NUMERICAL SIMULATIONS OF TRANSIENT GROUNDWATER MODELS IN DRY AREA

NATTAWUT PONGNU

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENT FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY (APPLIED MATHEMATICS)
DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG
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หัวข้อวิทยานิพนธ์	การจำลองเชิงตัวเลขของตัวแบบน้ำใต้ดินทรานส์เซียนท์ในพื้นที่แล้ง
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มหาวิทยาลัย	สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง (สจล.)
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บทคัดย่อ

การไหลของน้ำใต้ดินมีอิทธิพลต่อการศึกษาทางธรณีศาสตร์ ซึ่งสามารถอธิบายได้ด้วยตัวแบบสมการเชิงอนุพันธ์ย่อย ความดันชลศาสตร์เป็นตัวแปรสำคัญในการอธิบายการไหลของน้ำใต้ดิน ในวิทยานิพนธ์ฉบับนี้เราได้แสดงการจำลองแบบการไหลของน้ำใต้ดินแบบทรานส์เซียนท์โดยใช้วิธีผลต่างจำกัด ซึ่งแปรผันตามพจน์แหล่งปลายทางและแหล่งกำเนิด ขึ้นอยู่กับการเปลี่ยนแปลงตามฤดูกาล และความต้องการการใช้น้ำของแต่ละพื้นที่ การปรับขนาดกริดหรือตำแหน่งน้ำใต้ดิน การกำหนดพารามิเตอร์ของชั้นหินอุ้มน้ำ และพจน์แหล่งปลายทางและแหล่งกำเนิด นำมาจำลองสถานการณ์กับปัญหาที่มีพื้นที่ที่ซับซ้อน วิธีหาค่าเหมาะสมที่สุดโดยวิธีซิมเพล็กซ์ นำมาประยุกต์กับกระบวนการของสถานีสูบน้ำเพื่อเสียค่าใช้จ่ายในการสูบน้ำต่ำที่สุด โดยมีน้ำบาดาลใช้อย่างเพียงพอต่อความต้องการเพื่อการชลประทานอย่างยั่งยืน

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ABSTRACT

A groundwater flow is important in geosciences which is modeled through partial differential equations. The hydraulic head profiles are characterized the groundwater flow. In this thesis, we demonstrate the transient groundwater simulations using finite difference methods, the transient groundwater modeling with varied sink and source depend on the changing of seasonal water requirement and over water supply. The variable of grid sizes, aquifer parameters, sink and source terms are considered with complex geometries. The optimization technique using a simplex method, is applied to the process of the water injection system to get a minimum cost of groundwater needed for the sustainable groundwater use for irrigation.

Keywords : Hydraulic head, Groundwater flow model, Transient groundwater flow model, Optimization method, Finite difference method

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Nattawut Pongnu

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Chapter 1

Introduction

1.1 Motivation

The drought area in Thailand is main problem almost every year. The agricultural sector is most affected, as the quantities of many agricultural products fall such as rice, tapioca, sugarcane and sugar which may lead to high price improvements for some crop. The water requirement of agricultural sector in not enough. The inspiration of this thesis is how to make enough water in the dry area. The hydraulic head is used to measure water level in the groundwater. The hydraulic head profile in each area can be measured by the methods of the field measurement and the mathematical model. The field measurement is not suitable because the large area and high cost. On the other hand, the mathematical model is practical for the problem.

Groundwater modeling is a powerful tool for water resources management, groundwater protection and remediation. The models are decided by maker to predict the behavior of a groundwater system prior to implementation of a remediation plan. The significance of the utilization of water resources continues to grow due to the increasing require of water for irrigation as well as drinking, agriculture, commercial and industrial proposes. Although, the amount of groundwater resources have been decaying due population growth, uncontrolled and unplanned urbanization, industrialization, and agricultural activities. Hence, the sustainable management planning must be developed for the groundwater systems. The management planning have to limited in the case of legal well drilling and limited-pumpings.

On the other hand, the partial differential equations governing the system is solved by model. The groundwater models are solved by analytical and numerical solution techniques. Analytical methods are not suitable application to require much data and their application is limited to simple problem. Numerical methods can solve more complex problem than analytical solutions. Now, rapid development of computer processors and increasing speed, numerical modeling has become tools more effective an easy to use. The finite difference method and the finite element are most tools used numerical modeling approaches. Each method has its advantages and limitations. Selecting numerical modeling approach depend on the problem of concern and the objectives of modeling. Most of groundwater modeling has the aquifer systems with the heterogeneous structure. In the case of the steady-state groundwater model solutions can be obtained by the simply basic techniques. On the other hand, the case of transient ground water model is solved by the advanced techniques due to the difficult in terms of time dimension in the governing equations. Theoretical solution of the governing equation of groundwater model need general assumptions such as ideal solution domains and homogeneous geometries.

In this thesis, we propose simple and flexible groundwater modeling simulation using the using the implicit and explicit finite difference methods. The variable of grid size, aquifer parameters, sinks and source terms will be applied to problem having simple and complex geometries in the model. And the simplex method is therefore used to obtain the optimal management of the water injection stations to achieve minimum cost. The numerical experiments are also given.

1.2 Literature reviews

The finite difference [1-3] and finite elements [4-6] methods are the most popular numerical solution techniques. The case of free surface flows will consider in two groups: adaptive mesh methods is need a large number of calculation and they also require some convergence conditions [4]. It follows that the fixed mesh techniques are more popular than adaptive mesh techniques. If the variation of geometry and material in the third dimension is constant, then the two-dimensional modeling can be used. Although, If the material properties and/or geometry vary any the third dimension, then the three-dimension modeling may return better solutions than the two-dimension modeling. A useful spreadsheet for two and three dimensional steady-state and transient groundwater numerical simulation is proposed in [7-8].

In the recent years, there has been many research spreadsheets on the evolution of numerical models to the groundwater flow model, see for example [7-12].

Olsthoorn [7] proposed that spreadsheet is useful tool for two-and three-dimensional steady-state and transient groundwater modeling problems with homogeneous aquifer parameters and constant sinks and/or source term.

Karahan et al. [8] propose a transient groundwater flow modeling using spreadsheet simulation (TGMSS) model for solving groundwater problem. Result showed that the TGMSS AND MODFLOW were in good agreement in terms of resulting values of hydraulic heads in all cases.

Lam [9] described a spreadsheet program to solve the Laplace equation in three independent variables subject to constant Dirichlet boundary conditions by the Gauss-Seidel method and the successive over-relaxation (SOR method).

Kharab [10] used a single spreadsheet program for solving three-dimensional transient heat conduction problem by using multiple sheets.

Anderson et al. [11] showed that spreadsheets provide an easy way for understanding groundwater problems prior to using MODFLOW. They solve one-dimensional steady-state groundwater problems for homogeneous aquifer parameters and constant sinks and/or source terms.

Ayvaz et al. [12] proposed a simulation/optimization (S/O) model is proposed for the identification of unknow groundwater well locations and pumping rates for two-dimensional aquifer system.

1.3 Objectives of the thesis

1.3.1 To study the governing equations and the methods for solving in two-dimensional of the hydraulic head model

1.3.2 To study the governing equations and the methods for solving in two-dimensional of the hydraulic head model with varied sink and source terms depend on the changing of dry season.

1.3.3 To study the governing equations and the methods for solving in two-dimensional of the hydraulic head model with varied sink and source terms depend on the changing of dry season and apply with problem in simple and complex geometries in the model.

1.3.4 To study the governing equations and the methods for solving in two-dimensional of the hydraulic head model with varied sink and source terms depend on the changing of dry season and apply with problem in simple and complex geometries in the model and solve the problem with suitable numerical method.

1.3.5 To study the groundwater flow management model with drought area by using an optimization method.

1.3.6 To propose the optimal management of the water injection stations to achieve minimum cost.

1.4 Scopes of the thesis

To use the finite different methods (FDM) such as the alternating direction explicit method (ADEM), the alternating direction implicit method (ADIM), the explicit methods, the implicit methods to approximate two-dimensional of the transient groundwater model with varied sink and source terms depend on the changing of seasonal in dry area in simple and complex geometries.

To use finite difference method to approximate two-dimensional groundwater steady-flow model with drought area and using subjected to optimal management of the water injection stations to achieve minimum cost by using an optimization method.

1.5 Plan of the thesis

The thesis explains the mathematical modeling of transient ground water model in dry area and use numerical simulation that confirms the results of the technique. The processes of simulation are discussed domain of problem and setting aquifer parameters and using suitable numerical method for solving the problem.

The first part will study the basic knowledge about the hydraulic head model such as the governing equation and define the domain of problem in thesis.

The second part will study the numerical method for solving two-dimensional hydraulic head model such as the alternating direction explicit method (ADEM), the alternating direction implicit method (ADIM), the explicit methods, the implicit methods.

The third part is computation of the hydraulic head model with varied sink and source terms using finite difference methods. Variable grid size, aquifer parameters, sinks and source terms will be applied to problem having simple and complex geometries in the dry area.

The forth part will study the groundwater flow management model which model divided 2 models are the groundwater steady-state flow model and groundwater management model. The finite difference method is used to approximate groundwater steady-flow model.

Finally, the simplex method is used to obtain the optimal management of the water injection stations to achieve minimum cost.

1.6 Expected results

The expected results of thesis are the studied the behavior of groundwater flow from model simulation of the hydraulic head model with varied sink and source term depend on the changing of seasoning dry area and water requirement to get the lowest cost of groundwater management.

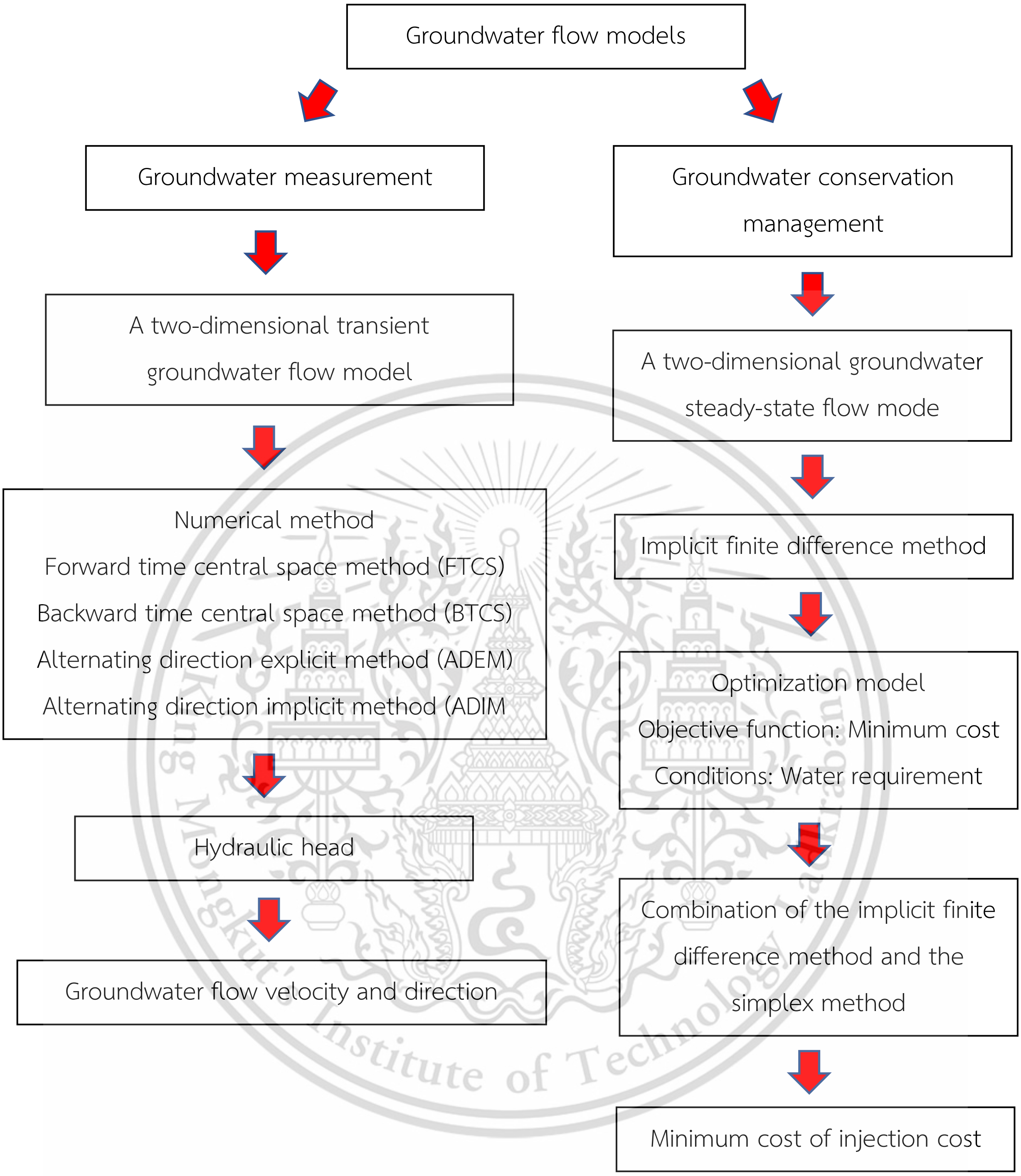

Figure 1.1 Process of numerical simulations to groundwater glow and groundwater management models.

Chapter 2

Basic Concepts and Preliminaries

2.1 The hydraulic head

The hydraulic head or total head is a measure of the potential of the water fluid at the measurement point. The groundwater motion will be starting from the high total head to the low total head.

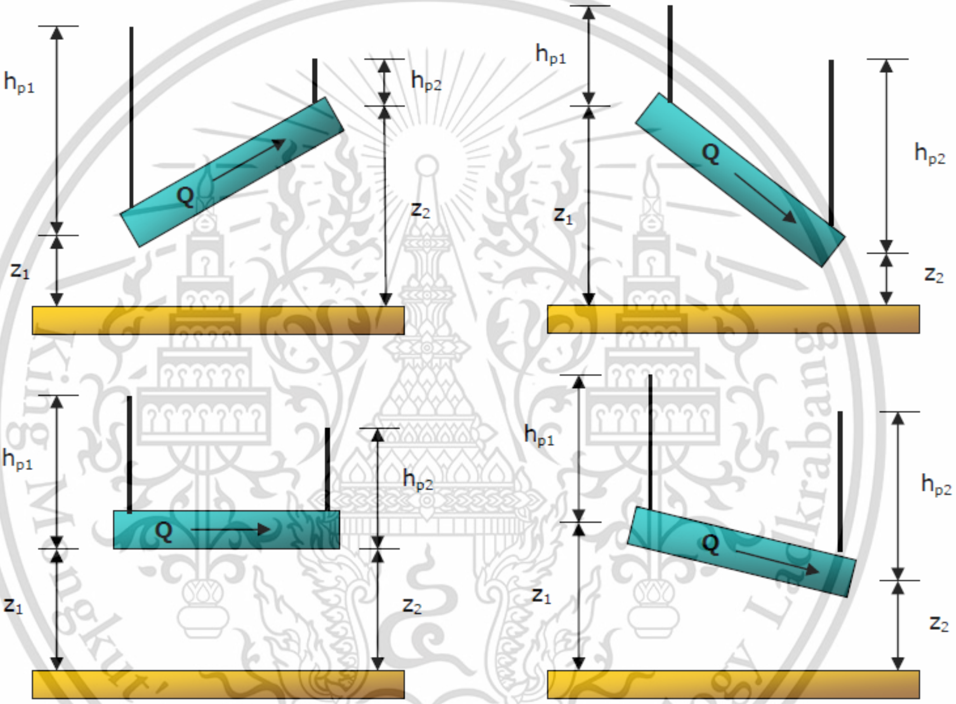

Figure 2.1 The comparing groundwater motion of elevation head difference and pressure head difference.

From Figure 2.1, the hydraulic head $h = h_p + z$ where h_p is a pressure head and z is the evaluation head. Although the elevation head and pressure head each point are equal or difference. The groundwater motion will be starting from the high total head to the low total head.

2.2 The governing equation of a groundwater flow model

The groundwater flow equation is used to describe the flow of groundwater through an aquifer. The transient flow of groundwater is described by a form of the diffusion equation. The steady-state flow of groundwater is described by a form of the Laplace equation. Let $(x,y,z,t)\in\Omega\subseteq R^3\times[0,T]$ where T is the stationary simulation time. Consider the mass balance equation,

$$\frac{\partial m}{\partial t}=m_{in}-m_{out}, \tag{2.1}$$

where m is the fluid mass,
t is time,
m_{in} is the mass flow rate entering of a surface,
m_{out} is the mass flow rate exiting of a surface.

From the Figure 1, the mass is the product of the density (ρ) and volume (V), we have

$$m_{xin}=\rho_w\left[q(x)\Delta y\Delta z\right]=\left[\rho_w q(x)\right]\Delta y\Delta z, \tag{2.2}$$

$$m_{xout}=\rho_w\left[q(x+\Delta x)\Delta y\Delta z\right]=\left[\rho_w q(x+\Delta x)\right]\Delta y\Delta z, \tag{2.3}$$

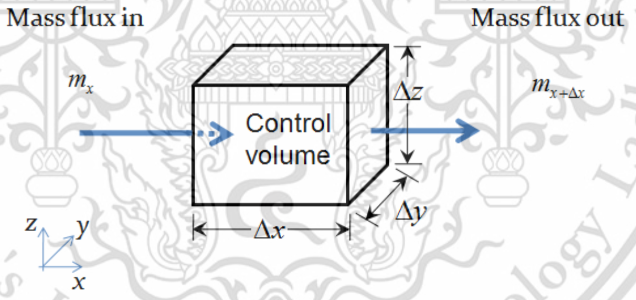

Figure 2.2 The mass flow in a control volume.

where $q(x)$ is the velocity of x-direction, m_{xin} and m_{xout} are the mass flow rate entering and exiting of x-direction respectively, in similarly

$$m_{yin}=\rho_w\left[q(y)\Delta x\Delta z\right]=\left[\rho_w q(y)\right]\Delta x\Delta z, \tag{2.4}$$

$$m_{yout}=\rho_w\left[q(y+\Delta y)\Delta x\Delta z\right]=\left[\rho_w q(y+\Delta y)\right]\Delta x\Delta z, \tag{2.5}$$

$$m_{zin}=\rho_w\left[q(z)\Delta x\Delta y\right]=\left[\rho_w q(z)\right]\Delta x\Delta y, \tag{2.6}$$

$$m_{zout}=\rho_w\left[q(z+\Delta z)\Delta x\Delta y\right]=\left[\rho_w q(z+\Delta z)\right]\Delta x\Delta y. \tag{2.7}$$

Substituting equations (2.4) – (2.7) into equation (2.1), we have

$$\frac{\partial m}{\partial t}=\left(m_{xin}-m_{xout}\right)+\left(m_{yin}-m_{yout}\right)+\left(m_{zin}-m_{zout}\right), \tag{2.8}$$

$$\begin{aligned}\frac{\partial m}{\partial t}=&\Big(\big[\rho_w q(x)\big]\Delta y\Delta z-\big[\rho_w q(x+\Delta x)\big]\Delta y\Delta z\Big)+\\&\Big(\big[\rho_w q(y)\big]\Delta x\Delta z-\big[\rho_w q(y+\Delta y)\big]\Delta x\Delta z\Big)+\\&\Big(\big[\rho_w q(z)\big]\Delta x\Delta y-\big[\rho_w q(z+\Delta z)\big]\Delta x\Delta y\Big),\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\frac{\partial m}{\partial t}=&-\rho_w\big(q(x+\Delta x)-q(x)\big)\Delta y\Delta z\\&-\rho_w\big(q(y+\Delta y)-q(y)\big)\Delta x\Delta z\\&-\rho_w\big(q(z+\Delta z)-q(z)\big)\Delta x\Delta y.\end{aligned} \tag{2.10}$$

Consider term $\frac{\partial m}{\partial t}$, we can obtain that

$$\frac{\partial m}{\partial t}=\frac{\partial}{\partial t}\left(\rho_w V_w\right), \qquad \text{where} \qquad m=\rho_w V_w. \tag{2.11}$$

Substituting the porosity $\phi=\frac{V_w}{V}$ and $V=\Delta x\Delta y\Delta z$, we will have

$$\frac{\partial m}{\partial t}=\frac{\partial}{\partial t}\left(\rho_w\phi V\right), \tag{2.12}$$

$$\frac{\partial m}{\partial t}=V\frac{\partial}{\partial t}\left(\rho_w\phi\right), \tag{2.13}$$

$$\frac{\partial m}{\partial t}=\Delta x\Delta y\Delta z\frac{\partial}{\partial t}\left(\rho_w\phi\right). \tag{2.14}$$

Substituting equation (2.14) into equation (2.10), we get

$$\begin{aligned}\Delta x\Delta y\Delta z\frac{\partial}{\partial t}(\rho_w\phi) &= -\rho_w\left(q(x+\Delta x)-q(x)\right)\Delta y\Delta z \\ &\quad -\rho_w\left(q(y+\Delta y)-q(y)\right)\Delta x\Delta z \\ &\quad -\rho_w\left(q(z+\Delta z)-q(z)\right)\Delta x\Delta y.\end{aligned} \tag{2.15}$$

Divide through by $\Delta x\Delta y\Delta z$, we will have

$$\begin{aligned}\frac{\partial}{\partial t}(\rho_w\phi) &= -\frac{\rho_w\left(q(x+\Delta x)-q(x)\right)}{\Delta x} \\ &\quad -\frac{\rho_w\left(q(y+\Delta y)-q(y)\right)}{\Delta y} \\ &\quad -\frac{\rho_w\left(q(z+\Delta z)-q(z)\right)}{\Delta z}.\end{aligned} \tag{2.16}$$

Consider term $\frac{\partial}{\partial t}(\rho_w\phi)$, we can obtain that

$$\frac{\partial}{\partial t}(\rho_w\phi) = \rho_w\frac{\partial\phi}{\partial t}+\phi\frac{\partial\rho_w}{\partial t}, \tag{2.17}$$

$$\frac{\partial}{\partial t}(\rho_w\phi) = \rho_w\frac{\partial\phi}{\partial p}\frac{\partial p}{\partial t}+\phi\frac{\partial\rho_w}{\partial p}\frac{\partial p}{\partial t}, \tag{2.18}$$

where p is pressure,

$$\frac{\partial}{\partial t}(\rho_w\phi) = \frac{\partial p}{\partial t}\left(\rho_w\frac{\partial\phi}{\partial p}+\phi\frac{\partial\rho_w}{\partial p}\right), \tag{2.19}$$

$$\frac{\partial}{\partial t}(\rho_w\phi) = \frac{\partial p}{\partial t}(\rho_w\beta+\phi\alpha), \tag{2.20}$$

where $\beta=\frac{\partial\phi}{\partial p}$ and $\alpha=\frac{\partial\rho_w}{\partial p}$. Since $p=\rho_w gh$, we will have

$$\frac{\partial}{\partial t}(\rho_w\phi)=\frac{\partial(\rho_w gh)}{\partial t}(\rho_w\beta+\phi\alpha), \tag{2.21}$$

$$\frac{\partial}{\partial t}(\rho_w\phi)=\rho_w(\rho_w g\beta+\phi g\alpha)\frac{\partial h}{\partial t}, \tag{2.22}$$

Substituting equation (2.22) into equation (2.16), we get

$$\begin{aligned}\rho_w(\rho_w g\beta+\phi g\alpha)\frac{\partial h}{\partial t}=&-\frac{\rho_w\left(q(x+\Delta x)-q(x)\right)}{\Delta x}\\&-\frac{\rho_w\left(q(y+\Delta y)-q(y)\right)}{\Delta y}\\&-\frac{\rho_w\left(q(z+\Delta z)-q(z)\right)}{\Delta z}.\end{aligned} \tag{2.23}$$

Assume that the density does not change much within the control volume, the density term is neglected on the left-hand side, we have

$$\begin{aligned}(\rho_w g\beta+\phi g\alpha)\frac{\partial h}{\partial t}=&-\frac{\left(q(x+\Delta x)-q(x)\right)}{\Delta x}\\&-\frac{\left(q(y+\Delta y)-q(y)\right)}{\Delta y}\\&-\frac{\left(q(z+\Delta z)-q(z)\right)}{\Delta z}.\end{aligned} \tag{2.24}$$

If the limit of $\Delta x,\Delta y,\Delta z$ tend to zero, we can transform the difference into the difference form. Recall that the definition of derivative,

$$\frac{df}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}. \tag{2.25}$$

From equation (2.24), we will have

$$\begin{aligned}\left(\rho_w g\beta+\phi g\alpha\right)\frac{\partial h}{\partial t}=&-\lim_{\Delta x\to 0}\frac{\left(q\left(x+\Delta x\right)-q\left(x\right)\right)}{\Delta x}\\&-\lim_{\Delta y\to 0}\frac{\left(q\left(y+\Delta y\right)-q\left(y\right)\right)}{\Delta y}\\&-\lim_{\Delta z\to 0}\frac{\left(q\left(z+\Delta z\right)-q\left(z\right)\right)}{\Delta z},\end{aligned}\tag{2.26}$$

$$\left(\rho_w g\beta+\phi g\alpha\right)\frac{\partial h}{\partial t}=-\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}-\frac{\partial q_z}{\partial z}.\tag{2.27}$$

If the Darcy's law is applied, we can see that

$$q_x=-K_x\frac{\partial h}{\partial x},\tag{2.28}$$

$$q_y=-K_y\frac{\partial h}{\partial y},\tag{2.29}$$

and

$$q_x=-K_z\frac{\partial h}{\partial z}.\tag{2.30}$$

Substituting equations (2.28) – (2.30) into equation (2.27), we have

$$\left(\rho_w g\beta+\phi g\alpha\right)\frac{\partial h}{\partial t}=-\frac{\partial}{\partial x}\left(-K_x\frac{\partial h}{\partial x}\right)-\frac{\partial}{\partial y}\left(-K_y\frac{\partial h}{\partial y}\right)-\frac{\partial}{\partial z}\left(-K_z\frac{\partial h}{\partial z}\right),\tag{2.31}$$

$$\left(\rho_w g\beta+\phi g\alpha\right)\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right).\tag{2.32}$$

Let S_s be specific storage, $S_s=\rho_w g\beta+\phi g\alpha$. Then

$$S_s\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right).\tag{2.33}$$

2.3 The boundary condition of a groundwater flow model

In this thesis, we consider the two-dimensional groundwater flow model, from equation (2.33), we have

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y} \right), \tag{2.34}$$

where $h = h(x, y, t)$ for $(x, y, t) \in \Omega \subseteq R^2 \times [0, T]$. The initial conditions are constant,

$$h(x, y, 0) = h_0. \tag{2.35}$$

The boundary condition for $t > 0$ are specified by

$$\frac{\partial h}{\partial n} = \cos\theta \frac{\partial h}{\partial x} + \sin\theta \frac{\partial h}{\partial y}. \tag{2.36}$$

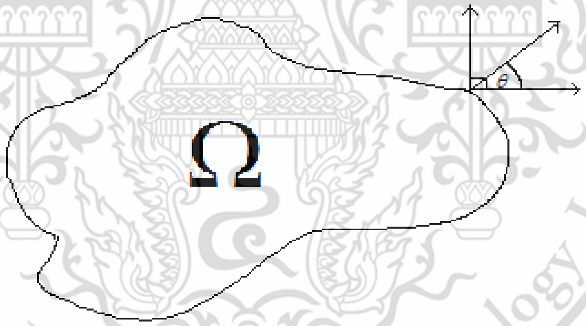

Figure 2.3 The boundary of groundwater flow model.

Chapter 3

Mathematical Simulation of a Groundwater in a Drought Area

In this chapter, the objective is to propose a simply and flexible groundwater simulation using the implicit and explicit finite difference methods. The complex geometry in the model is considered by variable grid sizes aquifer parameters, sinks and source terms.

3.1 The governing equation of two-dimensional transient groundwater flow model in drought area

The governing equation of vertically integrated Darcy's flow in a two-dimensional confined, compressible, isotropic, heterogeneous aquifer is [8],

$$S\frac{\partial H}{\partial t}=\frac{\partial}{\partial x}\left(K\frac{\partial H}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial H}{\partial y}\right)\pm W, \tag{3.1}$$

where H hydraulic head (metre), K hydraulic conductivity (metre/day), W sinks and/or source (1/day) and S matrix of specific storage (1/metre). We assume the hydraulic conductivity is constant. It obtains that

$$S\frac{\partial H}{\partial t}=K\left(\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}\right)\pm W, \tag{3.2}$$

with the initial conditions at $t=0,\ 0\le x\le L$ and $0\le y\le M$ where L, M are constant being specified,

$$H=H_0. \tag{3.3}$$

The boundary condition for $t > 0$ are specified,

$$\frac{\partial H}{\partial n} = B_N \qquad \text{at } 0 \le x \le L \text{ and } y = M, \tag{3.4}$$

$$\frac{\partial H}{\partial n} = B_S \qquad \text{at } 0 \le x \le L \text{ and } y = 0, \tag{3.5}$$

$$\frac{\partial H}{\partial n} = B_W \qquad \text{at } x = 0 \text{ and } 0 \le y \le M, \tag{3.6}$$

$$\frac{\partial H}{\partial n} = B_E \qquad \text{at } x = L \text{ and } 0 \le y \le M, \tag{3.7}$$

and given pumping well point each,

$$Q(x, y) = Q_i \quad \text{for } i = 1, 2, 3, \cdots, w, \tag{3.8}$$

in order to solve equation (3.2) in domain $\Omega \times [0, T]$ where $\Omega \in [0, \mathrm{L}] \times [0, \mathrm{M}]$ and w is the total number of sources and sinks.

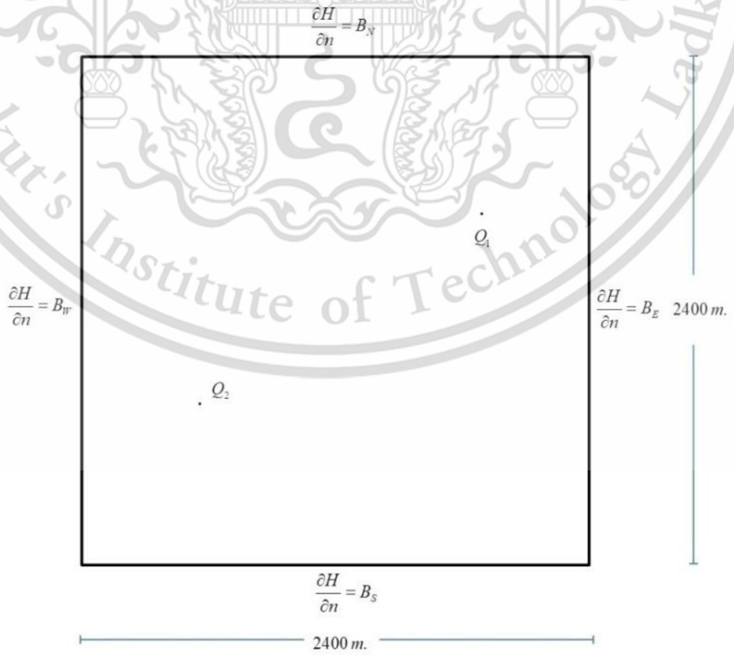

Figure 3.1 The boundary conditions of a two-dimensional transient groundwater flow model.

From Figure 3.1 the lengths of domain are $L \times M$, B_N, B_S, B_W and B_E are boundary conditions of model and Q_1, Q_2 are source and sinks term respectively.

3.2 Numerical methods for two-dimensional transient groundwater flow model in drought area

In this chapter, we will propose finite difference methods to the transient groundwater model such as the forward time central space method (FTCS), the backward time central space method (BTCS), the alternating direction explicit method (ADEM) and the alternating direction implicit method (ADIM). We now discretize (3.2) by dividing the interval $\left[0,\mathrm{L}\right]$ in x-direction into I subintervals such that $I\Delta x = L$, the interval $\left[0,\mathrm{M}\right]$ in y-direction into J subintervals such that $J\Delta y = M$ and the interval $\left[0,T\right]$ in time into N subintervals such that $N\Delta t = T$. We can then approximate $H\left(x,y,t\right)$ by $H_{i,j}^{n}$, value of the difference approximation of $H\left(x,y,t\right)$ at point $x = i\Delta x$, $y = j\Delta y$ and $t = n\Delta t$, where $0 \le i \le I$, $0 \le j \le J$ and $0 \le n \le N$ which I, J and N are positive integers.

3.2.1 Forward time central space method (FTCS)

Taking the central difference scheme in space and forward difference scheme in time into each terms of equation (3.2), then

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H_{i-1,j}^{n} - 2H_{i,j}^{n} + H_{i+1,j}^{n}}{\left(\Delta x\right)^2}, \tag{3.9}$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H_{i,j-1}^{n} - 2H_{i,j}^{n} + H_{i,j+1}^{n}}{\left(\Delta y\right)^2}, \tag{3.10}$$

$$\frac{\partial H}{\partial t} \approx \frac{H_{i,j}^{n+1} - H_{i,j}^{n}}{\Delta t}, \tag{3.11}$$

$$W_{i,j}^{n} = \pm \frac{Q_{i,j}^{n}}{\Delta x \Delta y H_{i,j}^{n}}. \tag{3.12}$$

Substituting equations (3.9) – (3.12) into equation (3.2), for $1 < i < I-1$ and $1 < j < J-1$ at $t > 0$,

$$\begin{aligned} H_{i,j}^{n+1} &= H_{i,j}^{n} + \xi\left(H_{i+1,j}^{n} - 2H_{i,j}^{n} + H_{i-1,j}^{n}\right) \\ &\quad + \eta\left(H_{i,j+1}^{n} - 2H_{i,j}^{n} + H_{i,j-1}^{n}\right) + \omega\left(\frac{Q_{i,j}^{n}}{\Delta x \Delta y H_{i,j}^{n}}\right), \end{aligned} \tag{3.13}$$

where $\xi = \dfrac{(\Delta t)K}{(\Delta x)^2 S}$, $\eta = \dfrac{(\Delta t)K}{(\Delta y)^2 S}$ and $\omega = \dfrac{\Delta t}{S}$. For $i = 1$ and $j = 1$ at $t > 0$, substituting the unknown value on the west boundary by forward difference approximation $H_{0,1}^{n} = H_{1,1}^{n} - (\Delta x)B_W$ and substituting the unknown value on the south boundary by forward difference approximation $H_{1,0}^{n} = H_{1,1}^{n} - (\Delta y)B_S$ into equation (3.13),

$$\begin{aligned} H_{1,1}^{n+1} &= H_{1,1}^{n} + \xi\left[H_{2,1}^{n} - H_{1,1}^{n} - (\Delta x)B_W\right] \\ &\quad + \eta\left[H_{1,2}^{n} - H_{1,1}^{n} - (\Delta y)B_S\right] + \omega\left(\frac{Q_{1,1}^{n}}{\Delta x \Delta y H_{1,1}^{n}}\right). \end{aligned} \tag{3.14}$$

For $1 < i < I-1$ and $j = 1$ at $t > 0$, substituting the unknown value on the south boundary by forward difference approximation $H_{i,0}^{n} = H_{i,1}^{n} - (\Delta y)B_S$ into equation (3.13),

$$\begin{aligned} H_{i,1}^{n+1} &= H_{i,1}^{n} + \xi\left(H_{i+1,1}^{n} - 2H_{i,1}^{n} + H_{i-1,1}^{n}\right) \\ &\quad + \eta\left[H_{i,2}^{n} - H_{i,1}^{n} - (\Delta y)B_S\right] + \omega\left(\frac{Q_{i,1}^{n}}{\Delta x \Delta y H_{i,1}^{n}}\right). \end{aligned} \tag{3.15}$$

For $i = I-1$ and $j = 1$ at $t > 0$, substituting the unknown value on the east boundary by backward difference approximation $H_{I,1}^{n} = H_{I-1,1}^{n} + (\Delta x)B_E$ and substituting the unknown value of the south boundary by forward difference approximation $H_{I-1,0}^{n} = H_{I-1,1}^{n} - (\Delta y)B_S$ into equation (3.13),

$$\begin{aligned} H_{I-1,1}^{n+1} &= H_{I-1,1}^{n} + \xi\left[(\Delta x)B_E - H_{I-1,1}^{n} + H_{I-2,1}^{n}\right] \\ &\quad + \eta\left[H_{I-1,2}^{n} - H_{I-1,1}^{n} - (\Delta y)B_S\right] + \omega\left(\frac{Q_{I-1,1}^{n}}{\Delta x \Delta y H_{I-1,1}^{n}}\right). \end{aligned} \tag{3.16}$$

For $i=1$ and $1<j<J-1$ at $t>0$, substituting the unknown value of the west boundary by forward difference approximation $H_{0,j}^{n}=H_{1,j}^{n}-(\Delta x)B_W$ into equation (3.13),

$$\begin{aligned} H_{1,j}^{n+1} = {} & H_{1,j}^{n}+\xi\left[H_{2,j}^{n}-H_{1,j}^{n}-(\Delta x)B_W\right] \\ & +\eta\left(H_{1,j+1}^{n}-2H_{1,j}^{n}+H_{1,j-1}^{n}\right)+\omega\left(\frac{Q_{1,j}^{n}}{\Delta x\Delta y H_{1,j}^{n}}\right). \end{aligned} \tag{3.17}$$

For $i=I-1$ and $1<j<J-1$ at $t>0$, substituting the unknown value on the east boundary by backward difference approximation $H_{I,j}^{n}=H_{I-1,j}^{n}+(\Delta x)B_E$ into equation (3.13),

$$\begin{aligned} H_{I-1,j}^{n+1} = {} & H_{I-1,j}^{n}+\xi\left[(\Delta x)B_E-H_{I-1,j}^{n}+H_{I-2,j}^{n}\right] \\ & +\eta\left(H_{I-1,j+1}^{n}-2H_{I-1,j}^{n}+H_{I-1,j-1}^{n}\right)+\omega\left(\frac{Q_{I-1,j}^{n}}{\Delta x\Delta y H_{I-1,j}^{n}}\right). \end{aligned} \tag{3.18}$$

For $i=1$ and $j=J-1$ at $t>0$, substituting the unknown value on the west boundary by forward difference approximation $H_{0,J-1}^{n}=H_{1,J-1}^{n}-(\Delta x)B_W$ and substituting the unknown value of the north boundary by backward difference approximation $H_{1,J}^{n}=H_{1,J-1}^{n}+(\Delta y)B_N$ into equation (3.13),

$$\begin{aligned} H_{1,J-1}^{n+1} = {} & H_{1,J-1}^{n}+\xi\left[H_{2,J-1}^{n}-H_{1,J-1}^{n}-(\Delta x)B_W\right] \\ & +\eta\left[(\Delta y)B_N-H_{1,J-1}^{n}+H_{1,J-2}^{n}\right]+\omega\left(\frac{Q_{1,J-1}^{n}}{\Delta x\Delta y H_{1,J-1}^{n}}\right). \end{aligned} \tag{3.19}$$

For $1<i<I-1$ and $j=J-1$ at $t>0$, substituting the unknown value on the north boundary by backward difference approximation $H_{i,J}^{n}=H_{i,J-1}^{n}+(\Delta y)B_N$ into equation (3.13),

$$\begin{aligned} H_{i,J-1}^{n+1} = {} & H_{i,J-1}^{n}+\xi\left(H_{i+1,J-1}^{n}-2H_{i,J-1}^{n}+H_{i-1,J-1}^{n}\right) \\ & +\eta\left[(\Delta y)B_N-H_{i,J-1}^{n}+H_{i,J-2}^{n}\right]+\omega\left(\frac{Q_{i,J-1}^{n}}{\Delta x\Delta y H_{i,J-1}^{n}}\right). \end{aligned} \tag{3.20}$$

For $i = I-1$ and $j = J-1$ at $t > 0$, substituting the unknown value on the east boundary by backward difference approximation $H^n_{I,J-1} = H^n_{I-1,J-1} + (\Delta x) B_E$ and substituting the unknown value on the north boundary by backward difference approximation $H^n_{I-1,J} = H^n_{I-1,J-1} + (\Delta y) B_N$ into equation (3.13),

$$\begin{aligned} H^{n+1}_{I-1,J-1} = {} & H^n_{I-1,J-1} + \xi\left[(\Delta x) B_E - H^n_{I-1,J-1} + H^n_{I-2,J-1}\right] \\ & + \eta\left[(\Delta y) B_N - H^n_{I-1,J-1} + H^n_{I-1,J-2}\right] + \omega\left(\frac{Q^n_{I-1,J-1}}{\Delta x \Delta y H^n_{I-1,J-1}}\right). \end{aligned} \tag{3.21}$$

3.2.2 Backward time central space method (BTCS)

Taking the central difference scheme in space and backward difference scheme in time into each terms of equation (3.2), then

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H^{n+1}_{i-1,j} - 2H^{n+1}_{i,j} + H^{n+1}_{i+1,j}}{(\Delta x)^2}, \tag{3.22}$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H^{n+1}_{i,j-1} - 2H^{n+1}_{i,j} + H^{n+1}_{i,j+1}}{(\Delta y)^2}, \tag{3.23}$$

$$\frac{\partial H}{\partial t} \approx \frac{H^{n+1}_{i,j} - H^n_{i,j}}{\Delta t}, \tag{3.24}$$

$$W^n_{i,j} = \pm \frac{Q^n_{i,j}}{\Delta x \Delta y H^n_{i,j}}. \tag{3.25}$$

Substituting equations (3.22) – (3.25) into equation (3.2), for $1 < i < I-1$ and $1 < j < J-1$ at $t > 0$,

$$\begin{aligned} & \xi H^{n+1}_{i-1,j} + \xi H^{n+1}_{i+1,j} + \eta H^{n+1}_{i,j-1} + \eta H^{n+1}_{i,j+1} - (1 + 2\xi + 2\eta) H^{n+1}_{i,j} \\ & \qquad = -H^n_{i,j} - \omega \frac{Q^n_{i,j}}{\Delta x \Delta y H^n_{i,j}}. \end{aligned} \tag{3.26}$$

where $\xi = \dfrac{(\Delta t) K}{(\Delta x)^2 S}$, $\eta = \dfrac{(\Delta t) K}{(\Delta y)^2 S}$ and $\omega = \dfrac{\Delta t}{S}$. For $i = 1$ and $j = 1$ at $t > 0$,

substituting the unknown value on the west boundary by forward difference approximation $H_{0,1}^{n+1} = H_{1,1}^{n+1} - (\Delta x) B_W$ and substituting the unknown value on the south boundary by forward difference approximation $H_{1,0}^{n+1} = H_{1,1}^{n+1} - (\Delta y) B_S$ into equation (3.26),

$$\begin{aligned}\xi H_{21}^{n+1} + \eta H_{1,2}^{n+1} - (1+\xi+\eta) H_{1,1}^{n+1}& \\ = -H_{1,1}^{n} - \omega \frac{Q_{1,1}^{n}}{\Delta x \Delta y H_{1,1}^{n}} + \xi(\Delta x) B_W + \eta(\Delta y) B_S.&\end{aligned} \tag{3.27}$$

For $1 < i < I-1$ and $j=1$ at $t>0$, substituting the unknown value on the south boundary by forward difference approximation $H_{i,0}^{n+1} = H_{i,1}^{n+1} - (\Delta y) B_S$ into equation (3.26),

$$\begin{aligned}\xi H_{i-1,1}^{n+1} + \xi H_{i+1,1}^{n+1} + \eta H_{i,2}^{n+1} - (1+2\xi+\eta) H_{i,1}^{n+1}& \\ = -H_{i,1}^{n} - \omega \frac{Q_{i,1}^{n}}{\Delta x \Delta y H_{i,1}^{n}} + \eta(\Delta y) B_S.&\end{aligned} \tag{3.28}$$

For $i = I-1$ and $j=1$ at $t>0$, substituting the unknown value on the east boundary by backward difference approximation $H_{I,1}^{n+1} = H_{I-1,1}^{n+1} + (\Delta x) B_E$ and substituting the unknown value on the south boundary by forward difference approximation $H_{I-1,0}^{n+1} = H_{I-1,1}^{n+1} - (\Delta y) B_S$ into equation (3.26),

$$\begin{aligned}\xi H_{I-2,1}^{n+1} + \eta H_{I-1,2}^{n+1} - (1+\xi+\eta) H_{I-1,1}^{n+1}& \\ = -H_{I-1,1}^{n} - \omega \frac{Q_{I-1,1}^{n}}{\Delta x \Delta y H_{I-1,1}^{n}} - \xi(\Delta x) B_E + \eta(\Delta y) B_S.&\end{aligned} \tag{3.29}$$

For $i=1$ and $1 < j < J-1$ at $t>0$, substituting the unknown value on the west boundary by forward difference approximation $H_{0,j}^{n+1} = H_{1,j}^{n+1} - (\Delta x) B_W$ into equation (3.26),

$$\begin{aligned}\xi H_{2,j}^{n+1} + \eta H_{1,j-1}^{n+1} + \eta H_{1,j+1}^{n+1} - (1+\xi+2\eta) H_{1,j}^{n+1}& \\ = -H_{1,j}^{n} - \omega \frac{Q_{1,j}^{n}}{\Delta x \Delta y H_{1,j}^{n}} + \xi(\Delta x) B_W.&\end{aligned} \tag{3.30}$$

For $i=I-1$ and $1<j<J-1$ at $t>0$, substituting the unknown value on the east boundary by backward difference approximation $H_{I,j}^{n+1}=H_{I-1,j}^{n+1}+(\Delta x)B_E$ into equation (3.26),

$$\begin{aligned}\xi H_{I-2,j}^{n+1}+\eta H_{I-1,j-1}^{n+1}+\eta H_{I-1,j+1}^{n+1}-(1+\xi+2\eta)H_{I-1,j}^{n+1}&\\=-H_{I-1,j}^{n}-\omega\frac{Q_{I-1,j}^{n}}{\Delta x\Delta y H_{I-1,j}^{n}}-\xi(\Delta x)B_E.&\end{aligned}\tag{3.31}$$

For $i=1$ and $j=J-1$ at $t>0$, substituting the unknown value on the west boundary by forward difference approximation $H_{0,J-1}^{n+1}=H_{1,J-1}^{n+1}-(\Delta x)B_W$ and substituting the unknown value of the north boundary by backward difference approximation $H_{1,J}^{n+1}=H_{1,J-1}^{n+1}+(\Delta y)B_N$ into equation (3.26),

$$\begin{aligned}\xi H_{2,J-1}^{n+1}+\eta H_{1,J}^{n+1}-(1+\xi+\eta)H_{1,J-1}^{n+1}&\\=-H_{1,J-1}^{n}-\omega\frac{Q_{1,J-1}^{n}}{\Delta x\Delta y H_{1,J-1}^{n}}+\xi(\Delta x)B_W-\eta(\Delta y)B_N.&\end{aligned}\tag{3.32}$$

For $1<i<I-1$ and $j=J-1$ at $t>0$, substituting the unknown value on the north boundary by backward difference approximation $H_{i,J}^{n+1}=H_{i,J-1}^{n+1}+(\Delta y)B_N$ into equation (3.26),

$$\begin{aligned}\xi H_{i-1,J-1}^{n+1}+\xi H_{i+1,J-1}^{n+1}+\eta H_{i,J-2}^{n+1}-(1+2\xi+\eta)H_{i,J-1}^{n+1}&\\=-H_{i,J-1}^{n}-\omega\frac{Q_{i,J-1}^{n}}{\Delta x\Delta y H_{i,J-1}^{n}}-\eta(\Delta y)B_N.&\end{aligned}\tag{3.33}$$

For $i=I-1$ and $j=J-1$ at $t>0$, substituting the unknown value on the east boundary by backward difference approximation $H_{I,J-1}^{n+1}=H_{I-1,J-1}^{n+1}+(\Delta x)B_E$ and substituting the unknown value on the north boundary by backward difference approximation $H_{I-1,J}^{n+1}=H_{I-1,J-1}^{n+1}+(\Delta y)B_N$ into equation (3.26),

$$\begin{aligned}\xi H_{I-2,J-1}^{n+1}+\eta H_{I-1,J-2}^{n+1}-(1+\xi+\eta)H_{I-1,J-1}^{n+1}&\\=-H_{I-1,J-1}^{n}-\omega\frac{Q_{I-1,J-1}^{n}}{\Delta x\Delta y H_{I-1,J-1}^{n}}-\xi(\Delta x)B_E-\eta(\Delta y)B_N.&\end{aligned}\tag{3.34}$$

The equations (3.26) – (3.34) can be written in matrix form as follow,

$$AH^{n+1} = B^{n}, \tag{3.35}$$

where
$$A = \begin{bmatrix} A_1 & B & & & \\ B & A_2 & B & & \\ & \ddots & \ddots & \ddots & \\ & & B & A_2 & B \\ & & & B & A_1 \end{bmatrix}, \tag{3.36}$$

$$A_1 = \begin{bmatrix} -(1+\xi+\eta) & & & & \\ \eta & -(1+2\xi+\eta) & \eta & & \\ & \ddots & \ddots & \ddots & \\ & & \eta & -(1+2\xi+\eta) & \eta \\ & & & \eta & -(1+\xi+\eta) \end{bmatrix}, \tag{3.37}$$

$$A_2 = \begin{bmatrix} -(1+\xi+2\eta) & \eta & & & \\ \eta & -(1+2\xi+2\eta) & \eta & & \\ & \ddots & \ddots & \ddots & \\ & & \eta & -(1+2\xi+2\eta) & \eta \\ & & & \eta & -(1+\xi+2\eta) \end{bmatrix}, \tag{3.38}$$

$$B = \begin{bmatrix} \xi & & & & \\ & \xi & & & \\ & & \ddots & & \\ & & & \xi & \\ & & & & \xi \end{bmatrix}, \tag{3.39}$$

$$H^{n+1} = \begin{bmatrix} H_{1,1} \\ H_{2,1} \\ \vdots \\ H_{I-2,J-1} \\ H_{I-1,J-1} \end{bmatrix}, \tag{3.40}$$

and

$$B^n = \begin{bmatrix} -H_{1,1}^n - \omega \dfrac{Q_{1,1}^n}{\Delta x \Delta y H_{1,1}^n} + \xi(\Delta x) B_W + \eta(\Delta y) B_S \\ -H_{2,1}^n - \omega \dfrac{Q_{2,1}^n}{\Delta x \Delta y H_{2,1}^n} + \eta(\Delta y) B_S \\ \vdots \\ -H_{I-2,1}^n - \omega \dfrac{Q_{I-2,1}^n}{\Delta x \Delta y H_{I-2,1}^n} + \eta(\Delta y) B_S \\ -H_{I-1,1}^n - \omega \dfrac{Q_{I-1,1}^n}{\Delta x \Delta y H_{I-1,1}^n} + \xi(\Delta x) B_E + \eta(\Delta y) B_S \\ -H_{1,2}^n - \omega \dfrac{Q_{1,2}^n}{\Delta x \Delta y H_{1,2}^n} + \xi(\Delta x) B_W \\ \vdots \\ -H_{i,j}^n - \omega \dfrac{Q_{i,j}^n}{\Delta x \Delta y H_{i,j}^n} \\ \vdots \\ -H_{I-1,J-2}^n - \omega \dfrac{Q_{I-1,J-2}^n}{\Delta x \Delta y H_{I-1,J-2}^n} + \xi(\Delta x) B_E \\ -H_{1,J-1}^n - \omega \dfrac{Q_{1,J-1}^n}{\Delta x \Delta y H_{1,J-1}^n} + \xi(\Delta x) B_W + \eta(\Delta y) B_N \\ -H_{2,J-1}^n - \omega \dfrac{Q_{2,J-1}^n}{\Delta x \Delta y H_{2,J-1}^n} + \eta(\Delta y) B_N \\ \vdots \\ -H_{I-2,J-1}^n - \omega \dfrac{Q_{I-2,J-1}^n}{\Delta x \Delta y H_{I-2,J-1}^n} + \eta(\Delta y) B_N \\ -H_{I-1,J-1}^n - \omega \dfrac{Q_{I-1,J-1}^n}{\Delta x \Delta y H_{I-1,J-1}^n} + \xi(\Delta x) B_E + \eta(\Delta y) B_N \end{bmatrix}, \tag{3.41}$$

for all $1 \le n \le N$. We will introduce splitting methods that require smaller CPU time than the BTCS method. For the alternating direction explicit (ADEM) and the alternating direction implicit methods (ADIM) are discussed in next the two subsections.

3.2.3 Alternating direction explicit method (ADEM)

The ADEM are extrapolative and need an easy algebraic solution for an unknown for all point in each time interval. ADEM equation can be written as [8],

$$S_{i,j}\left(\frac{H_{i,j}^{n+1}-H_{i,j}^{n}}{\Delta t}\right)=K_{i+\frac{1}{2},j}\left(\frac{H_{i+1,j}^{n}-H_{i,j}^{n}}{\left(\Delta x\right)^{2}}\right)+K_{i-\frac{1}{2},j}\left(\frac{H_{i-1,j}^{n+1}-H_{i,j}^{n+1}}{\left(\Delta x\right)^{2}}\right)$$
$$+K_{i,j+\frac{1}{2}}\left(\frac{H_{i,j+1}^{n}-H_{i,j}^{n}}{\left(\Delta y\right)^{2}}\right)+K_{i,j-\frac{1}{2}}\left(\frac{H_{i,j-1}^{n+1}-H_{i,j}^{n+1}}{\left(\Delta y\right)^{2}}\right)+W_{i,j}. \quad (3.42)$$

Second stage,

$$S_{i,j}\left(\frac{H_{i,j}^{n+1}-H_{i,j}^{n}}{\Delta t}\right)=K_{i-\frac{1}{2},j}\left(\frac{H_{i-1,j}^{n}-H_{i,j}^{n}}{\left(\Delta x\right)^{2}}\right)+K_{i+\frac{1}{2},j}\left(\frac{H_{i+1,j}^{n+1}-H_{i,j}^{n+1}}{\left(\Delta x\right)^{2}}\right)$$
$$+K_{i,j-\frac{1}{2}}\left(\frac{H_{i,j-1}^{n}-H_{i,j}^{n}}{\left(\Delta y\right)^{2}}\right)+K_{i,j+\frac{1}{2}}\left(\frac{H_{i,j+1}^{n+1}-H_{i,j}^{n+1}}{\left(\Delta y\right)^{2}}\right)+W_{i,j}, \quad (3.43)$$

where $W_{i,j}=\pm\frac{Q_{I,J}^{n+\frac{1}{2}}}{\Delta x\Delta y H_{i,j}^{n}}$. First stage, for convenient, we will letting that

$$a_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta x\right)^{2}}K_{i+\frac{1}{2},j}, \quad (3.44)$$

$$b_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta x\right)^{2}}K_{i-\frac{1}{2},j}, \quad (3.45)$$

$$c_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta y\right)^{2}}K_{i,j+\frac{1}{2}}, \quad (3.46)$$

$$d_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta y\right)^{2}}K_{i,j-\frac{1}{2}}, \quad (3.47)$$

and

$$e_{i,j}=\frac{\Delta t}{S_{i,j}}. \quad (3.48)$$

For $1 < i < I-1$ and $1 < j < J-1$ at $t > 0$, and by rearranging, equation (3.42) becomes,

$$\begin{aligned} H_{i,j}^{n+1} = {} & \frac{a_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(H_{i+1,j}^{n}-H_{i,j}^{n}\right)+\frac{b_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}H_{i-1,j}^{n+1} \\ & +\frac{c_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(H_{i,j+1}^{n}-H_{i,j}^{n}\right)+\frac{d_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}H_{i,j-1}^{n+1} \\ & +\frac{1}{\left(1+b_{i,j}+d_{i,j}\right)}H_{i,j}^{n}+\frac{e_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}W_{i,j}. \end{aligned} \tag{3.49}$$

For $i=1$ and $j=1$ at $t>0$, substituting the unknown value on the east boundary by forward difference approximation $H_{0,1}^{n+1}=H_{1,1}^{n+1}-\left(\Delta x\right)B_W$ and substituting the unknown value on the south boundary by forward difference approximation $H_{1,0}^{n+1}=H_{1,1}^{n+1}-\left(\Delta y\right)B_S$ into equation (3.49),

$$\begin{aligned} H_{1,1}^{n+1} = {} & a_{1,1}\left(H_{2,1}^{n}-H_{1,1}^{n}\right)-b_{1,1}\left(\Delta x\right)B_W \\ & +c_{1,1}\left(H_{1,2}^{n}-H_{1,1}^{n}\right)-d_{1,1}\left(\Delta y\right)B_S+H_{1,1}^{n}+e_{1,1}W_{1,1}. \end{aligned} \tag{3.50}$$

For $1<i<I-1$ and $j=1$ at $t>0$, substituting the unknown value on the south boundary by forward difference approximation $H_{i,0}^{n+1}=H_{i,1}^{n+1}-\left(\Delta y\right)B_S$ into equation (3.49),

$$\begin{aligned} H_{i,1}^{n+1} = {} & \frac{1}{\left(1+b_{i,1}\right)}\Big[a_{i,1}\left(H_{i+1,1}^{n}-H_{i,1}^{n}\right)+b_{i,1}H_{i-1,1}^{n+1} \\ & +c_{i,1}\left(H_{i,2}^{n}-H_{i,1}^{n}\right)-d_{i,1}\left(\Delta y\right)B_S+H_{i,1}^{n}+e_{i,j}W_{i,1}\Big]. \end{aligned} \tag{3.51}$$

For $i=I-1$ and $j=1$ at $t>0$, substituting the unknown value on the east boundary by backward difference approximation $H_{I,1}^{n}=H_{I-1,1}^{n}+\left(\Delta x\right)B_E$ and substituting the unknown value on the south boundary by forward difference approximation $H_{I-1,0}^{n+1}=H_{I-1,1}^{n+1}-\left(\Delta y\right)B_S$ into equation (3.49),

$$\begin{aligned} H_{I-1,1}^{n+1} = {} & \frac{1}{\left(1+b_{I-1,1}\right)}\Big[a_{I-1,1}\left(\Delta x\right)B_E+b_{I-1,1}H_{I-2,1}^{n+1} \\ & +c_{I-1,1}\left(H_{I-1,2}^{n}-H_{I-1,1}^{n}\right)-d_{I-1,1}\left(\Delta y\right)B_S+H_{I-1,1}^{n}+e_{I-1,1}W_{I-1,1}\Big]. \end{aligned} \tag{3.52}$$

For $i=1$ and $1<j<J-1$ at $t>0$, substituting the unknown value on the west boundary by forward difference approximation $H_{0,\mathrm{j}}^{n+1}=H_{1,\mathrm{j}}^{n+1}-(\Delta x)B_W$ into equation (3.49),

$$\begin{aligned} H_{1,j}^{n+1} = \frac{1}{\left(1+d_{1,j}\right)}\Big[& a_{1,j}\left(H_{2,j}^{n}-H_{1,j}^{n}\right)-b_{1,j}\left(\Delta x\right)B_W \\ &+c_{1,j}\left(H_{1,j+1}^{n}-H_{1,j}^{n}\right)+d_{1,j}H_{1,j-1}^{n+1}+H_{1,j}^{n}+e_{1,j}W_{1,j}\Big]. \end{aligned} \tag{3.53}$$

For $i=I-1$ and $1<j<J-1$ at $t>0$, substituting the unknown value on the east boundary by backward difference approximation $H_{I,j}^{n}=H_{I-1,j}^{n}+(\Delta x)B_E$ into equation (3.49),

$$\begin{aligned} H_{I-1,j}^{n+1} = {} & \frac{a_{I-1,j}}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}\left(\Delta x\right)B_E+\frac{b_{I-1,j}}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}H_{I-2,j}^{n+1} \\ &+\frac{c_{I-1,j}}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}\left(H_{I-1,j+1}^{n}-H_{I-1,j}^{n}\right)+\frac{d_{I-1,j}}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}H_{I-1,j-1}^{n+1} \\ &+\frac{1}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}H_{I-1,j}^{n}+\frac{e_{I-1,j}}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}W_{I-1,j}. \end{aligned} \tag{3.54}$$

For $i=1$ and $j=J-1$ at $t>0$, substituting the unknown value on the west boundary by forward difference approximation $H_{0,J-1}^{n+1}=H_{1,J-1}^{n+1}-(\Delta x)B_W$ and substituting the unknown value on the top boundary by backward difference approximation $H_{1,J}^{n}=H_{1,J-1}^{n}+(\Delta y)B_N$ into equation (3.49),

$$\begin{aligned} H_{1,J-1}^{n+1} = \frac{1}{\left(1+d_{1,J-1}\right)}\Big[& a_{1,J-1}\left(H_{2,J-1}^{n}-H_{1,J-1}^{n}\right)-b_{1,J-1}\left(\Delta x\right)B_W \\ &+c_{1,J-1}\left(\Delta y\right)B_N+d_{1,J-1}H_{1,J-2}^{n+1}+H_{1,J-1}^{n}+e_{1,J-1}W_{1,J-1}\Big]. \end{aligned} \tag{3.55}$$

For $1<i<I-1$ and $j=J-1$ at $t>0$, substituting the unknown value on the north boundary by backward difference approximation $H^n_{i,J}=H^n_{i,J-1}+(\Delta y)B_N$ into equation (3.49),

$$\begin{aligned}H^{n+1}_{i,J-1}={}&\frac{a_{i,J-1}}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}\left(H^n_{i+1,J-1}-H^n_{i,J-1}\right)+\frac{b_{i,J-1}}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}H^{n+1}_{i-1,J-1}\\&+\frac{c_{i,J-1}}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}(\Delta y)B_N+\frac{d_{i,J-1}}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}H^{n+1}_{i,J-2}\\&+\frac{1}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}H^n_{i,J-1}+\frac{e_{i,J-1}}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}W_{i,J-1}.\end{aligned}\tag{3.56}$$

For $i=I-1$ and $j=J-1$ at $t>0$, substituting the unknown value on the east boundary by backward difference approximation $H^n_{I,J-1}=H^n_{I-1,J-1}+(\Delta x)B_E$ and substituting the unknown value on the north boundary by backward difference approximation $H^n_{I-1,J}=H^n_{I-1,J-1}+(\Delta y)B_N$ into equation (3.49),

$$\begin{aligned}H^{n+1}_{I-1,J-1}={}&\frac{a_{I-1,J-1}}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}(\Delta x)B_E+\frac{b_{I-1,J-1}}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}H^{n+1}_{I-2,J-1}\\&+\frac{c_{I-1,J-1}}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}(\Delta y)B_N+\frac{d_{I-1,J-1}}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}H^{n+1}_{I-1,J-2}\\&+\frac{1}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}H^n_{I-1,J-1}+\frac{e_{I-1,J-1}}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}W_{I-1,J-1}.\end{aligned}\tag{3.57}$$

Second stage, we will letting that

$$a_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i-\frac{1}{2},j},\tag{3.58}$$

$$b_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i+\frac{1}{2},j},\tag{3.59}$$

$$c_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i,j-\frac{1}{2}},\tag{3.60}$$

$$d_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i,j+\frac{1}{2}},\tag{3.61}$$

and

$$e_{i,j}=\frac{\Delta t}{S_{i,j}}.\tag{3.62}$$

For $1<i<I-1$ and $1<j<J-1$ at $t>0$, and by rearranging, equation (3.43) become,

$$\begin{aligned} H_{i,j}^{n+1} = {} & \frac{a_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(H_{i-1,j}^{n}-H_{i,j}^{n}\right)+\frac{b_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}H_{i+1,j}^{n+1} \\ & +\frac{c_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(H_{i,j-1}^{n}-H_{i,j}^{n}\right)+\frac{d_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}H_{i,j+1}^{n+1} \\ & +\frac{1}{\left(1+b_{i,j}+d_{i,j}\right)}H_{i,j}^{n}+\frac{e_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}W_{i,j}. \end{aligned} \tag{3.63}$$

For $i=I-1$ and $j=J-1$ at $t>0$, substituting the unknown value on the east boundary by backward difference approximation $H_{I,J-1}^{n+1}=H_{I-1,J-1}^{n+1}+\left(\Delta x\right)B_E$ and substituting the unknown value on the north boundary by backward difference approximation $H_{I-1,J}^{n+1}=H_{I-1,J-1}^{n+1}+\left(\Delta y\right)B_N$ into equation (3.63),

$$\begin{aligned} H_{I-1,J-1}^{n+1} = {} & a_{I-1,J-1}\left(H_{I-2,J-1}^{n}-H_{I-1,J-1}^{n}\right)+b_{I-1,J-1}\left(\Delta x\right)B_E \\ & +c_{I-1,J-1}\left(H_{I-1,J-2}^{n}-H_{I-1,J-1}^{n}\right)+d_{I-1,J-1}\left(\Delta y\right)B_N \\ & +H_{I-1,J-1}^{n}+e_{I-1,J-1}W_{I-1,J-1}. \end{aligned} \tag{3.64}$$

For $1<i<I-1$ and $j=J-1$ at $t>0$, substituting the unknown value on the north boundary by backward difference approximation $H_{i,J}^{n+1}=H_{i,J-1}^{n+1}+\left(\Delta y\right)B_N$ into equation (3.63),

$$\begin{aligned} H_{i,J-1}^{n+1} = {} & \frac{1}{\left(1+b_{i,J-1}\right)}\Big[a_{i,J-1}\left(H_{i-1,J-1}^{n}-H_{i,J-1}^{n}\right)+b_{i,J-1}H_{i+1,J-1}^{n+1} \\ & +c_{i,J-1}\left(H_{i,J-2}^{n}-H_{i,J-1}^{n}\right)+d_{i,J-1}\left(\Delta y\right)B_N+H_{i,J-1}^{n}+e_{i,J-1}W_{i,J-1}\Big]. \end{aligned} \tag{3.65}$$

For $i=1$ and $j=J-1$ at $t>0$, substituting the unknown value on the west boundary by forward difference approximation $H_{0,J-1}^{n}=H_{1,J-1}^{n}-\left(\Delta x\right)B_W$ and substituting the unknown value on the north boundary by backward difference approximation $H_{1,J}^{n+1}=H_{1,J-1}^{n+1}+\left(\Delta y\right)B_N$ into equation (3.63),

$$\begin{aligned} H_{1,J-1}^{n+1} = {} & \frac{1}{\left(1+b_{1,J-1}\right)}\Big[-a_{1,J-1}\left(\Delta x\right)B_W+b_{1,J-1}H_{2,J-1}^{n+1} \\ & +c_{1,J-1}\left(H_{1,J-2}^{n}-H_{1,J-1}^{n}\right)+d_{1,J-1}\left(\Delta y\right)B_N+H_{1,J-1}^{n}+e_{1,J-1}W_{1,J-1}\Big]. \end{aligned} \tag{3.66}$$

For $i = I-1$ and $1 < j < J-1$ at $t > 0$, substituting the unknown value on the east boundary by backward difference approximation $H^{n+1}_{I,j} = H^{n+1}_{I-1,j} + (\Delta x) B_E$ into equation (3.63),

$$\begin{aligned} H^{n+1}_{I-1,j} = {} & \frac{1}{\left(1+d_{I-1,j}\right)} \Big[a_{I-1,j}\left(H^{n}_{I-2,j} - H^{n}_{I-1,j}\right) + b_{I-1,j}\left(\Delta x\right) B_E \\ & + c_{I-1,j}\left(H^{n}_{I-1,j-1} - H^{n}_{I-1,j}\right) + d_{I-1,j} H^{n+1}_{I-1,j+1} + H^{n}_{I-1,j} + e_{I-1,j} W_{I-1,j} \Big]. \end{aligned} \tag{3.67}$$

For $i = 1$ and $1 < j < J-1$ at $t > 0$, substituting the unknown value on the west boundary by forward difference approximation $H^{n}_{0,j} = H^{n}_{1,j} - (\Delta x) B_W$ into equation (3.63),

$$\begin{aligned} H^{n+1}_{1,j} = {} & -\frac{a_{1,j}}{\left(1+b_{1,j}+d_{1,j}\right)}\left(\Delta x\right) B_W + \frac{b_{1,j}}{\left(1+b_{1,j}+d_{1,j}\right)} H^{n+1}_{2,j} \\ & + \frac{c_{1,j}}{\left(1+b_{1,j}+d_{1,j}\right)}\left(H^{n}_{1,j-1} - H^{n}_{1,j}\right) + \frac{d_{1,j}}{\left(1+b_{1,j}+d_{1,j}\right)} H^{n+1}_{1,j+1} \\ & + \frac{1}{\left(1+b_{1,j}+d_{1,j}\right)} H^{n}_{1,j} + \frac{e_{1,j}}{\left(1+b_{1,j}+d_{1,j}\right)} W_{1,j}. \end{aligned} \tag{3.68}$$

For $i = I-1$ and $j = 1$ at $t > 0$, substituting the unknown value on the east boundary by backward difference approximation $H^{n+1}_{I,1} = H^{n+1}_{I-1,1} + (\Delta x) B_E$ and substituting the unknown value on the south boundary by forward difference approximation $H^{n}_{I-1,0} = H^{n}_{I-1,1} - (\Delta y) B_S$ into equation (3.63),

$$\begin{aligned} H^{n+1}_{I-1,1} = {} & \frac{1}{\left(1+d_{I-1,1}\right)} \Big[a_{I-1,1}\left(H^{n}_{I-2,1} - H^{n}_{I-1,1}\right) + b_{I-1,1}\left(\Delta x\right) B_E \\ & - c_{I-1,1}\left(\Delta y\right) B_S + d_{I-1,1} H^{n+1}_{I-1,2} + H^{n}_{I-1,1} + e_{I-1,1} W_{I-1,1} \Big]. \end{aligned} \tag{3.69}$$

For $1<i<I-1$ and $j=1$ at $t>0$, substituting the unknown value on the south boundary by forward difference approximation $H_{i,0}^{n}=H_{i,1}^{n}-\left(\Delta y\right)B_{S}$ into equation (3.63),

$$\begin{aligned} H_{i,1}^{n+1} = {} & \frac{a_{i,1}}{\left(1+b_{i,1}+d_{i,1}\right)}\left(H_{i-1,1}^{n}-H_{i,1}^{n}\right)+\frac{b_{i,1}}{\left(1+b_{i,1}+d_{i,1}\right)}H_{i+1,1}^{n+1} \\ & -\frac{c_{i,1}}{\left(1+b_{i,1}+d_{i,1}\right)}\left(\Delta y\right)B_{S}+\frac{d_{i,1}}{\left(1+b_{i,1}+d_{i,1}\right)}H_{i,2}^{n+1} \\ & +\frac{1}{\left(1+b_{i,1}+d_{i,1}\right)}H_{i,1}^{n}+\frac{e_{i,1}}{\left(1+b_{i,1}+d_{i,1}\right)}W_{i,1}. \end{aligned} \tag{3.70}$$

For $i=1$ and $j=1$ at $t>0$, substituting the unknown value on the west boundary by forward difference approximation $H_{0,1}^{n}=H_{1,1}^{n}-\left(\Delta x\right)B_{W}$ and substituting the unknown value on the south boundary by forward difference approximation $H_{1,0}^{n}=H_{1,1}^{n}-\left(\Delta y\right)B_{S}$ into equation (3.63),

$$\begin{aligned} H_{1,1}^{n+1} = {} & -\frac{a_{1,1}}{\left(1+b_{1,1}+d_{1,1}\right)}\left(\Delta x\right)B_{W}+\frac{b_{1,1}}{\left(1+b_{1,1}+d_{1,1}\right)}H_{2,1}^{n+1} \\ & -\frac{c_{1,1}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(\Delta y\right)B_{S}+\frac{d_{1,1}}{\left(1+b_{i,j}+d_{i,j}\right)}H_{1,2}^{n+1} \\ & +\frac{1}{\left(1+b_{i,j}+d_{i,j}\right)}H_{1,1}^{n}+\frac{e_{1,1}}{\left(1+b_{i,j}+d_{i,j}\right)}W_{1,1}. \end{aligned} \tag{3.71}$$

The stable solutions can be obtained only around the stability condition that [8],

$$\frac{KH}{S}\left[\frac{\Delta t}{\left(\Delta x\right)^{2}}+\frac{\Delta t}{\left(\Delta y\right)^{2}}\right]\le\frac{1}{2}. \tag{3.72}$$

3.2.4 Alternating direction implicit method (ADIM)

This method will divide into 2 stages. The first stage, taking the central difference scheme in space and forward difference scheme in time into each terms of equation (3.2), then

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H_{i-1,j}^{n+\frac{1}{2}} - 2H_{i,j}^{n+\frac{1}{2}} + H_{i+1,j}^{n+\frac{1}{2}}}{\left(\Delta x\right)^2}, \tag{3.73}$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H_{i,j-1}^{n} - 2H_{i,j}^{n} + H_{i,j+1}^{n}}{\left(\Delta y\right)^2}, \tag{3.74}$$

$$\frac{\partial h}{\partial t} \approx \frac{H_{i,j}^{n+\frac{1}{2}} - H_{i,j}^{n}}{\Delta t}, \tag{3.75}$$

$$W_{i,j}^{n} = \pm \frac{Q_{i,j}^{n}}{\Delta x \Delta y H_{i,j}^{n}}. \tag{3.76}$$

Substituting equations (3.73) – (3.76) into equation (3.2), for $1 < i < I-1$ and $1 < j < J-1$ at $t > 0$,

$$\begin{aligned} &-\alpha H_{i-1,j}^{n+\frac{1}{2}} + \left(1+2\alpha\right) H_{i,j}^{n+\frac{1}{2}} - \alpha H_{i+1,j}^{n+\frac{1}{2}} \\ &\qquad = \beta H_{i,j-1}^{n} + (1-2\beta) H_{i,j}^{n} + \beta H_{i,j+1}^{n} + \gamma \frac{Q_{i,j}^{n}}{\Delta x \Delta y H_{i,j}^{n}}. \end{aligned} \tag{3.77}$$

where $\xi = \frac{\left(\Delta t\right) K}{\left(\Delta x\right)^2 S}$, $\beta = \frac{\left(\Delta t\right) K}{\left(\Delta y\right)^2 S}$ and $\gamma = \frac{\Delta t}{S}$. For $i = 1$ and $1 < j < J-1$ at $t > 0$, substituting the unknown value on the west boundary by forward difference approximation $H_{0,j}^{n+\frac{1}{2}} = H_{1,j}^{n+\frac{1}{2}} - \left(\Delta x\right) B_W$ into equation (3.77),

$$\begin{aligned} &\left(1+\alpha\right) H_{1,j}^{n+\frac{1}{2}} - \alpha H_{2,j}^{n+\frac{1}{2}} \\ &\qquad = \beta H_{1,j-1}^{n} + (1-2\beta) H_{1,j}^{n} + \beta H_{1,j+1}^{n} + \gamma \frac{Q_{1,j}^{n}}{\Delta x \Delta y H_{1,j}^{n}} - \alpha\left(\Delta x\right) B_W. \end{aligned} \tag{3.78}$$

For $i = I-1$ and $1 < j < J-1$ at $t > 0$, substituting the unknown value on the east boundary by backward difference approximation $H_{I,j}^{n+\frac{1}{2}} = H_{I-1,j}^{n+\frac{1}{2}} + \left(\Delta x\right) B_E$ into equation (3.77),

$$\begin{aligned} &-\alpha H_{\mathrm{I}-2,j}^{n+\frac{1}{2}} + \left(1+\alpha\right) H_{\mathrm{I}-1,j}^{n+\frac{1}{2}} \\ &\qquad = \beta H_{\mathrm{I}-1,j-1}^{n} + (1-2\beta) H_{\mathrm{I}-1,j}^{n} + \beta H_{\mathrm{I}-1,j+1}^{n} + \gamma \frac{Q_{\mathrm{I}-1,j}^{n}}{\Delta x \Delta y H_{\mathrm{I}-1,j}^{n}} + \alpha\left(\Delta x\right) B_E. \end{aligned} \tag{3.79}$$

The equations (3.77) – (3.79) can be written in matrix form as follow,

$$AH^{n+\frac{1}{2}} = B^{n}, \tag{3.80}$$

where
$$A = \begin{bmatrix} 1+\alpha & -\alpha & & & \\ -\alpha & 1+2\alpha & -\alpha & & \\ & \vdots & \vdots & \vdots & \\ & & -\alpha & 1+2\alpha & -\alpha \\ & & & -\alpha & 1+\alpha \end{bmatrix}, \tag{3.81}$$

$$H^{n+\frac{1}{2}} = \begin{bmatrix} H_{1,j}^{n+\frac{1}{2}} \\ H_{1,j}^{n+\frac{1}{2}} \\ \vdots \\ H_{\mathrm{I}-2,j}^{n+\frac{1}{2}} \\ H_{\mathrm{I}-1,j}^{n+\frac{1}{2}} \end{bmatrix}, \tag{3.82}$$

$$B^{n} = \begin{bmatrix} \beta H_{1,j-1}^{n} + (1-2\beta) H_{1,j}^{n} + \beta H_{1,j+1}^{n} + \gamma \dfrac{Q_{1,j}^{n}}{\Delta x \Delta y H_{1,j}^{n}} - \alpha\left(\Delta x\right) B_W \\ \beta H_{2,j-1}^{n} + (1-2\beta) H_{2,j}^{n} + \beta H_{2,j+1}^{n} + \gamma \dfrac{Q_{2,j}^{n}}{\Delta x \Delta y H_{2,j}^{n}} \\ \vdots \\ \beta H_{\mathrm{I}-2,j-1}^{n} + (1-2\beta) H_{\mathrm{I}-2,j}^{n} + \beta H_{\mathrm{I}-2,j+1}^{n} + \gamma \dfrac{Q_{\mathrm{I}-2,j}^{n}}{\Delta x \Delta y H_{\mathrm{I}-2,j}^{n}} \\ \beta H_{\mathrm{I}-1,j-1}^{n} + (1-2\beta) H_{\mathrm{I}-1,j}^{n} + \beta H_{\mathrm{I}-1,j+1}^{n} + \gamma \dfrac{Q_{\mathrm{I}-1,j}^{n}}{\Delta x \Delta y H_{\mathrm{I}-1,j}^{n}} + \alpha\left(\Delta x\right) B_E \end{bmatrix}, \tag{3.83}$$

for all $1 \le n \le N$. The second stage, taking the central difference scheme in space and forward difference scheme in time into each terms of equation (3.2), then

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H_{i-1,j}^{n+\frac{1}{2}} - 2H_{i,j}^{n+\frac{1}{2}} + H_{i+1,j}^{n+\frac{1}{2}}}{\left(\Delta x\right)^2}, \tag{3.84}$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H_{i,j-1}^{n+1} - 2H_{i,j}^{n+1} + H_{i,j+1}^{n+1}}{\left(\Delta y\right)^2}, \tag{3.85}$$

$$\frac{\partial H}{\partial t} \approx \frac{H_{i,j}^{n+1} - H_{i,j}^{n+\frac{1}{2}}}{\Delta t}, \tag{3.86}$$

$$W_{i,j}^{n+\frac{1}{2}} = \pm \frac{Q_{i,j}^{n}}{\Delta x \Delta y H_{i,j}^{n+\frac{1}{2}}}. \tag{3.87}$$

Substituting equation (3.84) – (3.87) into equation (3.2), for $1 < i < I-1$ and $1 < j < J-1$ at $t > 0$,

$$\begin{aligned} -\beta H_{i,j-1}^{n+1} &+ \left(1+2\beta\right) H_{i,j}^{n+1} - \beta H_{i,j+1}^{n+1} \\ &= \alpha H_{i-1,j}^{n+\frac{1}{2}} + (1-2\alpha) H_{i,j}^{n+\frac{1}{2}} + \alpha H_{i+1,j}^{n+\frac{1}{2}} + \gamma \frac{Q_{i,j}^{n}}{\Delta x \Delta y H_{i,j}^{n+\frac{1}{2}}}. \end{aligned} \tag{3.88}$$

For $1 < i < I-1$ and $j = 1$ at $t > 0$, substituting the unknown value on the south boundary by forward difference approximation $H_{i,0}^{n+1} = H_{i,1}^{n+1} - \left(\Delta y\right) B_S$ into equation (3.88),

$$\begin{aligned} &\left(1+\beta\right) H_{i,1}^{n+1} - \beta H_{i,2}^{n+1} \\ &\quad = \alpha H_{i-1,1}^{n+\frac{1}{2}} + (1-2\alpha) H_{i,1}^{n+\frac{1}{2}} + \alpha H_{i+1,1}^{n+\frac{1}{2}} + \gamma \frac{Q_{i,1}^{n}}{\Delta x \Delta y H_{i,1}^{n+\frac{1}{2}}} - \beta\left(\Delta y\right) B_S. \end{aligned} \tag{3.89}$$

For $1 < i < I-1$ and $j = J-1$ at $t > 0$, substituting the unknown value on the north boundary by $H_{i,J}^{n+1} = H_{i,J-1}^{n+1} + \left(\Delta y\right) B_N$ into equation (3.88),

$$\begin{aligned} &-\beta H_{i,J-2}^{n+1} + \left(1+\beta\right) H_{i,J-1}^{n+1} \\ &\qquad = \alpha H_{i-1,J-1}^{n+\frac{1}{2}} + (1-2\alpha) H_{i,J-1}^{n+\frac{1}{2}} + \alpha H_{i+1,J-1}^{n+\frac{1}{2}} + \gamma \frac{Q_{i,J-1}^{n}}{\Delta x \Delta y H_{i,J-1}^{n+\frac{1}{2}}} + \beta\left(\Delta y\right) B_N . \end{aligned} \tag{3.90}$$

The equations (3.88) – (3.90) can be written in matrix form as follow,

$$AH^{n} = B^{n+\frac{1}{2}}, \tag{3.91}$$

where

$$A = \begin{bmatrix} 1+\beta & -\beta & & & \\ -\beta & 1+2\beta & -\beta & & \\ & \vdots & \vdots & \vdots & \\ & & -\beta & 1+2\beta & -\beta \\ & & & -\beta & 1+\beta \end{bmatrix}, \tag{3.92}$$

$$H^{n} = \begin{bmatrix} H_{\mathrm{i},1}^{m+1} \\ H_{i,2}^{m+1} \\ \vdots \\ H_{\mathrm{i,J}-2}^{m+1} \\ H_{\mathrm{i,J}-1}^{m+1} \end{bmatrix}, \tag{3.93}$$

$$B^{n+\frac{1}{2}} = \begin{bmatrix} \alpha H_{i-1,1}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,1}^{m+\frac{1}{2}} + \alpha H_{i+1,1}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,1}^{m}}{\Delta x \Delta y H_{i,1}^{m+\frac{1}{2}}} - \beta\left(\Delta y\right) B_S \\ \alpha H_{i-1,2}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,2}^{m+\frac{1}{2}} + \alpha H_{i+1,2}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,2}^{m}}{\Delta x \Delta y H_{i,2}^{m+\frac{1}{2}}} \\ \vdots \\ \alpha H_{i-1,J-2}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,\mathrm{J}-2}^{m+\frac{1}{2}} + \alpha H_{i+1,\mathrm{J}-2}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,\mathrm{J}-2}^{m}}{\Delta x \Delta y H_{i,\mathrm{J}-2}^{m+\frac{1}{2}}} \\ \alpha H_{i-1,J-1}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,J-1}^{m+\frac{1}{2}} + \alpha H_{i+1,J-1}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,J-1}^{m}}{\Delta x \Delta y H_{i,J-1}^{m+\frac{1}{2}}} + \beta\left(\Delta y\right) B_N \end{bmatrix}. \tag{3.94}$$

For all $1 \le n \le N$. The ADIM has an unconditionally stable scheme [13]. The calculated solution can be obtained in size of gridding.

3.3 Numerical experiments for two-dimensional transient groundwater flow model in drought area

A transient groundwater flow model is providing hydraulic head profile. The application of the numerical simulations of a transient groundwater flow model is tested using the hypothetical examples. From now, we assume the experimented groundwater area has dimension that $2.4\,km. \times 2.4\,km.$ The experimented area has homogeneous aquifer parameters, the initial hydraulic head is given $15m.$, the hydraulic conductivity $K = 15\,m/day$, storage capacity $S = 1\,m^{-1}$, grid spacing $\Delta x = \Delta y = 50\,m.$, number of grid spacing $I = J = 49$ and time step $\Delta t = 1\,day$ The five pumping wells is pumping the water from the ground.

3.3.1 Simulation 1 : Four pumping up and one injecting down wells

The pumping wells have the pumping rates as Table 3.1. This example will consider bounder line have no derivative boundary.

Table 3.1 The pumping rate each well (simulation 1).

Q(600m,600m)	Q(600m,1800m)	Q(1800m,600m)	Q(1800m,1800m)	Q(1200m,1200m)
-216 m^3/day	-216 m^3/day	-216 m^3/day	-216 m^3/day	864 m^3/day

Table 3.2 The hydraulic head at time $t = 3600$ day (FTCS - Simulation 1).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000
300	15.0000	14.9778	14.9472	14.9782	14.9948	14.9782	14.9472	14.9778	15.0000
600	15.0000	14.9472	14.4388	14.9583	15.0096	14.9583	14.4388	14.9472	15.0000
900	15.0000	14.9782	14.9583	15.0567	15.1795	15.0567	14.9583	14.9782	15.0000
1200	15.0000	14.9948	15.0096	15.1795	16.9293	15.1795	15.0096	14.9948	15.0000
1500	15.0000	14.9782	14.9583	15.0567	15.1795	15.0567	14.9583	14.9782	15.0000
1800	15.0000	14.9472	14.4388	14.9583	15.0096	14.9583	14.4388	14.9472	15.0000
2100	15.0000	14.9778	14.9472	14.9782	14.9948	14.9782	14.9472	14.9778	15.0000
2400	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000

Table 3.3 The hydraulic head at time $t = 3600$ day (BTCS - Simulation 1).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000
300	15.0000	14.9778	14.9472	14.9782	14.9948	14.9782	14.9472	14.9778	15.0000
600	15.0000	14.9472	14.4389	14.9583	15.0096	14.9583	14.4389	14.9472	15.0000
900	15.0000	14.9782	14.9583	15.0567	15.1795	15.0567	14.9583	14.9782	15.0000
1200	15.0000	14.9948	15.0096	15.1795	16.9292	15.1795	15.0096	14.9948	15.0000
1500	15.0000	14.9782	14.9583	15.0567	15.1795	15.0567	14.9583	14.9782	15.0000
1800	15.0000	14.9472	14.4389	14.9583	15.0096	14.9583	14.4389	14.9472	15.0000
2100	15.0000	14.9778	14.9472	14.9782	14.9948	14.9782	14.9472	14.9778	15.0000
2400	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000

Table 3.4 The hydraulic head at time $t = 3600$ day (ADEM - Simulation 1).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000
300	15.0000	14.9516	14.9127	14.9574	14.9854	14.9574	14.9127	14.9516	15.0000
600	15.0000	14.9127	14.4000	14.9565	15.0336	14.9565	14.4000	14.9127	15.0000
900	15.0000	14.9574	14.9565	15.1226	15.2887	15.1226	14.9565	14.9574	15.0000
1200	15.0000	14.9854	15.0336	15.2887	17.0766	15.2887	15.0336	14.9854	15.0000
1500	15.0000	14.9574	14.9565	15.1226	15.2887	15.1226	14.9565	14.9574	15.0000
1800	15.0000	14.9127	14.4000	14.9565	15.0336	14.9565	14.4000	14.9127	15.0000
2100	15.0000	14.9516	14.9127	14.9574	14.9854	14.9574	14.9127	14.9516	15.0000
2400	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000

Table 3.5 The hydraulic head at time $t = 3600$ day (ADIM - Simulation 1).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000
300	15.0000	14.9769	14.9469	14.9780	14.9947	14.9780	14.9469	14.9769	15.0000
600	15.0000	14.9461	14.4388	14.9583	15.0096	14.9583	14.4388	14.9461	15.0000
900	15.0000	14.9775	14.9583	15.0567	15.1795	15.0567	14.9583	14.9775	15.0000
1200	15.0000	14.9945	15.0096	15.1795	16.9292	15.1795	15.0096	14.9945	15.0000
1500	15.0000	14.9775	14.9583	15.0567	15.1795	15.0567	14.9583	14.9775	15.0000
1800	15.0000	14.9461	14.4388	14.9583	15.0096	14.9583	14.4388	14.9461	15.0000
2100	15.0000	14.9769	14.9469	14.9780	14.9947	14.9780	14.9469	14.9769	15.0000
2400	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000

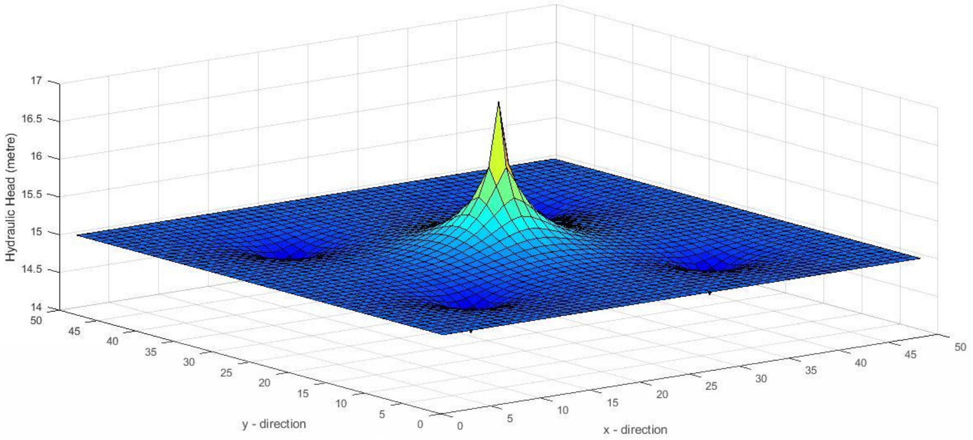

Figure 3.2 The surface graph of the hydraulic head (FTCS - Simulation 1).

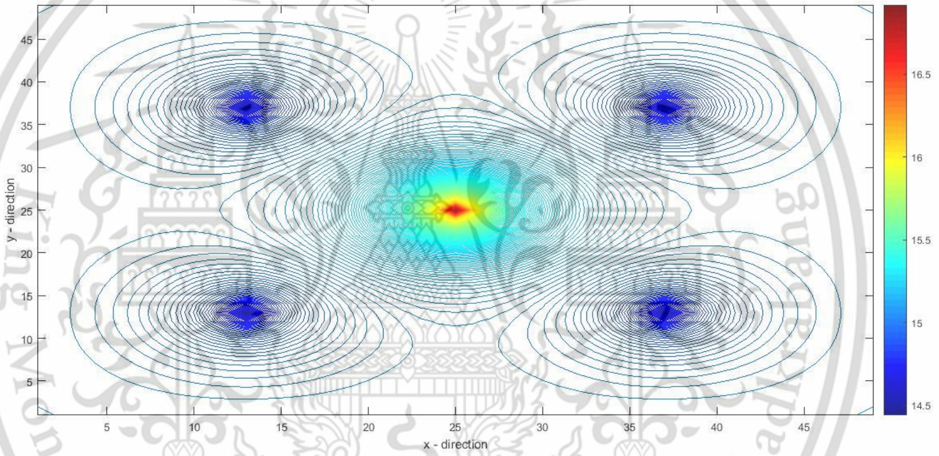

Figure 3.3 The contour graph of the hydraulic head (FTCS - Simulation 1).

The hydraulic head of FTCS technique can be seen in Table 3.2. The surface plot and contour plot of FTCS technique are shown in Figure 3.2 and Figure 3.3 respectively. Note that the hydraulic head near four pumping wells and are smaller than their farther point and the hydraulic head near an injection well are greater than their farther point.

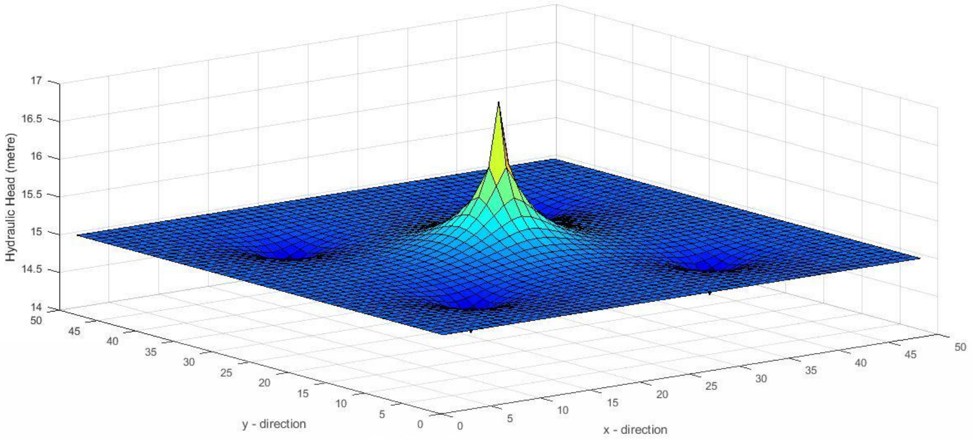

Figure 3.4 The surface graph of the hydraulic head (BTCS - Simulation 1).

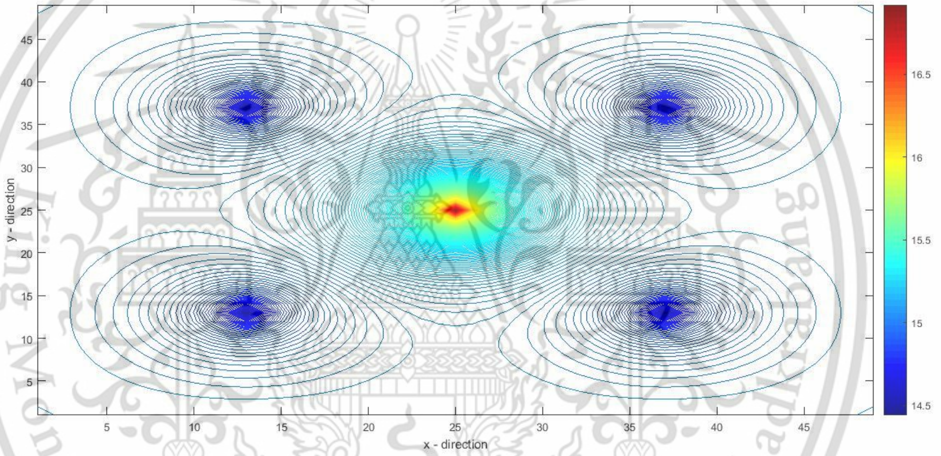

Figure 3.5 The contour graph of the hydraulic head value (BTCS - Simulation 1).

The hydraulic head of BTCS technique can be seen in Table 3.3. The surface plot and contour plot of FTCS technique are shown in Figure 3.4 and Figure 3.5 respectively. Note that the hydraulic head near four pumping wells and are smaller than their farther point and the hydraulic head near an injection well are greater than their farther point.

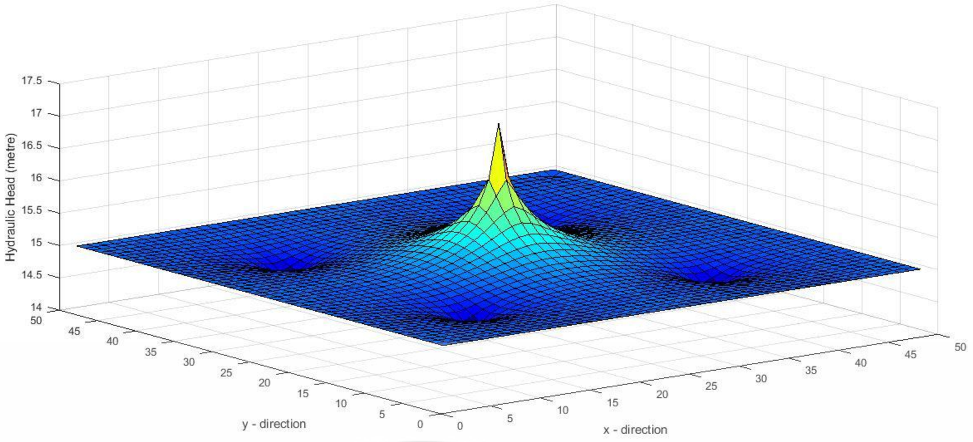

Figure 3.6 The surface graph of the hydraulic head (ADEM - Simulation 1).

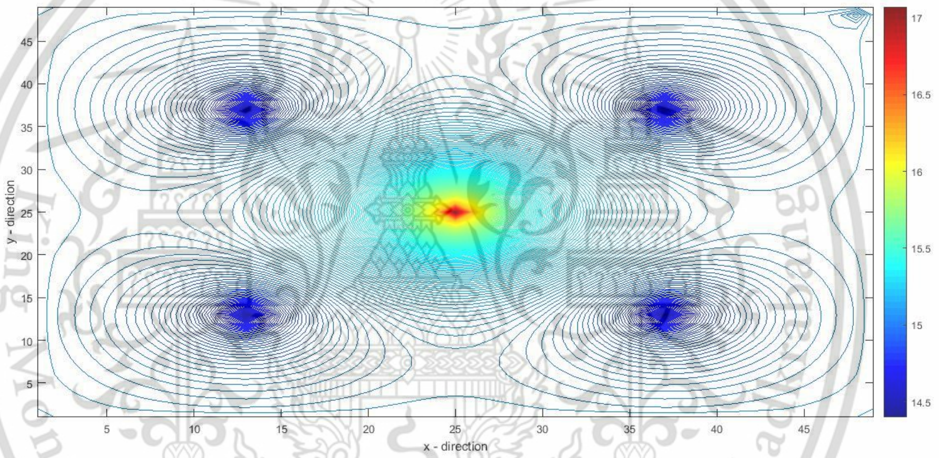

Figure 3.7 The contour graph of the hydraulic head (ADEM - Simulation 1).

The hydraulic head of ADEM technique can be seen in Table 3.4. The surface plot and contour plot of ADEM technique are shown in Figure 3.6 and Figure 3.7 respectively. Note that the hydraulic head near four pumping wells and are smaller than their farther point and the hydraulic head near an injection well are greater than their farther point.

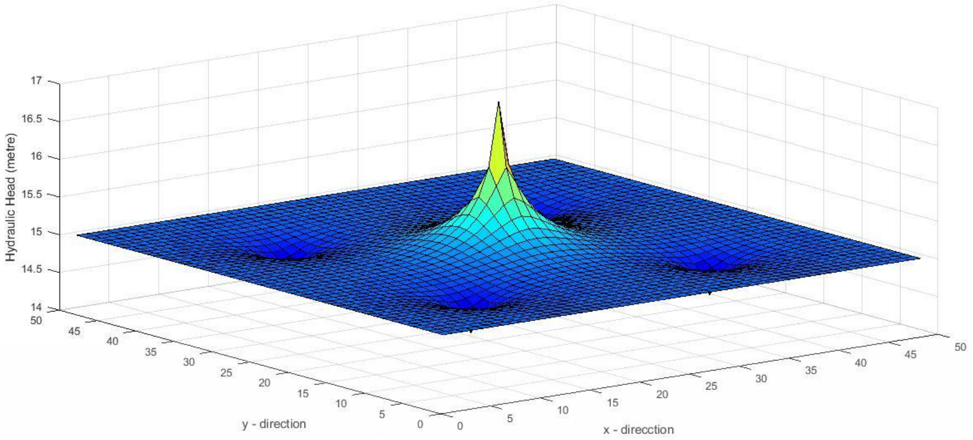

Figure 3.8 The surface graph of the hydraulic head (ADIM - Simulation 1).

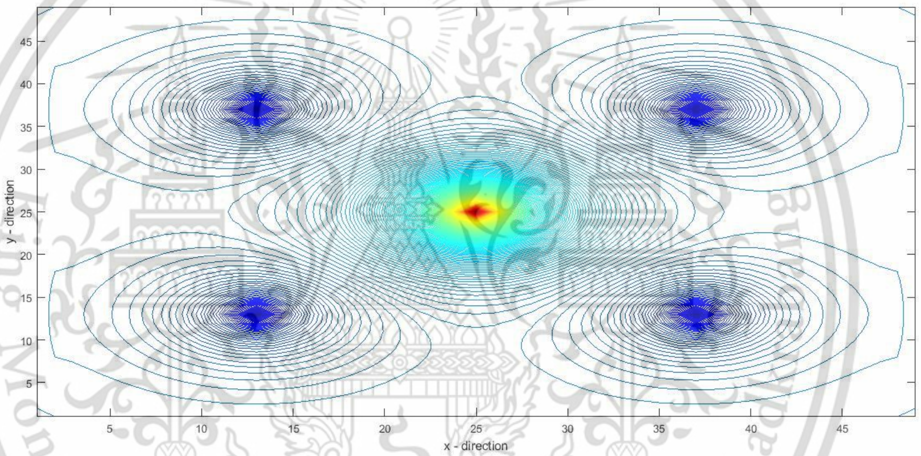

Figure 3.9 The contour graph of the hydraulic head (ADIM - Simulation 1).

The hydraulic head of ADIM technique can be seen in Table 3.5. The surface plot and contour plot of ADIM technique are shown in Figure 3.8 and Figure 3.9 respectively. Note that the hydraulic head near four pumping wells and are smaller than their farther point and the hydraulic head near an injection well are greater than their farther point.

Table 3.6 The total mass error (%) for each process technique.

Day\Methods	FTCS	BTCS	ADEM	ADIM
1	0.00×10^{0}	1.15×10^{-13}	1.21×10^{-7}	1.21×10^{-7}
3	3.64×10^{-7}	3.56×10^{-7}	1.75×10^{-6}	5.96×10^{-7}
5	1.19×10^{-6}	1.16×10^{-6}	5.08×10^{-6}	1.52×10^{-6}
10	5.14×10^{-6}	5.03×10^{-6}	1.98×10^{-5}	5.61×10^{-6}
30	4.23×10^{-5}	4.16×10^{-5}	1.40×10^{-4}	4.32×10^{-5}
50	1.04×10^{-4}	1.02×10^{-4}	3.17×10^{0}	1.05×10^{-4}
100	3.18×10^{-4}	3.16×10^{-4}	8.70×10^{-4}	3.19×10^{-4}
1000	6.80×10^{-3}	6.80×10^{-3}	1.49×10^{-2}	6.80×10^{-3}
1800	1.32×10^{-2}	1.32×10^{-2}	2.39×10^{-2}	1.32×10^{-2}
3000	2.02×10^{-2}	2.01×10^{-2}	2.13×10^{-2}	2.02×10^{-2}
3600	2.16×10^{-2}	2.16×10^{-2}	1.35×10^{-2}	2.16×10^{-2}

Table 3.7 The computing times (sec) for each proposed technique.

Day\Methods	FTCS	BTCS	ADEM	ADIM
1	0.04	0.49	0.05	0.32
3	0.04	1.42	0.05	0.41
5	0.04	2.31	0.05	0.54
10	0.04	4.75	0.07	0.83
30	0.04	13.95	0.13	1.94
50	0.05	22.87	0.11	2.89
100	0.06	45.99	0.18	5.54
1000	0.29	436.44	1.37	55.19
1800	0.49	785.26	2.48	98.87
3000	0.80	1312.12	4.03	165.63
3600	0.95	1551.12	4.82	202.02

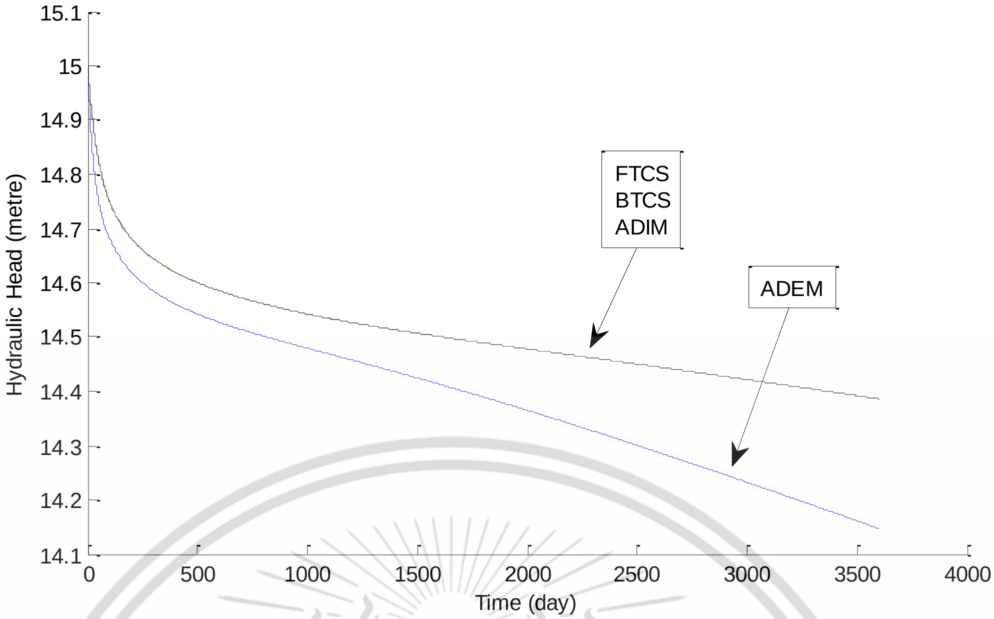

Figure 3.10 The variation of hydraulic head at $x = 600\,m.$ and $y = 600\,m.$ (all technique).

The line plot of all finite difference methods at $x = 600\,m.$ and $y = 600\,m.$ are shown in Figure 3.10. Note that the hydraulic head of FTCS, BTCS and ADIM technique have near value without ADEM technique.

Table 3.8 The stabilities for some grid spacing of FTCS, BTCS, ADEM and ADIM scheme.

Δx	Δy	Δt	FTCS	BTCS	ADEM	ADIM
100	100	150	stable	stable	stable	stable
		155	stable	stable	stable	stable
		160	stable	stable	stable	stable
		165	unstable	stable	stable	stable
		170	unstable	stable	stable	stable
		175	unstable	stable	stable	stable
		180	unstable	stable	stable	stable

3.3.2 Simulation 2 : Four pumping up wells with opened boundary

The four pumping wells is pumping the water from the ground at all times. There is no injection well. The pumping wells have the pumping rates as Table 3.9. Given boundary conditions as, $B_N = -0.005$, $B_S = 0.005$, $B_W = 0.005$ and $B_E = -0.005$.

Table 3.9 The pumping rate each well (simulation 2).

Q(600m,600m)	Q(600m,1800m)	Q(1800m,600m)	Q(1800m,1800m)	Q(1200m,1200m)
-216 m^3/day	-324 m^3/day	-432 m^3/day	-540 m^3/day	0 m^3/day

Table 3.10 The hydraulic head at time $t = 3600$ day (FTCS - Simulation 2).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.7095	14.1695	14.3161	14.3463	14.3525	14.3420	14.3081	14.1651	13.7075
300	14.1695	14.6193	14.7422	14.7960	14.8125	14.7840	14.7137	14.6069	14.1651
600	14.3161	14.7422	14.3846	14.9190	14.9591	14.8910	14.0848	14.7137	14.3081
900	14.3461	14.7957	14.9185	14.9725	14.9892	14.9608	14.8906	14.7837	14.3418
1200	14.3507	14.8086	14.9523	14.9855	14.9949	14.9856	14.9525	14.8087	14.3508
1500	14.3328	14.7584	14.8314	14.9356	14.9819	14.9482	14.8616	14.7713	14.3374
1800	14.2911	14.6533	13.4422	14.8310	14.9457	14.8612	13.7712	14.6841	14.2998
2100	14.1557	14.5808	14.6533	14.7581	14.8049	14.7710	14.6841	14.5941	14.1604
2400	13.7033	14.1557	14.2911	14.3327	14.3491	14.3373	14.2997	14.1604	13.7054

Table 3.11 The hydraulic head at time $t = 3600$ day (BTCS - Simulation 2).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.7118	14.1706	14.3173	14.3474	14.3536	14.3431	14.3092	14.1662	13.7098
300	14.1706	14.6192	14.7421	14.7960	14.8124	14.7840	14.7136	14.6068	14.1662
600	14.3173	14.7421	14.3846	14.9189	14.9590	14.8910	14.0847	14.7136	14.3092
900	14.3473	14.7958	14.9187	14.9726	14.9891	14.9607	14.8903	14.7835	14.3429
1200	14.3528	14.8105	14.9557	14.9873	14.9949	14.9837	14.9490	14.8067	14.3511
1500	14.3386	14.7714	14.8618	14.9483	14.9819	14.9355	14.8312	14.7582	14.3339
1800	14.3009	14.6841	13.7712	14.8611	14.9456	14.8310	13.4422	14.6533	14.2922
2100	14.1616	14.5940	14.6840	14.7710	14.8048	14.7580	14.6533	14.5807	14.1568
2400	13.7077	14.1616	14.3009	14.3384	14.3502	14.3338	14.2922	14.1568	13.7056

Table 3.12 The hydraulic head at time $t = 3600$ day (ADEM - Simulation 2).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.0697	13.6245	13.8804	13.9830	14.0078	13.9655	13.8524	13.6031	13.0527
300	13.6245	14.1672	14.3970	14.5256	14.5617	14.4969	14.3431	14.1338	13.6019
600	13.8801	14.3967	14.1355	14.7551	14.8164	14.7064	13.7972	14.3423	13.8504
900	13.9813	14.5235	14.7528	14.8820	14.9188	14.8538	14.6995	14.4906	13.9591
1200	14.0003	14.5517	14.8036	14.9107	14.9399	14.8946	14.7779	14.5319	13.9843
1500	13.9455	14.4649	14.6531	14.8243	14.8865	14.7936	14.5950	14.4292	13.9218
1800	13.8228	14.2865	13.4410	14.6462	14.7650	14.5927	13.0638	14.2267	13.7906
2100	13.5808	14.0989	14.2856	14.4585	14.5220	14.4271	14.2264	14.0624	13.5564
2400	13.0362	13.5796	13.8209	13.9391	13.9768	13.9200	13.7903	13.5564	13.0179

Table 3.13 The hydraulic head at time $t = 3600$ day (ADIM - Simulation 2).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.7106	14.1728	14.3203	14.3505	14.3568	14.3462	14.3122	14.1683	13.7086
300	14.1674	14.6192	14.7430	14.7970	14.8134	14.7850	14.7145	14.6069	14.1629
600	14.3132	14.7413	14.3846	14.9191	14.9592	14.8912	14.0847	14.7128	14.3051
900	14.3431	14.7949	14.9186	14.9726	14.9892	14.9607	14.8902	14.7826	14.3387
1200	14.3485	14.8095	14.9556	14.9873	14.9949	14.9837	14.9488	14.8058	14.3468
1500	14.3344	14.7704	14.8617	14.9483	14.9819	14.9355	14.8311	14.7572	14.3296
1800	14.2968	14.6832	13.7712	14.8613	14.9458	14.8311	13.4421	14.6524	14.2881
2100	14.1583	14.5940	14.6849	14.7720	14.8059	14.7590	14.6542	14.5807	14.1535
2400	13.7066	14.1637	14.3039	14.3415	14.3533	14.3369	14.2952	14.1589	13.7045

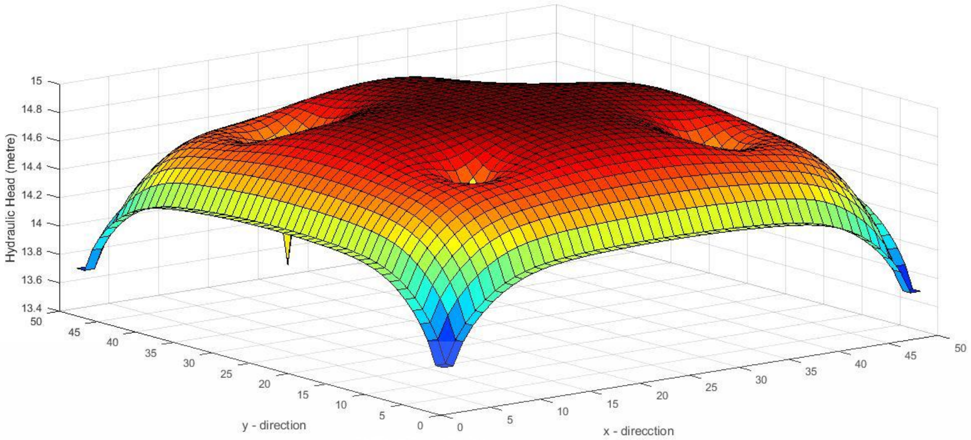

Figure 3.11 The surface graph of the hydraulic head (FTCS - Simulation 2).

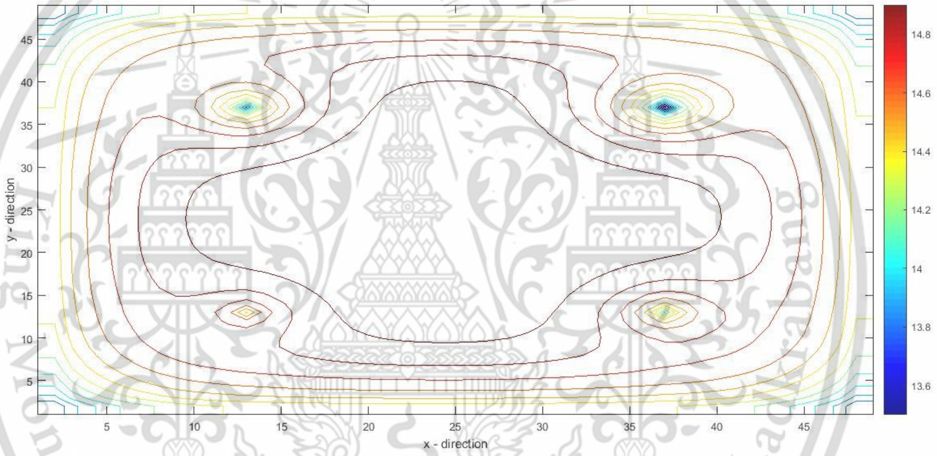

Figure 3.12 The contour graph of the hydraulic head (FTCS - Simulation 2).

The hydraulic head of FTCS technique can be seen in Table 3.10. The surface plot and contour plot are shown in Figure 3.11 and Figure 3.12 respectively. Note that the hydraulic head near four pumping wells and the border line are smaller than their farther point.

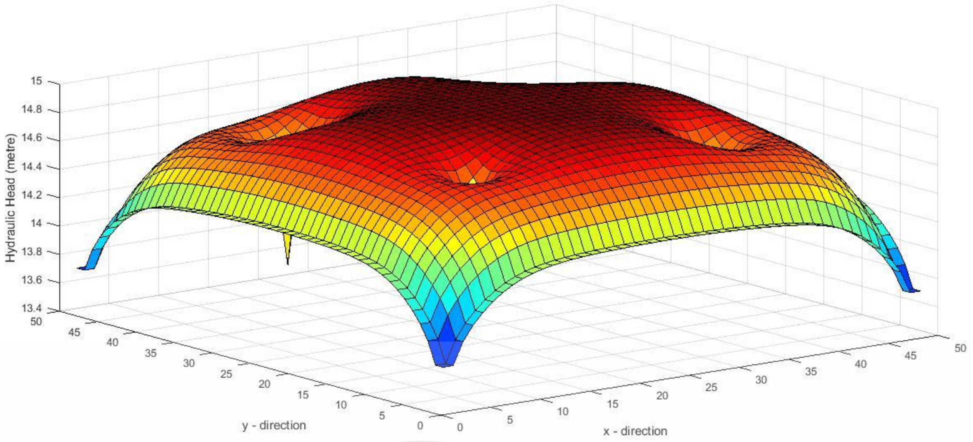

Figure 3.13 The surface graph of the hydraulic head (BTCS - Simulation 2).

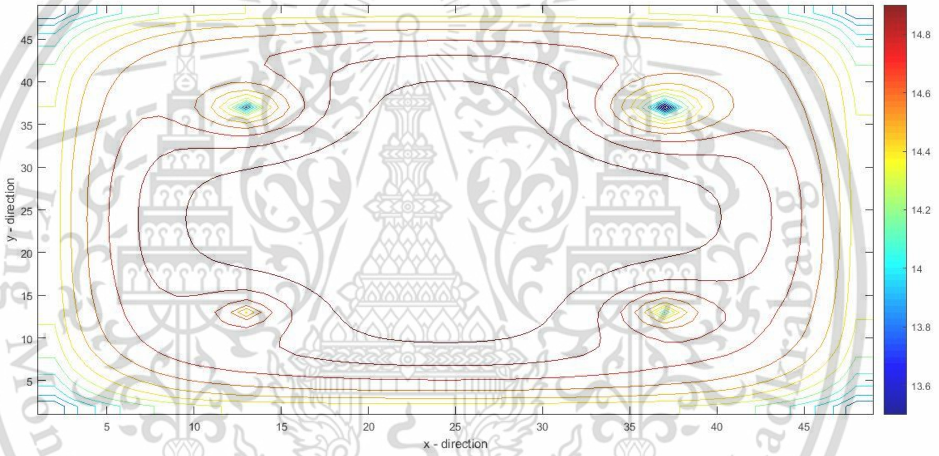

Figure 3.14 The contour graph of the hydraulic head (BTCS - Simulation 2).

The hydraulic head of BTCS technique can be seen in Table 3.11. The surface plot and contour plot are shown in Figure 3.13 and Figure 3.14 respectively. Note that the hydraulic head near four pumping wells and the border line are smaller than their farther point.

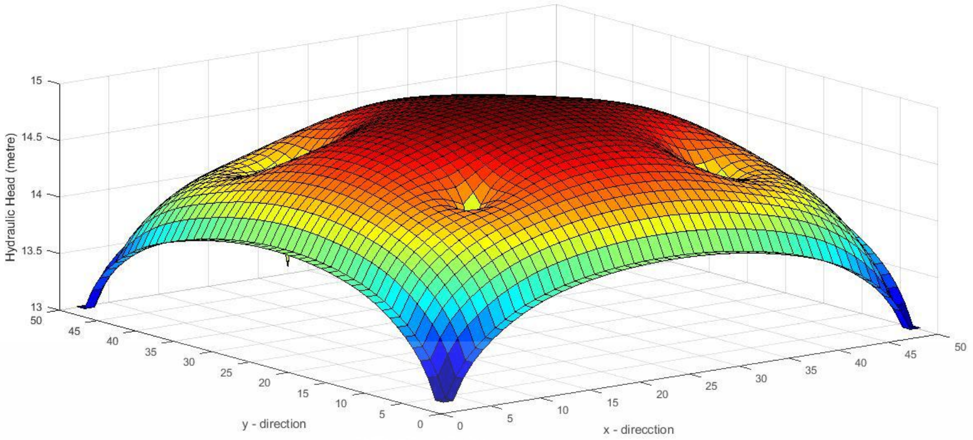

Figure 3.15 The surface graph of the hydraulic head (ADEM - Simulation 2).

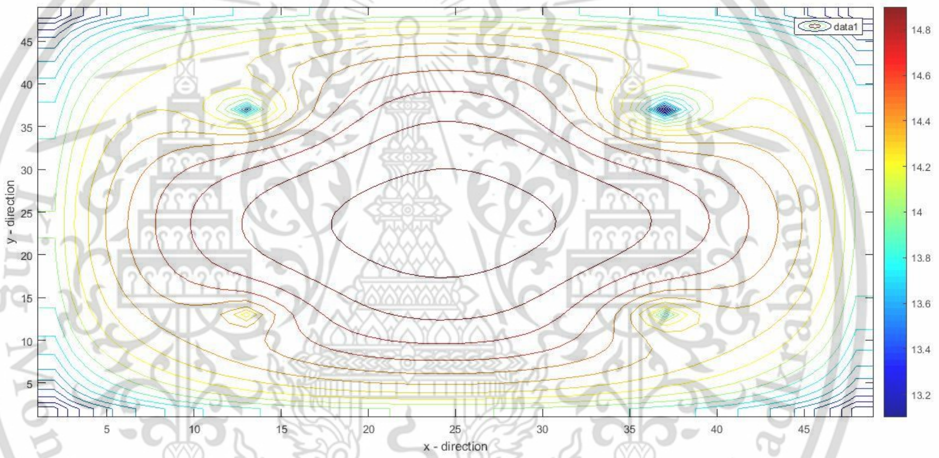

Figure 3.16 The contour graph of the hydraulic head (ADEM – Simulation 2).

The hydraulic head of ADEM technique can be seen in Table 3.12. The surface plot and contour plot are shown in Figure 3.15 and Figure 3.16 respectively. Note that the hydraulic head near four pumping wells and the border line are smaller than their farther point.

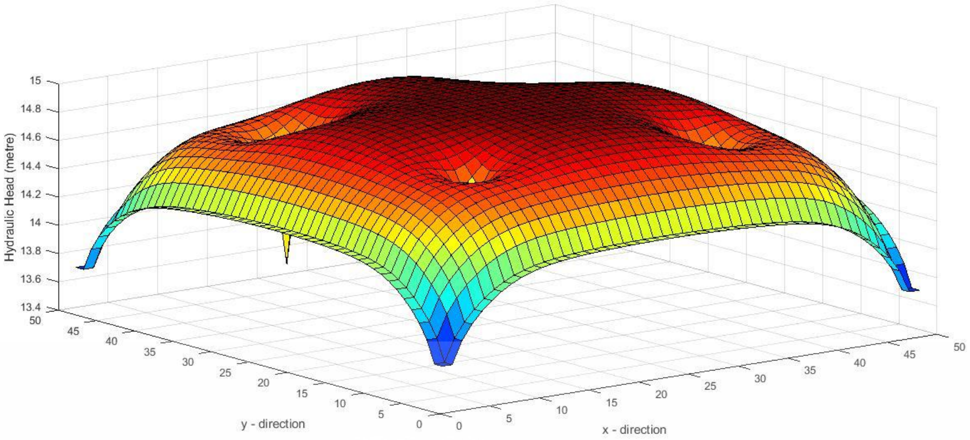

Figure 3.17 The surface graph of the hydraulic head (ADIM - Simulation 2).

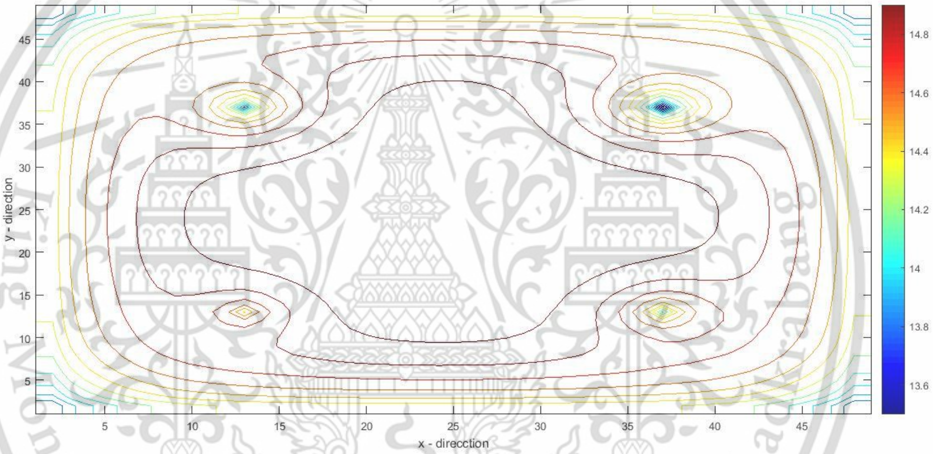

Figure 3.18 The contour graph of the hydraulic head (ADIM - Simulation 2)

The hydraulic head of ADIM technique can be seen in Table 3.13. The surface plot and contour plot are shown in Figure 3.17 and Figure 3.18 respectively. Note that the hydraulic head near four pumping wells and the border line are smaller than their farther point.

3.3.3 Simulation 3 : Four pumping up wells and one injecting down with opened boundary

The four pumping wells is pumping the water from the ground. There is an injection well which injection the water from a reservoir. The pumping wells and injection well have the pumping rates as table 3.11 at all times. Given boundary conditions as, $B_N = -0.005$, $B_S = 0.005$, $B_W = 0.005$ and $B_E = -0.005$.

Table 3.14 The pumping rate each well (simulation 3).

Q(600m,600m)	Q(600m,1800m)	Q(1800m,600m)	Q(1800m,1800m)	Q(1200m,1200m)
-216 m^3/day	-324 m^3/day	-432 m^3/day	-540 m^3/day	864 m^3/day

Table 3.15 The hydraulic head at time $t = 3600$ day (FTCS - Simulation 3).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.7095	14.1695	14.3162	14.3464	14.3526	14.3421	14.3081	14.1651	13.7075
300	14.1695	14.6193	14.7424	14.7969	14.8140	14.7849	14.7139	14.6069	14.1651
600	14.3162	14.7424	14.3871	14.9311	14.9811	14.9032	14.0872	14.7139	14.3081
900	14.3463	14.7968	14.9309	15.0526	15.1756	15.0406	14.9025	14.7845	14.3419
1200	14.3518	14.8121	14.9777	15.1738	16.9270	15.1702	14.9710	14.8083	14.3501
1500	14.3376	14.7723	14.8740	15.0283	15.1684	15.0154	14.8433	14.7591	14.3329
1800	14.2998	14.6843	13.7738	14.8733	14.9676	14.8431	13.4447	14.6535	14.2911
2100	14.1604	14.5941	14.6843	14.7719	14.8064	14.7590	14.6535	14.5808	14.1557
2400	13.7054	14.1604	14.2998	14.3374	14.3492	14.3328	14.2911	14.1557	13.7033

Table 3.16 The hydraulic head at time $t = 3600$ day (BTCS - Simulation 3).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.7118	14.1706	14.3173	14.3475	14.3538	14.3432	14.3092	14.1662	13.7098
300	14.1706	14.6192	14.7423	14.7969	14.8139	14.7849	14.7138	14.6068	14.1662
600	14.3173	14.7423	14.3870	14.9310	14.9810	14.9031	14.0872	14.7138	14.3092
900	14.3474	14.7967	14.9308	15.0526	15.1756	15.0406	14.9025	14.7844	14.3430
1200	14.3529	14.8120	14.9777	15.1738	16.9270	15.1701	14.9710	14.8082	14.3512
1500	14.3387	14.7723	14.8739	15.0283	15.1683	15.0154	14.8433	14.7591	14.3340
1800	14.3009	14.6843	13.7738	14.8733	14.9676	14.8431	13.4447	14.6535	14.2923
2100	14.1616	14.5940	14.6842	14.7719	14.8064	14.7589	14.6535	14.5807	14.1568
2400	13.7077	14.1616	14.3009	14.3385	14.3504	14.3339	14.2923	14.1568	13.7056

Table 3.17 The hydraulic head at time $t = 3600$ day (ADEM - Simulation 3).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.0699	13.6251	13.8825	13.9875	14.0136	13.9700	13.8545	13.6038	13.0530
300	13.6251	14.1687	14.4024	14.5380	14.5782	14.5093	14.3486	14.1353	13.6026
600	13.8822	14.4021	14.1577	14.8111	14.8966	14.7624	13.8199	14.3477	13.8525
900	13.9858	14.5358	14.8088	15.0651	15.2428	15.0368	14.7555	14.5030	13.9636
1200	14.0061	14.5683	14.8838	15.2348	17.0441	15.2186	14.8581	14.5484	13.9901
1500	13.9500	14.4772	14.7091	15.0074	15.2105	14.9767	14.6510	14.4416	13.9262
1800	13.8249	14.2919	13.4642	14.7023	14.8453	14.6487	13.0876	14.2322	13.7927
2100	13.5815	14.1005	14.2911	14.4709	14.5385	14.4395	14.2319	14.0640	13.5571
2400	13.0364	13.5803	13.8230	13.9436	13.9827	13.9245	13.7924	13.5570	13.0181

Table 3.18 The hydraulic head at time $t = 3600$ day (ADIM - Simulation 3).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.7106	14.1728	14.3203	14.3506	14.3569	14.3463	14.3122	14.1683	13.7086
300	14.1674	14.6192	14.7432	14.7979	14.8149	14.7859	14.7147	14.6069	14.1629
600	14.3132	14.7415	14.3871	14.9312	14.9812	14.9033	14.0872	14.7130	14.3051
900	14.3432	14.7958	14.9307	15.0526	15.1756	15.0406	14.9023	14.7835	14.3388
1200	14.3487	14.8111	14.9776	15.1738	16.9270	15.1702	14.9708	14.8073	14.3470
1500	14.3345	14.7713	14.8738	15.0283	15.1684	15.0154	14.8432	14.7581	14.3297
1800	14.2968	14.6834	13.7738	14.8734	14.9678	14.8432	13.4447	14.6526	14.2881
2100	14.1583	14.5941	14.6851	14.7729	14.8074	14.7600	14.6544	14.5808	14.1535
2400	13.7066	14.1637	14.3039	14.3416	14.3535	14.3370	14.2953	14.1589	13.7045

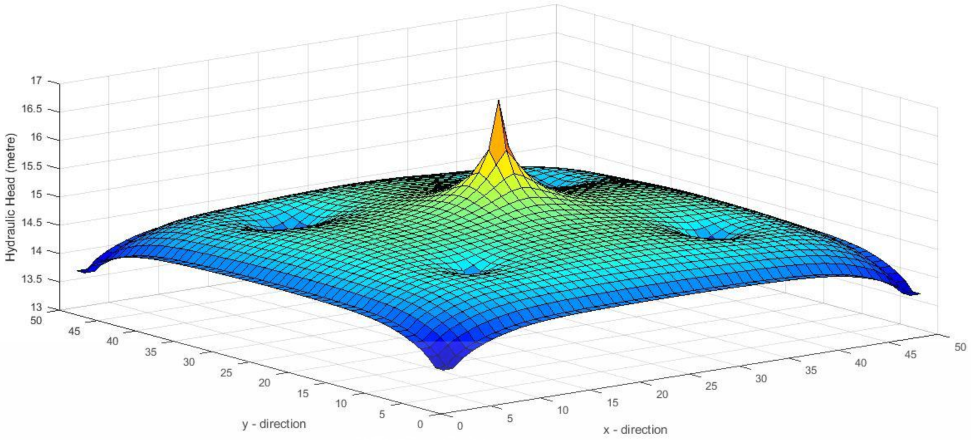

Figure 3.19 The surface graph of the hydraulic head (FTCS - Simulation 3).

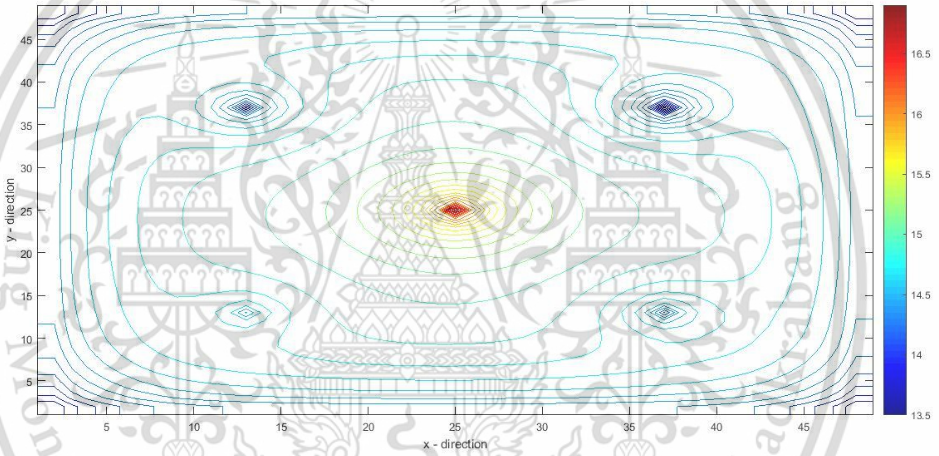

Figure 3.20 The contour graph of the hydraulic head (FTCS - Simulation 3).

The numerical hydraulic head values of FTCS technique can be seen in Table 3.15. The surface plot and contour plot are shown in Figure 3.19 and Figure 3.20 respectively. Note that the hydraulic head value near four pumping wells and the border line are smaller than their farther point and the hydraulic head value near an injection well are greater than their farther point.

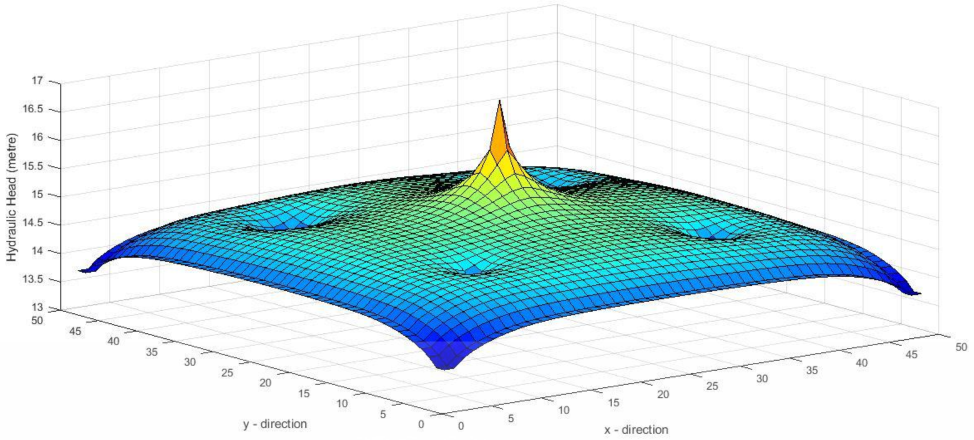

Figure 3.21 The surface graph of the hydraulic head (BTCS - Simulation 3).

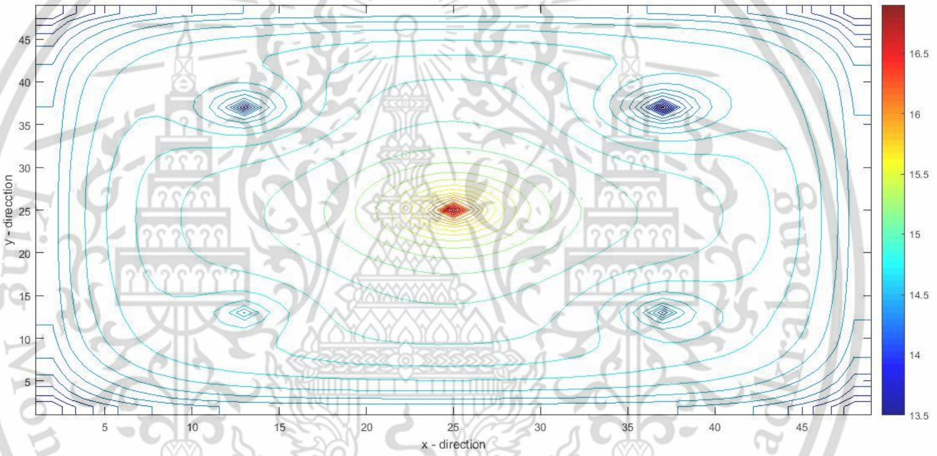

Figure 3.22 The contour graph of the hydraulic head (BTCS - Simulation 3).

The numerical hydraulic head of BTCS technique can be seen in Table 3.16. The surface plot and contour plot are shown in Figure 3.21 and Figure 3.22 respectively. Note that the hydraulic head near four pumping wells and the border line are smaller than their farther point and the hydraulic head near an injection well are greater than their farther point.

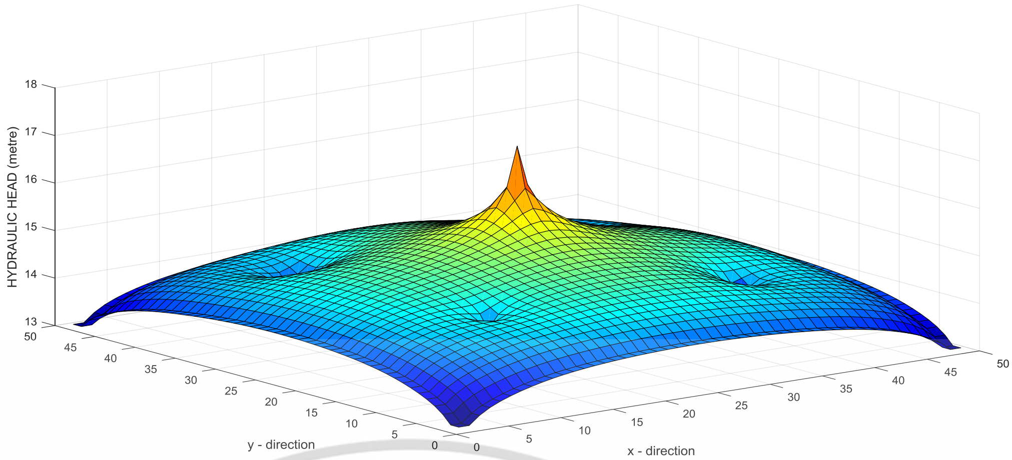

Figure 3.23 The surface graph of the hydraulic head (ADEM - Simulation 3)

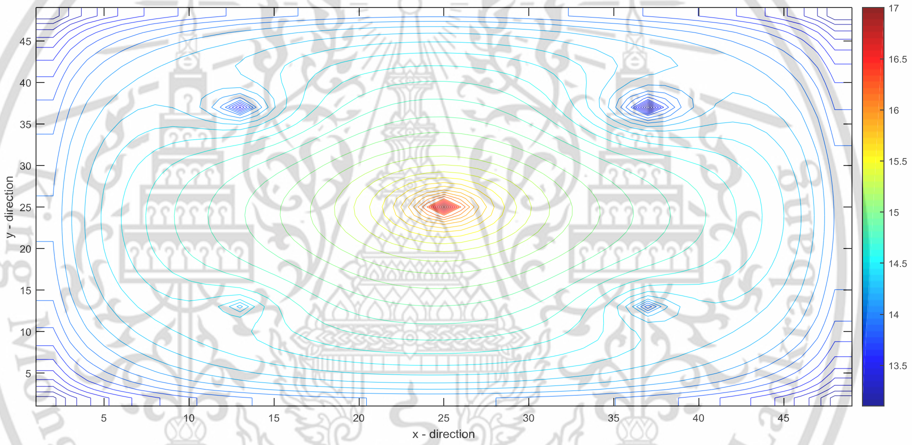

Figure 3.24 The contour graph of the hydraulic head (ADEM - Simulation 3).

The numerical hydraulic head of ADEM technique can be seen in Table 3.17. The surface plot and contour plot are shown in Figure 3.23 and Figure 3.24 respectively. Note that the hydraulic head near four pumping wells and the border line are smaller than their farther point and the hydraulic head near an injection well are greater than their farther point.

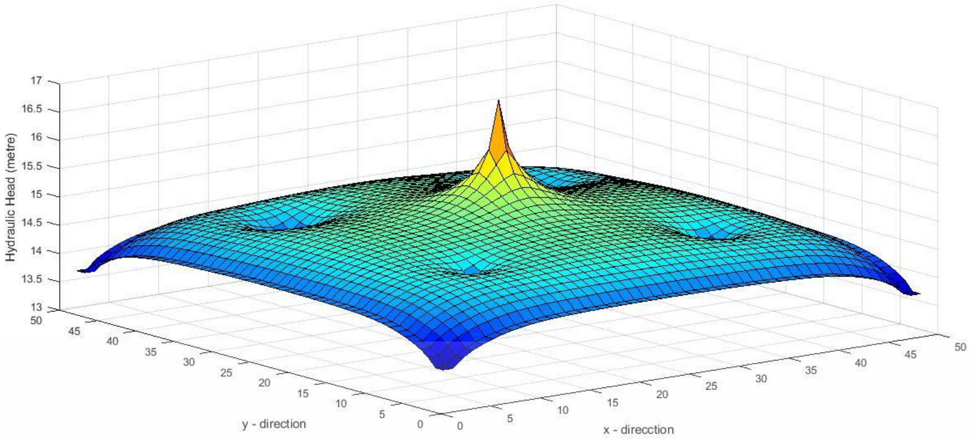

Figure 3.25 The surface graph of the hydraulic head (ADIM - Simulation 3).

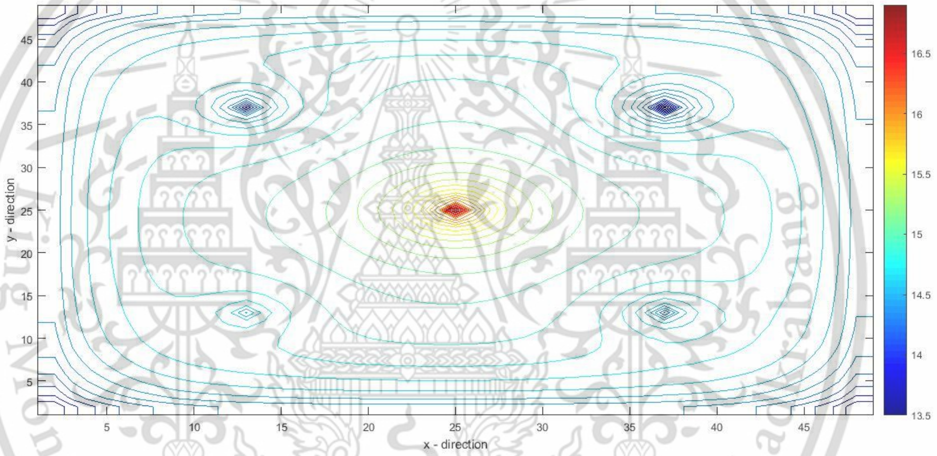

Figure 3.26 The contour graph of the hydraulic head (ADIM - Simulation 3).

The numerical hydraulic head values of ADIM technique can be seen in Table 3.18. The surface plot and contour plot are shown in Figure 3.25 and Figure 3.26 respectively. Note that the hydraulic head near four pumping wells and the border line are smaller than their farther point and the hydraulic head near an injection well are greater than their farther point.

3.3.4 Simulation 4 : Four pumping up wells and one period injecting down well with opened boundary

The four pumping wells is pumping the water from the ground at all times. There is an injection well is injection the water from a reservoir. After 6 months, the rainy season has coming, the injection well will inject the water along the rainy season for 6 months later. The pumping wells and injection well have the pumping rates as table 3.13. Given boundary conditions as, $B_N = -0.005$, $B_S = 0.005$, $B_W = 0.005$ and $B_E = -0.005$.

Table 3.19 The pumping rate each well (simulation 4).

Q(600m,600m)	Q(600m,1800m)	Q(1800m,600m)	Q(1800m,1800m)	Q(1200m,1200m)
-216 m^3/day	-324 m^3/day	-432 m^3/day	-540 m^3/day	864 m^3/day

Table 3.20 The hydraulic head at time $t = 3600$ day (FTCS - Simulation 4).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.7095	14.1695	14.3162	14.3463	14.3526	14.3420	14.3081	14.1651	13.7075
300	14.1695	14.6193	14.7423	14.7966	14.8133	14.7845	14.7138	14.6069	14.1651
600	14.3162	14.7423	14.3860	14.9257	14.9711	14.8978	14.0862	14.7138	14.3081
900	14.3462	14.7963	14.9252	15.0156	15.0885	15.0039	14.8973	14.7843	14.3419
1200	14.3508	14.8095	14.9643	15.0849	15.5387	15.0850	14.9645	14.8096	14.3509
1500	14.3329	14.7589	14.8381	14.9787	15.0813	14.9913	14.8683	14.7718	14.3375
1800	14.2911	14.6535	13.4436	14.8377	14.9577	14.8679	13.7727	14.6842	14.2998
2100	14.1557	14.5808	14.6535	14.7586	14.8058	14.7715	14.6842	14.5941	14.1604
2400	13.7033	14.1557	14.2911	14.3327	14.3492	14.3373	14.2998	14.1604	13.7054

Table 3.21 The hydraulic head at time $t = 3600$ day (BTCS - Simulation 4).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.7118	14.1706	14.3173	14.3475	14.3537	14.3432	14.3092	14.1662	13.7098
300	14.1706	14.6192	14.7422	14.7965	14.8133	14.7845	14.7137	14.6068	14.1662
600	14.3173	14.7422	14.3860	14.9256	14.9711	14.8977	14.0861	14.7137	14.3092
900	14.3473	14.7962	14.9252	15.0156	15.0885	15.0039	14.8973	14.7842	14.3430
1200	14.3520	14.8094	14.9643	15.0848	15.5399	15.0850	14.9645	14.8096	14.3520
1500	14.3340	14.7588	14.8381	14.9787	15.0813	14.9913	14.8683	14.7718	14.3386
1800	14.2923	14.6534	13.4436	14.8377	14.9577	14.8679	13.7727	14.6842	14.3009
2100	14.1568	14.5807	14.6534	14.7586	14.8057	14.7715	14.6842	14.5940	14.1616
2400	13.7056	14.1568	14.2922	14.3339	14.3503	14.3385	14.3009	14.1616	13.7077

Table 3.22 The hydraulic head at time $t = 3600$ day (ADEM - Simulation 4).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.0698	13.6249	13.8816	13.9856	14.0110	13.9681	13.8536	13.6035	13.0529
300	13.6248	14.1680	14.4000	14.5324	14.5708	14.5038	14.3462	14.1347	13.6023
600	13.8810	14.3994	14.1472	14.7850	14.8595	14.7369	13.8101	14.3457	13.8519
900	13.9820	14.5281	14.7804	14.9776	15.0909	14.9528	14.7324	14.4997	13.9635
1200	13.9958	14.5505	14.8333	15.0744	15.4841	15.0751	14.8344	14.5514	13.9953
1500	13.9271	14.4381	14.6277	14.8926	15.0585	14.9201	14.6809	14.4697	13.9452
1800	13.7932	14.2301	13.0772	14.6232	14.8082	14.6763	13.4534	14.2893	13.8227
2100	13.5579	14.0633	14.2295	14.4340	14.5311	14.4653	14.2887	14.0998	13.5801
2400	13.0191	13.5567	13.7915	13.9225	13.9801	13.9416	13.8221	13.5800	13.0352

Table 3.23 The hydraulic head at time $t = 3600$ day (ADIM - Simulation 4).

x\y	0	300	600	900	1200	1500	1800	2100	2400
0	13.7106	14.1728	14.3203	14.3506	14.3569	14.3463	14.3122	14.1683	13.7086
300	14.1674	14.6192	14.7431	14.7975	14.8143	14.7855	14.7146	14.6069	14.1629
600	14.3132	14.7414	14.3860	14.9258	14.9713	14.8979	14.0861	14.7129	14.3051
900	14.3431	14.7953	14.9251	15.0156	15.0885	15.0039	14.8972	14.7832	14.3388
1200	14.3477	14.8085	14.9642	15.0848	15.5392	15.0849	14.9644	14.8086	14.3478
1500	14.3298	14.7579	14.8380	14.9787	15.0813	14.9913	14.8682	14.7708	14.3344
1800	14.2881	14.6526	13.4436	14.8378	14.9578	14.8680	13.7727	14.6834	14.2968
2100	14.1535	14.5808	14.6543	14.7596	14.8067	14.7725	14.6851	14.5941	14.1583
2400	13.7045	14.1589	14.2952	14.3370	14.3534	14.3416	14.3039	14.1637	13.7066

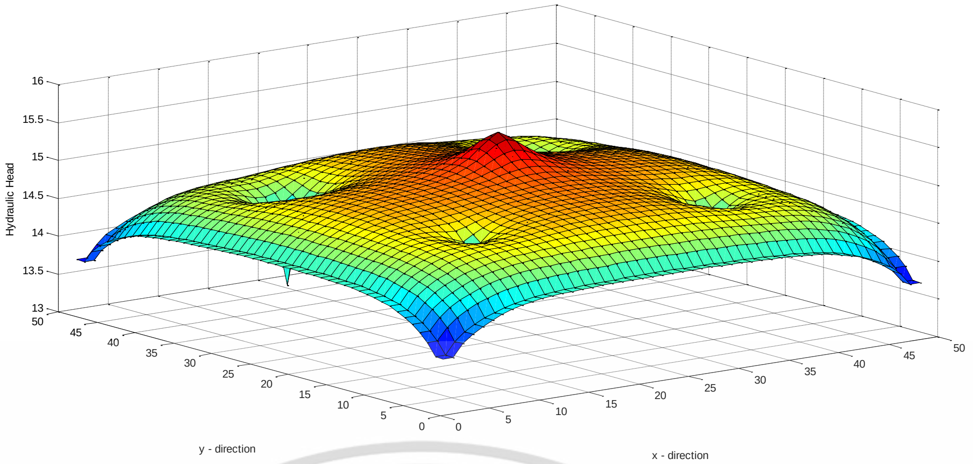

Figure 3.27 The surface graph of the hydraulic head (FTCS - Simulation 4).

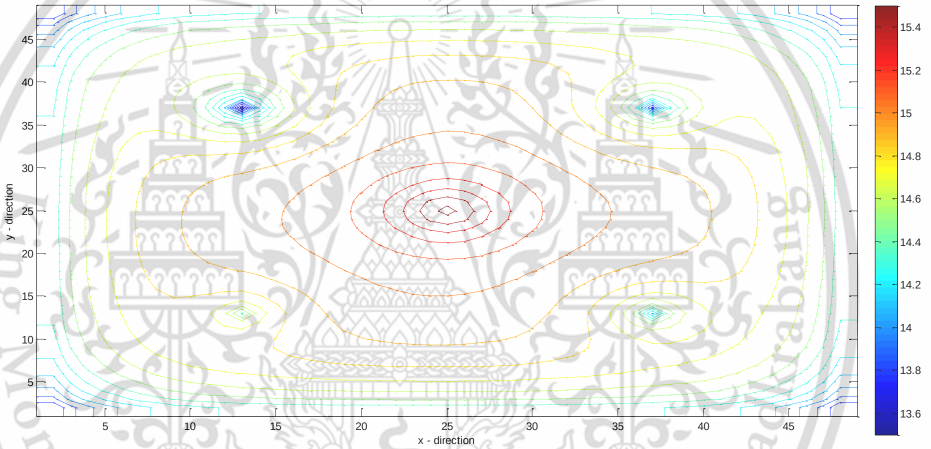

Figure 3.28 The contour graph of the hydraulic head (FTCS - Simulation 4).

The numerical hydraulic head values of FTCS technique can be seen in Table 3.20. The surface plot and contour plot are shown in Figure 3.27 and Figure 3.28 respectively. Note that the hydraulic head value near four pumping wells and the border line are smaller than their farther point and the hydraulic head value near an injection well are greater than their farther point.

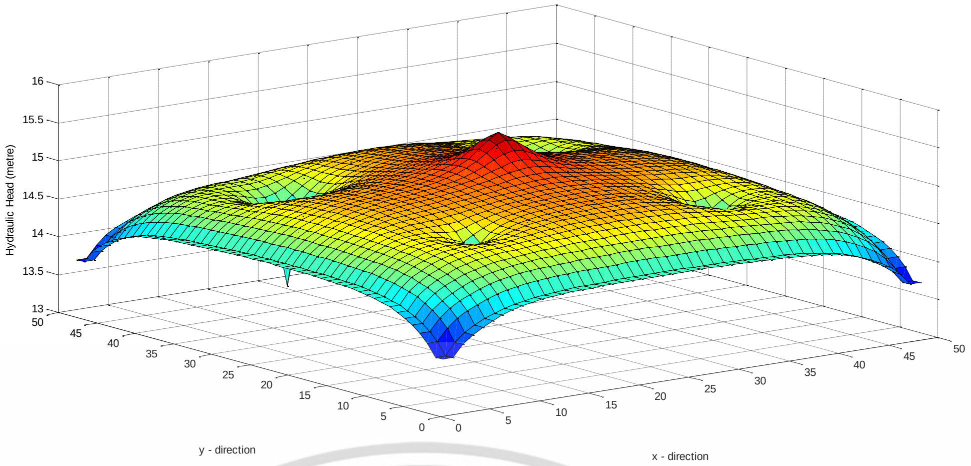

Figure 3.29 The surface graph of the hydraulic head (BTCS - Simulation 4).

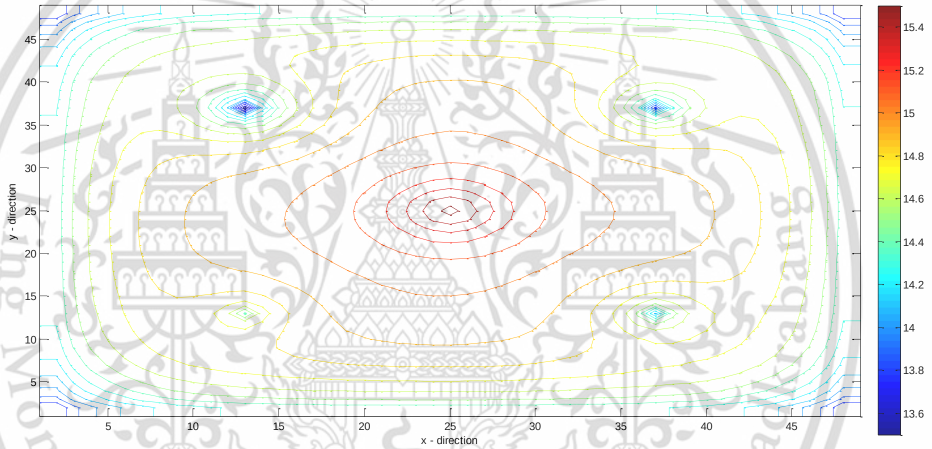

Figure 3.30 The contour graph of the hydraulic head (BTCS - Simulation 4).

The numerical hydraulic head of BTCS technique can be seen in Table 3.21. The surface plot and contour plot are shown in Figure 3.29 and Figure 3.30 respectively. Note that the hydraulic head near four pumping wells and the border line are smaller than their farther point and the hydraulic head near an injection well are greater than their farther point.

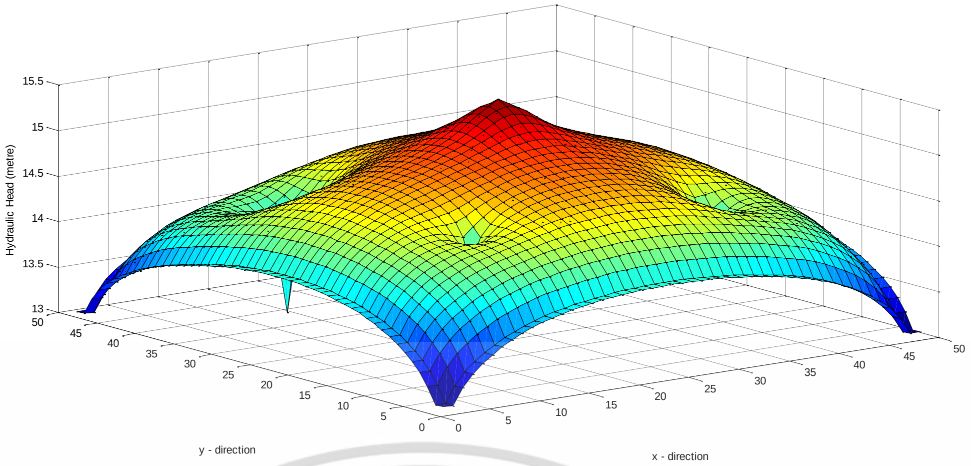

Figure 3.31 The surface graph of the hydraulic head (ADEM - Simulation 4).

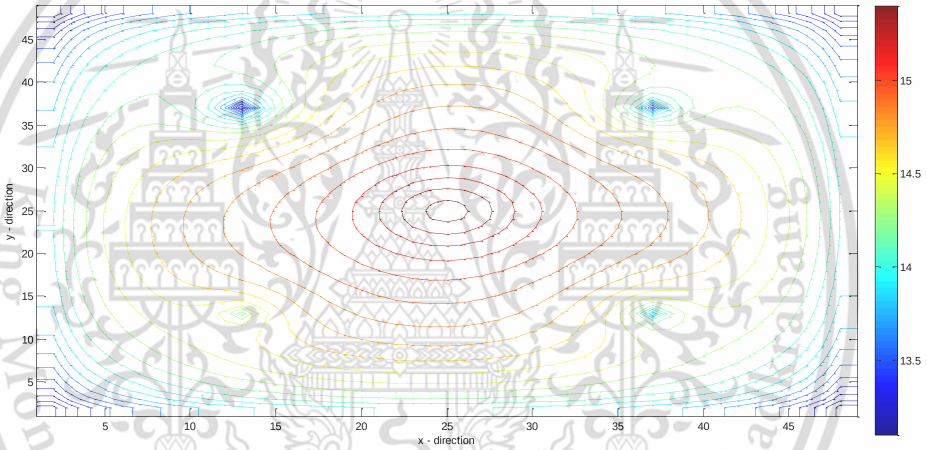

Figure 3.32 The contour graph of the hydraulic head (ADEM - Simulation 4).

The numerical hydraulic head of ADEM technique can be seen in Table 3.22. The surface plot and contour plot are shown in Figure 3.31 and Figure 3.32 respectively. Note that the hydraulic head near four pumping wells and the border line are smaller than their farther point and the hydraulic head near an injection well are greater than their farther point.

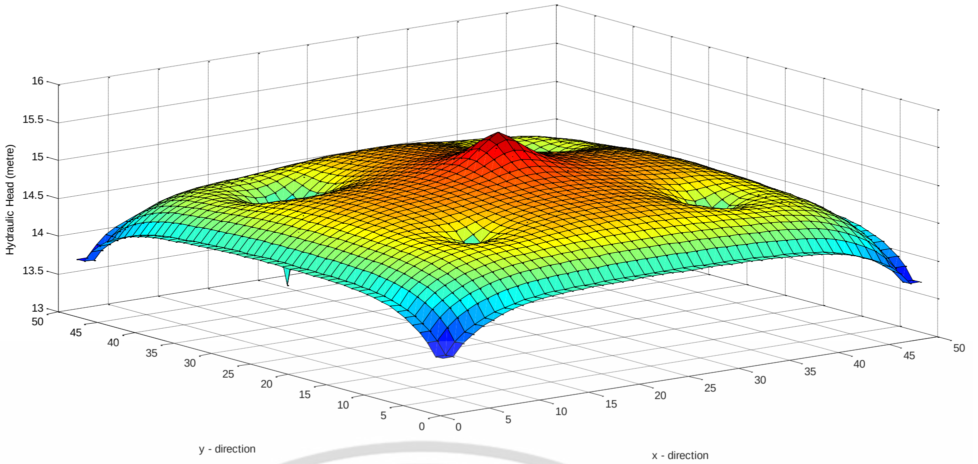

Figure 3.33 The surface graph of the hydraulic head (ADIM - Simulation 4).

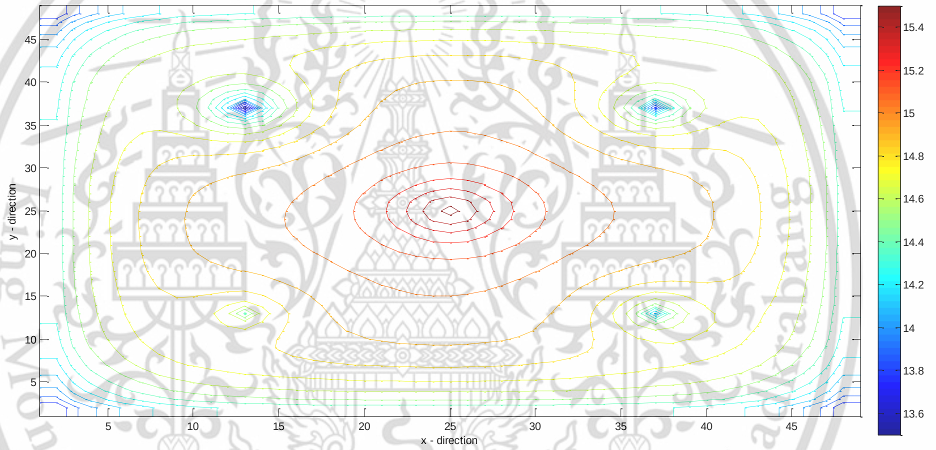

Figure 3.34 The contour graph of the hydraulic head (ADIM - Simulation 4).

The numerical hydraulic head of ADIM technique can be seen in Table 3.23. The surface plot and contour plot are shown in Figure 3.33 and Figure 3.34 respectively. Note that the hydraulic head near four pumping wells and the border line are smaller than their farther point and the hydraulic head near an injection well are greater than their farther point.

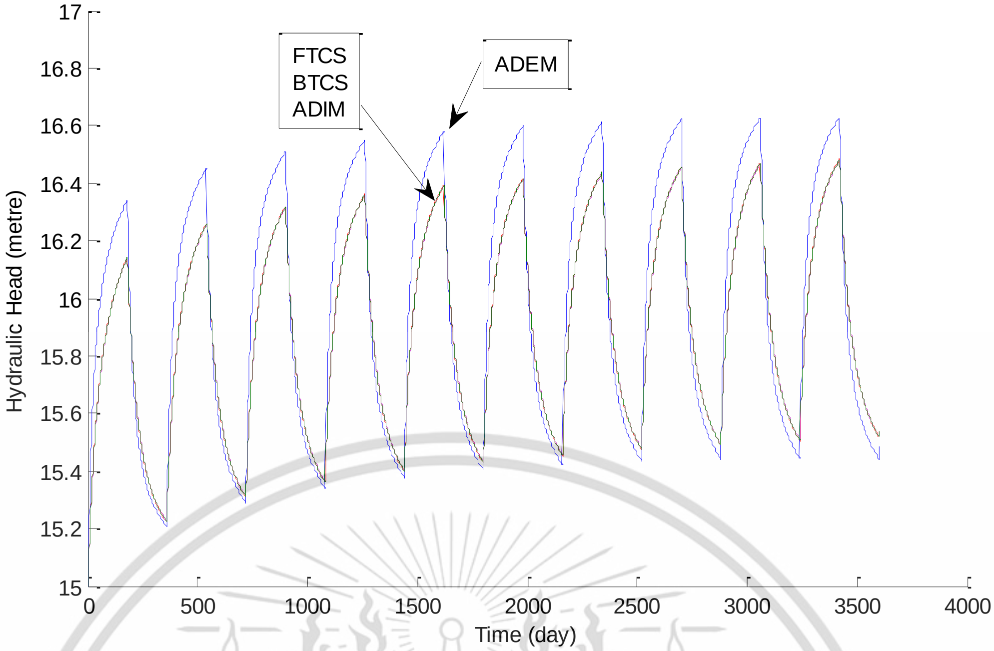

Figure 3.35 The variation of hydraulic head at $x = 1200\,m.$ and $y = 1200\,m.$ (all techniques).

The line plot of all finite difference methods at $x = 600\,m.$ and $y = 600\,m.$ are shown in Figure 3.10. We have the hydraulic head each technique which it's increasing and decreasing when changing time. Note that the hydraulic head of FTCS, BTCS and ADIM technique have near value without ADEM technique.

3.4 A two-dimensional transient groundwater flow model in drought area discussion

The two-dimensional transient groundwater flow model is used in the problem. The calculated results give the hydraulic head at each position in drought area. This study proposes a simply and flexible groundwater simulation using the implicit and explicit finite difference methods such as FTCS, BTCS, ADEM and ADIM. The complex geometry in the model considered by variable grid sizes aquifer parameters, sinks and source terms.

From the simulation 1, The four pumping wells are pumping the water from the surface water supply at all times. The values of hydraulic head near four pumping wells and the border line are smaller than their farther point. In the simulation 3, the more complicated model than the first simulation is proposed. The added injection well is injecting the water from a reservoir at all times. The overall hydraulic head is also increased. In fact, the problem must be corresponded with the season changing. In the simulation 4, we adapt the model from the simulation 3 by assuming that the rainy season has coming in after 6 months, the injection well will be injecting the water along that period of time. The hydraulic head at each four pumping well goes larger than the result in the simulation 2.

The results of the total mass errors and computing times for the simulation 1 are shown in Table 3.6 and 3.7. We can see that the finite difference methods give the accurately results for the simulation. The calculation speed by using the proposed finite difference methods are FTCS, ADEM, ADIM and BTCS, respectively. Note that the simulations have been calculated by a PC (Intel(R) core(TM) i3-2220 CPU 3.30 GHz processor and 4.0 GB RAM) and screen updating feature of each calculations was closed during each finish calculations. The stability results of each the finite difference methods are shown in Table 3.8. It can be concluded that the stability requirement is one of the disadvantages of the technique. We can see that the FTCS scheme is not good agreement for real-world application. However, the BTCS, ADEM and ADIM have an advantage over compare FTCS. It is also unconditionally stable methods as well.

These are then the considered area is drought area. For the real-world problem, the field measurement hydraulic head values, variable grid sizes aquifer parameters, the interpolated functions of sinks and source terms, must amend to be more suitable in puts for each considered terrains.

Chapter 4

Mathematical Simulations of a Groundwater Management in a Drought Area

In this chapter, the objective of groundwater flow management model is the minimum cost of injection rates. These are then subjected to optimal management of the water injection stations to achieve minimum cost. The numerical experiments are also given.

4.1 The governing equation of two-dimensional groundwater steady-flow model with drought area

Mathematical models are all based on the water balance principle. Combine the mass balance equation and Darcy's law produce the governing equation for groundwater flow. The general equation that governs two-dimensional groundwater steady-flow in isotropic, homogeneous porous media and vertically average [14],

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}=0, \qquad (4.1)$$

where $H(x,y)$ is hydraulic head (metre). We will introduce the affected term as sources and sinks due to the external inputs and outputs. Consequently, the equation (4.1) become

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}+W=0, \qquad (4.2)$$

where W is sinks and/or source (1/day). The boundary conditions are specified, for all $(x,y)\in[0,L]\times[0,M]$ where L,M are positive constant which represent the dimension of the rectangular domain,

$$\frac{\partial H}{\partial n}=B_N \qquad \text{for all } 0\le x\le L \text{ and } y=M, \qquad (4.3)$$

$$\frac{\partial H}{\partial n} = B_S \qquad \text{for all } 0 \le x \le L \text{ and } y = 0, \tag{4.4}$$

$$H = B_W \qquad \text{for all } x = 0 \text{ and } 0 \le y \le M, \tag{4.5}$$

$$H = B_E \qquad \text{for all } x = L \text{ and } 0 \le y \le M. \tag{4.6}$$

And the source terms W that represented by the rate of pumping well in each point,

$$W(x_s, y_s) = Q(x_s, y_s) = Q_s \qquad \text{for all } s = 1, 2, 3, \ldots, p, \tag{4.7}$$

where s is a number of pumping wells.

4.2 Numerical technique with two-dimensional groundwater steady flow model with drought area

The two-dimensional groundwater steady-flow not depend on time, the implicit finite difference method is used to solve groundwater flow model. The method is suitable for the model due to the linear systems of equations will be constructed. It is possible to implement with the groundwater management model. Taking the central difference scheme in space into each terms of equation (4.2), then

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H_{i-1,j} - 2H_{i,j} + H_{i+1,j}}{(\Delta x)^2}, \tag{4.8}$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H_{i,j-1} - 2H_{i,j} + H_{i,j+1}}{(\Delta y)^2}, \tag{4.9}$$

$$W_{i,j} = \pm \frac{Q_{i,j}}{\Delta x \Delta y H_{i,j}}. \tag{4.10}$$

Substituting equations (4.8) – (4.10) into equation (4.2), for $1 < i < I-1$ and $1 < j < J-1$,

$$-(2a+2b)H_{i,j} + aH_{i-1,j} + aH_{i+1,j} + bH_{i,j-1} + bH_{i,j+1} = -W_{i,j}, \tag{4.11}$$

where $a = \dfrac{1}{\left(\Delta x\right)^2}$ and $b = \dfrac{1}{\left(\Delta y\right)^2}$. For $i = 1$ and $j = 1$, substituting the unknown value on the south boundary by forward difference approximation $H_{1,0} = H_{1,1}$ into equation (4.11),

$$-\left(2a+b\right)H_{1,1} + aH_{2,1} + bH_{1,2} = -W_{1,1} - aB_W. \qquad (4.12)$$

For $1 < i < I-1$ and $j = 1$, substituting the unknown value on the south boundary by forward difference approximation $H_{i,0} = H_{i,1}$ into equation (4.11),

$$-\left(2a+b\right)H_{i,1} + aH_{i-1,1} + aH_{i+1,1} + bH_{i,2} = -W_{i,1}. \qquad (4.13)$$

For $i = I-1$ and $j = 1$, substituting the unknown value on the south boundary by forward difference approximation $H_{I-1,0} = H_{I-1,1}$ into equation (4.11),

$$-\left(2a+b\right)H_{I-1,1} + aH_{I-2,1} + bH_{I-1,2} = -W_{I-1,1} - aB_E. \qquad (4.14)$$

For $i = 1$ and $1 < j < J-1$,

$$-\left(2a+2b\right)H_{1,j} + aH_{2,j} + bH_{1j-1} + bH_{1,j+1} = -W_{1,j} - aB_W. \qquad (4.15)$$

For $i = I-1$ and $1 < j < J-1$,

$$-\left(2a+2b\right)H_{I-1,j} + aH_{I-2,j} + bH_{I-1,j-1} + bH_{I-1,j+1} = -W_{I-1,j} - aB_E. \qquad (4.16)$$

For $i = 1$ and $j = J-1$, substituting the approximate unknown value on the north boundary by backward difference approximation by $H_{1,J} = H_{1,J-1}$ into equation (4.11),

$$-\left(2a+b\right)H_{1,J-1} + aH_{2,J-1} + bH_{1,J-2} = -W_{1,J-1} - aB_W. \qquad (4.17)$$

For $1 < i < I-1$ and $j = J-1$, substituting the approximate unknown value on the north boundary by backward difference approximation $H_{i,J} = H_{i,J-1}$ into equation (4.11),

$$-(2a+b)H_{i,J-1} + aH_{i-1,J-1} + aH_{i+1,J-1} + bH_{i,J-2} = -W_{i,J-1}. \qquad (4.18)$$

For $i = I-1$ and $j = J-1$, substituting the unknown value on the north boundary by backward difference approximation $H_{I-1,J} = H_{I-1,J-1}$ into equation (4.11),

$$-(2a+b)H_{I-1,J-1} + aH_{I-2,J-1} + bH_{I-1,J-2} = -W_{I-1,J-1} - aB_E. \qquad (4.19)$$

The equations (4.11) – (4.19) can be written in matrix form as follow,

$$AH = B, \qquad (4.20)$$

where
$$A = \begin{bmatrix} A_1 & A_3 & & & \\ A_3 & A_2 & A_3 & & \\ & \ddots & \ddots & \ddots & \\ & & A_3 & A_2 & A_3 \\ & & & A_3 & A_1 \end{bmatrix}, \qquad (4.21)$$

$$A_1 = \begin{bmatrix} -(2a+b) & a & & & \\ a & -(2a+b) & a & & \\ & \ddots & \ddots & \ddots & \\ & & a & -(2a+b) & a \\ & & & a & -(2a+b) \end{bmatrix}, \qquad (4.22)$$

$$A_2 = \begin{bmatrix} -(2a+2b) & a & & & \\ a & -(2a+2b) & a & & \\ & \ddots & \ddots & \ddots & \\ & & a & -(2a+2b) & a \\ & & & a & -(2a+2b) \end{bmatrix}, \qquad (4.23)$$

$$A_3 = \begin{bmatrix} b & & & & \\ & b & & & \\ & & \ddots & & \\ & & & b & \\ & & & & b \end{bmatrix}, \qquad (4.24)$$

$$H = \begin{bmatrix} H_{1,1} \\ H_{2,1} \\ \vdots \\ H_{I-2,J-1} \\ H_{I-1,J-1} \end{bmatrix}, \tag{4.25}$$

and

$$B = \begin{bmatrix} -W_{1,1} - aB_W \\ -W_{2,1} \\ \vdots \\ -W_{I-2,1} \\ -W_{I-1,1} - aB_W \\ -W_{1,2} - aB_W \\ \vdots \\ -W_{i,j} \\ \vdots \\ -W_{I-1,J-2} - aB_E \\ -W_{1,J-1} - aB_W \\ -W_{2,J-1} \\ \vdots \\ -W_{I-2,J-1} \\ -W_{I-1,J-1} - aB_E \end{bmatrix}. \tag{4.26}$$

4.3 Groundwater management model with drought area using an optimization method.

The objective function C is the cost of all pump injection in the considered system,

$$C = \sum_{s=1}^{m} W_s Q_s . \tag{4.27}$$

where s is the number of pumping wells point, W_s is the cost of water pumping for each well s (Bath/ m^3) and Q_s is the injection rate for well s (m^3/day). The constraints are,

$$H_s \le H_{ST_s} , \tag{4.28}$$

where H_s is hydraulic head at monitoring point that measuring water requirement for each zone s and H_{ST_s} is the standard water requirement for each zone s. The upper bound of the injection rate for each pumping well are,

$$Q_s \le Q_{\max_s} , \tag{4.29}$$

the lower bound of the injection rate for each pumping well are,

$$Q_s \ge Q_{\min_s} , \tag{4.30}$$

and the hydraulic head at monitoring point s and the injection rate at pumping wells are non-negative, that are

$$H_s , Q_s \ge 0. \tag{4.31}$$

where $Q_{\min_s}$ and $Q_{\max_s}$ are the lower and upper bounds of the water injection rate for each point s, respectively. The optimal cost of them is solve by using the simplex method.

4.4 Numerical optimization experiment to a groundwater modeling conservation management in a drought area

We consider the area width 2400 m and length 2400 m which is between a pair of two rivers. The area is meshed by 100 grids points with grid space is 240 m. The boundary conditions of the area is specified Eqs.(4.3)-(4.6) where $B_N = 0$, $B_S = 0$, $B_W = 20$ and $B_E = 19$. The four injection wells are injecting the water to the ground. The injection wells have the difference lower bound of injection rates, difference upper bound of injection rates and the difference cost of injection wells for each zone as Table 4.1. There are 8 monitoring point for measuring water requirement. The hydraulic head at monitoring point have the difference standard water requirement each point as Table 4.2.

Table 4.1 The injection rates and the cost of each injection wells.

Position coordinate (x, y)	Q(1440m, 480m)	Q(480m, 720m)
Lower (m^3/day)	165	175
Upper (m^3/day)	250	230
Cost (Bath/m^3)	1.5	1.9
Position coordinate (x, y)	Q(1680m, 1440m)	Q(720m, 1680m)
Lower (m^3/day)	180	190
Upper (m^3/day)	300	270
Cost (Bath/m^3)	1.8	1.6

Table 4.2 The standard water requirement (SWR) each monitoring points.

Position coordinate (x, y)	(240,240)	(960,480)	(1920,720)	(720,1200)
SWR (m)	22	23	24	24
Position coordinate (x, y)	(1200,1200)	(240,1440)	(1200,1920)	(1920,1920)
SWR (m)	25	23	25	24

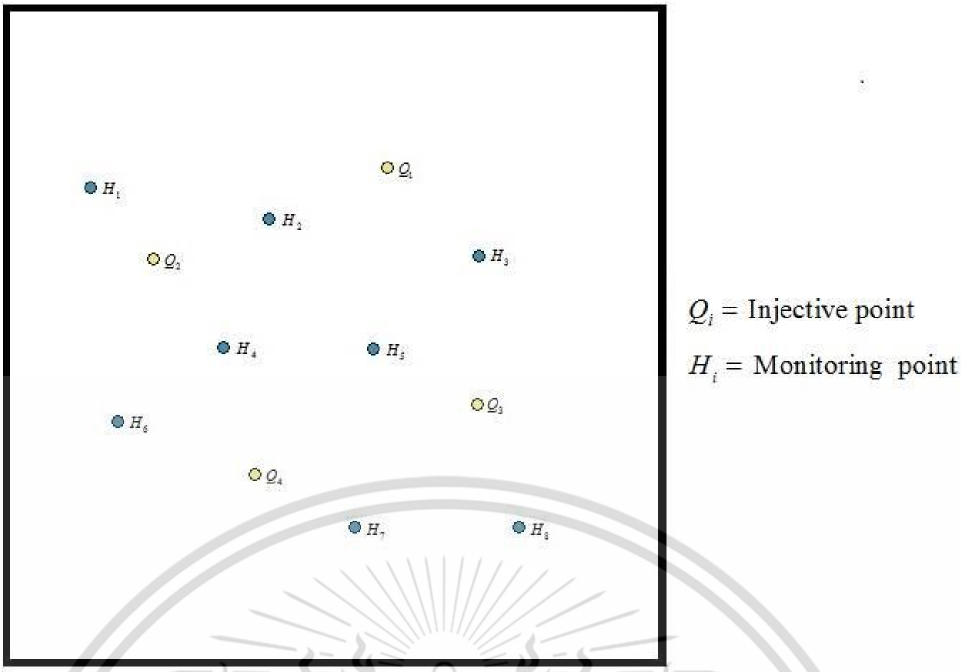

Figure 4.1 Simulation of groundwater management.

Table 4.3 Show the injection point and the monitoring point.

y\x	0	240	480	720	960	1200	1440	1680	1920	2160	2400
0											
240		H1									
480					H2		Q1				
720			Q2						H3		
960											
1200				H4		H5					
1440		H6						Q3			
1680				Q4							
1920						H7			H8		
2160											
2400											

Table 4.4 Table of the hydraulic head before optimal control of cost.

y\x	0	240	480	720	960	1200	1440	1680	1920	2160	2400
0	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
240	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
480	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
720	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
960	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1200	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1440	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1680	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1920	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
2160	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
2400	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000

Table 4.5 Table of the hydraulic head after optimal control of cost.

y\x	0	240	480	720	960	1200	1440	1680	1920	2160	2400
0	20.0000	22.4668	24.6064	25.9931	26.9362	27.5723	27.7052	25.9609	23.7603	21.4090	19.0000
240	20.0000	22.4668	24.6064	25.9931	26.9362	27.5723	27.7052	25.9609	23.7603	21.4090	19.0000
480	20.0000	22.7940	25.3593	26.4366	27.2432	28.0755	29.5824	26.4172	23.9110	21.4667	19.0000
720	20.0000	23.3498	27.6001	27.1510	27.5245	27.9041	27.8818	26.2143	24.0000	21.5467	19.0000
960	20.0000	23.0050	25.7905	27.0426	27.7999	28.1345	27.8265	26.5582	24.3280	21.7200	19.0000
1200	20.0000	22.8798	25.5142	27.4292	28.4979	29.0074	28.7315	27.8639	25.0340	22.0051	19.0000
1440	20.0000	23.0000	25.9574	28.6619	29.7552	30.6658	30.2281	31.1318	25.9390	22.2665	19.0000
1680	20.0000	23.1628	26.6534	31.5058	31.1952	33.6724	30.3834	28.5222	25.3236	22.1220	19.0000
1920	20.0000	22.9979	25.9875	28.7726	29.8473	30.4711	29.1111	27.2499	24.7114	21.8977	19.0000
2160	20.0000	22.8414	25.5262	27.7496	28.9502	29.2537	28.3398	26.6547	24.3745	21.7574	19.0000
2400	20.0000	22.8414	25.5262	27.7496	28.9502	29.2537	28.3398	26.6547	24.3745	21.7574	19.0000

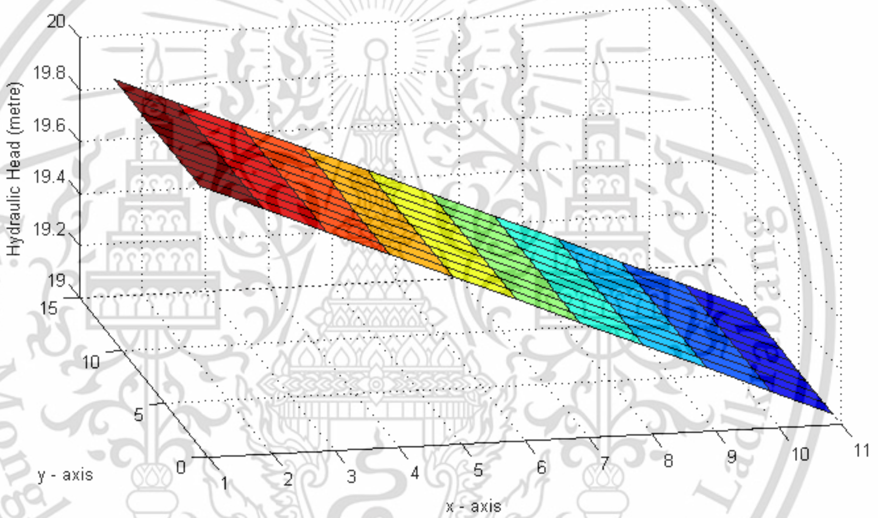

Figure 4.2 The hydraulic head before the proposed optimal control of cost will be activated.

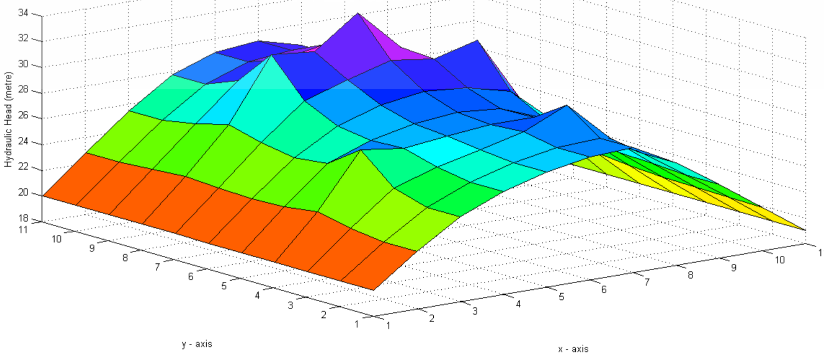

Figure 4.3 The hydraulic head after the optimal control of cost is activated.

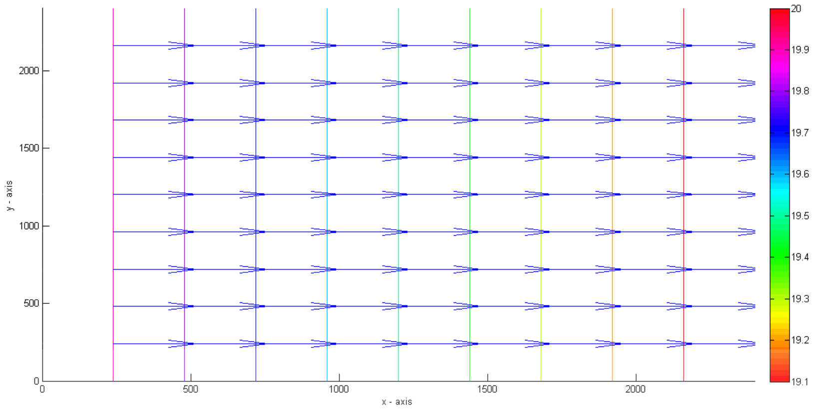

Figure 4.4 The groundwater flow directions before the optimal control of cost will be activated.

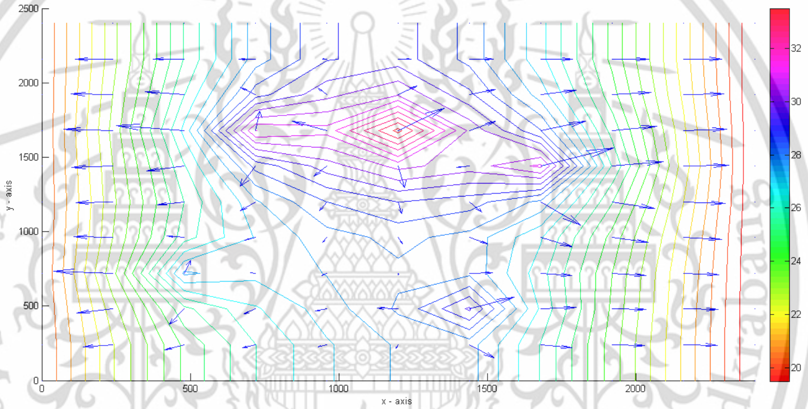

Figure 4.5 The hydraulic head and the groundwater flow directions after the proposed optimal control of cost is activated.

The surface graph before and after optimal control of cost can be seen in Figure 4.2 and Figure 4.3 respectively. The contour plot before and after optimal control of cost can be seen in Figure 4.4 and Figure 4.5 respectively. We can see direction flow that the water flow from high hydraulic head value into low hydraulic head value.

Table 4.6 The optimal injection rates of minimum cost in the system.

Position coordinate (x, y)	Injection rate (m^3/day)
Q(1440m, 480m)	165
Q(480m, 720m)	175
Q(1680m, 1440m)	239.48
Q(720m, 1680m)	214.81

4.5 A groundwater modeling conservation management in a drought area discussion

The injective point and monitoring point are located in the considered area as show in Figure 4.1. the monitored hydraulic head without and within controlled cost are shown in Figure 4.2 and Figure 4.3 respectively. The vector fields of groundwater flow velocity between both case are shown in Figure 4.4 and Figure 4.5. The hydraulic head before and after control of cost are shown in Table 4.4 and Table 4.5 respectively. The optimum injection rate at minimum cost of groundwater control is shown in Table 4.6.

Chapter 5

Conclusion

5.1 Conclusion

First, the objective is to propose a simply and flexible groundwater simulation using the implicit and explicit finite difference methods. The complex geometry in the model is considered by variable grid sizes aquifer parameters, sinks and source terms with drought area.

The transient groundwater flow model is used in the problem. The calculated results turn out the hydraulic head at each position/time in a drought are. This study proposes a simply and: flexible groundwater simulation using the implicit and explicit finite difference methods such as FTCS, BTCS, ADEM and ADIM. The complex geometry in the model is considered by variable grid sizes aquifer parameters, sinks and source terms. The results of the total mass errors and computing times for the example are shown in Tables 3.6 and 3.7. We will have, the finite difference methods have accuracy for this problem simulation and the calculation speed of each finite difference method order by fastest to slowest are FTCS, ADEM, ADIM and BTCS respectively. The result of the stability of each the finite difference method are shown on Table 3.8. It can be concluding that stability requirements are one of the disadvantages of the techniques. We can see that the both alternating direction methods are able to take in any sizes of grid spacing. These are then the methods are practical to simulate the ground water flow in many real-world cases. For the real-world problem, the real hydraulic head values, variable grid sizes aquifer parameters, sinks and source terms, must amend to be more suitable for each terrain.

Next, the objective of groundwater flow management model is the minimum cost of injection rates. These are then subjected to optimal management of the water injection stations to achieve minimum cost. The numerical experiments are also given.

We have established the groundwater management model, First, we will measure hydraulic head from the two-dimensional groundwater steady-flow model by using an implicit finite difference method. It will turn out that the system of linear equations is generated. We employ the system of linear equation to construct the groundwater management model to investigate the optimal least cost of the water injections in the system under the limitation conditions were required.

5.2 Further

In fact, the groundwater in each area have difference of the hydraulic conductivity. The hydraulic conductivity function depends on space, it will be use with the two-dimensional transient groundwater flow model in dry area.

Next, we will study the groundwater contamination that it is a problem in Thailand for many years. The dispersion model will be used to measure groundwater-quality which it will be use to combine the two-dimensional transient groundwater flow.

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Appendices

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Numerical simulation of groundwater measurement using alternating direction methods

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Numerical simulation of groundwater measurement using alternating direction methods

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Abstract

The groundwater measurement is obliged to take care of the issue of need water assets in numerous dry season ranges for agricultural use. In this study, we propose a groundwater stream model demonstrate that gives the pumping rates and the infusion rates individually. The objective are to propose a simply and flexible groundwater simulation using the implicit and explicit traditional finite difference methods and alternating direction methods. The groundwater model is giving the water driven head that gives the groundwater level. The understood limited distinction technique is utilized to surmise the groundwater flow. The complex geometry in the model is considered by variable grid sizes aquifer parameters, sinks and source terms. The proposed alternating direction methods are shown that they are able to use in groundwater simulation for the real-world cases.

Keywords: *Groundwater management, groundwater model, explicit alternating direction methods, implicit alternating direction method.*

2010 Mathematics Subject Classification: *35J15, 35Q93, 39A14, 90B50*

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1. Introduction

Groundwater simulation models are widely topics in the analysis and the management of groundwater systems. The important of the utilization of groundwater resources continues to grow due to the increasing require of water for irrigation as well as drinking, agriculture, commercial and industrial proposes. Although, the amount of groundwater resources have been decaying due population growth, uncontrolled and unplanned urbanization, industrialization, and agricultural activities. Hence, the sustainable management planning must be developed for the groundwater systems. The management planning has to limited in the case of legal well drilling and limited-pumping. On the other hand, numerical solutions have to be used if the aquifer system has complicated geometry of heterogeneous material properties. The models solve the partial differential equations governing the system. The groundwater model can be solved by analytical and numerical solution techniques. The simple and ideal cases with regularly shaped aquifers and homogeneous hydraulic properties can be solved by analytical methods. Most of groundwater modeling has the aquifer systems with the heterogeneous structure. In the case of the steady-state groundwater model solutions can be obtained by the simply basic techniques. On the other hand, the case of transient ground water model is solved by the advanced techniques due to the difficult in terms of time dimension in the governing equations. Theoretical solution of the governing equation of groundwater model needs general assumptions such as ideal solution domains and homogeneous geometries.

The finite difference [1-3] and finite elements [4-6] methods are the most popular numerical solution techniques. The case of free surface flows will consider in two groups: adaptive mesh methods is need a large number of calculation and they also require some convergence conditions [4]. It follows that the fixed mesh techniques are more popular than adaptive mesh techniques. If the variation of geometry and material in the third dimension is constant, then the two dimensional modeling can be used. Although, If the material properties and/or geometry vary any the third dimension, then the three-dimension modeling may return better solutions than the two-dimension modeling. A useful spreadsheet for two and three dimensional steady-state and transient groundwater numerical simulation is proposed in [7-8]. In this research, the objective is to propose a simply and flexible groundwater simulation using the implicit and explicit finite difference methods. The complex geometry in the model is considered by variable grid sizes aquifer parameters, sinks and source terms.

2. The governing equation

The governing equation of vertically integrated Darcy's flow in a two-dimensional confined, compressible, isotropic, heterogeneous aquifer is [8],

$$S\frac{\partial H}{\partial t}=\frac{\partial}{\partial x}\left(K\frac{\partial H}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial H}{\partial y}\right)\pm W, \tag{2.1}$$

where H hydraulic head (metre), K hydraulic conductivity (metre/day), W sinks and/or source (1/day) and S matrix of specific storage (1/metre). We assume the hydraulic conductivity is constant. It is obtain that

$$S\frac{\partial h}{\partial t}=K\left(\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}\right)\pm W, \tag{2.2}$$

with the initial conditions at $t=0$, $0\le x\le L$ and $0\le y\le M$ where L, M are constant being specified,

$$H=H_0. \tag{2.3}$$

The boundary condition for $t > 0$ are specified,

$$\frac{\partial H}{\partial n}=B_N \qquad \text{at } 0\le x\le L \text{ and } y=M, \tag{2.4}$$

$$\frac{\partial H}{\partial n}=B_S \qquad \text{at } 0\le x\le L \text{ and } y=0, \tag{2.5}$$

$$\frac{\partial H}{\partial n}=B_W \qquad \text{at } x=0 \text{ and } 0\le y\le M, \tag{2.6}$$

$$\frac{\partial H}{\partial n}=B_E \qquad \text{at } x=0 \text{ and } 0\le y\le M, \tag{2.7}$$

and given pumping well point each,

$$Q(x, y) = Q_i \qquad \text{for } i = 1, 2, 3, \ldots w, \tag{2.8}$$

in order to solve (2.2) in domain $\Omega \times [0, T]$ where $\Omega \in [0, L] \times [0, M]$ and w is the total number of sources and sinks.

From figure 1 the lengths of domain are $L \times M$, B_N, B_S, B_W and B_E are boundary conditions of model and Q_1, Q_2 are source and sinks term respectively.

3. Numerical Technique

In this paper, we will propose finite difference methods to the transient groundwater model such as the forward time central space method (FTCS), the backward time central space method (BTCS), the alternating direction explicit method (ADEM) and the alternating direction implicit method (ADIM). We now discretize (2.2) by dividing the interval $[0, L]$

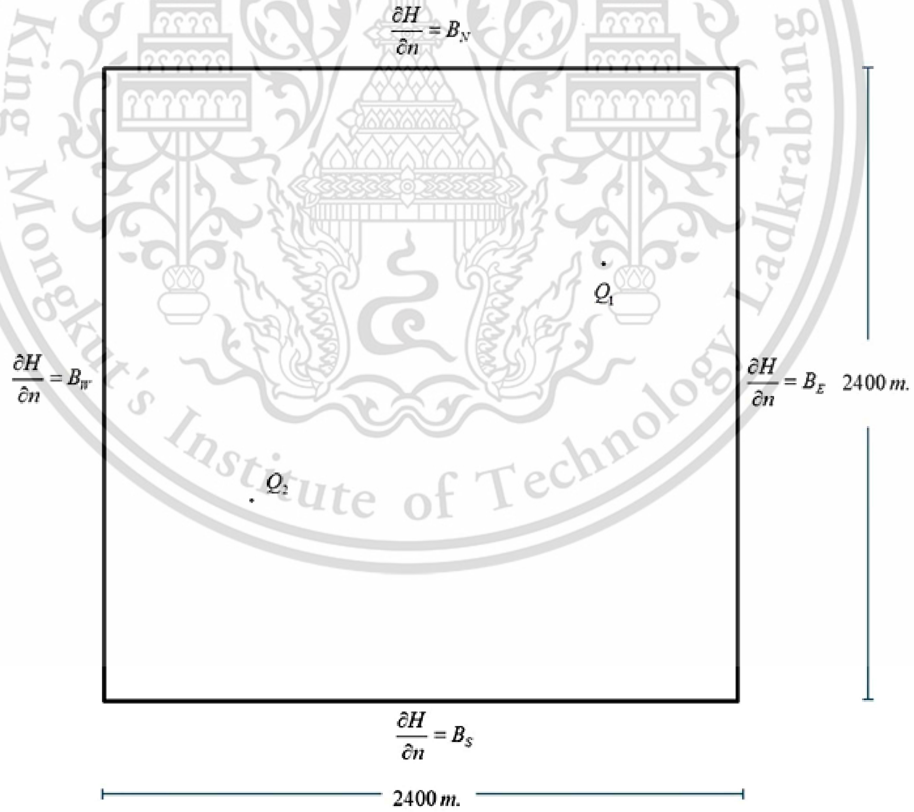

Figure 1

Boundary conditions of a transient groundwater flow model.

in x-direction into I subintervals such that $I\Delta x = L$, the interval $[0, M]$ in y-direction into J subintervals such that $J\,\Delta y = M$ and the interval $[0, T]$ in time into N subintervals such that $N\,\Delta t = T$. We can then approximate $H(x, y, t)$ by $H^n_{i,j}$, value of the difference approximation of $H(x, y, t)$ at point $x = i\Delta x$, $y = j\Delta y$ and $t = n\Delta t$, where $0 \le i \le I$, $0 \le j \le J$ and $0 \le n \le N$ which I, J and N are positive integers.

3.1 *Forward time central space method (FTCS)*

Taking the central difference scheme in space and forward difference scheme in time into each terms of equation (2.2), then

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H^n_{i-1,j} - 2H^n_{i,j} + H^n_{i+1,j}}{(\Delta x)^2}, \tag{3.1}$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H^n_{i,j-1} - 2H^n_{i,j} + H^n_{i,j+1}}{(\Delta y)^2}, \tag{3.2}$$

$$\frac{\partial H}{\partial t} \approx \frac{H^{n+1}_{i,j} - H^n_{i,j}}{\Delta t}, \tag{3.3}$$

$$W^n_{i,j} = \pm \frac{Q^n_{i,j}}{\Delta x \Delta y H^n_{i,j}}. \tag{3.4}$$

Substituting Eq.(3.1) – (3.4) into Eq.(2.2), for $1 < i < I - 1$ and $1 < j < J - 1$ at $t > 0$,

$$\begin{aligned} H^{n+1}_{i,j} &= H^n_{i,j} + \xi\left(H^n_{i+1,j} - 2H^n_{i,j} + H^n_{i-1,j}\right) \\ &\quad + \eta\left(H^n_{i,j+1} - 2H^n_{i,j} + H^n_{i,j-1}\right) + \omega\left(\frac{Q^n_{i,j}}{\Delta x \Delta y H^n_{i,j}}\right), \end{aligned} \tag{3.5}$$

where $\xi = \dfrac{(\Delta t)K}{(\Delta x)^2 S}$, $\eta = \dfrac{(\Delta t)K}{(\Delta y)^2 S}$ and $\omega = \dfrac{\Delta t}{S}$.

For $i = 1$ and $j = 1$ at $t > 0$, substituting the approximated unknown value on the left of the boundary by $H^n_{0,1} = H^n_{1,1} - (\Delta x)B_W$ and substituting the approximated unknown value on the bottom of the boundary by $H^n_{1,0} = H^n_{1,1} - (\Delta y)B_S$,

$$H_{1,1}^{n+1} = H_{1,1}^{n} + \xi\left[H_{2,1}^{n} - H_{1,1}^{n} - (\Delta x)B_W\right] + \eta\left[H_{1,2}^{n} - H_{1,1}^{n} - (\Delta y)B_S\right] + \omega\left(\frac{Q_{1,1}^{n}}{\Delta x \Delta y H_{1,1}^{n}}\right). \tag{3.6}$$

For $1 < i < I-1$ and $j = 1$ at $t = 0$, substituting the approximated unknown value on the bottom of the boundary by $H_{i,0}^{n} = H_{i,1}^{n} - (\Delta y)B_S$,

$$H_{i,1}^{n+1} = H_{i,1}^{n} + \xi\left(H_{i+1,1}^{n} - 2H_{i,1}^{n} + H_{i-1,1}^{n}\right) + \eta\left[H_{i,2}^{n} - H_{i,1}^{n} - (\Delta y)B_S\right] + \omega\left(\frac{Q_{i,1}^{n}}{\Delta x \Delta y H_{i,1}^{n}}\right). \tag{3.7}$$

For $i = I - 1$ and $j = 1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H_{I,1}^{n} = H_{I-1,1}^{n} + (\Delta x)B_E$ and substituting the approximated unknown value on the bottom of the boundary by $H_{I-1,0}^{n} = H_{I-1,1}^{n} - (\Delta y)B_S$,

$$H_{I-1,1}^{n+1} = H_{I-1,1}^{n} + \xi\left[(\Delta x)B_E - H_{I-1,1}^{n} + H_{I-2,1}^{n}\right] + \eta\left[H_{I-1,2}^{n} - H_{I-1,1}^{n} - (\Delta y)B_S\right] + \omega\left(\frac{Q_{I-1,1}^{n}}{\Delta x \Delta y H_{I-1,1}^{n}}\right). \tag{3.8}$$

For $i = 1$ and $1 < j < J - 1$ at $t > 0$, substituting the approximated unknown value on the left of the boundary by $H_{0,j}^{n} = H_{1,j}^{n} - (\Delta x)B_W$,

$$H_{1,j}^{n+1} = H_{1,j}^{n} + \xi\left[H_{2,j}^{n} - H_{1,j}^{n} - (\Delta x)B_W\right] + \eta\left(H_{1,j+1}^{n} - 2H_{1,j}^{n} + H_{1,j-1}^{n}\right) + \omega\left(\frac{Q_{1,j}^{n}}{\Delta x \Delta y H_{1,j}^{n}}\right). \tag{3.9}$$

For $i = I - 1$ and $1 < j < J - 1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H_{I,j}^{n} = H_{I-1,j}^{n} + (\Delta x)B_E$,

$$H_{I-1,j}^{n+1} = H_{I-1,j}^{n} + \xi\left[(\Delta x)B_E - H_{I-1,j}^{n} + H_{I-2,j}^{n}\right] + \eta\left(H_{I-1,j+1}^{n} - 2H_{I-1,j}^{n} + H_{I-1,j-1}^{n}\right) + \omega\left(\frac{Q_{I-1,j}^{n}}{\Delta x \Delta y H_{I-1,j}^{n}}\right). \tag{3.10}$$

For $i = 1$ and $j = J - 1$ at $t > 0$, substituting the approximated unknown value on the left of the boundary by $H^n_{0,J-1} = H^n_{1,J-1} - (\Delta x)B_W$ and substituting the approximate unknown value on the top of the boundary by $H^n_{1,J} = H^n_{1,J-1} + (\Delta y)B_N$,

$$H^{n+1}_{1,J-1} = H^n_{1,J-1} + \xi\left[H^n_{2,J-1} - H^n_{1,J-1} - (\Delta x)B_W\right] + \eta\left[(\Delta y)B_N - H^n_{1,J-1} + H^n_{1,J-2}\right] + \omega\left(\frac{Q^n_{1,J-1}}{\Delta x\Delta y H^n_{1,J-1}}\right). \tag{3.11}$$

For $1 < i < I - 1$ and $j = J - 1$ at $t > 0$, substituting the approximated unknown value on the top of the boundary by $H^n_{i,J} = H^n_{i,J-1} + (\Delta y)B_N$,

$$H^{n+1}_{i,J-1} = H^n_{i,J-1} + \xi\left(H^n_{i+1,J-1} - 2H^n_{i,J-1} + H^n_{i-1,J-1}\right) + \eta\left[(\Delta y)B_N - H^n_{i,J-1} + H^n_{i,J-2}\right] + \omega\left(\frac{Q^n_{i,J-1}}{\Delta x\Delta y H^n_{i,J-1}}\right). \tag{3.12}$$

For $i = I - 1$ and $j = J - 1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H^n_{I,J-1} = H^n_{I-1,J-1} + (\Delta x)B_E$ and substituting the approximated unknown value on the top of the boundary by $H^n_{I-1,J} = H^n_{I-1,J-1} + (\Delta y)B_N$,

$$H^{n+1}_{I-1,J-1} = H^n_{I-1,J-1} + \xi\left[(\Delta x)B_E - H^n_{I-1,J-1} + H^n_{I-2,J-1}\right] + \eta\left[(\Delta y)B_N - H^n_{I-1,J-1} + H^n_{I-1,J-2}\right] + \omega\left(\frac{Q^n_{I-1,J-1}}{\Delta x\Delta y H^n_{I-1,J-1}}\right). \tag{3.13}$$

3.2 *Backward time central space method (BTCS)*

Taking the central difference scheme in space and forward difference scheme in time into each terms of equation (2.2), then

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H^{n+1}_{i-1,j} - 2H^{n+1}_{i,j} + H^{n+1}_{i+1,j}}{(\Delta x)^2}, \tag{3.14}$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H^{n+1}_{i,j-1} - 2H^{n+1}_{i,j} + H^{n+1}_{i,j+1}}{(\Delta y)^2}, \tag{3.15}$$

$$\frac{\partial H}{\partial t} \approx \frac{H^{n+1}_{i,j} - H^{n}_{i,j}}{\Delta t}, \tag{3.16}$$

$$W^{n}_{i,j} = \pm \frac{Q^{n}_{i,j}}{\Delta x \Delta y H^{n}_{i,j}}. \tag{3.17}$$

Substituting Eq.(3.14) – (3.17) into Eq.(2.2), for $1 < i < I - 1$ and $1 < j < J - 1$ at $t > 0$,

$$\xi H^{n+1}_{i-1,j} + \xi H^{n+1}_{i+1,j} + \eta H^{n+1}_{i,j-1} + \eta H^{n+1}_{i,j+1} - \left(1 + 2\xi + 2\eta\right) H^{n+1}_{i,j} = -H^{n}_{i,j} - \omega \frac{Q^{n}_{i,j}}{\Delta x \Delta y H^{n}_{i,j}}, \tag{318}$$

where $\xi = \frac{(\Delta t)K}{(\Delta x)^2 S}$, $\eta = \frac{(\Delta t)K}{(\Delta y)^2 S}$ and $\omega = \frac{\Delta t}{S}$.

For $i = 1$ and $j = 1$ at $t > 0$, substituting the approximated unknown value on the left of the boundary by $H^{n+1}_{0,1} = H^{n+1}_{1,1} - (\Delta x) B_W$ and substituting the approximated unknown value on the bottom of the boundary by $H^{n+1}_{1,0} = H^{n+1}_{1,1} - (\Delta y) B_S$,

$$\xi H^{n+1}_{21} + \eta H^{n+1}_{1,2} - \left(1 + \xi + \eta\right) H^{n+1}_{1,1} = -H^{n}_{1,1} - \omega \frac{Q^{n}_{1,1}}{\Delta x \Delta y H^{n}_{1,1}} + \xi\left(\Delta x\right) B_W + \eta\left(\Delta y\right) B_S. \tag{3.19}$$

For $1 < i < I - 1$ and $j = 1$ at $t > 0$, substituting the approximated unknown value on the bottom boundary by $H^{n+1}_{i,0} = H^{n+1}_{i,1} - (\Delta y) B_S$,

$$\xi H^{n+1}_{i-1,1} + \xi H^{n+1}_{i+1,1} + \eta H^{n+1}_{i,2} - \left(1 + 2\xi + \eta\right) H^{n+1}_{i,1} = -H^{n}_{i,1} - \omega \frac{Q^{n}_{i,1}}{\Delta x \Delta y H^{n}_{i,1}} + \eta\left(\Delta y\right) B_S. \tag{3.20}$$

For $i = I - 1$ and $j = 1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H^{n+1}_{I,1} = H^{n+1}_{I-1,1} + (\Delta x) B_E$ and substituting the approximated unknown value on the bottom of the boundary by $H^{n+1}_{I-1,0} = H^{n+1}_{I-1,1} - (\Delta y) B_S$,

$$\xi H^{n+1}_{I-2,1} + \eta H^{n+1}_{I-1,2} - \left(1+\xi+\eta\right) H^{n+1}_{I-1,1} = -H^{n}_{I-1,1} - \omega \frac{Q^{n}_{I-1,1}}{\Delta x \Delta y H^{n}_{I-1,1}} - \xi\left(\Delta x\right) B_E + \eta\left(\Delta y\right) B_S. \qquad (3.21)$$

For $i = 1$ and $1 < j < J - 1$ at $t > 0$, substituting the approximated unknown value on the left of the boundary by $H^{n+1}_{0,j} = H^{n+1}_{1,j} - (\Delta x)B_W$,

$$\xi H^{n+1}_{2,j} + \eta H^{n+1}_{1,j-1} + \eta H^{n+1}_{1,j+1} - \left(1+\xi+2\eta\right) H^{n+1}_{1,j} = -H^{n}_{1,j} - \omega \frac{Q^{n}_{1,j}}{\Delta x \Delta y H^{n}_{1,j}} + \xi\left(\Delta x\right) B_W. \qquad (3.22)$$

For $i = I - 1$ and $1 < j < J - 1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H^{n+1}_{I,j} = H^{n+1}_{I-1,j} + (\Delta x)B_E$,

$$\xi H^{n+1}_{I-2,j} + \eta H^{n+1}_{I-1,j-1} + \eta H^{n+1}_{I-1,j+1} - \left(1+\xi+2\eta\right) H^{n+1}_{I-1,j} = -H^{n}_{I-1,j} - \omega \frac{Q^{n}_{I-1,j}}{\Delta x \Delta y H^{n}_{I-1,j}} - \xi\left(\Delta x\right) B_E. \qquad (3.23)$$

For $i = 1$ and $j = J - 1$ at $t > 0$, substituting the approximated unknown value on the left of the boundary by $H^{n+1}_{0,J-1} = H^{n+1}_{1,J-1} - (\Delta x)B_W$ and substituting the approximated unknown value on the top of the boundary by $H^{n+1}_{1,J} = H^{n+1}_{1,J-1} + (\Delta y)B_N$,

$$\xi H^{n+1}_{2,J-1} + \eta H^{n+1}_{1,J} - \left(1+\xi+\eta\right) H^{n+1}_{1,J-1} = -H^{n}_{1,J-1} - \omega \frac{Q^{n}_{1,J-1}}{\Delta x \Delta y H^{n}_{1,J-1}} + \xi\left(\Delta x\right) B_W - \eta\left(\Delta y\right) B_N. \qquad (3.24)$$

For $1 < i < I - 1$ and $j = J - 1$ at $t > 0$, substituting the approximated unknown value on the top of the boundary by $H^{n+1}_{i,J} = H^{n+1}_{i,J-1} + (\Delta y)B_N$,

$$\xi H^{n+1}_{i-1,J-1} + \xi H^{n+1}_{i+1,J-1} + \eta H^{n+1}_{i,J-2} - \left(1+2\xi+\eta\right) H^{n+1}_{i,J-1} = -H^{n}_{i,J-1} - \omega \frac{Q^{n}_{i,J-1}}{\Delta x \Delta y H^{n}_{i,J-1}} - \eta\left(\Delta y\right) B_N. \qquad (3.25)$$

For $i = I - 1$ and $j = J - 1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H^{n+1}_{I,J-1} = H^{n+1}_{I-1,J-1} + (\Delta x)B_E$

and substituting the approximated unknown value on the top of the boundary by $H^{n+1}_{I-1,J} = H^{n+1}_{I-1,J-1} + (\Delta y)B_N$,

$$\xi H^{n+1}_{I-2,J-1} + \eta H^{n+1}_{I-1,J-2} - \left(1+\xi+\eta\right) H^{n+1}_{I-1,J-1} = -H^{n}_{I-1,J-1} - \omega \frac{Q^{n}_{I-1,J-1}}{\Delta x \Delta y H^{n}_{I-1,J-1}} - \xi\left(\Delta x\right)B_E - \eta\left(\Delta y\right)B_N. \quad (3.26)$$

The equations (3.18) – (3.26) can be written in matrix form as follow,

$$AH^{n+1} = B^{n}, \quad (3.27)$$

where

$$A = \begin{bmatrix} A_1 & B & & & \\ B & A_2 & B & & \\ & \ddots & \ddots & \ddots & \\ & & B & A_2 & B \\ & & & B & A_1 \end{bmatrix}, \quad (3.28)$$

$$A_1 = \begin{bmatrix} -\left(1+\xi+\eta\right) & & & & \\ \eta & -\left(1+2\xi+\eta\right) & \eta & & \\ & \ddots & \ddots & \ddots & \\ & & \eta & -\left(1+2\xi+\eta\right) & \eta \\ & & & \eta & -\left(1+\xi+\eta\right) \end{bmatrix}, \quad (3.29)$$

$$A_2 = \begin{bmatrix} -\left(1+\xi+2\eta\right) & \eta & & & \\ \eta & -\left(1+2\xi+2\eta\right) & \eta & & \\ & \ddots & \ddots & \ddots & \\ & & \eta & -\left(1+2\xi+2\eta\right) & \eta \\ & & & \eta & -\left(1+\xi+2\eta\right) \end{bmatrix}, \quad (3.30)$$

$$B = \begin{bmatrix} \xi & & & & \\ & \xi & & & \\ & & \ddots & & \\ & & & \xi & \\ & & & & \xi \end{bmatrix}, \quad (3.31)$$

$$H^{n+1} = \begin{bmatrix} H_{1,1} \\ H_{2,1} \\ \vdots \\ H_{I-2,J-1} \\ H_{I-1,J-1} \end{bmatrix}, \tag{3.32}$$

and

$$B^{n} = \begin{bmatrix} -H^{n}_{1,1} - \omega \dfrac{Q^{n}_{1,1}}{\Delta x \Delta y H^{n}_{1,1}} + \xi(\Delta x) B_{W} + \eta(\Delta y) B_{S} \\ -H^{n}_{2,1} - \omega \dfrac{Q^{n}_{2,1}}{\Delta x \Delta y H^{n}_{2,1}} + \eta(\Delta y) B_{S} \\ \vdots \\ -H^{n}_{I-2,1} - \omega \dfrac{Q^{n}_{I-2,1}}{\Delta x \Delta y H^{n}_{I-2,1}} + \eta(\Delta y) B_{S} \\ -H^{n}_{I-1,1} - \omega \dfrac{Q^{n}_{I-1,1}}{\Delta x \Delta y H^{n}_{I-1,1}} + \xi(\Delta x) B_{E} + \eta(\Delta y) B_{S} \\ -H^{n}_{1,2} - \omega \dfrac{Q^{n}_{1,2}}{\Delta x \Delta y H^{n}_{1,2}} + \xi(\Delta x) B_{W} \\ \vdots \\ -H^{n}_{i,j} - \omega \dfrac{Q^{n}_{i,j}}{\Delta x \Delta y H^{n}_{i,j}} \\ \vdots \\ -H^{n}_{I-1,J-2} - \omega \dfrac{Q^{n}_{I-1,J-2}}{\Delta x \Delta y H^{n}_{I-1,J-2}} + \xi(\Delta x) B_{E} \\ -H^{n}_{1,J-1} - \omega \dfrac{Q^{n}_{1,J-1}}{\Delta x \Delta y H^{n}_{1,J-1}} + \xi(\Delta x) B_{W} + \eta(\Delta y) B_{N} \\ -H^{n}_{2,J-1} - \omega \dfrac{Q^{n}_{2,J-1}}{\Delta x \Delta y H^{n}_{2,J-1}} + \eta(\Delta y) B_{N} \\ \vdots \\ -H^{n}_{I-2,J-1} - \omega \dfrac{Q^{n}_{I-2,J-1}}{\Delta x \Delta y H^{n}_{I-2,J-1}} + \eta(\Delta y) B_{N} \\ -H^{n}_{I-1,J-1} - \omega \dfrac{Q^{n}_{I-1,J-1}}{\Delta x \Delta y H^{n}_{I-1,J-1}} + \xi(\Delta x) B_{E} + \eta(\Delta y) B_{N} \end{bmatrix}, \tag{3.33}$$

for all $1 \le n \le N$. We will introduce splitting methods that require smaller CPU time than the BTCS method. For the alternating direction explicit (ADEM) and the alternating direction implicit methods (ADIM) are discussed in next the two subsections.

3.3 *Alternating direction explicit method (ADEM)*

The ADEM are extrapolative and need an easy algebraic solution for an unknown for all point in each time interval. ADEM equation can be written as [8] ,

$$S_{i,j}\left(\frac{H_{i,j}^{n+1}-H_{i,j}^{n}}{\Delta t}\right)=K_{i+\frac{1}{2},j}\left(\frac{H_{i+1,j}^{n}-H_{i,j}^{n}}{(\Delta x)^{2}}\right)+K_{i-\frac{1}{2},j}\left(\frac{H_{i-1,j}^{n+1}-H_{i,j}^{n+1}}{(\Delta x)^{2}}\right)$$
$$+K_{i,j+\frac{1}{2}}\left(\frac{H_{i,j+1}^{n}-H_{i,j}^{n}}{(\Delta y)^{2}}\right)+K_{i,j-\frac{1}{2}}\left(\frac{H_{i,j-1}^{n+1}-H_{i,j}^{n+1}}{(\Delta y)^{2}}\right)+W_{i,j}. \quad (3.34)$$

Second stage,

$$S_{i,j}\left(\frac{H_{i,j}^{n+1}-H_{i,j}^{n}}{\Delta t}\right)=K_{i-\frac{1}{2},j}\left(\frac{H_{i-1,j}^{n}-H_{i,j}^{n}}{(\Delta x)^{2}}\right)+K_{i+\frac{1}{2},j}\left(\frac{H_{i+1,j}^{n+1}-H_{i,j}^{n+1}}{(\Delta x)^{2}}\right)$$
$$+K_{i,j-\frac{1}{2}}\left(\frac{H_{i,j-1}^{n}-H_{i,j}^{n}}{(\Delta y)^{2}}\right)+K_{i,j+\frac{1}{2}}\left(\frac{H_{i,j+1}^{n+1}-H_{i,j}^{n+1}}{(\Delta y)^{2}}\right)+W_{i,j}. \quad (3.35)$$

where $W_{i,j}=\pm\frac{Q_{I,J}^{n+\frac{1}{2}}}{\Delta x\Delta y H_{i,j}^{n}}$. First stage, for convenient, we will letting that

$$a_{i,j}=\frac{\Delta t}{S_{i,j}(\Delta x)^{2}}K_{i+\frac{1}{2},j}, \quad (3.36)$$

$$b_{i,j}=\frac{\Delta t}{S_{i,j}(\Delta x)^{2}}K_{i-\frac{1}{2},j}, \quad (3.37)$$

$$c_{i,j}=\frac{\Delta t}{S_{i,j}(\Delta y)^{2}}K_{i,j+\frac{1}{2}}, \quad (3.38)$$

$$d_{i,j} = \frac{\Delta t}{S_{i,j}\left(\Delta y\right)^2} K_{i,j-\frac{1}{2}}, \tag{3.39}$$

and

$$e_{i,j} = \frac{\Delta t}{S_{i,j}}. \tag{3.40}$$

For 1 < i < I – 1 and 1 < j < J – 1 at t > 0, and by rearranging, Eq.(3.34) becomes,

$$\begin{aligned} H_{i,j}^{n+1} = {} & \frac{a_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(H_{i+1,j}^{n} - H_{i,j}^{n}\right) + \frac{b_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)} H_{i-1,j}^{n+1} \\ & + \frac{c_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(H_{i,j+1}^{n} - H_{i,j}^{n}\right) + \frac{d_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)} H_{i,j-1}^{n+1} \\ & + \frac{1}{\left(1+b_{i,j}+d_{i,j}\right)} H_{i,j}^{n} + \frac{e_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)} W_{i,j}. \end{aligned} \tag{3.41}$$

For i = 1 and j = 1 at t > 0, substituting the approximated unknown value on the left of the boundary by $H_{0,1}^{n+1} = H_{1,1}^{n+1} - (\Delta x)B_W$ and substituting the approximated unknown value on the bottom of the boundary by $H_{1,0}^{n+1} = H_{1,1}^{n+1} - (\Delta y)B_S$,

$$\begin{aligned} H_{1,1}^{n+1} = {} & a_{1,1}\left(H_{2,1}^{n} - H_{1,1}^{n}\right) - b_{1,1}\left(\Delta x\right)B_W + c_{1,1}\left(H_{1,2}^{n} - H_{1,1}^{n}\right) \\ & - d_{1,1}\left(\Delta y\right)B_S + H_{1,1}^{n} + e_{1,1}W_{1,1}. \end{aligned} \tag{3.42}$$

For 1 < i < I – 1 and j = 1 at t > 0, substituting the approximated unknown value on the bottom of the boundary by $H_{i,0}^{n+1} = H_{i,1}^{n+1} - (\Delta y)B_S$,

$$\begin{aligned} H_{i,1}^{n+1} = {} & \frac{1}{\left(1+b_{i,1}\right)}\Big[a_{i,1}\left(H_{i+1,1}^{n} - H_{i,1}^{n}\right) + b_{i,1}H_{i-1,1}^{n+1} + c_{i,1}\left(H_{i,2}^{n} - H_{i,1}^{n}\right) \\ & - d_{i,1}\left(\Delta y\right)B_S + H_{i,1}^{n} + e_{i,j}W_{i,1}\Big]. \end{aligned} \tag{3.43}$$

For $i = I - 1$ and $j = 1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H_{I,1}^{n} = H_{I-1,1}^{n} + (\Delta x)B_E$ and substituting the approximated unknown value on the bottom of the boundary by $H_{I-1,0}^{n+1} = H_{I-1,1}^{n+1} - (\Delta y)B_S$,

$$H^{n+1}_{I-1,1} = \frac{1}{\left(1+b_{I-1,1}\right)}\Big[a_{I-1,1}\left(\Delta x\right)B_E + b_{I-1,1}H^{n+1}_{I-2,1} + c_{I-1,1}\left(H^n_{I-1,2} - H^n_{I-1,1}\right) - d_{I-1,1}\left(\Delta y\right)B_S + H^n_{I-1,1} + e_{I-1,1}W_{I-1,1}\Big]. \tag{3.44}$$

For $i = 1$ and $1 < j < J - 1$ at $t > 0$, substituting the approximated unknown value on the left of the boundary by $H^{n+1}_{0,j} = H^{n+1}_{1,j} - (\Delta x)B_W$,

$$H^{n+1}_{1,j} = \frac{1}{\left(1+d_{1,j}\right)}\Big[a_{1,j}\left(H^n_{2,j} - H^n_{1,j}\right) - b_{1,j}\left(\Delta x\right)B_W + c_{1,j}\left(H^n_{1,j+1} - H^n_{1,j}\right) + d_{1,j}H^{n+1}_{1,j-1} + H^n_{1,j} + e_{1,j}W_{1,j}\Big]. \tag{3.45}$$

For $i = I - 1$ and $1 < j < J - 1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H^n_{I,j} = H^n_{I-1,j} + (\Delta x)B_E$.

$$\begin{aligned} H^{n+1}_{I-1,j} = {} & \frac{a_{I-1,j}}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}\left(\Delta x\right)B_E + \frac{b_{I-1,j}}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}H^{n+1}_{I-2,j} \\ & + \frac{c_{I-1,j}}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}\left(H^n_{I-1,j+1} - H^n_{I-1,j}\right) + \frac{d_{I-1,j}}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}H^{n+1}_{I-1,j-1} \\ & + \frac{1}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}H^n_{I-1,j} + \frac{e_{I-1,j}}{\left(1+b_{I-1,j}+d_{I-1,j}\right)}W_{I-1,j}. \end{aligned} \tag{3.46}$$

For $i = 1$ and $j = J - 1$ at $t > 0$, substituting the approximated unknown value on the left of the boundary by $H^{n+1}_{0,J-1} = H^{n+1}_{1,J-1} - (\Delta x)B_W$ and substituting the approximated unknown value on the top of the boundary by $H^n_{1,J} = H^n_{1,J-1} + (\Delta y)B_N$,

$$H^{n+1}_{1,J-1} = \frac{1}{\left(1+d_{1,J-1}\right)}\Big[a_{1,J-1}\left(H^n_{2,J-1} - H^n_{1,J-1}\right) - b_{1,J-1}\left(\Delta x\right)B_W + c_{1,J-1}\left(\Delta y\right)B_N + d_{1,J-1}H^{n+1}_{1,J-2} + H^n_{1,J-1} + e_{1,J-1}W_{1,J-1}\Big]. \tag{3.47}$$

For $1 < i < I - 1$ and $j = J - 1$ at $t > 0$, substituting the approximated unknown value on the top of the boundary by $H^n_{i,J} = H^n_{i,J-1} + (\Delta y)B_N$.

$$\begin{aligned}H^{n+1}_{i,J-1} &= \frac{a_{i,J-1}}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}\left(H^{n}_{i+1,J-1}-H^{n}_{i,J-1}\right)+\frac{b_{i,J-1}}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}H^{n+1}_{i-1,J-1}\\ &+\frac{c_{i,J-1}}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}\left(\Delta y\right)B_N+\frac{d_{i,J-1}}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}H^{n+1}_{i,J-2}\\ &+\frac{1}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}H^{n}_{i,J-1}+\frac{e_{i,J-1}}{\left(1+b_{i,J-1}+d_{i,J-1}\right)}W_{i,J-1}.\end{aligned} \tag{3.48}$$

For $i = I - 1$ and $j = J - 1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H^{n}_{I,J-1} = H^{n}_{I-1,J-1} + (\Delta x)B_E$ and substituting the approximated unknown value on the top of the boundary by $H^{n}_{I-1,J} = H^{n}_{I-1,J-1} + (\Delta y)B_N$,

$$\begin{aligned}H^{n+1}_{I-1,J-1} &= \frac{a_{I-1,J-1}}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}\left(\Delta x\right)B_E+\frac{b_{I-1,J-1}}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}H^{n+1}_{I-2,J-1}\\ &+\frac{c_{I-1,J-1}}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}\left(\Delta y\right)B_N+\frac{d_{I-1,J-1}}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}H^{n+1}_{I-1,J-2}\\ &+\frac{1}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}H^{n}_{I-1,J-1}+\frac{e_{I-1,J-1}}{\left(1+b_{I-1,J-1}+d_{I-1,J-1}\right)}W_{I-1,J-1}.\end{aligned} \tag{3.49}$$

Second stage, we will letting that

$$a_{i,j} = \frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i-\frac{1}{2},j}, \tag{3.50}$$

$$b_{i,j} = \frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i+\frac{1}{2},j}, \tag{3.51}$$

$$c_{i,j} = \frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i,j-\frac{1}{2}}, \tag{3.52}$$

$$d_{i,j} = \frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i,j+\frac{1}{2}}, \tag{3.53}$$

and

$$e_{i,j} = \frac{\Delta t}{S_{i,j}}. \tag{3.54}$$

For $1 < i < I-1$ and $1 < j < J-1$ at $t > 0$, and by rearranging, Eq.(3.35) become,

$$\begin{aligned} H^{n+1}_{i,j} = {} & \frac{a_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(H^{n}_{i-1,j}-H^{n}_{i,j}\right)+\frac{b_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}H^{n+1}_{i+1,j} \\ & +\frac{c_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(H^{n}_{i,j-1}-H^{n}_{i,j}\right)+\frac{d_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}H^{n+1}_{i,j+1} \\ & +\frac{1}{\left(1+b_{i,j}+d_{i,j}\right)}H^{n}_{i,j}+\frac{e_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}W_{i,j}. \end{aligned} \tag{3.55}$$

For $i = I-1$ and $j = J-1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H^{n+1}_{I,J-1} = H^{n+1}_{I-1,J-1} + (\Delta x)B_E$ and substituting the approximated unknown value on the top of the boundary by $H^{n+1}_{I-1,J} = H^{n+1}_{I-1,J-1} + (\Delta y)B_N$,

$$\begin{aligned} H^{n+1}_{I-1,J-1} = {} & a_{I-1,J-1}(H^{n}_{I-2,J-1}-H^{n}_{I-1,J-1})+b_{I-1,J-1}(\Delta x)B_E+c_{I-1,J-1} \\ & (H^{n}_{I-1,J-2}-H^{n}_{I-1,J-1})+d_{I-1,J-1}(\Delta y)B_N+H^{n}_{I-1,J-1}+e_{I-1,J-1}W_{I-1,J-1}. \end{aligned} \tag{3.56}$$

For $1 < i < I-1$ and $j = J-1$ at $t > 0$, substituting the approximated unknown value on the top of the boundary by $H^{n+1}_{i,J} = H^{n+1}_{i,J-1} + (\Delta y)B_N$,

$$\begin{aligned} H^{n+1}_{i,J-1} = {} & \frac{1}{\left(1+b_{i,J-1}\right)}\Big[a_{i,J-1}\left(H^{n}_{i-1,J-1}-H^{n}_{i,J-1}\right)+b_{i,J-1}H^{n+1}_{i+1,J-1}+c_{i,J-1} \\ & \left(H^{n}_{i,J-2}-H^{n}_{i,J-1}\right)+d_{i,J-1}\left(\Delta y\right)B_N+H^{n}_{i,J-1}+e_{i,J-1}W_{i,J-1}\Big]. \end{aligned} \tag{3.57}$$

For $i = 1$ and $j = J-1$ at $t > 0$, substituting the approximated unknown value on the left of the boundary by $H^{n}_{0,J-1} = H^{n}_{1,J-1} - (\Delta x)B_W$ and substituting the approximated unknown value on the top of the boundary by $H^{n+1}_{1,J} = H^{n+1}_{1,J-1} + (\Delta y)B_N$,

$$H^{n+1}_{1,J-1} = \frac{1}{\left(1+b_{1,J-1}\right)}\Big[-a_{1,J-1}\left(\Delta x\right)B_W + b_{1,J-1}H^{n+1}_{2,J-1} + c_{1,J-1}\left(H^n_{1,J-2} - H^n_{1,J-1}\right) + d_{1,J-1}\left(\Delta y\right)B_N + H^n_{1,J-1} + e_{1,J-1}W_{1,J-1}\Big]. \tag{3.58}$$

For $i = I - 1$ and $1 < j < J - 1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H^{n+1}_{I,j} = H^{n+1}_{I-1,j} + (\Delta x)B_E$,

$$H^{n+1}_{I-1,j} = \frac{1}{\left(1+d_{I-1,j}\right)}\Big[a_{I-1,j}\left(H^n_{I-2,j} - H^n_{I-1,j}\right) + b_{I-1,j}\left(\Delta x\right)B_E + c_{I-1,j}\left(H^n_{I-1,j-1} - H^n_{I-1,j}\right) + d_{I-1,j}H^{n+1}_{I-1,j+1} + H^n_{I-1,j} + e_{I-1,j}W_{I-1,j}\Big]. \tag{3.59}$$

For $i = 1$ and $1 < j < J - 1$ at $t > 0$, substituting the approximated unknown value on the left of the boundary by $H^n_{0,j} = H^n_{1,j} - (\Delta x)B_W$,

$$\begin{aligned} H^{n+1}_{1,j} = &-\frac{a_{1,j}}{\left(1+b_{1,j}+d_{1,j}\right)}\left(\Delta x\right)B_W + \frac{b_{1,j}}{\left(1+b_{1,j}+d_{1,j}\right)}H^{n+1}_{2,j} \\ &+ \frac{c_{1,j}}{\left(1+b_{1,j}+d_{1,j}\right)}\left(H^n_{1,j-1} - H^n_{1,j}\right) + \frac{d_{1,j}}{\left(1+b_{1,j}+d_{1,j}\right)}H^{n+1}_{1,j+1} \\ &+ \frac{1}{\left(1+b_{1,j}+d_{1,j}\right)}H^n_{1,j} + \frac{e_{1,j}}{\left(1+b_{1,j}+d_{1,j}\right)}W_{1,j}. \end{aligned} \tag{3.60}$$

For $i = I - 1$ and $j = 1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H^{n+1}_{I,1} = H^{n+1}_{I-1,1} + (\Delta x)B_E$ and substituting the approximated unknown value on the bottom of the boundary by $H^n_{I-1,0} = H^n_{I-1,1} - (\Delta y)B_S$,

$$H^{n+1}_{I-1,1} = \frac{1}{\left(1+d_{I-1,1}\right)}\Big[a_{I-1,1}\left(H^n_{I-2,1} - H^n_{I-1,1}\right) + b_{I-1,1}\left(\Delta x\right)B_E - c_{I-1,1}\left(\Delta y\right)B_S + d_{I-1,1}H^{n+1}_{I-1,2} + H^n_{I-1,1} + e_{I-1,1}W_{I-1,1}\Big]. \tag{3.61}$$

For $1 < i < I - 1$ and $j = 1$ at $t > 0$, substituting the approximated unknown value on the bottom of the boundary by $H^n_{i,0} = H^n_{i,1} - (\Delta y)B_S$,

$$
\begin{aligned}
H_{i,1}^{n+1} = {} & \frac{a_{i,1}}{\left(1+b_{i,1}+d_{i,1}\right)}\left(H_{i-1,1}^{n}-H_{i,1}^{n}\right)+\frac{b_{i,1}}{\left(1+b_{i,1}+d_{i,1}\right)}H_{i+1,1}^{n+1} \\
& -\frac{c_{i,1}}{\left(1+b_{i,1}+d_{i,1}\right)}\left(\Delta y\right)B_S+\frac{d_{i,1}}{\left(1+b_{i,1}+d_{i,1}\right)}H_{i,2}^{n+1} \\
& +\frac{1}{\left(1+b_{i,1}+d_{i,1}\right)}H_{i,1}^{n}+\frac{e_{i,1}}{\left(1+b_{i,1}+d_{i,1}\right)}W_{i,1}. \qquad (3.62)
\end{aligned}
$$

For $i=1$ and $j=1$ at $t>0$, substituting the approximated unknown value on the left of the boundary by $H_{0,1}^{n}=H_{1,1}^{n}-(\Delta x)B_W$ and substituting the approximated unknown value on the bottom of the boundary by $H_{1,0}^{n}=H_{1,1}^{n}-(\Delta y)B_S$,

$$
\begin{aligned}
H_{1,1}^{n+1} = {} & -\frac{a_{1,1}}{\left(1+b_{1,1}+d_{1,1}\right)}\left(\Delta x\right)B_W+\frac{b_{1,1}}{\left(1+b_{1,1}+d_{1,1}\right)}H_{2,1}^{n+1} \\
& -\frac{c_{1,1}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(\Delta y\right)B_S+\frac{d_{1,1}}{\left(1+b_{i,j}+d_{i,j}\right)}H_{1,2}^{n+1} \\
& +\frac{1}{\left(1+b_{i,j}+d_{i,j}\right)}H_{1,1}^{n}+\frac{e_{1,1}}{\left(1+b_{i,j}+d_{i,j}\right)}W_{1,1}. \qquad (3.63)
\end{aligned}
$$

The stable solutions can be obtained only around the stability condition that [8]

$$
\frac{KH}{S}\left[\frac{\Delta t}{\left(\Delta x\right)^2}+\frac{\Delta t}{\left(\Delta y\right)^2}\right]\le\frac{1}{2}. \qquad (3.64)
$$

3.4 *Alternating direction implicit method (ADIM)*

This method will divide into 2 stages. The first stage, taking the central difference scheme in space and forward difference scheme in time into each terms of equation (2.2), then

$$
\frac{\partial^2 H}{\partial x^2}\approx\frac{H_{i-1,j}^{n+\frac{1}{2}}-2H_{i,j}^{n+\frac{1}{2}}+H_{i+1,j}^{n+\frac{1}{2}}}{\left(\Delta x\right)^2}, \qquad (3.65)
$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H^n_{i,j-1} - 2H^n_{i,j} + H^n_{i,j+1}}{(\Delta y)^2}, \tag{3.66}$$

$$\frac{\partial h}{\partial t} \approx \frac{H^{n+\frac{1}{2}}_{i,j} - H^n_{i,j}}{\Delta t}, \tag{3.67}$$

$$W^n_{i,j} = \pm \frac{Q^n_{i,j}}{\Delta x \Delta y H^n_{i,j}}. \tag{3.68}$$

Substituting Eq.(3.65) – (3.68) into Eq.(2.2), for $1 < i < I-1$ and $1 < j < J-1$ at $t > 0$,

$$-\alpha H^{n+\frac{1}{2}}_{i-1,j} + (1+2\alpha)H^{n+\frac{1}{2}}_{i,j} - \alpha H^{n+\frac{1}{2}}_{i+1,j} = \beta H^n_{i,j-1} + (1-2\beta)H^n_{i,j} + \beta H^n_{i,j+1} + \gamma \frac{Q^n_{i,j}}{\Delta x \Delta y H^n_{i,j}}. \tag{3.69}$$

where $\xi = \frac{(\Delta t)K}{(\Delta x)^2 S}$, $\beta = \frac{(\Delta t)K}{(\Delta y)^2 S}$ and $\gamma = \frac{\Delta t}{S}$.

For $i = 1$ and $1 < j < J-1$ at $t > 0$, substituting the approximated unknown value on the left of the boundary by $H^{n+\frac{1}{2}}_{0,j} = H^{n+\frac{1}{2}}_{1,j} - (\Delta x)B_W$,

$$(1+\alpha)H^{n+\frac{1}{2}}_{1,j} - \alpha H^{n+\frac{1}{2}}_{2,j} = \beta H^n_{1,j-1} + (1-2\beta)H^n_{1,j} + \beta H^n_{1,j+1} + \gamma \frac{Q^n_{1,j}}{\Delta x \Delta y H^n_{1,j}} - \alpha(\Delta x)B_W. \tag{3.70}$$

For $i = I-1$ and $1 < j < J-1$ at $t > 0$, substituting the approximated unknown value on the right of the boundary by $H^{n+\frac{1}{2}}_{I,j} = H^{n+\frac{1}{2}}_{I-1,j} + (\Delta x)B_E$,

$$-\alpha H^{n+\frac{1}{2}}_{I-2,j} + (1+\alpha)H^{n+\frac{1}{2}}_{I-1,j} = \beta H^n_{I-1,j-1} + (1-2\beta)H^n_{I-1,j} + \beta H^n_{I-1,j+1} + \gamma \frac{Q^n_{I-1,j}}{\Delta x \Delta y H^n_{I-1,j}} + \alpha(\Delta x)B_E. \tag{3.71}$$

The equations (3.69) – (3.71) can be written in matrix form as follow,

$$AH^{n+\frac{1}{2}} = B^n. \tag{3.72}$$

where

$$A = \begin{bmatrix} 1+\alpha & -\alpha & & & \\ -\alpha & 1+2\alpha & -\alpha & & \\ & \vdots & \vdots & \vdots & \\ & & -\alpha & 1+2\alpha & -\alpha \\ & & & -\alpha & 1+\alpha \end{bmatrix}, \tag{3.73}$$

$$H^{n+\frac{1}{2}} = \begin{bmatrix} H_{1,j}^{n+\frac{1}{2}} \\ H_{1,j}^{n+\frac{1}{2}} \\ \vdots \\ H_{I-2,j}^{n+\frac{1}{2}} \\ H_{I-1,j}^{n+\frac{1}{2}} \end{bmatrix}, \tag{3.74}$$

$$B^n = \begin{bmatrix} \beta H_{1,j-1}^n + (1-2\beta)H_{1,j}^n + \beta H_{1,j+1}^n + \gamma \dfrac{Q_{1,j}^n}{\Delta x \Delta y H_{1,j}^n} - \alpha(\Delta x)B_W \\ \beta H_{2,j-1}^n + (1-2\beta)H_{2,j}^n + \beta H_{2,j+1}^n + \gamma \dfrac{Q_{2,j}^n}{\Delta x \Delta y H_{2,j}^n} \\ \vdots \\ \beta H_{I-2,j-1}^n + (1-2\beta)H_{I-2,j}^n + \beta H_{I-2,j+1}^n + \gamma \dfrac{Q_{I-2,j}^n}{\Delta x \Delta y H_{I-2,j}^n} \\ \beta H_{I-1,j-1}^n + (1-2\beta)H_{I-1,j}^n + \beta H_{I-1,j+1}^n + \gamma \dfrac{Q_{I-1,j}^n}{\Delta x \Delta y H_{I-1,j}^n} + \alpha(\Delta x)B_E \end{bmatrix}. \tag{3.75}$$

for all $1 \le n \le N$. The second stage, taking the central difference scheme in space and forward difference scheme in time into each terms of equation (2.2), then

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H^{n+\frac{1}{2}}_{i-1,j} - 2H^{n+\frac{1}{2}}_{i,j} + H^{n+\frac{1}{2}}_{i+1,j}}{(\Delta x)^2}, \tag{3.76}$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H^{n+1}_{i,j-1} - 2H^{n+1}_{i,j} + H^{n+1}_{i,j+1}}{(\Delta y)^2}, \tag{3.77}$$

$$\frac{\partial H}{\partial t} \approx \frac{H^{n+1}_{i,j} - H^{n+\frac{1}{2}}_{i,j}}{\Delta t}, \tag{3.78}$$

$$W^{n+\frac{1}{2}}_{i,j} = \pm \frac{Q^{n}_{i,j}}{\Delta x \Delta y H^{n+\frac{1}{2}}_{i,j}}. \tag{3.79}$$

Substituting Eq.(3.76) – (3.6879) into Eq.(2.2), for $1 < i < I - 1$ and $1 < j < J - 1$ at $t > 0$,

$$\begin{aligned} -\beta H^{n+1}_{i,j-1} + (1+2\beta)H^{n+1}_{i,j} - \beta H^{n+1}_{i,j+1} &= \alpha H^{n+\frac{1}{2}}_{i-1,j} + \\ &(1-2\alpha)H^{n+\frac{1}{2}}_{i,j} + \alpha H^{n+\frac{1}{2}}_{i+1,j} + \gamma \frac{Q^{n}_{i,j}}{\Delta x \Delta y H^{n+\frac{1}{2}}_{i,j}}. \end{aligned} \tag{3.80}$$

For $1 < i < I - 1$ and $j = 1$ at $t > 0$, substituting the approximated unknown value on the bottom of the boundary by $H^{n+1}_{i,0} = H^{n+1}_{i,1} - (\Delta y)B_S$,

$$\begin{aligned} (1+\beta)H^{n+1}_{i,1} - \beta H^{n+1}_{i,2} &= \alpha H^{n+\frac{1}{2}}_{i-1,1} + (1-2\alpha)H^{n+\frac{1}{2}}_{i,1} + \\ &\alpha H^{n+\frac{1}{2}}_{i+1,1} + \gamma \frac{Q^{n}_{i,1}}{\Delta x \Delta y H^{n+\frac{1}{2}}_{i,1}} - \beta(\Delta y)B_S. \end{aligned} \tag{3.81}$$

For $1 < i < I - 1$ and $j = J - 1$ at $t > 0$, substituting the approximated unknown value on the top of the boundary by $H^{n+1}_{i,J} = H^{n+1}_{i,J-1} + (\Delta y)B_N$,

$$\begin{aligned} -\beta H^{n+1}_{i,J-2} + (1+\beta)H^{n+1}_{i,J-1} &= \alpha H^{n+\frac{1}{2}}_{i-1,J-1} + (1-2\alpha)H^{n+\frac{1}{2}}_{i,J-1} + \alpha H^{n+\frac{1}{2}}_{i+1,J-1} \\ &+ \gamma \frac{Q^{n}_{i,J-1}}{\Delta x \Delta y H^{n+\frac{1}{2}}_{i,J-1}} + \beta(\Delta y)B_N. \end{aligned} \tag{3.82}$$

The equations (3.80) – (3.82) can be written in matrix form as follow,

$$AH^{n} = B^{n+\frac{1}{2}}. \tag{3.83}$$

From (3.24) – (3.25), we write in matrix form, we will have

$$A = \begin{bmatrix} 1+\beta & -\beta & & & \\ -\beta & 1+2\beta & -\beta & & \\ & \vdots & \vdots & \vdots & \\ & & -\beta & 1+2\beta & -\beta \\ & & & -\beta & 1+\beta \end{bmatrix}, \tag{3.84}$$

$$H^{n} = \begin{bmatrix} H_{i,1}^{m+1} \\ H_{i,2}^{m+1} \\ \vdots \\ H_{i,J-2}^{m+1} \\ H_{i,J-1}^{m+1} \end{bmatrix}, \tag{3.85}$$

$$B^{n+\frac{1}{2}} = \begin{bmatrix} \alpha H_{i-1,1}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,1}^{m+\frac{1}{2}} + \alpha H_{i+1,1}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,1}^{m}}{\Delta x \Delta y H_{i,1}^{m+\frac{1}{2}}} - \beta(\Delta y) B_S \\ \alpha H_{i-1,2}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,2}^{m+\frac{1}{2}} + \alpha H_{i+1,2}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,2}^{m}}{\Delta x \Delta y H_{i,2}^{m+\frac{1}{2}}} \\ \vdots \\ \alpha H_{i-1,J-2}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,J-2}^{m+\frac{1}{2}} + \alpha H_{i+1,J-2}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,J-2}^{m}}{\Delta x \Delta y H_{i,J-2}^{m+\frac{1}{2}}} \\ \alpha H_{i-1,J-1}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,J-1}^{m+\frac{1}{2}} + \alpha H_{i+1,J-1}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,J-1}^{m}}{\Delta x \Delta y H_{i,J-1}^{m+\frac{1}{2}}} + \beta(\Delta y) B_N \end{bmatrix}. \tag{3.86}$$

for all $1 \le n \le N$. The ADIM has an unconditionally stable scheme [9]. The calculated solution can be obtained in size of gridding.

Table 1

The pumping rate each wells of example 1.

Q(600m, 600m)	Q(600m, 1800m)	Q(1800m, 600m)	Q(1800m, 1800m)	Q(1200m, 1200m)
-216 m^3/day	-216 m^3/day	-216 m^3/day	-216 m^3/day	864 m^3/day

Table 2

Hydraulic Head (metre) of FTCS technique at $t = 3600$ *day*.

$x\backslash y$	0	300	600	900	1200	1500	1800	2100	2400
0	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000
300	15.0000	14.9778	14.9472	14.9782	14.9948	14.9782	14.9472	14.9778	15.0000
600	15.0000	14.9472	14.4388	14.9583	15.0096	14.9583	14.4388	14.9472	15.0000
900	15.0000	14.9782	14.9583	15.0567	15.1795	15.0567	14.9583	14.9782	15.0000
1200	15.0000	14.9948	15.0096	15.1795	16.9293	15.1795	15.0096	14.9948	15.0000
1500	15.0000	14.9782	14.9583	15.0567	15.1795	15.0567	14.9583	14.9782	15.0000
1800	15.0000	14.9472	14.4388	14.9583	15.0096	14.9583	14.4388	14.9472	15.0000
1850	15.0000	14.9487	14.6865	14.9559	15.0025	14.9559	14.6865	14.9487	15.0000
2100	15.0000	14.9778	14.9472	14.9782	14.9948	14.9782	14.9472	14.9778	15.0000
2400	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000

4. Numerical Experiments

A transient groundwater flow model is providing hydraulic head profile. The application of the numerical simulations of a transient groundwater flow model is tested using the hypothetical examples. From now, we assume the experimented groundwater area has dimension that 2.4 *km*. × 2.4 *km*. The experimented area has homogeneous aquifer parameters, the initial hydraulic head is given 15*m*., the hydraulic conductivity $K = 15m/day$, storage capacity $S = 1m^{-1}$, grid spacing $\Delta x = \Delta y = 50m$, number of grid spacing $I = J = 49$ and time step $\Delta t = 1$ day. The five pumping wells is pumping the water from the ground. The pumping wells have the pumping rates as Table 1. This example will consider bounder line have no derivative boundary.

Table 3

Total mass error (%).

Day\Methods	FTCS	BTCS	ADEM	ADIM
1	0.00×10^{0}	1.15×10^{-13}	1.21×10^{-7}	1.21×10^{-7}
3	3.64×10^{-7}	3.56×10^{-7}	1.75×10^{-6}	5.96×10^{-7}
5	1.19×10^{-6}	1.16×10^{-6}	5.08×10^{-6}	$1.52 \times 10^{0-6}$
10	5.14×10^{-6}	5.03×10^{-6}	1.98×10^{-5}	5.61×10^{-6}
30	4.23×10^{-5}	4.16×10^{-5}	1.40×10^{-4}	4.32×10^{-5}
50	1.04×10^{-4}	1.02×10^{-4}	3.17×10^{0}	1.05×10^{-4}
100	3.18×10^{-4}	3.16×10^{-4}	8.70×10^{-4}	3.19×10^{-4}
1000	6.80×10^{-3}	6.80×10^{-3}	1.49×10^{-2}	6.80×10^{-3}
1800	1.32×10^{-2}	1.32×10^{-2}	2.39×10^{-2}	1.32×10^{-2}
3000	2.02×10^{-2}	2.01×10^{-2}	2.13×10^{-2}	2.02×10^{-2}
3600	2.16×10^{-2}	2.16×10^{-2}	1.35×10^{-2}	2.16×10^{-2}

Table 4

CPU times (sec)

Day\Methods	FTCS	BTCS	ADEM	ADIM
1	0.04	0.49	0.05	0.32
3	0.04	1.42	0.05	0.41
5	0.04	2.31	0.05	0.54
10	0.04	4.75	0.07	0.83
30	0.04	13.95	0.13	1.94
50	0.05	22.87	0.11	2.89
100	0.06	45.99	0.18	5.54
1000	0.29	436.44	1.37	55.19
1800	0.49	785.26	2.48	98.87
3000	0.80	1312.12	4.03	165.63
3600	0.95	1551.12	4.82	202.02

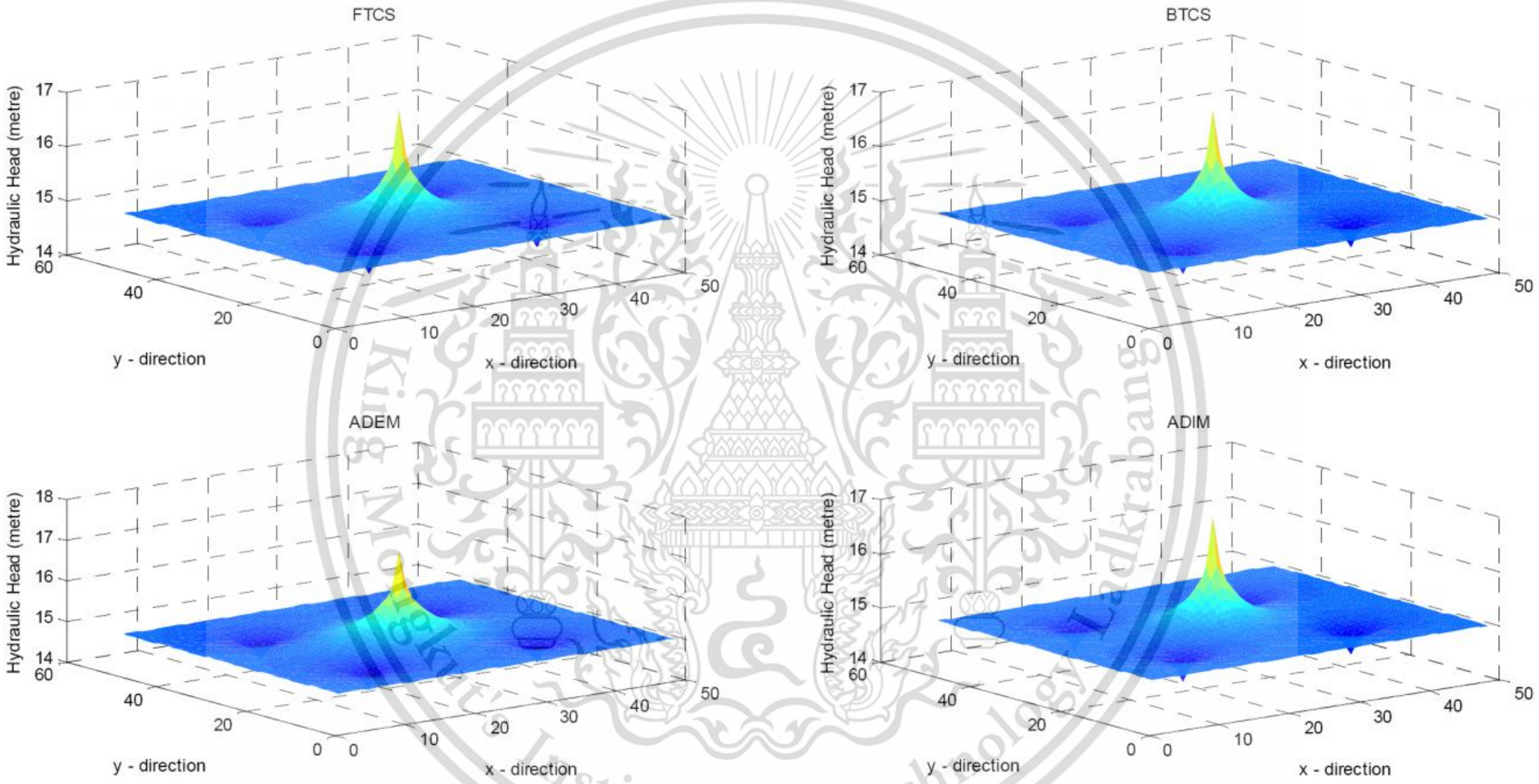

Figure 2

The surface graph of all finite difference techniques.

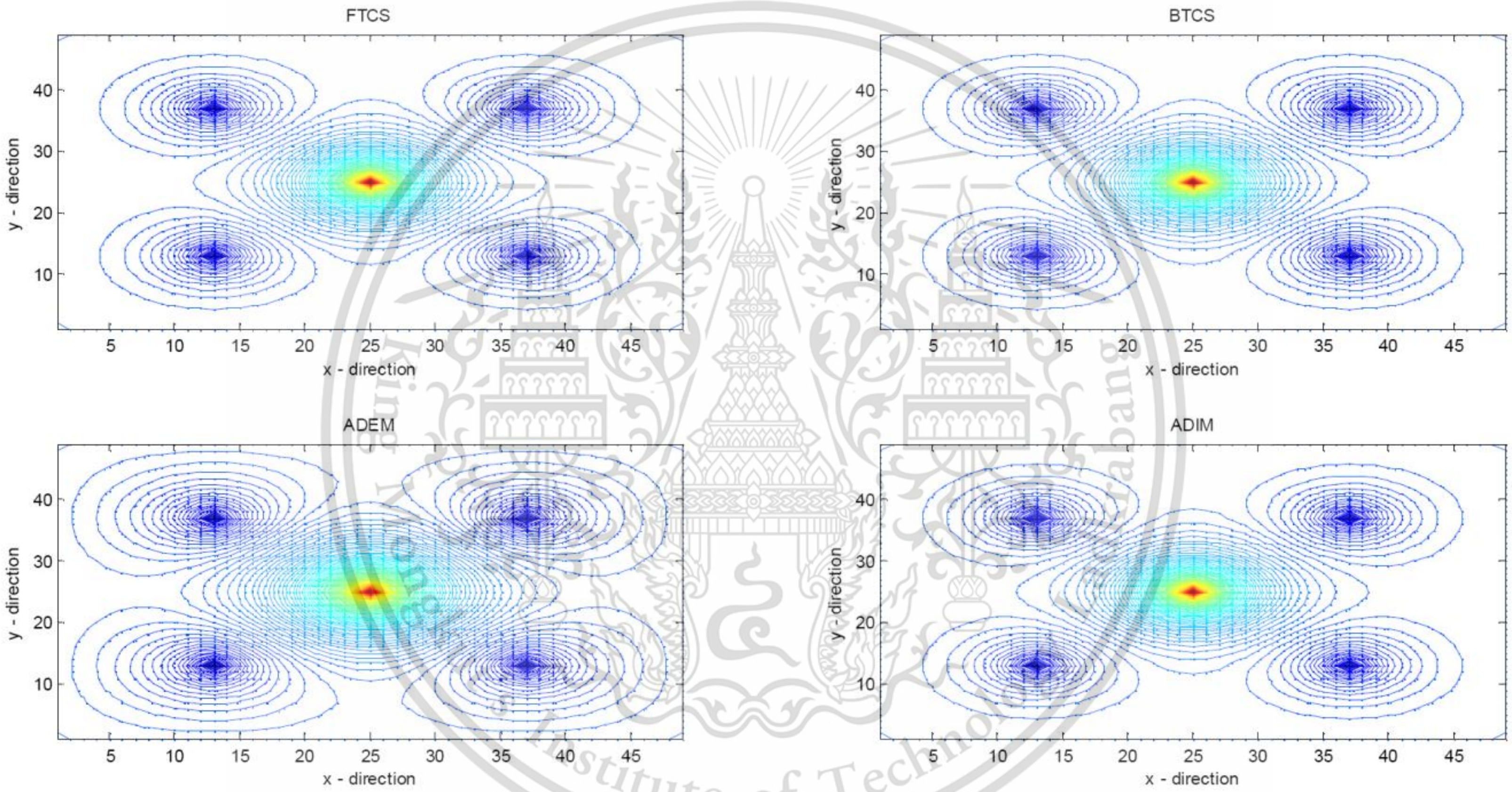

Figure 3

The contour graph of each finite difference methods.

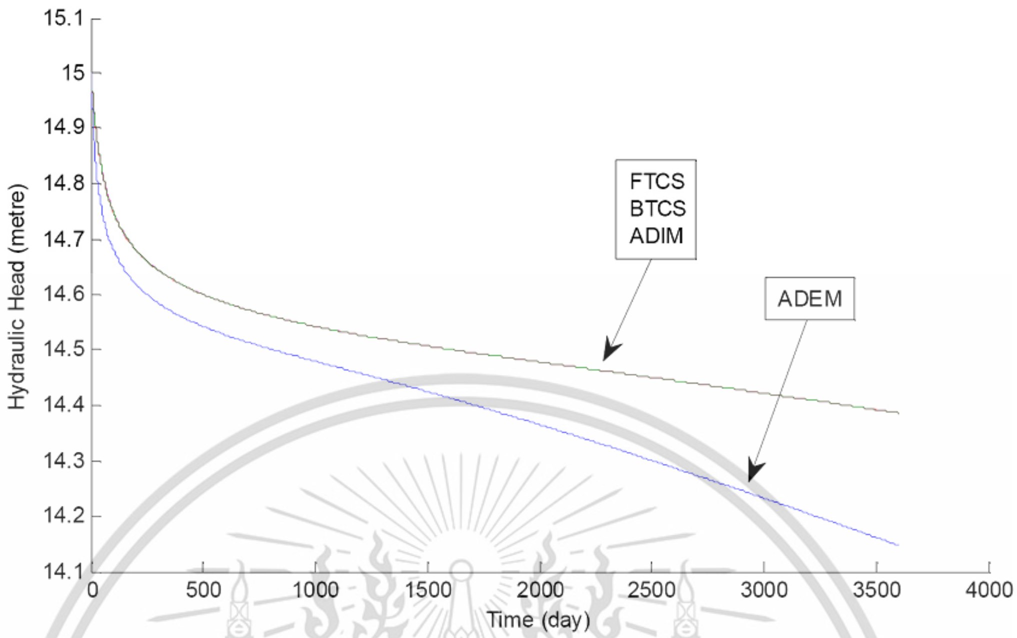

Figure 4

The variation of hydraulic head at $x = 600\,m$. and $y = 600\,m$. (FTCS technique).

Table 5

The stabilities for each grid spacing of FTCS, BTCS, ADEM and ADIM scheme.

Δx	Δy	Δt	FTCS	BTCS	ADEM	ADIM
100	100	150	stable	stable	stable	stable
		155	stable	stable	stable	stable
		160	stable	stable	stable	stable
		165	unstable	stable	stable	stable
		170	unstable	stable	stable	stable
		175	unstable	stable	stable	stable
		180	unstable	stable	stable	stable

5. Discussion and Conclusion

The transient groundwater flow model is used in the problem. The calculated results turn out the hydraulic head at each positions/times in a drought area (Table 2 and Figs.2-3). This study propose a simply and flexible groundwater simulation using the implicit and explicit finite difference methods such as FTCS, BTCS, ADEM and ADIM. The complex geometry in the model is considered by variable grid sizes aquifer parameters, sinks and source terms. The results of the total mass errors and CPU times for the example are shown in Tables 3-4. We will have, the finite difference methods have accuracy for this problem simulation and the calculation speed of each finite difference method order by fastest to slowest are FTCS, ADEM, ADIM and BTCS respectively. The result of the stability of each the finite difference methods are shown on Table 5. It can be concluding that stability requirements are one of the disadvantages of the techniques. We can see that the both alternating direction methods are able to take in any sizes of grid spacing. These are then the methods are practical to simulate the ground water flow in many real-world cases. For the real-world problem, the real hydraulic head values, variable grid sizes aquifer parameters, sinks and source terms, must amend to be more suitable for each terrains.

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Mathematical Simulation of a Groundwater Management in a Drought Area Using an Implicit Finite Difference Method

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Abstract : The groundwater management is required to solve the problem of lack water resources in many drought areas for agricultural usage. In this study, we propose a groundwater flow model and a groundwater management model that provide the pumping rates and the injection rates respectively. The groundwater model is providing the hydraulic head that gives the groundwater level. The implicit finite difference method is used to approximate the groundwater flow directions. The objective of groundwater flow management model is the minimum cost of injection rates. These are then subjected to optimal management of the water injection stations to achieve minimum cost. The numerical experiments are also given.

Keywords : groundwater management; groundwater model; implicit method.
2010 Mathematics Subject Classification : 35J15; 35Q93; 39A14; 90B50.

1 Introduction

Groundwater modeling is a powerful tool for water resources management, groundwater protection and remediation. The models are decided by maker to predict the behavior of a groundwater system prior to implementation of a remediation plan. The significance of the utilization of water resources continues to grow due to the increasing require of water for irrigation as well as drinking, agriculture, commercial and industrial proposes. Although, the amount of groundwater resources have been decaying due population growth, uncontrolled and unplanned urbanization,

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industrialization, and agricultural activities. Hence, the sustainable management planning must be developed for the groundwater systems. The management planning have to limited in the case of legal well drilling and limited-pumpings. On the other hand, the partial differential equations governing the system is solved by model. The groundwater model are solved by analytical and numerical solution techniques. Analytical methods is not suitable application to require much data and their application is limited to simple problem. Numerical methods can solve more complex problem than analytical solutions. Now, rapid development of computer processors and increasing speed, numerical modeling has become tools more effective an easy to use. The finite difference method and the finite element are most tools used numerical modeling approaches. Each method has its advantages and limitations. Selecting numerical modeling approach depend on the problem of concern and the objectives of modeling. Most of groundwater modeling has the aquifer systems with the heterogeneous structure. In the case of the steady-state groundwater model solutions can be obtained by the simply basic techniques. On the other hand, the case of transient ground water model is solved by the advanced techniques due to the difficult in terms of time dimension in the governing equations. Theoretical solution of the governing equation of groundwater model need general assumptions such as ideal solution domains and homogeneous geometries.

Groundwater models can be simple, analytical solutions of one-dimensional is like solutions of spreadsheet models [1], for very complicated three-dimensional models. It is always introduced to start with a simple model, as long as the model concept satisfies modeling objectives, and then the model complex can be increased [2]. The finite difference [3, 4, 5] and finite elements [6, 7, 8] methods are the most popular numerical solution techniques. A simulation/optimization model is proposed for the identification of unknown groundwater well locations and pumping rates for two-dimensions and model is combined with genetic algorithm based optimization model [9]. A useful spreadsheet for two and three dimensional steady-state and transient groundwater numerical simulation is proposed in [10].

In this research, the objective of groundwater flow management model is the minimum cost of injection rates. These are then subjected to optimal management of the water injection stations to achieve minimum cost. The numerical experiments are also given.

2 The governing equation of groundwater steady-flow model

Mathematical models of groundwater flow are all based on the water balance principle. The combination between the mass balance equation and Darcy's law produce the governing equation for groundwater flow. The general equation that governs two-dimensional groundwater steady-flow in isotropic, homogeneous porous media and vetically average [11],

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0, \tag{2.1}$$

where $H(x, y)$ is hydraulic head (metre). We will introduces the affected term as sources and sinks due to the external inputs and outputs. Consequently, the Eq.(2.1) becomes

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + W = 0, \tag{2.2}$$

where W is sinks and/or sources (1/day). The boundary conditions are specified, for all $(x, y) \in [0, L] \times [0, M]$ where L and M are positive constants which represent the dimension of the regtangular domain.

$$\frac{\partial H}{\partial n} = B_N \qquad \text{for all } 0 \le x \le L \text{ and } y = M, \tag{2.3}$$

$$\frac{\partial H}{\partial n} = B_S \qquad \text{for all } 0 \le x \le L \text{ and } y = 0, \tag{2.4}$$

$$H = B_W \qquad \text{for all } x = 0 \text{ and } 0 \le x \le M, \tag{2.5}$$

$$H = B_E \qquad \text{for all } x = L \text{ and } 0 \le x \le M. \tag{2.6}$$

and the source terms W that represented by the rate of pumping well in each point,

$$W(x_s, y_s) = Q(x_s, y_s) = Q_s \qquad \text{for all } s = 1, 2, 3, ..., p, \tag{2.7}$$

where s is a number of pumping wells.

3 Numerical techniques

Due to the groundwater steady-flow model is independent of time, the implicit finite difference method is used to solve the groundwater flow model. The method is suitable for the model due to the linear systems of equations will be constructed. It is possible to implement with the groundwater management model. Taking the central difference scheme in space into terms of equation (2.2), then

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H_{i-1,j} - 2H_{i,j} + H_{i+1,j}}{(\Delta x)^2}, \tag{3.1}$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H_{i,j-1} - 2H_{i,j} + H_{i,j+1}}{(\Delta y)^2}, \tag{3.2}$$

$$W_{i,j} = \pm \frac{Q_{i,j}}{\Delta x \Delta y H_{i,j}}. \tag{3.3}$$

Substituting Eq.(3.1) - (3.3) into Eq.(2.2), for $1 < i < I - 1$ and $1 < j < J - 1$,

$$-(2a + 2b) H_{i,j} + aH_{i-1,j} + aH_{i+1,j} + bH_{i,j-1} + bH_{i,j+1} = -W_{i,j}, \tag{3.4}$$

where $a = \frac{1}{(\Delta x)^2}$ and $b = \frac{1}{(\Delta y)^2}$. For $i = 1$ and $j = 1$, substituting the unknown value on the south boundary by forward difference approximation, $H_{1,0} = H_{1,1}$, into Eq.(3.4), we obtain

$$- (2a + b)\, H_{1,1} + aH_{2,1} + bH_{1,2} = -W_{1,1} - aB_W. \tag{3.5}$$

For $1 < i < I-1$ and $j = 1$, substituting the unknown value on the south boundary by forward difference approximation, $H_{i,0} = H_{i,1}$, into Eq.(3.4), we obtain

$$- (2a + b)\, H_{i,1} + aH_{i-1,1} + aH_{i+1,1} + bH_{i,2} = -W_{i,1}. \tag{3.6}$$

For $i = I - 1$ and $j = 1$, substituting the unknown value on the south boundary by forward difference approximation,$H_{I-1,0} = H_{I-1,1}$, into Eq.(3.4), we obtain

$$- (2a + b)\, H_{I-1,1} + aH_{I-2,1} + bH_{I-1,2} = -W_{I-1,1} - aB_E. \tag{3.7}$$

For $i = 1$ and $j = J - 1$, substituting the unknown value on the north boundary by backward difference approximation, $H_{1,J} = H_{1,J-1}$, into Eq.(3.4), we obtain

$$- (2a + b)\, H_{1,J-1} + aH_{2,J-1} + bH_{1,J-2} = -W_{1,J-1} - aB_W. \tag{3.8}$$

For $1 < i < I - 1$ and $j = J - 1$, substituting the unknown value on the north boundary by backward difference approximation, $H_{i,J} = H_{i,J-1}$, into Eq.(3.4), we obtain

$$- (2a + b)\, H_{i,J-1} + aH_{i-1,J-1} + aH_{i+1,J-1} + bH_{i,J-2} = -W_{i,J-1}. \tag{3.9}$$

For $i = I - 1$ and $j = J - 1$, substituting the unknown value on the north boundary by backward difference approximation, $H_{I-1,J} = H_{I-1,J-1}$, into Eq.(3.4), we obtain

$$- (2a + b)\, H_{I-1,J-1} + aH_{I-2,J-1} + bH_{I-1,J-2} = -W_{I-1,J-1} - aB_E. \tag{3.10}$$

The equations (3.4) - (3.10) can be written in matrix form as follow,

$$AH = B, \tag{3.11}$$

where

$$A = \begin{bmatrix} A_1 & A_3 & & & \\ A_3 & A_2 & A_3 & & \\ & \ddots & \ddots & \ddots & \\ & & A_3 & A_2 & A_3 \\ & & & A_3 & A_1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(2a+b) & a & & & \\ a & -(2a+b) & a & & \\ & \ddots & \ddots & \ddots & \\ & & a & -(2a+b) & a \\ & & & a & -(2a+b) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -(2a+2b) & a & & & \\ a & -(2a+2b) & a & & \\ & \ddots & \ddots & \ddots & \\ & & a & -(2a+2b) & a \\ & & & a & -(2a+2b) \end{bmatrix},$$

$$A_3 = \begin{bmatrix} b & & & & \\ & b & & & \\ & & \ddots & & \\ & & & b & \\ & & & & b \end{bmatrix},$$

$$H = \begin{bmatrix} H_{1,1} \\ H_{2,1} \\ \vdots \\ H_{I-2,J-1} \\ H_{I-1,J-1} \end{bmatrix},$$

and

$$B = \begin{bmatrix} -W_{1,1} - aB_W \\ -W_{2,1} \\ \vdots \\ -W_{I-2,1} \\ -W_{I-1,1} - aB_W \\ -W_{1,2} - aB_W \\ \vdots \\ -W_{i,j} \\ \vdots \\ -W_{I-1,J-2} - aB_E \\ -W_{1,J-1} - aB_W \\ -W_{2,J-1} \\ \vdots \\ -W_{I-2,J-1} \\ -W_{I-1,J-1} - aB_E \end{bmatrix}.$$

4 A groundwater management model

The objective function is the total cost of all pump injection in the considered system,

$$C = \sum_{s=1}^{m} W_s Q_s, \tag{4.1}$$

where s is the number of pumping wells, W_s is the cost of water pumping for each well s $(Baht/m^3)$ and Q_s is the injection rate for each well s (m^3/day). The constraint are

$$H_s \leq H_{ST_s}, \tag{4.2}$$

where H_s are the hydraulic head at monitoring point that measuring water requirement for each zone s and H_{ST_s} are the standard water requirement for each zone s. The upper bound of the injection rate for each pumping well are,

$$Q_s \leq Q_{\max_s}, \tag{4.3}$$

the lower bound of the injection rate for each pumping well are,

$$Q_s \geq Q_{\min_s}, \tag{4.4}$$

and the hydraulic head at monitoring point s and the injection rate at pumping wells are non-negative, that are

$$H_s, Q_s \geq 0, \tag{4.5}$$

where $Q_{\min_s}$ and $Q_{\max_s}$ are the lower and upper bounds of the water injection rate for each point s, respectively. The optimal cost of them is solved by using the simplex method.

5 Numerical Experiments

We consider the area width 2400 m and length 2400 m which is between a pair of two rivers. The area is meshed by 100 grids points with grid space is 240 m. The boundary conditions of the area are specified Eqs.(2.3) - (2.6) where $B_N = 0$, $B_S = 0$, $B_W = 20$ and $B_E = 19$. The four injection wells are injecting the water to the underground. The injection wells have the difference lower bound of injection rates, difference upper bound of injection rates and the difference cost of injection wells for each zone as Table 1. There are 8 monitoring point for measuring water requirement. The hydraulic head at monitoring point have the difference standard water requirement each point as Table 2.

Table 1: The injection rates and the cost of each pumping wells.

Position coordinate (x, y)	$Q(1440m, 480m)$	$Q(480m, 720m)$
Lower (m^3/day)	165	175
Upper (m^3/day)	250	230
Cost $(Bath/m^3)$	1.5	1.9
Position coordinate (x, y)	$Q(1680m, 1440m)$	$Q(720m, 1680m)$
Lower (m^3/day)	180	190
Upper (m^3/day)	300	270
Cost $(Bath/m^3)$	1.8	1.6

Table 2: The standard water requirement (SWR) for each monitoring points.

Position coordinate (x, y) (m, m)	(240,240)	(960,480)	(1920,720)	(720,1200)
SWR (m)	22	23	24	24
Position coordinate (x, y) (m, m)	(1200,1200)	(240,1440)	(1200,1920)	(1920,1920)
SWR (m)	25	23	25	24

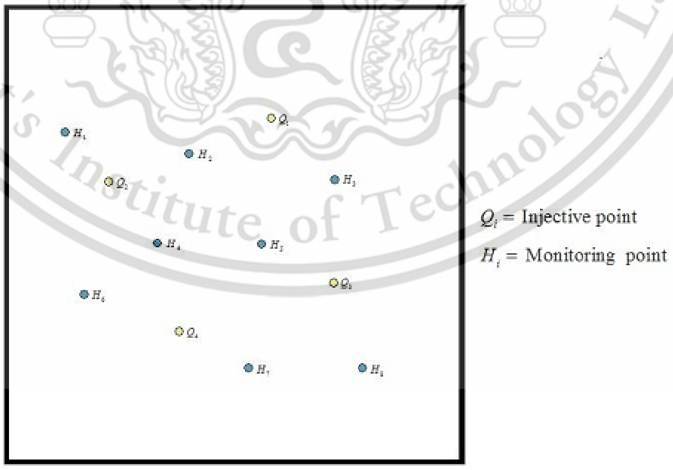

Figure 1: Simulation of groundwater management.

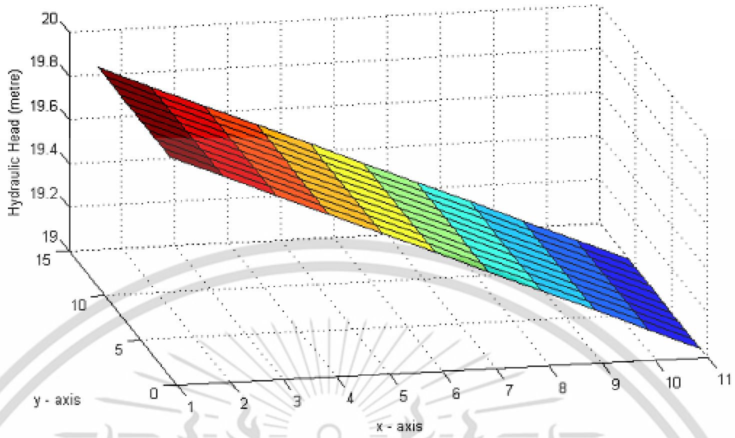

Figure 2: The surface graph before optimal control of cost.

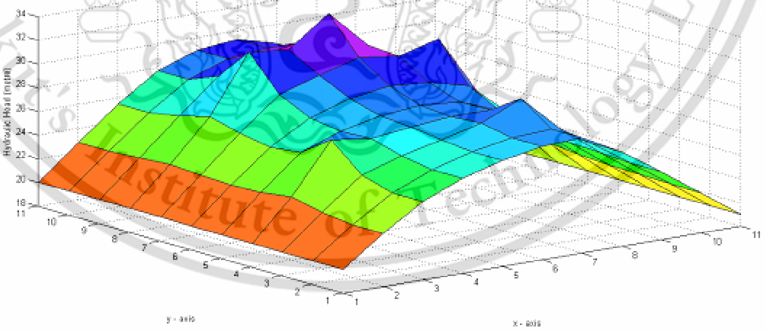

Figure 3: The surface graph after optimal control of cost.

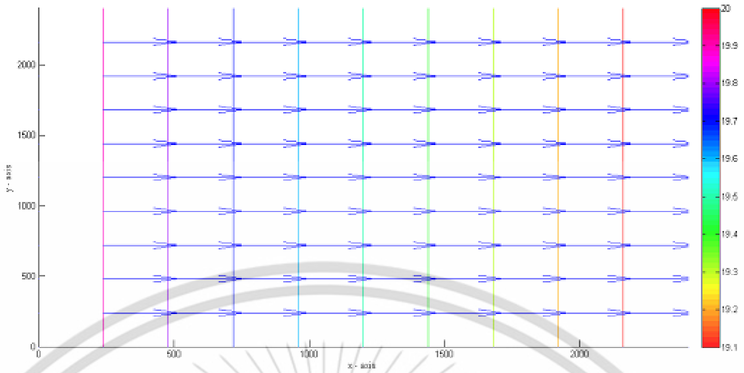

Figure 4: The contour graph and direction flow before optimal control of cost.

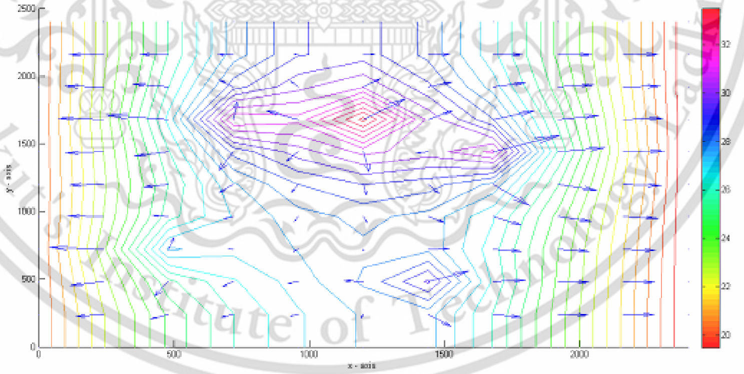

Figure 5: The contour graph and direction flow after optimal control of cost.

Table 3: Table of the hydraulic head before optimal control of cost.

y, x	0	240	480	720	960	1200	1440	1680	1920	2160	2400
0	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
240	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
480	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
720	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
960	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1200	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1440	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1680	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1920	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
2160	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
2400	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000

Table 4: Table of the hydraulic head after optimal control of cost.

y, x	0	240	480	720	960	1200	1440	1680	1920	2160	2400
0	20.0000	22.4668	24.6064	25.9931	26.9362	27.5723	27.7052	25.9609	23.7603	21.4090	19.0000
240	20.0000	22.4668	24.6064	25.9931	26.9362	27.5723	27.7052	25.9609	23.7603	21.4090	19.0000
480	20.0000	22.7940	25.3593	26.4366	27.2432	28.0755	29.5824	26.4172	23.9110	21.4667	19.0000
720	20.0000	23.3498	27.6001	27.1510	27.5245	27.9041	27.8818	26.2143	24.0000	21.5467	19.0000
960	20.0000	23.0050	25.7905	27.0426	27.7999	28.1345	27.8265	26.5582	24.3280	21.7200	19.0000
1200	20.0000	22.8798	25.5142	27.4292	28.4979	29.0074	28.7315	27.8639	25.0340	22.0051	19.0000
1440	20.0000	23.0000	25.9574	28.6619	29.7552	30.6658	30.2281	31.1318	25.9390	22.2665	19.0000
1680	20.0000	23.1628	26.6534	31.5058	31.1952	33.6724	30.3834	28.5222	25.3236	22.1220	19.0000
1920	20.0000	22.9979	25.9875	28.7726	29.8473	30.4711	29.1111	27.2499	24.7114	21.8977	19.0000
2160	20.0000	22.8414	25.5262	27.7496	28.9502	29.2537	28.3398	26.6547	24.3745	21.7574	19.0000
2400	20.0000	22.8414	25.5262	27.7496	28.9502	29.2537	28.3398	26.6547	24.3745	21.7574	19.0000

Table 5: The optimal injection rates of minimum cost in the systems.

Position Coordinate (x, y)	Injection Rate (m^3/day)
$Q(1440m, 480m)$	165
$Q(480m, 720m)$	175
$Q(1680m, 1440m)$	239.48
$Q(720m, 1680m)$	214.81

6 Discussion

The injective point and monitoring point are locate in the considered area as show in Figure 1. The monitored hydraulic head without and within controlled cost are shown in Figure 2 and Figure3 respectively. The vector fields of groundwater flow velocity between both case are are shown in Figure 4 and Figure 5 respectively. The hydraulic head before and after control of cost are shown in Table 3 and Table 4 respectively. The optimum injection rate at minimum cost of groundwater control is shown in Table 5.

7 Conclusion

We have established the groundwater management model. First, we will measure hydraulic head from the groundwater steady-flow model by using an implicit finite difference method. It will turn out that the system of linear equations is generated. We employ the system of linear equations to construct the groundwater management model to investigate the optimal least cost of the water injections in the system under the limitation conditions were required. Although, the water requirement and the injection cost of each monitoring points are unequal among injective stations. These are then subjected to the optimal presure of the groundwater injection station to achieve minimum cost. We have established a simulation process by means of which hydraulic head levels can be increase to agreed requirement levels at least cost.

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