FORMULAE OF THE FROBENIUS NUMBER FOR THREE RELATIVELY PRIME LUCAS NUMBERS

RATCHANOK BOKAEW

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
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MATHEMATICS
DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG
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Thesis Title	Formulae of the Frobenius number for three relatively prime Lucas numbers
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Student ID	59605022
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Abstract

The Frobenius number for positive integers $a_1, a_2, \ldots, a_n (n \geq 2)$ with $\gcd(a_1, \ldots, a_n) = 1$ is the largest positive integer that cannot be representable as a nonnegative integer combination of $a_1, \ldots, a_n$. In this work, we investigate the formulae of the Frobenius number for three relatively prime Lucas numbers by using the idea of Marín et al.'s work in 2007.

Keywords : Frobenius number, Lucas numbers, numerical semigroup

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Ratchanok Bokaew

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Chapter 1

Introduction

1.1 Research Motivation

Let $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ be integers. Any expression of the form $c_1a_1+c_2a_2+\ldots+c_na_n$ where $c_1, c_2, \ldots, c_n$ are integers, is called a linear combination of $a_1, a_2, \ldots, a_n$. Given positive integers $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ with gcd$(a_1, \ldots, a_n) = 1$, the Frobenius Problem is a problem to determine the largest positive integer $g(a_1, \ldots, a_n)$ that cannot be representable as a nonnegative integer combination of $a_1, \ldots, a_n$. The number $g(a_1, \ldots, a_n)$ is called the Frobenius number.

Definition 1.1. The Frobenius number of $a_1, a_2, \ldots, a_n$, denoted by $g\,(a_1, a_2, \ldots, a_n)$, is the largest integer Z such that $Z \neq c_1a_1 + c_2a_2 + \ldots + c_na_n$ for all nonnegative integers $c_1, c_2, \ldots, c_n$.

The Frobenius Problem is well known as the coin problem that asks for the largest monetary amount that cannot be obtained using only coins in the set of coin denominations which has no common dividor greater than 1. This problem is also referred to as the McNugget number, that was introduced by Henri Picciotto. For the applications of Frobenius Problem, we can use this problem to obtain upper bounds for the running time of the fundamental sorting Shellsort algorithm, to study tiling problem and to investigate Algebraic Geometric codes.

Let $a_1, \ldots, a_n$ be positive integers. We denote by $R = R(a_1, \ldots, a_n)$ the n-dimensional rectangle of sides a_i, that is, $R = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|0 \leq x_i \leq a_i,\ i = 1, \ldots, n\}$. A n-dimensional rectangle R is said to be tiled with tiles (n-dimensional rectangles) $R_1, \ldots, R_k$ if R can be filled entirely with copies of R_i, $1 \leq i \leq k$ (rotations are not allowed). The main result of this problem is given by Theorem 1.1 (below) stating that a sufficiently large n-dimensional rectangle can be tiled with a set of $n + k - 1$ tiles if any k-subset of the set of 1-coordinates (set of the first lengths) of the tiles are relatively primes and the set of j-coordinates (set of the j^{th} lengths) of the tiles are pairwise relatively prime for each $j = 2, \ldots, n$.

Theorem 1.1. ([8]). Let $k \geq 2$ and $n \geq 1$ be integers. Let $R_i(x_1^i, \ldots, x_n^i),\ i = 1, \ldots, n+k-1$ be integers formed with integers $x_j^i \geq 2$ such that

(a) gcd$(x_1^{i_1}, \ldots, x_1^{i_k}) = 1$ for any $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n + k - 1\}$ and

(b) gcd$(x_j^{i_1}, x_j^{i_2}) = 1$ for any $\{i_1, i_2\} \subset \{1, \ldots, n + k - 1\}$ and any $j = 2, \ldots, n$.

Let $g_1 = \max\{g(x^1_{i_1},\dots,x^1_{i_k})\,|\{i_1,\dots,i_k\}\subset\{1,\dots,n+k-1\}\}$ and

$$g_l = \max\left\{g(\frac{x_l^{i_1}\cdots x_l^{i_{l+k-2}}}{x_l^{i_{l+k-2}}},\dots,\frac{x_l^{i_1}\cdots x_l^{i_{l+k-2}}}{x_l^{i_1}})\,|\{i_1,\dots,i_{l+k-2}\}\subseteq\{1,\dots,n+k-1\}\right\}$$

for each $l = 2,\dots,n$.

Then $R(a_1,\dots,a_n)$ can be tiled with tiles $R_1,\dots,R_{n+k-1}$ if $a_i > \max_{1\le l\le n}\{g_l\}$ for all j.

Example 1.2. Let $R_1 = (22,3,3), R_2 = (14,5,5), R_3 = (21,2,2), R_4 = (15,7,7)$ and $R_5 = (55,11,11)$. In this case, we have $k = n = 3$.

$$\begin{aligned} g_1 &= \max\{g(22,14,21), g(22,14,15), g(22,14,55), g(22,21,15), g(22,21,55), g(22,15,55),\\ &\quad g(14,21,15), g(14,21,55), g(14,15,55), g(21,15,55)\}\\ &= \max\{139,91,173,181,243,97,288,151,179\} = 288. \end{aligned}$$

With $l = 2$ we obtain

$$\begin{aligned} g_2 &= \max\{g(3\cdot5,3\cdot2,5\cdot2), g(3\cdot5,3\cdot7,5\cdot7), g(3\cdot5,3\cdot11,5\cdot11), g(3\cdot2,3\cdot7,2\cdot7), g(3\cdot2,3\cdot11,2\cdot11),\\ &\quad g(3\cdot7,3\cdot11,7\cdot11), g(5\cdot2,5\cdot7,2\cdot7), g(5\cdot2,5\cdot11,2\cdot11), g(5\cdot7,5\cdot11,7\cdot11), g(2\cdot7,2\cdot11,7\cdot11)\}\\ &= \max\{g(15,6,10), g(15,21,35), g(15,33,55), g(6,21,14), g(6,33,22), g(21,33,77),\\ &\quad g(10,35,14), g(10,55,22), g(35,55,77), g(14,22,77)\}\\ &= \max\{29,139,227,43,71,331,81,133,603,195\} = 603. \end{aligned}$$

And with $l = 3$

$$\begin{aligned} g_3 &= \max\{g(3\cdot5\cdot2,3\cdot5\cdot7,3\cdot2\cdot7,5\cdot2\cdot7), g(3\cdot5\cdot2,3\cdot5\cdot11,3\cdot2\cdot11,5\cdot2\cdot11), g(3\cdot5\cdot7,3\cdot5\cdot11,3\cdot7\cdot11,5\cdot7\cdot11),\\ &\quad g(5\cdot2\cdot7,5\cdot2\cdot11,5\cdot7\cdot11,2\cdot7\cdot11)\}\\ &= \max\{g(30,105,42,70), g(30,165,66,110), g(105,165,231,385), g(70,110,385,154)\}\\ &= \max\{383,619,2579,1591\} = 2579. \end{aligned}$$

Therefore, Theorem 1.1 implies that $R(a_1,a_2,a_3)$ can be tiled with tiles $R_1,\dots,R_5$ if $a_1,a_2,a_3 > \max\{g_1,g_2,g_3\} = \{288,603,2579\} = 2579$.

The origin of this problem for $n = 2$ was proposed by Sylvester (1884) [15], which its solution was solved by Curran Sharp (1884) [2] :

$$g(a_1,a_2) = (a_1-1)(a_2-1)-1 = a_1a_2-a_1-a_2. \tag{1.1}$$

Roberts (1956) [13] found the Frobenius number of an arithmetical sequence :

$$g(a,a+d,\dots,a+kd) = a\left\lfloor\frac{a-2}{k}\right\rfloor + d(a-1). \tag{1.2}$$

For $n = 3$, Selmer and Beyer (1978) [14] solved Frobenius Problem by using a continued fraction algorithm. Then Rödseth (1978) [12] improved their result. Greenberg (1988) [5] found another algorithm.

In 21st century, the Frobenius Problem is still an interesting problem. There are several works about this problem as follows. Marín et al. (2007) [9] investigated the Frobenius number of Fibonacci numbers F_i, F_{i+2}, F_{i+k} for integers $i,k \ge 3$ where F_n is the nth term of the Fibonacci sequence defined by $F_n = F_{n-1}+F_{n-2}$, $n \ge 3$ with $F_1 = 1$

and $F_2 = 1$. They found that

$$g(F_i, F_{i+2}, F_{i+k}) = \begin{cases} (F_i - 1)F_{i+2} - F_i(rF_{k-2} + 1), & \text{if } r = 0 \text{ or } r \geq 1 \text{ and} \\ & F_{k-2}F_i < (F_i - rF_k)F_{i+2}, \\ (rF_k - 1)F_{i+2} - F_i((r-1)F_{k-2} + 1), & \text{otherwise.} \end{cases}$$

where $r = \left\lfloor \frac{F_i - 1}{F_k} \right\rfloor$ for $i, k \geq 3$. Later on, Ýlhan and Kýper (2008) [18] established the Frobenius number involving Lucas numbers L_n defined by $L_n = L_{n-1} + L_{n-2}$, $n \geq 3$ with $L_1 = 1$ and $L_2 = 3$. They found the formulae for

$$g(L_i, L_{i+1}, L_{i+k}) = L_i L_{i+1} - L_i - L_{i+1} \qquad \text{for } i, k \geq 2$$

$$g(L_i, L_{i+2}, L_{i+3}) = L_i \left\lfloor \frac{L_i - 2}{2} \right\rfloor + L_{i+1}(L_i - 1) \qquad \text{for } i \geq 3$$

$$g(L_{3i}, L_{3i} + 2, 2L_{3i} + 1) = \frac{(L_{3i})^2}{2} + L_{3i} - 1 \qquad \text{for } i \geq 1.$$

Moreover, Ong and Ponomarenko (2008) [10] solved the Frobenius Problem for sets of the form $\{m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k\}$, where m, n are relatively prime positive integers :

$$g(m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k) = n^{k-1}(mn - m - n) + \frac{(n-1)m^2(m^{k-1} - n^{k-1})}{m - n} \qquad (1.3)$$

for any integer k.

Gil et al. (2015) [4] found the Frobenius number of primitive Pythagorean triples :

$$g(m^2 - n^2, 2mn, m^2 + n^2) = (m-1)(m^2 - n^2) + (m-1)(2mn) - (m^2 + n^2). \qquad (1.4)$$

Recently, Tripathi (2017) [17] gave exact formulae for $g(a_1, a_2, a_3)$, where a_1, a_2, a_3 are pairwise coprime positive integers. His results are divided into several cases and are complicated, so we do not record them here.

In this work, we investigate the Frobenius number $g(L_i, L_{i+2}, L_{i+l})$ for integers $i \geq 3, l \geq 4$ by using the idea of Marín et al.'s work (2007) [9] and generalizes the work of Ýlhan and Kýper (2008)[18]. Our work needs the well-known Theorem of Brauer and Shockley (1962) [1] that states as following :

Theorem 1.3. ([1]). Let $1 < a_1 < \ldots < a_n$ be integers such that gcd$(a_1, \ldots, a_n) = 1$. Let $B = \{c_1a_1 + ... + c_na_n \mid c_i \in \mathbb{N} \cup \{0\}$ for all $i = 1, 2, \ldots, n\}$. Then

$$g(a_1, \ldots, a_n) = \max_{l \in \{1, 2, \ldots, a_1 - 1\}} \{t_l^*\} - a_1$$

where t_l^* is the smallest positive integers congruent to l modulo a_1 and $t_l^* \in B$.

Note that Theorem 1.3 can give the value for $g(a_1, \ldots, a_n)$; however, the formulae is not in the closed form and it is difficult to find t_l^* for each l. In our work, we are able to give an explicit formulae for $g(L_i, L_{i+2}, L_{i+l})$.

1.2 Objectives of the study

To prove the formulae that can be used to find the results on the Frobenius numbers of certain Lucas numerical semigroups which are generated by Lucas numbers.

1.3 Scopes of the study

1) Study the Frobenius problem.
2) Study the Frobenius number in three variables.
3) Investigate the Frobenius number in relatively prime three Lucas numbers.

1.4 Benefits of the study

To obtain a new formulae for solving the Frobenius number for numerical semigroups generated by Lucas numbers.

1.5 Research methodology

1) Study preliminaries of Frobenius problems.
2) Study research papers and textbook concerning Frobenius number.
3) Study definitions and properties of Fibonacci number, Lucas number and Frobenius number.
4) Determine the objectives and scopes of the research.
5) Study advanced topics in Frobenius number in three variables.
6) Study the works of Marín et al. [9] and Ýlhan and Kýper [18].
7) Prove the formulae for solving the Frobenius number in relatively prime three Lucas numbers.
8) Conclude the results, make suggestions for further works and write the thesis.

Table 1.1*:* The research schedule

<table>
<tr><th rowspan="3">Activity</th><th colspan="9">Time frame</th></tr>
<tr><th colspan="2">2016</th><th colspan="3">2017</th><th colspan="3">2018</th><th>2019</th></tr>
<tr><th>Aug.-Sep.</th><th>Oct.-Dec.</th><th>Jan.-May.</th><th>Jun.-Aug.</th><th>Sep.-Dec.</th><th>Jan.-May.</th><th>Jun.-Aug.</th><th>Sep.-Dec.</th><th>Jan.-Apr.</th></tr>
<tr><td>Step 1</td><td>⟶</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 2</td><td></td><td colspan="2">⟶</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 3</td><td></td><td></td><td>⟶</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 4</td><td></td><td></td><td></td><td>⟶</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 5</td><td></td><td></td><td></td><td></td><td colspan="2">⟶</td><td></td><td></td><td></td></tr>
<tr><td>Step 6</td><td></td><td></td><td></td><td></td><td></td><td colspan="2">⟶</td><td></td><td></td></tr>
<tr><td>Step 7</td><td></td><td></td><td></td><td></td><td></td><td></td><td colspan="2">⟶</td><td></td></tr>
<tr><td>Step 8</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>⟶</td></tr>
</table>

Chapter 2

Preliminaries

The purpose of this chapter is to collect lemmas, definitions, theorems and terminologys for used throughout of the thesis.

2.1 Necessary results

Definition 2.1. ***Fibonacci numbers*** are the numbers in the integer sequence defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

for integer $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$.

Definition 2.2. ***Lucas numbers*** are the numbers in the integer sequence defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}$$

for an integer $n \geq 2$ with $L_0 = 2$ and $L_1 = 1$.

Theorem 2.1. ([16]). For a nonnegative integer n, we have

(a) gcd$(L_n, L_{n+1}) = 1$,

(b) gcd$(L_n, L_{n+2}) = 1$.

Theorem 2.2. ([7]). If n and m are integers, then $L_n = L_m F_{n-m+1} + L_{m-1} F_{n-m}$.

A linear combination is an expression constructed from a set of terms by multiplying each term by a constant and adding the results.

Definition 2.3. Let $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ be integers. Any expression of the form $c_1a_1 + c_2a_2 + \ldots + c_na_n$ where $c_1, c_2, \ldots, c_n$ are integers, is called a linear combination of $a_1, a_2, \ldots, a_n$.

Definition 2.4. Let a and b be integers and let m be a positive integer. We say a is ***congruent*** to b modulo m , written

$$a \equiv b \pmod{m},$$

if and only if $m \,|(a-b)$.

Theorem 2.3. ([11]). Let m be a positive integer.

(a) For all integers a we have $a \equiv a$ (mod m).

(b) For all integers a and b we have $a \equiv b$ (mod m) if and only if $b \equiv a$ (mod m).

(c) For all integers a, b and c we have that if $a \equiv b$ (mod m) , $b \equiv c$ (mod m), then $a \equiv c$ (mod m).

(d) If a is any integer, then $m\,|a$ if and only if $a \equiv 0$ (mod m).

Definition 2.5. Let m be a positive integer. If $x \equiv a$ (mod m), then a is called a ***residue*** of x modulo m. If $0 \leq a \leq m-1$, then a is called the ***least residue*** of x modulo m.

Definition 2.6. Any set of m numbers that is congruent to exactly one of these least residue is called a ***complete residue system*** modulo m.

Definition 2.7. An integer g is said to be a ***common divisor*** of the integers a and b if $g\,|a$ and $g\,|b$. The largest one of these common divisors is called ***the greatest common divisor*** (gcd) and is denoted by gcd(a, b).

Moreover, the greatest common divisor of integers $a_1, ..., a_n$. The greatest common divisor of $n(\geq 2)$ positive integers $a_1, ..., a_n$ is the largest positive integer that divides each a_i. It is denoted by gcd$(a_1, ..., a_n)$.

Theorem 2.4. ([7]). Let $a_1, a_2, ..., a_n$ be $n(\geq 2)$ positive integers. Then

$$\text{gcd}(a_1, a_2, ..., a_n) = \text{gcd}(\text{gcd}(a_1, a_2, ..., a_{n-1}), a_n).$$

Definition 2.8. Two integers a and b are said to be ***relatively prime*** or ***coprime***, if the only positive integer that divides both of them is 1. Consequently, any prime number that divides one does not divide the other. This is equivalent to their gcd being 1.

Definition 2.9. The integers $a_1, a_2, \ldots, a_n$ are called ***pairwise relatively prime*** if gcd$(a_i, a_j) = 1$ for all $i \neq j$.

In the case of more than two integers, a distinction has to be made between relatively prime and pairwise relatively prime.

Definition 2.10. If x is a real number, then $\lfloor x \rfloor$ denoted ***the greatest integer*** function of x. It is the largest integer less than or equal to x. In other words, it is that integer n such that $n \leq x < n+1$.

Theorem 2.5. ([7]). Let a and b be two integers with $b > 0$. Then there exist unique integers q, r such that $a = qb + r$, where $0 \leq r < b$. The integer q and r are called the quotient and the remainder, respectively.

Definition 2.11. A ***semigroup*** is a pair $(S, *)$ where S is a non-empty set and $*$ is an associative binary operation on S. [i.e. $*$ is a function $S \times S \longrightarrow S$ with $(a, b) \longmapsto a * b$ and for all $a, b, c \in S$ we have $a * (b * c) = (a * b) * c$].

Definition 2.12. Let $a_1, \ldots, a_n$ be a set of positive integers ($n \geq 2$) such that gcd$(a_1, \ldots, a_n) = 1$. The ***numerical semigroup*** generated by $a_1, \ldots, a_n$ is the set $\{c_1a_1 + \ldots + c_na_n\}$ with $c_1, \ldots c_n$ be nonnegative integers, which we sometimes refer to as $\langle a_1, \ldots, a_n \rangle$.

2.2 Literature reviews

2.2.1 The work of Marín et al. ([9])

Theorem 2.6. Let $i, k \geq 3$ be integers and let $r = \left\lfloor \frac{F_i - 1}{F_k} \right\rfloor$,

$$g(F_i, F_{i+2}, F_{i+k}) = \begin{cases} (F_i - 1)F_{i+2} - F_i(rF_{k-2} + 1), & \text{if } r = 0 \text{ or } r \geq 1 \text{ and} \\ & F_{k-2}F_i < (F_i - rF_k)F_{i+2}, \\ (rF_k - 1)F_{i+2} - F_i((r-1)F_{l-2} + 1), & \text{otherwise.} \end{cases}$$

Example 2.7. Let $i = 3$ and $k = 5$. Then $r = \left\lfloor \frac{F_3 - 1}{F_5} \right\rfloor = 0$, and by Theorem 2.6 , we have

$$g(F_3, F_5, F_8) = g(2, 5, 21) = (F_3 - 1)F_5 - F_3((0)F_3 + 1) = 5 - 2(1) = 3.$$

We would like to confirm the value of $g(2, 5, 21)$ by the well-known Theorem 1.3.
Note that $g(F_3, F_5, F_8) = g(2, 5, 21) = \max_{k \in \{1\}} \{t_k^*\} - 2$.
Then we have to find t_k^* for $k = 1$, where t_k^* is the smallest positive integer congruent to k modulo $F_3 = 2$ and $t_k^* \in B$.
We get $t_1^* = 5$. Thus $g(F_3, F_5, F_8) = 5 - 2 = 3$ which is the same value obtained by Theorem 2.6.

Example 2.8. Take $i = 4$ and $k = 3$. Then $r = \left\lfloor \frac{F_4 - 1}{F_3} \right\rfloor = 1$, and $F_1F_4 < F_4 - ((1)F_3)F_6$.
Thus $g(F_4, F_6, F_7) = g(3, 8, 13) = (F_4 - 1)F_6 - F_4((1)F_1 + 1) = 10$.
On the other hand, by using Theorem 1.3, $g(F_4, F_6, F_7) = g(3, 8, 13) = \max_{l \in \{1,2\}} \{t_k^*\} - 3$.
We get $t_1^* = 13$ and $t_2^* = 8$. Thus $g(F_4, F_6, F_7) = \max\{13,\ 8\} - 3 = 10$ which is the same value as above.

Example 2.9. Take $i = 11$ and $k = 6$. Then $r = \left\lfloor \frac{F_{11} - 1}{F_6} \right\rfloor = 11$, and $F_4F_{11} > (F_{11} - 11F_6)F_{13}$.
Thus $g(F_{11}, F_{13}, F_{17}) = g(89, 233, 1597) = (11F_6 - 1)F_{13} - F_{11}(10F_4 + 1) = 17,512$.

2.2.2 The work of Ýlhan and Kýper ([18])

Theorem 2.10. Let $S = \langle L_n, L_{n+1}, L_{n+k} \rangle$ for $n, k \geq 2$. Then, the Frobenius number of Lucas numerical semigroup S is $g(S) = L_nL_{n+1} - L_n - L_{n+1}$.

Example 2.11. Let $n = 3$, $k = 2$ and we compute $g(L_3, L_4, L_5)$ by using Theorem 2.10.
We get $g(L_3, L_4, L_5) = g(4, 7, 11)$. Thus

$$g(L_3, L_4, L_5) = g(4, 7, 11) = L_3L_4 - L_3 - L_4 = 4(7) - 4 - 7 = 17.$$

Theorem 2.12. Let $S = \langle L_n, L_{n+2}, L_{n+3}\rangle$ for $n \geq 3$. Then, the Frobenius number of Lucas numerical semigroup S is $g(S) = L_n(\lfloor \frac{L_n-2}{2} \rfloor) + L_{n+1}(L_n - 1)$, where $\lfloor x \rfloor$ is the greatest integer and smaller than x, for a real number x.

Example 2.13. Let $n = 3$ and we compute $g(L_3, L_5, L_6)$ by using Theorem 2.12.
We get $g(L_3, L_5, L_6) = g(4, 11, 18)$. Thus

$$g(L_3, L_5, L_6 = g(4, 11, 18) = L_3\left(\left\lfloor \frac{L_3 - 2}{2} \right\rfloor\right) + L_4(L_3 - 1) = 4\left(\left\lfloor \frac{4-2}{2} \right\rfloor\right) + 7(4 - 1) = 25.$$

Theorem 2.14. Let $S = \langle L_{3n}, L_{3n} + 2, 2L_{3n} + 1\rangle$ for $n \geq 1$. Then, the Frobenius number of Lucas numerical semigroup S is $g(S) = \frac{(L_{3n})^2}{2} + L_{3n} - 1$.

Example 2.15. Let $n = 2$ and we compute $g(L_6, L_6 + 2, 2L_6 + 1)$ by using Theorem 2.14.
We get $g(L_6, L_6 + 2, 2L_6 + 1) = g(18, 20, 37)$. Thus

$$g(L_6, L_6 + 2, 2L_6 + 1) = g(18, 20, 37) = \frac{(L_6)^2}{2} + L_6 - 1 = \frac{(18)^2}{2} + 18 - 1 = 179.$$

Chapter 3

Necessary Lemmas

In this chapter, we give necessary lemmas to prove our main theorem. By Theorem 2.1 (a), we have gcd$(L_i, L_{i+1}) = 1$, thus gcd$(L_i, L_{i+1}, L_l) = 1$ for any integer $l \geq i+2$. Because L_l for $l \geq i+2$ can be written as a combination of L_i and L_{i+1} that is $L_l = L_{i+m} = L_{i+1}F_m + L_iF_{m-1}$, thus

$$g(L_i, L_{i+1}, L_l) = g(L_i, L_{i+1}) = L_iL_{i+1} - L_i - L_{i+1}$$

when $l = i + m$ for some integer $m \geq 2$.

In this work we will consider $g(L_i, L_{i+2}, L_{i+l})$ for integer $l \geq 3$, by Theorem 2.1 (b), we have gcd$(L_n, L_{n+2}) = 1$, hence gcd $(L_n, L_{n+2}, L_{i+l}) = 1$ for any integer $l \geq 3$.
In the case $l = 3$, we have

$$g(L_i, L_{i+2}, L_{i+3}) = g(L_i, L_i + L_{i+1}, L_i + 2L_{i+1})$$

for all $i \geq 1$.

Before investigating the value of $g(L_i, L_{i+2}, L_{i+l})$ for $l \geq 4$, we establish some lemmas. By Theorem 1.3, for fixed integer $l \geq 4$, we get

$$g(L_i, L_{i+2}, L_{i+l}) = \max_{k \in \{1,2,\ldots,L_i-1\}} \{t_k^*\} - L_i \tag{3.1}$$

where t_k^* is the smallest positive integer congruent to k modulo L_i and $t_k^* = xL_{i+2} + yL_{i+l}$ for some $x, y \geq 0$.

Then we shall construct the table, denoted by T_1, having entries $t_{x,y} = xL_{i+2} + yL_{i+l}$ for integers $x, y \geq 0$.

Table 3.1: Table T_1 : Showing $t_{x,y} = xL_{i+2} + yL_{i+l}$ for some $x, y \geq 0$

x\y	0	1	2	...
0	0	L_{i+l}	$2L_{i+l}$	...
1	L_{i+2}	$L_{i+2} + L_{i+l}$	$L_{i+2} + 2L_{i+l}$	...
2	$2L_{i+2}$	$2L_{i+2} + L_{i+l}$	$2L_{i+2} + 2L_{i+l}$	...
3	$3L_{i+2}$	$3L_{i+2} + L_{i+l}$	$3L_{i+2} + 2L_{i+l}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$F_l - 2$	$(F_l - 2)L_{i+2}$	$(F_l - 2)L_{i+2} + L_{i+l}$	$(F_l - 2)L_{i+2} + 2L_{i+l}$	...
$F_l - 1$	$(F_l - 1)L_{i+2}$	$(F_l - 1)L_{i+2} + L_{i+l}$	$(F_l - 1)L_{i+2} + 2L_{i+l}$	...
F_l	F_lL_{i+2}	$F_lL_{i+2} + L_{i+l}$	$F_lL_{i+2} + 2L_{i+l}$	...
$F_l + 1$	$(F_l + 1)L_{i+2}$	$(F_l + 1)L_{i+2} + L_{i+l}$	$(F_l + 1)L_{i+2} + 2L_{i+l}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 3.2*:* Table T_1 : Showing $t_{x,y} = xL_{i+2} + yL_{i+l}$ for some $x, y \geq 0$ (continuing)

x\y	$\dots$	$r-1$	r	$\dots$
0	$\dots$	$(r-1)L_{i+l}$	rL_{i+l}	$\dots$
1	$\dots$	$L_{i+2} + (r-1)L_{i+l}$	$L_{i+2} + rL_{i+l}$	$\dots$
2	$\dots$	$2L_{i+2} + (r-1)L_{i+l}$	$2L_{i+2} + rL_{i+l}$	$\dots$
3	$\dots$	$3L_{i+2} + (r-1)L_{i+l}$	$3L_{i+2} + rL_{i+l}$	$\dots$
$\vdots$		$\vdots$	$\vdots$	
$F_l - 2$	$\dots$	$(F_l - 2)L_{i+2} + (r-1)L_{i+l}$	$(F_l - 2)L_{i+2} + rL_{i+l}$	$\dots$
$F_l - 1$	$\dots$	$(F_l - 1)L_{i+2} + (r-1)L_{i+l}$	$(F_l - 1)L_{i+2} + rL_{i+l}$	$\dots$
F_l	$\dots$	$F_lL_{i+2} + (r-1)L_{i+l}$	$F_lL_{i+2} + rL_{i+l}$	$\dots$
$F_l + 1$	$\dots$	$(F_l + 1)L_{i+2} + (r-1)L_{i+l}$	$(F_l + 1)L_{i+2} + rL_{i+l}$	$\dots$
$\vdots$		$\vdots$	$\vdots$	

Since

$$L_{i+l} = L_{i+2}F_{l-1} + L_{i+1}F_{l-2} = L_{i+2}(F_l - F_{l-2}) + (L_{i+2} - L_i)F_{l-2} = F_lL_{i+2} - F_{l-2}L_i,$$

we get

$$t_{x,y} = xL_{i+2} + yL_{i+l} = xL_{i+2} + y(F_lL_{i+2} - F_{l-2}L_i) = (x + yF_l)L_{i+2} - yF_{l-2}L_i.$$

Thus the table T_1 can be represented as the table T_2.

Table 3.3*:* Table T_2 : Showing $t_{x,y} = (x + yF_l)L_{i+2} - yF_{l-2}L_i$ for some $x, y \geq 0$

x\y	0	1	2	$\dots$
0	0	$F_lL_{i+2} - F_{l-2}L_i$	$2F_lL_{i+2} - 2F_{l-2}L_i$	$\dots$
1	L_{i+2}	$(1 + F_l)L_{i+2} - F_{l-2}L_i$	$(1 + 2F_l)L_{i+2} - 2F_{l-2}L_i$	$\dots$
2	$2L_{i+2}$	$(2 + F_l)L_{i+2} - F_{l-2}L_i$	$(2 + 2F_l)L_{i+2} - 2F_{l-2}L_i$	$\dots$
3	$3L_{i+2}$	$(3 + F_l)L_{i+2} - F_{l-2}L_i$	$(3 + 2F_l)L_{i+2} - 2F_{l-2}L_i$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$F_l - 1$	$(F_l - 1)L_{i+2}$	$(2F_l - 1)L_{i+2} - F_{l-2}L_i$	$(3F_l - 1)L_{i+2} - 2F_{l-2}L_i$	$\dots$
F_l	F_lL_{i+2}	$2F_lL_{i+2} - F_{l-2}L_i$	$3F_lL_{i+2} - 2F_{l-2}L_i$	$\dots$
$F_l + 1$	$(F_l + 1)L_{i+2}$	$(2F_l + 1)L_{i+2} - F_{l-2}L_i$	$(3F_l + 1)L_{i+2} - 2F_{l-2}L_i$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 3.4: Table T_2 : Showing $t_{x,y} = (x + yF_l)L_{i+2} - yF_{l-2}L_i$ for some $x, y \geq 0$ (continuing)

x\y	...	$r-1$	r	...
0	...	$(r-1)F_lL_{i+2} - (r-1)F_{l-2}L_i$	$rF_lL_{i+2} - rF_{l-2}L_i$	...
1	...	$(1+(r-1)F_l)L_{i+2} - (r-1)F_{l-2}L_i$	$(1+rF_l)L_{i+2} - rF_{l-2}L_i$	...
2	...	$(2+(r-1)F_l)L_{i+2} - (r-1)F_{l-2}L_i$	$(2+rF_l)L_{i+2} - rF_{l-2}L_i$	...
3	...	$(3+(r-1)F_l)L_{i+2} - (r-1)F_{l-2}L_i$	$(3+rF_l)L_{i+2} - rF_{l-2}L_i$	...
$\vdots$		$\vdots$	$\vdots$	
$F_l - 1$	...	$(rF_l - 1)L_{i+2} - (r-1)F_{l-2}L_i$	$((r+1)F_l - 1)L_{i+2} - rF_{l-2}L_i$	...
F_l	...	$rF_lL_{i+2} - (r-1)F_{l-2}L_i$	$(r+1)F_lL_{i+2} - rF_{l-2}L_i$	...
$F_l + 1$	...	$(rF_l + 1)L_{i+2} - (r-1)F_{l-2}L_i$	$(r+1)F_lL_{i+2} - rF_{l-2}L_i$	...
$\vdots$		$\vdots$	$\vdots$	

From now on, we define the set $T_{F_l-1,\infty}$ to contain the first $F_l - 1$ entries of all columns in the table T_2. That is

$$T_{F_l-1,\infty} = \{\ t_{x,y} \mid 0 \leq x \leq F_l - 1 \text{ and } y \geq 0\ \}.$$

Throughout the paper, we set $r = \left\lfloor \frac{L_i - 1}{F_l} \right\rfloor$ and $L_i - 1 = rF_l + q$ for some integer $0 \leq q \leq F_l - 1$. Let $T_{F_l-1,r}$ be the set that contains the first $F_l - 1$ entries of columns $0, 1, 2, \ldots, r-1$ and the first q entries of column r, i.e.,

$$T_{F_l-1,r} = \{\ t_{x,y} \mid 0 \leq x \leq F_l - 1 \text{ and } 0 \leq y \leq r-1\ \} \cup \{t_{0,r}, t_{1,r}, \ldots, t_{q,r}\}.$$

Lemma 3.1. (i) The set $T_{F_l-1,r}$ is a complete residue system modulo L_i.

(ii) In the table T_1, $t_{m,n} \leq t_{j,k}$ for all $m \leq j$ and $n \leq k$. Moreover, $t_{m+1,n} < t_{m,n+1}$ for all $0 \leq m, n \leq F_l - 2$.

Proof. (i) For each $t_{x,y} = (x + yF_l)L_{i+2} - yF_{l-2}L_i \in T_{F_l-1,r}$, we have $0 \leq x + yF_l \leq q + rF_l = L_i - 1$. Since $\gcd(L_i, L_{i+2}) = 1$, $T_{F_l-1,r}$ is a complete residue system modulo L_i.

(ii) Recall that $t_{m,n} = mL_{i+2} + nL_{i+l}$ and $t_{j,k} = jL_{i+2} + kL_{i+l}$.
It is obvious that for $m \leq j$, then

$$mL_{i+2} \leq jL_{i+2},$$

$$mL_{i+2} + nL_{i+l} \leq jL_{i+2} + nL_{i+l},$$

$$t_{m,n} \leq t_{j,n},$$

and for $n \leq k$, then

$$nL_{i+l} \leq kL_{i+l},$$

$$mL_{i+2} + nL_{i+l} \leq mL_{i+2} + nL_{i+l},$$

$$t_{m,n} \leq t_{m,k}.$$

Therefore, $t_{m,n} \le t_{j,k}$ for $m \le j$ and $n \le k$.
For $0 \le m, n \le F_l - 2$, we have

$$\begin{aligned} t_{m,n+1} - t_{m+1,n} &= (mL_{i+2} + (n+1)L_{i+l}) - ((m+1)L_{i+2} + nL_{i+l}) \\ &= (mL_{i+2} - (m+1)L_{i+2}) + ((n+1)L_{i+l} - nL_{i+l}) \\ &= L_{i+l} - L_{i+2} > 0. \end{aligned}$$

Therefore, $t_{m+1,n} < t_{m,n+1}$ for all $0 \le m, n \le F_l - 2$.

□

We define t_x ,for $x = 0, 1, 2, \ldots$, as follows:

$$\begin{array}{cccccc} t_0 = t_{0,0} & t_{F_l} = t_{0,1} & t_{2F_l} = t_{0,2} & \cdots & t_{rF_l} = t_{0,r} & \cdots \\ t_1 = t_{1,0} & t_{F_l+1} = t_{1,1} & t_{2F_l+1} = t_{1,2} & \cdots & t_{rF_l+1} = t_{1,r} & \cdots \\ t_2 = t_{2,0} & t_{F_l+2} = t_{2,1} & t_{2F_l+2} = t_{2,2} & \cdots & t_{rF_l+2} = t_{2,r} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ t_{F_l-1} = t_{F_l-1,0} & t_{2F_l-1} = t_{F_l-1,1} & t_{3F_l-1} = t_{F_l-1,2} & \cdots & t_{(r+1)F_l-1} = t_{F_l-1,r} & \cdots \end{array}$$

The elements of $T_{F_l-1,\infty}$ can be represented as $t_x = xL_{i+2} - \left\lfloor \frac{x}{F_l} \right\rfloor F_{l-2}L_i$ for $x = 0, 1, \ldots$.

Lemma 3.2. Let $t_{u,v}$ be an entry of T_1 and $t_{u,v} \notin T_{F_l-1,r}$. Then there exist $t_{x,y} \in T_{F_l-1,r}$ such that $t_{u,v} \equiv t_{x,y} \pmod{L_i}$ and $t_{u,v} > t_{x,y}$.

Proof. By the definition of t_x as defined above, the set $T_{F_l-1,r}$ can be written as

$$T_{F_l-1,r} = \{t_0, \ldots, t_{F_l-1}, t_{F_l}, \ldots, t_{2F_l-1}, t_{2F_l}, \ldots, t_{3F_l-1}, \ldots, t_{rF_l}, \ldots, t_{rF_l+q} = t_{L_i-1}\}.$$

We will consider two cases as follows.
Case 1: $t_{u,v} \in T_{F_l-1,\infty} \backslash T_{F_l-1,r}$
Then $t_{u,v} = t_{aL_i+b}$ for some integer $a \ge 1$ and $0 \le b \le L_i - 1$. We see that

$$\begin{aligned} t_{aL_i+b} &= (aL_i + b)L_{i+2} - \left\lfloor \frac{aL_i + b}{F_l} \right\rfloor F_{l-2}L_i \\ &\equiv bL_{i+2} - \left\lfloor \frac{b}{F_l} \right\rfloor F_{l-2}L_i \pmod{L_i} \\ &\equiv t_b \pmod{L_i}. \end{aligned}$$

Since $0 \le b \le L_i - 1$, $t_b = t_{x,y} \in T_{F_l-1,r}$ for some x, y.
That is, $t_{u,v} \equiv t_{x,y} \pmod{L_i}$.
Next, we will show that $t_{u,v} > t_{x,y}$, i.e., $t_{aL_i+b} > t_b$.
Since $t_{aL_i+b} \ge t_{L_i+b}$ for $a \ge 1$, it is enough to show only that $t_{L_i+b} > t_b$.
Recall that $r = \left\lfloor \frac{L_i-1}{F_l} \right\rfloor$ and $L_i - 1 = rF_l + q$ for some $0 \le q \le F_l - 1$.
We will consider into two subcases depending on the value of r.

Subcase 1.1: Suppose $r = 0$. Since $L_i - 1 < F_l$, it implies that $L_i \le F_l$.

Since $0 \leq b \leq L_i - 1$, we have $L_i + b \leq L_i + L_i - 1$ so $L_i + b \leq 2F_l - 1$.
If $0 \leq L_i + b \leq F_l - 1$, then both t_b and t_{L_i+b} are in the first column of the table T_1.
By Lemma 3.1 (ii), we obtain $t_{L_i+b} > t_b$.

Suppose that $F_l \leq L_i + b \leq 2F_l - 1$. Then t_b and t_{L_i+b} are in the first and second columns of the table T_1, respectively.
If $L_i < \frac{F_l}{2}$, then $L_i + b < \frac{F_l}{2} + \frac{F_l}{2} = F_l$, a contradiction.
Hence we have $F_{l-2} \leq \frac{F_l}{2} \leq L_i$.
Finally, we have

$$\begin{aligned} t_{L_i+b} - t_b &= L_i L_{i+2} - F_{l-2} L_i \\ &= L_i \left(L_{i+2} - F_{l-2}\right) \\ &> L_i \left(L_i - F_{l-2}\right) \\ &> 0. \end{aligned}$$

Subcase 1.2: Suppose that $r \geq 1$.
Consider

$$\begin{aligned} t_{L_i+b} - t_b &= \left((L_i + b)L_{i+2} - \left\lfloor \frac{L_i + b}{F_l} \right\rfloor F_{l-2} L_i\right) - \left(bL_{i+2} - \left\lfloor \frac{b}{F_l} \right\rfloor F_{l-2} L_i\right) \\ &= ((L_i + b)L_{i+2} - bL_{i+2}) - \left(\left\lfloor \frac{L_i + b}{F_l} \right\rfloor F_{l-2} L_i - \left\lfloor \frac{b}{F_l} \right\rfloor F_{l-2} L_i\right) \\ &= ((L_i + b) - b)\, L_{i+2} - \left(\left\lfloor \frac{L_i + b}{F_l} \right\rfloor - \left\lfloor \frac{b}{F_l} \right\rfloor\right) F_{l-2} L_i \\ &= L_i \left(L_{i+2} - F_{l-2} \left(\left\lfloor \frac{L_i + b}{F_l} \right\rfloor - \left\lfloor \frac{b}{F_l} \right\rfloor\right)\right). \end{aligned}$$

Write $b = mF_l + n$ where $0 \leq n \leq F_l - 1$.
Since $L_i - 1 = rF_l + q$ with $0 \leq q \leq F_l - 1$, it follows that

$$\begin{aligned} \left\lfloor \frac{L_i + b}{F_l} \right\rfloor - \left\lfloor \frac{b}{F_l} \right\rfloor &= \left\lfloor \frac{L_i - 1 + b + 1}{F_l} \right\rfloor - m \\ &= \left\lfloor \frac{rF_l + q + mF_l + n + 1}{F_l} \right\rfloor - m \\ &\leq r + 1. \end{aligned}$$

It is enough to show that $L_{i+2} > (r + 1)\, F_{l-2}$.
To this end, we see that, for $r \geq 1$,

$$\begin{aligned} L_{i+2} - (r + 1)\, F_{l-2} &= L_i + L_{i+1} - (r + 1)\, F_{l-2} \\ &= rF_l + q + 1 + L_{i+1} - (r + 1)\, F_{l-2} \\ &= r\left(F_l - F_{l-2}\right) - F_{l-2} + q + 1 + L_{i+1} \\ &= rF_{l-1} - F_{l-2} + q + 1 + L_{i+1} > 0. \end{aligned}$$

Case 2: $t_{u,v} \notin T_{F_l-1,\infty}$

By Lemma 3.1 (i), $T_{F_l-1,r}$ is a complete system of residue modulo L_i, there exists $t_{x,y} \in T_{F_l-1,r}$ such that $t_{u,v} \equiv t_{x,y} \pmod{L_i}$.

Then $0 \leq x \leq F_l - 1 < u$.

If $v \geq y$, by Lemma 3.1 (ii), $t_{x,y} \leq t_{x,v} < t_{u,v}$.

Suppose $v < y$. Then $t_{u,v} \equiv t_{x,y} \pmod{L_i}$ implies $u + vF_l \equiv x + yF_l \pmod{L_i}$.

From Lemma 3.1 (i), $0 \leq x + yF_l \leq L_i - 1$, thus $u + vF_l > x + yF_l$.

Since $-vF_{l-2}L_i > -yF_{l-2}L_i$, we have $t_{u,v} > t_{x,y}$. □

Chapter 4

Formulae of the Frobenius number for three relatively prime Lucas numbers

In this chapter, we prove the formulae for solving the Frobenius number for numerical semigroups generated by L_i, L_{i+2} and L_{i+l}.

Theorem 4.1. Let $i \geq 3$, $l \geq 4$ be integers and $r = \left\lfloor \frac{L_i - 1}{F_l} \right\rfloor$. Then

$$g(L_i,\, L_{i+2},\, L_{i+l}) = \begin{cases} (L_i - 1)L_{i+2} - (1 + rF_{l-2})L_i, & \text{if } 1.)\ r = 0, \\ & \text{or } 2.)\ r \geq 1 \text{ and } (L_i - rF_l)L_{i+2} > F_{l-2}L_i, \\ (rF_l - 1)L_{i+2} - (1 + (r-1)F_{l-2})L_i, & \text{otherwise.} \end{cases}$$

Proof. Let $T^* = \{t_0^*, t_1^*, ..., t_{F_i-1}^*\}$. Now we will consider t_k^* for $k = 1, 2, \ldots, L_i - 1$ when t_k^* is the smallest positive integer congruent to k modulo L_i and t_k^* can be written as $xL_{i+2} + yL_{i+l}$ for some integers $x, y \geq 0$.
Recall that $t_x = xL_{i+2} - \left\lfloor \frac{x}{F_l} \right\rfloor F_{l-2}L_i$ for $x = 0, 1, \ldots$.
If $r = 0$, by Lemma 3.2, we have that t_x is the smallest positive integer congruent to k modulo L_i for some integer $0 \leq k \leq L_i - 1$ and we see that t_x can be represented as a linear combination of L_{i+2} and L_{i+l}.
Hence $T_{F_l-1,r} = T^*$.
If $r \geq 1$, by Lemma 3.1 (ii), then

$$t_{F_l-1,i} = \max_{0 \leq x \leq F_l - 1} \{t_{x,i} \mid t_{x,i} \in T_{F_l-1,r}\} \quad \text{for each } i = 0, 1, \ldots, r-1,$$

$$t_{F_l-1,r-1} = \max_{0 \leq i \leq r-1} \{t_{F_l-1,i} \mid t_{F_l-1,i} \in T_{F_l-1,r}\},$$

and

$$t_{k,r} = \max_{0 \leq x \leq k} \{t_{x,r} \mid t_{x,r} \in T_{F_l-1,r}\}.$$

We will find the necessary condition for $t_{k,r} > t_{F_l-1,r-1}$.
It is true if and only if $(L_i - 1)L_{i+2} - F_{l-2}L_i > (rF_l - 1)L_{i+2} - (r-1)F_{l-2}L_i$ that is $(L_i - rF_l)L_{i+2} > F_{l-2}L_i$.
Hence we can conclude the result of this theorem. □

Example 4.2. Let $i = 3$ and $l = 5$. Then $r = \left\lfloor \frac{L_3 - 1}{F_5} \right\rfloor = 0$, and by our main theorem, we have

$$g(L_3, L_5, L_8) = g(4, 11, 47) = (L_3 - 1)L_5 - (1 + (0)F_3)L_3 = 3(11) - 1(4) = 29.$$

We would like to confirm the value of $g(4, 11, 47)$ by the well-known Theorem 1.3.
Since $g(L_3, L_5, L_8) = g(4, 11, 47) = \max_{k \in \{1,2,3\}} \{t_k^*\} - 4.$

Then we have to find t_k^* for each $k = 1, 2, 3,$ that t_k^* is the smallest positive integer congruent to k modulo $L_3 = 4$ and $t_k^* \in B$.
We get $t_1^* = 33\,,\ t_2^* = 22$ and $t_3^* = 11$.
Thus $g(L_3, L_5, L_8) = \max\{33, 22, 11\} - 4 = 29$ which is the same value obtained by our result.

Example 4.3. Take $i = 4$ and $l = 4$. Then $r = \left\lfloor \frac{L_4 - 1}{F_4} \right\rfloor = 2$, and $(L_4 - 2F_4)L_6 > F_2 L_4$.
Thus $g(L_4, L_6, L_8) = g(7, 18, 47) = (L_4 - 1)L_6 - (1 + 2F_2)L_4 = 87$.
On the other hand, by using Theorem 1.3,

$$g(L_4, L_6, L_8) = g(7, 18, 47) = \max_{l \in \{1,2,3,4,5,6\}} \{t_k^*\} - 7.$$

We get $t_1^* = 36, t_2^* = 65, t_3^* = 94, t_4^* = 18, t_5^* = 47$ and $t_6^* = 83$.
Thus $g(L_4, L_6, L_8) = \max\{36,\ 65,\ 94,\ 18,\ 47,\ 83\} - 7 = 87$ which is the same value as above.

Example 4.4. Take $i = 13$ and $l = 7$. Then $r = \left\lfloor \frac{L_{13} - 1}{F_7} \right\rfloor = 40$, and $(L_{13} - 40F_7)L_{15} < F_5 L_{13}$.
Thus $g(L_{13}, L_{15}, L_{20}) = g(521, 1364, 15127) = (40F_7 - 1)L_{15} - (1 + 39F_5)L_{13} = 605,800$.

Chapter 5

Conclusions

In this thesis, we find the formulae of the Frobenius number for three relatively prime Lucas numbers L_i, L_{i+2} and L_{i+l} for integers $i \geq 3$ and $l \geq 4$ denoted by $g(L_i,\, L_{i+2},\, L_{i+l})$. We use the idea of Marín et al.'s work [9].

Let $r = \left\lfloor \frac{L_i - 1}{F_l} \right\rfloor$. If the value of r is either $r = 0$ or $r \geq 1$ and $(L_i - rF_l)L_{i+2} > F_{l-2}L_i$, then $g(L_i,\, L_{i+2},\, L_{i+l}) = (L_i - 1)L_{i+2} - (1 + rF_{l-2})L_i$. Otherwise , $g(L_i,\, L_{i+2},\, L_{i+l}) = (rF_l - 1)L_{i+2} - (1 + (r-1)F_{l-2})L_i$.

We note that our work extends one theorem in Ýlhan and Kýper's work [18] as shown in Theorem 2.12 in Chapter 2.

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Appendix

Original Article

Formulae of the Frobenius number in relatively prime three Lucas numbers

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Abstract

In this paper, we find the explicit formulae of the Frobenius number for numerical semigroups generated by relatively prime three Lucas numbers L_i , L_{i+2} and L_{i+l} for given integers $i \geq 3$, $l \geq 4$.

Keywords: Frobenius number, Lucas numbers, Fibonacci numbers.

1. Introduction

Let $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ be integers. Any expression of the form $c_1a_1 + c_2a_2 + \ldots + c_na_n$ where $c_1, c_2, \ldots, c_n$ are integers, is called a linear combination of $a_1, a_2, \ldots, a_n$. Given positive integers $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ with $\gcd(a_1, \ldots, a_n) = 1$, the Frobenius Problem is a problem to determine the largest positive integer that cannot be representable as a nonnegative integer combination of $a_1, \ldots, a_n$.

Definition The Frobenius number of $a_1, a_2, \ldots, a_n$, denoted by $g(a_1, a_2, \ldots, a_n)$, is the largest integer Z such that $Z \neq c_1a_1 + c_2a_2 + \ldots + c_na_n$ for all nonnegative integers $c_1, c_2, \ldots, c_n$.

For example, $g(3,5) = 7$, $g(6,9,20) = 43$.

The Frobenius Problem is well known as the coin problem that asks for the largest monetary amount that cannot be obtained using only coins in the set of coin denominations which has no common divisor greater than 1. This problem is also referred to as the McNugget number introduced by Henri Picciotto. There are several applications of the Frobenius Problem, for example, obtaining upper bounds for the running time of the Shell-sort algorithm, studying partitions of vector spaces and investigating algebraic geometric codes; see Ramíres Alfonsín (2005).

The origin of this problem for $n=2$ was proposed by Sylvester (1884), which its solution was solved by Sharp (1884):

$$g(a_1,a_2)=(a_1-1)(a_2-1)-1=a_1a_2-a_1-a_2.$$

Roberts (1956) found the Frobenius number of an arithmetical sequence:

$$g(a,a+d,\ldots,a+kd)=a\left\lfloor\frac{a-2}{k}\right\rfloor+d(a-1).$$

For $n=3$, Selmer and Beyer (1978) solved Frobenius Problem by using a continued fraction algorithm. Then Rödseth (1978) improved their result. Greenberg (1988) found another algorithm.

In 21st century, the Frobenius Problem is still an interesting problem. There are several works about this problem as follows. Marín et al. (2007) investigated the Frobenius number of Fibonacci numbers F_i,F_{i+2},F_{i+k} for integers $i,k\geq 3$ where F_n is the n^{th} term of the Fibonacci sequence defined by $F_n=F_{n-1}+F_{n-2}$, $n\geq 3$ with $F_1=1$ and $F_2=1$. They found that

$$g(F_i,F_{i+2},F_{i+k})=\begin{cases}(F_i-1)F_{i+2}-F_i(rF_{k-2}+1), & \text{if } r=0 \text{ or } r\geq 1 \text{ and}\\ & F_{k-2}F_i<(F_i-rF_k)F_{i+2},\\ (rF_k-1)F_{i+2}-F_i((r-1)F_{k-2}+1), & \text{otherwise,}\end{cases}$$

where $r=\left\lfloor\frac{F_i-1}{F_k}\right\rfloor$ for $r,k\geq 3$. Later on, Ýlhan and Kýper (2008) established the Frobenius number involving Lucas numbers L_n defined by $L_n=L_{n-1}+L_{n-2}$, $n\geq 3$ with $L_1=1$ and $L_2=3$. They found following the formulae :

$$g(L_i, L_{i+1}, L_{i+k}) = L_i L_{i+1} - L_i - L_{i+1} \qquad \text{for } i,k \ge 2,$$

$$g(L_i, L_{i+2}, L_{i+3}) = L_i \left\lfloor \frac{L_i - 2}{2} \right\rfloor + L_{i+1}(L_i - 1) \qquad \text{for } i \ge 3,$$

and

$$g\left(L_{3i}, L_{3i}+2, 2L_{3i}+1\right) = \frac{L_{3i}^2}{2} + L_{3i} - 1 \qquad \text{for } i \ge 1.$$

Moreover, Ong and Ponomarenko (2008) solved the Frobenius Problem for sets of the form $\{m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k\}$, where m,n are relatively prime positive integers:

$$g(m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k) = n^{k-1}(mn - m - n) + \frac{(n-1)m^2(m^{k-1} - n^{k-1})}{m-n}$$

for any positive integer k. Gil et al. (2015) found the Frobenius number of primitive Pythagorean triples:

$$g(m^2 - n^2, 2mn, m^2 + n^2) = (m-1)(m^2 - n^2) + (m-1)(2mn) - (m^2 + n^2).$$

Recently, Tripathi (2017) gave an exact formula for $g(a_1, a_2, a_3)$, where a_1, a_2, a_3 are pairwise coprime positive integers. His results are divided into several cases and are complicated, so we do not record them here.

In the recent paper, we investigate the Frobenius number $g(L_i, L_{i+2}, L_{i+l})$ for integers $i \ge 3$, $l \ge 4$ by using the idea of Marín et al.'s work (2007) and generalize the work of Ýlhan and Kýper (2008). Our work needs the well-known Theorem of Brauer and Shockley (1962) that states as follows:

Theorem A Let $1 < a_1 < \ldots < a_n$ be integers such that $\gcd(a_1, \ldots, a_n) = 1$. Let $B = \{a_1x_1 + \ldots + a_nx_n \mid x_i \in \mathbb{N} \cup \{0\} \text{ for all } i = 1,2,\ldots,n\}$. Then

$$g(a_1, \ldots, a_n) = \max_{l \in \{1,2,\ldots,a_1-1\}} \{t_l\} - a_1,$$

where t_l is the smallest positive integers congruent to l modulo a_1 and $t_l \in B$.

Note that Theorem A can give the value for $g(a_1, \ldots, a_n)$; however, the formula is not in the closed form and it is difficult to find t_l for each l. In our work, we are able to give an explicit formula for $g\left(L_i, L_{i+2}, L_{i+l}\right)$.

2. Necessary Lemmas

Before investigating the value of $g(L_i, L_{i+2}, L_{i+l})$ for $i \ge 3$, $l \ge 4$, we establish some lemmas. By Theorem A, for fixed integer $i \ge 3$, $l \ge 4$, we get

$$g(L_i, L_{i+2}, L_{i+l}) = \max_{k\in\{1,2,\ldots,L_i-1\}}\{t_k^*\} - L_i$$

where t_k^* is the smallest positive integer congruent to k modulo L_i and $t_k^* = xL_{i+2} + yL_{i+l}$ for some $x, y \geq 0$. Then we shall construct the table, denoted by T_1, having entries $t_{x,y} = xL_{i+2} + yL_{i+l}$ for integers $x, y \geq 0$. Since

$$L_{i+l} = L_{i+2}F_{l-1} + L_{i+1}F_{l-2} = L_{i+2}(F_l - F_{l-2}) + (L_{i+2} - L_i)F_{l-2} = F_lL_{i+2} - F_{l-2}L_i\,,$$

we get

$$t_{x,y} = xL_{i+2} + yL_{i+l} = xL_{i+2} + y(F_lL_{i+2} - F_{l-2}L_i) = (x + yF_l)L_{i+2} - yF_{l-2}L_i.$$

Thus the table T_1 can be represented as the table T_2.

From now on, we define the set $T_{F_l-1,\infty}$ to contain the first $F_l - 1$ entries of all columns in the table T_2. That is

$$T_{F_l-1,\infty} = \{\, t_{x,y} \mid 0 \leq x \leq F_l - 1 \text{ and } y \geq 0 \,\}.$$

Throughout the paper, we set $r = \left\lfloor \dfrac{L_i - 1}{F_l} \right\rfloor$ and $L_i - 1 = rF_l + q$ for some integer $0 \leq q \leq F_l - 1$. Let $T_{F_l-1,r}$ be the set that contains the first $F_l - 1$ entries of columns $0, 1, 2, \ldots, r-1$ and the first q entries of column r, i.e.,

$$T_{F_l-1,r} = \{\, t_{x,y} \mid 0 \leq x \leq F_l - 1 \text{ and } 0 \leq y \leq r - 1 \,\} \cup \{t_{0,r}, t_{1,r}, \ldots, t_{q,r}\}.$$

Lemma 1 (i) The set $T_{F_l-1,r}$ is a complete system of residues modulo L_i.

(ii) In the table T_1, $t_{m,n} \leq t_{j,k}$ for all $m \leq j$ and $n \leq k$. Moreover, $t_{m+1,n} < t_{m,n+1}$ for all $0 \leq m, n \leq F_l - 2$.

Proof. (i) For each $t_{x,y} = (x + yF_l)L_{i+2} - yF_{l-2}L_i \in T_{F_l-1,r}$, we have $0 \leq x + yF_l \leq q + rF_l = L_i - 1$. Since $\gcd(L_i, L_{i+2}) = 1$, $T_{F_l-1,r}$ is a complete system of residues modulo L_i.

(ii) Recall that $t_{m,n} = mL_{i+2} + nL_{i+l}$ and $t_{j,k} = jL_{i+2} + kL_{i+l}$. It is obvious that for $m \leq j$, $t_{m,n} \leq t_{j,n}$ and for $n \leq k$, $t_{m,n} \leq t_{m,k}$. Therefore, $t_{m,n} \leq t_{j,k}$ for all $m \leq j$ and $n \leq k$. For $0 \leq m, n \leq F_l - 2$, we have $t_{m,n+1} - t_{m+1,n} = (F_l - 1)L_{i+2} - F_{l-2}L_i > 0$. □

We define t_x as follows:

$$\begin{array}{lllll} t_0 = t_{0,0} & t_{F_l} = t_{0,1} & t_{2F_l} = t_{0,2} \quad \cdots & t_{rF_l} = t_{0,r} & \cdots \\ t_1 = t_{1,0} & t_{F_l+1} = t_{1,1} & t_{2F_l+1} = t_{1,2} \quad \cdots & t_{rF_l+1} = t_{1,r} & \cdots \\ t_2 = t_{2,0} & t_{F_l+2} = t_{2,1} & t_{2F_l+2} = t_{2,2} \quad \cdots & t_{rF_l+2} = t_{2,r} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ t_{F_l-1} = t_{F_l-1,0} & t_{2F_l-1} = t_{F_l-1,1} & t_{3F_l-1} = t_{F_l-1,2} \quad \cdots & t_{(r+1)F_l-1} = t_{F_l-1,r} & \cdots . \end{array}$$

The elements of $T_{F_l-1,\infty}$ can be represented as $t_x = xL_{i+2} - \left\lfloor \dfrac{x}{F_l} \right\rfloor F_{l-2}L_i$ for $x = 0,1,\ldots$.

Lemma 2 Let $t_{u,v}$ be an entry of T_1 and $t_{u,v} \notin T_{F_l-1,r}$. Then there exist $t_{x,y} \in T_{F_l-1,r}$ such that $t_{u,v} \equiv t_{x,y} \pmod{L_i}$ and $t_{u,v} > t_{x,y}$.

Proof. By the definition of t_x as defined above, the set $T_{F_l-1,r}$ can be written as

$$T_{F_l-1,r} = \{t_0,\ldots,t_{F_l-1},t_{F_l},\ldots,t_{2F_l-1},t_{2F_l},\ldots,t_{3F_l-1},\ldots,t_{rF_l},\ldots,t_{rF_l+q} = t_{L_i-1}\}.$$

We will consider two cases as follows.

Case 1: $t_{u,v} \in T_{F_l-1,\infty} \setminus T_{F_l-1,r}$

Then $t_{u,v} = t_{aL_i+b}$ for some integer $a \ge 1$ and $0 \le b \le L_i - 1$. We see that

$$t_{aL_i+b} = (aL_i + b)L_{i+2} - \left\lfloor \frac{aL_i + b}{F_l} \right\rfloor F_{l-2}L_i \equiv bL_{i+2} - \left\lfloor \frac{b}{F_l} \right\rfloor F_{l-2}L_i = t_b \pmod{L_i}.$$

Since $0 \le b \le L_i - 1$, $t_b = t_{x,y} \in T_{F_l-1,r}$ for some x, y. That is, $t_{u,v} \equiv t_{x,y} \pmod{L_i}$. Next, we will show that $t_{u,v} > t_{x,y}$, i.e., $t_{aL_i+b} > t_b$. Since $t_{aL_i+b} \ge t_{L_i+b}$ for $a \ge 1$, it is enough to show only that $t_{L_i+b} > t_b$. Recall that $r = \left\lfloor \dfrac{L_i - 1}{F_l} \right\rfloor$ and $L_i - 1 = rF_l + q$ for some $0 \le q \le F_l - 1$. We will consider into two subcases depending on the value of r.

Subcase 1.1: If $r = 0$, then $L_i - 1 < F_l$, so $L_i + b \le 2F_l - 1$. If $0 \le L_i + b \le F_l - 1$, then both t_b and t_{L_i+b} are in the first column of the table T_1. By Lemma 1(ii), we obtain $t_{L_i+b} > t_b$.

Suppose that $F_l \le L_i + b \le 2F_l - 1$. Then t_b and t_{L_i+b} are in the first and second columns of the table T_1, respectively. If $L_i < \frac{F_l}{2}$, then $L_i + b < \frac{F_l}{2} + \frac{F_l}{2} = F_l$, a contradiction. Hence we have $F_{l-2} \le \frac{F_l}{2} \le L_i$. Finally, we have

$$t_{L_i+b} - t_b = L_i L_{i+2} - F_{l-2} L_i = L_i \left(L_{i+2} - F_{l-2} \right) > L_i \left(L_i - F_{l-2} \right) > 0.$$

Subcase 1.2: Suppose that $r \ge 1$. Consider

$$t_{L_i+b} - t_b = L_i \left(L_{i+2} - F_{l-2} \left(\left\lfloor \frac{L_i + b}{F_l} \right\rfloor - \left\lfloor \frac{b}{F_l} \right\rfloor \right) \right).$$

Write $b = mF_l + n$ where $0 \le n \le F_l - 1$. Since $L_i - 1 = rF_l + q$ with $0 \le q \le F_l - 1$, it follows that

$$\left\lfloor \frac{L_i + b}{F_l} \right\rfloor - \left\lfloor \frac{b}{F_l} \right\rfloor = \left\lfloor \frac{L_i - 1 + b + 1}{F_l} \right\rfloor - m = \left\lfloor \frac{rF_l + q + mF_l + n + 1}{F_l} \right\rfloor - m \le r + 1.$$

It is enough to show that $L_{i+2} > (r+1)F_{l-2}$. To this end, we see that

$$\begin{aligned} L_{i+2} - (r+1)F_{l-2} &= L_i + L_{i+1} - (r+1)F_{l-2} \\ &= rF_l + q + 1 + L_{i+1} - (r+1)F_{l-2} \\ &= r\left(F_l - F_{l-2}\right) - F_{l-2} + q + 1 + L_{i+1} \\ &= rF_{l-1} - F_{l-2} + q + 1 + L_{i+1} > 0 \end{aligned}$$

since $r \ge 1$.

Case 2: $t_{u,v} \notin T_{F_l-1,\infty}$

Since $T_{F_l-1,r}$ is a complete system of residue modulo L_i, there exists $t_{x,y} \in T_{F_l-1,r}$ such that $t_{u,v} \equiv t_{x,y} \pmod{L_i}$. Then $0 \le x \le F_l - 1 < u$. If $v \ge y$, by Lemma 1(ii), $t_{x,y} \le t_{x,v} < t_{u,v}$. Suppose $v < y$. Then $t_{u,v} \equiv t_{x,y} \pmod{L_i}$ implies $u + vF_l \equiv x + yF_l \pmod{L_i}$. From Lemma 1(i), $0 \le x + yF_l \le L_i - 1$, and thus $u + vF_l = m(x + yF_l)$ for some integer $m \ge 1$. Hence $u + vF_l \ge x + yF_l$. Since $-vF_{l-2}L_i > -yF_{l-2}L_i$, we have $t_{u,v} > t_{x,y}$. □

3. Main Theorem

Theorem Let $i \geq 3$, $l \geq 4$ be integers and $r = \left\lfloor \frac{L_i - 1}{F_l} \right\rfloor$. Then

$$g(L_i, L_{i+2}, L_{i+l}) = \begin{cases} (L_i - 1)L_{i+2} - (1 + rF_{l-2})L_i, & \text{if 1.) } r = 0, \\ & \text{or 2.) } r \geq 1 \text{ and } (L_i - rF_l)L_{i+2} > F_{l-2}L_i, \\ (rF_l - 1)L_{i+2} - (1 + (r-1)F_{l-2})L_i, & \text{otherwise.} \end{cases}$$

Proof. From Theorem A, now we have to consider t_k^* for $k = 1, 2, \ldots, L_i - 1$ when t_k^* is the smallest positive integer congruent to k modulo L_i and t_k^* can be written as $xL_{i+2} + yL_{i+l}$ for some integers $x, y \geq 0$. Since $t_x = xL_{i+2} - \left\lfloor \frac{x}{F_l} \right\rfloor F_{l-2}L_i$ for $x = 0, 1, \ldots$.

If $r = 0$, by Lemma 2, we have that t_x is the smallest positive integer congruent to k modulo L_i for some integer $0 \leq k \leq L_i - 1$. And we see that t_x can be represented as a linear combination of L_{i+2} and L_{i+l}. Hence $T_{F_l-1,r} = \{t_k^* \mid k = 1, 2, \ldots, L_i - 1\}$. If $r \geq 1$, by Lemma 1(ii), then

$$t_{F_l-1,i} = \max_{0 \leq x \leq F_l - 1} \{t_{x,i} \mid t_{x,i} \in T_{F_l-1,r}\} \quad \text{for each } i = 0, 1, \ldots, r-1,$$

$$t_{F_l-1,r-1} = \max_{0 \leq i \leq r-1} \{t_{F_l-1,i} \mid t_{F_l-1,i} \in T_{F_l-1,r}\},$$

and

$$t_{k,r} = \max_{0 \leq x \leq k} \{t_{x,r} \mid t_{x,r} \in T_{F_l-1,r}\}.$$

We will find the necessary condition for $t_{k,r} > t_{F_l-1,r-1}$. It is true if and only if $(L_i - 1)L_{i+2} - F_{l-2}L_i > (rF_l - 1)L_{i+2} - (r-1)F_{l-2}L_i$ that is $(L_i - rF_l)L_{i+2} > F_{l-2}L_i$. Hence we can conclude the result of this theorem. □

Example 1 Let $i=3$ and $l=5$. Then $r=\left\lfloor \frac{L_3-1}{F_5} \right\rfloor=0$, and by our main theorem, we have

$$g(L_3,L_5,L_8)=g(4,11,47)=(L_3-1)L_5-(1+(0)F_3)L_3=3(11)-1(4)=29.$$

We would like to confirm the value of $g(4,11,47)$ by the well-known Theorem A. Since $g(L_3,L_5,L_8)=g(4,11,47)=\max_{k\in\{1,2,3\}}\{t_k^*\}-4$. Then we have to find t_k^* for each $k=1,2,3$, that t_k^* is the smallest positive integer congruent to k modulo $L_3=4$ and $t_k^*\in B$. We get $t_1^*=33$, $t_2^*=22$ and $t_3^*=11$. Thus $g(L_3,L_5,L_8)=\max\{33,22,11\}-4=29$ which is the same value obtained by our result.

Example 2 Take $i=4$ and $l=4$. Then $r=\left\lfloor \frac{L_4-1}{F_4} \right\rfloor=2$, and $(L_4-2F_4)L_6>F_2L_4$.

Thus $g(L_4,L_6,L_8)=g(7,18,47)=(L_4-1)L_6-(1+2F_2)L_4=87$. On the other hand, by using Theorem A, $g(L_4,L_6,L_8)=g(7,18,47)=\max_{k\in\{1,2,3,4,5,6\}}\{t_k^*\}-7$. We get $t_1^*=36, t_2^*=65, t_3^*=94$, $t_4^*=18$, $t_5^*=47$ and $t_6^*=83$. Thus $g(L_4,L_6,L_8)=\max\{36, 65, 94, 18, 47, 83\}-7=87$ which is the same value as above.

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$x \backslash y$	0	1	2	…	r	…
0	0	L_{i+l}	$2L_{i+l}$	…	rL_{i+l}	
1	L_{i+2}	$L_{i+2}+L_{i+l}$	$L_{i+2}+2L_{i+l}$	…	$L_{i+2}+rL_{i+l}$	
2	$2L_{i+2}$	$2L_{i+2}+L_{i+l}$	$2L_{i+2}+2L_{i+l}$	…	$2L_{i+2}+rL_{i+l}$	
3	$3L_{i+2}$	$3L_{i+2}+L_{i+l}$	$3L_{i+2}+2L_{i+l}$	…	$3L_{i+2}+rL_{i+l}$	
⋮	⋮	⋮	⋮		⋮	
F_l-2	$(F_l-2)L_{i+2}$	$(F_l-2)L_{i+2}+L_{i+l}$	$(F_l-2)L_{i+2}+2L_{i+l}$	…	$(F_l-2)L_{i+2}+rL_{i+l}$	
F_l-1	$(F_l-1)L_{i+2}$	$(F_l-1)L_{i+2}+L_{i+l}$	$(F_l-1)L_{i+2}+2L_{i+l}$	…	$(F_l-1)L_{i+2}+rL_{i+l}$	
F_l	F_lL_{i+2}	$F_lL_{i+2}+L_{i+l}$	$F_lL_{i+2}+2L_{i+l}$	…	$F_lL_{i+2}+rL_{i+l}$	
F_l+1	$(F_l+1)L_{i+2}$	$(F_l+1)L_{i+2}+L_{i+l}$	$(F_l+1)L_{i+2}+2L_{i+l}$	…	$(F_l+1)L_{i+2}+rL_{i+l}$	
⋮	⋮	⋮	⋮		⋮	

Table T_1 : $t_{x,y}=xL_{i+2}+yL_{i+l}$ for $x,y\geq 0$

$x \backslash y$	0	1	2	…	r	…
0	0	$F_lL_{i+2}-F_{l-2}L_i$	$2F_lL_{i+2}-2F_{l-2}L_i$	…	$rF_lL_{i+2}-rF_{l-2}L_i$	…
1	L_{i+2}	$(1+F_l)L_{i+2}-F_{l-2}L_i$	$(1+2F_l)L_{i+2}-2F_{l-2}L_i$	…	$(1+rF_l)L_{i+2}-rF_{l-2}L_i$	…
2	$2L_{i+2}$	$(2+F_l)L_{i+2}-F_{l-2}L_i$	$(2+2F_l)L_{i+2}-2F_{l-2}L_i$	…	$(2+rF_l)L_{i+2}-rF_{l-2}L_i$	…
3	$3L_{i+2}$	$(3+F_l)L_{i+2}-F_{l-2}L_i$	$(3+2F_l)L_{i+2}-2F_{l-2}L_i$	…	$(3+rF_l)L_{i+2}-rF_{l-2}L_i$	…
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
F_l-1	$(F_l-1)L_{i+2}$	$(2F_l-1)L_{i+2}-F_{l-2}L_i$	$(3F_l-1)L_{i+2}-2F_{l-2}L_i$	…	$\left((r+1)F_l-1\right)L_{i+2}-rF_{l-2}L_i$	…
F_l	F_lL_{i+2}	$2F_lL_{i+2}-F_{l-2}L_i$	$3F_lL_{i+2}-2F_{l-2}L_i$	…	$(r+1)F_lL_{i+2}-rF_{l-2}L_i$	…
F_l+1	$(F_l+1)L_{i+2}$	$(2F_l+1)L_{i+2}-F_{l-2}L_i$	$(3F_l+1)L_{i+2}-2F_{l-2}L_i$	…	$\left((r+1)F_l+1\right)L_{i+2}-rF_{l-2}L_i$	…
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

Table T_2 : $t_{x,y}=(x+yF_l)L_{i+2}-yF_{l-2}L_i$ for $x,y\geq 0$

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