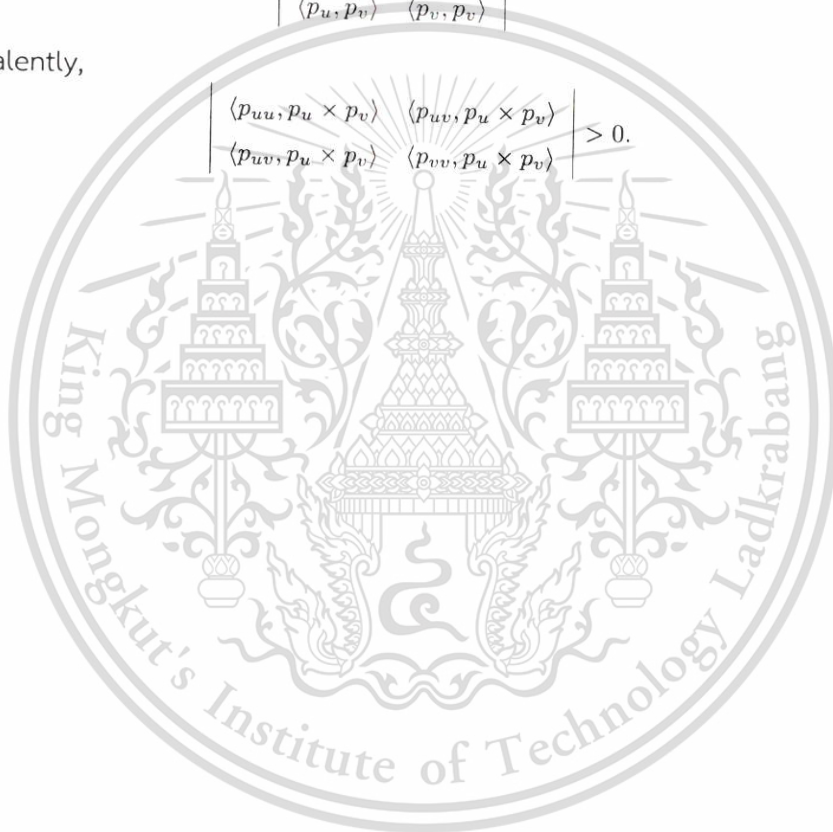
$$\begin{vmatrix} L & M \\ M & N \end{vmatrix} = \begin{vmatrix} \langle p_{uu}, V \rangle & \langle p_{uv}, V \rangle \\ \langle p_{uv}, V \rangle & \langle p_{vv}, V \rangle \end{vmatrix},$$

where  $V := \frac{p_u \times p_v}{\|p_u \times p_v\|}$ . This determinant can be explicitly written down, but the formulas are too long to be written here. The convexity condition of the pedal surface is

$$\frac{\begin{vmatrix} \langle p_{uu}, V \rangle & \langle p_{uv}, V \rangle \\ \langle p_{uv}, V \rangle & \langle p_{vv}, V \rangle \end{vmatrix}}{\begin{vmatrix} \langle p_u, p_u \rangle & \langle p_u, p_v \rangle \\ \langle p_u, p_v \rangle & \langle p_v, p_v \rangle \end{vmatrix}} > 0,$$

or equivalently,

$$\begin{vmatrix} \langle p_{uu}, p_u \times p_v \rangle & \langle p_{uv}, p_u \times p_v \rangle \\ \langle p_{uv}, p_u \times p_v \rangle & \langle p_{vv}, p_u \times p_v \rangle \end{vmatrix} > 0. \tag{2.34}$$



## Chapter 3

### The slope metric on a surface of revolution

In section 2.5 we introduce the slope metric for a general case. In the same way, by similar computation with the general construction of the slope metric show that we can define the slope metric for surface of revolution. Let us consider the surface of revolution  $M$  with the parametrization (2.1) and induced Riemannian metric (2.2). Indeed surface of revolution is parametrized by

$$(u, v) \mapsto (m(u) \cos v, m(u) \sin v, u).$$

The tangent plane  $T_p M$  at point  $p \in S$  is spanned by the vector fields

$$\frac{\partial}{\partial u} := (m'(u) \cos v, m'(u) \sin v, 1), \quad \frac{\partial}{\partial v} := (-m(u) \sin v, m(u) \cos v, 0),$$

with the normal vector as a point  $p$

$$\vec{n}_p = (-m(u) \cos v, -m(u) \sin v, m(u)m'(u) \cos^2 v + m(u)m'(u) \sin^2 v).$$

Following again Matsumoto's slope principle, same with the general case, the gravitational  $\vec{F}$  acts perpendicular on  $xy$ -plane and it is given by

$$e_3 = (0, 0, -1).$$

Then the projection of  $e_3$  on  $T_p M$  is given by

$$u_p = \vec{n}_p \times (e_3 \times \vec{n}_p) = (-m^2 m' \cos v, -m^2 m' \sin v, -m^2) = -m^2 \frac{\partial}{\partial u}.$$

Observe that the orthonormal frame in  $T_p M$  at a given  $p$  is

$$e_1 = -\frac{1}{\sqrt{(m')^2 + 1}} \cdot \frac{\partial}{\partial u}, \quad e_2 = \frac{1}{m} \cdot \frac{\partial}{\partial v}$$

and here, the relation between the coordinates  $(X, Y)$  of  $T_p M$  with respect to  $\{e_1, e_2\}$  and the canonical coordinates  $(\dot{u}, \dot{v})$  is

$$X = -\sqrt{1 + (m')^2} \cdot \dot{u}, \quad Y = m \cdot \dot{v}.$$

The limaçon implicit equation (2.20) reads now

$$[1 + (m')^2] \dot{u}^2 + m^2 \cdot \dot{v}^2 = c \sqrt{[1 + (m')^2] \dot{u}^2 + m^2 \dot{v}^2} - a \sqrt{1 + (m')^2} \cdot \dot{u}.$$

We obtain the slope metric in the form (2.22) with

$$\alpha = \sqrt{[1 + (m')^2] \dot{u}^2 + m^2 \dot{v}^2}, \quad \beta = \dot{u}. \tag{3.1}$$

Moreover, the parametrization (2.13) of an arbitrary surface can be adapted to the case of surfaces of revolution by putting

$$S \rightarrow \mathbb{R}^3, \quad (x, y) \mapsto (x, y, z = \phi(\sqrt{x^2 + y^2})). \tag{3.2}$$

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The induced Riemannian metric is

$$\begin{aligned} a_{ij}(x, y) &:= \begin{pmatrix} 1 + \phi_x^2 & \phi_x \phi_y \\ \phi_x \phi_y & 1 + \phi_y^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{\phi'(\sqrt{x^2+y^2})^2 x^2}{x^2+y^2} & \frac{\phi'(\sqrt{x^2+y^2})^2 xy}{x^2+y^2} \\ \frac{\phi'(\sqrt{x^2+y^2})^2 xy}{x^2+y^2} & 1 + \frac{\phi'(\sqrt{x^2+y^2})^2 y^2}{x^2+y^2} \end{pmatrix}, \end{aligned}$$

provided  $x^2 + y^2 \neq 0$ , where

$$\phi_x = \frac{\phi'(\sqrt{x^2+y^2})x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \phi_y = \frac{\phi'(\sqrt{x^2+y^2})y}{\sqrt{x^2+y^2}},$$

and  $\beta = \phi_x \dot{x} + \phi_y \dot{y}$ . Thus we can define the slope metric  $F$  for surface of revolution in term of function  $\phi$ .

### 3.1 Main Result

**Theorem 3.1.** Let  $S \rightarrow \mathbb{R}^3$  be a surface of revolution. Then the following statements are equivalent:

- (i)  $S$  admits a strongly convex slope metric;
- (ii)  $[\phi'(s)]^2 < \frac{1}{3}$  where  $S \rightarrow \mathbb{R}^3 : (x, y) \mapsto (x, y, z = \phi(s))$  and  $s = \sqrt{x^2 + y^2}$ ;
- (iii)  $[m'(u)]^2 > 3$  where  $S \rightarrow \mathbb{R}^3 : (u, v) \mapsto (m(u) \cos v, m(u) \sin v, u)$ .

**Proof.** First, we shall prove (i)  $\Leftrightarrow$  (ii). Note that for the surface of revolution  $S : z = \phi(\sqrt{x^2 + y^2})$ , we have

$$\begin{aligned} (\phi_x)^2 + (\phi_y)^2 &= \left( \phi'(\sqrt{x^2+y^2}) \frac{x}{\sqrt{x^2+y^2}} \right)^2 + \left( \phi'(\sqrt{x^2+y^2}) \frac{y}{\sqrt{x^2+y^2}} \right)^2 \\ &= \left( \phi'(\sqrt{x^2+y^2}) \right)^2. \end{aligned}$$

Recall from 2.23 that an equivalent condition for  $S$  to admit a strongly convex slope metric is the inequality

$$(\phi_x)^2 + (\phi_y)^2 < \frac{1}{3}.$$

Thus the assertions (i) and (ii) are equivalent.

Next, we shall show the equivalence between (ii) and (iii). Indeed, we can parametrize the surface of revolution by using trigonometric functions, that is

$$S \rightarrow \mathbb{R}^3, \quad (x = m(u) \cdot \cos v, y = m(u) \cdot \sin v, z = u). \quad (3.3)$$

This surface of revolution is obtained by rotating the curve  $x = m(z)$  along the  $z$  axis. Applying the parametrization (3.2) to (3.3) yields

$$x^2 + y^2 = m^2(u) \cos^2 v + m^2(u) \sin^2 v = m^2(u),$$

and hence

$$z = \phi(\sqrt{x^2 + y^2}) = \phi(|m(u)|).$$

For the sake of simplicity we consider here only the case  $+m(u)$ . Since  $z = u$  it follows that  $\phi(m(u)) = u$ . Substituting  $s = \sqrt{x^2 + y^2}$  yields  $s = m(u)$  and thus

$$m(\phi(s)) = m(\phi(m(u))) = m(u) = s.$$

Hence,  $\phi$  and  $m$  are inverse function of each other. From the inverse function theorem, we have

$$m'(u) = \frac{1}{\phi'(s)} \Big|_{s=m(u)}$$

or, equivalently

$$[m^{-1}]'(s) = \frac{1}{m'(m^{-1}(s))} \Leftrightarrow \phi'(s) = \frac{1}{m'(u)} \Big|_{u=\phi(s)}$$

and from here it is clear that

$$[m'(u)]^2 = \left[ \frac{1}{\phi'(s)} \right]^2.$$

The condition  $[\phi'(s)]' < \frac{1}{3}$  is now equivalent to  $[m'(u)]^2 > 3$ .

□

## 3.2 Examples

### 3.2.1 Two-sheeted hyperboloid

Two-sheeted hyperboloid of revolution  $\phi(s)$  can be defined by

$$\phi(s) = a\sqrt{s^2 + b}, \quad a > 0, \quad b \geq 0.$$

we can see that

$$\phi'(t) = a \frac{t}{\sqrt{s^2 + b}}.$$

Since  $s < \sqrt{s^2 + b}$  and  $-t < \sqrt{s^2 + b}$  or  $t > -\sqrt{s^2 + b}$ , it follows that

$$-a < a \frac{s}{\sqrt{s^2 + b}} < a.$$

Therefore this surface admit strongly convex slope metric for  $a \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ .

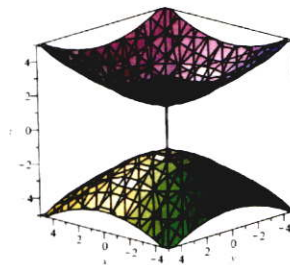


Figure 3.1: Hyperboloid of two sheeted

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### 3.2.2 Cone

Cone  $M(\phi)$  can be defined by

$$\phi(s) = as, \quad a > 0.$$

Since  $\phi'(s) = a$ , by Theorem 3.1 its slope metric is strongly convex if and only if  $a \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ .

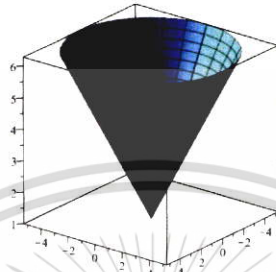


Figure 3.2: Cone

### 3.2.3 Ellipsoid

Ellipsoid can be defined by

$$\phi(s) = \frac{c}{a} \sqrt{a^2 - s^2}, \quad a > 0, \quad c > 0.$$

Then  $\phi'(s) = \frac{c}{a} \frac{-s}{\sqrt{a^2 - s^2}}$ . Therefore this surface admit strongly convex slope metric on the interval  $s \in (0, \frac{a^2}{\sqrt{a^2 + 3c^2}})$ .

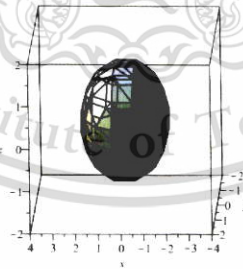


Figure 3.3: Ellipsoid

### 3.2.4 One-sheeted hyperboloid

One-sheeted hyperboloid can be defined by

$$\phi(s) = a\sqrt{s^2 - b}$$

Putting  $\phi(s) = u$ , we have

$$m(u) = \pm \sqrt{\frac{u^2}{a^2} + b}$$

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It follows that

$$m'(u)^2 = \frac{1}{a^2} \left[ 1 - \frac{b}{\left(\frac{u^2}{a^2} + b\right)} \right]$$

and thus the strongly convex condition for this surface is  $a \in (-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$ .

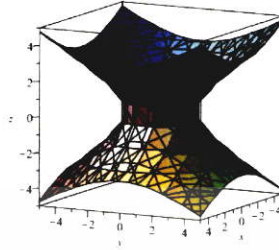


Figure 3.4: Hyperboloid of one sheeted

### 3.2.5 The elliptic paraboloid

Consider the elliptic paraboloid  $S \rightarrow \mathbb{R}^3$ ,  $(x, y) \mapsto (x, y, z = f(x, y) = 100 - x^2 - y^2)$ . In this parametrization, the convexity condition is  $f_x^2 + f_y^2 = 4(x^2 + y^2) < \frac{1}{3}$ , i.e.  $x^2 + y^2 < \frac{1}{12}$ , hence the strongly convexity condition is satisfied only on a circular vicinity of height  $\frac{1}{12}$  units of the hilltop.

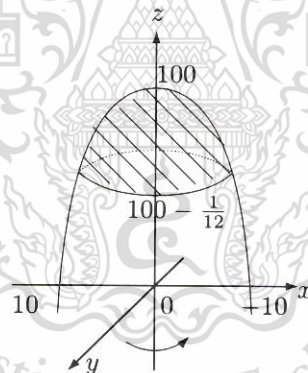


Figure 3.5: The elliptic paraboloid (a)

If we consider the parametrization (3.2) then we get  $(x, y) \mapsto (x, y, z = \phi(\sqrt{x^2 + y^2}))$ , where  $\phi(s) = 100 - s^2$ . The strongly convexity condition  $[\phi'(s)]^2 < \frac{1}{3}$  is equivalent to  $s^2 < \frac{1}{12}$ , that is  $-\frac{1}{\sqrt{12}} < s < \frac{1}{\sqrt{12}}$  and hence we obtain the same result as above.

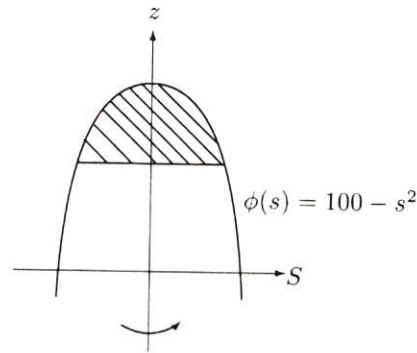


Figure 3.6: The elliptic paraboloid (b)

Finally, in the trigonometric parametrization we have

$$s \in \mathbb{R}^3, \quad (u, v) \mapsto (x = m(u) \cos v, y = m(u) \sin v, z = u) \quad \text{for} \quad u \in (-\infty, a], \quad v \in [0, 2\pi)$$

in this case  $a = 100$ .

This surface of revolution is obtained by rotating the curve  $x = m(z)$  around the  $z$  axis. We compute the inverse of  $\phi(s) = 100 - s^2 = u \Leftrightarrow s = \pm\sqrt{100 - u}$ , with  $u \in (-\infty, 100]$ . For simplicity we consider only  $s = \sqrt{100 - u}$ ,

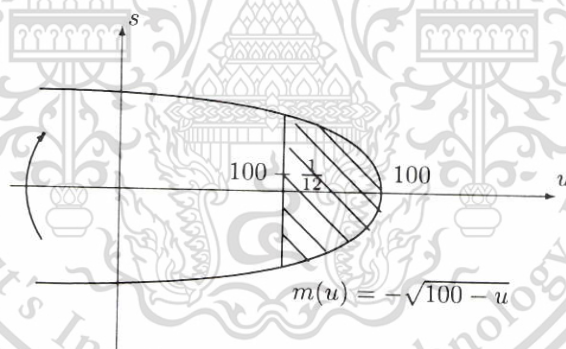


Figure 3.7: The elliptic paraboloid (c)

here

$$m'(u) = (\sqrt{100 - u})' = -\frac{1}{2\sqrt{100 - u}} \Rightarrow [m'(u)]^2 = \frac{1}{4(100 - u)} > 3,$$

that is

$$100 - \frac{1}{12} < u \leq 100,$$

i.e. the convexity domain is the same as before.

### 3.2.6 The Gauss function

Let us consider the surface of revolution

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$$(x, y) \rightarrow \left(x, y, z = f(x, y) = \frac{1}{2\sqrt{6}}e^{-x^2-y^2}\right)$$

or

$$(x, y) \rightarrow (x, y, z = \phi(\sqrt{x^2 + y^2})), \quad \phi(s) = \frac{1}{2\sqrt{6}}e^{-s^2}.$$

We have  $\phi'(s)^2 = \frac{1}{3}e^{-s^2}(s^2)$ . Since  $e^{s^2} = 1 + s^2 + \frac{s^4}{2!} + \dots > s$  thus  $1 > se^{-s^2}$  therefore the slope metric is globally strong convex on this surface.

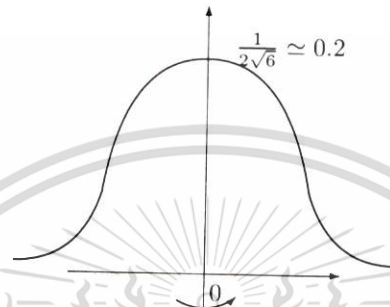


Figure 3.8: The Gauss function (a)

Observe that  $\phi(s)$  is bijective for  $s \in (0, \infty)$ , and by putting  $\phi(s) = u$  we solve this equation and get

$$m(u) = \pm \frac{\sqrt{-2 \log(24u^2)}}{2}, \quad \text{for } u \in \left(0, \frac{1}{2\sqrt{6}}\right),$$

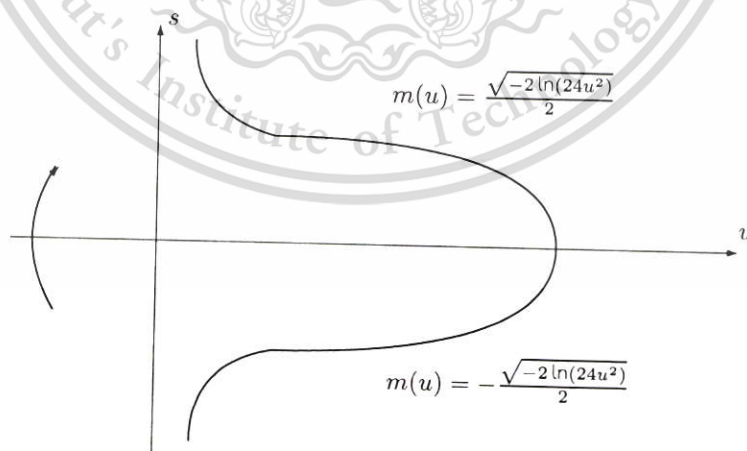


Figure 3.9: The Gauss function (b)

and compute

$$m'(u) = -\frac{1}{u\sqrt{-2\log(24u^2)}}.$$

We denote  $\mu(u) = [m'(u)]^2$ , for  $u \in (0, \frac{1}{2\sqrt{6}})$  and compute the min of  $\mu(u)$ .

The solution of  $\mu'(u) = 0$  are  $\pm \frac{1}{2\sqrt{6}e^{-\frac{1}{2}}}$  and observe that only  $\frac{\sqrt{6}}{12} \cdot e^{-\frac{1}{2}} \simeq 0.12 < 0.2$  is in the interval  $(0, \frac{1}{2\sqrt{6}})$ . Same computation shows that

$$\mu\left(\frac{1}{2\sqrt{6}}e^{-\frac{1}{2}}\right) \approx 32.6 > 3,$$

i.e. the convexity condition is globally verified.



## Chapter 4

### Pedals and Finsler metrics

#### 4.1 Some properties of the pedal on a plane curve

**Proposition 4.1.** If  $(\mathcal{C})$  is a continuously differentiable closed curve in plane, then its pedal curve  $(\mathcal{P}) : p = p(t)$  with respect to a point  $P(x_0, y_0)$  is also a continuously differentiable closed curve in plane.

**Proof.** From hypothesis, after some rescaling of the parameter  $t$ , we have

$$c(0) = c(2\pi), \quad \dot{c}(0) = \dot{c}(2\pi),$$

and hence

$$T(0) = T(2\pi), \quad N(0) = N(2\pi).$$

Using now (2.27) it follows

$$p(0) = p(2\pi),$$

i.e.  $p$  is also periodic with the same period as  $(\mathcal{C})$ . Moreover,  $\dot{p}(0) = \dot{p}(2\pi)$ , where by derivation of (2.27) we get

$$\dot{p}(t) = k_c \cdot |c'| \cdot [\langle r_0 - c, T \rangle \cdot N + \langle r_0 - c, N \rangle \cdot T]. \quad (4.1)$$

□

We will ask now the question if the pedal curve  $p(t)$  goes through the origin  $O(0, 0)$  of  $\mathbb{R}^2$ . This is equivalent to asking if the vectorial equation

$$p(t) = \langle c, N \rangle \cdot N + \langle r_0, T \rangle \cdot T = (0, 0)^t$$

has a solution. Since  $N$  and  $T$  are linearly independent, this equation is equivalent to the system of equations,

$$\begin{aligned} \langle c, N \rangle &= 0, \\ \langle r_0, N \rangle &= 0. \end{aligned} \quad (4.2)$$

From here we have the following conclusion:

**Lemma 4.2.** Let the pedal point  $P$  be the origin. Then  $(\mathcal{P})$  passes through origin if and only if  $(\mathcal{C})$  also passes through origin.

**Proof.** Assume that  $(\mathcal{P})$  passes the origin. Then there is a  $t_0 \in \mathbb{R}$  such that  $p(t_0) = (0, 0)$ . By (2.27) we get

$$\langle c(t_0), N \rangle = - \begin{vmatrix} c_1 & c_2 \\ \dot{c}_1 & \dot{c}_2 \end{vmatrix} = 0.$$

Conversely, assume that  $(\mathcal{C})$  passes through origin, i.e.,  $c(t_0) = (0, 0)$  for some  $t_0 \in \mathbb{R}$ . From (2.27), we get  $p(t_0) = (0, 0)$ . That is,  $c(t_0) = (0, 0)$ , or  $(\mathcal{C})$  passes through origin. □

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**Lemma 4.3.** Let the pedal point  $P$  be a point in  $\mathbb{R}^2$  which is not the origin. Suppose  $(\mathcal{C})$  passes through the origin, i.e, there is a  $t_0 \in \mathbb{R}$  such that  $c(t_0) = (0,0)$ . Then  $(\mathcal{P})$  passes through origin if and only if  $r_0$  and  $T(t_0)$  are orthogonal.

**Proof.** Let the pedal point  $P$  be not the origin, i.e.  $x_0 \neq 0$  or  $y_0 \neq 0$ . The curve  $(\mathcal{C})$  passes through origin, i.e. there exists  $t_0$  to such that  $c(t_0) = (0,0)$ . We have to show that  $p(t_0) = 0$  if and only if  $\langle r_0, T(t_0) \rangle = 0$ . Note that

$$\begin{aligned} p(t_0) &= \langle c(t_0), N(t_0) \rangle N(t_0) + \langle r_0, T(t_0) \rangle T(t_0) \\ &= \langle r_0, T(t_0) \rangle T(t_0). \end{aligned}$$

Since  $T(t_0) \neq (0,0)$ , we have  $p(t_0) = 0$  if and only if  $\langle r_0, T(t_0) \rangle = 0$ .  $\square$

**Lemma 4.4.** If  $P$  is not the origin and  $(\mathcal{C})$  does not pass through origin, then  $(\mathcal{P})$  does not pass origin either.

**Proof.** Suppose  $(\mathcal{P})$  passes origin. Then there exists a  $t_0 \in \mathbb{R}$  such that  $p(t_0) = 0$ . Then (2.27) reads

$$\langle c, N \rangle N + \langle r_0, T \rangle T = (0,0)$$

Since  $N$  and  $T$  are linearly independent, this equation is equivalent to the system of equation,

$$\langle c, N \rangle = 0, \quad \langle r_0, T \rangle = 0.$$

This is impossible, since  $(\mathcal{C})$  is smooth curve,  $c(t)$  cannot be parallel to  $T(t)$ .  $\square$

**Theorem 4.5.** The pedal curve  $(\mathcal{P})$  is strongly convex if and only if

$$\frac{k_c \{-2k_c \langle r_0 - c, T \rangle^2 + \langle r_0 - c, N \rangle - 2k_c \langle r_0 - c, N \rangle^2\}}{\langle c, N \rangle \langle r_0, N \rangle + \langle c, T \rangle \langle r_0, T \rangle - \langle c, N \rangle^2 + \langle r_0, T \rangle^2} > 0 \quad (4.3)$$

for all  $t \in [0, 2\pi)$ .

**Proof.** By formula (2.12) we know that a curve  $p = p(t)$  is strongly convex if and only if

$$\frac{\dot{p}(t) \times \ddot{p}(t)}{p(t) \times \dot{p}(t)} > 0. \quad (4.4)$$

Here the cross product of two vectors  $u = (a, b)$  and  $v = (c, d)$  in  $\mathbb{R}^2$  is given by

$$u \times v = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Observe that the strongly convex condition is independent on the parametrization of  $p(t)$ . Recall from Theorem 2.5 we have

$$\begin{aligned} p(t) &= \langle c, N \rangle \cdot N + \langle r_0, T \rangle \cdot T \\ \dot{p}(t) &= -k_c \cdot |\dot{c}|^2 \{ \langle c, T \rangle \cdot N + \langle c, N \rangle \cdot T \} \\ \ddot{p}(t) &= [(k_c |\dot{c}|)'] \langle r_0 - c, T \rangle + k_c |\dot{c}|^2 [-1 + 2k_c \langle r_0 - c, N \rangle] \cdot N \\ &\quad + [(k_c |\dot{c}|)'] \langle r_0 - c, N \rangle - 2k_c^2 |\dot{c}|^2 \cdot \langle r_0 - c, T \rangle \cdot T. \end{aligned}$$

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For convenience, we denote

$$\begin{aligned}
 v_0 &= \langle c, N \rangle, \\
 u_0 &= \langle r_0, T \rangle, \\
 v_1 &= -k_c \cdot |\dot{c}|^2 \langle c, T \rangle, \\
 u_1 &= -k_c \cdot |\dot{c}|^2 \langle c, N \rangle, \\
 v_2 &= [(k_c |\dot{c}|)'] \langle r_0 - c, T \rangle + k_c |\dot{c}|^2 [-1 + 2k_c \langle r_0 - c, N \rangle], \\
 u_2 &= [(k_c |\dot{c}|)'] \langle r_0 - c, N \rangle - 2k_c^2 |\dot{c}|^2 \cdot \langle r_0 - c, T \rangle.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \frac{\dot{p} \times \ddot{p}}{p \times \dot{p}} &= \frac{\begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix}}{\begin{vmatrix} v_0 & v_1 \\ u_0 & u_1 \end{vmatrix}} \\
 &= \frac{k_c |\dot{c}|^2 \begin{vmatrix} \langle r_0 - c, T \rangle & -1 + 2k_c \langle r_0 - c, N \rangle \\ \langle r_0 - c, N \rangle & -2k_c \langle r_0 - c, T \rangle \end{vmatrix}}{\begin{vmatrix} \langle c, N \rangle & \langle r_0 - c, N \rangle \\ \langle r_0, T \rangle & \langle r_0 - c, T \rangle \end{vmatrix}} \\
 &= \frac{k_c |\dot{c}|^2 [-2k_c \langle r_0 - c, T \rangle^2 + \langle r_0 - c, N \rangle - 2k_c \langle r_0 - c, N \rangle^2]}{\langle c, N \rangle \langle r_0 - c, N \rangle - \langle r_0, T \rangle \langle r_0 - c, T \rangle}.
 \end{aligned}$$

Therefore, the pedal curve ( $\mathcal{P}$ ) is strongly convex if and only if

$$\frac{k_c \{-2k_c \langle r_0 - c, T \rangle^2 + \langle r_0 - c, N \rangle - 2k_c \langle r_0 - c, N \rangle^2\}}{\langle c, N \rangle \langle r_0, N \rangle + \langle c, T \rangle \langle r_0, T \rangle - \langle c, N \rangle^2 - \langle r_0, T \rangle^2} > 0$$

for all  $t \in [0, 2\pi)$ . □

**Remark 4.6.** In the case when the pedal point  $P$  is the origin, formula (4.3) simplifies to

$$k_c [2k_c \langle c, T \rangle^2 + \langle c, N \rangle + 2k_c \langle c, N \rangle^2] > 0. \quad (4.5)$$

Moreover, observe that the position vector  $c(t)$  can be decomposed in the orthonormal basis  $\{T, N\}$  as

$$c(t) = \langle c(t), T(t) \rangle T(t) + \langle c(t), N(t) \rangle N(t),$$

hence

$$\langle c(t), c(t) \rangle = \langle c(t), T(t) \rangle^2 + \langle c(t), N(t) \rangle^2.$$

Thus the formula (4.5) is equivalent to

$$k_c [2k_c \|c\|^2 + \langle c, N \rangle] > 0.$$

## 4.2 Examples of pedals and their corresponding Finsler metrics

### 4.2.1 The slope metric whose indicatrix is a limaçon

Let us consider the curve  $(C)$  is the circle  $(x - k)^2 + y^2 = a^2$  with parametric equations

$$(C) : \begin{cases} x(t) = k + a \cos t \\ y(t) = a \sin t \end{cases}, \quad t \in [0, 2\pi),$$

then the Frenet frame along  $(C)$  is

$$T(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \quad N(t) = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix},$$

and  $|c'| = a$ ,  $k_c = \frac{1}{a}$ . Using  $r_0 = (0, 0)$ , the equation (2.27) gives the pedal curve

$$p(t) = \langle c, N \rangle \cdot N,$$

with parametric equations

$$\begin{cases} p_1(t) = (a + k \cos t) \cdot \cot t \\ p_2(t) = (a + k \cos t) \cdot \sin t. \end{cases}$$

This is equivalent with the polar equation

$$r = a + k \cdot \cos \theta,$$

where  $(r, \theta)$  are polar coordinate in  $\mathbb{R}^2$ , or the implicit equation

$$(x^2 + y^2 - kx)^2 = a^2(x^2 + y^2). \quad (4.6)$$

The curvature of the limaçon reads now

$$k_p(t) = \frac{a^2 + 3ak \cos t + 2k^2}{(a + k \cos t)^2},$$

and the condition  $k_p(t) > 0$  is therefore equivalent to

$$a^2 + 3ak \cos t + 2k^2 > 0.$$

Observe that the minimum of this expression is obtained for  $\cos t = -1$ , hence

$$a^2 + 3ab \cos t + 2k^2 > a^2 - 3ab + 2k^2 > 0,$$

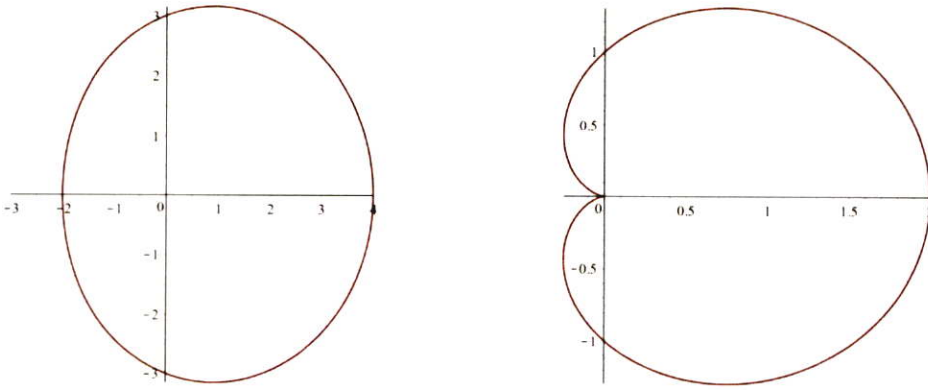
for  $a > 2k$ . This is the convexity condition for the pedal  $p(t)$  in this case.

By applying the Okubo method for the pedal  $(p)$  it follows that the corresponding Minkowski norm in  $\mathbb{R}^2$  is

$$F(x, y) = \frac{x^2 + y^2}{b\sqrt{x^2 + y^2} + ax},$$

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**Figure 4.1:** A limaçon curve for  $a = 1, b = 3$ . (left), and the non-convex case for  $a = 1, b = 1$  (right).

that is a Minkowski slope metric (see [8]). By smoothly moving this Minkowski norm on a 2-dimensional smooth manifold  $M$  we get the usual slope metric on  $M$

$$F = \frac{\alpha^2}{\alpha - \beta},$$

where  $\alpha$  is a Riemannian metric  $M$  and  $\beta$  a linear 1-form

In conclusion ([5], [8]):

**Proposition 4.7.** The Finsler metric on a two-dimensional manifold  $M$  whose indicatrix is given by the pedal curve of a circle  $(x - k)^2 + y^2 = a^2$  with origin as pedal point is a slope type metric  $F = \frac{\alpha^2}{\alpha - \beta}$ . This Finsler metric is strongly convex for  $a > 2k$ .

#### 4.2.2 The pedal curve of an ellipse

Consider is an ellipse  $(C)$ , i.e. i.e.

$$(C) : \begin{cases} x = k + a \cos t \\ y = b \sin t \end{cases}, \quad k > 0, b > 0, a > 0, a \neq b,$$

and  $P(x_0, y_0)$  an arbitrary point.

The Frenet frame is

$$T(t) = \frac{1}{|c'|} \begin{pmatrix} -a \sin t \\ b \cos t \end{pmatrix}, \quad N(t) = \frac{1}{|c'|} \begin{pmatrix} -b \cos t \\ -a \sin t \end{pmatrix},$$

$$|c'|^2 = b^2 \cdot \cos^2 t + a^2 \cdot \sin^2 t, \quad k_c = \frac{ab}{(b^2 \cdot \cos^2 t + a^2 \cdot \sin^2 t)^{3/2}}.$$

The pedal curve of  $(C)$  with respect to the pedal point  $P(x_0, y_0)$  is

$$p(t) = \frac{1}{|c'|^2} \left\{ b(k \cos t + a) \begin{pmatrix} b \cos t \\ a \sin t \end{pmatrix} + (-x_0 \cdot a \sin t + y_0 b \cdot \cos t) \begin{pmatrix} -a \sin t \\ b \cos t \end{pmatrix} \right\}.$$

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For simplicity, we consider the case when  $P \equiv O$  is the origin. Therefore the pedal curve has the parametric equations

$$\begin{cases} p_1(t) = \frac{1}{|c|^2} b^2 (k \cos t + a) \cos t \\ p_2(t) = \frac{1}{|c|^2} ab (k \cos t + a) \sin t, \end{cases} \quad (4.7)$$

and from here it follows the implicit equation

$$a^2 x^2 + b^2 y^2 = (x^2 + y^2 - kx)^2.$$

Recall that a curve  $p = p(t)$  is convex if and only if  $k_c[2k_c|c|^2 + \langle c, N \rangle] > 0$ . So, we compute this equation and get the condition

$$\cos^2 t (k \cos t + 3a)(a^2 - b^2) + 3d^2 k \cos t - a^3 + 2ak^2 + 2ab^2 > 0.$$

Finally we obtain the strongly convexity

$$a > b > \frac{1}{\sqrt{2}} \sqrt{3ak + a^2 - 2k^2}.$$

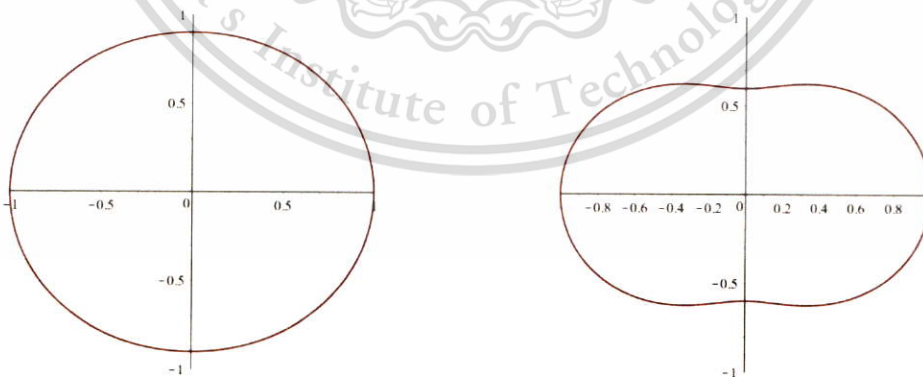
By apply Okubo's method, we obtain the Minkowski norm

$$F = \frac{x^2 + y^2}{\sqrt{a^2 x^2 + b^2 y^2 + kx}}, \quad (4.8)$$

with the strongly convexity condition  $a > b > \frac{1}{\sqrt{2}} \sqrt{3ak + a^2 - 2k^2}$ . Observe that this gives the Finsler metric on  $M$

$$F = \frac{\alpha_1^2}{\alpha_2 - \beta}, \quad (4.9)$$

where  $\alpha_1, \alpha_2$  are two different Riemannian metrics. In the case  $\alpha_1 = \alpha_2$ , we obtain the usual slope metric.



**Figure 4.2:** The convex curve in (4.7) for  $a = 10$ ,  $b = 9$ ,  $k = 2$  (left), and the non-convex case for  $a = 10$ ,  $b = 6$ ,  $k = 2$  (right).

### 4.2.3 The pedal surface of a 2-sphere

Consider the 2-sphere  $(\sigma)$  in  $\mathbb{R}^3$  with center  $(k, 0, 0)$  and radius  $r$ , i.e.

$$(\sigma) : \begin{cases} x = k + r \sin v \cos u \\ y = r \sin v \sin u \\ z = r \cos v. \end{cases} \quad (4.10)$$

Then the unit normal vector is

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \left( \frac{r^2 \sin^2 v \cos u}{\sqrt{r^4 \sin^2 v}}, \frac{r^2 \sin^2 v \sin u}{\sqrt{r^4 \sin^2 v}}, \frac{r^2 \sin^2 v \cos v}{\sqrt{r^4 \sin^2 v}} \right)^t,$$

and hence the pedal surface of the 2-sphere  $(\sigma)$  centered at  $(k, 0, 0)$  with respect to the pedal point  $P \equiv O$  (origin of  $\mathbb{R}^3$ ) is

$$p(u, v) = \begin{cases} x(u, v) = \sin v(r + k \cos u \sin v) \cos u \\ y(u, v) = \sin v(r + k \cos u \sin v) \sin u \\ z(u, v) = \cos v(r + k \cos u \sin v). \end{cases}$$

The implicit equation of  $p(u, v)$  is

$$x^2 + y^2 + z^2 = r \sqrt{x^2 + y^2 + z^2} + kx.$$

A straightforward computation gives

$$p_v(u, v) = \begin{cases} x(u, v) = \cos u \left( \frac{\partial}{\partial v} R(u, v) \right) \sin v + R(u, v) \cos v \\ y(u, v) = \sin u \left( \frac{\partial}{\partial v} R(u, v) \right) \sin v + R(u, v) \sin v \\ z(u, v) = \left( \frac{\partial}{\partial v} R(u, v) \right) \cos v - R(u, v) \sin v, \end{cases}$$

when we put  $R(u, v) = r + k \cos u \sin v$ , and  $R_u := \frac{\partial R(u, v)}{\partial u}$ , etc. We obtain

$$\begin{aligned} E &:= \langle p_u, p_u \rangle = -R(u, v)^2 \cos^2 v + \left( \frac{\partial}{\partial u} R(u, v) \right)^2 + R(u, v)^2 \\ F &:= \langle p_u, p_v \rangle = \left( \frac{\partial}{\partial u} R(u, v) \right) \left( \frac{\partial}{\partial v} R(u, v) \right) \\ G &:= \langle p_v, p_v \rangle = \left( \frac{\partial}{\partial v} R(u, v) \right)^2 + R(u, v)^2, \end{aligned}$$

and hence

$$EG - F^2 = R(u, v)^4 \sin^2 v + R(u, v)^2 \left( \frac{\partial}{\partial v} R(u, v) \right)^2 \sin^2 v + R(u, v)^2 \left( \frac{\partial}{\partial u} R(u, v) \right)^2 + R(u, v)^4.$$

Therefore  $EG - F^2 > 0$ . Likewise, we compute

$$p_{uu}(u, v) = \begin{cases} x(u, v) = -\sin(v) \{4 \sin(v) \cos(u)^2 k + \cos(u)r - 2 \sin(v)k\} \\ y(u, v) = -\sin(v) \sin(u) \{4 \cos(u) \sin(v)k + r\} \\ z(u, v) = -\cos(v) \cos(u) \sin(v), \end{cases}$$

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$$p_{uv}(u, v) = \begin{cases} x(u, v) = -\cos(v) \sin(u) \{4 \cos(u) \sin(v)k + r\} \\ y(u, v) = \cos(v) \{4 \sin(v) \cos(u)^2k + \cos(u)r - 2 \sin(v)k\} \\ z(u, v) = -\sin(u)k \{2 \cos(v)^2 - 1\}, \end{cases}$$

$$p_{vv}(u, v) = \begin{cases} x(u, v) = \cos(u) \{4 \cos(v)^2 \cos(u)k - 2 \cos(u)k - \sin(v)r\} \\ y(u, v) = \sin(u) \{4 \cos(v)^2 \cos(u)k - 2 \cos(u)k - \sin(v)r\} \\ z(u, v) = -\cos(v) \{4 \cos(u) \sin(v)k + r\}. \end{cases}$$

If we denote  $V := \frac{p_u \times p_v}{\|p_u \times p_v\|}$  it follows

$$\begin{aligned} L &:= \langle p_{uu}, V \rangle \\ &= - [\sin(v)^2 (2 \cos(v)^4 \cos(u)^3 k^3 - 2 \cos(v)^2 \cos(u)^3 k^3 - 5 \cos(v)^2 \cos(u)^2 \sin(v) k^2 r - \\ &2 \cos(v)^2 \cos(u) k^3 - 4 \cos(v)^2 \cos(u) k r^2 + 3 \cos(u)^2 \sin(v) k^2 r + 2 \cos(u) k^3 + \\ &4 \cos(u) k r^2 + 2 \sin(v) k^2 r + \sin(v) r^3)] / \\ &[-\sin(v)^2 (2 \cos(v)^2 \cos(u)^3 \sin(v) k^3 r + \cos(v)^2 \cos(u)^2 k^4 + 5 \cos(v)^2 \cos(u)^2 k^2 r^2 - \\ &2 \cos(u)^3 \sin(v) k^3 r - \cos(u)^2 k^4 - 5 \cos(u)^2 k^2 r^2 - 2 \cos(u) \sin(v) k^3 r \\ &- 4 \cos(u) \sin(v) k r^3 - k^2 r^2 - r^4)]^{1/2}, \end{aligned}$$

$$\begin{aligned} M &:= \langle p_{uv}, V \rangle \\ &= - [2 \cos(v) \sin(u) \sin(v) \cos(u) k^2 (\cos(v)^2 \cos(u)k - \cos(u)k - \sin(v)r)] / \\ &[-\sin(v)^2 (2 \cos(v)^2 \cos(u)^3 \sin(v) k^3 r + \cos(v)^2 \cos(u)^2 k^4 + 5 \cos(v)^2 \cos(u)^2 k^2 r^2 - \\ &2 \cos(u)^3 \sin(v) k^3 r - \cos(u)^2 k^4 - 5 \cos(u)^2 k^2 r^2 - 2 \cos(u) \sin(v) k^3 r \\ &- 4 \cos(u) \sin(v) k r^3 - k^2 r^2 - r^4)]^{1/2}, \end{aligned}$$

$$\begin{aligned} N &:= \langle p_{vv}, V \rangle \\ &= [\sin(v) (3 \cos(v)^2 \cos(u)^2 k^2 r - 2 \cos(u)^3 \sin(v) k^3 - 5 \cos(u)^2 k^2 r \\ &- 4 \cos(u) \sin(v) k r^2 - r^3)] / \\ &[-\sin(v)^2 (2 \cos(v)^2 \cos(u)^3 \sin(v) k^3 r + \cos(v)^2 \cos(u)^2 k^4 + 5 \cos(v)^2 \cos(u)^2 k^2 r^2 - \\ &2 \cos(u)^3 \sin(v) k^3 r - \cos(u)^2 k^4 - 5 \cos(u)^2 k^2 r^2 - 2 \cos(u) \sin(v) k^3 r \\ &- 4 \cos(u) \sin(v) k r^3 - k^2 r^2 - r^4)]^{1/2}, \end{aligned}$$

$$\begin{aligned} LN - M^2 &= (-6 \cos(v)^2 \cos(u)^3 \sin(v) k^3 r - 4 \cos(v)^2 \cos(u)^2 k^4 - 11 \cos(v)^2 \cos(u)^2 k^2 r^2 \\ &+ 6 \cos(u)^3 \sin(v) k^3 r + 4 \cos(u)^2 k^4 + 11 \cos(u)^2 k^2 r^2 + 6 \cos(u) \sin(v) k^3 r \\ &+ 6 \cos(u) \sin(v) k r^3 + 2 k^2 r^2 + r^4) \sin(v)^2 / (2 \sin(v) k r \cos(u) + k^2 + r^2) \\ &= (6 \cos(u)^3 \sin(v)^3 k^3 r - 4 \cos(u)^2 k^4 (\cos(v)^2 - 1) - 11 \cos(u)^2 k^2 r^2 (\cos(v)^2 - 1) \\ &+ 6 \cos(u) \sin(v) k r (k^2 + r^2) + 2 k^2 r^2 + r^4) \sin(v)^2 / (2 \sin(v) k r \cos(u) + k^2 + r^2) \\ &= \frac{(2 \cos(u) \sin(v) k + r)(\cos(u) \sin(v) k + r)(3 \sin(v) k r \cos(u) + 2 k^2 + r^2)}{2 \sin(v) k r \cos(u) + k^2 + r^2}. \end{aligned}$$

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By assuming that the pedal point  $O$  is inside the sphere, i.e.  $k < r$ , it follows

$$2 \sin(v)kr \cos(u) + k^2 + r^2 > 0, \quad \cos(u) \sin(v)k + r > 0.$$

Hence, the strong convexity reads

$$(2 \cos(u) \sin(v)k + r)(3 \sin(v)kr \cos(u) + 2k^2 + r^2) > 0.$$

Denote  $\cos(u) \sin(v) = A$ . Then

$$(2Ak + r)(3Akr + 2k^2 + r^2) > 0.$$

Taking into account that  $-1 \leq A \leq 1$ , and  $r, k > 0$  it results that the pedal surface is strongly convex for  $r > 2k$ .

The Minkowski metric associated is obtained by Okubo's method

$$F = \frac{x^2 + y^2 + z^2}{r \sqrt{x^2 + y^2 + z^2 + kx}},$$

that is a slope metric are  $F = \frac{\alpha^2}{\alpha - \beta}$  on the surface  $M$ .

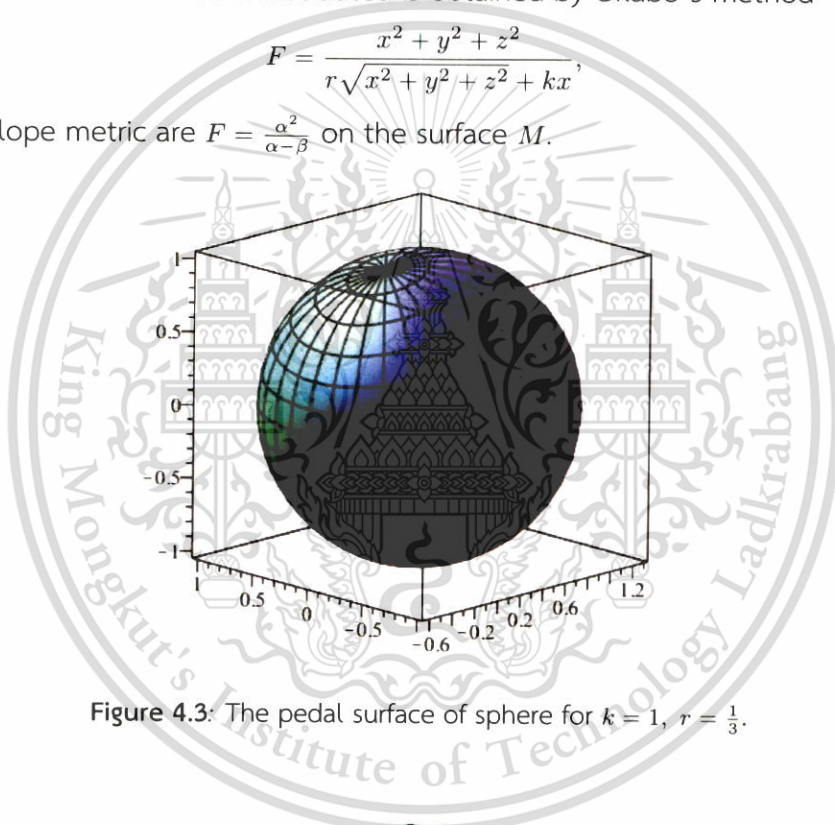


Figure 4.3: The pedal surface of sphere for  $k = 1$ ,  $r = \frac{1}{3}$ .

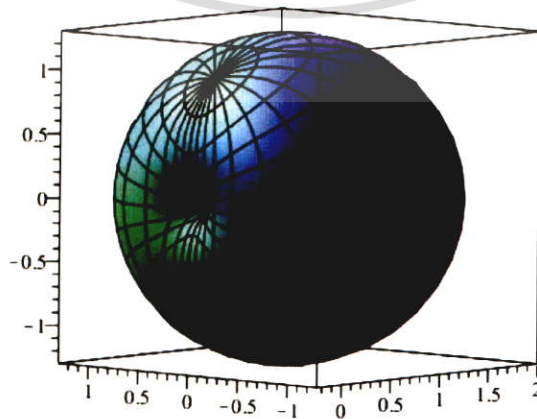


Figure 4.4: The pedal surface of sphere for  $k = 1$ ,  $r = 1$ . This surface is not convex.

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By smoothly moving this Minkowski norm on a 3-dimensional smooth manifold  $M$  we get the usual slope metric on  $M$   $F = \frac{\alpha^2}{\alpha - \beta}$ , where  $\alpha$  is a Riemannian metric on  $M$  and  $\beta$  a linear 1-form.

#### 4.2.4 The Pedal surface of an ellipsoid

As more general of the sphere case, we can consider the Finsler metric on a surface  $M$  whose indicatrix is the pedal surface of an ellipsoid.

The parametric equations of an ellipsoid can be written as

$$(\mathcal{S}) : \begin{cases} x = k + a \sin v \cos u, \\ y = b \sin v \sin u, \\ z = c \cos v \end{cases}, \quad k > 0, a > 0, b > 0, c > 0.$$

The unit normal reads

$$N = \frac{\mathcal{S}_v \times \mathcal{S}_u}{\|\mathcal{S}_v \times \mathcal{S}_u\|} = \frac{1}{\|\mathcal{S}_v \times \mathcal{S}_u\|} \begin{pmatrix} bc \sin v \cos u \\ ac \sin v \sin u \\ ab \cos v \end{pmatrix},$$

where

$$\|\mathcal{S}_v \times \mathcal{S}_u\| = \sqrt{c^2 b^2 \sin^2 v - c^2 b^2 \sin^2 v \sin^2 u + c^2 a^2 \sin^2 v \sin^2 u + a^2 b^2 - a^2 b^2 \sin^2 v}$$

By formula (2.33), the pedal surface of  $\mathcal{S}$  with respect the pedal point  $P(x_0, y_0, z_0)$  is

$$p(u, v) = \sin v \frac{(bc(k - x_0) \sin v \cos u + a(-cy_0 \sin v \sin u - bz_0 \cos v + bc))}{\|\mathcal{S}_v \times \mathcal{S}_u\|^2} \begin{pmatrix} bc \sin v \cos u \\ ac \sin v \sin u \\ ab \cos v \end{pmatrix} + (x_0, y_0, z_0)^t. \quad (4.11)$$

Finding the explicit strongly convexity condition of the surface (4.11) implies some long computations, so we will consider the case when  $P \equiv O$  is the origin. In this case we obtain

$$p(u, v) : \begin{cases} x(u, v) = \frac{b^2 c^2 (k \cos u \sin v + a)}{\|\mathcal{S}_v \times \mathcal{S}_u\|^2} \sin v \cos u \\ y(u, v) = \frac{abc^2 (k \cos u \sin v + a)}{\|\mathcal{S}_v \times \mathcal{S}_u\|^2} \sin v \sin u \\ z(u, v) = \frac{ab^2 c (k \cos u \sin v + a)}{\|\mathcal{S}_v \times \mathcal{S}_u\|^2} \cos v. \end{cases} \quad (4.12)$$

From (4.12) we can see that

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = \frac{a^2 b^4 c^4 (k \cos u \sin v + a)^2}{\|\mathcal{S}_v \times \mathcal{S}_u\|^4}. \quad (4.13)$$

On the other hand,

$$x^2 + y^2 + z^2 - kx = \frac{ab^2c^2(k \cos u \sin v + a)}{\|\mathcal{S}_v \times \mathcal{S}_u\|^2}, \quad (4.14)$$

and by comparing (4.13) and (4.14) we get the implicit equation of the pedal surface:

$$(x^2 + y^2 + z^2 - kx)^2 = a^2x^2 + b^2y^2 + c^2z^2. \quad (4.15)$$

Finding general conditions for the strongly convexity of the ellipsoid pedal is to complicate. Some numerical simulations show that for instance, in the case  $k = 1/3$ ,  $a = b = 2$ ,  $c = \sqrt{6}$ , we indeed obtain a strongly convex surface; see Figure 4.

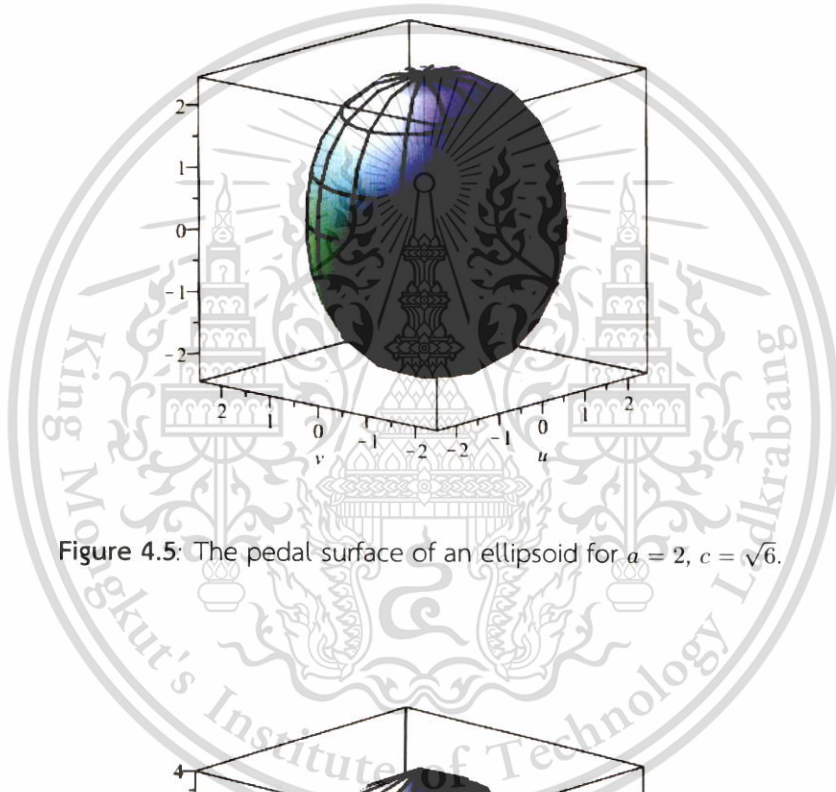


Figure 4.5: The pedal surface of an ellipsoid for  $a = 2$ ,  $c = \sqrt{6}$ .

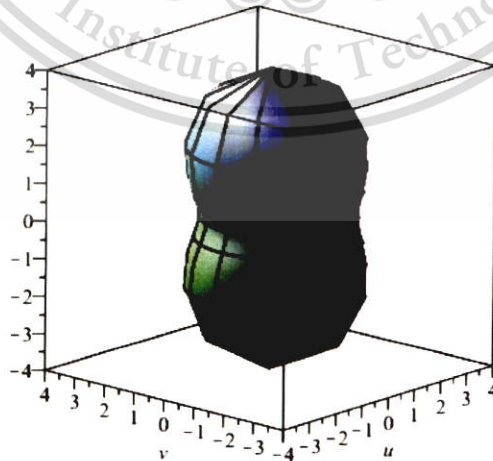


Figure 4.6: The pedal surface of an ellipsoid for  $k = 1/3$ ,  $a = b = 2$ ,  $c = 4$ . This surface is not convex.

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By applying Okubo's method to (4.15) we obtain the Minkowski norm

$$F = \frac{x^2 + y^2 + z^2}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2 + kx}}$$

that is clearly the generalization of (4.8).

The Finsler metric corresponding to this indicatrix surface is of the type (4.9), where  $\alpha_1, \alpha_2$  are two different Riemannian metrics.



## Chapter 5

# Conclusions and Suggestions

### 5.1 Conclusions

From a slope metric on a surface of revolution section, we have obtained the conclusion that we can define the slope metric  $F$  for surface of revolution in term of function  $\phi$ . Then We obtain conditions for any surface of revolution to admit a strongly convex slope metric.

From the pedal and Finsler metric section, we have obtained the following conclusions:

1. The Finsler metric on a two dimensional manifold  $M$  whose indicatrix is given by the pedal curve of the ellipse with origin as pedal point is of type  $F = \frac{\alpha_1^2}{\alpha_2 - \beta}$ .
2. The Finsler metric on a three dimensional manifold  $M$  whose indicatrix is given by the pedal surface of a sphere with origin as pedal point is a slope type metric  $F = \frac{\alpha^2}{\alpha - \beta}$ . This Finsler metric is strongly convex for  $r > 2k$ . Observe that a quick look at the computation show that the same is true for the arbitrary dimensional case as well.
3. The Finsler metric on a three dimensional manifold  $M$  whose indicatrix is given by the pedal surface of the ellipsoid with origin as pedal point is of type  $F = \frac{\alpha_1^2}{\alpha_2 - \beta}$ , where  $\alpha_1, \alpha_2$  are two different Riemannian metrics and  $\beta$  is a linear form in  $TM$ . This Finsler metric is strongly convex subject to some conditions for  $a, b, c$  and  $k$ .
4. The Finsler metric on an  $n$ -dimensional manifold  $M$  whose indicatrix is given by the pedal hypersurface of an ellipsoid with origin as pedal point is a slope type metric  $F = \frac{\alpha_1^2}{\alpha_2 - \beta}$ , where  $\alpha_1, \alpha_2$  are two different Riemannian metrics on  $M$  and  $\beta$  a linear one form on  $TM$ . This Finsler metric is strongly convex for some further conditions on the constants giving the axes of the ellipsoid and the coordinates of its center.

### 5.2 Suggestions

There are some open problems following from this thesis, e.g.

1. We need to know more about geometrical meaning for equation (4.3).
2. We can study more property of the Finsler metric that obtain by pedal curve and surface to know about new Finlser metric and high dimensional slope metric.

## References

- [1] Antonelli, P.L. Ingarden, R. and Matsumoto, M. 1993. **The theory of sprays and Finsler spaces with applications in physics and biology.** 58. Kluwer Academic Publishers Group. Dordrecht.
- [2] Bacso, S. Chern, X. and Shen, Z. 2007 "Curvature properties of  $(\alpha, \beta)$ -metric." *Advanced Studies in Pure Mathematics.* 48. Math. Soc. of Japan: 73-110.
- [3] Bao, D. Chern, S.S. and Shen, Z. 2000. **The Introduction to Riemann-Finsler Geometry.** Springy.
- [4] Bao, D. Robles, C. and Shen, Z. 2004. "Zermelo navigation on Riemannian manifolds." *J. Diff. Geom.* 66: 377-435.
- [5] Carathéodory, C. 1999. **Calculus of Variations and Partial Differential Equations of the First Order.** San Francisco: Holden-day.
- [6] Chern, S.S. Chen, W.H. and Lam, K.S. 1999. **Lectures on differential geometry. Series on University Mathematics.** New Jersey : World Scientific Publishing Co.
- [7] Gilbson, C.G. 1998. **Elementary Geometry of Algebraic curve.** Cambridge Univ. Press.
- [8] Markvorsen, S. 2015. "A Finsler geodesic spray paradigm for wildfire spread modelling." *Nonlinear Analysis: Real World Applications.* 28: 208-228.
- [9] Matsumoto, M. 1989. "A slope of mountain is a Finsler surface with respect to a time measure." *J. Math. Kyoto Univ.* 29: 17-25.
- [10] Matsumoto, M. 2003. "Finsler Geometry in the 20<sup>th</sup> Century." Antonelli, P.L. **Handbook of Finsler geometry.** Kluwer Acad. Publ: 557-966.
- [11] O'Neill, B. 2006. **Elementary Differential Geometry.** Academic Press Inc.
- [12] Sabau, S.V. Shibuya, K. and Yoshikawa, R. 2017. "Geodesics on strong Kropina manifolds." *European J. Math.* vol.3: 1172-1224.
- [13] Shen, Z. 2003. "Finsler Metrics with  $K=0$  and  $S=0$ ", *Canadian J. Math.* Vol. 55: 112-132.
- [14] Shiohama, K. Shioya, T. and Tanaka, M. 2003. **The Geometry of Total Curvature on Complete Open Surfaces.** Cambridge : Cambridge University Press.
- [15] Tu, L.W. 2010. **An Introduction to Manifolds Second Edition.** New York Dordrecht Heidelberg London.

- [16] Yoshikawa, R. and Sabau, S.V. 2014. "Kropina metrics and Zermelo navigation on Riemannian manifolds." *Geometriae Dedicata*. Vol. 171 (1): 199-148.





## Appendix A

Surfaces of revolution admitting strongly convex slope metrics



# SURFACES OF REVOLUTION ADMITTING STRONGLY CONVEX SLOPE METRICS

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## Abstract

This paper discusses the geometry of a surface endowed with a slope metric. We obtain necessary and sufficient conditions for any surface of revolution to admit a strongly convex slope metric. Such conditions involve certain inequalities for the derivative of the associated function on the Cartesian coordinate and the polar coordinate. In particular, we apply this result to certain well-known surface of revolution.

**Keywords :** slope metric; surface of revolution; strong convexity

**2000 Mathematics Subject Classification :** 53A04; 53A05 (2000 MSC )

## 1 Introduction and Preliminaries

Optimal transport and optimal control problems are important topics in pure and applied mathematics. For instance, it is well-known that the minimizing time travel in the Euclidean space is the straightline. Moreover, when traveling from a point  $A$  to a point  $B$  in the Euclidean space, the distance and hence the time is the same as when traveling from  $B$  to  $A$ .

However, in real life problems, applications to engineering, industry, etc., there are external forces acting on the traveling object in Euclidean space, like magnetic fields, gravitational fields, winds, etc. For the sake of simplicity, let us assume we need to travel by a ship in an open sea from a point  $A$  to a point  $B$ , say from the pier to an island. In the absence of any wind or marine currents the shortest traveling time is the straightline, however if a mild wind comes up, then we arrive to the following control problem.

*Consider a ship sailing in the open sea in calm waters. Suppose a mild wind comes up. How must the ship be steered in order to reach a given destination in the shortest time?*

This problem was considered for the first time by E. Zermelo in 1931 ([10]) when he assumed that the open sea was the Euclidean plane  $\mathbb{R}^2$  with the Euclidean metric.

The first thing to remark is that, when travelling with constant speed under the action of a mild wind, the minimal time needed to travel from  $A$  to  $B$  is different from the time needed to travel from  $B$  to  $A$ . Indeed, the time minimal paths and the travel times are different when sailing against the wind and when sailing in the same direction with the

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wind. This is called nowadays the *Zermelo's navigation problem* (see [3] for details and generalizations).

The simple example above suggests that there are other types of distances and metrics, called Randers metrics, that give different minimal time trajectories from the canonical Euclidean metric. Randers metrics are widely used in optimal control, Physics, Biology and many other fields of pure and applied mathematics ([1], [8]). Randers metrics belong to a more general family of metrics, called Finsler metrics (see [1], [2] for basics on Finsler metrics).

This paper has no intention to be an introduction to Finsler metrics. We will recall only that *Finsler geometry is just the Riemannian geometry without the quadratic restriction* ([4]). Indeed, what we call today a Finsler norm was actually introduced by B. Riemann in his famous Habilitation dissertation from 1854, namely a metric function

$$ds = F(x^1, x^2, \dots, x^n; dx^1, dx^2, \dots, dx^n)$$

that depends on the position  $(x^1, x^2, \dots, x^n)$  of the point and direction. Such a metric is determined by a function  $F$  defined on the tangent bundle  $TM$  of an  $n$ -dimensional smooth manifold  $M$  having the properties:

1.  $F(x, y)$  is positive on  $TM \setminus \{0\}$ ;
2.  $F(x, y)$  it is 1-homogeneous in  $y$ , that is  $F(x, \lambda y) = \lambda F(x, y)$ , for all constants  $\lambda > 0$ ;
3. the Hessian matrix

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \quad (1.1)$$

is positive definite on  $TM \setminus \{0\}$ .

The most important special case is the case when  $F^2 = g_{ij}(x)y^i y^j$ , in other words  $F$  is a quadratic form in the variable  $y$ . This is what we call today a *Riemannian metric*. Therefore, rather than regarding the Finsler geometry as just a generalization of Riemannian geometry, it is more appropriate to say that Finsler geometry is just the Riemannian geometry without the quadratic restriction above.

The Randers metrics naturally appearing from the Zermelo's navigation problem are deformations of Riemannian metrics in the sense  $F = \alpha + \beta$ , where  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ , and  $\beta = b^i(x)y^i$ , where  $a_{ij}$  is a Riemannian metric.

Another type of Finsler metric is the so-called *slope metric* introduced by M. Matsumoto in 1989 (see [6]) based on a letter of P. Finsler, the founder of Finsler geometry. The control problem is the following

*Suppose a person walking on a horizontal plane with velocity  $c$ , while the gravitational force is acting perpendicularly on this plane. The person is almost ignorant of the action of this force. Imagine the person walks now with same velocity on the inclined plane of angle  $\epsilon$  to the horizontal sea level. Under the influence of gravitational forces, what is the trajectory the person should walk in the center to reach a given destination in the shortest time?*

The metric giving the shortest travelling paths was called therefore the *slope metric* ([6], [7]). Obviously the problem described above it is important for applications to the

real world in industry, constructions, or when transporting something on a slope. It was shown by M. Matsumoto (see [6], [7]) that the corresponding Finsler metric is also a deformation of a Riemannian metric  $\alpha$  by a linear form  $\beta$  by the formula

$$F = \frac{\alpha^2}{\alpha - \beta}.$$

Another important reason to consider the slope metrics is the fight against wild fires ([5]). Almost everyday we hear news about wildfires in different regions of the Earth. In order to deal with it, the firefighters need to act promptly based on a proper understanding, predicting and modelling the evolution of wildfires. The slope metric is one of the most appropriate mathematical models for predicting the evolution of wildfires, hence the mathematical study of such metrics might become in the future vital for predicting, controlling and successfully fighting the wildfires (see [5] and references within for details).

Let us remind that the study of shortest paths in a Riemannian or Finsler space is related to the calculus of variations. A variational problem determined by a Finsler metric  $F$  is non-degenerate when the Hessian (1.1) is regular matrix. Moreover, the extremal paths are minimizing when the metric is positive definite, and this happens in the case when the unit circle in the tangent space  $T_x M$  is strongly convex (see [2] for a very detailed exposition of the calculus of variations for Finsler spaces).

Based on the discussion above, we can formulate the problem we are going to study in the present paper.

*The wildfires can happen across geographically very complicated terrain including forests, grasslands, rice fields, etc. On what kind of surfaces, the wildfire evolution can be modeled as the variational problem of the slope metric  $F = \alpha^2/(\alpha - \beta)$ ? In other words, on which surfaces there are naturally induced slope metrics whose Hessian matrices (1.1) are positive defined?*

In the present paper we will give some general conditions for a slope metric to have positive definite Hessian and we will give some examples of inclined surfaces where the wildfires behaviour can be predicted and eventually controlled.



(a) The unit circle of the slope metric in the case of walking in the slope of a mountain. (b) the slope of a wildfire on slope of a mountain.

Figure 1

Let us formulate the above problem in the Riemannian geometry formalism. Consider a surface  $S$  embedded in the Euclidean space  $\mathbb{R}^3$  parametrized by

$$S \rightarrow \mathbb{R}^3, (x, y) \mapsto (x, y, z = f(x, y)), \quad (1.2)$$

for a smooth function  $f$ , that is  $S$  is the graph of the function  $z = f(x, y)$ . It is well known that  $S$  is a 2-dimensional differential manifold. It follows that the induced Riemannian metric from  $\mathbb{R}^3$  on the surface  $S$  is represented by the matrix

$$a_{ij} = \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix},$$

where  $f_x$  and  $f_y$  are the partials derivative of  $f$  with respect to  $x$  and  $y$ , respectively. We consider the tangent plane  $\pi_p$  spanned by two vectors

$$\partial_x := (1, 0, f_x), \quad \partial_y := (0, 1, f_y).$$

We can construct an  $a$ -orthogonal basis  $e_1, e_2$  in  $\pi_p$  by choosing  $e_1$  to point on the steepest downhill direction of  $\pi_p$ . In the plane  $\pi_p$  with origin  $p = (0, 0)$  and  $(X, Y)$  in the basis  $\{e_1, e_2\}$ , the slope principle by Matsumoto [6] tells us that the time minimizing trajectory of a hiker on the plane  $\pi_p$  is given by walking in the direction given by the limaçon

$$r = v + a \cdot \cos \theta, \quad (1.3)$$

where  $(r, \theta)$  are the polar coordinate of the  $XY$  plane,  $v$  is the velocity of the hiker on the flat plane  $xy$ ,  $a = \frac{g}{2} \cdot \sin \varepsilon$  where  $g$  is gravitational constant for fixed  $p \in S$  and  $\varepsilon$  is the incline angle to the sea level. Moreover any vector of  $\pi_p$  can be written as  $\dot{x}\partial_x + \dot{y}\partial_y$ . By using the relation between the coordinates  $(X, Y)$  in the basis  $\{e_1, e_2\}$  and the canonical component  $(\dot{x}, \dot{y})$  we obtain the implicit limaçon equation  $h(\dot{x}, \dot{y}) = 0$  where

$$h(\dot{x}, \dot{y}) = \dot{x}^2 + \dot{y}^2 + (f_x \dot{x} + f_y \dot{y})^2 - v \cdot \sqrt{\dot{x}^2 + \dot{y}^2 + (f_x \dot{x} + f_y \dot{y})^2} - \frac{g}{2} \cdot (f_x \dot{x} + f_y \dot{y}).$$

From this, by use Okubo's method (see [1] for detail), we can describe the surface  $S$  via the *fundamental function*  $F : TS \rightarrow [0, \infty)$  such that

$$h\left(\frac{\dot{x}}{F}, \frac{\dot{y}}{F}\right) = 0.$$

By solving this equation we received

$$F = \frac{\alpha^2}{v\alpha - \frac{g}{2}\beta},$$

where

$$\begin{cases} \alpha &= \sqrt{a_{11}\dot{x}^2 + 2a_{12}\dot{x}\dot{y} + a_{22}\dot{y}^2} \text{ is the induced Riemannian metric on } S, \text{ and} \\ \beta &= f_x \dot{x} + f_y \dot{y}. \end{cases} \quad (1.4)$$

By normalization, we get the usual form of the *slope metric*

$$F = \frac{\alpha^2}{\alpha - \beta}. \quad (1.5)$$

It is well known that the function  $F$  induces a Finsler norm on the surface  $S$  if and only if the  $h$  is a convex (see [2] for detail). In this case  $(S, F)$  becomes a Finsler manifold. The next theorem provides a necessity and sufficiency condition for  $h$  to be strongly convex.

**Theorem 1.1** [9] *A 2-dimensional differential manifold  $M$  endowed with the fundamental function (1.5) is a Finsler manifold if and only if  $f_x^2 + f_y^2 < \frac{1}{3}$ .*

In the present work, we will focus on a well known class of 2-dimensional differential manifolds generated by rotating a curve (called the profile curve) around an axis of rotation, called a surface of revolution. Indeed, for each smooth function  $\phi : [0, \infty) \rightarrow \mathbb{R}$  that is extensible to a smooth even function around 0, the surface of revolution in the 3-dimensional Euclidean space is defined by the equation  $z = \phi(\sqrt{x^2 + y^2})$ .

There are several ways of introducing local coordinates on such a surface. The parametrization (1.2) of an arbitrary surface can be adapted to the case of surfaces of revolution by putting

$$S \rightarrow \mathbb{R}^3, \quad (x, y) \mapsto (x, y, z = \phi(\sqrt{x^2 + y^2})) \quad (1.6)$$

The induced Riemannian metric is

$$\begin{aligned} a_{ij}(x, y) &:= \begin{pmatrix} 1 + \phi_x^2 & \phi_x \phi_y \\ \phi_x \phi_y & 1 + \phi_y^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{(\phi'(\sqrt{x^2+y^2}))^2 x^2}{x^2+y^2} & \frac{(\phi'(\sqrt{x^2+y^2}))^2 xy}{x^2+y^2} \\ \frac{(\phi'(\sqrt{x^2+y^2}))^2 xy}{x^2+y^2} & 1 + \frac{(\phi'(\sqrt{x^2+y^2}))^2 y^2}{x^2+y^2} \end{pmatrix}, \end{aligned}$$

provide  $x^2 + y^2 \neq 0$ , where

$$\phi_x = \frac{\phi'(\sqrt{x^2+y^2})x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \phi_y = \frac{\phi'(\sqrt{x^2+y^2})y}{\sqrt{x^2+y^2}},$$

and  $\beta = \phi_x \dot{x} + \phi_y \dot{y}$ , then we can define the slope metric  $F$  for surface of revolution in term of function  $\phi$ .

Our main work in this paper is to investigate necessary and sufficient conditions for any surface of revolution to admit a strongly convex slope metric. Such conditions rely on certain inequalities for the derivative of the profile curve on the Cartesian coordinate and the polar coordinate. In particular, we apply this result to classical well-known surfaces of revolution.

## 2 Convexity conditions for the slope metric on surface of revolution

**Theorem 2.1** *Let  $S \rightarrow \mathbb{R}^3$  be a surface of revolution. Then the following statements are equivalent:*

- (i)  $S$  admits a strongly convex slope metric;
- (ii)  $[\phi'(s)]^2 < \frac{1}{3}$  where  $S \rightarrow \mathbb{R}^3 : (x, y) \mapsto (x, y, z = \phi(s))$  and  $s = \sqrt{x^2 + y^2}$ ;
- (iii)  $[m'(u)]^2 > 3$  where  $S \rightarrow \mathbb{R}^3 : (u, v) \mapsto (m(u) \cos v, m(u) \sin v, u)$ .

*Proof.* First, we shall prove (i)  $\Leftrightarrow$  (ii). Note that for the surface of revolution  $S : z = \phi(\sqrt{x^2 + y^2})$ , we have

$$\begin{aligned} (\phi_x)^2 + (\phi_y)^2 &= \left( \phi'(\sqrt{x^2 + y^2}) \frac{x}{\sqrt{x^2 + y^2}} \right)^2 + \left( \phi'(\sqrt{x^2 + y^2}) \frac{y}{\sqrt{x^2 + y^2}} \right)^2 \\ &= \left( \phi'(\sqrt{x^2 + y^2}) \right)^2. \end{aligned}$$

Recall from Theorem 1.1 that an equivalent condition for  $S$  to admit a strongly convex slope metric is the inequality

$$(\phi_x)^2 + (\phi_y)^2 < \frac{1}{3}.$$

Thus the assertions (i) and (ii) are equivalent.

Next, we shall show the equivalence between (ii) and (iii). Indeed, we can parametrize the surface of revolution by using trigonometric functions, that is

$$S \rightarrow \mathbb{R}^3, \quad (x = m(u) \cdot \cos v, y = m(u) \cdot \sin v, z = u). \quad (2.1)$$

This surface of revolution is obtained by rotating the curve  $x = m(z)$  along the  $z$  axis. Applying the parametrization (1.6) to (2.1) yields

$$x^2 + y^2 = m^2(u) \cos^2 v + m^2(u) \sin^2 v = m^2(u),$$

and hence

$$z = \phi(\sqrt{x^2 + y^2}) = \phi(|m(u)|).$$

For the sake of simplicity we consider here only the case  $+m(u)$ . Since  $z = u$  it follows that  $\phi(m(u)) = u$ . Substituting  $s = \sqrt{x^2 + y^2}$  yields  $s = m(u)$  and thus

$$m(\phi(s)) = m(\phi(m(u))) = m(u) = s.$$

Hence,  $\phi$  and  $m$  are inverse function of each other. From the inverse function theorem, we have

$$m'(u) = \frac{1}{\phi'(s)} \Big|_{s=m(u)}$$

or, equivalently

$$[m^{-1}]'(s) = \frac{1}{m'(m^{-1}(s))} \Leftrightarrow \phi'(s) = \frac{1}{m'(u)} \Big|_{u=\phi(s)}$$

and from here it is clear that

$$[m'(u)]^2 = \left[ \frac{1}{\phi'(s)} \right]^2.$$

The condition  $[\phi'(s)]^2 < \frac{1}{3}$  is now equivalent to  $[m'(u)]^2 > 3$ .

### 3 Classical surfaces of revolution admitting strongly convex slope metrics

In this section, we investigate several classical surfaces of revolution to admit convex slope metrics. Here are our main worked out examples.

#### 3.1 The Elliptic paraboloid

Consider the elliptic paraboloid  $S \rightarrow \mathbb{R}^3$ ,  $(x, y) \mapsto (x, y, z = f(x, y) = 100 - x^2 - y^2)$  this surface was firstly studied by Bao, Robles (see [3]). In this parametrization, the convexity condition is  $f_x^2 + f_y^2 = 4(x^2 + y^2) < 1/3$ , i.e.  $x^2 + y^2 < 1/12$ , hence the strongly convexity condition is satisfied only on a circular vicinity of height  $1/12$  units of the hilltop.

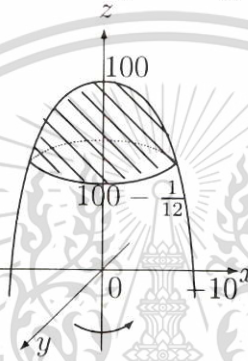


Figure 2

Putting this surface into the parametrization (1.6), we have  $z = \phi(s)$  where  $\phi(s) = 100 - s^2$ . The strongly convexity condition  $[\phi'(s)]^2 < 1/3$  is equivalent to  $-1/\sqrt{12} < s < 1/\sqrt{12}$ .

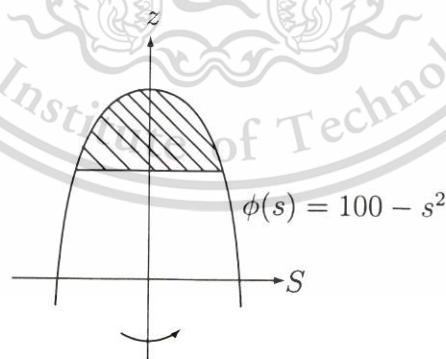


Figure 3

Finally, in the trigonometric parametrization we have

$$(u, v) \mapsto (x = m(u) \cos v, y = m(u) \sin v, z = u) \text{ for } u \in (-\infty, 100], v \in [0, 2\pi).$$

This surface of revolution is obtained by rotating the curve  $x = m(z)$  around the  $z$  axis. The inverse of  $\phi$  is given by the function  $s = \pm\sqrt{100 - u}$  for  $u \in (-\infty, 100]$ . For simplicity, we consider only  $s = \sqrt{100 - u}$ . We have  $[m'(u)]^2 = 1/(4(100 - u)) > 3$ , or

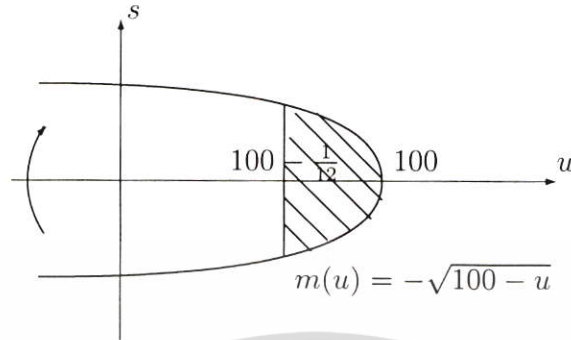


Figure 4

$100 - (1/12) < u \leq 100$ , i.e. the convexity domain is the same as before.

**Corollary 3.1** For the elliptic paraboloid  $z = \phi(s) = 100 - s^2$ , its slope metric (1.5) is strongly convex on the open domain  $\{(x, y) \in S \mid \sqrt{x^2 + y^2} < 1/\sqrt{12}\}$ .

### 3.2 The Cone

Consider a cone, that is, a surface of revolution defined by

$$S \rightarrow \mathbb{R}^3, \quad (x, y) \rightarrow \left(x, y, z = a\sqrt{x^2 + y^2}\right), \quad \text{for } a > 0. \quad (3.1)$$

We can write  $z = \phi(s) := as$ . Since  $\phi'(s) = a$ , by Theorem 2.1 its slope metric is strongly convex on the whole surface if and only if  $a \in (0, 1/\sqrt{3})$ . We obtain

**Corollary 3.2** The slope metric (1.5) defined on the cone  $S$  given by (3.1) is strongly convex if and only if  $a \in (0, 1/\sqrt{3})$ .

### 3.3 The Ellipsoid

Consider an Ellipsoid, that is, a surface of revolution defined by

$$(x, y) \rightarrow \left(x, y, z = \frac{c}{a}\sqrt{a^2 - x^2 - y^2}\right), \quad \text{for } a > 0. \quad (3.2)$$

Write  $z = \phi(s) := c\sqrt{a^2 - s^2}/a$ . Then  $\phi'(s) = -cs/(a\sqrt{a^2 - s^2})$ , and thus the strongly convexity condition is satisfied only for  $s \in (-\frac{a^2}{\sqrt{a^2+3c^2}}, \frac{a^2}{\sqrt{a^2+3c^2}})$ .

**Corollary 3.3** The slope metric (1.5) defined on the ellipsoid  $S$  given by (3.2) is strongly convex on the open domain  $\{(x, y) \in S \mid \sqrt{x^2 + y^2} < \frac{a^2}{\sqrt{a^2+3c^2}}\}$ .

### 3.4 The Two-sheeted hyperboloid

Consider the surface of revolution

$$(x, y) \rightarrow \left( x, y, z = a\sqrt{x^2 + y^2 + b^2} \right), \quad \text{for } a > 0. \quad (3.3)$$

We can write  $z = \phi(s) := a\sqrt{s^2 + b^2}$ . Then  $\phi'(s) = as/(\sqrt{s^2 + b^2})$ , and thus the strongly convexity condition of the slope metric for this surface is  $a \in (0, 1/\sqrt{3})$ .

**Corollary 3.4** *The slope metric (1.5) defined on the hyperboloid  $S$  given by (3.3) is strongly convex if and only if  $a \in (0, 1/\sqrt{3})$ .*

### 3.5 The One-sheeted hyperboloid

Consider the surface of revolution

$$(x, y) \rightarrow \left( x, y, z = a\sqrt{x^2 + y^2 - b^2} \right), \quad \text{for } a > 0. \quad (3.4)$$

Putting  $z = \phi(s) := a\sqrt{s^2 - b^2}$ , we get  $\phi'(s) = as/(\sqrt{s^2 - b^2})$ . Thus, the strongly convexity condition is satisfied only for  $s \in (-\infty, -\frac{|b|}{\sqrt{1-3a^2}}) \cup (\frac{|b|}{\sqrt{1-3a^2}}, \infty)$  where  $1 - 3a^2 > 0$ .

**Corollary 3.5** *The slope metric (1.5) defined on the hyperboloid  $S$  given by (3.4) is strongly convex on the open domain  $\{(x, y) \in S \mid \sqrt{x^2 + y^2} > |b|/\sqrt{1-3a^2}\}$ , where  $0 < a < 1/\sqrt{3}$ .*

### 3.6 The surface of revolution generated by the Gauss function

Recall that the Gauss function  $h(x) = e^{-x^2}$  is frequently used in statistics and it gives the shape of the normal distribution. Let us consider the surface of revolution

$$(x, y) \rightarrow \left( x, y, z = f(x, y) = \frac{1}{2\sqrt{6}}e^{-x^2 - y^2} \right). \quad (3.5)$$

We can write  $z = \phi(s) := e^{-s^2}/(2\sqrt{6})$ . Since  $se^{-s^2} < 1$  for any  $s > 0$ , the slope metric is globally strong convex on this surface.

Observe that  $\phi(s)$  is bijective for  $s \in (0, \infty)$ , and by putting  $\phi(s) = u$  and we get

$$m(u) = \pm \frac{\sqrt{-2\ln(24u^2)}}{2} \quad \text{for } u \in \left( 0, \frac{1}{2\sqrt{6}} \right),$$

We denote  $\mu(u) = [m'(u)]^2$  for  $(0, 1/(2\sqrt{6}))$  and compute the minimum of  $\mu(u)$ . The only critical point of  $\mu$  in  $(0, 1/(2\sqrt{6}))$  is  $e^{-\frac{1}{2}}/(2\sqrt{6})$ . Since

$$\mu\left(\frac{1}{2\sqrt{6}}e^{-\frac{1}{2}}\right) \approx 32.6 > 3,$$

the convexity condition is globally verified.

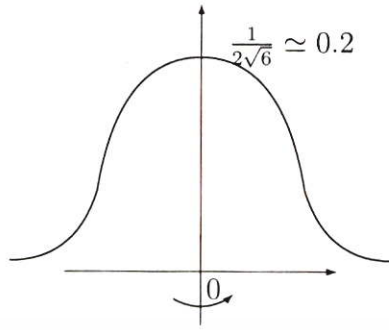


Figure 5

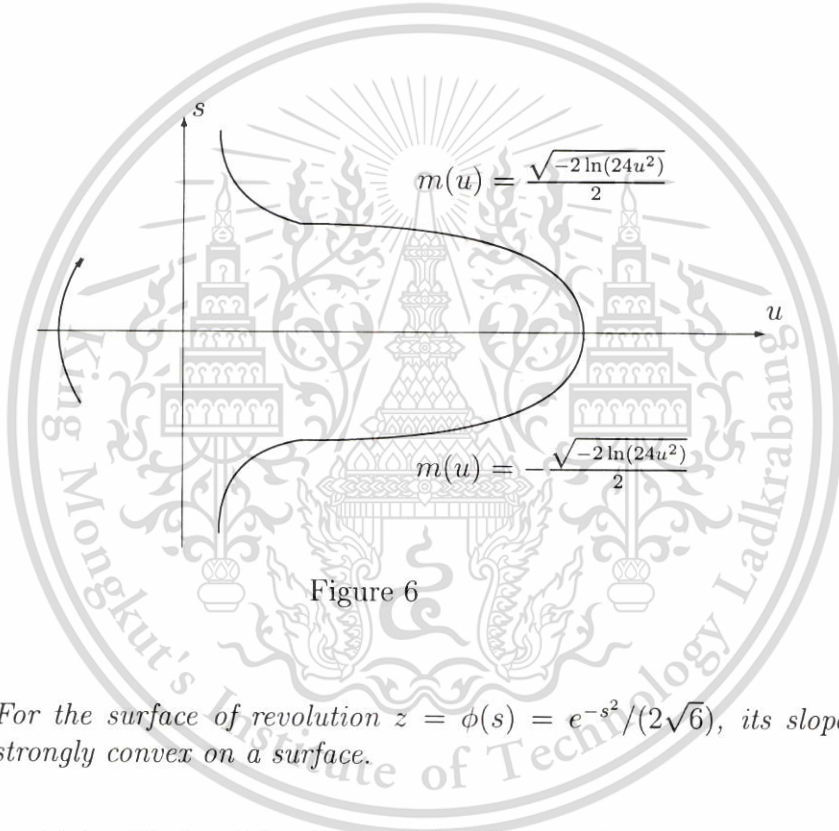


Figure 6

**Corollary 3.6** For the surface of revolution  $z = \phi(s) = e^{-s^2}/(2\sqrt{6})$ , its slope metric (1.5) is globally strongly convex on a surface.

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## References

- [1] P.L. Antonelli, R. Ingarden, M. Matsumoto, The theory of sprays and Finsler spaces with applications in physics and biology. Fundamental Theories of Physics, 58.

- Kluwer Academic Publishers Group, Dordrecht, 1993. xvi+308 pp. ISBN: 0-7923-2577-X.
- [2] D. Bao, S.S. Chern, Z. Shen, The Introduction to Riemann-Finsler Geometry, Springy, 2000.
  - [3] D. Bao, C. Robles, Ricci and Flag Curvatures in Finsler Geometry, Riemann-Finsler Geometry, MSRI Publications, Vol. 50 (2004), 197–259.
  - [4] S. S. Chern, Finsler geometry is just the Riemannian Geometry without the quadratic restriction, Notices of AMS, September 1996, 959–962.
  - [5] S. Markvorsen, A Finsler geodesic spray paradigm for wildfire spread modelling, Nonlinear Analysis: Real World Applications, 28 (2015), 208–228.
  - [6] M. Matsumoto, A slope of mountain is a Finsler surface with respect to a time measure, J.Math. Kyoto Univ., 29(1989), 17–25.
  - [7] M. Matsumoto, Finsler Geometry in the 20<sup>th</sup> Century, in Handbook of Finsler geometry Ed.P.L.Antonelli, Kluwer Acad. Publ., 2003, 557–966.
  - [8] S. V. Sabau, K. Shibuya, H. Shimada, Metric structures associated to Finsler metrics, Publ. Math. Debrecen, 84 (2014), no. 1-2, 89–103.
  - [9] H. Shimada, S. V. Sabau, Introduction to Matsumoto metric, Nonlinear Analysis 63 (2005), e165–e168.
  - [10] E. Zermelo, Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung, Z. Angew. Math. Mech. 11 (1931), 114–124.



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## Letter of acceptance

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## Appendix B

Finslerian indicatrices as algebraic curves and surfaces



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# Finslerian indicatrices as algebraic curves and surfaces

P. Chansri, P. Chansangiam and S. V. Sabau

**Abstract.** We show how to construct new Finsler metrics, in two and three dimensions, whose indicatrices are pedal curves or pedal surfaces of some other curves or surfaces. These Finsler metrics are generalizations of the famous slope metric, also called Matsumoto metric.

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## 1 Introduction

Finsler manifolds are natural generalizations of Riemannian manifolds in the same respect as normed spaces and Minkowski spaces are generalizations of Euclidean spaces.

In the case of the Euclidean space, or more general, of Riemannian manifolds, the space looks uniform and isotropic, that is, the same in all direction. However, our daily experiences as well as the metrics and distances naturally appearing in applications to real life problems in Physics, Computer science, biology, etc. show that the space is not isotropic, there exists some preferred directions (see [1], [5], [7], [11]).

To be more precise, we recall that a *Finsler metric*  $(M, F)$  is given by specifying a Finsler norm  $F : TM \rightarrow \mathbb{R}$  defined on the tangent space  $(TM, M)$  of an  $n$ -dimensional manifold  $M$ . A Finsler norm has the following properties

1.  $F$  is  $C^\infty$  on  $\widetilde{TM} := TM \setminus \{O\}$ , where  $O$  is the zero section;
2.  $F$  is 1- positive homogeneous, i.e.  $F(x, \lambda y) = \lambda \cdot F(x, y)$ ,  $\forall \lambda > 0$ ,  $(x, y) \in TM$ ;
3.  $F$  is strongly convex, i.e. the Hessian  $g_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$  is positive defined for any  $(x, y) \in \widetilde{TM}$ .

Observe that the fundamental function  $F$  determines and it is determined by its indicatrix (the unit tangent bundle)  $\Sigma_F := \{(x, y) \in TM : F(x, y) = 1\}$ , which is a smooth hypersurface of  $TM$ . For each point  $x \in M$ , we can define the indicatrix

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at  $x$  as  $\Sigma_x := \{y \in TM : F(x, y) = 1\} = \Sigma_F \cap T_x M$ , which is a smooth, closed, strictly convex hypersurface in  $T_x M$ . It is therefore important to remark that to give a Finsler structure  $F$  on a manifold  $M$  is equivalent to giving a smooth hypersurface  $\Sigma \hookrightarrow TM$  for which, the canonical projection  $\pi : \Sigma \rightarrow M$  is a surjective submersion with the property that, for each  $x \in M$ , the  $\pi$ -fiber  $\Sigma = \pi^{-1}(x)$  is a strictly convex hypersurface in  $T_x M$  enclosing the origin. If  $(M, F)$  is a Finsler manifold, then the restriction of  $F$  to each tangent space  $T_x M$  induce a Minkowski norm in  $T_x M$ . To give such a Minkowski norm is equivalent to giving the indicatrix  $\Sigma_x$  at  $x$ . A Finsler structure on  $M$  is a family of Minkowski norms  $(F_x, T_x M)$  moving smoothly on the manifold.

From now on, we are going to regard Finsler and Minkowski norms as hypersurfaces in  $TM$  and  $T_x M$ , respectively. With this image in mind, constructing examples of Finsler manifolds or Minkowski norms reduce to the effective construction of the hypersurfaces  $\Sigma$  and  $\Sigma_x$ , respectively. Observe that the central symmetric spheres or ellipsoids give Riemannian metrics since they are all quadratic forms in the fiber coordinate  $y$  of  $T_x M$ , hence we need to construct simple hypersurface which are not quadratic forms in  $y$ 's.

Even though there exists already a lot of literature about Finsler manifolds and indicatrices ([10]), as well as about the pedal curves ([9]) and algebraic curves in general ([4]), our approach, we reconsider this topic in modern terminology, aiming to provide new insights into the theory of Finsler spaces.

For instance, recall that the Randers and Kropina metrics are obtained by a rigid translation of the unit sphere such that the origin is enclosed by it or it is included in its boundary, respectively. We point out that Kropina metrics are actually conic Finsler metrics (see [11], [3] for details).

Another similar example of Finsler metric is the slope metric (see [3], [6]), where, in the 2 dimensional case, the curve indicatrix is a *limaçon*. The associated Finsler norm is written in the general form  $F = \frac{\alpha^2}{\alpha - \beta}$  and called the *slope metric* (or a Matsumoto metric), where  $\alpha = \sqrt{a_{ij}y^i y^j}$  is a Riemannian metric and  $\beta = b_i y^i$  is a linear 1-form. In [3] we have studied the geometry of the slope metric globally induced on a surface of revolution.

On the other hand, let us observe that a *limaçon* is an algebraic curve obtained as the pedal curve of a circle with respect to the origin. This insight opens a new perspective on indicatrices i.e. Finsler metrics, as algebraic curves. In the three (or higher) dimensional case it is also possible to regard indicatrices as hyper-surfaces.

In the present paper we study the following problems:

1. How to construct two dimensional Finsler metrics whose indicatrices are pedal curves of some algebraic curves as generalization of the slope metric and point out the convexity conditions of the pedal curves. In special we will consider the case of pedal of conics.
2. How to extend this study to the three dimensional case (and arbitrary dimensional case). This study is new in the sense that indicatrices of three dimensional slope metrics are studied for the first time. From algebraic point of view the geometry of pedal surfaces is also an interesting topic.

Arbitrary dimension Finsler metrics whose indicatrices are pedal hypersurfaces can be studied in a similar manner, but the concrete computations can be quite messy.

Finally, we point out that our approach is important because it illustrates and clarifies the geometrical meaning of three (and higher) dimensional slope metrics, called Matsumoto metrics in the arbitrary dimensional case. Indeed, initially, the two-dimensional slope metric was defined by Matsumoto as the Finsler metric whose indicatrix is a limaçon (see [6]). After seeing that these Finsler metrics are of the type  $F = \frac{\alpha^2}{\alpha - \beta}$ , where  $\alpha$  is a Riemannian metric and  $\beta$  a linear one form, they were simply generalized to the arbitrary dimensional case without any further considerations on the geometrical meaning. By using our pedal curves and surfaces approach one can see that the higher dimensional Matsumoto metrics are those Finsler metrics whose indicatrices are pedal hypersurfaces of spheres.

## 2 The pedal curve of a plane algebraic curve

Let us consider a plane algebraic curve  $(C)$  given in parametric form

$$(2.1) \quad (C) : \quad x = x(t), \quad y = y(t),$$

then, at regular values of the parameter  $t$ , the tangent line to  $(C)$  is

$$(2.2) \quad (\ell) : \quad \dot{y}(t) \cdot x - \dot{x}(t) \cdot y + \{\dot{x}(t) \cdot y(t) - x(t) \cdot \dot{y}(t)\} = 0,$$

and the orthogonal line to  $(\ell)$  through a point  $P(x_0, y_0)$  is given by

$$(\ell)^\perp : \quad y - y_0 = -\frac{\dot{x}(t)}{\dot{y}(t)}(x - x_0),$$

where dots represent the derivative of a function of one variable with respect to  $t$ .

For a regular plane algebraic curve  $(C)$ , and a fixed point  $P(x_0, y_0)$ , called the *pedal point*, the *pedal curve* of the curve  $(C)$  with respect to  $P$  is the parameterized curve obtained by associating to the parameter  $t$  the orthogonal projection  $p(t)$  of  $P$  onto the tangent line at  $t$  (see [4], [9] for details on algebraic curves). The pedal curves are considered important in geometrical optics and kinematics.

We recall that the moving equation of the Frenet frame  $(T(t), N(t))$  along  $(C)$  are given by

$$\begin{cases} \frac{dT}{dt} = |c'(t)| \cdot k_c \cdot N(t) \\ \frac{dN}{dt} = -|c'(t)| \cdot k_c \cdot T(t), \end{cases}$$

where  $(T(t), N(t))$  are the unit tangent and normal vectors along  $c$ , respectively,  $|c'(t)| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$  is the speed of  $(C)$  and

$$k_c := \frac{\langle c''(t), N(t) \rangle}{|c'(t)|^2}$$

is the curvature of the curve  $(C)$ . Here  $\langle \cdot, \cdot \rangle$  is the usual inner product of the Euclidean plane.

A straightforward computation shows that the pedal curve of  $(\mathcal{C})$  with respect to the point  $P$  is given by

$$(2.3) \quad p(t) = \langle c - r_0, N \rangle \cdot N + r_0,$$

where we denote by  $r_0$  the position vectors of the point  $P(x_0, y_0)$ .

**Proposition 2.1.** *If  $(\mathcal{C})$  is a continuously differentiable closed curve in plane, then its pedal curve  $(\mathcal{P}) : p = p(t)$  with respect to a point  $P(x_0, y_0)$  is also a continuously differentiable closed curve in plane.*

*Proof.* From hypothesis, after some rescaling of the parameter  $t$ , we have  $c(0) = c(2\pi)$ ,  $\dot{c}(0) = \dot{c}(2\pi)$ , and hence  $T(0) = T(2\pi)$ ,  $N(0) = N(2\pi)$ . Using now (2.3) it follows  $p(0) = p(2\pi)$ , i.e.  $p$  is also periodic with the same period as  $(\mathcal{C})$ . Moreover,  $\dot{p}(0) = 2\pi$ , where by derivation of (2.3) we get

$$(2.4) \quad \dot{p}(t) = k_c \cdot |c'| \cdot [\langle r_0 - c, T \rangle \cdot N + \langle r_0 - c, N \rangle \cdot T].$$

□

We will ask now the question if the pedal curve  $p(t)$  goes through the origin  $O(0, 0)$  of  $\mathbb{R}^2$ . This is equivalent to asking if the vectorial equation

$$p(t) = \langle c, N \rangle \cdot N + \langle r_0, T \rangle \cdot T = (0, 0)^t, \text{ where } t \text{ denote the transposed matrix,}$$

has solution. Since  $N$  and  $T$  linearly independent, this equation is equivalent to the system of equation,

$$(2.5) \quad \langle c, N \rangle = 0, \quad \langle r_0, N \rangle = 0.$$

We consider

Case 1. Assume the pedal point is origin, i.e.  $r_0 = (0, 0)$ . In this case we get only the equation

$$\langle c, N \rangle = - \begin{vmatrix} c_1 & c_2 \\ \dot{c}_1 & \dot{c}_2 \end{vmatrix} = 0,$$

and observe that for a continuous differentiable curve this is possible if and only if  $(\mathcal{C})$  passes through origin.

Hence, in this case  $(\mathcal{P})$  passes through origin if and only if  $(\mathcal{C})$  passes through origin.

Case 2. Assume the pedal point  $P$  is not the origin, i.e.  $x_0 \neq 0$  or  $y_0 \neq 0$ . Then we consider further the cases:

- 2.1 The curve  $(\mathcal{C})$  passes through origin, i.e. there exists  $t_0$  to such that  $c(t_0) = (0, 0)$ ,  $\dot{c}(t_0) \neq (0, 0)$ . In this case we obtain  $\langle r_0, T(t_0) \rangle = 0$ , i.e.  $r_0$  and  $T(t_0)$  are orthogonal.
- 2.2 The curve  $(\mathcal{C})$  do not pass through the origin, i.e. both conditions in (2.5) must be simultaneously verified, but this is impossible. Indeed, since  $(\mathcal{C})$  is continuously differentiable curve,  $c(t)$  cannot be collinear to  $T(t)$ , nor  $r_0$  can be always orthogonal to  $T$  since  $(\mathcal{C})$  is a closed curve.

We conclude:

**Theorem 2.2.** *Let  $C$  be a plane algebraic curve with parametric equation (2.1) and  $P(x_0, y_0)$  a point in  $\mathbb{R}^2$ ,  $P \notin (C)$ .*

- *If  $P$  is the origin, then  $(\mathcal{P})$  passes through origin if and only if  $(C)$  also passes through origin.*
- *If  $P$  is not the origin, and  $(C)$  passes through the origin then  $(\mathcal{P})$  passes through origin if and only if  $r_0$  and  $T(t_0)$  are orthogonal, where  $t_0$  is the value of the parameter  $t$  such that  $c(t_0) = (0, 0)$ .*
- *If  $P$  is not the origin, and  $(C)$  does not pass through origin, then  $(\mathcal{P})$  does not pass origin either.*

Next, we will study the convexity condition of the pedal curve  $(\mathcal{P})$  in (2.3). We recall that a curve  $p = p(t)$  is strongly convex if and only if

$$(2.6) \quad \frac{\dot{p}(t) \times \ddot{p}(t)}{p(t) \times \dot{p}(t)} > 0,$$

where the cross product of two vector  $u = (a, b)$ ,  $v = (c, d)$  is given by  $u \times v = ad - bc$ , (see for instance [2], page 88). Observe that this condition is independent on the parameterization of  $p(t)$ . A straightforward computation gives

$$\begin{aligned} \frac{\dot{p} \times \ddot{p}}{p \times \dot{p}} &= \frac{\begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix}}{\begin{vmatrix} v_0 & v_1 \\ u_0 & u_1 \end{vmatrix}} = \frac{k_c |\dot{c}|^2 \begin{vmatrix} \langle r_0 - c, T \rangle & -1 + 2k_c \langle r_0 - c, N \rangle \\ \langle r_0 - c, N \rangle & -2k_c \langle r_0 - c, T \rangle \end{vmatrix}}{\begin{vmatrix} \langle c, N \rangle & \langle r_0 - c, N \rangle \\ \langle r_0, T \rangle & \langle r_0 - c, T \rangle \end{vmatrix}} \\ &= \frac{k_c |\dot{c}|^2 [-2k_c \langle r_0 - c, T \rangle^2 + \langle r_0 - c, N \rangle - 2k_c \langle r_0 - c, N \rangle^2]}{\langle c, N \rangle \langle r_0 - c, N \rangle - \langle r_0, T \rangle \langle r_0 - c, T \rangle}, \end{aligned}$$

where we have used  $\dot{p}(t) = -k_c \cdot |\dot{c}|^2 \{ \langle c, T \rangle \cdot N + \langle c, N \rangle \cdot T \}$ . Hence, we obtain

**Theorem 2.3.** *The pedal curve  $(\mathcal{P})$  is strongly convex if and only if*

$$(2.7) \quad k_p(t) := \frac{k_c \{-2k_c \langle r_0 - c, T \rangle^2 + \langle r_0 - c, N \rangle - 2k_c \langle r_0 - c, N \rangle^2\}}{\langle c, N \rangle \langle r_0, N \rangle + \langle c, T \rangle \langle r_0, T \rangle - \langle c, N \rangle^2 - \langle r_0, T \rangle^2} > 0,$$

for all  $t \in [0, 2\pi)$ .

**Remark 2.1.** In the case when the pedal point  $P$  is the origin, formula (2.7) simplifies to

$$(2.8) \quad k_c [2k_c \langle c, T \rangle^2 + \langle c, N \rangle + 2k_c \langle c, N \rangle^2] > 0.$$

Moreover, observe that the position vector  $c(t)$  can be decomposed in the orthonormal basis  $\{T, N\}$  as  $c(t) = \langle c(t), T(t) \rangle T(t) + \langle c(t), N(t) \rangle N(t)$ , hence  $\langle c(t), c(t) \rangle = \langle c(t), T(t) \rangle^2 + \langle c(t), N(t) \rangle^2$ . It is now easy to see now that the formula (2.8) is equivalent to  $k_c [2k_c |c|^2 + \langle c, N \rangle] > 0$ .

### 3 Some remarkable pedal curves and their corresponding Finsler metrics

#### 3.1 The slope metric whose indicatrix is a limaçon

##### 3.1.1 The pedal curve

It is easy to see that the pedal curve of the circle with center  $(0, k)$  and radius  $a$  with respect to the origin of  $\mathbb{R}^2$  is a limaçon.

Indeed, the curve  $(\mathcal{C})$  is the circle  $(x - k)^2 + y^2 = a^2$ , then the equation (2.3) gives the pedal curve

$$\begin{cases} p_1(t) = (a + k \cos t) \cdot \cot t \\ p_2(t) = (a + k \cos t) \cdot \sin t. \end{cases}$$

This is equivalent with the polar equation  $r = a + k \cdot \cos \theta$ , where  $(r, \theta)$  are the polar coordinates in  $\mathbb{R}^2$ , or the implicit equation

$$(3.1) \quad (x^2 + y^2 - kx)^2 = a^2(x^2 + y^2).$$

Observe that the pedal curve  $(\mathcal{P})$  is not passing through origin (Theorem 2.2). Moreover, the curvature of the limaçon reads now

$$k_p(t) = \frac{a^2 + 3ak \cos t + 2k^2}{a^2 + 2ak \cos t + k^2},$$

and since  $a^2 + 2ak \cos t + k^2 \geq a^2 - 2ak + k^2 = (a - k)^2 > 0$ , for  $a \neq k$ , the condition  $k_p(t) > 0$  is therefore equivalent to  $a^2 + 3ak \cos t + 2k^2 > 0$ . Observe that the minimum of this expression is obtained for  $\cos t = -1$ , hence  $a^2 + 3ak \cos t + 2k^2 > a^2 - 3ak + 2k^2 = (a - k)(a - 2k) > 0$ , for  $a > 2k$ .

This is the convexity condition for the pedal  $p(t)$  in this case (see Figure 1).

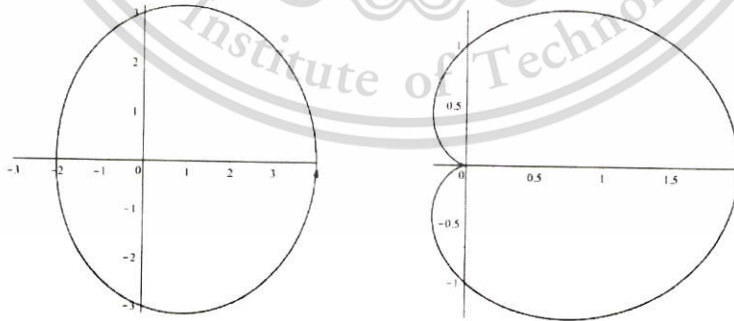


Figure 1: A convex limaçon curve for  $a = 3$ ,  $k = 1$  (left) and a non-convex limaçon curve for  $a = 1$ ,  $k = 1$  (right).

### 3.1.2 The Finsler metric

The Finsler metric whose indicatrix is given as a curve in each tangent space can easily determined. Indeed, observe that the limaçon implicit equation (3.1) is equivalent to

$$\frac{x^2 + y^2}{a\sqrt{x^2 + y^2 + kx}} = 1,$$

hence the corresponding Minkowski norm in  $\mathbb{R}^2$  is

$$F(x, y) = \frac{x^2 + y^2}{a\sqrt{x^2 + y^2 + kx}},$$

that is a Minkowski slope metric (see [6], this approach is sometimes called the Okubo's method). By smoothly moving this Minkowski norm on a 2- dimensional smooth manifold  $M$  we get the usual slope metric on  $M$   $F = \frac{\alpha^2}{\alpha - \beta}$ , where  $\alpha$  is a Riemannian metric  $M$  and  $\beta$  a linear 1-form (see our recent paper [3] for a study of the slope metric on a surface of revolution).

In conclusion ([2], [6], [3]):

**Proposition 3.1.** *The Finsler metric on a two dimensional manifold  $M$  whose indicatrix is given by the pedal curve of a circle  $(x - k)^2 + y^2 = a^2$  with origin as pedal point is a slope type metric  $F = \frac{\alpha^2}{\alpha - \beta}$ . This Finsler metric is strongly convex for  $a > 2k$ .*

**Remark 3.1.** We observe that a similar result can be obtained when  $(C)$  is the unit circle and  $P(a, 0)$  a point on the  $x$ -axis. By a similar computation as in the case above, we obtain the pedal curve parametric equations

$$(3.2) \quad \begin{cases} p_1(t) = (1 - a \cos t) \cdot \cos t + a \\ p_2(t) = (1 - a \cos t) \cdot \sin t. \end{cases}$$

Observe that this curve can be regarded as a limaçon with parameters  $(-a, 1)$  with center translated from origin to  $(a, 1)$ . The convexity condition reads  $-\frac{1}{2} < a < \frac{1}{2}$ .

We obtain that the Finsler metric on a two dimensional manifold  $M$  whose indicatrix is given by the pedal curve of a unit circle  $x^2 + y^2 = 1$  with pedal point  $(a, 0)$  is of type

$$F = \frac{\alpha^2}{\beta_1 - \sqrt{(\alpha - \beta)^2 + \beta_2^2}},$$

where  $\alpha$  is a Riemannian metric and  $\beta_1, \beta_2$  are the linear form in  $TM$ . This Finsler metric is strongly convex for  $a \in (-\frac{1}{2}, \frac{1}{2})$ .

## 3.2 The pedal curve of an ellipse

### 3.2.1 The pedal curve

**Remark 3.2 (Motivation).** Let us recall that two polynomials  $P, Q$  in  $x, y$  with real coefficients are equivalent if there exists a non zero  $\lambda \in \mathbb{R}$  such that  $P = \lambda \cdot Q$ . This is an equivalence relation on the set of polynomials and an equivalence class

is called an *affine plane curve*. Moreover, two affine curves  $(c_1) : f(x, y) = 0$ ,  $(c_2) : g(x, y) = 0$  are called *affinely equivalent* if there exists an affine map  $\phi$  on  $\mathbb{R}^2$  and a scalar  $\lambda \neq 0$  such that  $g(x, y) = \lambda \cdot f(\phi(x, y))$ . Since the set of affine maps on  $\mathbb{R}^2$  is a group  $(Aff(2), 0)$ , with the operation of composition, affine equivalence defines an equivalence relation for plane curves in  $\mathbb{R}^2$ . Observe that the degree  $d$  of a curves is an affine invariant. Clearly  $d = 1$  gives straight lines, so they are not interesting for us.

The next simple case is  $d = 2$ , i.e. conic, the circle being affinely equivalent to real ellipse, which is the only closed and convex conic.

It is therefore naturally to consider the general case when the curve  $(C)$  is an ellipse, i.e.

$$(C) : x = k + a \cos t, \quad y = b \sin t, \quad k > 0, b > 0, a > 0, a \neq b,$$

and  $P(x_0, y_0)$  an arbitrary point.

The pedal curve of  $(C)$  with respect to the pedal point  $P(x_0, y_0)$  is

$$p(t) = \frac{1}{|c|^2} \left\{ b(k \cos t + a) \begin{pmatrix} b \cos t \\ a \sin t \end{pmatrix} + (-x_0 \cdot a \sin t + y_0 b \cdot \cos t) \begin{pmatrix} -a \sin t \\ b \cos t \end{pmatrix} \right\}.$$

For the sake of simplicity, we consider the case when  $P \equiv O$  is the origin. In this case, the pedal curve has the parametric equations

$$(3.3) \quad \begin{cases} p_1(t) = \frac{1}{|c|^2} b^2 (k \cos t + a) \cos t \\ p_2(t) = \frac{1}{|c|^2} ab (k \cos t + a) \sin t, \end{cases}$$

and from here it follows the implicit equation  $a^2 x^2 + b^2 y^2 = (x^2 + y^2 - kx)^2$ .

Recall that a curve  $p = p(t)$  is convex if and only if  $k_c [2k_c |c|^2 + \langle c, N \rangle] > 0$ , so we compute this equation and get the condition

$$-a^3 + 2ak^2 + 2ab^2 + (a^2 - b^2) \cos^2 t (k \cos t + 3a) + 3ka^2 \cos t > 0.$$

Again for simplicity we can consider  $a > b$ , then we have to check that

$$3a^2 k \cos t - a^3 + 2ak^2 + 2ab^2 > -3a^2 k - a^3 + 2ak^2 + 2ab^2 > 0,$$

then strongly convexity reads (see Figure 2)

$$(3.4) \quad a > b > \frac{1}{\sqrt{2}} \sqrt{3ak + a^2 - 2k^2}.$$

### 3.2.2 The Finsler metric

If we apply Okubo's method we obtain the Minkowski norm

$$(3.5) \quad F = \frac{x^2 + y^2}{\sqrt{a^2 x^2 + b^2 y^2 + kx}},$$

with the strongly convexity condition  $a > b > \frac{1}{\sqrt{2}}\sqrt{3ak + a^2 - 2k^2}$ . Observe that this gives the Finsler metric on  $M$

$$(3.6) \quad F = \frac{\alpha_1^2}{\alpha_2 - \beta},$$

where  $\alpha_1, \alpha_2$  are two different Riemannian metrics. In the case  $\alpha_1 = \alpha_2$  we obtain the usual slope metric.

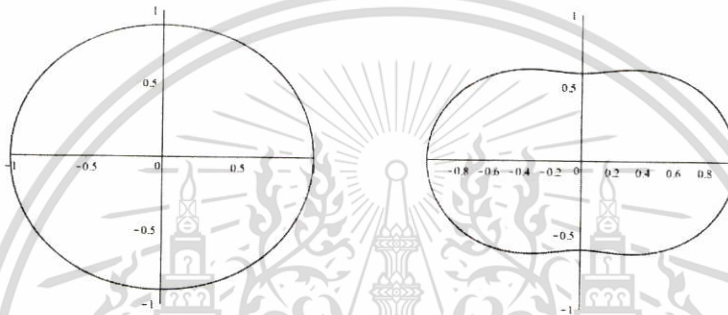


Figure 2: The convex curve in (3.3) for  $a = 10, b = 9, k = 2$  (left), and the non-convex case for  $a = 10, b = 6, k = 2$  (right).

We summarize

**Theorem 3.2.** *The Finsler metric on a two dimensional manifold  $M$  whose indicatrix is given by the pedal curve of the ellipse  $(\frac{x-k}{a})^2 + (\frac{y}{b})^2 = 1$  with origin as pedal point is of type  $F = \frac{\alpha_1^2}{\alpha_2 - \beta}$ , where  $\alpha_1, \alpha_2$  are two different Riemannian metrics and  $\beta$  is a linear form in  $TM$ . This Finsler metric is strongly convex for  $a > b > \frac{1}{\sqrt{2}}\sqrt{3ak + a^2 - 2k^2}$ .*

**Remark 3.3.** Observe that if put  $a = b$  in theorem (3.2) we obtained the Finsler metric  $F = \frac{x^2 + y^2}{a\sqrt{x^2 + y^2 + kx}}$ , with strongly convexity condition  $a > \frac{1}{\sqrt{2}}\sqrt{3ak + a^2 - 2k^2}$ , or, equivalently

$$(3.7) \quad 2a^2 > 3ak + a^2 - 2k^2 = (a - k)(a - 2k) > 0.$$

Therefore we obtain  $a > 2k$ , that is same result as in Proposition 3.1.

## 4 The pedal surface

We are going to extend our considerations from curves of surfaces. Instead of the curve  $(C)$ , we are going to consider a smooth surface  $\mathcal{S} \hookrightarrow \mathbb{R}^3$  embedded in  $\mathbb{R}^3$  with parametric equations

$$(4.1) \quad (\mathcal{S}) : x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

and observe that, at any regular vector  $(u, v)$  of the parameters, the tangent plane to  $(\mathcal{S})$  at  $(x(u, v), y(u, v), z(u, v)) \in \mathcal{S}$  is given by

$$(\pi) : \frac{\partial(y, z)}{\partial(u, v)}(x - x(u, v)) + \frac{\partial(z, x)}{\partial(u, v)}(y - y(u, v)) + \frac{\partial(x, y)}{\partial(u, v)}(z - z(u, v)) = 0,$$

where

$$\frac{\partial(y, z)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix}$$

and so on. The normal to  $(\pi)$  at a point  $(u, v)$  is given by

$$(\pi^\perp) : \frac{x}{\frac{\partial(y, z)}{\partial(u, v)}} = \frac{y}{\frac{\partial(z, x)}{\partial(u, v)}} = \frac{z}{\frac{\partial(x, y)}{\partial(u, v)}}.$$

Let  $(\mathcal{S})$  be a regular surface parameterized on in (4.1) and let  $P(x_0, y_0, z_0)$  be a fixed point, the *pedal point*. Then the *pedal surface* of the surface  $(\mathcal{S})$  with respect to the point  $P$  is the parameterized surface obtained by associating to the parameter  $(u, v)$  the orthogonal projection  $p(u, v)$  of  $P$  onto the tangent plane  $(\pi)$  at  $\mathcal{S}(u, v)$ .

The tangent plane  $(\pi)$  is generated by the vectors  $\mathcal{S}_u = (x_u, y_u, z_u)^t$ , and  $\mathcal{S}_v = (x_v, y_v, z_v)^t$ , while the unit normal vector to  $(\mathcal{S})$  is

$$N = \frac{\mathcal{S}_u \times \mathcal{S}_v}{\|\mathcal{S}_u \times \mathcal{S}_v\|} = \frac{1}{\|\mathcal{S}_u \times \mathcal{S}_v\|} \left( \frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)} \right)^t.$$

Similarly with the plane curve's case, a straightforward computation shows that the pedal surface of the surface  $(\mathcal{S})$  with respect to the point  $P$  is given by

$$(4.2) \quad p(u, v) = (\mathcal{S} - r_0, N) \cdot N + r_0.$$

The convexity condition of the pedal surface is given by the condition  $K > 0$ , there  $K$  is the Gauss curvature, that is,

$$(4.3) \quad \begin{vmatrix} \langle p_{uu}, p_u \times p_v \rangle & \langle p_{uv}, p_u \times p_v \rangle \\ \langle p_{uv}, p_u \times p_v \rangle & \langle p_{vv}, p_u \times p_v \rangle \end{vmatrix} > 0.$$

The formula can be quite complicate in the general case, but we will consider some examples.

**Remark 4.1.** In the same way we can define the pedal hypersurface of an  $n$ -dimensional surface  $\mathcal{S} \hookrightarrow \mathbb{R}^{n+1}$ . The formula (4.2) is clearly true for any dimensions, but the sectional curvature computations and the determination of the strongly convexity condition becomes more difficult. Nevertheless, in the case of the  $n$ -sphere the computations are quite straightforward as we shall see.

## 5 Some remarkable pedal surfaces and the corresponding Finsler metrics

### 5.1 The pedal surface of a 2-sphere

#### 5.1.1 The pedal surface

The easiest case is in the case where  $(\mathcal{S})$  is the 2-sphere  $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$  with center  $(k, 0, 0)$  and radius  $r$ , i.e.

$$(\mathcal{S}) : x = k + r \sin v \cos u, \quad y = r \sin v \sin u, \quad z = r \cos v, \quad k > 0, r > 0.$$

Then the exterior oriented unit normal vector is

$$N = \frac{\mathcal{S}_v \times \mathcal{S}_u}{\|\mathcal{S}_v \times \mathcal{S}_u\|} = (\cos u \sin v, \sin u \sin v, \cos v)^t,$$

and hence the pedal surface of the 2-sphere  $(\mathcal{S})$  center at  $(k, 0, 0)$  with respect to the pedal point  $P \equiv O$  (origin of  $\mathbb{R}^3$ ) is

$$p(u, v) : \begin{cases} x(u, v) = \sin v(r + k \cos u \sin v) \cos u \\ y(u, v) = \sin v(r + k \cos u \sin v) \sin u \\ z(u, v) = \cos v(r + k \cos u \sin v). \end{cases}$$

The implicit equation of  $p(u, v)$  can be written in the form  $f(x, y, z) = 0$  where

$$f(x, y, z) = x^2 + y^2 + z^2 - r\sqrt{x^2 + y^2 + z^2} - kx.$$

This surface can be called the *limaçon surface*, or the two dimensional limaçon.

We recall that a surface is called strongly convex when  $LN - M^2 > 0$ , where  $L = \langle p_{uu}, V \rangle$ ,  $N = \langle p_{vv}, V \rangle$  and  $M = \langle p_{uv}, V \rangle$ . Then, the unit normal vector is given by  $V := \nabla f / \|\nabla f\|$ , hence the strongly convexity condition reads  $\langle p_{uu}, \nabla f \rangle \langle p_{vv}, \nabla f \rangle - \langle p_{uv}, \nabla f \rangle^2 > 0$ , and a straightforward computation gives

$$\nabla f : \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)^t = \begin{pmatrix} (2Ak + r)A - k \\ (2Ak + r) \sin u \sin v \\ (2Ak + r) \cos v \end{pmatrix},$$

where  $A := \cos u \sin v$ .

Moreover, we have

$$\langle p_{uu}, \nabla f \rangle = [(-4kA^2 - rA + 2k \sin^2 v)(2A^2k + rA - k) - (4kA + r)(2kA + r) \sin^2 v \sin^2 u - kA(2Ak + r) \cos^2 v]$$

$$\langle p_{vv}, \nabla f \rangle = [(-4kA^2 - rA + 2k \cos^2 u)(2A^2k + rA - k) + 2kA(2Ak + r) \sin^2 u - (4kA + r)(2kA + r)(1 - A^2)]$$

$$\langle p_{uv}, \nabla f \rangle = 2Ak^2 \cos v \sin u.$$

The strongly convexity condition reads now

$$\langle p_{uu}, \nabla f \rangle \langle p_{vv}, \nabla f \rangle - \langle p_{uv}, \nabla f \rangle^2 = \sin^2 v(3Akr + 2k^2 + r^2)(Ak + r)(2Ak + r) > 0.$$

Taking into account that  $-1 \leq A \leq 1$ , and  $r, k > 0$  it results that the pedal surface is strongly convex for  $r > 2k$ . This condition is consistent with the condition obtained in the case of the pedal curve of the circle (see Figure 3).

### 5.1.2 The Finsler metric

The Minkowski metric associated can be easily obtained by Okubo's method

$$F = \frac{x^2 + y^2 + z^2}{r\sqrt{x^2 + y^2 + z^2 + kx}},$$

that is a slope metric  $F = \frac{\alpha^2}{\alpha - \beta}$  on the surface  $M$ .

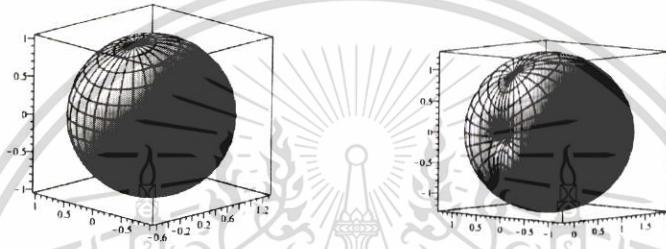


Figure 3: The convex pedal surface of the sphere, with pedal point in origin, for  $k = \frac{1}{3}$ ,  $r = 1$  (left), and the non-convex case for  $k = 1$ ,  $r = 1$  (right).

By smoothly moving this Minkowski norm on a 3-dimensional smooth manifold  $M$  we get the usual slope metric on  $M$   $F = \frac{\alpha^2}{\alpha - \beta}$ , where  $\alpha$  is a Riemannian metric on  $M$  and  $\beta$  a linear 1-form.

In conclusion we get:

**Theorem 5.1.** *The Finsler metric on a three dimensional manifold  $M$  whose indicatrix is given by the pedal surface of a sphere  $(x - k)^2 + y^2 + z^2 = r^2$  with origin as pedal point is a slope type metric  $F = \frac{\alpha^2}{\alpha - \beta}$ , where  $\alpha$  is a Riemannian metric on  $M$  and  $\beta$  a linear one form on  $TM$ . This Finsler metric is strongly convex for  $r > 2k$ .*

**Remark 5.1.** Without giving here the concrete computations, a quick look at the formulas above show that the same is true for the arbitrary dimensional case as well. We only formulate here without proof the following

**Theorem 5.2.** *The Finsler metric on an  $n \geq 2$ -dimensional manifold  $M$  whose indicatrix is given by the pedal hypersurface of an  $n-1$ -sphere  $(x_1 - k)^2 + x_2^2 + \dots + x_n^2 = r^2$  with origin as pedal point is a slope type metric  $F = \frac{\alpha^2}{\alpha - \beta}$ , where  $\alpha$  is a Riemannian metric on  $M$  and  $\beta$  a linear one form on  $TM$ . This Finsler metric is strongly convex for  $r > 2k$ .*

## 5.2 The Pedal surface of an ellipsoid

As a generalization of the Section 3.2, we can consider the Finsler metric on a surface  $M$  whose indicatrix is the pedal surface of an ellipsoid. The computations are quite long, so we give only some ideas of the construction in this section.

### 5.2.1 The pedal surface

The parametric equations of an ellipsoid can be written as

$$(\mathcal{S}) : x = k + a \sin v \cos u, \quad y = b \sin v \sin u, \quad z = c \cos v, \quad k > 0, a > 0, b > 0, c > 0.$$

The exterior oriented unit normal reads

$$N = \frac{\mathcal{S}_v \times \mathcal{S}_u}{\|\mathcal{S}_v \times \mathcal{S}_u\|} = \frac{1}{\|\mathcal{S}_v \times \mathcal{S}_u\|} \begin{pmatrix} bc \sin v \cos u \\ ac \sin v \sin u \\ ab \cos v \end{pmatrix},$$

where  $\|\mathcal{S}_v \times \mathcal{S}_u\| = \sqrt{c^2 b^2 \sin^2 v \cos^2 u + c^2 a^2 \sin^2 v \sin^2 u + a^2 b^2 \cos^2 v}$ .

By formula (4.2), the pedal surface of  $\mathcal{S}$  with respect the pedal point  $P(x_0, y_0, z_0)$  is

$$(5.1) \quad p(u, v) = \sin v \frac{(bc(k - x_0) \sin v \cos u + a(-cy_0 \sin v \sin u - bz_0 \cos v + bc))}{\|\mathcal{S}_v \times \mathcal{S}_u\|^2} \begin{pmatrix} bc \sin v \cos u \\ ac \sin v \sin u \\ ab \cos v \end{pmatrix} + (x_0, y_0, z_0)^t.$$

This general case implies some long computations, but we can again consider the case when  $P \equiv O$  is the origin. In this case we obtain

$$(5.2) \quad p(u, v) : \begin{cases} x(u, v) = \frac{b^2 c^2 (k \cos u \sin v + a)}{\|\mathcal{S}_v \times \mathcal{S}_u\|^2} \sin v \cos u \\ y(u, v) = \frac{abc^2 (k \cos u \sin v + a)}{\|\mathcal{S}_v \times \mathcal{S}_u\|^2} \sin v \sin u \\ z(u, v) = \frac{ab^2 c (k \cos u \sin v + a)}{\|\mathcal{S}_v \times \mathcal{S}_u\|^2} \cos v. \end{cases}$$

From (3.4) we can see that

$$(5.3) \quad a^2 x^2 + b^2 y^2 + c^2 z^2 = \frac{a^2 b^4 c^4 (k \cos u \sin v + a)^2}{\|\mathcal{S}_v \times \mathcal{S}_u\|^4}.$$

On the other hand,

$$(5.4) \quad x^2 + y^2 + z^2 - kx = \frac{ab^2 c^2 (k \cos u \sin v + a)}{\|\mathcal{S}_v \times \mathcal{S}_u\|^2},$$

and by comparing (5.3) and (5.4) we get the implicit equation of the pedal surface:

$$(5.5) \quad (x^2 + y^2 + z^2 - kx)^2 = a^2 x^2 + b^2 y^2 + c^2 z^2.$$

Finding an explicit form of general conditions for the strongly convexity of the ellipsoid pedal involves some long computations, that we omit. Some numerical simulations show that for instance, in the case  $k = 1/3$ ,  $a = b = 2$ ,  $c = \sqrt{6}$ , we indeed obtain a strongly convex surface, see Figure 4.

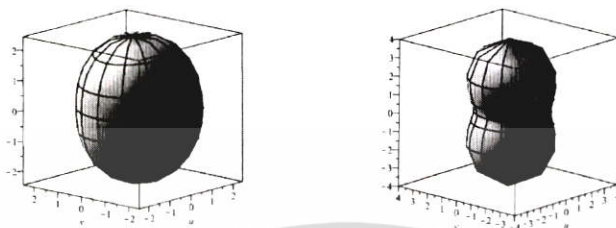


Figure 4: The convex pedal surface of an ellipsoid, with pedal point at origin, for  $k = 1/3$ ,  $a = b = 2$ ,  $c = \sqrt{6}$  and the non-convex case for  $k = 1/3$ ,  $a = b = 2$ ,  $c = 4$ .

### 5.2.2 The Finsler metric

Applying Okubo's method to (5.5) we obtain the Minkowski norm  $F = \frac{x^2 + y^2 + z^2}{\sqrt{a^2 x^2 + b^2 y^2 + c^2 z^2 + kx}}$  that is clearly the generalization of (3.5),

The Finsler metric corresponding to this indicatrix surface is of the type (3.6), where  $\alpha_1, \alpha_2$  are two different Riemannian metrics. This the generalization of the discussion in Section 3.2.

We can summarize

**Theorem 5.3.** *The Finsler metric on a three dimensional manifold  $M$  whose indicatrix is given by the pedal surface of the ellipsoid  $(\frac{x-k}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$  with origin as pedal point is of type  $F = \frac{\alpha_1^2}{\alpha_2 - \beta}$ , where  $\alpha_1, \alpha_2$  are two different Riemannian metrics and  $\beta$  is a linear form in  $TM$ . This Finsler metric is strongly convex subject to some conditions for  $a, b, c$  and  $k$ .*

**Remark 5.2.** Similarly with the sphere case, without giving here the concrete computations, one can easily see that the same formulas are true for the arbitrary dimensional case as well.

The Finsler metric on an  $n$ -dimensional manifold  $M$  whose indicatrix is given by the pedal hypersurface of an ellipsoid with origin as pedal point is a slope type metric  $F = \frac{\alpha_1^2}{\alpha_2 - \beta}$ , where  $\alpha_1, \alpha_2$  are two different Riemannian metrics on  $M$  and  $\beta$  a linear one form on  $TM$ . This Finsler metric is strongly convex for some further conditions on the constants giving the axes of the ellipsoid and the coordinates of its center.

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## References

- [1] P. L. Antonelli, R. S. Ingarden, M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Acad. Publ., FTP Vol. 58, 1993.

- [2] D. Bao, S. S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer, Graduate texts in Mathematics, 2000.
- [3] P. Chansri, P. Chansangiam, S. V. Sabau, *The geometry on the slope of a mountain*, arXiv:1811.02123 [math.DG], 2018.
- [4] C. G. Gibson, *Elementary Geometry of Algebraic Curves*, Cambridge University Press, 1998.
- [5] S. Markvorsen, *A Finsler geodesic spray paradigm for wildfire spread modeling*, Nonlinear Analysis: Real World Applications (2015), 28, 208–228.
- [6] M. Matsumoto, *A slope of mountain is a Finsler surface with respect to a time measure*, J. Math. Kyoto Univ., 29(1989), 17–25.
- [7] S. V. Sabau, K. Shibuya, H. Shimada, *Metric structures associated to Finsler metrics*, Publ. Math. Debrecen, 84 (2014), no. 1-2, 89–103.
- [8] S. V. Sabau, K. Shibuya, R. Yoshikawa, *Geodesics on strong Kropina manifolds*, European J. Math vol.3 (2017), 1172–1224.
- [9] I. D. Teodorescu, S. D. Teodorescu, *Collection of problems of elementary geometry, Didactical and Pedagogical Editorial House*, Bucharest, 1975, 61–65.
- [10] C. Udriste, *Finsler-Lagrange-Hamilton structures associated to control systems*, Eds: M. Anastasiei, P.L. Antonelli, Proceedings of a Conference held on August 26-31, 2001, Iasi, Romania, Kluwer Academic Publishers, 2003, 233–243.
- [11] R. Yoshikawa, S. V. Sabau, *Kropina metrics and Zermelo navigation on Riemannian manifolds*, Geometriae Dedicata, August 2014, Vol. 171 (1), 199–148.

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### Academic Publications

1. with P. Chansangiam, S. V. Sabau, Surface of revolution admitting strongly convex slope metrics, bulletin of the transilvania university of brasov series iii mathematics informatics physics, 2020.
2. with P. Chansangiam, S. V. Sabau, Finslerian indicatrices as algebraic curves and surfaces, Balkan Journal of Geometry and Its Applications, 2020.