INEQUALITIES INVOLVING TRACY-SINGH PRODUCTS AND KHATRI-RAO PRODUCTS FOR BOUNDED LINEAR OPERATORS ON A HILBERT SPACE

ARNON PLOYMUKDA

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
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อสมการเกี่ยวกับผลคูณเทรซี-ซิงห์และผลคูณคาทรี-ราวสำหรับ ตัวดำเนินการเชิงเส้นที่มีขอบเขตบนปริภูมิฮิลเบิร์ต

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บทคัดย่อ

ในงานวิจัยนี้ เราศึกษาความโค้งและความเว้าของการส่งต่างๆที่เกี่ยวข้องกับฟังก์ชันทางเดียวเชิงตัวดำเนินการ ผลคูณเทรซี-ซิงห์ และผลคูณคาทรี-ราวของตัวดำเนินการที่มีขอบเขตบนปริภูมิฮิลเบิร์ต เราได้อสมการต่างๆ สำหรับตัวดำเนินซึ่งเกี่ยวข้องกับผลคูณเทรซี-ซิงห์ และผลคูณคาทรี-ราว โดยเฉพาะอย่างยิ่ง เราได้อสมการเชบีเชฟสำหรับตัวดำเนินการที่เกี่ยวข้องกับผลคูณเทรซี-ซิงห์ ผลคูณคาทรี-ราว และค่าเฉลี่ยพีทาโกรัสถ่วงน้ำหนัก ผลลัพธ์ในงานวิจัยนี้ครอบคลุมผลลัพธ์สำหรับผลคูณเทรซี-ซิงห์ ผลคูณคาทรี-ราว ผลคูณโครเนคเคอร์ และผลคูณฮาดามาร์ดของเมทริกซ์ และผลคูณเทนเซอร์ของตัวดำเนินการ

คำสำคัญ : ค่าเฉลี่ยพีทาโกรัสถ่วงน้ำหนัก ตัวดำเนินการเชิงเส้นที่มีขอบเขต ปริภูมิฮิลเบิร์ต ฟังก์ชันทางเดียวเชิงตัวดำเนินการ ผลคูณคาทรี-ราว ผลคูณเทรซี-ซิงห์ อสมการเชบีเชฟ

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Abstract

In this research, we investigate concavity and convexity of certain maps related to operator-monotone functions, Tracy-Singh products and Khatri-Rao products of bounded linear operators on a Hilbert space. We also establish several inequalities for operators involving Tracy-singh products and Khatri-Rao products. In particular, we obtain Chebyshev-type inequalities for operators concerning Tracy- Singh products, Khatri-Rao products and weighted Pythagorean mean. Our results include the results for Tracy-Singh/Khatri-Rao/Kronecker/Hadamard products of matrices and tensor products of operators.

Keywords : weighted Pythagorean mean, bounded linear operator, Hilbert space, operator-monotone function, Khatri-Rao product, Tracy-Singh product, Chebyshev inequality,

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Arnon Ploymukda

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Chapter 1

Introduction

1.1 Inception and Importance

Matrix product is a binary operation that produces a matrix from two matrices. There are various matrix products which are of interest in both theory and applications, such as the Kronecker product, Hadamard product, Tracy-Singh product and Khatri-Rao product.

The Kronecker product is an operation on two matrices of arbitrary size resulting in a block matrix. For any complex matrices $A = [a_{ij}]$ and B of arbitrary sizes, the Kronecker product of A and B is given by the block matrix

$$A \hat{\otimes} B \;=\; [a_{ij}B]_{ij}. \tag{1.1}$$

This kind of matrix product has wide applications in matrix theory, system theory, physics, statistics, computer science, signal processing and other special fields. The Kronecker product has an important role in the linear matrix equation theory, for example, in solving the Sylvester equation, the Lyapunov equation, and the commutativity equation.

The Hadamard product is a binary operation that takes two matrices of the same dimensions. This product is known as the entry-wise or Schur product. The Hadamard product for two complex matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size is defined by

$$A \hat{\odot} B \;=\; [a_{ij}b_{ij}]. \tag{1.2}$$

The concept of Hadamard product is well known in linear algebra. This product has been used in combinatorial analysis, finite geometry, group theory, number theory, and regular graphs. Applications of the Hadamard product can also be found in other fields, for example, in correcting codes in satellite transmissions and cryptography, signal processing and pattern recognition, and lossy compression algorithms as images in JPEG format. Faliva [15] seems to be the first one to observe the connection between the Kronecker and Hadamard products based on two permutation matrices.

A generalization of the Kronecker product was defined for partitioned matrices by Tracy and Singh [44]. Consider partitioned matrices A and B such that the (i, j)th block of A is A_{ij} and the (k, l)th block of B is B_{kl}. The Tracy-Singh product of A and B is defined by

$$A \hat{\boxtimes} B \;=\; \big[\, [A_{ij} \hat{\otimes} B_{kl}]_{kj} \,\big]_{ij}.$$

The Tracy-Singh product is studied and applied widely in matrix theory and statistics.

In the same way, the Khatri-Rao product for partitioned matrices, introduced by Khatri and Rao [21], is a generalized Hadamard product. The Khatri-Rao product is defined for two partitioned matrices $A = [A_{ij}]$ and $B = [B_{kl}]$ as follows

$$A \hat{\boxdot} B \;=\; \left[A_{ij} \hat{\otimes} B_{ij}\right]_{ij}. \tag{1.3}$$

In [26], Liu established a connection between the Khatri-Rao and Tracy-Singh producs via selection matrices. This connection play an important role to give inequalities involving the two products.

As a natural generalization of a matrix, a Hilbert-space operator is a bounded linear transformation of a Hilbert space into itself. The tensor product of two Hilbert-space operators can be viewed as an infinite-dimensional extension of the Kronecker product of matrices. Recall that the tensor product of two operators $A \in \mathfrak{B}(\mathbb{H}, \mathbb{H}')$ and $B \in \mathfrak{B}(\mathbb{K}, \mathbb{K}')$ is the unique bounded linear operator from $\mathbb{H} \otimes \mathbb{K}$ into $\mathbb{H}' \otimes \mathbb{K}'$ such that

$$(A \otimes B)(x \otimes y) \;=\; Ax \otimes By \tag{1.4}$$

for all $x \in \mathbb{H}$ and $y \in \mathbb{K}$. By filtering the tensor product through a positive linear map, the Hadamard product of Hilbert-space operators can be presented (see [16]). The Hadamard product of $A, B \in \mathfrak{B}(\mathbb{H})$ can represented by

$$A \odot B \;=\; U^*(A \otimes B)U, \tag{1.5}$$

where $U : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ is the isometry defined by $Ue_i = e_i \otimes e_i$, where $\{e_i\}$ is an orthonormal basis of $\mathbb{H}$.

Recently, the notion of tensor product was extended to the Tracy-Singh product for Hilbert-space operators in [34]. This kind of product provides a natural extension for both Tracy-Singh product of matrices and tensor product of operators. Fundamental algebraic, order, and analytic properties of Tracy-Singh product of operator were investigated in [34, 35]. In [36], the authors generalized the tensor product of operators and Khatri-Rao product of matrices to the Khatri-Rao product for operators acting on a direct sum of Hilbert spaces. They also provided a construction of a unital linear map taking the Tracy-Singh product of operators to their Khatri-Rao product.

On the other hand, it has long been known that in the case $0 \leqslant p \leqslant 1$ the map $A \mapsto A^p$ is concave on the space of positive definite matrices. Lieb [25] succeeded in proving the concavity of the map $(A, B) \mapsto A^{1-p} \hat{\otimes} B^p$ in the case $0 < p \leqslant 1$. In [5], Ando established concavity and convexity theorems of this type in full generality, and to apply them to obtain unusual estimates for Hadamard products of positive definite matrices. He showed that if f is a positive operator-monotone function on $(0, \infty)$ and if ϕ_1 and ϕ_2 are concave maps, then the map

$$(A, B) \;\mapsto\; f[\phi_1(A)^{-1} \hat{\otimes} \phi_2(B)](\phi_1(A) \otimes I) \tag{1.6}$$

is concave where A and B are positive definite. This theorem leads to the Lieb's result as corollary. If ϕ_1 is affine in the above, then the map

$$(A, B) \mapsto f[\phi_1(A)\hat{\otimes}\phi_2(B)^{-1}](\phi_1(A) \otimes I) \tag{1.7}$$

is convex. Moreover, Hölder type inequalities involving Hadamard products of positive definite matrices are also obtained. The work of Al-Zhour [4] extends some results in [5] to Tracy-Singh products and Khatri-Rao products of positive definite matrices. Hölder type inequalities for Tracy-Singh and Khatri-Rao products of matrices, generalizations of Hölder type inequalities in [5], were obtained in [3]. Mond and Pečarić [30] extended the matrix results in [5] to Hilbert-space operators and obtained concavity and convexity theorems associated with positive operator-monotone functions.

One of the fundamental inequalities in mathematics is the Chebyshev sum inequality which states that

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \geqslant \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \tag{1.8}$$

for all real numbers a_i, b_i $(1 \leqslant i \leqslant n)$ such that $a_1 \leqslant \cdots \leqslant a_n$ and $b_1 \leqslant \cdots \leqslant b_n$, or $a_1 \geqslant \cdots \geqslant a_n$ and $b_1 \geqslant \cdots \geqslant b_n$. This inequality can be generalized to

$$\sum_{i=1}^{n} w_i a_i b_i \geqslant \left(\sum_{i=1}^{n} w_i a_i\right)\left(\sum_{i=1}^{n} w_i b_i\right) \tag{1.9}$$

where $w_i \geqslant 0$ for all $1 = 1, \ldots, n$. This inequality was generalized by Matharu and Aujla [27] to the case of positive semidefinite involving Hadamard products: for any matrices $A_1 \geqslant \cdots \geqslant A_n \geqslant 0$ and $B_1 \geqslant \cdots \geqslant B_n \geqslant 0$, and any nonnegative numbers $w_1, \ldots, w_n$, we have

$$\sum_{i=1}^{n} w_i (A_i \hat{\odot} B_i) \geqslant \left(\sum_{i=1}^{n} w_i A_i\right) \hat{\odot} \left(\sum_{i=1}^{n} w_i B_i\right). \tag{1.10}$$

A continuous version of the Chebyshev sum inequality (1.9) says that if $f, g : [a, b] \to \mathbb{R}$ are synchronous on $[a, b]$,

$$(f(x) - f(y))(g(x) - g(y)) \geqslant 0 \tag{1.11}$$

for all $x, y \in [a, b]$, then

$$\int_a^b \alpha(x)dx \int_a^b \alpha(x)f(x)g(x)dx \geqslant \int_a^b \alpha(x)f(x)dx \cdot \int_a^b \alpha(x)g(x)dx, \tag{1.12}$$

where $\alpha : [a, b] \to [0, \infty)$ is an integrable function (see [29]). In [31], Moslehian and Bakherad extended this inequality to the case of continuous fields of Hilbert-space operators with a bounded measurable function involving Hadamard products by using the notion of synchronous Hadamard property. They proved that if two continuous fields $(A_t)_{t\in\Omega}, (B_t)_{t\in\Omega}$ of operators, parametrized by a compact Hausdorff space Ω equipped with a Radon measure μ, have the synchronous Hadamard property, then

$$\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \odot B_t)d\mu(t) \geqslant \int_\Omega \alpha(t)A_t d\mu(t) \odot \int_\Omega \alpha(s)B_s d\mu(s), \tag{1.13}$$

where $\alpha : \Omega \to [0, \infty)$ is a bounded measurable function. Moreover, they presented Chebyshev-type inequalities regarding operator means.

A complement of (1.12) was introduced by Grüss [18], providing an estimate of the difference between the integral of the product and the product of the integrals for two functions. For each integrable function $f : [a, b] \to \mathbb{R}$, let us denote

$$\mathcal{I}(f) = \frac{1}{b-a}\int_a^b f(x)dx, \qquad T_{\mathcal{I}}(f,g) = \mathcal{I}(f \cdot g) - \mathcal{I}(f) \cdot \mathcal{I}(g).$$

For any integrable functions $f, g : [a, b] \to \mathbb{R}$ satisfying the conditions $k \leqslant f(x) \leqslant K$, $l \leqslant g(x) \leqslant L$ for all $x \in [a, b]$ and k, K, l, L are real constants, we have

$$|T_{\mathcal{I}}(f,g)| \leqslant \frac{1}{4}(K-k)(L-l). \tag{1.14}$$

This inequality is known as Grüss-type inequality. In [17], Gonska, Raşa and Rusu used the terminology Chebyshev-Grüss-type inequalities referring to Grss-type inequalities for (special cases of) generalized Chebyshev functionals $T_{\mathcal{I}}$ which have a general form

$$|T_{\mathcal{I}}(f,g)| \leqslant E(\mathcal{I}, f, g), \tag{1.15}$$

where E is an expression in terms of certain properties of $\mathcal{I}$ and some kind of oscillations of f and g. They also established new Chebyshev-Grüss inequalities via discrete oscillations.

In this thesis, we investigate concavity and convexity of certain maps related to operator-monotone functions, Tracy-Singh products and Khatri-Rao products of bounded linear operators on a Hilbert space. We also establish a number of inequalities for operators involving Tracy-Singh products and Khatri-Rao products. In particular, we obtain Chebyshev-type inequalities for operators dealing with Tracy-singh products and Khatri-Rao products. Our results generalize the results known so far in the literature for Tracy-Singh/Khatri-Rao/Kronecker/Hadamard products of matrices and tensor products of operators.

1.2 Objectives of the Study

1) To investigate the concavity and convexity of several maps involving Tracy-Singh products, Khatri-Rao products, and operator-monotone functions of positive operators

2) To investigate Chebyshev-type inequalities for operators involving Tracy-Singh products.

3) To investigate Chebyshev-type inequalities for operators involving Khatri--Rao products.

1.3 Scope of the Study

We consider bounded linear operators on a Hilbert space. All Hilbert spaces considered here are complex Hilbert spaces.

1.4 Benefits of the Study

1) To develop further theory for operators on a Hilbert space.
2) To obtain inequalities for operator on a Hilbert space.

1.5 Research Methodology

1) Study advanced topics in matrix theory.
2) Study background in functional analysis.
3) Collect and study research papers and textbooks concerning Tracy-Singh and Khatri-Rao products of matrices, and tensor product of operators.
4) Collect and study research papers and textbooks concerning inequalities for numbers, matrices and operators.
5) Investigate concavity and convexity of certain maps related to operator-monotone functions, Tracy-Singh products and Khatri-Rao products of operators.
6) Investigate inequalities of Chebyshev-type for operators involving Tracy-Singh products.
7) Investigate inequalities of Chebyshev-type for operators involving Khatri-Rao products.
8) Conclude the results, make suggestions for further works and write the thesis.

Table 1.1: The research schedule

Activity	Time frame									
	2016	2017			2018			2019		2020
	9-12	1-4	5-8	9-12	1-4	5-8	9-12	1-4	5-8	9-12	1-4
Step 1	⟷										
Step 2		⟷									
Step 3			⟷								
Step 4				⟷							
Step 5					⟷	⟷	⟷				
Step 6								⟷	⟷		
Step 7									⟷	⟷	
Step 8											⟷

Chapter 2

Preliminaries

The purpose of this chapter is to provide basic concepts and tools in topology, functional analysis and operator theory used in this research.

2.1 Preliminaries on Topology

In this section, we recall some definitions concerning topological spaces. See [6, 14, 33] for more details.

Definition 2.1. A **topology** τ on a set X is a collection of subsets of X satisfying:

(i) $\emptyset, X \in \tau$.

(ii) τ is closed under arbitrary unions.

(iii) τ is closed under finite intersections.

We call the pair (X, τ) a **topological space**. The elements of τ are called the **open sets** of X. We often omit specific mention of τ if no confusion will arise.

Definition 2.2. Let (X, τ) be a topological space and $x \in X$. A set $U \subseteq X$ is said to be a **neighborhood** of x if there is $G \in \tau$ such that $x \in G \subseteq U$. Denote by $\mathfrak{N}(x)$ the collection of all neighborhoods of x.

Definition 2.3. A topological space X is called **Hausdorff** if any two distinct points can be separated by disjoint neighborhoods of the points. That is, for each pair $x, y \in X$ with $x \neq y$, there exist $U \in \mathfrak{N}(x)$ and $V \in \mathfrak{N}(y)$ such that $U \cap V = \emptyset$.

Definition 2.4. A **cover** of a topological space X is a family $\mathcal{C}$ of subsets of X whose union is X. A **subcover** of a cover $\mathcal{C}$ is a subfamily of $\mathcal{C}$ which is a cover of X. An **open cover** of X is a cover consisting of open sets.

Definition 2.5. A topological space X is said to be **compact** if every open cover of X has a finite subcover. That is, X is compact if every family $\{G_i : i \in I\}$ of open sets satisfying $X = \bigcup_{i \in I} G_i$ has a finite subfamily $G_{i_1}, \dots, G_{i_n}$ such that $X = \bigcup_{j=1}^{n} G_{i_j}$.

Definition 2.6. A topological space X is **locally compact** if every point has a compact neighborhood in X.

2.2 Preliminaries on Measure Spaces

In this section, we recall some basic definitions on measure theory. See [2, 8] for more information.

Definition 2.7. A **σ-algebra** $\mathcal{A}$ of a set X is a collection of subsets of X satisfying:

(i) $\emptyset \in \mathcal{A}$.

(ii) If $E \in \mathcal{A}$, then $X - E \in \mathcal{A}$.

(iii) If $E_i \in \mathcal{A}$ for all $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a **measurable space** and elements of $\mathcal{A}$ are called **measurable sets**.

Definition 2.8. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a **measure** if:

(i) $\mu(\emptyset) = 0$;

(ii) if E_i is a countable collection of pairwise disjoint sets in $\mathcal{A}$, then

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i).$$

If $\mu(X) < \infty$, then we call μ a finite measure. We call $(X, \mathcal{A}, \mu)$ a **measure space**.

Definition 2.9. Let X be a topological space. The **Borel σ-algebra** of X is the σ-algebra generated by the family of all open sets in X. The Borel σ-algebra of X will be denoted by $\mathcal{B}_X$. Elements of $\mathcal{B}_X$ are called **Borel sets**.

Definition 2.10. Let X be a Hausdorff space. A measure $\mu : \mathcal{B}_X \to [0, \infty]$ is called a **Radon measure** if it satisfies the followings:

(i) $\mu(K) < \infty$ for every compact set K.

(ii) If E is a borel set of X, then

$$\mu(E) = \inf\{\mu(G) : G \supseteq E,\ G \text{ open}\}.$$

(iii) If E is an open set of X, then

$$\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}.$$

2.3 Operators on a Hilbert Space

In this section, we provide prerequisites about operators on a Hilbert space. More details can be found in [22, 43].

From now on, $\mathbb{F}$ stands for the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For two operators S and T, we write ST for the composition $S \circ T$ (if it exists).

Definition 2.11. A **normed space** is a vector space $\mathbb{X}$ over a field $\mathbb{F}$ together with a mapping $\|\cdot\| : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$, called a **norm**, satisfying the following conditions:

(i) $\|x\| \geqslant 0$ for all $x \in \mathbb{X}$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in \mathbb{X}$ and $\alpha \in \mathbb{F}$.

(iii) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in \mathbb{X}$.

Definition 2.12. Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces, and let $T : \mathbb{X} \to \mathbb{Y}$ be a linear operator. Then T is said to be a **bounded linear operator** if there exists a constant M such that

$$\|T(x)\| \leqslant M\|x\| \quad \text{for all } x \in \mathbb{X}. \tag{2.1}$$

From now on, we will write Tx instead of $T(x)$.

Definition 2.13. Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces. Let $T : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator. Then the **operator norm** of T is defined as

$$\|T\| = \inf\{M \geqslant 0 : \|Tx\| \leqslant M\|x\|, \ x \in \mathbb{X}\}. \tag{2.2}$$

Definition 2.14. An **inner product space** is a vector space $\mathbb{X}$ over a field $\mathbb{F}$ together with a mapping $\langle \cdot, \cdot \rangle : \mathbb{X} \times \mathbb{X} \to \mathbb{F}$, called an **inner product**, satisfying the following conditions:

(i) $\langle x, x \rangle \geqslant 0$ for all $x \in \mathbb{X}$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

(ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in \mathbb{X}$.

(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in \mathbb{X}$ and $\alpha \in \mathbb{F}$.

(iv) $\overline{\langle x, y \rangle} = \langle y, x \rangle$ for all $x, y \in \mathbb{X}$.

Definition 2.15. A **Hilbert space** is a complete inner product space.

When $\mathbb{H}$ and $\mathbb{K}$ are Hilbert spaces, denote by $\mathfrak{B}(\mathbb{H},\mathbb{K})$ the set of all bounded linear operator from $\mathbb{H}$ into $\mathbb{K}$ and abbreviate $\mathfrak{B}(\mathbb{H},\mathbb{H})$ to $\mathfrak{B}(\mathbb{H})$.

Definition 2.16. Let $T \in \mathfrak{B}(\mathbb{H},\mathbb{K})$. The **adjoint** of T is the unique linear operator $T^* \in \mathfrak{B}(\mathbb{K},\mathbb{H})$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x \in \mathbb{H}$ and $y \in \mathbb{K}$.

Definition 2.17. Let $\mathbb{H}$ be a complex Hilbert space. Then $T \in \mathfrak{B}(\mathbb{H})$ is called

- **self-adjoint** if $T^* = T$,
- **projection** if $T^2 = T = T^*$,
- **isometry** if $T^*T = I$.

Definition 2.18. Let T be a self-adjoint operator on a complex Hilbert $\mathbb{H}$ into itself. Then T is said to be **positive**, denoted by $T \geqslant 0$, if

$$\langle Tx, x\rangle \geqslant 0 \text{ for all } x \in \mathbb{H}.$$

We denote $T > 0$ if T is positive and invertible. The set of all positive operators on $\mathbb{H}$ is denoted by $\mathfrak{B}(\mathbb{H})^+$. We denote $\mathfrak{B}(\mathbb{H})^{++}$ the set of all positive invertible operators on $\mathbb{H}$.

Definition 2.19. Let S and T be two self-adjoint operators on a complex Hilbert $\mathbb{H}$. We write $S \geqslant T$ if $S - T$ is positive and write $S > T$ if $S - T$ is positive and invertible.

Definition 2.20. Let $T \in \mathfrak{B}(\mathbb{H})$ be a positive operator on a complex Hilbert space $\mathbb{H}$. A positive operator $S \in \mathfrak{B}(\mathbb{H})$ such that $S^2 = T$ is called a **positive square root** of T, denoted by $T^{1/2}$.

2.4 Functions of Operators

In this section, we explain how to define functions of operators. A more detailed description can be found in [10, 22, 42].

Throughout this section, a Hilbert space $\mathbb{H}$ means complex Hilbert space.

Definition 2.21. Let $T \in \mathfrak{B}(\mathbb{H})$. The **spectrum** Sp$(T)$ of T is the collection of complex numbers λ such that $T-\lambda I$ is not invertible.

Theorem 2.22. Let $T \in \mathfrak{B}(\mathbb{H})$ be a self-adjoint operator. Then the spectrum Sp(T) lies in the closed interval $[m, M]$ where

$$m = \inf_{\|x\|=1} \langle Tx, x\rangle, \qquad M = \sup_{\|x\|=1} \langle Tx, x\rangle. \tag{2.3}$$

Theorem 2.23 (Spectral family). Let $T \in \mathfrak{B}(\mathbb{H})$ be a self-adjoint operator. Set m and M as in Theorem 2.22. Then there is a family $\{E_\lambda\}$ of projection operators on $\mathbb{H}$ depending on a real parameter λ and such that

(i) $E_{\lambda_1} \leqslant E_{\lambda_2}$ for $\lambda_1 \leqslant \lambda_2$,

(ii) $E_\lambda x \to E_{\lambda_0} x$ as $\lambda_0 < \lambda \to \lambda_0,\ x \in \mathbb{H}$,

(iii) $E_\lambda = 0$ for $\lambda < m,\ E_\lambda = I$ for $\lambda \geqslant M$,

(iv) $TE_\lambda = E_\lambda T$.

The family $\{E_\lambda\}$ is called the **spectral family** associated with the operator T.

Theorem 2.24 (Spectral representation). Let $T \in \mathfrak{B}(\mathbb{H})$ be a self-adjoint operator. Set m and M where m and M as in Theorem 2.22. Then T has the spectral representation

$$T = \int_{m-0}^{M} \lambda \, dE_\lambda, \tag{2.4}$$

where $\{E_\lambda\}$ is the spectral family associated with T. More generally, if p is polynomial in λ with real coefficients, then

$$p[T] = \int_{m-0}^{M} p(\lambda)\, dE_\lambda. \tag{2.5}$$

Remark 2.25. Notation $m-0$ indicates that one must take into account a contribution at $\lambda = m$ which occurs if $E_m \neq 0$ (and $m \neq 0$). We can write

$$\int_{m-0}^{M} \lambda \, dE_\lambda = mE_m + \int_{m}^{M} \lambda \, dE_\lambda.$$

Similarly,

$$\int_{m-0}^{M} p(\lambda)\, dE_\lambda = p(m)E_m + \int_{m}^{M} p(\lambda)\, dE_\lambda.$$

Theorem 2.26 (Spectral Theorem)**.** Let $T \in \mathfrak{B}(\mathbb{H})$ be a self-adjoint operator. Let f be a continuous real-valued function on $[m, M]$ where m and M defined as in Theorem 2.22. Then $f[T]$ has the spectral representation

$$f[T] \; = \; \int_{m-0}^{M} f(\lambda)\, dE_\lambda, \tag{2.6}$$

where $\{E_\lambda\}$ is the spectral family associated with T.

Definition 2.27. A function $f : (0, \infty) \to (0, \infty)$ is said to be **operator-monotone** if $f[S] \geqslant f[T]$ whenever $S \geqslant T > 0$.

Example 2.28. The followings are operator-monotone functions on $(0, \infty)$:

(i) $t \mapsto \alpha t + \beta$, where $\alpha > 0$ and $\beta \geqslant 0$;

(ii) $t \mapsto -t^{-1}$;

(iii) $t \mapsto \log(t)$;

(iv) $t \mapsto t^\alpha$, where $\alpha \in [0, 1]$.

Theorem 2.29 ([11])**.** Let $f : (0, \infty) \to (0, \infty)$ be an operator-monotone function. Then there is a finite Borel measure μ on $[0, 1]$ such that

$$f(x) \; = \; \int_0^1 1 \,!_t\, x \; d\mu(t), \qquad x > 0. \tag{2.7}$$

Here, $!_t$ is the t-weighted harmonic mean defined by

$$a \,!_t\, b \; = \; \left[(1-t)a^{-1} + tb^{-1}\right]^{-1}, \qquad a, b > 0.$$

Definition 2.30. Let $\mathbb{H}_1, \ldots, \mathbb{H}_k, \mathbb{K}$ be Hilbert spaces. For each $i = 1, \ldots, k$, let E_i be a convex subset of $\mathfrak{B}(\mathbb{H}_i)$. A function $\phi : E_1 \times \cdots \times E_k \to \mathfrak{B}(\mathbb{K})$ is said to be **concave** if

$$\phi((1-\alpha)S_1 + \alpha T_1, \ldots, (1-\alpha)S_k + \alpha T_k) \; \geqslant \; (1-\alpha)\phi(S_1, \ldots, S_k) + \alpha\phi(T_1, \ldots, T_k)$$

for any $S_i, T_i \in E_i$ $(i = 1, \ldots, k)$ and $\alpha \in (0, 1)$. A function ϕ is **convex** if $-\phi$ is concave. A function ϕ is **affine** if it is both concave and convex.

Example 2.31. (i) The map $X \mapsto X^\alpha$, where $0 < \alpha < 1$, is concave on $\mathfrak{B}(\mathbb{H})^+$.

(ii) The map $X \mapsto \log[X]$ is concave on $\mathfrak{B}(\mathbb{H})^{++}$.

(iii) The map $X \mapsto X^{-1}$ is convex on $\mathfrak{B}(\mathbb{H})^{++}$.

(iv) The map $(X,Y) \mapsto YX^{-1}Y$ is convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{H})^{++}$.

(v) If $\Phi : \mathfrak{B}(\mathbb{H})^{++} \to \mathfrak{B}(\mathbb{K})^{++}$ is concave and $\Psi : \mathfrak{B}(\mathbb{H})^{++} \to \mathfrak{B}(\mathbb{K})^{++}$ is affine, then the map $X \mapsto \Psi(X)\Phi(X)^{-1}\Psi(X)$ is convex on $\mathfrak{B}(\mathbb{H})^{++}$.

2.5 Operator Matrices

In this section, we recall standard results on operator matrices which can be found, e.g., in [13, 19, 45].

For any vector space $\mathbb{V}$, we say that $\mathbb{V}$ is the direct sum of subspaces $\mathbb{V}_1, \ldots, \mathbb{V}_n$ and we write $\mathbb{V} = \bigoplus_{i=1}^n \mathbb{V}_i$, if for every $v \in \mathbb{V}$, there exist unique vectors $v_i \in \mathbb{V}_i$ for $i = 1, \ldots, n$ such that $v = v_1 + \cdots + v_n$.

For Hilbert spaces $\mathbb{H}_1, \ldots, \mathbb{H}_n$, consider the direct sum Hilbert space

$$\bigoplus_{i=1}^{n} \mathbb{H}_i = \{x_1 \oplus \ldots \oplus x_n \mid x_i \in \mathbb{H}_i, i = 1, \ldots, n\}$$

equipped with the inner product

$$\langle x_1 \oplus \ldots \oplus x_n, y_1 \oplus \ldots \oplus y_n \rangle = \langle x_1, y_1 \rangle + \ldots + \langle x_n, y_n \rangle,$$

where $x_i, y_i \in \mathbb{H}_i$ for $i = 1, \ldots, n$ with addition and multiplication defined (in usual way) as follows:

$$(x_1 \oplus \ldots \oplus x_n) + (y_1 \oplus \ldots \oplus y_n) = (x_1 + y_1) \oplus \ldots \oplus (x_n + y_n),$$
$$\alpha (x_1 \oplus \ldots \oplus x_n) = (\alpha x_1) \oplus \ldots \oplus (\alpha x_n).$$

Let $\mathbb{H}_1, \ldots, \mathbb{H}_n$ and $\mathbb{K}_1, \ldots, \mathbb{K}_m$ be complex Hilbert spaces. Denote

$$\mathbb{H} = \bigoplus_{j=1}^{n} \mathbb{H}_j, \quad \mathbb{K} = \bigoplus_{i=1}^{m} \mathbb{K}_i.$$

For each $i = 1, \ldots, m$, let P_i be the natural projection from $\mathbb{K}$ onto $\mathbb{K}_i$, defined by

$$P_i(x_1 \oplus \ldots \oplus x_i \oplus \ldots, x_m) = x_i.$$

For each $j = 1, \ldots, n$, let E_j be the canonical embedding from $\mathbb{H}_j$ into $\mathbb{H}$, defined by

$$E_j(x_j) = 0 \oplus \ldots \oplus 0 \oplus x_j \oplus 0 \oplus \ldots \oplus 0.$$

An operator $A \in \mathfrak{B}(\mathbb{H}, \mathbb{K})$ can be represented uniquely as a block operator matrix

$$A = [A_{ij}]_{i,j=1}^{m,n} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix},$$

where $A_{ij} = P_i A E_j \in \mathfrak{B}(\mathbb{H}_j, \mathbb{K}_i)$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$. An operator $A \in \mathfrak{B}(\mathbb{H}, \mathbb{K})$ acts on $\mathbb{H}$ in the form

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + \ldots + A_{1n}x_n \\ \vdots \\ A_{m1}x_1 + \ldots + A_{mn}x_n \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{H}_1 \oplus \ldots \oplus \mathbb{H}_n.$$

2.6 Operator Means

In this section, we introduce the notions of operator mean in the sense of Kubo and Ando [23]. See [5, 20, 23] for more details.

Definition 2.32. An **operator mean** is a binary operation $\sigma : \mathfrak{B}(\mathbb{H})^+ \times \mathfrak{B}(\mathbb{H})^+ \to \mathfrak{B}(\mathbb{H})^+$ which satisfying the following axioms:

(i) Monotonicity: $A_1 \leqslant A_2$ and $B_1 \leqslant B_2$ imply $A_1 \sigma B_1 \leqslant A_2 \sigma B_2$.

(ii) Continuity from above: $A_n \downarrow A$ and $B_n \downarrow B$ imply $(A_n \sigma B_n) \downarrow (A \sigma B)$ where $A_n \downarrow A$ means that A_n is a decreasing sequence and $A_n \to A$ as $n \to \infty$.

(iii) Transformer inequality: $C^*(A\sigma B)C \leqslant (C^*AC)\sigma(C^*BC)$ for every $C \in \mathfrak{B}(\mathbb{H})^+$.

(iv) Normalization condition: $A \sigma A = A$.

Example 2.33. Let $t \in [0, 1]$.

(i) The ***t*-weighted arithmetic mean** of $A, B \in \mathfrak{B}(\mathbb{H})$ is defined by

$$A \nabla_t B = (1-t)A + tB. \tag{2.8}$$

(ii) The ***t*-weighted geometric mean** of $A, B \in \mathfrak{B}(\mathbb{H})^{++}$ is defined by

$$A \sharp_t B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}}. \tag{2.9}$$

For arbitrary $A, B \in \mathfrak{B}(\mathbb{H})^+$, we define the t-weighted geometric mean of A and B to be

$$A \sharp_t B = \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \sharp_t (B + \varepsilon I). \tag{2.10}$$

(iii) The ***t*-weighted harmonic mean** of $A, B \in \mathfrak{B}(\mathbb{H})^{++}$ is defined by

$$A \,!_t\, B = \left[(1-t)A^{-1} + tB^{-1} \right]^{-1}. \tag{2.11}$$

For any $A, B \in \mathfrak{B}(\mathbb{H})^+$, we define

$$A \,!_t\, B = \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \,!_t\, (B + \varepsilon I). \tag{2.12}$$

Theorem 2.34 (see e.g. [23]). The weighted geometric means, weighted arithmetic means and weighted harmonic means for operators are monotone in the sense that if $A_1 \leqslant A_2$ and $B_1 \leqslant B_2$, then $A_1 \sigma B_1 \leqslant A_2 \sigma B_2$ where σ is any of $\triangledown_t, !_t, \sharp_t$.

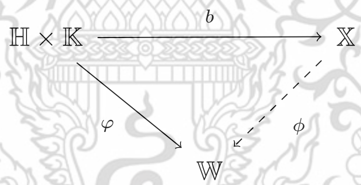
Theorem 2.35. For any $A, B \in \mathfrak{B}(\mathbb{H})^{++}$, we have $(A \sharp_t B)^{-1} = A^{-1} \sharp_t B^{-1}$.

Theorem 2.36. For each $t \in [0,1]$, the map $(A,B) \mapsto A \,!_t\, B$ is concave on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{H})^{++}$.

2.7 Tensor Product of Hilbert Spaces

In this section, we review basic knowledge about tensor product of Hilbert spaces. See [24, 41] for more information.

Definition 2.37. Let $\mathbb{H}$ and $\mathbb{K}$ be Hilbert spaces. Then a **tensor product** of $\mathbb{H}$ and $\mathbb{K}$ is a Hilbert space $\mathbb{X}$ together with a bounded bilinear map $b : \mathbb{H} \times \mathbb{K} \to \mathbb{X}$ such that for any Hilbert space $\mathbb{W}$ and a bounded bilinear map $\varphi : \mathbb{H} \times \mathbb{K} \to \mathbb{W}$, there exists a unique linear map $\phi : \mathbb{X} \to \mathbb{W}$ such that $\phi \circ b = \varphi$.



Theorem 2.38. Let $\mathbb{H}$ and $\mathbb{K}$ be Hilbert spaces. Then $\mathbb{H}$ and $\mathbb{K}$ has a tensor product.

If $(\mathbb{X}, b)$ is a tensor product of $\mathbb{H}$ and $\mathbb{K}$, it is customary to write $x \otimes y$ in place of $b(x,y)$ and $\mathbb{H} \otimes \mathbb{K}$ in place of $\mathbb{X}$.

Theorem 2.39. Let $A \in \mathfrak{B}(\mathbb{H}, \mathbb{H}')$ and $B \in \mathfrak{B}(\mathbb{K}, \mathbb{K}')$. Then there exists a unique bounded linear operator $\psi : \mathbb{H} \otimes \mathbb{K} \to \mathbb{H}' \otimes \mathbb{K}'$ such that for any $x \in \mathbb{H}$ and $y \in \mathbb{K}$,

$$\psi(x \otimes y) \;=\; Ax \otimes By. \tag{2.13}$$

The unique linear map ψ is said to be the tensor product of A and B denoted by $A \otimes B$.

2.8 Tracy-Singh Product of Operators

In this section, we give the definitions and some properties including algebraic,order and analytic properties of Tracy-Singh product for block operator matrices on a Hilbert space.

We decompose the complex Hilbert spaces $\mathbb{H}, \mathbb{H}', \mathbb{K}$ and $\mathbb{K}'$ as direct sums of certain Hilbert spaces as follows:

$$\mathbb{H} = \bigoplus_{j=1}^{n} \mathbb{H}_j, \quad \mathbb{H}' = \bigoplus_{i=1}^{m} \mathbb{H}'_i, \quad \mathbb{K} = \bigoplus_{l=1}^{q} \mathbb{K}_l, \quad \mathbb{K}' = \bigoplus_{k=1}^{p} \mathbb{K}'_k.$$

Each operator $A \in \mathfrak{B}(\mathbb{H}, \mathbb{H}')$ can be expressed uniquely as a block operator matrix

$$A = [A_{ij}]_{i,j=1}^{m,n} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix},$$

where $A_{ij} \in \mathfrak{B}(\mathbb{H}_j, \mathbb{H}'_i)$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Similarly, an operator $B \in \mathfrak{B}(\mathbb{K}, \mathbb{K}')$ can be represented uniquely as a block operator matrix

$$B = [B_{kl}]_{k,l=1}^{p,q} = \begin{bmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & \ddots & \vdots \\ B_{p1} & \cdots & B_{pq} \end{bmatrix},$$

where $B_{kl} \in \mathfrak{B}(\mathbb{K}_l, \mathbb{K}'_k)$ for each $k = 1, \ldots, p$ and $l = 1, \ldots, q$.

Definition 2.40. Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathfrak{B}(\mathbb{H}, \mathbb{H}')$ and $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathfrak{B}(\mathbb{K}, \mathbb{K}')$. The **Tracy-Singh product** of A and B, denoted as $A \boxtimes B$, is defined to be

$$A \boxtimes B = \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij}, \tag{2.14}$$

which is a bounded linear operator from $\bigoplus_{j,l=1}^{n,q} \mathbb{H}_j \otimes \mathbb{K}_l$ into $\bigoplus_{i,k=1}^{m,p} \mathbb{H}'_i \otimes \mathbb{K}'_k$.

Remark 2.41. If both A and B are 1×1 block operator matrices, their Tracy-Singh product $A \boxtimes B$ is the tensor product $A \otimes B$.

Example 2.42. Consider complex Hilbert spaces

$$\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2 \oplus \mathbb{H}_3, \quad \mathbb{H}' = \mathbb{H}'_1 \oplus \mathbb{H}'_2, \quad \mathbb{K} = \mathbb{K}_1 \oplus \mathbb{K}_2, \quad \mathbb{K}' = \mathbb{K}'_1 \oplus \mathbb{K}'_2.$$

Let $A \in \mathfrak{B}(\mathbb{H}, \mathbb{H}')$ and $B \in \mathfrak{B}(\mathbb{K}, \mathbb{K}')$. Then A and B can be written as block operator matrices

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $A_{ij} \in \mathfrak{B}(\mathbb{H}_j, \mathbb{H}'_i)$ for each $i = 1, 2,\ j = 1, 2, 3$ and $B_{kl} \in \mathfrak{B}(\mathbb{K}_l, \mathbb{K}'_k)$ for each $k, l = 1, 2$. The Tracy-Singh product of A and B is

$$A \boxtimes B \ = \ \begin{bmatrix} A_{11} \otimes B_{11} & A_{11} \otimes B_{12} & A_{12} \otimes B_{11} & A_{12} \otimes B_{12} & A_{13} \otimes B_{11} & A_{13} \otimes B_{12} \\ A_{11} \otimes B_{21} & A_{11} \otimes B_{22} & A_{12} \otimes B_{21} & A_{22} \otimes B_{21} & A_{21} \otimes B_{22} & A_{13} \otimes B_{22} \\ A_{21} \otimes B_{11} & A_{21} \otimes B_{12} & A_{22} \otimes B_{11} & A_{22} \otimes B_{12} & A_{23} \otimes B_{11} & A_{23} \otimes B_{12} \\ A_{21} \otimes B_{21} & A_{21} \otimes B_{22} & A_{22} \otimes B_{21} & A_{22} \otimes B_{22} & A_{23} \otimes B_{21} & A_{23} \otimes B_{22} \end{bmatrix},$$

where $A_{ij} \otimes B_{kl} \in \mathfrak{B}(\mathbb{H}_j \otimes \mathbb{K}_l, \mathbb{H}'_i \otimes \mathbb{K}'_k)$.

Theorem 2.43 ([34, 35]). The Tracy-Singh product of operators satisfies the following properties (provide that each term is well-defined):

(i) $\alpha(A \boxtimes B) = (\alpha A) \boxtimes B = A \boxtimes (\alpha B)$ for any $\alpha \in \mathbb{C}$.

(ii) $A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C$.

(iii) $(B + C) \boxtimes A = B \boxtimes A + C \boxtimes A$.

(iv) $(A \boxtimes B)^* = A^* \boxtimes B^*$.

(v) $(A \boxtimes B)(C \boxtimes D) = (AC) \boxtimes (BD)$.

(vi) If A and B are invertible, then $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$.

(vii) If $A \geqslant 0$ and $B \geqslant 0$, then $A \boxtimes B \geqslant 0$.

(viii) If $A \geqslant C \geqslant 0$ and $B \geqslant D \geqslant 0$, then $A \boxtimes B \ \geqslant \ C \boxtimes D$.

(ix) If $A \geqslant 0$ and $B \geqslant 0$, then $(A \boxtimes B)^{\alpha} = A^{\alpha} \boxtimes B^{\alpha}$ for any positive real α.

Theorem 2.44 ([35]). The Tracy-Singh product of operators is jointly continuous.

Theorem 2.45 ([35]). Let $A \in \mathfrak{B}(\mathbb{H})$. If f is an analytic function on a region containing the spectra of A, then $f[A \boxtimes I] = f[A] \boxtimes I$ and $f[I \boxtimes A] = I \boxtimes f[A]$.

2.9 Khatri-Rao Products of Operators

Fix the following orthogonal decompositions of complex Hilbert spaces:

$$\mathbb{H} \ = \ \bigoplus_{j=1}^{n} \mathbb{H}_j, \quad \mathbb{H}' \ = \ \bigoplus_{i=1}^{m} \mathbb{H}'_i, \quad \mathbb{K} \ = \ \bigoplus_{j=1}^{n} \mathbb{K}_j, \quad \mathbb{K}' \ = \ \bigoplus_{i=1}^{m} \mathbb{K}'_i. \tag{2.15}$$

That is, we fix how to partition any operator matrix in $\mathfrak{B}(\mathbb{H}, \mathbb{H}')$ and $\mathfrak{B}(\mathbb{K}, \mathbb{K}')$.

Definition 2.46. Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathfrak{B}(\mathbb{H}, \mathbb{H}')$ and $B = [B_{ij}]_{i,j=1}^{m,n} \in \mathfrak{B}(\mathbb{K}, \mathbb{K}')$ be operators partitioned into matrices according to the decomposition (2.15). We define the **Khatri-Rao product** of A and B to be

$$A \boxdot B \;=\; [A_{ij} \otimes B_{ij}]_{i,j=1}^{m,n} . \tag{2.16}$$

which is a bounded linear operator from $\bigoplus_{j=1}^{n} \mathbb{H}_j \otimes \mathbb{K}_j$ into $\bigoplus_{i=1}^{m} \mathbb{H}'_i \otimes \mathbb{K}'_i$.

Remark 2.47. If both A and B are 1×1 block operator matrices, then $A \boxdot B$ is $A \otimes B$. When $\mathbb{H}_i = \mathbb{K}_i = \mathbb{C}$ and $\mathbb{H}'_j = \mathbb{K}'_j = \mathbb{C}$ for all i, j, the Khatri-Rao product is the Hadamard product of complex matrices.

Example 2.48. Consider complex Hilbert spaces

$$\mathbb{H} \;=\; \mathbb{H}_1 \oplus \mathbb{H}_2, \qquad \mathbb{H}' \;=\; \mathbb{H}'_1 \oplus \mathbb{H}'_2, \qquad \mathbb{K} \;=\; \mathbb{K}_1 \oplus \mathbb{K}_2, \qquad \mathbb{K}' \;=\; \mathbb{K}'_1 \oplus \mathbb{K}'_2.$$

Let $A \in \mathfrak{B}(\mathbb{H}, \mathbb{H}')$ and $B \in \mathfrak{B}(\mathbb{K}, \mathbb{K}')$. Then A and B can be written as block operator matrices

$$A \;=\; \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B \;=\; \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $A_{ij} \in \mathfrak{B}(\mathbb{H}_j, \mathbb{H}'_i)$ and $B_{ij} \in \mathfrak{B}(\mathbb{K}_j, \mathbb{K}'_i)$ for each $i, j = 1, 2$. The Khatri-Rao product of A and B is

$$A \boxdot B \;=\; \begin{bmatrix} A_{11} \otimes B_{11} & A_{12} \otimes B_{12} \\ A_{21} \otimes B_{21} & A_{22} \otimes B_{22} \end{bmatrix},$$

where $A_{ij} \otimes B_{ij} \in \mathfrak{B}(\mathbb{H}_j \otimes \mathbb{K}_j, \mathbb{H}'_i \otimes \mathbb{K}'_i)$.

Theorem 2.49 ([36])**.** The Khatri-Rao product for operators satisfies the following properties (provided that each term is well-defined):

(i) $(\alpha A) \boxdot B = \alpha(A \boxdot B) = A \boxdot (\alpha B)$ for any $\alpha \in \mathbb{C}$.

(ii) $A \boxdot (B + C) = A \boxdot B + A \boxdot C$.

(iii) $(B + C) \boxdot A = B \boxdot A + C \boxdot A$.

(iv) $(A \boxdot B)^* = A^* \boxdot B^*$.

(v) If $A \geqslant 0$ and $B \geqslant 0$, then $A \boxdot B \geqslant 0$.

(vi) If $A \geqslant C \geqslant 0$ and $B \geqslant D \geqslant 0$, then $A \boxdot B \geqslant C \boxdot D$.

Theorem 2.50 ([39])**.** The Khatri-Rao product of operators is jointly continuous.

Theorem 2.51 ([36])**.** There are isometries Z_1 and Z_2 such that

$$A \boxdot B \;=\; Z_1^*(A \boxtimes B)Z_2, \tag{2.17}$$

for all $A \in \mathfrak{B}(\mathbb{H}, \mathbb{H}')$ and $B \in \mathfrak{B}(\mathbb{K}, \mathbb{K}')$. For the case $\mathbb{H} = \mathbb{H}'$ and $\mathbb{K} = \mathbb{K}'$, we have $Z_1 = Z_2 := Z$.

Definition 2.52. A linear map $\Phi : \mathfrak{B}(\mathbb{H}) \to \mathfrak{B}(\mathbb{K})$ is said to be **positive** if $\Phi(A) \geqslant 0$ whenever $A \geqslant 0$. It is said to be **unital** if $\Phi(I) = I$.

Theorem 2.53 ([36])**.** There is a unital positive linear map

$$\Phi \;:\; \mathfrak{B}\left(\bigoplus_{i,j=1}^{n,n} \mathbb{H}_i \otimes \mathbb{K}_j\right) \;\to\; \mathfrak{B}\left(\bigoplus_{i=1}^{n} \mathbb{H}_i \otimes \mathbb{K}_i\right)$$

such that $\Phi(A \boxtimes B) = A \boxdot B$ for any $A \in \mathfrak{B}(\mathbb{H})$ and $B \in \mathfrak{B}(\mathbb{K})$.

2.10 Bochner Integration

In this section, we recall basic definitions and results related to the Bochner integral. See [1] for more details.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\mathbb{X}$ a vector space over $\mathbb{C}$. A function $\varphi : \Omega \to \mathbb{X}$ that assumes only a finite number of values, say $x_1, \ldots, x_k$, is called a **$\mathbb{X}$-simple function** if $E_i = \varphi^{-1}(\{x_i\}) \in \mathcal{A}$ for each i. The formula

$$\varphi \;=\; \sum_{i=1}^{k} \chi_{E_i} x_i \tag{2.18}$$

is called the standard representation of φ. If $\mu(E_i) < \infty$ for each nonzero x_i, then φ is called an **$\mathbb{X}$-step function**. The integral of an $\mathbb{X}$-valued step function φ is the vector

$$\int_\Omega \varphi \, d\mu \;=\; \sum_{i=1}^{k} \mu(E_i) x_i. \tag{2.19}$$

Definition 2.54. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\mathbb{X}$ a Banach space. Let $f : \Omega \to \mathbb{X}$ be a vector function. We define the **norm function** of f to be the real valued function $\|f\| : \Omega \to \mathbb{R}$ such that $\|f\|(t) = \|f(t)\|$ for every $t \in \Omega$.

Definition 2.55. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\mathbb{X}$ a Banach space. We say that $f : \Omega \to \mathbb{X}$ is **strongly μ-measurable** if there exists a sequence $(\varphi_n)_{n=1}^{\infty}$ of $\mathbb{X}$-simple functions such that

$$\lim_{n\to\infty} \|f(t) - \varphi_n(t)\| \;=\; 0 \tag{2.20}$$

for μ-almost all $t \in \Omega$.

Definition 2.56. A strongly μ-measurable function $f : \Omega \to \mathbb{X}$ is **Bochner integrable** if there exists a sequence of $\mathbb{X}$-step functions $(\varphi_n)_{n=1}^{\infty}$ such that the real measurable function $\|f - \varphi_n\|$ is Lebesgue integrable for each n and

$$\lim_{n\to\infty} \int_{\Omega} \|f - \varphi_n\| \, d\mu \;=\; 0. \tag{2.21}$$

In this case, the Bochner integral of f over Ω is defined by

$$\int_{\Omega} f \, d\mu \;=\; \lim_{n\to\infty} \int_{\Omega} \varphi_n \, d\mu. \tag{2.22}$$

Theorem 2.57. Let $f : \Omega \to \mathbb{X}$ be Bochner integrable and let $\mathbb{Y}$ be another Banach space. If $T : \mathbb{X} \to \mathbb{Y}$ is a bounded operator, then the function $Tf : \Omega \to \mathbb{Y}$ is also Bochner integrable and

$$\int_{\Omega} Tf \, d\mu \;=\; T\Big(\int_{\Omega} f \, d\mu\Big). \tag{2.23}$$

Theorem 2.58. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and $\mathbb{X}$ a Banach space. Then a measurable function $f : \Omega \to \mathbb{X}$ is Bochner integrable if and only if its norm function $\|f\|$ is Lebesgue integrable.

Chapter 3

Concavity and Convexity of Several Maps Involving Tracy-Singh Products, Khatri-Rao Products, and Operator-Monotone Functions

In this chapter, we investigate concavity and convexity of certain maps related to Tracy-Singh products and Khatri-Rao products of oprators. The main tools we use are operator means and suitable integral representations of certain operator-monotone functions. Our results in this chapter generalize the results known so far for Tracy-Singh and Khatri-Rao products of matrices and tensor products of operators. Furthermore, we develop new concavity/convexity theorems.

Throughout this chapter, let $\mathbb{H}, \mathbb{H}', \mathbb{K}$ and $\mathbb{K}'$ be complex Hilbert spaces and decompose

$$\mathbb{H} = \bigoplus_{i=1}^{n} \mathbb{H}_i, \quad \mathbb{H}' = \bigoplus_{i=1}^{m} \mathbb{H}'_i, \quad \mathbb{K} = \bigoplus_{i=1}^{q} \mathbb{K}_i, \quad \mathbb{K}' = \bigoplus_{i=1}^{p} \mathbb{K}'_i.$$

3.1 Concavity and Convexity of Certain Maps Involving Operator-Monotone Functions

In this section, we provide concavity and convexity theorems related to Tracy-Singh products of operators and operator-monotone functions.

Theorem 3.1. Let $f : (0,\infty) \to (0,\infty)$ be an operator-monotone function. If $\phi_1 : \mathfrak{B}(\mathbb{H})^{++} \to \mathfrak{B}(\mathbb{H}')^{++}$ and $\phi_2 : \mathfrak{B}(\mathbb{K})^{++} \to \mathfrak{B}(\mathbb{K}')^{++}$ are concave maps, then the maps

$$(A,B) \mapsto f[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (I \boxtimes \phi_2(B)), \tag{3.1}$$

$$(A,B) \mapsto f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot (\phi_1(A) \boxtimes I) \tag{3.2}$$

are concave on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

Proof. Let $A \in \mathfrak{B}(\mathbb{H})^{++}$ and $B \in \mathfrak{B}(\mathbb{K})^{++}$. Since ϕ_1 and ϕ_2 are positive linear maps, we have that $\phi_1(A) > 0$ and $\phi_2(B) > 0$. Using Theorem 2.43, we have $\phi_1(A) \boxtimes \phi_2(B)^{-1} > 0$ and $\phi_1(A)^{-1} \boxtimes \phi_2(B) > 0$. Then $f[\phi_1(A) \boxtimes \phi_2(B)^{-1}]$ and $f[\phi_1(A)^{-1} \boxtimes \phi_2(B)]$ are well-defined operators. By Theorem 2.29, there is a finite Borel measure μ on $[0,1]$ such that

$$f(x) = \int_0^1 1 \,!_t\, x \; d\mu(t), \qquad x > 0.$$

Then

$$f\left[\phi_1(A)\boxtimes\phi_2(B)^{-1}\right]\cdot(I\boxtimes\phi_2(B))$$
$$= \left\{\int_0^1 (I\boxtimes I)\ !_t\ \left(\phi_1(A)\boxtimes\phi_2(B)^{-1}\right)\ d\mu(t)\right\}\cdot(I\boxtimes\phi_2(B))$$
$$= \int_0^1 \left\{(I\boxtimes I)\ !_t\ \left(\phi_1(A)\boxtimes\phi_2(B)^{-1}\right)\right\}(I\boxtimes\phi_2(B))\ d\mu(t).$$

For each $t\in[0,1]$, by Theorem 2.43, we get

$$\left\{(I\boxtimes I)\ !_t\ \left(\phi_1(A)\boxtimes\phi_2(B)^{-1}\right)\right\}\cdot(I\boxtimes\phi_2(B))$$
$$= \left\{(1-t)(I\boxtimes I)^{-1}+t\left(\phi_1(A)\boxtimes\phi_2(B)^{-1}\right)^{-1}\right\}^{-1}\cdot(I\boxtimes\phi_2(B))$$
$$= \left\{(1-t)(I\boxtimes I)+t\left(\phi_1(A)^{-1}\boxtimes\phi_2(B)\right)\right\}^{-1}\cdot(I\boxtimes\phi_2(B)^{-1})^{-1}$$
$$= \left[(I\boxtimes\phi_2(B)^{-1})\left\{(1-t)I\boxtimes I+t\phi_1(A)^{-1}\boxtimes\phi_2(B)\right\}\right]^{-1}$$
$$= \left\{(1-t)(I\boxtimes\phi_2(B)^{-1})(I\boxtimes I)+t(I\boxtimes\phi_2(B)^{-1})(\phi_1(A)^{-1})\boxtimes\phi_2(B)\right\}^{-1}$$
$$= \left\{(1-t)\left(I\boxtimes\phi_2(B)^{-1}\right)+t\left(\phi_1(A)^{-1}\boxtimes I\right)\right\}^{-1}$$
$$= \left\{(1-t)\left(I\boxtimes\phi_2(B)\right)^{-1}+t\left(\phi_1(A)\boxtimes I\right)^{-1}\right\}^{-1}$$
$$= (I\boxtimes\phi_2(B))\ !_t\ (\phi_1(A)\boxtimes I).$$

Since the weighted harmonic mean is concave (Theorem 2.36), so is the map

$$(A,B)\ \mapsto\ \left\{(I\boxtimes I)\ !_t\ \left(\phi_1(A)\boxtimes\phi_2(B)^{-1}\right)\right\}\cdot(I\boxtimes\phi_2(B)).$$

It is well-known that any nonnegative linear combination of concave maps is concave. As the integral is the limit of nononegative linear combinations, the map

$$(A,B)\ \mapsto\int_0^1 \left\{(I\boxtimes I)\ !_t\ \left(\phi_1(A)\boxtimes\phi_2(B)^{-1}\right)\right\}\cdot(I\boxtimes\phi_2(B))\ d\mu(t)$$

is concave. That is the map (3.1) is concave. Similarly, the map (3.2) is concave. □

Remark 3.2. Since $\phi_1(A)\boxtimes\phi_2(B)^{-1}$ commutes with $I\boxtimes\phi_2(B)$, we have

$$f[\phi_1(A)\boxtimes\phi_2(B)^{-1}]\cdot(I\boxtimes\phi_2(B))\ =\ (I\boxtimes\phi_2(B))\cdot f[\phi_1(A)\boxtimes\phi_2(B)^{-1}].$$

Similarly,

$$f[\phi_1(A)^{-1}\boxtimes\phi_2(A)]\cdot(\phi_1(A)\boxtimes I)\ =\ (\phi_1(A)\boxtimes I)\cdot f[\phi_1(A)^{-1}\boxtimes\phi_2(B)].$$

Remark 3.2 tells me that the following maps are concave on $\mathfrak{B}(\mathbb{H})^{++}\times\mathfrak{B}(\mathbb{K})^{++}$:

$$(A,B)\ \mapsto\ (I\boxtimes\phi_2(B))\cdot f[\phi_1(A)\boxtimes\phi_2(B)^{-1}],$$
$$(A,B)\ \mapsto\ (\phi_1(A)\boxtimes I)\cdot f[\phi_1(A)^{-1}\boxtimes\phi_2(B)].$$

Example 3.3. Recall that the function $t \mapsto t^p$ is operator-monotone for any $0 \leqslant p \leqslant 1$. Given two concave maps $\phi_1 : \mathfrak{B}(\mathbb{H})^{++} \to \mathfrak{B}(\mathbb{H}')^{++}$ and $\phi_2 : \mathfrak{B}(\mathbb{K})^{++} \to \mathfrak{B}(\mathbb{K}')^{++}$, by Theorem 3.1, the maps

$$(A, B) \mapsto [\phi_1(A) \boxtimes \phi_2(B)^{-1}]^p \cdot (I \boxtimes \phi_2(B)), \tag{3.3}$$

$$(A, B) \mapsto [\phi_1(A)^{-1} \boxtimes \phi_2(B)]^p \cdot (\phi_1(A) \boxtimes I) \tag{3.4}$$

are concave on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

Corollary 3.4. Let $f : (0, \infty) \to (0, \infty)$ be an operator-monotone function. If $\phi_1 : \mathfrak{B}(\mathbb{H})^{++} \to \mathfrak{B}(\mathbb{H}')^{++}$ and $\phi_2 : \mathfrak{B}(\mathbb{K})^{++} \to \mathfrak{B}(\mathbb{K}')^{++}$ are concave maps, then the maps

$$(A, B) \mapsto f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot (I \boxtimes \phi_2(B)^{-1}), \tag{3.5}$$

$$(A, B) \mapsto f[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (\phi_1(A)^{-1} \boxtimes I), \tag{3.6}$$

are convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

Proof. We know that the function $g(x) = f(x^{-1})^{-1}$ is operator-monotone. By Theorem 2.43, we have $\left(\phi_1(A) \boxtimes \phi_2(B)^{-1}\right)^{-1} = \phi_1(A)^{-1} \boxtimes \phi_2(B)$ and then

$$f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot \left(I \boxtimes \phi_2(B)^{-1}\right) = g[\phi_1(A) \boxtimes \phi_2(B)^{-1}]^{-1} \cdot \left(I \boxtimes \phi_2(B)^{-1}\right).$$

Since $\phi_1(A) \boxtimes \phi_2(B)^{-1}$ and $I \boxtimes \phi_2(B)$ are commute, we have

$$\begin{aligned} g[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (I \boxtimes \phi_2(B)) &= (I \boxtimes \phi_2(B)) \cdot g[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \\ &= \left\{g[\phi_1(A) \boxtimes \phi_2(B)^{-1}]^{-1} \cdot (I \boxtimes \phi_2(B))^{-1}\right\}^{-1} \\ &= \left\{f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot \left(I \boxtimes \phi_2(B)^{-1}\right)\right\}^{-1}. \end{aligned}$$

Theorem 3.1 implies the concavity of the map

$$\begin{aligned} (A, B) &\mapsto g[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (I \boxtimes \phi_2(B)) \\ &= \left\{f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot \left(I \boxtimes \phi_2(B)^{-1}\right)\right\}^{-1}. \end{aligned}$$

Thus, the map

$$(A, B) \mapsto f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot \left(I \boxtimes \phi_2(B)^{-1}\right)$$

is convex. Similarly, the map (3.6) is convex. □

Theorem 3.5. Let $f : (0, \infty) \to (0, \infty)$ be an operator-monotone function. If $\phi_1 : \mathfrak{B}(\mathbb{H})^{++} \to \mathfrak{B}(\mathbb{H}')^{++}$ is a concave map and $\phi_2 : \mathfrak{B}(\mathbb{K})^{++} \to \mathfrak{B}(\mathbb{K}')^{++}$ is an affine map, then the map

$$(A, B) \mapsto f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot (I \boxtimes \phi_2(B)) \tag{3.7}$$

is convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

Proof. By Theorem 2.29, there is a finite Borel measure μ on $[0,1]$ such that (2.7) holds. Then

$$\begin{aligned} f\left[\phi_1(A)^{-1}\boxtimes\phi_2(B)\right]\cdot(I\boxtimes\phi_2(B)) &= \left\{\int_0^1 (I\boxtimes I)\ !_t\ \left(\phi_1(A)^{-1}\boxtimes\phi_2(B)\right)\ d\mu(t)\right\}(I\boxtimes\phi_2(B)) \\ &= \int_0^1 \left\{(I\boxtimes I)\ !_t\ \left(\phi_1(A)^{-1}\boxtimes\phi_2(B)\right)\right\}(I\boxtimes\phi_2(B))\ d\mu(t). \end{aligned}$$

For each $t\in[0,1]$, it follows from Theorem 2.43 that

$$\begin{aligned} &\left\{(I\boxtimes I)\ !_t\ \left(\phi_1(A)^{-1}\boxtimes\phi_2(B)\right)\right\}(I\boxtimes\phi_2(B)) \\ &\quad= \left\{(1-t)(I\boxtimes I)+t\left(\phi_1(A)^{-1}\boxtimes\phi_2(B)\right)^{-1}\right\}^{-1}(I\boxtimes\phi_2(B)) \\ &\quad= (I\boxtimes\phi_2(B))\,(I\boxtimes\phi_2(B))^{-1}\left\{(1-t)(I\boxtimes I)+t\left(\phi_1(A)\boxtimes\phi_2(B)^{-1}\right)\right\}^{-1}(I\boxtimes\phi_2(B)) \\ &\quad= (I\boxtimes\phi_2(B))\left\{\left[(1-t)(I\boxtimes I)+t\left(\phi_1(A)\boxtimes\phi_2(B)^{-1}\right)\right](I\boxtimes\phi_2(B))\right\}^{-1}(I\boxtimes\phi_2(B)) \\ &\quad= (I\boxtimes\phi_2(B))\left\{(1-t)\,(I\boxtimes\phi_2(B))+t\,(\phi_1(A)\boxtimes I)\right\}^{-1}(I\boxtimes\phi_2(B))\,. \end{aligned}$$

The concavity of the map $(A,B)\mapsto(1-t)\,(I\boxtimes\phi_2(B))+t\,(\phi_1(A)\boxtimes I)$ and the affinity of the map $(A,B)\mapsto I\boxtimes\phi_2(B)$ together yield the convexity of the map

$$\begin{aligned} (A,B) &\mapsto (I\boxtimes\phi_2(B))\left\{(1-t)I\boxtimes\phi_2(B)+t\phi_1(A)\boxtimes I\right\}^{-1}(I\boxtimes\phi_2(B)) \\ &= \left\{(I\boxtimes I)\ !_t\ \left(\phi_1(A)^{-1}\boxtimes\phi_2(B)\right)\right\}(I\boxtimes\phi_2(B)). \end{aligned}$$

Hence, the map (3.7) is convex. □

Example 3.6. Let $\phi_1:\mathfrak{B}(\mathbb{H})^{++}\to\mathfrak{B}(\mathbb{H}')^{++}$ is a concave map and $\phi_2:\mathfrak{B}(\mathbb{K})^{++}\to\mathfrak{B}(\mathbb{K}')^{++}$ is an affine map. For any $0\leqslant p\leqslant 1$, we have by Theorem 3.5 that the map

$$(A,B)\ \mapsto\ \left[\phi_1(A)^{-1}\boxtimes\phi_2(B)\right]^p\cdot(I\boxtimes\phi_2(B)) \tag{3.8}$$

is convex on $\mathfrak{B}(\mathbb{H})^{++}\times\mathfrak{B}(\mathbb{K})^{++}$.

Corollary 3.7. The following map is convex on $\mathfrak{B}(\mathbb{H})^{++}\times\mathfrak{B}(\mathbb{K})^{++}$:

$$(A,B)\ \mapsto\ I\boxtimes(B\log[B])-\log[A]\boxtimes B. \tag{3.9}$$

Proof. Using Theorems 2.43 and 2.45, we get

$$\begin{aligned} I\boxtimes(B\log[B])-\log[A]\boxtimes B &= \left\{I\boxtimes\log[B]-\log[A]\boxtimes I\right\}\cdot(I\boxtimes B) \\ &= \left\{\log[I\boxtimes B]-\log[A\boxtimes I]\right\}\cdot(I\boxtimes B) \\ &= \log[(I\boxtimes B)(A\boxtimes I)^{-1}]\cdot(I\boxtimes B) \\ &= \log[A^{-1}\boxtimes B]\cdot(I\boxtimes B). \end{aligned}$$

Since $\log x$ is operator-monotone, by Theorem 3.5 we obtain that the map

$$(A,B) \mapsto \log[A^{-1} \boxtimes B] \cdot (I \boxtimes B)$$

is convex. Hence, the map (3.9) is convex. □

Theorem 3.8. Let $f : (0,\infty) \to (0,\infty)$ be an operator-monotone function. If $\phi_1 : \mathfrak{B}(\mathbb{H})^{++} \to \mathfrak{B}(\mathbb{H}')^{++}$ is an affine map and $\phi_2 : \mathfrak{B}(\mathbb{K})^{++} \to \mathfrak{B}(\mathbb{K}')^{++}$ is a concave map, then the map

$$(A,B) \;\mapsto\; f[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (\phi_1(A) \boxtimes I) \tag{3.10}$$

is convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

Proof. By Theorem 2.29, there is a finite Borel measure μ on $[0,1]$ such that (2.7) holds. Then

$$\begin{aligned} & f[\phi_2(B) \boxtimes \phi_1(A)^{-1}] \cdot (\phi_2(B) \boxtimes I) \\ &\qquad = \int_0^1 \left\{(I \boxtimes I) \,!_t\, \left(\phi_1(A) \boxtimes \phi_2(B)^{-1}\right)\right\} (\phi_1(A) \boxtimes I)\; d\mu(t). \end{aligned}$$

For each $t \in [0,1]$, by proving in the same way of the proof in Theorem 3.5, we obtain

$$\begin{aligned} & \left\{(I \boxtimes I) \,!_t\, \left(\phi_1(A) \boxtimes \phi_2(B)^{-1}\right)\right\} (\phi_1(A) \boxtimes I) \\ &\qquad = (\phi_1(A) \boxtimes I) \left\{(1-t)\,(\phi_1(A) \boxtimes I) + t\,(I \boxtimes \phi_2(B))\right\}^{-1} (\phi_1(A) \boxtimes I)\,. \end{aligned}$$

The concavity of the map $(A,B) \mapsto (1-t)\,(\phi_1(A) \boxtimes I) + t\,(I \boxtimes \phi_2(B))$ and the affinity of the map $(A,B) \mapsto \phi_1(A) \boxtimes I$ together yield the convexity of the map

$$(A,B) \;\mapsto\; (\phi_1(A) \boxtimes I) \left\{(1-t)\,(\phi_1(A) \boxtimes I) + t\,(I \boxtimes \phi_2(B))\right\}^{-1} (\phi_1(A) \boxtimes I)\,.$$

Hence, the map (3.10) is convex. □

Example 3.9. Let $\phi_1 : \mathfrak{B}(\mathbb{H})^{++} \to \mathfrak{B}(\mathbb{H}')^{++}$ is an affine map and $\phi_2 : \mathfrak{B}(\mathbb{K})^{++} \to \mathfrak{B}(\mathbb{K}')^{++}$ is a concave map. For any $0 \leqslant p \leqslant 1$, we have by Theorem 3.8 that the map

$$(A,B) \;\mapsto\; \left[\phi_2(B) \boxtimes \phi_1(A)^{-1}\right]^p \cdot (\phi_2(B) \boxtimes I) \tag{3.11}$$

is convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

Corollary 3.10. The following map is convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$:

$$(A,B) \;\mapsto\; (A \log[A]) \boxtimes I - A \boxtimes \log[B]. \tag{3.12}$$

Proof. The proof is similar to Corollary 3.7. $\square$

We mention that the maps (3.2), (3.6), (3.10) and (3.12) extend the matrix results in [5] to the case of operators. Moreover, the maps (3.2), (3.6) and (3.10) generalize the operator results in [30].

3.2 Concavity and Convexity Theorems for Tracy-Singh Products of Operators

In this section, we present concavity and convexity theorems for Tracy-Singh products of operators.

For each $i = 1, \ldots, k$, let $\mathbb{H}_i$ and $\mathbb{H}'_i$ be Hilbert spaces and decompose

$$\mathbb{H}_i = \bigoplus_{r=1}^{n_i} \mathbb{H}_{i,r}, \qquad \mathbb{H}'_i = \bigoplus_{s=1}^{m_i} \mathbb{H}'_{i,s},$$

where all $\mathbb{H}_{i,r}$ and $\mathbb{H}'_{i,s}$ are Hilbert spaces. For $k \in \mathbb{N} - \{1\}$ and a finite number of operators $A_i \in \mathfrak{B}(\mathbb{H}_i, \mathbb{H}'_i)$ for $i = 1, \ldots, k$, we denote

$$\begin{aligned} \boxtimes_{i=1}^{1} A_i &= A_1, \\ \boxtimes_{i=1}^{k} A_i &= ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{k-1}) \boxtimes A_k. \end{aligned}$$

The following theorem generalizes Corollary 6.2 of [5] to the case of Tracy-Singh product of operators.

Theorem 3.11. Let $0 \leqslant p_i \leqslant 1$ for $i = 1, \ldots, k$ be such that $\sum_{i=1}^{k} p_i \leqslant 1$. Then the map

$$(A_1, \ldots, A_k) \mapsto \boxtimes_{i=1}^{k} A_i^{p_i} \tag{3.13}$$

is concave on $\mathfrak{B}(\mathbb{H}_1)^{++} \times \cdots \times \mathfrak{B}(\mathbb{H}_k)^{++}$.

Proof. We proceed by induction on k. Clearly, the map $A_1 \mapsto A_1^{p_1}$ is concave. Suppose the assertion is generally true for the case $k - 1$. If $p_k = 0$, then the map becomes

$$(A_1, \ldots, A_k) \mapsto \left(\boxtimes_{i=1}^{k-1} A_i^{p_i} \right) \boxtimes I$$

which is concave. If $p_k = 1$, then $p_i = 0$ for all $i = 1, \ldots, k-1$ and the map

$$(A_1, \ldots, A_k) \mapsto \left(\boxtimes_{i=1}^{k} I \right) \boxtimes A_k$$

is concave. Now suppose $0 < p_k < 1$. By the induction assumption, the map

$$\phi(A_1, \ldots, A_{k-1}) = \boxtimes_{i=1}^{k-1} A_i^{p_i/(1-p_k)}$$

is concave. By applying Theorem 3.1 with $f(x) = x^{p_k}$, the map

$$(A_1, \ldots, A_k) \mapsto f\left[\phi(A_1, \ldots, A_{k-1})^{-1} \boxtimes A_k\right] \cdot (\phi(A_1, \ldots, A_{k-1}) \boxtimes I)$$

is concave. We obtain the concavity of the map (3.13) since

$$\begin{aligned} & f\left[\phi(A_1, \ldots, A_{k-1})^{-1} \boxtimes A_k\right] \cdot (\phi(A_1, \ldots, A_{k-1}) \boxtimes I) \\ &= \left(\phi(A_1, \ldots, A_{k-1})^{-1} \boxtimes A_k\right)^{p_k} (\phi(A_1, \ldots, A_{k-1}) \boxtimes I) \\ &= \left(\phi(A_1, \ldots, A_{k-1})^{-p_k} \boxtimes A_k^{p_k}\right) (\phi(A_1, \ldots, A_{k-1}) \boxtimes I) \\ &= \phi(A_1, \ldots, A_{k-1})^{1-p_k} \boxtimes A_k^{p_k} \\ &= \left(\boxtimes_{i=1}^{k-1} A_i^{p_i}\right) \boxtimes A_k^{p_k} \\ &= \boxtimes_{i=1}^{k} A_i^{p_i}. \end{aligned}$$

□

A simple special case of Theorem 3.11 is that for $k = 2$.

Corollary 3.12. For each $r \in (0, 1)$, the map

$$(A, B) \mapsto A^{1-r} \boxtimes B^r \tag{3.14}$$

is concave on $\mathfrak{B}(\mathbb{H})^+ \times \mathfrak{B}(\mathbb{K})^+$.

Proof. Theorem 3.11 implies that the map (3.14) is concave on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$. Since the Tracy-Singh product is jointly continuous Theorem 2.43), this map is also concave on $\mathfrak{B}(\mathbb{H})^+ \times \mathfrak{B}(\mathbb{K})^+$. □

Corollary 3.13. For each $r \in (0, 1)$, the map

$$(A, B) \mapsto A^{1-r} \boxtimes B^r + A^r \boxtimes B^{1-r} \tag{3.15}$$

is concave on $\mathfrak{B}(\mathbb{H})^+ \times \mathfrak{B}(\mathbb{K})^+$.

Lemma 3.14 ([37]). For each $i = 1, \ldots, k$, let $A_i, B_i \in \mathfrak{B}(\mathbb{H})^+$. Then

$$\left(\boxtimes_{i=1}^{k} A_i\right) \sharp_t \left(\boxtimes_{i=1}^{k} B_i\right) = \boxtimes_{i=1}^{k} (A_i \sharp_t B_i). \tag{3.16}$$

Theorem 3.15. For each $i = 1, \dots, k$, let $\phi_i : \mathfrak{B}(\mathbb{H}_i)^{++} \to \mathfrak{B}(\mathbb{H}_i')^{++}$ be a concave map. Then the map

$$(A_1, \dots, A_k) \mapsto \boxtimes_{i=1}^{k} \phi_i(A_i)^{-1}. \tag{3.17}$$

is convex on $\mathfrak{B}(\mathbb{H}_1)^{++} \times \cdots \times \mathfrak{B}(\mathbb{H}_k)^{++}$.

Proof. To show that the map $(A_1, \dots, A_k) \mapsto \boxtimes_{i=1}^{k} \phi_i(A_i)^{-1}$ is convex, let $t \in [0,1]$ and $A_i, B_i \in \mathfrak{B}(\mathbb{H}_i)^{++}$ for all $i = 1, \dots, k$. Applying Theorems 2.35 and 2.43, and Lemma 3.14, and the arithmetic-geometric means inequality for operators, we have

$$\begin{aligned}
\boxtimes_{i=1}^{k} \phi_i((1-t)A_i + tB_i)^{-1} &\leqslant \boxtimes_{i=1}^{k} ((1-t)\phi_i(A_i) + t\phi_i(B_i))^{-1} \\
&= \boxtimes_{i=1}^{k} (\phi_i(A_i) \nabla_t \phi_i(B_i))^{-1} \\
&\leqslant \boxtimes_{i=1}^{k} (\phi_i(A_i) \sharp_t \phi_i(B_i))^{-1} \\
&= \boxtimes_{i=1}^{k} \left(\phi_i(A_i)^{-1} \sharp_t \phi_i(B_i)^{-1}\right) \\
&= \boxtimes_{i=1}^{k} \phi_i(A_i)^{-1} \sharp_t \boxtimes_{i=1}^{k} \phi_i(B_i)^{-1} \\
&\leqslant \left(\boxtimes_{i=1}^{k} \phi_i(A_i)^{-1}\right) \nabla_t \left(\boxtimes_{i=1}^{k} \phi_i(B_i)^{-1}\right) \\
&= (1-t) \boxtimes_{i=1}^{k} \phi_i(A_i)^{-1} + t \boxtimes_{i=1}^{k} \phi_i(B_i)^{-1}.
\end{aligned}$$

Hence, the map (3.17) is convex. $\square$

Corollary 3.16. For each $i = 1, \dots, k$, let $0 \leqslant p_i \leqslant 1$. Then the map

$$(A_1, \dots, A_k) \mapsto \boxtimes_{i=1}^{k} A_i^{-p_i} \tag{3.18}$$

is convex on $\mathfrak{B}(\mathbb{H}_1)^{++} \times \cdots \times \mathfrak{B}(\mathbb{H}_k)^{++}$.

Proposition 3.17. For each $i = 1, \dots, k$, let $0 \leqslant p_i \leqslant 1$ and $1 \leqslant q \leqslant 2$ be such that $\sum_{i=1}^{k} p_i \leqslant q - 1$. Then the map

$$(A_1, \dots, A_{k+1}) \mapsto \left(\boxtimes_{i=1}^{k} A_i^{-p_i}\right) \boxtimes A_{k+1}^{q} \tag{3.19}$$

is convex on $\mathfrak{B}(\mathbb{H}_1)^{++} \times \cdots \times \mathfrak{B}(\mathbb{H}_{k+1})^{++}$.

Proof. By Theorem 3.11, the map

$$(A_1, \ldots, A_{k+1}) \mapsto \Big(\boxtimes_{i=1}^{k} A_i^{p_i} \Big) \boxtimes A_{k+1}^{2-q}$$

is concave on $\mathfrak{B}(\mathbb{H}_1)^{++} \times \cdots \times \mathfrak{B}(\mathbb{H}_{k+1})^{++}$. Clearly, the map

$$(A_1, \ldots, A_{k+1}) \mapsto \Big(\boxtimes_{i=1}^{k} I \Big) \boxtimes A_{k+1}$$

is affine. It follows from Theorem 2.43 that the map

$$\begin{aligned}(A_1, \ldots, A_{k+1}) \mapsto & \Big\{ \Big(\boxtimes_{i=1}^{k} I \Big) \boxtimes A_{k+1} \Big\} \Big\{ \Big(\boxtimes_{i=1}^{k} A_i^{p_i} \Big) \boxtimes A_{k+1}^{2-q} \Big\}^{-1} \Big(\boxtimes_{i=1}^{k} I \Big) \boxtimes A_{k+1} \\ &= \Big(\boxtimes_{i=1}^{k} A_i^{-p_i} \Big) \boxtimes A_{k+1}^{q}\end{aligned}$$

is convex. □

Corollary 3.18. For each $r \in (0,1)$, the maps

$$(A, B) \mapsto A^{-r} \boxtimes B^{1+r}, \tag{3.20}$$

$$(A, B) \mapsto A^{1+r} \boxtimes B^{-r} \tag{3.21}$$

are convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

Proof. The convexity of the map (3.20) follows from Proposition 3.17. Now, we will prove that the map (3.21) is convex. Let $r \in (0,1)$. Theorem 2.43 implies that

$$A^{1+r} \boxtimes B^{-r} = \big(A^r \boxtimes B^{-r}\big)(A \boxtimes I) = \big(A \boxtimes B^{-1}\big)^r (A \boxtimes I).$$

Using Theorem 3.58 with $f(x) = x^r$, we reach the convexity of the map (3.21). □

Corollary 3.19. For each $r \in (0,1)$, the map

$$(A, B) \mapsto A^{-r} \boxtimes B^{1+r} + A^{1+r} \boxtimes B^{-r} \tag{3.22}$$

is convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

We mention that Theorems 3.11 and 3.15, Corollary 3.16, and Proposition 3.17 generalize the matrix results involving Kronecker products provided in [5] and Tracy-Singh products provided in [4]. Moreover, Theorem 3.15 can be viewed an extension of Theorem 4 in [30].

Now, we focus on convexity of certain operator-valued functions involving Tracy-Singh products.

Lemma 3.20 ([12])**.** For each $A \in \mathfrak{B}(\mathbb{H})^+$, the map

$$\alpha \;\mapsto\; A^{\alpha} + A^{-\alpha} \tag{3.23}$$

is convex on $\mathbb{R}$, increasing on $[0,\infty)$, decreasing on $(-\infty,0]$ and has its minimum at $\alpha = 0$.

Theorem 3.21. Let $A \in \mathfrak{B}(\mathbb{H})^+$ and $B \in \mathfrak{B}(\mathbb{K})^+$. Then the map

$$\alpha \;\mapsto\; A^{1+\alpha} \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^{1+\alpha} \tag{3.24}$$

is convex on $[-1,1]$, decreasing on $[-1,0]$, increasing on $[0,1]$, and attains its minimum at $\alpha = 0$.

Proof. To show that the map

$$\phi(\alpha) \;=\; A^{1+\alpha} \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^{1+\alpha}$$

is increasing on $[0,1]$, let $0 \leqslant \alpha \leqslant \beta \leqslant 1$ and $\beta \neq 0$. In Lemma 3.20, replacing α by $0 \leqslant \frac{\alpha}{\beta} \leqslant 1$ and A by $A^{\beta} \boxtimes B^{-\beta}$, we get

$$\begin{aligned} A^{\alpha} \boxtimes B^{-\alpha} + A^{-\alpha} \boxtimes B^{\alpha} &= \left(A^{\beta} \boxtimes B^{-\beta}\right)^{\frac{\alpha}{\beta}} + \left(A^{\beta} \boxtimes B^{-\beta}\right)^{-\frac{\alpha}{\beta}} \\ &\leqslant A^{\beta} \boxtimes B^{-\beta} + \left(A^{\beta} \boxtimes B^{-\beta}\right)^{-1} \\ &= A^{\beta} \boxtimes B^{-\beta} + A^{-\beta} \boxtimes B^{\beta}. \end{aligned}$$

Multiply both sides by $A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}$, we have by Theorem 2.43 that

$$\begin{aligned} A^{1+\alpha} \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^{1+\alpha} &= \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right)\left(A^{\alpha} \boxtimes B^{-\alpha} + A^{-\alpha} \boxtimes B^{\alpha}\right)\left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right) \\ &\leqslant \left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right)\left(A^{\beta} \boxtimes B^{-\beta} + A^{-\beta} \boxtimes B^{\beta}\right)\left(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}\right) \\ &= A^{1+\beta} \boxtimes B^{1-\beta} + A^{1-\beta} \boxtimes B^{1+\beta}. \end{aligned}$$

This implies that $\phi(\alpha) \leqslant \phi(\beta)$. Thus, ϕ is increasing on $[0,1]$. Note that $\phi(\alpha) = \phi(-\alpha)$, so ϕ is decreasing on $[-1,0]$. Consequently, ϕ is convex on $[-1,1]$ and has its minimum at $\alpha = 0$. □

Corollary 3.22. Let $A \in \mathfrak{B}(\mathbb{H})^+$ and $B \in \mathfrak{B}(\mathbb{K})^+$. Then the map

$$\alpha \;\mapsto\; A^{\alpha} \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^{\alpha} \tag{3.25}$$

is convex on $[0,1]$, decreasing on $[0,\frac{1}{2}]$, increasing on $[\frac{1}{2},1]$, and attains its minimum at $\alpha = \frac{1}{2}$.

Proof. Replacing A, B by $A^{\frac{1}{2}}, B^{\frac{1}{2}}$ in Theorem 3.21, respectively, and then replacing $(1+\alpha)/2$ by α, we get the desired result. $\square$

As a consequence, we obtain an operator version of the arithmetic-geometric mean inequality as follows.

Corollary 3.23. Let $A \in \mathfrak{B}(\mathbb{H})^+$ and $B \in \mathfrak{B}(\mathbb{K})^+$. For any $t \in [\frac{1}{2}, 1]$, we have

$$2(A^{\frac{1}{2}} \boxtimes B^{\frac{1}{2}}) \leqslant A^t \boxtimes B^{1-t} + A^{1-t} \boxtimes B^t \leqslant A \boxplus B.$$

Here, $\boxplus$ denotes the Tracy-Singh sum [40] defined by $A \boxplus B = A \boxtimes I + I \boxtimes B$.

Notice that Theorem 3.21 and Corollary 3.22 can be viewed generalizations of [32, Theorem 3.2 and Corollary 3.3] to the case of operators.

3.3 Concavity and Convexity Theorems for Khatri-Rao Products of Operators

In this section, we establish concavity and convexity theorems for Tracy-Singh products and Khatri-Rao products of operators. The results in this section are established by using the concavity and convexity theorems for Tracy-Singh products and the connection between the Khatri-Rao and Tracy-Singh products.

For each $i = 1, \ldots, k$, let $\mathbb{H}_i$ and $\mathbb{H}'_i$ be Hilbert spaces and decompose

$$\mathbb{H}_i = \bigoplus_{r=1}^{n} \mathbb{H}_{i,r}, \qquad \mathbb{H}'_i = \bigoplus_{s=1}^{m} \mathbb{H}'_{i,s}$$

where all $\mathbb{H}_{i,r}$ and $\mathbb{H}'_{i,s}$ are Hilbert spaces. We set $\boxdot_{i=1}^{k} A_i = A_1$. For $k \in \mathbb{N} - \{1\}$ and a finite number of operators $A_i \in \mathfrak{B}(\mathbb{H}_i, \mathbb{H}'_i)$ for $i = 1, \ldots, k$, we denote

$$\boxdot_{i=1}^{k} A_i = ((A_1 \boxdot A_2) \boxdot \cdots \boxdot A_{k-1}) \boxdot A_k.$$

Lemma 3.24. There are isometries Z_1 and Z_2 such that

$$\boxdot_{i=1}^{k} A_i = Z_1^* \Big(\boxtimes_{i=1}^{k} A_i \Big) Z_2 \tag{3.26}$$

for any $A_i \in \mathfrak{B}(\mathbb{H}_i, \mathbb{H}'_i)$, $i = 1, \ldots, k$. If $\mathbb{H}_i$ and $\mathbb{H}'_i$ are the same space for all i, the $Z_1 = Z_2 := Z$.

Proof. We proceed by induction on k. If $k=1$, the property (3.26) is true by using $Z_1 = I$ and $Z_2 = I$. Suppose that there exist isometries R_1 and R_2 such that

$$\boxdot_{i=1}^{k-1} A_i \;=\; R_1^* \Big(\boxtimes_{i=1}^{k-1} A_i \Big) R_2.$$

By Theorem 2.51, there are isometries S_1, S_2 such that

$$\Big(\boxdot_{i=1}^{k-1} A_i \Big) \boxdot A_k \;=\; S_1^* \Big[\Big(\boxdot_{i=1}^{k-1} A_i \Big) \boxtimes A_k \Big] S_2.$$

Then

$$\begin{aligned}
\boxdot_{i=1}^{k} A_i \;&=\; \Big(\boxdot_{i=1}^{k-1} A_i \Big) \boxdot A_k \\
&=\; S_1^* \Big[\Big(\boxdot_{i=1}^{k-1} A_i \Big) \boxtimes A_k \Big] S_2 \\
&=\; S_1^* \Big\{ \Big[R_1^* \Big(\boxtimes_{i=1}^{k-1} A_i \Big) R_2 \Big] \boxtimes A_k \Big\} S_2 \\
&=\; S_1^* (R_1^* \boxtimes I) \Big[\Big(\boxtimes_{i=1}^{k-1} A_i \Big) \boxtimes A_k \Big] (R_2 \boxtimes I) S_2 \\
&=\; [(R_1 \boxtimes I) S_1]^* \Big(\boxtimes_{i=1}^{k} A_i \Big) (R_2 \boxtimes I) S_2.
\end{aligned}$$

Set $Z_1 = (R_1 \boxtimes I) S_1$ and $Z_2 = (R_2 \boxtimes I) S_2$. For each $i = 1, 2$,

$$\begin{aligned}
Z_i^* Z_i \;&=\; [(R_i \boxtimes I) S_i]^* (R_i \boxtimes I) S_i \;=\; S_i^* (R_i^* \boxtimes I)(R_i \boxtimes I) S_i \\
&=\; S_i^* [(R_i^* R_i) \boxtimes I] S_i \;=\; S_i^* (I \boxtimes I) S_i \;=\; S_i^* S_i \;=\; I.
\end{aligned}$$

Hence, Z_1 and Z_2 are isometries. When $\mathbb{H}_i = \mathbb{H}_i'$ for all $i = 1, \ldots, k$, we have $Z_1 = Z_2$ from the construction. □

Next, we develop concavity theorems for Khatri-Rao products of operators.

Theorem 3.25. Let $0 \leqslant p_i \leqslant 1$ for $i = 1, \ldots, k$ be such that $\sum_{i=1}^{k} p_i \leqslant 1$. Then the map

$$(A_1, \ldots, A_k) \;\mapsto\; \boxdot_{i=1}^{k} A_i^{p_i} \tag{3.27}$$

is concave on $\mathfrak{B}(\mathbb{H}_1)^{++} \times \cdots \times \mathfrak{B}(\mathbb{H}_k)^{++}$.

Proof. From Lemma 3.24, the map $X \mapsto Z^* X Z$, taking the Tracy-Singh product $\boxtimes_{i=1}^{k} A_i$ to the Khatri-Rao product $\boxdot_{i=1}^{k} A_i$, is linear and preserves positivity. Recall that the composition between a linear map and a concave map results in a concave map. Since the map

$$(A_1, \ldots, A_k) \mapsto \boxtimes_{i=1}^{k} A_i^{p_i}$$

is concave by Theorem 3.11, we have the concavity of the map

$$(A_1, \dots, A_k) \mapsto Z^*\Big(\boxtimes_{i=1}^{k} A_i^{p_i}\Big) Z = \boxdot_{i=1}^{k} A_i^{p_i}. \qquad \square$$

Corollary 3.26. For each $r \in (0,1)$, the map

$$(A, B) \mapsto A^{1-r} \boxdot B^r \tag{3.28}$$

is concave on $\mathfrak{B}(\mathbb{H})^+ \times \mathfrak{B}(\mathbb{K})^+$.

Proof. It follows from the case $k = 2$ in Theorem 3.25 together with the continuity of the Khatri-Rao product (Theorem 2.50). □

Corollary 3.27. For each $r \in (0,1)$, the map

$$(A, B) \mapsto A^{1-r} \boxdot B^r + A^r \boxdot B^{1-r} \tag{3.29}$$

is concave on $\mathfrak{B}(\mathbb{H})^+ \times \mathfrak{B}(\mathbb{K})^+$.

Proposition 3.28. For each $i = 1, \dots, k$, let $\phi_i : B(\mathbb{H}_i)^{++} \to B(\mathbb{H}_i')^{++}$ be a concave map. Then the map

$$(A_1, \dots, A_k) \mapsto \boxdot_{i=1}^{k} \phi_i(A_i)^{-1} \tag{3.30}$$

is convex on $\mathfrak{B}(\mathbb{H}_1)^{++} \times \cdots \times \mathfrak{B}(\mathbb{H}_k)^{++}$.

Proof. It follows from Lemma 3.24 and Theorem 3.15. □

Corollary 3.29. Let $0 \leqslant p_i \leqslant 1$ for each $i = 1, \dots, k$. Then the map

$$(A_1, \dots, A_k) \mapsto \boxdot_{i=1}^{k} A_i^{-p_i} \tag{3.31}$$

is convex on $\mathfrak{B}(\mathbb{H}_1)^{++} \times \cdots \times \mathfrak{B}(\mathbb{H}_k)^{++}$.

Proof. It follows from Proposition 3.28 by putting $\phi_i(A_i) = A_i^{p_i}$ for each i. □

Corollary 3.30. For each $r \in (0,1)$, the maps

$$(A,B) \mapsto A^{-r} \boxdot B^{1+r}, \tag{3.32}$$

$$(A,B) \mapsto A^{1+r} \boxdot B^{-r} \tag{3.33}$$

are convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

Proof. It follows from Theorem 2.51 and Corollary 3.18. □

Corollary 3.31. For each $r \in (0,1)$, the map

$$(A,B) \mapsto A^{-r} \boxdot B^{1+r} + A^{1+r} \boxdot B^{-r} \tag{3.34}$$

is convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

Proposition 3.32. For each $i = 1,\dots,k$, let $0 \leqslant p_i \leqslant 1$ and $1 \leqslant q \leqslant 2$ be such that $\sum_{i=1}^{k} p_i \leqslant q-1$. Then the map

$$(A_1,\dots,A_{k+1}) \mapsto \left(\boxdot_{i=1}^{k} A_i^{-p_i} \right) \boxdot A_{k+1}^{q} \tag{3.35}$$

is convex on $\mathfrak{B}(\mathbb{H}_1)^{++} \times \cdots \times \mathfrak{B}(\mathbb{H}_{k+1})^{++}$.

Proof. Applying Proposition 3.17 with Lemma 3.24, we get the result. □

Corollary 3.33. For each $r \in (0,1)$, the maps

$$(A,B) \mapsto A^{-r} \boxdot B^{1+r}, \tag{3.36}$$

$$(A,B) \mapsto A^{1+r} \boxdot B^{-r} \tag{3.37}$$

are convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

Proof. It follows from Corollary 3.18. □

Corollary 3.34. For each $r \in (0,1)$, the map

$$(A,B) \mapsto A^{-r} \boxdot B^{1+r} + A^{1+r} \boxdot B^{-r} \tag{3.38}$$

is convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$.

We mention that Theorem 3.25, Propositions 3.28 and 3.32, and Corollary 3.29 are extensions of matrix results in [5]. in addition, Proposition 3.28 is a generalization of [30, Thoerem 4].

Now, we focus on convexity of certain operator-valued functions involving Khatri-Rao products.

Proposition 3.35. Let $A \in \mathfrak{B}(\mathbb{H})^+$ and $\mathfrak{B}(\mathbb{K})^+$. Then the map

$$\alpha \mapsto A^{1+\alpha} \boxdot B^{1-\alpha} + A^{1-\alpha} \boxdot B^{1+\alpha} \tag{3.39}$$

is convex on $[-1,1]$, decreasing on $[-1,0]$, increasing on $[0,1]$, and attains its minimum at $\alpha = 0$.

Proof. From Theorem 2.51, the map $X \mapsto Z^*XZ$, taking the Tracy-Singh product $A \boxtimes B$ to the Khatri-Rao product $A \boxdot B$, is linear and preserves positivity. We know that the composition between a linear map and a convex map results in a convex map. Since the map

$$\alpha \mapsto A^{1+\alpha} \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^{1+\alpha}$$

is concave by Theorem 3.21, we have the convexity of the map

$$\alpha \mapsto Z^*\Big(A^{1+\alpha} \boxtimes B^{1-\alpha} + A^{1-\alpha} \boxtimes B^{1+\alpha}\Big)Z = A^{1+\alpha} \boxdot B^{1-\alpha} + A^{1-\alpha} \boxdot B^{1+\alpha}.$$ □

Corollary 3.36. Let $A \in \mathfrak{B}(\mathbb{H})^+$ and $B \in \mathfrak{B}(\mathbb{K})^+$. Then the map

$$\alpha \mapsto A^{\alpha} \boxdot B^{1-\alpha} + A^{1-\alpha} \boxdot B^{\alpha} \tag{3.40}$$

is convex on $[0,1]$, decreasing on $[0,\frac{1}{2}]$, increasing on $[\frac{1}{2},1]$, and attains its minimum at $\alpha = \frac{1}{2}$.

Proof. The proof is similar to Corollary 3.22. □

As a consequence, we obtain an operator version of the arithmetic-geometric mean inequality as follows.

Corollary 3.37. Let $A \in \mathfrak{B}(\mathbb{H})^+$ and $B \in \mathfrak{B}(\mathbb{K})^+$. For any $t \in [\frac{1}{2},1]$, we have

$$2(A^{1/2} \boxdot B^{1/2}) \leqslant A^t \boxdot B^{1-t} + A^{1-t} \boxdot B^t \leqslant A \circledast B.$$

Here, ⊞ denotes the Khatri-Rao sum [38] defined by $A \circledast B = A \boxdot I + I \boxdot B$.

Notice that Proposition 3.35 and Corollary 3.36 are operator extensions of [32, Theorem 3.2 and Corollary 3.3].

Chapter 4

Chebyshev-Type Inequalities for Operators Involving Tracy-Singh Products and Weighted Pythagorean Means

In this chapter, we investigate Chebyshev-type inequalities for bounded continuous fields of operators involving Tracy-Singh products. The continuous field considered here is parametrized by a locally compact Hausdorff space equipped with a finite Radon measure. We also establish inequalities of Chebyshev-type for bounded continuous fields of operators relating Tracy-Singh products and weighted Pythagorean means. The weighted Pythagorean means considered here are the weighted arithmetic mean, the weighted geometric mean, and the weighted harmonic mean. Moreover, Tracy-Singh product versions of the Chebyshev-Grüss-type inequality via oscillations are also obtained. Our results include Chebyshev-type inequalities for tensor product of operators and Tracy-Singh/Kronecker products of matrices.

Throughout this chapter, let $\mathbb{H}, \mathbb{K}$ be complex Hilbert spaces such that

$$\mathbb{H} = \bigoplus_{i=1}^{m} \mathbb{H}_i, \qquad \mathbb{K} = \bigoplus_{i=1}^{n} \mathbb{K}_i,$$

and let Ω be a locally compact Hausdorff space equipped with a finite Radon measure μ. A family $\mathcal{A} = (A_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$ is said to be bounded if there is a constant $M > 0$ such that $\|A_t\| \leqslant M$ for all $t \in \Omega$. The family $\mathcal{A}$ is said to be a continuous field if parametrization $t \mapsto A_t$ is norm-continuous on Ω. Every continuous field $\mathcal{A} = (A_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$ can have the Bochner integral $\int_\Omega A_t d\mu(t)$ if the norm function $t \mapsto \|A_t\|$ process the Lebesgue integrability. In this case, the resulting integral is a unique element in $\mathfrak{B}(\mathbb{H})$ such that

$$\phi\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega \phi(A_t) d\mu(t)$$

for every bounded linear functional ϕ on $\mathfrak{B}(\mathbb{H})$.

Proposition 4.1. Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})$. Then for any $X \in \mathfrak{B}(\mathbb{K})$,

$$\int_\Omega A_t d\mu(t) \boxtimes X = \int_\Omega (A_t \boxtimes X) d\mu(t), \tag{4.1}$$

$$X \boxtimes \int_\Omega A_t d\mu(t) = \int_\Omega (X \boxtimes A_t) d\mu(t). \tag{4.2}$$

Proof. Since the map $t \mapsto A_t$ is continuous and bounded, the norm function $t \mapsto \|A_t\|$ is Lebesgue integrable. By Theorem 2.58, the map $t \mapsto A_t$ is Bochner integrable on Ω. Let $X \in \mathfrak{B}(\mathbb{K})$. Using Theorem 2.43, the map

$$\psi \;:\; \mathfrak{B}(\mathbb{H}) \;\to\; \mathfrak{B}\Big(\bigoplus_{i,j=1}^{m,n} \mathbb{H}_i \otimes \mathbb{K}_j\Big), \qquad T \mapsto T \boxtimes X$$

is linear and bounded. Theorem 2.57 implies that the map $t \mapsto A_t \boxtimes X$ is Bochner integrable on Ω and

$$\int_\Omega A_t d\mu(t) \boxtimes X \;=\; \psi\Big(\int_\Omega A_t d\mu(t)\Big) \;=\; \int_\Omega \psi(A_t) d\mu(t) \;=\; \int_\Omega (A_t \boxtimes X) d\mu(t).$$

In the same way, we reach the assertion (4.2) by using tha fact that the map

$$\varphi \;:\; \mathfrak{B}(\mathbb{H}) \;\to\; \mathfrak{B}\Big(\bigoplus_{i,j=1}^{n,m} \mathbb{K}_i \otimes \mathbb{H}_j\Big), \qquad T \mapsto X \boxtimes T$$

is linear and bounded □

4.1 Chebyshev-Type Inequalities Involving Tracy-Singh Products of Operators

From now on, let Ω be a locally compact Hausdorff space equipped with a finite Radon measure μ.

From the definition of the synchronous functions in (1.11), we define synchronous Tracy-Singh property for fields of operators as follow:

Definition 4.2. The fields $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$ and $\mathfrak{B}(\mathbb{K})$, respectively, are said to have the **synchronous Tracy-Singh property** if

$$(A_t - A_s) \boxtimes (B_t - B_s) \;\geqslant\; 0 \tag{4.3}$$

for all $s, t \in \Omega$. They are said to have the **opposite-synchronous Tracy-Singh property** if the reverse of (4.3) holds for all $s, t \in \Omega$.

The following result is an extension of the Chebyshev integral inequality (1.12) to the case of operators.

Theorem 4.3. Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be bounded continuous fields of self-adjoint operators in $\mathfrak{B}(\mathbb{H})$ and $\mathfrak{B}(\mathbb{K})$, respectively, and let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.

(i) If $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Tracy-Singh property, then

$$\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \boxtimes B_t)d\mu(t) \geqslant \int_\Omega \alpha(t)A_t d\mu(t) \boxtimes \int_\Omega \alpha(s)B_s d\mu(s). \tag{4.4}$$

(ii) If $\mathcal{A}$ and $\mathcal{B}$ have the opposite-synchronous Tracy-Singh property, then the reverse of (4.4) holds.

Proof. By using Theorem 2.43, and Proposition 4.1, we have

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \boxtimes B_t)d\mu(t) - \int_\Omega \alpha(t)A_t d\mu(t) \boxtimes \int_\Omega \alpha(s)B_s d\mu(s) \\
&\quad = \iint_{\Omega^2} \alpha(s)\alpha(t)(A_t \boxtimes B_t)d\mu(t)d\mu(s) - \iint_{\Omega^2} \alpha(t)\alpha(s)(A_t \boxtimes B_s)d\mu(t)d\mu(s) \\
&\quad = \iint_{\Omega^2} \alpha(s)\alpha(t)[A_t \boxtimes (B_t - B_s)]d\mu(t)d\mu(s) \\
&\quad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)\left[(A_t \boxtimes (B_t - B_s)\right]d\mu(t)d\mu(s) \\
&\qquad + \frac{1}{2}\iint_{\Omega^2} \alpha(t)\alpha(s)\left[A_s \boxtimes (B_s - B_t)\right]d\mu(s)d\mu(t) \\
&\quad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)\left[(A_t \boxtimes (B_t - B_s)\right]d\mu(t)d\mu(s) \\
&\qquad - \frac{1}{2}\iint_{\Omega^2} \alpha(t)\alpha(s)\left[A_s \boxtimes (B_t - B_s)\right]d\mu(s)d\mu(t) \\
&\quad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)\left[(A_t - A_s) \boxtimes (B_t - B_s)\right]d\mu(t)d\mu(s).
\end{aligned}$$

Here, we have used Fubini's Theorem [7] to interchange the order of integrals. For the case (i), we have

$$\iint_{\Omega^2} \alpha(s)\alpha(t)\left[(A_t - A_s) \boxtimes (B_t - B_s)\right]d\mu(t)d\mu(s) \geqslant 0 \tag{4.5}$$

and thus (4.4) holds. For another case, we get

$$\iint_{\Omega^2} \alpha(s)\alpha(t)\left[(A_t - A_s) \boxtimes (B_t - B_s)\right]d\mu(t)d\mu(s) \leqslant 0$$

and thus the reverse of (4.4) holds. □

Remark 4.4. In Theorem 4.3 and other results in this chapter, we may assume that Ω is a compact Hausdorff space. In this case, every continuous field on Ω is automatically bounded.

For the case $m = n = 1$, i.e., $\mathbb{H}$ and $\mathbb{K}$ are not decomposed, the synchronous Tracy-Singh property in Definition 4.2 reduces to the synchronous tensor property:

$$(A_t - A_s) \otimes (B_t - B_s) \geqslant 0 \tag{4.6}$$

for all $s, t \in \Omega$. If two fields $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$ have the synchronous tensor property, then $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Hadamard property [31, Definition 2.1], i.e.,

$$(A_t - A_s) \odot (B_t - B_s) \geqslant 0 \tag{4.7}$$

for all $s, t \in \Omega$. The following result gives Chebyshev-type inequalities concerning tensor products and Hadamard products.

Corollary 4.5. Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be bounded continuous fields of self-adjoint operators in $\mathfrak{B}(\mathbb{H})$ and $\mathfrak{B}(\mathbb{K})$, respectively, and let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.

(i) If $\mathcal{A}$ and $\mathcal{B}$ have the synchronous tensor property, then

$$\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \otimes B_t)d\mu(t) \geqslant \int_\Omega \alpha(t)A_t d\mu(t) \otimes \int_\Omega \alpha(s)B_s d\mu(s), \tag{4.8}$$

$$\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \odot B_t)d\mu(t) \geqslant \int_\Omega \alpha(t)A_t d\mu(t) \odot \int_\Omega \alpha(s)B_s d\mu(s). \tag{4.9}$$

(ii) If $\mathcal{A}$ and $\mathcal{B}$ have the opposite-synchronous tensor property, then the reverse of (4.8) and (4.9) hold.

Proof. For the case $m = n = 1$, the Tracy-Singh product in Theorem 4.3 reduces to the tensor product. Assume that A and B have the synchronous tensor property. Note that for any $X \in \mathfrak{B}(\mathbb{H})$,

$$X^* \left(\int_\Omega A_t d\mu(t) \right) X = \int_\Omega X^* A_t X d\mu(t).$$

Using the fact that the Hadamard product is expressed as the deformation of tensor product via the isometry U defined in (1.5), we obtain

$$\begin{aligned}
\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \odot B_t)d\mu(t) &= \int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)U^*(A_t \otimes B_t)U d\mu(t) \\
&= U^* \left(\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \otimes B_t)d\mu(t) \right) U \\
&\geqslant U^* \left(\int_\Omega \alpha(t)A_t d\mu(t) \otimes \int_\Omega \alpha(s)B_s d\mu(s) \right) U \\
&= \int_\Omega \alpha(t)A_t d\mu(t) \odot \int_\Omega \alpha(s)B_s d\mu(s).
\end{aligned}$$

The case (ii) for Hadamard products can be similarly treated. □

We can see that the inequality (4.9) is the same as [31, Theorem 2.2], but they hold under different hypothesis.

The next corollary is a discrete version of Theorem 4.3.

Corollary 4.6. Let $A_i \in \mathfrak{B}(\mathbb{H}), B_i \in \mathfrak{B}(\mathbb{K})$ be self-adjoint operators and ω_i be nonnegative number for each $i = 1, \ldots, k$. Let $\mathcal{A} = (A_1, \ldots, A_k)$ and $\mathcal{B} = (B_1, \ldots, B_k)$.

(i) If $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Tracy-Singh property, then

$$\sum_{i=1}^{k} \omega_i \sum_{i=1}^{k} \omega_i (A_i \boxtimes B_i) \geqslant \left(\sum_{i=1}^{k} \omega_i A_i \right) \boxtimes \left(\sum_{i=1}^{k} \omega_i B_i \right). \tag{4.10}$$

(ii) If $\mathcal{A}$ and $\mathcal{B}$ have the opposite-synchronous Tracy-Singh property, then the reverse of (4.10) holds.

Proof. When we set $\Omega = \{1, \ldots, k\}$ equipped with the counting measure, the integral $\int_\Omega A_t d\mu(t)$ reduces to the finite sum, i.e.,

$$\int_\Omega A_t d\mu(t) = \sum_{i=1}^{k} A_i.$$

From Theorem 4.4, setting $\alpha(i) = \omega_i$ for all $i = 1, \ldots, k$, we get the result. □

4.2 Chebyshev-Type Inequalities Involving Weighted Pythagorean Means of Operators

In this section, the space Ω is equipped with a total ordering $\preccurlyeq$.

Definition 4.7. We say that a field $\mathcal{A} = (A_t)_{t \in \Omega}$ of self-adjoint operators is increasing (decreasing, resp.) whenever $s \preccurlyeq t$ implies $A_s \leqslant A_t$ ($A_s \geqslant A_t$, resp.).

Definition 4.8. Two ordered pairs (X_1, X_2) and (Y_1, Y_2) of self-adjoint operators are said to have the **synchronous property** if either

(i) $X_i \leqslant Y_i$ for $i = 1, 2$, or

(ii) $X_i \geqslant Y_i$ for $i = 1, 2$.

The pairs (X_1, X_2) and (Y_1, Y_2) are said to have the **opposite-synchronous property** if either

(i) $X_1 \leqslant Y_1$ and $X_2 \geqslant Y_2$, or

(ii) $X_1 \geqslant Y_1$ and $X_2 \leqslant Y_2$.

Definition 4.9. Let $\mathcal{A} = (A_t)_{t\in\Omega}, \mathcal{B} = (B_t)_{t\in\Omega}, \mathcal{C} = (C_t)_{t\in\Omega}, \mathcal{D} = (D_t)_{t\in\Omega}$ be continuous fields of self-adjoint operators. Two ordered pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ are said to have the **synchronous monotone property** if (A_t, B_t) and (C_t, D_t) have the synchronous property for all $t \in \Omega$. They are said to have the **opposite-synchronous monotone property** if (A_t, B_t) and (C_t, D_t) have the opposite-synchronous property for all $t \in \Omega$.

4.2.1 Inequalities on Weighted Geometric Means

Lemma 4.10 ([37]). Let $A, B, C, D \in \mathfrak{B}(\mathbb{H})^+$ and $w \in [0, 1]$. Then

$$(A \boxtimes B)\sharp_w(C \boxtimes D) \;=\; (A\sharp_w C) \boxtimes (B\sharp_w D).$$

Theorem 4.11. Let $\mathcal{A} = (A_t)_{t\in\Omega}, \mathcal{B} = (B_t)_{t\in\Omega}, \mathcal{C} = (C_t)_{t\in\Omega}, \mathcal{D} = (D_t)_{t\in\Omega}$ be bounded continuous fields in $\mathfrak{B}(\mathbb{H})^+$, $\alpha : \Omega \to [0, \infty)$ a bounded measurable function and $w \in [0, 1]$.

(i) If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either all increasing, or all decreasing, then

$$\begin{aligned}\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t)\sharp_w(C_t \boxtimes D_t)]d\mu(t) \\ \geqslant \int_\Omega \alpha(t)(A_t\sharp_w C_t)d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s\sharp_w D_s)d\mu(s).\end{aligned} \tag{4.11}$$

(ii) The reverse of (4.11) holds if either

(a) $\mathcal{A}, \mathcal{C}$ are increasing and $\mathcal{B}, \mathcal{D}$ are decreasing, or

(b) $\mathcal{A}, \mathcal{C}$ are decreasing and $\mathcal{B}, \mathcal{D}$ are increasing.

Proof. Let $s, t \in \Omega$ and assume without loss of generality that $s \preccurlyeq t$. Applying Theorem

2.43, Lemma 4.10, Proposition 4.1, and Fubini's Theorem [7], we have

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)[(A_t\boxtimes B_t)\sharp_w(C_t\boxtimes D_t)]d\mu(t)\\
&-\int_\Omega \alpha(t)(A_t\sharp_w C_t)d\mu(t)\boxtimes\int_\Omega \alpha(s)(B_s\sharp_w D_s)d\mu(s)\\
&\qquad= \iint_{\Omega^2}\alpha(s)\alpha(t)[(A_t\boxtimes B_t)\sharp_w(C_t\boxtimes D_t)]d\mu(t)d\mu(s)\\
&\qquad\quad-\iint_{\Omega^2}\alpha(t)\alpha(s)[(A_t\sharp_w C_t)\boxtimes(B_s\sharp_w D_s)]d\mu(t)d\mu(s)\\
&\qquad= \iint_{\Omega^2}\alpha(s)\alpha(t)[(A_t\sharp_w C_t)\boxtimes(B_t\sharp_w D_t)]d\mu(t)d\mu(s)\\
&\qquad\quad-\iint_{\Omega^2}\alpha(t)\alpha(s)[(A_t\sharp_w C_t)\boxtimes(B_s\sharp_w D_s)]d\mu(t)d\mu(s)\\
&\qquad= \iint_{\Omega^2}\alpha(s)\alpha(t)[(A_t\sharp_w C_t)\boxtimes(B_t\sharp_w D_t-B_s\sharp_w D_s)]d\mu(t)d\mu(s)\\
&\qquad= \frac{1}{2}\iint_{\Omega^2}\alpha(s)\alpha(t)[(A_t\sharp_w C_t)\boxtimes(B_t\sharp_w D_t-B_s\sharp_w D_s)]d\mu(t)d\mu(s)\\
&\qquad\quad+\frac{1}{2}\iint_{\Omega^2}\alpha(t)\alpha(s)[(A_s\sharp_w C_s)\boxtimes(B_s\sharp_w D_s-B_t\sharp_w D_t)]d\mu(s)d\mu(t)\\
&\qquad= \frac{1}{2}\iint_{\Omega^2}\alpha(s)\alpha(t)[(A_t\sharp_w C_t)\boxtimes(B_t\sharp_w D_t-B_s\sharp_w D_s)]d\mu(t)d\mu(s)\\
&\qquad\quad-\frac{1}{2}\iint_{\Omega^2}\alpha(t)\alpha(s)[(A_s\sharp_w C_s)\boxtimes(B_t\sharp_w D_t-B_s\sharp_w D_s)]d\mu(s)d\mu(t)\\
&\qquad= \frac{1}{2}\iint_{\Omega^2}\alpha(s)\alpha(t)[A_t\sharp_w C_t-A_s\sharp_w C_s]\boxtimes[B_t\sharp_w D_t-B_s\sharp_w D_s]d\mu(t)d\mu(s).
\end{aligned}$$

If $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are all increasing, we have by the monotonicity of geometric means (Theorem 2.34) that

$$A_t\sharp_w C_t \geqslant A_s\sharp_w C_s \quad\text{and}\quad B_t\sharp_w D_t \geqslant B_s\sharp_w D_s.$$

If $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are all decreasing, we have

$$A_t\sharp_w C_t \leqslant A_s\sharp_w C_s \quad\text{and}\quad B_t\sharp_w D_t \leqslant B_s\sharp_w D_s.$$

Both cases lead to the same conclusion that

$$(A_t\sharp_w C_t-A_s\sharp_w C_s)\boxtimes(B_t\sharp_w D_t-B_s\sharp_w D_s) \geqslant 0,$$

and hence (4.11) holds. The cases (ii) yield the same conclusion that

$$(A_t\sharp_w C_t-A_s\sharp_w C_s)\boxtimes(B_t\sharp_w D_t-B_s\sharp_w D_s) \leqslant 0.$$

and hence the reverse of (4.11) holds. □

Recall that a continuous function $J\to\mathbb{R}$ is **super-multiplicative** if

$$f(xy) \geqslant f(x)f(y) \quad\text{for all}\ \ x,y\in J.$$

In [31], Moslehian and Bakherad established Chebyshev-type inequality regarding operator means. For increasing fields $\mathcal{A}=(A_t)_{t\in\Omega},\mathcal{B}=(B_t)_{t\in\Omega},\mathcal{C}=(C_t)_{t\in\Omega},\mathcal{D}=(D_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})^+$ and a bounded measurable function $\alpha : \Omega \to [0,\infty)$, we have [31, Theorem 3.1]

$$\begin{aligned}\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \odot B_t)\sigma(C_t \odot D_t)]d\mu(t) \\ \geqslant \int_\Omega \alpha(t)(A_t\sigma C_t)d\mu(t) \odot \int_\Omega \alpha(s)(B_s\sigma D_s)d\mu(s),\end{aligned} \tag{4.12}$$

where σ is an operator mean with the super-multiplicative representing function. We can see that Inequality (4.11) is a Tracy-Singh product version of (4.12) for the case of the representing function $f = t^w$.

The next corollary is a discrete version of Theorem 4.11.

Corollary 4.12. Let $A_i, B_i, C_i, D_i \in \mathfrak{B}(\mathbb{H})^+$ and $\omega \geqslant 0$ for each $i = 1,\dots,k$. Let $\mathcal{A} = (A_1,\dots,A_k)$, $\mathcal{B} = (B_1,\dots,B_k)$, $\mathcal{C} = (C_1,\dots,C_k)$, $\mathcal{D} = (D_1,\dots,D_k)$.

(i) If $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are either all increasing, or all decreasing, then

$$\left(\sum_{i=1}^k \omega_i\right)\left(\sum_{i=1}^k \omega_i[(A_i \boxtimes B_i)\sharp_w(C_i \boxtimes D_i)]\right) \geqslant \left(\sum_{i=1}^k \omega_i(A_i\sharp_w C_i)\right) \boxtimes \left(\sum_{i=1}^k \omega_i(B_i\sharp_w D_i)\right). \tag{4.13}$$

(ii) The reverse of (4.13) holds if either

(a) $\mathcal{A},\mathcal{C}$ are increasing and $\mathcal{B},\mathcal{D}$ are decreasing, or

(b) $\mathcal{A},\mathcal{C}$ are decreasing and $\mathcal{B},\mathcal{D}$ are increasing.

Proof. Setting $\Omega = \{1,\dots,k\}$ equipped with the counting measure and $\alpha(i) = \omega_i$ for all $i = 1,\dots,k$ in Theorem 4.11, we get the result. □

4.2.2 Inequalities on Weighted Arithmetic Means

Lemma 4.13. Let A, B, C, D be self-adjoint operators in $\mathfrak{B}(\mathbb{H})$ and $w \in [0,1]$.

(i) If (A,B) and (C,D) have the synchronous property, then

$$(A \boxtimes B)\nabla_w(C \boxtimes D) \geqslant (A\nabla_w C) \boxtimes (B\nabla_w D). \tag{4.14}$$

(ii) If (A,B) and (C,D) have the opposite-synchronous property, then the reverse of (4.14) holds.

Proof. For the synchronous case, we have by using positivity of the Tracy-Singh product (Theorem 2.43) that $(A - C) \boxtimes (B - D) \geqslant 0$. Applying Theorem 2.43, we obtain

$$\begin{aligned} 0 &\leqslant w(1-w)\,[(A-C)\boxtimes(B-D)] \\ &= w(1-w)\,[A\boxtimes B - A\boxtimes D - C\boxtimes B + C\boxtimes D] \\ &= [(1-w)(A\boxtimes B) + w(C\boxtimes D)] - [(1-w)A + wC]\boxtimes[(1-w)B + wD] \\ &= [(A\boxtimes B)\,\triangledown_w\,(C\boxtimes D)] - [(A\,\triangledown_w\,C)\boxtimes(B\,\triangledown_w\,D)]. \end{aligned}$$

Thus $(A \triangledown_w C) \boxtimes (B \triangledown_w D) \leqslant (A \boxtimes B) \triangledown_w (C \boxtimes D)$.

For the opposite-synchronous case, we have $(A - C) \boxtimes (B - D) \leqslant 0$ and hence the reverse of inequality (4.14) holds. □

Theorem 4.14. Let $\mathcal{A} = (A_t)_{t\in\Omega}, \mathcal{B} = (B_t)_{t\in\Omega}, \mathcal{C} = (C_t)_{t\in\Omega}, \mathcal{D} = (D_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$, $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function and $w \in [0, 1]$.

(i) If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the synchronous monotone property and all of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either increasing or decreasing, then

$$\begin{aligned} &\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t)\triangledown_w(C_t \boxtimes D_t)]d\mu(t) \\ &\qquad\qquad \geqslant \int_\Omega \alpha(t)(A_t \triangledown_w C_t)d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s \triangledown_w D_s)d\mu(s). \end{aligned} \tag{4.15}$$

(ii) If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the opposite-synchronous monotone property and if either

(a) $\mathcal{A}, \mathcal{C}$ are increasing and $\mathcal{B}, \mathcal{D}$ are decreasing, or

(b) $\mathcal{A}, \mathcal{C}$ are decreasing and $\mathcal{B}, \mathcal{D}$ are increasing,

then the reverse of (4.15) holds.

Proof. Let $s, t \in \Omega$ and assume without loss of generality that $s \preccurlyeq t$. First, we consider the case (i). We have by using Theorem 2.43, Lemma 4.13, Proposition 4.1, and Fubini's

Theorem [7] that

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)[(A_t \boxtimes B_t)\nabla_w(C_t\boxtimes D_t)]d\mu(t)\\
&\quad -\int_\Omega \alpha(t)(A_t\nabla_w C_t)d\mu(t)\boxtimes \int_\Omega \alpha(s)(B_s\nabla_w D_s)d\mu(s)\\
&\qquad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxtimes B_t)\nabla_w(C_t\boxtimes D_t)]d\mu(t)d\mu(s)\\
&\qquad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t\nabla_w C_t)\boxtimes(B_s\nabla_w D_s)]d\mu(t)d\mu(s)\\
&\qquad \geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\nabla_w C_t)\boxtimes(B_t\nabla_w D_t)]d\mu(t)d\mu(s)\\
&\qquad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t\nabla_w C_t)\boxtimes(B_s\nabla_w D_s)]d\mu(t)d\mu(s)\\
&\qquad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\nabla_w C_t)\boxtimes(B_t\nabla_w D_t - B_s\nabla_w D_s)]d\mu(t)d\mu(s)\\
&\qquad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\nabla_w C_t)\boxtimes(B_t\nabla_w D_t - B_s\nabla_w D_s)]d\mu(t)d\mu(s)\\
&\qquad\quad + \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_s\nabla_w C_s)\boxtimes(B_s\nabla_w D_s - B_t\nabla_w D_t)]d\mu(s)d\mu(t)\\
&\qquad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\nabla_w C_t - A_s\nabla_w C_s)\boxtimes(B_t\nabla_w D_t - B_s\nabla_w D_s)]d\mu(t)d\mu(s).
\end{aligned}$$

If all of $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are increasing, we have by Theorem 2.34 that

$$A_t\nabla_w C_t \geqslant A_s\nabla_w C_s \quad \text{and} \quad B_t\nabla_w D_t \geqslant B_s\nabla_w D_s.$$

If all of $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are decreasing, we have

$$A_t\nabla_w C_t \leqslant A_s\nabla_w C_s \quad \text{and} \quad B_t\nabla_w D_t \leqslant B_s\nabla_w D_s.$$

Using the positivity of the Tracy-Singh product (Theorem 2.43), both cases lead to the same conclusion that

$$(A_t\nabla_w C_t - A_s\nabla_w C_s)\boxtimes(B_t\nabla_w D_t - B_s\nabla_w D_s) \geqslant 0$$

and hence (4.15) holds.

Now, we consider the case (ii). Since $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ have the opposite-synchronous monotone property, we have by applying Theorem 2.43, Lemma 4.13, Proposition 4.1, and Fubini's Theorem [7] that

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)[(A_t \boxtimes B_t)\nabla_w(C_t\boxtimes D_t)]d\mu(t)\\
&\quad -\int_\Omega \alpha(t)(A_t\nabla_w C_t)d\mu(t)\boxtimes \int_\Omega \alpha(s)(B_s\nabla_w D_s)d\mu(s)\\
&\qquad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxtimes B_t)\nabla_w(C_t\boxtimes D_t)]d\mu(t)d\mu(s)\\
&\qquad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t\nabla_w C_t)\boxtimes(B_s\nabla_w D_s)]d\mu(t)d\mu(s)\\
&\qquad \leqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\nabla_w C_t)\boxtimes(B_t\nabla_w D_t)]d\mu(t)d\mu(s)\\
&\qquad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t\nabla_w C_t)\boxtimes(B_s\nabla_w D_s)]d\mu(t)d\mu(s)\\
&\qquad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\nabla_w C_t - A_s\nabla_w C_s)\boxtimes(B_t\nabla_w D_t - B_s\nabla_w D_s)]d\mu(t)d\mu(s).
\end{aligned}$$

The cases (a) and (b) yield the same conclusion that

$$(A_t \nabla_w C_t - A_s \nabla_w C_s) \boxtimes (B_t \nabla_w D_t - B_s \nabla_w D_s) \leqslant 0$$

and hence the reverse of (4.15) holds. □

Corollary 4.15. Let $A_i, B_i, C_i, D_i \in \mathfrak{B}(\mathbb{H})^+$ and $\omega \geqslant 0$ for each $i = 1, \ldots, k$. Let $\mathcal{A} = (A_1, \ldots, A_k)$, $\mathcal{B} = (B_1, \ldots, B_k)$, $\mathcal{C} = (C_1, \ldots, C_k)$, $\mathcal{D} = (D_1, \ldots, D_k)$.

(i) If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the synchronous monotone property and all of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either increasing or decreasing, then

$$\left(\sum_{i=1}^{k} \omega_i\right) \left(\sum_{i=1}^{k} \omega_i [(A_i \boxtimes B_i) \nabla_w (C_i \boxtimes D_i)]\right) \geqslant \left(\sum_{i=1}^{k} \omega_i (A_i \nabla_w C_i)\right) \boxtimes \left(\sum_{i=1}^{k} \omega_i (B_i \nabla_w D_i)\right). \tag{4.16}$$

(ii) If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the opposite-synchronous monotone property and if either

(a) $\mathcal{A}, \mathcal{C}$ are increasing and $\mathcal{B}, \mathcal{D}$ are decreasing, or

(b) $\mathcal{A}, \mathcal{C}$ are decreasing and $\mathcal{B}, \mathcal{D}$ are increasing,

then reverse of (4.16) holds.

4.2.3 Inequalities on Weighted Harmonic Means

Lemma 4.16. Let A, B, C, D be operators in $\mathfrak{B}(\mathbb{H})^+$ and $w \in [0, 1]$.

(i) If (A, B) and (C, D) have the synchronous property, then

$$(A \boxtimes B) \,!_w\, (C \boxtimes D) \leqslant (A \,!_w\, C) \boxtimes (B \,!_w\, D). \tag{4.17}$$

(ii) If (A, B) and (C, D) have the opposite-synchronous property, then the reverse of (4.17) holds.

Proof. By continuity, we may assume that $A, B, C, D > 0$. Assume that (A, B) and (C, D) have the synchronous property. The case $A \geqslant C$ and $B \geqslant D$ leads to $A^{-1} \leqslant C^{-1}$ and $B^{-1} \leqslant D^{-1}$, and thus

$$(A^{-1} - C^{-1}) \boxtimes (B^{-1} - D^{-1}) \geqslant 0. \tag{4.18}$$

The case $A \leqslant C$ and $B \leqslant D$ also leads to the inequality (4.18). Using Theorem 2.43 and (4.18), we get

$$\begin{aligned}
0 &\leqslant w(1-w)\left(A^{-1}-C^{-1}\right) \boxtimes \left(B^{-1}-D^{-1}\right) \\
&= w(1-w)A^{-1} \boxtimes B^{-1} + w(1-w)C^{-1} \boxtimes D^{-1} - w(1-w)A^{-1} \boxtimes D^{-1} - w(1-w)C^{-1} \boxtimes B^{-1} \\
&= \left[(1-w)-(1-w)^2\right] A^{-1} \boxtimes B^{-1} + (w-w^2)C^{-1} \boxtimes D^{-1} - w(1-w)A^{-1} \boxtimes D^{-1} \\
&\quad - w(1-w)C^{-1} \boxtimes B^{-1} \\
&= \left(A^{-1} \boxtimes B^{-1}\right) \triangledown_w \left(C^{-1} \boxtimes D^{-1}\right) - \left(A^{-1} \triangledown_w C^{-1}\right) \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right).
\end{aligned}$$

This implies that

$$\left(A^{-1} \boxtimes B^{-1}\right) \triangledown_w \left(C^{-1} \boxtimes D^{-1}\right) \geqslant \left(A^{-1} \triangledown_w C^{-1}\right) \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right).$$

Hence,

$$\begin{aligned}
(A \boxtimes B) \,!_w\, (C \boxtimes D) &= \left\{(A \boxtimes B)^{-1} \triangledown_w (C \boxtimes D)^{-1}\right\}^{-1} \\
&= \left\{\left(A^{-1} \boxtimes B^{-1}\right) \triangledown_w \left(C^{-1} \boxtimes D^{-1}\right)\right\}^{-1} \\
&\leqslant \left\{\left(A^{-1} \triangledown_w C^{-1}\right) \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right)\right\}^{-1} \\
&= \left(A^{-1} \triangledown_w C^{-1}\right)^{-1} \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right)^{-1} \\
&= (A \,!_w\, C) \boxtimes (B \,!_w\, D).
\end{aligned}$$

For the opposite-synchronous case, we have

$$(A^{-1} - C^{-1}) \boxtimes (B^{-1} - D^{-1}) \leqslant 0$$

and hence the reverse of (4.17) holds. For arbitrary $A, B, C, D \geqslant 0$, perturb each of them with εI and then take limit as $\varepsilon \to 0^+$. □

Theorem 4.17. Let $\mathcal{A} = (A_t)_{t\in\Omega}, \mathcal{B} = (B_t)_{t\in\Omega}, \mathcal{C} = (C_t)_{t\in\Omega}, \mathcal{D} = (D_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$, $\alpha : \Omega \to [0,\infty)$ be a bounded measurable function and $w \in [0,1]$.

(i) If $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ have the opposite-synchronous monotone property and if all of $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are either increasing or decreasing, then

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t) \,!_w\, (C_t \boxtimes D_t)]d\mu(t) \\
&\qquad \geqslant \int_\Omega \alpha(t)(A_t \,!_w\, C_t)d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s \,!_w\, D_s)d\mu(s).
\end{aligned} \tag{4.19}$$

(ii) If $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ have synchronous monotone property and if either

(a) $\mathcal{A},\mathcal{C}$ are both increasing, and $\mathcal{B},\mathcal{D}$ are both decreasing, or

(b) $\mathcal{A},\mathcal{C}$ are both decreasing and $\mathcal{B},\mathcal{D}$ are both increasing, then the reverse of (4.19) holds.

Proof. Let $s,t \in \Omega$ with $s \preccurlyeq t$. If the pairs $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ are opposite-synchronous, then we have by applying Theorem 2.43, Lemma 4.16, Proposition 4.1, and Fubini's Theorem [7] that

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t) \,!_w\, (C_t \boxtimes D_t)]d\mu(t) \\
&\quad - \int_\Omega \alpha(t)(A_t \,!_w\, C_t)d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s \,!_w\, D_s)d\mu(s) \\
&\qquad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxtimes B_t) \,!_w\, (C_t \boxtimes D_t)]d\mu(t)d\mu(s) \\
&\qquad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \,!_w\, C_t) \boxtimes (B_s \,!_w\, D_s)]d\mu(t)d\mu(s) \\
&\qquad \geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \,!_w\, C_t) \boxtimes (B_t \,!_w\, D_t)]d\mu(t)d\mu(s) \\
&\qquad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \,!_w\, C_t) \boxtimes (B_s \,!_w\, D_s)]d\mu(t)d\mu(s) \\
&\qquad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \,!_w\, C_t) \boxtimes (B_t \,!_w\, D_t - B_s \,!_w\, D_s)]d\mu(t)d\mu(s) \\
&\qquad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \,!_w\, C_t) \boxtimes (B_t \,!_w\, D_t - B_s \,!_w\, D_s)]d\mu(t)d\mu(s) \\
&\qquad\quad + \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_s \,!_w\, C_s) \boxtimes (B_s \,!_w\, D_s - B_t \,!_w\, D_t)]d\mu(s)d\mu(t) \\
&\qquad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[A_t \,!_w\, C_t - A_s \,!_w\, C_s] \boxtimes [B_t \,!_w\, D_t - B_s \,!_w\, D_s]d\mu(t)d\mu(s).
\end{aligned}$$

If all of $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are increasing, we have by the monotonicity of the weighted harmonic mean (Theorem 2.34) that

$$A_t \,!_w\, C_t \;\geqslant\; A_s \,!_w\, C_s \quad \text{and} \quad B_t \,!_w\, D_t \;\geqslant\; B_s \,!_w\, D_s.$$

If all of $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are decreasing, we have

$$A_t \,!_w\, C_t \;\leqslant\; A_s \,!_w\, C_s \quad \text{and} \quad B_t \,!_w\, D_t \;\leqslant\; B_s \,!_w\, D_s.$$

Both ceases lead to the same conclusion that

$$(A_t \,!_w\, C_t - A_s \,!_w\, C_s) \boxtimes (B_t \,!_w\, D_t - B_s \,!_w\, D_s) \;\geqslant\; 0$$

and hence (4.19) holds. Another assertion can be proved in a similar manner to that of the second assertion in Theorem 4.14. □

Corollary 4.18. Let $A_i, B_i, C_i, D_i \in \mathfrak{B}(\mathbb{H})^+$ and $\omega \geqslant 0$ for each $i = 1, \ldots, k$. Let $\mathcal{A} = (A_1, \ldots, A_k)$, $\mathcal{B} = (B_1, \ldots, B_k)$, $\mathcal{C} = (C_1, \ldots, C_k)$, $\mathcal{D} = (D_1, \ldots, D_k)$.

(i) If $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ have the opposite-synchronous monotone property and if all of $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are either increasing or decreasing, then

$$\left(\sum_{i=1}^{k}\omega_i\right)\left(\sum_{i=1}^{k}\omega_i[(A_i\boxtimes B_i)\,!_w\,(C_i\boxtimes D_i)]\right) \geqslant \left(\sum_{i=1}^{k}\omega_i(A_i\,!_w\,C_i)\right)\boxtimes\left(\sum_{i=1}^{k}\omega_i(B_i\,!_w\,D_i)\right). \tag{4.20}$$

(ii) If $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ have synchronous monotone property and if either

(a) $\mathcal{A},\mathcal{C}$ are both increasing, and $\mathcal{B},\mathcal{D}$ are both decreasing, or

(b) $\mathcal{A},\mathcal{C}$ are both decreasing and $\mathcal{B},\mathcal{D}$ are both increasing,

then the reverse of (4.20) holds.

4.3 Chebyshev-Grüss-Type Inequaities via Oscillations

Throughout this section, let Ω be a compact Housedorff space equipped with a probability Radon measure μ. For any continuous field $\mathcal{A}=(A_t)_{t\in\Omega}$ in $\mathfrak{B}(\mathbb{H})$ and $\mathcal{B}=(B_t)_{t\in\Omega}$ in $\mathfrak{B}(\mathbb{K})$, we define

$$\mathcal{A}\boxtimes\mathcal{B} = (A_t\boxtimes B_t)_{t\in\Omega}, \quad \mathcal{I}(\mathcal{A}) = \int_\Omega A_t\,d\mu(t),$$

$$\mathrm{osc}(\mathcal{A}) = \max\{\|A_t-A_s\| : (t,s)\in \mathrm{supp}(\mu\times\mu)\}.$$

Here, we recall that the **support** of the product measure $\mu\times\mu$ is defined by

$$\mathrm{supp}(\mu\times\mu) = \{(t,s)\in\Omega^2 : (\mu\times\mu)(G)>0 \text{ for all open sets } G\subseteq\Omega^2 \text{ containing } (t,s)\}.$$

We call $\mathrm{osc}(\mathcal{A})$ the **oscillation** of the field $\mathcal{A}$.

In the next theorem, we generalize Grüss-type inequality (1.14) to the case of operators involving Tracy-Singh products.

Theorem 4.19. Let $\mathcal{A}=(A_t)_{t\in\Omega}$ and $\mathcal{B}=(B_t)_{t\in\Omega}$ be bounded continuous fields of self-adjoint operators in $\mathfrak{B}(\mathbb{H})$ and $\mathfrak{B}(\mathbb{K})$, respectively. Then

$$\mathcal{I}(\mathcal{A}\boxtimes\mathcal{B})-\mathcal{I}(\mathcal{A})\boxtimes\mathcal{I}(\mathcal{B}) \leqslant \frac{1}{2}\mathrm{osc}(\mathcal{A})\cdot\mathrm{osc}(\mathcal{B})(\mu\times\mu)(\Omega^2\backslash\Delta)I, \tag{4.21}$$

where $\Delta=\{(t,t): t\in\Omega\}$.

Proof. We have by using Theorem 2.43, Proposition 4.1 and Fubini's Theorem [7] that

$$\begin{aligned}
&\mathcal{I}(\mathcal{A}\boxtimes\mathcal{B}) - \mathcal{I}(\mathcal{A})\boxtimes\mathcal{I}(\mathcal{B}) \\
&\quad= \int_\Omega A_t\boxtimes B_t d\mu(t) - \int_\Omega A_t d\mu(t)\boxtimes\int_\Omega B_s d\mu(s) \\
&\quad= \int_\Omega d\mu(s)\int_\Omega A_t\boxtimes B_t d\mu(t) - \int_\Omega A_t d\mu(t)\boxtimes\int_\Omega B_s d\mu(s) \\
&\quad= \iint_{\Omega^2} A_t\boxtimes B_t d\mu(t)d\mu(s) - \iint_{\Omega^2} A_t\boxtimes B_s d\mu(t)d\mu(s) \\
&\quad= \iint_{\Omega^2} A_t\boxtimes(B_t - B_s)d\mu(t)d\mu(s) \\
&\quad= \frac{1}{2}\iint_{\Omega^2} A_t\boxtimes(B_t - B_s)d\mu(t)d\mu(s) + \frac{1}{2}\iint_{\Omega^2} A_s\boxtimes(B_s - B_t)d\mu(s)d\mu(t) \\
&\quad= \frac{1}{2}\iint_{\Omega^2}(A_t - A_s)\boxtimes(B_t - B_s)d\mu(t)d\mu(s) \\
&\quad= \frac{1}{2}\iint\limits_{\Omega^2\setminus\Delta}(A_t - A_s)\boxtimes(B_t - B_s)d\mu(t)d\mu(s) \\
&\quad\leqslant \frac{1}{2}\operatorname{osc}(\mathcal{A})\cdot\operatorname{osc}(\mathcal{B})(\mu\times\mu)(\Omega^2\setminus\Delta)I.
\end{aligned}$$

□

We mention that Theorem 4.19 can be viewed as an operator version of [17, Theorem 7].

Corollary 4.20. For each $i = 1,\ldots,k$, let $A_i \in \mathfrak{B}(\mathbb{H})$ and $B_i \in \mathfrak{B}(\mathbb{K})$ be self-adjoint operators. Then

$$\sum_{i=1}^{k}(A_i\boxtimes B_i) - \left(\sum_{i=1}^{k}A_i\right)\boxtimes\left(\sum_{i=1}^{k}B_i\right) \leqslant \frac{k(k-1)}{2}\max_{1\leqslant i,j\leqslant n}\|A_i - A_j\|\cdot\max_{1\leqslant i,j\leqslant k}\|B_i - B_j\|I.$$

Proof. Let $\mathcal{A} = (A_1,\ldots,A_k)$ and $\mathcal{B} = (B_1,\ldots,B_k)$. Set $\Omega = \{1,\ldots,k\}$ equipped with the counting measure. We have

$$(\mu\times\mu)(\Omega^2\setminus\Delta) = \frac{k(k-1)}{2}, \qquad \operatorname{supp}(\mu\times\mu) = \Omega\times\Omega$$

and thus

$$\operatorname{osc}(\mathcal{A}) = \max_{1\leqslant i,j\leqslant k}\|A_i - A_j\|, \qquad \operatorname{osc}(\mathcal{B}) = \max_{1\leqslant i,j\leqslant k}\|B_i - B_j\|.$$

□

Example 4.21. Let $\Omega = [0,1]$, $w \in \Omega$ and $0 < \alpha \leqslant 1$. Consider the probability Radon measure $\mu = \alpha\lambda + (1-\alpha)\delta_w$, where λ is Lebesgue measure on Ω and δ_w is the Dirac measure at w. Set

$$\mathcal{I}(A) := \int_0^1 A_t d\mu(t) = \alpha\int_0^1 A_t d\lambda(t) + (1-\alpha)A_w.$$

We have

$$\begin{aligned}\mu \times \mu &= (\alpha\lambda + (1-\alpha)\delta_w) \times (\alpha\lambda + (1-\alpha)\delta_w) \\ &= \alpha^2(\lambda \times \lambda) + \alpha(1-\alpha)\,(\lambda \times \delta_w) + (1-\alpha)\alpha\,(\delta_w \times \lambda) + (1-\alpha)^2\,(\delta_w \times \delta_w)\,.\end{aligned}$$

Then

$$\begin{aligned}(\mu \times \mu)\left([0,1]^2 \backslash \Delta\right) &= \alpha^2(\lambda \times \lambda)\left([0,1]^2 \backslash \Delta\right) + \alpha(1-\alpha)\,(\lambda \times \delta_w)\left([0,1]^2 \backslash \Delta\right) \\ &\quad + (1-\alpha)\alpha\,(\delta_w \times \lambda)\left([0,1]^2 \backslash \Delta\right) + (1-\alpha)^2\,(\delta_w \times \delta_w)\left([0,1]^2 \backslash \Delta\right) \\ &= \alpha(2-\alpha).\end{aligned}$$

For any continuous fields $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ of self-adjoint operators, the inequality (4.21) becomes

$$\mathcal{I}(\mathcal{A} \boxtimes \mathcal{B}) - \mathcal{I}(\mathcal{A}) \boxtimes \mathcal{I}(\mathcal{B}) \leqslant \frac{1}{2}\alpha(2-\alpha) \max_{0\leqslant s,t\leqslant 1} \|A_t - A_s\| \cdot \max_{0\leqslant t,s\leqslant 1} \|B_t - B_s\| I.$$

Chapter 5

Chebyshev-Type Inequalities for Operators Involving Khatri-Rao Products and Weighted Pythagorean Means

In this chapter, we obtain Chebyshev-type integral inequalities of bounded continuous fields of operators concerning Khatri-Rao products and weighted Pythagorean means. The fields considered here are parametrized by a locally compact Hausdorff space Ω equipped with a finite Radon measure μ. We also derive Chebyshev-Grüss-type integral inequality via oscillations when μ is a probability Radon measure. These integral inequalities can be reduced to discrete inequalities by setting Ω to be a finite space equipped with the counting measure. Our results include Chebyshev-type inequalities for tensor product of operators and Khatri-Rap/Kronecker/Hadamard products of matrices.

Throughout this chapter, let Ω be a locally compact Hausdorff space endowed with a finite Radon measure μ and let $\mathbb{H}, \mathbb{K}$ be complex Hilbert spaces. We fix the following orthoganal decompositions:

$$\mathbb{H} = \bigoplus_{i=1}^{n} \mathbb{H}_i, \quad \mathbb{K} = \bigoplus_{i=1}^{n} \mathbb{K}_i,$$

where all $\mathbb{H}_i$ and $\mathbb{K}_i$ are Hilbert spaces.

Proposition 5.1. Let $\mathcal{A} = (A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})$. Then for any $X \in \mathfrak{B}(\mathbb{K})$,

$$\int_\Omega A_t d\mu(t) \boxdot X = \int_\Omega (A_t \boxdot X) d\mu(t), \tag{5.1}$$

$$X \boxdot \int_\Omega A_t d\mu(t) = \int_\Omega (X \boxdot A_t) d\mu(t). \tag{5.2}$$

Proof. The map $t \mapsto A_t$ is Bochner integrable on Ω because it is continuous. Let $X \in \mathfrak{B}(\mathbb{K})$. Since the map

$$\psi : \mathfrak{B}(\mathbb{H}) \to \mathfrak{B}\Big(\bigoplus_{i=1}^{n} \mathbb{H}_i \otimes \mathbb{K}_i\Big), \quad T \mapsto T \boxdot X,$$

is bounded linear operator, we have by Theorem 2.57 that the map $t \mapsto \psi(A_t)$ is Bochner integrable and

$$\int_\Omega A_t d\mu(t) \boxdot X = \psi\Big(\int_\Omega A_t d\mu(t)\Big) = \int_\Omega \psi(A_t) d\mu(t) = \int_\Omega (A_t \boxdot X) d\mu(t).$$

Similarly, we reach the assertion (5.2) by using tha fact that the map

$$\varphi \;:\; \mathfrak{B}(\mathbb{H}) \;\to\; \mathfrak{B}\Big(\bigoplus_{i=1}^{n} \mathbb{K}_i \otimes \mathbb{H}_i\Big), \qquad T \mapsto X \boxdot T,$$

is linear and bounded. □

5.1 Chebyshev-Type Inequalities Involving Khatri-Rao Products of Operators

From Theorem 4.3, we obtain Chebyshev-type inequalities involving Khatri-Rao products under the assumption on the synchronous Tracy-Singh property as follow:

Proposition 5.2. Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be bounded continuous fields of self-adjoint operators in $\mathfrak{B}(\mathbb{H})$ and $\mathfrak{B}(\mathbb{K})$, respectively, and let $\alpha : \Omega \to [0,\infty)$ be a bounded measurable function.

(i) If $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Tracy-Singh property, then

$$\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \boxdot B_t)d\mu(t) \geqslant \int_\Omega \alpha(t)A_t d\mu(t) \boxdot \int_\Omega \alpha(s)B_s d\mu(s). \qquad (5.3)$$

(ii) If $\mathcal{A}$ and $\mathcal{B}$ have the opposite-synchronous Tracy-Singh property, then the reverse of (5.3) holds.

Proof. Assume that $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Tracy-Singh property. Note that the Khatri-Rao product is expressed as the deformation of Tracy-Singh product via the isometry Z defined in Theorem 2.51. Since the map $X \mapsto Z^*XZ$ is a bounded linear operator, we have, by Theorem 4.3,

$$\begin{aligned}
\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \boxdot B_t)d\mu(t) &= \int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)Z^*(A_t \boxtimes B_t)Z d\mu(t) \\
&= Z^*\left(\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \boxtimes B_t)d\mu(t)\right) Z \\
&\geqslant Z^*\left(\int_\Omega \alpha(t)A_t d\mu(t) \boxtimes \int_\Omega \alpha(s)B_s d\mu(s)\right) Z \\
&= \int_\Omega \alpha(t)A_t d\mu(t) \boxdot \int_\Omega \alpha(s)B_s d\mu(s).
\end{aligned}$$

The case (ii) can be similarly proven. □

From the definition of the synchronous functions in (1.11), we define synchronous Khatri-Rao property for fields of operators as follow:

Definition 5.3. The fields $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$ and $\mathfrak{B}(\mathbb{K})$, respectively, are said to have the **synchronous Khatri-Rao property** if

$$(A_t - A_s) \boxdot (B_t - B_s) \geqslant 0 \tag{5.4}$$

for all $s, t \in \Omega$. They are said to have the **opposite-synchronous Khatri-Rao property** if the reverse of (5.4) holds for all $s, t \in \Omega$.

Note that if two fields $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Tracy-Singh property, then $\mathcal{A}$ and $\mathcal{B}$ have the synchonous Khatri-Rao property.

The next theorem gives Chebyshev-type inequalities involving Khatri-Rao products under the assumption on synchronous Khatri-Rao property.

Theorem 5.4. Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be bounded continuous fields of self-adjoint operators in $\mathfrak{B}(\mathbb{H})$ and $\mathfrak{B}(\mathbb{K})$, respectively, and let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.

(i) If $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Khatri-Rao property, then

$$\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \boxdot B_t)d\mu(t) \geqslant \int_\Omega \alpha(t)A_t d\mu(t) \boxdot \int_\Omega \alpha(s)B_s d\mu(s). \tag{5.5}$$

(ii) If $\mathcal{A}$ and $\mathcal{B}$ have the opposite-synchronous Khatri-Rao property, then the reverse of (5.5) holds.

Proof. Using Theorem 2.49 and Proposition 5.1, we have

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \boxdot B_t)d\mu(t) - \int_\Omega \alpha(t)A_t d\mu(t) \boxdot \int_\Omega \alpha(s)B_s d\mu(s) \\
&\quad = \iint_{\Omega^2} \alpha(s)\alpha(t)(A_t \boxdot B_t)d\mu(t)d\mu(s) - \iint_{\Omega^2} \alpha(s)\alpha(t)(A_t \boxdot B_s)d\mu(t)d\mu(s) \\
&\quad = \iint_{\Omega^2} \alpha(s)\alpha(t)[A_t \boxdot (B_t - B_s)]d\mu(t)d\mu(s) \\
&\quad = \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[A_t \boxdot (B_t - B_s)]d\mu(t)d\mu(s) \\
&\qquad + \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[A_s \boxdot (B_s - B_t)]d\mu(s)d\mu(t) \\
&\quad = \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)\left[(A_t - A_s) \boxdot (B_t - B_s)\right] d\mu(t)d\mu(s).
\end{aligned}$$

Here, we have used Fubini's Theorem [7] to interchange the order of integrals. For the case (i), we have

$$\iint_{\Omega^2} \alpha(s)\alpha(t)\left[(A_t - A_s) \boxdot (B_t - B_s)\right] d\mu(t)d\mu(s) \geqslant 0 \tag{5.6}$$

and thus (5.5) holds. For another case, we get the reverse of (5.6) and, thus, the reverse of (5.5) holds. □

The next corollary is a discrete version of Theorem 5.4.

Corollary 5.5. For each $i = 1,\dots,k$, let $A_i \in \mathfrak{B}(\mathbb{H})$ and $B_i \in \mathfrak{B}(\mathbb{K})$ be self-adjoint operators and ω_i be a nonnegative number. Let $\mathcal{A} = (A_1,\dots,A_k)$ and $\mathcal{B} = (B_1,\dots,B_k)$.

(i) If $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Khatri-Rao property, then

$$\sum_{i=1}^{n} \omega_i \sum_{i=1}^{n} \omega_i (A_i \boxdot B_i) \;\geqslant\; \left(\sum_{i=1}^{n} \omega_i A_i\right) \boxdot \left(\sum_{i=1}^{n} \omega_i B_i\right). \tag{5.7}$$

(ii) If $\mathcal{A}$ and $\mathcal{B}$ have the opposite-synchronous Khatri-Rao property, then the reverse of (5.7) holds.

Recall that, for each $A \in \mathbb{M}_n(\mathbb{C})$, the induced map,

$$T_A : \mathbb{C}^n \to \mathbb{C}^n, \qquad x \mapsto Ax,$$

is bounded linear operator. For any complex matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ partitioned in block-matrix form, we have [36]

$$T_A \boxdot T_B \;=\; T_{A \boxdot B}. \tag{5.8}$$

Example 5.6. Consider $\omega_1 = \omega_2 = \frac{1}{2}$, $\mathcal{A} = (T_{A_1}, T_{A_2})$, and $\mathcal{B} = (T_{B_1}, T_{B_2})$, where

$$A_1 = \left[\begin{array}{c|cc} -1 & 1 & -1 \\ \hline 1 & 2 & 0 \\ -1 & 0 & -3 \end{array}\right], \quad A_2 = \left[\begin{array}{c|cc} 2 & 1 & 0 \\ \hline 1 & 3 & -1 \\ 0 & -1 & 1 \end{array}\right],$$

$$B_1 = \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 1 & -1 & 0 \\ \hline -1 & 0 & 1 \end{array}\right], \quad B_2 = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 4 & 0 \\ \hline 0 & 0 & 2 \end{array}\right].$$

Since $A_2 - A_1 \geqslant 0$, we have $T_{A_2} - T_{A_1} \geqslant 0$. Similarly, $T_{B_2} - T_{B_1} \geqslant 0$. By the positivity of the Khatri-Rao product, we get

$$(T_{A_2} - T_{A_1}) \boxdot (T_{B_2} - T_{B_1}) \;\geqslant\; 0,$$

i.e., $\mathcal{A} = (T_{A_1}, T_{A_2})$, and $\mathfrak{B} = (T_{B_1}, T_{B_2})$ have the synchronous Khatri-Rao product property. Thus (5.7) holds for $\mathcal{A}$ and $\mathfrak{B}$, that is,

$$T_{A_1} \boxdot T_{B_1} + T_{A_2} \boxdot T_{B_2} \;\geqslant\; \frac{1}{2}(T_{A_1} + T_{A_2}) \boxdot (T_{B_1} + T_{B_2}). \tag{5.9}$$

Applying (5.8), we have

$$T_{A_1 \boxdot B_1 + A_2 \boxdot B_2} \;\geqslant\; T_{\frac{1}{2}(A_1 + A_2) \boxdot (B_1 + B_2)}$$

or equivalent to

$$A_1 \boxdot B_1 + A_2 \boxdot B_2 \ \geqslant \ \frac{1}{2}(A_1 + A_2) \boxdot (B_1 + B_2).$$

By computing, we obtain

$$\begin{aligned} A_1 \boxdot B_1 + A_2 \boxdot B_2 - \frac{1}{2}(A_1 + A_2) \boxdot (B_1 + B_2) \\ &= \left[\begin{array}{cc|cc} 3 & 1 & -1 & 1 \\ 1 & 9 & 0 & 0 \\ \hline -1 & 0 & 8 & -2 \\ 1 & 0 & -2 & -1 \end{array}\right] - \frac{1}{2} \left[\begin{array}{cc|cc} 3 & 2 & -2 & 1 \\ 2 & 3 & 0 & 0 \\ \hline -2 & 0 & 15 & -3 \\ 1 & 0 & -3 & -6 \end{array}\right] \\ &= \frac{1}{2} \left[\begin{array}{cc|cc} 3 & 0 & 0 & 1 \\ 0 & 15 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 4 \end{array}\right] \\ &\geqslant 0. \end{aligned}$$

Therefore (5.9) holds.

We mention that Theorem 5.4 and Corollary 5.5 are extensions of Chebyshev integral inequality (1.12) and ChebyShev sum inequality (1.8), respectively. Moreover, Corollary 5.5 generalizes Chebyshev inequality for Hadamard product of matrices in [27, Theorem 2.1].

5.2 Chebyshev-Type Inequalities Involving Weighted Pythagorean Means of Operators

Throughout this section, we assume that the space Ω is equipped with a total ordering $\preccurlyeq$.

5.2.1 Inequalities on Weighted Geometric Means

Lemma 5.7 ([37]). Let $A, B, C, D \in \mathfrak{B}(\mathbb{H})^+$ and $w \in [0, 1]$. Then

$$(A \boxdot B)\sharp_w(C \boxdot D) \ \geqslant \ (A\sharp_w C) \boxdot (B\sharp_w D). \tag{5.10}$$

Theorem 5.8. Let $\mathcal{A} = (A_t)_{t\in\Omega}, \mathcal{B} = (B_t)_{t\in\Omega}, \mathcal{C} = (C_t)_{t\in\Omega}, \mathcal{D} = (D_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$, $\alpha : \Omega \to [0,\infty)$ be a bounded measurable function and $w \in [0,1]$. If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either all increasing, or all decreasing, then

$$\begin{aligned}\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)[(A_t \boxdot B_t)\sharp_w(C_t \boxdot D_t)]d\mu(t) \\ \geqslant \int_\Omega \alpha(t)(A_t\sharp_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s\sharp_w D_s)d\mu(s).\end{aligned} \tag{5.11}$$

Proof. Let $s,t \in \Omega$. Without loss of generally, assume that $s \preccurlyeq t$. Applying Theorem 2.49, Proposition 5.1, Lemma 5.7, and Fubini's Theorem [7], we have

$$\begin{aligned}&\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)[(A_t \boxdot B_t)\sharp_w(C_t \boxdot D_t)]d\mu(t) \\ &- \int_\Omega \alpha(t)(A_t\sharp_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s\sharp_w D_s)d\mu(s) \\ &= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxdot B_t)\sharp_w(C_t \boxdot D_t)]d\mu(t)d\mu(s) \\ &\quad - \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\sharp_w C_t) \boxdot (B_s\sharp_w D_s)]d\mu(t)d\mu(s) \\ &\geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\sharp_w C_t) \boxdot (B_t\sharp_w D_t)]d\mu(t)d\mu(s) \\ &\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t\sharp_w C_t) \boxdot (B_s\sharp_w D_s)]d\mu(t)d\mu(s) \\ &= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\sharp_w C_t) \boxdot (B_t\sharp_w D_t - B_s\sharp_w D_s)]d\mu(t)d\mu(s) \\ &= \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\sharp_w C_t) \boxdot (B_t\sharp_w D_t - B_s\sharp_w D_s)]d\mu(t)d\mu(s) \\ &\quad + \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_s\sharp_w C_s) \boxdot (B_s\sharp_w D_s - B_t\sharp_w D_t)]d\mu(s)d\mu(t) \\ &= \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[A_t\sharp_w C_t - A_s\sharp_w C_s] \boxdot [B_t\sharp_w D_t - B_s\sharp_w D_s]d\mu(t)d\mu(s).\end{aligned}$$

If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are all increasing, we have by Lemma 2.34 that

$$A_t\sharp_w C_t \geqslant A_s\sharp_w C_s \quad \text{and} \quad B_t\sharp_w D_t \geqslant B_s\sharp_w D_s.$$

If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are all decreasing, we have

$$A_t\sharp_w C_t \leqslant A_s\sharp_w C_s \quad \text{and} \quad B_t\sharp_w D_t \leqslant B_s\sharp_w D_s.$$

By Theorem 2.49, both cases lead to the same conclusion that

$$(A_t\sharp_w C_t - A_s\sharp_w C_s) \boxdot (B_t\sharp_w D_t - B_s\sharp_w D_s) \geqslant 0,$$

and hence (5.11) holds. □

Note that Inequality (5.11) is a Khatri-Rao product version of (4.12) for the case of the representing function $f = t^w$.

Corollary 5.9. For each $i = 1, \dots, k$, let $A_i, B_i, C_i, D_i \in \mathfrak{B}(\mathbb{H})^+$ and ω_i be a nonnegative number. Let $\mathcal{A} = (A_1, \dots, A_k), \mathcal{B} = (B_1, \dots, B_k), \mathcal{C} = (C_1, \dots, C_k), \mathcal{D} = (D_1, \dots, D_k)$ and $w \in [0, 1]$. If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either all increasing, or all decreasing, then

$$\sum_{i=1}^{k} \omega_i \sum_{i=1}^{k} \omega_i (A_i \boxdot B_i) \sharp_w (C_i \boxdot D_i)] \ \geqslant \ \left(\sum_{i=1}^{k} \omega_i (A_i \sharp_w C_i) \right) \boxdot \left(\sum_{i=1}^{k} \omega_i (B_i \sharp_w D_i) \right). \qquad (5.12)$$

The next goal is to establish a reverse version of Theorem 5.8.

Lemma 5.10. Let $A, B, C, D \in \mathfrak{B}(\mathbb{H})^+$ be such that $0 < m_1 I \leqslant A \boxtimes B \leqslant M_1 I$ and $0 < m_2 I \leqslant C \boxtimes D \leqslant M_2 I$. Denote $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and

$$\lambda \ = \ \frac{Mm^w - mM^w}{(1-w)(M-m)} \cdot \left(\frac{1-w}{w} \cdot \frac{M^w - m^w}{Mm^w - mM^w} \right)^w . \qquad (5.13)$$

Then for any $w \in (0, 1)$, we have

$$\lambda (A \boxdot B) \sharp_w (C \boxdot D) \ \leqslant \ (A \sharp_w C) \boxdot (B \sharp_w D). \qquad (5.14)$$

Proof. Consider a map $\Phi : X \mapsto Z^* X Z$, where Z is the isometry in Theorem 2.51. Since Φ is a unital positive linear map, we have by [28, Corollary 3.5] that

$$\lambda [\Phi (A \boxtimes B) \sharp_w \Phi (C \boxtimes D)] \ \leqslant \ \Phi \left((A \boxtimes B) \sharp_w (C \boxtimes D) \right).$$

Using Lemma 4.10, we get

$$\begin{aligned} \lambda [(A \boxdot B) \sharp_w (C \boxdot D)] \ &= \ \lambda [\Phi (A \boxtimes B) \sharp_w \Phi (C \boxtimes D)] \\ &\leqslant \ \Phi \left((A \boxtimes B) \sharp_w (C \boxtimes D) \right) \\ &= \ \Phi \left((A \sharp_w C) \boxtimes (B \sharp_w D) \right) \\ &= \ (A \sharp_w C) \boxdot (B \sharp_w D). \end{aligned}$$

□

Theorem 5.11. Let $\mathcal{A} = (A_t)_{t \in \Omega}, \mathcal{B} = (B_t)_{t \in \Omega}, \mathcal{C} = (C_t)_{t \in \Omega}, \mathcal{D} = (D_t)_{t \in \Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$ with $0 < m_1 I \leqslant A_t \boxtimes B_t \leqslant M_1 I$ and $0 < m_2 I \leqslant C_t \boxtimes D_t \leqslant M_2 I$ for all $t \in \Omega$, and let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function. Let m, M, w, λ as in Lemma 5.10. If either

(i) $\mathcal{A}, \mathcal{C}$ are increasing and $\mathcal{B}, \mathcal{D}$ are decreasing, or

(ii) $\mathcal{A}, \mathcal{C}$ are decreasing and $\mathcal{B}, \mathcal{D}$ are increasing,

then

$$\begin{aligned} &\lambda \int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t) [(A_t \boxdot B_t) \sharp_w (C_t \boxdot D_t)] d\mu(t) \\ &\qquad \leqslant \int_\Omega \alpha(t) (A_t \sharp_w C_t) d\mu(t) \boxdot \int_\Omega \alpha(s) (B_s \sharp_w D_s) d\mu(s). \end{aligned} \qquad (5.15)$$

Proof. Applying Theorem 2.49, Proposition 5.1, Lemma 5.10 and Fubini's Theorem [7], we have

$$\begin{aligned}
&\lambda\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)[(A_t \boxdot B_t)\sharp_w(C_t\boxdot D_t)]d\mu(t)\\
&\quad -\int_\Omega \alpha(t)(A_t\sharp_w C_t)d\mu(t)\boxdot\int_\Omega \alpha(s)(B_s\sharp_w D_s)d\mu(s)\\
&\qquad = \iint_{\Omega^2}\alpha(s)\alpha(t)[\lambda(A_t\boxdot B_t)\sharp_w(C_t\boxdot D_t)]d\mu(t)d\mu(s)\\
&\qquad\quad -\iint_{\Omega^2}\alpha(s)\alpha(t)[(A_t\sharp_w C_t)\boxdot(B_s\sharp_w D_s)]d\mu(t)d\mu(s)\\
&\qquad \leqslant \iint_{\Omega^2}\alpha(s)\alpha(t)[(A_t\sharp_w C_t)\boxdot(B_t\sharp_w D_t)]d\mu(t)d\mu(s)\\
&\qquad\quad -\iint_{\Omega^2}\alpha(t)\alpha(s)[(A_t\sharp_w C_t)\boxdot(B_s\sharp_w D_s)]d\mu(t)d\mu(s)\\
&\qquad = \iint_{\Omega^2}\alpha(s)\alpha(t)[(A_t\sharp_w C_t)\boxdot(B_t\sharp_w D_t - B_s\sharp_w D_s)]d\mu(t)d\mu(s)\\
&\qquad = \frac{1}{2}\iint_{\Omega^2}\alpha(s)\alpha(t)[(A_t\sharp_w C_t)\boxdot(B_t\sharp_w D_t - B_s\sharp_w D_s)]d\mu(t)d\mu(s)\\
&\qquad\quad +\frac{1}{2}\iint_{\Omega^2}\alpha(s)\alpha(t)[(A_s\sharp_w C_s)\boxdot(B_s\sharp_w D_s - B_t\sharp_w D_t)]d\mu(s)d\mu(t)\\
&\qquad = \frac{1}{2}\iint_{\Omega^2}\alpha(s)\alpha(t)[A_t\sharp_w C_t - A_s\sharp_w C_s]\boxdot[B_t\sharp_w D_t - B_s\sharp_w D_s]d\mu(t)d\mu(s).
\end{aligned}$$

We have by the monotonicity of the geometric mean (Theorem 2.34) with Theorem 2.49 that

$$(A_t\sharp_w C_t - A_s\sharp_w C_s)\boxdot(B_t\sharp_w D_t - B_s\sharp_w D_s)\ \leqslant\ 0$$

and hence (5.15) holds. □

Corollary 5.12. For each $i = 1,\ldots,k$, let $A_i, B_i, C_i, D_i \in \mathfrak{B}(\mathbb{H})^+$ be such that $0 < m_1 I \leqslant A_t \boxtimes B_t \leqslant M_1 I$ and $0 < m_2 I \leqslant C_t \boxtimes D_t \leqslant M_2 I$ for all $t \in \Omega$, and ω_i be a nonnegative number. Let $\mathcal{A} = (A_1,\ldots,A_k), \mathcal{B} = (B_1,\ldots,B_k), \mathcal{C} = (C_1,\ldots,C_k), \mathcal{D} = (D_1,\ldots,D_k)$ and m, M, w, λ as in Lemma 5.10. If either

(i) $\mathcal{A},\mathcal{C}$ are increasing and $\mathcal{B},\mathcal{D}$ are decreasing, or

(ii) $\mathcal{A},\mathcal{C}$ are decreasing and $\mathcal{B},\mathcal{D}$ are increasing,

then

$$\lambda\sum_{i=1}^{k}\omega_i\sum_{i=1}^{k}\omega_i(A_i\boxdot B_i)\sharp_w(C_i\boxdot D_i)] \quad\leqslant\ \left(\sum_{i=1}^{k}\omega_i(A_i\sharp_w C_i)\right)\boxdot\left(\sum_{i=1}^{k}\omega_i(B_i\sharp_w D_i)\right). \quad (5.16)$$

5.2.2 Inequalities on Weighted Arithmetic Means

Lemma 5.13. Let $A, B, C, D \in \mathfrak{B}(\mathbb{H})^+$ and $w \in [0,1]$.

(i) If (A, B) and (C, D) have the synchronous property, then

$$(A \boxdot B)\nabla_w (C \boxdot D) \geqslant (A\nabla_w C) \boxdot (B\nabla_w D). \tag{5.17}$$

(ii) If (A, B) and (C, D) have the opposite-synchronous property, then the reverse of (5.17) holds.

Proof. For the case of synchronous, we have by applying Theorem 2.51 and Lemma 4.13 that

$$\begin{aligned}(A \boxdot B)\,\nabla_w\,(C \boxdot D) &= [Z^*(A \boxtimes B)Z]\,\nabla_w\,[Z^*(C \boxtimes D)Z] \\ &= Z^*[(A \boxtimes B)\,\nabla_w\,(C \boxtimes D)]Z \\ &\geqslant Z^*[(A\,\nabla_w\,C) \boxtimes (B\,\nabla_w\,D)]Z \\ &= (A\,\nabla_w\,C) \boxdot (B\,\nabla_w\,D).\end{aligned}$$

The case (ii) can be similarly proven. □

Theorem 5.14. Let $\mathcal{A} = (A_t)_{t\in\Omega}, \mathcal{B} = (B_t)_{t\in\Omega}, \mathcal{C} = (C_t)_{t\in\Omega}, \mathcal{D} = (D_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$. Let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function and $w \in [0, 1]$.

(i) If $(\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D})$ have the synchronous monotone property and all of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either increasing or decreasing, then

$$\begin{aligned}&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t)\nabla_w (C_t \boxdot D_t)]d\mu(t) \\ &\qquad \geqslant \int_\Omega \alpha(t)(A_t\nabla_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s\nabla_w D_s)d\mu(s).\end{aligned} \tag{5.18}$$

(ii) If $(\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D})$ have the opposite-synchronous monotone property and if either

(a) $\mathcal{A}, \mathcal{C}$ are increasing and $\mathcal{B}, \mathcal{D}$ are decreasing, or

(b) $\mathcal{A}, \mathcal{C}$ are decreasing and $\mathcal{B}, \mathcal{D}$ are increasing,

then the reverse of (5.18) holds.

Proof. First, we consider the case (i). We have by using Theorem 2.49, Proposition 5.1, Lemma 5.13 and Fubini's Theorem [7] that

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)[(A_t \boxdot B_t)\nabla_w(C_t \boxdot D_t)]d\mu(t) \\
&-\int_\Omega \alpha(t)(A_t\nabla_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s\nabla_w D_s)d\mu(s) \\
&\qquad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxdot B_t)\nabla_w(C_t \boxdot D_t)]d\mu(t)d\mu(s) \\
&\qquad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t\nabla_w C_t) \boxdot (B_s\nabla_w D_s)]d\mu(t)d\mu(s) \\
&\qquad \geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\nabla_w C_t) \boxdot (B_t\nabla_w D_t)]d\mu(t)d\mu(s) \\
&\qquad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t\nabla_w C_t) \boxdot (B_s\nabla_w D_s)]d\mu(t)d\mu(s) \\
&\qquad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\nabla_w C_t) \boxdot (B_t\nabla_w D_t - B_s\nabla_w D_s)]d\mu(t)d\mu(s) \\
&\qquad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\nabla_w C_t) \boxdot (B_t\nabla_w D_t - B_s\nabla_w D_s)]d\mu(t)d\mu(s) \\
&\qquad\quad + \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_s\nabla_w C_s) \boxdot (B_s\nabla_w D_s - B_t\nabla_w D_t)]d\mu(s)d\mu(t) \\
&\qquad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[A_t\nabla_w C_t - A_s\nabla_w C_s] \boxdot [B_t\nabla_w D_t - B_s\nabla_w D_s]d\mu(t)d\mu(s).
\end{aligned}$$

By Theorems 2.34 and 2.49, we have

$$(A_t\nabla_w C_t - A_s\nabla_w C_s) \boxdot (B_t\nabla_w D_t - B_s\nabla_w D_s) \ \geqslant\ 0$$

and hence (5.18) holds. The case (ii) can be similarly treated. □

Corollary 5.15. For each $i = 1,\ldots,k$, let $A_i, B_i, C_i, D_i \in \mathfrak{B}(\mathbb{H})^+$ be such that $0 < m_1 I \leqslant A_t \boxtimes B_t \leqslant M_1 I$ and $0 < m_2 I \leqslant C_t \boxtimes D_t \leqslant M_2 I$ for all $t \in \Omega$, and ω_i be a nonnegative number. Let $\mathcal{A} = (A_1,\ldots,A_k), \mathcal{B} = (B_1,\ldots,B_k), \mathcal{C} = (C_1,\ldots,C_k), \mathcal{D} = (D_1,\ldots,D_k)$ and $w \in [0,1]$.

(i) If $(\mathcal{A},\mathcal{B}), (\mathcal{C},\mathcal{D})$ have the synchronous monotone property and all of $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are either increasing or decreasing, then

$$\sum_{i=1}^k \omega_i \sum_{i=1}^k \omega_i[(A_i \boxdot B_i)\nabla_w(C_i \boxdot D_i)] \ \geqslant\ \left(\sum_{i=1}^k \omega_i(A_i\nabla_w C_i)\right) \boxdot \left(\sum_{i=1}^k \omega_i(B_i\nabla_w D_i)\right). \tag{5.19}$$

(ii) If $(\mathcal{A},\mathcal{B}), (\mathcal{C},\mathcal{D})$ have the opposite-synchronous monotone property and if either

(a) $\mathcal{A},\mathcal{C}$ are increasing and $\mathcal{B},\mathcal{D}$ are decreasing, or

(b) $\mathcal{A},\mathcal{C}$ are decreasing and $\mathcal{B},\mathcal{D}$ are increasing,

then the reverse of (5.19) holds.

We can illustrate Corollary 5.15 for the case of operators induced from matrices as follows.

Example 5.16. Consider $\mathcal{A} = (T_{A_1}, T_{A_2}), \mathcal{B} = (T_{B_1}, T_{B_2}), \mathcal{C} = (T_{C_1}, T_{C_2})$ and $\mathcal{D} = (T_{D_1}, T_{D_2})$, where

$$A_1 = \left[\begin{array}{c|cc} 2 & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right], \quad A_2 = \left[\begin{array}{c|cc} 3 & 1 & 0 \\ \hline 1 & 2 & 1 \\ 0 & 1 & 4 \end{array}\right],$$

$$B_1 = \left[\begin{array}{c|cc} 3 & 2 & 0 \\ \hline 2 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad B_2 = \left[\begin{array}{c|cc} 5 & 3 & 0 \\ \hline 3 & 6 & 2 \\ 0 & 2 & 4 \end{array}\right],$$

$$C_1 = \left[\begin{array}{c|cc} 2 & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad C_2 = \left[\begin{array}{c|cc} 3 & 1 & 0 \\ \hline 1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right],$$

$$D_1 = \left[\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad D_2 = \left[\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 1 & 3 & 2 \\ 0 & 2 & 2 \end{array}\right].$$

We can check the hypothesis of Corollary 5.15: (i) all of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are increasing, and (ii) $(\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D})$ have the synchronous monotone property. Set $\Omega = \{1, 2\}$ equipped with the counting measure and $\omega_1 = \omega_2 = \frac{1}{2}$. Let us denote $\triangledown = \triangledown_{1/2}$. Now, a direct computation reveals that

$$(A_1 \boxdot B_1)\triangledown(C_1 \boxdot D_1) + (A_2 \boxdot B_2)\triangledown(C_2 \boxdot D_2) - \frac{1}{2}(A_1\triangledown C_1 + A_2\triangledown C_2) \boxdot (B_1\triangledown D_1 + B_2\triangledown D_2)$$

$$= \frac{1}{4}\left[\begin{array}{c|cccc} 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 6 & 2 & 5 & 2 \\ 0 & 2 & 4 & 2 & 4 \\ 0 & 5 & 2 & 14 & 8 \\ 0 & 2 & 4 & 8 & 10 \end{array}\right] \geqslant 0,$$

or equivalently,

$$(A_1 \boxdot B_1)\triangledown(C_1 \boxdot D_1) + (A_2 \boxdot B_2)\triangledown(C_2 \boxdot D_2) \geqslant \frac{1}{2}(A_1\triangledown C_1 + A_2\triangledown C_2) \boxdot (B_1\triangledown D_1 + B_2\triangledown D_2).$$

Passing through the induced linear maps and applying the properties $T_A \triangledown T_B = T_{A\triangledown B}$ and (5.8), we obtain

$$(T_{A_1} \boxdot T_{B_1})\triangledown(T_{C_1} \boxdot T_{D_1}) + (T_{A_2} \boxdot T_{B_2})\triangledown(T_{C_2} \boxdot T_{D_2})$$
$$\geqslant \frac{1}{2}(T_{A_1}\triangledown T_{C_1} + T_{A_2}\triangledown T_{C_2}) \boxdot (T_{B_1}\triangledown T_{D_1} + T_{B_2}\triangledown T_{D_2}).$$

Thus, the equality (5.19) holds in this case.

5.2.3 Inequalities on Weighted Harmonic Means

Lemma 5.17. Let $A, B, C, D \in \mathfrak{B}(\mathbb{H})^+$ and $w \in [0,1]$. If (A, B) and (C, D) have the opposite-synchronous property, then

$$(A \boxdot B) \,!_w\, (C \boxdot D) \;\geqslant\; (A \,!_w\, C) \boxdot (B \,!_w\, D). \tag{5.20}$$

Proof. Recall that for any $X \in \mathfrak{B}(\mathbb{H})$, we have

$$X^*(A \,!_w\, B)X \;=\; (X^*AX) \,!_w\, (X^*BX). \tag{5.21}$$

Applying Theorem 2.51, Lemma 4.16 and (5.21), we obtain

$$\begin{aligned}
(A \boxdot B) \,!_w\, (C \boxdot D) \;&=\; [Z^*(A \boxtimes B)Z] \,!_w\, [Z^*(C \boxtimes D)Z] \\
&\geqslant\; Z^*[(A \boxtimes B) \,!_w\, (C \boxtimes D)]Z \\
&\geqslant\; Z^*[(A \,!_w\, C) \boxtimes (B \,!_w\, D)]Z \\
&=\; (A \,!_w\, C) \boxdot (B \,!_w\, D).
\end{aligned}$$

Here, Z is the selection operator defined in Theorem 2.51. □

Theorem 5.18. Let $\mathcal{A} = (A_t)_{t\in\Omega}, \mathcal{B} = (B_t)_{t\in\Omega}, \mathcal{C} = (C_t)_{t\in\Omega}, \mathcal{D} = (D_t)_{t\in\Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$ and let and $\alpha : \Omega \to [0, \infty)$ a bounded measurable function. If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the opposite-synchronous monotone property and all of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either increasing or decreasing, then

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t) \;!_w\; (C_t \boxdot D_t)]d\mu(t) \\
&\qquad \geqslant \int_\Omega \alpha(t)(A_t \;!_w\; C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s \;!_w\; D_s)d\mu(s).
\end{aligned} \tag{5.22}$$

Proof. Using Theorem 2.49, Proposition 5.1, Lemma 5.17 and Fubini's theorem [7], we get

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)[(A_t \boxdot B_t)\ !_w\ (C_t \boxdot D_t)]d\mu(t)\\
&-\int_\Omega \alpha(t)(A_t\ !_w\ C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s\ !_w\ D_s)d\mu(s)\\
&\qquad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxdot B_t)\ !_w\ (C_t \boxdot D_t)]d\mu(t)d\mu(s)\\
&\qquad - \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\ !_w\ C_t) \boxdot (B_s\ !_w\ D_s)]d\mu(t)d\mu(s)\\
&\qquad \geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\ !_w\ C_t) \boxdot (B_t\ !_w\ D_t)]d\mu(t)d\mu(s)\\
&\qquad - \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\ !_w\ C_t) \boxdot (B_s\ !_w\ D_s)]d\mu(t)d\mu(s)\\
&\qquad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\ !_w\ C_t) \boxdot (B_t\ !_w\ D_t - B_s\ !_w\ D_s)]d\mu(t)d\mu(s)\\
&\qquad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\ !_w\ C_t) \boxdot (B_t\ !_w\ D_t - B_s\ !_w\ D_s)]d\mu(t)d\mu(s)\\
&\qquad\quad + \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_s\ !_w\ C_s) \boxdot (B_s\ !_w\ D_s - B_t\ !_w\ D_t)]d\mu(s)d\mu(t)\\
&\qquad = \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\,!_w\,C_t - A_s\,!_w\,C_s) \boxdot (B_t!_w\,D_t - B_s\,!_w\ D_s)]d\mu(t)d\mu(s).
\end{aligned}$$

Applying the positivity of the Khatri-Rao product (Theorem 2.49) and the monotonicity of the arithmetic mean (Theorem 2.34), we have

$$(A_t\,!_w\,C_t - A_s\,!_w\,C_s) \boxdot (B_t\,!_w\,D_t - B_s\,!_w\,D_s) \geqslant 0$$

and hence (5.22) holds. □

Corollary 5.19. For each $i = 1, \dots, k$, let $A_i, B_i, C_i, D_i \in \mathfrak{B}(\mathbb{H})^+$ and ω_i be a nonnegative number. Let $\mathcal{A} = (A_1, \dots, A_k), \mathcal{B} = (B_1, \dots, B_k), \mathcal{C} = (C_1, \dots, C_k), \mathcal{D} = (D_1, \dots, D_k)$ and $w \in [0,1]$. If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the opposite-synchronous monotone property and all of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either increasing or decreasing, then

$$\sum_{i=1}^k \omega_i \sum_{i=1}^k \omega_i[(A_i \boxdot B_i)\,!_w\,(C_i \boxdot D_i)] \geqslant \left(\sum_{i=1}^k \omega_i(A_i\,!_w\,C_i)\right) \boxdot \left(\sum_{i=1}^k \omega_i(B_i\,!_w\,D_i)\right). \quad (5.23)$$

Lemma 5.20. Let $A, B, C, D \in \mathfrak{B}(\mathbb{H})^+$ with $0 < m_1 I \leqslant A \boxtimes B \leqslant M_1 I$ and $0 < m_2 I \leqslant C \boxtimes D \leqslant M_2 I$. Denote

$$\lambda = \left[\frac{(m_1 \sharp M_1)\nabla_{1-w}(m_2 \sharp M_2)}{(m_1 \nabla_{1-w} M_2)\sharp(M_1 \nabla_{1-w} m_2)}\right]^2. \quad (5.24)$$

If (A, B) and (C, D) have the synchronous property, then for any $w \in [0,1]$,

$$\lambda[(A \boxdot B)\,!_w\,(C \boxdot D)] \leqslant (A\,!_w\,C) \boxdot (B\,!_w\,D).$$

Proof. The cases $w = 0$ and $w = 1$ are trivial. Now, let $w \in (0, 1)$. We have by Lemma 5.17 that

$$(A \boxtimes B) \,!_w\, (C \boxtimes D) \leqslant (A \,!_w\, C) \boxtimes (B \,!_w\, D).$$

Consider the map $\Phi : X \mapsto Z^* X Z$. Using the result of [28, Corollary 3.8], we get

$$\Phi(X \,!\, Y) \geqslant \frac{\left(\sqrt{M_1 m_1} + \sqrt{M_2 m_2}\right)^2}{(M_1 + m_2)(m_1 + M_2)} \left[\Phi(X) \,!\, \Phi(Y)\right],$$

where $X, Y \in \mathfrak{B}(\mathbb{H})^+$ with $0 < m_1 I \leqslant X \leqslant M_1 I$ and $0 < m_2 I \leqslant Y \leqslant M_2 I$. Using Theorems 2.49 and 2.51, we have

$$\begin{aligned}
&(A \,!_w\, C) \boxdot (B \,!_w\, D) \\
&\quad = Z^* \left[(A \,!_w\, C) \boxtimes (B \,!_w\, D)\right] Z \\
&\quad \geqslant Z^* \left[(A \boxtimes B) \,!_w\, (C \boxtimes D)\right] Z \\
&\quad = \frac{1}{2} Z^* \left\{ \left[\frac{1}{1-w}(A \boxtimes B)\right] ! \left[\frac{1}{w}(C \boxtimes D)\right] \right\} Z \\
&\quad \geqslant \frac{1}{2} \frac{\left(\frac{\sqrt{M_1 m_1}}{1-w} + \frac{\sqrt{M_2 m_2}}{w}\right)^2}{\left(\frac{M_1}{1-w} + \frac{m_2}{w}\right)\left(\frac{m_1}{1-w} + \frac{M_2}{w}\right)} \left\{ \left[\frac{1}{1-w} Z^*(A \boxtimes B) Z\right] ! \left[\frac{1}{w} Z^*(C \boxtimes D) Z\right] \right\} \\
&\quad = \frac{\left(w\sqrt{M_1 m_1} + (1-w)\sqrt{M_2 m_2}\right)^2}{(wM_1 + (1-w)m_2)(wm_1 + (1-w)M_2)} \left[(A \boxdot B) \,!_w\, (C \boxdot D)\right] \\
&\quad = \left(\frac{(m_1 \# M_1) \nabla_{1-w} (m_2 \# M_2)}{(M_1 \nabla_{1-w} m_2) \# (m_1 \nabla_{1-w} M_2)} \right)^2 \left[(A \boxdot B) \,!_w\, (C \boxdot D)\right] \\
&\quad = \lambda \left[(A \boxdot B) \,!_w\, (C \boxdot D)\right].
\end{aligned}$$

□

Theorem 5.21. Let $\mathcal{A} = (A_t)_{t \in \Omega}, \mathcal{B} = (B_t)_{t \in \Omega}, \mathcal{C} = (C_t)_{t \in \Omega}, \mathcal{D} = (D_t)_{t \in \Omega}$ be bounded continuous fields of operators in $\mathfrak{B}(\mathbb{H})^+$ such that $0 < m_1 I \leqslant A_t \boxtimes B_t \leqslant M_1 I$ and $0 < m_2 I \leqslant C_t \boxtimes D_t \leqslant M_2 I$ for all $t \in \Omega$. Let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function and w, λ as in Lemma 5.20. If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the synchronous monotone property and if either

(i) $\mathcal{A}, \mathcal{C}$ are increasing and $\mathcal{B}, \mathcal{D}$ are decreasing, or

(ii) $\mathcal{A}, \mathcal{C}$ are decreasing and $\mathcal{B}, \mathcal{D}$ are increasing,

then

$$\begin{aligned}
\lambda \int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t) [(A_t \boxdot B_t) \,!_w\, (C_t \boxdot D_t)] d\mu(t) & \\
\leqslant \int_\Omega \alpha(t)(A_t \,!_w\, C_t) d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s \,!_w\, D_s) d\mu(s). &
\end{aligned} \qquad (5.25)$$

Proof. Applying Theorem 2.49, Proposition 5.1, Lemma 5.20, and Fubini's Theorem [7], we get

$$
\begin{aligned}
\lambda \int_{\Omega} \alpha(s) d\mu(s) & \int_{\Omega} \alpha(t)[(A_t \boxdot B_t) \, !_w \, (C_t \boxdot D_t)] d\mu(t) \\
- \int_{\Omega} \alpha(t)(A_t \, !_w \, C_t) d\mu(t) & \boxdot \int_{\Omega} \alpha(s)(B_s \, !_w \, D_s) d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)[\lambda(A_t \boxdot B_t) \, !_w \, (C_t \boxdot D_t)] d\mu(t) d\mu(s) \\
&\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \, !_w \, C_t) \boxdot (B_s \, !_w \, D_s)] d\mu(t) d\mu(s) \\
&\leqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \, !_w \, C_t) \boxdot (B_t \, !_w \, D_t)] d\mu(t) d\mu(s) \\
&\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \, !_w \, C_t) \boxdot (B_s \, !_w \, D_s)] d\mu(t) d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \, !_w \, C_t) \boxdot (B_t \, !_w \, D_t - B_s \, !_w \, D_s)] d\mu(t) d\mu(s) \\
&= \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \, !_w \, C_t) \boxdot (B_t \, !_w \, D_t - B_s \, !_w \, D_s)] d\mu(t) d\mu(s) \\
&\quad + \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_s \, !_w \, C_s) \boxdot (B_s \, !_w \, D_s - B_t \, !_w \, D_t)] d\mu(s) d\mu(t) \\
&= \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[A_t \, !_w \, C_t - A_s \, !_w \, C_s] \boxdot [B_t \, !_w \, D_t - B_s \, !_w \, D_s] d\mu(t) d\mu(s).
\end{aligned}
$$

We have, by Theorems 2.34 and 2.49,

$$
(A_t \, !_w \, C_t - A_s \, !_w \boxdot (B_t \, !_w \, D_t - B_s \, !_w \, D_s) \leqslant 0
$$

and hence (5.25) holds. □

Corollary 5.22. For each $i = 1, \ldots, k$, let $A_i, B_i, C_i, D_i \in \mathfrak{B}(\mathbb{H})^+$ and ω_i be a nonnegative number. Let $\mathcal{A} = (A_1, \ldots, A_k), \mathcal{B} = (B_1, \ldots, B_k), \mathcal{C} = (C_1, \ldots, C_k), \mathcal{D} = (D_1, \ldots, D_k)$ and $w \in [0, 1]$ and m, M, w, λ as in Lemma 5.20. If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the synchronous monotone property and if either

(i) $\mathcal{A}, \mathcal{C}$ are increasing and $\mathcal{B}, \mathcal{D}$ are decreasing, or

(ii) $\mathcal{A}, \mathcal{C}$ are decreasing and $\mathcal{B}, \mathcal{D}$ are increasing,

then

$$
\lambda \sum_{i=1}^{k} \omega_i \sum_{i=1}^{k} \omega_i [(A_i \boxdot B_i) \, !_w \, (C_i \boxdot D_i)] \leqslant \left(\sum_{i=1}^{k} \omega_i (A_i \, !_w \, C_i) \right) \boxdot \left(\sum_{i=1}^{k} \omega_i (B_i \, !_w \, D_i) \right). \quad (5.26)
$$

5.3 Chebyshev-Grüss-Type Inequalities via Oscillations

Throughout this section, let Ω be a compact Hausdorff space equipped with a probability Radon measure μ. For any continuous fields $\mathcal{A} = (A_t)_{t \in \Omega}$ and $\mathcal{B} = (B_t)_{t \in \Omega}$

of operators in $\mathfrak{B}(\mathbb{H})$ and $\mathfrak{B}(\mathbb{K})$, respectively, we define

$$\mathcal{A} \boxdot \mathcal{B} \;=\; (A_t \boxdot B_t)_{t\in\Omega}.$$

In the next theorem, we generalize Grüss-type inequality (1.14) to the case of operators involving Khatri-Rao products.

Theorem 5.23. Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be bounded continuous fields of self-adjoint operators in $\mathfrak{B}(\mathbb{H})$ and $\mathfrak{B}(\mathbb{K})$, respectively. Then

$$\mathcal{I}(\mathcal{A}\boxdot\mathcal{B}) - \mathcal{I}(\mathcal{A})\boxdot\mathcal{I}(\mathcal{B}) \;\leqslant\; \frac{1}{2}\,\mathrm{osc}(\mathcal{A})\,\mathrm{osc}(\mathcal{B})(\mu\times\mu)(\Omega^2\backslash\Delta)I, \tag{5.27}$$

where $\Delta = \{(t,t) : t\in\Omega\}$.

Proof. By using Theorem 2.49 and Proposition 5.1, and Fubini's theorem [7], we have

$$\begin{aligned}
&\mathcal{I}(\mathcal{A}\boxdot\mathcal{B}) - \mathcal{I}(\mathcal{A})\boxdot\mathcal{I}(\mathcal{B})\\
&\quad= \int_\Omega A_t\boxdot B_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t)\boxdot\int_\Omega B_s\,d\mu(s)\\
&\quad= \int_\Omega d\mu(s)\int_\Omega A_t\boxdot B_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t)\boxdot\int_\Omega B_s\,d\mu(s)\\
&\quad= \iint_{\Omega^2} A_t\boxdot B_t\,d\mu(t)d\mu(s) - \iint_{\Omega^2} A_t\boxdot B_s\,d\mu(t)d\mu(s)\\
&\quad= \iint_{\Omega^2} A_t\boxdot(B_t-B_s)\,d\mu(t)d\mu(s)\\
&\quad= \frac{1}{2}\iint_{\Omega^2} A_t\boxdot(B_t-B_s)\,d\mu(t)d\mu(s) + \frac{1}{2}\iint_{\Omega^2} A_s\boxdot(B_s-B_t)\,d\mu(s)d\mu(t)\\
&\quad= \frac{1}{2}\iint_{\Omega^2}(A_t-A_s)\boxdot(B_t-B_s)\,d\mu(t)d\mu(s)\\
&\quad= \frac{1}{2}\iint_{\Omega^2\backslash\Delta}(A_t-A_s)\boxdot(B_t-B_s)\,d\mu(t)d\mu(s)\\
&\quad\leqslant \frac{1}{2}\,\mathrm{osc}(\mathcal{A})\,\mathrm{osc}(\mathcal{B})(\mu\times\mu)(\Omega^2\backslash\Delta)I.
\end{aligned}$$

□

We mention that Theorem 5.23 can be viewed as an operator version of [17, Theorem 7].

Corollary 5.24. For each $i = 1,\ldots,k$, let $A_i \in \mathfrak{B}(\mathbb{H})$ and $B_i \in \mathfrak{B}(\mathbb{K})$ be self-adjoint operators. Then

$$\sum_{i=1}^{k}(A_i\boxdot B_i) - \left(\sum_{i=1}^{k}A_i\right)\boxdot\left(\sum_{i=1}^{k}B_i\right) \;\leqslant\; \frac{k(k-1)}{2}\max_{1\leqslant i,j\leqslant k}\|A_i-A_j\|\cdot\max_{1\leqslant i,j\leqslant k}\|B_i-B_j\|I.$$

Proof. Let $\mathcal{A} = (A_1, \dots, A_k)$ and $\mathcal{B} = (B_1 \dots, B_k)$. Set Ω to be the finite space $\{1, \dots, k\}$ equipped with the counting measure in Theorem 5.23, we have

$$(\mu \times \mu)(\Omega^2 \backslash \Delta) \; = \; \frac{k(k-1)}{2}, \qquad \text{supp}(\mu \times \mu) \; = \; \Omega^2$$

and thus

$$\text{osc}(\mathcal{A}) \; = \; \max_{1 \leqslant i,j \leqslant k} \|A_i - A_j\|, \qquad \text{osc}(\mathcal{B}) \; = \; \max_{1 \leqslant i,j \leqslant k} \|B_i - B_j\|. \qquad \square$$

Example 5.25. Let $\Omega = [0,1]$, $w \in \Omega$ and $0 < \alpha \leqslant 1$. Consider the probability Radon measure $\mu = \alpha\lambda + (1-\alpha)\delta_w$, where λ is Lebesgue measure on Ω and δ_w is the Dirac measure at w. Set

$$\mathcal{I}(A) \; := \; \int_0^1 A_t d\mu(t) \; = \; \alpha \int_0^1 A_t d\lambda(t) + (1-\alpha) A_w.$$

For any bounded continuous fields $\mathcal{A} = (A_t)_{t \in \Omega}$ and $\mathcal{B} = (B_t)_{t \in \Omega}$ of self-adjoint operators, the inequality (5.27) becomes

$$\mathcal{I}(\mathcal{A} \boxdot \mathcal{B}) - \mathcal{I}(\mathcal{A}) \boxdot \mathcal{I}(\mathcal{B}) \; \leqslant \; \frac{1}{2}\alpha(2-\alpha) \max_{0 \leqslant s,t \leqslant 1} \|A_t - A_s\| \cdot \max_{0 \leqslant t,s \leqslant 1} \|B_t - B_s\| I.$$

Chapter 6

Conclusions and Suggestions

6.1 Conclusions

In this research, we establish several inequalities for bounded linear operators involving Tracy-Singh products and Khatri-Rao products. Firstly, we investigate concavity and convexity of certain maps dealing with Tracy-Singh products, Khatri-Rao products and operator-monotone functions. The main tools we use are operator means and suitable integral representations of certain operator-monotone functions on the unit interval with respect to finite Borel measures.

Next, we establish a number of integral inequalities of Chebyshev-type for bounded continuous fields of operators involving Tracy-Singh products and weighted Pythagorean means. The Pythagorean means considered here are the arithmetic mean, the geometric mean and the harmonic mean. The bounded continuous fields considered here are parametrized by a locally compact Hausdorff space Ω equipped with a finite Radon measure μ. When μ is a probability Radon measure, we obtain the Chebyshev-Grüss-type integral inequality involving Tracy-Singh products via oscillations.

Finally, we provide integral inequalities of Chebyshev-type for bounded continuous fields of operators which are parametrized by a locally compact Hausdorff space equipped with a finite Radon measure. Under certain assumptions on synchronous Khatri-Rao property of the fields of operators, we obtain Chebyshev-type inequalities concerning Khatri-Rao products. We also establish Chebyshev-type inequalities involving Khatri-Rao products and weighted Pythagorean means under the assumption of synchonous monotone property of the fields of operators. Moreover, we derive Chebyshev-Grüss-type integral inequality for Khatri-Rao products via oscillations with respect to a probability Radon measure.

Integral inequalities in our research can be reduced to discrete inequalities by setting Ω to be a finite space equipped with the counting measure. Our results include inequalities for tensor product of operators. Matrix analogues of our results can be obtained by setting the Hilbert spaces $\mathbb{H} = \mathbb{C}^m$ and $\mathbb{K} = \mathbb{C}^n$. In this case, our results include the results for Tracy-Singh products, Khatri-Rao products, Kronecker products and Hadamard products of matrices.

6.2 Suggestions for Further Works

For future work, we may investigate Chebyshev-type inequalities for Tracy-Singh products and Khatri-Rao products under different hypotheses. Furthermore, we may study Chebyshev-type inequalities when replacing weighted Pythagorean means by quasi-arithmetic power mean, the logarithmic mean, or general operator means.

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Appendices

Appendix A

Concavity and Convexity of Several Maps Involving Tracy-Singh Products, Khatri-Rao Products, and Operator-Monotone Functions of Positive Operators

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Concavity and convexity of several maps involving Tracy-Singh products, Khatri-Rao products, and operator-monotone functions of positive operators

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ABSTRACT: We establish concavity and convexity theorems for a number of operator-valued maps involving Tracy-Singh products and Khatri-Rao products of positive operators on a Hilbert space. Operator means serve as useful tools for some convexity results. We also investigate certain maps dealing with positive operator-monotone functions. In this case, the concavity and the convexity of such maps are examined through suitable integral representations of the operator-monotone functions on the unit interval with respect to finite Borel measures.

KEYWORDS: positive operator, Tracy-Singh product, Khatri-Rao product, operator-monotone function, Bochner integration

MSC2010: 46G10 47A05 47A63 47A64 47A80

INTRODUCTION

This paper focuses on concavity and convexity of certain maps dealing with Tracy-Singh products and Khatri-Rao products of operators. Such operator products are generalizations of famous matrix products in the literature, namely, the Kronecker product, the Hadamard product, the Tracy-Singh product, and the Khatri-Rao product.

Recall that the Kronecker product is defined for two matrices $A = [a_{ij}]$ and B of arbitrary sizes resulting in a block matrix

$$A * B = [a_{ij}B]_{ij}.$$

The Hadamard product is defined for two matrices A and B of the same size

$$A \circ B = [a_{ij}b_{ij}].$$

Concavity and convexity properties of several matrix-valued maps involving Kronecker products and Hadamard products were collected in Refs. 1–3. As a generalization of the Kronecker product, the Tracy-Singh product[4] is defined for partitioned matrices $A = [A_{ij}]$ and $B = [B_{kl}]$ by

$$A \circledast B = [[A_{ij} * B_{kl}]_{kl}]_{ij}.$$

The work of Al-Zhour[5] extends some results of Ando[1] to Tracy-Singh products of positive definite matrices. The Khatri-Rao product[6], as a generalized Hadamard product, for $A = [A_{ij}]$ and $B = [B_{ij}]$ in the same block-matrix form, is defined by

$$A \odot B = [A_{ij} * B_{ij}]_{ij}.$$

In functional analysis aspect, the tensor product of Hilbert space operators can be viewed as an infinite-dimensional extension of the Kronecker product. Mond and Pečarić[7] extended the matrix results of Ando[1] to Hilbert space operators and obtained concavity/convexity theorems associated with positive operator-monotone functions. Ref. 8 extended the notion of tensor product for operators and Tracy-Singh product for matrices to the Tracy-Singh product for Hilbert space operators, and supply its algebraic and order properties. Analytic properties of the Tracy-Singh product were discussed in Ref. 9. Ref. 10 introduced the Khatri-Rao product of Hilbert space operators and gave a relationship between the Khatri-Rao product and the Tracy-Singh product of two operators via isometric selection operators.

In this study, we investigate concavity and convexity of certain maps related to Tracy-Singh products and Khatri-Rao products of oprators. The main tools we use are operator means and suitable integral representations of certain operator-monotone functions. Our results in this paper generalize the results known so far for Tracy-Singh and Khatri-Rao products of matrices and tensor products of operators. Furthermore, we develop new concavity/convexity theorems.

PRELIMINARIES ON TRACY-SINGH AND KHATRI-RAO PRODUCTS

Throughout this paper, let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{K}$ and $\mathcal{K}'$ be complex Hilbert spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, the symbol $\mathbb{B}(\mathcal{X},\mathcal{Y})$ stands for the algebra of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and when $\mathcal{X}=\mathcal{Y}$, we write $\mathbb{B}(\mathcal{X})$ instead of $\mathbb{B}(\mathcal{X},\mathcal{X})$. The cone of positive operators on $\mathcal{H}$ is denoted by $\mathbb{B}(\mathcal{H})^+$. For self-adjoint operators A and B on the same space, the situation $A \geqslant B$ means that $A-B$ is positive. Denote the set of all positive invertible operators on $\mathcal{H}$ by $\mathbb{B}(\mathcal{H})^{++}$. If $A \in \mathbb{B}(\mathcal{H})^{++}$, we write $A>0$. The identity operator and the zero operator are denoted by I and 0, respectively.

To define the Tracy-Singh product and the Khatri-Rao product for operators, we decompose

$$\mathcal{H}=\bigoplus_{j=1}^{n}\mathcal{H}_j,\quad \mathcal{H}'=\bigoplus_{i=1}^{m}\mathcal{H}_i',$$
$$\mathcal{K}=\bigoplus_{l=1}^{q}\mathcal{K}_l,\quad \mathcal{K}'=\bigoplus_{k=1}^{p}\mathcal{K}_k',$$

where all $\mathcal{H}_j$, $\mathcal{H}_i'$, $\mathcal{K}_l$ and $\mathcal{K}_k'$ are Hilbert spaces. For each j, let $U_j:\mathcal{H}_j\to\mathcal{H}$ be the canonical embedding

$$(0,\dots,0,x_j,0,\dots,0)\mapsto x_j.$$

Similarly, for each l, let $V_l:\mathcal{K}_l\to\mathcal{K}$ be the canonical embedding. For each i and k, let $P_i:\mathcal{H}'\to\mathcal{H}_i'$ and $Q_k:\mathcal{K}'\to\mathcal{K}_k'$ be the orthogonal projections. Each $A\in\mathbb{B}(\mathcal{H},\mathcal{H}')$ and $B\in\mathbb{B}(\mathcal{K},\mathcal{K}')$ can be expressed uniquely as operator matrices

$$A=[A_{ij}]_{i,j=1}^{m,n},\quad B=[B_{kl}]_{k,l=1}^{p,q},$$

where $A_{ij}=P_iAU_j\in\mathbb{B}(\mathcal{H}_j,\mathcal{H}_i')$ and $B_{kl}=Q_kBV_l\in\mathbb{B}(\mathcal{K}_l,\mathcal{K}_k')$ for each i, j, k, l.

Definition 1 Let $A=[A_{ij}]_{i,j=1}^{m,n}\in\mathbb{B}(\mathcal{H},\mathcal{H}')$ and $B=[B_{kl}]_{k,l=1}^{p,q}\in\mathbb{B}(\mathcal{K},\mathcal{K}')$. We define the *Tracy-Singh product* of A and B to be the bounded linear operator

$$A\boxtimes B=[[A_{ij}\otimes B_{kl}]_{kl}]_{ij},$$
$$A\boxtimes B:\bigoplus_{j=1}^{n}\bigoplus_{l=1}^{q}\mathcal{H}_j\otimes\mathcal{K}_l\to\bigoplus_{i=1}^{m}\bigoplus_{k=1}^{p}\mathcal{H}_i'\otimes\mathcal{K}_k'.$$

When $m=p$ and $n=q$, we define the *Khatri-Rao product* of A and B to be the bounded linear operator

$$A\boxdot B=[A_{ij}\otimes B_{ij}]_{ij}:\bigoplus_{i=1}^{n}\mathcal{H}_i\otimes\mathcal{K}_i\to\bigoplus_{j=1}^{m}\mathcal{H}_j'\otimes\mathcal{K}_j'.$$

Lemma 1 (Refs. 8, 9) *Let A, B, C, D be compatible operators. Then*
(i) *The map* $(A,B)\mapsto A\boxtimes B$ *is bilinear and jointly continuous.*
(ii) $(A\boxtimes B)(C\boxtimes D)=(AC)\boxtimes(BD)$.
(iii) *If A and B are invertible, then* $(A\boxtimes B)^{-1}=A^{-1}\boxtimes B^{-1}$.
(iv) *If A and B are positive, then* $(A\boxtimes B)^{\alpha}=A^{\alpha}\boxtimes B^{\alpha}$ *for any* $\alpha>0$.
(v) *If* $A\geqslant C\geqslant 0$ *and* $B\geqslant D\geqslant 0$, *then* $A\boxtimes B\geqslant C\boxtimes D\geqslant 0$.
(vi) *If* $A>0$ *and* $B>0$, *then* $A\boxtimes B>0$.

Lemma 2 (Ref. 9) *Let* $A\in\mathbb{B}(\mathcal{H})$.
(i) *If f is an analytic function on a region containing the spectra of A and* $I\boxtimes A$, *then* $f(I\boxtimes A)=I\boxtimes f(A)$.
(ii) *If f is an analytic function on a region containing the spectra of A and* $A\boxtimes I$, *then* $f(A\boxtimes I)=f(A)\boxtimes I$.

Lemma 3 (Ref. 10) *Let* $A\in\mathbb{B}(\mathcal{H})$ *and* $B\in\mathbb{B}(\mathcal{K})$. *If* $A\geqslant 0$ *and* $B\geqslant 0$, *then* $A\boxdot B\geqslant 0$.

Lemma 4 (Ref. 10) *There are isometries* Z_1 *and* Z_2 *such that*

$$A\boxdot B=Z_1^*(A\boxtimes B)Z_2 \tag{1}$$

for all $A\in\mathbb{B}(\mathcal{H},\mathcal{H}')$ *and* $B\in\mathbb{B}(\mathcal{K},\mathcal{K}')$. *For the case* $\mathcal{H}=\mathcal{H}'$ *and* $\mathcal{K}=\mathcal{K}'$, *we have* $Z_1=Z_2:=Z$.

Lemma 5 *The Khatri-Rao product of operators is jointly continuous.*

Proof: It follows from (1) and the continuity of the Tracy-Singh product (Lemma 1). □

For each $i=1,\dots,k$, let $\mathcal{H}_i$ and $\mathcal{H}_i'$ be Hilbert spaces and decompose

$$\mathcal{H}_i=\bigoplus_{r=1}^{n_i}\mathcal{H}_{i,r},\quad \mathcal{H}_i'=\bigoplus_{s=1}^{m_i}\mathcal{H}_{i,s}',$$

where all $\mathcal{H}_{i,r}$ and $\mathcal{H}_{i,s}'$ are Hilbert spaces. For a finite number of operator matrices $A_i\in\mathbb{B}(\mathcal{H}_i,\mathcal{H}_i')$ for $i=1,\dots,k$, we use the following notations,

$$\mathop{\boxtimes}_{i=1}^{k}A_i=((A_1\boxtimes A_2)\boxtimes\cdots\boxtimes A_{k-1})\boxtimes A_k,$$
$$\mathop{\boxdot}_{i=1}^{k}A_i=((A_1\boxdot A_2)\boxdot\cdots\boxdot A_{k-1})\boxdot A_k.$$

Lemma 6 *There are isometries Z_1 and Z_2*

$$\mathop{\boxdot}_{i=1}^{k} A_i = Z_1^* \Big(\mathop{\boxtimes}_{i=1}^{k} A_i \Big) Z_2 \tag{2}$$

for any $A_i \in \mathbb{B}(\mathscr{H}_i, \mathscr{H}_i')$, $i = 1, \dots, k$. If $\mathscr{H}_i$ and $\mathscr{H}_i'$ are the same space for all i, the $Z_1 = Z_2 := Z$.

Proof: We proceed by induction on k. If $k = 2$, the property (2) is true by Lemma 4. Suppose that there exist isometries R_1 and R_2 such that

$$\mathop{\boxdot}_{i=1}^{k-1} A_i = R_1^* \Big(\mathop{\boxtimes}_{i=1}^{k-1} A_i \Big) R_2.$$

By Lemma 4, there are isometries S_1, S_2 such that

$$\Big(\mathop{\boxdot}_{i=1}^{k-1} A_i \Big) \boxdot A_k = S_1^* [(\mathop{\boxdot}_{i=1}^{k-1} A_i) \boxtimes A_k] S_2.$$

Then

$$\begin{aligned} \mathop{\boxdot}_{i=1}^{k} A_i &= \Big(\mathop{\boxdot}_{i=1}^{k-1} A_i \Big) \boxdot A_k \\ &= S_1^* [(\mathop{\boxdot}_{i=1}^{k-1} A_i) \boxtimes A_k] S_2 \\ &= S_1^* [R_1^* (\mathop{\boxtimes}_{i=1}^{k-1} A_i) R_2 \boxtimes A_k] S_2 \\ &= S_1^* (R_1^* \boxtimes I) \Big[\Big(\mathop{\boxtimes}_{i=1}^{k-1} A_i \Big) \boxtimes A_k \Big] (R_2 \boxtimes I) S_2 \\ &= [(R_1 \boxtimes I) S_1]^* \Big(\mathop{\boxtimes}_{i=1}^{k} A_i \Big) (R_2 \boxtimes I) S_2. \end{aligned}$$

Set $Z_1 = (R_1 \boxtimes I) S_1$ and $Z_2 = (R_2 \boxtimes I) S_2$. Then Z_1 and Z_2 are isometries. When $\mathscr{H}_i = \mathscr{H}_i'$ for all $i = 1, \dots, k$, we have $Z_1 = Z_2$ from the construction. □

CONCAVITY AND CONVEXITY

In this section, we provide concavity and convexity theorems related to Tracy-Singh products of operators. First of all, recall the following terminologies.

Definition 2 A function $f : (0, \infty) \to (0, \infty)$ is said to be operator-monotone if $f[A] \geqslant f[B]$ whenever $A \geqslant B > 0$. Here, $f[A]$ is the (continuous) functional calculus of f defined on the spectrum of A.

Definition 3 Let $\mathscr{H}_1, \dots, \mathscr{H}_k, \mathscr{K}$ be Hilbert spaces. For each $i = 1, \dots, k$, let E_i be a convex subset of $\mathbb{B}(\mathscr{H}_i)$. A function $\phi : E_1 \times \cdots \times E_k \to \mathbb{B}(\mathscr{K})$ is said to be concave if

$$\begin{aligned} &\phi((1-t)A_1 + tB_1, \dots, (1-t)A_k + tB_k) \\ &\qquad \leqslant (1-t)\phi(A_1, \dots, A_k) + t\phi(B_1, \dots, B_k) \end{aligned}$$

for any $A_i, B_i \in E_i$ $(i = 1, \dots, k)$ and $t \in (0, 1)$. A function ϕ is convex if $-\phi$ is concave. A map between two convex sets is said to be affine if it preserves convex combinations.

Recall that, for each $t \in (0, 1)$, the t-weighted harmonic mean and the t-weighted geometric mean of $A, B \in \mathbb{B}(\mathscr{H})^{++}$ is defined respectively by

$$\begin{aligned} A \,!_t\, B &= [(1-t)A^{-1} + tB^{-1}]^{-1}, \\ A \#_t B &= A^{1/2} \big(A^{-1/2} B A^{-1/2} \big)^t A^{1/2}. \end{aligned}$$

For arbitrary $A, B \in \mathbb{B}(\mathscr{H})^{+}$, we define the t-weighted geometric mean of A and B to be

$$A \#_t B = \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \#_t (B + \varepsilon I),$$

where the limit is taken in the strong-operator topology.

Lemma 7 (Ref. 11) *For each $t \in [0, 1]$, the map $(A, B) \mapsto A \,!_t\, B$ is concave on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{H})^{++}$.*

The next lemma gives an integral representation of operator-monotone functions on $(0, \infty)$ in terms of Borel measures on $[0, 1]$.

Lemma 8 (Ref. 12) *Let $f : (0, \infty) \to (0, \infty)$ be an operator-monotone function. Then there is a finite Borel measure μ on $[0, 1]$ such that*

$$f(x) = \int_0^1 1 \,!_t\, x \, \mathrm{d}\mu(t), \quad x > 0. \tag{3}$$

Theorem 1 *Let $f : (0, \infty) \to (0, \infty)$ be an operator-monotone function. If $\phi_1 : \mathbb{B}(\mathscr{H})^{++} \to \mathbb{B}(\mathscr{H}')^{++}$ and $\phi_2 : \mathbb{B}(\mathscr{K})^{++} \to \mathbb{B}(\mathscr{K}')^{++}$ are concave maps, then the maps*

$$\begin{aligned} (A, B) &\mapsto f[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (I \boxtimes \phi_2(B)), \quad (4) \\ (A, B) &\mapsto f[\phi_1(A)^{-1} \boxtimes \phi_2(B)] \cdot (\phi_1(A) \boxtimes I) \quad (5) \end{aligned}$$

are concave on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{K})^{++}$.

Proof: Let $A \in \mathbb{B}(\mathscr{H})^{++}$ and $B \in \mathbb{B}(\mathscr{K})^{++}$. Then $\phi_1(A) > 0$ and $\phi_2(B) > 0$. Lemma 1 implies that $f[\phi_1(A) \boxtimes \phi_2(B)^{-1}]$ and $f[\phi_1(A)^{-1} \boxtimes \phi_2(B)]$ are well-defined operators. By Lemma 8, there is a finite Borel measure μ on $[0, 1]$ such that (3) holds. Using Bochner integration, we have

$$\begin{aligned} &f[\phi_1(A) \boxtimes \phi_2(B)^{-1}] \cdot (I \boxtimes \phi_2(B)) \\ &\quad = \int_0^1 \{ (I \boxtimes I) \,!_t\, (\phi_1(A) \boxtimes \phi_2(B)^{-1}) \} (I \boxtimes \phi_2(B)) \, \mathrm{d}\mu(t). \end{aligned}$$

For each $t \in [0,1]$, by Lemma 1 we obtain

$$\begin{aligned}&\{(I \boxtimes I)\ !_t(\phi_1(A)\boxtimes\phi_2(B)^{-1})\}\cdot(I\boxtimes\phi_2(B))\\ &= \big[(1-t)(I\boxtimes I)+t(\phi_1(A)\boxtimes\phi_2(B)^{-1})^{-1}\big]^{-1}\\ &\quad\cdot(I\boxtimes\phi_2(B))\\ &= \big[(I\boxtimes\phi_2(B)^{-1})\\ &\quad\cdot\big\{(1-t)I\boxtimes I+t\phi_1(A)^{-1}\boxtimes\phi_2(B)\big\}\big]^{-1}\\ &= \big[(1-t)(I\boxtimes\phi_2(B))^{-1}+t(\phi_1(A)\boxtimes I)^{-1}\big]^{-1}\\ &= (I\boxtimes\phi_2(B))\ !_t(\phi_1(A)\boxtimes I).\end{aligned}$$

Since the weighted harmonic mean is concave (Lemma 7), so is the map

$$(A,B)\mapsto\{(I\boxtimes I)\ !_t(\phi_1(A)\boxtimes\phi_2(B)^{-1})\}\cdot(I\boxtimes\phi_2(B)).$$

Thus the map (4) is concave. Similarly, the map (5) is concave. □

Remark 1 Since $\phi_1(A)\boxtimes\phi_2(B)^{-1}$ commutes with $I\boxtimes\phi_2(B)$, we have

$$\begin{aligned}&f[\phi_1(A)\boxtimes\phi_2(B)^{-1}]\cdot(I\boxtimes\phi_2(B))\\ &\qquad= (I\boxtimes\phi_2(B))\cdot f[\phi_1(A)\boxtimes\phi_2(B)^{-1}].\end{aligned}$$

Similarly,

$$\begin{aligned}&f[\phi_1(A)^{-1}\boxtimes\phi_2(A)]\cdot(\phi_1(A)\boxtimes I)\\ &\qquad= (\phi_1(A)\boxtimes I)\cdot f[\phi_1(A)^{-1}\boxtimes\phi_2(B)].\end{aligned}$$

Example 1 Recall that the function $t\mapsto t^p$ is operator-monotone for any $0\leqslant p\leqslant 1$. Given two concave maps $\phi_1:\mathbb{B}(\mathscr{H})^{++}\to\mathbb{B}(\mathscr{H}')^{++}$ and $\phi_2:\mathbb{B}(\mathscr{K})^{++}\to\mathbb{B}(\mathscr{K}')^{++}$, by Theorem 1 the maps

$$\begin{aligned}(A,B)&\mapsto[\phi_1(A)\boxtimes\phi_2(B)^{-1}]^p\cdot(I\boxtimes\phi_2(B)),\\ (A,B)&\mapsto[\phi_1(A)^{-1}\boxtimes\phi_2(B)]^p\cdot(\phi_1(A)\boxtimes I)\end{aligned}$$

are concave on $\mathbb{B}(\mathscr{H})^{++}\times\mathbb{B}(\mathscr{K})^{++}$.

Corollary 1 *Let $f:(0,\infty)\to(0,\infty)$ be operator-monotone. If $\phi_1:\mathbb{B}(\mathscr{H})^{++}\to\mathbb{B}(\mathscr{H}')^{++}$ and $\phi_2:\mathbb{B}(\mathscr{K})^{++}\to\mathbb{B}(\mathscr{K}')^{++}$ are concave maps, then the maps*

$$(A,B)\mapsto f[\phi_1(A)^{-1}\boxtimes\phi_2(B)]\cdot(I\boxtimes\phi_2(B)^{-1}),\quad(6)$$
$$(A,B)\mapsto f[\phi_1(A)\boxtimes\phi_2(B)^{-1}]\cdot(\phi_1(A)^{-1}\boxtimes I)\quad(7)$$

are convex on $\mathbb{B}(\mathscr{H})^{++}\times\mathbb{B}(\mathscr{K})^{++}$.

Proof: Note that the function $g(x):=f(x^{-1})^{-1}$ is operator-monotone. By Lemma 1, we have

$$\begin{aligned}&f[\phi_1(A)^{-1}\boxtimes\phi_2(B)](I\boxtimes\phi_2(B)^{-1})\\ &\qquad= g[\phi_1(A)^{-1}\boxtimes\phi_2(B)]^{-1}(I\boxtimes\phi_2(B)^{-1}).\end{aligned}$$

Theorem 1 implies the concavity of the map

$$\begin{aligned}(A,B)&\mapsto g[\phi_1(A)\boxtimes\phi_2(B)^{-1}]\cdot(I\boxtimes\phi_2(B))\\ &= \{(I\boxtimes\phi_2(B)^{-1})\cdot f[\phi_1(A)^{-1}\boxtimes\phi_2(B)]\}^{-1}\\ &= \{f[\phi_1(A)^{-1}\boxtimes\phi_2(B)]\cdot(I\boxtimes\phi_2(B)^{-1})\}^{-1}.\end{aligned}$$

Thus the map (6) is convex. Similarly, the map (7) is convex. □

Theorem 2 *Let $f:(0,\infty)\to(0,\infty)$ be an operator-monotone function. If $\phi_1:\mathbb{B}(\mathscr{H})^{++}\to\mathbb{B}(\mathscr{H}')^{++}$ is a concave map and $\phi_2:\mathbb{B}(\mathscr{K})^{++}\to\mathbb{B}(\mathscr{K}')^{++}$ is an affine map, then the maps*

$$(A,B)\mapsto f[\phi_1(A)^{-1}\boxtimes\phi_2(B)]\cdot(I\boxtimes\phi_2(B)),\quad(8)$$
$$(A,B)\mapsto f[\phi_2(B)\boxtimes\phi_1(A)^{-1}]\cdot(\phi_2(B)\boxtimes I)\quad(9)$$

are convex on $\mathbb{B}(\mathscr{H})^{++}\times\mathbb{B}(\mathscr{K})^{++}$.

Proof: By Lemma 8, there is a finite Borel measure μ on $[0,1]$ such that (3) holds. Then

$$\begin{aligned}&f[\phi_1(A)^{-1}\boxtimes\phi_2(B)]\cdot(I\boxtimes\phi_2(B))\\ &\quad=\int_0^1\{(I\boxtimes I)\ !_t(\phi_1(A)^{-1}\boxtimes\phi_2(B))\}(I\boxtimes\phi_2(B))\,\mathrm{d}\mu(t).\end{aligned}$$

For each $t\in[0,1]$, it follows from Lemma 1 that

$$\begin{aligned}&\{(I\boxtimes I)\ !_t(\phi_1(A)^{-1}\boxtimes\phi_2(B))\}\\ &= \big[(1-t)(I\boxtimes I)+t(\phi_1(A)^{-1}\boxtimes\phi_2(B))^{-1}\big]^{-1}\\ &= \big[(1-t)(I\boxtimes I)+t(\phi_1(A)\boxtimes\phi_2(B)^{-1})\big]^{-1}\\ &= (I\boxtimes\phi_2(B))\big[(1-t)(I\boxtimes\phi_2(B))+t(\phi_1(A)\boxtimes I)\big]^{-1}.\end{aligned}$$

The concavity of the map $(A,B)\mapsto(1-t)(I\boxtimes\phi_2(B))+t(\phi_1(A)\boxtimes I)$ and the affinity of the map $(A,B)\mapsto I\boxtimes\phi_2(B)$ together yield the convexity of the map

$$\begin{aligned}&(A,B)\mapsto\\ &(I\boxtimes\phi_2(B))\{(1-t)I\boxtimes\phi_2(B)+t\phi_1(A)\boxtimes I\}^{-1}(I\boxtimes\phi_2(B))\\ &\qquad= \{(I\boxtimes I)\ !_t(\phi_1(A)^{-1}\boxtimes\phi_2(B))\}(I\boxtimes\phi_2(B)).\end{aligned}$$

Hence the map (8) is convex. Similarly, the map (9) is convex. □

Corollary 2 *The maps*

$$(A,B)\mapsto I\boxtimes(B\log[B])-\log[A]\boxtimes B,\quad(10)$$
$$(A,B)\mapsto(A\log[A])\boxtimes I-A\boxtimes\log[B]\quad(11)$$

are convex on $\mathbb{B}(\mathscr{H})^{++}\times\mathbb{B}(\mathscr{H})^{++}$.

Proof: Using Lemmas 1 and 2, we obtain

$$\begin{aligned} I \boxtimes (B\log[B]) - \log[A]\boxtimes B &= \{I\boxtimes \log[B] - \log[A]\boxtimes I\}\cdot(I\boxtimes B) \\ &= \{\log[I\boxtimes B] - \log[A\boxtimes I]\}\cdot(I\boxtimes B) \\ &= \log[(I\boxtimes B)(A\boxtimes I)^{-1}]\cdot(I\boxtimes B) \\ &= (I\boxtimes B)\cdot\log[A^{-1}\boxtimes B]. \end{aligned}$$

Since $\log x$ is operator-monotone, by Theorem 2 we obtain that the map

$$(A,B)\mapsto \log[A^{-1}\boxtimes B]\cdot(I\boxtimes B)$$

is convex. Hence the map (10) is convex. Similarly, the map (11) is convex. □

Example 2 Let $\phi_1 : \mathbb{B}(\mathscr{H})^{++}\to\mathbb{B}(\mathscr{H}')^{++}$ be a concave map and $\phi_2 : \mathbb{B}(\mathscr{K})^{++}\to\mathbb{B}(\mathscr{K}')^{++}$ an affine map. For any $0\leqslant p\leqslant 1$, we have by Theorem 2 that the maps

$$\begin{aligned} (A,B) &\mapsto [\phi_1(A)^{-1}\boxtimes\phi_2(B)]^p\cdot(I\boxtimes\phi_2(B)), \\ (A,B) &\mapsto [\phi_2(B)\boxtimes\phi_1(A)^{-1}]^p\cdot(\phi_2(B)\boxtimes I) \end{aligned}$$

are convex on $\mathbb{B}(\mathscr{H})^{++}\times\mathbb{B}(\mathscr{K})^{++}$.

We mention that the maps (5), (7), (9) and (11) are extensions of results discussed in Ref. 7.

CONCAVITY THEOREMS FOR TRACY-SINGH AND KHATRI-RAO PRODUCTS

In this section, we present concavity theorems for Tracy-Singh products of operators. Concavity theorems for Khatri-Rao products of operators are established by using the concavity theorems for Tracy-Singh products and the connection between the Khatri-Rao and Tracy-Singh products.

The next result generalizes Corollary 6.2 of Ref. 1 to the case of Tracy-Singh product of operators.

Theorem 3 *Let $0\leqslant p_i\leqslant 1$, $i=1,\dots,k$, be such that $\sum_{i=1}^k p_i\leqslant 1$. Then the map*

$$(A_1,\dots,A_k)\mapsto \mathop{\boxtimes}_{i=1}^{k} A_i^{p_i} \tag{12}$$

is concave on $\mathbb{B}(\mathscr{H}_1)^{++}\times\cdots\times\mathbb{B}(\mathscr{H}_k)^{++}$.

Proof: We proceed by induction on k. Clearly, the map $A_1\mapsto A_1^{p_1}$ is concave. Suppose the assertion is generally true for the case $k-1$. If $p_k=0$, then the map becomes

$$(A_1,\dots,A_k)\mapsto((A_1\boxtimes A_2)\boxtimes\cdots\boxtimes A_{k-1})\boxtimes I,$$

which is concave. If $p_k=1$, then $p_i=0$ for all $i=1,\dots,k-1$ and the map is clearly concave. Now suppose $0<p_k<1$. By the induction assumption, the map

$$\phi(A_1,\dots,A_{k-1}) = \mathop{\boxtimes}_{i=1}^{k-1} A_i^{p_i/(1-p_k)}$$

is concave. By applying Theorem 1 with $f(x)=x^{p_k}$, the map

$$(A_1,\dots,A_k)\mapsto f(\phi(A_1,\dots,A_{k-1})^{-1}\boxtimes A_k)\cdot(\phi(A_1,\dots,A_{k-1})\boxtimes I)$$

is concave. We obtain the concavity of the map (12),

$$\begin{aligned} f(\phi(A_1,\dots,A_{k-1})^{-1}\boxtimes A_k)&(\phi(A_1,\dots,A_{k-1})\boxtimes I) \\ &= (\phi(A_1,\dots,A_{k-1})^{-p_k}\boxtimes A_k^{p_k})(\phi(A_1,\dots,A_{k-1})\boxtimes I) \\ &= \phi(A_1,\dots,A_{k-1})^{1-p_k}\boxtimes A_k^{p_k} = \mathop{\boxtimes}_{i=1}^{k} A_i^{p_i}. \end{aligned}$$

□

A special case of Theorem 3 is when $k=2$.

Corollary 3 *For each $r\in(0,1)$, the map*

$$(A,B)\mapsto A^{1-r}\boxtimes B^r \tag{13}$$

is concave on $\mathbb{B}(\mathscr{H})^+\times\mathbb{B}(\mathscr{K})^+$.

Proof: Theorem 3 implies that the map (13) is concave on $\mathbb{B}(\mathscr{H})^{++}\times\mathbb{B}(\mathscr{K})^{++}$. Since the Tracy-Singh product is jointly continuous (Lemma 1), this map is also concave on $\mathbb{B}(\mathscr{H})^+\times\mathbb{B}(\mathscr{K})^+$. □

Next, we develop concavity theorems for Khatri-Rao products of operators.

Theorem 4 *Let $0\leqslant p_i\leqslant 1$, $i=1,\dots,k$, be such that $\sum_{i=1}^k p_i\leqslant 1$. Then the map*

$$(A_1,\dots,A_k)\mapsto \mathop{\boxdot}_{i=1}^{k} A_i^{p_i} \tag{14}$$

is concave on $\mathbb{B}(\mathscr{H}_1)^{++}\times\cdots\times\mathbb{B}(\mathscr{H}_k)^{++}$.

Proof: From Lemma 6, the map $X\mapsto Z^*XZ$, taking the Tracy-Singh product $\boxtimes_{i=1}^k A_i$ into the Khatri-Rao product $\boxdot_{i=1}^k A_i$, is linear and preserves positivity. Recall that the composition between a linear map and a concave map results in a concave map. Since the map $(A_1,\dots,A_k)\mapsto\boxtimes_{i=1}^k A_i^{p_i}$ is concave by Theorem 3, we have the concavity of the map is concave. We obtain the concavity of the map from (12), since

$$(A_1,\dots,A_k)\mapsto Z^*(\mathop{\boxtimes}_{i=1}^{k} A_i^{p_i})Z = \mathop{\boxdot}_{i=1}^{k} A_i^{p_i}.$$

□

Corollary 4 *For each $r \in (0,1)$, the map*

$$(A,B) \mapsto A^{1-r} \boxdot B^r,$$

is concave on $\mathbb{B}(\mathscr{H})^+ \times \mathbb{B}(\mathscr{K})^+$.

Proof: It follows from Theorem 4 when $k=2$ together with the continuity of the Khatri-Rao product, Lemma 5. □

CONVEXITY THEOREMS FOR TRACY-SINGH AND KHATRI-RAO PRODUCTS

In this section, we establish convexity theorems for Tracy-Singh products and Khatri-Rao products of operators. Weighted arithmetic/geometric/harmonic means of operators serve as useful tools.

Lemma 9 (Ref. 13) *Let $A_i, B_i \in \mathbb{B}(\mathscr{H})^+$, $1 \leqslant i \leqslant k$. Then*

$$\Big(\boxtimes_{i=1}^{k} A_i\Big)\#_t\Big(\boxtimes_{i=1}^{k} B_i\Big) = \boxtimes_{i=1}^{k}\big(A_i \#_t B_i\big).$$

Theorem 5 *Let ϕ_i, $i=1,\dots,k$, be a concave map from $B(\mathscr{H}_i)^{++}$ to $B(\mathscr{H}_i')^{++}$. Then the map*

$$(A_1,\dots,A_k) \mapsto \boxtimes_{i=1}^{k} \phi_i(A_i)^{-1} \tag{15}$$

is convex on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++}$.

Proof: Let $t \in [0,1]$. By continuity, we may assume that A_i and B_i are positive invertible operators. Applying Lemmas 1 and 9 and the arithmetic-geometric means inequality for operators, we have

$$\begin{aligned}
&\boxtimes_{i=1}^{k} \phi_i((1-t)A_i + tB_i)^{-1}\\
&\leqslant \boxtimes_{i=1}^{k} ((1-t)\phi_i(A_i) + t\phi_i(B_i))^{-1}\\
&\leqslant \boxtimes_{i=1}^{k} (\phi_i(A_i) \#_t \phi_i(B_i))^{-1}\\
&= \boxtimes_{i=1}^{k} \phi_i(A_i)^{-1} \#_t \boxtimes_{i=1}^{k} \phi_i(B_i)^{-1}\\
&\leqslant (1-t)\boxtimes_{i=1}^{k} \phi_i(A_i)^{-1} + t\boxtimes_{i=1}^{k} \phi_i(B_i)^{-1}.
\end{aligned}$$

Hence the map (15) is convex. □

Corollary 5 *Let $0 < p_i \leqslant 1$, $i=1,\dots,k$. Then the map*

$$(A_1,\dots,A_k) \mapsto \boxtimes_{i=1}^{k} A_i^{-p_i}$$

is convex on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++}$.

Proposition 1 *Let $0 \leqslant p_i \leqslant 1$, $i=1,\dots,k$, and $1 \leqslant q \leqslant 2$ be such that $\sum_{i=1}^{k} p_i \leqslant q-1$. Then the map*

$$(A_1,\dots,A_{k+1}) \mapsto \Big(\boxtimes_{i=1}^{k} A_i^{-p_i}\Big) \boxtimes A_{k+1}^{q}$$

is convex on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_{k+1})^{++}$.

Proof: By Theorem 3, the map

$$(A_1,\dots,A_{k+1}) \mapsto \Big(\boxtimes_{i=1}^{k} A_i^{p_i}\Big) \boxtimes A_{k+1}^{2-q}$$

is concave on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_{k+1})^{++}$. Clearly, the map

$$(A_1,\dots,A_{k+1}) \mapsto \Big(\boxtimes_{i=1}^{k} I\Big) \boxtimes A_{k+1}$$

is affine. It follows from Lemma 1 that the map

$$\begin{aligned}
&(A_1,\dots,A_{k+1}) \mapsto\\
&\Big[\Big(\boxtimes_{i=1}^{k} I\Big)\boxtimes A_{k+1}\Big]\Big[\Big(\boxtimes_{i=1}^{k} A_i^{p_i}\Big)\boxtimes A_{k+1}^{2-q}\Big]^{-1}\Big(\boxtimes_{i=1}^{k} I\Big)\boxtimes A_{k+1}\\
&\qquad = \Big(\boxtimes_{i=1}^{k} A_i^{-p_i}\Big)\boxtimes A_{k+1}^{q}
\end{aligned}$$

is convex. □

Theorem 6 *For each $r \in (0,1)$, the maps*

$$(A,B) \mapsto A^{-r} \boxtimes B^{1+r}, \tag{16}$$

$$(A,B) \mapsto A^{1+r} \boxtimes B^{-r} \tag{17}$$

are convex on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{K})^{++}$.

Proof: The convexity of the map (16) follows from Proposition 1. By continuity, we may assume that A and B are invertible. Lemma 1 implies that

$$A^{1+r} \boxtimes B^{-r} = (A^r \boxtimes B^{-r})(A \boxtimes I) = (A \boxtimes B^{-1})^r (A \boxtimes I).$$

It follows from Lemmas 1 and 8 that

$$\begin{aligned}
A^{1+r} \boxtimes B^{-r} &= \int_0^1 ((I \boxtimes I)\,!_t\,(A \boxtimes B^{-1}))\,\mathrm{d}\mu(t)\,(A \boxtimes I)\\
&= \int_0^1 \big[(1-t)(I \boxtimes I) + t(A \boxtimes B^{-1})^{-1}\big]^{-1}(A \boxtimes I)\,\mathrm{d}\mu(t)\\
&= \int_0^1 \big[(1-t)(I \boxtimes I) + t(A^{-1} \boxtimes B)\big]^{-1}(A \boxtimes I)\,\mathrm{d}\mu(t)\\
&= \int_0^1 (A \boxtimes I)\,[(1-t)(A \boxtimes I) + t(I \boxtimes B)]^{-1}(A \boxtimes I)\,\mathrm{d}\mu(t).
\end{aligned}$$

Since the map $A \mapsto A^{-1}$ is convex and the map $(A,B) \mapsto (1-t)(A \boxtimes I) + t(I \boxtimes B)$ is affine, the map

$$(A,B) \mapsto (I \boxtimes B)\{(1-t)(A \boxtimes I) + t(I \boxtimes B)\}^{-1}(I \boxtimes B)$$

is convex. Thus the map $(A,B) \mapsto A^{1+r} \boxtimes B^{-r}$ is convex. □

Proposition 2 *Let ϕ_i, $i = 1, \dots, k$, be concave maps from $B(\mathscr{H}_i)^{++}$ to $B(\mathscr{H}_i')^{++}$. Then the map*

$$(A_1, \dots, A_k) \mapsto \mathop{\boxdot}_{i=1}^{k} \phi_i(A_i)^{-1}$$

is convex on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++}$.

Proof: It follows from Lemma 6 and Theorem 5. □

Corollary 6 *Let $0 < p_i \leqslant 1$ for each $i = 1, \dots, k$. Then the map*

$$(A_1, \dots, A_k) \mapsto \mathop{\boxdot}_{i=1}^{k} A_i^{-p_i}$$

is convex on $\mathbb{B}(\mathscr{H}_1)^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++}$.

Proof: It follows from Proposition 2 by putting $\phi_i(A_i) = A_i^{p_i}$ for each i. □

Proposition 3 *For each $r \in (0,1)$, the maps*

$$(A,B) \mapsto A^{-r} \boxdot B^{1+r},$$
$$(A,B) \mapsto A^{1+r} \boxdot B^{-r}$$

are convex on $\mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{K})^{++}$.

Proof: It follows from Lemma 4 and Theorem 6. □

Recall that the Moore-Penrose inverse of an operator $T \in \mathbb{B}(\mathscr{H}, \mathscr{H}')$ is the operator $T^\dagger \in \mathbb{B}(\mathscr{H}', \mathscr{H})$ satisfying the conditions $TT^\dagger T = T$, $T^\dagger T T^\dagger = T$, $(TT^\dagger)^* = TT^\dagger$, and $(T^\dagger T)^* = T^\dagger T$. It is well known that $T^\dagger$ exists if and only if the range of T is closed[14].

Lemma 10 (Ref. 15) *Let*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{bmatrix} \in \mathbb{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$$

be a self-adjoint operator. Suppose that T_{11} has a closed range. Then $T \geqslant 0$ if and only if $T_{11} \geqslant 0$, $T_{12} = T_{11} T_{11}^\dagger T_{12}$, and $T_{22} \geqslant T_{12}^* T_{11}^\dagger T_{12}$.

Recall that for any interval J, a continuous function $f : J \to \mathbb{R}$ is convex if and only if $f(x+h) + f(x-h) - 2f(x) \geqslant 0$ for all $x \in J$ and $h > 0$ such that $x \pm h \in J$.

Theorem 7 *Let $A \in \mathbb{B}(\mathscr{H})^+$ and $B \in \mathbb{B}(\mathscr{K})^+$ have closed ranges. Then the operator-valued function*

$$\phi : [-1,1] \to \mathbb{B}\Big(\bigoplus_{i=1}^{n} \mathscr{H}_i \otimes \mathscr{K}_i\Big),$$
$$\phi(t) = A^{1+t} \boxdot B^{1-t} + A^{1-t} \boxdot B^{1+t} \tag{18}$$

is convex on $[-1,1]$, decreasing on $[-1,0]$, increasing on $[0,1]$, attains minimality at $t = 0$, and attains maximality at $t = -1, 1$.

Proof: Let $s \in [-1,1]$ and $t > 0$ be such that $s \pm t \in [-1,1]$. Consider the operator matrices

$$T_1 = \begin{bmatrix} A^{1+s+t} & A^{1+s} \\ A^{1+s} & A^{1+s-t} \end{bmatrix}, \quad T_2 = \begin{bmatrix} A^{1-s-t} & A^{1-s} \\ A^{1-s} & A^{1-s+t} \end{bmatrix},$$
$$T_3 = \begin{bmatrix} B^{1+s+t} & B^{1+s} \\ B^{1+s} & B^{1+s-t} \end{bmatrix}, \quad T_4 = \begin{bmatrix} B^{1-s-t} & B^{1-s} \\ B^{1-s} & B^{1-s+t} \end{bmatrix}.$$

Note that

$$A^{1+s} = (AA^\dagger A)^{1+s+t} A^{-t} = A^{1+s+t} (A^{1+s+t})^\dagger A^{1+s},$$
$$A^{1+s-t} = A^{-t} (AA^\dagger A)^{1+s+t} A^{-t} = A^{1+s} (A^{1+s+t})^\dagger A^{1+s}.$$

We have by Lemma 10 that T_i is positive for all $i = 1,2,3,4$. By the monotonicity of Khatri-Rao product, Lemma 3, we have that the operator $X \equiv T_1 \boxdot T_4 + T_2 \boxdot T_3$ is

$$\begin{bmatrix} A^{1+s+t} \boxdot B^{1-s-t} + A^{1-s-t} \boxdot B^{1+s+t} & A^{1+s} \boxdot B^{1-s} + A^{1-s} \boxdot B^{1+s} \\ A^{1+s} \boxdot B^{1-s} + A^{1-s} \boxdot B^{1+s} & A^{1+s-t} \boxdot B^{1-s+t} + A^{1-s+t} \boxdot B^{1+s-t} \end{bmatrix},$$

which is positive. Similarly, the operator Y,

$$\begin{bmatrix} A^{1+s-t} \boxdot B^{1-s+t} + A^{1-s+t} \boxdot B^{1+s-t} & A^{1+s} \boxdot B^{1-s} + A^{1-s} \boxdot B^{1+s} \\ A^{1+s} \boxdot B^{1-s} + A^{1-s} \boxdot B^{1+s} & A^{1+s+t} \boxdot B^{1-s-t} + A^{1-s+t} \boxdot B^{1+s+t} \end{bmatrix}$$

is also positive. It follows that

$$0 \leqslant X + Y = \begin{bmatrix} \phi(s+t) + \phi(s-t) & 2\phi(s) \\ 2\phi(s) & \phi(s+t) + \phi(s-t) \end{bmatrix}$$
$$= U \begin{bmatrix} \phi(s+t) + \phi(s-t) + 2\phi(s) & 0 \\ 0 & \phi(s+t) + \phi(s-t) - 2\phi(s) \end{bmatrix} U^*,$$

where

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix}.$$

Again, Lemma 10 guarantees that

$$\phi(s+t) + \phi(s-t) \geqslant 2\phi(s).$$

This means that ϕ is convex. The fact that $\phi(t) = \phi(-t)$ for all $t \in [-1,1]$ and the convexity of ϕ implies that ϕ has the minimal value at 0. Hence ϕ is decreasing on $[-1,0]$ and increasing on $[0,1]$. □

Corollary 7 *Let $A \in \mathbb{B}(\mathscr{H})^+$ and $B \in \mathbb{B}(\mathscr{K})^+$ have closed ranges. Then the parameterization*

$$\psi : [0,1] \to \mathbb{B}\Big(\bigoplus_{i=1}^{n} \mathscr{H}_i \otimes \mathscr{K}_i\Big),$$
$$\psi(t) = A^t \boxdot B^{1-t} + A^{1-t} \boxdot B^t$$

is convex on $[0,1]$, decreasing on $[0,1/2]$, increasing on $[1/2,1]$, attains minimality at $t = 1/2$, and attains maximality at $t = 0$, 1.

Proof: Let $f : [0,1] \to [-1,1]$ be defined by $f(t) = 2t - 1$. Then $\psi = \phi \circ f$ where ϕ is given by (18). Now, the desired results follow from Theorem 7 by using $f([0,1]) = [-1,1]$, $f([0,1/2]) = [-1,0]$, $f([1/2,0]) = [0,1]$, and $f(1/2) = 0$. □

As a consequence, we obtain an operator version of the arithmetic-geometric mean inequality as follows.

Corollary 8 *Let $A \in \mathbb{B}(\mathscr{H})^+$ and $B \in \mathbb{B}(\mathscr{K})^+$ have closed ranges. For any $t \in [1/2,1]$, we have*

$$2(A^{1/2} \boxdot B^{1/2}) \leqslant A^t \boxdot B^{1-t} + A^{1-t} \boxdot B^t \leqslant A \boxplus B,$$

where $\boxplus$ denotes the Khatri-Rao sum[16] defined by $A \boxplus B = A \boxdot I + I \boxdot B$.

We mention that Theorem 5, Corollary 5, and Proposition 1 generalize the matrix results involving Tracy-Singh products provided in Ref. 5.

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Appendix B

Integral Inequalities of Chebyshev Type for Continuous Fields of Hermitian Operators Involving Tracy–Singh Products and Weighted Pythagorean Means

symmetry

MDPI

Article

Integral Inequalities of Chebyshev Type for Continuous Fields of Hermitian Operators Involving Tracy–Singh Products and Weighted Pythagorean Means

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Abstract: In this paper, we establish several integral inequalities of Chebyshev type for bounded continuous fields of Hermitian operators concerning Tracy-Singh products and weighted Pythagorean means. The weighted Pythagorean means considered here are parametrization versions of three symmetric means: the arithmetic mean, the geometric mean, and the harmonic mean. Every continuous field considered here is parametrized by a locally compact Hausdorff space equipped with a finite Radon measure. Tracy-Singh product versions of the Chebyshev-Grüss inequality via oscillations are also obtained. Such integral inequalities reduce to discrete inequalities when the space is a finite space equipped with the counting measure. Moreover, our results include Chebyshev-type inequalities for tensor product of operators and Tracy-Singh/Kronecker products of matrices.

Keywords: Chebyshev inequality; Tracy-Singh product; continuous field of operators; Bochner integral; weighted Pythagorean mean

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1. Introduction

One of the fundamental inequalities in mathematics is the Chebyshev inequality, named after P.L. Chebyshev, which states that

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \geqslant \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \tag{1}$$

for all real numbers a_i, b_i $(1 \leqslant i \leqslant n)$ such that $a_1 \leqslant \ldots \leqslant a_n$ and $b_1 \leqslant \ldots \leqslant b_n$, or $a_1 \geqslant \ldots \geqslant a_n$ and $b_1 \geqslant \ldots \geqslant b_n$. This inequality can be generalized to

$$\sum_{i=1}^{n} w_i a_i b_i \geqslant \left(\sum_{i=1}^{n} w_i a_i\right)\left(\sum_{i=1}^{n} w_i b_i\right) \tag{2}$$

where $w_i \geqslant 0$ for all $1 = 1, \ldots, n$. A matrix version of (2) involving the Hadamard product was obtained in [1].

A continuous version of the Chebyshev inequality [2] says that if $f, g : [a, b] \to \mathbb{R}$ are monotone functions in the same sense and $p : [a, b] \to [0, \infty)$ is an integrable function, then

$$\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx \geqslant \int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx. \tag{3}$$

If f and g are monotone in the opposite sense, the reverse inequality holds. In [3], Moslehian and Bakherad extended this inequality to Hilbert space operators related with the Hadamard product by using the notion of synchronous Hadamard property. They also presented integral Chebyshev inequalities respecting operator means.

The Grüss inequality, first introduced by G. Grüss in 1935 [4], is a complement of the Chebyshev inequality. This inequality gives a bound of the difference between the product of the integrals and the integral of the product for two integrable functions. For each integral function $f : [a, b] \to \mathbb{R}$, let us denote

$$\mathcal{I}(f) = \frac{1}{b-a} \int_a^b f(x)dx.$$

The Grüss inequality states that if $f, g : [a, b] \to \mathbb{R}$ are integrable functions and there exist real constants k, K, l, L such that $k \leqslant f(x) \leqslant K$ and $l \leqslant g(x) \leqslant L$ for all $x \in [a, b]$, then

$$|\mathcal{I}(fg) - \mathcal{I}(f)\mathcal{I}(g)| \leqslant \frac{1}{4}(K-k)(L-l). \tag{4}$$

This inequality has been studied and generalized by several authors; see [5–7]. In [7], the term Chebyshev-Grüss inequalities is used mentioning to Grüss inequalities for Chebyshev functions $T_{\mathcal{I}}$ which defined as

$$T_{\mathcal{I}}(f, g) = \mathcal{I}(f \cdot g) - \mathcal{I}(f) \cdot \mathcal{I}(g).$$

A general form of Chebyshev-Grüss inequalities is given by

$$|T_{\mathcal{I}}(f, g)| \leqslant E(\mathcal{I}, f, g)$$

where E is an expression depending on the arithmetic integral mean $\mathcal{I}$ and oscillations of f and g. Chebyshev-Grüss inequalities for some kind of operator via discrete oscillations is presented by Gonska, Raça and Rusu [7].

On the other hand, the notion of tensor product of operators is a key concept in functional analysis and its applications particularly in quantum mechanics. The theory of tensor product of operators has been investigate in the literature; see, e.g., [8,9]. In [10,11], the authors extend the notion of tensor product to the Tracy-Singh product for operators on a Hilbert space, and supply algebraic/order/analytic properties of this product.

In this paper, we establish a number of integral inequalities of Chebyshev type for continuous fields of Hermitian operators relating Tracy-singh products and weighted Pythagorean means. The Pythagorean means considered here are three classical means -the geometric mean, the arithmetic mean, and the harmonic mean. The continuous field considered here is parametrized by a locally compact Hausdorff space Ω endowed with a finite Radon measure. In Section 2, we give basic results on Tracy-Singh products for Hilbert space operators and Bochner integrability of continuous field of operators on a locally compact Housdorff space. In Section 3, we provide Chebyshev type inequalities involving Tracy-Singh products of operators under the assumption of synchronous Tracy-Singh property. In Section 4, we establish Chebyshev integral inequalities concerning operator means and Tracy-Singh products under the assumption of synchronous monotone property. Finally, we prove Chebyshev-Grüss inequalities via oscillations for continuous fields of operators in Section 5. In the case that Ω is a finite space with the counting measure, such integral inequalities reduce to discrete inequalities. Our results include Chebyshev-type inequalities concerning tensor product of operators and Tracy-Singh/Kronecker products of matrices.

2. Preliminaries

In this paper, we consider complex Hilbert spaces $\mathbb{H}$ and $\mathbb{K}$. The symbol $\mathbb{B}(\mathbb{X})$ stands to the Banach space of bounded linear operators on a Hilbert space $\mathbb{X}$. The cone of positive operators on $\mathbb{X}$ is denoted by $\mathbb{B}(\mathbb{X})^+$. For Hermitian operators A and B in $\mathbb{B}(\mathbb{X})$, the situation $A \geq B$ means that $A - B \in \mathbb{B}(\mathbb{X})^+$. Denote the set of all positive invertible operators on $\mathbb{X}$ by $\mathbb{B}(\mathbb{X})^{++}$.

We fix the following orthogonal decompositions:

$$\mathbb{H} = \bigoplus_{i=1}^{m} \mathbb{H}_i, \quad \mathbb{K} = \bigoplus_{k=1}^{n} \mathbb{K}_k$$

where all $\mathbb{H}_i$ and $\mathbb{K}_j$ are Hilbert spaces. Such decompositions lead to a unique representation for each operator $A \in \mathbb{B}(\mathbb{H})$ and $B \in \mathbb{B}(\mathbb{K})$ as a block-matrix form:

$$A = [A_{ij}]_{i,j=1}^{m,m} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$ and $B_{kl} \in \mathbb{B}(\mathbb{K}_l, \mathbb{K}_k)$ for each i, j, k, l.

2.1. Tracy-Singh Product for Operators

Let $A \in \mathbb{B}(\mathbb{H})$ and $B \in \mathbb{B}(\mathbb{K})$. Recall that the tensor product of A and B, denoted by $A \otimes B$, is a unique bounded linear operator on the tensor product space $\mathbb{H} \otimes \mathbb{K}$ such that

$$(A \otimes B)(x \otimes y) = Ax \otimes By, \quad \forall x \in \mathbb{H}, \forall y \in \mathbb{K}.$$

When $\mathbb{H} = \mathbb{K} = \mathbb{C}$, the tensor product of operators becomes the Kronecker product of matrices.

Definition 1. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathbb{K})$. The Tracy-Singh product of A and B is defined to be in the form*

$$A \boxtimes B = \Big[\big[A_{ij} \otimes B_{kl} \big]_{kl} \Big]_{ij}, \tag{5}$$

which is a bounded linear operator from $\bigoplus_{i=1}^{m} \bigoplus_{k=1}^{n} \mathbb{H}_i \otimes \mathbb{K}_k$ into itself.

When $m = n = 1$, the Tracy-Singh product $A \boxtimes B$ is the tensor product $A \otimes B$. If $\mathbb{H}_i = \mathbb{K}_j = \mathbb{C}$ for all i, j, the above definition becomes the usual Tracy-Singh product for complex matrices.

Lemma 1 ([10,11]). *Let A, B, C, D be compatible operators. Then*

1. $(\alpha A) \boxtimes B = A \boxtimes (\alpha B) = \alpha(A \boxtimes B)$ *for any* $\alpha \in \mathbb{C}$.
2. $(A + B) \boxtimes (C + D) = A \boxtimes C + A \boxtimes D + B \boxtimes C + B \boxtimes D$.
3. $(A \boxtimes B)(C \boxtimes D) = (AC) \boxtimes (BD)$.
4. *If A and B are Hermitian, then so is $A \boxtimes B$.*
5. *If A and B are positive and invertible, then $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ for any $\alpha \in \mathbb{R}$.*
6. *If $A \geqslant C \geqslant 0$ and $B \geqslant D \geqslant 0$, then $A \boxtimes B \geqslant C \boxtimes D \geqslant 0$.*

2.2. Bochner Integration

Let Ω be a locally compact Hausdorff (LCH) space equipped with a finite Radon measure μ. A family $\mathcal{A} = (A_t)_{t \in \Omega}$ of operators in $\mathbb{B}(\mathbb{H})$ is said to be bounded if there is a constant $M > 0$ for which $\|A_t\| \leqslant M$ for all $t \in \Omega$. The family $\mathcal{A}$ is said to be a continuous field if parametrization $t \mapsto A_t$ is norm-continuous on Ω. Every continuous field $\mathcal{A} = (A_t)_{t\in\Omega}$ can have the Bochner integral $\int_\Omega A_t d\mu(t)$ if the norm function $t \mapsto \|A_t\|$ possess the Lebesgue integrability. In this case, the resulting integral is a unique element in $\mathbb{B}(\mathbb{H})$ such that

$$\phi\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega \phi(A_t)\, d\mu(t)$$

for every bounded linear functional ϕ on $\mathbb{B}(\mathbb{H})$.

Lemma 2 (e.g., [12]). *Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space and (Γ, v) a finite measure space. Then a measurable function $f : \Gamma \to \mathbb{X}$ is Bochner integrable if and only if its norm function $\|f\|$ is Lebesgue integrable.*

Lemma 3 (e.g., [12]). *Let $f : \Gamma \to \mathbb{X}$ be a Bochner integrable function. If $\varphi : \mathbb{X} \to \mathbb{Y}$ is a bounded linear operator, then the composition $\varphi \circ f$ is Bochner integrable and*

$$\int_\Gamma (\varphi \circ f) dv = \varphi \int_\Gamma f dv.$$

Proposition 1. *Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathbb{B}(\mathbb{H})$. Then for any $X \in \mathbb{B}(\mathbb{K})$,*

$$\int_\Omega A_t d\mu(t) \boxtimes X = \int_\Omega (A_t \boxtimes X) d\mu(t).$$

Proof. Since the map $t \mapsto A_t$ is continuous and bounded, it is Bochner integrable on Ω. Note that the map $T \mapsto T \boxtimes X$ is linear and bounded by Lemma 1. Now, Lemma 3 implies that the map $t \mapsto A_t \boxtimes X$ is Bochner integrable on Ω and

$$\int_\Omega A_t d\mu(t) \boxtimes X = \int_\Omega (A_t \boxtimes X) d\mu(t).$$

for all $X \in \mathbb{B}(\mathbb{K})$. □

3. Chebyshev Type Inequalities Involving Tracy-Singh Products of Operators

From now on, let Ω be an LCH space equipped with a finite Radon measure μ. Let $\mathcal{A} = (A_t)_{t\in\Omega}$, $\mathcal{B} = (B_t)_{t\in\Omega}$, $\mathcal{C} = (C_t)_{t\in\Omega}$ and $\mathcal{D} = (D_t)_{t\in\Omega}$ be continuous fields of Hilbert space operators.

Definition 2. *The fields $\mathcal{A}$ and $\mathcal{B}$ are said to have the synchronous Tracy-Singh property if, for all $s, t \in \Omega$,*

$$(A_t - A_s) \boxtimes (B_t - B_s) \geqslant 0. \tag{6}$$

They are said to have the opposite-synchronous Tracy-Singh property if the reverse of (6) holds for all $s, t \in \Omega$.

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be bounded continuous fields of Hermitian operators in $\mathbb{B}(\mathbb{H})$ and $\mathbb{B}(\mathbb{K})$, respectively, and let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.*

1. *If $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Tracy-Singh property, then*

$$\int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t)(A_t \boxtimes B_t) d\mu(t) \geqslant \int_\Omega \alpha(t) A_t d\mu(t) \boxtimes \int_\Omega \alpha(s) B_s d\mu(s). \tag{7}$$

2. *If $\mathcal{A}$ and $\mathcal{B}$ have the opposite-synchronous Tracy-Singh property, then the reverse of (7) holds.*

Proof. By using Lemma 1, Proposition 1 and Fubini's Theorem [13], we have

$$
\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)(A_t \boxtimes B_t)d\mu(t) - \int_\Omega \alpha(t)A_t d\mu(t) \boxtimes \int_\Omega \alpha(s)B_s d\mu(s)\\
&\quad= \iint_{\Omega^2} \alpha(s)\alpha(t)(A_t \boxtimes B_t)d\mu(t)d\mu(s) - \iint_{\Omega^2} \alpha(t)\alpha(s)(A_t \boxtimes B_s)d\mu(t)d\mu(s)\\
&\quad= \frac{1}{2}\iint_{\Omega^2} [\alpha(s)\alpha(t)(A_t \boxtimes B_t) - \alpha(t)\alpha(s)(A_t \boxtimes B_s)]\, d\mu(t)d\mu(s)\\
&\qquad+ \frac{1}{2}\iint_{\Omega^2} [\alpha(t)\alpha(s)(A_s \boxtimes B_s) - \alpha(s)\alpha(t)(A_s \boxtimes B_t)]\, d\mu(s)d\mu(t)\\
&\quad= \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)\,[(A_t - A_s)\boxtimes(B_t - B_s)]\, d\mu(t)d\mu(s).
\end{aligned}
$$

For the case 1, we have

$$\iint_{\Omega^2} \alpha(s)\alpha(t)\,[(A_t - A_s)\boxtimes(B_t - B_s)]\, d\mu(t)d\mu(s) \ \geqslant\ 0 \tag{8}$$

and thus (7) holds. For another case, we get the reverse of (8) and, thus, the reverse of (7) holds. □

Remark 1. *In Theorem 1 and other results in this paper, we may assume that Ω is a compact Hausdorff space. In this case, every continuous field on Ω is automatically bounded.*

The next corollary is a discrete version of Theorem 1.

Corollary 1. *Let A_i, B_i be Hermitian operators and let ω_i be nonnegative numbers for each $i = 1, \dots, n$. Let $\mathcal{A} = (A_1, \dots, A_n)$ and $\mathcal{B} = (B_1, \dots, B_n)$.*

1. *If $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Tracy-Singh property, then*

$$\sum_{i=1}^n \omega_i \sum_{i=1}^n \omega_i (A_i \boxtimes B_i) \ \geqslant\ \left(\sum_{i=1}^n \omega_i A_i\right) \boxtimes \left(\sum_{i=1}^n \omega_i B_i\right). \tag{9}$$

2. *If $\mathcal{A}$ and $\mathcal{B}$ have the opposite-synchronous Tracy-Singh property, then the reverse of (9) holds.*

Proof. From the previous theorem, set $\Omega = \{1, \dots, n\}$ equipped with the counting measure and $\alpha(i) = \omega_i$ for all $i = 1, \dots, n$. □

4. Chebyshev Integral Inequalities Concerning Weighted Pythagorean Means of Operators

Throughout this section, the space Ω is equipped with a total ordering $\preccurlyeq$.

Definition 3. *We say that a field $\mathcal{A}$ is increasing (decreasing, resp.) whenever $s \preccurlyeq t$ implies $A_s \leqslant A_t$ ($A_s \geqslant A_t$, resp.).*

Definition 4. *Two ordered pairs (X_1, X_2) and (Y_1, Y_2) of Hermitian operators are said to have the synchronous property if either*

$$X_i \ \leqslant\ Y_i \ \textit{for}\ \ i = 1, 2, \ \ \textit{or}\ \ X_i \ \geqslant\ Y_i \ \textit{for}\ \ i = 1, 2.$$

The pairs (X_1, X_2) and (Y_1, Y_2) are said to have the opposite-synchronous property if either

$$X_1 \ \leqslant\ Y_1 \ \textit{and}\ \ X_2 \ \geqslant\ Y_2, \ \ \textit{or}\ \ X_1 \ \geqslant\ Y_1 \ \textit{and}\ \ X_2 \ \leqslant\ Y_2.$$

Definition 5. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be continuous fields of Hermitian operators. Two ordered pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ are said to have the synchronous monotone property if (A_t, B_t) and (C_t, D_t) have the synchronous property for all $t \in \Omega$. They are said to have the opposite-synchronous monotone property if (A_t, B_t) and (C_t, D_t) have the opposite-synchronous property for all $t \in \Omega$.*

Let us recall the notions of weighted classical Pythagorean means for operators. Indeed, they are generalizations of three famous symmetric operator means as follows. For any $w \in [0,1]$, the w-weighted arithmetic mean of $A, B \in \mathbb{B}(\mathbb{H})$ is defined by

$$A \triangledown_w B = (1-w)A + wB.$$

The w-weighted geometric mean and w-weighted harmonic mean of $A, B \in \mathbb{B}(\mathbb{H})^{++}$ are defined respectively by

$$\begin{aligned} A \sharp_w B &= A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^w A^{\frac{1}{2}}, \\ A \,!_w\, B &= \left[(1-w) A^{-1} + w B^{-1} \right]^{-1}. \end{aligned}$$

For any $A, B \in \mathbb{B}(\mathbb{H})^+$, we define the w-weighted geometric mean and w-weighted harmonic mean of A and B to be

$$\begin{aligned} A \sharp_w B &= \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \sharp_w (B + \varepsilon I), \\ A \,!_w\, B &= \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \,!_w\, (B + \varepsilon I), \end{aligned}$$

respectively. Here, the limits are taken in the strong-operator topology.

Lemma 4 (see e.g., [14]). *The weighted geometric means, weighted arithmetic means and weighted harmonic means for operators are monotone in the sense that if $A_1 \leqslant A_2$ and $B_1 \leqslant B_2$, then $A_1 \sigma B_1 \leqslant A_2 \sigma B_2$ where σ is any of $\triangledown_w, !_w, \sharp_w$.*

Lemma 5 ([15]). *Let $A, B, C, D \in \mathbb{B}(\mathbb{H})^+$ and $w \in [0,1]$. Then*

$$(A \boxtimes B) \sharp_w (C \boxtimes D) = (A \sharp_w C) \boxtimes (B \sharp_w D).$$

Theorem 2. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be bounded continuous fields in $\mathcal{B}(\mathbb{H})^+$ and let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.*

1. *If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either all increasing, or all decreasing then*

$$\begin{aligned} \int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t) [(A_t \boxtimes B_t) \sharp_w (C_t \boxtimes D_t)] d\mu(t) \\ \geqslant \int_\Omega \alpha(t) (A_t \sharp_w C_t) d\mu(t) \boxtimes \int_\Omega \alpha(s) (B_s \sharp_w D_s) d\mu(s). \end{aligned} \tag{10}$$

2. *The reverse of (10) holds if either*

   2.1 *$\mathcal{A}, \mathcal{C}$ are increasing and $\mathcal{B}, \mathcal{D}$ are decreasing, or*
   2.2 *$\mathcal{A}, \mathcal{C}$ are decreasing and $\mathcal{B}, \mathcal{D}$ are increasing.*

Proof. Let $s,t \in \Omega$ and assume without loss of generally that $s \preccurlyeq t$. By applying Lemmas 1 and 5, Proposition 1, and Fubini's Theorem [13], we have

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t)\sharp_w(C_t \boxtimes D_t)]d\mu(t) - \int_\Omega \alpha(t)(A_t\sharp_w C_t)d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s\sharp_w D_s)d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxtimes B_t)\sharp_w(C_t \boxtimes D_t)]d\mu(t)d\mu(s) \\
&\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t\sharp_w C_t) \boxtimes (B_s\sharp_w D_s)]d\mu(t)d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\sharp_w C_t) \boxtimes (B_t\sharp_w D_t)]d\mu(t)d\mu(s) \\
&\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t\sharp_w C_t) \boxtimes (B_s\sharp_w D_s)]d\mu(t)d\mu(s) \\
&= \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\sharp_w C_t) \boxtimes (B_t\sharp_w D_t) - (A_t\sharp_w C_t) \boxtimes (B_s\sharp_w D_s)]d\mu(t)d\mu(s) \\
&\quad + \frac{1}{2}\iint_{\Omega^2} \alpha(t)\alpha(s)[(A_s\sharp_w C_s) \boxtimes (B_s\sharp_w D_s) - (A_s\sharp_w C_s) \boxtimes (B_t\sharp_w D_t)]d\mu(s)d\mu(t) \\
&= \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[A_t\sharp_w C_t - A_s\sharp_w C_s] \boxtimes [B_t\sharp_w D_t - B_s\sharp_w D_s]d\mu(t)d\mu(s).
\end{aligned}$$

If $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are all increasing, we have by Lemma 4 that $A_t\sharp_w C_t \geqslant A_s\sharp_w C_s$ and $B_t\sharp_w D_t \geqslant B_s\sharp_w D_s$. If $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are all decreasing, we have $A_t\sharp_w C_t \leqslant A_s\sharp_w C_s$ and $B_t\sharp_w D_t \leqslant B_s\sharp_w D_s$. Both cases lead to the same conclusion that

$$(A_t\sharp_w C_t - A_s\sharp_w C_s) \boxtimes (B_t\sharp_w D_t - B_s\sharp_w D_s) \geqslant 0,$$

and hence (10) holds. The cases 2.1 and 2.2 yield the same conclusion that

$$(A_t\sharp_w C_t - A_s\sharp_w C_s) \boxtimes (B_t\sharp_w D_t - B_s\sharp_w D_s) \leqslant 0.$$

and hence the reverse of (10) holds. □

Lemma 6. *Let A,B,C,D be Hermitian operators in $\mathbb{B}(\mathbb{H})$ and $w \in [0,1]$.*

1. *If (A,B) and (C,D) have the synchronous property, then*

$$(A \boxtimes B)\nabla_w(C \boxtimes D) \geqslant (A\nabla_w C) \boxtimes (B\nabla_w D). \tag{11}$$

2. *If (A,B) and (C,D) have the opposite-synchronous property, then the reverse of (11) holds.*

Proof. For the synchronous case, we have by using positivity of the Tracy-Singh product (Lemma 1) that $(A-C) \boxtimes (B-D) \geqslant 0$. Applying Lemma 1, we obtain

$$\begin{aligned}
0 &\leqslant w(1-w)\,[(A_1 - B_1) \boxtimes (A_2 - B_2)] \\
&= w(1-w)\,[A_1 \boxtimes A_2 - A_1 \boxtimes B_2 - B_1 \boxtimes A_2 + B_1 \boxtimes B_2] \\
&= [(1-w)(A_1 \boxtimes A_2) + w(B_1 \boxtimes B_2)] - [(1-w)A_1 + wB_1] \boxtimes [(1-w)A_2 + wB_2] \\
&= [(A_1 \boxtimes A_2)\nabla_w (B_1 \boxtimes B_2)] - [(A_1 \nabla_w B_1) \boxtimes (A_2 \nabla_w B_2)].
\end{aligned}$$

Thus $(A_1 \nabla_w B_1) \boxtimes (A_2 \nabla_w B_2) \leqslant (A_1 \boxtimes A_2)\nabla_w (B_1 \boxtimes B_2)$.

For the opposite-synchronous case, we have $(A_1 - B_1) \boxtimes (A_2 - B_2) \leqslant 0$ and hence the reverse of inequality (11) holds. □

Theorem 3. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be bounded continuous fields of operators in $\mathcal{B}(\mathbb{H})^+$, let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.*

1. *If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the synchronous monotone property and all of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either increasing or decreasing, then*

$$\begin{aligned}\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t)\nabla_w(C_t \boxtimes D_t)]d\mu(t)& \\ \geqslant \int_\Omega \alpha(t)(A_t \nabla_w C_t)d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s \nabla_w D_s)d\mu(s).&\end{aligned} \tag{12}$$

2. *If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the opposite-synchronous monotone property and if either*

   2.1 *$\mathcal{A}, \mathcal{C}$ are increasing and $\mathcal{B}, \mathcal{D}$ are decreasing, or*
   2.2 *$\mathcal{A}, \mathcal{C}$ are decreasing and $\mathcal{B}, \mathcal{D}$ are increasing,*

   then the reverse of (12) holds.

Proof. Let $s, t \in \Omega$ and assume without loss of generally that $s \preccurlyeq t$. First, we consider the case 1. We have by using Lemmas 1 and 6, proposition 1, and Fubini's Theorem [13] that

$$\begin{aligned}&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t)\nabla_w(C_t \boxtimes D_t)]d\mu(t) - \int_\Omega \alpha(t)(A_t \nabla_w C_t)d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s \nabla_w D_s)d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxtimes B_t)\nabla_w(C_t \boxtimes D_t)]d\mu(t)d\mu(s) \\
&\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \nabla_w C_t) \boxtimes (B_s \nabla_w D_s)]d\mu(t)d\mu(s) \\
&\geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \nabla_w C_t) \boxtimes (B_t \nabla_w D_t)]d\mu(t)d\mu(s) \\
&\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \nabla_w C_t) \boxtimes (B_s \nabla_w D_s)]d\mu(t)d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \nabla_w C_t) \boxtimes (B_t \nabla_w D_t) - (A_t \nabla_w C_t) \boxtimes (B_s \nabla_w D_s)]d\mu(t)d\mu(s) \\
&= \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \nabla_w C_t) - (A_s \nabla_w C_s)] \boxtimes [(B_t \nabla_w D_t) - (B_s \nabla_w D_s)]d\mu(t)d\mu(s).\end{aligned}$$

Now, by Lemmas 1 and 4, we have

$$(A_t \nabla_w C_t - A_s \nabla_w C_s) \boxtimes (B_t \nabla_w D_t - B_s \nabla_w D_s) \ \geqslant \ 0$$

and hence (12) holds. The case 2 can be similarly proven. □

Lemma 7. *Let A, B, C, D be positive operators in $\mathbb{B}(\mathbb{H})$ and $w \in [0, 1]$.*

1. *If (A, B) and (C, D) are synchronous, then*

$$(A \boxtimes B) \,!_w\, (C \boxtimes D) \ \leqslant \ (A \,!_w\, C) \boxtimes (B \,!_w\, D). \tag{13}$$

2. *If (A, B) and (C, D) are opposite-synchronous, then the reverse of (13) holds.*

Proof. Assume that (A, B) and (C, D) are synchronous. By continuity, we may assume that $A, B, C, D > 0$. We have

$$(A^{-1} - C^{-1}) \boxtimes (B^{-1} - D^{-1}) \geqslant 0. \tag{14}$$

Using Lemma 1 and (14), we get

$$\begin{aligned} 0 &\leqslant w(1-w)A^{-1} \boxtimes B^{-1} + w(1-w)C^{-1} \boxtimes D^{-1} - w(1-w)A^{-1} \boxtimes D^{-1} - w(1-w)C^{-1} \boxtimes B^{-1} \\ &= \left[(1-w) - (1-w)^2\right] A^{-1} \boxtimes B^{-1} + (w - w^2)C^{-1} \boxtimes D^{-1} - w(1-w)A^{-1} \boxtimes D^{-1} \\ &\quad - w(1-w)C^{-1} \boxtimes B^{-1} \\ &= \left(A^{-1} \boxtimes B^{-1}\right) \triangledown_w \left(C^{-1} \boxtimes D^{-1}\right) - \left(A^{-1} \triangledown_w C^{-1}\right) \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right). \end{aligned}$$

This implies that

$$\left(A^{-1} \boxtimes B^{-1}\right) \triangledown_w \left(C^{-1} \boxtimes D^{-1}\right) \geqslant \left(A^{-1} \triangledown_w C^{-1}\right) \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right).$$

Hence,

$$\begin{aligned} (A \boxtimes B) \,!_w\, (C \boxtimes D) &= \left\{(A \boxtimes B)^{-1} \triangledown_w (C \boxtimes D)^{-1}\right\}^{-1} \\ &= \left\{\left(A^{-1} \boxtimes B^{-1}\right) \triangledown_w \left(C^{-1} \boxtimes D^{-1}\right)\right\}^{-1} \\ &\leqslant \left\{\left(A^{-1} \triangledown_w C^{-1}\right) \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right)\right\}^{-1} \\ &= \left(A^{-1} \triangledown_w C^{-1}\right)^{-1} \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right)^{-1} \\ &= (A \,!_w\, C) \boxtimes (B \,!_w\, D). \end{aligned}$$

For the opposite-synchronous case, we have

$$(A^{-1} - C^{-1}) \boxtimes (B^{-1} - D^{-1}) \leqslant 0$$

and hence the reverse of (13) holds. □

Theorem 4. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be bounded continuous fields of operators in $\mathbb{B}(\mathbb{H})^+$ and $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.*

1. *If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the opposite-synchronous monotone property and if all of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either increasing or decreasing, then*

$$\begin{aligned} &\int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t) \,!_w\, (C_t \boxtimes D_t)] d\mu(t) \\ &\qquad \geqslant \int_\Omega \alpha(t)(A_t \,!_w\, C_t) d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s \,!_w\, D_s) d\mu(s). \end{aligned} \tag{15}$$

2. *If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have synchronous monotone property and if either*
   - 2.1 *$\mathcal{A}, \mathcal{C}$ are both increasing, and $\mathcal{B}, \mathcal{D}$ are both decreasing, or*
   - 2.2 *$\mathcal{A}, \mathcal{C}$ are both decreasing and $\mathcal{B}, \mathcal{D}$ are both increasing,*

   then the reverse of (15) holds.

Proof. Let $s, t \in \Omega$ with $s \preccurlyeq t$. If the pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ are opposite-synchronous, then we have by applying Lemmas 1 and 7, Proposition 1, and Fubini's Theorem [13] that

$$\begin{aligned}
&\int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t) \,!_w\, (C_t \boxtimes D_t)] d\mu(t) - \int_\Omega \alpha(t)(A_t \,!_w\, C_t) d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s \,!_w\, D_s) d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxtimes B_t) \,!_w\, (C_t \boxtimes D_t)] d\mu(t) d\mu(s) \\
&\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \,!_w\, C_t) \boxtimes (B_s \,!_w\, D_s)] d\mu(t) d\mu(s) \\
&\geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \,!_w\, C_t) \boxtimes (B_t \,!_w\, D_t)] d\mu(t) d\mu(s) \\
&\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \,!_w\, C_t) \boxtimes (B_s \,!_w\, D_s)] d\mu(t) d\mu(s) \\
&= \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[A_t \,!_w\, C_t - A_s \,!_w\, C_s] \boxtimes [B_t \,!_w\, D_t - B_s \,!_w\, D_s] d\mu(t) d\mu(s).
\end{aligned}$$

For the case 1, we have, by Lemmas 1 and 4,

$$(A_t \,!_w\, C_t - A_s \,!_w\, C_s) \boxtimes (B_t \,!_w\, D_t - B_s \,!_w\, D_s) \geqslant 0$$

and hence (15) holds. Another assertion can be proved in a similar manner to that of the second assertion in Theorem 3. □

5. Chebyshev-Grüss Inequaities via Oscillations

Throughout this section, let Ω be an LCH space equipped with a probability Radon measure μ. For any continuous field $\mathcal{A} = (A_t)_{t\in\Omega}$ in $\mathbb{B}(\mathbb{H})$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ in $\mathbb{B}(\mathbb{K})$, we define

$$\begin{aligned}
\mathcal{A} \boxtimes \mathcal{B} &= (A_t \boxtimes B_t)_{t\in\Omega}, \quad \mathcal{I}(\mathcal{A}) = \int_\Omega A_t d\mu(t), \\
\mathrm{osc}(\mathcal{A}) &= \max\{\|A_t - A_s\| : (t,s) \in \mathrm{supp}(\mu \times \mu)\}.
\end{aligned}$$

Here, we recall that the support of the product measure $\mu \times \mu$ is defined by

$$\mathrm{supp}(\mu \times \mu) = \{(t,s) \in \Omega^2 : (\mu \times \mu)(G) > 0 \text{ for all open sets } G \subseteq \Omega^2 \text{ containing } (t,s)\}.$$

We call $\mathrm{osc}(\mathcal{A})$ the oscillation of the field $\mathcal{A}$.

Theorem 5. *Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be continuous fields of Hermitian operators in $\mathbb{B}(\mathbb{H})$ and $\mathbb{B}(\mathbb{K})$, respectively. Then*

$$\mathcal{I}(\mathcal{A} \boxtimes \mathcal{B}) - \mathcal{I}(\mathcal{A}) \boxtimes \mathcal{I}(\mathcal{B}) \leqslant \frac{1}{2} \mathrm{osc}(\mathcal{A}) \cdot \mathrm{osc}(\mathcal{B}) (\mu \times \mu)(\Omega^2 \backslash \Delta)(I_\mathbb{H} \boxtimes I_\mathbb{K}), \qquad (16)$$

where $\Delta = \{(t,t) : t \in \Omega\}$.

Proof. We have by using Lemma 1, Proposition 1 and Fubini's Theorem [13] that

$$\begin{aligned}\mathcal{I}(\mathcal{A}\boxtimes\mathcal{B})-\mathcal{I}(\mathcal{A})\boxtimes\mathcal{I}(\mathcal{B}) &= \int_\Omega d\mu(s)\int_\Omega A_t\boxtimes B_t d\mu(t)-\int_\Omega A_t d\mu(t)\boxtimes\int_\Omega B_s d\mu(s)\\ &= \iint_{\Omega^2} A_t\boxtimes B_t d\mu(t)d\mu(s)-\iint_{\Omega^2} A_t\boxtimes B_s d\mu(t)d\mu(s)\\ &= \frac{1}{2}\iint_{\Omega^2}(A_t\boxtimes B_t-A_t\boxtimes B_s+A_s\boxtimes B_s-A_s\boxtimes B_t)d\mu(t)d\mu(s)\\ &= \frac{1}{2}\iint\limits_{\Omega^2\backslash\Delta}(A_t-A_s)\boxtimes(B_t-B_s)d\mu(t)d\mu(s)\\ &\leqslant \frac{1}{2}\operatorname{osc}(\mathcal{A})\cdot\operatorname{osc}(\mathcal{B})(\mu\times\mu)(\Omega^2\backslash\Delta)(I_{\mathbb{H}}\boxtimes I_{\mathbb{K}}). \quad\square\end{aligned}$$

Corollary 2. *Let $A_i \in \mathbb{B}(\mathbb{H})$ and $B_i \in \mathbb{B}(\mathbb{K})$ be Hermitian operators for all $i = 1, \ldots, n$. Then*

$$\sum_{i=1}^n (A_i\boxtimes B_i)-\left(\sum_{i=1}^n A_i\right)\boxtimes\left(\sum_{i=1}^n B_i\right) \leqslant \frac{n(n-1)}{2}\max_{1\leqslant i,j\leqslant n}\|A_i-A_j\|\cdot\max_{1\leqslant i,j\leqslant n}\|B_i-B_j\|(I_{\mathbb{H}}\boxtimes I_{\mathbb{K}}).$$

Proof. Set $\Omega = \{1, \ldots, n\}$ equipped with the counting measure. We have

$$(\mu\times\mu)(\Omega^2\backslash\Delta)=\frac{n(n-1)}{2}, \quad \operatorname{supp}(\mu\times\mu)=\Omega\times\Omega$$

and thus

$$\operatorname{osc}(A_1,\ldots,A_n) = \max_{1\leqslant i,j\leqslant n}\|A_i-A_j\|, \quad \operatorname{osc}(B_1,\ldots,B_n) = \max_{1\leqslant i,j\leqslant n}\|B_i-B_j\|. \quad\square$$

Example 1. *Let $\Omega = [0, 1]$, $w \in \Omega$ and $0 < \alpha \leqslant 1$. Consider the probability Radon measure $\mu = \alpha\lambda + (1-\alpha)\delta_w$, where λ is Lebesgue measure on Ω and δ_w is the Dirac measure at w. Set*

$$\mathcal{I}(A) := \int_0^1 A_t d\mu(t) = \alpha\int_0^1 A_t d\lambda(t)+(1-\alpha)A_w.$$

We have

$$\mu\times\mu = \alpha^2(\lambda\times\lambda)+\alpha(1-\alpha)(\lambda\times\delta_w)+(1-\alpha)\alpha(\delta_w\times\lambda)+(1-\alpha)^2(\delta_w\times\delta_w).$$

Then $\operatorname{supp}(\mu\times\mu) = [0,1]\times[0,1]$ and $(\mu\times\mu)([0,1]^2\backslash\Delta) = \alpha(2-\alpha)$. For any continuous fields $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ of Hermitian operators, the inequality (16) becomes

$$\mathcal{I}(\mathcal{A}\boxtimes\mathcal{B})-\mathcal{I}(\mathcal{A})\boxtimes\mathcal{I}(\mathcal{B}) \leqslant \frac{1}{2}\alpha(2-\alpha)\max_{0\leqslant s,t\leqslant 1}\|A_t-A_s\|\cdot\max_{0\leqslant t,s\leqslant 1}\|B_t-B_s\|(I_{\mathbb{H}}\boxtimes I_{\mathbb{K}}).$$

6. Conclusions

We establish several integral inequalities of Chebyshev type for continuous fields of Hermitian operators which are parametrized by an LCH space equipped with a finite Radon measure. We also obtain the Chebyshev-Grüss integral inequality via oscillations with respect to a probability Radon measure. These inequalities involve Tracy-Singh products and weighted versions of famous symmetric means. For a particular case that the LCH space is a finite space equipped with the counting measure, such integral inequalities reduce to discrete inequalities. Our results include Chebyshev-type inequalities for tensor product of operators and Tracy-Singh/Kronecker products of matrices.

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Appendix C

Chebyshev-Type Integral Inequalities for Continuous Fields of Operators Concerning Khatri-Rao Products and Synchronous Properties

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Article

Chebyshev-Type Integral Inequalities for Continuous Fields of Operators Concerning Khatri–Rao Products and Synchronous Properties

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Abstract: We consider bounded continuous fields of self-adjoint operators which are parametrized by a locally compact Hausdorff space Ω equipped with a finite Radon measure μ. Under certain assumptions on synchronous Khatri–Rao property of the fields of operators, we obtain Chebyshev-type inequalities concerning Khatri–Rao products. We also establish Chebyshev-type inequalities involving Khatri–Rao products and weighted Pythagorean means under certain assumptions of synchronous monotone property of the fields of operators. The Pythagorean means considered here are three classical symmetric means: the geometric mean, the arithmetic mean, and the harmonic mean. Moreover, we derive the Chebyshev–Grüss integral inequality via oscillations when μ is a probability Radon measure. These integral inequalities can be reduced to discrete inequalities by setting Ω to be a finite space equipped with the counting measure. Our results provide analog results for matrices and integrable functions. Furthermore, our results include the results for tensor products of operators, and Khatri–Rao/Kronecker/Hadamard products of matrices, which have been not investigated in the literature.

Keywords: Chebyshev sum inequality; Khatri–Rao product; Bochner integral; weighted Pythagorean mean

MSC: 47A63; 47A64; 47A80

1. Introduction

In mathematical analysis and applications, analytic inequalities serve as fundamental tools for comparison, approximation, and optimization. The classical Chebyshev sum inequality states that for any real tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ both are increasing or decreasing, we have

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \geqslant \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right). \tag{1}$$

This inequality was generalized by Matharu and Aujla [1] to the case of positive semidefinite matrices involving the Hadamard (entrywise) product $\circ$: for any matrices $A_1 \geqslant \cdots \geqslant A_n \geqslant 0$ and $B_1 \geqslant \cdots \geqslant B_n \geqslant 0$, and any positive numbers $\omega_1 \ldots, \omega_n$, we have

$$\sum_{i=1}^{n} \omega_i \sum_{i=1}^{n} \omega_i (A_i \circ B_i) \geqslant \left(\sum_{i=1}^{n} \omega_i A_i\right) \circ \left(\sum_{i=1}^{n} \omega_i B_i\right). \tag{2}$$

To discuss a continuous version of this inequality in a compact form, let us denote for each integrable function $f, g : [a, b] \to \mathbb{R}$,

$$\mathcal{I}(f) = \frac{1}{b-a}\int_a^b f(x)dx, \tag{3}$$

$$T_{\mathcal{I}}(f,g) = \mathcal{I}(f \cdot g) - \mathcal{I}(f)\cdot\mathcal{I}(g). \tag{4}$$

The latter is called the general Chebyshev functional (see [2]). The Chebyshev functional (4) has many applications in numerical quadrature, probability and statistics, and existence for solutions to certain differential equations. It was obtained in [3] that if such f and g are synchronous on $[a, b]$, that is,

$$(f(x) - f(y))(g(x) - g(y)) \geqslant 0 \tag{5}$$

for all $x, y \in [a, b]$, then

$$T_{\mathcal{I}}(f,g) \geqslant 0. \tag{6}$$

The opposite inequality of (6) holds if both f and g are opposite-synchronous on $[a, b]$. Operator extensions of this inequality were presented by Moslehian and Bakherad [4]. They generalized the Chebyshev integral inequality (6) to the case of continuous fields of Hilbert space operators with a bounded measurable function involving Hadamard products by using the notion of synchronous Hadamard property. They proved that if two continuous fields $(A_t)_{t\in\Omega}$, $(B_t)_{t\in\Omega}$ of operators, parametrized by a compact Hausdorff space Ω equipped with a Radon measure μ, have the synchronous Hadamard property, then

$$\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)(A_t\odot B_t)d\mu(t) \geqslant \int_\Omega \alpha(t)A_t d\mu(t)\odot\int_\Omega \alpha(s)B_s d\mu(s), \tag{7}$$

where $\alpha : \Omega \to [0, \infty)$ is a bounded measurable function. Moreover, they gave some Chebyshev-type inequalities concerning operator means and Hadamard products.

A complement of (6) was introduced by Grüss [5], providing an estimate of the difference between the integral of the product and the product of the integrals for two functions. For any integrable functions $f, g : [a, b] \to \mathbb{R}$ satisfying the conditions $k \leqslant f(x) \leqslant K$, $l \leqslant g(x) \leqslant L$ for all $x \in [a, b]$ and k, K, l, L are real constants, we have

$$|T_{\mathcal{I}}(f,g)| \leqslant \frac{1}{4}(K-k)(L-l). \tag{8}$$

We can apply Grüss inequalities to estimate error bounds for some integral means and numerical quadrature rules; see e.g., [6,7]. In [8], Gonska, Raşa and Rusu used the terminology Chebyshev–Grüss inequalities referring to Grüss inequalities for (special cases of) generalized Chebyshev functionals $T_{\mathcal{I}}$ which have a general form

$$|T_{\mathcal{I}}(f,g)| \leqslant E(\mathcal{I}, f, g), \tag{9}$$

where E is an expression in terms of certain properties of $\mathcal{I}$ and some kind of oscillations of f and g. They also established new Chebyshev–Grüss inequalities via discrete oscillations.

On the other hand, in the theory of operator product, the notion of tensor product for Hilbert space operators was extended to the Tracy–Singh product for such operators [9]. Algebraic, order, and analytic properties of the Tracy–Singh product for operator were discussed in [9,10]. The notion of tensor product was also generalized to the Khatri–Rao product for Hilbert space operators in [11].

The work [11] shows that the Khatri–Rao product and the Tracy–Singh product are related via isometric selection operators.

In this paper, we establish new several integral inequalities of Chebyshev-type for continuous fields of self-adjoint operators involving Khatri–Rao products and operator means. In Section 2, we give preliminaries on Khatri–Rao and Tracy–Singh products for operators, and Bochner integrability of continuous field of operators on a locally compact Hausdorff space. In Section 3, we provide Chebyshev-type inequalities involving Khatri–Rao products of operators under the assumption of synchronous Khatri–Rao property. In Section 4, we establish Chebyshev integral inequalities concerning weighted Pythagorean means and Khatri–Rao products under the assumption of synchronous monotone property. We prove Chebyshev–Grüss inequalities via oscillations for continuous fields of operators in Section 5. Our results generalize the matrix result [1] and the result for integrable functions [8]. Moreover, our results include the results for tensor products of operators, and Khatri–Rao/Kronecker/Hadamard products of matrices, which have been not investigated in the literature. Finally, we summarize our work in Section 6.

2. Preliminaries

Throughout this paper, let $\mathbb{H}$, and $\mathbb{K}$ be complex Hilbert spaces. The symbol $\mathcal{B}(\mathbb{H})$ stands to the Banach space of bounded linear operators from $\mathbb{H}$ into itself. The vector space of self-adjoint operators on $\mathbb{H}$ is denoted by $\mathcal{B}(\mathbb{H})^{sa}$. Denote the set of all positive (positive invertible, respectively) operators on $\mathbb{H}$ by $\mathcal{B}(\mathbb{H})^{+}$ ($\mathcal{B}(\mathbb{H})^{++}$, resp.). For any $A, B \in \mathcal{B}(\mathbb{H})^{sa}$, the situation $A \geqslant B$ ($A > B$, resp.) means that $A - B \in \mathcal{B}(\mathbb{H})^{+}$ ($A - B \in \mathcal{B}(\mathbb{H})^{++}$, resp.).

Through this paper, we apply the projection theorem to decompose

$$\mathbb{H} \;=\; \bigoplus_{i=1}^{n} \mathbb{H}_i, \quad \mathbb{K} \;=\; \bigoplus_{i=1}^{n} \mathbb{K}_i,$$

where all $\mathbb{H}_i$ and $\mathbb{K}_i$ are Hilbert spaces. Each operator $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ can be expressed uniquely as operator matrices

$$A \;=\; [A_{ij}]_{i,j=1}^{n,n} \quad \text{and} \quad B \;=\; [B_{ij}]_{i,j=1}^{n,n},$$

where $A_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$ and $B_{ij} \in \mathbb{B}(\mathbb{K}_j, \mathbb{K}_i)$ for each i, j.

2.1. Khatri–Rao Product and Tracy–Singh Product for Operators

Recall that the tensor product of $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ is a unique bounded linear operator $A \otimes B \in \mathcal{B}(\mathbb{H} \otimes \mathbb{K})$ such that for all $x \in \mathbb{H}$ and all $y \in \mathbb{K}$,

$$(A \otimes B)(x \otimes y) \;=\; Ax \otimes By.$$

Fix a countable orthonormal basis $\mathbb{E}$ of $\mathbb{H}$. Recall that the Hadamard product of $A, B \in \mathcal{B}(\mathbb{H})$ is defined to be the operator $A \odot B \in \mathcal{B}(\mathbb{H})$ such that for all $e \in \mathbb{E}$,

$$\langle (A \odot B)e, e\rangle \;=\; \langle Ae, e\rangle \langle Be, e\rangle.$$

It is known that the Hadamard product of $A, B \in \mathcal{B}(\mathbb{H})$ can be expressed as

$$A \odot B \;=\; U^*(A \otimes B)U, \tag{10}$$

where $U : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ is the isometry defined by $Ue = e \otimes e$ for all $e \in \mathbb{E}$ (see e.g., [12]).

From the previous setting, we define the Khatri–Rao product of A and B to be the bounded linear operator from $\bigoplus_{i=1}^{n} \mathbb{H}_i \otimes \mathbb{K}_i$ into itself represented by an operator matrix

$$A \boxdot B \;=\; \left[A_{ij} \otimes B_{ij}\right]_{ij}. \tag{11}$$

We define the Tracy–Singh product of A and B to be the bounded linear operator from $\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \mathbb{H}_i \otimes \mathbb{K}_j$ into itself represented by an operator matrix

$$A \boxtimes B \;=\; \left[\left[A_{ij} \otimes B_{kl}\right]_{kl}\right]_{ij}. \tag{12}$$

The maps $(A, B) \mapsto A \boxdot B$ and $(A, B) \mapsto A \boxtimes B$ are bilinear. Moreover, we have:

Lemma 1 ([9–11]). *Let A, B, C, D be compatible operators. Then*

1. *If $A \geqslant C \geqslant 0$ and $B \geqslant D \geqslant 0$, then $A \boxdot B \geqslant C \boxdot D \geqslant 0$.*
2. *If $A \geqslant C \geqslant 0$ and $B \geqslant D \geqslant 0$, then $A \boxtimes B \geqslant C \boxtimes D \geqslant 0$.*
3. *$(A \boxtimes B)(C \boxtimes D) = (AC) \boxtimes (BD)$.*
4. *$(A + C) \boxtimes (B + D) \;=\; A \boxtimes B + A \boxtimes D + C \boxtimes B + C \boxtimes D$.*
5. *If A and B are invertible, then $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$.*
6. *If A and B are positive, then $(A \boxtimes B)^{\alpha} = A^{\alpha} \boxtimes B^{\alpha}$ for any $\alpha \geqslant 0$.*

Lemma 2 ([11]). *There is an isometry Z such that $A \boxdot B = Z^*(A \boxtimes B)Z$ for any operators $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$.*

2.2. Bochner Integration for Operator-Valued Maps

Throughout this paper, let Ω be a locally compact Hausdorff space endowed with a finite Radon measure μ. A continuous map $A : \Omega \to \mathbb{X} \subseteq \mathcal{B}(\mathbb{H})$ is called a continuous field of operators in $\mathbb{X}$ parametrized by Ω, denoted by $A \in \mathfrak{C}(\Omega; \mathbb{X})$. For convenience, for each $t \in \Omega$, we may write A_t instead of $A(t)$. The field A is said to be bounded if there is a constant $M > 0$ such that $\|A_t\| \leqslant M$ for all $t \in \Omega$. The set of all bounded continuous fields of operators in $\mathbb{X}$ parametrized by Ω is denoted by $\mathfrak{BC}(\Omega; \mathbb{X})$. If $A \in \mathfrak{C}(\Omega; \mathbb{X})$ is such that the norm function $t \mapsto \|A_t\|$ is Lebergue integrable on Ω (e.g., $A \in \mathfrak{BC}(\Omega; \mathbb{X})$), then we can form the Bochner integral $\int_\Omega A_t d\mu(t)$.

Lemma 3 (see e.g., [13]). *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces, and $\varphi : \mathbb{X} \to \mathbb{Y}$ a bounded linear operator. For any Bochner integrable function $f : \Gamma \to \mathbb{X}$, the composition $\varphi \circ f$ is also Bochner integrable and*

$$\int_\Gamma (\varphi \circ f) dv \;=\; \varphi \int_\Gamma f dv.$$

Proposition 1. *For any $A \in \mathfrak{BC}(\Omega, \mathcal{B}(\mathbb{H}))$ and $X \in \mathcal{B}(\mathbb{K})$, we have*

$$\int_\Omega A_t d\mu(t) \boxdot X \;=\; \int_\Omega (A_t \boxdot X) d\mu(t). \tag{13}$$

Proof. The map $t \mapsto A_t$ is Bochner integrable on Ω because it is continuous. Since the map $T \mapsto T \boxdot X$ is bounded linear operator, we have by Lemma 3 that the map $t \mapsto A_t \boxdot X$ is Bochner integrable on Ω and (13) holds. □

3. Chebyshev-Type Inequalities Involving Khatri–Rao Products of Operators

We introduce the following property, and prove Chebyshev-type inequalities involving Khatri–Rao products of operators.

Definition 1. *The fields A and B of operators parametrized by Ω are said to have the synchronous Khatri–Rao property if, for all $s, t \in \Omega$,*

$$(A_t - A_s) \boxdot (B_t - B_s) \geqslant 0. \tag{14}$$

They are said to have the opposite-synchronous Khatri–Rao property if the reverse of (14) holds for all $s, t \in \Omega$.

The following result is an extension of the Chebyshev integral inequality (6) to the case of operators involving Khatri–Rao products.

Theorem 1. *Let $A \in \mathfrak{BC}(\Omega, \mathcal{B}(\mathbb{H})^{sa})$ and $B \in \mathfrak{BC}(\Omega, \mathcal{B}(\mathbb{K})^{sa})$, and let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.*

1. *If A and B have the synchronous Khatri–Rao property, then*

$$\int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t)(A_t \boxdot B_t) d\mu(t) \geqslant \int_\Omega \alpha(t) A_t d\mu(t) \boxdot \int_\Omega \alpha(s) B_s d\mu(s). \tag{15}$$

2. *If A and B have the opposite-synchronous Khatri–Rao property, then then the reverse of (15) holds.*

Proof. By using Lemma 1 and Proposition 1, we have

$$\begin{aligned}
&\int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t)(A_t \boxdot B_t) d\mu(t) - \int_\Omega \alpha(t) A_t d\mu(t) \boxdot \int_\Omega \alpha(s) B_s d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)(A_t \boxdot B_t) d\mu(t) d\mu(s) - \iint_{\Omega^2} \alpha(t)\alpha(s)(A_t \boxdot B_s) d\mu(t) d\mu(s) \\
&= \iint_{\Omega^2} [\alpha(s)\alpha(t)(A_t \boxdot B_t) - \alpha(t)\alpha(s)(A_t \boxdot B_s)] \, d\mu(t) d\mu(s) \\
&= \frac{1}{2} \iint_{\Omega^2} [\alpha(s)\alpha(t)(A_t \boxdot B_t) - \alpha(t)\alpha(s)(A_t \boxdot B_s)] \, d\mu(t) d\mu(s) \\
&\quad + \frac{1}{2} \iint_{\Omega^2} [\alpha(t)\alpha(s)(A_s \boxdot B_s) - \alpha(s)\alpha(t)(A_s \boxdot B_t)] \, d\mu(s) d\mu(t) \\
&= \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t) \, [(A_t - A_s) \boxdot (B_t - B_s)] \, d\mu(t) d\mu(s).
\end{aligned}$$

Here, we have used Fubini's Theorem [14] to interchange the order of integrals. For the case 1, we have

$$\iint_{\Omega^2} \alpha(s)\alpha(t) \, [(A_t - A_s) \boxdot (B_t - B_s)] \, d\mu(t) d\mu(s) \geqslant 0 \tag{16}$$

and thus (15) holds. For another case, we get the reverse of (16) and, thus, the reverse of (15) holds. □

For the case $n = 1$, i.e., $\mathbb{H}$ and $\mathbb{K}$ are not decomposed, the synchronous Khatri–Rao property in Definition 1 reduces to the synchronous tensor property:

$$(A_t - A_s) \otimes (B_t - B_s) \geqslant 0.$$

If two fields A and B of operators parametrized by Ω have the synchronous tensor property, then A and B have the synchronous Hadamard property ([4], Definition 2.1), i.e.,

$$(A_t - A_s) \odot (B_t - B_s) \geqslant 0$$

for all $s, t \in \Omega$. The following result gives Chebyshev-type inequalities involving tensor products and Hadamard products.

Corollary 1. *Let $A, B \in \mathfrak{BC}(\Omega, \mathcal{B}(\mathbb{H})^{sa})$ and let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.*

1. *If A and B have the synchronous tensor property, then*

$$\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \otimes B_t)d\mu(t) \geqslant \int_\Omega \alpha(t)A_t d\mu(t) \otimes \int_\Omega \alpha(s)B_s d\mu(s), \tag{17}$$
$$\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \odot B_t)d\mu(t) \geqslant \int_\Omega \alpha(t)A_t d\mu(t) \odot \int_\Omega \alpha(s)B_s d\mu(s). \tag{18}$$

2. *If A and B have the opposite-synchronous tensor property, then then the reverses of (17) and (18) holds.*

Proof. For the case $n = 1$, the Khatri–Rao product in Theorem 1 reduces to the tensor product. Assume that A and B have the synchronous tensor property. Using the fact that the Hadamard product is expressed as the deformation of tensor product via the isometry U defined in (10), we obtain

$$\begin{aligned}
\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \odot B_t)d\mu(t) &= \int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)U^*(A_t \otimes B_t)U d\mu(t) \\
&= U^* \left(\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)(A_t \otimes B_t)d\mu(t) \right) U \\
&\geqslant U^* \left(\int_\Omega \alpha(t)A_t d\mu(t) \otimes \int_\Omega \alpha(s)B_s d\mu(s) \right) U \\
&= \int_\Omega \alpha(t)A_t d\mu(t) \odot \int_\Omega \alpha(s)B_s d\mu(s).
\end{aligned}$$

Case 2 for Hadamard products can be similarly treated. □

We can see that the inequality (18) is the same as (7), but they hold under different hypothesis. The next corollary is a discrete version of Theorem 1.

Corollary 2. *Let $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ where $A_i \in \mathcal{B}(\mathbb{H})^{sa}$, $B_i \in \mathcal{B}(\mathbb{K})^{sa}$ and ω_i is a nonnegative number for each $i = 1, \ldots, k$.*

1. *If A and B have the synchronous Khatri–Rao property, then*

$$\sum_{i=1}^{k} \omega_i \sum_{i=1}^{k} \omega_i (A_i \boxdot B_i) \geqslant \left(\sum_{i=1}^{k} \omega_i A_i \right) \boxdot \left(\sum_{i=1}^{k} \omega_i B_i \right). \tag{19}$$

2. *If A and B have the opposite-synchronous Khatri–Rao property, then the reverse of (19) holds.*

Proof. From the previous theorem, consider the finite space $\Omega = \{1, \ldots, k\}$ equipped with the counting measure and $\alpha(i) = \omega_i$ for all $i = 1, \ldots, k$. □

This corollary generalizes Chebyshev sum inequalities for the case of real numbers in inequality (1) and for Hadamard product of matrices in [1].

Next, we illustrate Chebyshev-type inequalities for bounded linear operators induced from matrices. Recall that with each $A \in \mathbb{M}_n(\mathbb{C})$ one can naturally associate a bounded linear operator

$$T_A : \mathbb{C}^n \to \mathbb{C}^n, \quad x \mapsto Ax.$$

For any complex matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ partitioned in block-matrix form, we have [11]

$$T_A \boxdot T_B = T_{A \boxdot B}. \tag{20}$$

Example 1. *Consider* $\omega_1 = \omega_2 = \frac{1}{2}$, $A = (T_{A_1}, T_{A_2})$, *and* $B = (T_{B_1}, T_{B_2})$, *where*

$$A_1 = \left[\begin{array}{c|cc} -1 & 1 & -1 \\ \hline 1 & 2 & 0 \\ -1 & 0 & -3 \end{array}\right], \quad A_2 = \left[\begin{array}{c|cc} 2 & 1 & 0 \\ \hline 1 & 3 & -1 \\ 0 & -1 & 1 \end{array}\right],$$

$$B_1 = \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 1 & -1 & 0 \\ \hline -1 & 0 & 1 \end{array}\right], \quad B_2 = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 4 & 0 \\ \hline 0 & 0 & 2 \end{array}\right].$$

First, we check the hypothesis of Corollary 2. Since $A_2 - A_1 \geqslant 0$, *we have* $T_{A_2} - T_{A_1} \geqslant 0$. *Similarly,* $T_{B_2} - T_{B_1} \geqslant 0$. *By the positivity of the Khatri–Rao product, we get*

$$(T_{A_2} - T_{A_1}) \boxdot (T_{B_2} - T_{B_1}) \geqslant 0,$$

i.e., the fields $A = (T_{A_1}, T_{A_2})$ *and* $B = (T_{B_1}, T_{B_2})$ *have the synchronous Khatri–Rao product property.*

Now, we can check that the following matrix is positive semidefinite:

$$\begin{aligned} & A_1 \boxdot B_1 + A_2 \boxdot B_2 - \frac{1}{2}(A_1 + A_2) \boxdot (B_1 + B_2) \\ &= \left[\begin{array}{cc|cc} 3 & 1 & -1 & 1 \\ 1 & 9 & 0 & 0 \\ \hline -1 & 0 & 8 & -2 \\ 1 & 0 & -2 & -1 \end{array}\right] - \frac{1}{2}\left[\begin{array}{cc|cc} 3 & 2 & -2 & 1 \\ 2 & 3 & 0 & 0 \\ \hline -2 & 0 & 15 & -3 \\ 1 & 0 & -3 & -6 \end{array}\right] \\ &= \frac{1}{2}\left[\begin{array}{cc|cc} 3 & 0 & 0 & 1 \\ 0 & 15 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 4 \end{array}\right] \\ &\geqslant 0. \end{aligned}$$

Thus,

$$A_1 \boxdot B_1 + A_2 \boxdot B_2 \geqslant \frac{1}{2}(A_1 + A_2) \boxdot (B_1 + B_2).$$

Passing through the induced linear maps, we get

$$T_{A_1 \boxdot B_1 + A_2 \boxdot B_2} \geqslant T_{\frac{1}{2}(A_1 + A_2) \boxdot (B_1 + B_2)}.$$

Finally, applying the property (20), we have

$$T_{A_1} \boxdot T_{B_1} + T_{A_2} \boxdot T_{B_2} \geqslant \frac{1}{2}(T_{A_1} + T_{A_2}) \boxdot (T_{B_1} + T_{B_2}),$$

i.e., the inequality (19) in Corollary 2 holds.

4. Chebyshev Integral Inequalities Concerning Weighted Pythagorean Means of Operators

We start this section by introducing order assumptions on continuous fields and supplying preliminaries on weighted arithmetic/geometric/harmonic means of operators. The main part is to establish Chebyshev-type inequalities involving Khatri–Rao products concerning such operator means and order assumptions.

Throughout this section, the space Ω is equipped with a total ordering $\preccurlyeq$. Consider the following definitions:

Definition 2. *We say that a field A is increasing (decreasing, resp.) whenever $s \preccurlyeq t$ implies $A_s \leqslant A_t$ ($A_s \geqslant A_t$, respectivley).*

Definition 3. *Two ordered pairs (X_1, X_2) and (Y_1, Y_2) of self-adjoint operators are said to have the synchronous property if either*

$$X_i \leqslant Y_i \text{ for } i = 1,2, \text{ or } X_i \geqslant Y_i \text{ for } i = 1,2.$$

The pairs (X_1, X_2) and (Y_1, Y_2) are said to have the opposite-synchronous property if either

$$X_1 \leqslant Y_1 \text{ and } X_2 \geqslant Y_2, \text{ or } X_1 \geqslant Y_1 \text{ and } X_2 \leqslant Y_2.$$

Definition 4. *Let A, B, C, D be continuous fields of self-adjoint operators parametrized by Ω. Two ordered pairs (A, B) and (C, D) are said to have the synchronous monotone property if (A_t, B_t) and (C_t, D_t) have the synchronous property for all $t \in \Omega$. The pairs (A, B) and (C, D) are said to have the opposite-synchronous monotone property if (A_t, B_t) and (C_t, D_t) have the opposite-synchronous property for all $t \in \Omega$.*

Recall that the three classical Pythagorean means are the following symmetric means: the arithmetic mean, the harmonic mean, and the geometric mean. For each $w \in [0,1]$, the w-weighted versions of such means are respectively defined for any $A, B \in \mathcal{B}(\mathbb{H})^{++}$ by

$$\begin{aligned} A \triangledown_w B &= (1-w)A + wB, \\ A \,!_w\, B &= \left[(1-w)A^{-1} + wB^{-1}\right]^{-1}, \\ A \sharp_w B &= A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{w} A^{\frac{1}{2}}. \end{aligned}$$

These means can be defined for arbitrary positive operators by the following continuity argument with respect to the strong-operator topology:

$$A \sharp_w B = \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \sharp_w (B + \varepsilon I).$$

For brevity, we write $A \sharp B$ for $A \sharp_{1/2} B$. The Pythagorean means have the following remarkable property: for any $w \in [0,1]$, $T \in \mathcal{B}(\mathbb{H})$, and $A, B \in \mathcal{B}(\mathbb{H})^+$, we have

$$T^*(A \,\sigma_w\, B)T \leqslant (T^*AT) \,\sigma_w\, (T^*BT), \tag{21}$$

here σ is anyone of $\triangledown, !, \#$.

Lemma 4 (see e.g., [15]). *The weighted geometric means, weighted arithmetic means and weighted harmonic means for operators are (jointly) monotone in the sense that if $X_1 \leqslant X_2$ and $Y_1 \leqslant Y_2$, then $X_1 \sigma Y_1 \leqslant X_2 \sigma Y_2$ where σ is any of $\triangledown_w, !_w, \sharp_w$.*

4.1. Inequalities on Weighted Geometric Means

Recall that a linear map between two operator algebras is said to be positive if it maps positive operators to positive operators.

Lemma 5 ([16]). *Let $A, B, C, D \in \mathcal{B}(\mathbb{H})^+$ and $w \in [0,1]$. Then*

$$(A \boxdot B) \sharp_w (C \boxdot D) \geqslant (A \sharp_w C) \boxdot (B \sharp_w D).$$

Theorem 2. *Let $A, B, C, D \in \mathfrak{BC}(\Omega; \mathcal{B}(\mathbb{H})^+)$ and let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function. If A, B, C, D are either all increasing, or all decreasing, then*

$$\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t)\sharp_w (C_t \boxdot D_t)]d\mu(t) \geqslant \int_\Omega \alpha(t)(A_t \sharp_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s \sharp_w D_s)d\mu(s). \tag{22}$$

Proof. Let $s, t \in \Omega$. Without loss of generality, assume that $s \preccurlyeq t$. By applying Lemmas 1 and 5, Proposition 1, and Fubini's Theorem [14], we have

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t)\sharp_w (C_t \boxdot D_t)]d\mu(t) - \int_\Omega \alpha(t)(A_t \sharp_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s \sharp_w D_s)d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxdot B_t)\sharp_w (C_t \boxdot D_t)]d\mu(t)d\mu(s) - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \sharp_w C_t) \boxdot (B_s \sharp_w D_s)]d\mu(t)d\mu(s) \\
&\geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \sharp_w C_t) \boxdot (B_t \sharp_w D_t)]d\mu(t)d\mu(s) - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \sharp_w C_t) \boxdot (B_s \sharp_w D_s)]d\mu(t)d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \sharp_w C_t) \boxdot (B_t \sharp_w D_t) - (A_t \sharp_w C_t) \boxdot (B_s \sharp_w D_s)]d\mu(t)d\mu(s) \\
&= \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \sharp_w C_t) \boxdot (B_t \sharp_w D_t) - (A_t \sharp_w C_t) \boxdot (B_s \sharp_w D_s)]d\mu(t)d\mu(s) \\
&\quad + \frac{1}{2} \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_s \sharp_w C_s) \boxdot (B_s \sharp_w D_s) - (A_s \sharp_w C_s) \boxdot (B_t \sharp_w D_t)]d\mu(s)d\mu(t) \\
&= \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[A_t \sharp_w C_t - A_s \sharp_w C_s] \boxdot [B_t \sharp_w D_t - B_s \sharp_w D_s]d\mu(t)d\mu(s).
\end{aligned}$$

If A, B, C, D are all increasing, we have by Lemma 4 that $A_t \sharp_w C_t \geqslant A_s \sharp_w C_s$ and $B_t \sharp_w D_t \geqslant B_s \sharp_w D_s$. If A, B, C, D are all decreasing, we have $A_t \sharp_w C_t \leqslant A_s \sharp_w C_s$ and $B_t \sharp_w D_t \leqslant B_s \sharp_w D_s$. By Lemma 1, both cases lead to the same conclusion that $(A_t \sharp_w C_t - A_s \sharp_w C_s) \boxdot (B_t \sharp_w D_t - B_s \sharp_w D_s) \geqslant 0$, and hence (22) holds. □

The next corollary is a discrete version of Theorem 1.

Corollary 3. *Let $A = (A_1, \ldots, A_k), B = (B_1, \ldots, B_k), C = (C_1, \ldots, C_k)$ and $D = (D_1, \ldots, D_k)$ where $A_i, B_i, C_i, D_i \in \mathcal{B}(\mathbb{H})^+$ for each $i = 1, \ldots, k$. If A, B, C, D are either all increasing, or all decreasing, then*

$$\frac{1}{k} \sum_{i=1}^{k} [(A_i \boxdot B_i)\sharp_w (C_i \boxdot D_i)] \geqslant \left(\frac{1}{k} \sum_{i=1}^{k} (A_i \sharp_w C_i) \right) \boxdot \left(\frac{1}{k} \sum_{i=1}^{k} (B_i \sharp_w D_i) \right). \tag{23}$$

Proof. Setting $\Omega = \{1, \ldots, k\}$ equipped with the counting measure and $\alpha(i) = \frac{1}{k}$ for all $i = 1, \ldots, k$ in Theorem 2, we get the result. □

Operator inequality (23) can be regarded as a generalization of the Chebyshev sum inequality (1). The next goal is to establish a reverse version of Theorem 2.

Lemma 6. *Let $A, B, C, D \in \mathcal{B}(\mathbb{H})^+$ be such that $0 < m_1 I \leqslant A \boxtimes B \leqslant M_1 I$ and $0 < m_2 I \leqslant C \boxtimes D \leqslant M_2 I$. Denote $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and*

$$\lambda = \frac{Mm^w - mM^w}{(1-w)(M-m)} \cdot \left(\frac{1-w}{w} \cdot \frac{M^w - m^w}{Mm^w - mM^w} \right)^w. \tag{24}$$

Then for any $w \in (0, 1)$, we have

$$\lambda (A \boxdot B)\sharp_w (C \boxdot D) \leqslant (A \sharp_w C) \boxdot (B \sharp_w D).$$

Proof. Consider a map $\Phi : T \mapsto Z^*TZ$, where Z is the isometry in Lemma 2. Since Φ is a unital positive linear map, we have by ([17], Corollary 3.5) that

$$\lambda[\Phi(A \boxtimes B)\, \#_w\, \Phi(C \boxtimes D)] \leqslant \Phi\left((A \boxtimes B)\, \#_w\, (C \boxtimes D)\right).$$

From ([16], Theorem 1), we get

$$(A \boxtimes B)\, \#_w\, (C \boxtimes D) = (A\, \#_w\, C) \boxtimes (B\, \#_w\, D).$$

Hence,

$$\begin{aligned} \lambda[(A \boxdot B)\, \#_w\, (C \boxdot D)] &= \lambda[\Phi(A \boxtimes B)\, \#_w\, \Phi(C \boxtimes D)] \\ &\leqslant \Phi\left((A \boxtimes B)\, \#_w\, (C \boxtimes D)\right) \\ &= \Phi\left((A\, \#_w\, C) \boxtimes (B\, \#_w\, D)\right) \\ &= (A\, \#_w\, C) \boxdot (B\, \#_w\, D). \quad \square \end{aligned}$$

Theorem 3. *Let $A, B, C, D \in \mathfrak{BC}(\Omega; \mathcal{B}(\mathbb{H})^+)$ with $0 < m_1 I \leqslant A_t \boxtimes B_t \leqslant M_1 I$ and $0 < m_2 I \leqslant C_t \boxtimes D_t \leqslant M_2 I$ for all $t \in \Omega$, and $\alpha : \Omega \to [0, \infty)$ a bounded measurable function. Let m, M, w, λ as in Lemma 6. If either*

1. *A, C are increasing and B, D are decreasing, or*
2. *A, C are decreasing and B, D are increasing,*

then

$$\begin{aligned} \lambda \int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t)\sharp_w(C_t \boxdot D_t)]d\mu(t) \\ \leqslant \int_\Omega \alpha(t)(A_t \sharp_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s \sharp_w D_s)d\mu(s). \end{aligned} \tag{25}$$

Proof. Let $s, t \in \Omega$. Without loss of generality, assume that $s \preccurlyeq t$. By applying Lemmas 1 and 6, Proposition 1, and Fubini's Theorem [14], we have

$$\begin{aligned} &\lambda \int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t)\sharp_w(C_t \boxdot D_t)]d\mu(t) - \int_\Omega \alpha(t)(A_t \sharp_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s \sharp_w D_s)d\mu(s) \\ &\quad = \iint_{\Omega^2} \alpha(s)\alpha(t)[\lambda(A_t \boxdot B_t)\sharp_w(C_t \boxdot D_t)]d\mu(t)d\mu(s) \\ &\qquad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \sharp_w C_t) \boxdot (B_s \sharp_w D_s)]d\mu(t)d\mu(s) \\ &\quad \leqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \sharp_w C_t) \boxdot (B_t \sharp_w D_t)]d\mu(t)d\mu(s) \\ &\qquad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \sharp_w C_t) \boxdot (B_s \sharp_w D_s)]d\mu(t)d\mu(s) \\ &\quad = \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \sharp_w C_t) \boxdot (B_t \sharp_w D_t) - (A_t \sharp_w C_t) \boxdot (B_s \sharp_w D_s)]d\mu(t)d\mu(s) \\ &\qquad + \frac{1}{2} \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_s \sharp_w C_s) \boxdot (B_s \sharp_w D_s) - (A_s \sharp_w C_s) \boxdot (B_t \sharp_w D_t)]d\mu(s)d\mu(t) \\ &\quad = \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[A_t \sharp_w C_t - A_s \sharp_w C_s] \boxdot [B_t \sharp_w D_t - B_s \sharp_w D_s]d\mu(t)d\mu(s). \end{aligned}$$

We have by Lemmas 1 and 4 that $(A_t \sharp_w C_t - A_s \sharp_w C_s) \boxdot (B_t \sharp_w D_t - B_s \sharp_w D_s) \leqslant 0$, and hence (25) holds. $\square$

4.2. Inequalities on Weighted Arithmetic Means

Lemma 7 ([18]). *Let $A, B, C, D \in \mathcal{B}(\mathbb{H})^{sa}$ and $w \in [0,1]$.*

1. *If (A, B) and (C, D) have the synchronous property, then*

$$(A \boxdot B)\nabla_w(C \boxdot D) \geqslant (A\nabla_w C) \boxdot (B\nabla_w D). \tag{26}$$

2. *If (A, B) and (C, D) have the opposite-synchronous property, then the reverse of (26) holds.*

Theorem 4. *Let $A, B, C, D \in \mathfrak{BC}(\Omega; \mathcal{B}(\mathbb{H})^+)$ and let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.*

1. *If $(A, B), (C, D)$ have the synchronous monotone property and all of A, B, C, D are either increasing or decreasing, then*

$$\begin{aligned}\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t)\nabla_w(C_t \boxdot D_t)]d\mu(t)& \\ \geqslant \int_\Omega \alpha(t)(A_t\nabla_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s\nabla_w D_s)d\mu(s).&\end{aligned} \tag{27}$$

2. *If $(A, B), (C, D)$ have the opposite-synchronous monotone property and if either*
   - *(a) A, C are increasing and B, D are decreasing, or*
   - *(b) A, C are decreasing and B, D are increasing,*

   then the reverse of (27) holds.

Proof. Let $s, t \in \Omega$. Without loss of generality, assume that $s \preccurlyeq t$. First, we consider the case 1. We have by using Lemmas 1 and 7, proposition 1, and Fubini's Theorem that

$$\begin{aligned}&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t)\nabla_w(C_t \boxdot D_t)]d\mu(t) - \int_\Omega \alpha(t)(A_t\nabla_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s\nabla_w D_s)d\mu(s)\\
&\quad= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxdot B_t)\nabla_w(C_t \boxdot D_t)]d\mu(t)d\mu(s)\\
&\quad\quad- \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t\nabla_w C_t) \boxdot (B_s\nabla_w D_s)]d\mu(t)d\mu(s)\\
&\quad\geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\nabla_w C_t) \boxdot (B_t\nabla_w D_t)]d\mu(t)d\mu(s)\\
&\quad\quad- \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t\nabla_w C_t) \boxdot (B_s\nabla_w D_s)]d\mu(t)d\mu(s)\\
&\quad= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t\nabla_w C_t) \boxdot (B_t\nabla_w D_t) - (A_t\nabla_w C_t) \boxdot (B_s\nabla_w D_s)]d\mu(t)d\mu(s)\\
&\quad= \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)[A_t\nabla_w C_t - A_s\nabla_w C_s] \boxdot [B_t\nabla_w D_t - B_s\nabla_w D_s]d\mu(t)d\mu(s).\end{aligned}$$

By Lemmas 1 and 4, we have $(A_t\nabla_w C_t - A_s\nabla_w C_s) \boxdot (B_t\nabla_w D_t - B_s\nabla_w D_s) \geqslant 0$ and hence (27) holds. The case 2 can be similarly treated. □

We can illustrate Theorem 4 for the case of operators induced from matrices as follows.

Example 2. *Consider the following pairs of induced bounded linear operators:* $A = (T_{A_1}, T_{A_2})$, $B = (T_{B_1}, T_{B_2})$, $C = (T_{C_1}, T_{C_2})$ *and* $D = (T_{D_1}, T_{D_2})$, *where*

$$A_1 = \left[\begin{array}{c|cc} 2 & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right], \quad A_2 = \left[\begin{array}{c|cc} 3 & 1 & 0 \\ \hline 1 & 2 & 1 \\ 0 & 1 & 4 \end{array}\right], \quad B_1 = \left[\begin{array}{c|cc} 3 & 2 & 0 \\ \hline 2 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

$$B_2 = \left[\begin{array}{c|cc} 5 & 3 & 0 \\ \hline 3 & 6 & 2 \\ 0 & 2 & 4 \end{array}\right], \quad C_1 = \left[\begin{array}{c|cc} 2 & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad C_2 = \left[\begin{array}{c|cc} 3 & 1 & 0 \\ \hline 1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right],$$

$$D_1 = \left[\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad D_2 = \left[\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 1 & 3 & 2 \\ 0 & 2 & 2 \end{array}\right].$$

We can check the hypothesis of Theorem 4: (i) all of A, B, C, D *are increasing, and (ii)* $(A, B), (C, D)$ *have the synchronous monotone property. Set* $\Omega = \{1, 2\}$ *equipped with the counting measure and* $\alpha(i) = \frac{1}{2}$ *for all* $i = 1, 2$. *Let us denote* $\triangledown = \triangledown_{1/2}$. *Now, a direct computation reveals that*

$$(A_1 \boxdot B_1)\triangledown(C_1 \boxdot D_1) + (A_2 \boxdot B_2)\triangledown(C_2 \boxdot D_2) - \frac{1}{2}(A_1\triangledown C_1 + A_2\triangledown C_2) \boxdot (B_1\triangledown D_1 + B_2\triangledown D_2)$$
$$= \frac{1}{4}\left[\begin{array}{c|cccc} 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 6 & 2 & 5 & 2 \\ 0 & 2 & 4 & 2 & 4 \\ 0 & 5 & 2 & 14 & 8 \\ 0 & 2 & 4 & 8 & 10 \end{array}\right] \geqslant 0,$$

or equivalently,

$$(A_1 \boxdot B_1)\triangledown(C_1 \boxdot D_1) + (A_2 \boxdot B_2)\triangledown(C_2 \boxdot D_2) \geqslant \frac{1}{2}(A_1\triangledown C_1 + A_2\triangledown C_2) \boxdot (B_1\triangledown D_1 + B_2\triangledown D_2).$$

Passing through the induced linear maps and applying the properties $T_{A\triangledown B} = T_A \triangledown T_B$ *and (20), we obtain*

$$(T_{A_1} \boxdot T_{B_1})\triangledown(T_{C_1} \boxdot T_{D_1}) + (T_{A_2} \boxdot T_{B_2})\triangledown(T_{C_2} \boxdot T_{D_2})$$
$$\geqslant \frac{1}{2}(T_{A_1}\triangledown T_{C_1} + T_{A_2}\triangledown T_{C_2}) \boxdot (T_{B_1}\triangledown T_{D_1} + T_{B_2}\triangledown T_{D_2}).$$

Thus, the equality (27) holds in this case.

4.3. Inequalities on Weighted Harmonic Means

Lemma 8 ([18]). *Let* $A, B, C, D \in \mathcal{B}(\mathbb{H})^+$ *and* $w \in [0, 1]$. *If* (A, B) *and* (C, D) *have the opposite-synchronous property, then*

$$(A \boxdot B) \,!_w\, (C \boxdot D) \geqslant (A \,!_w\, C) \boxdot (B \,!_w\, D).$$

Theorem 5. *Let* $A, B, C, D \in \mathfrak{BC}(\Omega; \mathcal{B}(\mathbb{H})^+)$ *and* $\alpha : \Omega \to [0, \infty)$ *a bounded measurable function. If* (A, B) *and* (C, D) *have the opposite-synchronous monotone property and all of* A, B, C, D *are either increasing or decreasing, then*

$$\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t) \,!_w\, (C_t \boxdot D_t)]d\mu(t)$$
$$\geqslant \int_\Omega \alpha(t)(A_t \,!_w\, C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s \,!_w\, D_s)d\mu(s). \quad (28)$$

Proof. Let $s, t \in \Omega$ with $s \preccurlyeq t$. By applying Lemmas 1 and 8, Proposition 1, and Fubini's theorem, we get

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t) !_w (C_t \boxdot D_t)]d\mu(t) - \int_\Omega \alpha(t)(A_t !_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s !_w D_s)d\mu(s) \\
&\quad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxdot B_t) !_w (C_t \boxdot D_t)]d\mu(t)d\mu(s) \\
&\quad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t !_w C_t) \boxdot (B_s !_w D_s)]d\mu(t)d\mu(s) \\
&\quad \geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t !_w C_t) \boxdot (B_t !_w D_t)]d\mu(t)d\mu(s) \\
&\quad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t !_w C_t) \boxdot (B_s !_w D_s)]d\mu(t)d\mu(s) \\
&\quad = \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[A_t !_w C_t - A_s !_w C_s] \boxdot [B_t !_w D_t - B_s !_w D_s]d\mu(t)d\mu(s).
\end{aligned}$$

By Lemmas 1 and 4, $(A_t !_w C_t - A_s !_w C_s) \boxdot (B_t !_w D_t - B_s !_w D_s) \geqslant 0$ and hence (28) holds. □

Lemma 9 ([18]). *Let $A, B, C, D \in \mathcal{B}(\mathbb{H})^+$ with $0 < m_1 I \leqslant A \boxtimes B \leqslant M_1 I$ and $0 < m_2 I \leqslant C \boxtimes D \leqslant M_2 I$. Denote*

$$\lambda = \left[\frac{(m_1 \sharp M_1) \nabla_{1-w} (m_2 \sharp M_2)}{(m_1 \nabla_{1-w} M_2) \sharp (M_1 \nabla_{1-w} m_2)} \right]^2. \tag{29}$$

If (A, B) and (C, D) have the synchronous property, then for any $w \in [0, 1]$,

$$\lambda (A \boxdot B) !_w (C \boxdot D) \leqslant (A !_w C) \boxdot (B !_w D).$$

Theorem 6. *Let $A, B, C, D \in \mathfrak{BC}(\Omega; \mathcal{B}(\mathbb{H})^+)$ be such that for all $t \in \Omega$, $0 < m_1 I \leqslant A_t \boxtimes B_t \leqslant M_1 I$ and $0 < m_2 I \leqslant C_t \boxtimes D_t \leqslant M_2 I$. Let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function. If (A, B) and (C, D) have the synchronous monotone property and if either*

1. *A, C are increasing and B, D are decreasing, or*
2. *A, C are decreasing and B, D are increasing,*

then with the constant λ in (29) we have

$$\begin{aligned}
\lambda \int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t) !_w (C_t \boxdot D_t)]d\mu(t)& \\
\leqslant \int_\Omega \alpha(t)(A_t !_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s !_w D_s)d\mu(s).&
\end{aligned} \tag{30}$$

Proof. Let $s, t \in \Omega$ with $s \preccurlyeq t$. We have by using Lemmas 1 and 9, Proposition 1, and Fubini's Theorem that

$$\begin{aligned}
&\lambda \int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxdot B_t) !_w (C_t \boxdot D_t)]d\mu(t) - \int_\Omega \alpha(t)(A_t !_w C_t)d\mu(t) \boxdot \int_\Omega \alpha(s)(B_s !_w D_s)d\mu(s) \\
&\quad = \iint_{\Omega^2} \alpha(s)\alpha(t)[\lambda(A_t \boxdot B_t) !_w (C_t \boxdot D_t)]d\mu(t)d\mu(s) \\
&\quad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t !_w C_t) \boxdot (B_s !_w D_s)]d\mu(t)d\mu(s) \\
&\quad \leqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t !_w C_t) \boxdot (B_t !_w D_t)]d\mu(t)d\mu(s) \\
&\quad\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t !_w C_t) \boxdot (B_s !_w D_s)]d\mu(t)d\mu(s) \\
&\quad = \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[A_t !_w C_t - A_s !_w C_s] \boxdot [B_t !_w D_t - B_s !_w D_s]d\mu(t)d\mu(s).
\end{aligned}$$

We have, by Lemmas 1 and 4, $(A_t \,!_w\, C_t - A_s \,!_w\, C_s) \boxdot (B_t \,!_w\, D_t - B_s \,!_w\, D_s) \leqslant 0$ and hence (30) holds. □

Remark 1. *When we set $\Omega = \{1, \dots, k\}$ equipped with the counting measure, we get discrete versions of Theorems 3–6. Matrix analogs of our results can be obtained particularly by setting $\mathbb{H} = \mathbb{C}^n$. In this case, our results include Chebyshev-type inequalities for Khatri–Rao products, Kronecker products and Hadamard products of matrices.*

5. Chebyshev–Grüss Inequalities via Oscillations

In this section, we prove a Chebyshev–Grüss inequality via oscillations for continuous fields of operators. The basic setup is as follows.

Let Ω be a compact Hausdorff space equipped with a probability Radon measure μ. For any $A \in \mathfrak{C}(\Omega; \mathcal{B}(\mathbb{H}))$ and $B \in \mathfrak{C}(\Omega; \mathcal{B}(\mathbb{K}))$, we define

$$\begin{aligned} A \boxdot B &= (A_t \boxdot B_t)_{t \in \Omega}, \\ \mathcal{I}(A) &= \int_\Omega A_t d\mu(t), \\ \operatorname{osc}(A) &= \max\{\|A_t - A_s\| : (t,s) \in \operatorname{supp}(\mu \times \mu)\}. \end{aligned}$$

Here, we recall that the support of the product measure $\mu \times \mu$ is defined by

$$\operatorname{supp}(\mu \times \mu) = \{(t,s) \in \Omega^2 \mid (\mu \times \mu)(G) > 0 \text{ for all open sets } G \subseteq \Omega^2 \text{ containing } (t,s)\}$$

We call $\operatorname{osc}(A)$ the oscillation of the field A.

In the next theorem, we generalize Grüss inequality (8) to the case of operators concerning Khatri–Rao products.

Theorem 7. *For any $A \in \mathfrak{C}(\Omega; \mathcal{B}(\mathbb{H})^{sa})$ and $B \in \mathfrak{C}(\Omega; \mathcal{B}(\mathbb{K})^{sa})$, we have*

$$\mathcal{I}(A \boxdot B) - \mathcal{I}(A) \boxdot \mathcal{I}(B) \leqslant \frac{1}{2} \operatorname{osc}(A) \operatorname{osc}(B) (\mu \times \mu)(\Omega^2 \backslash \Delta) I, \tag{31}$$

where $\Delta = \{(t,t) : t \in \Omega\}$.

Proof. By using Lemma 1, Proposition 1 and Fubini's theorem, we have

$$\begin{aligned} \mathcal{I}(A \boxdot B) - \mathcal{I}(A) \boxdot \mathcal{I}(B) &= \int_\Omega d\mu(s) \int_\Omega A_t \boxdot B_t d\mu(t) - \int_\Omega A_t d\mu(t) \boxdot \int_\Omega B_s d\mu(s) \\ &= \iint_{\Omega^2} A_t \boxdot B_t d\mu(t) d\mu(s) - \iint_{\Omega^2} A_t \boxdot B_s d\mu(t) d\mu(s) \\ &= \frac{1}{2} \iint_{\Omega^2} (A_t - A_s) \boxdot (B_t - B_s) d\mu(t) d\mu(s) \\ &= \frac{1}{2} \iint_{\Omega^2 \backslash \Delta} (A_t - A_s) \boxdot (B_t - B_s) d\mu(t) d\mu(s) \\ &\leqslant \frac{1}{2} \operatorname{osc}(A) \operatorname{osc}(B) (\mu \times \mu)(\Omega^2 \backslash \Delta) I. \quad \square \end{aligned}$$

Theorem 7 can be viewed as an operator version of ([8], Theorem 7).

Corollary 4. *For any $A_i \in \mathcal{B}(\mathbb{H})^{sa}$ and $B_i \in \mathcal{B}(\mathbb{K})^{sa}$, $i = 1, \dots, k$, we have*

$$\sum_{i=1}^k (A_i \boxdot B_i) - \left(\sum_{i=1}^k A_i \right) \boxdot \left(\sum_{i=1}^k B_i \right) \leqslant \frac{k(k-1)}{2} \max_{1 \leqslant i,j \leqslant k} \|A_i - A_j\| \cdot \max_{1 \leqslant i,j \leqslant k} \|B_i - B_j\| I.$$

Proof. Let $A = (A_1, \ldots, A_k)$ and $B = (B_1 \ldots, B_k)$. Set Ω to be the finite space $\{1, \ldots, k\}$ equipped with the counting measure in Theorem 7, we have

$$(\mu \times \mu)(\Omega^2 \backslash \Delta) = \frac{k(k-1)}{2}, \quad \mathrm{supp}(\mu \times \mu) = \Omega^2$$

and thus $\mathrm{osc}(A) = \max_{1 \leqslant i,j \leqslant k} \|A_i - A_j\|$, $\mathrm{osc}(B) = \max_{1 \leqslant i,j \leqslant k} \|B_i - B_j\|$. □

6. Conclusions and Future Work

We investigate integral inequalities of Chebyshev-type for bounded continuous fields of self-adjoint operators which are parametrized by a locally compact Hausdorff space Ω equipped with a finite Radon measure μ. Under certain assumptions on the synchronous Khatri–Rao property of the fields of operators, we obtain Chebyshev-type inequalities concerning Khatri–Rao products. We also establish Chebyshev-type inequalities involving Khatri–Rao products and weighted Pythagorean means under the assumption of synchronous monotone property of the fields of operators. Moreover, we derive Chebyshev–Grüss integral inequality via oscillations when μ is a probability Radon measure. These integral inequalities can be reduced to discrete inequalities by setting Ω to be a finite space equipped with the counting measure. Our results include Chebyshev-type inequalities for the tensor product of operators and Khatri–Rao/Kronecker/Hadamard products of matrices.

For future work, we may investigate Chebyshev-type inequalities when replacing weighted Pythagorean means by quasi-arithmetic power means, the logarithmic mean, or general operator means.

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1. Ploymukda, A. and Chansangiam, P. 2019. Concavity and Convexity of Several Maps Involving Tracy-Singh Products, Khatri-Rao Products, and Operator-Monotone Functions of Positive Operators, *ScienceAsia* **45**, 194–201.
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