THE METHODS FOR SOLVING THE GENERAL SPLIT FEASIBILITY PROBLEM AND FIXED POINT PROBLEMS

SIRAWIT PREMJITPRAPHAN

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN APPLIED MATHEMATICS
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วิธีสำหรับการแก้ปัญหาความเป็นไปได้แบ่งแยกทั่วไปและปัญหาจุดตรึง

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บทคัดย่อ

วัตถุประสงค์ของวิทยานิพนธ์นี้คือแนะนำทฤษฎีบทสองทฤษฎี ขั้นแรกเป็นการแนะนำระเบียบวิธีการทำซ้ำสำหรับแก้ปัญหาความเป็นไปได้แบ่งแยกของจุดตรึงสำหรับการส่งแบบกึ่งหดตัวเทียมลิพชิทซ์ จากนั้นได้ทำการพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มสำหรับการหาค่าผลเฉลยร่วมของปัญหาดังกล่าว ขั้นที่สองเป็นการแนะนำระเบียบวิธีการทำซ้ำสำหรับแก้ปัญหาความเท่าเทียมกันแบ่งแยกของจุดตรึงของการส่งกึ่งไม่ขยาย จากนั้นได้ทำการพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มของกระบวนการทำซ้ำดังกล่าวภายใต้เงื่อนไขที่เหมาะสม

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Abstract

The objective of this thesis is to introduce two main theorems. Firstly, we introduce the iterative algorithms for solving the split feasibility problem and fixed point problem for Lipschitzian quasi-pseudo contractive mapping. Then, we prove strong convergence theorems to find a common solution of these problems under appropriate conditions. Finally, we introduce iterative algorithms for solving the split equality fixed point problem for quasi nonexpansive mappings. After that, we prove a strong convergence theorem for a proposed iterative scheme under some appropriate conditions.

Keywords : Quasi-nonexpansive mapping, Lipschitzian quasi-pseudo-contractive mapping, Nonexpansive mapping, Fixed point problem, Split feasibility problem, Split Equality Fixed point problem

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Chapter 1

Introduction

1.1 Background and signification of the research

Nowadays, many problems have many solutions. Mathematical is a very important tool to solve these problems. The fixed point theory is efficient and preferable for use. This theory can guarantee that the answers are existence and unique. Many mathematicians are interested in developing iterative algorithm of fixed point theory of various mappings.

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, the norm $\|\cdot\|$ and C is a nonempty closed convex subset of H. Using the notations of weak and strong convergence by "$\rightharpoonup$" and "$\rightarrow$", respectively.

Let $T : C \to C$ be a mapping, we denote $F(T)$ by the set of all *fixed points* of T i.e.,

$$F(T) = \{x \in C : Tx = x\}.$$

Example 1.1.

1. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = \frac{4x+1}{3}$, then $F(T) = \{-1\}$.
2. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = \frac{x^2+1}{2}$, then $F(T) = \{1\}$.
3. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = \frac{x^2+2}{3}$, then $F(T) = \{1, 2\}$.
4. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = x - 5$, then $F(T) = \emptyset$.
5. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = x$, then $F(T) = \mathbb{R}$.

Let C be a nonempty closed convex subset of H and $A : C \to H$ be a nonlinear mapping. The *variational inequality problem* is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \tag{1.1}$$

for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C, A)$.

Variational inequalities were introduced by Stampacchia [1] and provide a useful tool for researching a large variety of interesting problems arising in physics, economics, finance, optimization and medical images [1]-[12].

The (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following property which is very useful.

Lemma 1.2. Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Given $x \in H$ and $y \in C$, then following holds

$$P_C x = y \Leftrightarrow \langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

Definition 1.1. Let $T : C \to C$ is said to be:

1. *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

2. *α-contraction* if there is a constant $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \quad \forall x, y \in C;$$

3. *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C;$$

[25] proved that the mapping T is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C;$$

4. *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C \quad \text{and} \quad p \in F(T);$$

It can be easily seen that every nonexpansive mapping is quasi-nonexpansive mapping, where $F(T) \neq \emptyset$.

5. *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \quad \forall x, y \in C;$$

[38] proved that the mapping T is equivalent to

$$\langle Tx - Ty, (I - T)x - (I - T)y \rangle \geq 0, \quad \forall x, y \in C;$$

and

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(\mathcal{I} - T)x - (\mathcal{I} - T)y\|^2, \quad \forall x, y \in C.$$

6. *pseudo-contractive* if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C;$$

7. *quasi-pseudo-contractive* if

$$\|Tx - y\|^2 \leq \|x - y\|^2 + \|Tx - x\|^2 \quad \forall x \in C \quad \text{and} \quad y \in F(T).$$

A mapping $T : C \to H$ is called *$\mathcal{L}$-Lipschitzian* if

$$\|Tx - Ty\| \leq \mathcal{L}\, \|x - y\|\,, \qquad \forall x, y \in C$$

for some constant $\mathcal{L} > 0$. It easy to see that nonexpansive mapping is 1-Lipschitzian. A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\, \|Ax - Ay\|^2\,,$$

for all $x, y \in C$. It is obvious that any α-inverse strongly monotone mapping A is $\frac{1}{\alpha}$ -Lipschitzian.

Let C and Q be nonempty closed convex of two Hilbert space H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. In 1994, Censor and Elfving [5] introduced the *split feasibility problem* (in short, SFP) is formulated as finding a point x^* with the property

$$x^* \in C \text{ and } Ax^* \in Q. \tag{1.2}$$

The set of all solutions of split feasibility problem is denoted by $\varphi = \{x^* \in C : Ax^* \in Q\}$, the SFP in finite dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction.

Assuming that SFP is consistent, it is easy to see that $x^* \in C$ is a solution of (1.1) if and only if it solves the following fixed point equation

$$x^* = P_C(I - \gamma A^*(I - P_Q))x^*, \tag{1.3}$$

where P_C and P_Q are the metric projections from H_1 onto C and from H_2 onto Q, respectively, γ is a positive constant and A^* denotes by adjoint of A.

The popular algorithm used in approximating the solution of the SFP (1.2) is the CQ-algorithm of Byrne [6]:

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q))x_n, \tag{1.4}$$

for all $n \in \mathbb{N}$, where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the *spectral radius* of the operator A^*A and A^* is the adjoint of A.

Let $U : C \to C$ and $T : Q \to Q$ be two nonlinear operators. The *split common fixed points problem* (SCFPP) [7, 8] is to find a point $p^* \in C$ such that

$$p^* \in F(U) \text{ and } Ap^* \in F(T).$$

The solution set of SCFPP is denoted by $\Phi = \{p^* \in F(U) : Ap^* \in F(T)\}$. The split common fixed point problem is a natural extension of the split feasibility problem, if $I_C = U$ and $I_Q = T$ where $I_C : C \to C$ and $I_Q : Q \to Q$ are identity mapping.

In 2015, Hamdi, Liou, Yao and Luo [4] proved a strong convergence theorem as following algorithm : $x_0 \in H_1$ and

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n S\left((1-\eta_n) z_n + \eta_n S z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + (\mathcal{I} - \alpha_n \mathcal{B})\left(x_n - \delta A^*\left(Ax_n - v_n\right)\right), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n T\left((1-\gamma_n) u_n + \gamma_n T u_n\right) \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$, $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, $f : C \to H_1$ is ρ-contraction, $\mathcal{B}$ is strongly positive bounded linear operator on H_1, $T : Q \to Q$ is an $\mathcal{L}_1$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_1 > 1$, $U : C \to C$ is an $\mathcal{L}_2$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_2 > 1$. They showed that the sequence $\{x_n\}$ converges strongly to the unique fixed point of the contraction mapping $P_\Phi\left(\gamma f + \mathcal{I} - \mathcal{B}\right)$.

The split feasibility problem and fixed point problem is to find

$$u^* \in C \cap F(U) \text{ and } Au^* \in Q \cap F(T), \tag{1.5}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two nonlinear mapping. The set of solution of (1.5) is denoted by $\mathbf{\Gamma}$, that is,

$$\mathbf{\Gamma} = \{x \mid x \in C \cap F(U), Ax \in Q \cap F(T)\}.$$

It is immediately evident that (1.5) can be derived from SFP and SCFPP.

In 2013, Moudafi [24] introduced the following *split equality feasibility problem* (SEFP) to find x^* and y^* with the property

$$x^* \in C, y^* \in Q \quad \text{s.t.} \quad Ax^* = By^*, \tag{1.6}$$

where H_1, H_2 and H_3 be real Hilbert spaces. $C \subset H_1$, $Q \subset H_2$ be two non-empty closed convex sets, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators.
It is easy to see that the problem (1.6) could be reduced to the problem (1.2) where $H_3 \equiv H_2$ and $B \equiv I$ (I be the identity mappings on $H_2 \to H_2$).

In order to solve SEFP (1.6), Moudafi [24] introduced the following simultaneous iterative method:

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma A^*\left(Ax_n - By_n\right)), \\ y_{n+1} = P_Q(y_n + \beta B^*\left(Ax_n - By_n\right)), \quad \forall n \geq 0, \end{cases}$$

under suitable conditions, he proved the weak convergence of sequence $\{(x_n, y_n)\}$ to (x^*, y^*) where $(x^*, y^*) \in C \times Q$ is a solution of (1.6).

In 2014, Zhao [26] introduced the following algorithm for solving problem (1.6):

$$\begin{cases} u_n = x_n - \gamma_n A^*\left(Ax_n - By_n\right), \\ x_{n+1} = \beta_n u_n + (1-\beta_n) S u_n, \\ w_n = y_n + \gamma_n B^*\left(Ax_n - By_n\right), \\ y_{n+1} = \beta_n w_n + (1-\beta_n) T w_n, \quad \forall n \geq 0, \end{cases}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive mappings, A^* and B^* are the adjoints of A and B respectively, $\{\gamma_n\} \in (\varepsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \varepsilon)$ (for ε small enough). Under some conditions, the authors obtained the sequence $\{(x_n, y_n)\}$ converge weakly to (x^*, y^*) in (1.6).

In 2012, Dong and He [33] introduced following projection algorithm for SEFP (1.6) :

$$\begin{cases} u_n = x_n - \gamma_n A^* (Ax_n - By_n), \\ x_{n+1} = P_C u_n, \\ w_n = y_n + \gamma_n B^* (Ax_n - By_n), \\ y_{n+1} = P_Q w_n, \quad \forall n \geq 0. \end{cases}$$

where the stepsizes do not depend on the operator norms $\|A\|$ and $\|B\|$.

In 2013, Moudafi [32] introduced the following split equality fixed point problem (SEFPP); let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be non-linear operators such that $F(U) \neq \emptyset$ and $F(T) \neq \emptyset$, where $F(U)$ and $F(T)$ denote the sets of fixed point of U and T respectively. In (1.6), if $C := F(U)$ and $Q := F(T)$, then SEFP (1.6) could be reduced to the SEFPP, to find x^* and y^* with the property

$$x^* \in F(U), y^* \in F(T) \quad \text{s.t.} \quad Ax^* = By^*, \tag{1.7}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, which allows asymmetric and partial relations between x^* and y^*. This can further be used to cover many situations, such as decomposition methods for PDEs, applications in the game theory, in intensity-modulated radiation therapy (see [9]).

In 2015, Che and Li [27] proposed the following iterative algorithm for finding a solution of SEFPP (1.7):

$$\begin{cases} u_n = x_n - \gamma_n A^* (Ax_n - By_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) T u_n, \\ v_n = y_n + \gamma_n B^* (Ax_n - By_n), \\ y_{n+1} = \beta_n y_n + (1 - \beta_n) S w_n, \quad \forall n \geq 0, \end{cases} \tag{1.8}$$

and under suitable conditions, they also established the weak convergence of the scheme (1.8).

In 2009, Kangtunyakarn and Suantai [2] introduced the S-mapping generated by

$T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. They defined a mapping $S : C \to C$ as follows:

$$
\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\vdots\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I,
\end{aligned}
$$

where $\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings of C into itself and $\alpha_j = \left(\alpha_1^j, \alpha_2^j, \alpha_3^j\right) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, for every $j = 1, 2, \ldots, N$.

For the special cases of S-mapping, we have

1. If we put $\alpha_1^j = \beta_j$ and $\alpha_2^j = 0$, for all $j = 1, 2, \ldots, N$, then the S-mapping is reduced to the W-mapping [36].
2. If we put $\alpha_1^j = \lambda_j$ and $\alpha_3^j = 0$, for all $j = 1, 2, \ldots, N$, then the S-mapping is reduced to the K-mapping [37].

1.2 Objectives of the research

1) To propose new iterative schemes for finding the solutions of split feasibility problem and fixed point problems of Lipschitzian quasi-pseudo-contractive mapping in a framework of Hilbert space.
2) To propose new iterative schemes for finding the solutions of split equality fixed point problem of quasi nonexpansive mapping in a framework of Hilbert spaces.

1.3 Scope of the research

1) Split feasibility problems, split common fixed point problems, split equality feasibility problems and split equality fixed point problems are focused in a Hilbert space.
2) The fixed point problems of nonlinear mappings such as quasi-nonexpansive mappings, Lipschitzian mapping and quasi-pseudo contractive mapping are focused in a Hilbert space.
3) All strong convergence theorems are considered and proved in a Hilbert spaces.

1.4 Research methodology

1) We introduce the iterative algorithms for solving the split feasibility problem and fixed point problem for Lipschitzian quasi-pseudo contractive mapping.

2) We prove strong convergence theorems to find a common solution of these problems under appropriate conditions.

3) We introduce iterative algorithms for solving the split equality fixed point problem for quasi nonexpansive mappings.

4) We prove a strong convergence theorem for a proposed iterative scheme under some appropriate conditions.

1.5 Expected benefits

1) Obtain some algorithms for solving the split feasibility problem and fixed point problem for Lipschitzian quasi-contractive mappings and obtain some sufficient for strong convergence of the proposed algorithm.

2) Obtain some algorithms for solving the split equality fixed point problem for quasi nonexpansive mappings and obtain some sufficient for strong convergence of the proposed algorithm.

3) The new knowledges obtained in this work can be applied to another split problems, for example, split feasibility problems, split equality problems, split fixed point problems, etc., and can also be used in many real world applications, for instance, image recovery, signal processing etc..

This thesis consists of five chapters as follows:

In chapter 1, we introduce the background of this thesis such as various mappings, iterative methods for various mappings, the definitions and the relation of nonlinear mappings.

In chapter 2, we give the definitions, lemmas, remarks and some results to prove our main theorems.

In chapter 3, we prove the strong convergence theorems for finding common solutions of the split feasibility problems and fixed point problems and the strong convergence theorems for finding solutions of split equality fixed point problems. Moreover, we use S-mapping applied to our main results.

In chapter 4, we describe the conclusion of the thesis.

Chapter 2

Preliminaries

The purpose of this chapter is to explain fundamental concepts and definitions used throughout this thesis. Moreover, we give some lemmas, remarks and useful results used in the later chapters. Throughout this chapter, we use the letter $\mathbb{R}$ for the set of all real numbers, $\mathbb{C}$ for the set of all complex numbers and $\mathbb{F}$ for the set of all real or complex numbers.

2.1 Linear spaces

A vector space (also called a linear space) over a field $\mathbb{F}$ is a set E together with two operators that satisfy the eight axioms listed below. We present the definitions and some properties of linear spaces.

Definition 2.1. [10] Let E be a nonempty set, and assume that each pair of elements x and y in E can be combined by a process called addition to yield an element z in E denoted by $z = x+y$. Assume also that this operation of addition satisfies the following conditions (v1) $\sim$ (v4):

(v1) $(x+y)+z = x+(y+z)$,

(v2) $x+y = y+x$,

(v3) there exists a unique element in E denoted by 0 and called the zero element, or the origin, such that $x+0 = x$ for all $x \in E$,

(v4) to each $x \in E$ there corresponds a unique element in E denoted by $-x$ and called the negative of x such that $x+(-x) = 0$.

We also assume that each scalar $\alpha \in \mathbb{R}$ and each element x in E can be combined by a process called scalar multiplication to yield an element y in E denoted by $y = \alpha x$ satisfying $(\tilde{v}1)$ $\sim$ $(\tilde{v}4)$:

$(\tilde{v}1)$ $\alpha(\beta x) = (\alpha\beta)x$,

$(\tilde{v}2)$ $1 \cdot x = x$,

$(\tilde{v}3)$ $(\alpha+\beta)x = \alpha x + \beta x$,

$(\tilde{v}4)$ $\alpha(x+y) = \alpha x + \alpha y$.

The algebraic system E defined by these operations and axioms is called a *linear space*. A linear space is often called a vector space.

Definition 2.2. [11] A set E in a vector space is called *convex* if for any $x, y \in E$ and $\alpha \in [0,1]$, we have $\alpha x + (1-\alpha)y \in E$.

2.2 Properties of Hilbert spaces

In this section, the definition and some properties of Hilbert space are as follows.

Definition 2.3. [16] Let X be a linear space (or vector space) over the field $\mathbb{F}$. A *norm* on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(I1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

(I2) $\|\alpha x\| = |\alpha| \, \|x\|$,

(I3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

The ordered pair $(X, \|\cdot\|)$ is called a *normed space or normed vector space or normed linear space*.

Definition 2.4. (**Cauchy sequence** [11]) A sequence of vectors $\{x_n\}$ in a normed space X is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n > N$.

Theorem 2.1. [11] A subset S of a normed space X is *closed* if and only if every sequence of elements of S convergent in X has its limit in S, i.e.,

$$\{x_n\} \subseteq S \text{ and } x_n \to x \text{ implies } x \in S.$$

Definition 2.5. [17] An inner product on a vector space K over the field $\mathbb{F}$ is a function $\langle \cdot, \cdot \rangle : K \times K \to \mathbb{F}$, that assigns a scalar $\langle x, y \rangle$ for every $x, y \in K$, such that for all $x, y, z \in K$ and $\alpha \in \mathbb{F}$:

(I1) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$,

(I2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$,

(I3) $\overline{\langle x, y \rangle} = \langle y, x \rangle$,

(I4) $\langle x, x \rangle > 0 \Leftrightarrow x \neq 0$,

A vector space K over $\mathbb{F}$ with a specific inner product is called an inner product space. If $\mathbb{F} = \mathbb{C}$ is a complex inner product space, and if $\mathbb{F} = \mathbb{R}$, K is a real inner product space.

Theorem 2.2. [17] For an inner product space K, $x, y, z \in K$ and $\alpha \in \mathbb{F}$:

(J1) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$,

(J2) $\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$,

(J3) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$,

(J4) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$, (J5) If $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in K$ then $y = z$.

Remark 2.3. [10] An inner product space is called a real inner product space for the case when the scalars are the real numbers and $\langle x, y\rangle$ is a real number. For the case, (I3) means

$$\langle x, y\rangle = \langle y, x\rangle.$$

Remark 2.4. [10] Using (J1) and (J2), we obtain that for $x, y \in K$ and $\alpha, \beta \in \mathbb{C}$

$$\langle x, \alpha y + \beta z\rangle = \bar{\alpha}\,\langle x, y\rangle + \bar{\beta}\,\langle x, z\rangle\,.$$

Definition 2.6. [10] A complete inner product space is called a *Hilbert space*.

Theorem 2.5. [10] The inner product in an inner product space K is jointly continuous:

$$x_n \to x \;\text{ and }\; y_n \to y \Rightarrow \langle x_n, y_n\rangle \to \langle x, y\rangle.$$

Remark 2.6. [10] We of course obtain from Theorem 2.5 that if $x_n \to x$, then for a fixed $y \in K$,

$$\langle x_n, y\rangle \to \langle x, y\rangle \;\text{ and }\; \langle y, x_n\rangle \to \langle y, x\rangle$$

Remark 2.7. [10] Let K be an inner product space. For each x in K, we define its *norm* $\|x\|$ by

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}}\,.$$

Theorem 2.8. (**Schwarz inequality** [10]) Let K be an inner product space and let x and y be elements in K. Then the following holds:

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Definition 2.7. (**Strong convergence** [11]) A sequence $\{x_n\}$ of vectors in an inner product space K is called *strongly convergent* to x in K if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.8. (**Weak convergence** [11]) A sequence $\{x_n\}$ of vectors in an inner product space K is called *weakly convergent* to x in K if

$$\langle x_n, y\rangle \to \langle x, y\rangle \;\text{ as }\; n \to \infty \;\text{ for all }\; y \in X.$$

Theorem 2.9. [11] A strongly convergence sequence is weakly convergence (to the same limit), *i.e.*, $x_n \to x$ implies $x_n \rightharpoonup x$.

Remark 2.10. [10] If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Lemma 2.11. [10] Let $\{x_n\}$ be a Cauchy sequence of an inner product space K such that $x_n \rightharpoonup x$. Then $x_n \to x$.

Theorem 2.12. [10] Let H be a Hilbert space and let C be a nonempty closed convex subset of H with $\{x_n\} \subset C$ and $x_n \rightharpoonup x$, then $x \in C$.

Theorem 2.13. (**Opial's theorem** [10]) Let H be a Hilbert space and suppose $x_n \rightharpoonup x$. Then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for any $y \in H$ with $x \neq y$.

Theorem 2.14. [10] Let $\{a_n\}$ be a bounded of real numbers. Then, there exists subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Remark 2.15. [10] Let H be an inner product space. Then we know that the following (1) and (2) are equivalent:

(1) H is complete,

(2) each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Theorem 2.16. [11] Weakly convergent sequences $\{x_n\}$ in a Hilbert space H are bounded, i.e., if $\{x_n\}$ is a weakly convergent sequence, then there exists a number M such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Theorem 2.17. (**Double extract subsequence principle** [16]) Let $\{x_n\}$ be a sequence in a Hilbert space H and $x \in H$. If every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a further subsequence $\left\{x_{n_{k_l}}\right\}$ such that $\lim_{l\to\infty} x_{n_{k_l}} = x$, then $\lim_{n\to\infty} x_n = x$.

Definition 2.9. (**Lower semicontinuous** [10]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\} \text{ is closed.}$$

Moreover, f is called convex if for any $x_1, x_2 \in C$ and $t \in [0,1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Similarly, f is called concave if for any $x_1, x_2 \in C$ and $t \in [0,1]$,

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2).$$

Theorem 2.18. [10] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightharpoonup x_0$. Then

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

2.3 Properties of bounded linear operators

The properties of bounded linear operators are as follows.

Definition 2.10. [11] Let C be a nonempty closed convex subset of a real Hilbert space H. Then

(1) $T : H \to H$ is called a *linear operator* if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

for all $x, y \in H$ and $\alpha, \beta \in \mathbb{R}$.

(2) $T : H \to H$ is called *bounded* if there exists $K \geq 0$ such that

$$\|Tx\| \leq K\|x\|,$$

for all $x \in H$.

Definition 2.11. (**Adjoint Operator** [11]) Let T be a bounded operator on a Hilbert space H. The operator $T^* : H \to H$ defined by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad \text{for all} \quad x, y \in H$$

is called the *adjoint operator* of A.

Theorem 2.19. [11] The adjoint operator T^* of a bounded operator T is bounded. Moreover, we have $\|T\| = \|T^*\|$ and $\|T^*T\| = \|T\|^2$.

Definition 2.12. (**Self-Adjoint Operator** [11]) Let T be a bounded operator on a Hilbert space H. If $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x \in H$, then T is called the *self-adjoint operator*.

Remark 2.20. If T is a bounded operator on a Hilbert space H, then T^*T is self-adjoint operator.

Theorem 2.21. [11] Let T be a bounded linear self-adjoint operator on a Hilbert space H. Then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle| \,.$$

Definition 2.13. (**Normal Operator** [11]) A bounded operator T on a Hilbert space H is called a *normal operator* if $TT^* = T^*T$.

Theorem 2.22. [11] A bounded operator T on a Hilbert space H is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.

Definition 2.14. [11] An operator T is called *positive* if it is self-adjoint and

$$\langle Tx, x\rangle \geq 0, \quad \text{for all} \quad x \in H.$$

Definition 2.15. [20] Let T be a bounded linear operator on a Hilbert space H. The *spectral radius* of T, denoted by $r_\sigma(T)$, is the number defined by

$$r_\sigma(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

where $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)(x) = 0, \text{ for some } 0 \neq x \in H\}$.

Theorem 2.23. [20] Let T be a normal bounded linear operator on a Hilbert space H. Then T is is self-adjoint operator if and only if $\sigma(T) \subset \mathbb{R}$.

Definition 2.16. [21] A self-adjoint operator T is a strongly positive operator on H if there is a constant $\gamma > 0$ with property

$$\langle Tx, x\rangle \geq \gamma \|x\|^2, \text{ for all } x \in H.$$

The following example shows that A is a bounded linear operator.

Example 2.24. Let $\mathbb{R}^2$ be the two dimensional space of real numbers with an inner product $\langle \cdot,\cdot\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\langle u, v\rangle = u \cdot v = u_1v_1 + u_2v_2$ and a usual norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ given by $\|u\| = \sqrt{u_1^2 + u_2^2}$, for all $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$. Let an operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Ax = (x_1 - x_2, x_1 + x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then A is a bounded linear on $\mathbb{R}^2$.

Solution. Let $\alpha, \beta \in \mathbb{R}$ and $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Thus we derive

$$\begin{aligned} A(\alpha x + \beta y) &= (\alpha x_1 + \beta y_1 - (\alpha x_2 + \beta y_2), \alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2) \\ &= (\alpha(x_1 - x_2) + \beta(y_1 - y_2), \alpha(x_1 + x_2) + \beta(y_1 + y_2)) \\ &= \alpha(x_1 - x_2, x_1 + x_2) + \beta(y_1 - y_2, y_1 + y_2) \\ &= \alpha Ax + \beta Ay. \end{aligned}$$

It implies that A is linear. Consider that

$$\begin{aligned} \|Ax\| &= \|(x_1 - x_2, x_1 + x_2)\| \\ &= \sqrt{(x_1 - x_2)^2 + (x_1 + x_2)^2} \\ &= \sqrt{x_1^2 - 2x_1x_2 + x_2^2 + x_1^2 + 2x_1x_2 + x_2^2} \\ &= \sqrt{2(x_1^2 + x_2^2)} \\ &= \sqrt{2}\,\|x\|. \end{aligned}$$

Hence A is bounded.

2.4 Metric projection and Variational inequality problems

In this part, we propose further the properties of projectors and relation of Metric projection and Variational inequality problems in Hilbert space.

Definition 2.17. (**Metric projection** [10]) The (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.25. [18] Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Given $x \in H$ and $y \in C$, then following holds

$$P_C x = y \Leftrightarrow \langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Lemma 2.26. [19] Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Then the following hods:

(1) $\|P_C x - P_C y\| \leq \|x - y\|$, for all $x, y \in H$,

(2) $\langle y - x, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$, for all $x, y \in H$.

It is well-known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \forall x, y \in H.$$

Obviously, it implies that

$$\|(x - y) - (P_C x - P_C y)\|^2 \leq \|x - y\|^2 - \|P_C x - P_C y\|^2, \forall x, y \in H.$$

The variational inequalities theory, which was introduced by Stampacchia [1], arise in various models for a large number of mathematical, engineering, physical and other problems. The property of variational inequality problems is follows :

Theorem 2.27. [10] Let H be a real Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $\alpha > 0$ and let $A : C \to H$ be α-inverse strongly monotone. Then $VI(C, A) \neq \emptyset$.

Lemma 2.28. [19] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

2.5 Fixed points of nonexpansive mappings, nonspreading mappings and quasi-nonexpansive mappings

In this section, we study about the existence and properties of nonexpansive mappings, nonspreading mappings and quasi-nonexpansive mappings.

Theorem 2.29. [10] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Theorem 2.30. [10] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.

Lemma 2.31. (**Demiclosedness principle** [22]) Assume that T is a nonexpansive self-mapping of closed convex subset C of a Hilbert space H. If T has a fixed point, then $I-T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y it follows that $(I-T)x = y$. Here, I is the identity mapping of H.

Example 2.32. Let $T : [-3,3] \to [-3,3]$ be defined by $Tx = \frac{2x-1}{3}$. Then T is a nonexpansive mapping.

Solution. Let $x, y \in [-3,3]$. Thus we get

$$|Tx - Ty| = \left|\frac{2x-1}{3} - \frac{2y-1}{3}\right| = \left|\frac{2}{3}(x-y)\right| \le |x-y|$$

Hence T is a nonexpansive mapping.

By the concept of quasi-nonexpansive mapping and nonexpansive mapping, we know that a nonexpansive mapping with at least one fixed point in C is quasi-nonexpansive mapping but the inverse may be not true. It implies that the class of quasi-nonexpansive mapping generalizes the class of nonexpansive mapping. Next is the property of quasi-nonexpansive mapping.

Theorem 2.33. [23] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a quasi-nonexpansive mapping of C into itself. Then $F(T)$ is a nonempty closed convex set on which T is continuous.

Example 2.34. Let $T : [-3,3] \to [-3,3]$ be defined by $Tx = \frac{2x-1}{3}$. Then T is a quasi-nonexpansive mapping.

Solution. Let $x, y \in [-3,3]$. We observe that $F(T) = \{-1\}$. For every $v \in [-3,3]$ and $-1 \in F(T)$, we have

$$|T(v) - T(-1)| = \left|\frac{2v-1}{3} - (-1)\right| = \frac{2}{3}|v+1| \le |v+1|.$$

Therefore T is a quasi-nonexpansive mapping.

Example 2.35. Let ℓ_2 be the set of all sequences of complex numbers $(x_1, x_2, \cdots, x_i, \cdots)$ with $\sum_{i=1}^{\infty}|x_i|^2 < \infty$ is defined by $\|x\|_2 = (\sum_{i=1}^{\infty}|x_i|^2)^{\frac{1}{2}}$ and $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ for all $x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in \ell_2$. Suppose that

$$Tx = \frac{1}{2}(x_1, x_2, \ldots, x_i, \ldots).$$

Then T is quasi-nonexpansive mapping.

Solution. We observe that $F(T) = 0$. Consider that

$$\begin{aligned}\|Tx - x^*\| &= \|(\frac{1}{2}x_1 - 0, \frac{1}{2}x_2 - 0, \ldots, \frac{1}{2}x_i - 0, \ldots)\| \\ &= \frac{1}{2}\|(x_1 - 0, x_2 - 0, \ldots, x_i - 0, \ldots)\| \\ &\leq \|(x_1 - 0, x_2 - 0, \ldots, x_i - 0, \ldots)\| \\ &= \|x - x^*\|\end{aligned}$$

Hence T is quasi-nonexpansive mapping.

A mapping T on a closed convex subset C of a Hilbert space H is said to be nonspreading mapping if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|Tx - y\|^2 \tag{2.1}$$

for all $x, y \in C$.

Lemma 2.36. [3] Let H be a Hilbert space, let C be a nonempty closed convex subset of H , and let T be a nonspreading mapping of C into itself. Then $F(T)$ is closed and convex.

Remark 2.37. A nonspreading mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive mapping. But the converse is not true.

Example 2.38. Let $T : [1, 2] \to [1, 2]$ be defined by $Tx = \frac{1}{x}$. Then T is a nonspreading mapping.
Solution. Let $x, y \in [1, 2]$. Thus we get

$$|Tx - Ty|^2 = \left|\frac{1}{x} - \frac{1}{y}\right|^2 = \left|\frac{y - x}{xy}\right|^2 \leq |x - y|^2,$$

and

$$\begin{aligned}2\langle x - Tx, y - Ty\rangle =& 2\left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right) \\ \geq& 0, \quad (\text{ Since } x \geq \frac{1}{x}, y \geq \frac{1}{y}).\end{aligned}$$

Hence

$$\begin{aligned}|x - y|^2 + 2\langle x - Tx, y - Ty\rangle &\geq |x - y|^2 \\ &\geq |Tx - Ty|^2.\end{aligned}$$

Hence T is a nonspreading mapping.

2.6 Some useful Lemmas and Theorems

This section is an important section to prove our main results. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. In this thesis, we represent weak and strong convergence by " $\rightharpoonup$" and " $\to$", respectively.

Lemma 2.39. Let H be a real Hilbert space. Then there holds the following well-known results:

(1) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$,

(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,

(3) $\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2$,

for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.40. [12] Let $\{\mathcal{Q}_n\} \subset [0, +\infty], \{v_n\} \subset [0, 1]$ and $\{\eta_n\}$ be three real number sequences. Suppose that $\{\mathcal{Q}_n\}, \{v_n\}$ and $\{\eta_n\}$ satisfy the following three conditions:

(i) $\mathcal{Q}_{n+1} \leq (1 - v_n)\mathcal{Q}_n + \eta_n v_n$,

(ii) $\sum_{n=1}^{\infty} v_n = \infty$,

(iii) $\limsup_{n\to\infty} \eta_n \leq 0$ or $\sum_{n=1}^{\infty} |\eta_n v_n| < \infty$.

Then, $\lim_{n\to\infty} \mathcal{Q}_n = 0$.

Lemma 2.41. [40] Let $\{\rho_n\}$ be a sequences of real numbers. Assume that there exists a subsequence $\{\rho_{n_k}\}$ of $\{\rho_n\}$ such that $\rho_{n_k} \leq \rho_{n_k+1}$ for all $k \geq 0$. For every $n \geq N_0$, define an integer sequence $\{\tau(n)\}$ as

$$\tau(n) = max\{i \leq n : \rho_{n_i} < \rho_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and

$$max\{\rho_{\tau(n)}, \rho_n\} \leq \rho_{\tau(n)+1},$$

for all $n \geq N_0$.

Lemma 2.42. [14] Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let A_i be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\gamma_i > 0$ and $\bar{\gamma} = min_{i=1,2,...,N}\gamma_i$. Let $\{a_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N a_i = 1$. Then the following properties hold:

(i) $\left\|\mathcal{I} - \rho\sum_{i=1}^N a_i A_i\right\| \leq 1 - \rho\bar{\gamma}$ and $\mathcal{I} - \rho\sum_{i=1}^N a_i A_i$ is a nonexpansive mapping where $0 < \rho < \|A_i\|^{-1}$ and for all $i = 1, 2, ..., N$.

(ii) $VI(C, \sum_{i=1}^N a_i A_i) = \bigcap_{i=1}^N VI(C, A_i)$ where $\bigcap_{i=1}^N VI(C, A_i) \neq \emptyset$.

Proposition 2.43. [15] Let H be a real Hilbert space. Let $\mathcal{U} : H \to H$ be an $\mathcal{L}$-Lipschitzian operator with $\mathcal{L} > 1$. Then

$$F\left(((1 - \zeta)\mathcal{I} + \zeta\mathcal{U})\mathcal{U}\right) = F\left(\mathcal{U}((1 - \zeta)\mathcal{I} + \zeta\mathcal{U})\right) = F(\mathcal{U})$$

for all $\zeta \in (0, \frac{1}{\mathcal{L}})$.

Proposition 2.44. [15] Let H be a real Hilbert space. Let $\mathcal{U}: H \to H$ be an $\mathcal{L}$-Lipschitzian quasi-pseudo-contractive operator. Then we have

$$\|\mathcal{U}((1-\eta)x+\eta\mathcal{U}x)-u^*\|^2 \leq \|x-u^*\|^2+(1-\eta)\|x-\mathcal{U}((1-\eta)x+\eta\mathcal{U}x)\|^2,$$

and the operator $(1-\xi)\mathcal{I}+\xi\mathcal{U}((1-\eta)\mathcal{I}+\eta\mathcal{U})$ is quasi-nonexpansive when $0<\xi<\eta<\frac{1}{\sqrt{1+\mathcal{L}^2}+1}$, that is,

$$\|(1-\xi)x+\xi\mathcal{U}((1-\eta)x+\eta\mathcal{U}x)-u^*\| \leq \|x-u^*\|$$

for all $x \in H$ and $u^* \in F(\mathcal{U})$.

Proposition 2.45. [15] Let H be a real Hilbert space. Let $\mathcal{U}: H \to H$ be an $\mathcal{L}$-Lipschitzian operator with $\mathcal{L}>1$. If $\mathcal{I}-\mathcal{U}$ is demiclosed at 0, then $\mathcal{I}-\mathcal{U}((1-\zeta)\mathcal{I}+\zeta\mathcal{U})$ is also demiclosed at 0 when $\zeta \in (0, \frac{1}{\mathcal{L}})$.

Lemma 2.46. Let C be a nonempty closed convex subset of a real Hilbert space H and let $T: C \to C$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then $VI(C, \mathcal{I}-T) = F(T)$.

Proof. It is easy to see that $F(T) \subseteq VI(C, \mathcal{I}-T)$.
Let $u \in VI(C, \mathcal{I}-T)$, then we have

$$\langle v-u, (\mathcal{I}-T)u\rangle \geq 0, \ \forall v \in C. \tag{2.2}$$

Let $v^* \in F(T)$, then we have

$$\|Tu-v^*\|^2 \leq \|u-v^*\|^2. \tag{2.3}$$

On the other hand

$$\begin{aligned} &\|Tu-v^*\|^2 \\ &= \|(u-v^*)-(\mathcal{I}-T)u\|^2 \\ &= \|u-v^*\|^2 - 2\langle u-v^*, (\mathcal{I}-T)u\rangle + \|(\mathcal{I}-T)u\|^2. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we have

$$\|u-v^*\|^2 - 2\langle u-v^*, (\mathcal{I}-T)u\rangle + \|(\mathcal{I}-T)u\|^2 \leq \|u-v^*\|^2. \tag{2.5}$$

It implies that from (2.2) and (2.5) that

$$\|(\mathcal{I}-T)u\|^2 \leq 2\langle u-v^*, (\mathcal{I}-T)u\rangle \leq 0. \tag{2.6}$$

It follows that $u \in F(T)$. Hence $VI(C, \mathcal{I}-T) \subseteq F(T)$. □

Remark 2.47. From Lemma 2.28 and 2.46, we have

$$F(T) = VI(C, \mathcal{I}-T) = F(P_C(\mathcal{I}-\lambda(\mathcal{I}-T))),$$

for all $\lambda > 0$.

Chapter 3

Convergence theorems in Hilbert space and its application

3.1 Strong convergence theorems for finding common elements of split feasibility problem and fixed point problem

In this section, we introduce strong convergence theorems for finding a common element of split feasibility problem and fixed point problem in Hilbert spaces. In addition, our results improve existing results.

Theorem 3.1. Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*. For each $i = 1, 2, ..., N$, let D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,...,N} \gamma_i$. Let $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is an $\mathcal{L}_1$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_1 > 1$ and let $T : C \to C$ is an $\mathcal{L}_2$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_2 > 1$. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1 - \xi_n) z_n + \xi_n S ((1 - \eta_n) z_n + \eta_n S z_n), \\ y_n = \alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right) (x_n - \delta A^* (A x_n - v_n)), \\ u_n = P_C y_n, \\ x_{n+1} = (1 - \beta_n) u_n + \beta_n T ((1 - \gamma_n) u_n + \gamma_n T u_n), \qquad for\ n \geq 1, \end{cases} \tag{3.1}$$

where parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0, 1]$, δ and γ are two positive constants.
Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1 + \mathcal{L}_1^2} + 1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \dfrac{1}{\sqrt{1 + \mathcal{L}_2^2} + 1}$,

(iv) $0 < \delta, \gamma < \dfrac{1}{\|A\|^2}$ and $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ for $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}} \left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right) z$.

Proof. Let $z^* = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right) z^*$, we have $z^* \in C \cap F(T)$ and $Az^* \in Q \cap F(S)$. From P_Q is firmly nonexpansive, thus

$$\begin{aligned}\|z_n - Az^*\|^2 &= \|P_Q A x_n - P_Q A z^*\|^2 \\ &\leq \|Ax_n - Az^*\|^2 - \|(\mathcal{I} - P_Q)Ax_n - (\mathcal{I} - P_Q)Az^*\|^2 \\ &= \|Ax_n - Az^*\|^2 - \|Ax_n - z_n\|^2 . \end{aligned} \tag{3.2}$$

Applying Proposition 2.43, condition (ii) and (iii), we have

$$F\left(S((1-\eta_n)\mathcal{I} + \eta_n S)\right) = F(S)$$

and

$$F\left(T((1-\gamma_n)\mathcal{I} + \gamma_n T)\right) = F(T)$$

for all $n \in \mathbb{N}$.

By Proposition 2.44 and condition (ii), we have

$$\begin{aligned}\|v_n - Az^*\| &= \|[(1-\xi_n)\mathcal{I} + \xi_n S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)] z_n - Az^*\| \\ &\leq \|z_n - Az^*\| . \end{aligned} \tag{3.3}$$

This together with (3.2), it implies that

$$\begin{aligned}\|v_n - Az^*\|^2 &\leq \|z_n - Az^*\|^2 \\ &\leq \|Ax_n - Az^*\|^2 - \|Ax_n - z_n\|^2 \end{aligned} \tag{3.4}$$

By proposition 2.44 and condition (iii), we have

$$\begin{aligned}\|x_{n+1} - z^*\| &= \|[(1-\beta_n)\mathcal{I} + \beta_n T\left((1-\gamma_n)\mathcal{I} + \gamma_n T\right)] u_n - z^*\| \\ &\leq \|u_n - z^*\| . \end{aligned} \tag{3.5}$$

Since P_C is nonexpansive, we have

$$\begin{aligned}\|u_n - z^*\| &= \|P_C y_n - P_C z^*\| \\ &\leq \|y_n - z^*\| . \end{aligned} \tag{3.6}$$

From definition of $\{y_n\}$ and using lemma (2.42), we obtain

$$
\begin{aligned}
\|y_n - z^*\| = &\left\|\alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)(x_n - \delta A^*(Ax_n - v_n)) - z^*\right\| \\
=&\|\alpha_n \gamma f(x_n) - \alpha_n \gamma f(z^*) + \alpha_n \gamma f(z^*) - \alpha_n \sum_{i=1}^{N} a_i D_i z^* + x_n - \delta A^*(Ax_n - v_n) \\
&- \alpha_n \sum_{i=1}^{N} a_i D_i (x_n - \delta A^*(Ax_n - v_n)) + \alpha_n \sum_{i=1}^{N} a_i D_i z^* - z^*\| \\
=&\|\alpha_n \gamma (f(x_n) - f(z^*)) + \alpha_n \left(\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right) \\
&+ \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)(x_n - z^* - \delta A^*(Ax_n - v_n))\| \\
\leq& \alpha_n \gamma \|f(x_n) - f(z^*)\| + \alpha_n \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| \\
&+ \left\|\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right\| \|x_n - z^* + \delta A^*(v_n - Ax_n)\| \\
\leq& \alpha_n \gamma \rho \|x_n - z^*\| + \alpha_n \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| \\
&+ (1 - \alpha_n \bar{\gamma}) \|x_n - z^* + \delta A^*(v_n - Ax_n)\|.
\end{aligned} \tag{3.7}
$$

Observe that

$$
\begin{aligned}
&\langle x_n - z^*, A^*(v_n - Ax_n)\rangle \\
=&\langle Ax_n - Az^*, v_n - Ax_n\rangle \\
=&\langle Ax_n - Az^* + v_n - Ax_n - (v_n - Ax_n), v_n - Ax_n\rangle \\
=&\langle Ax_n - Az^* + v_n - Ax_n, v_n - Ax_n\rangle - \langle v_n - Ax_n, v_n - Ax_n\rangle \\
=&\langle v_n - Az^*, v_n - Ax_n\rangle - \|v_n - Ax_n\|^2.
\end{aligned} \tag{3.8}
$$

and

$$
\langle v_n - Az^*, v_n - Ax_n\rangle = \frac{1}{2}\left(\|v_n - Az^*\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Az^*\|^2\right). \tag{3.9}
$$

From (3.4), (3.8) and (3.9), we obtain

$$
\begin{aligned}
&\langle x_n - z^*, A^*(v_n - Ax_n)\rangle \\
=&\frac{1}{2}\left(\|v_n - Az^*\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Az^*\|^2\right) - \|v_n - Ax_n\|^2 \\
\leq&\frac{1}{2}\left(\|Ax_n - Az^*\|^2 - \|Ax_n - z_n\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Az^*\|^2\right) - \|v_n - Ax_n\|^2 \\
=&-\frac{1}{2}\|z_n - Ax_n\|^2 - \frac{1}{2}\|v_n - Ax_n\|^2.
\end{aligned} \tag{3.10}
$$

From (3.10), we have

$$\begin{aligned}
&\left\|x_n - z^* + \delta A^*\left(v_n - Ax_n\right)\right\|^2 \\
&= \left\|x_n - z^*\right\|^2 + \delta^2\left\|A^*\left(v_n - Ax_n\right)\right\|^2 + 2\delta\langle x_n - z^*, A^*(v_n - Ax_n)\rangle \\
&\le \left\|x_n - z^*\right\|^2 + \delta^2\left\|A^*\right\|^2\left\|v_n - Ax_n\right\|^2 + 2\delta\left(-\frac{1}{2}\left\|z_n - Ax_n\right\|^2 - \frac{1}{2}\left\|v_n - Ax_n\right\|^2\right) \\
&= \left\|x_n - z^*\right\|^2 + \delta^2\left\|A\right\|^2\left\|v_n - Ax_n\right\|^2 - \delta\left\|z_n - Ax_n\right\|^2 - \delta\left\|v_n - Ax_n\right\|^2 \\
&= \left\|x_n - z^*\right\|^2 + \delta\left(\delta\left\|A\right\|^2 - 1\right)\left\|v_n - Ax_n\right\|^2 - \delta\left\|z_n - Ax_n\right\|^2. \qquad (3.11)
\end{aligned}$$

From (3.11) and condition (iv), we have

$$\left\|x_n - z^* + \delta A^*\left(v_n - Ax_n\right)\right\|^2 \le \left\|x_n - z^*\right\|^2.$$

So,

$$\left\|x_n - z^* + \delta A^*\left(v_n - Ax_n\right)\right\| \le \left\|x_n - z^*\right\|. \qquad (3.12)$$

From (3.7) and (3.12), we get

$$\begin{aligned}
&\left\|y_n - z^*\right\| \\
&\le \alpha_n\gamma\rho\left\|x_n - z^*\right\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| + (1 - \alpha_n\bar{\gamma})\left\|x_n - z^* + \delta A^*\left(v_n - Ax_n\right)\right\| \\
&\le \alpha_n\gamma\rho\left\|x_n - z^*\right\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| + (1 - \alpha_n\bar{\gamma})\left\|x_n - z^*\right\| \\
&= [1 - \alpha_n(\bar{\gamma} - \gamma\rho)]\left\|x_n - z^*\right\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|. \qquad (3.13)
\end{aligned}$$

By definition of $\{x_n\}$, (3.5), (3.6) and (3.13), we get

$$\begin{aligned}
\left\|x_{n+1} - z^*\right\| &\le [1 - \alpha_n(\bar{\gamma} - \gamma\rho)]\left\|x_n - z^*\right\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| \\
&= [1 - \alpha_n(\bar{\gamma} - \gamma\rho)]\left\|x_n - z^*\right\| + \alpha_n(\bar{\gamma} - \gamma\rho)\frac{\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|}{\bar{\gamma} - \gamma\rho}.
\end{aligned}$$

By induction, we get

$$\left\|x_{n+1} - z^*\right\| \le max\left\{\left\|x_0 - z^*\right\|, \frac{\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|}{\bar{\gamma} - \gamma\rho}\right\}.$$

Hence, the sequence $\{x_n\}$ is bounded.

Since P_C is firmly nonexpansive, we have

$$\begin{aligned}
\left\|u_n - z^*\right\|^2 &= \left\|P_C y_n - z^*\right\|^2 \\
&= \left\|P_C y_n - P_C z^*\right\|^2 \\
&\le \left\|y_n - z^*\right\|^2 - \left\|\left(\mathcal{I} - P_C\right)y_n - \left(\mathcal{I} - P_C\right)z^*\right\|^2 \\
&= \left\|y_n - z^*\right\|^2 - \left\|y_n - P_C y_n\right\|^2 \\
&= \left\|y_n - z^*\right\|^2 - \left\|u_n - y_n\right\|^2. \qquad (3.14)
\end{aligned}$$

From (3.5), (3.13) and (3.14), we have

$$\begin{aligned}
&\|x_{n+1}-z^*\|^2\\
&\leq \|u_n-z^*\|^2\\
&\leq \|y_n-z^*\|^2-\|u_n-y_n\|^2\\
&=\left([1-\alpha_n(\bar{\gamma}-\gamma\rho)]\|x_n-z^*\|+\alpha_n\left\|\gamma f(z^*)-\sum_{i=1}^{N}a_iD_iz^*\right\|\right)^2-\|u_n-y_n\|^2\\
&=(1-\alpha_n(\bar{\gamma}-\gamma\rho))^2\|x_n-z^*\|^2+2\alpha_n[1-\alpha_n(\bar{\gamma}-\gamma\rho)]\|x_n-z^*\|\left\|\gamma f(z^*)-\sum_{i=1}^{N}a_iD_iz^*\right\|\\
&\quad+\alpha_n^2\left\|\gamma f(z^*)-\sum_{i=1}^{N}a_iD_iz^*\right\|^2-\|u_n-y_n\|^2.
\end{aligned}$$

That is,

$$\begin{aligned}
\|u_n-y_n\|^2\leq{}&\|x_n-z^*\|^2-\|x_{n+1}-z^*\|^2+\alpha_n^2\left\|\gamma f(z^*)-\sum_{i=1}^{N}a_iD_iz^*\right\|^2\\
&+2\alpha_n[1-\alpha_n(\bar{\gamma}-\gamma\rho)]\|x_n-z^*\|\left\|\gamma f(z^*)-\sum_{i=1}^{N}a_iD_iz^*\right\|. \qquad (3.15)
\end{aligned}$$

Next, we focus our analysis on the fact that the sequence $\{\|x_n-z^*\|\}$ is either monotone decreasing at infinity(Case 1) or not(Case 2).

<u>*Case1.*</u> There exists $n_0\in\mathbb{N}$ such that the sequence $\{\|x_n-z^*\|\}_{n\geq n_0}$ is a monotone decreasing.

<u>*Case2.*</u> For any $\bar{n_0}\in\mathbb{N}$, there exists an integer $\bar{m}\geq\bar{n_0}$ such that

$$\|x_{\bar{m}}-z^*\|\leq\|x_{\bar{m}+1}-z^*\|.$$

<u>*Case1.*</u>, we assume that there exists some integer $m>0$ such that $\{\|x_n-z^*\|\}$ is decreasing for all $n\geq m$.

In this case, we get $\lim_{n\to\infty}\|x_n-z^*\|$ exists. From (3.15) and condition (i), we deduce

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \qquad (3.16)$$

From (3.7) and condition (iv), we have

$$\begin{aligned}
\|y_n-z^*\|\leq{}&\alpha_n\gamma\rho\|x_n-z^*\|+\alpha_n\left\|\gamma f(z^*)-\sum_{i=1}^{N}a_iD_iz^*\right\|\\
&+(1-\alpha_n\bar{\gamma})\|x_n-z^*+\delta A^*(v_n-Ax_n)\|\\
={}&\alpha_n\bar{\gamma}\left(\frac{\gamma\rho\|x_n-z^*\|+\left\|\gamma f(z^*)-\sum_{i=1}^{N}a_iD_iz^*\right\|}{\bar{\gamma}}\right)\\
&+(1-\alpha_n\bar{\gamma})\|x_n-z^*+\delta A^*(v_n-Ax_n)\|. \qquad (3.17)
\end{aligned}$$

Since $\{x_n\}$ is bounded, then there exists a constant $M>0$ such that

$$\sup_{n\in\mathbb{N}}\left\{\frac{\gamma\rho\|x_n-z^*\|+\left\|\gamma f(z^*)-\sum_{i=1}^{N}a_iD_iz^*\right\|}{\bar{\gamma}}\right\}<M.$$

By convexity of $(\cdot)^2$ and (3.17), we have

$$\|y_n - z^*\|^2 \leq \alpha_n \bar{\gamma} M^2 + (1 - \alpha_n \bar{\gamma}) \|x_n - z^* + \delta A^* (v_n - Ax_n)\|^2 . \tag{3.18}$$

From (3.5), (3.6), (3.11) and (3.18), thus

$$\begin{aligned}
&\|x_{n+1} - z^*\|^2 \\
\leq& \|u_n - z^*\|^2 \\
\leq& \|y_n - z^*\|^2 \\
\leq& \alpha_n \bar{\gamma} M^2 + (1 - \alpha_n \bar{\gamma}) \|x_n - z^* + \delta A^* (v_n - Ax_n)\|^2 \\
\leq& \alpha_n \bar{\gamma} M^2 + (1 - \alpha_n \bar{\gamma}) \left(\|x_n - z^*\|^2 + \delta \left(\delta \|A\|^2 - 1 \right) \|v_n - Ax_n\|^2 - \delta \|z_n - Ax_n\|^2 \right) \\
=& (1 - \alpha_n \bar{\gamma}) \|x_n - z^*\|^2 + (1 - \alpha_n \bar{\gamma}) \delta \left(\delta \|A\|^2 - 1 \right) \|v_n - Ax_n\|^2 \\
&- \delta (1 - \alpha_n \bar{\gamma}) \|z_n - Ax_n\|^2 + \alpha_n \bar{\gamma} M^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(1 - \alpha_n \bar{\gamma}) \delta \left(1 - \delta \|A\|^2 \right) \|v_n - Ax_n\|^2 + \delta (1 - \alpha_n \bar{\gamma}) \|z_n - Ax_n\|^2 \\
\leq& (1 - \alpha_n \bar{\gamma}) \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + \alpha_n \bar{\gamma} M^2 \\
\leq& \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + \alpha_n \bar{\gamma} M^2.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|z_n - Ax_n\| = 0. \tag{3.19}$$

Consider that

$$\begin{aligned}
\|v_n - z_n\| &= \|v_n - Ax_n + Ax_n - z_n\| \\
&\leq \|v_n - Ax_n\| + \|z_n - Ax_n\| .
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{3.20}$$

Note that

$$\begin{aligned}
v_n - z_n &= (1 - \xi_n) z_n + \xi_n S ((1 - \eta_n) z_n + \eta_n S z_n) - z_n \\
&= \xi_n [S ((1 - \eta_n) \mathcal{I} + \eta_n S) z_n - z_n] .
\end{aligned}$$

From (3.20), then

$$\lim_{n\to\infty} \|z_n - S ((1 - \eta_n) \mathcal{I} + \eta_n S) z_n\| = 0. \tag{3.21}$$

Consider that

$$\begin{aligned}
&\|S ((1 - \eta_n) \mathcal{I} + \eta_n S) z_n - S ((1 - \eta_n) \mathcal{I} + \eta_n S) Ax_n\| \\
\leq& \mathcal{L}_1 \|((1 - \eta_n) \mathcal{I} + \eta_n S) z_n - ((1 - \eta_n) \mathcal{I} + \eta_n S) Ax_n\| \\
=& \mathcal{L}_1 \|(1 - \eta_n) (z_n - Ax_n) + \eta_n (S z_n - S Ax_n)\| \\
\leq& \mathcal{L}_1 ((1 - \eta_n) \|z_n - Ax_n\| + \eta_n \|S z_n - S Ax_n\|) \\
\leq& \mathcal{L}_1 ((1 - \eta_n) \|z_n - Ax_n\| + \eta_n \mathcal{L}_1 \|z_n - Ax_n\|) \\
=& \mathcal{L}_1 (1 - \eta_n (1 - \mathcal{L}_1)) \|z_n - Ax_n\| .
\end{aligned} \tag{3.22}$$

From (3.22), thus

$$
\begin{aligned}
&\|Ax_n - S((1-\eta_n)\mathcal{I}+\eta_n S)Ax_n\| \\
&\le \|Ax_n - z_n\| + \|z_n - S((1-\eta_n)\mathcal{I}+\eta_n S)z_n\| \\
&\quad + \|S((1-\eta_n)\mathcal{I}+\eta_n S)z_n - S((1-\eta_n)\mathcal{I}+\eta_n S)Ax_n\| \\
&\le \|Ax_n - z_n\| + \|z_n - S((1-\eta_n)\mathcal{I}+\eta_n S)z_n\| + \mathcal{L}_1(1-\eta_n(1-\mathcal{L}_1))\|z_n - Ax_n\|. \qquad (3.23)
\end{aligned}
$$

From (3.19), (3.21) and (3.23), then we have

$$\lim_{n\to\infty}\|Ax_n - S((1-\eta_n)\mathcal{I}+\eta_n S)Ax_n\| = 0. \qquad (3.24)$$

Since

$$
\begin{aligned}
\|Ax_n - SAx_n\| &= \|Ax_n - S((1-\eta_n)\mathcal{I}+\eta_n S)Ax_n + S((1-\eta_n)\mathcal{I}+\eta_n S)Ax_n - SAx_n\| \\
&\le \|Ax_n - S((1-\eta_n)\mathcal{I}+\eta_n S)Ax_n\| + \|S((1-\eta_n)\mathcal{I}+\eta_n S)Ax_n - SAx_n\| \\
&\le \|Ax_n - S((1-\eta_n)\mathcal{I}+\eta_n S)Ax_n\| + \mathcal{L}_1\|((1-\eta_n)\mathcal{I}+\eta_n S)Ax_n - Ax_n\| \\
&= \|Ax_n - S((1-\eta_n)\mathcal{I}+\eta_n S)Ax_n\| + \mathcal{L}_1\eta_n\|Ax_n - SAx_n\|,
\end{aligned}
$$

it implies that

$$\|Ax_n - SAx_n\| \le \frac{1}{1-\mathcal{L}_1\eta_n}\|Ax_n - S((1-\eta_n)\mathcal{I}+\eta_n S)Ax_n\|.$$

By (3.24), we obtain

$$\lim_{n\to\infty}\|Ax_n - SAx_n\| = 0. \qquad (3.25)$$

Consider that

$$
\begin{aligned}
\|y_n - x_n\| &= \left\|\alpha_n\gamma f(x_n) + \left(\mathcal{I} - \alpha_n\sum_{i=1}^{N}a_iD_i\right)(x_n - \delta A^*(Ax_n - v_n)) - x_n\right\| \\
&= \left\|\alpha_n\gamma f(x_n) - \delta A^*(Ax_n - v_n) - \alpha_n\sum_{i=1}^{N}a_iD_ix_n + \delta\alpha_n\sum_{i=1}^{N}a_iD_iA^*(Ax_n - v_n)\right\| \\
&= \left\|\alpha_n\left(\gamma f(x_n) - \sum_{i=1}^{N}a_iD_ix_n + \delta\sum_{i=1}^{N}a_iD_iA^*(Ax_n - v_n)\right) + \delta A^*(v_n - Ax_n)\right\| \\
&= \left\|\alpha_n\left(\gamma f(x_n) - \sum_{i=1}^{N}a_iD_i(x_n - \delta A^*(Ax_n - v_n))\right) + \delta A^*(v_n - Ax_n)\right\| \\
&\le \alpha_n\left\|\gamma f(x_n) - \sum_{i=1}^{N}a_iD_i(x_n - \delta A^*(Ax_n - v_n))\right\| + \delta\|A^*(v_n - Ax_n)\| \\
&\le \alpha_n\left\|\gamma f(x_n) - \sum_{i=1}^{N}a_iD_i(x_n - \delta A^*(Ax_n - v_n))\right\| + \delta\|A^*\|\|v_n - Ax_n\| \\
&= \alpha_n\left\|\gamma f(x_n) - \sum_{i=1}^{N}a_iD_i(x_n - \delta A^*(Ax_n - v_n))\right\| + \delta\|A\|\|v_n - Ax_n\|.
\end{aligned}
$$

It follows from (3.19) and condition (i) that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \qquad (3.26)$$

From definition of $\{x_n\}$, we have

$$\begin{aligned}\|x_{n+1}-z^*\|^2 &= \|(1-\beta_n)u_n+\beta_n T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-z^*\|^2\\ &= \|(1-\beta_n)(u_n-z^*)+\beta_n\left[T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-z^*\right]\|^2\\ &=(1-\beta_n)\|u_n-z^*\|^2+\beta_n\|T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-z^*\|^2\\ &\quad -\beta_n(1-\beta_n)\|T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-u_n\|^2. \end{aligned} \tag{3.27}$$

Applying proposition 2.44, we have

$$\|T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-z^*\|^2 \leq \|u_n-z^*\|^2+(1-\gamma_n)\|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|^2. \tag{3.28}$$

From (3.6), (3.12), (3.18), (3.27) and (3.28), thus

$$\begin{aligned}\|x_{n+1}-z^*\|^2 &=(1-\beta_n)\|u_n-z^*\|^2+\beta_n\|T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-z^*\|^2\\ &\quad -\beta_n(1-\beta_n)\|T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-u_n\|^2\\ &\leq(1-\beta_n)\|u_n-z^*\|^2+\beta_n(\|u_n-z^*\|^2+(1-\gamma_n)\|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|^2)\\ &\quad -\beta_n(1-\beta_n)\|T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-u_n\|^2\\ &=\|u_n-z^*\|^2+\beta_n(1-\gamma_n)\|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|^2\\ &\quad -\beta_n(1-\beta_n)\|T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-u_n\|^2\\ &\leq\|y_n-z^*\|^2+\beta_n(1-\gamma_n)\|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|^2\\ &\quad -\beta_n(1-\beta_n)\|T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-u_n\|^2\\ &\leq\alpha_n\bar{\gamma}M^2+(1-\alpha_n\bar{\gamma})\|x_n-z^*+\delta A^*(v_n-Ax_n)\|^2\\ &\quad +\beta_n(1-\gamma_n)\|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|^2\\ &\quad -\beta_n(1-\beta_n)\|T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-u_n\|^2\\ &=\alpha_n\bar{\gamma}M^2+(1-\alpha_n\bar{\gamma})\|x_n-z^*+\delta A^*(v_n-Ax_n)\|^2\\ &\quad -\beta_n(\gamma_n-\beta_n)\|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|^2\\ &\leq\alpha_n\bar{\gamma}M^2+\|x_n-z^*\|^2-\beta_n(\gamma_n-\beta_n)\|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|^2. \end{aligned}$$

It implies that

$$\beta_n(\gamma_n-\beta_n)\|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|^2 \leq \alpha_n\bar{\gamma}M^2+\|x_n-z^*\|^2-\|x_{n+1}-z^*\|^2.$$

By condition (i) and (iii), we get

$$\lim_{n\to\infty}\|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|=0. \tag{3.29}$$

Observe that

$$\begin{aligned}\|u_n-Tu_n\| &\leq \|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|+\|T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)-Tu_n\|\\ &\leq \|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|+\mathcal{L}_2\|(1-\gamma_n)\,u_n+\gamma_n Tu_n-u_n\|\\ &= \|u_n-T\left((1-\gamma_n)\,u_n+\gamma_n Tu_n\right)\|+\mathcal{L}_2\gamma_n\|u_n-Tu_n\|. \end{aligned}$$

Thus,

$$\|u_n - Tu_n\| \leq \frac{1}{1-\mathcal{L}_2\gamma_n}\left\|u_n - T\left((1-\gamma_n)\,u_n + \gamma_n Tu_n\right)\right\|.$$

This together with (3.29) implies that,

$$\lim_{n\to\infty}\|u_n - Tu_n\| = 0. \tag{3.30}$$

Next, we will show that

$$\limsup_{n\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \leq 0,$$

where $z^* = P_{\mathbf{\Gamma}}(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i)z^*$.

Choose a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle = \lim_{j\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{n_j} - z^*\rangle. \tag{3.31}$$

Since the sequence $\{y_n\}$ is bounded, without loss of generality, we have a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup z$ as $n \to \infty$. Subsequently, we derive from above conclusion that

$$\begin{cases} x_{n_i} \rightharpoonup z, \\ y_{n_i} \rightharpoonup z, \quad \text{as} \quad n \to \infty \\ u_{n_i} \rightharpoonup z \end{cases} \tag{3.32}$$

and

$$\begin{cases} Ax_{n_i} \rightharpoonup Az, \\ Ay_{n_i} \rightharpoonup Az, \quad \text{as} \quad n \to \infty \\ Au_{n_i} \rightharpoonup Az. \end{cases} \tag{3.33}$$

Note that $u_{n_i} = P_C y_{n_i} \in C$ and (3.32), thus $z \in C$.

From demiclosedness of $(\mathcal{I} - T)$ and $(\mathcal{I} - T)u_{n_i} \to 0$, then $z \in F(T)$.

Therefore, $z \in C \cap F(T)$.

Note that $z_{n_i} = P_Q Ax_{n_i} \in Q$ and from (3.19) and (3.33), we have $z_{n_i} \rightharpoonup Az$.

Thus, $Az \in Q$.

From demiclosedness of $(\mathcal{I} - S)$ and $(\mathcal{I} - S)Ax_{n_i} \to 0$, then $Az \in F(S)$.

Therefore, $Az \in Q \cap F(S)$. That is $z \in \mathbf{\Gamma}$.

Consequently,

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle &= \lim_{j\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{n_j} - z^*\rangle \\ &= \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, z - z^*\rangle \\ &\leq 0. \end{aligned} \tag{3.34}$$

Consider that

$$
\begin{aligned}
\|y_n - z^*\|^2 \leq & \left\|\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right\|^2 \|x_n - z^* - \delta A^*(Ax_n - v_n)\|^2 \\
& + 2\langle \alpha_n \gamma (f(x_n) - f(z^*)) + \alpha_n \left(\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right), y_n - z^*\rangle \\
= & \left\|\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right\|^2 \|x_n - z^* - \delta A^*(Ax_n - v_n)\|^2 \\
& + 2\alpha_n \gamma \langle f(x_n) - f(z^*), y_n - z^*\rangle + 2\alpha_n \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
\leq & \left\|\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right\|^2 \|x_n - z^*\|^2 + 2\alpha_n \gamma \|f(x_n) - f(z^*)\| \|y_n - z^*\| \\
& + 2\alpha_n \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
\leq & (1 - \alpha_n \bar{\gamma})^2 \|x_n - z^*\|^2 + 2\alpha_n \gamma \rho \|x_n - z^*\| \|y_n - z^*\| \\
& + 2\alpha_n \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
\leq & (1 - \alpha_n \bar{\gamma})^2 \|x_n - z^*\|^2 + \alpha_n \gamma \rho \left(\|x_n - z^*\|^2 + \|y_n - z^*\|^2\right) \\
& + 2\alpha_n \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
= & (1 - \alpha_n \bar{\gamma})^2 \|x_n - z^*\|^2 + \alpha_n \gamma \rho \|x_n - z^*\|^2 + \alpha_n \gamma \rho \|y_n - z^*\|^2 \\
& + 2\alpha_n \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle.
\end{aligned}
$$

It follow that

$$
\begin{aligned}
& (1 - \alpha_n \gamma \rho) \|y_n - z^*\|^2 \\
\leq & \left(1 - 2\alpha_n \bar{\gamma} + {\alpha_n}^2 \bar{\gamma}^2 + \alpha_n \gamma \rho\right) \|x_n - z^*\|^2 + 2\alpha_n \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
= & (1 + \alpha_n \gamma \rho - 2\alpha_n \bar{\gamma}) \|x_n - z^*\|^2 + {\alpha_n}^2 \bar{\gamma}^2 \|x_n - z^*\|^2 \\
& + 2\alpha_n \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
= & (1 - \alpha_n \gamma \rho + 2\alpha_n \gamma \rho - 2\alpha_n \bar{\gamma}) \|x_n - z^*\|^2 + {\alpha_n}^2 \bar{\gamma}^2 \|x_n - z^*\|^2 \\
& + 2\alpha_n \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle
\end{aligned}
$$

then,

$$
\begin{aligned}
\|y_n - z^*\|^2 \leq & \left[1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_n}\right] \|x_n - z^*\|^2 + \frac{\bar{\gamma}^2 {\alpha_n}^2}{1 - \gamma\rho\alpha_n} \|x_n - z^*\|^2 \\
& + \frac{2\alpha_n}{1 - \gamma\rho\alpha_n} \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\|x_{n+1}-z^*\|^2 \\
\le\ &\|y_n-z^*\|^2 \\
\le\ &\left[1-\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{1-\gamma\rho\alpha_n}\right]\|x_n-z^*\|^2+\frac{\bar{\gamma}^2{\alpha_n}^2}{1-\gamma\rho\alpha_n}\|x_n-z^*\|^2 \\
&+\frac{2\alpha_n}{1-\gamma\rho\alpha_n}\langle\gamma f(z^*)-\sum_{i=1}^{N}a_iD_iz^*,y_n-z^*\rangle \\
=\ &\left[1-\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{1-\gamma\rho\alpha_n}\right]\|x_n-z^*\|^2 \\
&+\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{1-\gamma\rho\alpha_n}\left[\frac{\bar{\gamma}^2\alpha_n}{2(\bar{\gamma}-\gamma\rho)}\|x_n-z^*\|^2+\frac{1}{\bar{\gamma}-\gamma\rho}\langle\gamma f(z^*)-\sum_{i=1}^{N}a_iD_iz^*,y_n-z^*\rangle\right]. \qquad (3.35)
\end{aligned}$$

Applying (3.34), (3.35) and Lemma 2.40, we obtain $x_n \to z^*$ as $n \to \infty$.
Case 2, we assume that there exists some integer $\bar{n_0}$ such that

$$\|x_{\bar{n_0}}-z^*\| \le \|x_{\bar{n_0}+1}-z^*\|.$$

Setting $w_n = \|x_n - z^*\|$, then

$$w_{\bar{n_0}} \le w_{\bar{n_0}+1}.$$

Define an integer sequence $\{\tau_n\}$ for all $n \ge n_0$ as follows:

$$\tau(n) = max\{l \in \mathbb{N} \mid l \le n, w_l \le w_{l+1}\}.$$

It is clear that τ_n is a nondecreasing sequence satisfying

$$\lim_{n\to\infty} \tau(n) = \infty.$$

and

$$w_{\tau(n)} \le w_{\tau(n)+1}$$

for all $n \ge n_0$.
By a similar argument of Case 1, that is

$$\lim_{n\to\infty}\|u_{\tau(n)}-y_{\tau(n)}\|=0,$$
$$\lim_{n\to\infty}\|x_{\tau(n)}-y_{\tau(n)}\|=0,$$
$$\lim_{n\to\infty}\|SAx_{\tau(n)}-Ax_{\tau(n)}\|=0$$

and

$$\lim_{n\to\infty}\|u_{\tau(n)}-Tu_{\tau(n)}\|=0.$$

This implies that $w_w(y_{\tau(n)}) \subset \mathbf{\Gamma}$.
We obtain

$$\limsup_{n\to\infty}\langle\gamma f(z^*)-\sum_{i=1}^{N}a_iD_iz^*,y_{\tau_n}-z^*\rangle \le 0. \qquad (3.36)$$

From $w_{\tau(n)} \leq w_{\tau(n)+1}$ and (3.35), we have

$$\begin{aligned} {w_{\tau(n)}}^2 \leq & {w_{\tau(n)+1}}^2 \\ & \leq \left[1 - \frac{2\alpha_{\tau(n)}(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_{\tau(n)}}\right] w^2_{\tau(n)} + \frac{\bar{\gamma}^2\alpha^2_{\tau(n)}}{1 - \gamma\rho\alpha_{\tau(n)}} w^2_{\tau(n)} \\ & \quad + \frac{2\alpha_{\tau(n)}}{1 - \gamma\rho\alpha_{\tau(n)}} \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{\tau(n)} - z^* \rangle. \end{aligned} \tag{3.37}$$

By Lemma 2.40, we have

$$\lim_{n\to\infty} w_{\tau(n)+1} = 0.$$

Applying Lemma 2.41 ,we have

$$max\{w_{\tau(n)}, w_n\} \leq w_{\tau(n)+1}.$$

It implies that

$$w_n \leq w_{\tau(n)+1}. \tag{3.38}$$

Since w_n is nondecreasing sequence and $n \leq \tau(n)$,

$$w_n \leq w_{\tau(n)}. \tag{3.39}$$

From (3.38) and (3.39), we obtain

$$0 \leq w_n \leq max\{w_{\tau(n)}, w_{\tau(n)+1}\}.$$

Therefore, $w_n \to 0$. That is, $x_n \to z^*$. This complete the proof. □

By using our main result, we obtain the following results in Hilbert spaces.

Corollary 3.2. Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, D is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\ldots,N} \gamma_i$. Let $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is an $\mathcal{L}_1$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_1 > 1$ and let $T : C \to C$ is an $\mathcal{L}_2$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_2 > 1$. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1 - \xi_n) z_n + \xi_n S\left((1 - \eta_n) z_n + \eta_n S z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + (\mathcal{I} - \alpha_n D)\left(x_n - \delta A^*\left(A x_n - v_n\right)\right), \\ u_n = P_C y_n, \\ x_{n+1} = (1 - \beta_n) u_n + \beta_n T\left((1 - \gamma_n) u_n + \gamma_n T u_n\right), \qquad for\ n \geq 1, \end{cases} \tag{3.40}$$

where parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0, 1]$, δ and γ are two positive constants.

We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.4) that is,

$$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}.$$

Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \frac{1}{\sqrt{1+\mathcal{L}_1^2}+1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \frac{1}{\sqrt{1+\mathcal{L}_2^2}+1}$,

(iv) $0 < \delta, \gamma < \frac{1}{\|A\|^2}$ and $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D\|^{-1}$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}} (\gamma f + \mathcal{I} - D) z$.

Proof. Putting $D = D_1 = D_2 = D_3 = ... = D_N$ in Theorem 3.1, we get the desired conclusions. □

Corollary 3.3. Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*. For each $i = 1, 2, ..., N$, let D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\ldots,N} \gamma_i$. Let $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is an $\mathcal{L}$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L} > 1$. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n S((1-\eta_n) z_n + \eta_n S z_n), \\ x_{n+1} = P_C \left[\alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)(x_n - \delta A^*(A x_n - v_n))\right], \qquad for\ n \geq 1, \end{cases} \tag{3.41}$$

where parameters $\{\alpha_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0, 1]$, δ and γ are two positive constants.

We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.4), that is,

$$\mathbf{\Gamma} = \{x \mid x \in C, Ax \in Q \cap F(S)\}.$$

Suppose that $S - \mathcal{I}$ is demiclosed at 0. Assume that the following conditions are satisfied

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \frac{1}{\sqrt{1+\mathcal{L}_1^2}+1}$,

(iii) $0 < \delta < \frac{1}{\|A\|^2}$ and $\bar{\gamma} > \gamma\rho$,

(iv) $0 < \gamma < \frac{1}{\|A\|^2}$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ for $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\Gamma}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right) z$.

Proof. Putting $T \equiv \mathcal{I}$ in Theorem 3.1, we get the desired conclusions. □

3.2 Strong convergence theorems for finding the set of solutions of the split equality fixed point problem for quasi-nonexpansive mappings

Theorem 3.4. For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be nonempty closed convex subset of H_1 and H_2, respectively. Let $T_i : C_i \to C_i$ be quasi-nonexpansive mapping for all $i = 1, 2$ and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 \mid x \in F(T_1), y \in F(T_2) \text{ and } Ax = By\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases} u_n = x_n - \gamma_n A^* (Ax_n - By_n), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) P_{C_1} (I - \lambda_n^1 (I - T_1)) u_n, \\ v_n = y_n + \gamma_n B^* (Ax_n - By_n), \\ y_{n+1} = \alpha_n v + (1 - \alpha_n) P_{C_2} (I - \lambda_n^2 (I - T_2)) v_n, \end{cases} \tag{3.42}$$

for all $n \geq 1$,where $\{\alpha_n\} \subset [0, 1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0, 1)$ for all $i = 1, 2$ and $\gamma_n \in (a, b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon\right)$ with $\epsilon > 0$ small enough for all $n \in \mathbb{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{F(T_1)} u$ and $\bar{y^*} = P_{F(T_2)} v$.

Proof. Let $(x^*, y^*) \in \Omega$, then $x^* \in F(T_1)$, $y^* \in F(T_2)$ and $Ax^* = By^*$. From 2.6 in Lemma 2.46, we have

$$\|A^1 x\|^2 \leq 2\langle x - x^*, A^1 x\rangle, \tag{3.43}$$

where $A^1 = I - T_1$ and for all $x \in C_1$. Similarly, we have

$$\|A^2 y\|^2 \leq 2\langle y - y^*, A^2 y\rangle, \tag{3.44}$$

where $A^2 = I - T_2$ and for all $y \in C_2$.
By Remark 2.47, we have $x^* \in F\left(P_{C_1}\left(I - \lambda_n^1 A^1\right)\right)$ and $y^* \in F\left(P_{C_2}\left(I - \lambda_n^2 A^2\right)\right)$.

Since P_{C_1} is a nonexpansive mapping, we have

$$\begin{aligned}
&\|P_{C_1}\left(I-\lambda_n^1 A^1\right)x-x^*\|^2\\
=&\|P_{C_1}\left(I-\lambda_n^1 A^1\right)x-P_{C_1}\left(I-\lambda_n^1 A^1\right)x^*\|^2\\
\leq&\|x-x^*-\lambda_n^1(A^1x-A^1x^*)\|^2\\
\leq&\|x-x^*-\lambda_n^1 A^1x\|^2\\
=&\|x-x^*\|^2+(\lambda_n^1)^2\|A^1x\|^2-2\lambda_n^1\langle x-x^*,A^1x\rangle\\
\leq&\|x-x^*\|^2+(\lambda_n^1)^2\|A^1x\|^2-\lambda_n^1\|A^1x\|^2\\
=&\|x-x^*\|^2-(\lambda_n^1)(1-\lambda_n^1)\|A^1x\|^2\\
\leq&\|x-x^*\|^2,
\end{aligned}$$

for all $x\in C_1$. Similarly, we obtain

$$\|P_{C_2}\left(I-\lambda_n^2 A^2\right)y-y^*\|^2\leq\|y-y^*\|^2,$$

for all $y\in C_2$.
From definition of $\{u_n\}$, we have

$$\begin{aligned}
\|u_n-x^*\|^2=&\|x_n-x^*-\gamma_n A^*\left(Ax_n-By_n\right)\|^2\\
=&\|x_n-x^*\|^2+\gamma_n^2\|A^*\left(Ax_n-By_n\right)\|^2\\
&-2\gamma_n\langle x_n-x^*,A^*\left(Ax_n-By_n\right)\rangle.
\end{aligned}\tag{3.45}$$

Consider that

$$\begin{aligned}
&\|A^*\left(Ax_n-By_n\right)\|^2\\
=&\langle A^*\left(Ax_n-By_n\right),A^*\left(Ax_n-By_n\right)\rangle\\
=&\langle Ax_n-By_n,AA^*\left(Ax_n-By_n\right)\rangle\\
\leq&\lambda_A\|Ax_n-By_n\|^2
\end{aligned}\tag{3.46}$$

and

$$\begin{aligned}
&-2\langle x_n-x^*,A^*\left(Ax_n-By_n\right)\rangle\\
=&-2\langle Ax_n-Ax^*,Ax_n-By_n\rangle\\
=&-\|Ax_n-Ax^*\|^2-\|Ax_n-By_n\|^2+\|Ax^*-By_n\|^2.
\end{aligned}\tag{3.47}$$

Substitute (3.46) and (3.47) into (3.45), we have

$$\begin{aligned}
&\|u_n-x^*\|^2\\
\leq&\|x_n-x^*\|^2+\gamma_n^2\lambda_A\|Ax_n-By_n\|^2-\gamma_n\|Ax_n-Ax^*\|^2\\
&-\gamma_n\|Ax_n-By_n\|^2+\gamma_n\|Ax^*-By_n\|^2\\
=&\|x_n-x^*\|^2-\gamma_n(1-\lambda_A\gamma_n)\|Ax_n-By_n\|^2\\
&-\gamma_n\|Ax_n-Ax^*\|^2+\gamma_n\|Ax^*-By_n\|^2.
\end{aligned}\tag{3.48}$$

By using the same method as (3.48), we have

$$\begin{aligned}\|v_n - y^*\|^2 \leq & \|y_n - y^*\|^2 - \gamma_n(1-\lambda_B\gamma_n)\|Ax_n - By_n\|^2 \\ & - \gamma_n\|By_n - By^*\|^2 + \gamma_n\|By^* - Ax_n\|^2. \end{aligned} \tag{3.49}$$

From (3.48) and (3.49), we have

$$\begin{aligned} & \|u_n - x^*\|^2 + \|v_n - y^*\|^2 \\ \leq & \|x_n - x^*\|^2 + \|y_n - y^*\|^2 - \gamma_n(1-\lambda_A\gamma_n)\|Ax_n - By_n\|^2 \\ & - \gamma_n(1-\lambda_B\gamma_n)\|Ax_n - By_n\|^2 \\ & - \gamma_n\|Ax_n - Ax^*\|^2 + \gamma_n\|Ax^* - By_n\|^2 \\ & - \gamma_n\|By_n - By^*\|^2 + \gamma_n\|By^* - Ax_n\|^2 \\ \leq & \|x_n - x^*\|^2 + \|y_n - y^*\|^2 \\ & - \gamma_n\left(2 - \gamma_n\left(\lambda_A + \lambda_B\right)\right)\|Ax_n - By_n\|^2. \end{aligned} \tag{3.50}$$

From the definition of $\{x_n\}$, we have

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ = & \|\alpha_n u + (1-\alpha_n) P_{C_1}\left(I - \lambda_n^1(I - T_1)\right) u_n - x^*\|^2 \\ \leq & \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|P_{C_1}\left(I - \lambda_n^1(I - T_1)\right) u_n - x^*\|^2 \\ \leq & \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2. \end{aligned} \tag{3.51}$$

By using the same method as (3.51), we have

$$\|y_{n+1} - y^*\|^2 \leq \alpha_n\|v - y^*\|^2 + (1-\alpha_n)\|v_n - y^*\|^2. \tag{3.52}$$

From (3.50), (3.51) and (3.52), we have

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \\ \leq & \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\ & + \alpha_n\|v - y^*\|^2 + (1-\alpha_n)\|v_n - y^*\|^2 \\ = & \alpha_n(\|u - x^*\|^2 + \|v - y^*\|^2) \\ & + (1-\alpha_n)(\|u_n - x^*\|^2 + \|v_n - y^*\|^2) \\ \leq & \alpha_n(\|u - x^*\|^2 + \|v - y^*\|^2) \\ & + (1-\alpha_n)(\|x_n - x^*\|^2 + \|y_n - y^*\|^2 \\ & - \gamma_n\left(2 - \gamma_n\left(\lambda_A + \lambda_B\right)\right)\|Ax_n - By_n\|^2) \\ \leq & \alpha_n(\|u - x^*\|^2 + \|v - y^*\|^2) \\ & + (1-\alpha_n)(\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\ \leq & \max\{\|u - x^*\|^2 + \|v - y^*\|^2, \|x_1 - x^*\|^2 + \|y_1 - y^*\|^2\}. \end{aligned} \tag{3.53}$$

From mathematical induction, we have

$$\|x_n - x^*\|^2 + \|y_n - y^*\|^2 \leq \max\{\|u - x^*\|^2 + \|v - y^*\|^2, \|x_1 - x^*\|^2 + \|y_1 - y^*\|^2\}.$$

So, we have $\{x_n\}$ and $\{y_n\}$ are bounded. Furthermore, $\{u_n\}$ and $\{v_n\}$ are bounded. From (3.53), we have

$$\begin{aligned}&\gamma_n(1-\alpha_n)\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2\\ \leq&\alpha_n(\|u-x^*\|^2+\|v-y^*\|^2)+C_n-C_{n+1},\end{aligned} \tag{3.54}$$

where $C_n=\|x_n-x^*\|^2+\|y_n-y^*\|^2$, for all $x^*\in F(T_1)$, $y^*\in F(T_2)$ and $n\in\mathbb{N}$.
From (3.54), we separate the proof into two cases.
<u>*Case1.*</u> Suppose that $C_{n+1}\leq C_n$ for all $n\geq n_0$ (for n_0 large enough). Since the sequence $\{C_n\}$ is bounded, we get $\lim_{n\to\infty}C_n=c$, for some $c\in\mathbb{R}$.
From (3.54) and properties of γ_n and α_n, we obtain

$$\lim_{n\to\infty}\|Ax_n-By_n\|=0. \tag{3.55}$$

From the definition of $\{u_n\}$ and $\{v_n\}$, we have

$$\|u_n-x_n\|=\gamma_n\|A^*\left(Ax_n-By_n\right)\| \tag{3.56}$$

and

$$\|v_n-y_n\|=\gamma_n\|B^*\left(Ax_n-By_n\right)\|. \tag{3.57}$$

From (3.55), (3.56) and (3.57), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|v_n-y_n\|=0. \tag{3.58}$$

By using properties of P_{C_1}, we have

$$\begin{aligned}&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\ \leq&\|\left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\|^2\\ =&\|u_n-x^*-\lambda_n^1(I-T_1)(u_n-x^*)\|^2\\ =&\|u_n-x^*\|^2-2\lambda_n^1\langle u_n-x^*,(I-T_1)u_n\rangle\\ &+(\lambda_n^1)^2\|(I-T_1)u_n\|^2\\ \leq&\|u_n-x^*\|^2-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2.\end{aligned} \tag{3.59}$$

By using the same method as (3.59), we have

$$\begin{aligned}&\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\ \leq&\|v_n-y^*\|^2-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2.\end{aligned} \tag{3.60}$$

From (3.50), (3.59) and (3.60), we have

$$\begin{aligned}
&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad+\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\
\leq&\|u_n-x^*\|^2+\|v_n-y^*\|^2\\
&\qquad-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2\\
&\quad-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2\\
\leq&\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&\quad-\gamma_n\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2\\
&\quad-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2\\
&\quad-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2.
\end{aligned} \tag{3.61}$$

From the definition of $\{x_n\}$, $\{y_n\}$ and (3.61), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\
&\leq\alpha_n\|u-x^*\|^2+\alpha_n\|v-y^*\|^2\\
&\quad+(1-\alpha_n)\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad+(1-\alpha_n)\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\
&=\alpha_n\big(\|u-x^*\|^2+\|v-y^*\|^2\big)\\
&\quad+(1-\alpha_n)\big(\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad+\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\big)
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
&\leq\alpha_n\big(\|u-x^*\|^2+\|v-y^*\|^2\big)\\
&\quad+(1-\alpha_n)\big(\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&\quad-\gamma_n\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2\\
&\quad-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2\\
&\quad-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2\big).
\end{aligned} \tag{3.63}$$

By using properties of P_{C_1}, we have

$$\begin{aligned}
&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\leq \langle\left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\\
&\qquad ,P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\rangle\\
&=\frac{1}{2}\big(\|\left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\|^2\\
&\quad +\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad -\|\left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\\
&\quad -P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n+x^*\|^2\big)\\
&\leq\frac{1}{2}\big(\|u_n-x^*\|^2+\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad -\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\\
&\quad -\lambda_n^1((I-T_1)u_n-(I-T_1)x^*)\|^2\big)\\
&=\frac{1}{2}\big(\|u_n-x^*\|^2+\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad -\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad -\left(\lambda_n^1\right)^2\|(I-T_1)u_n-(I-T_1)x^*\|^2\\
&\quad +2\lambda_n^1\langle u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\\
&\qquad ,(I-T_1)u_n-(I-T_1)x^*\rangle\big).
\end{aligned}\tag{3.64}$$

From (3.64), we have

$$\begin{aligned}
&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\leq\|u_n-x^*\|^2-\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad -\left(\lambda_n^1\right)^2\|(I-T_1)u_n-(I-T_1)x^*\|^2\\
&\quad +2\lambda_n^1\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|\\
&\quad \cdot\|(I-T_1)u_n-(I-T_1)x^*\|.
\end{aligned}\tag{3.65}$$

Similarly, we have

$$\begin{aligned}
&\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\
&\leq\|v_n-y^*\|^2-\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|^2\\
&\quad -\left(\lambda_n^2\right)^2\|(I-T_2)v_n-(I-T_2)y^*\|^2\\
&\quad +2\lambda_n^2\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|\\
&\quad \cdot\|(I-T_2)v_n-(I-T_2)y^*\|.
\end{aligned}\tag{3.66}$$

From (3.62), (3.65) and (3.66), we have

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\
&\le \alpha_n\big(\|u-x^*\|^2+\|v-y^*\|^2\big)\\
&\quad+(1-\alpha_n)\big(\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad+\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\big)\\
&\le \alpha_n\big(\|u-x^*\|^2+\|v-y^*\|^2\big)+\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&\quad-(1-\alpha_n)\big(\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad+\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|^2\big)\\
&\quad+2\lambda_n^1\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|\\
&\quad\cdot\|(I-T_1)u_n-(I-T_1)x^*\|\\
&\quad+2\lambda_n^2\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|\\
&\quad\cdot\|(I-T_2)v_n-(I-T_2)y^*\|.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
&(1-\alpha_n)\big(\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad+\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|^2\big)\\
&\quad\le \alpha_n\big(\|u-x^*\|^2+\|v-y^*\|^2\big)\\
&\qquad+2\lambda_n^1\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|\\
&\qquad\cdot\|(I-T_1)u_n-(I-T_1)x^*\|\\
&\qquad+2\lambda_n^2\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|\\
&\qquad\cdot\|(I-T_2)v_n-(I-T_2)y^*\|\\
&\qquad+C_n-C_{n+1}.
\end{aligned}
$$

From $\sum_{n=1}^{\infty}\lambda_n^i<\infty$, for all $i=1,2$ and $\lim_{n\to\infty}C_n=c$, we have

$$
\begin{aligned}
&\lim_{n\to\infty}\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-u_n\|\\
&=\lim_{n\to\infty}\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-v_n\|=0.
\end{aligned}
\tag{3.67}
$$

From (3.58) and (3.67), we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x_n\|\\
&=\lim_{n\to\infty}\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y_n\|=0.
\end{aligned}
\tag{3.68}
$$

Since

$$
\begin{aligned}
&x_{n+1}-x_n\\
&=\alpha_n(u-x_n)+(1-\alpha_n)\left(P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x_n\right),
\end{aligned}
$$

$$
\begin{aligned}
&y_{n+1}-y_n\\
&=\alpha_n(v-y_n)+(1-\alpha_n)\left(P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y_n\right),
\end{aligned}
$$

and (3.68), we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.69}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, we get $W_w(x_n) \neq \emptyset$ and $W_w(y_n) \neq \emptyset$.
Since $W_w(x_n)$ and $W_w(y_n)$ are nonempty sets, then there exists $\hat{x} \in C_1, \hat{y} \in C_2$ such that $\hat{x} \in W_w(x_n)$ and $\hat{y} \in W_w(y_n)$. We may assume, there exists subsequences $\{x_{n_k}\}$, $\{y_{n_k}\}$ of $\{x_n\}$, $\{y_n\}$ such that

$$x_{n_k} \rightharpoonup \hat{x} \quad \text{as} \quad k \to \infty. \tag{3.70}$$

and

$$y_{n_k} \rightharpoonup \hat{y} \quad \text{as} \quad k \to \infty. \tag{3.71}$$

Next, we will show that $(\hat{x}, \hat{y}) \in \Omega$.
From (3.58), (3.70) and (3.71), we obtain $u_{n_k} \rightharpoonup \hat{x}$ and $v_{n_k} \rightharpoonup \hat{y}$ as $k \to \infty$.
Assume that $\hat{x} \notin F(T_1)$.
Since $F(T_1) = F(P_{C_1}(I - \lambda^1_{n_k}(I - T_1)))$, we have $\hat{x} \neq P_{C_1}(I - \lambda^1_{n_k}(I - T_1))\hat{x}$. From Opial's condition, $\lim_{k\to\infty} \lambda^1_{n_k} = 0$ and condition (i), we have

$$\begin{aligned}
&\liminf_{k\to\infty} \|u_{n_k} - \hat{x}\| \\
&< \liminf_{k\to\infty} \|u_{n_k} - P_{C_1}(I - \lambda^1_{n_k}(I - T_1))\hat{x}\| \\
&\leq \liminf_{k\to\infty} (\|u_{n_k} - P_{C_1}(I - \lambda^1_{n_k}(I - T_1))u_{n_k}\| \\
&\quad + \|P_{C_1}(I - \lambda^1_{n_k}(I - T_1))u_{n_k} \\
&\quad - P_{C_1}(I - \lambda^1_{n_k}(I - T_1))\hat{x}\|) \\
&\leq \liminf_{k\to\infty} (\|u_{n_k} - P_{C_1}(I - \lambda^1_{n_k}(I - T_1))u_{n_k}\| \\
&\quad + \|u_{n_k} - \hat{x}\| + \lambda^1_{n_k}\|(I - T_1)u_{n_k} - (I - T_1)\hat{x}\|) \\
&= \liminf_{k\to\infty} \|u_{n_k} - \hat{x}\|.
\end{aligned}$$

This is a contradiction. Thus $\hat{x} \in F(T_1)$.
From $v_{n_k} \rightharpoonup \hat{y}$ as $k \to \infty$ and using the same method as $\hat{x} \in F(T_1)$, we have $\hat{y} \in F(T_2)$.
Since $A\hat{x} - B\hat{y} \in W_w(Ax_n - By_n)$, (3.55) and weakly lower semi-continuous of norm, we get

$$\|A\hat{x} - B\hat{y}\| \leq \liminf_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0.$$

Then $A\hat{x} = B\hat{y}$. Hence $(\hat{x}, \hat{y}) \in \Omega$.
Since $\hat{x} \in F(T_1)$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \langle u - \hat{x^*}, x_n - \hat{x^*} \rangle \\
&= \limsup_{k\to\infty} \langle u - \hat{x^*}, x_{n_k} - \hat{x^*} \rangle \\
&= \langle u - \hat{x^*}, \hat{x} - \hat{x^*} \rangle \\
&\leq 0,
\end{aligned}$$

where $\hat{x^*} = P_{F(T_1)}u$. Since $\hat{y} \in F(T_2)$, we have

$$
\begin{aligned}
&\limsup_{n\to\infty}\langle v - \hat{y^*}, y_n - \hat{y^*}\rangle \\
&= \limsup_{k\to\infty}\langle v - \hat{y^*}, y_{n_k} - \hat{y^*}\rangle \\
&= \langle v - \hat{y^*}, \hat{y} - \hat{y^*}\rangle \\
&\leq 0,
\end{aligned}
$$

where $\hat{y^*} = P_{F(T_2)}v$.

Next, we show that a sequence $\{(x_n, y_n)\}$ converges strongly to $(\hat{x^*}, \hat{y^*}) \in \Omega$, where $\hat{x^*} = P_{F(T_1)}u$ and $\hat{y^*} = P_{F(T_2)}v$.

From the definitions of $\{x_n\}$ and $\{y_n\}$, we have

$$\|x_{n+1} - \hat{x^*}\|^2 \leq (1-\alpha_n)\|x_n - \hat{x^*}\|^2 + 2\alpha_n\langle u - \hat{x^*}, x_{n+1} - \hat{x^*}\rangle$$

and

$$\|y_{n+1} - \hat{y^*}\|^2 \leq (1-\alpha_n)\|y_n - \hat{y^*}\|^2 + 2\alpha_n\langle v - \hat{y^*}, y_{n+1} - \hat{y^*}\rangle.$$

Then

$$
\begin{aligned}
&\|x_{n+1} - \hat{x^*}\|^2 + \|y_{n+1} - \hat{y^*}\|^2 \\
&\leq (1-\alpha_n)\big(\|x_n - \hat{x^*}\|^2 + y_n - \hat{y^*}\|^2\big) \\
&\quad + 2\alpha_n\big(\langle u - \hat{x^*}, x_{n+1} - \hat{x^*}\rangle + \langle v - \hat{y^*}, y_{n+1} - \hat{y^*}\rangle\big),
\end{aligned}
$$

or

$$C_{n+1} \leq (1-\alpha_n)C_n + 2\alpha_n\varrho_n, \tag{3.72}$$

where $\varrho_n = \langle u - \hat{x^*}, x_{n+1} - \hat{x^*}\rangle + \langle v - \hat{y^*}, y_{n+1} - \hat{y^*}\rangle$, for all $n \in \mathbb{N}$.

From Lemma 2.28, thus

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty}\big(\|x_n - \hat{x^*}\|^2 + \|y_n - \hat{y^*}\|^2\big) = 0.$$

Therefore (x_n, y_n) converges strongly to $(\hat{x^*}, \hat{y^*})$.

Since $A\hat{x^*} - B\hat{y^*} \in W_w(Ax_n - By_n)$, (3.55) and weakly lower semi-continuous of norm, we get

$$\|A\hat{x^*} - B\hat{y^*}\| \leq \liminf_{k\to\infty}\|Ax_{n_k} - By_{n_k}\| = 0.$$

Then $A\hat{x^*} = B\hat{y^*}$. Hence $(\hat{x^*}, \hat{y^*}) \in \Omega$.

<u>*Case*2.</u> Suppose that C_n is not monotone decreasing sequence, then there exists an integer n_0 such that $C_{n_0} \leq C_{n_0+1}$.

Define the integer sequence $\tau(n)$ for all $n \geq n_0$ as follows,

$$\tau(n) = \max\{k \leq n : C_k < C_{k+1}\}.$$

It is clear that $\tau(n)$ is a nondecreasing with

$\lim_{n\to\infty}\tau(n) = \infty$ and $C_{\tau(n)} < C_{\tau(n)+1}$.

From (3.72), we have

$$C_{\tau(n)+1} \leq (1-\alpha_{\tau(n)})C_{\tau(n)} + 2\alpha_{\tau(n)}\varrho_{\tau(n)}.$$

From Lemma 2.28, thus

$$\lim_{n\to\infty} C_{\tau(n)} = 0.$$

Applying (3.69), we have

$$\lim_{n\to\infty} C_{\tau(n)+1} = 0.$$

By Lemma 2.41, we have

$$C_n \leq \max\{C_n, C_{\tau(n)}\} \leq C_{\tau(n)+1}.$$

From above inequality and $\lim_{n\to\infty} C_{\tau(n)+1} = 0$, we obtain

$$\lim_{n\to\infty} \left(\|x_n - \hat{x^*}\|^2 + \|y_n - \hat{y^*}\|^2\right) = \lim_{n\to\infty} C_n = 0.$$

That implies $\{(x_n, y_n)\}$ converges strongly to $(\hat{x^*}, \hat{y^*})$. By using the same methods as case 1, we have
$(\hat{x^*}, \hat{y^*}) \in \Omega$, where $\hat{x^*} = P_{F(T_1)}u$ and $\hat{y^*} = P_{F(T_2)}v$. This is complete the proof. □

Corollary 3.5. For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $T_i : C_i \to C_i$ be quasi-nonexpansive mapping for all $i = 1, 2$ and let $A : H_1 \to H_3$ be bounded linear operator with adjoints A^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 : x \in F(T_1), y \in F(T_2)$ and $Ax = y\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases} u_n = x_n - \gamma_n A^*\left(Ax_n - y_n\right), \\ x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1}\left(I - \lambda_n^1 (I - T_1)\right) u_n, \\ v_n = (1-\gamma_n) y_n + \gamma_n A x_n, \\ y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2}\left(I - \lambda_n^2 (I - T_2)\right) v_n, \end{cases} \tag{3.73}$$

for all $n \geq 1$ where $\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0$,$\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0,1)$ for all $i = 1, 2$ and $\gamma_n \in (a, b) \subset \left(\epsilon, \frac{2}{\lambda_A} - \epsilon\right)$ with $\epsilon > 0$ small enough for all $n \in \mathbf{N}$ and λ_A be spectral radius of A^*A. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{F(T_1)}u$ and $\bar{y^*} = P_{F(T_2)}v$.

Proof. By using Theorem 3.4 and taking $B \equiv I$, we obtain the conclusion. □

Chapter 4

Application

In this section, to show the application of section 3.1 and section 3.2 as follows:

4.1 Strong convergence theorems for finding common elements of split feasibility problem and fixed point problem

Definition 4.1. Let $S : C \to C$ is called κ-strictly pseudo-contractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \|(I - S)x - (I - S)y\|^2, \forall x, y \in C. \tag{4.1}$$

Lemma 4.1. ([29]) Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C. If S is a κ-strict pseudo-contractive mapping, then S satisfies the Lipschitz condition

$$\|Sx - Sy\| \leq \frac{1 + \kappa}{1 - \kappa} \|x - y\|, \quad \forall x, y \in C.$$

By Lemma 4.1, applying T, S are κ, $\bar{\kappa}$-strict and quasi pseudo-contractive mappings, respctively, we obtain this theorem.

Theorem 4.2. Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*. For each $i = 1, 2, 3, ..., N$, let D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,...,N} \gamma_i$. Let $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is a $\bar{\kappa}$-strict and quasi pseudo-contractive mapping, $T : C \to C$ is a κ-strict and quasi pseudo-contractive mapping. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1 - \xi_n) z_n + \xi_n S ((1 - \eta_n) z_n + \eta_n S z_n), \\ y_n = \alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^N a_i D_i\right) (x_n - \delta A^* (A x_n - v_n)), \\ u_n = P_C y_n, \\ x_{n+1} = (1 - \beta_n) u_n + \beta_n T ((1 - \gamma_n) u_n + \gamma_n T u_n), \qquad for\ n \geq 1, \end{cases} \tag{4.2}$$

where parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0, 1]$, δ and γ are two positive constants.

We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.4) ,that is,

$$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}.$$

Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1+\left(\frac{1+\bar{\kappa}}{1-\bar{\kappa}}\right)^2}+1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \dfrac{1}{\sqrt{1+\left(\frac{1+\kappa}{1-\kappa}\right)^2}+1}$,

(iv) $0 < \delta, \gamma < \dfrac{1}{\|A\|^2}$ and $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ for $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\Gamma}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right) z$.

Proof. By using Theorem 3.1 and Lemma 4.1, we obtain the conclusion. □

In 2009, Kangtunyakarn and Suantai([31]) introduced the S-mapping generated by a finite family of κ-strictly pseudo contractive mappings and real numbers as follows:

Definition 4.2. Let C be a nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ_i-strict pseudo contractions of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\vdots\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 4.3. ([31]) Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ-strict pseudo contractions of C into C with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $\kappa = max\{\kappa_i : i = 1, 2, ..., N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (\kappa, 1]$, $\alpha_3^N \in [\kappa, 1)$ $\alpha_2^j \in [\kappa, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^{N} F(T_i)$ and S is a nonexpansive mapping.

Theorem 4.4. Let C and Q are nonempty closed convex subset of real Hilbert spaces. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ_i-strict pseudo contractions of C into C with $\bigcap_{i=1}^{N} F(T_i) \neq$

$\emptyset$ and $\kappa = max\{\kappa_i : i = 1,2,...,N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I,\ j = 1,2,...,N,$ where $I = [0,1]$ $,\alpha_1^j + \alpha_2^j + \alpha_3^j = 1,\ \alpha_1^j,\ \alpha_3^j \in (\kappa, 1)$ for all $j = 1,2,...,N-1$ and $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in [\kappa, 1)$ $\alpha_2^j \in [\kappa, 1)$ for all $j = 1,2,...,N$. Let S be the S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{\bar{T}_i\}_{i=1}^N$ be a finite family of $\bar{\kappa}_i$-strict pseudo contractions of Q into Q with $\bigcap_{i=1}^N F(\bar{T}_i) \neq \emptyset$ and $\bar{\kappa} = max\{\bar{\kappa}_i : i = 1,2,...,N\}$ and let $\beta_j = (\beta_1^j, \beta_2^j, \beta_3^j) \in I \times I \times I,\ j = 1,2,...,N,$ where $I = [0,1]$ $,\beta_1^j + \beta_2^j + \beta_3^j = 1,\ \beta_1^j,\ \beta_3^j \in (\bar{\kappa}, 1)$ for all $j = 1,2,...,N-1$ and $\beta_1^N \in (\bar{\kappa}, 1], \beta_3^N \in [\bar{\kappa}, 1)$ $\beta_2^j \in [\bar{\kappa}, 1)$ for all $j = 1,2,...,N$. Let $\bar{S}$ be the S-mapping generated by $\bar{T}_1, \bar{T}_2, ..., \bar{T}_N$ and $\beta_1, \beta_2, ..., \beta_N$. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, let D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,...,N} \gamma_i$. Let $f : C \to H_1$ is a ρ-contraction. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n \bar{S}\left((1-\eta_n) z_n + \eta_n \bar{S} z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^N a_i D_i\right)(x_n - \delta A^*(A x_n - v_n)), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n S\left((1-\gamma_n) u_n + \gamma_n S u_n\right), \quad for\ n \geq 1, \end{cases} \tag{4.3}$$

where parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$, δ and γ are two positive constants.

We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.4) ,that is,

$$\mathbf{\Gamma} = \{x \mid x \in C \cap \bigcap_{i=1}^N F(T_i), Ax \in Q \cap \bigcap_{i=1}^N F(\bar{T}_i)\}.$$

Suppose that $S - \mathcal{I}$ and $\bar{S} - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \frac{1}{\sqrt{2}+1},$

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \frac{1}{\sqrt{2}+1},$

(iv) $0 < \delta, \gamma < \frac{1}{\|A\|^2}$ and $\bar{\gamma} > \gamma\rho,$

(v) $0 < \alpha_n < \|D_i\|^{-1}$ for $i = 1,2,...,N.$

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^N a_i D_i\right) z.$

Proof. By using Theorem 3.1 and Lemma 4.3, we obtain the conclusion. □

4.2 Strong convergence theorems for finding the set of solutions of the split equality fixed point problem for quasi-nonexpansive mappings

The following lemma will be used to prove in the application.

Lemma 4.5. [3] Let H be a Hilbert space, let C be a non-empty closed convex subset of H, and let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.

In 2009, Kangtunyakarn and Suantai [31] introduced the S-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ as follows.

Definition 4.3. Let C be a non-empty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows;

$$
\begin{aligned}
U_0 &= I, \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\
&\vdots \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}
$$

This mapping is called an *S-mapping* generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 4.6. [39] Let C be a non-empty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (0, 1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (0, 1]$, $\alpha_3^N \in [0, 1)$ $\alpha_2^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a quasi-nonexpansive mapping.

By using these results, we obtain the following theorem.

Theorem 4.7. For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C_1 into C_1 with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (0, 1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (0, 1]$, $\alpha_3^N \in [0, 1)$ $\alpha_2^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{\bar{T}_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C_2 into C_2 with $\bigcap_{i=1}^{N} F(\bar{T_i}) \neq \emptyset$, and let $\beta_j = (\beta_1^j, \beta_2^j, \beta_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\beta_1^j + \beta_2^j + \beta_3^j = 1$, β_1^j, $\beta_3^j \in (0, 1)$ for all $j = 1, 2, ..., N-1$ and $\beta_1^N \in (0, 1], \beta_3^N \in [0, 1)$ $\beta_2^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let $\bar{S}$ be S-mapping generated by $\bar{T_1}, \bar{T_2}, ..., \bar{T_N}$ and $\beta_1, \beta_2, ..., \beta_N$. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 : x \in \bigcap_{i=1}^{N} F(T_i), y \in \bigcap_{i=1}^{N} F(\bar{T_i})$ and $Ax = By\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases} u_n = x_n - \gamma_n A^* \left(Ax_n - By_n \right), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) P_{C_1} \left(I - \lambda_n^1 (I - S) \right) u_n, \\ v_n = y_n + \gamma_n B^* \left(Ax_n - By_n \right), \\ y_{n+1} = \alpha_n v + (1 - \alpha_n) P_{C_2} \left(I - \lambda_n^2 (I - \bar{S}) \right) v_n, \end{cases} \tag{4.4}$$

for all $n \geq 1$ where $\{\alpha_n\} \subset [0, 1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0, 1)$ for all $i = 1, 2$ and $\gamma_n \in (a, b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon \right)$ with $\epsilon > 0$ small enough for all $n \in \mathbf{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{F(S)} u$ and $\bar{y^*} = P_{F(\bar{S})} v$.

Proof. By using Theorem 3.4 and 4.6, we obtain the conclusion. □

Moreover, if we put $F(T_1) = C_1$ and $F(T_2) = C_2$ in Theorem 3.4, we obtain the SEFPP reduced to the SEFP.

Theorem 4.8. For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 : Ax = By\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases} u_n = x_n - \gamma_n A^* \left(Ax_n - By_n \right), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) P_{C_1} u_n, \\ v_n = y_n + \gamma_n B^* \left(Ax_n - By_n \right), \\ y_{n+1} = \alpha_n v + (1 - \alpha_n) P_{C_2} v_n, \end{cases} \tag{4.5}$$

for all $n \geq 1$,where $\{\alpha_n\} \subset [0, 1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0, 1)$ for all $i = 1, 2$ and $\gamma_n \in (a, b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon \right)$ with $\epsilon > 0$ small enough for all $n \in \mathbf{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{C_1} u$ and $\bar{y^*} = P_{C_2} v$.

Proof. Put $T_i = I$ for all $i = 1, 2, 3, ..., N$ in Theorem 3.4, we get $F(T_1) = C_1$ and $F(T_2) = C_2$. Using Theorem 3.4, we obtain the desired results. □

Chapter 5

Conclusions

In this section, we conclude all main results obtained in this thesis.

5.1 Strong convergence theorems for finding common elements of split feasibility problem and fixed point problem

1. Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*. For each $i = 1, 2, ..., N$, let D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,...,N} \gamma_i$. Let $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is an $\mathcal{L}_1$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_1 > 1$ and let $T : C \to C$ is an $\mathcal{L}_2$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_2 > 1$. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n S\left((1-\eta_n) z_n + \eta_n S z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^N a_i D_i\right)\left(x_n - \delta A^*\left(A x_n - v_n\right)\right), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n T\left((1-\gamma_n) u_n + \gamma_n T u_n\right), \qquad for\ n \geq 1, \end{cases} \tag{5.1}$$

where parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0, 1]$, δ and γ are two positive constants.

Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0\ and \sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1 + \mathcal{L}_1^2} + 1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \dfrac{1}{\sqrt{1 + \mathcal{L}_2^2} + 1}$,

(iv) $0 < \delta, \gamma < \dfrac{1}{\|A\|^2}$ and $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ for $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^N a_i D_i\right) z$.

2. Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, D is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$

and $\bar{\gamma} = \min_{i=1,2,\ldots,N} \gamma_i$. Let $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is an $\mathcal{L}_1$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_1 > 1$ and let $T : C \to C$ is an $\mathcal{L}_2$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_2 > 1$. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n S\left((1-\eta_n) z_n + \eta_n S z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + (\mathcal{I} - \alpha_n D)\left(x_n - \delta A^*\left(Ax_n - v_n\right)\right), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n T\left((1-\gamma_n) u_n + \gamma_n T u_n\right), \qquad for\ n \geq 1, \end{cases} \tag{5.2}$$

where parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$, δ and γ are two positive constants.

We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.4) ,that is,

$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}$.

Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \frac{1}{\sqrt{1+\mathcal{L}_1^2}+1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \frac{1}{\sqrt{1+\mathcal{L}_2^2}+1}$,

(iv) $0 < \delta, \gamma < \frac{1}{\|A\|^2}$ and $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D\|^{-1}$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - D\right) z$.

3. Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*. For each $i = 1, 2, \ldots, N$, let D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\ldots,N} \gamma_i$. Let $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is an $\mathcal{L}$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L} > 1$. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n S\left((1-\eta_n) z_n + \eta_n S z_n\right), \\ x_{n+1} = P_C\left[\alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)\left(x_n - \delta A^*\left(Ax_n - v_n\right)\right)\right], \qquad for\ n \geq 1, \end{cases} \tag{5.3}$$

where parameters $\{\alpha_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$, δ and γ are two positive constants.

We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.4) ,that is,

$\mathbf{\Gamma} = \{x \mid x \in C, Ax \in Q \cap F(S)\}$.

Suppose that $S - \mathcal{I}$ is demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim\limits_{n\to\infty} \alpha_n = 0 \ and \ \sum\limits_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1+\mathcal{L}_1^2}+1},$

(iii) $0 < \delta < \dfrac{1}{\|A\|^2}$ and $\bar{\gamma} > \gamma\rho,$

(iv) $0 < \gamma < \dfrac{1}{\|A\|^2},$

(v) $0 < \alpha_n < \|D_i\|^{-1}$ for $i = 1, 2, ..., N.$

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right) z.$

4. Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*. For each $i = 1, 2, 3, ..., N$, let D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min\limits_{i=1,2,...,N} \gamma_i$. Let $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is a $\bar{\kappa}$-strict and quasi pseudo-contractive mapping, $T : C \to C$ is a κ-strict and quasi pseudo-contractive mapping. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n S\left((1-\eta_n) z_n + \eta_n S z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)\left(x_n - \delta A^*\left(A x_n - v_n\right)\right), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n T\left((1-\gamma_n) u_n + \gamma_n T u_n\right), \qquad for \ n \geq 1, \end{cases} \tag{5.4}$$

where parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0, 1]$, δ and γ are two positive constants.

We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.4) ,that is,

$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}$.

Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim\limits_{n\to\infty} \alpha_n = 0 \ and \ \sum\limits_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1+\left(\frac{1+\bar{\kappa}}{1-\bar{\kappa}}\right)^2}+1},$

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \dfrac{1}{\sqrt{1+\left(\frac{1+\kappa}{1-\kappa}\right)^2}+1},$

(iv) $0 < \delta, \gamma < \dfrac{1}{\|A\|^2}$ and $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ for $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}} \left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i \right) z$.

5. Let C and Q are nonempty closed convex subset of real Hilbert spaces. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ_i-strict pseudo contractions of C into C with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $\kappa = max\{\kappa_i : i = 1, 2, ..., N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in [\kappa, 1)$ $\alpha_2^j \in [\kappa, 1)$ for all $j = 1, 2, ..., N$. Let S be the S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{\bar{T_i}\}_{i=1}^{N}$ be a finite family of $\bar{\kappa_i}$-strict pseudo contractions of Q into Q with $\bigcap_{i=1}^{N} F(\bar{T_i}) \neq \emptyset$ and $\bar{\kappa} = max\{\bar{\kappa_i} : i = 1, 2, ..., N\}$ and let $\beta_j = (\beta_1^j, \beta_2^j, \beta_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\beta_1^j + \beta_2^j + \beta_3^j = 1$, β_1^j, $\beta_3^j \in (\bar{\kappa}, 1)$ for all $j = 1, 2, ..., N-1$ and $\beta_1^N \in (\bar{\kappa}, 1], \beta_3^N \in [\bar{\kappa}, 1)$ $\beta_2^j \in [\bar{\kappa}, 1)$ for all $j = 1, 2, ..., N$. Let $\bar{S}$ be the S-mapping generated by $\bar{T_1}, \bar{T_2}, ..., \bar{T_N}$ and $\beta_1, \beta_2, ..., \beta_N$. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, let D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,...,N} \gamma_i$. Let $f : C \to H_1$ is a ρ-contraction. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n \bar{S} \left((1-\eta_n) z_n + \eta_n \bar{S} z_n \right), \\ y_n = \alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i \right) (x_n - \delta A^* (A x_n - v_n)), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n S \left((1-\gamma_n) u_n + \gamma_n S u_n \right), \quad for\ n \geq 1, \end{cases} \tag{5.5}$$

where parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0, 1]$, δ and γ are two positive constants.

We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.4) ,that is,

$\mathbf{\Gamma} = \{x \mid x \in C \cap \bigcap_{i=1}^{N} F(T_i), Ax \in Q \cap \bigcap_{i=1}^{N} F(\bar{T_i})\}$.

Suppose that $S - \mathcal{I}$ and $\bar{S} - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{2}+1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \dfrac{1}{\sqrt{2}+1}$,

(iv) $0 < \delta, \gamma < \dfrac{1}{\|A\|^2}$ and $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ for $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\Gamma}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right) z.$

5.2 Strong convergence theorems for finding the set of solutions of the split equality fixed point problem for quasi-nonexpansive mappings

1. For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $T_i : C_i \to C_i$ be quasi-nonexpansive mapping for all $i = 1, 2$ and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 \mid x \in F(T_1), y \in F(T_2)$ and $Ax = By\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases} u_n = x_n - \gamma_n A^*\left(Ax_n - By_n\right), \\ x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1}\left(I - \lambda_n^1 (I - T_1)\right) u_n, \\ v_n = y_n + \gamma_n B^*\left(Ax_n - By_n\right), \\ y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2}\left(I - \lambda_n^2 (I - T_2)\right) v_n, \end{cases} \tag{5.6}$$

for all $n \geq 1$,where $\{\alpha_n\} \subset [0, 1]$ with $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0, 1)$ for all $i = 1, 2$ and $\gamma_n \in (a, b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon\right)$ with $\epsilon > 0$ small enough for all $n \in \mathbf{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{F(T_1)} u$ and $\bar{y^*} = P_{F(T_2)} v$.

2. For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $T_i : C_i \to C_i$ be quasi-nonexpansive mapping for all $i = 1, 2$ and let $A : H_1 \to H_3$ be bounded linear operator with adjoints A^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 : x \in F(T_1), y \in F(T_2)$ and $Ax = y\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases} u_n = x_n - \gamma_n A^*\left(Ax_n - y_n\right), \\ x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1}\left(I - \lambda_n^1 (I - T_1)\right) u_n, \\ v_n = (1-\gamma_n) y_n + \gamma_n A x_n, \\ y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2}\left(I - \lambda_n^2 (I - T_2)\right) v_n, \end{cases} \tag{5.7}$$

for all $n \geq 1$ where $\{\alpha_n\} \subset [0, 1]$ with $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0, 1)$ for all $i = 1, 2$ and $\gamma_n \in (a, b) \subset \left(\epsilon, \frac{2}{\lambda_A} - \epsilon\right)$ with $\epsilon > 0$ small enough for all $n \in \mathbf{N}$ and λ_A be spectral radius of A^*A. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{F(T_1)} u$ and $\bar{y^*} = P_{F(T_2)} v$.

3. For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C_1 into C_1 with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0,1]$ $, \alpha_1^j + \alpha_2^j + \alpha_3^j = 1,\ \alpha_1^j,\ \alpha_3^j \in (0,1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (0,1], \alpha_3^N \in [0,1)$ $\alpha_2^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{\bar{T}_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C_2 into C_2 with $\bigcap_{i=1}^N F(\bar{T}_i) \neq \emptyset$, and let $\beta_j = (\beta_1^j, \beta_2^j, \beta_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0,1]$ $, \beta_1^j + \beta_2^j + \beta_3^j = 1,\ \beta_1^j,\ \beta_3^j \in (0,1)$ for all $j = 1, 2, ..., N-1$ and $\beta_1^N \in (0,1], \beta_3^N \in [0,1)$ $\beta_2^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let $\bar{S}$ be S-mapping generated by $\bar{T_1}, \bar{T_2}, ..., \bar{T_N}$ and $\beta_1, \beta_2, ..., \beta_N$. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 : x \in \bigcap_{i=1}^N F(T_i), y \in \bigcap_{i=1}^N F(\bar{T}_i)$ and $Ax = By\}$ is a non-empty set and let $\{x_n\}, \{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases} u_n = x_n - \gamma_n A^* (Ax_n - By_n), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) P_{C_1} \left(I - \lambda_n^1 (I - S)\right) u_n, \\ v_n = y_n + \gamma_n B^* (Ax_n - By_n), \\ y_{n+1} = \alpha_n v + (1 - \alpha_n) P_{C_2} \left(I - \lambda_n^2 (I - \bar{S})\right) v_n, \end{cases} \tag{5.8}$$

for all $n \geq 1$ where $\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty \lambda_n^i < \infty$ and $\lambda_n^i \in (0,1)$ for all $i = 1, 2$ and $\gamma_n \in (a,b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon\right)$ with $\epsilon > 0$ small enough for all $n \in \mathbb{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{F(S)} u$ and $\bar{y^*} = P_{F(\bar{S})} v$.

4. For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 : Ax = By\}$ is a non-empty set and let $\{x_n\}, \{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases} u_n = x_n - \gamma_n A^* (Ax_n - By_n), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) P_{C_1} u_n, \\ v_n = y_n + \gamma_n B^* (Ax_n - By_n), \\ y_{n+1} = \alpha_n v + (1 - \alpha_n) P_{C_2} v_n, \end{cases} \tag{5.9}$$

for all $n \geq 1$,where $\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty \lambda_n^i < \infty$ and $\lambda_n^i \in (0,1)$ for all $i = 1, 2$ and $\gamma_n \in (a,b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon\right)$ with $\epsilon > 0$ small enough for all $n \in \mathbb{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{C_1} u$ and $\bar{y^*} = P_{C_2} v$.

5.3 Suggestions

In our thesis, we obtain some results of split feasibility, split equality and fixed point problem in a Hilbert space. For those who would like to extend these results, we suggest that proving our results in a Banach space would be better.

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Appendix

Appendix A

The research papers

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The Theory of the Feasibility Problems and Fixed Point Problems of Nonlinear Mappings

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Abstract : In this paper, the authors extend and improve some results of Hamdi [Hamdi, A., Liou, Y.C., Yao, Y. and Luo, C.: The common solutions of the split feasibility problems and fixed point problems. Journal of Inequalities and Applications (2015) 2015:385 DOI10.1186/s13660-015-0870-6] by using the concept of lemma 2.11 Suwannaut and Kangtunyakarn [Suwannaut, S. and Kangtunyakarn, A.: The combination of the set of solutions of equilibrium problem for convergence theorem of the set of fixed points of strictly pseudo-contractive mappings and variational inequalities problem. Fixed Point Theory and its Applications (2013) 2013:291]. Then they prove strong convergence theorem of the proposed iteration under some control condition. Moreover, we use S-mapping in application with our main result.

Keywords : the split feasibility problem; fixed point problem; $\mathcal{L}$-Lipschitzian; strongly positive.

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1 Introduction

The split feasibility problem has become the inspiration in pure and applied mathematics. It attracted the author's attention due to its application in signal processing. The problem was introduced by Censor and Elfving(1994)([1]).

Let C and Q be nonempty closed convex subsets of real Hilbert space H_1 and H_2, respectively.

The split feasibility problem(SFP) was formulated so as to find a point u^* satisfy the properties :

$$u^* \in C \text{ and } Au^* \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator.

The split common fixed point problem(SCFP) was formulated such that

$$u^* \in F(T) \text{ and } Au^* \in F(S), \tag{1.2}$$

where $F(T)$ and $F(S)$ are fixed point sets of the operators $T : H_1 \to H_1$ and $S : H_2 \to H_2$.

Recently, the study of the split common fixed point problem(SCFP) has become popular among mathematicians. The problem, first analysed by Censor and Segal([2]), is a natural extension of the SFP and the convex feasibility problem.

In ([3]) Hamdi, Liou, Yao and Luo proved strong convergence theorem as following algorithm : $x_0 \in H_1$ and

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n S((1-\eta_n) z_n + \eta_n S z_n), \\ y_n = \alpha_n \gamma f(x_n) + (\mathcal{I} - \alpha_n \mathcal{B})(x_n - \delta A^*(Ax_n - v_n)), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n T((1-\gamma_n) u_n + \gamma_n T u_n) \end{cases}$$

for all $n \in \mathbb{N}$,
where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$, $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, $f : C \to H_1$ is ρ-contraction, $\mathcal{B}$ is strongly positive bounded linear operator on H_1, $S : Q \to Q$ is an $\mathcal{L}_1$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_1 > 1$, $T : C \to C$ is an $\mathcal{L}_2$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_2 > 1$. They showed that the sequence $\{x_n\}$ converges strongly to the unique fixed point of the contraction mapping $P_{\mathbf{\Gamma}}(\gamma f + \mathcal{I} - \mathcal{B})$.

The purpose of this paper was to study the following split feasibility problem and fixed point problem :

$$\text{Find } u^* \in C \cap F(T) \text{ and } Au^* \in Q \cap F(S). \tag{1.3}$$

The set of solution of (1.3) is denoted by $\mathbf{\Gamma}$, that is,
$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}$.

It is immediately evident that (1.3) can be derived from SFP(1.1) and SCFP(1.2).

In this paper, we're motivated and inspired by Hamdi, Liou, Yao and Luo ([3]), we modified the split feasibility problem and fixed point problem by Hamdi, Liou, Yao and Luo ([3]) and used the concept from Lemma 2.11. we will introduce a new iteration to approach the solution of (1.3).
The proof of the strong convergence result is given later in the paper.

2 Preliminaries

Throughout this paper, we always assume that H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Using the notations of weak and strong convergence by "$\rightharpoonup$" and "$\rightarrow$", respectively.

Recall that a mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. The set of all elements of fixed point of a mapping T is denoted by $F(T) = \{x \in C : Tx = x\}$. Goebel and Kirk ([4]) showed that $F(T)$ is closed and convex. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2, \quad \lambda \in [0, 1]$$

and

$$\|x + y\|^2 = \|x\|^2 + 2 \langle x, y \rangle + \|y\|^2$$

for all $x, y \in H$.

Lemma 2.1. [5] *Let H be a real Hilbert space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle, \quad \forall x, y \in H.$$

Definition 2.2. An operator A is a *strongly positive bounded linear operator* on H if there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H.$$

Definition 2.3. An operator $A : C \to H$ is called *$\mathcal{L}$-Lipschitzian* if

$$\|Ax - Ay\| \leq \mathcal{L} \|x - y\|, \quad \forall x, y \in C$$

for some constant $\mathcal{L} > 0$. If $\mathcal{L} \in [0, 1]$, then A is called $\mathcal{L}$-contraction.

Definition 2.4. An operator $A : C \to C$ is called *pseudo-contractive* if

$$\langle Ax - Ay, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

Definition 2.5. An operator $A : C \to C$ is called *quasi-pseudo-contractive* if

$$\|Ax - y\|^2 \leq \|x - y\|^2 + \|Ax - x\|^2$$

for all $x \in C$ and $y \in F(A)$.

Definition 2.6. An operator $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \qquad \forall x, y \in C.$$

It is obvious that any α-inverse strongly monotone mapping A is $\frac{1}{\alpha} - Lipschitzian$.

Definition 2.7. An operator $A : C \to C$ is called *firmly nonexpansive* if

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 - \|(\mathcal{I} - A)x - (\mathcal{I} - A)y\|^2, \qquad \forall x, y \in C.$$

Definition 2.8. An operator A is said to be *demiclosed* if $\forall x_n \rightharpoonup \bar{u}$ and $A(x_n) \to u$ imply that $A(\bar{u}) = u$

Lemma 2.9. [6] *Let $\{\mathcal{Q}_n\} \subset [0, +\infty], \{v_n\} \subset [0, 1]$ and $\{\eta_n\}$ be three real number sequences. Suppose that $\{\mathcal{Q}_n\}, \{v_n\}$ and $\{\eta_n\}$ satisfy the following three conditions:*

(i) $\mathcal{Q}_{n+1} \leq (1 - v_n)\mathcal{Q}_n + \eta_n v_n$,

(ii) $\sum_{n=1}^{\infty} v_n = \infty$,

(iii) $\limsup_{n\to\infty} \eta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\eta_n v_n| < \infty$.

Then, $\lim_{n\to\infty} \mathcal{Q}_n = 0$.

Lemma 2.10. [7] *Let $\{\rho_n\}$ be a sequences of real numbers. Assume that there exists a subsequence $\{\rho_{n_k}\}$ of $\{\rho_n\}$ such that $\rho_{n_k} \leq \rho_{n_k+1}$ for all $k \geq 0$. For every $n \geq N_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = max\{i \leq n : \rho_{n_i} < \rho_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and

$$max\{\rho_{\tau(n)}, \rho_n\} \leq \rho_{\tau(n)+1},$$

for all $n \geq N_0$.

Lemma 2.11. [8] *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let A_i be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\gamma_i > 0$ and $\bar{\gamma} = min_{i=1,2,...,N}\gamma_i$. Let $\{a_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N a_i = 1$. Then the following properties hold:*

(i) $\left\|\mathcal{I} - \rho \sum_{i=1}^{N} a_i A_i\right\| \leq 1 - \rho\bar{\gamma}$ *and* $\mathcal{I} - \rho \sum_{i=1}^{N} a_i A_i$ *is a nonexpansive mapping for every* $0 < \rho < \|A_i\|^{-1}$ *for* $i = 1, 2, ..., N$.

(ii) $VI(C, \sum_{i=1}^{N} a_i A_i) = \bigcap_{i=1}^{N} VI(C, A_i)$.

Proposition 2.12. [9] *Let* H *be a real Hilbert space. Let* $\mathcal{U} : H \to H$ *be an* $\mathcal{L}$*-Lipschitzian operator with* $\mathcal{L} > 1$*. Then*

$$F\left(((1-\zeta)\mathcal{I} + \zeta\mathcal{U})\mathcal{U}\right) = F\left(\mathcal{U}((1-\zeta)\mathcal{I} + \zeta\mathcal{U})\right) = F(\mathcal{U})$$

for all $\zeta \in (0, \frac{1}{\mathcal{L}})$.

Proposition 2.13. [9] *Let* H *be a real Hilbert space. Let* $\mathcal{U} : H \to H$ *be an* $\mathcal{L}$*-Lipschitzian quasi-pseudo-contractive operator. Then we have*

$$\|\mathcal{U}\left((1-\eta)x + \eta\mathcal{U}x\right) - u^*\|^2 \leq \|x - u^*\|^2 + (1-\eta)\|x - \mathcal{U}\left((1-\eta)x + \eta\mathcal{U}x\right)\|^2,$$

and the operator $(1-\xi)\mathcal{I} + \xi\mathcal{U}\left((1-\eta)\mathcal{I} + \eta\mathcal{U}\right)$ *is quasi-nonexpansive when* $0 < \xi < \eta < \frac{1}{\sqrt{1+\mathcal{L}^2}+1}$*, that is,*

$$\|(1-\xi)x + \xi\mathcal{U}\left((1-\eta)x + \eta\mathcal{U}x\right) - u^*\| \leq \|x - u^*\|$$

for all $x \in H$ *and* $u^* \in F(\mathcal{U})$.

Proposition 2.14. [9] *Let* H *be a real Hilbert space. Let* $\mathcal{U} : H \to H$ *be an* $\mathcal{L}$*-Lipschitzian operator with* $\mathcal{L} > 1$*. If* $\mathcal{I} - \mathcal{U}$ *is demiclosed at* 0*, then* $\mathcal{I} - \mathcal{U}((1-\zeta)\mathcal{I} + \zeta\mathcal{U})$ *is also demiclosed at* 0 *when* $\zeta \in (0, \frac{1}{\mathcal{L}})$.

3 Main Results

Theorem 3.1. *Let* H_1 *and* H_2 *are two real Hilbert space, let* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *are two nonempty closed convex sets. Let* $A : H_1 \to H_2$ *is a bounded linear operator with its adjoint* A^**,* D_i *is strongly positive bounded linear operator on* H_1 *with coefficient* $\gamma_i > 0$ *and* $\bar{\gamma} = \min_{i=1,2,\ldots,N} \gamma_i$*,* $f : C \to H_1$ *is a* ρ*-contraction,* $S : Q \to Q$ *is an* $\mathcal{L}_1$*-Lipschitzian quasi-pseudo-contractive operator with* $\mathcal{L}_1 > 1$ *,* $T : C \to C$ *is an* $\mathcal{L}_2$*-Lipschitzian quasi-pseudo-contractive operator with* $\mathcal{L}_2 > 1$*. Assume that* $\mathbf{\Gamma} \neq \emptyset$ *and let* $\{x_n\}$ *be a sequences generated by* $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n)\, z_n + \xi_n S\left((1-\eta_n)\, z_n + \eta_n S z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)\left(x_n - \delta A^*\left(A x_n - v_n\right)\right), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n T\left((1-\gamma_n)\, u_n + \gamma_n T u_n\right), \quad \text{for } n \geq 1, \end{cases} \tag{3.1}$$

The parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ *and* $\{\eta_n\}$ *are real sequences in* $[0,1]$ *,* δ *and* γ *are two positive constants.*

We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.3)*, that is,*
$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}$.
Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0*. Assume that the following conditions are satisfied :*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1+\mathcal{L}_1^2}+1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \dfrac{1}{\sqrt{1+\mathcal{L}_2^2}+1}$,

(iv) $0 < \delta, \gamma < \dfrac{1}{\|A\|^2}$ *and* $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ *for* $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right) z$.

Proof. Let $z^* = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right) z^*$, we have $z^* \in C \cap F(T)$ and $Az^* \in Q \cap F(S)$. From P_Q is firmly nonexpansive, thus

$$\begin{aligned}\|z_n - Az^*\|^2 &= \|P_Q Ax_n - P_Q Az^*\|^2 \\ &\le \|Ax_n - Az^*\|^2 - \|(\mathcal{I} - P_Q)Ax_n - (\mathcal{I} - P_Q)Az^*\|^2 \\ &= \|Ax_n - Az^*\|^2 - \|Ax_n - z_n\|^2 . \end{aligned} \tag{3.2}$$

Applying Proposition 2.12, condition (ii) and (iii), we have

$$F\left(S((1-\eta_n)\mathcal{I} + \eta_n S)\right) = F(S)$$

and

$$F\left(T((1-\gamma_n)\mathcal{I} + \gamma_n T)\right) = F(T)$$

for all $n \in \mathbb{N}$.
By Proposition 2.13 and condition (ii), we have

$$\begin{aligned}\|v_n - Az^*\| &= \|[(1-\xi_n)\mathcal{I} + \xi_n S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)] z_n - Az^*\| \\ &\le \|z_n - Az^*\| . \end{aligned} \tag{3.3}$$

This together with (3.2), it implies that

$$\begin{aligned}\|v_n - Az^*\|^2 &\le \|z_n - Az^*\|^2 \\ &\le \|Ax_n - Az^*\|^2 - \|Ax_n - z_n\|^2 \end{aligned} \tag{3.4}$$

By Proposition 2.13 and condition (iii), we have

$$\begin{aligned}\|x_{n+1}-z^*\| &= \|[(1-\beta_n)\mathcal{I}+\beta_n T((1-\gamma_n)\mathcal{I}+\gamma_n T)]u_n - z^*\| \\ &\leq \|u_n - z^*\|. \end{aligned} \tag{3.5}$$

Since P_C is nonexpansive, we have

$$\begin{aligned}\|u_n-z^*\| &= \|P_C y_n - P_C z^*\| \\ &\leq \|y_n - z^*\|. \end{aligned} \tag{3.6}$$

From definition of $\{y_n\}$, we obtain

$$\begin{aligned}\|y_n-z^*\| &= \left\|\alpha_n\gamma f(x_n) + \left(\mathcal{I}-\alpha_n\sum_{i=1}^N a_iD_i\right)(x_n-\delta A^*(Ax_n-v_n)) - z^*\right\| \\ &= \|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(z^*) + \alpha_n\gamma f(z^*) - \alpha_n\sum_{i=1}^N a_iD_iz^* + x_n - \delta A^*(Ax_n-v_n) \\ &\quad - \alpha_n\sum_{i=1}^N a_iD_i(x_n-\delta A^*(Ax_n-v_n)) + \alpha_n\sum_{i=1}^N a_iD_iz^* - z^*\| \\ &= \|\alpha_n\gamma(f(x_n)-f(z^*)) + \alpha_n\left(\gamma f(z^*) - \sum_{i=1}^N a_iD_iz^*\right) \\ &\quad + \left(\mathcal{I}-\alpha_n\sum_{i=1}^N a_iD_i\right)(x_n - z^* - \delta A^*(Ax_n-v_n))\| \\ &\leq \alpha_n\gamma\|f(x_n)-f(z^*)\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^N a_iD_iz^*\right\| \\ &\quad + \left\|\mathcal{I}-\alpha_n\sum_{i=1}^N a_iD_i\right\|\|x_n-z^*+\delta A^*(v_n-Ax_n)\| \\ &\leq \alpha_n\gamma\rho\|x_n-z^*\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^N a_iD_iz^*\right\| \\ &\quad + (1-\alpha_n\bar{\gamma})\|x_n-z^*+\delta A^*(v_n-Ax_n)\|. \end{aligned} \tag{3.7}$$

Observe that

$$\begin{aligned}&\langle x_n-z^*, A^*(v_n-Ax_n)\rangle \\ =&\langle Ax_n-Az^*, v_n-Ax_n\rangle \\ =&\langle Ax_n-Az^*+v_n-Ax_n-(v_n-Ax_n), v_n-Ax_n\rangle \\ =&\langle Ax_n-Az^*+v_n-Ax_n, v_n-Ax_n\rangle - \langle v_n-Ax_n, v_n-Ax_n\rangle \\ =&\langle v_n-Az^*, v_n-Ax_n\rangle - \|v_n-Ax_n\|^2. \end{aligned} \tag{3.8}$$

and

$$\langle v_n - Az^*, v_n - Ax_n\rangle = \frac{1}{2}\left(\|v_n - Az^*\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Az^*\|^2\right). \quad (3.9)$$

From (3.4), (3.8) and (3.9), we obtain

$$\begin{aligned}
&\langle x_n - z^*, A^*(v_n - Ax_n)\rangle \\
=&\frac{1}{2}\left(\|v_n - Az^*\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Az^*\|^2\right) - \|v_n - Ax_n\|^2 \\
\leq&\frac{1}{2}\left(\|Ax_n - Az^*\|^2 - \|Ax_n - z_n\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Az^*\|^2\right) - \|v_n - Ax_n\|^2 \\
=&-\frac{1}{2}\|z_n - Ax_n\|^2 - \frac{1}{2}\|v_n - Ax_n\|^2. \quad (3.10)
\end{aligned}$$

From (3.10), we have

$$\begin{aligned}
&\|x_n - z^* + \delta A^*(v_n - Ax_n)\|^2 \\
=&\|x_n - z^*\|^2 + \delta^2\|A^*(v_n - Ax_n)\|^2 + 2\delta\langle x_n - z^*, A^*(v_n - Ax_n)\rangle \\
\leq&\|x_n - z^*\|^2 + \delta^2\|A^*\|^2\|v_n - Ax_n\|^2 + 2\delta\left(-\frac{1}{2}\|z_n - Ax_n\|^2 - \frac{1}{2}\|v_n - Ax_n\|^2\right) \\
=&\|x_n - z^*\|^2 + \delta^2\|A\|^2\|v_n - Ax_n\|^2 - \delta\|z_n - Ax_n\|^2 - \delta\|v_n - Ax_n\|^2 \\
=&\|x_n - z^*\|^2 + \delta\left(\delta\|A\|^2 - 1\right)\|v_n - Ax_n\|^2 - \delta\|z_n - Ax_n\|^2. \quad (3.11)
\end{aligned}$$

From (3.11) and condition (iv), we have

$$\|x_n - z^* + \delta A^*(v_n - Ax_n)\|^2 \leq \|x_n - z^*\|^2.$$

So,

$$\|x_n - z^* + \delta A^*(v_n - Ax_n)\| \leq \|x_n - z^*\|. \quad (3.12)$$

From (3.7) and (3.12), we get

$$\begin{aligned}
&\|y_n - z^*\| \\
\leq&\alpha_n\gamma\rho\|x_n - z^*\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| + (1 - \alpha_n\bar{\gamma})\|x_n - z^* + \delta A^*(v_n - Ax_n)\| \\
\leq&\alpha_n\gamma\rho\|x_n - z^*\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| + (1 - \alpha_n\bar{\gamma})\|x_n - z^*\| \\
=&[1 - \alpha_n(\bar{\gamma} - \gamma\rho)]\|x_n - z^*\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|. \quad (3.13)
\end{aligned}$$

By definition of $\{x_n\}$, (3.5), (3.6) and (3.13), we get

$$\begin{aligned}\|x_{n+1}-z^*\| &\leq [1-\alpha_n(\bar{\gamma}-\gamma\rho)]\,\|x_n-z^*\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| \\ &= [1-\alpha_n(\bar{\gamma}-\gamma\rho)]\,\|x_n-z^*\| + \alpha_n(\bar{\gamma}-\gamma\rho)\frac{\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|}{\bar{\gamma}-\gamma\rho}.\end{aligned}$$

By induction, we get

$$\|x_{n+1}-z^*\| \leq \max\left\{\|x_0-z^*\|, \frac{\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|}{\bar{\gamma}-\gamma\rho}\right\}.$$

Hence, the sequence $\{x_n\}$ is bounded.
Since P_C is firmly nonexpansive, we have

$$\begin{aligned}\|u_n-z^*\|^2 &= \|P_C y_n - z^*\|^2 \\ &= \|P_C y_n - P_C z^*\|^2 \\ &\leq \|y_n - z^*\|^2 - \|(\mathcal{I}-P_C)\,y_n - (\mathcal{I}-P_C)\,z^*\|^2 \\ &= \|y_n-z^*\|^2 - \|y_n - P_C y_n\|^2 \\ &= \|y_n-z^*\|^2 - \|u_n-y_n\|^2. \end{aligned} \tag{3.14}$$

From (3.5), (3.13) and (3.14), we have

$$\begin{aligned}&\|x_{n+1}-z^*\|^2 \\ &\leq \|u_n-z^*\|^2 \\ &\leq \|y_n-z^*\|^2 - \|u_n-y_n\|^2 \\ &= \left([1-\alpha_n(\bar{\gamma}-\gamma\rho)]\,\|x_n-z^*\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|\right)^2 - \|u_n-y_n\|^2 \\ &= (1-\alpha_n(\bar{\gamma}-\gamma\rho))^2\,\|x_n-z^*\|^2 \\ &\quad + 2\alpha_n\,[1-\alpha_n(\bar{\gamma}-\gamma\rho)]\,\|x_n-z^*\|\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| \\ &\quad + \alpha_n^2\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|^2 - \|u_n-y_n\|^2.\end{aligned}$$

That is,

$$\|u_n - y_n\|^2 \le \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + \alpha_n^2 \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|^2 + 2\alpha_n \left[1 - \alpha_n(\bar{\gamma} - \gamma\rho)\right] \|x_n - z^*\| \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|. \quad (3.15)$$

Next, we focus our analysis on the fact that the sequence $\{\|x_n - z^*\|\}$ is either monotone decreasing at infinity(Case 1) or not(Case 2).
Case1. There exists $n_0 \in \mathbb{N}$ such that the sequence $\{\|x_n - z^*\|\}_{n \ge n_0}$ is decreasing.
Case2. For any $\bar{n_0} \in \mathbb{N}$, there exists an integer $\bar{m} \ge \bar{n_0}$ such that

$$\|x_{\bar{m}} - z^*\| \le \|x_{\bar{m}+1} - z^*\|.$$

In *Case*1, we assume that there exists some integer $m > 0$ such that $\{\|x_n - z^*\|\}$ is decreasing for all $n \ge m$.
In this case, we get $\lim_{n\to\infty} \|x_n - z^*\|$ exists. From (3.15) and condition (i), we deduce

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \quad (3.16)$$

From (3.7) and condition (iv), we have

$$\begin{aligned}\|y_n - z^*\| \le& \alpha_n \gamma\rho \|x_n - z^*\| + \alpha_n \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| \\ &+ (1 - \alpha_n\bar{\gamma}) \|x_n - z^* + \delta A^* (v_n - Ax_n)\| \\ =& \alpha_n\bar{\gamma}\left(\frac{\gamma\rho \|x_n - z^*\| + \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|}{\bar{\gamma}}\right) \\ &+ (1 - \alpha_n\bar{\gamma}) \|x_n - z^* + \delta A^* (v_n - Ax_n)\|. \end{aligned} \quad (3.17)$$

Since $\{x_n\}$ is bounded, then there exists a constant $M > 0$ such that

$$\sup_n \left\{\frac{\gamma\rho \|x_n - z^*\| + \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|}{\bar{\gamma}}\right\} < M.$$

By using property of convex function of $\|\cdot\|^2$ and (3.17), we have

$$\|y_n - z^*\|^2 \le \alpha_n\bar{\gamma}M^2 + (1 - \alpha_n\bar{\gamma}) \|x_n - z^* + \delta A^* (v_n - Ax_n)\|^2. \quad (3.18)$$

From (3.5), (3.6), (3.11) and (3.18), thus

$$\begin{aligned}
&\|x_{n+1} - z^*\|^2 \\
\leq& \|u_n - z^*\|^2 \\
\leq& \|y_n - z^*\|^2 \\
\leq& \alpha_n \bar{\gamma} M^2 + (1 - \alpha_n \bar{\gamma}) \|x_n - z^* + \delta A^* (v_n - Ax_n)\|^2 \\
\leq& \alpha_n \bar{\gamma} M^2 + (1 - \alpha_n \bar{\gamma}) \left(\|x_n - z^*\|^2 + \delta \left(\delta \|A\|^2 - 1 \right) \|v_n - Ax_n\|^2 - \delta \|z_n - Ax_n\|^2 \right) \\
=& (1 - \alpha_n \bar{\gamma}) \|x_n - z^*\|^2 + (1 - \alpha_n \bar{\gamma}) \delta \left(\delta \|A\|^2 - 1 \right) \|v_n - Ax_n\|^2 \\
&- \delta (1 - \alpha_n \bar{\gamma}) \|z_n - Ax_n\|^2 + \alpha_n \bar{\gamma} M^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(1 - \alpha_n \bar{\gamma}) \delta \left(1 - \delta \|A\|^2 \right) \|v_n - Ax_n\|^2 + \delta (1 - \alpha_n \bar{\gamma}) \|z_n - Ax_n\|^2 \\
\leq& (1 - \alpha_n \bar{\gamma}) \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + \alpha_n \bar{\gamma} M^2 \\
\leq& \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + \alpha_n \bar{\gamma} M^2.
\end{aligned}$$

This implies that

$$\lim_{n \to \infty} \|v_n - Ax_n\| = \lim_{n \to \infty} \|z_n - Ax_n\| = 0. \tag{3.19}$$

Consider that

$$\begin{aligned}
\|v_n - z_n\| &= \|v_n - Ax_n + Ax_n - z_n\| \\
&\leq \|v_n - Ax_n\| + \|z_n - Ax_n\|.
\end{aligned}$$

Thus

$$\lim_{n \to \infty} \|v_n - z_n\| = 0. \tag{3.20}$$

Note that

$$\begin{aligned}
v_n - z_n &= (1 - \xi_n) z_n + \xi_n S ((1 - \eta_n) z_n + \eta_n S z_n) - z_n \\
&= \xi_n [S ((1 - \eta_n) \mathcal{I} + \eta_n S) z_n - z_n].
\end{aligned}$$

From (3.20), then

$$\lim_{n \to \infty} \|z_n - S ((1 - \eta_n) \mathcal{I} + \eta_n S) z_n\| = 0. \tag{3.21}$$

Consider that

$$\begin{aligned}
&\|S ((1 - \eta_n) \mathcal{I} + \eta_n S) z_n - S ((1 - \eta_n) \mathcal{I} + \eta_n S) Ax_n\| \\
\leq& \mathcal{L}_1 \|((1 - \eta_n) \mathcal{I} + \eta_n S) z_n - ((1 - \eta_n) \mathcal{I} + \eta_n S) Ax_n\| \\
=& \mathcal{L}_1 \|(1 - \eta_n) (z_n - Ax_n) + \eta_n (S z_n - S Ax_n)\| \\
\leq& \mathcal{L}_1 ((1 - \eta_n) \|z_n - Ax_n\| + \eta_n \|S z_n - S Ax_n\|) \\
\leq& \mathcal{L}_1 ((1 - \eta_n) \|z_n - Ax_n\| + \eta_n \mathcal{L}_1 \|z_n - Ax_n\|) \\
=& \mathcal{L}_1 (1 - \eta_n (1 - \mathcal{L}_1)) \|z_n - Ax_n\|.
\end{aligned} \tag{3.22}$$

From (3.22), thus

$$\begin{aligned}
&\left\|Ax_n - S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)Ax_n\right\| \\
&\leq \|Ax_n - z_n\| + \left\|z_n - S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)z_n\right\| \\
&\quad + \left\|S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)z_n - S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)Ax_n\right\| \\
&\leq \|Ax_n - z_n\| + \left\|z_n - S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)z_n\right\| + \mathcal{L}_1\left(1 - \eta_n(1-\mathcal{L}_1)\right)\|z_n - Ax_n\|.
\end{aligned} \tag{3.23}$$

From (3.19), (3.21) and (3.23), then we have

$$\lim_{n\to\infty} \left\|Ax_n - S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)Ax_n\right\| = 0. \tag{3.24}$$

Since

$$\begin{aligned}
&\|Ax_n - SAx_n\| \\
&= \left\|Ax_n - S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)Ax_n + S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)Ax_n - SAx_n\right\| \\
&\leq \left\|Ax_n - S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)Ax_n\right\| + \left\|S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)Ax_n - SAx_n\right\| \\
&\leq \left\|Ax_n - S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)Ax_n\right\| + \mathcal{L}_1\left\|\left((1-\eta_n)\mathcal{I} + \eta_n S\right)Ax_n - Ax_n\right\| \\
&= \left\|Ax_n - S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)Ax_n\right\| + \mathcal{L}_1\eta_n\|Ax_n - SAx_n\|.
\end{aligned}$$

It implies that

$$\|Ax_n - SAx_n\| \leq \frac{1}{1-\mathcal{L}_1\eta_n}\left\|Ax_n - S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)Ax_n\right\|.$$

By (3.24), we obtain

$$\lim_{n\to\infty}\|Ax_n - SAx_n\| = 0. \tag{3.25}$$

Consider that

$$\begin{aligned}
\|y_n - x_n\| &= \left\|\alpha_n\gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)(x_n - \delta A^*(Ax_n - v_n)) - x_n\right\| \\
&= \left\|\alpha_n\gamma f(x_n) - \delta A^*(Ax_n - v_n) - \alpha_n \sum_{i=1}^{N} a_i D_i x_n + \delta\alpha_n \sum_{i=1}^{N} a_i D_i A^*(Ax_n - v_n)\right\| \\
&= \left\|\alpha_n\left(\gamma f(x_n) - \sum_{i=1}^{N} a_i D_i x_n + \delta \sum_{i=1}^{N} a_i D_i A^*(Ax_n - v_n)\right) + \delta A^*(v_n - Ax_n)\right\| \\
&= \left\|\alpha_n\left(\gamma f(x_n) - \sum_{i=1}^{N} a_i D_i (x_n - \delta A^*(Ax_n - v_n))\right) + \delta A^*(v_n - Ax_n)\right\| \\
&\le \alpha_n \left\|\gamma f(x_n) - \sum_{i=1}^{N} a_i D_i (x_n - \delta A^*(Ax_n - v_n))\right\| + \delta \|A^*(v_n - Ax_n)\| \\
&\le \alpha_n \left\|\gamma f(x_n) - \sum_{i=1}^{N} a_i D_i (x_n - \delta A^*(Ax_n - v_n))\right\| + \delta \|A^*\| \|v_n - Ax_n\| \\
&= \alpha_n \left\|\gamma f(x_n) - \sum_{i=1}^{N} a_i D_i (x_n - \delta A^*(Ax_n - v_n))\right\| + \delta \|A\| \|v_n - Ax_n\|.
\end{aligned}$$

It follows from (3.19) and condition (i) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.26}$$

From definition of $\{x_n\}$, we have

$$\begin{aligned}
\|x_{n+1} - z^*\|^2 &= \|(1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n T u_n) - z^*\|^2 \\
&= \|(1-\beta_n)(u_n - z^*) + \beta_n [T((1-\gamma_n)u_n + \gamma_n T u_n) - z^*]\|^2 \\
&= (1-\beta_n)\|u_n - z^*\|^2 + \beta_n \|T((1-\gamma_n)u_n + \gamma_n T u_n) - z^*\|^2 \\
&\quad - \beta_n(1-\beta_n)\|T((1-\gamma_n)u_n + \gamma_n T u_n) - u_n\|^2. \qquad (3.27)
\end{aligned}$$

Applying proposition 2.13, we have

$$\begin{aligned}
&\|T((1-\gamma_n)u_n + \gamma_n T u_n) - z^*\|^2 \\
&\le \|u_n - z^*\|^2 + (1-\gamma_n)\|u_n - T((1-\gamma_n)u_n + \gamma_n T u_n)\|^2. \qquad (3.28)
\end{aligned}$$

From (3.6),(3.12), (3.18), (3.27) and (3.28), thus

$$\begin{aligned}
\|x_{n+1}-z^*\|^2 =&(1-\beta_n)\|u_n-z^*\|^2+\beta_n\|T((1-\gamma_n)u_n+\gamma_nTu_n)-z^*\|^2\\
&-\beta_n(1-\beta_n)\|T((1-\gamma_n)u_n+\gamma_nTu_n)-u_n\|^2\\
\le&(1-\beta_n)\|u_n-z^*\|^2+\beta_n(\|u_n-z^*\|^2\\
&+(1-\gamma_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2)\\
&-\beta_n(1-\beta_n)\|T((1-\gamma_n)u_n+\gamma_nTu_n)-u_n\|^2\\
=&\|u_n-z^*\|^2+\beta_n(1-\gamma_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2\\
&-\beta_n(1-\beta_n)\|T((1-\gamma_n)u_n+\gamma_nTu_n)-u_n\|^2\\
\le&\|y_n-z^*\|^2+\beta_n(1-\gamma_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2\\
&-\beta_n(1-\beta_n)\|T((1-\gamma_n)u_n+\gamma_nTu_n)-u_n\|^2\\
\le&\alpha_n\bar{\gamma}M^2+(1-\alpha_n\bar{\gamma})\|x_n-z^*+\delta A^*(v_n-Ax_n)\|^2\\
&+\beta_n(1-\gamma_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2\\
&-\beta_n(1-\beta_n)\|T((1-\gamma_n)u_n+\gamma_nTu_n)-u_n\|^2\\
=&\alpha_n\bar{\gamma}M^2+(1-\alpha_n\bar{\gamma})\|x_n-z^*+\delta A^*(v_n-Ax_n)\|^2\\
&-\beta_n(\gamma_n-\beta_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2\\
\le&\alpha_n\bar{\gamma}M^2+\|x_n-z^*\|^2\\
&-\beta_n(\gamma_n-\beta_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2.
\end{aligned}$$

It implies that

$$\beta_n(\gamma_n-\beta_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2\le\alpha_n\bar{\gamma}M^2+\|x_n-z^*\|^2-\|x_{n+1}-z^*\|^2.$$

By condition (i) and (iii), we get

$$\lim_{n\to\infty}\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|=0. \tag{3.29}$$

Observe that

$$\begin{aligned}
\|u_n-Tu_n\|\le&\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|+\|T((1-\gamma_n)u_n+\gamma_nTu_n)-Tu_n\|\\
\le&\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|+\mathcal{L}_2\|(1-\gamma_n)u_n+\gamma_nTu_n-u_n\|\\
=&\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|+\mathcal{L}_2\gamma_n\|u_n-Tu_n\|.
\end{aligned}$$

Thus,

$$\|u_n-Tu_n\|\le\frac{1}{1-\mathcal{L}_2\gamma_n}\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|.$$

This together with (3.29) implies that,

$$\lim_{n\to\infty}\|u_n-Tu_n\|=0. \tag{3.30}$$

Next, we will show that

$$\limsup_{n\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \leq 0,$$

where $z^* = P_{\mathbf{\Gamma}}(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i)z^*$.
Choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle = \lim_{i\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{n_i} - z^*\rangle. \quad (3.31)$$

Since the sequence $\{y_n\}$ is bounded, without loss of generality, we have a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup z$. Subsequently, we derive from above conclusion that

$$\begin{cases} x_{n_i} \rightharpoonup z, \\ y_{n_i} \rightharpoonup z, \\ u_{n_i} \rightharpoonup z \end{cases} \quad (3.32)$$

and

$$\begin{cases} Ax_{n_i} \rightharpoonup Az, \\ Ay_{n_i} \rightharpoonup Az, \\ Au_{n_i} \rightharpoonup Az. \end{cases} \quad (3.33)$$

Note that $u_{n_i} = P_C y_{n_i} \in C$ and (3.32), thus $z \in C$.
From demiclosedness of $(\mathcal{I} - T)$ and $(\mathcal{I} - T)u_{n_i} \to 0$, then $z \in F(T)$.
Therefore, $z \in C \cap F(T)$.
Note that $z_{n_i} = P_Q Ax_{n_i} \in Q$ and from (3.19) and (3.33), we have $z_{n_i} \rightharpoonup Az$.
Thus, $Az \in Q$.
From demiclosedness of $(\mathcal{I} - S)$ and $(\mathcal{I} - S)Ax_{n_i} \to 0$, then $Az \in F(S)$.
Therefore, $Az \in Q \cap F(S)$. That is $z \in \mathbf{\Gamma}$.
Consequently,

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle &= \lim_{i\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{n_i} - z^*\rangle \\ &= \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, z - z^*\rangle \\ &\leq 0. \end{aligned} \quad (3.34)$$

Consider that

$$\begin{aligned}
\|y_n - z^*\|^2 \leq & \left\|\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right\|^2 \|x_n - z^* - \delta A^*(Ax_n - v_n)\|^2 \\
& + 2\langle \alpha_n \gamma (f(x_n) - f(z^*)) + \alpha_n \left(\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right), y_n - z^*\rangle \\
= & \left\|\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right\|^2 \|x_n - z^* - \delta A^*(Ax_n - v_n)\|^2 \\
& + 2\alpha_n\gamma\langle f(x_n) - f(z^*), y_n - z^*\rangle + 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
\leq & \left\|\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right\|^2 \|x_n - z^*\|^2 + 2\alpha_n\gamma \|f(x_n) - f(z^*)\| \|y_n - z^*\| \\
& + 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
\leq & (1 - \alpha_n\bar{\gamma})^2 \|x_n - z^*\|^2 + 2\alpha_n\gamma\rho \|x_n - z^*\| \|y_n - z^*\| \\
& + 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
\leq & (1 - \alpha_n\bar{\gamma})^2 \|x_n - z^*\|^2 + \alpha_n\gamma\rho \left(\|x_n - z^*\|^2 + \|y_n - z^*\|^2\right) \\
& + 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
= & (1 - \alpha_n\bar{\gamma})^2 \|x_n - z^*\|^2 + \alpha_n\gamma\rho \|x_n - z^*\|^2 + \alpha_n\gamma\rho \|y_n - z^*\|^2 \\
& + 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle.
\end{aligned}$$

It follow that

$$\begin{aligned}
& (1 - \alpha_n\gamma\rho) \|y_n - z^*\|^2 \\
& \leq \left(1 - 2\alpha_n\bar{\gamma} + {\alpha_n}^2\bar{\gamma}^2 + \alpha_n\gamma\rho\right) \|x_n - z^*\|^2 + 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
& = (1 + \alpha_n\gamma\rho - 2\alpha_n\bar{\gamma}) \|x_n - z^*\|^2 + {\alpha_n}^2\bar{\gamma}^2 \|x_n - z^*\|^2 \\
& \quad + 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle
\end{aligned}$$

$$= (1 - \alpha_n\gamma\rho + 2\alpha_n\gamma\rho - 2\alpha_n\bar{\gamma})\,\|x_n - z^*\|^2 + \alpha_n{}^2\bar{\gamma}^2\,\|x_n - z^*\|^2 + 2\alpha_n\langle\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle$$

then,

$$\|y_n - z^*\|^2 \leq \left[1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_n}\right]\|x_n - z^*\|^2 + \frac{\bar{\gamma}^2\alpha_n{}^2}{1 - \gamma\rho\alpha_n}\,\|x_n - z^*\|^2 + \frac{2\alpha_n}{1 - \gamma\rho\alpha_n}\langle\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle.$$

Therefore,

$$\begin{aligned}
&\|x_{n+1} - z^*\|^2 \\
&\leq \|y_n - z^*\|^2 \\
&\leq \left[1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_n}\right]\|x_n - z^*\|^2 + \frac{\bar{\gamma}^2\alpha_n{}^2}{1 - \gamma\rho\alpha_n}\,\|x_n - z^*\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \gamma\rho\alpha_n}\langle\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
&= \left[1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_n}\right]\|x_n - z^*\|^2 \\
&\quad + \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_n}\left[\frac{\bar{\gamma}^2\alpha_n}{2(\bar{\gamma} - \gamma\rho)}\,\|x_n - z^*\|^2 + \frac{1}{\bar{\gamma} - \gamma\rho}\langle\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle\right].
\end{aligned} \tag{3.35}$$

Applying (3.34), (3.35) and Lemma 2.9, we obtain $x_n \to z^*$ as $n \to \infty$.
In *Case*2, we assume that there exists some integer $\bar{n_0}$ such that

$$\|x_{\bar{n_0}} - z^*\| \leq \|x_{\bar{n_0}+1} - z^*\|.$$

Setting $w_n = \|x_n - z^*\|$, then

$$w_{\bar{n_0}} \leq w_{\bar{n_0}+1}.$$

Define an integer sequence $\{\tau_n\}$ for all $n \geq n_0$ as follows:

$$\tau(n) = max\{l \in \mathbb{N} \mid n_0 \leq l \leq n, w_l \leq w_{l+1}\}.$$

It is clear that τ_n is a nondecreasing sequence satisfying

$$\lim_{n\to\infty} \tau(n) = \infty.$$

and

$$w_{\tau(n)} \leq w_{\tau(n)+1}$$

for all $n \geq n_0$.
By a similar argument of Case 1, that is

$$\lim_{n\to\infty} \|u_{\tau(n)} - y_{\tau(n)}\| = 0,$$
$$\lim_{n\to\infty} \|x_{\tau(n)} - y_{\tau(n)}\| = 0,$$
$$\lim_{n\to\infty} \|SAx_{\tau(n)} - Ax_{\tau(n)}\| = 0$$

and

$$\lim_{n\to\infty} \|u_{\tau(n)} - Tu_{\tau(n)}\| = 0.$$

This implies that $w_w(y_{\tau(n)}) \subset \mathbf{\Gamma}$.
We obtain

$$\limsup_{n\to\infty} \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{\tau_n} - z^* \rangle \leq 0. \tag{3.36}$$

From $w_{\tau(n)} \leq w_{\tau(n)+1}$ and (3.35), we have

$$\begin{aligned} w^2_{\tau(n)} \leq & w^2_{\tau(n)+1} \\ & \leq \left[1 - \frac{2\alpha_{\tau(n)}(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_{\tau(n)}}\right] w^2_{\tau(n)} + \frac{\bar{\gamma}^2\alpha^2_{\tau(n)}}{1 - \gamma\rho\alpha_{\tau(n)}} w^2_{\tau(n)} \\ & \quad + \frac{2\alpha_{\tau(n)}}{1 - \gamma\rho\alpha_{\tau(n)}} \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{\tau(n)} - z^* \rangle. \end{aligned} \tag{3.37}$$

It implies that

$$w^2_{\tau(n)} \leq \frac{2}{2(\bar{\gamma} - \gamma\rho) - \bar{\gamma}^2\alpha_{\tau(n)}} \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{\tau(n)} - z^* \rangle. \tag{3.38}$$

Combining (3.36) and (3.38), we have

$$\limsup_{n\to\infty} w_{\tau(n)} \leq 0,$$

and hence

$$\lim_{n\to\infty} w_{\tau(n)} = 0, \tag{3.39}$$

From (3.39), implies that

$$\lim_{n\to\infty} w_{\tau(n)+1} = 0.$$

Applying Lemma 2.10 ,we have

$$max\{w_{\tau(n)}, w_n\} \leq w_{\tau(n)+1}.$$

It implies that

$$w_n \leq w_{\tau(n)+1}. \tag{3.40}$$

Since w_n is nondecreasing sequence and $n \leq \tau(n)$,

$$w_n \leq w_{\tau(n)}. \tag{3.41}$$

From (3.40) and (3.41), we obtain

$$0 \leq w_n \leq max\{w_{\tau(n)}, w_{\tau(n)+1}\}.$$

Therefore, $w_n \to 0$. That is, $x_n \to z^*$. This complete the proof. □

By using our main result, we obtain the following results in Hilbert spaces.

Corollary 3.2. *Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, D is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\ldots,N} \gamma_i$, $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is an $\mathcal{L}_1$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_1 > 1$, $T : C \to C$ is an $\mathcal{L}_2$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_2 > 1$. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$*

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n)\, z_n + \xi_n S\left((1-\eta_n)\, z_n + \eta_n S z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + (\mathcal{I} - \alpha_n D)\left(x_n - \delta A^*\left(A x_n - v_n\right)\right), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n T\left((1-\gamma_n) u_n + \gamma_n T u_n\right), \quad for\ n \geq 1, \end{cases} \tag{3.42}$$

The parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$, δ and γ are two positive constants.
We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.3), that is,
$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}.$
Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \frac{1}{\sqrt{1+\mathcal{L}_1^2}+1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \frac{1}{\sqrt{1+\mathcal{L}_2^2}+1}$,

(iv) $0 < \delta, \gamma < \frac{1}{\|A\|^2}$ *and* $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D\|^{-1}$.

Then the sequence $\{x_n\}$ *converge strongly to the unique fixed point of the contraction mapping* $z = P_{\mathbf{\Gamma}}(\gamma f + \mathcal{I} - D)\, z$.

Proof. Putting $D = D_1 = D_2 = D_3 = ... = D_N$ in Theorem 3.1, we get the desired conclusions. □

Corollary 3.3. *Let* H_1 *and* H_2 *are two real Hilbert space, let* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *are two nonempty closed convex sets. Let* $A : H_1 \to H_2$ *is a bounded linear operator with its adjoint* A^*, D_i *is strongly positive bounded linear operator on* H_1 *with coefficient* $\gamma_i > 0$ *and* $\bar{\gamma} = \min_{i=1,2,\ldots,N} \gamma_i$, $f : C \to H_1$ *is a* ρ*-contraction,* $S : Q \to Q$ *is an* $\mathcal{L}$*-Lipschitzian quasi-pseudo-contractive operator with* $\mathcal{L} > 1$. *Assume that* $\mathbf{\Gamma} \neq \emptyset$ *and let* $\{x_n\}$ *be sequences generated by* $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1 - \xi_n)\, z_n + \xi_n S\left((1 - \eta_n)\, z_n + \eta_n S z_n\right), \\ x_{n+1} = P_C\left[\alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^N a_i D_i\right)(x_n - \delta A^*(A x_n - v_n))\right], for\, n \geq 1 \end{cases} \tag{3.43}$$

The parameters $\{\alpha_n\}, \{\xi_n\}$ *and* $\{\eta_n\}$ *are real sequences in* $[0, 1]$, δ *and* γ *are two positive constants.*
We use $\mathbf{\Gamma}$ *to denote the set of solution of problem* (1.3), *that is,*
$\mathbf{\Gamma} = \{x \mid x \in C, Ax \in Q \cap F(S)\}$.
Suppose that $S - \mathcal{I}$ *is demiclosed at* 0. *Assume that the following conditions are satisfied :*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1 + \mathcal{L}_1^2} + 1}$,

(iii) $0 < \delta < \dfrac{1}{\|A\|^2}$ *and* $\bar{\gamma} > \gamma\rho$,

(iv) $0 < \gamma < \dfrac{1}{\|A\|^2}$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ *for* $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ *converge strongly to the unique fixed point of the contraction mapping* $z = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^N a_i D_i\right)$.

Proof. Putting $T \equiv \mathcal{I}$ in Theorem 3.1, we get the desired conclusions. □

4 Application

Lemma 4.1. [10] *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C. If S is a κ-strict pseudo-contractive mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \le \frac{1+\kappa}{1-\kappa}\|x-y\|, \quad \forall x, y \in C.$$

By Lemma 4.1, applying T, S are κ,$\bar{\kappa}$-strict pseudo-contractive mappings, we obtain this theorem.

Theorem 4.2. *Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\ldots,N} \gamma_i$, $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is a $\bar{\kappa}$-strict pseudo-contractive mapping, $T : C \to C$ is a κ-strict pseudo-contractive mapping. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$*

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n S\left((1-\eta_n) z_n + \eta_n S z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)(x_n - \delta A^*(Ax_n - v_n)), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n T\left((1-\gamma_n) u_n + \gamma_n T u_n\right), \quad \text{for } n \ge 1, \end{cases} \tag{4.1}$$

The parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$, δ and γ are two positive constants.
We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.3), that is,
$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}$.
Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1 + \left(\frac{1+\bar{\kappa}}{1-\bar{\kappa}}\right)^2} + 1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \dfrac{1}{\sqrt{1 + \left(\frac{1+\kappa}{1-\kappa}\right)^2} + 1}$,

(iv) $0 < \delta, \gamma < \dfrac{1}{\|A\|^2}$ *and* $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ *for* $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ *converge strongly to the unique fixed point of the contraction mapping* $z = P_{\Gamma}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right)$.

Proof. By using Theorem 3.1 and Lemma 4.1, we obtain the conclusion. □

In 2009, Kangtunyakarn and Suantai([11]) introduced the S-mapping generated by a finite family of κ-strictly pseudo contractive mappings and real numbers as follows:

Definition 4.3. Let C be a nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ_i-strict pseudo contractions of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$
\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\ \ \vdots\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}
$$

This mapping is called S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 4.4. [11] *Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ-strict pseudo contractions of C into C with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $\kappa = max\{\kappa_i : i = 1, 2, ..., N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in [\kappa, 1)$ $\alpha_2^j \in [\kappa, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^{N} F(T_i)$ and S is a nonexpansive mapping.*

Theorem 4.5. *Let C and Q are nonempty closed convex subset of real Hilbert spaces. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ_i-strict pseudo contractions of C into C with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $\kappa = max\{\kappa_i : i = 1, 2, ..., N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in [\kappa, 1)$ $\alpha_2^j \in [\kappa, 1)$ for all $j = 1, 2, ..., N$. Let S be the S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{\bar{T}_i\}_{i=1}^{N}$ be a finite family of $\bar{\kappa}_i$-strict pseudo contractions of Q into Q with*

$\bigcap_{i=1}^{N} F(\bar{T}_i) \neq \emptyset$ and $\bar{\kappa} = max\{\bar{\kappa}_i : i = 1, 2, ..., N\}$ and let $\beta_j = (\beta_1^j, \beta_2^j, \beta_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\beta_1^j + \beta_2^j + \beta_3^j = 1$, β_1^j, $\beta_3^j \in (\bar{\kappa}, 1)$ for all $j = 1, 2, ..., N-1$ and $\beta_1^N \in (\bar{\kappa}, 1], \beta_3^N \in [\bar{\kappa}, 1)$ $\beta_2^j \in [\bar{\kappa}, 1)$ for all $j = 1, 2, ..., N$. Let $\bar{S}$ be the S-mapping generated by $\bar{T}_1, \bar{T}_2, ..., \bar{T}_N$ and $\beta_1, \beta_2, ..., \beta_N$. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,...,N} \gamma_i$, $f : C \to H_1$ is a ρ-contraction. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n \bar{S}\left((1-\eta_n) z_n + \eta_n \bar{S} z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)(x_n - \delta A^*(Ax_n - v_n)), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n S\left((1-\gamma_n) u_n + \gamma_n S u_n\right), \quad \text{for } n \geq 1, \end{cases} \tag{4.2}$$

The parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0, 1]$, δ and γ are two positive constants.
We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.3), that is,

$$\mathbf{\Gamma} = \{x \mid x \in C \cap \bigcap_{i=1}^{N} F(T_i), Ax \in Q \cap \bigcap_{i=1}^{N} F(\bar{T}_i)\}.$$

Suppose that $S - \mathcal{I}$ and $\bar{S} - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \frac{1}{\sqrt{2}+1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \frac{1}{\sqrt{2}+1}$,

(iv) $0 < \delta, \gamma < \frac{1}{\|A\|^2}$ *and* $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ *for* $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right)$.

Proof. By using Theorem 3.1 and Lemma 4.4, we obtain the conclusion. □

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Strong Convergence Theorem for the Split Equality Fixed Point Problem for Quasi-nonexpansive Mapping and Application

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Abstract—Motivated by the work of Zhao [9], [10], [11] and by reducing some of his conditions, we consider a split equality fixed point problem for quasi-nonexpansive mappings which includes split feasibility problem, split equality problem, split fixed point problem, etc. The strong convergence theorem of the proposed iterative scheme could be obtained, under some control conditions. Furthermore, we use S-mapping applied to our main result to prove strong convergence theorems.

Index Terms—Split equality fixed point problem, Split equality problem, Split feasibility problem, Fixed Point problem, Quasi-nonexpansive mapping.

I. Introduction

Let C and Q be the non-empty closed convex subsets of the Hilbert spaces H_1 and H_2 respectively, and $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) is formulated as finding a point x^* with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{1}$$

The SFP in finite-dimensional spaces was firstly introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and medical image reconstruction [6]. The SFP has drawn attention from many researchers due to its applications in many branches of engineering and medical sciences. Many iterative algorithms have been suggested, ([7], [8], [12], [16], etc).

Assuming that SFP (1) is consistent (that is, (1) has a solution), it is easy to see that $x^* \in C$ is a solution of (1) if and only if it solves the following fixed point equation

$$x^* = P_C(I - \gamma A^*(I - P_Q))x^*, \tag{2}$$

where P_C and P_Q are the metric projections from H_1 onto C and from H_2 onto Q respectively, γ is a positive constant and A^* denotes by adjoint of A.

The popular algorithm used in approximating the solution of the SFP (1) is the CQ-algorithm, which was firstly proposed by Byrne [6]:

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q))x_n, \tag{3}$$

for all $n \in \mathbf{N}$, where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.

Recently, Moudafi [8] introduced the following split equality feasibility problem (SEFP) to find x^* and y^* with the property

$$x^* \in C, y^* \in Q \quad \text{s.t.} \quad Ax^* = By^*, \tag{4}$$

where H_1, H_2 and H_3 be real Hilbert spaces. $C \subset H_1$, $Q \subset H_2$ be two non-empty closed convex sets, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators.
It is easy to see that the problem (4) could be reduced to the problem (1) where $H_3 \equiv H_2$ and $B \equiv I$ (I be the identity mappings on $H_2 \to H_2$).

In order to solve SEFP (4), Moudafi [8] introduced the following simultaneous iterative method:

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma A^*(Ax_n - By_n)), \\ y_{n+1} = P_C(y_n + \beta B^*(Ax_{n+1} - By_n)), \quad \forall n \geq 0. \end{cases}$$

Under suitable conditions, he proved the weak convergence of sequence $\{(x_n, y_n)\}$ to a solution of (4) in Hilbert spaces.

Zhao [9] introduced the following algorithm for solving problem (4):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \beta_n u_n + (1-\beta_n)Su_n, \\ w_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \beta_n w_n + (1-\beta_n)Tw_n, \quad \forall n \geq 0, \end{cases}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive mappings, A^* and B^* are the adjoints of A and B respectively, $\{\gamma_n\} \in (\varepsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \varepsilon)$ (for ε small enough). Under some conditions, the authors obtained the sequence $\{(x_n, y_n)\}$ converged weakly to (x^*, y^*) in (4).

Dong and He [10] introduced following projection algorithm for SEFP (4) where the stepsizes do not depend on the operator norms $\|A\|$ and $\|B\|$:

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = P_C u_n, \\ w_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = P_Q w_n, \quad \forall n \geq 0. \end{cases}$$

Subsequently, Moudafi [7] introduced the following split equality fixed point problem (SEFPP); let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be non-linear operators such that

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$F(U) \neq \emptyset$ and $F(T) \neq \emptyset$, where $F(U)$ and $F(T)$ denote the sets of fixed point of U and T respectively. In (4), if $C := F(U)$ and $Q := F(T)$, then SEFP (4) could be reduced to the SEFPP, to find x^* and y^* with the property

$$x^* \in F(U), y^* \in F(T) \quad \text{s.t.} \quad Ax^* = By^*, \tag{5}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, which allows asymmetric and partial relations between x and y. This can further be used to cover many situations, such as decomposition methods for PDEs, applications in the game theory, in intensity-modulated radiation therapy(see [17]).

Very recently, Che and Li [11] proposed the following iterative algorithm for finding a solution of SEFPP (5):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Tu_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \beta_n y_n + (1-\beta_n) Sw_n, \quad \forall n \geq 0, \end{cases} \tag{6}$$

and under suitable conditions, they also established the weak convergence of the scheme (6).

In this work, we established the following iterative algorithm to solve the split equality fixed point problem (SEFPP),

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1}\left(I - \lambda_n^1 (I - T_1)\right) u_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2}\left(I - \lambda_n^2 (I - T_2)\right) v_n, \end{cases}$$

where $T_1 : C_1 \to C_1$, $T_2 : C_2 \to C_2$ are two quasi-nonexpansive mappings. Under suitable conditions, we proved strong convergence theorems of the iterative scheme (10) to a solution of the split equality fixed point problem (5) in the real Hilbert spaces.

II. Preliminaries

Throughout this paper, we always assume that H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let C be a non-empty closed convex subset of H. Recall that a mapping T of C into itself is called quasi-nonexpansive if

$$\|Tx - y^*\| \leq \|x - y^*\|,$$

for all $x \in C$ and $y^* \in F(T)$. The set of all elements of fixed point of a mapping T is denoted by $F(T) = \{x \in C : Tx = x\}$. Goebel and Kirk [5] showed that $F(T)$ is closed and convex. For $\lambda \in [0, 1]$,

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

and

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$$

for all $x, y \in H$. Let P_C be the metric projection of H onto C i.e.,for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Remark 2.1: It is well-known that metric projection P_C has the following properties:

1) P_C is firmly nonexpansive, i.e.,
$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \ \forall x, y \in H.$$
2) For each $x \in H$,
$$z = P_C(x) \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.2: [4] Let H be a real Hilbert space. Then

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

Lemma 2.3: [2] Let $\{\mathcal{Q}_n\} \subset [0, +\infty], \{v_n\} \subset [0, 1]$ and $\{\eta_n\}$ be three real number sequences. Suppose that $\{\mathcal{Q}_n\}, \{v_n\}$ and $\{\eta_n\}$ satisfy the following three conditions:

(i) $\mathcal{Q}_{n+1} \leq (1 - v_n)\mathcal{Q}_n + \eta_n v_n$,

(ii) $\sum_{n=1}^{\infty} v_n = \infty$,

(iii) $\limsup_{n\to\infty} \eta_n \leq 0$ or $\sum_{n=1}^{\infty} |\eta_n v_n| < \infty$.

Then, $\lim_{n\to\infty} \mathcal{Q}_n = 0$.

Lemma 2.4: [4] Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.5: Let C be a non-empty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then $VI(C, I-T) = F(T)$.

Proof: It is easy to see that $F(T) \subseteq VI(C, I-T)$. Let $u \in VI(C, I-T)$, then we have

$$\langle v - u, (I-T)u\rangle \geq 0, \ \forall v \in C. \tag{7}$$

Let $v^* \in F(T)$, then we have

$$\|Tu - v^*\|^2 \leq \|u - v^*\|^2. \tag{8}$$

On the other hand

$$\begin{aligned} &\|Tu - v^*\|^2 \\ &= \|(u - v^*) - (I-T)u\|^2 \\ &= \|u - v^*\|^2 - 2\langle u - v^*, (I-T)u\rangle + \|(I-T)u\|^2. \end{aligned} \tag{9}$$

From (8) and (9), we have

$$\|u - v^*\|^2 - 2\langle u - v^*, (I-T)u\rangle + \|(I-T)u\|^2 \leq \|u - v^*\|^2.$$

From (7), we have

$$\|(I-T)u\|^2 \leq 2\langle u - v^*, (I-T)u\rangle.$$

It follows that $v^* \in F(T)$. Hence $VI(C, I-T) \subseteq F(T)$. ■

Remark 2.6: From Lemma 2.4 and 2.5, we have

$$F(T) = VI(C, I-T) = F(P_C(I - \lambda(I-T))),$$

for all $\lambda > 0$.

Lemma 2.7: [3] Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbf{N}$. Then there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbf{N}$ such that $\tau(n) \to \infty$ and

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the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbf{N}$;

$$t_{\tau(n)} \leq t_{\tau(n)+1} \quad , t_n \leq t_{\tau(n)+1}.$$

In fact

$$\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}.$$

III. Main result

Theorem 3.1: For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $T_i : C_i \to C_i$ be quasi-nonexpansive mapping for all $i = 1, 2$ and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 \mid x \in F(T_1), y \in F(T_2) \text{ and } Ax = By\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases} u_n = x_n - \gamma_n A^* \left(Ax_n - By_n\right), \\ x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1}\left(I - \lambda_n^1 (I - T_1)\right) u_n, \\ v_n = y_n + \gamma_n B^* \left(Ax_n - By_n\right), \\ y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2}\left(I - \lambda_n^2 (I - T_2)\right) v_n, \end{cases} \tag{10}$$

for all $n \geq 1$,where $\{\alpha_n\} \subset [0, 1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0, 1)$ for all $i = 1, 2$ and $\gamma_n \in (a, b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon\right)$ for all $n \in \mathbf{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively, ϵ is a small enough. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $x^* = P_{F(T_1)}u$ and $y^* = P_{F(T_2)}v$.

Proof: Let $(x^*, y^*) \in \Omega$, then $x^* \in F(T_1)$, $y^* \in F(T_2)$ and $Ax^* = By^*$. From Lemma 2.5, we have

$$\|A^1 x\|^2 \leq 2 < x - x^*, A^1 x >, \tag{11}$$

where $A^1 = I - T_1$ and for all $x \in C_1$. Similarly, we have

$$\|A^2 y\|^2 \leq 2 < y - y^*, A^2 y >, \tag{12}$$

where $A^2 = I - T_2$ and for all $y \in C_2$.
Since $x^* \in F(T_1), y^* \in F(T_2)$ and $Ax^* = By^*$.
By Remark 2.6, we have $x^* \in F\left(P_{C_1}\left(I - \lambda_n^1 A^1\right)\right)$ and $y^* \in F\left(P_{C_2}\left(I - \lambda_n^2 A^2\right)\right)$.
Since P_{C_1} is a nonexpansive mapping, we have

$$\begin{aligned} &\|P_{C_1}\left(I - \lambda_n^1 A^1\right) x - x^*\|^2 \\ =&\|P_{C_1}\left(I - \lambda_n^1 A^1\right) x - P_{C_1}\left(I - \lambda_n^1 A^1\right) x^*\|^2 \\ \leq&\|x - x^* - \lambda_n^1 (A^1 x - A^1 x^*)\|^2 \\ \leq&\|x - x^* - \lambda_n^1 A^1 x\|^2 \\ =&\|x - x^*\|^2 + (\lambda_n^1)^2 \|A^1 x\|^2 - 2\lambda_n^1 \langle x - x^*, A^1 x\rangle \\ \leq&\|x - x^*\|^2 + (\lambda_n^1)^2 \|A^1 x\|^2 - \lambda_n^1 \|A^1 x\|^2 \\ =&\|x - x^*\|^2 - (\lambda_n^1)(1 - \lambda_n^1)\|A^1 x\|^2 \\ \leq&\|x - x^*\|^2, \end{aligned}$$

for all $x \in C_1$. Similarly, we obtain

$$\|P_{C_2}\left(I - \lambda_n^2 A^2\right) y - y^*\|^2 \leq \|y - y^*\|^2,$$

for all $y \in C_2$.
From definition of $\{u_n\}$, we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|x_n - x^* - \gamma_n A^* \left(Ax_n - By_n\right)\|^2 \\ &= \|x_n - x^*\|^2 + \gamma_n^2 \|A^* \left(Ax_n - By_n\right)\|^2 \\ &\quad - 2\gamma_n \langle x_n - x^*, A^* \left(Ax_n - By_n\right)\rangle. \end{aligned} \tag{13}$$

Consider that

$$\begin{aligned} &\|A^* \left(Ax_n - By_n\right)\|^2 \\ =&\langle A^* \left(Ax_n - By_n\right), A^* \left(Ax_n - By_n\right)\rangle \\ =&\langle Ax_n - By_n, AA^* \left(Ax_n - By_n\right)\rangle \\ \leq&\lambda_A \|Ax_n - By_n\|^2 \end{aligned} \tag{14}$$

and

$$\begin{aligned} &- 2\langle x_n - x^*, A^* \left(Ax_n - By_n\right)\rangle \\ =&- 2\langle Ax_n - Ax^*, Ax_n - By_n\rangle \\ =&- \|Ax_n - Ax^*\|^2 - \|Ax_n - By_n\|^2 + \|Ax^* - By_n\|^2. \end{aligned} \tag{15}$$

Substitute (14) and (15) into (13), we have

$$\begin{aligned} &\|u_n - x^*\|^2 \\ \leq&\|x_n - x^*\|^2 + \gamma_n^2 \lambda_A \|Ax_n - By_n\|^2 - \gamma_n \|Ax_n - Ax^*\|^2 \\ &- \gamma_n \|Ax_n - By_n\|^2 + \gamma_n \|Ax^* - By_n\|^2 \\ =&\|x_n - x^*\|^2 - \gamma_n (1 - \lambda_A \gamma_n)\|Ax_n - By_n\|^2 \\ &- \gamma_n \|Ax_n - Ax^*\|^2 + \gamma_n \|Ax^* - By_n\|^2. \end{aligned} \tag{16}$$

By using the same method as (16), we have

$$\begin{aligned} \|v_n - y^*\|^2 \leq&\|y_n - y^*\|^2 - \gamma_n (1 - \lambda_B \gamma_n)\|Ax_n - By_n\|^2 \\ &- \gamma_n \|By_n - By^*\|^2 + \gamma_n \|By^* - Ax_n\|^2. \end{aligned} \tag{17}$$

From (16) and (17), we have

$$\begin{aligned} &\|u_n - x^*\|^2 + \|v_n - y^*\|^2 \\ \leq&\|x_n - x^*\|^2 + \|y_n - y^*\|^2 - \gamma_n (1 - \lambda_A \gamma_n)\|Ax_n - By_n\|^2 \\ &- \gamma_n (1 - \lambda_B \gamma_n)\|Ax_n - By_n\|^2 \\ &- \gamma_n \|Ax_n - Ax^*\|^2 + \gamma_n \|Ax^* - By_n\|^2 \\ &- \gamma_n \|By_n - By^*\|^2 + \gamma_n \|By^* - Ax_n\|^2 \\ =&\|x_n - x^*\|^2 + \|y_n - y^*\|^2 \\ &- \gamma_n \left(2 - \gamma_n \left(\lambda_A + \lambda_B\right)\right) \|Ax_n - By_n\|^2. \end{aligned} \tag{18}$$

From the definition of $\{x_n\}$, we have

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ =&\|\alpha_n u + (1-\alpha_n) P_{C_1}\left(I - \lambda_n^1 (I - T_1)\right) u_n - x^*\|^2 \\ \leq&\alpha_n \|u - x^*\|^2 + (1-\alpha_n)\|P_{C_1}\left(I - \lambda_n^1 (I - T_1)\right) u_n - x^*\|^2 \\ \leq&\alpha_n \|u - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2. \end{aligned} \tag{19}$$

By using the same method as (19), we have

$$\|y_{n+1} - y^*\|^2 \leq \alpha_n \|v - y^*\|^2 + (1-\alpha_n)\|v_n - y^*\|^2. \tag{20}$$

From (18), (19) and (20), we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\ \le&\alpha_n\|u-x^*\|^2+(1-\alpha_n)\|u_n-x^*\|^2\\ &+\alpha_n\|v-y^*\|^2+(1-\alpha_n)\|v_n-y^*\|^2\\ =&\alpha_n(\|u-x^*\|^2+\|v-y^*\|^2)\\ &+(1-\alpha_n)(\|u_n-x^*\|^2+\|v_n-y^*\|^2)\\ \le&\alpha_n(\|u-x^*\|^2+\|v-y^*\|^2)\\ &+(1-\alpha_n)(\|x_n-x^*\|^2+\|y_n-y^*\|^2\\ &-\gamma_n\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2) \qquad (21)\\ \le&\alpha_n(\|u-x^*\|^2+\|v-y^*\|^2)\\ &+(1-\alpha_n)(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\ \le&\max\{\|u-x^*\|^2+\|v-y^*\|^2,\|x_1-x^*\|^2+\|y_1-y^*\|^2\}.\end{aligned}$$

From mathematical induction, we have $\{x_n\}$ and $\{y_n\}$ are bounded. Furthermore, $\{u_n\}$ and $\{v_n\}$ are bounded. From (21), we have

$$\begin{aligned}&\gamma_n(1-\alpha_n)\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2\\ \le&\alpha_n(\|u-x^*\|^2+\|v-y^*\|^2)+C_n-C_{n+1}, \qquad (22)\end{aligned}$$

where $C_n=\|x_n-x^*\|^2+\|y_n-y^*\|^2$, for all $x^*\in F(T_1)$, $y^*\in F(T_2)$ and $n\in\mathbf{N}$.
From (22), we separate the proof into two cases.
<u>*Case1.*</u> Suppose that $C_{n+1}\le C_n$ for all $n\ge n_0$ (for n_0 large enough). Since the sequence $\{C_n\}$ is bounded, we get $\lim_{n\to\infty}C_n=c$, for some $c\in\mathbf{R}$.
From (22) and properties of γ_n and α_n, we obtain

$$\lim_{n\to\infty}\|Ax_n-By_n\|=0. \qquad (23)$$

From the definition of $\{u_n\}$ and $\{v_n\}$, we have

$$\|u_n-x_n\|=\gamma_n\|A^*\left(Ax_n-By_n\right)\| \qquad (24)$$

and

$$\|v_n-y_n\|=\gamma_n\|B^*\left(Ax_n-By_n\right)\|. \qquad (25)$$

From (23), (24) and (25), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|v_n-y_n\|=0. \qquad (26)$$

By using properties of P_{C_1}, we have

$$\begin{aligned}&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\ \le&\|\left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\|^2\\ =&\|u_n-x^*-\lambda_n^1(I-T_1)(u_n-x^*)\|^2\\ =&\|u_n-x^*\|^2-2\lambda_n^1\langle u_n-x^*,(I-T_1)u_n\rangle\\ &+(\lambda_n^1)^2\|(I-T_1)u_n\|^2\\ \le&\|u_n-x^*\|^2-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2. \qquad (27)\end{aligned}$$

By using the same method as (27), we have

$$\begin{aligned}&\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\ \le&\|v_n-y^*\|^2-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2. \qquad (28)\end{aligned}$$

From (18), (27) and (28), we have

$$\begin{aligned}&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\ &+\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\ \le&\|u_n-x^*\|^2+\|v_n-y^*\|^2\\ &-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2\\ &-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2\\ \le&\|x_n-x^*\|^2+\|y_n-y^*\|^2\\ &-\gamma_n\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2\\ &-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2\\ &-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2. \qquad (29)\end{aligned}$$

From the definition of $\{x_n\}$ and $\{y_n\}$, we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\ \le&\ \alpha_n\|u-x^*\|^2\\ &+(1-\alpha_n)\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\ &+(1-\alpha_n)\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\ &+\alpha_n\|v-y^*\|^2\\ =&\ \alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)\\ &+(1-\alpha_n)\big(\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\ &+\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\big)\\ \le&\ \alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)\\ &+(1-\alpha_n)\big(\|x_n-x^*\|^2+\|y_n-y^*\|^2\\ &-\gamma_n\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2\\ &-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2\\ &-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2\big). \qquad (30)\end{aligned}$$

It implies that

$$\begin{aligned}&(1-\alpha_n)\big(\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2\\ &+\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2\big)\\ \le&C_n-C_{n+1}+\alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right). \qquad (31)\end{aligned}$$

From (31) and $\lim_{n\to\infty}C_n=c$, we have

$$\lim_{n\to\infty}\|(I-T_1)u_n\|=\lim_{n\to\infty}\|(I-T_2)v_n\|=0. \qquad (32)$$

By using properties of P_{C_1}, we have

$$\begin{aligned}
&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\le \langle\left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\\
&\quad ,P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\rangle\\
&=\frac{1}{2}\big(\|\left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\|^2\\
&\quad +\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad -\|\left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\\
&\quad -P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n+x^*\|^2\big)\\
&\le\frac{1}{2}\big(\|u_n-x^*\|^2+\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad -\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\\
&\quad -\lambda_n^1\left((I-T_1)u_n-(I-T_1)x^*\right)\|^2\big)\\
&=\frac{1}{2}\big(\|u_n-x^*\|^2+\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad -\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad -\left(\lambda_n^1\right)^2\|(I-T_1)u_n-(I-T_1)x^*\|^2\\
&\quad +2\lambda_n^1\langle u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\\
&\quad ,(I-T_1)u_n-(I-T_1)x^*\rangle\big).
\end{aligned}\tag{33}$$

From (33), we have

$$\begin{aligned}
&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\le\|u_n-x^*\|^2-\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad -\left(\lambda_n^1\right)^2\|(I-T_1)u_n-(I-T_1)x^*\|^2\\
&\quad +2\lambda_n^1\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|\\
&\quad \cdot\|(I-T_1)u_n-(I-T_1)x^*\|.
\end{aligned}\tag{34}$$

Similarly, we have

$$\begin{aligned}
&\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\
&\le\|v_n-y^*\|^2-\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|^2\\
&\quad -\left(\lambda_n^2\right)^2\|(I-T_2)v_n-(I-T_2)y^*\|^2\\
&\quad +2\lambda_n^2\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|\\
&\quad \cdot\|(I-T_2)v_n-(I-T_2)y^*\|.
\end{aligned}\tag{35}$$

From the definition of $\{x_n\}$, $\{y_n\}$, (34) and (35), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\
&\le\alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)\\
&\quad +(1-\alpha_n)\big(\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad +\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\big)\\
&\le\alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)+\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&\quad -(1-\alpha_n)\big(\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad +\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|^2\big)\\
&\quad +2\lambda_n^1\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|\\
&\quad \cdot\|(I-T_1)u_n-(I-T_1)x^*\|\\
&\quad +2\lambda_n^2\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|\\
&\quad \cdot\|(I-T_2)v_n-(I-T_2)y^*\|.
\end{aligned}$$

It implies that

$$\begin{aligned}
&(1-\alpha_n)\big(\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad +\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|^2\big)\\
&\le\alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)\\
&\quad +2\lambda_n^1\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|\\
&\quad \cdot\|(I-T_1)u_n-(I-T_1)x^*\|\\
&\quad +2\lambda_n^2\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|\\
&\quad \cdot\|(I-T_2)v_n-(I-T_2)y^*\|\\
&\quad +C_n-C_{n+1}.
\end{aligned}$$

From (32) and $\lim_{n\to\infty}C_n=c$, we have

$$\lim_{n\to\infty}\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-u_n\| \tag{36}$$

$$=\lim_{n\to\infty}\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-v_n\|=0. \tag{37}$$

From (26) and (36), we obtain

$$\lim_{n\to\infty}\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x_n\| \tag{38}$$

$$=\lim_{n\to\infty}\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y_n\|=0. \tag{39}$$

Since

$$\begin{aligned}
&x_{n+1}-x_n\\
&=\alpha_n(u-x_n)+(1-\alpha_n)\left(P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x_n\right),
\end{aligned}$$

$$\begin{aligned}
&y_{n+1}-y_n\\
&=\alpha_n(v-y_n)+(1-\alpha_n)\left(P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y_n\right),
\end{aligned}$$

and (38), we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{40}$$

Since $W_w(x_n)$ and $W_w(y_n)$ are non-empty sets, then there exists $\hat{x}\in C_1,\hat{y}\in C_2$ such that $\hat{x}\in W_w(x_n)$ and $\hat{y}\in W_w(y_n)$.

We may assume, there exists subsequences $\{x_{n_k}\}$, $\{y_{n_k}\}$ of $\{x_n\}$, $\{y_n\}$ such that

$$x_{n_k} \rightharpoonup \hat{x} \quad \text{as} \quad k \to \infty. \tag{41}$$

and

$$y_{n_k} \rightharpoonup \hat{y} \quad \text{as} \quad k \to \infty. \tag{42}$$

Next, we will show that $(\hat{x}, \hat{y}) \in \Omega$.
From (26), (41) and (42), we obtain $u_{n_k} \rightharpoonup \hat{x}$ and $v_{n_k} \rightharpoonup \hat{y}$ as $k \to \infty$.
Assume that $\hat{x} \notin F(T_1)$.
Since $F(T_1) = F(P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right))$, we have $\hat{x} \neq P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)\hat{x}$. From Opial's condition, $\lim_{k\to\infty} \lambda^1_{n_k} = 0$ and condition i), we have

$$\begin{aligned}
&\liminf_{k\to\infty} \|u_{n_k} - \hat{x}\| \\
&< \liminf_{k\to\infty} \|u_{n_k} - P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)\hat{x}\| \\
&\le \liminf_{k\to\infty} \big(\|u_{n_k} - P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)u_{n_k}\| \\
&\quad + \|P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)u_{n_k} \\
&\quad - P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)\hat{x}\|\big) \\
&\le \liminf_{k\to\infty} \big(\|u_{n_k} - P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)u_{n_k}\| \\
&\quad + \|u_{n_k} - \hat{x}\| + \lambda^1_{n_k}\|(I - T_1)u_{n_k} - (I - T_1)\hat{x}\|\big) \\
&= \liminf_{k\to\infty} \|u_{n_k} - \hat{x}\|.
\end{aligned}$$

This is a contradiction. Thus $\hat{x} \in F(T_1)$.
From $v_{n_k} \rightharpoonup \hat{y}$ as $k \to \infty$ and using the same method as $\hat{x} \in F(T_1)$, we have $\hat{y} \in F(T_2)$.
Since $A\hat{x} - B\hat{y} \in W_w(Ax_n - By_n)$ and weakly lower semi-continuous of norm, we get

$$\|A\hat{x} - B\hat{y}\| \le \liminf_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0.$$

Then $A\hat{x} = B\hat{y}$. Hence $(\hat{x}, \hat{y}) \in \Omega$.
Consider that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle u - \hat{x^*}, x_n - \hat{x^*}\rangle \\
&= \limsup_{k\to\infty} \langle u - \hat{x^*}, x_{n_k} - \hat{x^*}\rangle \\
&= \langle u - \hat{x^*}, \hat{x} - \hat{x^*}\rangle \\
&\le 0,
\end{aligned}$$

where $\hat{x^*} = P_{F(T_1)}u$ and

$$\begin{aligned}
&\limsup_{n\to\infty} \langle v - \hat{y^*}, y_n - \hat{y^*}\rangle \\
&= \limsup_{k\to\infty} \langle v - \hat{y^*}, y_{n_k} - \hat{y^*}\rangle \\
&= \langle v - \hat{y^*}, \hat{y} - \hat{y^*}\rangle \\
&\le 0,
\end{aligned}$$

where $\hat{y^*} = P_{F(T_2)}v$.
Next, we show that a sequence $\{(x_n, y_n)\}$ converges strongly to $(\hat{x^*}, \hat{y^*}) \in \Omega$, where $\hat{x^*} = P_{F(T_1)}u$ and $\hat{y^*} = P_{F(T_2)}v$.
From the definitions of $\{x_n\}$ and $\{y_n\}$, we have

$$\|x_{n+1} - \hat{x^*}\|^2 \le (1-\alpha_n)\|x_n - \hat{x^*}\|^2 + 2\alpha_n\langle u - \hat{x^*}, x_{n+1} - \hat{x^*}\rangle$$

and

$$\|y_{n+1} - \hat{y^*}\|^2 \le (1-\alpha_n)\|y_n - \hat{y^*}\|^2 + 2\alpha_n\langle v - \hat{y^*}, y_{n+1} - \hat{y^*}\rangle.$$

Then

$$\begin{aligned}
&\|x_{n+1} - \hat{x^*}\|^2 + \|y_{n+1} - \hat{y^*}\|^2 \\
&\le (1 - \alpha_n)\big(\|x_n - \hat{x^*}\|^2 + y_n - \hat{y^*}\|^2\big) \\
&\quad + 2\alpha_n\big(\langle u - \hat{x^*}, x_{n+1} - \hat{x^*}\rangle + \langle v - \hat{y^*}, y_{n+1} - \hat{y^*}\rangle\big),
\end{aligned}$$

or

$$C_{n+1} \le (1 - \alpha_n)C_n + 2\alpha_n\varrho_n, \tag{43}$$

where $\varrho_n = \langle u - \hat{x^*}, x_{n+1} - \hat{x^*}\rangle + \langle v - \hat{y^*}, y_{n+1} - \hat{y^*}\rangle$, for all $n \in \mathbf{N}$.
From Lemma 2.3, thus

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \left(\|x_n - \hat{x^*}\|^2 + \|y_n - \hat{y^*}\|^2\right) = 0.$$

Therefore (x_n, y_n) converges strongly to $(\hat{x^*}, \hat{y^*})$.
Since $A\hat{x^*} - B\hat{y^*} \in W_w(Ax_n - By_n)$ and weakly lower semi-continuous of norm, we get

$$\|A\hat{x^*} - B\hat{y^*}\| \le \liminf_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0.$$

Then $A\hat{x^*} = B\hat{y^*}$. Hence $(\hat{x^*}, \hat{y^*}) \in \Omega$.
Case2. Suppose that C_n is not monotone sequence, then there exists an integer n_0 such that $C_{n_0} \le C_{n_0+1}$.
Define the integer sequence $\tau(n)$ for all $n \ge n_0$ as follows,

$$\tau(n) = \max\{k \le n : C_k < C_{k+1}\}.$$

It is clear that $\tau(n)$ is a nondecreasing with $\lim_{n\to\infty} \tau(n) = \infty$ and $C_{\tau(n)} < C_{\tau(n)+1}$.
From (43), we have

$$C_{\tau(n)+1} \le (1 - \alpha_{\tau(n)})C_{\tau(n)} + 2\alpha_{\tau(n)}\varrho_{\tau(n)}.$$

From Lemma 2.3, thus

$$\lim_{n\to\infty} C_{\tau(n)} = 0.$$

Applying (40), we have

$$\lim_{n\to\infty} C_{\tau(n)+1} = 0.$$

By Lemma (2.7), we have

$$C_n \le \max\{C_n, C_{\tau(n)}\} \le C_{\tau(n)+1}.$$

From above inequality and $\lim_{n\to\infty} C_{\tau(n)+1} = 0$, we obtain

$$\lim_{n\to\infty} \left(\|x_n - \hat{x^*}\|^2 + \|y_n - \hat{y^*}\|^2\right) = \lim_{n\to\infty} C_n = 0.$$

That implies $\{(x_n, y_n)\}$ converges strongly to $\left(\hat{x^*}, \hat{y^*}\right)$.
By using the same methods as case 1, we have $\left(\hat{x^*}, \hat{y^*}\right) \in \Omega$, where $\hat{x^*} = P_{F(T_1)}u$ and $\hat{y^*} = P_{F(T_2)}v$.
This is complete the proof. ■

Corollary 3.2: For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $T_i : C_i \to C_i$ be quasi-nonexpansive mapping for all $i = 1, 2$ and let $A : H_1 \to H_3$ be bounded linear operator with adjoints A^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 : x \in F(T_1), y \in F(T_2) \text{ and } Ax = y\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases}
u_n = x_n - \gamma_n A^*\left(Ax_n - y_n\right), \\
x_{n+1} = \alpha_n u + (1 - \alpha_n)\, P_{C_1}\left(I - \lambda^1_n(I - T_1)\right)u_n, \\
v_n = (1 - \gamma_n)y_n + \gamma_n Ax_n, \\
y_{n+1} = \alpha_n v + (1 - \alpha_n)\, P_{C_2}\left(I - \lambda^2_n(I - T_2)\right)v_n,
\end{cases} \tag{44}$$

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for all $n \geq 1$ where $\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0,1)$ for all $i = 1,2$ and $\gamma_n \in (a,b) \subset \left(\epsilon, \frac{2}{\lambda_A} - \epsilon\right)$ for all $n \in \mathbf{N}$ and λ_A be spectral radius of A^*A, ϵ is a small enough. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{F(T_1)}u$ and $\bar{y^*} = P_{F(T_2)}v$.

Proof: By using Theorem 3.1 and taking $B \equiv I$, we obtain the conclusion. ■

IV. Application

A mapping $T : C \to C$ is called nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \ \forall x, y \in C.$$

Such mapping is defined by Kohsaka and Takahashi [13]. The following lemma will be used to prove in the application.

Lemma 4.1: [13] Let H be a Hilbert space, let C be a non-empty closed convex subset of H, and let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.

In 2009, Kangtunyakarn and Suantai [14] introduced the S-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ as follows.

Definition 4.1: Let C be a non-empty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows;

$$\begin{aligned}
U_0 &= I, \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\
&\ \ \vdots \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called an S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 4.2: [15] Let C be a non-empty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0,1], \alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (0,1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (0,1], \alpha_3^N \in [0,1)$ $\alpha_2^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a quasi-nonexpansive mapping.

By using these results, we obtain the following theorem.

Theorem 4.3: For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C_1 into C_1 with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0,1]$ $, \alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (0,1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (0,1], \alpha_3^N \in [0,1)$ $\alpha_2^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{\bar{T}_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C_2 into C_2 with $\bigcap_{i=1}^N F(\bar{T}_i) \neq \emptyset$, and let $\beta_j = (\beta_1^j, \beta_2^j, \beta_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0,1]$ $, \beta_1^j + \beta_2^j + \beta_3^j = 1$, β_1^j, $\beta_3^j \in (0,1)$ for all $j = 1, 2, ..., N-1$ and $\beta_1^N \in (0,1], \beta_3^N \in [0,1)$ $\beta_2^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let $\bar{S}$ be the mapping generated by $\bar{T}_1, \bar{T}_2, ..., \bar{T}_N$ and $\beta_1, \beta_2, ..., \beta_N$. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 : x \in \bigcap_{i=1}^N F(T_i), y \in \bigcap_{i=1}^N F(\bar{T}_i)$ and $Ax = By\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases}
u_n = x_n - \gamma_n A^* (Ax_n - By_n), \\
x_{n+1} = \alpha_n u + (1 - \alpha_n) P_{C_1} \left(I - \lambda_n^1 (I - S)\right) u_n, \\
v_n = y_n + \gamma_n B^* (Ax_n - By_n), \\
y_{n+1} = \alpha_n v + (1 - \alpha_n) P_{C_2} \left(I - \lambda_n^2 (I - \bar{S})\right) v_n,
\end{cases} \tag{45}$$

for all $n \geq 1$ where $\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0,1)$ for all $i = 1,2$ and $\gamma_n \in (a,b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon\right)$ for all $n \in \mathbf{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively, ϵ is a small enough. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{F(S)}u$ and $\bar{y^*} = P_{F(\bar{S})}v$.

Proof: By using Theorem 3.1 and 4.2, we obtain the conclusion. ■

Moreover, if we put $F(T_1) = C_1$ and $F(T_2) = C_2$ in Theorem 3.1, we obtain the SEFPP reduced to the SEFP.

Theorem 4.4: For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $T_i : C_i \to C_i$ be quasi-nonexpansive mapping for all $i = 1, 2$ and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 : Ax = By\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases}
u_n = x_n - \gamma_n A^* (Ax_n - By_n), \\
x_{n+1} = \alpha_n u + (1 - \alpha_n) P_{C_1} u_n, \\
v_n = y_n + \gamma_n B^* (Ax_n - By_n), \\
y_{n+1} = \alpha_n v + (1 - \alpha_n) P_{C_2} v_n,
\end{cases} \tag{46}$$

for all $n \geq 1$,where $\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0,1)$ for all $i = 1,2$ and $\gamma_n \in (a,b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon\right)$ for all $n \in \mathbf{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively, ϵ is a small enough. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{C_1}u$ and $\bar{y^*} = P_{C_2}v$.

Proof: By using Theorem 3.1, we put $F(T_1) = C_1$ and $F(T_2) = C_2$, we obtain the conclusion. ■

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V. Conclusion

We have proposed an algorithm for solving a new split equality fixed point problem for quasi-nonexpansive mapping, and proved its converges in the Hilbert spaces. In Application, we used S-mapping applied to our main result to prove the strong convergence theorems.

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