COMPOSITE DIFFERENTIAL TRANSFORM FORMULAE APPLIED TO NONLINEAR PLANE AUTONOMOUS SYSTEMS AND DIFFERENTIAL TRANSFORMATION METHOD FOR CIRCULAR MEMBRANE VIBRATIONS

URAIWAN SOMBOON

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN APPLIED MATHEMATICS

DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

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สูตรการแปลงเชิงอนุพันธ์ฟังก์ชันประกอบประยุกต์ใช้กับระบบอิสระบนระนาบไม่เชิงเส้นและวิธีการแปลงเชิงอนุพันธ์สำหรับการสั่นของเมมเบรนที่มีโดเมนเป็นวงกลม

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บทคัดย่อ

ในงานวิจัยนี้ได้แบ่งงานออกเป็นสองส่วนดังนี้ ส่วนที่หนึ่งได้นำเสนอที่มาการสร้างสูตรใหม่ของวิธีการแปลงเชิงอนุพันธ์ของผลคูณของฟังก์ชันประกอบ สูตรการแปลงที่ได้นี้ถูกนำไปประยุกต์ใช้กับระบบอิสระบนระนาบโดยใช้วิธีการแปลงเชิงอนุพันธ์ และวิธีการแปลงเชิงอนุพันธ์แบบหลายชั้น อนุกรมผลเฉลยที่ได้จากการประมาณโดยใช้ทั้งสองวิธีดังกล่าวถูกนำไปเปรียบเทียบกับผลเฉลยที่ได้จากทิศทางการไหลของสนามเวกเตอร์ที่หาได้จากระบบสมการตั้งต้นและเปรียบเทียบกับผลเฉลยเชิงวิเคราะห์ที่คำนวณโดยใช้วิธีระนาบเฟส ผู้วิจัยพบว่าผลเฉลยที่ได้จากวิธีการแปลงเชิงอนุพันธ์แบบหลายชั้น มีผลเฉลยที่ใกล้เคียงกับผลเฉลยเชิงวิเคราะห์และสอดคล้องกับทิศทางการไหลของสนามเวกเตอร์มากกว่าวิธีการแปลงเชิงอนุพันธ์ ในส่วนของงานส่วนที่สองนั้น ผู้วิจัยได้นำเสนอขั้นตอนของวิธีการแปลงเชิงอนุพันธ์แบบ 1 มิติ เพื่อที่จะประมาณอนุกรมผลเฉลยสำหรับปัญหาการสั่นของเมมเบรนที่มีโดเมนเป็นวงกลมภายใต้เงื่อนไขเริ่มต้นและเงื่อนไขขอบที่กำหนด โดยปัญหานี้ได้ศึกษาทั้งหมดสองกรณีด้วยกันคือ การสั่นที่ขึ้นกับรัศมีเพียงอย่างเดียว และการสั่นที่ขึ้นกับทั้งรัศมีและมุม ซึ่งในงานนี้ได้แสดงให้เห็นว่าการหาผลเฉลยโดยใช้วิธีการแปลงเชิงอนุพันธ์ง่ายกว่าผลเฉลยเชิงวิเคราะห์ในแง่ของการนำไปใช้เขียนโปรแกรม

คำสำคัญ : การสั่นของเมมเบรนที่มีโดเมนเป็นวงกลม การแปลงเชิงอนุพันธ์ การแปลงเชิงอนุพันธ์แบบหลายชั้น ระบบอิสระบนระนาบไม่เชิงเส้น

Thesis Title	Composite differential transform formulae applied to nonlinear plane autonomous systems and differential transformation method for circular membrane vibrations
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Abstract

In this research, the work is divided into two parts. The first part is to present a new derivation technique for new differential transfrom formulae of a product of composite functions. The composite formulae are applied to nonlinear plane autonomous systems by using the differential transformation method (DTM) and the multistep differential transformation method (MsDTM). The approximate series solutions estimated by the DTM and the MsDTM are compared with the flow direction of the vector fields defined by the original system and an analytical solution calculated by phase plane method. The MsDTM results are in better agreement with the analytical solution than the DTM ones are founded. The second part is to present the steps of one-dimensional DTM to find the series solutions for the vibrations of a circular membrane under the specified initial and boundary conditions. The problems will be studied in the both cases of vibrations depending only on radius and of the vibrations depending on both radius and angle. This work show that the DTM is easier to use than the analytical method from the point of view of programming.

Keywords : circular membrane vibrations, differential transformation method, nonlinear plane autonomous system

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Uraiwan Somboon

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Chapter 1

Introduction

1.1 Inception and importance

The differential transformation method (DTM) is an alternative procedure for obtaining an approximate Taylor series solution or the semi-analytical solution of differential equations and differential equation systems. The one dimensional DTM has been successfully applied to a wide class of non-linear ordinary differential equations (ODEs) arising in many areas of science and engineering such as viscous flow [26], predictive control [6], vibration ([5], [8], [15]) and steady heat conduction problems [7]. The main advantage of this method is that it can be applied directly to nonlinear differential equations without the requiring linearization and discretization. The concept of the DTM was first proposed by Zhou in 1986 (see [27]), who solved linear and nonlinear problems in electrical circuits and many other problems related to differential equations (see also [9], [17], [18], [19], [20] and [22]). To illustrate the DTM for solving differential equations, the basic definitions of the DTM are introduced as follows. The one-dimensional differential transform of the k-th derivative of a function $x(t)$ is defined as

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}, \quad k \in \mathbb{I}^+ \cup \{0\}. \tag{1.1}$$

In Eq.(1.1), $x(t)$ is called the original function and $X(k)$ is called the transformed function. The inverse one-dimensional differential transforms of $X(k)$ is defined as

$$x(t) = \sum_{k=0}^{\infty} X(k)(t-t_0)^k, \tag{1.2}$$

that is,

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}. \tag{1.3}$$

Although the DTM series solution gives a good approximation for some problems, in some cases, the seires solution diverges in wider domain. Due to this reason the muti-step differential transformation method (MsDTM) is used. The MsDTM is based on the DTM, but compared with other methods, it does not need small parameters, auxiliary functions and parameters, discretization. In this technique, the solution domain is divided in subdomains. The MsDTM is advantageous for applications in Physics. For instance, due to small time steps the MsDTM has a powerful accuracy especially for initial value problem (IVP) (see more [10], [11], [21], [23], [29]). In actual applications of the DTM, the approximate series solution of the IVP can be express by the finite series,

$$x(t) = \sum_{k=0}^{N} X(k)(t-t_0)^k, t \in [0,T]. \tag{1.4}$$

In fact, the MsDTM gives us the solution in the form,

$$x(t) = \begin{cases} x_1(t), & t \in [0,t_1] \\ x_2(t), & t \in [t_1,t_2] \\ \vdots \\ x_m(t), & t \in [t_{m-1},t_m], \end{cases} \tag{1.5}$$

where $t_i = ih$ $(h = \frac{T}{m})$, $x_i(t) = \sum\limits_{k=0}^{N} X_i(k)(t-t_i)^k$ and the initial condition $x_i^{(k)}(t_{i-1}) = x_{i-1}^{(k)}(t_{i-1})$. In particular, we are interested in the technique introduced by Chang [4], for calculating the DTM of nonlinear functions. The developed technique depends only on the fundamental operation properties of differential transform and calculus.

An autonomous system is a system of first-order differential equations of the form

$$\frac{dx_i}{dt} = g_i(x_1, x_2, ..., x_n), \ \ i = 1, 2, 3, ..., n, \tag{1.6}$$

which the right hand side does not explicitly depend on the independent variable. When the independent variable is time, they are also called time-invariant systems. Historically, autonomous systems first appeared in descriptions of physical processes with a finite number of degrees of freedom. Many laws in Physics, where the independent variable is usually assumed to be time, are expressed as autonomous systems because it is assumed the laws of nature which hold now are identical to those for any point in the past or future. Autonomous systems are closely related to dynamical systems. In the case $n = 2$, the system is called a plane autonomous system. The autonomous systems can be linear or nonlinear (see more [28]). In this research, we are interested in the case of nonlinear.

In this research, a derivation technique of new differential transform formulae for the product of composite functions is presented. The computation consists of three steps. The first step is finding the differential transformation of the product of two composite functions. The next step is finding the differential transformation for the higher order derivative of a power function. The last step is the derivation of the new differential transformation, calculated by using the general formulae of higher order derivatives of composite functions studied in [25] combined with the results from previous step. Then, the new differential transform formulae obtained are used to transform the nonlinear plane autonomous systems for finding the DTM and the MsDTM approximate solutions of the problem.

Circular membranes are important parts of drums, pumps, microphones, and other devices. This accounts for their great importance in engineering. We consider the case when the circular membrane is plane and its material is elastic, but offers no resistance to bending (this excludes thin metallic membranes). Then the vibrations of the circular membrane is given in the form of two-dimensional wave equation in polar coordinates,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right), \tag{1.7}$$
$$u(R,\theta,t) = 0 \text{ for all } t \geq 0, u(r,\theta,0) = f(r,\theta), u_t(r,\theta,0) = g(r,\theta).$$

where $0 \leq r \leq R, 0 \leq \theta \leq 2\pi, c^2 = T/\rho$ in term of the membrane's tension T and density ρ, R is a radius of a membrane, a membrane is fixed along the boundary circle radius $R, f(r,\theta)$ is the initial shape at time $t = 0$ and $g(r,\theta)$ is the initial velocity (see [16]).

In this research, we will show how to extend the method of differential transformation to the problem of vibrations of a circular membrane. The computation consists of three steps. The first step is using the method of separation of variables to obtain ODEs from the wave equation in Eq.(2.69). The next step is applying the DTM to ODEs from the previous step to obtain recursive relations. The last step is to find the coefficients of the series solutions for ODEs using the recursive relations.

1.2 Objectives of the study

1) To present a new derivation technique for new differential transfrom formulae of a product of composite functions.
2) To find series solutions of nonlinear plane autonomous systems by using differential transformation method (DTM) with composite transform formulae.
3) To present the steps of one-dimensional DTM for the vibration of a circular membrane.
4) To find the series solutions of the vibrations of a circular membrane under the specified initial and boundary conditions by using the DTM.

1.3 Scope of the study

1) The new derivation technique for new differential transform formulae of a product of composite functions are shown and these composite formulae are applied to approximate series solutions of nonlinear plane autonomous systems by using the differential transformation method (DTM) and the multi-step differential transformation method (MsDTM).
2) The one-dimensional DTM is applied to find the series solutions for the vibrations of a circular membrane under the specied initial and boundary conditions. This problems will be studied in the both cases of vibrations depending only on radius and of the vibrations depending on both radius and angle.

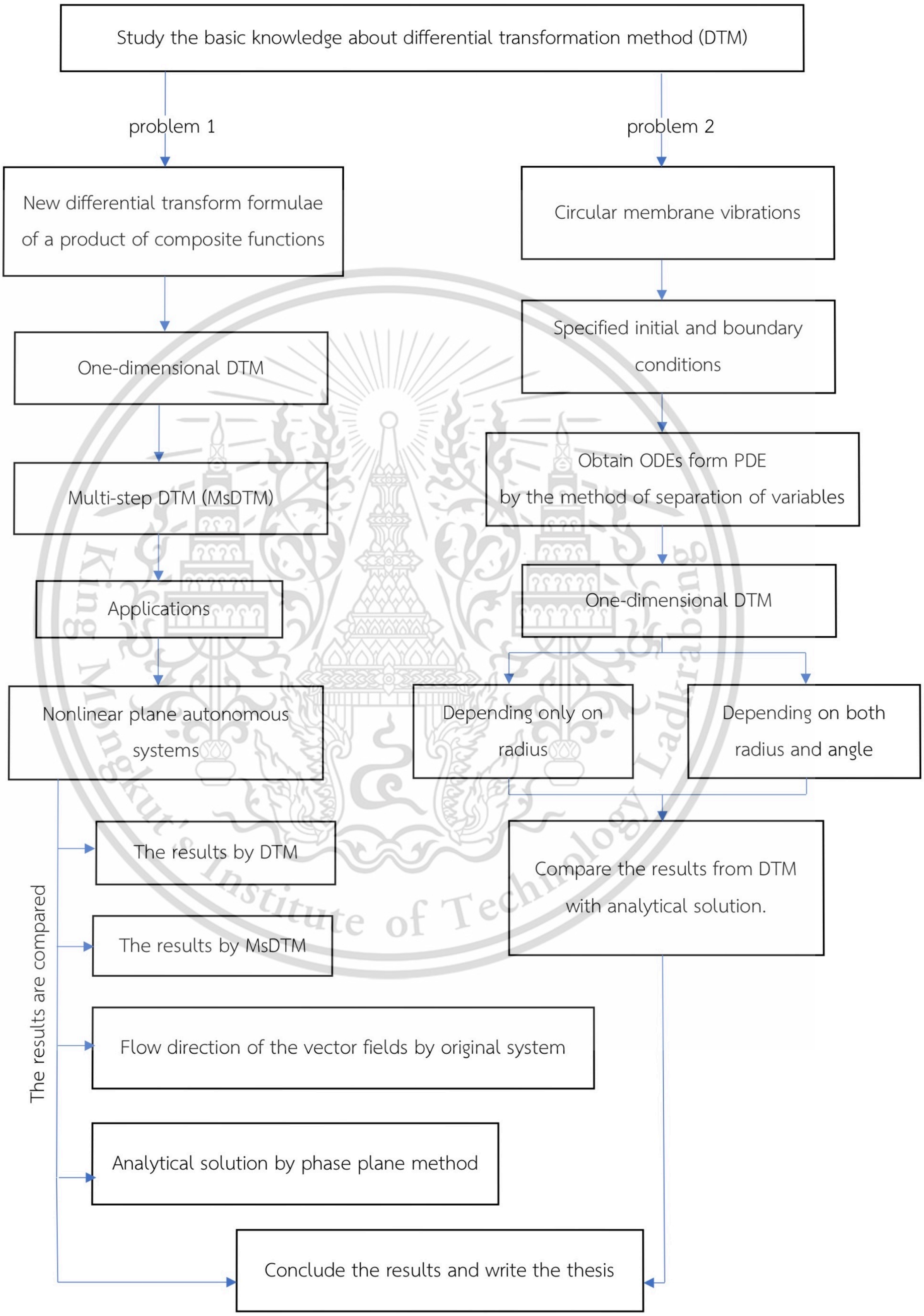
1.4 Benefits of the Study

1) Provide the new transform fomulae for differential transformation method of composite functions.

2) The multi-step differential transformation method combined with composite transform formulae have been successfully applied to solving nonlinear plane autonomous systems.

3) Develop a new mathematical method for solved the solutions of circular membrane vibrations.

1.5 Plan of study



Chapter 2

Preliminaries

The purpose of this chapter is to provide basic concept and tools in one-dimensional differential transformation method, multi-step differential transformation method, nonlinear plane autonomous system and the vibrations of a circular membrane used in the research.

2.1 One-dimensional differential transformation method

The basic definitions and fundamental operations of the differential transform are introduced as follows.

Definition 2.1. The one-dimensional differential transform of the function $x(t)$ is defined as

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}, \quad k \in \mathbb{I}^+ \cup \{0\}. \tag{2.1}$$

In Eq.(2.1), $x(t)$ is called the original function and $X(k)$ is called the transformed function.

Definition 2.2. The inverse one-dimensional differential transforms of $X(k)$ is defined as

$$x(t) = \sum_{k=0}^{\infty} X(k)(t-t_0)^k, \tag{2.2}$$

that is,

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}. \tag{2.3}$$

Equation (2.3) implies that the concept of differential transformation method is derived from Taylor series expansion. From the definition of Eqs. (2.1) and (2.2), it is easy to prove that the transformed functions comply with the following fundamental operations as shown in Table **??**. Actually, in concrete applications, the function $x(t)$ is expressed by a finite series and Eq.(2.2) becomes

$$x(t) = \sum_{k=0}^{N} X(k)(t-t_0)^k. \tag{2.4}$$

2.2 Multi-step differential transformation method

The multi-step differential transformation method (MsDTM) is advantageous for applications in Physics. For instance, due to small time steps the MsDTM has a powerful accuracy especially for initial value problem (IVP). Also, because it is based on the DTM, does not need to small parameter, auxiliary function and parameter, discretization, etc.,

Table 2.1: The fundamental operations of one-dimensional DTM.

Original function $x(t)$	Transformed function $X(k)$
$u(t) \pm v(t)$	$U(k) \pm V(k)$
$\lambda u(t)$	$\lambda U(k)$, λ is constant.
$u(t)v(t)$	$\sum_{r=0}^{k} U(r)V(k-r)$
$u(t)v(t)w(t)$	$\sum_{r=0}^{k}\sum_{l=0}^{r} U(l)V(r-l)W(k-r)$
$u_1(t)u_2(t)\cdots u_n(t)$	$\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_1=0}^{k_2} U_1(k_1)U_2(k_2-k_1)\cdots U_n(k-k_{n-1})$
$\frac{d^r}{dt^r}u(t)$	$\frac{(k+r)!}{k!}U(k+r)$
t^n	$\delta(k-n) = \begin{cases} 1, & \text{if } k=n \\ 0, & \text{if } k \neq n \end{cases}$
$e^{\lambda u(t)}$	$\frac{\lambda^k}{k!}$
$\sin(\omega u(t)+\alpha)$	$\frac{\omega^k}{k!}\sin\left(\frac{k\pi}{2}+\alpha\right)$
$\cos(\omega u(t)+\alpha)$	$\frac{\omega^k}{k!}\cos\left(\frac{k\pi}{2}+\alpha\right)$

versus other analytical methods. For perception of the Ms-DTM basic idea, consider a general equation of n-th order ordinary differential equation [14],

$$f\left(t, x, x', x'', ...x^{(n)}\right) = 0, \tag{2.5}$$

subject to the initial conditions

$$x^{(k)} = d_k, \quad k = 0, 1, 2, ..., n-1. \tag{2.6}$$

To illustrate the DTM for solving differential equations, the basic definition of the DTM are introduced as in section 2.1.

Let $[0,T]$ be the interval over which we want to find the solution of the IVP. In accual applications of the DTM, the approximate solution of the IVP can be expressed by the finite series

$$x(t)=\sum_{k=0}^{N}X(k)t^k,\quad t\in[0,T]. \tag{2.7}$$

Let us assume that the interval $[0,T]$ in divided into m subintervals $[t_{i-1},t_i],\ i=1,2,3,...,m$ of the equal step size $h=\dfrac{T}{m}$ by using the nodes $t_i=ih$. The main ideas of the Ms-DTM are as follows. First, we apply the DTM to Eq.(2.5) over the interval $[0,t_1]$, we will obtain the following approximate solution,

$$x_1(t)=\sum_{k=0}^{N}X_1(k)t^k,\ \ t\in[0,t_1] \tag{2.8}$$

using the initial conditions $x_1^{(k)}(0)=d_k$. For $i\geq 2$, at each subinterval $[t_{i-1},t_i]$ we will use the initial conditions $x_i^{(k)}(t_{i-1})=x_{i-1}^{(k)}(t_{i-1})$ and apply the DTM to Eq.(2.5) over the interval $[t_{i-1},t_i]$, where t_0 in Eq.(2.7) is replaced by t_{i-1}. The process is repeated and generates a sequence of approximate solutions $x_i(t),\ i=1,2,3,...,m$ for the solution $x(t)$,

$$x_i(t)=\sum_{k=0}^{N}X_i(k)(t-t_i)^k,\quad t\in[t_{i-1},t_i]. \tag{2.9}$$

In fact, the MsDTM gives us the solution in the form,

$$x(t)=\begin{cases}x_1(t), & t\in[0,t_1]\\ x_2(t), & t\in[t_1,t_2]\\ \vdots & \\ x_m(t), & t\in[t_{m-1},t_m],\end{cases} \tag{2.10}$$

where $t_i=ih,\ x_i(t)=\sum\limits_{k=0}^{N}X_i(k)(t-t_i)^k$ and the initial condition $x_i^{(k)}(t_{i-1})=x_{i-1}^{(k)}(t_{i-1})$ (see more [21]).

2.3 Autonomous systems

Autonomous systems are the systems of first-order DEs of the form

$$\begin{aligned}\frac{dx_1}{dt}&=g_1(x_1,x_2,...,x_n)\\ \frac{dx_2}{dt}&=g_2(x_1,x_2,...,x_n)\\ &\vdots\\ \frac{dx_n}{dt}&=g_n(x_1,x_2,...,x_n),\end{aligned}$$

such that the independent variable does not explicitly appear on the right hand side of each DE. In the case $n=2$, the system is called a plane autonomous system and

$V(x_1, x_2) = \big(g_1(x_1, x_2), g_2(x_1, x_2)\big)$ is a vector field in the plane that indicates the movement direction. If the parameter t is interpreted as time, then $X(t) = \big(x(t), y(t)\big)$ indicates the position of the particle in the plane at time t and a solution of the system is interpreted as a path of this particle starting from $X(0,0) = \big(x(0), y(0)\big)$ (see Figure 2.1). The autonomous systems can be linear or nonlinear (see more [28]).

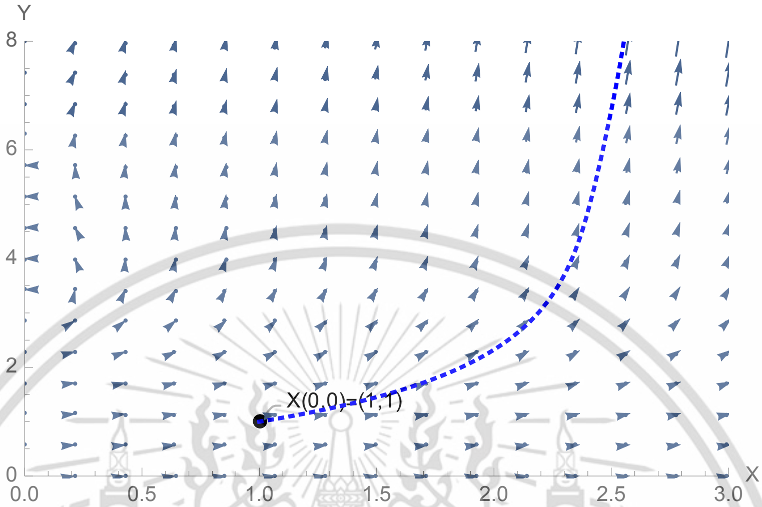


Figure 2.1: Vector fields and solution of plane autonomous system starting from $X(0,0) = (1,1)$.

Many applications from physics give rise to nonlinear autonomous second-order differential equations, that is differential equations of the form $\frac{d^2x}{dt^2} = g\Big(x, \frac{dx}{dt}\Big)$. Any autonomous second-order differential equation can be written as an autonomous system. If we let $y = \frac{dx}{dt}$, then $\frac{d^2x}{dt^2} = g\Big(x, \frac{dx}{dt}\Big)$ becomes $\frac{dy}{dt} = g(x,y)$. Hence the second-order differential equation becomes the system of two first-order equations, that is

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = g(x,y).$$

Some examples of autonomous systems in the plane are as follows:

1. *Simple predator-prey system*,

$$\frac{dx}{dt} = x\Big(1 - \frac{x}{k}\Big) - axy, \quad \frac{dy}{dt} = bxy - y, \quad x(t) \geq 0,\ y(t) \geq 0,$$

   $x(t)$ is the prey population, $y(t)$ is the predator population.

2. *Predator-prey with logistic growth for predator*. A different model allows for survival of the predator species in the absence of prey. Consider the system

$$\frac{dx}{dt} = x(k_1 - x - c_1 y), \quad \frac{dy}{dt} = y(k_2 - y + c_2 x), \quad x(t) \geq 0,\ y(t) \geq 0,$$

   where k_1, k_2 are carrying capacities and c_1, c_2 are interaction coefficients.

3. *Competing species (Lotka-Volterra)*. Consider the system for two populations

$$\frac{dx}{dt} = x(k_1 - x - c_1 y), \quad \frac{dy}{dt} = y(k_2 - y - c_2 x), \quad x(t) \geq 0,\ y(t) \geq 0,$$

here $k_1, k_2 > 0$ are carrying capacities and c_1, c_2 are interaction coefficients.

4. *The frictionless pendulum*. The second-order equation of motion is

$$\frac{d^2x}{dt^2} + \sin x = 0,$$

or as a first-order hamiltonian system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x.$$

5. *Pendulum with friction*. The second-order equation of motion is

$$\frac{d^2x}{dt^2} + r\frac{dx}{dt} + \sin x = 0, \quad r > 0.$$

The equivalent first-order system,

$$\frac{dx}{dt} = y, \quad \frac{dx}{dt} = -\sin x - ry.$$

2.4 Bessel's equation and Bessel function $J_\nu(x)$.

Bessel's equation,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0, \tag{2.11}$$

where the parameter ν is a given real number which is positive or zero. Bessel's equation often appears if a problem shows cylindrical symmetry. For example, the circular membrane. Hence, according to the Frobenius theory, it has the solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}, \quad (a \neq 0). \tag{2.12}$$

The first and second derivatives of Eq.(2.12) are

$$y'(x) = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} \tag{2.13}$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2} \tag{2.14}$$

Substituting Eqs.(2.12)-(2.14) into Eq.(2.11), we obtain

$$
\begin{aligned}
&x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y\\
&=x^2\sum_{m=0}^{\infty}(m+r)(m+r-1)a_mx^{m+r-2}+x\sum_{m=0}^{\infty}(m+r)a_mx^{m+r-1}+(x^2-\nu^2)\sum_{m=0}^{\infty}a_mx^{m+r}\\
&=\sum_{m=0}^{\infty}(m+r)(m+r-1)a_mx^{m+r}+\sum_{m=0}^{\infty}(m+r)a_mx^{m+r}+\sum_{m=0}^{\infty}a_mx^{m+r+2}-\nu^2\sum_{m=0}^{\infty}a_mx^{m+r}\\
&=a_0(r)(r-1)x^r+\sum_{m=1}^{\infty}(m+r)(m+r-1)a_mx^{m+r}+a_0(r)x^r+\sum_{m=1}^{\infty}(m+r)a_mx^{m+r}+\\
&\quad\sum_{m=0}^{\infty}a_mx^{m+r+2}-\nu^2a_0x^r-\nu^2\sum_{m=1}^{\infty}a_mx^{m+r}\\
&=a_0\big(r(r-1)+r-\nu^2\big)x^r+\sum_{m=1}^{\infty}a_m\big((m+r)(m+r-1)+(m+r)-\nu^2\big)x^{m+r}+\sum_{m=0}^{\infty}a_mx^{m+r+2}\\
&=a_0\big(r^2-\nu^2\big)x^r+x^r\sum_{m=1}^{\infty}a_m\big((m+r)^2-\nu^2\big)x^m+x^r\sum_{m=0}^{\infty}a_mx^{m+2}=0.
\end{aligned}
$$

We consider in the case $a_0\neq 0$, we have $r^2-\nu^2=0\rightarrow(r-\nu)(r+\nu)=0$.
The roots are $r_1=\nu$ and $r_2=-\nu$.
Then,

$$
\begin{aligned}
&x^\nu\sum_{m=1}^{\infty}a_m\big((m+\nu)^2-\nu^2\big)x^m+x^\nu\sum_{m=0}^{\infty}a_mx^{m+2}\\
&=x^\nu\sum_{m=1}^{\infty}a_m\big(m(m+2\nu)\big)x^m+x^\nu\sum_{m=0}^{\infty}a_mx^{m+2}\\
&=x^\nu\left((1+2\nu)a_1+\sum_{m=2}^{\infty}a_m\big(m(m+2\nu)\big)x^m+\sum_{m=0}^{\infty}a_mx^{m+2}\right)\\
&=x^\nu\left((1+2\nu)a_1+\sum_{m=0}^{\infty}a_{m+2}\big((m+2)((m+2)+2\nu)\big)x^{m+2}+\sum_{m=0}^{\infty}a_mx^{m+2}\right)\\
&=x^\nu\left((1+2\nu)a_1+\sum_{m=0}^{\infty}\left(\Big((m+2)((m+2+2\nu)\Big)a_{m+2}+a_m\right)x^{m+2}\right).
\end{aligned}
$$

Then, by undetermined coefficient, we obtain

$$(1+2\nu)a_1=0$$

$$(m+2)(m+2+2\nu)a_{m+2}+a_m=0,\qquad m=0,1,2,3,...$$

or

$$a_{m+2}=\frac{-a_m}{(m+2)(m+2+2\nu)},\qquad m=0,1,2,3,... \tag{2.15}$$

Since $1+2\nu\neq 0, a_1=0$, it follows that $a_3=a_5=a_7=\cdots=0$. Hence, we have to deal only with a_0,a_2,a_4..... For $m=2k$, Eq.(2.15) becomes

$$a_{2k}=-\frac{a_{2k-2}}{2^2k(k+\nu)},\qquad k=1,2,3,... \tag{2.16}$$

From Eq.(2.16) we can now determine $a_2, a_4, ...$ successively. This gives

$$a_2 = -\frac{a_0}{2^2(1+\nu)}$$
$$a_4 = -\frac{a_2}{2^2(2)(2+\nu)} = \frac{a_0}{2^4 2!(1+\nu)(2+\nu)}$$
$$\vdots$$

and in the general

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(\nu+1)(\nu+2)\cdots(\nu+k)}, \quad k = 1, 2, 3, ... \tag{2.17}$$

2.4.1 Bessel function $J_n(x)$ for integer $\nu = n$.

For $\nu = n$ the Eq.(2.17) become

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(n+1)(n+2)\cdots(n+k)}, \quad k = 1, 2, 3, ... \tag{2.18}$$

a_0 still arbitrary, so that Eq.(2.12) with these coefficients would contain this arbitrary factor a_0. We choose

$$a_0 = \frac{1}{2^n n!}, \tag{2.19}$$

because then $n!(n+1)(n+2)\cdots(n+k) = (n+k)!$ in Eq.(2.18), so that Eq.(2.18) becomes

$$a_{2k} = \frac{(-1)^k}{2^{2k+n} k!(n+k)!}, \quad k = 1, 2, 3, ... \tag{2.20}$$

By inserting these coefficients into Eq.(2.12) and remembering that $a_1 = a_3 = \cdots = 0$, we obtain a particular solution of Bessel's equation that is denoted by $J_n(x)$:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n} k!(n+k)!}, \quad (n \geq 0). \tag{2.21}$$

$J_n(x)$ is called the *Bessel function of the first kind* of order n. The series in Eq.(2.21) converges for all x, as the ratio test shows. Hence $J_n(x)$ is defined for all x. The series converges very rapidly because of the factorials in the denominator.

2.4.2 Bessel function $J_\nu(x)$ for any $\nu \geq 0$ and Gamma function.

We choose

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)} \tag{2.22}$$

with the *gamma function* $\Gamma(\nu+1)$ defined by

$$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu dt, \quad (\nu > -1). \tag{2.23}$$

Integration by parts gives

$$\Gamma(\nu+1) = -e^{-t} t^\nu \Big|_0^\infty + \nu \int_0^\infty e^{-t} t^{\nu-1} dt = 0 + \nu\Gamma(\nu). \tag{2.24}$$

This is the basic functional relation of the gamma function

$$\Gamma(\nu+1) = \nu\Gamma(\nu). \tag{2.25}$$

Now form Eq.(2.23) with $\nu = 0$ and then by Eq.(2.25) we obtain

$$\Gamma(1) = \int_0^\infty e^{-t}dt = -e^{-t}\Big|_0^\infty = 0 - (-1) = 1 \tag{2.26}$$

and then $\Gamma(2) = 1 \cdot \Gamma(1) = 1!, \Gamma(3) = 2\Gamma(1) = 2!$ and in the general

$$\Gamma(n+1) = n!, \quad (n = 0, 1, 2, \ldots). \tag{2.27}$$

Hence, *the gamma function generalizes the factorial function to arbitrary positive* ν. Thus, Eq.(2.22) with $\nu = n$ agree with Eq.(2.19).

Furthermore, from Eq.(2.17) with a_0 given by Eq.(2.22) we first have

$$a_{2k} = \frac{(-1)^k}{2^{2k}k!(\nu+1)(\nu+2)\cdots(\nu+k)2^\nu\Gamma(\nu+1)}. \tag{2.28}$$

Now Eq.(2.27) gives $(\nu+1)\Gamma(\nu+1) = \Gamma(\nu+2)$, $(\nu+2)\Gamma(\nu+2) = \Gamma(\nu+3)$ and so on, so that

$$(\nu+1)(\nu+2)\cdots(\nu+k)\Gamma(\nu+1) = \Gamma(\nu+k+1). \tag{2.29}$$

Hence, because of Eq. (2.22) of a_0 the coefficient in Eq.(2.17) are simply

$$a_{2k} = \frac{(-1)^k}{2^{2k+\nu}k!\Gamma(\nu+k+1)}. \tag{2.30}$$

With these coefficients and $r = r_1 = \nu$ we get from Eq.(2.12) a particular of Eq.(2.11), denoted by $J_\nu(x)$ and given by

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k+\nu}k!\Gamma(\nu+k+1)}. \tag{2.31}$$

$J_\nu(x)$ is called the *Bessel function of the first kind of order* ν. The series in Eq.(2.31) converges for all x, as one can verify by the ratio test.

2.4.3 Fourier-Bessel series

The series solutions of the presented problems consist of the coefficients of the Fourier-Bessel series corresponding to the Bessel functions of the first kind (see also [1]). The following theorem explain the meaning of the Fourier-Bessel series based on the orthogonality relations.

Theorem 2.1 (Orthogonality of the Bessel Functions [16]). For each fixed nonnegative integer n the sequence of the Bessel functions of the first kind $J_n(h_{n1}r), J_n(h_{n2}r), \ldots$ with $h_{nm} = \dfrac{\alpha_{nm}}{R}$ where α_{nm} is the mth positive zero of $J_n, (m = 1, 2, 3, \ldots)$, forms an orthogonal set on the interval $0 \le r \le R$ with respect to the weight function r, that is

$$\langle J_n(h_{nm}r), J_n(h_{nj}r)\rangle = \int_0^R rJ_n(h_{nm}r)J_n(h_{nj}r)dr = 0 \quad (j \neq m, n \text{ fixed}). \tag{2.32}$$

Theorem 2.2. The functions $\phi_{nm}(r,\theta) = J_n(h_{nm}r)\cos(n\theta)$ and $\psi_{nm}(r,\theta) = J_n(h_{nm}r)\sin(n\theta)$ for $n = 0, 1, 2, ..., m = 1, 2, 3, ...$ form a complete orthogonal set of functions relative to the inner product

$$\langle \phi_{nm}(r,\theta), \phi_{pj}(r,\theta)\rangle = \int_0^{2\pi}\int_0^R \phi_{nm}(r,\theta)\phi_{pj}(r,\theta) r dr d\theta = 0 \quad (\text{for } nm \neq pj), \tag{2.33}$$

$$\langle \psi_{nm}(r,\theta), \psi_{pj}(r,\theta)\rangle = \int_0^{2\pi}\int_0^R \psi_{nm}(r,\theta)\psi_{pj}(r,\theta) r dr d\theta = 0 \quad (\text{for } nm \neq pj), \tag{2.34}$$

$$\langle \phi_{nm}(r,\theta), \psi_{pj}(r,\theta)\rangle = \int_0^{2\pi}\int_0^R \phi_{nm}(r,\theta)\psi_{pj}(r,\theta) r dr d\theta = 0 \quad (\text{for all } nm \text{ and } pj). \tag{2.35}$$

We are interested in taking a function $f(r), f(r,\theta)$ and expanding it using Fourier eigenfunction expansion.

1. The expansion of $f(r)$ with Bessel functions J_n (n fixed),

$$f(r) = \sum_{m=1}^{\infty} A_{nm} J_n(h_{nm}r) = A_{n1}J_n(h_{n1}r) + A_{n2}J_n(h_{n2}r) + A_{n3}J_n(h_{n3}r) + \cdots, \tag{2.36}$$

where $h_{nm} = \dfrac{\alpha_{nm}}{R}$, is called *Fourier-Bessel series*. We multiply both sides of Eq. (2.36) by $rJ_n(h_{nj}r)$ and then integrate on both sides from 0 to R, we obtain

$$\begin{aligned}\langle f(r), J_n(h_{nj}r)\rangle = \int_0^R rf(r)J_n(h_{nj}r)dr &= \int_0^R r\left(\sum_{m=1}^{\infty} A_{nm}J_n(h_{nm}r)\right)J_n(h_{nj}r)dr \\ &= \sum_{m=0}^{\infty} A_{nm}\int_0^R rJ_n(h_{nm}r)J_n(h_{nj}r)dr \\ &= \sum_{m=0}^{\infty} A_{nm}\langle J_n(h_{nm}r), J_n(h_{nj}r)\rangle.\end{aligned}$$

Because of the orthogonality all the integrals on the right are zero, except when $m = j$. Then, we have

$$A_{nm}\langle J_n(h_{nm}r), J_n(h_{nm}r)\rangle = A_{nm}\|J_n(h_{nm}r)\|^2.$$

Thus

$$\langle f(r), J_n(h_{nm}r)\rangle = A_{nm}\|J_n(h_{nm}r)\|^2.$$

Here the coefficients are

$$\begin{aligned}A_{nm} &= \frac{\langle f(r), J_n(h_{nm}r)\rangle}{\|J_n(h_{nm}r)\|^2} \\ &= \frac{1}{\|J_n(h_{nm}r)\|^2}\int_0^R rf(r)J_n(h_{nm}r)dr \\ &= \frac{2}{R^2J_{n+1}^2(\alpha_{nm})}\int_0^R rf(r)J_n(h_{nm}r)dr,\end{aligned} \tag{2.37}$$

where the square of the norm is $\|J_n(h_{nm}r)\|^2 = \int\limits_0^R rJ_n^2(h_{nm}r)dr = \dfrac{R^2}{2}J_{n+1}^2(\alpha_{nm})$, $J_n(h_{nm}r) = \sum\limits_{l=0}^{\infty}\dfrac{(-1)^l(h_{nm}r)^{2l+n}}{2^{2l+n}l!(n+l)!}$ is the Bessel function of order n of the first kind and α_{nm} is the mth positive zero of J_n, $(m = 1, 2, 3, ...)$.

2. The expansion of $f(r,\theta)$ with Bessel functions J_n

$$f(r,\theta)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}J_n(h_{nm}r)\Big(A_{nm}\cos(n\theta)+B_{nm}\sin(n\theta)\Big), \tag{2.38}$$

is called the *Fourier-Bessel expansion of* f. It requires us to write $f(r,\theta)$ as a linear combination of the functions

$$\phi_{nm}(r,\theta)=J_n(h_{nm}r)\cos(n\theta)\quad\text{and}\quad\psi_{nm}(r,\theta)=J_n(h_{nm}r)\sin(n\theta)$$

for $n=0,1,2,...,m=1,2,3,...$. As usual, we can use orthogonality to express the coefficients in this combination as ratios of inner products (integrals). If $nm\neq pj$, we have

$$\begin{aligned}\langle\phi_{nm}(r,\theta),\phi_{pj}(r,\theta)\rangle&=\int_0^{2\pi}\int_0^R\phi_{nm}(r,\theta)\phi_{pj}(r,\theta)rdrd\theta\\&=\int_0^{2\pi}\int_0^R J_n(h_{nm}r)\cos(n\theta)J_p(h_{pj}r)\cos(p\theta)rdrd\theta\\&=\int_0^{2\pi}\cos(n\theta)\cos(p\theta)d\theta\int_0^R rJ_n(h_{nm}r)J_p(h_{pj}r)dr.\end{aligned}\tag{2.39}$$

By orthogonality of the functions $\{\cos(n\theta)\}$ on $[0,2\pi]$, the first integral is equal to zero if $n\neq p$. If $n=p$, then the first integral is not equal to zero. So, we have

$$\int_0^R J_n(h_{nm}r)J_n(h_{nj}r)dr=0,\quad\text{if } m\neq j. \tag{2.40}$$

Thus,

$$\begin{aligned}\langle\phi_{nm}(r,\theta),\phi_{nm}(r,\theta)\rangle&=\int_0^{2\pi}\cos^2(n\theta)d\theta\int_0^R rJ_n^2(h_{nm}r)dr\\&=\int_0^{2\pi}\int_0^R J_n^2(h_{nm}r)\cos^2(n\theta)rdrd\theta.\end{aligned}\tag{2.41}$$

Here the coefficients are

$$\begin{aligned}A_{nm}&=\frac{\langle f(r,\theta),\phi_{nm}(r,\theta)\rangle}{\langle\phi_{nm}(r,\theta),\phi_{nm}(r,\theta)\rangle}\\&=\frac{\int_0^{2\pi}\int_0^R f(r,\theta)J_n(h_{nm}r)\cos(n\theta)rdrd\theta}{\int_0^{2\pi}\int_0^R J_n^2(h_{nm}r)\cos^2(n\theta)rdrd\theta},\end{aligned}\tag{2.42}$$

for $n=0,1,2,...,m=1,2,3,...$, where

$$\int_0^{2\pi}\int_0^R J_n^2(h_{nm}r)\cos^2(n\theta)rdrd\theta=\begin{cases}\pi R^2J_1^2(\alpha_{0m}), & n=0,\\ \dfrac{\pi R^2}{2}J_{n+1}^2(\alpha_{nm}), & n\geq 1.\end{cases}$$

We finally find that

$$A_{0m}=\frac{1}{\pi R^2J_1^2(\alpha_{0m})}\int_0^{2\pi}\int_0^R f(r,\theta)J_0(h_{0m}r)rdrd\theta \tag{2.43}$$

$$A_{nm}=\frac{2}{\pi R^2J_{n+1}^2(\alpha_{nm})}\int_0^{2\pi}\int_0^R f(r,\theta)J_n(h_{nm}r)\cos(n\theta)rdrd\theta. \tag{2.44}$$

We now consider, if $nm \neq pj$, we have

$$\begin{aligned} \langle \psi_{nm}(r,\theta), \psi_{pj}(r,\theta) \rangle &= \int_0^{2\pi} \int_0^R \psi_{nm}(r,\theta)\psi_{pj}(r,\theta) r dr d\theta \\ &= \int_0^{2\pi} \int_0^R J_n(h_{nm}r)\sin(n\theta) J_p(h_{pj}r)\sin(p\theta) r dr d\theta \\ &= \int_0^{2\pi} \sin(n\theta)\sin(p\theta) d\theta \int_0^R r J_n(h_{nm}r) J_p(h_{pj}r) dr. \end{aligned} \tag{2.45}$$

By orthogonality of the functions $\{\sin(n\theta)\}$ on $[0, 2\pi]$, the first integral is equal to zero if $n \neq p$. If $n = p$, then the first integral is not equal to zero. So, we have

$$\int_0^R J_n(h_{nm}r) J_n(h_{nj}r) dr = 0, \quad \text{if } m \neq j. \tag{2.46}$$

Thus,

$$\begin{aligned} \langle \psi_{nm}(r,\theta), \psi_{nm}(r,\theta) \rangle &= \int_0^{2\pi} \sin^2(n\theta) d\theta \int_0^R r J_n^2(h_{nm}r) dr \\ &= \int_0^{2\pi} \int_0^R J_n^2(h_{nm}r)\sin^2(n\theta) r dr d\theta. \end{aligned} \tag{2.47}$$

Here the coefficients are

$$\begin{aligned} B_{nm} &= \frac{\langle f(r,\theta), \psi_{nm}(r,\theta) \rangle}{\langle \psi_{nm}(r,\theta), \psi_{nm}(r,\theta) \rangle} \\ &= \frac{\int_0^{2\pi} \int_0^R f(r,\theta) J_n(h_{nm}r)\sin(n\theta) r dr d\theta}{\int_0^{2\pi} \int_0^R J_n^2(h_{nm}r)\sin^2(n\theta) r dr d\theta}, \end{aligned} \tag{2.48}$$

for $n = 0, 1, 2, ..., m = 1, 2, 3, ...,$ where

$$\int_0^{2\pi} \int_0^R J_n^2(h_{nm}r)\sin^2(n\theta) r dr d\theta = \frac{\pi R^2}{2} J_{n+1}^2(\alpha_{nm})$$

We finally find that

$$B_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})} \int_0^{2\pi} \int_0^R f(r,\theta) J_n(h_{nm}r)\sin(n\theta) r dr d\theta, \tag{2.49}$$

where $J_n(h_{nm}r) = \sum_{l=0}^{\infty} \frac{(-1)^l (h_{nm}r)^{2l+n}}{2^{2l+n} l!(n+l)!}$ is the Bessel function of order n of the first kind and $h_{nm} = \frac{\alpha_{nm}}{R}$, α_{nm} is the mth positive zero of $J_n, (n = 0, 1, 2, ..., m = 1, 2, 3, ...)$.

2.5 The vibrations of a circular membrane

2.5.1 Derivation of the PDE of the model from forces

The PDE will be obtained by considering the forces acting on a small portion of the physical system, the membrane in Figure. 2.2, as it is moving up and down. Since the deflections of the membrane and the angles of the inclination are small, the sides of the portion are approximately equal to $\triangle x$ and $\triangle y$. The tension T is the force per unit length. Hence the forces acting on the sides of the portion are approximately $T\triangle x$

and $T\triangle y$. Since the membrane is perfectly flexible, these forces are tangent to the moving membrane at every instant.

We first consider the horizontal component of the forces. These components are obtained by multiplying the forces by the cosines of the angles of inclination. Since these angles are small, their cosines are close to 1. Hence, the horizontal components of the forces at the opposite sides are approximately equal. Therefore, the motion of the particles of the membrane in a horizontal direction will be negligibly small. From this we conclude that we may regard the motion of the membrane as transversal, that is each particle moves vertically.

These components along the right side and the left side are $T\triangle y \sin\beta$ and $T\triangle x \sin\alpha$, respectively. Here α and β are the values of the angle of the inclination (which varies slightly along the edges) in the middle of the edges, and the minus sign appear because the force on the left side is directed downward. Since the angle are small, we may replace their sines by their tangents. Hence the resultant of those two vertical components is

$$\begin{aligned} T\triangle y(\sin\beta - \sin\alpha) &\approx T\triangle y(\tan\beta - \tan\alpha) \\ &= T\triangle y\Big(\frac{\partial u}{\partial x}\Big|_{(x+\triangle x, y_1)} - \frac{\partial u}{\partial x}\Big|_{(x,y_2)}\Big) \end{aligned} \tag{2.50}$$

where y_1 and y_2 are values between y and $y+\triangle y$. Similarly, the resultant of the vertical components of the forces acting on the other two sides of the portion is

$$T\triangle x\Big(\frac{\partial u}{\partial y}\Big|_{(x_1, y+\triangle y)} - \frac{\partial u}{\partial y}\Big|_{(x_2,y)}\Big) \tag{2.51}$$

where x_1 and x_2 are values between x and $x+\triangle x$.

From Newton's second law, the sum of the forces given by Eqs.(2.50) and (2.51) is equal to the mass $\rho\triangle A$ of that small portion times the acceleration $\dfrac{\partial^2 u}{\partial t^2}$; here ρ is the mass of the undeflected membrane per unit area, and $\triangle A = \triangle x \triangle y$ is the area of that portion when it is undeflected. Thus

$$\begin{gathered} \text{Mass} \times \text{Acceleration} = \text{Force}, \\ \rho\triangle x\triangle y\frac{\partial^2 u}{\partial t^2} = T\triangle y\Big(\frac{\partial u}{\partial x}\Big|_{(x+\triangle x, y_1)} - \frac{\partial u}{\partial x}\Big|_{(x,y_2)}\Big) + T\triangle x\Big(\frac{\partial u}{\partial y}\Big|_{(x_1, y+\triangle y)} - \frac{\partial u}{\partial y}\Big|_{(x_2,y)}\Big), \end{gathered} \tag{2.52}$$

where the derivatives on the left is evaluated at some suitable point $(\tilde{x}, \tilde{y})$ corresponding to the portion. Division by $\rho\triangle x\triangle y$ gives

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left(\frac{\frac{\partial u}{\partial x}\Big|_{(x+\triangle x, y_1)} - \frac{\partial u}{\partial x}\Big|_{(x,y_2)}}{\triangle x} + \frac{\frac{\partial u}{\partial y}\Big|_{(x_1, y+\triangle y)} - \frac{\partial u}{\partial y}\Big|_{(x_2,y)}}{\triangle y}\right). \tag{2.53}$$

If we let $\triangle x$ and $\triangle y$ approach to zero, we obtain the PDE of the model

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad c^2 = \frac{T}{\rho}. \tag{2.54}$$

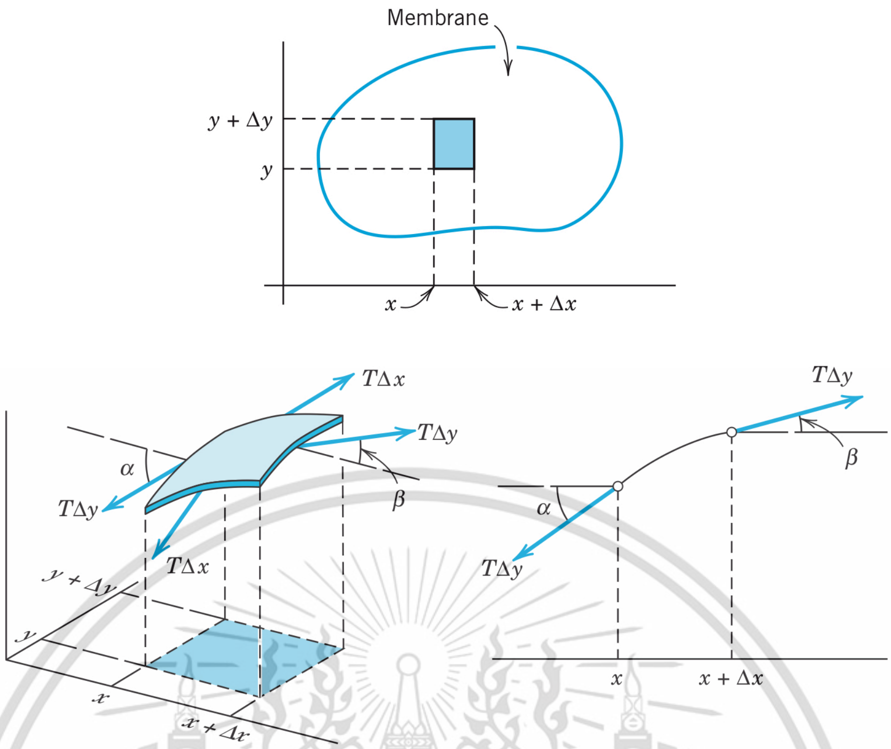


Figure 2.2: Vibrating membrane (picture form [16]).

This PDE is called the *two dimensional wave equation* and $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ is the Laplacian $\nabla^2 u$ of u.

2.5.2 Laplacian in polar coordinates

It is general principal in boundary value problems for PDEs to choose coordinates that make the formula for the boundary as simple as possible. Here polar coordinates are used for this purpose as follows. Since we want to discuss circular membrane. We first transform the Laplacian in the wave equation Eq. (2.54) into polar coordinates r, θ, where $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(\frac{y}{x})$. By the chain rule, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{2.55}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\right) \tag{2.56}$$

$$= \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta \partial r}\frac{\partial \theta}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}\left(\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right)\frac{\partial \theta}{\partial x}. \tag{2.57}$$

By differentiation of r and θ, we obtain

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}\frac{\partial}{\partial x}(x^2+y^2) = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r} \tag{2.58}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\Big(\frac{x}{r}\Big) = \frac{1}{r^2}\Big(r - x\frac{\partial r}{\partial x}\Big) = \frac{r - x(\frac{x}{r})}{r^2} = \frac{r^2-x^2}{r^3} = \frac{y^2}{r^3} \tag{2.59}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(\frac{y}{x})^2}\Big(-\frac{y}{x}\Big) = -\frac{y}{x^2+y^2} = -\frac{y}{r^2} \tag{2.60}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\Big(-\frac{y}{r^2}\Big) = -y(-2r^{-3})\frac{\partial r}{\partial x} = -y\Big(-\frac{2}{r^3}\Big)\frac{\partial r}{\partial x} = \frac{2y(\frac{x}{r})}{r^3} = \frac{2xy}{r^4}. \tag{2.61}$$

We substitute Eqs.(2.58)-(2.61) into Eq.(2.57). Assuming continuity of the first and second partial derivatives, we have $\frac{\partial^2 u}{\partial r\partial\theta} = \frac{\partial^2 u}{\partial\theta\partial r}$. Then

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial r}\Big(\frac{y^2}{r^3}\Big) + \left(\frac{\partial^2 u}{\partial r}\Big(\frac{x}{r}\Big) + \frac{\partial^2 u}{\partial\theta\partial r}\Big(-\frac{y}{r^2}\Big)\right)\Big(\frac{x}{r}\Big) + \\ &\quad \frac{\partial u}{\partial\theta}\Big(\frac{2xy}{r^4}\Big)\left(\frac{\partial^2 u}{\partial r\partial\theta}\Big(\frac{x}{r}\Big) + \frac{\partial^2 u}{\partial\theta^2}\Big(-\frac{y}{r^2}\Big)\right)\Big(-\frac{y}{r^2}\Big) \\ &= \frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{2xy}{r^3}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{y^2}{r^4}\frac{\partial^2 u}{\partial\theta^2} + \frac{y^2}{r^3}\frac{\partial u}{\partial r} + \frac{2xy}{r^4}\frac{\partial u}{\partial\theta}\end{aligned} \tag{2.62}$$

In the same way, by the chain rule, we obtain

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y} \tag{2.63}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y}\right) \tag{2.64}$$

$$= \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial y^2} + \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial y} + \frac{\partial^2 u}{\partial\theta\partial r}\frac{\partial\theta}{\partial y}\right)\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial^2\theta}{\partial y^2}\left(\frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial y} + \frac{\partial^2 u}{\partial\theta^2}\frac{\partial\theta}{\partial y}\right)\frac{\partial\theta}{\partial y}. \tag{2.65}$$

By differentiation of r and θ with respect to y, we obtain

$$\frac{\partial r}{\partial y} = \frac{y}{r},\ \ \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3},\ \ \frac{\partial\theta}{\partial y} = \frac{x}{r^2} \text{ and } \frac{\partial^2\theta}{\partial y^2} = -\frac{2xy}{r^4}. \tag{2.66}$$

We substitute Eq.(2.66) into Eq.(2.65). Assuming continuity of the first and second partial derivatives, we have $\frac{\partial^2 u}{\partial r\partial\theta} = \frac{\partial^2 u}{\partial\theta\partial r}$. Then

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial u}{\partial r}\Big(\frac{x^2}{r^3}\Big) + \left(\frac{\partial^2 u}{\partial r}\Big(\frac{y}{r}\Big) + \frac{\partial^2 u}{\partial\theta\partial r}\Big(\frac{x}{r^2}\Big)\right)\Big(\frac{y}{r}\Big) + \\ &\quad \frac{\partial u}{\partial\theta}\Big(-\frac{2xy}{r^4}\Big)\left(\frac{\partial^2 u}{\partial r\partial\theta}\Big(\frac{y}{r}\Big) + \frac{\partial^2 u}{\partial\theta^2}\Big(\frac{x}{r^2}\Big)\right)\Big(\frac{x}{r^2}\Big) \\ &= \frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{2xy}{r^3}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{x^2}{r^4}\frac{\partial^2 u}{\partial\theta^2} + \frac{x^2}{r^3}\frac{\partial u}{\partial r} - \frac{2xy}{r^4}\frac{\partial u}{\partial\theta}\end{aligned} \tag{2.67}$$

By adding Eqs.(2.62) and (2.67) we see that the Laplacian of u in polar coordinates is

$$\begin{aligned}\nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\\ &= \left(\frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{2xy}{r^3}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{y^2}{r^4}\frac{\partial^2 u}{\partial\theta^2} + \frac{y^2}{r^3}\frac{\partial u}{\partial r} + \frac{2xy}{r^4}\frac{\partial u}{\partial\theta}\right) +\\ &\quad \left(\frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{2xy}{r^3}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{x^2}{r^4}\frac{\partial^2 u}{\partial\theta^2} + \frac{x^2}{r^3}\frac{\partial u}{\partial r} - \frac{2xy}{r^4}\frac{\partial u}{\partial\theta}\right)\\ &= \left(\frac{x^2+y^2}{r^2}\right)\frac{\partial^2 u}{\partial r^2} + \left(\frac{-2xy+2xy}{r^3}\right)\frac{\partial^2 u}{\partial r\partial\theta} + \left(\frac{y^2+x^2}{r^4}\right)\frac{\partial^2 u}{\partial\theta^2} + \left(\frac{y^2+x^2}{r^3}\right)\frac{\partial u}{\partial r} + \left(\frac{2xy-2xy}{r^4}\right)\frac{\partial u}{\partial\theta}\\ &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}.\end{aligned} \tag{2.68}$$

2.5.3 Circular membrane

Circular membranes are important parts of drums, pumps, microphones, and other devices. This accounts for their great importance in engineering. We consider the case when the circular membrane is plane and its material is elastic, but offers no resistance to bending (this excludes thin metallic membranes). Then the vibrations of the circular membrane is given in the form of two-dimensional wave equation in polar coordinates,

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right),\\ u(R,\theta,t) = 0 \text{ for all } t \geq 0, u(r,\theta,0) = f(r,\theta), u_t(r,\theta,0) = g(r,\theta).\end{gathered} \tag{2.69}$$

where $0 \leq r \leq R, 0 \leq \theta \leq 2\pi, c^2 = T/\rho$ in term of the membrane's tension T and density ρ (per unit area), R is a radius of a membrane, a membrane is fixed along the boundary circle radius R, $f(r,\theta)$ is the initial shape at time $t = 0$ and $g(r,\theta)$ is the initial velocity (see [16]).

2.5.4 Vibrations of a circular membrane independent of angle θ

We shall be concerned with a membrane of radius R which exhibit circular symmetry. That is, we are considering solutions $u(x,t)$ that have the symmetry of the boundary conditions. The circularly symmetric solutions to the wave equation do not depend on θ, they depend only on r and t. Then $u_{\theta\theta}$ is vanished in Eq.(2.69) and the model of the problem reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{2.70}$$

$$u(R,t) = 0, \quad \text{for all } t \geq 0 \tag{2.71}$$

$$u(r,0) = f(r) \quad \text{(initial deflection)} \tag{2.72}$$

$$u_t(r,0) = g(r) \quad \text{(initial velocity)}. \tag{2.73}$$

STEP 1: Two ordinary differential equations (ODEs) from the wave equation (Eq.(2.70)**).** Using the method of separation variables, we first determine solutions

$$u(r,t) = w(r)g(t). \tag{2.74}$$

Differentiating Eq.(2.74), we obtain

$$\frac{\partial^2 u}{\partial t^2} = w\frac{d^2 g}{dt^2}, \quad \frac{\partial u}{\partial r} = \frac{dw}{dr}g \text{ and } \frac{\partial^2 u}{\partial r^2} = \frac{d^2 w}{dr^2}g. \tag{2.75}$$

By substituting Eq.(2.75) into Eq.(2.70), we get

$$w\frac{d^2 g}{dt^2} = c^2\left(\frac{\partial^2 w}{\partial r^2}g + \frac{1}{r}\frac{dw}{dr}g\right). \tag{2.76}$$

Dividing the result by c^2wg, we get

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{w}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right). \tag{2.77}$$

The variables are now separated. Here the left and right hand sides are independent, that is both sides must be constant and this constant must be negative. For convenience we write this constant as $-k^2$ in order to obtain solutions that satisfy the boundary conditions. Thus,

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{w}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = -h^2. \tag{2.78}$$

Eq.(2.78) gives us two ordinary differential equations (ODEs),

$$\frac{d^2 g}{dt^2} + \lambda^2 g = 0, \quad \lambda = ch \tag{2.79}$$

$$\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + h^2 w = 0. \tag{2.80}$$

The first equation in Eq.(2.79) for the times function, it is the simple harmonic oscillator equation. The second equation in Eq.(2.80) for the radial function. We can reduce Eq.(2.80) to Bessel's equation if we set $s = hr$, then $\frac{1}{r} = \frac{h}{s}$ and retaining the notation w for simplicity, we obtain the chain rule

$$\frac{dw}{dr} = \frac{dw}{ds}\frac{ds}{dr} = \frac{dw}{ds}h \text{ and } \frac{d^2 w}{dr^2} = \frac{d^2 w}{ds^2}h^2. \tag{2.81}$$

By substituting Eq.(2.81) into Eq.(2.80) and omitting the common factor h^2, we have

$$\frac{d^2 w}{ds^2} + \frac{1}{s}\frac{dw}{ds} + w = 0, \tag{2.82}$$

or

$$s^2\frac{d^2 w}{ds^2} + s\frac{dw}{ds} + s^2 w = 0, \tag{2.83}$$

this equation is Bessel's equation with parameter $n = 0$.

STEP 2: Satisfying the boundary condition Eq. (2.71)**.** Solutions of Eq.(2.83) are the Bessel functions J_0 and Y_0 of the first and second kind (these functions are plotted in Figure. 2.3). However, we see that in this problem, the Y_0 functions are not allowed as

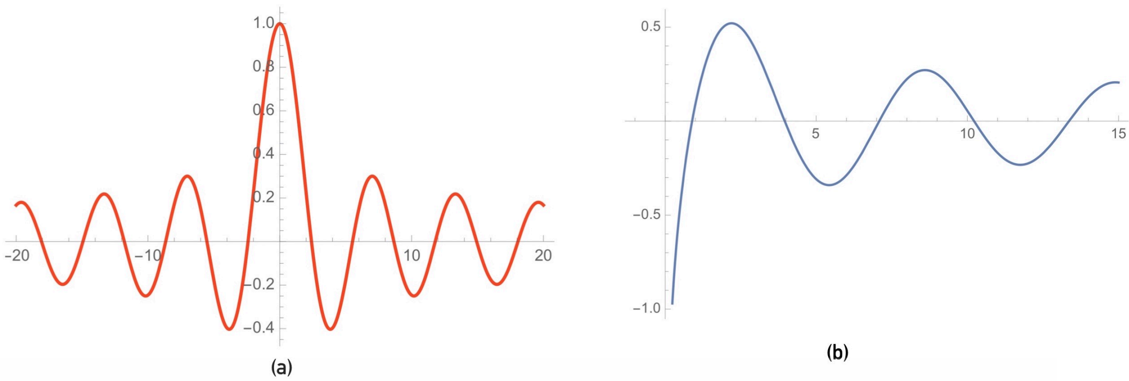


Figure 2.3: Bessel function (a) J_0 and (b) Y_0.

they becomes minus infinite at 0, so that we cannot use it because the deflection of the membrane must always remain finite, only the J_0 functions are permitted as they are finite at $r = 0$. This leaves us with

$$w(r) = J_0(s) = J_0(hr) \qquad (s = hr). \tag{2.84}$$

On the boundary $r = R$ from Eq.(2.71) is fixed so that it cannot be displaced, we get

$$w(r) = J_0(hR) = 0. \tag{2.85}$$

We can satisfy this condition because J_0 has positive zeros. This determines the allowed values of h since hR must correspond to a zero of the Bessel function. In other words, if α_m are the values of s for which J_0 is zero, that is $s = \alpha_1, \alpha_2, ...$, with numerical values

$$\alpha_1 = 2.4048, \alpha_2 = 5.5201, \alpha_3 = 8.6537, \alpha_4 = 11.7915, \alpha_5 = 14.9309$$

and so on. Then, it follows that the allowed values of h are

$$hR = \alpha_m \quad \text{or} \quad h = h_m = \frac{\alpha_m}{R}, \quad m = 1, 2, ... \tag{2.86}$$

Hence, a whole set of the radial functions

$$w_m(r) = J_0(h_m r) = J_0\Big(\frac{\alpha_m}{R} r\Big), \quad m = 1, 2, ... \tag{2.87}$$

are the solution of Eq.(2.80) that are zero on the boundary circle $r = R$.

For $w_m(r)$ in Eq.(2.87), a corresponding general solution of Eq.(2.79) with $\lambda = \lambda_m = ch_m = \dfrac{c\alpha_m}{R}$ is

$$g_m(t) = A_m \cos(\lambda_m t) + B_m \sin(\lambda_m t). \tag{2.88}$$

Hence, we have the mth solution as

$$u_m(r,t) = w_m(r) g_m(t) = J_0(k_m r)\Big(A_m \cos(\lambda_m t) + B_m \sin(\lambda_m t)\Big), \quad m = 1, 2, ... \tag{2.89}$$

are solution of wave equation Eq.(2.70) satisfying the condition Eq.(2.71). These are the eigenfunctions of our problem. The corresponding eigenvalues are λ_m.

STEP 3: Solution of the entire problem. The general solution of the problem is a linear combination of all these solutions in Eq.(2.89), that is

$$u(r,t) = \sum_{m=1}^{\infty} w_m(r)g_m(t) = \sum_{m=1}^{\infty} J_0\Big(\frac{\alpha_m}{R}r\Big)\Big(A_m \cos(\lambda_m t) + B_m \sin(\lambda_m t)\Big). \tag{2.90}$$

To obtain a solution $u(r,t)$ that also satisfies the initial conditions Eqs. (2.72) and (2.73). Setting $t = 0$ and using Eq.(2.72), we obtain

$$u(r,0) = \sum_{m=1}^{\infty} A_m J_0\Big(\frac{\alpha_m}{R}r\Big) = f(r). \tag{2.91}$$

Thus, for the series in Eq.(2.90) to satisfy the condition Eq.(2.72), the constants A_m must be the coefficients of the *Fourier-Bessel series* in Eq.(2.91) that represents $f(r)$ in terms of $J_0\Big(\frac{\alpha_m}{R}r\Big)$, that is

$$A_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R r f(r) J_0\Big(\frac{\alpha_m}{R}r\Big)dr \qquad (m = 1, 2, ...). \tag{2.92}$$

The coefficients B_m in Eq.(2.90) can be determined from the initial velocity in Eq.(2.73) in a similar fashion. That is,

$$B_m = \frac{2}{c\alpha_m R J_1^2(\alpha_m)} \int_0^R r g(r) J_0\Big(\frac{\alpha_m}{R}r\Big)dr \qquad (m = 1, 2, ...). \tag{2.93}$$

2.5.5 Vibrations of a circular membrane depending on both radius and angle.

We now consider a circular membrane of radius R which is fixed along the boundary circle, with the initial shape $f(r,\theta)$ and the initial velocity $g(r,\theta)$. The model of the problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\Big) \tag{2.94}$$

$$u(R,\theta,t) = 0, \quad \text{for all } t \geq 0 \tag{2.95}$$

$$u(r,,\theta,0) = f(r,\theta) \quad \text{(initial deflection)} \tag{2.96}$$

$$u_t(r,,\theta,0) = g(r,\theta) \quad \text{(initial velocity)}. \tag{2.97}$$

STEP 1: Three ordinary differential equations (ODEs) from the wave equation (Eq.(2.94)**).** Using the method of separation of variables, we first determine solutions

$$u(r,\theta,t) = z(r,\theta)g(t) \tag{2.98}$$

Differentiating Eq.(2.98), we obtain

$$\frac{\partial^2 u}{\partial t^2} = z\frac{d^2 g}{dt^2}, \quad \frac{\partial u}{\partial r} = \frac{\partial z}{\partial r}g, \quad \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 z}{\partial r^2}g \text{ and } \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 z}{\partial \theta^2}g. \tag{2.99}$$

Substituting Eq.(2.99) into Eq.(2.94) gives

$$z\frac{d^2 g}{dt^2} = c^2\Big(\frac{\partial^2 z}{\partial r^2}g + \frac{1}{r}\frac{\partial z}{\partial r}g + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}g\Big). \tag{2.100}$$

Dividing both sides by c^2zg, we get

$$\frac{1}{c^2g}\frac{d^2g}{dt^2} = \frac{1}{z}\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right). \tag{2.101}$$

The variables are now separated. Hence, both sides must equal a constant, that is

$$\frac{1}{c^2g}\frac{d^2g}{dt^2} = \frac{1}{z}\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right) = -h^2. \tag{2.102}$$

Eq.(2.102) gives an ODE and a PDE,

$$\frac{d^2g}{dt^2} + \lambda^2 g = 0, \quad \lambda = ch \tag{2.103}$$

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial r^2} + h^2 z = 0. \tag{2.104}$$

The PDE in Eq.(2.104) can be separated by substituting $z = w(r)q(\theta)$ and its derivative into Eq.(2.104), we obtain

$$\frac{d^2w}{dr^2}q + \frac{1}{r}\frac{dw}{dr}q + \frac{1}{r^2}w\frac{d^2q}{d\theta^2} + h^2wq = 0. \tag{2.105}$$

Dividing both sides by r^2wq, we get

$$\frac{r^2}{w}\frac{d^2w}{dr^2} + \frac{r}{w}\frac{dw}{dr} + \frac{1}{q}\frac{d^2q}{d\theta^2} + h^2r^2 = 0 \tag{2.106}$$

$$\frac{1}{q}\frac{d^2q}{d\theta^2} = -\frac{1}{w}\left(r^2\frac{d^2w}{dr^2} + r\frac{dw}{dr}\right) - h^2r^2. \tag{2.107}$$

The variables are now separated. Hence, both sides must be constant, that is

$$\frac{1}{q}\frac{d^2q}{d\theta^2} = -\frac{1}{w}\left(r^2\frac{d^2w}{dr^2} + r\frac{dw}{dr}\right) - h^2r^2 = -n^2. \tag{2.108}$$

Eq.(2.108) gives two ODEs,

$$\frac{d^2q}{d\theta^2} + n^2q = 0 \tag{2.109}$$

$$r^2\frac{d^2w}{dr^2} + r\frac{dw}{dr} + (h^2r^2 - n^2)w = 0. \tag{2.110}$$

STEP 2: Satisfying the boundary condition Eq.(2.95)**.** Solutions of Eq.(2.110) are the Bessel functions $w(r) = J_n(hr)$. Since the zero boundary condition in Eq.(2.95) yields

$$w(r) = J_n(hR) = 0. \tag{2.111}$$

This means that

$$hR = \alpha_{nm} \quad \text{or} \quad h = h_{nm} = \frac{\alpha_{nm}}{R},$$

where α_{nm} is the mth positive zero of J_n. Hence, a whole set of the radial functions

$$w_{nm}(r) = J_n(h_{nm}r) = J_n\left(\frac{\alpha_{nm}}{R}r\right), \tag{2.112}$$

for any $n = 0, 1, 2, 3, ...$ and $m = 1, 2, 3, ...$

For $w_{nm}(r)$ in Eq.(2.112), a corresponding general solution of Eq.(2.103) with $\lambda = \lambda_{mn} = ch_{nm} = c\dfrac{\alpha_{nm}}{R}$ is

$$g_{nm}(t) = a_{nm}\cos(\lambda_{nm}t) + b_{nm}\sin(\lambda_{nm}t), \tag{2.113}$$

and a corresponding general solution of Eq.(2.109) is

$$q_{nm}(\theta) = a^*_{nm}\cos(n\theta) + b^*_{nm}\sin(n\theta) \tag{2.114}$$

Hence, we have the mth solution as

$$\begin{aligned} u_{nm}(r,\theta,t) &= w_{nm}(r)q_{nm}(\theta)g_{nm}(t) \\ &= J_n(h_{nm}r)\Big(a*_{nm}\cos(n\theta) + b*_{nm}\sin(n\theta)\Big)\Big(a_{nm}\cos(\lambda_{nm}t) + b_{nm}\sin(\lambda_{nm}t)\Big), \end{aligned} \tag{2.115}$$

where, $m = 1, 2, 3, \ldots$ are solution of wave equation Eq.(2.94) satisfying the condition Eq.(2.95).

STEP 3: Solution of the entire problem. The general solution of the problem is a linear combination of all these solutions in Eq.(2.94), that is

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} w_{nm}(r)q_{nm}(\theta)g_{nm}(t) \\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(h_{nm}r)\Big(a^*_{nm}\cos(n\theta) + b^*_{nm}\sin(n\theta)\Big)\Big(a_{nm}\cos(\lambda_{nm}t) + b_{nm}\sin(\lambda_{nm}t)\Big) \\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(h_{nm}r)\Big(A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)\Big)\cos(\lambda_{nm}t) + \\ &\quad \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(h_{nm}r)\Big(A^*_{nm}\cos(n\theta) + B^*_{nm}\sin(n\theta)\Big)\sin(\lambda_{nm}t). \end{aligned} \tag{2.116}$$

To obtain a solution $u(r,\theta,t)$ that also satisfies the initial conditions Eqs.(2.96). Setting $t = 0$ and using Eq.(2.97), we obtain

$$u(r,\theta,0) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(h_{nm}r)\Big(A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)\Big) = f(r,\theta). \tag{2.117}$$

Thus, for the series in Eq.(2.116) to satisfy the condition Eq.(2.96), the constants A_{nm} and B_{nm} must be the coefficients of the *Fourier-Bessel series* in Eq.(2.117) that represents $f(r,\theta)$ in terms of $J_n\Big(h_{nm}r\Big)$, that is

$$A_{0m} = \frac{1}{\pi R^2 J_1^2(\alpha_{0m})}\int_0^{2\pi}\int_0^R f(r,\theta)J_0(h_{0m}r)rdrd\theta, \tag{2.118}$$

$$A_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})}\int_0^{2\pi}\int_0^R f(r,\theta)J_n(h_{nm}r)\cos(n\theta)rdrd\theta, \tag{2.119}$$

$$B_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})}\int_0^{2\pi}\int_0^R f(r,\theta)J_n(h_{nm}r)\sin(n\theta)rdrd\theta. \tag{2.120}$$

The coefficients A^*_{nm} and B^*_{nm} in Eq.(2.116) can be determined from the initial velocity in Eq.(2.97) in a similar fashion. That is,

$$A*_{0m} = \frac{1}{\pi c\alpha_{0m} R J_1^2(\alpha_{0m})} \int_0^{2\pi} \int_0^R g(r,\theta) J_0(h_{0m} r) r dr d\theta, \tag{2.121}$$

$$A*_{nm} = \frac{2}{\pi c\alpha_{nm} R J_{n+1}^2(\alpha_{nm})} \int_0^{2\pi} \int_0^R g(r,\theta) J_n(h_{nm} r) \cos(n\theta) r dr d\theta, \tag{2.122}$$

$$B*_{nm} = \frac{2}{\pi c\alpha_{nm} R J_{n+1}^2(\alpha_{nm})} \int_0^{2\pi} \int_0^R g(r,\theta) J_n(h_{nm} r) \sin(n\theta) r dr d\theta, \tag{2.123}$$

for $n = 0, 1, 2, ..., m = 1, 2, 3, ...$

Chapter 3

Composite transform formulae for differential transformation method with application to the nonlinear plane autonomous systems

In this chapter, we obtain 8 new formulae of the product of composite functions for differential transformation method and then we are using them in nonlinear plane autonomous system problems.

3.1 Derivation of the composite transform formulae

This section introduces our derivation technique of the new differential transform formulae for the product of composite functions derived in Formulae $1-8$. To obtain these new formulae, the derivation is shown in the following steps.

Step 1. The differential transformation for the product of two composite functions in represented by $f(y(t))g(y(t))$, which are the original functions. By the definition given in Section 2.1 of the DTM combined with Leibniz formula, we obtain

$$\begin{aligned} \frac{1}{k!}\left[\frac{d^k}{dt^k}f(y(t))g(y(t))\right]_{t=t_0} &= \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{k!}{(k-r)!r!}\frac{d^r}{dt^r}f(y(t))\frac{d^{k-r}}{dt^{k-r}}g(y(t))\right]_{t=t_0}, \\ &= \sum_{r=0}^{k}F(r)G(k-r), \end{aligned} \tag{3.1}$$

where $F(r)=\dfrac{1}{r!}\left[\dfrac{d^r}{dt^r}f(y(t))\right]_{t=t_0}$, $G(k-r)=\dfrac{1}{(k-r)!}\left[\dfrac{d^{k-r}}{dt^{k-r}}g(y(t))\right]_{t=t_0}$.

Step 2. This step finds the differential transformation for the higher order derivative of the power function, then we obtain the following Lemma.

Lemma 3.1. If $k,r,m\in\mathbb{I}^+\cup\{0\}$ and let $w=r-m=0,...,r$ where $r=0,...,k$ and $m=0,...,r$, then

$$\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^w\right]_{t=t_0}=1, \quad \text{if } k=0, \tag{3.2}$$

$$\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^w\right]_{t=t_0}=\sum_{k_{w-1}=0}^{k}\sum_{k_{w-2}=0}^{k_{w-1}}\cdots\sum_{k_1=0}^{k_2}Y(k_1)Y(k_2-k_1)\cdots Y(k-k_{w-1}), \quad \text{if } k>0. \tag{3.3}$$

Proof. Assume that $k,r,m\in\mathbb{I}^+\cup\{0\}$ and let $w=r-m=0,...,r$ where $r=0,...,k$ and $m=0,...,r$.

Case $k=0$; we have $r=0$ and $m=0$, then $\left[\dfrac{1}{0!}\dfrac{d^0}{dt^0}y(t)^0\right]_{t=t_0}=1.$

Case $k > 0$; we will prove by mathematical induction. Let $P(w)$ be Eq.(3.3).
First, we will show that the statement holds for $w = 0$, that is

$$P(0) = \left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^0\right]_{t=t_0} = \left[\frac{1}{k!}\frac{d^k}{dt^k}1\right]_{t=t_0} = 0.$$

Next, we assume that the statement is true for $w = r-1$, that is

$$P(r-1) = \left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-1}\right]_{t=t_0} = \sum_{k_{r-2}=0}^{k}\sum_{k_{r-3}=0}^{k_{r-2}}\cdots\sum_{k_1=0}^{k_2} Y(k_1)Y(k_2-k_1)\cdots Y(k-k_{r-2}). \quad (3.4)$$

We will show that the statement is also true for $w = r$. This can be seen as follows

$$\begin{aligned}
P(r) &= \left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0} = \left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-1}y(t)\right]_{t=t_0}, \\
&= \left[\frac{1}{k!}\sum_{k_{r-1}=0}^{k}\frac{k!}{(k-k_{r-1})!k_{r-1}!}\frac{d^{k_{r-1}}}{dt^{k_{r-1}}}y(t)^{r-1}\frac{d^{k-k_{r-1}}}{dt^{k-k_{r-1}}}y(t)\right]_{t=t_0}, \\
&= \sum_{k_{r-1}=0}^{k}\left[\frac{1}{k_{r-1}!}\frac{d^{k_{r-1}}}{dt^{k_{r-1}}}y(t)^{r-1}\frac{1}{(k-k_{r-1})!}\frac{d^{k-k_{r-1}}}{dt^{k-k_{r-1}}}y(t)\right]_{t=t_0}, \\
&= \sum_{k_{r-1}=0}^{k}\left[\sum_{k_{r-2}=0}^{k_{r-1}}\sum_{k_{r-3}=0}^{k_{r-2}}\cdots\sum_{k_1=0}^{k_2}Y(k_1)Y(k_2-k_1)\cdots Y(k-k_{r-2})\frac{1}{(k-k_{r-1})!}\frac{d^{k-k_{r-1}}}{dt^{k-k_{r-1}}}y(t)\right]_{t=t_0} \\
&= \sum_{k_{r-1}=0}^{k}\left[\sum_{k_{r-2}=0}^{k_{r-1}}\sum_{k_{r-3}=0}^{k_{r-2}}\cdots\sum_{k_1=0}^{k_2}Y(k_1)Y(k_2-k_1)\cdots Y(k-k_{r-2})Y(k-k_{r-1})\right]_{t=t_0} \\
&= \sum_{k_{r-1}=0}^{k}\sum_{k_{r-2}=0}^{k_{r-1}}\sum_{k_{r-3}=0}^{k_{r-2}}\cdots\sum_{k_1=0}^{k_2}Y(k_1)Y(k_2-k_1)\cdots Y(k-k_{r-2})Y(k-k_{r-1}).
\end{aligned}$$

Therefore, the statement holds for $w = r$, and the proof is completed. □

Step 3. The functions $f(y(t))$ and $g(y(t))$ in Step 1 are considered as the original functions in the formulae $1-8$. To obtain these new differential transform formulae, the general formulae of higher order derivatives of some composite functions are used together with Lemma 3.1 in the following calculations.

Formula 1. If $f(y(t)) = e^{y(t)}$ is the original function, then

$$\begin{aligned}
F(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}e^{y(t)}\right]_{t=t_0} &= \frac{1}{k!}\left[e^{y(t)}\sum_{r=0}^{k}\frac{1}{r!}\sum_{m=0}^{r}\frac{(-1)^m r!}{(r-m)!m!}y(t)^m\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t0}, \\
&= e^{y(t_0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m y(t_0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0}, \\
&= e^{Y(0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m Y(0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0},
\end{aligned}$$

where $Y(0) = \frac{1}{0!}\left[\frac{d^0}{dt^0}y(t)\right]_{t=t_0} = y(t_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0}$.

formula 2. If $f(y(t)) = \ln(y(t)), y(t) > 0$ is the original function, then

$$F(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\ln(y(t))\right]_{t=t_0} = \frac{1}{k!}\left[\delta_k \ln(y(t)) + \sum_{r=1}^{k}\frac{(-1)^{r-1}}{ry(t)^r}\binom{k}{r}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0},$$

$$= \frac{1}{k!}\delta_k \ln(y(t_0)) + \sum_{r=1}^{k}\frac{(-1)^{r-1}}{ry(t_0)^r}\binom{k}{r}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0},$$

$$= \frac{1}{k!}\delta_k \ln(Y(0)) + \sum_{r=1}^{k}\frac{(-1)^{r-1}}{rY(0)^r}\binom{k}{r}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0},$$

where $\binom{k}{r} = \frac{k!}{(k-r)!r!}$ are the binomial coefficient, $\delta_k = \begin{cases} 1, & \text{if } k=0 \\ 0, & \text{if } k>0 \end{cases}$, $Y(0) = \frac{1}{0!}\left[\frac{d^0}{dt^0}y(t)\right]_{t=t_0} = y(t_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0}$.

Formula 3. If $f(y(t)) = \sin(y(t))$ is the original function, then

$$F(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\sin(y(t))\right]_{t=t_0} = \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\sin(t)\Big|_{t=y(t)}\Big(\sum_{m=0}^{r}\frac{(-1)^m r! y(t)^m}{(r-m)!m!}\frac{d^k}{dt^k}y(t)^{r-m}\Big)\right]_{t=t_0},$$

$$= \sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\sin(t)\Big|_{t=y(t_0)}\sum_{m=0}^{r}\frac{(-1)^m y(t_0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0},$$

$$= \sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\sin(t)\Big|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y(0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0},$$

where $Y(0) = \frac{1}{0!}\left[\frac{d^0}{dt^0}y(t)\right]_{t=t_0} = y(t_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0}$.

Formula 4. If $f(y(t)) = \cos(y(t))$ is the original function, then

$$F(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\cos(y(t))\right]_{t=t_0} = \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\cos(t)\Big|_{t=y(t)}\Big(\sum_{m=0}^{r}\frac{(-1)^m r! y(t)^m}{(r-m)!m!}\frac{d^k}{dt^k}y(t)^{r-m}\Big)\right]_{t=t_0},$$

$$= \sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\cos(t)\Big|_{t=y(t_0)}\sum_{m=0}^{r}\frac{(-1)^m y(t_0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0},$$

$$= \sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\cos(t)\Big|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y(0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0},$$

where $Y(0) = \frac{1}{0!}\left[\frac{d^0}{dt^0}y(t)\right]_{t=t_0} = y(t_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0}$.

Formula 5. If $f(y(t)) = \sinh(y(t))$ is the original function, then

$$F(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\sinh(y(t))\right]_{t=t_0} = \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\sinh(t)\Big|_{t=y(t)}\Big(\sum_{m=0}^{r}\frac{(-1)^m r! y(t)^m}{(r-m)!m!}\frac{d^k}{dt^k}y(t)^{r-m}\Big)\right]_{t=t_0},$$

$$= \sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\sinh(t)\Big|_{t=y(t_0)}\sum_{m=0}^{r}\frac{(-1)^m y(t_0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0},$$

$$= \sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\sinh(t)\Big|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y(0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0},$$

where $Y(0) = \frac{1}{0!}\left[\frac{d^0}{dt^0}y(t)\right]_{t=t_0} = y(t_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0}$.

Formula 6. If $f(y(t)) = \cosh(y(t))$ is the original function, then

$$\begin{aligned}
F(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\cosh(y(t))\right]_{t=t_0} &= \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\cosh(t)\Big|_{t=y(t)}\Big(\sum_{m=0}^{r}\frac{(-1)^m r! y(t)^m}{(r-m)!m!}\frac{d^k}{dt^k}y(t)^{r-m}\Big)\right]_{t=t_0},\\
&= \sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\cosh(t)\Big|_{t=y(t_0)}\sum_{m=0}^{r}\frac{(-1)^m y(t_0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0},\\
&= \sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\cosh(t)\Big|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y(0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0},
\end{aligned}$$

where $Y(0) = \frac{1}{0!}\left[\frac{d^0}{dt^0}y(t)\right]_{t=t_0} = y(t_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=t_0}$.

Formula 7. If $f(y(t)) = \sqrt{y(t)}$ is the original function, then

$$\begin{aligned}
F(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\sqrt{y(t)}\right]_{t=t_0} &= \frac{1}{k!}\left[\frac{\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)\Gamma(\frac{1}{2})}\sum_{r=0}^{k}\frac{(-1)^r}{(\frac{1}{2}-r)}\binom{k}{r}y(t)^{\frac{1}{2}-r}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0},\\
&= \frac{\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)\Gamma(\frac{1}{2})}\sum_{r=0}^{k}\frac{(-1)^r}{(\frac{1}{2}-r)}\binom{k}{r}y(t_0)^{\frac{1}{2}-r}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0},\\
&= \frac{\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)\Gamma(\frac{1}{2})}\sum_{r=0}^{k}\frac{(-1)^r}{(\frac{1}{2}-r)}\binom{k}{r}Y(0)^{\frac{1}{2}-r}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0},
\end{aligned}$$

where $\binom{k}{r} = \frac{k!}{(k-r)!r!}$ are the binomial coefficient, $\Gamma(1+z) = z\Gamma(z)$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(n) = (n-1)!, n \in \mathbb{I}^+$, $Y(0) = y(t_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0}$.

Formula 8. If $f(y(t)) = \frac{1}{y(t)}$ is the original function, then

$$\begin{aligned}
F(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\frac{1}{y(t)}\right]_{t=t_0} &= \frac{1}{k!}\left[(k+1)\sum_{r=0}^{k}\frac{(-1)^r}{(r+1)}\binom{k}{r}y(t)^{-r-1}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0},\\
&= (k+1)\sum_{r=0}^{k}\frac{(-1)^r}{(r+1)}\binom{k}{r}y(t_0)^{-r-1}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0},\\
&= (k+1)\sum_{r=0}^{k}\frac{(-1)^r}{(r+1)}\binom{k}{r}Y(0)^{-r-1}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0},
\end{aligned}$$

where $\binom{k}{r} = \frac{k!}{(k-r)!r!}$ are the binomial coefficient, $Y(0) = y(t_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^r\right]_{t=t_0}$. The transformed functions are shown in Table **??**.

Table 3.1*:* The fundamental operations of one-dimensional DTM.

Original function $f(y(t))$	Transformed function $F(k)$
$e^{y(t)}$	$e^{Y(0)} \sum_{r=0}^{k} \sum_{m=0}^{r} \frac{(-1)^m Y(0)^m}{(r-m)!m!} \left[\frac{1}{k!}\frac{d^k}{dt^k} y(t)^{r-m}\right]_{t=t_0}$
$\ln(y(t))$	$\frac{1}{k!}\delta_k \ln(Y(0)) + \sum_{r=1}^{k} \frac{(-1)^{r-1}}{rY(0)^r}\binom{k}{r} \left[\frac{1}{k!}\frac{d^k}{dt^k} y(t)^{r}\right]_{t=t_0}$
$\sin(y(t))$	$\sum_{r=0}^{k} \frac{1}{r!}\frac{d^r}{dt^r}\sin(t)\Big\vert_{t=Y(0)} \sum_{m=0}^{r} \frac{(-1)^m Y(0)^m}{(r-m)!m!} \left[\frac{1}{k!}\frac{d^k}{dt^k} y(t)^{r-m}\right]_{t=t_0}$
$\cos(y(t))$	$\sum_{r=0}^{k} \frac{1}{r!}\frac{d^r}{dt^r}\cos(t)\Big\vert_{t=Y(0)} \sum_{m=0}^{r} \frac{(-1)^m Y(0)^m}{(r-m)!m!} \left[\frac{1}{k!}\frac{d^k}{dt^k} y(t)^{r-m}\right]_{t=t_0}$
$\sinh(y(t))$	$\sum_{r=0}^{k} \frac{1}{r!}\frac{d^r}{dt^r}\sinh(t)\Big\vert_{t=Y(0)} \sum_{m=0}^{r} \frac{(-1)^m Y(0)^m}{(r-m)!m!} \left[\frac{1}{k!}\frac{d^k}{dt^k} y(t)^{r-m}\right]_{t=t_0}$
$\cosh(y(t))$	$\sum_{r=0}^{k} \frac{1}{r!}\frac{d^r}{dt^r}\cosh(t)\Big\vert_{t=Y(0)} \sum_{m=0}^{r} \frac{(-1)^m Y(0)^m}{(r-m)!m!} \left[\frac{1}{k!}\frac{d^k}{dt^k} y(t)^{r-m}\right]_{t=t_0}$
$\sqrt{y(t)}$	$\frac{\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)\Gamma(\frac{1}{2})} \sum_{r=0}^{k} \frac{(-1)^r}{(\frac{1}{2}-r)}\binom{k}{r} Y(0)^{\frac{1}{2}-r} \left[\frac{1}{k!}\frac{d^k}{dt^k} y(t)^{r}\right]_{t=t_0}$
$\frac{1}{y(t)}$	$(k+1)\sum_{r=0}^{k} \frac{(-1)^r}{(r+1)}\binom{k}{r} Y(0)^{-r-1} \left[\frac{1}{k!}\frac{d^k}{dt^k} y(t)^{r}\right]_{t=t_0}$

3.2 Applications

In this section, we extended the application of the DTM to nonlinear plane autonomous systems. To demonstrate the formulae introduced in the previous section, three examples are studied here. The accuracy of the method is assessed by graphical and data value comparisons.

Example 3.2. Consider the following system of nonlinear plane autonomous

$$x' = e^y \tag{3.5}$$

$$y' = e^x, \quad \text{for } t \in [0, 1.25], \tag{3.6}$$

subject to the initial conditions $x(0) = 0, y(0) = 0.$

Applying the DTM of Eqs.(3.5) and (3.6) and using the initial conditions $x(0) =$

$0, y(0) = 0$, it follows

$$X(k+1) = \frac{1}{(k+1)} e^{Y(0)} \sum_{r=0}^{k} \sum_{m=0}^{r} \frac{(-1)^m Y(0)^m}{(r-m)!m!} \left[\frac{1}{k!} \frac{d^k}{dt^k} y(t)^{r-m} \right]_{t=0},$$
$$Y(k+1) = \frac{1}{(k+1)} e^{X(0)} \sum_{r=0}^{k} \sum_{m=0}^{r} \frac{(-1)^m X(0)^m}{(r-m)!m!} \left[\frac{1}{k!} \frac{d^k}{dt^k} x(t)^{r-m} \right]_{t=0},$$
$$X(0) = 0, Y(0) = 0.$$

By substituting $k = 0, 1, 2, ..., 11$ we obtain the coefficients of the series solution as follows

$$X(1) = Y(1) = 1, \quad X(2) = Y(2) = \frac{1}{2}, \quad X(3) = Y(3) = \frac{1}{3}, \quad X(4) = Y(4) = \frac{1}{4}$$
$$X(5) = Y(5) = \frac{1}{5}, \quad X(6) = Y(6) = \frac{1}{6}, \quad X(7) = Y(7) = \frac{1}{7}, \quad X(8) = Y(8) = \frac{1}{8}$$
$$X(9) = Y(9) = \frac{1}{9}, \quad X(10) = Y(10) = \frac{1}{10}, \quad X(11) = Y(11) = \frac{1}{11}, \quad X(12) = Y(12) = \frac{1}{12}.$$

Hence, the series solution reads

$$y(t) = x(t) = t + \frac{t^2}{2} + \frac{t^3}{3} + + \frac{t^4}{4} + \frac{t^5}{5} + \frac{t^6}{6} + \frac{t^7}{7} + \frac{t^8}{8} + \frac{t^9}{9} + \frac{t^{10}}{10} + \frac{t^{11}}{11} + \frac{t^{12}}{12}, \quad t \in [0, 1.25].$$

On the other hand, by applying the MsDTM to Eqs.(3.5) and (3.6) with the same initial conditions, it follows

$$X_i(k+1) = \frac{1}{(k+1)} e^{Y_i(0)} \sum_{r=0}^{k} \sum_{m=0}^{r} \frac{(-1)^m Y_i(0)^m}{(r-m)!m!} \left[\frac{1}{k!} \frac{d^k}{dt^k} y(t)^{r-m} \right]_{t=t_i},$$
$$Y_i(k+1) = \frac{1}{(k+1)} e^{X_i(0)} \sum_{r=0}^{k} \sum_{m=0}^{r} \frac{(-1)^m X_i(0)^m}{(r-m)!m!} \left[\frac{1}{k!} \frac{d^k}{dt^k} x(t)^{r-m} \right]_{t=t_i},$$
$$X_0(0) = 0, X_i(0) = x_{i-1}(t_i), Y_0(0) = 0, Y_i(0) = y_{i-1}(t_i), \quad i = 1, 2, 3, 4, 5.$$

Thus, we obtain the series solution

$$y(t) = x(t) = \begin{cases} t + 0.5t^2 + 0.3333t^3 + 0.25t^4 + 0.2t^5 + 0.16667t^6 + 0.14268t^7 \\ +0.125t^8 + 0.1111t^9 + 0.1t^{10} + 0.0909t^{11} + 0.0833t^{12}, & t \in [0, 0.25], \\ 0.286682 + 1.33333(t-0.25) + 0.88889(t-0.25)^2 + 0.79012(t-0.25)^3 \\ +0.79012(t-0.25)^4 + 0.842798(t-0.25)^5 + 0.936433(t-0.25)^6 \\ +1.07022(t-0.25)^7 + 1.24859(t-0.25)^8 + 1.47981(t-0.25)^9 \\ +1.77577(t-0.25)^{10} + 2.15245(t-0.25)^{11} + 2.63078(t-0.25)^{12}, & t \in [0.25, 0.5], \\ 0.693147 + 2(t-0.5) + 2(t-0.5)^2 + 2.66667(t-0.5)^3 + 4(t-0.5)^4 + 6.4(t-0.5)^5 \\ +10.66667(t-0.5)^6 + 18.2857(t-0.5)^7 + 32(t-0.5)^8 + 56.8889(t-0.5)^9 \\ +102.4(t-0.5)^{10} + 186.182(t-0.5)^{11} + 341.333(t-0.5)^{12}, & t \in [0.5, 0.75], \\ 1.38628 + 3.99993(t-0.75) + 7.99972(t-0.75)^2 + 21.3322(t-0.75)^3 \\ +63.9955(t-0.75)^4 + 204.782(t-0.75)^5 + 682.595(t-0.75)^6 + 2340.28(t-0.75)^7 \\ +8190.85(t-0.75)^8 + 29122.5(t-0.75)^9 + 104839(t-0.75)^{10} \\ +381227(t-0.75)^{11} + 1.39781 \times 10^6 (t-0.75)^{12}, & t \in [0.75, 1], \end{cases}$$

$$y(t) = x(t) = \begin{cases} 4.05773 + 57.843(t-1) + 1672.91(t-1)^2 + 64510.45(t-1)^3 + 2.79861 \times 10^6 (t-1)^4 \\ +1.56645 \times 10^{13} (t-1)^8 + 8.05403 \times 10^{14} (t-1)^9 + 4.19282 \times 10^{16} (t-1)^{10} \\ +2.20478 \times 10^{18} (t-1)^{11} + 1.16903 \times 10^{20} (t-1)^{12}, & t \in [1, 1.25]. \end{cases}$$

This problem with the initial conditions $x(0) = 0, y(0) = 0$ can be solved analytically by the phase-plane method to obtain the analytical solution $y(x) = x$. As seen in Figure 3.1, the approximate series solutions calculated by the DTM and the MsDTM are the same as the analytical solution and they have the same direction with the flow of the vector fields. Moreover, the DTM and the MsDTM gave data results similar to analytical results (see Table 3.2).

Figure 3.1*:* The MsDTM, the DTM and numerical solution compared with vector fields flow directions with the initial conditions $x(0) = 0, y(0) = 0$.

However, if we consider the problem with the initial conditions of $x(0) = -2, y(0) = 1$, the analytical solution obtained is $y(x) = \ln(e^x + e - e^{-2})$. The data values of the approximate solutions of the DTM and the MsDTM were compared with analytical solution are shown in Table 3.3. We can see that the MsDTM results are much more similar to the analytical results than the DTM results.

Table 3.2*:* DTM and MsDTM values compared with the analytical solutions.

t	$x(t)$	MsDTM	Analytical	Error
0.2	0.2231436	0.2231436	0.2231436	0
0.4	0.5108257	0.5108257	0.5108257	0
0.6	0.9162908	0.9162908	0.9162908	0
0.8	1.6094160	1.6094160	1.6094160	0
t	$x(t)$	DTM	Analytical	Error
0.2	0.2231436	0.2231436	0.2231436	0
0.4	0.5108257	0.5108257	0.5108257	0
0.6	0.9162908	0.9162908	0.9162908	0
0.8	1.6094160	1.6094160	1.6094160	0

Table 3.3*:* DTM and MsDTM values compared with the analytical solutions.

t	$x(t)$	MsDTM	Analytical	Error
0.2	-1.4473324	1.0360783	1.0360783	0
0.4	-0.8671829	1.0996385	1.0996384	1×10^{-7}
0.6	-0.2340544	1.2161777	1.2161776	1×10^{-7}
0.8	0.5147103	1.4483531	1.4483533	2×10^{-7}
t	$x(t)$	DTM	Analytical	Error
0.2	-1.4473324	1.0360783	1.0360783	0
0.4	-0.8671831	1.0996383	1.0996384	2×10^{-7}
0.6	-0.2340822	1.2161499	1.2161171	2.12×10^{-5}
0.8	0.5130110	1.4466537	1.4476856	1.0319×10^{-3}

Figure 3.2*:* The MsDTM, the DTM and numerical solution compared with vector fields flow directions with initial conditions $x(0) = -2, y(0) = 1$.

The following two examples show that the proposed new transformed functions of the product of composite functions can be applied effectively to the nonlinear plane autonomous system when the analytical solutions are unavailable.

Example 3.3. Let us consider the following system of nonlinear plane autonomous system

$$x' = x^2 e^x \tag{3.7}$$

$$y' = ye^x - y, \text{ for } t \in [0, 0.2], \tag{3.8}$$

subject to the initial conditions $x(0) = 1, y(0) = 1$.

Applying the DTM to Eqs.(3.7) and (3.8) and with the initial conditions $x(0) =$

$1, y(0) = 1$, it follows that

$$X(k+1) = \frac{1}{k+1}\sum_{r=0}^{k} F(r) \sum_{l=0}^{k-r} X(l)X(k-r-l),$$
$$Y(k+1) = \frac{1}{k+1}\Big(\sum_{r=0}^{k} Y(k-r)G(r) - Y(k)\Big),$$

and the initial conditions becomes $X(0) = 1, Y(0) = 1$, where

$$F(r) = e^{Y(0)} \sum_{l=0}^{r}\sum_{m=0}^{l} \frac{(-1)^m Y(0)^m}{(l-m)!m!}\left[\frac{1}{r!}\frac{d^r}{dt^r} y(t)^{l-m}\right]_{t=0},$$
$$G(r) = e^{X(0)} \sum_{l=0}^{r}\sum_{m=0}^{l} \frac{(-1)^m X(0)^m}{(l-m)!m!}\left[\frac{1}{r!}\frac{d^r}{dt^r} x(t)^{l-m}\right]_{t=0}.$$

Hence, we obtain the series solution by the DTM

$$\begin{aligned} x(t) &= 1 + 2.71828t + 9.72444t^2 + 38.8048t^3 + 164.329t^4 + 722.872t^5 + 3265.98t^6 + 15052.5t^7 \\ &\quad + 7045.9t^8 + 333873t^9 + 1.59826 \times 10^6 t^{10} + 7.71516 \times 10^6 t^{11} + 3.75161 \times 10^7 t^{12}, \ \ t \in [0, 0.2], \\ y(t) &= 1 + 1.71828t + 5.17077t^2 + 19.3526t^3 + 80.1435t^4 + 351.093t^5 + 1593.47t^6 + 7409.3t^7 \\ &\quad + 35062.4t^8 + 168146t^9 + 814830t^{10} + 3.98192 \times 10^6 t^{11} + 1.95937 \times 10^7 t^{12}, \ \ t \in [0, 0.2]. \end{aligned}$$

On the other hand, by applying the MsDTM to Eqs.(3.7) and (3.8), we obtain

$$X_i(k+1) = \frac{1}{k+1}\sum_{r=0}^{k} F_i(r) \sum_{l=0}^{k-r} X_i(l)X_i(k-r-l),$$
$$Y_i(k+1) = \frac{1}{k+1}\Big(\sum_{r=0}^{k} Y + i(k-r)G_i(r) - Y_i(k)\Big),$$
$$X_0(0) = 1, X_i(0) = x_{i-1}(t_i), Y_0(0) = 1, Y_i(0) = y_{i-1}(t_i), i = 1, 2, 3, 4,$$

where

$$F_i(r) = e^{Y_i(0)} \sum_{l=0}^{r}\sum_{m=0}^{l} \frac{(-1)^m Y_i(0)^m}{(l-m)!m!}\left[\frac{1}{r!}\frac{d^r}{dt^r} y_i(t)^{l-m}\right]_{t=t_i},$$
$$G_i(r) = e^{X_i(0)} \sum_{l=0}^{r}\sum_{m=0}^{l} \frac{(-1)^m X_i(0)^m}{(l-m)!m!}\left[\frac{1}{r!}\frac{d^r}{dt^r} x_i(t)^{l-m}\right]_{t=t_i}.$$

The following approximate series solution is the result.

$$x(t) = \begin{cases} 1 + 2.71828t + 9.72444t^2 + 38.8048t^3 + 164.329t^4 + 722.872t^5 + 3265.98t^6 + 15052.5t^7 \\ +7045.9t^8 + 333873t^9 + 1.59826 \times 10^6 t^{10} + 7.71516 \times 10^6 t^{11} + 3.75161 \times 10^7 t^{12}, \ \ t \in [0, 0.05], \\ 1.1664 + 4.09489(t-0.05) + 19.3628(t-0.05)^2 + 103.094(t-0.05)^3 \\ +585.693(t-0.05)^4 + 3468.59(t-0.05)^5 + 21148.6(t-0.05)^6 + 131763(t-0.05)^7 \\ +834742(t-0.05)^8 + 5.35899 \times 10^6 (t-0.05)^9 + 3.47789 \times 10^7 (t-0.05)^{10} \\ +2.27748 \times 10^8 (t-0.05)^{11} + 1.50275 \times 10^9 (t-0.05)^{12}, \ \ t \in [0.05, 0.1], \end{cases}$$

$$x(t)=\begin{cases}1.43766+7.26784(t-0.1)+51.4129(t-0.1)^2+416.14(t-0.1)^3+3626.29(t-0.1)^4\\+33122.1(t-0.1)^5+312581(t-0.1)^6+3.02155\times10^6(t-0.1)^7\\+2.9749\times10^7(t-0.1)^8+2.97167\times10^8(t-0.1)^9+3.00338\times10^9(t-0.1)^{10}\\+3.06482\times10^{10}(t-0.1)^{11}+3.15286\times10^{11}(t-0.1)^{12},\ \ t\in[0.1,0.15]\\2.02412+19.7164(t-0.15)+293.806(t-0.15)^2+5196.51(t-0.15)^3+100769(t-0.15)^4\\+2.06843\times10^6(t-0.15)^5+4.41157\times10^7(t-0.15)^6+9.67037\times10^8(t-0.15)^7\\+2.16368\times10^{10}(t-0.15)^8+4.91847\times10^{11}(t-0.15)^9+1.13227\times10^{13}(t-0.15)^{10}\\+2.63346\times10^{14}(t-0.15)^{11}+6.17734\times10^{15}(t-0.15)^{12},\ \ t\in[0.15,0.2].\end{cases}$$

$$y(t)=\begin{cases}1+1.71828t+5.17077t^2+19.3526t^3+80.1435t^4+351.093t^5+1593.47t^6+7409.3t^7\\+35062.4t^8+168146t^9+814830t^{10}+3.98192\times10^6t^{11}+1.95937\times10^7t^{12},\ \ t\in[0,0.05],\\1.1019+2.43565(t-0.05)+9.93481(t-0.05)^2+50.7118(t-0.05)^3\\+286.334(t-0.05)^4+1708.87(t-0.05)^5+10559.3(t-0.05)^6+66814.5(t-0.05)^7\\+430122(t-0.05)^8+2.8059\times10^6(t-0.05)^9+1.84864\times10^7(t-0.05)^{10}\\+1.2283\times10^8(t-0.05)^{11}+8.21708\times10^8(t-0.05)^{12},\ \ t\in[0.05,0.1]\\1.25743+4.03738(t-0.1)+25.7226(t-0.1)^2+206.07(t-0.1)^3\\+1823.14(t-0.1)^4+17024.4(t-0.1)^5+164483(t-0.1)^6+1.62549\times10^6(t-0.1)^7\\+1.63413\times10^7(t-0.1)^8+1.66403\times10^8(t-0.1)^9+1.71169\times10^9(t-0.1)^{10}\\+1.75515\times10^{10}(t-0.1)^{11}+1.85343\times10^{11}(t-0.1)^{12},\ \ t\in[0.1,0.15]\\1.57118+10.3218(t-0.15)+151.148(t-0.15)^2+2779.76(t-0.15)^3+56212.2(t-0.15)^4\\+1.19708\times10^6(t-0.15)^5+2.63404\times10^7(t-0.15)^6+5.92862\times10^8(t-0.15)^7\\+1.35679\times10^{10}(t-0.15)^8+3.145\times10^{11}(t-0.15)^9+7.36425\times^{12}(t-0.15)^{10}\\+1.73866\times10^{14}(t-0.15)^{11}+4.13307\times10^{15}(t-0.15)^{12},\ \ t\in[0.15,0.2].\end{cases}$$

The approximate series solution obtained by the MsDTM and the DTM are compared graphically with the flow direction of the vector fields (Figure 3.3). We can see that the MsDTM result is in better agreement with vector fields than the DTM results. As seen in Figure 3.4, we show the approximate series solutions calculated by the MsDTM in many initial conditions such as (-2,1), (-2,0), (-2,-2), (-2, -2) and (0,-2).

Example 3.4. Let us consider the following system of nonlinear plane autonomous system

$$x' = 2x + \sin y \tag{3.9}$$

$$y' = x(y^2+1),\ \text{for } t\in[0,0.4], \tag{3.10}$$

subject to the initial conditions $x(0)=1, y(0)=1$.

Applying the DTM to Eqs.(3.9) and (3.10) and with the initial conditions $x(0) =$

Figure 3.3*:* The MsDTM and the DTM compared with vector fields flow directions with initial conditions $x(0) = 1, y(0) = 1$.

Figure 3.4*:* The solutions from MsDTM in initial conditions (-2,1), (-2,0), (-2,-2), (-2, -2), (0,-2) with vector fields flow directions.

$1, y(0) = 1$, it follows that

$$X(k+1) = \frac{1}{k+1}\left(2X(k) + \sum_{r=0}^{k} \frac{d^r}{dt^r}\,\text{sin}(t)\Big|_{t=Y(0)} \sum_{m=0}^{r} \frac{(-1)^m Y(0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k}y(t)^{r-m}\right]_{t=0}\right),$$

$$Y(k+1) = \frac{1}{k+1}\left(X(k) + \sum_{r=0}^{k}\sum_{l=0}^{r} Y(l)Y(r-l)X(k-r)\right),$$

and the initial conditions becomes $X(0) = 1, Y(0) = 1$. Then, we obtain the series solution

$$\begin{aligned} x(t) &= 1 + 2.84147t + 3.38177t^2 + 2.56549t^3 + 0.498022t^4 - 3.62605t^5 - 13.1978t^6 \\ &\quad - 36.6427t^7 - 93.1066t^8 - 224.685t^9 - 520.598t^{10} - 1160.66t^{11} - 2481.54t^{12}, \;\; t \in [0, 0.4], \\ y(t) &= 1 + 2t + 4.84147t^2 + 10.6041t^3 + 24.528t^4 + 57.5466t^5 + 134.91t^6 \\ &\quad + 315.16t^7 + 733.76t^8 + 1702.66t^9 + 3938.03t^{10} + 9079.66t^{11} + 20873.9t^{12}, \;\; t \in [0, 0.4]. \end{aligned}$$

On the other hand, by applying the MsDTM to Eqs. (3.9) and (3.10), it follows

$$
\begin{aligned}
X_i(k+1) &= \frac{1}{k+1}\left(2X_i(k) + \sum_{r=0}^{k} \frac{d^r}{dt^r}\sin(t)\Big|_{t=Y_i(0)} \sum_{m=0}^{r} \frac{(-1)^m Y_i(0)^m}{(r-m)!m!}\left[\frac{1}{k!}\frac{d^k}{dt^k} y(t)^{r-m}\right]_{t=0}\right),\\
Y_i(k+1) &= \frac{1}{k+1}\left(X_i(k) + \sum_{r=0}^{k}\sum_{l=0}^{r} Y_i(l)Y_i(r-l)X_i(k-r)\right),\\
&X_0(0) = 1, X_i(0) = x_{i-1}(t_i), Y_0(0) = 1, Y_i(0) = y_{i-1}(t_i), i = 1,2,3,4,5.
\end{aligned}
$$

Hence, we obtain the series solution

$$
x(t) = \begin{cases}
1 + 2.84147t + 3.38177t^2 + 2.56549t^3 + 0.498022t^4 - 3.62605t^5 - 13.1978t^6 \\
-36.6427t^7 - 93.1066t^8 - 224.685t^9 - 520.598t^{10} - 1160.66t^{11} - 2481.54t^{12}, \;\; t \in [0, 0.08], \\
1.25028 + 3.43174(t-0.08) + 3.98652(t-0.08)^2 + 2.28135(t-0.08)^3 \\
-3.27511(t-0.08)^4 - 19.3406(t-0.08)^5 - 67.1434(t-0.08)^6 \\
-205.262(t-0.08)^7 - 584.413(t-0.08)^8 - 1563.52(t-0.08)^9 \\
-3898.43(t-0.08)^{10} - 8808.52(t-0.08)^{11} - 16669.2(t-0.08)^{12}, \;\; t \in [0.08, 0.16], \\
1.55128 + 4.10089(t-0.16) + 4.24803(t-0.16)^2 - 1.14073(t-0.16)^3 \\
-23.7465(t-0.16)^4 - 108.548(t-0.16)^5 - 404.828(t-0.16)^6 \\
-1336.03(t-0.16)^7 - 2807.58(t-0.16)^8 - 8159.94(t-0.16)^9 \\
-2936.53(t-0.16)^{10} + 104910(t-0.16)^{11} + 850945(t-0.16)^{12}, \;\; t \in [0.16, 0.24], 1.90448 + 4.67667(t - \\
-142.022(t-0.24)^4 - 585.377(t-0.24)^5 - 1394.54(t-0.24)^6 + 4460.95(t-0.24)^7 \\
+93830.2(t-0.24)^8 + 865886(t-0.24)^9 + 6.29241 \times 10^6 (t-0.24)^{10} \\
+3.98717 \times 10^7 (t-0.24)^{11} + 2.27165 \times 10^8 (t-0.24)^{12}, \;\; t \in [0.24, 0.32], \\
2.27311 + 420097(t-0.32) + 9.88838(t-0.32)^2 - 38.0005(t-0.32)^3 + 1179.65(t-0.32)^4 \\
+24086.5(t-0.32)^5 + 281349(t-0.32)^6 + 2.24486(t-0.32)^7 \\
+8.20097 \times 10^6 (t-0.32)^8 - 1.15647 \times 10^8 (t-0.32)^9 - 3.29081 \times 10^9 (t-0.32)^{10} \\
-5.19089 \times 10^{10} (t-0.32)^{11} - 6.53363 \times 10^{11} (t-0.32)^{12}, \;\; t \in [0.32, 0.4].
\end{cases}
$$

$$
y(t) = \begin{cases}
1 + 2t + 4.84147t^2 + 10.6041t^3 + 24.528t^4 + 57.5466t^5 + 134.91t^6 \\
+315.16t^7 + 733.76t^8 + 1702.66t^9 + 3938.03t^{10} + 9079.66t^{11} + 20873.9t^{12}, \;\; t \in [0, 0.08], \\
1.19765 + 3.04364(t-0.08) + 8.7346(t-0.08)^2 + 24.1547(t-0.08)^3 \\
+69.2557(t-0.08)^4 + 199.329(t-0.08)^5 + 571.305(t-0.08)^6 \\
+1629.29(t-0.08)^7 + 4624.49(t-0.08)^8 + 13067.8(t-0.08)^9 \\
+36779.9(t-0.08)^{10} + 103170(t-0.08)^{11} + 288632(t-0.08)^{12}, \;\; t \in [0.08, 0.16], \\
1.5131 + 5.10287(t-0.16) + 18.7225(t-0.16)^2 + 68.529(t-0.16)^3 \\
+254.775(t-0.16)^4 + 942.871(t-0.16)^5 + 3463.75(t-0.16)^6 + 12638.4(t-0.16)^7 \\
+45855(t-0.16)^8 + 165580(t-0.16)^9 + 597192(t-0.16)^{10} \\
+2.15185 \times 10^6 (t-0.16)^{11} + 7.76857 \times 10^6 (t-0.16)^{12}, \;\; t \in [0.16, 0.24],
\end{cases}
$$

$$y(t)=\begin{cases} 2.09104+10.2318(t-0.24)+53.3091(t-0.24)^2+278.516(t-0.24)^3 \\ +1449.56(t-0.24)^4+7466.42(t-0.24)^5+38168.7(t-0.24)^6+194558(t-0.24)^7 \\ +994211(t-0.24)^8+5.12151\times10^6(t-0.24)^9+2.67247\times10^7(t-0.24)^{10} \\ +1.41713\times10^8(t-0.24)^{11}+7.64272\times10^8(t-0.24)^{12}, \quad t\in[0.24,0.32], \\ 3.4941+30.0249(t-0.32)+266.217(t-0.32)^2+2342.96(t-0.32)^3+20645.6(t-0.32)^4 \\ +185021(t-0.32)^5+1.70474\times10^6(t-0.32)^6+1.61372\times10^7(t-0.32)^7 \\ +1.55164\times10^8(t-0.32)^8+1.49064\times10^9(t-0.32)^9+1.40754\times10^{10}(t-0.32)^{10} \\ +1.28807\times10^{11}(t-0.32)^{11}+1.12816\times10^{12}(t-0.32)^{12}, \quad t\in[0.32,0.4]. \end{cases}$$

Similar to the previous examples, the MsDTM results is in better agreement with the flow of the vector fields than the DTM result (see Figure 3.5). As seen in Figure 3.6, we show the approximate series solutions calculated by the MsDTM in many initial conditions such as (-1,1), (-1,-1), (0,1), (1, 1), (0,-1) and (1, -1).

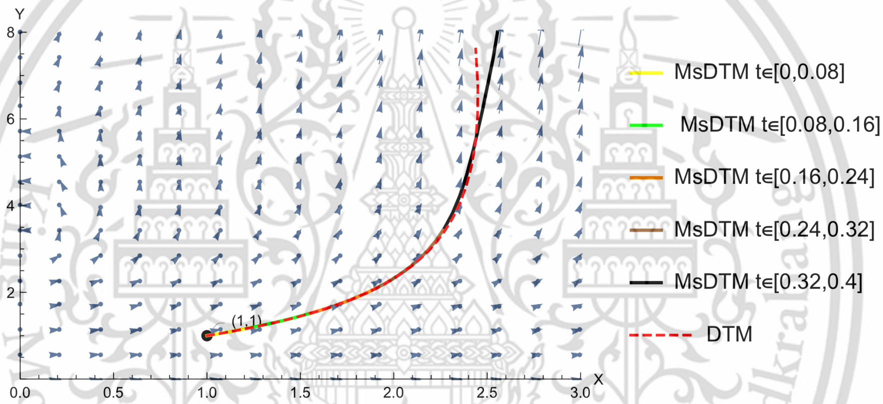


Figure 3.5*:* The MsDTM and the DTM compared with vector fields flow directions.

Figure 3.6*:* The solutions from MsDTM in initial conditions (-1,1), (-1,-1), (0,1), (1, 1), (0,-1), (1, -1) with vector fields flow directions.

Chapter 4

Differential transformation method for circular membrane vibrations

In this chapter, we obtain the step for solve the problems of vibrations of a circular membrane by using the differential transformation method (DTM).

4.1 Derivation of the method

In this section, we will show how to use the DTM to the problems of vibrations of a circular membrane. The presented problems include the vibrations of a circular membrane independent of angle studied in subsection 2.5.4 and the vibrations of a circular membrane depending on both radius and angle studied in subsection 2.5.5

4.1.1 Vibrations of a circular membrane independent of angle θ.

In this section, we consider a circular membrane of radius R which is fixed along the boundary circle, the initial shape $f(r)$, and assume that the initial velocity $g(r)$ is equal to zero. The model of the problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{4.1}$$

$$u(R,t) = 0 \text{ for all } t \geq 0, u(r,0) = f(r), u_t(r,0) = 0. \tag{4.2}$$

The calculation consists of the following three steps.

Step 1 By the method of separation of variables, we obtain two linear ODEs form the wave equation in Eq.(4.1).
The method of separation of variables uses the substitution

$$u(r,t) = w(r)g(t). \tag{4.3}$$

Differentiating Eq.(4.3), we obtain

$$\frac{\partial^2 u}{\partial t^2} = w\frac{d^2 g}{dt^2}, \frac{\partial u}{\partial r} = \frac{dw}{dr}g, \text{ and } \frac{\partial^2 u}{\partial r^2} = \frac{d^2 w}{dr^2}g. \tag{4.4}$$

By substituting Eq.(4.4) into Eq.(4.1), we obtain

$$w\frac{d^2 g}{dt^2} = c^2\left(\frac{d^2 w}{dr^2}g + \frac{1}{r}\frac{dw}{dr}g\right). \tag{4.5}$$

Dividing the result by $c^2 wg$, we have

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{w}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right). \tag{4.6}$$

The variables are now separated. Hence, both sides are independent and this can only be so if they are equal to a constant. This constant must be negative in order to obtain solutions that satisfy the boundary condition without being identically zero. Thus,

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{w}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = -h_{nm}^2. \tag{4.7}$$

Eq.(4.7) gives the two linear ODEs,

$$\frac{d^2 g}{dt^2} + \lambda_{nm}^2 g = 0, \quad \lambda_{nm} = ch_{nm}, \tag{4.8}$$

by defining $s = h_{nm} r$ we reduce $\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + h_{nm}^2 w = 0$ to the Bessel equation, that is

$$s^2\frac{d^2 w}{ds^2} + s\frac{dw}{ds} + s^2 w = 0. \tag{4.9}$$

Step 2 We apply the differential transform method to ODEs in Eqs.(4.8) and (4.9) to obtain the recursive formulas.

Applying the fundamental operations of DTM in Table 1 to Eqs.(4.8) and (4.9), respectively, we obtain

$$G(k+2) = -\frac{\lambda_{nm}^2 G(k)}{(k+1)(k+2)} \tag{4.10}$$

$$\begin{aligned}&\sum_{l=0}^{k}\delta(l-2)(k-l+1)(k-l+2)W(k-l+2)+\\&\sum_{l=0}^{k}\delta(l-1)(k-l+1)W(k-l+1)+\sum_{l=0}^{k}\delta(l-2)W(k-l)=0,\end{aligned} \tag{4.11}$$

with the initial values $G(0) = A_{nm}, G(1) = 0, W(0) = 1$ and $W(1) = 0$, where A_{nm} are the coefficients of the Fourier-Bessel series in Eq.(2.37).

Step 3 Using the recursive formulas in Eqs.(4.10) and (4.11) we find the coefficients of the series solutions of the Eqs. (4.8) and (4.9), respectively.

To find the series solution of Eq.(4.8), we substitute $k = 0, 1, 2, ...$ into Eq.(4.10), then we obtain

$$\begin{aligned}
G(2) &= -\frac{-\lambda_{nm}^2 G(0)}{2} = -\frac{\lambda_{nm}^2 A_{nm}}{2}, & G(3) &= -\frac{-\lambda_{nm}^2 G(1)}{6} = 0,\\
G(4) &= -\frac{-\lambda_{nm}^2 G(2)}{12} = \frac{\lambda_{nm}^2 A_{nm}}{24}, & G(5) &= -\frac{-\lambda_{nm}^2 G(3)}{20} = 0,\\
G(6) &= -\frac{-\lambda_{nm}^2 G(4)}{30} = -\frac{\lambda_{nm}^2 A_{nm}}{720}, & G(7) &= -\frac{-\lambda_{nm}^2 G(5)}{42} = 0,\\
&\vdots & &\vdots
\end{aligned}$$

where, $G(0), G(1), G(2), ...$ are the coefficients of the series solution.

Hence, we obtain the solution corresponding to Eq.(4.8), by substituting $\lambda_{nm} = c\frac{\alpha_{nm}}{R}$,

$$g_{nm}(t) = A_{nm}\left(1 - \frac{2c^2\alpha_{nm}^2 t^2}{R^2} + \frac{2c^4\alpha_{nm}^4 t^4}{3R^4} - \frac{4c^6\alpha_{nm}^6 t^6}{45R^6} + \frac{2c^8\alpha_{nm}^8 t^8}{315R^8} - \dots\right). \tag{4.12}$$

To find the series solution of Eq.(4.9), we substitute $k = 1, 2, \ldots$ into Eq.(4.11) and we obtain

$$\begin{aligned}
k = 1; \quad & \delta(0-2)(1-0+1)(1-0+2)W(1-0+2) + \delta(0-1)(1-0+1)W(1-0+1) + \\
& \delta(0-2)W(1-0) + \delta(1-2)(1-1+1)(1-1+2)W(1-1+2) + \\
& \delta(1-1)(1-1+1)W(1-1+1) + \delta(1-2)W(1-1) = 0 \\
& W(1) = 0 \\
k = 2; \quad & W(2) = \frac{-W(0)}{4} = -\frac{1}{4}, \qquad k = 3; \quad W(3) = \frac{-W(1)}{9} = 0 \\
k = 4; \quad & W(4) = \frac{-W(2)}{16} = \frac{1}{64}, \qquad k = 5; \quad W(5) = \frac{-W(3)}{25} = 0 \\
& \vdots \qquad\qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

Since, $W(k) = \dfrac{-W(k-2)}{k^2}$, $k = 2, 3, 4, \ldots$ or $W(k+2) = \dfrac{-W(k)}{(k+2)^2}$, $k = 0, 1, 2, 3, \ldots$ and $W(0), W(1), W(2), \ldots$ are the coefficients of the series solution. Hence, we obtain the series solution corresponding to Eq.(4.9), by substituting $s = h_{nm}r = \dfrac{\alpha_{nm}}{R}r$,

$$w_{nm}(r) = 1 - \frac{\alpha_{nm}^2 r^2}{4R^2} + \frac{\alpha_{nm}^4 r^4}{64R^4} - \frac{\alpha_{nm}^6 r^6}{2304R^6} + \frac{\alpha_{nm}^8 r^8}{147456R^8} - \frac{\alpha_{nm}^{10} r^{10}}{14745600R^{10}} + \cdots . \tag{4.13}$$

Therefore, the series solution of vibrations of a circular membrane is

$$u(r,t) = \sum_{m=1}^{\infty} w_{nm}(r) g_{nm}(t) = \sum_{m=1}^{\infty} A_{nm}\Big(1 - \frac{2c^2\alpha_{nm}^2 t^2}{R^2} + \cdots\Big)\Big(1 - \frac{\alpha_{nm}^2 r^2}{4R^2} + \cdots\Big), \tag{4.14}$$

where A_{nm} are the coefficients of the Fourier-Bessel series corresponding to J_n which can be calculated by Eq.(2.37) and α_{nm} is the mth positive zero of J_n.
Next, let us consider the general case, when the solution can also depend on angle θ.

4.1.2 Vibrations of a circular membrane depending on both radius and angle.

We now consider a circular membrane of radius R which is fixed along the boundary circle, with the initial shape $f(r,\theta)$, and the initial velocity $g(r,\theta)$ equal zero. The model of the problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right), \tag{4.15}$$

$$u(R,\theta,t) = 0 \text{ for all } t \geq 0, u(r,\theta,0) = f(r,\theta), u_t(r,\theta,0) = 0. \tag{4.16}$$

The calculation consists of the following three steps.

Step 1 Three ODEs form the wave equation in Eq.(4.15) using the method of separation variables.
We define a solution in the method of separation of variables,

$$u(r,\theta,t) = z(r,\theta)g(t). \tag{4.17}$$

Differentiating Eq.(4.17), we obtain

$$\frac{\partial^2 u}{\partial t^2} = z\frac{d^2 g}{dt^2}, \quad \frac{\partial u}{\partial r} = \frac{\partial z}{\partial r}g, \quad \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 z}{\partial r^2}g \text{ and } \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 z}{\partial \theta^2}g. \tag{4.18}$$

Substituting Eq.(4.18) into Eq.(4.15) gives

$$z\frac{d^2 g}{dt^2} = c^2\left(\frac{\partial^2 z}{\partial r^2}g + \frac{1}{r^2}\frac{\partial z}{\partial r}g + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}g\right). \tag{4.19}$$

Dividing both sides by $c^2 zg$ yields

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{z}\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right). \tag{4.20}$$

The variables are now separated. Hence, both sides must equal a constant, that is

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{z}\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right) = -h_{nm}^2, \tag{4.21}$$

Eq.(4.21) gives an ODE and a PDE, as follows:

$$\frac{d^2 g}{dt^2} + \lambda_{nm}^2 g = 0, \quad \lambda_{nm} = ch_{nm} \tag{4.22}$$

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + h_{nm}^2 z = 0. \tag{4.23}$$

The PDE as Eq.(4.23) can be separated by substituting $z = w(r)q(\theta)$ and its derivatives into Eq.(4.23), we obtain

$$\frac{d^2 w}{dr^2}q + \frac{1}{r}\frac{dw}{dr}q + \frac{1}{r^2}w\frac{d^2 q}{d\theta^2} + h_{nm}^2 wq = 0. \tag{4.24}$$

On the both sides, multiplying by r^2/wq and then rearranging the equation, we obtain

$$\frac{1}{q}\frac{d^2 q}{d\theta^2} = -\frac{1}{w}\left(r^2\frac{d^2 w}{dr^2} + r\frac{dw}{dr}\right) - h_{nm}^2 r^2. \tag{4.25}$$

The variables are now separated. The expressions on both sides must equal a constant, that is

$$\frac{1}{q}\frac{d^2 q}{d\theta^2} = -\frac{1}{w}\left(r^2\frac{d^2 w}{dr^2} + r\frac{dw}{dr}\right) - h_{nm}^2 r^2 = -n^2, \tag{4.26}$$

Eq.(4.26) gives two ODEs, as follows:

$$\frac{d^2 q}{d\theta^2} + n^2 q = 0, \tag{4.27}$$

$$r^2\frac{d^2 w}{dr^2} + r\frac{dw}{dr} + (h_{nm}^2 r^2 - n^2)w = 0, \tag{4.28}$$

Eq.(4.28) is known as the Bessel equation of order n where $n = 1, 2, 3, ...$. The nonnegative integer n in Eqs.(4.27) and (4.28) depends on the initial shape as shown in Table 4.1.

Table 4.1: The values of nonnegative integer n corresponding to given initial shape $f(r,\theta)$, and the initial values $G(0), G(1), Q(0)$ and $Q(1)$.

Initial shape $f(r,\theta)$	Value of n	G(0)	G(1)	Q(0)	Q(1)
$w(r)$	0	A_{0m}	0	1	0
$w(r)\sin(N\theta), N=0,1,...$	N	B_{Nm}	0	0	N
$w(r)\cos(N\theta), N=0,1,...$	N	A_{Nm}	0	1	0

Step 2 We apply the differential transform method to ODEs in Eqs.(4.22), (4.27) and (4.28) to obtain recursive formulas.

Taking the DTM of Eqs. (4.22), (4.27) and (4.28), respectively, we obtain

$$G(k+2) = -\frac{-\lambda_{nm}^2 G(k)}{(k+1)(k+2)}, \tag{4.29}$$

$$Q(k+2) = -\frac{n^2 Q(k)}{(k+1)(k+2)}, \tag{4.30}$$

$$\begin{aligned} n^2 W(k) = &\sum_{l=0}^{k} \delta(l-2)(k-l+1)(k-l+2)W(k-l+2)+ \\ &\sum_{l=0}^{k} \delta(l-1)(k-l+1)W(k-l+1) + \frac{\alpha_{nm}^2}{R^2}\sum_{l=0}^{k}\delta(l-2)W(k-l). \end{aligned} \tag{4.31}$$

Step 3 Using the recursive formulas in Eqs.(4.29), (4.30) and (4.31) to find the coefficients of the series solutions of ODEs.

Substituting $k = 0,1,2,...$ into Eq.(4.29), we obtain

$$\begin{aligned} G(2) &= -\frac{-\lambda_{nm}^2 G(0)}{2}, & G(3) &= -\frac{-\lambda_{nm}^2 G(1)}{6} = 0, \\ G(4) &= -\frac{-\lambda_{nm}^2 G(2)}{12} = \frac{\lambda_{nm}^2 G(0)}{24}, & G(5) &= -\frac{-\lambda_{nm}^2 G(3)}{20} = 0, \\ G(6) &= -\frac{-\lambda_{nm}^2 G(4)}{30} = -\frac{\lambda_{nm}^2 G(0)}{720}, & G(7) &= -\frac{-\lambda_{nm}^2 G(5)}{42} = 0, \\ &\vdots & &\vdots \end{aligned}$$

where, $G(0), G(1), G(2), ...$ are the coefficients of the series solution. Hence, we obtain the solution corresponding to Eq.(4.22),

$$g_{nm}(t) = G(0)\left(1 - \frac{c^2\alpha_{nm}^2 t^2}{2R^2} + \frac{c^4\alpha_{nm}^4 t^4}{24R^4} - \frac{c^6\alpha_{nm}^6 t^6}{720R^6} + \frac{c^8\alpha_{nm}^8 t^8}{40320R^8} - \cdots\right). \tag{4.32}$$

where the values of $G(0)$ depend on the initial shape, as shown in Table 4.1.

Substituting $k = 0,1,2,...$ into Eq.(4.27), we obtain

$$\begin{aligned} Q(2) &= -\frac{n^2 Q(0)}{2}, & Q(3) &= -\frac{n^2 Q(1)}{6}, \\ Q(4) &= -\frac{n^2 G(2)}{12} = \frac{n^4 Q(0)}{24}, & Q(5) &= -\frac{n^2 Q(3)}{20} = \frac{n^4 Q(1)}{120}, \\ Q(6) &= -\frac{n^2 Q(4)}{30} = -\frac{n^2 Q(0)}{720}, & Q(7) &= -\frac{n^2 Q(5)}{42} = -\frac{n^6 Q(1)}{5040}, \\ &\vdots & &\vdots \end{aligned}$$

where, $Q(0), Q(1), Q(2), \ldots$ are the coefficients of the series solution. Hence, we obtain the solution corresponding to Eq.(4.30),

$$q_n(\theta) = Q(0) + Q(1)\theta - \frac{n^2Q(0)}{2}\theta^2 - \frac{n^2Q(1)}{6}\theta^3 + \frac{n^4Q(0)}{24}\theta^4 + \frac{n^4Q(0)}{120}\theta^5 - \cdots . \tag{4.33}$$

Substituting $k = 1, 2, 3, \ldots$ into Eq.(4.28), we obtain

$$k = 1;\ \ n^2W(1) = W(1), \qquad k = 2;\ \ W(2) = \frac{\alpha_{nm}^2 W(0)}{n^2 - 4},$$
$$k = 3;\ \ W(3) = \frac{\alpha_{nm}^2 W(1)}{n^2 - 9}, \qquad k = 4;\ \ W(4) = \frac{\alpha_{nm}^2 W(2)}{n^2 - 16},$$
$$k = 5;\ \ W(3) = \frac{\alpha_{nm}^2 W(3)}{n^2 - 25}, \qquad k = 6;\ \ W(6) = \frac{\alpha_{nm}^2 W(4)}{n^2 - 36},$$
$$\vdots \qquad\qquad \vdots$$

Since, $W(k) = \dfrac{\alpha_{nm}^2 W(k-2)}{n^2 - k^2}, k = 2, 3, 4, \ldots,$ or $W(k+2) = \dfrac{\alpha_{nm}^2 W(k)}{n^2 - (k+2)^2}, k = 0, 1, 2, \ldots.$ Hence, we obtain the solution corresponding to Eq.(4.31),

$$w_{nm}(r) = W(0) + W(1)r + W(2)r^2 + W(3)r^3 + \cdots \tag{4.34}$$

Therefore, the solutions of vibrations of a circular membrane is

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} w_{nm}(r)q_{nm}(t)q_n(\theta) \\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} G(0)\Big(W(0) + W(1)r + W(2)r^2 + \cdots\Big)\Big(1 - \frac{c^2\alpha_{nm}^2 t^2}{2R^2} + \frac{c^4\alpha_{nm}^4 t^4}{24R^4} - \cdots\Big) \\ &\quad \Big(Q(0) + Q(1)\theta - \frac{n^2Q(0)}{2}\theta^2 - \frac{n^2Q(1)}{6}\theta^3 + \cdots\Big), \end{aligned} \tag{4.35}$$

where the values of $G(0)$ depend on the initial shape function, it can be A_{nm} or B_{nm} (see Table 4.1), here A_{nm} and B_{nm} are calculated by Eqs.(2.44) and (2.49), respectively, and α_{nm} is the mth positive zero of J_n.

The following shows the calculations of A_{nm} and B_{nm} corresponding to the value of nonnegative integer n. Let us recall Eqs.(2.44) and (2.49), we have

$$A_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})}\int_0^{2\pi}\int_0^R f(r,\theta)J_n(h_{nm}r)\cos(n\theta)r\,dr\,d\theta, \text{ and}$$
$$B_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})}\int_0^{2\pi}\int_0^R f(r,\theta)J_n(h_{nm}r)\sin(n\theta)r\,dr\,d\theta,$$

As we can see Eqs.(2.44) and (2.49) consist of the integral forms of the initial shape function $f(r,\theta) = w(r)q(\theta)$. Here, we illustrate the three cases of $q(\theta)$, thai is $q(\theta) = 1$, $q(\theta) = \cos(N\theta)$ and $q(\theta) = \sin(N\theta)$, here the calculations of A_{nm} and B_{nm} are as follows:

1. If $q(\theta) = 1$, then $\int_0^{2\pi}\cos(n\theta)d\theta = 0$ for $n \geq 1$ except $n = 0$. Thus A_{nm} are available for $n = 0$. That is A_{0m} are obtainable when $q(\theta) = 1$.

2. If $q(\theta) = \cos(N\theta), N = 0, 1, 2, \ldots$, then $\int_0^{2\pi}\cos(N\theta)\cos(n\theta)d\theta = 0$ for $n \neq N$. Thus $A_{nm} = 0$ when $n \neq N$. Besides $\int_0^{2\pi}\cos(N\theta)\sin(n\theta)d\theta = 0$ for all n. Thus $B_{nm} = 0$,

for all n. Therefore A_{nm} are available for $n = N$. That is A_{Nm} are obtainable when $q(\theta) = \cos(N\theta), N = 0, 1, 2, \ldots$.

3. If $q(\theta) = \sin(N\theta), N = 0, 1, 2, \ldots$, then $\int_0^{2\pi} \sin(N\theta)\cos(n\theta)d\theta = 0$ for all n. Thus $A_{nm} = 0$, for all n. Besides $\int_0^{2\pi} \sin(N\theta)\sin(n\theta)d\theta = 0$ for $n \neq N$. Thus $B_{nm} = 0$ when $n \neq N$. Therefore B_{nm} are available for $n = N$. That is B_{Nm} are obtainable when $q(\theta) = \sin(N\theta), N = 0, 1, 2, \ldots$.

As summarized in the Table 4.1, if we know the value of n then we obtain the initial values $G(0), G(1), Q(0), Q(1)$, and the similar for the initial values of the recursive relation in Eq.(4.31). Observe that $W(k), k = 0, 1, 2, \ldots$ also depend on the value of n as shown in Table 4.2.

Table 4.2*:* The initial values $W(k), k = 0, 1, 2, \ldots$ depending on n of Bessel equation in Eq.(4.28).

Value of n	$W(0)$	$W(1)$	$W(2)$	$W(3)$	$\cdots$	$W(n-1)$	$W(n)$
0	1	0					
1	0	$\frac{\alpha_{1m}}{2}$					
2		0	$\frac{\alpha_{2m}}{8}$				
3			0	$\frac{\alpha_{3m}}{48}$			
$\vdots$							
n						0	$\frac{\alpha_{nm}}{2^n n!}$

Table 4.3*:* The values of α_{nm} where the mth is positive zero of Bessel function J_n.

n \ m	1	2	3	4	5	6	$\cdots$
0	2.40483	5.52008	8.65373	11.79153	14.93092	18.07106	$\cdots$
1	03.83171	7.01559	10.1735	13.3237	16.4706	19.6159	$\cdots$
2	5.13562	8.41724	11.6198	14.796	17.9598	21.117	$\cdots$
3	6.38016	9.76102	13.0152	16.2235	19.4094	22.5827	$\cdots$
4	7.58834	11.0647	14.3725	17.616	20.8269	24.019	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

4.2 Applications

In this section, four examples of the problem are illustrated corresponding to the method in the previous section. Example 4.1 shows the vibrations of a circular membrane independent on angle corresponding to the problem in the subsection 2.5.4. Examples 4.2, 4.3 and 4.4 show the vibrations of a circular membrane depending on both and angle corresponding to the problem in the subsection 2.5.5. The accuracy of the method is assessed by data value comparisons with the analytical solutions.

Example 4.1. Consider the problem of vibrations of a circular membrane depending only on radius with $R = 1, c = 2$, the initial shape $f(r) = 1 - r^2$ and the initial velocity equal to zero $g(r) = 0$. That is

$$\frac{\partial^2 u}{\partial t^2} = 4\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{4.36}$$

$$u(1,t) = 0 \text{ for all } t \geq 0, u(r,0) = 1 - r^2, u_t(r,0) = 0. \tag{4.37}$$

Hence, the series solutions obtained according to the Eqs.(4.12) and (4.13) are as follows:

$$g_{0m}(t) = A_{0m}\left(1 - 2\alpha_{0m}^2 t^2 + \frac{2\alpha_{0m}^4 t^4}{3} - \frac{4\alpha_{0m}^6 t^6}{45} + \frac{2\alpha_{0m}^8 t^8}{315} - \frac{4\alpha_{0m}^{10} t^{10}}{14175} + \cdots\right),$$

$$w_{0m}(r) = 1 - \frac{\alpha_{0m}^2 r^2}{4} + \frac{\alpha_{0m}^4 r^4}{64} - \frac{\alpha_{0m}^6 r^6}{2304} + \frac{\alpha_{0m}^8 r^8}{147456} - \frac{\alpha_{0m}^{10} r^{10}}{147456000} + \cdots,$$

where A_{0m} are calculated by Eq.(2.37) and α_{0m} is the mth positive zero of J_0 as shown in Table 4.3. Therefore, the series solution of vibrations of a circular membrane is

$$\begin{aligned}
u(r,t) &= \sum_{m=1}^{\infty} w_{0m}(r) g_{0m}(t) \\
&= \sum_{m=1}^{\infty} A_{0m}\left(1 - \frac{\alpha_{0m}^2 r^2}{4} + \frac{\alpha_{0m}^4 r^4}{64} - \cdots\right)\left(1 - 2\alpha_{0m}^2 t^2 + \frac{2\alpha_{0m}^4 t^4}{3} - \cdots\right) \\
&= (1 - 1.44558r^2 + 0.552586r^4 - \cdots)(1.10802 - 12.8158t^2 + 24.7055t^4 - \cdots) \\
&\quad - (1 - 7.61782)r^2 + 14.5078r^4 - \cdots)(0.139777 - 8.51839t^2 + 86.5221t^4 - \cdots) \\
&\quad + (1 - 18.722r^2 + 87.6281r^4 - \cdots)(0.045476 - 6.81119t^2 + 170.025t^4 - \cdots) - \cdots.
\end{aligned}$$

The analytical solution of a circular membrane in this example is

$$\begin{aligned}
u(r,t) &= \sum_{m=1}^{\infty} \frac{8}{\alpha_{0m} J_1(\alpha_{0m})} J_0(\alpha_{0m} r)\cos(2\alpha_{0m} t) \\
&= 1.10801 J_0(2.40483r)\cos(4.80966t) \\
&\quad - 0.13978 J_0(5.52008r)\cos(11.04016t) + 0.04548 J_0(8.65373r)\cos(17.30746t) - \cdots.
\end{aligned}$$

Figure 4.1 shows the motion of the series solution for the first term $(m = 1, \alpha_{01} = 2.40483)$, the second term $(m = 2, \alpha_{02} = 5.52008)$ and the third term $(m = 3, \alpha_{03} = 8.65373)$ at the initial time. As show in Table 4.4, the approximate series solutions of the vibrations of a circular membrane obtained by using the DTM are compared with the analytical solution.

Figure 4.1: Normal modes of the vibrations of a circular membrane independent of the angle for Example 4.1.

Table 4.4: The comparison results for Example 4.1 between DTM solutions and analytical solutions at the initial time.

α_{nm}	r	DTM	Analytical	|DTM-Analytical|
$\alpha_{01} = 2.40483$	0	1.10786	1.10786	0
	0.10	1.08872	1.08872	0
	0.20	1.03849	1.03849	0
$\alpha_{02} = 5.52008$	0	0.13967	0.13967	0
	0.10	0.12719	0.12719	0
	0.20	0.09662	0.09662	1.38778×10^{-17}
$\alpha_{03} = 8.65373$	0	0.04539	0.04539	6.93889×10^{-18}
	0.10	0.03574	0.03574	6.93889×10^{-18}
	0.20	0.01500	0.01500	8.67362×10^{-18}

Example 4.2. Consider the problem of vibrations of a circular membrane depending on both r and θ with $R = 1, c = 1$, the initial shape $f(r,\theta) = 1 - r^4$ and initial velocity equal to zero $g(r,\theta) = 0$. That is,

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right), \tag{4.38}$$

$$u(1,\theta,t) = 0 \text{ for all } t \geq 0, u(r,\theta,0) = 1 - r^4, u_t(r,\theta,0) = 0. \tag{4.39}$$

From the initial shape, the nonnegative integer n can only be zero (see Table 4.1) because $A_{nm} = 0$ when $n \geq 1$ (i.e., $A_{0m} \neq 0$) and B_{nm} in Eq.(2.49) is always zero. The initial values are $G(0) = A_{0m}, G(1) = 0, Q(0) = 1, Q(1) = 0, W(0) = 1$ and $W(1) = 0$.
Hence, we obtain the series solutions corresponding to Eq.(4.32), (4.33) and (4.34), as

follows:

$$g_{0m} = A_{0m}\left(1 - \frac{\alpha_{0m}^2 t^2}{2} + \frac{\alpha_{0m}^4 t^4}{24} - \frac{\alpha_{0m}^6 t^6}{720} + \frac{\alpha_{0m}^8 t^8}{40320} - \frac{\alpha_{0m}^{10} t^{10}}{3628800} + \cdots\right)$$

$$q_0(\theta) = 1$$

$$w_{0m}(r) = 1 - \frac{\alpha_{0m}^2 r^2}{4} + \frac{\alpha_{0m}^4 r^4}{64} - \frac{\alpha_{0m}^6 r^6}{2304} + \frac{\alpha_{0m}^8 r^8}{147456} - \frac{\alpha_{0m}^{10} r^{10}}{147456000} + \cdots,$$

where A_{0m} are calculated by Eq.(2.44) and α_{0m} is the mth positive zero of J_0 as shown in Table 4.3. Therefore, the series solution of vibrations of a circular membrane is

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=0}^{\infty} q_n(\theta) \sum_{m=1}^{\infty} w_{nm}(r) g_{nm}(t) = q_0(\theta) \sum_{m=1}^{\infty} w_{0m}(r) g_{0m}(t) \\ &= (1 - 1.4458r^2 + 0.522586r^4 - \cdots)(2.73318 - 7.90328t^2 + 3.80886t^4 - \cdots) \\ &\quad - (1 - 7.61782r^2 + 14.5078r^4 - \cdots)(0.971432 - 14.8004t^2 + 37.5822t^4 - \cdots) \\ &\quad + (1 - 18.7218r^2 + 87.6261r^4 - \cdots)(0.344381 - 12.8948t^2 + 80.4712t^4 - \cdots) - \cdots. \end{aligned}$$

The analytical solution of a circular membrane in this example is

$$\begin{aligned} u(r,\theta,t) &= \sum_{m=1}^{\infty} \frac{16(\alpha_{0m}J_2(\alpha_{0m}) - 2J_3(\alpha_{0m}))}{\alpha_{0m}^3 J_1^2(\alpha_{0m})} J_0(\alpha_{0m}r)\cos(\alpha_{0m}t) \\ &= 2.73318J_0(2.40483r)\cos(2.40483t) - 0.971432J_0(5.52008r)\cos(5.52008t) \\ &\quad - 0.344381J_0(8.65373r)\cos(8.65373t) - \cdots. \end{aligned}$$

Figure 4.2 shows the motion of the series solution for the first term ($m = 1, \alpha_{01} = 2.40483$), the second term ($m = 2, \alpha_{02} = 5.52008$) and the third term ($m = 3, \alpha_{03} = 8.65373$) at the initial time. As show in Table 4.5, the approximate series solutions of the vibrations of a circular membrane obtained by using the DTM are compared with the analytical solution.

Figure 4.2: Normal modes of the vibrations of a circular membrane depending on both r and θ for Example 4.2, when $n = 0$.

Example 4.3. Let us consider the Example 4.2 with the initial shape defined by $f(r,\theta) = r(1 - r^4)\cos(\theta)$. From the initial shape, we obtain only $n = 1$ (see Table 4.1) because

Table 4.5: The comparison results for Example 4.2 between DTM solutions and analytical solutions at the initial time.

α_{nm}	r	DTM	Analytical	|DTM-Analytical|
	0	2.73318	2.73318	0
$\alpha_{01} = 2.40483$	0.10	2.69381	2.69381	0
	0.20	2.57739	2.57739	0
	0	0.97143	0.97143	0
$\alpha_{02} = 5.52008$	0.10	0.89827	0.89827	0
	0.20	0.69722	0.69722	1.11022×10^{-16}
	0	0.34438	0.34438	0
$\alpha_{03} = 8.65373$	0.10	0.28286	0.28286	0
	0.20	0.13093	0.13093	2.77556×10^{-17}

$A_{nm} = 0$, when $n = 0$ and $n > 1$, and B_{nm} in Eq.(2.49) is always zero. The initial values $G(0) = A_{1m}, G(1) = 0, Q(0) = 1, Q(1) = 0, W(0) = 0$ and $W(1) = \frac{\alpha_{1m}}{2}$. Then, we obtain the series solutions in Eqs.(4.32), (4.33) and (4.34) as follows:

$$g_{1m} = A_{1m}\left(1 - \frac{\alpha_{1m}^2 t^2}{2} + \frac{\alpha_{1m}^4 t^4}{24} - \frac{\alpha_{1m}^6 t^6}{720} + \frac{\alpha_{1m}^8 t^8}{40320} - \frac{\alpha_{1m}^{10} t^{10}}{3628800} + \cdots\right)$$
$$q_1(\theta) = 1 - \frac{\theta^2}{2} - \frac{\theta^4}{24} - \frac{\theta^6}{720} + \frac{\theta^8}{40320} - \cdots$$
$$w_{1m}(r) = \frac{\alpha_{1m} r}{2} - \frac{\alpha_{1m}^3 r^3}{16} + \frac{\alpha_{1m}^5 r^5}{384} - \frac{\alpha_{1m}^7 r^7}{18432} + \frac{\alpha_{1m}^9 r^9}{1474560} - \frac{\alpha_{1m}^{11} r^{11}}{176947200} + \cdots,$$

where A_{1m} are calculated by Eq.(2.44) and α_{1m} is the mth positive zero of J_1 as shown in Table 4.3. Therefore, the series solution of vibrations of a circular membrane is

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=0}^{\infty} q_n(\theta) \sum_{m=1}^{\infty} w_{nm}(r) g_{nm}(t) = q_1(\theta) \sum_{m=1}^{\infty} w_{1m}(r) g_{1m}(t) \\ &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots\right)\Big[(1.91586r - 3.51607r^3 + \cdot)(0.964141 - 7.07776t^2 + \cdots) \\ &\quad - (3.5078r - 21.5811r^3 + \cdots)(0.387905 - 9.54606t^2 + \cdots) \\ &\quad + (5.8099r - 98.0564r^3 + \cdots)(1.3701 \times 10^8 - 9.24954 \times 10^9 t^2 + \cdots) - \cdots\Big]. \end{aligned}$$

The analytical solution of a circular membrane in this example is

$$\begin{aligned} u(r,\theta,t) &= \cos\theta \sum_{m=1}^{\infty} \frac{8(\alpha_{1m} J_3(\alpha_{1m}) - 2J_4(\alpha_{1m}))}{\alpha_{1m}^3 J_2^2(\alpha_{1m})} J_1(\alpha_{1m} r)\cos(\alpha_{1m} t) \\ &= \cos\theta\Big[0.96414 J_1(3.83171r)\cos(3.83171t) - 0.387905 J_1(7.01559r)\cos(7.01559t) \\ &\quad + 1.3701 \times 10^8 J_1(11.6198r)\cos(11.6198t) - \cdots\Big]. \end{aligned}$$

Figure 4.3 shows the motion of the series solution for the first term $(m = 1, \alpha_{11} = 3.83171)$, the second term $(m = 2, \alpha_{12} = 7.01559)$ and the third term $(m = 3, \alpha_{13} = 10.1735)$ at the initial time.

As show in Table 4.6, the approximate series solutions of the vibrations of a circular membrane obtained by using the DTM are compared with the analytical solution. The results given in Table 4.6 showed that, the series solutions obtained by the DTM are equal to the analytical solutions.

Figure 4.3: Normal modes of the vibrations of a circular membrane depending on both r and θ for Example 4.3, when $n = 1$.

Table 4.6: The comparison results for Example 4.3 between DTM solutions and analytical solutions at the initial time.

α_{nm}	θ \ r	0		π		2π	
		DTM	Analytical	DTM	Analytical	DTM	Analytical
$\alpha_{11} = 3.83171$	0	0	0	0	0	0	0
	0.01	0.01847	0.01847	-0.01847	-0.01847	0.01847	0.01847
	0.02	0.03692	0.03692	-0.03692	-0.03692	0.03692	0.03692
$\alpha_{12} = 7.01559$	0	0	0	0	0	0	0
	0.01	0.01036	0.01036	-0.01036	-0.01036	0.01036	0.01036
	0.02	0.02715	0.02715	-0.02715	-0.02715	0.02715	0.02715
$\alpha_{13} = 10.1735$	0	0	0	0	0	0	0
	0.01	0.00856	0.00856	-0.00856	-0.00856	0.00856	0.00856
	0.02	0.00170	0.00170	-0.00170	-0.00170	0.00170	0.00170

Example 4.4. Let us consider the Example 4.2 with the initial shape defined by $f(r,\theta) = r(1-r^4)\cos(2\theta)$. From the initial shape, we obtain only $n = 2$ (see Table 4.1) because $A_{nm} = 0$, when $n = 0, 1$ and $n > 2$, and B_{nm} in Eq.(2.49) is always zero. The initial values are $G(0) = A_{2m}, G(1) = 0, Q(1) = 0, Q(1) = 0, W(1) = 0$ and $W(2) = \dfrac{\alpha_{2m}}{8}$. Therefore, the

series solution of vibrations of a circular membrane is

$$
\begin{aligned}
u(r,\theta,t) &= \sum_{n=0}^{\infty} q_n(\theta) \sum_{m=1}^{\infty} w_{nm}(r) g_{nm}(t) = q_2(\theta) \sum_{m=1}^{\infty} w_{2m}(r) g_{2m}(t) \\
&= \left(1 - 2\theta^2 + \frac{2\theta^4}{3} - \cdots\right)\Big[(0.641953r^2 - 1.41094r^4 + \cdots)(1.1876 - 15.6612 + \cdots) \\
&\quad - (1.05216r^2 - 6.21209r^4 + \cdots)(0.145193 - 5.14345t^2 + \cdots) \\
&\quad + (1.45248r^2 - 16.3427r^4 + \cdots)(0.197975 - 13.3653t^2 + \cdots) - \cdots\Big].
\end{aligned}
$$

where A_{2m} are calculated by Eq.(2.44) and α_{2m} is the mth positive zero of J_2 as shown in Table 4.3.

Chapter 5

Conclusions

In this section we will summarize all the main results obtained from the two problems research, which are as follows:

5.1 Composite transform formulae for differential transformation method with application to the nonlinear plane autonomous systems

In the first problem, the MsDTM combined with our new formulae have been successfully applied to solve nonlinear plane autonomous systems. Three different examples were solved and the series solutions of the DTM and the MsDTM were obtained. These are compared with the analytical solutions calculated by the phase-plane method in the first example and compared to the vector fields flow directions in the second and the third examples. The results of the MsDTM were more similar to the analytical solution and to the vector field flow direction than the DTM results. Therefore, this method based on our new transformed functions is a reliable and efficient mathematical tool for solving nonlinear plane autonomous systems.

5.2 Differential transformation method for circular membrane vibrations

In the second problem, the importance of the differential transformation method (DTM) lies in the initial values of the recursive formulas derived from the conversion of the ODEs problem, and the recursive formulas used to find the coefficients of the series solution of the problem. In this work, solving of the vibration problem of a circular membrane, the initial values of the recursive formulas can be calculated from the coefficients of the Fourier-Bessel series which is in the integral form of the initial shape of the membrane. The comparison of the DTM series solutions and the analytical solutions show that these are the same, hence it follows that the DTM method is suitable for this type of problems. Moreover, we consider that DTM is easier to use from the programming point of view.

5.3 Suggestions

In future research, we may apply our new transform formulae studied in the first problems to other problems aside from nonlinear plane autonomous systems and try to derive other transform formulae besides our 8 composite transform formulae.

The second problem, the some cases of the nonlinear vibrating circular membrane problems are difficult to be unsolvable analytically, we intend to modify and apply the steps of the DTM method presented in this work to these problems, in a future research.

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Appendices

Appendix A

The first research paper

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DIFFERENTIAL TRANSFORMATION METHOD FOR CIRCULAR MEMBRANE VIBRATIONS

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Abstract

The purpose of this research is to present the steps of one-dimensional differential transformation method (DTM) to find the series solutions for the vibrations of a circular membrane under the specified initial and boundary conditions. The problems will be studied in the both cases of vibrations depending only on radius and of the vibrations depending on both radius and angle. We illustrate four examples of problems which the exact solutions can be solve analytically and compare them to the DTM results, to show that the DTM is reliable and of high accuracy. This work shows that the DTM is easier to use than the analytical method from the point of view of programming.

2000 *Mathematics Subject Classification:* Primary: 35L05, Secondary: 35L20

Key words: analytical solution, approximate series solution, differential transform method, vibrations of a circular membrane.

1 Introduction

Circular membranes are important parts of drums, pumps, microphones, and other devices. This accounts for their great importance in engineering. We consider the case when the circular membrane is plane and its material is elastic, but offers no resistance to bending (this excludes thin metallic membranes). Then the vibrations of the circular membrane is given in the form of two-dimensional wave equation in polar coordinates,

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$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right),$$
$$u(R,\theta,t) = 0 \text{ for all } t \geq 0, u(r,\theta,0) = f(r,\theta), u_t(r,\theta,0) = g(r,\theta). \tag{1}$$

where $0 \leq r \leq R, 0 \leq \theta \leq 2\pi, c^2 = T/\rho$ in term of the membranes tension T and density ρ, R is a radius of a membrane, a membrane is fixed along the boundary circle radius $R, f(r,\theta)$ is the initial shape at time $t = 0$ and $g(r,\theta)$ is the initial velocity (see [4]).

The differential transformation method (DTM) is an alternative procedure for obtaining an approximate Taylor series solution or the semi-analytical solution of differential equations. The main advantage of this method is that it can be applied directly to nonlinear differential equations without the requiring linearization and discretization. The concept of the DTM was introduced by Zhou [10], who solved linear and nonlinear problems in electrical circuits and many other problems related to differential equations (see also [3], [5], [6], [7], [8] and [9])

In the present paper, we will show how to extend the method of differential transformation to the problem of vibrations of a circular membrane. The computation consists of three steps. The first step is using the method of separation of variables to obtain ODEs from the wave equation in Eq.(1). The next step is applying the DTM to ODEs from the previous step to obtain recursive relations. The last step is to find the coefficients of the series solutions for ODEs using the recursive relations.

The present paper has been organized as follows. In the section 2, the one-dimensional differential transformation method is introduced, and the Fourier Bessel series are described. In the section 3, the analysis of the method for the vibrations of a circular membrane both the vibrations depending on only radius and the vibrations depending on both radius and angle are described, as shown in subsection 3.1 and subsection 3.2, respectively. Four examples of vibrations of a circular membrane with different conditions corresponding to the three steps in the section 3, have been presented in the section 4. The conclusion is given at the end of the paper in the section 5.

2 Preliminaries

The basic definitions and fundamental operations of the differential transform are introduced as follows.

2.1 The one-dimensional differential transformation

Definition 1. *The one-dimensional differential transform of the function $x(t)$ is defined as*

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0}, \quad k \in \mathbb{I}^+ \cup \{0\}. \tag{2}$$

In Eq.(2), $x(t)$ is called the original function and $X(k)$ is called the transformed function.

Definition 2. *The inverse one-dimensional differential transforms of $X(k)$ is defined as*

$$x(t) = \sum_{k=0}^{\infty} X(k)t^k, \tag{3}$$

that is,

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0}. \tag{4}$$

Equation (4) implies that the concept of differential transformation method is derived from Taylor series expansion. Actually, in concrete applications, the function $x(t)$ is expressed by a truncated series and Eq.(3) becomes

$$x(t) = \sum_{k=0}^{N} X(k)t^k. \tag{5}$$

The fundamental operations of one-dimensional the DTM are shown in Table 1.

Table 1: The fundamental operations of one-dimensional DTM.

Original function $x(t)$	Transformed function $X(k)$
$x(t) \pm y(t)$	$X(k) \pm Y(k)$
$\lambda x(t)$	$\lambda X(k)$
$x(t)y(t)$	$\sum_{r=0}^{k} X(r)Y(k-r)$
$x(t)y(t)z(t)$	$\sum_{r=0}^{k}\sum_{l=0}^{r} X(l)Y(r-l)Z(k-r)$
$\frac{d^r}{dt^r}x(t)$	$\frac{(k+r)!}{k!}X(k+r)$

2.2 Fourier-Bessel series

The series solutions of the presented problems consist of the coefficients of the Fourier-Bessel series corresponding to the Bessel functions of the first kind (see also [1]). The following theorem explain the meaning of the Fourier-Bessel series based on the orthogonality relations.

Theorem 1 (Orthogonality of the Bessel Functions [3])**.** *For each fixed nonnegative integer n the sequence of the Bessel functions of the first kind $J_n(h_{n1}r)$, $J_n(h_{n2}r), \ldots$ with $h_{nm} = \alpha_n m/R$ where $\alpha_n m$ is the mth positive zero of $J_n, (m = 1, 2, 3, \ldots)$, forms an orthogonal set on the interval $0 \leq r \leq R$ with respect to the weight function r, that is*

$$\int_0^R rJ_n(h_{nm}r)J_n(h_{nj}r)dr = 0 \quad (j \neq m, n \text{ fixed}). \tag{6}$$

The Fourier-Bessel series for vibrations of a circular membrane independent of angle in subsection 3.1 corresponding to J_n (n fixed) is $f(r) = \sum_{m=1}^{\infty} A_{nm} J_n(h_{nm} r)$, (with $h_{nm} = \alpha_{nm}/R$). Here the coefficients are

$$A_{nm} = \frac{2}{R^2 J_{n+1}^2(\alpha_{nm})} \int_0^R r f(r) J_n(h_{nm} r) dr, \tag{7}$$

where $J_n(h_{nm}r) = \sum_{l=0}^{\infty} \dfrac{(-1)^l (h_{nm}r)^{2l+n}}{2^{2l+n} l!(n+l)!}$ is the Bessel function of order n of the first kind and α_{nm} is the mth positive zero of $J_n, (m = 1, 2, 3, ...)$,

The Fourier-Bessel series for vibrations of a circular membrane depending on both radius and angle in subsection 3.2 corresponding to J_n is

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(h_{nm}r)\Big(A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)\Big).$$

Here the coefficients are

$$A_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})} \int_0^{2\pi}\int_0^R f(r,\theta) J_n(h_{nm}r)\cos(n\theta) r dr d\theta, \tag{8}$$

$$B_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})} \int_0^{2\pi}\int_0^R f(r,\theta) J_n(h_{nm}r)\sin(n\theta) r dr d\theta, \tag{9}$$

where $J_n(h_{nm}r) = \sum_{l=0}^{\infty} \dfrac{(-1)^l (h_{nm}r)^{2l+n}}{2^{2l+n} l!(n+l)!}$ is the Bessel function of order n of the first kind and α_{nm} is the mth positive zero of $J_n, (n = 0, 1, 2, ..., m = 1, 2, 3, ...)$.

3 Analysis of method

In this section, we will show how to use the DTM to the problems of vibrations of a circular membrane. The presented problems include the vibrations of a circular membrane independent of angle studied in subsection 3.1 and the vibrations of a circular membrane depending on both radius and angle studied in subsection 3.2.

3.1 Vibrations of a circular membrane independent of angle θ.

In this section, we consider a circular membrane of radius R which is fixed along the boundary circle, the initial shape $f(r)$, and assume that the initial velocity $g(r)$ is equal to zero. The model of the problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{10}$$

$$u(R,t) = 0 \text{ for all } t \geq 0, u(r,0) = f(r), u_t(r,0) = 0. \tag{11}$$

The calculation consists of the following three steps.

Step 1 By the method of separation of variables, we obtain two linear ODEs form the wave equation in Eq.(10).
The method of separation of variables uses the substitution

$$u(r,t) = w(r)g(t). \tag{12}$$

Differentiating Eq.(12), we obtain

$$\frac{\partial^2 u}{\partial t^2} = w\frac{d^2 g}{dt^2}, \frac{\partial u}{\partial r} = \frac{dw}{dr}g, \text{ and } \frac{\partial^2 u}{\partial r^2} = \frac{d^2 w}{dr^2}g. \tag{13}$$

By substituting Eq.(13) into Eq.(10), we obtain

$$w\frac{d^2 g}{dt^2} = c^2\left(\frac{d^2 w}{dr^2}g + \frac{1}{r}\frac{dw}{dr}g\right). \tag{14}$$

Dividing the result by c^2wg, we have

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{w}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right). \tag{15}$$

The variables are now separated. Hence, both sides are independent and this can only be so if they are equal to a constant. This constant must be negative in order to obtain solutions that satisfy the boundary condition without being identically zero. Thus,

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{w}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = -h_{nm}^2. \tag{16}$$

Eq.(16) gives the two linear ODEs,

$$\frac{d^2 g}{dt^2} + \lambda_{nm}^2 g = 0, \quad \lambda_{nm} = ch_{nm}, \tag{17}$$

by defining $s = h_{nm}r$ we reduce $\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + h_{nm}^2 w = 0$ to the Bessel equation, that is

$$s^2\frac{d^2 w}{ds^2} + s\frac{dw}{ds} + s^2 w = 0. \tag{18}$$

Step 2 We apply the differential transform method to ODEs in Eqs.(17) and (18) to obtain the recursive formulas.
Applying the fundamental operations of DTM in Table 1 to Eqs.(17) and (18), respectively, we obtain

$$G(k+2) = -\frac{\lambda_{nm}^2 G(k)}{(k+1)(k+2)} \tag{19}$$

$$\sum_{l=0}^{k}\delta(l-2)(k-l+1)(k-l+2)W(k-l+2)+ \\ \sum_{l=0}^{k}\delta(l-1)(k-l+1)W(k-l+1)+\sum_{l=0}^{k}\delta(l-2)W(k-l)=0, \tag{20}$$

with the initial values $G(0) = A_{nm}, G(1) = 0, W(0) = 1$ and $W(1) = 0$, where A_{nm} are the coefficients of the Fourier-Bessel series in Eq.(7).

Step 3 Using the recursive formulas in Eqs.(19) and (20) we find the coefficients of the series solutions of the Eqs. (17) and (18), respectively.
To find the series solution of Eq.(17), we substitute into Eq.(19), then we obtain

$$\begin{aligned}
G(2) &= -\frac{-\lambda_{nm}^2 G(0)}{2} = -\frac{\lambda_{nm}^2 A_{nm}}{2}, & G(3) &= -\frac{-\lambda_{nm}^2 G(1)}{6} = 0,\\
G(4) &= -\frac{-\lambda_{nm}^2 G(2)}{12} = \frac{\lambda_{nm}^2 A_{nm}}{24}, & G(5) &= -\frac{-\lambda_{nm}^2 G(3)}{20} = 0,\\
G(6) &= -\frac{-\lambda_{nm}^2 G(4)}{30} = -\frac{\lambda_{nm}^2 A_{nm}}{720}, & G(7) &= -\frac{-\lambda_{nm}^2 G(5)}{42} = 0,\\
&\vdots & &\vdots
\end{aligned}$$

where, $G(0), G(1), G(2), \ldots$ are the coefficients of the series solution.
Hence, we obtain the solution corresponding to Eq.(17), by substituting $\lambda_{nm} = c\dfrac{\alpha_{nm}}{R}$,

$$g_{nm}(t) = A_{nm}\left(1-\frac{2c^2\alpha_{nm}^2 t^2}{R^2}+\frac{2c^4\alpha_{nm}^4 t^4}{3R^4}-\frac{4c^6\alpha_{nm}^6 t^6}{45R^6}+\frac{2c^8\alpha_{nm}^8 t^8}{315R^8}-\cdots\right). \tag{21}$$

To find the series solution of Eq.(18), we substitute $k = 1, 2, \ldots$ into Eq.(20) and we obtain

$$\begin{aligned}
k=1;\quad & \delta(0-2)(1-0+1)(1-0+2)W(1-0+2)+\\
& \delta(0-1)(1-0+1)W(1-0+1)+\\
& \delta(0-2)W(1-0)+\delta(1-2)(1-1+1)(1-1+2)W(1-1+2)+\\
& \delta(1-1)(1-1+1)W(1-1+1)+\delta(1-2)W(1-1)=0\\
& W(1)=0\\
k=2;\quad & W(2)=\frac{-W(0)}{4}=-\frac{1}{4}, \qquad k=3;\quad W(3)=\frac{-W(1)}{9}=0\\
k=4;\quad & W(4)=\frac{-W(2)}{16}=\frac{1}{64}, \qquad k=5;\quad W(5)=\frac{-W(3)}{25}=0\\
& \vdots \qquad\qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

Since, $W(k) = \dfrac{-W(k-2)}{k^2},\ k = 2, 3, 4, \ldots$ or $W(k+2) = \dfrac{-W(k)}{(k+2)^2},\ k = 0, 1, 2, 3, \ldots$ and $W(0), W(1), W(2), \ldots$ are the coefficients of the series solution. Hence, we obtain the series solution corresponding to Eq.(18), by substituting $s = h_{nm}r = \frac{\alpha_{nm}}{R}r$,

$$w_{nm}(r) = 1 - \frac{\alpha_{nm}^2 r^2}{4R^2} + \frac{\alpha_{nm}^4 r^4}{64R^4} - \frac{\alpha_{nm}^6 r^6}{2304R^6} + \frac{\alpha_{nm}^8 r^8}{147456R^8} - \frac{\alpha_{nm}^{10} r^{10}}{14745600R^{10}} + \cdots . \quad (22)$$

Therefore, the series solution of vibrations of a circular membrane is

$$u(r,t) = \sum_{m=1}^{\infty} w_{nm}(r)g_{nm}(t) = \sum_{m=1}^{\infty} A_{nm}\Big(1 - \frac{2c^2\alpha_{nm}^2 t^2}{R^2} + \cdots\Big)\Big(1 - \frac{\alpha_{nm}^2 r^2}{4R^2} + \cdots\Big), \quad (23)$$

where A_{nm} are the coefficients of the Fourier-Bessel series corresponding to J_n which can be calculated by Eq.(7) and α_{nm} is the mth positive zero of J_n.
Next, let us consider the general case, when the solution can also depend on angle θ.

3.2 Vibrations of a circular membrane depending on both radius and angle.

We now consider a circular membrane of radius R which is fixed along the boundary circle, with the initial shape $f(r,\theta)$, and the initial velocity $g(r,\theta)$ equal zero. The model of the problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right), \quad (24)$$

$$u(R,\theta,t) = 0 \text{ for all } t \geq 0, u(r,\theta,0) = f(r,\theta), u_t(r,\theta,0) = 0. \quad (25)$$

The calculation consists of the following three steps.

Step 1 Three ODEs form the wave equation in Eq.(24) using the method of separation variables.
We define a solution in the method of separation of variables,

$$u(r,\theta,t) = z(r,\theta)g(t). \quad (26)$$

Differentiating Eq.(26), we obtain

$$\frac{\partial^2 u}{\partial t^2} = z\frac{d^2 g}{dt^2}, \quad \frac{\partial u}{\partial r} = \frac{\partial z}{\partial r}g, \quad \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 z}{\partial r^2}g \text{ and } \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 z}{\partial \theta^2}g. \quad (27)$$

Substituting Eq.(27) into Eq.(24) gives

$$z\frac{d^2 g}{dt^2} = c^2\left(\frac{\partial^2 z}{\partial r^2}g + \frac{1}{r^2}\frac{\partial z}{\partial r}g + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}g\right). \quad (28)$$

Dividing both sides by c^2zg yields

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{z}\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right). \quad (29)$$

The variables are now separated. Hence, both sides must equal a constant, that is

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{z}\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right) = -h_{nm}^2, \tag{30}$$

Eq.(30) gives an ODE and a PDE, as follows:

$$\frac{d^2 g}{dt^2} + \lambda_{nm}^2 g = 0, \quad \lambda_{nm} = ch_{nm} \tag{31}$$

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + h_{nm}^2 z = 0. \tag{32}$$

The PDE as Eq.(32) can be separated by substituting $z = w(r)q(\theta)$ and its derivatives into Eq.(32), we obtain

$$\frac{d^2 w}{dr^2}q + \frac{1}{r}\frac{dw}{dr}q + \frac{1}{r^2}w\frac{d^2 q}{d\theta^2} + h_{nm}^2 wq = 0. \tag{33}$$

On the both sides, multiplying by r^2/wq and then rearranging the equation, we obtain

$$\frac{1}{q}\frac{d^2 q}{d\theta^2} = -\frac{1}{w}\left(r^2\frac{d^2 w}{dr^2} + r\frac{dw}{dr}\right) - h_{nm}^2 r^2. \tag{34}$$

The variables are now separated. The expressions on both sides must equal a constant, that is

$$\frac{1}{q}\frac{d^2 q}{d\theta^2} = -\frac{1}{w}\left(r^2\frac{d^2 w}{dr^2} + r\frac{dw}{dr}\right) - h_{nm}^2 r^2 = -n^2, \tag{35}$$

Eq.(35) gives two ODEs, as follows:

$$\frac{d^2 q}{d\theta^2} + n^2 q = 0, \tag{36}$$

$$r^2\frac{d^2 w}{dr^2} + r\frac{dw}{dr} + (h_{nm}^2 r^2 - n^2)w = 0, \tag{37}$$

Eq.(37) is known as the Bessel equation of order n where $n = 1, 2, 3, \ldots$. The nonnegative integer n in Eqs.(36) and (37) depends on the initial shape as shown in Table 2.

Table 2: The values of nonnegative integer n corresponding to given initial shape $f(r,\theta)$, and the initial values $G(0), G(1), Q(0)$ and $Q(1)$.

Initial shape $f(r,\theta)$	Value of n	G(0)	G(1)	Q(0)	Q(1)
$w(r)$	0	A_{0m}	0	1	0
$w(r)\sin(N\theta), N = 0, 1, \ldots$	N	B_{Nm}	0	0	N
$w(r)\cos(N\theta), N = 0, 1, \ldots$	N	A_{Nm}	0	1	0

Step 2 We apply the differential transform method to ODEs in Eqs.(31), (36) and (37) to obtain recursive formulas.
Taking the DTM of Eqs. (31), (36) and (37),respectively, we obtain

$$G(k+2) = -\frac{-\lambda_{nm}^2 G(k)}{(k+1)(k+2)}, \tag{38}$$

$$Q(k+2) = -\frac{n^2 Q(k)}{(k+1)(k+2)}, \tag{39}$$

$$\begin{aligned} n^2 W(k) = \sum_{l=0}^{k} \delta(l-2)(k-l+1)(k-l+2)W(k-l+2)+ \\ \sum_{l=0}^{k} \delta(l-1)(k-l+1)W(k-l+1) + \frac{\alpha_{nm}^2}{R^2}\sum_{l=0}^{k} \delta(l-2)W(k-l). \end{aligned} \tag{40}$$

Step 3 Using the recursive formulas in Eqs.(38), (39) and (40) to find the coefficients of the series solutions of ODEs.
Substituting $k = 0, 1, 2, ...$ into Eq.(38), we obtain

$$\begin{aligned} G(2) &= -\frac{-\lambda_{nm}^2 G(0)}{2}, & G(3) &= -\frac{-\lambda_{nm}^2 G(1)}{6} = 0, \\ G(4) &= -\frac{-\lambda_{nm}^2 G(2)}{12} = \frac{\lambda_{nm}^2 G(0)}{24}, & G(5) &= -\frac{-\lambda_{nm}^2 G(3)}{20} = 0, \\ G(6) &= -\frac{-\lambda_{nm}^2 G(4)}{30} = -\frac{\lambda_{nm}^2 G(0)}{720}, & G(7) &= -\frac{-\lambda_{nm}^2 G(5)}{42} = 0, \\ &\vdots & &\vdots \end{aligned}$$

where, $G(0), G(1), G(2), ...$ are the coefficients of the series solution. Hence, we obtain the solution corresponding to Eq.(31),

$$g_{nm}(t) = G(0)\left(1 - \frac{c^2\alpha_{nm}^2 t^2}{2R^2} + \frac{c^4\alpha_{nm}^4 t^4}{24R^4} - \frac{c^6\alpha_{nm}^6 t^6}{720R^6} + \frac{c^8\alpha_{nm}^8 t^8}{40320R^8} - \cdots\right). \tag{41}$$

where the values of $G(0)$ depend on the initial shape, as shown in Table 2.

Substituting $k = 0, 1, 2, ...$ into Eq.(36), we obtain

$$\begin{aligned} Q(2) &= -\frac{n^2 Q(0)}{2}, & Q(3) &= -\frac{n^2 Q(1)}{6}, \\ Q(4) &= -\frac{n^2 G(2)}{12} = \frac{n^4 Q(0)}{24}, & Q(5) &= -\frac{n^2 Q(3)}{20} = \frac{n^4 Q(1)}{120}, \\ Q(6) &= -\frac{n^2 Q(4)}{30} = -\frac{n^2 Q(0)}{720}, & Q(7) &= -\frac{n^2 Q(5)}{42} = -\frac{n^6 Q(1)}{5040}, \\ &\vdots & &\vdots \end{aligned}$$

where, $Q(0), Q(1), Q(2), \ldots$ are the coefficients of the series solution. Hence, we obtain the solution corresponding to Eq.(39),

$$q_n(\theta) = Q(0)+Q(1)\theta-\frac{n^2Q(0)}{2}\theta^2-\frac{n^2Q(1)}{6}\theta^3+\frac{n^4Q(0)}{24}\theta^4+\frac{n^4Q(0)}{120}\theta^5-\cdots. \quad (42)$$

Substituting $k = 1, 2, 3, \ldots$ into Eq.(37), we obtain

$$\begin{aligned}
&k=1;\; n^2W(1)=W(1), && k=2;\; W(2)=\frac{\alpha_{nm}^2W(0)}{n^2-4},\\
&k=3;\; W(3)=\frac{\alpha_{nm}^2W(1)}{n^2-9}, && k=4;\; W(4)=\frac{\alpha_{nm}^2W(2)}{n^2-16},\\
&k=5;\; W(3)=\frac{\alpha_{nm}^2W(3)}{n^2-25}, && k=6;\; W(6)=\frac{\alpha_{nm}^2W(4)}{n^2-36},\\
&\vdots && \vdots
\end{aligned}$$

Since, $W(k) = \frac{\alpha_{nm}^2W(k-2)}{n^2-k^2}, k = 2, 3, 4, \ldots$, or $W(k+2) = \frac{\alpha_{nm}^2W(k)}{n^2-(k+2)^2}, k = 0, 1, 2, \ldots$. Hence, we obtain the solution corresponding to Eq.(40),

$$w_{nm}(r) = W(0) + W(1)r + W(2)r^2 + W(3)r^3 + \cdots \quad (43)$$

Therefore, the solutions of vibrations of a circular membrane is

$$\begin{aligned}
u(r,\theta,t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} w_{nm}(r)q_{nm}(t)q_n(\theta)\\
&= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} G(0)\Big(W(0)+W(1)r+W(2)r^2+\cdots\Big)\\
&\quad \left(1-\frac{c^2\alpha_{nm}^2t^2}{2R^2}+\frac{c^4\alpha_{nm}^4t^4}{24R^4}-\cdots\right)\\
&\quad \left(Q(0)+Q(1)\theta-\frac{n^2Q(0)}{2}\theta^2-\frac{n^2Q(1)}{6}\theta^3+\cdots\right),
\end{aligned} \quad (44)$$

where the values of $G(0)$ depend on the initial shape function, it can be A_{nm} or B_{nm} (see Table 2), here A_{nm} and B_{nm} are calculated by Eqs.(8) and (9), respectively, and α_{nm} is the mth positive zero of J_n.

The following shows the calculations of A_{nm} and B_{nm} corresponding to the value of nonnegative integer n. Let us recall Eqs.(8) and (9), we have

$$A_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})}\int_0^{2\pi}\int_0^R f(r,\theta)J_n(h_{nm}r)\cos(n\theta)r\,dr\,d\theta, \text{ and}$$

$$B_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})}\int_0^{2\pi}\int_0^R f(r,\theta)J_n(h_{nm}r)\sin(n\theta)r\,dr\,d\theta,$$

As we can see Eqs.(8) and (9) consist of the integral forms of the initial shape function $f(r,\theta) = w(r)q(r)$. Here, we illustrate the three cases of $q(\theta)$ i.e., (i) $q(\theta) = 1$ (ii) $q(\theta) = \cos(N\theta)$ and (iii) $q(\theta) = \sin(N\theta)$, here the calculations of A_{nm} and B_{nm} are as follows:

1. If $q(\theta) = 1$, then $\int_0^{2\pi} \cos(n\theta)d\theta = 0$ for $N \leq 1$ except $n = 0$. Thus A_{nm} are available for $n = 0$. That is A_{0m} are obtainable when $q(\theta) = 1$.

2. If $q(\theta) = \cos(N\theta), N = 0, 1, 2, ...$, then $\int_0^{2\pi} \cos(N\theta)\cos(n\theta)d\theta = 0$ for $n \neq N$. Thus $A_{nm} = 0$ when $n \neq N$. Besides $\int_0^{2\pi} \cos(N\theta)\sin(n\theta)d\theta = 0$ for all n. Thus $B_{nm} = 0$, for all n. Therefore A_{nm} are available for $n = N$. That is A_{Nm} are obtainable when $q(\theta) = \cos(N\theta), N = 0, 1, 2, ...$

3. If $q(\theta) = \sin(N\theta), N = 0, 1, 2, ...$, then $\int_0^{2\pi} \sin(N\theta)\cos(n\theta)d\theta = 0$ for all n. Thus $A_{nm} = 0$, for all n. Besides $\int_0^{2\pi} \sin(N\theta)\sin(n\theta)d\theta = 0$ for $n \neq N$. Thus $B_{nm} = 0$ when $n \neq N$. Therefore B_{nm} are available for $n = N$. That is B_{Nm} are obtainable when $q(\theta) = \sin(N\theta), N = 0, 1, 2, ...$

As summarized in the Table 2, if we know the value of n then we obtain the initial values $G(0), G(1), Q(0), Q(1)$, and the similar for the initial values of the recursive relation in Eq.(40). Observe that $W(k), k = 0, 1, 2, ...$ also depend on the value of n as shown in Table 3.

Table 3: The initial values $W(k), k = 0, 1, 2, ...$ depending on n of Bessel equation in Eq.(37).

Value of n	$W(0)$	$W(1)$	$W(2)$	$W(3)$	$\cdots$	$W(n-1)$	$W(n)$
0	1	0					
1	0	$\frac{\alpha_{1m}}{2}$					
2		0	$\frac{\alpha_{2m}}{8}$				
3			0	$\frac{\alpha_{3m}}{48}$			
$\vdots$							
n						0	$\frac{\alpha_{nm}}{2^n n!}$

Table 4: The values of α_{nm} where the mth is positive zero of Bessel function J_n.

n \ m	1	2	3	4	5	6	$\cdots$
0	2.40483	5.52008	8.65373	11.79153	14.93092	18.07106	$\cdots$
1	03.83171	7.01559	10.1735	13.3237	16.4706	19.6159	$\cdots$
2	5.13562	8.41724	11.6198	14.796	17.9598	21.117	$\cdots$
3	6.38016	9.76102	13.0152	16.2235	19.4094	22.5827	$\cdots$
4	7.58834	11.0647	14.3725	17.616	20.8269	24.019	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

4 Applications

In this section, four examples of the problem are illustrated corresponding to the method in the previous section. Example 1 shows the vibrations of a circular membrane independent on angle corresponding to the problem in the subsection 3.1. Examples 2, 3 and 4 show the vibrations of a circular membrane depending on both and angle corresponding to the problem in the subsection 3.2. The accuracy of the method is assessed by data value comparisons with the analytical solutions.

Example 1. *Consider the problem of vibrations of a circular membrane depending on radius in Eqs.*(10) *and* (11) *with radius 1,* $c = 2$*, the initial shape* $f(r) = 1 - r^2$ *and the initial velocity equal to zero. Hence, the series solutions in Eqs.*(21) *and* (22) *are as follows:*

$$g_{0m}(t) = A_{0m}\left(1 - 2\alpha_{0m}^2 t^2 + \frac{2\alpha_{0m}^4 t^4}{3} - \frac{4\alpha_{0m}^6 t^6}{45} + \frac{2\alpha_{0m}^8 t^8}{315} - \frac{4\alpha_{0m}^{10} t^{10}}{14175} + \cdots\right),$$

$$w_{0m}(r) = 1 - \frac{\alpha_{0m}^2 r^2}{4} + \frac{\alpha_{0m}^4 r^4}{64} - \frac{\alpha_{0m}^6 r^6}{2304} + \frac{\alpha_{0m}^8 r^8}{147456} - \frac{\alpha_{0m}^{10} r^{10}}{147456000} + \cdots,$$

where A_{0m} *are calculated by Eq.*(7) *and* α_{0m} *is the mth positive zero of* J_0 *as shown in Table 4. Therefore, the series solution of vibrations of a circular membrane is*

$$\begin{aligned}
u(r,t) &= \sum_{m=1}^{\infty} w_{0m}(r) g_{0m}(t) \\
&= \sum_{m=1}^{\infty} A_{0m}\left(1 - \frac{\alpha_{0m}^2 r^2}{4} + \frac{\alpha_{0m}^4 r^4}{64} - \cdots\right)\left(1 - 2\alpha_{0m}^2 t^2 + \frac{2\alpha_{0m}^4 t^4}{3} - \cdots\right) \\
&= (1 - 1.44558r^2 + 0.552586r^4 - \cdots)(1.10802 - 12.8158t^2 + 24.7055t^4 - \cdots) - \\
&\quad (1 - 7.61782)r^2 + 14.5078r^4 - \cdots)(0.139777 - 8.51839t^2 + 86.5221t^4 - \cdots) + \\
&\quad (1 - 18.722r^2 + 87.6281r^4 - \cdots)(0.045476 - 6.81119t^2 + 170.025t^4 - \cdots) - \cdots .
\end{aligned}$$

The analytical solution of a circular membrane in this example is

$$\begin{aligned}
u(r,t) &= \sum_{m=1}^{\infty} A_m J_0(\alpha_m r)\cos(2\alpha_m t) = 1.10801 J_0(2.40483r)\cos(4.80966t) - \\
&\quad 0.13978 J_0(5.52008r)\cos(11.04016t) + 0.04548 J_0(8.65373r)\cos(17.30746t) - \cdots .
\end{aligned}$$

Figure 1 shows the motion of the series solution for the first term ($m = 1, \alpha_{01} = 2.40483$)*, the second term* ($m = 2, \alpha_{02} = 5.52008$) *and the third term* ($m = 3, \alpha_{03} = 8.65373$) *at the initial time.*

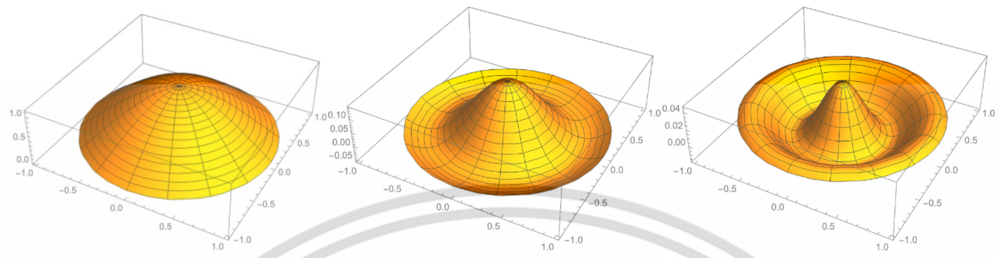


Figure 1: Normal modes of the vibrations of a circular membrane independent of the angle for Example 1.

Example 2. *Consider the problem of vibrations of a circular membrane depending on both r and θ in Eqs.(24) and (25) with radius 1, $c = 1$, the initial shape $f(r,\theta) = 1 - r^4$ and initial velocity equal to zero. From the initial shape, the nonnegative integer n can only be zero (see Table 2) because $A_{nm} = 0$ when $n \geq 1$ (i.e., $A_{0m} \neq 0$) and B_{nm} in Eq.(9) is always zero. The initial values are $G(0) = A_{0m}, G(1) = 0, Q(0) = 1, Q(1) = 0, W(0) = 1$ and $W(1) = 0$.*
Hence, we obtain the series solutions corresponding to Eq.(41), (42) and (43), as follows:

$$g_{0m} = A_{0m}\left(1 - \frac{\alpha_{0m}^2 t^2}{2} + \frac{\alpha_{0m}^4 t^4}{24} - \frac{\alpha_{0m}^6 t^6}{720} + \frac{\alpha_{0m}^8 t^8}{40320} - \frac{\alpha_{0m}^{10} t^{10}}{3628800} + \cdots\right)$$

$$q_0(\theta) = 1$$

$$w_{0m}(r) = 1 - \frac{\alpha_{0m}^2 r^2}{4} + \frac{\alpha_{0m}^4 r^4}{64} - \frac{\alpha_{0m}^6 r^6}{2304} + \frac{\alpha_{0m}^8 r^8}{147456} - \frac{\alpha_{0m}^{10} r^{10}}{147456000} + \cdots,$$

where A_{0m} are calculated by Eq.(8) and α_{0m} is the mth positive zero of J_0 as shown in Table 4. Therefore, the series solution of vibrations of a circular membrane is

$$u(r,\theta,t) = \sum_{n=0}^{\infty} q_n(\theta) \sum_{m=1}^{\infty} w_{nm}(r) g_{nm}(t) = q_0(\theta) \sum_{m=1}^{\infty} w_{0m}(r) g_{0m}(t)$$

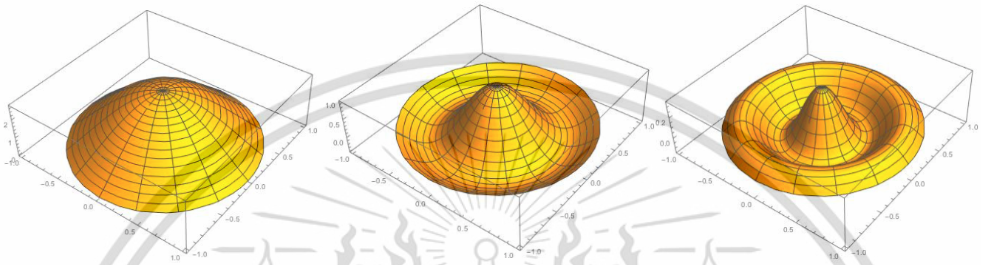
$$= (1 - 1.4458r^2 + 0.522586r^4 - \cdots)(2.73318 - 7.90328t^2 + 3.80886t^4 - \cdots) -$$
$$(1 - 7.61782r^2 + 14.5078r^4 - \cdots)(0.971432 - 14.8004t^2 + 37.5822t^4 - \cdots) +$$
$$(1 - 18.7218r^2 + 87.6261r^4 - \cdots)(0.344381 - 12.8948t^2 + 80.4712t^4 - \cdots) - \cdots.$$

The analytical solution of a circular membrane in this example is

$$u(r,\theta,t) = \sum_{m=1}^{\infty} \frac{16(\alpha_{0m} J_2(\alpha_{0m}) - 2J_3(\alpha_{0m}))}{\alpha_{0m}^3 J_1^2(\alpha_{0m})} J_0(\alpha_{0m} r)\cos(\alpha_{0m} t)$$

$$= 2.73318 J_0(2.40483r)\cos(2.40483t) - 0.971432 J_0(5.52008r)\cos(5.52008t) +$$
$$0.344381 J_0(8.65373r)\cos(8.65373t) - \cdots.$$

Figure 2 shows the motion of the series solution for the first term ($m = 1, \alpha_{01} = 2.40483$)*, the second term* ($m = 2, \alpha_{02} = 5.52008$) *and the third term* ($m = 3, \alpha_{03} = 8.65373$) *at the initial time.*



(a) $\alpha_{01} = 2.40483$ (b) $\alpha_{02} = 5.52008$ (c) $\alpha_{03} = 8.65373$

Figure 2: Normal modes of the vibrations of a circular membrane depending on both r and θ for Example 2, when $n = 0$.

Example 3. *Let us consider the Example 2 with the initial shape defined by* $f(r,\theta) = r(1-r^4)\cos(\theta)$. *From the initial shape, we obtain only* $n = 1$ *(see Table 2) because* $A_{nm} = 0$, *when* $n = 0$ *and* $n > 1$, *and* B_{nm} *in Eq.*(9) *is always zero. The initial values* $G(0) = A_{1m}, G(1) = 0, Q(0) = 1, Q(1) = 0, W(0) = 0$ *and* $W(1) = \dfrac{\alpha_{1m}}{2}$. *Then, we obtain the series solutions in Eqs.*(41), (42) *and* (43) *as follows:*

$$g_{1m} = A_{1m}\left(1 - \frac{\alpha_{1m}^2 t^2}{2} + \frac{\alpha_{1m}^4 t^4}{24} - \frac{\alpha_{1m}^6 t^6}{720} + \frac{\alpha_{1m}^8 t^8}{40320} - \frac{\alpha_{1m}^{10} t^{10}}{3628800} + \cdots\right)$$

$$q_1(\theta) = 1 - \frac{\theta^2}{2} - \frac{\theta^4}{24} - \frac{\theta^6}{720} + \frac{\theta^8}{40320} - \cdots$$

$$w_{1m}(r) = \frac{\alpha_{1m} r}{2} - \frac{\alpha_{1m}^3 r^3}{16} + \frac{\alpha_{1m}^5 r^5}{384} - \frac{\alpha_{1m}^7 r^7}{18432} + \frac{\alpha_{1m}^9 r^9}{1474560} - \frac{\alpha_{1m}^{11} r^{11}}{176947200} + \cdots,$$

where A_{1m} *are calculated by Eq.*(8) *and* α_{1m} *is the* m*th positive zero of* J_1 *as shown in Table 4. Therefore, the series solution of vibrations of a circular membrane is*

$$u(r,\theta,t) = \sum_{n=0}^{\infty} q_n(\theta) \sum_{m=1}^{\infty} w_{nm}(r) g_{nm}(t) = q_1(\theta) \sum_{m=1}^{\infty} w_{1m}(r) g_{1m}(t)$$

$$= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots\right)\Big[(1.91586r - 3.51607r^3 + \cdot)(0.964141 - 7.07776t^2 + \cdots) - (3.5078r - 21.5811r^3 + \cdots)(0.387905 - 9.54606t^2 + \cdots) + (5.8099r - 98.0564r^3 + \cdots)(1.3701 \times 10^8 - 9.24954 \times 10^9 t^2 + \cdots) - \cdots\Big].$$

The analytical solution of a circular membrane in this example is

$$u(r,\theta,t) = \cos\theta \sum_{m=1}^{\infty} \frac{8(\alpha_{1m}J_3(\alpha_{1m}) - 2J_4(\alpha_{1m}))}{\alpha_{1m}^3 J_2^2(\alpha_{1m})} J_1(\alpha_{1m}r)\cos(\alpha_{1m}t)$$

$$= \cos\theta\Big[0.96414J_1(3.83171r)\cos(3.83171t) - 0.387905J_1(7.01559r)\cos(7.01559t) +$$

$$1.3701\times 10^8 J_1(11.6198r)\cos(11.6198t) - \cdots\Big].$$

Figure 3 shows the motion of the series solution for the first term ($m = 1, \alpha_{11} = 3.83171$)*, the second term* ($m = 2, \alpha_{12} = 7.01559$) *and the third term* ($m = 3, \alpha_{13} = 10.1735$) *at the initial time.*

As show in Table 5, the approximate series solutions of the vibrations of a circular membrane obtained by using the DTM are compared with the analytical solution. The results given in Table 5 showed that, the series solutions obtained by the DTM are equal to the analytical solutions.

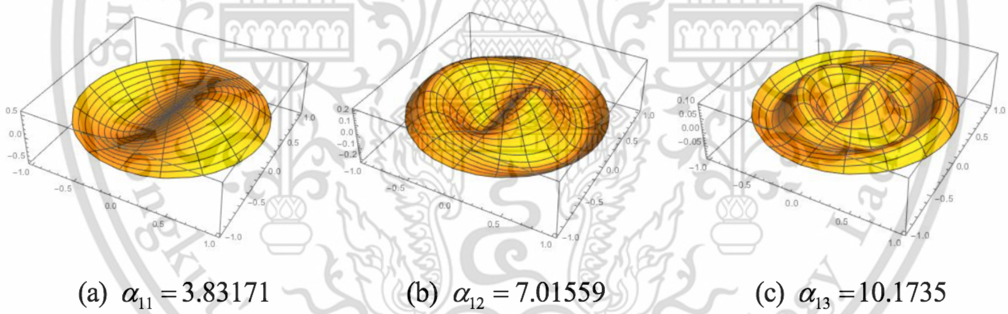


Figure 3: Normal modes of the vibrations of a circular membrane depending on both r and θ for Example 3, when $n = 1$.

Table 5: The comparison results for Example 3 between DTM solutions and analytical solutions at the initial time.

α_{nm}	r \ θ	0		π		2π	
		DTM	Analytical	DTM	Analytical	DTM	Analytical
$\alpha_{11} = 3.83171$	0	0	0	0	0	0	0
	0.01	0.01847	0.01847	-0.01847	-0.01847	0.01847	0.01847
	0.02	0.03692	0.03692	-0.03692	-0.03692	0.03692	0.03692
$\alpha_{12} = 7.01559$	0	0	0	0	0	0	0
	0.01	0.01036	0.01036	-0.01036	-0.01036	0.01036	0.01036
	0.02	0.02715	0.02715	-0.02715	-0.02715	0.02715	0.02715
$\alpha_{13} = 10.1735$	0	0	0	0	0	0	0
	0.01	0.00856	0.00856	-0.00856	-0.00856	0.00856	0.00856
	0.02	0.00170	0.00170	-0.00170	-0.00170	0.00170	0.00170

Example 4. *Let us consider the Example 2 with the initial shape defined by* $f(r,\theta) = r(1-r^4)\cos(2\theta)$. *From the initial shape, we obtain only* $n = 2$ *(see Table 2) because* $A_{nm} = 0$, *when* $n = 0, 1$ *and* $n > 2$, *and* B_{nm} *in Eq.*(9) *is always zero. The initial values are* $G(0) = A_{2m}, G(1) = 0, Q(1) = 0, Q(1) = 0, W(1) = 0$ *and* $W(2) = \dfrac{\alpha_{2m}}{8}$. *Therefore, the series solution of vibrations of a circular membrane is*

$$u(r,\theta,t) = \sum_{n=0}^{\infty} q_n(\theta) \sum_{m=1}^{\infty} w_{nm}(r) g_{nm}(t) = q_2(\theta) \sum_{m=1}^{\infty} w_{2m}(r) g_{2m}(t)$$

$$= \left(1 - 2\theta^2 + \frac{2\theta^4}{3} - \cdots\right)\Big[(0.641953r^2 - 1.41094r^4 + \cdot)(1.1876 - 15.6612 + \cdots) -$$
$$(1.05216r^2 - 6.21209r^4 + \cdots)(0.145193 - 5.14345t^2 + \cdots) +$$
$$(1.45248r^2 - 16.3427r^4 + \cdots)(0.197975 - 13.3653t^2 + \cdots) - \cdots\Big].$$

where A_{2m} *are calculated by Eq.*(8) *and* α_{2m} *is the* m*th positive zero of* J_2 *as shown in Table 4.*

5 Conclusions

The importance of the differential transformation method (DTM) lies in the initial values of the recursive formulas derived from the conversion of the ODEs problem, and the recursive formulas used to find the coefficients of the series solution of the problem. In this work, solving of the vibration problem of a circular membrane, the initial values of the recursive formulas can be calculated from the coefficients of the Fourier-Bessel series which is in the integral form of the initial shape of the membrane. The comparison of the DTM series solutions and the analytical solutions show that these are the same, hence it follows that the DTM method is suitable for this type of problems. Moreover, we consider that DTM is easier to use from the programming point of view. Since in the case of the nonlinear vibrating circular membrane problems are difficult to be unsolvable analytically, we intend to modify and apply the steps of the DTM method presented in this work to these problems, in a future research.

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Appendix B

The second research paper

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Original Article

New transform formulae for differential transformation method with applications to the nonlinear plane autonomous systems

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Abstract

This work presents a new derivation technique for new differential transform formulae of a product of composite functions. The new formulae are applied to nonlinear plane autonomous systems to demonstrate their efficiency and reliability. The approximate series solutions estimated by the differential transform method (DTM) and the multistep differential transform method (MsDTM) are then compared with the flow direction of the vector fields defined by the original system and an analytical solution calculated by the phase-plane method. We found that the MsDTM results are in better agreement with the analytical solution than the DTM ones. Moreover, the MsDTM can be applied to systems whose analytical solutions are unobtainable. The approximate solutions by the MsDTM have the same direction to the flow of the vector field of the system. It follows that the proposed new formulae are reliable and efficient.

Keywords: differential transform method, nonlinear plane autonomous systems, multistep differential transform method, phase-plane method

1. Introduction

Autonomous systems are systems of first-order differential equations of the form

$$\frac{dx_1}{dt} = g_1(x_1,\ldots,x_n)$$
$$\frac{dx_2}{dt} = g_2(x_1,\ldots,x_n)$$
$$\vdots$$
$$\frac{dx_n}{dt} = g_n(x_1,\ldots,x_n)$$

such that the independent variable does not explicitly appear on the right hand side of each differential equation. In the case of $n=2$, the system is called a plane autonomous system and $V(x_1,x_2)=\left(g_1(x_1,x_2),g_2(x_1,x_2)\right)$ is a vector field in the plane that indicates the movement direction. If the parameter t is interpreted as time, then $X(t)=\left(x(t),y(t)\right)$ indicates the position of the particle in the plane at time t and a solution of the system is interpreted as a path of this particle starting from $X(0,0)=\left(x(0),y(0)\right)$ (Zill & Wright, 2014).

The differential transformation method (DTM) is an alternative procedure for obtaining an approximate Taylor series solution of differential equations. The main advantage of this method is that it can be applied directly to nonlinear differential equations without requiring linearization and discretization. The concept of the differential transform method was introduced by Zhou (Zhou, 1986), who solved linear and nonlinear problems in electrical circuits and many

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other problems related to differential equations (Damirchi & Shamami, 2016; Mahgoub & Alshikh, 2017; Methi, 2016; Mirzaee, 2011; Moon, Bhosale, Gajbhiye, & Lonare, 2014; Patil & Khambayat, 2014).

Although the DTM series solution gives a good approximation for some problems, in some cases, the series solution diverges in a wider domain. Due to this reason the multistep differential transform method (MsDTM) is used. The MsDTM is based on the DTM, but compared with other methods it does not need small parameters, auxiliary functions and parameters, or discretization. In this technique, the solution domain is divided in subdomains (Ebenezer, Freihet, Khan & Khan, 2016; Ertürk, Odibat & Momami, 2012; Odibat, Bertelle, Aziz-Alaoui & Duchamp, 2010; Rashidi, Chamkha, & Keimanesh, 2011; Zurigat & Ababneh, 2015). In particular, we are interested in the technique introduced by Chang (Change & Chang, 2008), to calculate the DTM of nonlinear functions.

In this paper, a derivation technique of new differential transform formulae for the product of composite functions is presented. The computation consists of three steps. The first step is finding the differential transformation of the product of two composite functions in the general form as shown in Equation 3.1. The next step is finding the differential transformation for the higher order derivative of a power function as shown in Lemma 3.1. The last step is the derivation of the new differential transformation shown in Formulae 1–8, calculated by using the general formulae of higher order derivatives of composite functions studied in (Weisstein, n.d.) combined with Lemma 3.1. Then, the new differential transform formulae obtained are used to transform the nonlinear plane autonomous systems to find the DTM and MsDTM approximate solutions of the problem. By comparing graphically the results, we obtain the approximate series solutions calculated by the DTM and the MsDTM that have the same direction with the vector fields flow and they are also similar to the analytical solution obtained by the phase-plane method.

Here is the structure of the paper. In section 2, the one-dimensional differential transformation method is described. In section 3, the analysis of the method and new formulae calculation are proposed. In section 4, the new differential transform formulae proposed are applied to three examples of nonlinear plane autonomous systems to show the reliability and efficiency of the method. The conclusion is given at the end of the paper in section 5.

2. Basic Definitions and Fundamental Operations of the One-Dimensional Differential Transform Method

2.1 Definition

The one-dimensional differential transform of the function $x(t)$ is defined as

$$X(\mathrm{k})=\frac{1}{k!}\left[\frac{d^k x(\mathrm{t})}{dt^k}\right]_{t=t_0},\ k\geq 0. \tag{2.1}$$

In Equation 2.1, $x(\mathrm{t})$ is called the original function and $X(\mathrm{k})$ is called the transformed function.

2.2 Definition

The inverse one-dimensional differential transform of $X(\mathrm{k})$ is defined as

$$x(\mathrm{t})=\sum_{k=0}^{\infty}X(\mathrm{k})\left(\mathrm{t}-\mathrm{t}_0\right)^k, \tag{2.2}$$

that is,

$$x(\mathrm{t})=\sum_{k=0}^{\infty}\frac{\left(t-t_0\right)^k}{k!}\left[\frac{d^k x(\mathrm{t})}{dt^k}\right]_{t=t_0}. \tag{2.3}$$

Equation 2.3 implies that the concept of differential transformation method is derived from Taylor series expansion. Actually, in concrete applications, the function $x(\mathrm{t})$ is expressed by a truncated series and Equation 2.2 becomes

$$\mathrm{x}(\mathrm{t})=\sum_{k=0}^{N}X(\mathrm{k})\left(\mathrm{t}-\mathrm{t}_0\right)^k. \tag{2.4}$$

2.3 Fundamental operations

The fundamental operations of the one-dimensional DTM are shown in Table 1. The multistep differential transformation method (MsDTM) is advantageous for applications in physics. For instance, due to small time steps the MsDTM has a powerful accuracy especially for an initial value problem (IVP).

Let $[0,\mathrm{T}]$ be the interval over which we want to find the solution of the IVP. In actual applications of the DTM, the approximate solution of the IVP can be expressed by the finite series

$$\mathrm{x}(\mathrm{t})=\sum_{k=0}^{N}X(\mathrm{k})\left(\mathrm{t}\right)^k,\ \mathrm{t}\in[0,\mathrm{T}]. \tag{2.5}$$

Let us assume that the interval $[0,\mathrm{T}]$ in divided into n subintervals $[\mathrm{t}_{i-1},\mathrm{t}_i],\ \mathrm{i}=1,\ldots,\mathrm{m}$ of equal step size $h=T/m$ by using the nodes $t_i=ih.$ The main ideas of the MsDTM can be found in (Odibat, Bertelle, Aziz-Alaoui, & Duchamp, 2010). In fact, the MsDTM gives the solution in the form,

$$x(\mathrm{t})=\begin{cases}\mathrm{x}_0(\mathrm{t}), & t\in[0,\mathrm{t}_1]\\ \mathrm{x}_1(\mathrm{t}), & t\in[\mathrm{t}_1,\mathrm{t}_2]\\ \vdots & \\ \mathrm{x}_m(\mathrm{t}), & t\in[t_m,\mathrm{t}_{m-1}],\end{cases} \tag{2.6}$$

Table 1. Fundamental operations of one-dimensional DTM.

Original function $x(\mathrm{t})$	Transformed functions $X(\mathrm{k})$
$x(\mathrm{t}) \pm \mathrm{y(t)}$	$X(\mathrm{k}) \pm \mathrm{Y(k)}$
$\lambda x(\mathrm{t})$	$\lambda X(\mathrm{k})$
$x(\mathrm{t})\,\mathrm{y(t)}$	$\sum_{r=0}^{k} X(\mathrm{r})\,\mathrm{Y(k-r)}$
$x(\mathrm{t})\,\mathrm{y(t)}\,\mathrm{z(t)}$	$\sum_{r=0}^{k}\sum_{l=0}^{r} X(\mathrm{l})\,\mathrm{Y(r-l)}\,\mathrm{Z(k-r)}$
$\frac{d^r}{dt^r} x(\mathrm{t})$	$\frac{(\mathrm{k+r})!}{k!} X(\mathrm{k+r})$

where $x_i(\mathrm{t}) = \sum_{k=0}^{N} X_i(\mathrm{k})(\mathrm{t}-\mathrm{t}_i)^k$ and the initial condition $x_i^{(\mathrm{k})}(\mathrm{t}_{i-1}) = \mathrm{x}_{i-1}^{(\mathrm{k})}(\mathrm{t}_{i-1})$.

3. Analysis of Method

This section introduces our derivation technique of the new differential transform formulae for the product of composite functions derived in Formulae 1–8. To obtain these new formulae, the derivation is shown in the following steps.

Step 1. The differential transformation for the product of two composite functions is represented by $f\left(y(\mathrm{t})\right) g\left(y(\mathrm{t})\right)$, which are the original functions. By the definition given in Section 2.1 of the DTM combined with Leibniz formula, we obtain

$$\frac{1}{k!}\left[\frac{d^k}{dt^k} f\left(y(\mathrm{t})\right) g\left(y(\mathrm{t})\right)\right] = \frac{1}{k!}\left[\sum_{r=0}^{k} \frac{k!}{(\mathrm{k-r})!\mathrm{r}!} \frac{d^r}{dt^r} f\left(y(\mathrm{t})\right) \frac{d^{k-r}}{dt^{k-r}} g\left(y(\mathrm{t})\right)\right]_{t=t_0}$$
$$= \sum_{r=0}^{k} F(\mathrm{r})\,\mathrm{G(k-r)}, \tag{3.1}$$

where $F(\mathrm{r}) = \frac{1}{r!}\left[\frac{d^r}{dt^r} f\left(y(\mathrm{t})\right)\right]_{t=t_0}$, $\mathrm{G(k-r)} = \frac{1}{(k-r)!}\left[\frac{d^{k-r}}{dt^{k-r}} g\left(y(\mathrm{t})\right)\right]_{t=t_0}$.

Step 2. This step finds the differential transformation for the higher order derivative of the power function that is used in Step 3.

Lemma 3.1. If $k,r,m \in \mathrm{I}^+ \cup \{0\}$ and let $w = r - m = 0,...,r$ where $r = 0,...,k,\ m = 0,...,r$, then

$$\left[\frac{1}{k!}\frac{d^k y(\mathrm{t})^w}{dt^k}\right]_{t=t_0} = 1, \quad k = 0,$$

$$\left[\frac{1}{k!}\frac{d^k y(\mathrm{t})^w}{dt^k}\right]_{t=t_0} = \sum_{k_{w-1}=0}^{k}\sum_{k_{w-2}=0}^{k_{w-1}} \cdots \sum_{k_1=0}^{k_2} Y(\mathrm{k}_1)Y(\mathrm{k}_2-\mathrm{k}_1)\cdots \mathrm{Y}(\mathrm{k}-\mathrm{k}_{w-1}), \quad \mathrm{k} > 0. \tag{3.2}$$

Proof. Assume that $k,r,m \in \mathrm{I}^+ \cup \{0\}$ and let $w = r - m = 0,...,r$ where $r = 0,...,k,\ m = 0,...,r$

Case $k = 0$; we have $r = 0$ and $m = 0$, then $\left[\frac{1}{0!}\frac{d^0 y(\mathrm{t})^0}{dt^0}\right]_{t=t_0} = 1.$

Case $k > 0$; we will prove by mathematical induction. Let $P(\mathrm{w})$ be Equation 3.2.
First, we will show that the statement holds for $w = 0$, that is

$$P(0) = \left[\frac{1}{k!}\frac{d^k y(\mathrm{t})^0}{dt^k}\right]_{t=t_0} = 0.$$

Next, we assume that the statement is true for $w = r-1$, that is

$$P(\mathrm{r}-1) = \left[\frac{1}{k!}\frac{d^k y(\mathrm{t})^{r-1}}{dt^k}\right]_{t=t_0} = \sum_{k_{r-2}=0}^{k}\sum_{k_{r-3}=0}^{k_{r-2}}\cdots\sum_{k_1=0}^{k_2} Y(\mathrm{k}_1)Y(\mathrm{k}_2-\mathrm{k}_1)\cdots \mathrm{Y}(\mathrm{k}-\mathrm{k}_{r-2}).$$

We will show that the statement is also true for $w = r$. This can be seen as follows

$$\begin{aligned}P(\mathrm{r}) &= \left[\frac{1}{k!}\frac{d^k y(\mathrm{t})^r}{dt^k}\right]_{t=t_0} = \left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})^{r-1}y(\mathrm{t})\right)\right] \\
&= \left[\frac{1}{k!}\sum_{k_{r-1}=0}^{k}\frac{k!}{(\mathrm{k}-\mathrm{k}_{r-1})!\mathrm{k}_{r-1}!}\frac{d^{k_{r-1}}}{dt^{k_{r-1}}}y(\mathrm{t})^{r-1}\frac{d^{k-k_{r-1}}}{dt^{k-k_{r-1}}}y(\mathrm{t})\right]_{t=t_0} \\
&= \sum_{k_{r-1}=0}^{k}\left[\frac{1}{\mathrm{k}_{r-1}!}\frac{d^{k_{r-1}}}{dt^{k_{r-1}}}y(\mathrm{t})^{r-1}\frac{1}{(\mathrm{k}-\mathrm{k}_{r-1})!}\frac{d^{k-k_{r-1}}}{dt^{k-k_{r-1}}}y(\mathrm{t})\right]_{t=t_0} \\
&= \sum_{k_{r-1}=0}^{k}\sum_{k_{r-2}=0}^{k_{r-1}}\cdots\sum_{k_1=0}^{k_2} Y(\mathrm{k}_1)Y(\mathrm{k}_2-\mathrm{k}_1)\cdots \mathrm{Y}(\mathrm{k}-\mathrm{k}_{r-1}).\end{aligned}$$

Therefore, the statement holds for $w = r$, and the proof is completed.

Step 3. The functions $f\left(y(\mathrm{t})\right)$ and $g\left(y(\mathrm{t})\right)$ in Step 1 are considered as the original functions in the Formulae 1–8. To obtain these new differential transform formulae, the general formulae of higher order derivatives of some composite functions are used together with Lemma 3.1 in the following calculations.

Formula 1. If $f\left(y(\mathrm{t})\right) = e^{y(\mathrm{t})}$ is the original function, then

$$\begin{aligned}F(\mathrm{k}) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}e^{y(\mathrm{t})}\right]_{t=t_0} = \frac{1}{k!}\left[e^{y(\mathrm{t})}\sum_{r=0}^{k}\frac{1}{r!}\sum_{m=0}^{r}\frac{(-1)^m r!}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}y^m(\mathrm{t})\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0} \\
&= \mathrm{e}^{y(\mathrm{t}_0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0} \\
&= \mathrm{e}^{Y(0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0},\end{aligned}$$

where $Y(0) = y(\mathrm{t}_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$.

Formula 2. If $f\left(y(\mathrm{t})\right) = \ln\left(y(\mathrm{t})\right)$, $\mathrm{y(t)} > 0$ is the original function, then

$$\begin{aligned}F(\mathrm{k}) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\ln\left(y(\mathrm{t})\right)\right]_{t=t_0} = \frac{1}{k!}\left[\delta_k \ln\left(y(\mathrm{t})\right) + \sum_{r=1}^{k}\frac{(-1)^{r-1}}{ry^r(\mathrm{t})}\binom{k}{r}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0} \\
&= \frac{1}{k!}\delta_k \ln\left(Y(0)\right) + \sum_{r=1}^{k}\frac{(-1)^{r-1}}{rY^r(\mathrm{t}_0)}\binom{k}{r}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0},\end{aligned}$$

where $\binom{k}{r} = \frac{\mathrm{k!}}{(k-r)!\mathrm{r!}}$ are the binomial coefficients, $\delta_k = \begin{cases} 1, & k=0 \\ 0, & k=1,2,3,\dots \end{cases}$ and

$$\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0} = \sum_{k_{r-1}=0}^{k}\sum_{k_{r-2}=0}^{k_{r-1}}\dots\sum_{k_1=0}^{k_2} Y(\mathrm{k}_1)Y(\mathrm{k}_2-\mathrm{k}_1)\dots \mathrm{Y}(\mathrm{k}-\mathrm{k}_{r-1}).$$

Formula 3. If $f\left(y(\mathrm{t})\right) = \sin\left(y(\mathrm{t})\right)$ is the original function, then

$$\begin{aligned} F(\mathrm{k}) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\sin\left(y(\mathrm{t})\right)\right]_{t=t_0} = \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\sin(\mathrm{t})\Big|_{t=y(\mathrm{t})}\left(\sum_{m=0}^{r}\frac{(-1)^m r!}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}y^m(\mathrm{t})\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right)\right]_{t=t_0} \\ &= \sum_{r=0}^{k}\frac{d^r}{dt^r}\sin(\mathrm{t})\Big|_{t=y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0} \\ &= \sum_{r=0}^{k}\frac{d^r}{dt^r}\sin(\mathrm{t})\Big|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0} \end{aligned}$$

where $Y(0) = y(\mathrm{t}_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$.

Formula 4. If $f\left(y(\mathrm{t})\right) = \cos\left(y(\mathrm{t})\right)$ is the original function, then

$$\begin{aligned} F(\mathrm{k}) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\cos\left(y(\mathrm{t})\right)\right]_{t=t_0} = \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\cos(\mathrm{t})\Big|_{t=y(\mathrm{t})}\left(\sum_{m=0}^{r}\frac{(-1)^m r!}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}y^m(\mathrm{t})\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right)\right]_{t=t_0} \\ &= \sum_{r=0}^{k}\frac{d^r}{dt^r}\cos(\mathrm{t})\Big|_{t=y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0} \\ &= \sum_{r=0}^{k}\frac{d^r}{dt^r}\cos(\mathrm{t})\Big|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0} \end{aligned}$$

where $Y(0) = y(\mathrm{t}_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$.

Formula 5. If $f\left(y(\mathrm{t})\right) = \sinh\left(y(\mathrm{t})\right)$ is the original function, then

$$\begin{aligned} F(\mathrm{k}) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\sinh\left(y(\mathrm{t})\right)\right]_{t=t_0} = \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\sinh(\mathrm{t})\Big|_{t=y(\mathrm{t})}\left(\sum_{m=0}^{r}\frac{(-1)^m r!}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}y^m(\mathrm{t})\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right)\right]_{t=t_0} \\ &= \sum_{r=0}^{k}\frac{d^r}{dt^r}\sinh(\mathrm{t})\Big|_{t=y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0} \\ &= \sum_{r=0}^{k}\frac{d^r}{dt^r}\sinh(\mathrm{t})\Big|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0} \end{aligned}$$

where $Y(0) = y(\mathrm{t}_0)$, and we have used 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$.

Formula 6. If $f\left(y(\mathrm{t})\right)=\cosh\left(y(\mathrm{t})\right)$ is the original function, then

$$F(\mathrm{k})=\frac{1}{k!}\left[\frac{d^k}{dt^k}\cosh\left(y(\mathrm{t})\right)\right]_{t=t_0}=\frac{1}{k!}\left[\sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dt^r}\cosh(\mathrm{t})\bigg|_{t=y(\mathrm{t})}\left(\sum_{m=0}^{r}\frac{(-1)^m r!}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}y^m(\mathrm{t})\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right)\right]_{t=t_0}$$

$$=\sum_{r=0}^{k}\frac{d^r}{dt^r}\cosh(\mathrm{t})\bigg|_{t=y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$$

$$=\sum_{r=0}^{k}\frac{d^r}{dt^r}\cosh(\mathrm{t})\bigg|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$$

where $Y(0)=y(\mathrm{t}_0)$, and we have used Lemma 3.1 to transform $\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$.

Formula 7. If $f\left(y(\mathrm{t})\right)=\sqrt{y(\mathrm{t})}$ is the original function, then

$$F(\mathrm{k})=\frac{1}{k!}\left[\frac{d^k}{dt^k}\sqrt{y(\mathrm{t})}\right]_{t=t_0}=\frac{1}{k!}\left[\frac{\Gamma(\mathrm{k}+\frac{1}{2})}{2\Gamma(\mathrm{k}+1)\Gamma(\frac{1}{2})}\sum_{r=0}^{k}\frac{(-1)^r}{(\frac{1}{2}-\mathrm{r})}\binom{k}{r}\left(y(\mathrm{t})\right)^{\frac{1}{2}-r}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}$$

$$=\frac{\Gamma(\mathrm{k}+\frac{1}{2})}{2\Gamma(\mathrm{k}+1)\Gamma(\frac{1}{2})}\sum_{r=0}^{k}\frac{(-1)^r}{(\frac{1}{2}-\mathrm{r})}\binom{k}{r}\left(\mathrm{Y}(0)\right)^{\frac{1}{2}-r}\left[\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}$$

where $\binom{k}{r}=\frac{\mathrm{k}!}{(k-r)!\mathrm{r}!}$ are the binomial coefficients,

$\Gamma(1+\mathrm{z})=\mathrm{z}\,\Gamma(\mathrm{z}),\ \mathrm{z}\in\mathbb{C},\ \Gamma(\frac{1}{2})=\sqrt{\pi},\ \Gamma(\mathrm{n})=(\mathrm{n}-1)!,\ \mathrm{n}\in\mathbf{Z}^+$, and

$$\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}=\sum_{k_{r-1}=0}^{k}\sum_{k_{r-2}=0}^{k_{r-1}}\cdots\sum_{k_1=0}^{k_2}Y(\mathrm{k}_1)Y(\mathrm{k}_2-\mathrm{k}_1)\ldots\mathrm{Y}(\mathrm{k}-\mathrm{k}_{r-1}).$$

Formula 8. If $f\left(y(\mathrm{t})\right)=\frac{1}{y(\mathrm{t})}$ is the original function, then

$$F(\mathrm{k})=\frac{1}{k!}\left[\frac{d^k}{dt^k}\frac{1}{y(\mathrm{t})}\right]_{t=t_0}=\frac{1}{k!}\left[(\mathrm{k}+1)\sum_{r=0}^{k}\frac{(-1)^r}{(\mathrm{r}+1)}\binom{k}{r}\left(y(\mathrm{t})\right)^{-r-1}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}$$

$$=(\mathrm{k}+1)\sum_{r=0}^{k}\frac{(-1)^r}{(\mathrm{r}+1)}\binom{k}{r}\left(\mathrm{Y}(\mathrm{t}_0)\right)^{-r-1}\left[\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}$$

where $\binom{k}{r}=\frac{\mathrm{k}!}{(k-r)!\mathrm{r}!}$ are the binomial coefficients, and

$$\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^r\right]_{t=t_0}=\sum_{k_{r-1}=0}^{k}\sum_{k_{r-2}=0}^{k_{r-1}}\cdots\sum_{k_1=0}^{k_2}Y(\mathrm{k}_1)Y(\mathrm{k}_2-\mathrm{k}_1)\ldots\mathrm{Y}(\mathrm{k}-\mathrm{k}_{r-1}).$$

The transformed functions are shown in Table 2.

Table 2. Transformed functions of some nonlinear functions.

Original function $f\left(y(\mathrm{t})\right)$	Transformed functions $F(\mathrm{k})$
$e^{y(\mathrm{t})}$	$\mathrm{e}^{Y(\mathrm{t}_0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m Y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$
$\ln\left(y(\mathrm{t})\right)$	$\frac{1}{k!}\delta_k \ln\left(Y(\mathrm{t}_0)\right)+\sum_{r=1}^{k}\frac{(-1)^{r-1}}{rY^r(\mathrm{t}_0)}\binom{k}{r}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r}\right]_{t=t_0}$
$\sin\left(\mathrm{y}(\mathrm{t})\right)$	$\sum_{r=0}^{k}\frac{d^r}{dt^r}\sin(\mathrm{t})\Big\rvert_{t=Y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$
$\cos\left(y(\mathrm{t})\right)$	$\sum_{r=0}^{k}\frac{d^r}{dt^r}\cos(\mathrm{t})\Big\rvert_{t=Y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$
$\sinh\left(y(\mathrm{t})\right)$	$\sum_{r=0}^{k}\frac{d^r}{dt^r}\sinh(\mathrm{t})\Big\rvert_{t=Y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$
$\cosh\left(y(\mathrm{t})\right)$	$\sum_{r=0}^{k}\frac{d^r}{dt^r}\cosh(\mathrm{t})\Big\rvert_{t=Y(\mathrm{t}_0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(\mathrm{t}_0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_0}$
$\sqrt{y(\mathrm{t})}$	$\frac{\Gamma(\mathrm{k}+\frac{1}{2})}{2\Gamma(\mathrm{k}+1)\Gamma(\frac{1}{2})}\sum_{r=0}^{k}\frac{(-1)^r}{(\frac{1}{2}-\mathrm{r})}\binom{k}{r}\left(\mathrm{Y}(\mathrm{t}_0)\right)^{\frac{1}{2}-r}\left[\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r}\right]_{t=t_0}$
$\frac{1}{y(\mathrm{t})}$	$(\mathrm{k}+1)\sum_{r=0}^{k}\frac{(-1)^r}{(\mathrm{r}+1)}\binom{k}{r}\left(\mathrm{Y}(\mathrm{t}_0)\right)^{-r-1}\left[\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r}\right]_{t=t_0}$

4. Applications

In this section, we extended the application of the DTM to nonlinear plane autonomous systems. To demonstrate the formulae introduced in the previous section, three examples are studied here. The accuracy of the method is assessed by graphical and data value comparisons.

Example 4.1. Consider the following system of nonlinear plane autonomous

$$x' = e^{y} \tag{4.1}$$

$$y' = e^{x},\ \text{for } t\in[0,1.25], \tag{4.2}$$

subject to the initial conditions $x(0)=0,\ \mathrm{y}(0)=0.$
Applying the DTM of Equations 4.1 and 4.2 and using the initial conditions $x(0)=0,\ \mathrm{y}(0)=0.$ it follows

$$\mathrm{X}(\mathrm{k}+1)=\frac{1}{k+1}\left(\mathrm{e}^{Y(0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=0}\right)$$

$$\mathrm{Y}(\mathrm{k}+1)=\frac{1}{k+1}\left(\mathrm{e}^{\mathrm{X}(0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m X^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(\mathrm{x}(\mathrm{t})\right)^{r-m}\right]_{t=0}\right),$$

$$X(0)=0,\ \mathrm{Y}(0)=0.$$

By substituting $k=0,\ldots,11$ we obtain the coefficients of the series solution as follows

$$X(1)=\mathrm{Y}(1)=1,\ \mathrm{X}(2)=\mathrm{Y}(2)=\frac{1}{2},\ \mathrm{X}(3)=\mathrm{Y}(3)=\frac{1}{3},\ \mathrm{X}(4)=\mathrm{Y}(4)=\frac{1}{4},$$
$$\mathrm{X}(5)=\mathrm{Y}(5)=\frac{1}{5},\ \mathrm{X}(6)=\mathrm{Y}(6)=\frac{1}{6},\ \mathrm{X}(7)=\mathrm{Y}(7)=\frac{1}{7},\ \mathrm{X}(8)=\mathrm{Y}(8)=\frac{1}{8},$$
$$\mathrm{X}(9)=\mathrm{Y}(9)=\frac{1}{9},\ \mathrm{X}(10)=\mathrm{Y}(10)=\frac{1}{10},\ \mathrm{X}(11)=\mathrm{Y}(11)=\frac{1}{11},\ \mathrm{X}(12)=\mathrm{Y}(12)=\frac{1}{12}.$$

Hence, the series solution reads

$$y(\mathrm{t})=\mathrm{x}(\mathrm{t})=\mathrm{t}+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{4}+\frac{t^5}{5}+\frac{t^6}{6}+\frac{t^7}{7}+\frac{t^8}{8}+\frac{t^9}{9}+\frac{t^{10}}{10}+\frac{t^{11}}{11}+\frac{t^{12}}{12},\ t\in[0,1.25].$$

On the other hand, by applying the MsDTM to Equations 4.1 and 4.2 with same initial conditions, it follows

$$\mathrm{X}_i(\mathrm{k}+1)=\frac{1}{k+1}\left(\mathrm{e}^{Y_i(0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m Y_i^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_i}\right)$$

$$\mathrm{Y}_i(\mathrm{k}+1)=\frac{1}{k+1}\left(\mathrm{e}^{\mathrm{X}_i(0)}\sum_{r=0}^{k}\sum_{m=0}^{r}\frac{(-1)^m X_i^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(\mathrm{x}(\mathrm{t})\right)^{r-m}\right]_{t=t_i}\right),$$

$$X_0(0)=0,\mathrm{X}_i(0)=\mathrm{x}_{i-1}(\mathrm{t}_i),\ \ \mathrm{Y}_0(0)=0,\ Y_i(0)=\mathrm{y}_{i-1}(\mathrm{t}_i),\ \mathrm{i}=1,2,3,4,5.$$

Thus, we obtain the series solution

$$y(\mathrm{t})=\mathrm{x}(\mathrm{t})=\begin{cases}\mathrm{t}+0.5\,\mathrm{t}^2+0.3333\,\mathrm{t}^3+0.25\,\mathrm{t}^4+0.2\,\mathrm{t}^5+0.16667\,\mathrm{t}^6+0.14286\,\mathrm{t}^7\\ +0.125t^8+0.1111t^9+0.1t^{10}+0.0909t^{11}+0.0833t^{12},\quad t\in[0,0.25]\\ 0.286682+1.33333(\mathrm{t}-0.25)+0.888889(\mathrm{t}-0.25)^2+0.790124(\mathrm{t}-0.25)^3\\ +0.790124(\mathrm{t}-0.25)^4+0.842798(\mathrm{t}-0.25)^5+0.936443(\mathrm{t}-0.25)^6\\ +1.07022(\mathrm{t}-0.25)^7+1.24859(\mathrm{t}-0.25)^8+1.47981(\mathrm{t}-0.25)^9\\ +1.77577(\mathrm{t}-0.25)^{10}+2.15245(\mathrm{t}-0.25)^{11}+2.63078(\mathrm{t}-0.25)^{12},\ t\in[0.25,0.5]\\ 0.693147+2(\mathrm{t}-0.5)+2(\mathrm{t}-0.5)^2+2.66667(\mathrm{t}-0.5)^3+4(\mathrm{t}-0.5)^4+6.4(\mathrm{t}-0.5)^5\\ +10.66667(\mathrm{t}-0.5)^6+18.2857(\mathrm{t}-0.5)^7+32(\mathrm{t}-0.5)^8+56.8889(\mathrm{t}-0.5)^9\\ +102.4(\mathrm{t}-0.5)^{10}+186.182(\mathrm{t}-0.5)^{11}+341.333(\mathrm{t}-0.5)^{12},\ t\in[0.5,0.75]\\ 1.38628+3.99993(\mathrm{t}-0.75)+7.99972(\mathrm{t}-0.75)^2+21.3322(\mathrm{t}-0.75)^3\\ +63.9955(\mathrm{t}-0.75)^4+204.782(\mathrm{t}-0.75)^5+682.595(\mathrm{t}-0.75)^6+2340.28(\mathrm{t}-0.75)^7\\ 8190.85(\mathrm{t}-0.75)^8+29122.5(\mathrm{t}-0.75)^9+104839(\mathrm{t}-0.75)^{10}+381227(\mathrm{t}-0.75)^{11}\\ +1.39781\times10^6(\mathrm{t}-0.75)^{12},\ t\in[0.75,1]\\ 4.05773+57.843(\mathrm{t}-1)+1672.91(\mathrm{t}-1)^2+64510.45(\mathrm{t}-1)^3+2.79861\times10^6(\mathrm{t}-1)^4\\ +1.56645\times10^{13}(\mathrm{t}-1)^8+8.05403\times10^{14}(\mathrm{t}-1)^9+4.19282\times10^{16}(\mathrm{t}-1)^{10}\\ +2.20478\times10^{18}(\mathrm{t}-1)^{11}+1.16903\times10^{20}(\mathrm{t}-1)^{12},\ t\in[1,1.25].\end{cases}$$

This problem with the initial conditions $x(0)=0,\ \mathrm{y}(0)=0$ can be solved analytically by the phase-plane method to obtain the analytical solution $y(\mathrm{x})=\mathrm{x}$. As seen in Figure 1, the approximate series solutions calculated by the DTM and the MsDTM are the same as the analytical solution and they have the same direction with the flow of the vector fields. Moreover, the DTM and the MsDTM gave data results similar to the analytical results (Table 3).

However, if we consider the problem with the initial conditions of $x(0)=-2,\ \mathrm{y}(0)=1.$, the analytical solution obtained is $y(\mathrm{x})=\ln\left(e^x+e-e^{-2}\right)$. The data values of the approximate solutions of the DTM and the MsDTM were compared

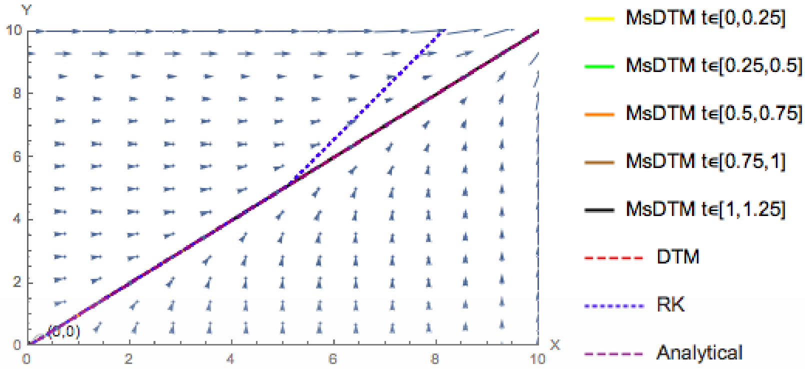

Figure 1. MsDTM: The DTM and numerical solution compared with vector field flow directions.

Table 3. DTM and MsDTM values compared with the analytical solutions.

t	x(t)	MsDTM	Analytical	Error
0.2	0.2231436	0.2231436	0.2231436	0
0.4	0.5108257	0.5108257	0.5108257	0
0.6	0.9162908	0.9162908	0.9162908	0
0.8	1.6094160	1.6094160	1.6094160	0
t	x(t)	DTM	Analytical	Error
0.2	0.2231436	0.2231436	0.2231436	0
0.4	0.5108248	0.5108248	0.5108248	0
0.6	0.9160622	0.9160622	0.9160622	0
0.8	1.5924103	1.5924103	1.5924103	0

with analytical solution and are shown in Table 4. We can see that the MsDTM results are much more similar to the analytical results than the DTM results.

The following two examples show that the proposed new transformed functions of the product of composite functions can be applied effectively to the nonlinear plane autonomous system when the analytical solutions are unavailable.

Example 4.2. Let us consider the following system of nonlinear plane autonomous

$$x' = x^2 e^y \tag{4.3}$$

$$y' = ye^x - y, \text{ for } t \in [0, 0.2], \tag{4.4}$$

subject to the initial conditions $x(0) = 1,\ y(0) = 1.$

Applying the DTM to Equations 4.3 and 4.4 and with the initial conditions $x(0) = 1, y(0) = 1,$ it follows that

$$X(k+1) = \frac{1}{k+1}\sum_{r=0}^{k} F(r) \sum_{l=0}^{k-r} X(l)\, X(k-r-l)$$

$$Y(k+1) = \frac{1}{k+1}\left(\sum_{r=0}^{k} Y(k-r)\, G(r) - Y(k)\right),$$

and the initial condition becomes $X(0) = 1,\ Y(0) = 1,$

where

$$F(r) = e^{Y(0)} \sum_{l=0}^{r}\sum_{m=0}^{l} \frac{(-1)^m Y^m(0)}{(l-m)!m!}\left[\frac{1}{r!}\frac{d^r}{dt^r}\left(y(t)\right)^{l-m}\right]_{t=0},$$

$$G(r) = e^{X(0)} \sum_{l=0}^{r}\sum_{m=0}^{l} \frac{(-1)^m X^m(0)}{(l-m)!m!}\left[\frac{1}{r!}\frac{d^r}{dt^r}\left(x(t)\right)^{l-m}\right]_{t=0}.$$

Table 4. DTM and MsDTM values compared with the analytical solutions.

t	*x*(t)	MsDTM	Analytical	Error
0.2	-1.4473324	1.0360783	1.0360783	0
0.4	-0.8671829	1.0996385	1.0996384	1×10^{-7}
0.6	-0.2340544	1.2161777	1.2161776	1×10^{-7}
0.8	0.5147103	1.4483531	1.4483533	2×10^{-7}
t	*x*(t)	DTM	Analytical	Error
0.2	-1.4473324	1.0360783	1.0360783	0
0.4	-0.8671831	1.0996383	1.0996384	2×10^{-7}
0.6	-0.2340822	1.2161499	1.2161711	2.12×10^{-5}
0.8	0.5130110	1.4466537	1.4476856	1.0319×10^{-3}

Hence, we obtain the series solution by the DTM

$$x(t) = 1 + 2.71828\,t + 9.72444t^2 + 38.8048t^3 + 164.329t^4 + 722.872t^5 + 3265.98t^6 + 15052.5t^7 + 7045.9t^8 + 333873t^7 + 1.59826\times 10^6 t^{10} + 7.71576\times 10^6 t^{11} + 3.75161\times 10^7 t^{12},\ t \in [0, 0.2]$$

$$y(t) = 1 + 1.71828\,t + 5.17077\,t^2 + 19.3526t^3 + 80.1435t^4 + 351.093t^5 + 1593.47t^6 + 7409.3t^7 + 35062.4t^8 + 168146t^9 + 814830t^{10} + 3.98192\times 10^6 t^{11} + 1.95937\times 10^7 t^{12},\ \ t \in [0, 0.2].$$

On the other hand, by applying the MsDTM to Equations 4.3 and 4.4 we obtain

$$X_i(\mathrm{k}+1) = \frac{1}{k+1}\sum_{r=0}^{k} F_i(\mathrm{r}) \sum_{l=0}^{k-r} X_i(l)\,\mathrm{X}_i(\mathrm{k}-\mathrm{r}-l)$$

$$Y_i(\mathrm{k}+1) = \frac{1}{k+1}\left(\sum_{r=0}^{k} Y_i(\mathrm{k}-\mathrm{r})\mathrm{G}_i(\mathrm{r}) - \mathrm{Y}_i(\mathrm{k})\right),\ \ X_0(0) = 1,\ \mathrm{X}_i(0) = \mathrm{x}_{i-1}(\mathrm{t}_i),\ \mathrm{Y}_0(0) = 1,\ \mathrm{Y}_i(0) = \mathrm{y}_{i-1}(\mathrm{t}_i),\ \ \mathrm{i} = 1,2,3,4$$

where

$$F_i(\mathrm{r}) = \mathrm{e}^{Y_i(0)} \sum_{l=0}^{r}\sum_{m=0}^{l} \frac{(-1)^m Y_i^m(0)}{(\mathrm{l}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{\mathrm{r}!}\frac{d^r}{dt^r}\big(y(\mathrm{t})\big)^{l-m}\right]_{t=t_i},\ \ \mathrm{G}(\mathrm{r}) = \mathrm{e}^{\mathrm{X}_i(0)} \sum_{l=0}^{r}\sum_{m=0}^{l} \frac{(-1)^m X_i^{\,m}(0)}{(\mathrm{l}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{\mathrm{r}!}\frac{d^r}{dt^r}\big(\mathrm{x}(\mathrm{t})\big)^{l-m}\right]_{t=t_i}.$$

The following approximate series solution is the result.

$$x(t) = \begin{cases} 1 + 2.71828t + 9.72444t^2 + 38.8048t^3 + 164.329t^4 + 722.872t^5 + 3265.98t^6 + 15052.5t^7 \\ +70454.9t^8 + 333873t^9 + 1.59826\times 10^6 t^{10} + 7.71576\times 10^6 t^{11} + 3.75176\times 10^7 t^{12},\ t \in [0, 0.05] \\ 1.1664 + 4.09489(\mathrm{t}-0.05) + 19.3628(\mathrm{t}-0.05)^2 + 103.094(\mathrm{t}-0.05)^3 \\ +585.693(\mathrm{t}-0.05)^4 + 3468.59(\mathrm{t}-0.05)^5 + 21148.6(\mathrm{t}-0.05)^6 + 131763(\mathrm{t}-0.05)^7 \\ +834742(\mathrm{t}-0.06)^8 + 5.35899\times 10^6(\mathrm{t}-0.06)^9 + 3.47789\times 10^7(\mathrm{t}-0.06)^{10} \\ +2.27748\times 10^8(\mathrm{t}-0.06)^{11} + 1.50275\times 10^9(\mathrm{t}-0.06)^{12},\ \ t \in [0.05, 0.1] \\ 1.43766 + 7.26784(\mathrm{t}-0.1) + 51.4129(\mathrm{t}-0.1)^2 + 416.14(\mathrm{t}-0.12)^3 \\ +3626.29(\mathrm{t}-0.1)^4 + 33122.1(\mathrm{t}-0.1)^5 + 312581(\mathrm{t}-0.12)^6 + 3.02155\times 10^6(\mathrm{t}-0.1)^7 \\ +2.9749\times 10^7(\mathrm{t}-0.1)^8 + 2.97167\times 10^8(\mathrm{t}-0.1)^9 + 3.00338\times 10^9(\mathrm{t}-0.1)^{10} \\ +3.06482\times 10^{10}(\mathrm{t}-0.1)^{11} + 3.15286\times 10^{11}(\mathrm{t}-0.1)^{12},\ t \in [0.1, 0.15] \\ 2.02412 + 19.7164(\mathrm{t}-0.15) + 293.806(\mathrm{t}-0.15)^2 + 5196.51(\mathrm{t}-0.15)^3 + 100769(\mathrm{t}-0.15)^4 \\ +2.06843\times 10^6(\mathrm{t}-0.15)^5 + 4.41157\times 10^7(\mathrm{t}-0.15)^6 + 9.67037\times 10^8(\mathrm{t}-0.15)^7 \\ +2.16368\times 10^{10}(\mathrm{t}-0.15)^8 + 4.91847\times 10^{11}(\mathrm{t}-0.15)^9 + 1.13227\times 10^{13}(\mathrm{t}-0.15)^{10} \\ +2.63346\times 10^{14}(\mathrm{t}-0.15)^{11} + 6.17734\times 10^{15}(\mathrm{t}-0.15)^{12},\ t \in [0.15, 0.2]. \end{cases}$$

$$y(\mathrm{t})=\begin{cases}1+1.71828t+5.17077t^2+19.3536t^4+80.1435t^5+351.093t^6+7409.3t^7\\ +35062.4t^8+168146t^9+814830t^{10}+3.98192\times10^6t^{11}+1.95937\times10^7t^{12},\ t\in[0,0.05]\\ 1.1019+2.43565(\mathrm{t}-0.05)+9.93481(\mathrm{t}-0.05)^2+50.7118(\mathrm{t}-0.05)^3\\ +286.334(\mathrm{t}-0.05)^4+1708.87(\mathrm{t}-0.05)^5+10559.3(\mathrm{t}-0.05)^6+66814.5(\mathrm{t}-0.05)^7\\ +430122(\mathrm{t}-0.05)^8+2.8059\times10^6(\mathrm{t}-0.05)^9+1.84864\times10^7(\mathrm{t}-0.05)^{10}\\ +1.2283\times10^8(\mathrm{t}-0.05)^{11}+8.21708\times10^8(\mathrm{t}-0.05)^{12},\ t\in[0.05,0.1]\\ 1.25743+4.03738(\mathrm{t}-0.1)+25.7226(\mathrm{t}-0.1)^2+206.07(\mathrm{t}-0.1)^3\\ +1823.14(\mathrm{t}-0.1)^4+17024.4(\mathrm{t}-0.1)^5+164483(\mathrm{t}-0.1)^6+1.62549\times10^6(\mathrm{t}-0.1)^7\\ +1.63413\times10^7(\mathrm{t}-0.1)^8+1.66403\times10^8(\mathrm{t}-0.1)^9+1.71169\times10^9(\mathrm{t}-0.11)^{10}\\ +1.75515\times10^{10}(\mathrm{t}-0.1)^{11}+1.85343\times10^{11}(\mathrm{t}-0.1)^{12},\ t\in[0.1,0.15]\\ 1.57118+10.3218(\mathrm{t}-0.15)+151.148(\mathrm{t}-0.15)^2+2.779.76(\mathrm{t}-0.15)^3+56212.2(\mathrm{t}-0.15)^4\\ +1.19708\times10^6(\mathrm{t}-0.15)^5+2.63404\times10^7(\mathrm{t}-0.15)^6+5.92862\times10^8(\mathrm{t}-0.15)^7\\ +1.35679\times10^{10}(\mathrm{t}-0.15)^8+3.145\times10^{11}(\mathrm{t}-0.15)^9+7.36425\times10^{12}(\mathrm{t}-0.15)^{10}\\ +1.73866\times10^{14}(\mathrm{t}-0.15)^{11}+4.13307\times10^{15}(\mathrm{t}-0.15)^{12},\ t\in[0.15,0.2].\end{cases}$$

The approximate series solution obtained by the MsDTM and the DTM are compared graphically with the flow direction of the vector fields (Figure 2). We can see that the MsDTM result is in better agreement with vector field than the DTM result.

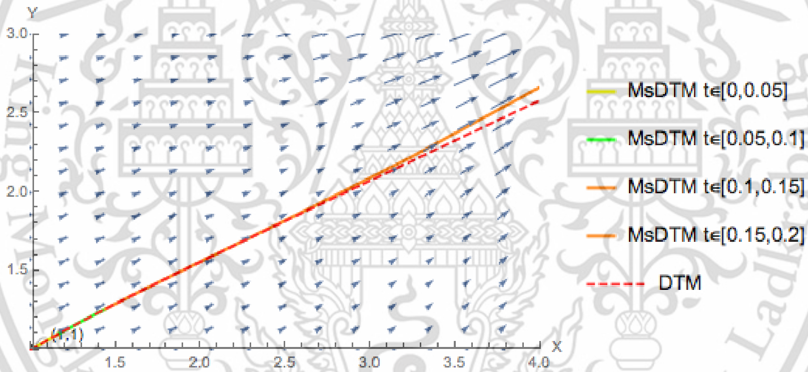

Figure 2. MsDTM and DTM compared with vector field flow directions.

Example 4.3. Let us consider the following system of nonlinear plane autonomous

$$x'=2x+\sin y \tag{4.5}$$

$$y'=x(\mathrm{y}^2+1),\ \text{for } t\in[0,0.4], \tag{4.6}$$

subject to the initial condition $x(0)=1,\ \mathrm{y}(0)=1.$

Applying the DTM of Equations 4.5 and 4.6, we obtain

$$X(\mathrm{k}+1)=\frac{1}{k+1}\left(2X(\mathrm{k})+\sum_{r=0}^{k}\frac{d^r}{dt^r}\sin(\mathrm{t})\Big|_{t=Y(0)}\sum_{m=0}^{r}\frac{(-1)^m Y^m(0)}{(\mathrm{r}-\mathrm{m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=0}\right)$$

$$Y(\mathrm{k}+1)=\frac{1}{k+1}\left(X(\mathrm{k})+\sum_{r=0}^{k}\sum_{l=0}^{r}Y(l)\,\mathrm{Y}(\mathrm{r}-l)X(\mathrm{k}-\mathrm{r})\right),$$

and the initial condition becomes $x(0)=1,\ \mathrm{y}(0)=1.$

Then, we obtain the series solution

$$x(\mathrm{t}) = 1 + 2.84147\mathrm{t} + 3.38177\mathrm{t}^2 + 2.56549t^3 + 0.498022t^4 - 3.62605t^5 - 13.1978t^6 - 36.6427t^7 - 93.1066t^8 - 224.685t^9 - 520.598t^{10} - 1160.66t^{11} - 2481.54t^{12},\ t \in [0, 0.4]$$

$$y(\mathrm{t}) = 1 + 2\mathrm{t} + 4.84147\mathrm{t}^2 + 10.6041t^3 + 24.528t^4 + 57.5466t^5 + 134.91t^6 + 315.164t^7 + 733.76t^8 + 1702.66t^9 + 3938.03t^{10} + 9079.66t^{11} + 20873.9t^{12},\ t \in [0, 0.4].$$

On the other hand, by applying the MsDTM to Equationa 4.5 and 4.6, it follows

$$X_i(\mathrm{k}+1) = \frac{1}{k+1}\left(2X_i(\mathrm{k}) + \sum_{r=0}^{k} \frac{d^r}{dt^r}\sin(\mathrm{t})\bigg|_{t=Y_i(0)} \sum_{m=0}^{r} \frac{(-1)^m Y_i^m(0)}{(\mathrm{r-m})!\mathrm{m}!}\left[\frac{1}{k!}\frac{d^k}{dt^k}\left(y(\mathrm{t})\right)^{r-m}\right]_{t=t_i}\right)$$

$$Y_i(\mathrm{k}+1) = \frac{1}{k+1}\left(X_i(\mathrm{k}) + \sum_{r=0}^{k}\sum_{l=0}^{r} Y_i(l)\,\mathrm{Y}_i(\mathrm{r}-l)X_i(\mathrm{k-r})\right),$$

$$X_0(0) = 1,\ \mathrm{X}_i(0) = \mathrm{x}_{i-1}(\mathrm{t}_i),\ \mathrm{Y}_0(0) = 1,\ Y_i(0) = y_{i-1}(\mathrm{t}_i),\ \mathrm{i} = 1,2,3,4,5.$$

Hence, we obtain the series solution

$$x(\mathrm{t}) = \begin{cases} 1 + 2.84147\mathrm{t} + 3.38177\mathrm{t}^2 + 2.56549t^3 + 0.498022t^4 - 3.62605t^5 - 13.1978t^6 \\ -36.6427t^7 - 93.1066t^8 - 224.685t^9 - 520.598t^{10} - 1160.66t^{11} - 2481.54t^{12},\ t \in [0, 0.08] \\ 1.25028 + 3.43174(\mathrm{t}-0.08) + 3.98652(\mathrm{t}-0.08)^2 + 2.28135(\mathrm{t}-0.08)^3 \\ -3.27511(\mathrm{t}-0.08)^4 - 19.3406(\mathrm{t}-0.08)^5 - 67.1434(\mathrm{t}-0.08)^6 \\ -205.262(\mathrm{t}-0.08)^7 - 584.413(\mathrm{t}-0.08)^8 - 1563.52(\mathrm{t}-0.08)^9 \\ -3898.43(\mathrm{t}-0.08)^{10} - 8808.52(\mathrm{t}-0.08)^{11} - 16669.2(\mathrm{t}-0.08)^{12},\ t \in [0.08, 0.16] \\ 1.55128 + 4.10089(\mathrm{t}-0.16) + 4.24803(\mathrm{t}-0.16)^2 - 1.14073(\mathrm{t}-0.16)^3 \\ -23.7465(\mathrm{t}-0.16)^4 - 108.548(\mathrm{t}-0.16)^5 - 404.828(\mathrm{t}-0.16)^6 \\ -1336.03(\mathrm{t}-0.16)^7 - 3807.58(\mathrm{t}-0.16)^8 - 8159.94(\mathrm{t}-0.16)^9 \\ -2936.53(\mathrm{t}-0.16)^{10} + 104910(\mathrm{t}-0.16)^{11} + 850945(\mathrm{t}-0.16)^{12},\ t \in [0.16, 0.24] \\ 1.90448 + 4.67667(\mathrm{t}-0.24) + 2.1336(\mathrm{t}-0.24)^2 - 22.5505(\mathrm{t}-0.24)^3 \\ -142.022(\mathrm{t}-0.24)^4 - 585.377(\mathrm{t}-0.24)^5 - 1394.54(\mathrm{t}-0.24)^6 + 4460.95(\mathrm{t}-0.24)^7 \\ +93830.2(\mathrm{t}-0.24)^8 + 865886(\mathrm{t}-0.24)^9 + 6.29241\times10^6(\mathrm{t}-0.24)^{10} \\ +3.98717\times10^7(\mathrm{t}-0.24)^{11} + 2.27165\times10^8(\mathrm{t}-0.24)^{12},\ t \in [0.24, 0.32] \\ 2.27311 + 420097(\mathrm{t}-0.32) - 9.88838(\mathrm{t}-0.32)^2 - 38.0005(\mathrm{t}-0.32)^3 + 1179.65(\mathrm{t}-0.32)^4 \\ +24086.5(\mathrm{t}-0.32)^5 + 281349(\mathrm{t}-0.32)^6 + 2.24486\times10^6(\mathrm{t}-0.32)^7 \\ +8.20097\times10^6(\mathrm{t}-0.32)^8 - 1.15647\times10^8(\mathrm{t}-0.32)^9 - 3.29081\times10^9(\mathrm{t}-0.32)^{10} \\ -5.19089\times10^{10}(\mathrm{t}-0.32)^{11} - 6.53363\times10^{11}(\mathrm{t}-0.32)^{12},\ t \in [0.32, 0.4]. \end{cases}$$

$$y(\mathrm{t}) = \begin{cases} 1 + 2\mathrm{t} + 4.84147\mathrm{t}^2 + 10.6041t^3 + 24.528t^4 + 57.5466t^5 + 134.91t^6 + 315.164t^7 \\ +733.76t^8 + 1702.66t^9 + 3938.03t^{10} + 9079.66t^{11} + 20873.9t^{12},\ t \in [0, 0.08] \\ 1.19765 + 3.04364(\mathrm{t}-0.08) + 8.7346(\mathrm{t}-0.08)^2 + 24.1547(\mathrm{t}-0.08)^3 \\ +69.2557(\mathrm{t}-0.08)^4 + 199.329(\mathrm{t}-0.08)^5 + 571.305(\mathrm{t}-0.08)^6 \\ +1629.29(\mathrm{t}-0.08)^7 + 4624.49(\mathrm{t}-0.08)^8 + 13067.8(\mathrm{t}-0.08)^9 \\ +36779.9(\mathrm{t}-0.08)^{10} + 103170(\mathrm{t}-0.08)^{11} + 288632(\mathrm{t}-0.08)^{12},\ t \in [0.08, 0.16] \\ 1.5131 + 5.10287(\mathrm{t}-0.16) + 18.7225(\mathrm{t}-0.16)^2 + 68.529(\mathrm{t}-0.16)^3 \\ +254.775(\mathrm{t}-0.16)^4 + 942.871(\mathrm{t}-0.16)^5 + 3463.75(\mathrm{t}-0.16)^6 + 12638.4(\mathrm{t}-0.16)^7 \\ +45855(\mathrm{t}-0.16)^8 + 165680(\mathrm{t}-0.16)^9 + 597192(\mathrm{t}-0.16)^{10} \\ +2.15185\times10^6(\mathrm{t}-0.16)^{11} + 7.76857\times10^6(\mathrm{t}-0.16)^{12},\ t \in [0.16, 0.24] \end{cases}$$

$$y(t)=\begin{cases}2.09104+10.2318(t-0.24)+53.3091(t-0.24)^2+278.516(t-0.24)^3\\ +1449.56(t-0.24)^4+7466.42(t-0.24)^5+38168.7(t-0.24)^6+194558(t-0.24)^7\\ +994211(t-0.24)^8+5.12151\times10^6(t-0.24)^9+2.67247\times10^7(t-0.24)^{10}\\ +1.41713\times10^8(t-0.24)^{11}+7.64272\times10^8(t-0.24)^{12},\ t\in[0.24,0.32]\\ 3.4941+30.0249(t-0.32)+266.217(t-0.32)^2+2342.96(t-0.32)^3+20645.6(t-0.32)^4\\ +185021(t-0.32)^5+1.70474\times10^6(t-0.32)^6+1.61372\times10^7(t-0.32)^7\\ +1.55164\times10^8(t-0.32)^8+1.49064\times10^9(t-0.32)^9+1.40754\times10^{10}(t-0.32)^{10}\\ +1.28807\times10^{11}(t-0.32)^{11}+1.12816\times10^{12}(t-0.32)^{12},\ t\in[0.32,0.4].\end{cases}$$

Similar to the previous examples, the MsDTM result is in better agreement with the flow of the vector field than the DTM result (Figure 3).

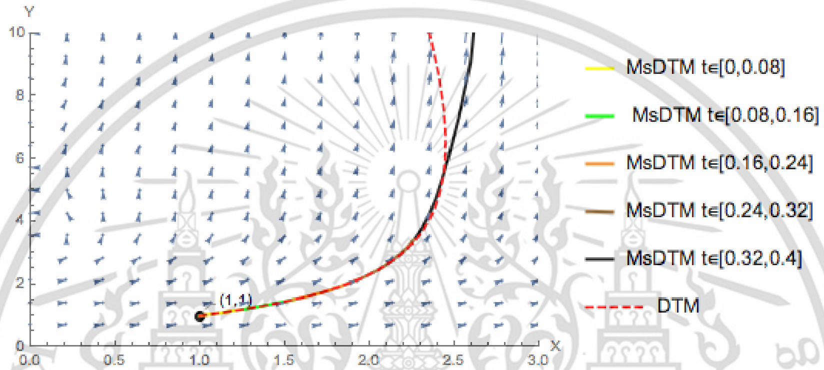

Figure 3. MsDTM and DTM compared with vector field flow directions.

5. Conclusions

The MsDTM combined with our new formulae have been successfully applied to solve nonlinear plane autonomous systems. Three different examples were solved and the series solutions of the DTM and the MsDTM were obtained. These are compared with the analytical solutions calculated by the phase-plane method in the first example and compared to the vector fields flow directions in the second and the third examples. The results of the MsDTM were more similar to the analytical solution and to the vector field flow direction than the DTM results. Therefore, this method based on our new transformed functions is a reliable and efficient mathematical tool for solving nonlinear plane autonomous systems.

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Academic Publication

1. Kasemsuwan, J., Sabau, S.V., Somboon, U. 2019. "Differential transformation method for circular membrane vibrations". Bulletin of the Transilvania University of Brasov, Series III: Mathematics, Informatics, Physics, 12(2), 333-350.
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