

**MATHEMATICAL SIMULATION TO A MOTION OF SUSPENDED
STRING AND VIBRATING MEMBRANE USING DIFFERENTIAL
TRANSFORMATION METHOD**



**A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENT FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY (APPLIED MATHEMATICS)
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG**

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KMITL-2018-SC-M-001-036

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หัวข้อวิทยานิพนธ์	แบบจำลองทางคณิตศาสตร์ของการเคลื่อนที่ของเส้นลวด ในแนวตั้งและการสั่นของเมมเบรนโดยใช้วิธีการแปลงเชิงอนุพันธ์
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บทคัดย่อ

ในงานวิจัยนี้ได้ทำการศึกษาปัญหาออกเป็น 3 ขั้นตอน ขั้นตอนแรก เพื่อแสดงที่มาของทฤษฎีบทใหม่สำหรับการแปลงเทอมไม่เชิงเส้นของวิธีการแปลงเชิงอนุพันธ์ และประยุกต์ใช้วิธีการแปลงเชิงอนุพันธ์ในการหาผลเฉลยประมาณค่าสำหรับปัญหาค่าขอบของการเคลื่อนที่ของสมการการสั่นเส้นลวดในแนวตั้งที่มีความหนาแน่นไม่สม่ำเสมอ ปัญหาการสั่นในเส้นลวดในแนวตั้งที่มีแรงภายนอกกระทำกับเส้นลวด และปัญหาการสั่นในเส้นลวดในแนวตั้งที่มีความหน่วง สามารถแสดงการหาผลเฉลยโดยง่ายได้โดยวิธีการแปลงเชิงอนุพันธ์ ขั้นตอนที่สองเราใช้วิธีการแปลงเชิงอนุพันธ์แบบสามมิติเพื่อประมาณค่าสำหรับปัญหาสมการการสั่นในเมมเบรนเราได้แสดงการหาผลเฉลยเชิงวิเคราะห์ของการสั่นเมมเบรนด้วยวิธีแยกตัวแปร เราพบว่าผลการเปรียบเทียบระหว่างผลเฉลยเชิงวิเคราะห์กับผลเฉลยประมาณค่ามีความสอดคล้องกันในกรณีที่ไม่มีความหน่วงและแรงภายนอกกระทำ นอกจากนี้วิธีการแปลงเชิงอนุพันธ์ สามารถนำมาใช้เพื่อหาผลเฉลยประมาณค่าโดยคำนึงถึงแรงภายนอกและมีความหน่วงซึ่งไม่สามารถหาผลเฉลยเชิงวิเคราะห์ได้ เราได้นำเสนอวิธีการแปลงเชิงอนุพันธ์สำหรับฟังก์ชันแบบไม่เชิงเส้น ที่มาของสมการเวียนบังเกิดของการแปลงเชิงอนุพันธ์ถูกแสดงและนำไปประยุกต์เพื่อหาสัมประสิทธิ์ของอนุกรมเทย์เลอร์ที่ถูกนำไปใช้ประมาณค่าฟังก์ชันไม่เชิงเส้น เราพบว่าการเปรียบเทียบผลระหว่างค่าประมาณของอนุกรมเทย์เลอร์และฟังก์ชันไม่เชิงเส้นมีความสอดคล้องกัน

คำสำคัญ : การแปลงเชิงอนุพันธ์ ผลเฉลยประมาณค่า สมการการสั่นในเส้นลวดในแนวตั้ง สมการการสั่นในเมมเบรน

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ABSTRACT

In this research is concerned with three parts. The first part is to derive the general form for the non-linear term transformation of the differential transformation method (DTM) and apply this method to find the approximated solution for the initial boundary value problem of the suspended string equation with non-uniform densities, with nonlinear external force, with damping term and without damping term. We then apply the three-dimensional differential transform method to estimate solutions to the vibrating membrane. We also find the analytical solution of the vibrating membrane by separable variable method. We find that the comparison results between the analytical solution and the estimated solution are in good agreements in case of no damping and the external force. Furthermore, the differential transform method can also be used to find approximated solutions with both an external force and damping term which cannot be achieved by an analytical solution. We also present the modified differential transformation method for the nonlinear function. The derivation of the differential transformed recurrence formulas are shown and applied to find coefficients of the Taylor's series which are the approximant for the nonlinear functions. We find that the comparison results between the approximate Taylor's series and the nonlinear function are acceptable.

Keywords: Differential Transformation Method, Approximated Solution, Suspended String Equation, the Vibrating of Membrane.



Acknowledgements

Frist, I would like to express my sincere gratitude to my advisor, Assistant Professor Dr. Jaipong Kasemsuwan, for giving me an opportunity to do this thesis, for providing me all support and guidance, for understanding, patience and encouragement until the completion of the thesis.

I am fully indebted to my committees, Assistant Professor Dr. Kanognude Wuttanachamsri, Assistant Professor Dr. Nopparat Pochai, Assistant Professor Dr. Kanchana Kamnungkit, for their useful comments. I also heartily thank my external examiner, Associate Professor Dr. Suwon Tangmanee for his kindness.

I gratefully acknowledge the financial support from Rajamangala University of Technology Tawan-ok Chanthaburi Campus.

To all the senior Ph.D. Students in the department of applied mathematics with their consistent guidance and ample time spent as my proof readers. Also with their unwavering support and help to bring this thesis into success. I would also like to thank Mrs. Kanyarat Cheawchan for her help with computer programming.

Finally, I would like to express my gratitude and thanks to my parent, older sister, sister in law, daughter and husband who have always been tremendous support and encouragement throughout my entire study.

Kamonpad Mansilp

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Chapter 1

Introduction

1.1 Research Motivation

Many researchers have attempted to understand several phenomena occurring in nature by applying knowledge from different fields such as mechanical engineering, electrical engineering, industrial engineering, energy and medicine. Most of these problems have been studied by employing some form of mathematical modeling using both ordinary differential equations (ODE) and partial differential equations (PDE). These problems require an accurate solution, either as an analytical solutions or approximate solutions.

In the study of horizontal vibration which is in the form of partial differential equations widely such as water wave, standing acoustic wave, X-ray standing wave and ultrasonic wave. It can be seen that researches and studies of waves are very beneficial to humans. The suspended string equation was studied firstly by a Russian mathematician [1]. The suspended string equation is shown in equation (1.1) where $u(x,t)$ denotes the horizontal displacement of the heavy and flexible string suspended with the upper end fixed and the lower end free.

$$\begin{aligned} \partial_t^2 u(x,t) - \frac{x}{m+1} \partial_x u(x,t) - \partial_x u(x,t) &= f(x,u(x,t)), \quad (x,t) \in \Omega, \\ u(a,t) &= 0, \quad t \in [0,T], \\ u(x,0) &= \phi(x), \quad \partial_t u(x,0) = \psi(x), \quad x \in [0,a], \end{aligned} \quad (1.1)$$

where Ω is a cylindrical domain $(0,a) \times (0,T)$ and the power density is denoted by $\rho x = \alpha x^m$ where α and m are constants.

An acoustic membrane under tension can encourage transverse vibrations. A thin layer of a vibration, is used in acoustics to produce or transfer sound, such as a drum, microphone, or loudspeaker. The characteristics of an idealized drumhead can be modeled by the vibrations of a circular membrane of uniform thickness, attached to a rigid frame. Due to the phenomenon of resonance, at certain vibration frequencies, its resonant frequencies, the membrane can store vibrational energy, the surface moving in a characteristic pattern of standing waves.

The equation of motion for the forced transvers vibration of the membrane in [2] is as follows:

$$P \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + f(x, y, t) = \rho(x, y) \frac{\partial^2 w}{\partial t^2}, \quad (1.2)$$

where $f(x, y, t)$ is the pressure force loading acting in the z direction (external force), P is intensity of tension at a point that is equal to production of the tensile stress and the thickness of the membrane and $\rho(x, y)$ is the mass per unit area. We assume that $f(x, y, t) = 0$, $P = 1$, and $\rho(x, y) = 1$. Then Equation (1.2) leads to

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial t^2}. \quad (1.3)$$

The differential transformation method (DTM) is a very powerful and efficient tool applied in so many research works to solve partial differential equations (PDE) such as done in [3]- [17]. The DTM applied to second order PDE under boundary conditions were studied in [6] without initial condition.

In practice, several realistic problems are PDE which are subject to both initial and boundary conditions. Therefore, a solution to the problems taking into account those conditions using DTM method are the primary objective of this research.

The suspended string equation was previously studied using numerical method in [5] - [12]. The numerical solution excluding an external force has been shown in [12] using the Crank-Nicolson method. [10] and [11] have shown the numerical simulation including both linear and nonlinear damping terms yet without an external force. The numerical simulation including an external force has been considered in [12]. However, all of those works considered suspended string only in the uniform density case and defined form of the nonlinear external force including the linear and nonlinear damping cases.

In this work, we extended the study in [5] and [6] using the two dimensional DTM and then develop a new theorem for the cubic transformation. A suspended string equation is used which involves both boundary conditions as a case study.

Differential transform is the transformation of function $c(x)$ so called original function to be new function $C(k)$ so called transform function.

Differential Transform of $c(x)$ is defined by [38]

$$C(k) = \frac{1}{k!} \left[\frac{d^k}{dx^k} c(x) \right]_{x=x_0} \quad (1.4)$$

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and differential inverse transform of $C(k)$ is defined by [38]

$$c(x) = \sum_{k=0}^{\infty} C(k)(x-x_0)^k \quad (1.5)$$

1.2 Literature Review

The differential transform method (DTM) is based on the high-order Taylor series expansion. This method is a powerful tool for solving linear and non-linear ordinary differential equations [22] - [24] and [29], and for solving two and three-dimensional partial differential equations in both linear and non-linear problems. DTM can be used to solve the differential equation subject to initial and boundary conditions having both linear and non-linear terms within an acceptable error range.

The two-dimensional differential transform method (2D-DTM) is used to find the solutions of both linear PDEs [23] - [36].

Additionally, the three-dimensional differential transform method (3D-DTM) is applied to find the solutions of linear and non-linear PDEs [25] and [35] respectively. It is noted that the differential transform method can be used to solve multidimensional PDEs, such as the Westervelt equation [31], heat-like and wave-like equations [35] and fuzzy partial differential equations [33], as well as the linear and non-linear system of PDEs [24] and [36].

There are many engineering studies which have examined the dynamical analysis of membrane by various engineering tools such as loudspeakers, microphones, membrane valves, pressure regulators, and antennas for space communications [14] - [12]. In this research, [15] we have studied the oscillation of a membrane-like plate, which is determined by the tension and insignificant resistance to the bending, we have applied the differential transform method to find solutions in the vibrating equation of a membrane, and the comparison result is in good agreement in the simple case.

In addition, this work is also concerned with a new algorithm for calculating one-dimensional differential transform of nonlinear functions first presented in [37] applying the Taylor's series about the origin (Maclaurin series) to approximate the nonlinear functions in the basic function forms such as the exponential function, the logarithm function, the trigonometric function and the hyperbolic trigonometric function. However, we tried to apply the Maclaurin series to approximate the

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nonlinear functions, we found that the approximate Maclaurin series were not in good agreement with the nonlinear functions. In this work, we should study further from the previous work [37] to modify the differential transformation method for finding the approximate Taylor's series about any points for the nonlinear functions.

In solving these examples, recurrence systems are first obtained and then a routine computer programming is used to find an approximated solutions in the Taylor's series form.

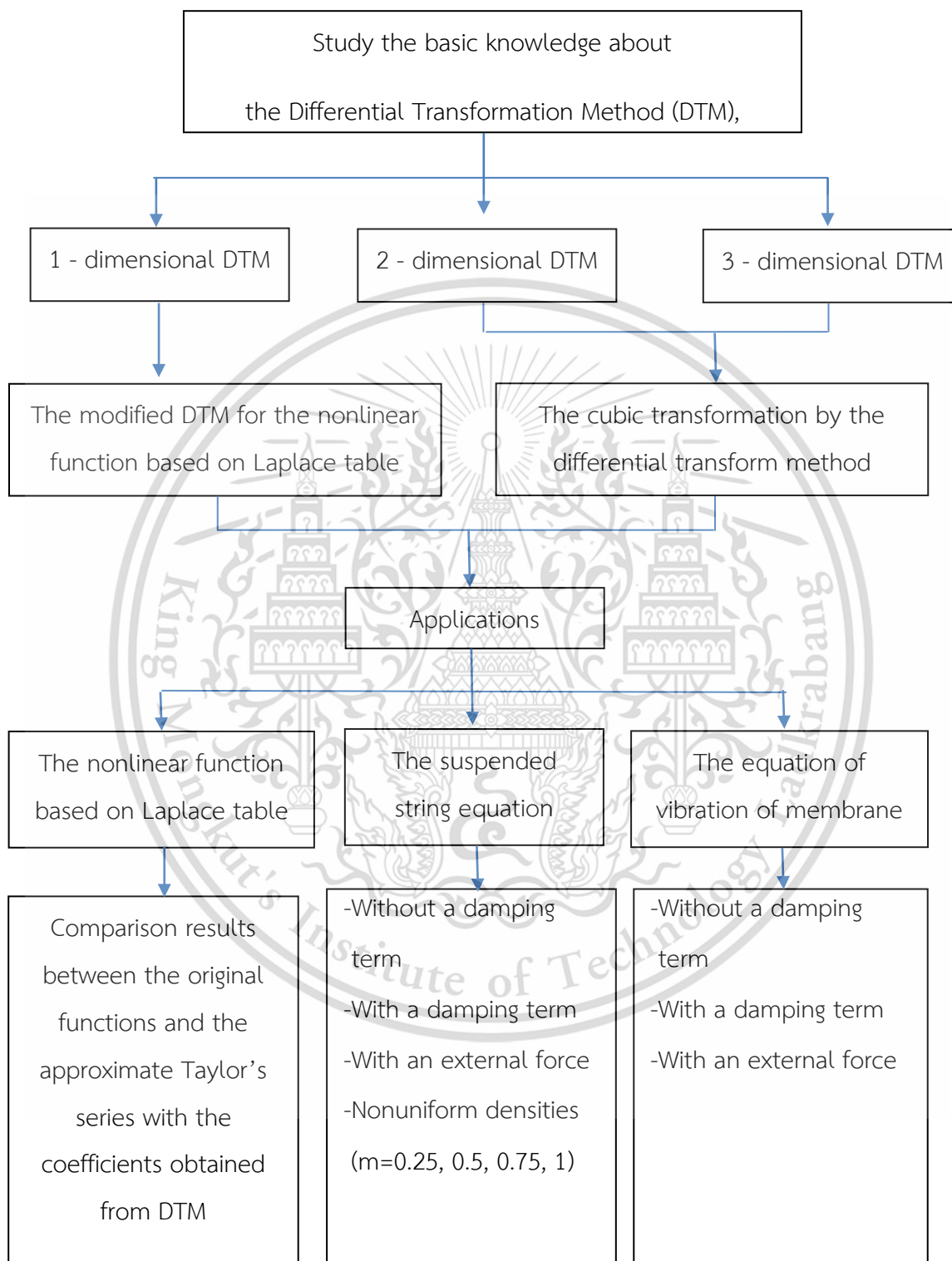
1.3 Objectives of the Study

- 1) To derive of the formulas involed the nonlinear transformation of the differential transformation method (DTM).
- 2) To use the differential transform method for solving the suspended string equation without a damping term, with a damping term, with an external force and a nonuniform densities.
- 3) To use the differential transform method for solving the vibrating membrane equation without a damping term, with a damping term, with an external force, and with an external force and a damping term.
- 4) To use the modified differential transform method for finding the approximate Taylor's series of nonlinear functions.

1.4 Scope of the Study

The derivation of new formulas for the nonlinear cubic function by the differential transformation method (DTM) are shown and apply this method to the approximate solution to the initial and boundary value problems of the suspended string and the vibrating membrane. In addition, the modified differential transformation method for some nonlinear functions will be studied.

1.5 Plan of the Study



1.6 Expected Results

The expected results of thesis are as follows the modified theorem of the differential transform of the nonlinear term is presented. The proof of the theorem is also derived and the theorem can be applied in several applications. The non-linear term of differential transform can be applied efficiently to the nonlinear terms of suspended string equation and the equation of motion of the membrane. The differential transform method can be applied to find the approximate Taylor's series for the nonlinear functions.



Chapter 2

Basic Concepts and Preliminaries

In this chapter, we introduce two equations which are the suspended string equation and the equation of motion of membrane in Section 2.1 and 2.2 respectively. In Section 2.3, we indicate the basic definitions and fundamental operations of differential transform.

2.1 The Suspended String Equation

The suspended string equation as shown in Equation (2.1) where $u(x,t)$ denotes the horizontal displacement of the heavy and flexible string suspended with the upper end fixed and the lower end free [1].

$$\begin{aligned} \partial_t^2 u(x,t) - \frac{x}{m+1} \partial_x u(x,t) - \partial_x u(x,t) &= f(x,u(x,t)), \quad (x,t) \in \Omega, \\ u(a,t) &= 0, \quad t \in [0,T], \\ u(x,0) &= \phi(x), \quad \partial_t u(x,0) = \psi(x), \quad x \in [0,a], \end{aligned} \quad (2.1)$$

where Ω is a cylindrical domain $(0,a) \times (0,T)$ and the power density is denoted by $\rho(x) = \alpha x^m$ where α and m are constant.

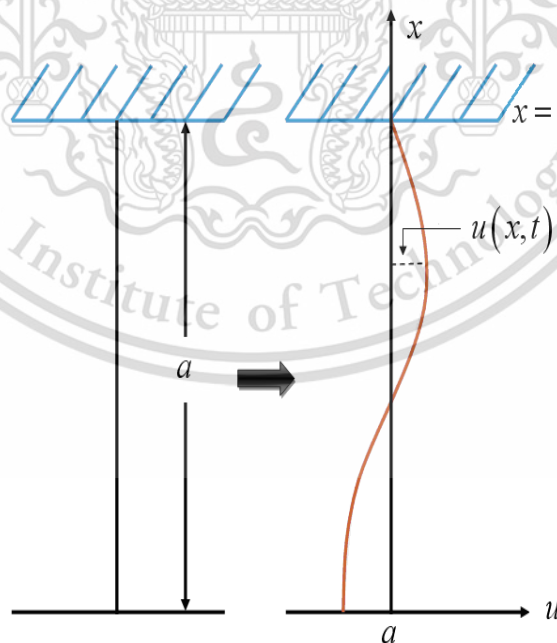


Figure 2.1 The suspended string.

N. S. Koshlyakov [1] studied the suspended string which is flexible, heavy, limited length (a) and mass density at position x . The suspended string was hung vertically with the top end being pinned to the ceiling, the bottom end was released independently under the force of gravity. Set the vertical line through the fixed point to the ceiling as the x -axis and origin is determined that the position of the top end. When the suspended string is in normal state without external force, except for gravity. At the bottom end is the position $x=a$ as shown in Figure 2.1. We define $u = u(x,t)$ instead of vibration (the displacement) of the suspended string and assume that the hanging of the suspended string vibrates in a plane. In that direction, all the points of the suspended string vibrate horizontally under the external linear force h . The action on the suspended string with h in the form of linear $h(x,t)$ and the suspended string vibrates slightly on $\Omega = \{(x,t) | [0,a] \times [0,t]\}$ Newton's second law. The suspended string equation is defined by

$$\rho(x) \frac{\partial^2 u}{\partial t^2} - g \left(\frac{\partial}{\partial x} \left(\int_0^x \rho(s) ds \frac{\partial u}{\partial x} \right) \right) = h, \quad (x,t) \in \Omega \quad (2.2)$$

where gravity(g) is 9.8 m/s^2 and a linear outer forcing form (h) N/m for this study approximate g is 1 m/s^2 .

The derivation of suspended string equation can be considered as follows:

we consider that part of the suspended string equation with x and $x + \Delta x$ is both ends of the string and Δx is distance between x and $x + \Delta x$ by Δx have a little value.

By $F(x,t) = F_1$ instead of the force at the point x at t and $F(x + \Delta x, t) = F_2$ instead of the force at the point $x + \Delta x$ at t and define $\theta_1(x), \theta_2(x + \Delta x)$ are the angle with the vertical as shown in Figure 2.2.

F_1 is the force acting on the suspended string at the point x on the horizontal force and vertical force are $F_1 \sin \theta_1$ and $F_1 \cos \theta_1$, respectively Figure 2.2 F_2 is the force acting on the suspended string at the point $x + \Delta x$ on the horizontal force and vertical force is $F_2 \sin \theta_2$ and $F_2 \cos \theta_2$, respectively as shown in Figure 2.2

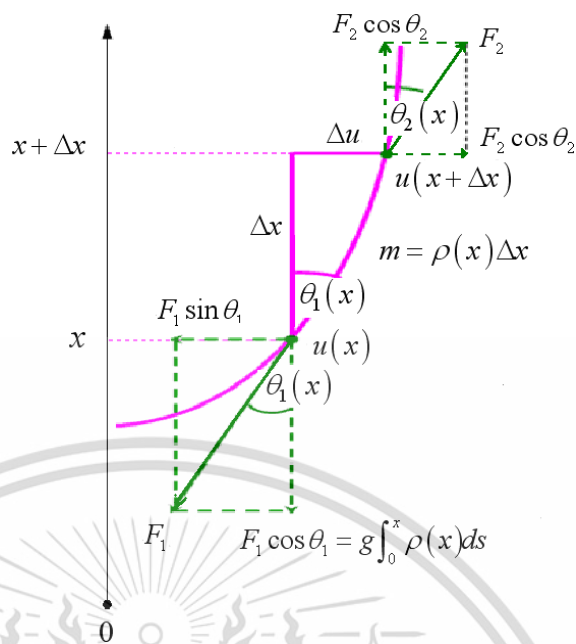


Figure 2.2 The section of the suspended string in the considered range.

Obtaining $\rho(x)$ is mass of density, s is distance and m is mass which $m = \rho(x) \times s$

$$\text{at } x + \Delta x \text{ we have, } m(x + \Delta x) = \int_0^{x + \Delta x} \rho(s) ds \quad (2.3)$$

$$\text{at } x \text{ we have, } m(x) = \int_0^x \rho(s) ds \quad (2.4)$$

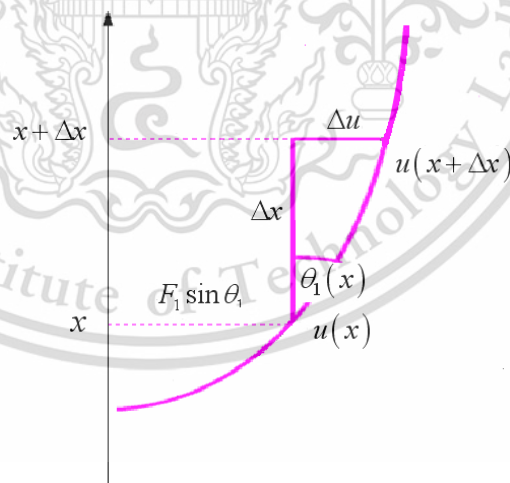


Figure 2.3 The value of the tangent.

Figure 2.3, we have

$$\tan \theta = \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x},$$

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that

$$\frac{\sin \theta}{\cos \theta} = \frac{\partial u}{\partial x}, \quad (2.5)$$

due to the horizontal force is

$$\begin{aligned} F(x + \Delta x, t) - F(x, t) &= F_2 \sin \theta_2 - F_1 \sin \theta_1 \\ &= F_2 \cos \theta_2 \frac{\sin \theta_2}{\cos \theta_2} - F_1 \cos \theta_1 \frac{\sin \theta_1}{\cos \theta_1} \\ &= F_2 \cos \theta_2 \tan \theta_2 - F_1 \cos \theta_1 \tan \theta_1, \end{aligned}$$

since θ has very small and close to zero, we have

$$= F_2 \tan \theta_2 - F_1 \tan \theta_1,$$

from $F = mg$ and $\tan \theta = \frac{\partial u}{\partial x}$, we have

$$F(x + \Delta x, t) - F(x, t) = g \left\{ \int_0^{x+\Delta x} \rho(s) ds \frac{\partial u(x + \Delta x, t)}{\partial x} - \int_0^x \rho(s) ds \frac{\partial u(x, t)}{\partial x} \right\}, \quad (2.6)$$

since

$$\frac{F(x + \Delta x, t) - F(x, t)}{\Delta x} = \frac{\partial F(x + \theta \Delta x, t)}{\partial x}, \quad 0 < \theta < 1$$

therefore

$$\begin{aligned} F(x + \Delta x, t) - F(x, t) &= \Delta x \frac{\partial F(x + \theta \Delta x, t)}{\partial x} \\ &= \Delta x \frac{\partial}{\partial x} \left[g \int_0^{x+\theta \Delta x} \rho(s) ds \frac{\partial u}{\partial x}(x + \theta \Delta x, t) \right], \end{aligned} \quad (2.7)$$

then

$$m = \rho(x) \Delta x, \quad (2.8)$$

from Newton's second law

$$ma = \sum F,$$

$$\rho(x) \Delta x \frac{\partial^2 u}{\partial t^2} = F(x + \Delta x, t) - F(x, t), \quad (2.9)$$

substituting Equation (2.7) in Equation (2.9), we have

$$\rho(x) \Delta x \frac{\partial^2 u}{\partial t^2} = g \left\{ \int_0^{x+\Delta x} \rho(s) ds \frac{\partial u(x + \Delta x, t)}{\partial x} - \int_0^x \rho(s) ds \frac{\partial u(x, t)}{\partial x} \right\}, \quad (2.10)$$

and

Equation (2.10),

we

have

$$\rho(x) \Delta x \frac{\partial^2 u}{\partial t^2} = \Delta x \frac{\partial}{\partial x} F(x + \theta \Delta x, t), \quad 0 < \theta < 1$$

$$\text{hence } \rho(x) \Delta x \frac{\partial^2 u}{\partial t^2} = \Delta x \frac{\partial}{\partial x} \left(g \int_0^{x+\theta \Delta x} \rho(s) ds \frac{\partial u(x + \theta \Delta x, t)}{\partial x} \right), \quad 0 < \theta < 1 \quad (2.11)$$

from Equation (2.11) with ξ value between x and $x + \Delta x$, we have

$$\rho(x)\Delta x \frac{\partial^2 u}{\partial t^2} = \Delta x g \frac{\partial}{\partial x} \int_0^\xi \rho(s) ds \frac{\partial u}{\partial x}(\xi, t), \quad x < \xi < x + \Delta x. \quad (2.12)$$

Considering Equation (2.12) when $g \approx 1$

when the external force $h(x, t)$ act on the suspended string horizontally, we have

$$\rho(x) \frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial}{\partial x} \left(\int_0^x \rho(s) ds \frac{\partial u(x, t)}{\partial x} \right) + h(x, t) = 0. \quad (2.13)$$

Since the mass of density depends on each point x therefore equation of density ρ is in the form of a power function of x or the power density is

$\rho(x) = \alpha x^\mu$ when α and m is constant
hence

$$\begin{aligned} \alpha x^m \frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial}{\partial x} \left(\int_0^x \alpha x^m dx \frac{\partial u(x, t)}{\partial x} \right) + h(x, t) &= 0, \\ \alpha x^m \frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial}{\partial x} \left(\alpha \frac{x^{m+1}}{m+1} \frac{\partial u(x, t)}{\partial x} \right) + h(x, t) &= 0, \\ x^m \frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial}{\partial x} \left(\frac{x^{m+1}}{m+1} \frac{\partial u(x, t)}{\partial x} \right) + \frac{h(x, t)}{\alpha} &= 0, \\ x^m \frac{\partial^2 u(x, t)}{\partial t^2} - \left(\frac{x^{m+1}}{m+1} \frac{\partial^2 u(x, t)}{\partial x^2} + (m+1) \frac{x^m}{m+1} \frac{\partial u(x, t)}{\partial x} \right) + \frac{h(x, t)}{\alpha} &= 0. \end{aligned}$$

Divide all of equation with, we have

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \left(\frac{x}{m+1} \frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial u(x, t)}{\partial x} \right) + \frac{h(x, t)}{\alpha x^\mu} = 0, \quad (2.14)$$

obtaining L is operation of differential order 2 we have,

$$L = \frac{x}{m+1} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}, \quad (2.15)$$

$$Lu = \frac{x}{m+1} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}, \quad (2.16)$$

substituting Equation (2.16) in Equation (2.14), we obtain the suspended string equation

$$\frac{\partial^2 u(x, t)}{\partial t^2} - Lu(x, t) + \frac{h(x, t)}{\alpha x^m} = 0, \quad (2.17)$$

$$\frac{\partial^2 u(x, t)}{\partial t^2} - Lu(x, t) + f(x, u(x, t)) = 0, \quad (2.18)$$

by $f(x, u(x, t)) = \frac{h(x, t)}{\alpha x^m}$ is external force.

In this research, we study the initial and boundary value problem of the suspended string Equation (2.18).

Since the top end of the suspended string is pinned to the ceiling.

$$\text{The boundary condition } u(0,t) = 0 \text{ and } \frac{\partial u(x,t)}{\partial x} = 0, \quad t \in [0, T]. \quad (2.19)$$

Since the suspended string depends on the initial shape of the string before it is allowed to vibrate and the initial speed of the points on the string.

$$\text{The initial conditions } u(x,0) = f_1(x) \text{ and } \frac{\partial u(x,0)}{\partial t} = f_2(x), \quad x \in [0, a]. \quad (2.20)$$

Analytical Solution

The suspended string equation of the initial and boundary value problem

$$\frac{\partial^2 u(x,t)}{\partial t^2} - Lu(x,t) = 0, \quad (2.21)$$

$$\text{with the boundary conditions } u(0,t) = 0, \quad t \in (0, T) \quad (2.22)$$

$$\text{and the initial conditions } u(x,0) = f_1(x) \text{ and } \frac{\partial u(x,0)}{\partial t} = f_2(x), \quad x \in (0, a) \quad (2.23)$$

L the second order differential, we have

$$L = \left(\frac{x}{m+1} \right) \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}. \quad (2.24)$$

Assuming the solution in the form of power series, we have

$$u(x,t) = \sum_{j=1}^{\infty} u_j(t) \phi_j(x),$$

$$f_1(x) = \sum_{j=1}^{\infty} p_j \phi_j(x), \quad (2.25)$$

$$f_2(x) = \sum_{j=1}^{\infty} q_j \phi_j(x),$$

$$\begin{aligned} \text{then } \frac{\partial^2 u(x,t)}{\partial t^2} &= \sum_{j=1}^{\infty} \frac{\partial^2 u_j(t)}{\partial t^2} \phi_j(x) \\ &= \sum_{j=1}^{\infty} \ddot{u}_j(t) \phi_j(x), \end{aligned} \quad (2.26)$$

$$Lu(x,t) = \sum_{j=1}^{\infty} u_j(t) L\phi_j(x), \quad (2.27)$$

$$\text{and } L\phi_j = \left(\frac{x}{m+1} \right) \frac{\partial^2 \phi_j}{\partial x^2} + \frac{\partial \phi_j}{\partial x},$$

$$\text{since } Lu(x,t) = \sum_{j=1}^{\infty} u_j(t) \left(\left(\frac{x}{m+1} \right) \frac{\partial^2 \phi_j}{\partial x^2} + \frac{\partial \phi_j}{\partial x} \right),$$

$$\sum_{j=1}^{\infty} \ddot{u}_j(t) \phi_j(x) - \sum_{j=1}^{\infty} u_j(t) L \phi_j(x) = 0.$$

Due to define $\phi_j(x)$ consistent with the solution of the eigenvalues problem then $L \phi_j(x) = \lambda \phi_j(x)$

$$\begin{aligned} \sum_{j=1}^{\infty} \ddot{u}_j(t) \phi_j(x) - \sum_{j=1}^{\infty} u_j(t) \lambda \phi_j(x) &= 0, \\ \left[\sum_{j=1}^{\infty} \ddot{u}_j(t) - \lambda \sum_{j=1}^{\infty} u_j(t) \right] \phi_j(x) &= 0, \\ \sum_{j=1}^{\infty} [\ddot{u}_j(t) - \lambda u_j(t)] \phi_j(x) &= 0. \end{aligned} \quad (2.28)$$

Consider of the solution Equation (2.28) as follows:

$$\begin{aligned} \ddot{u}_j + \lambda_j u_j &= 0 \\ u_j(0) = p_j, \quad \dot{u}_j(0) &= q_j. \end{aligned} \quad (2.29)$$

Find the solution by auxiliary Equation in the form

$$r^2 + \lambda_j = 0, \quad r = \pm \sqrt{\lambda_j} i.$$

The solutions are complex in general form

$$\begin{aligned} u_j &= c_1 \cos \sqrt{\lambda_j} t + c_2 \sin \sqrt{\lambda_j} t, \\ \dot{u}_j &= c_1 (-\sin \sqrt{\lambda_j} t) (\sqrt{\lambda_j}) + c_2 (\cos \sqrt{\lambda_j} t) (\sqrt{\lambda_j}), \end{aligned}$$

from the boundary condition $u_j(0) = p_j$ we have,

$$\begin{aligned} p_j &= c_1 \cos 0 + c_2 \sin 0, \\ c_1 &= p_j. \end{aligned}$$

From the boundary condition $\dot{u}_j(0) = q_j$ we have,

$$\begin{aligned} q_j &= c_1 (-\sin 0) (\sqrt{\lambda_j}) + c_2 (\cos 0) (\sqrt{\lambda_j}), \\ c_2 &= \frac{q_j}{\sqrt{\lambda_j}}, \end{aligned}$$

then

$$u_j = p_j \cos \sqrt{\lambda_j} t + \frac{q_j}{\sqrt{\lambda_j}} \sin \sqrt{\lambda_j} t.$$

That is

$$u(x, t) = \sum_{j=1}^{\infty} \left(p_j \cos \sqrt{\lambda_j} t + \frac{q_j}{\sqrt{\lambda_j}} \sin \sqrt{\lambda_j} t \right) \phi_j(x) \quad (2.30)$$

with $\phi_j(x)$ the solution of the eigenvalues problem for L

$$L \phi(x) = \left(\left(\frac{x}{m+1} \right) \frac{d^2}{dx^2} + \frac{d}{dx} \right) \phi(x) = \lambda \phi(x) \text{ in } (0, a), \quad (2.31)$$

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with the boundary condition

$$\phi(a) = 0, \quad (2.32)$$

we obtain $y = \sqrt{x}$ and $y^m = x^{\frac{m}{2}}$,

then $\psi(y) = x^{\frac{m}{2}} \phi(x) = y^m \phi(x)$,

and $\phi(x) = y^{-m} \psi(y) = x^{-\frac{m}{2}} \psi(\sqrt{x})$,

$$\begin{aligned} \partial_x \phi(x) &= \left(\frac{-m}{2} \right) x^{-\frac{m}{2}-1} \psi(\sqrt{x}) + x^{-\frac{m}{2}} \psi'(\sqrt{x}) \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2} x^{-\frac{m}{2}-1} \left\{ -m\psi(y) + x^{\frac{1}{2}} \psi'(y) \right\}, \\ \partial_x^2 \phi(x) &= \frac{1}{2} x^{-\frac{m}{2}-2} \left(\frac{-m}{2} - 1 \right) \left\{ -m\psi(\sqrt{x}) + x^{\frac{1}{2}} \psi'(\sqrt{x}) \right\} \\ &\quad + \frac{1}{2} x^{-\frac{m}{2}-1} \left\{ -m\psi'(\sqrt{x}) \frac{1}{2\sqrt{x}} + \frac{1}{2} x^{\frac{1}{2}} \psi'(\sqrt{x}) + x^{\frac{1}{2}} \psi''(\sqrt{x}) \frac{1}{2\sqrt{x}} \right\} \\ &= \frac{1}{4} x^{-\frac{m}{2}-2} \left[x\psi''(y) + (-2m-1)x^{\frac{1}{2}}\psi'(y) + m(m+2)\psi(y) \right], \end{aligned}$$

since

$$\frac{x}{m+1} \partial_x^2 \phi(x) = \frac{1}{4(m+1)} x^{-\frac{m}{2}-1} \left[x\psi''(y) - (2m+1)x^{\frac{1}{2}}\psi'(y) + m(m+2)\psi(y) \right],$$

from Equation (2.21), we have

$$\begin{aligned} \frac{x^{\frac{m}{2}-1}}{4(m+1)} \left[x\psi''(y) + x^{\frac{1}{2}}\psi'(y) - m^2\psi(y) \right] &= -\lambda\phi(x), \\ x\psi''(y) + x^{\frac{1}{2}}\psi'(y) - m^2\psi(y) &= -\lambda\phi(x) x^{\frac{m+2}{2}} \times 4(m+1), \end{aligned}$$

we obtain $y = x^{\frac{1}{2}}$, $\psi(y) = y^m \phi(x)$,

$$(y^2 = x), \quad (\phi(x) = y^{-m} \psi(y)),$$

then

$$y^2 \psi''(y) + y\psi'(y) + \{4(m+1)\lambda y^2 - m^2\} \psi(y) = 0. \quad (2.32)$$

Considering Equation (2.32) in the form of Bessel Equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0,$$

with the general solution $y = c_1 J_\nu(x) + c_2 Y_\nu(x)$,

where $J_\nu(x)$ is first of Bessel function

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{k! \Gamma(\nu+k+1)},$$

and $Y_\nu(x)$ is the second of Bessel function

$$Y_\nu(x) = \frac{J_\nu(x) \cos \nu\pi - J_{-\nu}(x)}{\sin \nu\pi}$$

$$= \frac{2}{\pi} J_\nu(x) \log \frac{x}{2} - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{-n+2k} \\ - \dots - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k}}{(k!)^2} \frac{\tau'(k+1)}{\tau(k+1)}.$$

Equation (2.32) match the form of the Bessel equation.

Let $4(\mu+1)\lambda = a^2,$

and $\psi(y) = F(ay) = F(Y),$

then $\frac{\partial \psi(y)}{\partial y} = a \frac{\partial F(Y)}{\partial Y}, \quad \frac{\partial^2 \psi(y)}{\partial y^2} = a^2 \frac{\partial^2 F(Y)}{\partial^2 Y},$

since

$$y^2 \psi''(y) + y \psi'(y) + \{a^2 y^2 - \mu^2\} \psi(y) = 0 \\ a^2 y^2 \frac{d^2 F(Y)}{dY^2} + ay \frac{dF(Y)}{dY^2} + \{Y^2 - \mu^2\} F(Y) = 0 \\ Y^2 \frac{d^2 F(Y)}{dY^2} + Y \frac{d}{dY} F(Y) + \{Y^2 - \mu^2\} F(Y) = 0. \quad (2.33)$$

From boundary condition $\psi(\sqrt{a}) = 0,$

that is $\psi(y) = F(Y) = c_1 J_\mu(Y) + c_2 Y_\mu(Y) \\ = c_1 J_\mu(2\sqrt{\lambda(\mu+1)}y) + c_2 Y_\mu(2\sqrt{\lambda(\mu+1)}y). \quad (2.34)$

Form boundary condition, we have

$$\psi(0) = Y_\mu(0) \quad \text{and} \quad \lim_{y \rightarrow 0} Y_\mu(y) = -\infty,$$

then $c_2 = 0,$

Equation (2.34), we have

$$\psi(y) = c_1 J_\mu(2\sqrt{\lambda(\mu+1)}y),$$

and boundary condition $\phi(a) = 0,$ we have

$$\psi(\sqrt{a}) = (\sqrt{a})^\mu (\phi(\sqrt{a}))^2 = 0,$$

due to $c_1 \neq 0,$ we have

$$J_\mu(2\sqrt{\lambda(\mu+1)}a) = 0,$$

since

$$\phi(x) = y^{-\mu} \psi(y) \\ = x^{-\frac{\mu}{2}} C J_\mu(2\sqrt{\lambda(\mu+1)}y). \quad (2.35)$$

Let $2\sqrt{\lambda_k(\mu+1)}a = \gamma_k,$

we have $\lambda_k = \frac{\gamma_k^2}{4(\mu+1)a}$

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by $\{ \gamma_k : k = 1, 2, \dots \}$ is set of all positive centers of the Bessel function $J_\mu(x)$
and $\gamma_1 < \gamma_2 < \gamma_3 \dots$

from Equation (2.35), we have

$$2\sqrt{\lambda(\mu+1)} = 2\sqrt{\frac{\gamma_k^2(\mu+1)}{4(\mu+1)a}} = \frac{\gamma_k}{\sqrt{a}},$$

then $\phi_k(x) = Cx^{-\frac{\mu}{2}} J_\mu\left(\frac{\gamma_k}{\sqrt{a}}\sqrt{x}\right),$

$$C = \frac{1}{\sqrt{a}J_{\mu+1}(\gamma_k)},$$

the specific function

$$\phi_k(x) = \frac{1}{\sqrt{a}J_{\mu+1}(\gamma_k)} \cdot \frac{1}{x^{\frac{\mu}{2}}} J_\mu\left(\gamma_k \sqrt{\frac{x}{a}}\right),$$

$$u(x,t) = \sum_{j=1}^{\infty} \left(p_j \cos \sqrt{\lambda_j} t + \frac{q_j}{\sqrt{\lambda_j}} \sin \sqrt{\lambda_j} t \right) \phi_j(x).$$

2.2 The Equation of Motion of Membrane

Rao S. [2] studied vibration of membrane and equation of motion of membrane, respectively.

2.2.1 Vibration of Membrane

A membrane is a plate that is subjected to tension and has negligible bending resistance. Thus a membrane bears the same relationship to a plate as a string bears to a beam A drumhead is an example of a membrane.

2.2.2 Equation of Motion of Membrane

To derive the equation of motion of a membrane, consider the membrane to be bounded by a plane curve S in the xy -plane, as shown in Figure 2.4. Let $f(x,y,t)$ denote the pressure loading acting in the z direction and P the intensity of tension at a point that is equal to the product of the tensile stress and the thickness of the membrane. The magnitude of P is usually

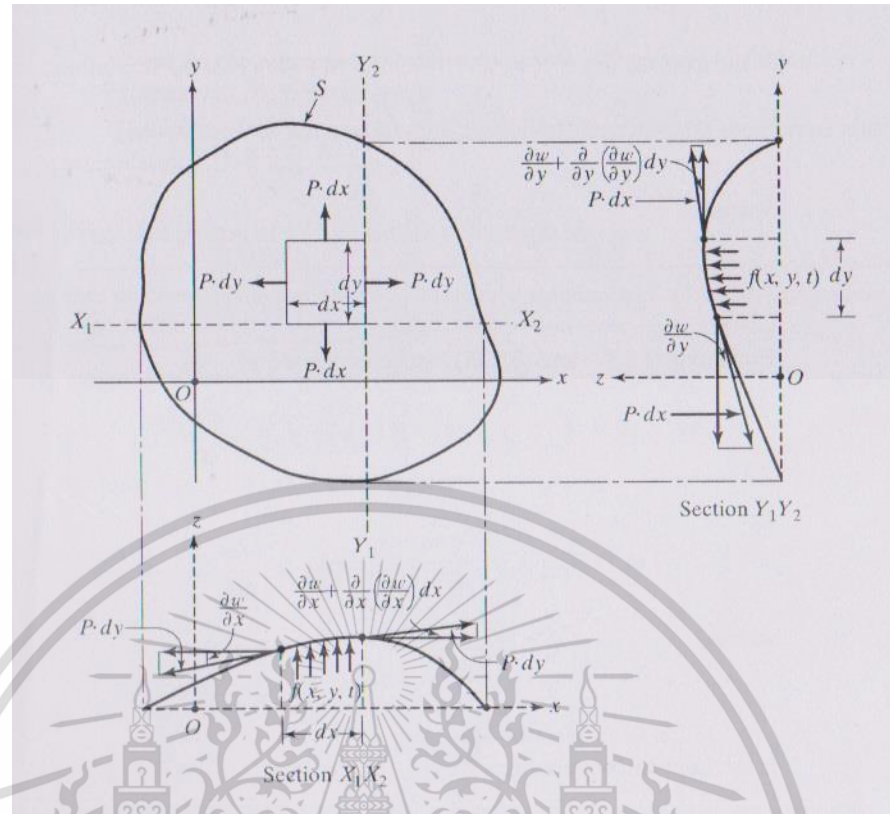


Figure 2.4 A membrane under uniform tension.

constant throughout the membrane, as in a drumhead. If we consider an elemental area $dx dy$, forces of magnitude $P dx$ and $P dy$ act on the sides parallel to the y -axes and x -axes, respectively, as shown in Figure 2.4. The net forces acting along the z direction due to these forces are

$$\left(P \frac{\partial^2 w}{\partial y^2} dx dy \right) \text{ and } \left(P \frac{\partial^2 w}{\partial x^2} dx dy \right). \quad (2.36)$$

The pressure force along the z direction is $f(x, y, t) dx dy$,

and the inertia force is $\rho(x, y) \frac{\partial^2 w}{\partial t^2} dx dy$,

where $\rho(x, y)$ is the mass per unit area. The equation of motion for the forced transverse vibration of the membrane can be obtained as

$$P \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + f = \rho \frac{\partial^2 w}{\partial t^2}, \quad (2.37)$$

If the external force $f(x, y, t) = 0$, Equation (2.37) gives the free-vibration equation

$$c^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \frac{\partial^2 w}{\partial t^2}, \quad (2.38)$$

where

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$$c = \left(\frac{P}{\rho} \right)^{1/2}.$$

The equation of motion for the forced transvers vibration of the membrane in [2] as follows:

$$P \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + f(x, y, t) = \rho(x, y) \frac{\partial^2 w}{\partial t^2}, \quad (2.39)$$

2.3 The Fundamental Operation on Differential Transformation Method

Method

The fundamental of one-dimensional, two-dimensional and three-dimensional operation on differential transformation method as follows.

2.3.1 One-Dimensional Differential Transformation Method

The basic definitions and fundamental operations of one-dimensional differential transformation are defined

Definition 2.1. If $f(t)$ is analytical solution in time domain T , then it is differential continuously with respect to time t .

For $t=0$ and k belongs to the set of non-negative integers, denotes as K-domain.

$$F(k) = \frac{1}{k!} \left[\frac{d^k}{dt^k} f(t) \right]_{t=0}, \quad \forall k \in K, \quad (2.51)$$

where $f(t)$ is called the original function and $F(k)$ is called the transform function.

Definition 2.2 The differential inverse transform of $F(k)$ is defined as

$$f(t) = \sum_{k=0}^{\infty} F(k) t^k \quad (2.53)$$

In principal application, function $f(t)$ is shown by finite numbers of terms can be written as

$$f(t) = \sum_{k=0}^N F(k) t^k \quad (2.54)$$

The fundamental mathematical operations performed by differential transform method are listed in Table 2.1 [37].

Table 2.1 The fundamental operations of one-dimension differential transformation method.

Original function	Transform function
2.1 $f(t) = y(t) \pm z(t)$	$F(k) = Y(k) \pm Z(k)$
2.2 $f(t) = cy(t)$	$F(k) = cY(k)$
2.3 $f(t) = \frac{dy(t)}{dt}$	$F(k) = (k+1)Y(k+1)$
2.4 $f(t) = \frac{d^n y(t)}{dt^n}$	$F(k) = \frac{(k+n)!}{k!} Y(k+n)$
2.5 $f(t) = y'(t)z(t)$	$F(k) = \sum_{m=0}^k Y(m)Z(k-m)$
2.6 $f(t) = y(t)z'(t)$	$F(k) = \sum_{m=0}^k (k+1-m)Y(m)Z(k+1-m)$
2.7 $f(t) = c$	$F(k) = \begin{cases} c, & k = 0 \\ 0, & k \geq 1 \end{cases}$
2.8 $f(t) = t^m, m = -1, -2, \dots$	$F(k) = \frac{1}{k!} [m(m-1)(m-2)\dots(m-k+1)]$

2.3.2 Two-dimensional Differential Transformation Method

The basic definitions and fundamental operations of two-dimensional differential transform method are defined as follows.

Definition 2.3 The two-dimensional differential transform of function $w(x, y)$ is defined as

$$W(k, h) = \frac{1}{k!h!} \left. \frac{\partial^{k+h} w(x, y)}{\partial x^k \partial y^h} \right|_{(0,0)}, k \geq 0 \text{ and } h \geq 0. \quad (2.55)$$

For k and h belong to set of non negative integers.

Definition 2.4 The inverse two-dimensional differential transform of sequence $\{W(k, h)\}_{k,h=0}^{\infty}$ is defined as

$$w(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k, h) x^k y^h. \quad (2.56)$$

In principal application, function $w(x, t)$ is shown by finite numbers of terms can be written as

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$$w(x, y) = \sum_{k=0}^N \sum_{h=0}^N W(k, h) x^k y^h. \quad (2.54)$$

The fundamental mathematical operations performed by differential transform method are listed in Table 3.1 The fundamental operations of two-dimension differential transform method [5].

2.3.3 Three-Dimensional Differential Transformation Method

The basic definitions and fundamental operations of three-dimensional differential transform are defined.

Definition 2.5 The three-dimensional differential transform of function $w(x, y, t)$ is define as

$$W(k, h, m) = \frac{1}{k!h!m!} \frac{\partial^{k+h+m} w(x, y, t)}{\partial x^k \partial y^h \partial t^m} \Big|_{(0,0,0)}, \quad k \geq 0, h \geq 0 \text{ and } m \geq 0. \quad (2.57)$$

Definition 2.6 The inverse three-dimensional differential transform of sequence $\{W(k, h, m)\}_{k,h,m=0}^{\infty}$ is define as

$$w(x, y, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \sum_{m=0}^{\infty} W(k, h, m) x^k y^h t^m. \quad (2.58)$$

In principal application, function $w(x, y, t)$ is shown by finite numbers of terms can be written as

$$w(x, y, t) = \sum_{k=0}^N \sum_{h=0}^N \sum_{m=0}^N W(k, h, m) x^k y^h t^m. \quad (2.54)$$

Chapter 3

Differential Transformation Method for the Suspended String Equation and the Vibrating Membrane Equation

In this chapter, we propose the derivation of the analytical solution for the equation of motion of membrane in Section 3.1. The fundamental operations of two-dimension and three-dimension differential transform method show in Section 3.2-3.3, respectively. DTM is applied to find approximated solutions. In solving these examples for the suspended string equation and the vibrating membrane equation are shown in Section 3.4 and 3.5, recurrence systems are first obtained and then a routine computer programming is used to find an approximated solution in the Taylor series form.

3.1 Analytical Solution for the Vibrating Membrane Equation

The following is the derivation of the analytical solution. The equation of motion for the forced transverse vibration of the membrane in [2] as follows:

$$P \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + f(x, y, t) = \rho(x, y) \frac{\partial^2 w}{\partial t^2}, \quad (3.1)$$

where $f(x, y, t)$ is the pressure force loading acting in the z direction (external force), P is intensity of tension at a point that is equal to production of the tensile stress and the thickness of the membrane, and $\rho(x, y)$ is the mass per unit area. We assume that $f(x, y, t) = 0$, $P = 1$, and $\rho(x, y) = 1$. Then Equation (3.1) leads to

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial t^2}. \quad (3.2)$$

The initial conditions of the equation of motion of a membrane [2] are assumed

$$w(x, y, 0) = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b,$$
$$\frac{\partial w}{\partial t}(x, y, 0) = 0, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b.$$

We assume $a = 1$ and $b = 1$. The boundary conditions of the equation of motion of a membrane are assumed

$$\begin{aligned}
w(x,0,t) &= 0, \quad 0 \leq x \leq 1, \\
w(0,y,t) &= 0, \quad 0 \leq y \leq 1, \\
w(x,1,t) &= 0, \quad 0 \leq x \leq 1, \\
w(1,y,t) &= 0, \quad 0 \leq y \leq 1, t \geq 0.
\end{aligned} \tag{3.3}$$

Considering the initial conditions $w(x,y,0) = \sin \pi x \sin \pi y$ and $\frac{\partial w}{\partial t}(x,y,0) = 0$.

The boundary conditions as follow

$$\begin{aligned}
w(x,0,t) &= 0, \quad 0 \leq x \leq 1, \\
w(0,y,t) &= 0, \quad 0 \leq y \leq 1, \\
w(x,1,t) &= 0, \quad 0 \leq x \leq 1, \\
w(1,y,t) &= 0, \quad 0 \leq y \leq 1, t \geq 0.
\end{aligned}$$

The general solution can be written by the separation of variable technique.

$$w(x,y,t) = X(x)Y(y)T(t), \tag{3.4}$$

subject to the eigenvalue, $-\omega^2, \alpha^2$ and β^2 , can be written Equation (3.4)

$$\begin{aligned}
\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{T''(t)}{T(t)} &= -\omega^2, \\
\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} &= -\omega^2
\end{aligned}$$

and

$$\begin{aligned}
-\frac{X''(x)}{X(x)} &= \frac{Y''(y)}{Y(y)} + \omega^2 = \alpha^2, \\
-\frac{X''(x)}{X(x)} &= \alpha^2
\end{aligned}$$

then

$$\begin{aligned}
X''(x) + \alpha^2 X(x) &= 0, \\
\frac{Y''(y)}{Y(y)} &= \alpha^2 - \omega^2,
\end{aligned}$$

assumed

$$(-\alpha^2 + \omega^2) = \beta^2$$

$$Y''(y) + (-\alpha^2 + \omega^2)Y(y) = 0$$

then,

$$Y''(y) + \beta^2 Y(y) = 0$$

and,

$$\frac{T''(t)}{T(t)} = \omega^2 \text{ then, } T''(t) + \omega^2 T(t) = 0,$$

$$X''(x) + \alpha^2 X(x) = 0, \tag{3.5}$$

$$Y''(y) + \beta^2 Y(y) = 0, \tag{3.6}$$

$$T''(t) + \omega^2 T(t) = 0. \tag{3.7}$$

We get solution of Equation (3.5) as $X(x) = C_1 \cos \alpha x + C_2 \sin \alpha x$ where C_1 and C_2 are arbitrary constants. Subject to the boundary conditions $X(0) = 0$,

we have $C_1 = 0$.

Then $X(x) = C_2 \sin \alpha x$,

the boundary condition $X(1) = 0$.

Let $C_2 \neq 0$ give $\alpha = m\pi; m \in I$,

we have that

$$X_m(x) = C_m \sin m\pi x. \quad (3.8)$$

boundary conditions $Y(0) = 0$, we get solution of Equation (3.6) as

$$Y(y) = C_3 \cos \beta y + C_4 \sin \beta y.$$

According to we have $C_3 = 0$.

Then $Y(y) = C_4 \sin \beta y$,

the boundary conditions. Then $Y(y) = 0$.

Let $C_4 \neq 0$ give $\beta = n\pi; n \in I$, we have that

$$Y_n(y) = C_n \sin n\pi y. \quad (3.9)$$

Let

$$\omega^2 = \beta^2 + \alpha^2,$$

$$\omega = \pi \sqrt{m^2 + n^2}.$$

We get solution of Equation (3.7) as $T_{mn}(t) = A_{mn} \cos \omega t + B_{mn} \sin \omega t$.

Then

$$T_{mn}(t) = A_{mn} \cos \pi \sqrt{m^2 + n^2} t + B_{mn} \sin \pi \sqrt{m^2 + n^2} t. \quad (3.10)$$

According to Equations (2.43) - (2.44) and (2.45) can be written

$$\begin{aligned} W_{mn}(x, y, t) &= X(x)Y(y)T(t), \\ &= \left(A_{mn} C_m C_n \sin m\pi x \sin n\pi y \cos \pi \sqrt{m^2 + n^2} t \right) \\ &\quad + \left(B_{mn} C_m C_n \sin m\pi x \sin n\pi y \sin \pi \sqrt{m^2 + n^2} t \right), \end{aligned} \quad (3.11)$$

where $F_{mn} = A_{mn} C_m C_n$, $H_{mn} = B_{mn} C_m C_n$ and for all $m, n = 1, 2, 3, \dots$. By using the superposition principle, Equation (3.11) become

$$\begin{aligned} w(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}(x, y, t), \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(F_{mn} \sin m\pi x \sin n\pi y \cos \pi \sqrt{m^2 + n^2} t \right), \\ &\quad + \left(H_{mn} \sin m\pi x \sin n\pi y \sin \pi \sqrt{m^2 + n^2} t \right), \end{aligned} \quad (3.12)$$

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according to initial condition $w(x, y, 0) = \sin \pi x \sin \pi y$ and $\frac{\partial w}{\partial t}(x, y, 0) = 0$. By using

Fourier series, we have

$$F_{mn} = 1 \quad ; \quad m = 1 \quad \text{and} \quad n = 1,$$

$$H_{mn} = 0.$$

Substituting $F_{mn} = 1$ and $H_{mn} = 0$ in Equation (3.12), we obtain that analytical solution is

$$w(x, y, t) = \sin \pi x \sin \pi y \cos \sqrt{2}t.$$

3.2 The Fundamental Operations of Two-dimensional Differential transformation Method

Formulas 3.1 – 3.6 are derived in [4] and [5], respectively. The following is the derivation of Formula 3.7.

Table 3.1 The fundamental operations of two-dimensional differential Transformation Method.

Original function	Transform function
3.1 $v(x, t) = a(x, t) \frac{\partial^2 u(x, t)}{\partial x^2}$	$V(k, h) = \sum_{i=0}^k \sum_{j=0}^h (k-i+2)(k-i+1)A(i, j)U(k-i+2, h-j)$
3.2 $v(x, t) = b(x, t) \frac{\partial^2 u(x, t)}{\partial x \partial t}$	$V(k, h) = \sum_{i=0}^k \sum_{j=0}^h (k-i+1)(h-i+1)B(i, j)U(k-i+1, h-j+1)$
3.3 $v(x, t) = c(x, t) \frac{\partial^2 u(x, y)}{\partial t^2}$	$V(k, h) = \sum_{i=0}^k \sum_{j=0}^h (h-i+2)(h-i+1)C(i, j)U(k-i, h-j+2)$
3.4 $v(x, t) = d(x, t) \frac{\partial u(x, y)}{\partial x}$	$V(k, h) = \sum_{i=0}^k \sum_{j=0}^h (k-i+1)D(i, j)U(k-i+1, h-j)$
3.5 $v(x, t) = e(x, t) \frac{\partial u(x, t)}{\partial t}$	$V(k, h) = \sum_{i=0}^k \sum_{j=0}^h (h-i+1)E(i, j)U(k-i, h-j+1)$
3.6 $v(x, t) = x^m t^n$	$V(k, h) = \delta(k-m, h-n) = \delta(k-m)\delta(h-n)$, where $\delta(k-m) = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases} \quad \delta(h-n) = \begin{cases} 1, & h = n, \\ 0, & h \neq n, \end{cases}$
3.7 $v(x, t) = u^n(x, t)$	$V(k, h) = \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} \dots \sum_{k_3=0}^{k_2} \sum_{k_1=0}^{k_3} \sum_{h_3=0}^{h_2} \sum_{h_1=0}^{h_3} \dots U_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2})$ $\times U_n(k - k_{n-1}, h - h_{n-1})$
3.8 $v(x, t) = u^3(x, t)$	$V(k, h) = \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} U_1(k_1, h_1)U_2(k_2 - k_1, h_2 - h_1)$ $\times U_3(k - k_2, h - h_2)$

Show that,

for $V : I^+ \rightarrow R$, $U_n : I^+ \rightarrow R$; $v : R \rightarrow R$, $u_n : R \rightarrow R$; $n \in I^+$, $x \in R$ and $k = 0, 1, 2, \dots$

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If $v(x, t) = u_1(x, t)u_2(x, t)..u_{n-1}(x, t)u_n(x, t)$ then,

$$V(k, h) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \sum_{h_2=0}^{h_3} \sum_{h_1=0}^{h_2} U_1(k_1, h_1)U_2(k_2 - k_1, h_2 - h_1) \dots \\ \times U_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2})U_n(k - k_{n-1}, h - h_{n-1}).$$

By definition 2.3 of two dimensional differential transformation, we have

$$V(k, h) = \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} v(x, t) \right]_{(0,0)}.$$

Thus for $v(x, t) = u_1(x, t)u_2(x, t)..u_{n-1}(x, t)u_n(x, t)$ and using definition (2.4), we then have

$$V(k, h) = \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} u_1(x, t)u_2(x, t)..u_{n-1}(x, t)u_n(x, t) \right]_{(0,0)} \\ = \frac{1}{k!h!} \sum_{k_{n-1}=0}^k \sum_{h_{n-1}=0}^h \frac{k!}{(k_{n-1})!(k - k_{n-1})!} \frac{h!}{(h_{n-1})!(h - h_{n-1})!} \\ \times \left[\frac{\partial^{k_{n-1}+h_{n-1}}}{\partial x^{k_{n-1}} \partial t^{h_{n-1}}} u_1(x, t)u_2(x, t)..u_{n-1}(x, t) \frac{\partial^{k-k_{n-1}+h-h_{n-1}}}{\partial x^{k-k_{n-1}} \partial t^{h-h_{n-1}}} u_n(x, t) \right]_{(0,0)} \\ = \sum_{k_{n-1}=0}^k \sum_{h_{n-1}=0}^h \left[\frac{1}{(k_{n-1})!(h_{n-1})!} \frac{\partial^{k_{n-1}+h_{n-1}}}{\partial x^{k_{n-1}} \partial t^{h_{n-1}}} u_1(x, t)u_2(x, t)..u_{n-1}(x, t) \right]_{(0,0)} \\ \times \left[\frac{1}{(k - k_{n-1})!(h - h_{n-1})!} \frac{\partial^{1-k_{n-1}+1-h_{n-1}}}{\partial x^{k-k_{n-1}} \partial t^{h-h_{n-1}}} u_n(x, t) \right]_{(0,0)} \\ = \sum_{k_{n-1}=0}^k \sum_{h_{n-1}=0}^h \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-2}=0}^{h_{n-1}} \left[\frac{1}{(k_{n-2})!(h_{n-2})!} \frac{\partial^{k_{n-2}+h_{n-2}}}{\partial x^{k_{n-2}} \partial t^{h_{n-2}}} u_1(x, t)u_2(x, t)..u_{n-2}(x, t) \right]_{(0,0)} \\ \times \left[\frac{1}{(k_{n-1} - k_{n-2})!(h_{n-1} - h_{n-2})!} \frac{\partial^{k_{n-1}-k_{n-2}+h_{n-1}-h_{n-2}}}{\partial x^{k_{n-1}-k_{n-2}} \partial t^{h_{n-1}-h_{n-2}}} u_{n-1}(x, t) \right]_{(0,0)} \\ \times \left[\frac{1}{(k - k_{n-1})!(h - h_{n-1})!} \frac{\partial^{k-k_{n-1}+h-h_{n-1}}}{\partial x^{k-k_{n-1}} \partial t^{h-h_{n-1}}} u_n(x, t) \right]_{(0,0)}$$

$$\begin{aligned}
 &= \sum_{k_{n-1}=0}^k \sum_{h_{n-1}=0}^h \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{k_{n-3}=0}^{k_{n-2}} \sum_{h_{n-3}=0}^{h_{n-2}} \left[\frac{1}{(k_{n-3})!(h_{n-3})!} \frac{\partial^{k_{n-3}+h_{n-3}}}{\partial x^{k_{n-3}} \partial t^{h_{n-3}}} u_1(x,t) u_2(x,t) \dots u_{n-3}(x,t) \right]_{(0,0)} \\
 &\times \left[\frac{1}{(k_{n-2}-k_{n-3})!(h_{n-2}-h_{n-3})!} \frac{\partial^{k_{n-2}-k_{n-3}+h_{n-2}-h_{n-3}}}{\partial x^{k_{n-2}-k_{n-3}} \partial t^{h_{n-2}-h_{n-3}}} u_{n-2}(x,t) \right]_{(0,0)} \\
 &\times \left[\frac{1}{(k_{n-1}-k_{n-2})!(h_{n-1}-h_{n-2})!} \frac{\partial^{k_{n-1}-k_{n-2}+h_{n-1}-h_{n-2}}}{\partial x^{k_{n-1}-k_{n-2}} \partial t^{h_{n-1}-h_{n-2}}} u_{n-1}(x,t) \right]_{(0,0)} \\
 &\times \left[\frac{1}{(k-k_{n-1})!(1-h_{n-1})!} \frac{\partial^{1-k_{n-1}+1-h_{n-1}}}{\partial x^{1-k_{n-1}} \partial t^{1-h_{n-1}}} u_n(x,t) \right]_{(0,0)}. \\
 V(k,h) &= \sum_{k_{n-1}=0}^k \sum_{h_{n-1}=0}^h \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{k_{n-3}=0}^{k_{n-2}} \sum_{h_{n-3}=0}^{h_{n-2}} \dots \sum_{k_4=0}^{k_3} \sum_{h_4=0}^{h_3} \sum_{k_3=0}^{k_2} \sum_{h_3=0}^{h_2} \sum_{k_2=0}^{k_1} \sum_{h_2=0}^{h_1} \\
 &\times U_1(k_1, h_1) U_2(k_2 - k_1, h_2 - h_1) U_3(k_3 - k_2, h_3 - h_2) \dots \\
 &\times U_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}) U_n(k - k_{n-1}, h - h_{n-1}).
 \end{aligned}$$

Since $v(x,t) = u_1(x,t)u_2(x,t)\dots u_{n-1}(x,t)u_n(x,t)$,

$$\begin{aligned}
 V(k,h) &= \sum_{k_{n-1}=0}^k \sum_{h_{n-1}=0}^h \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-2}=0}^{h_{n-1}} \dots \sum_{k_3=0}^{k_2} \sum_{h_3=0}^{h_2} \sum_{k_2=0}^{k_1} \sum_{h_2=0}^{h_1} U_1(k_1, h_1) U_2(k_2 - k_1, h_2 - h_1) \dots \\
 &\times U_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}) U_n(k - k_{n-1}, h - h_{n-1}).
 \end{aligned}$$

The fundamental mathematical operations performed by differential transform method are listed in Table 3.1 The fundamental operations of two-dimension differential transform method [23].

3.3 The Fundamental Operations of Three-dimensional Differential Transformation Method

Formulas 3.9 – 3.13 shown in [23]. The following is the derivation of the formula 3.14.

Table 3.2 The fundamental operations of three-dimensional differential transformation method

Original function	Transform function
3.9 $v(x,y,t) = a(x,y,t) \frac{\partial^2 w(x,y,t)}{\partial x^2}$	$V(k,h,m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (k-i+2)(k-i+1) \times A(i,j,r) W(k-i+2, h-j, m-r).$
3.10 $v(x,y,t) = b(x,y,t) \frac{\partial^2 w(x,y,t)}{\partial y^2}$	$V(k,h,m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+2)(h-i+1) \times B(i,j,r) U(k-i, h-j+2, m-r).$

Original function	Transform function
3.11 $v(x, y, t) = c(x, y, t) \frac{\partial^2 w(x, y, t)}{\partial t^2}$	$V(k, h, m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (m-i+2)(m-i+1) \\ \times C(i, j, r) W(k-i, h-j, m-r+2).$
3.12 $v(x, y, t) = d(x, y, t) \frac{\partial w(x, y, t)}{\partial t}$	$V(k, h, m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+1) \\ \times D(i, j, r) W(k-i, h-j+1, m-r).$
3.13 $v(x, y, t) = x^n y^l t^s$	$V(k, h, m) = \delta(k-n, h-l, m-s) \\ = \delta((k-n)\delta(h-l)\delta(m-s)), \text{ where}$ $\delta(k-n) = \begin{cases} 1 & k = n, \\ 0 & k \neq n, \end{cases}$ $\delta(h-l) = \begin{cases} 1 & h = l, \\ 0 & h \neq l, \end{cases}$ $\delta(m-p) = \begin{cases} 1 & m = s, \\ 0 & m \neq s, \end{cases}$
3.14 $v(x, y, t) = w^n(x, y, t)$	$V(k, h, m) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \dots \\ \sum_{k_2=0}^{k_1} \sum_{k_1=0}^{k_2} \sum_{h_1=0}^h \sum_{h_2=0}^{h_1} \sum_{m_1=0}^m \\ \times W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots \\ \times W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2}) \\ \times W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).$
3.15 $v(x, y, t) = w^3(x, y, t)$	$V(k, h, m) = \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} \sum_{m_2=0}^m \sum_{m_1=0}^{m_2} W_1(k_1, h_1, m_1) \\ \times W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \\ \times W_3(k - k_2, h - h_2, m - m_2).$

Show that,

for $V : I^+ \rightarrow \mathbb{R}$, $W_n : I^+ \rightarrow \mathbb{R}$; $v : \mathbb{R} \rightarrow \mathbb{R}$, $w_n : \mathbb{R} \rightarrow \mathbb{R}$; $n \in I^+$

and $k, h, m = 0, 1, 2, 3, \dots$

If $v(x, y, t) = w_1(x, y, t)w_2(x, y, t) \dots w_{n-1}(x, y, t)w_n(x, y, t)$ then,

$$V(k, h, m) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \dots \sum_{k_2=0}^{k_1} \sum_{k_1=0}^{k_2} \sum_{h_1=0}^h \sum_{h_2=0}^{h_1} \sum_{m_1=0}^m \\ \times W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots \\ \times W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2}) W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).$$

By definition 2.5 of three-dimensional differential transformation, we have

$$V(k, h, m) = \frac{1}{k!h!m!} \left[\frac{\partial^{k+h+m} v(x, y, t)}{\partial x^k \partial y^h \partial t^m} \right]_{(0,0,0)}.$$

Thus for $v(x, y, t) = w_1(x, y, t)w_2(x, y, t)...w_{n-1}(x, y, t)w_n(x, y, t)$ and using definition 2.5, we then have

$$\begin{aligned}
V(k, h, m) &= \frac{1}{k!h!m!} \left[\frac{\partial^{k+h+m}}{\partial x^k \partial y^h \partial t^m} w_1(x, y, t)w_2(x, y, t)...w_{n-1}(x, y, t)w_n(x, y, t) \right]_{(0,0,0)} \\
&= \frac{1}{k!h!m!} \sum_{k_{n-1}=0}^k \sum_{h_{n-1}=0}^h \sum_{m_{n-1}=0}^m \frac{k!}{(k_{n-1})!(k-k_{n-1})!} \frac{h!}{(h_{n-1})!(h-h_{n-1})!} \frac{m!}{(m_{n-1})!(m-m_{n-1})!} \\
&\quad \left[\frac{\partial^{k_{n-1}+h_{n-1}+m_{n-1}}}{\partial x^{k_{n-1}} \partial y^{h_{n-1}} \partial t^{m_{n-1}}} w_1(x, y, t)w_2(x, y, t)... \right. \\
&\quad \left. \times w_{n-1}(x, y, t) \frac{\partial^{k-k_{n-1}+h-h_{n-1}+m-m_{n-1}}}{\partial x^{k-k_{n-1}} \partial y^{h-h_{n-1}} \partial t^{m-m_{n-1}}} w_n(x, y, t) \right]_{(0,0,0)} \\
&= \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \\
&\quad \left[\frac{1}{(k_{n-2})!(h_{n-2})!(m_{n-2})!} \frac{\partial^{k_{n-2}+h_{n-2}+m_{n-2}}}{\partial x^{k_{n-2}} \partial y^{h_{n-2}} \partial t^{m_{n-2}}} w_1(x, y, t)w_2(x, y, t)...w_{n-2}(x, y, t) \right]_{(0,0,0)} \\
&\quad \left[\frac{1}{(k_{n-1}-k_{n-2})!(h_{n-1}-h_{n-2})!(m_{n-1}-m_{n-2})!} \right. \\
&\quad \left. \times \frac{\partial^{k_{n-1}-k_{n-2}+h_{n-1}-h_{n-2}+m_{n-1}-m_{n-2}}}{\partial x^{k_{n-1}-k_{n-2}} \partial y^{h_{n-1}-h_{n-2}} \partial t^{m_{n-1}-m_{n-2}}} w_{n-1}(x, y, t) \right]_{(0,0,0)} \\
&\quad \left[\frac{1}{(k-k_{n-1})!(h-h_{n-1})!(m-m_{n-1})!} \frac{\partial^{k-k_{n-1}+h-h_{n-1}+m-m_{n-1}}}{\partial x^{k-k_{n-1}} \partial y^{h-h_{n-1}} \partial t^{m-m_{n-1}}} w_n(x, y, t) \right]_{(0,0,0)} \\
&= \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{k_{n-3}=0}^{k_{n-2}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{h_{n-3}=0}^{h_{n-2}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \sum_{m_{n-3}=0}^{m_{n-2}} \\
&\quad \left[\frac{1}{(k_{n-3})!(h_{n-3})!(m_{n-3})!} \frac{\partial^{k_{n-2}+h_{n-2}+m_{n-2}}}{\partial x^{k_{n-2}} \partial y^{h_{n-2}} \partial t^{m_{n-2}}} w_1(x, y, t)w_2(x, y, t)...w_{n-3}(x, y, t) \right]_{(0,0,0)} \\
&\quad \left[\frac{1}{(k_{n-2}-k_{n-3})!(h_{n-2}-h_{n-3})!(m_{n-2}-m_{n-3})!} \right. \\
&\quad \left. \times \frac{\partial^{k_{n-2}-k_{n-3}+h_{n-2}-h_{n-3}+m_{n-2}-m_{n-3}}}{\partial x^{k_{n-2}-k_{n-3}} \partial y^{h_{n-2}-h_{n-3}} \partial t^{m_{n-2}-m_{n-3}}} w_{n-2}(x, y, t) \right]_{(0,0,0)} \\
&\quad \left[\frac{1}{(k_{n-1}-k_{n-2})!(h_{n-1}-h_{n-2})!(m_{n-1}-m_{n-2})!} \right. \\
&\quad \left. \times \frac{\partial^{k_{n-1}-k_{n-2}+h_{n-1}-h_{n-2}+m_{n-1}-m_{n-2}}}{\partial x^{k_{n-1}-k_{n-2}} \partial y^{h_{n-1}-h_{n-2}} \partial t^{m_{n-1}-m_{n-2}}} w_{n-1}(x, y, t) \right]_{(0,0,0)} \\
&\quad \left[\frac{1}{(1-k_{n-1})!(1-h_{n-1})!(1-m_{n-1})!} \frac{\partial^{1-k_{n-1}+1-h_{n-1}+1-m_{n-1}}}{\partial x^{1-k_{n-1}} \partial y^{1-h_{n-1}} \partial t^{1-m_{n-1}}} w_n(x, y, t) \right]_{(0,0,0)},
\end{aligned}$$

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$$\begin{aligned}
V(k, h, m) = & \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{k_{n-3}=0}^{k_{n-2}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{h_{n-3}=0}^{h_{n-2}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \sum_{m_{n-3}=0}^{m_{n-2}} \dots \\
& \sum_{k_3=0}^{k_4} \sum_{k_2=0}^{k_3} \sum_{h_3=0}^{h_4} \sum_{h_2=0}^{h_3} \sum_{m_3=0}^{m_4} \sum_{m_2=0}^{m_3} \\
& \times W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots \\
& \times W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2}) W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).
\end{aligned}$$

since $v(x, y, t) = w_1(x, y, t)w_2(x, y, t)\dots w_{n-1}(x, y, t)w_n(x, y, t)$,

$$\begin{aligned}
V(k, h, m) = & \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \sum_{h_2=0}^{h_3} \sum_{h_1=0}^{h_2} \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} \\
& \times W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots \\
& \times W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2}) W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).
\end{aligned}$$

3.4 Differential Transformation Method for the Suspended String Equation

In this Section, the seven examples of the suspended string equation will be presented and used to demonstrate how DTM is applied to find approximated solutions. In solving seven examples, recurrence systems are first obtained and then a routine computer programming is used to find an approximated solution in the Taylor series form.

Example 3.1 Consider suspended string equation without damping term.

$$c(x, t) \frac{\partial^2 u(x, t)}{\partial t^2} + a(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} + d(x, t) \frac{\partial u(x, t)}{\partial x} = 0, \quad (3.13)$$

with the initial and boundary conditions,

$$\begin{aligned}
u(x, 0) &= \sin x, \quad x \in \mathbb{R}, \\
u(1, t) &= 0, \quad \text{and} \quad \frac{\partial u(0, t)}{\partial x} = 0, \quad t \in \mathbb{R},
\end{aligned} \quad (3.14)$$

we use the following Kronecker symbols:

$$\delta(x, y) = \delta(x)\delta(y) = \begin{cases} 1 & ; x = y = 0 \\ 0 & \text{otherwise,} \end{cases}$$

comparing equation (3.13) to general terms of PDEs in 3.6 Table 3.1 the fundamental operations of two-dimension differential transform method, we have

$$a(x, t) = -x, \quad c(x, t) = 1, \quad \text{and} \quad d(x, t) = -1.$$

Then for all $i \geq 0$ and $j \geq 0$, we have

$$A(i, j) = -\delta(i-1, j), \quad C(i, j) = \delta(i, j) \quad \text{and} \quad D(i, j) = -\delta(i, j).$$

From Equation (3.13) by applying DTM and Kronecker symbols to the given suspended string equation, we have

$$\begin{aligned} & \sum_{i=0}^k \sum_{j=0}^h (h-i+2)(h-i+1)\delta(i,j)U(k-i,h-j+2) \\ &= \sum_{i=0}^k \sum_{j=0}^h (k-i+2)(k-i+1)(-\delta(i-1,j))U(k-i+2,h-j) \\ &+ \sum_{i=0}^k \sum_{j=0}^h (k-i+1)(-\delta(i,j))U(k-i+1,h-j), \end{aligned}$$

then

$$\begin{aligned} U(k,h+2) &= -\frac{1}{(h+2)(h+1)\delta(0,0)} \\ &\times \left[\sum_{i=0}^k \sum_{j=0}^h (k-i+2)(k-i+1)(-\delta(i-1,j))U(k-i+2,h-j) \right. \\ &\quad \left. + \sum_{i=0}^k (k-i+1)(-\delta(i,j))U(k-i+1,h-j) \right]. \\ &= \frac{1}{(h+2)(h+1)} [(k+1)(k+1)U(k+1,h)], \end{aligned}$$

that is

$$(h+2)(h+1)U(k,h+2) = (k+1)^2 U(k+1,h). \quad (3.15)$$

From the initial and boundary value problem (3.14) and definition (2.3), we found that

$$\sin x \cong x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\sum_{k=0}^{\infty} U(k,0)x^k = U(0,0)x^0 + U(1,0)x + U(2,0)x^2 + U(3,0)x^3 + U(4,0)x^4 + \dots$$

Then, we have

$$U(k,0) = \begin{cases} 0, & \text{if } k = 0, 2, 4, \dots \\ \frac{1}{k!}, & \text{if } k = 1, 5, 9, \dots \\ \frac{(-1)^k}{k!}, & \text{if } k = 3, 7, 11, \dots \end{cases} \quad (3.16)$$

and

$$U(k,1) = 0, \quad k = 0, 1, 2, \dots \quad (3.17)$$

From the boundary condition (3.14), we have

$$U(0,h) = 0, \quad h = 0, 1, 2, \dots \quad (3.18)$$

From (3.16) - (3.18) and by recursive formula (3.15), we obtain the coefficients $U(k,h)$ for the series solution

That is

$$u(x,t) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots - \frac{3x^2t^2}{4} + \frac{5x^4t^2}{48} - \frac{7x^6t^2}{1440} + \frac{9x^8t^2}{80640} \\ + \dots - \frac{xt^4}{4} + \frac{5x^3t^4}{36} - \frac{7x^5t^4}{480} + \frac{x^7t^4}{1680} + \dots$$

Example 3.2 Consider suspended string equation with damping term

$$c(x,t) \frac{\partial^2 u(x,t)}{\partial t^2} + a(x,t) \frac{\partial^2 u(x,t)}{\partial x^2} + d(x,t) \frac{\partial u(x,t)}{\partial x} + e(x,t) \frac{\partial u(x,t)}{\partial t} = 0, \quad (3.19)$$

with the initial and boundary conditions,

$$u(x,0) = \sin x, \quad x \in \mathbb{R}, \\ u(1,t) = 0, \quad \text{and} \quad \frac{\partial u(0,t)}{\partial x} = 0, \quad t \geq 0,$$

comparing Equation (3.19) to general terms of PDEs in Table 3.1 the fundamental operations of two-dimension differential transform method, we have

$$a(x,t) = -x, \quad c(x,t) = 1 \quad \text{and} \quad d(x,t) = e(x,t) = -1.$$

Then for all $i \geq 0$ and $j \geq 0$, we have

$$A(i,j) = -\delta(i-1,j), \quad C(i,j) = \delta(i,j) \quad \text{and} \quad D(i,j) = E(i,j) = -\delta(i,j).$$

From Equation (3.19) by applying DTM and Kronecker symbols to the given suspended string equation, we have

$$\sum_{i=0}^k \sum_{j=0}^h (h-i+2)(h-i+1)\delta(i,j)U(k-i,h-j+2) \\ = \sum_{i=0}^k \sum_{j=0}^h (k-i+2)(k-i+1)(-\delta(i-1,j))U(k-i+2,h-j) \\ + \sum_{i=0}^k \sum_{j=0}^h (k-i+1)(-\delta(i,j))U(k-i+1,h-j) \\ + \sum_{i=0}^k \sum_{j=0}^h (h-i+1)(-\delta(i,j))U(k-i,h-j+1),$$

then

$$U(k,h+2) = -\frac{1}{(h+2)(h+1)\delta(0,0)} \\ \times \left[\sum_{i=0}^k \sum_{j=0}^h (k-i+2)(k-i+1)(-\delta(i-1,j))U(k-i+2,h-j) \right. \\ \left. + \sum_{i=0}^k (k-i+1)(-\delta(i,j))U(k-i+1,h-j) \right] \\ \left. + \sum_{i=0}^k (h-i+1)(-\delta(i,j))U(k-i,h-j+1) \right] \\ = \frac{1}{(h+2)(h+1)} \left[(k+1)^2 U(k+1,h) - (h+1)U(k,h+1) \right],$$

that is

$$(h+2)(h+1)U(k,h+2) = (k+1)^2 U(k+1,h) - (h+1)U(k,h+1). \quad (3.20)$$

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From (3.16) - (3.18) and by recursive formula (3.20), we obtain the coefficients $U(k, h)$ for the series solution

$$u(x, t) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + 0.0833xt^2 + 0.1042x^2t^2 - 0.0042x^3t^2 - 0.0049x^4t^2 + \dots$$

Example 3.3 Consider suspended string equation with external force

$$c(x, t) \frac{\partial^2 u(x, t)}{\partial t^2} + a(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} + d(x, t) \frac{\partial u(x, t)}{\partial x} + u^3 = 0, \quad (3.21)$$

with the initial and boundary conditions,

$$u(x, 0) = \sin x, \quad x \in \mathbb{R},$$

$$u(1, t) = 0, \quad \text{and} \quad \frac{\partial u(0, t)}{\partial x} = 0, \quad t \geq 0,$$

comparing (3.21) to general terms of PDEs in Table 3.1 the fundamental operations of two-dimension differential transform method, we have

$$a(x, t) = -x, \quad c(x, t) = 1 \quad \text{and} \quad d(x, t) = -1.$$

Then for all $i \geq 0$ and $j \geq 0$, we have

$$A(i, j) = -\delta(i-1, j), \quad C(i, j) = \delta(i, j) \quad \text{and} \quad D(i, j) = -\delta(i, j).$$

From Equation (3.21) by applying DTM and Kronecker symbols to the given suspended string equation, we have

$$\begin{aligned} & \sum_{i=0}^k \sum_{j=0}^h (h-i+2)(h-i+1)\delta(i, j)U(k-i, h-j+2) \\ &= \sum_{i=0}^k \sum_{j=0}^h (k-i+2)(k-i+1)(-\delta(i-1, j))U(k-i+2, h-j) \\ &+ \sum_{i=0}^k \sum_{j=0}^h (k-i+1)(-\delta(i, j))U(k-i+1, h-j) \\ &- \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} U_1(k_1, h_1)U_2(k_2-k_1, h_2-h_1)U_3(k-k_2, h-h_2), \end{aligned}$$

then

$$\begin{aligned} U(k, h+2) &= -\frac{1}{(h+2)(h+1)\delta(0, 0)} \\ &\times \left[\sum_{i=0}^k \sum_{j=0}^h (k-i+2)(k-i+1)(-\delta(i-1, j))U(k-i+2, h-j) \right. \\ &\quad \left. + \sum_{i=0}^k (k-i+1)(-\delta(i, j))U(k-i+1, h-j) \right] \\ &\quad - \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} U_1(k_1, h_1)U_2(k_2-k_1, h_2-h_1)U_3(k-k_2, h-h_2) \\ &= \frac{1}{(h+2)(h+1)} \left[(k+1)^2 U(k+1, h) - \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} U_1(k_1, h_1) \right. \\ &\quad \left. \times U_2(k_2-k_1, h_2-h_1)U_3(k-k_2, h-h_2) \right], \end{aligned}$$

that is

$$(h+2)(h+1)U(k, h+2) = (k+1)^2 U(k+1, h) - \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} U_1(k_1, h_1) \times U_2(k_2 - k_1, h_2 - h_1) U_3(k - k_2, h - h_2). \quad (3.22)$$

From (3.16) - (3.18) and by recursive formula (3.22), we obtain the coefficients $U(k, h)$ for the series solution

$$u(x, t) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + 0.5xt^2 - 0.75x^2t^2 + 0.0833x^3t^2 + 0.1042x^4t^2 + \dots$$

Example 3.4 Consider suspended string equation with non uniform density $\left(m = \frac{1}{4}\right)$

$$c(x, t) \frac{\partial^2 u(x, t)}{\partial t^2} + a(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} + d(x, t) \frac{\partial u(x, t)}{\partial x} = 0, \quad (3.23)$$

with the initial and boundary conditions,

$$u(x, 0) = \sin x, \quad x \in \mathbb{R}, \\ u(1, t) = 0, \quad \text{and} \quad \frac{\partial u(0, t)}{\partial x} = 0, \quad t \geq 0,$$

comparing Equation (3.23) to general terms of PDEs in Table 3.1 the fundamental operations of two-dimension differential transform method, we have

$$a(x, t) = \frac{-x}{\frac{1}{4} + 1} = \frac{-x}{5/4}, \quad c(x, t) = 1 \quad \text{and} \quad d(x, t) = -1$$

Then for all $i \geq 0$ and $j \geq 0$, we have

$$A(i, j) = -\frac{4}{5} \delta(i-1, j), \quad C(i, j) = \delta(i, j) \quad \text{and} \quad D(i, j) = -\delta(i, j).$$

From Equation (3.23) by applying DTM and Kronecker symbols to the given suspended string equation, we have

$$\sum_{i=0}^k \sum_{j=0}^h (h-i+2)(h-i+1) \delta(i, j) U(k-i, h-j+2) \\ = \sum_{i=0}^k \sum_{j=0}^h (k-i+2)(k-i+1) \left(-\frac{4}{5} \delta(i-1, j) \right) U(k-i+2, h-j) \\ + \sum_{i=0}^k \sum_{j=0}^h (k-i+1) (-\delta(i, j)) U(k-i+1, h-j),$$

then

$$U(k, h+2) = -\frac{1}{(h+2)(h+1)\delta(0,0)} \\ \times \left[\sum_{i=0}^k \sum_{j=0}^h (k-i+2)(k-i+1) (-\delta(i-1, j)) U(k-i+2, h-j) \right. \\ \left. + \sum_{i=0}^k (k-i+1) \left(-\frac{4}{5} \delta(i-1, j) \right) U(k-i+1, h-j) \right].$$

$$= \frac{1}{(h+2)(h+1)} \left[(k+1) \left(\frac{4}{5}k+1 \right) U(k+1, h) \right],$$

that is

$$(h+2)(h+1)U(k, h+2) = (k+1) \left(\frac{4}{5}k+1 \right) U(k+1, h). \quad (3.24)$$

From (3.16) - (3.18) and by recursive formula (3.24), we obtain the coefficients $U(k, h)$ for the series solution

$$u(x, t) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots - 2025x^2t^2 + 0.3542x^4t^2 - 0.0174x^6t^2 + \dots$$

Example 3.5 Consider suspended string equation with non uniform density $\left(m = \frac{1}{2} \right)$

$$c(x, t) \frac{\partial^2 u(x, t)}{\partial t^2} + a(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} + d(x, t) \frac{\partial u(x, t)}{\partial x} = 0, \quad (3.25)$$

with the initial and boundary conditions,

$$u(x, 0) = \sin x, \quad x \in \mathbb{R},$$

$$u(1, t) = 0, \quad \text{and} \quad \frac{\partial u(0, t)}{\partial x} = 0, \quad t \geq 0,$$

comparing Equation (3.25) to general terms of PDEs in Table 3.1 the fundamental operations of two-dimension differential transform method, we have

$$a(x, t) = \frac{-x}{\frac{1}{2}+1} = \frac{-x}{3/2}, \quad c(x, t) = 1 \quad \text{and} \quad d(x, t) = -1$$

Then for all $i \geq 0$ and $j \geq 0$, we have

$$A(i, j) = -\frac{2}{3} \delta(i-1, j), \quad C(i, j) = \delta(i, j) \quad \text{and} \quad D(i, j) = -\delta(i, j).$$

By applying DTM to the given suspended string equation, we have

$$(h+2)(h+1)U(k, h+2) = (k+1) \left(\frac{2}{3}k+1 \right) U(k+1, h). \quad (3.26)$$

From (3.16) - (3.18) and by recursive formula (3.26), we obtain the coefficients $U(k, h)$ for the series solution

$$u(x, t) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots - 1.25x^2t^2 + 0.1875x^4t^2 - 0.00903x^6t^2 + \dots$$

Example 3.6 Consider suspended string equation with non uniform density $\left(m = \frac{3}{4} \right)$

$$c(x, t) \frac{\partial^2 u(x, t)}{\partial t^2} + a(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} + d(x, t) \frac{\partial u(x, t)}{\partial x} = 0, \quad (3.27)$$

with the initial and boundary conditions,

$$u(x, 0) = \sin x, \quad x \in \mathbb{R},$$

$$u(1,t) = 0, \text{ and } \frac{\partial u(0,t)}{\partial x} = 0, \quad t \geq 0,$$

comparing Equation (3.27) to general terms of PDEs in Table 3.1 the fundamental operations of two-dimension differential transform method, we have

$$a(x,t) = \frac{-x}{\frac{3}{4}+1} = \frac{-x}{7/4}, \quad c(x,t) = 1 \text{ and } d(x,t) = -1$$

Then for all $i \geq 0$ and $j \geq 0$, we have

$$A(i,j) = -\frac{4}{7}\delta(i-1,j), \quad C(i,j) = \delta(i,j) \text{ and } D(i,j) = -\delta(i,j).$$

By applying DTM to the given suspended string equation, we have

$$(h+2)(h+1)U(k,h+2) = (k+1)\left(\frac{4}{7}k+1\right)U(k+1,h). \quad (3.28)$$

From (3.16) - (3.18) and by recursive formula (3.28), we obtain the coefficients $U(k,h)$ for the series solution

$$u(x,t) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots - 0.9167x^2t^2 + 0.1319x^4t^2 - 0.0063x^6t^2 + \dots$$

Example 3.7 Consider suspended string equation with non uniform density ($m=1$)

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{x}{2} \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} = 0, \quad (3.29)$$

with the initial and boundary conditions,

$$u(x,0) = \sin x, \quad x \in \mathbb{R}, \\ u(1,t) = 0, \text{ and } \frac{\partial u(0,t)}{\partial x} = 0, \quad t \geq 0,$$

comparing Equation (3.29) to general terms of PDEs in Table 3.1 the fundamental operations of two-dimension differential transform method, we have

$$a(x,t) = \frac{-x}{1+1} = \frac{-x}{2}, \quad c(x,t) = 1 \text{ and } d(x,t) = -1$$

Then for all $i \geq 0$ and $j \geq 0$, we have

$$A(i,j) = \frac{-1}{2}\delta(i-1,j), \quad C(i,j) = \delta(i,j) \text{ and } D(i,j) = -\delta(i,j).$$

By applying DTM to the given suspended string equation, we have

$$(h+2)(h+1)U(k,h+2) = (k+1)\left(\frac{k}{2}+1\right)U(k+1,h). \quad (3.30)$$

From (3.16) - (3.18) and by recursive formula (3.30), we obtain the coefficients $U(k,h)$ for the series solution

$$u(x,t) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots - 0.5x^2t^2 + 0.0625x^4t^2 - 0.00278x^6t^2 + \dots - 0.125xt^4 \dots$$

3.5 Differential Transformation Method for The Vibrating Membrane Equation

In this Section, the four examples of the vibrating membrane equation will be presented and used to demonstrate how DTM is applied to find approximated solutions. In solving these examples, recurrence systems are first obtained and then a routine computer programming is used to find an approximated solution in the Taylor series form.

Example 3.8 Consider the vibrating equation of membrane

$$c(x, y, t) \frac{\partial^2 w}{\partial t^2} + a(x, y, t) \frac{\partial^2 w}{\partial x^2} + b(x, y, t) \frac{\partial^2 w}{\partial y^2} = 0, \quad (3.31)$$

with the initial and boundary conditions,

$$\begin{aligned} w(x, y, 0) &= \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \\ \frac{\partial w}{\partial t}(x, y, 0) &= 0, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \\ w(x, 0, t) &= 0, \quad 0 \leq x \leq 1, \\ w(0, y, t) &= 0, \quad 0 \leq y \leq 1, \\ w(x, 1, t) &= 0, \quad 0 \leq x \leq 1, \\ w(1, y, t) &= 0, \quad 0 \leq y \leq 1, t \geq 0. \end{aligned} \quad (3.32)$$

Comparing Equation (3.31) to general terms of PDEs in Table 3.2 the fundamental operations of three-dimension differential transform method, we have

$$a(x, y, t) = b(x, y, t) = -1 \quad \text{and} \quad c(x, y, t) = 1.$$

Then for all $i \geq 0, j \geq 0$ and $r \geq 0$, we have

$$A(i, j, r) = B(i, j, r) = -\delta(i, j, r) \quad \text{and} \quad C(i, j, r) = \delta(i, j, r).$$

The following Kronecker symbols:

$$\delta(x, y, t) = \delta(x)\delta(y)\delta(t) = \begin{cases} 1 & ; x = y = t = 0 \\ 0 & ; \text{otherwise,} \end{cases}$$

from Equation (3.31) by applying DTM and Kronecker symbols to the given suspended string equation, we have

$$\begin{aligned} & \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (m-i+2)(m-i+1) \delta(i, j, r) W(k-i, h-j, m-r+2) \\ &= \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (k-i+2)(k-i+1) - \delta(i, j, r) W(k-i+2, h-j, m-r) \\ &+ \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+1) - \delta(i, j, r) W(k-i, h-j+1, m-r), \end{aligned}$$

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then

$$\begin{aligned}
 W(k, h, m+2) &= \frac{1}{(m+2)(m+1)\delta(0,0,0)} \\
 &\left[\sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (k-i+2)(k-i+1) - \delta(i, j, r) W(k-i+2, h-j, m-r) \right. \\
 &\quad \left. + \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+2)(h-i+1) - \delta(i, j, r) U(k-i, h-j+2, m-r) \right] \\
 &= \frac{1}{(m+2)(m+1)} (k+2)(k+1)(h+2)(h+1) W(k+2, h, m) W(k, h+2, m),
 \end{aligned}$$

that is

$$\begin{aligned}
 (m+2)(m+1)W(k, h, m+2) &= (k+2)(k+1)(h+2)(h+1) \\
 &\quad \times W(k+2, h, m) W(k, h+2, m).
 \end{aligned} \tag{3.33}$$

Comparing between the coefficient of Taylor's series of sine and the series in the definition (2.5)

$$\begin{aligned}
 \sin(\pi x) \sin(\pi y) &\cong \frac{2\pi^2}{2!} xt - \frac{4\pi^4}{4!} x^3 t + \frac{6\pi^6}{6!} x^5 t - \frac{8\pi^8}{8!} x^7 t + \frac{10\pi^{10}}{10!} x^9 t - \frac{12\pi^{12}}{12!} x^{11} t + \dots \\
 \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k, h, 0) x^k y^h &= W(0, 0, 0) x^0 y^0 + W(1, 1, 0) xy \\
 &\quad + W(2, 1, 0) x^2 y + W(3, 1, 0) x^3 y + \dots \\
 &\quad + W(0, 1, 0) xy + W(1, 2, 0) xy^2 + W(2, 2, 0) x^2 y^2 + \dots
 \end{aligned} \tag{3.34}$$

Then, we have

$$W(k, h, 0) = \begin{cases} 0 & \text{if } k = 0, 1, 2, \dots \text{ and } h = 0, 2, 4, 6, \dots \\ \frac{(k+h)\pi^{k+h}}{(k+h)!} & \text{if } k = 1, 5, 9, \dots \text{ and } h = 1, 3, 5, 7, \dots \\ -\frac{(k+h)\pi^{k+h}}{(k+h)!} & \text{if } k = 3, 7, 11, \dots \text{ and } h = 1, 3, 5, 7, \dots \end{cases} \tag{3.35}$$

and from the initial condition (3.32)

$$W(k, h, 1) = 0, \quad k, h = 0, 1, 2, 3, \dots \tag{3.36}$$

from the boundary condition (3.32), we have

$$\begin{aligned}
 W(k, 0, m) &= 0, \quad k, m = 0, 1, 2, 3, \dots \\
 W(0, h, m) &= 0, \quad h, m = 0, 1, 2, 3, \dots \\
 W(k, 1, m) &= 0, \quad k, m = 0, 1, 2, 3, \dots \\
 W(1, h, m) &= 0, \quad h, m = 0, 1, 2, 3, \dots
 \end{aligned} \tag{3.37}$$

From (3.35) - (3.37) and by recursive formula (3.33), we obtain the coefficients $W(k, h, m)$ for the series solution.

That is

$$u(x, y, t) = \pi^2 xy - \pi^4 t^2 xy + \frac{1}{6} \pi^6 t^4 xy - \frac{1}{90} \pi^8 t^6 xy - \frac{1}{6} \pi^4 x^3 y + \frac{1}{6} \pi^6 t^2 x^3 y + \dots$$

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Example 3.9 Consider the vibrating membrane equation with damping term

$$c(x, y, t) \frac{\partial^2 w}{\partial t^2} + a(x, y, t) \frac{\partial^2 w}{\partial x^2} + b(x, y, t) \frac{\partial^2 w}{\partial y^2} + d(x, y, t) \frac{\partial w}{\partial t} = 0, \quad (3.38)$$

with the initial and boundary conditions,

$$w(x, y, 0) = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1,$$

$$\frac{\partial w}{\partial t}(x, y, 0) = 0, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1,$$

$$w(x, 0, t) = 0, \quad 0 \leq x \leq 1,$$

$$w(0, y, t) = 0, \quad 0 \leq y \leq 1,$$

$$w(x, 1, t) = 0, \quad 0 \leq x \leq 1,$$

$$w(1, y, t) = 0, \quad 0 \leq y \leq 1, t \geq 0,$$

comparing Equation (3.38) to general terms of PDEs in Table 3.2 the fundamental operations of three-dimension differential transform method, we have

$$c(x, y, t) = 1 \text{ and } a(x, y, t) = b(x, y, t) = d(x, y, t) = -1.$$

Then for all $i \geq 0, j \geq 0$ and $r \geq 0$, we have

$$C(i, j, r) = \delta(i, j, r) \text{ and } A(i, j, r) = B(i, j, r) = D(i, j, r) = -\delta(i, j, r).$$

from Equation (3.38) by applying DTM and Kronecker symbols to the given suspended string equation, we have

$$\begin{aligned} & \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (m-i+2)(m-i+1) \delta(i, j, r) W(k-i, h-j, m-r+2) \\ &= \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (k-i+2)(k-i+1) - \delta(i, j, r) W(k-i+2, h-j, m-r) \\ &+ \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+1) - \delta(i, j, r) W(k-i, h-j+1, m-r), \\ &+ \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+1) - \delta(i, j, r) W(k-i, h-j+1, m-r), \\ W(k, h, m+2) &= \frac{1}{(m+2)(m+1) \delta(0, 0, 0)} \end{aligned}$$

then

$$\begin{aligned} & \left[\sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (k-i+2)(k-i+1) - \delta(i, j, r) W(k-i+2, h-j, m-r) \right. \\ &+ \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+2)(h-i+1) - \delta(i, j, r) U(k-i, h-j+2, m-r) \\ &+ \left. \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+1) - \delta(i, j, r) W(k-i, h-j+1, m-r) \right] \\ &= \frac{1}{(m+2)(m+1)} (k+2)(k+1)(h+2)(h+1) W(k+2, h, m) W(k, h+2, m), \end{aligned}$$

that is

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$$\begin{aligned}
(m+2)(m+1)W(k, h, m+2) &= (k+2)(k+1)(h+2)(h+1) \\
&\times W(k+2, h, m)W(k, h+2, m) \\
&- (h+1)W(k, h+1, m).
\end{aligned} \tag{3.39}$$

From (3.35) - (3.37) and by recursive formula (3.39), we obtain the coefficients $W(k, h, m)$ for the series solution.

That is

$$w(x, y, t) = -\frac{5}{2}\pi^2 t^2 x + \frac{5}{6}\pi^6 t^4 x + \frac{5}{12}\pi^4 t^2 x^3 - \frac{5}{36}\pi^6 t^4 x^3 - \frac{1}{48}\pi^6 t^2 x^5 + \frac{1}{144}\pi^8 t^4 x^5 + \dots$$

Example 3.10 Consider the vibrating equation of membrane with external force

$$c(x, y, t) \frac{\partial^2 w}{\partial t^2} + a(x, y, t) \frac{\partial^2 w}{\partial x^2} + b(x, y, t) \frac{\partial^2 w}{\partial y^2} - w^3 = 0, \tag{3.40}$$

with the initial and boundary conditions,

$$w(x, y, 0) = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1,$$

$$\frac{\partial w}{\partial t}(x, y, 0) = 0, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1,$$

$$w(x, 0, t) = 0, \quad 0 \leq x \leq 1,$$

$$w(0, y, t) = 0, \quad 0 \leq y \leq 1,$$

$$w(x, 1, t) = 0, \quad 0 \leq x \leq 1,$$

$$w(1, y, t) = 0, \quad 0 \leq y \leq 1, t \geq 0,$$

comparing Equation (3.40) to general terms of PDEs in Table 3.2 the fundamental operations of three-dimension differential transform method, we have

$$a(x, y, t) = b(x, y, t) = -1 \quad \text{and} \quad c(x, y, t) = 1.$$

Then for all $i \geq 0, j \geq 0$ and $r \geq 0$, we have

$$A(i, j, r) = B(i, j, r) = -\delta(i, j, r) \quad \text{and} \quad C(i, j, r) = \delta(i, j, r).$$

from Equation (3.40) by applying DTM and Kronecker symbols to the given suspended string equation, we have

$$\begin{aligned}
&\sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (m-i+2)(m-i+1)\delta(i, j, r)W(k-i, h-j, m-r+2) \\
&= \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (k-i+2)(k-i+1) - \delta(i, j, r)W(k-i+2, h-j, m-r) \\
&+ \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+1) - \delta(i, j, r)W(k-i, h-j+1, m-r), \\
&+ \sum_{k_2=0}^k \sum_{h_2=0}^h \sum_{h_1=0}^h \sum_{m_2=0}^m \sum_{m_1=0}^{m_2} W_1(k_1, h_1, m_1)W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1)W_3(k - k_2, h - h_2, m - m_2),
\end{aligned}$$

then

$$\begin{aligned}
W(k, h, m+2) &= \frac{1}{(m+2)(m+1)\delta(0,0,0)} \\
&\left[\sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (k-i+2)(k-i+1) - \delta(i, j, r) W(k-i+2, h-j, m-r) \right. \\
&+ \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+2)(h-i+1) - \delta(i, j, r) U(k-i, h-j+2, m-r) \\
&+ \left. \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} \sum_{m_2=0}^m \sum_{m_1=0}^{m_2} W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) W_3(k - k_2, h - h_2, m - m_2) \right] \\
&= \frac{1}{(m+2)(m+1)} (k+2)(k+1)(h+2)(h+1) W(k+2, h, m) W(k, h+2, m) \\
&+ \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} \sum_{m_2=0}^m \sum_{m_1=0}^{m_2} W_1(k_1, h_1, m_1) \times W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) W_3(k - k_2, h - h_2, m - m_2),
\end{aligned}$$

that is

$$\begin{aligned}
(m+2)(m+1)W(k, h, m+2) &= (k+2)(k+1)(h+2)(h+1)W(k+2, h, m)W(k, h+2, m) \\
&+ \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} \sum_{m_2=0}^m \sum_{m_1=0}^{m_2} W_1(k_1, h_1, m_1) \\
&W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) W_3(k - k_2, h - h_2, m - m_2).
\end{aligned} \tag{3.41}$$

From (3.35) - (3.37) and by recursive formula (3.41), we obtain the coefficients $W(k, h, m)$ for the series solution.

That is

$$\begin{aligned}
w(x, y, t) &= \pi^2 xy - \pi^4 t^2 xy + \frac{1}{6} \pi^6 t^4 xy + \frac{1}{30} \left(-\frac{\pi^8}{6} + 3 \left(-\frac{\pi^8}{18} - \frac{\pi^{10}}{36} \right) \right) t^6 xy \\
&- \frac{1}{6} \pi^4 x^3 y + \frac{1}{6} \pi^6 t^2 x^3 y + \dots
\end{aligned}$$

Example 3.11 Consider the vibrating membrane equation with external force and damping term.

$$c(x, y, t) \frac{\partial^2 w}{\partial t^2} + a(x, y, t) \frac{\partial^2 w}{\partial x^2} + b(x, y, t) \frac{\partial^2 w}{\partial y^2} + d(x, y, t) \frac{\partial w}{\partial t} - w^3 = 0, \tag{3.42}$$

with the initial and boundary conditions,

$$w(x, y, 0) = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1,$$

$$\frac{\partial w}{\partial t}(x, y, 0) = 0, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1,$$

$$w(x, 0, t) = 0, \quad 0 \leq x \leq 1,$$

$$w(0, y, t) = 0, \quad 0 \leq y \leq 1,$$

$$w(x, 1, t) = 0, \quad 0 \leq x \leq 1,$$

$$w(1, y, t) = 0, \quad 0 \leq y \leq 1, t \geq 0,$$

comparing Equation (3.42) to general terms of PDEs in Table 3.2 the fundamental operations of three-dimension differential transform method, we have, we have

$$a(x, y, t) = b(x, y, t) = -1 \text{ and } c(x, y, t) = d(x, y, t) = 1.$$

Then for all $i \geq 0, j \geq 0$ and $r \geq 0$, we have

$$A(i, j, r) = B(i, j, r) = -\delta(i, j, r) \text{ and } D(i, j, r) = C(i, j, r) = \delta(i, j, r).$$

Applying DTM and Kronecker symbols to the vibrating membrane equation, we have

$$\begin{aligned} (m+2)(m+1)W(k, h, m+2) &= (k+2)(k+1)(h+2)(h+1)W(k+2, h, m)W(k, h+2, m) \\ &\quad - (h+1)W(k, h+1, m) + \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} \sum_{m_2=0}^m \sum_{m_1=0}^{m_2} W_1(k_1, h_1, m_1) \\ &\quad W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) W_3(k - k_2, h - h_2, m - m_2). \end{aligned} \quad (3.43)$$

From (3.35) - (3.37) and by recursive formula (3.43), we obtain the coefficients $W(k, h, m)$ for the series solution.

That is

$$w(x, y, t) = -\frac{1}{2}\pi^2 t^2 x + 1/6\pi^4 t^4 x + \frac{1}{30}(-(\pi^3)^6) + \frac{1}{12}(\pi^2 - 2\pi^6)t^6 x + \frac{1}{12}\pi^4 t^2 x^3 - \frac{1}{36}\pi^6 t^4 x^3 + \dots$$

Chapter 4

Applications and Numerical Examples

In this chapter, we show graphical comparison between result without a damping term, with a damping term, with an external force and a nonuniform densities for the suspended string equations in Section 4.1. We also propose the graphical result of the amplitude of the approximate solution from DTM of the vibration of membrane without a damping term, with a damping term, with an external force in Section 4.2.

4.1 The Suspended String Equation

The comparisons result of the suspended string vibration without a damping term (a) and with a damping term (b) are shown in Figure 4.1 Graphical comparison between the vibration displacements without a damping term (a) calculated in Example 3.1 and with a damping term (b) calculated in Example 3.2 when $m=0$.

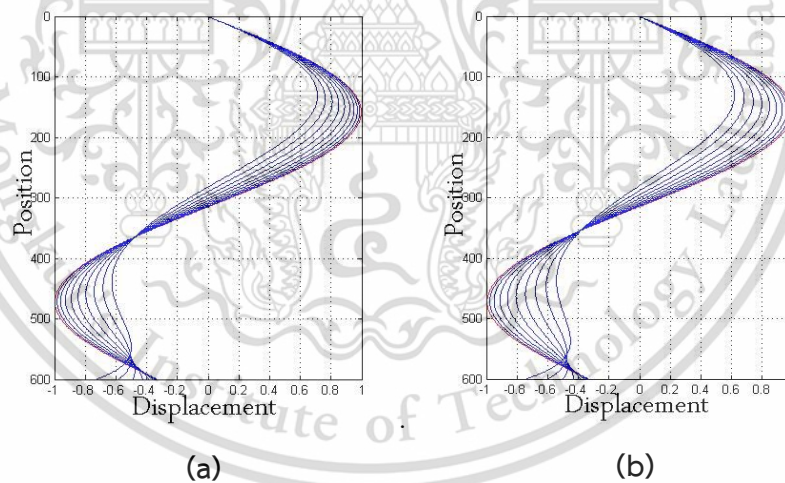


Figure 4.1 Graphical comparison between the vibration displacements without a damping term (a) calculated in Example 3.1 and with a damping term (b) calculated in Example 3.2 when $m=0$.

We find that the amplitude of vibration with a damping term (b) is less than that of the vibration without a damping term (a), as the data values shown in

Table 4.1 Data Value of the suspended string vibration without a damping term

Example 3.1 (u_1) with a damping term Example 3.2 (u_2), with an external force

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Example 3.3 (u_3) and error for the vibration of the suspended string equations at $x = 0.20$.

The comparisons result of the suspended string vibration without an external force (a) and with an external force (b) are shown in Figure 4.2 Graphical comparison between the vibration displacements between the without an external force (a) in calculated Example 3.1 and with an external force (b) in calculated Example 3.3 when $m=0$.

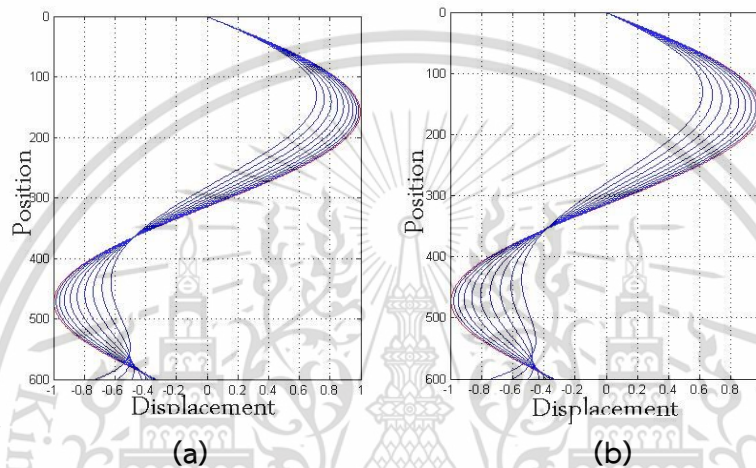


Figure 4.2 Graphical comparison between the vibration displacements between the without an external force (a) calculated in Example 3.1 and with an external force (b) calculated in Example 3.3 when $m=0$.

We find that the amplitude of vibration with an external force (b) is less than that of vibration without an external force (a), as the data values shown in Table 4.1

Table 4.1 Data Value of the approximate solution of vibration displacement without a damping term in Example 3.1. (u_1) with a damping term in Example 3.2 calculated in (u_2), with an external force in Example 3.3 (u_3) and the errors for the vibrating suspended string equations at $x = 0.20$

t (sec)	u_1 (without damping and external force)	u_2 (with damping)	u_3 (with external force)	$ u_1 - u_2 $	$ u_1 - u_3 $
$t = 0.00$	0.9134	0.9134	0.9134	0.0000	0.0000
$t = 0.10$	0.8973	0.8928	0.8928	0.0045	0.0345
$t = 0.15$	0.8772	0.8673	0.8671	0.0099	0.0101
$t = 0.20$	0.8493	0.8320	0.8315	0.0173	0.0178
$t = 0.25$	0.8135	0.7874	0.7861	0.0261	0.0274

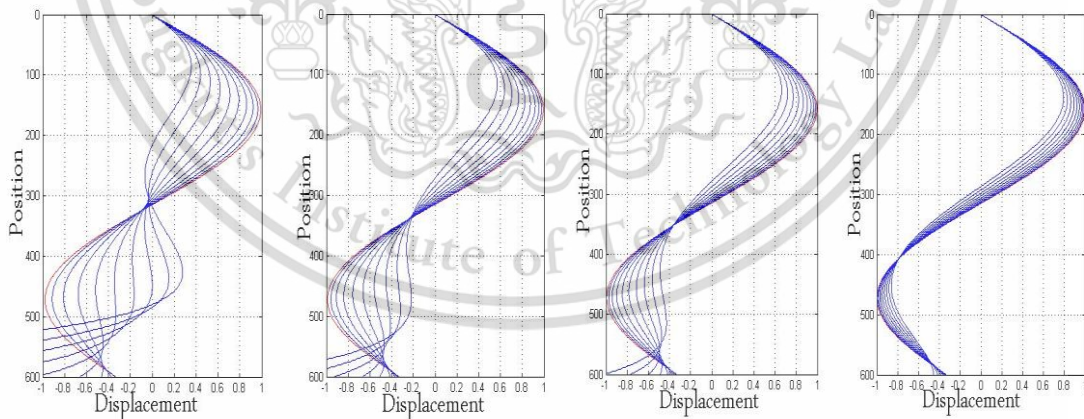
(a) $m=0.25$ (b) $m=0.5$ (c) $m=0.75$ (d) $m=1$

Figure 4.3 Graphical comparison between the vibration displacements with various nonuniform densities ($m=0.25, 0.5, 0.75, 1$) calculated in Examples 3.4 - 3.7.

We find that the shape of vibration (blue line) approach to initial shape of vibration (red line) when the densities and time increase, as the data value shown in Table 4.2.

The data values of the vibration displacements with the increasing densities (i.e, $m=0.25, 0.50, 0.75, 1.00$) calculated in Examples 3.4 - 3.7 and the errors when fixed $x=0.2$ under different times t are shown in Table 4.2.

Table 4.2 Data values of the vibration displacements with the increasing densities (i.e, $m=0.25, 0.50, 0.75, 1.00$) calculated in Examples 3.4 - 3.7 and the errors at $x = 0.20$.

$t(\text{sec})$ / $w(DTM)$	$t = 0.00$	$t = 0.10$	$t = 0.15$	$t = 0.20$	$t = 0.25$	$ t_{0.00} - t_{0.25} $
$w(m = 0.25)$	0.9134	0.8704	0.8176	0.7456	0.6561	0.2573
$w(m = 0.50)$	0.9134	0.8883	0.8571	0.8140	0.7549	0.1585
$w(m = 0.75)$	0.9134	0.8943	0.8705	0.8374	0.7952	0.1182
$w(m = 1.00)$	0.9134	0.9018	0.8832	0.8671	0.8412	0.0722

4.2 The Vibrating Membrane Equation

Next, we show the results from the Examples 3.8-3.11 as the following Figures 4.4 - 4.7,

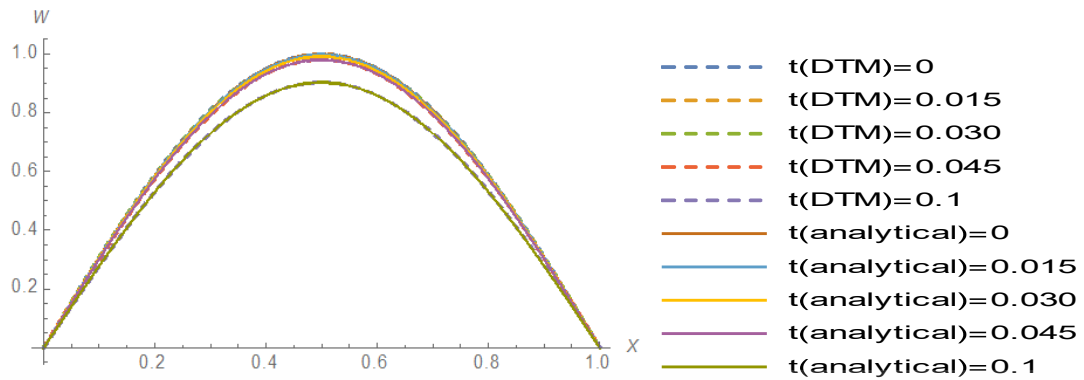


Figure 4.4 Graphical comparison between the amplitude of the analytical solution and the approximate solution from DTM of the vibrating membrane equation calculated in Example 3.8

Table 4.3 Data value of analytical solution and approximate solution from DTM calculated in Example 3.8 and error for the vibrating membrane equation at $x = 0.5$ and $y = 0.5$.

$t(\text{sec})$	$w(\text{analytical solution})$	$w(\text{DTM})$	$\text{error} = w(\text{analytical}) - w(\text{DTM}) $
$t = 0.000$	1.0000	1.0001	1.00×10^{-4}
$t = 0.015$	0.999978	1.00008	3.00×10^{-4}
$t = 0.030$	0.999911	1.00001	9.9×10^{-5}
$t = 0.045$	0.9998	0.999901	1.01×10^{-4}
$t = 0.100$	0.999013	0.999115	1.02×10^{-4}

From the graphical comparison between the amplitude of the analytical solution and the approximate solution from DTM calculated in Example 3.8 for the vibration of membrane as shown in the Figure 4.4 and the value analytical solution and approximate solution from calculated in Example 3.8 and error for the vibrating membrane equation at $x = 0.5$ and $y = 0.5$ as shown in Tables 4.3, we can see that the amplitude of the analytical solution and the approximate solution from DTM in Example 3.8 of the vibration of membrane are close to each other.

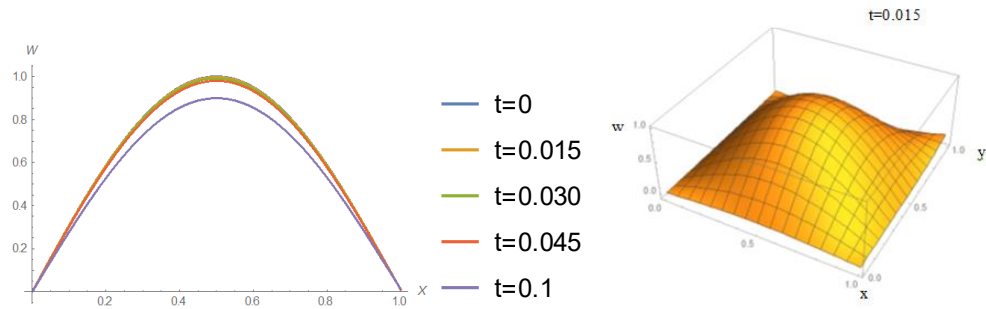


Figure 4.5 Graphical result of the amplitude of the approximate solution from DTM of the vibration of membrane with a damping term in Example 3.9

Table 4.4 Data value of DTM calculated in Example 3.8, DTM with a damping calculated in Example 3.9 and the errors for the vibrating membrane equation at $x = 0.5$ and $y = 0.5$.

$t(\text{sec})$	$w(DTM)$	$w(DTM \text{ with damping})$	$error = \left \frac{w(DTM)}{-w(DTM \text{ with damping})} \right $
$t = 0.000$	1.0001	1.00011	1.00×10^{-5}
$t = 0.015$	1.00008	1.00009	1.00×10^{-5}
$t = 0.030$	1.00001	1.00002	1.00×10^{-5}
$t = 0.045$	0.999901	0.999904	3.00×10^{-6}
$t = 0.100$	0.999115	0.998968	1.47×10^{-4}

From the graphical result of the amplitude of the approximate solution from DTM of the vibration of membrane with a damping term in Example 3.9 as shown in Figure 4.5 and the data value DTM in Example 3.8, DTM with damping in Example 3.9 and error of the vibrating membrane equation at $x = 0.5$ and $y = 0.5$ as shown in Tables 4.4, they show that the amplitude of motion for the vibrating of membrane decreases when damping term is added to the problem.

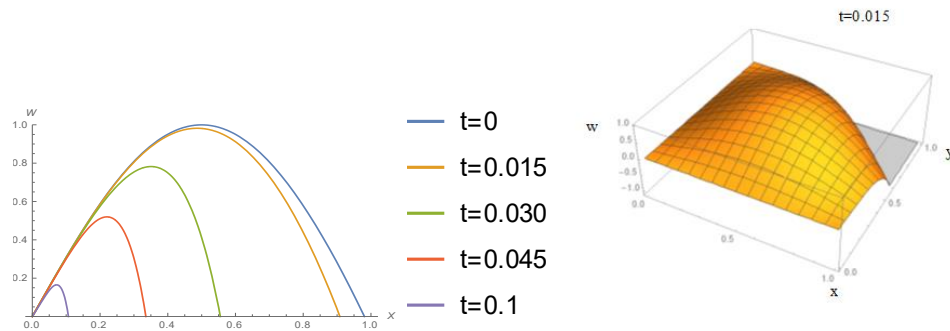


Figure 4.6 Graphical result of the amplitude of the approximate solution from DTM of the vibration of membrane with an external force in Example 3.10

From the graphical result of the amplitude of the approximate solution from DTM of the vibration of membrane with external force Example 3.10 shown in Figure 4.6 and the value DTM in Example 3.8, DTM with a damping in Example 3.9 and error of equation of the motion for the vibration of membrane at $x = 0.5$ and $y = 0.5$, they show that the amplitude of motion for the vibration of membrane decreases when external force is added to the problem.

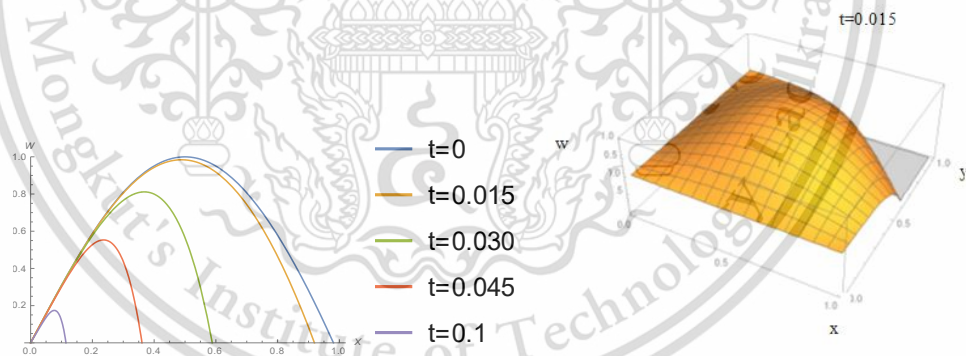


Figure 4.7 Graphical result of the amplitude of the approximate solution from DTM of the vibration of membrane with a damping term and an external force in Example 3.11

From the graphical result of the amplitude of the approximate solution from DTM of the vibration of membrane with a damping term and external force in Example 3.11 shown in Figure 4.7 and the value DTM Example 3.8, DTM with a damping in Example 3.9 and error of vibrating membrane equation $x = 0.5$ and $y = 0.5$, they show that the amplitude of motion for the vibration of membrane decreases when external force is added to the problem.

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$\gamma = 0.5$ as shown in Table 4.4, they show that the amplitude of motion for the vibration of membrane decreases when a damping term and external force are added to the problem.



Chapter 5

Differential Transformation Method for the nonlinear functions, Applications and Numerical Examples

In this chapter, DTM is applied to find approximated solutions for the nonlinear functions. The two examples in following Section 5.1, recurrence systems are first obtained and then a routine computer programming is used to find an approximated solution in the Taylor series form. We showed the comparison results obtained from the recurrence relation of the differential transformation to [37] and modified recurrence relation of the differential transformation in this work.

5.1 Differential Transformation Method for the Nonlinear Functions

In this Section, DTM is applied to find approximated solutions for the nonlinear functions.

Example 5.1 Consider $f_1(t) = e^{at}$ from definition 2.1, we have $k = 0$

$$\begin{aligned} F_1(0) &= \frac{1}{0!} \left[\frac{d^0 f_1(t)}{dx^0} \right]_{t=t_0} \\ &= f_1(t_0) \\ &= e^{at_0}. \end{aligned}$$

Differentiating $f_1(t_0) = e^{at_0}$ with respect on t ,

$$\frac{df_1(t_0)}{dt} = ae^{at_0} = af_1(t_0), \quad (5.1)$$

differential transformation method Table 2.1 The fundamental operations of one-dimension differential transformation method, we have

$$(k+1)F_1(k+1) = aF_1(k),$$

we obtain $k+1$ with k , we have

$$\begin{aligned} kF_1(k) &= aF_1(k-1) \\ F_1(k) &= \frac{aF_1(k-1)}{k}, \quad k \geq 1, \end{aligned}$$

and recursive method of T-function for $f_1(t_0) = e^{at_0}$, we have

$$F_1(k) = \begin{cases} F_1[0] = e^{at_0}, & k = 0, \\ \frac{aF_1(0)}{k!}, & k \geq 1. \end{cases} \quad (5.2)$$

Example 5.2 Consider $f_2(t) = e^{at} \cos \omega t$ by $h(t) = e^{at} \sin \omega t$.

From definition 2.1, considering $k = 0$

$$F_2(t_0) = e^{at_0} \cos \omega t_0, \quad (5.3)$$

$$H(t_0) = e^{at_0} \sin \omega t_0, \quad (5.4)$$

Differentiating $f_2(t) = e^{at} \cos \omega t$ and $h(t) = e^{at} \sin \omega t$ with respect on t ,

$$\begin{aligned} \frac{df_2(t)}{dt} &= -\omega(e^{at} \sin \omega t) + a(e^{at} \cos \omega t) = -\omega h(t) + af_3(t), \\ \frac{dh(t)}{dt} &= \omega(e^{at} \cos \omega t) + a(e^{at} \sin \omega t) = \omega f_3(t) + ah(t), \end{aligned} \quad (5.5)$$

recursive method of T-function for $f_2(t) = e^{at} \cos \omega t$, we have

$$F_2(k) = \begin{cases} e^{at_0} \cos \omega t_0, & k = 0, \\ \frac{-\omega H(0) + aF_2(0)}{k}, & k \geq 1, \end{cases} \quad (5.6)$$

and recursive method of T-function for $h(t) = e^{at} \sin \omega t$, we have

$$H(k) = \begin{cases} e^{at_0} \sin \omega t_0, & k = 0, \\ \frac{\omega F_3(0) + aH(0)}{k}, & k \geq 1, \end{cases} \quad (5.7)$$

5.2 Applications and Numerical Examples

In this Section, we will show the comparison results obtained from the recurrence relation of the differential transformation in [37] and modified recurrence relation of the differential transformation in this work.

The following recursive formulas of the differential transformation of $f_1(t) = e^{at}$ is obtained by definition 2.1 when $t_0 = 0$ as applied in [37]

$$F_1(k) = \begin{cases} 1, & k = 0 \\ \frac{aF_1(k-1)}{k}, & k \geq 1. \end{cases} \quad (5.8)$$

Equation (5.8) is the recurrence relation for finding the coefficient of the approximate Maclaurin series of the original nonlinear function

$$f_1(t) = e^{at}.$$

We find that for the given $k=600$ and $t \in [-0.2, 0.2]$, the graphical comparison results between $f_1(t) = e^{at}$ and the approximate Maclaurin series using *DTM* are the same for defined $-450 < a < 450$ as shown in Figure 5.1. However the approximate Maclaurin series is diverge for defined $a \leq -450$ and $a \geq 450$ as shown in Figure 5.2.

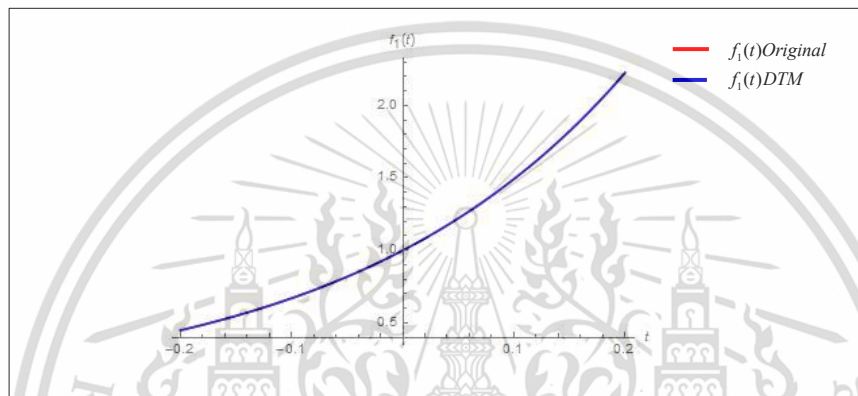


Figure 5.1 The graphical comparison results between $f_1(t) = e^{at}$ and

$f_1(t) = 1t^0 + 7t^1 + \frac{49}{2}t^2 + \frac{343}{6}t^3 + \frac{2401}{24}t^4 + \frac{1680}{120}t^5 + \dots + 9 \times 10^{-902}t^{600}$ using *DTM* when $t \in [-0.2, 0.2]$, $a = 4$ and $k = 600$.

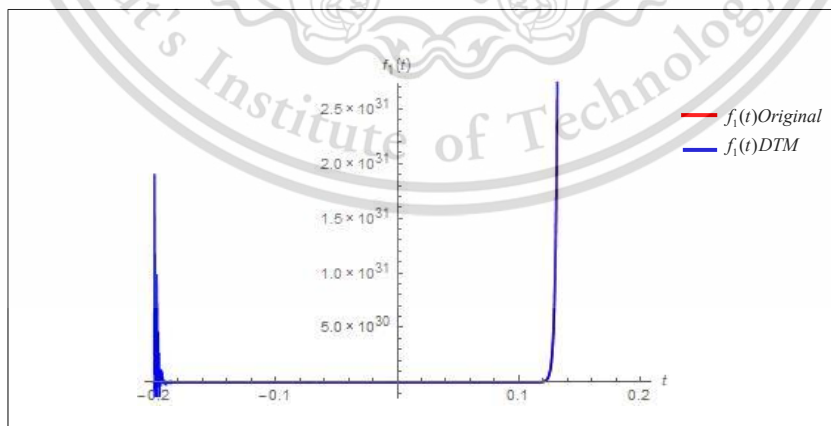


Figure 5.2 The graphical comparison results between $f_1(t) = e^{at}$ and

$f_1(t) = 1t^0 + 550t^1 + 151250t^2 + \frac{83187500}{3}t^3 + \dots + 1.3 \times 10^{236}t^{600}$ using *DTM* when $t \in [-0.2, 0.2]$,

$a = 550$ and $k = 600$.

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By the similar way to Equation (5.8), the following recursive formulas of the differential transformation of $f_2(t) = e^{at} \cos \omega t$ is obtained by definition 2.1 when $t_0 = 0$ as applied in [37]

$$F_2(k) = \begin{cases} 1, & k = 1, \\ \frac{-\omega H(k-1) + aF_2(k-1)}{k}, & k \geq 1, \end{cases} \quad (5.9)$$

and recursive method of T-function for $h(t) = e^{at} \sin \omega t$, we have

$$H(k) = \begin{cases} 0, & k = 0 \\ \frac{\omega F_2(k-1) + aH(k-1)}{k}, & k \geq 1, \end{cases} \quad (5.10)$$

We find that for the given $k = 600$ and $t \in [-0.5, 0.5]$, the graphical comparison results between $f_2(t) = e^{at} \cos \omega t$ and the approximate Maclaurin series using *DTM* are the same for defined $-180 < a < 180$ and $-70 < \omega < 70$ as shown in Figure 5.3 However the approximate Maclaurin series is diverge for defined $a \leq -180, a \geq 180$ and $\omega \leq -70, \omega \geq 70$ as shown in Figure 5.4.

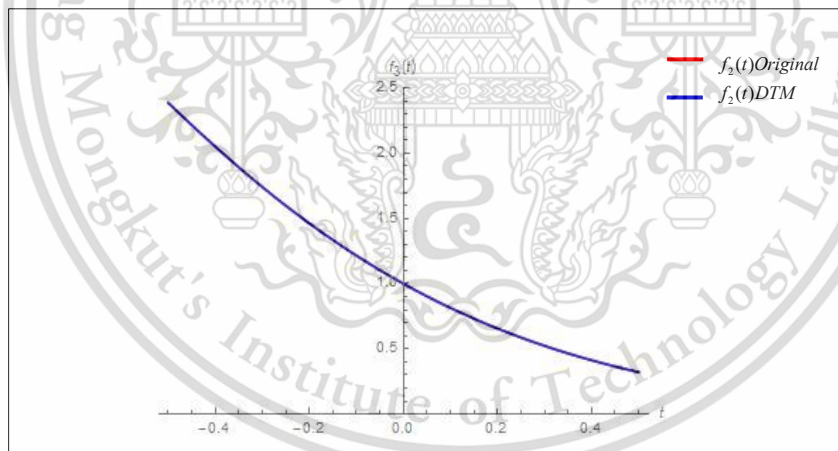


Figure 5.3 The graphical comparison results between $f_2(t) = e^{at} \cos \omega t$ and $f_2(t) = 1t^0 - 2t^1 + \frac{3}{2}t^2 - \frac{1}{3}t^3 - \frac{7}{24}t^4 + \frac{19}{60}t^5 + \dots - 6.1 \times 10^{-1200}t^{600}$ using *DTM* when $t \in [-0.5, 0.5]$, $a = -2$, $k = 600$ and $\omega = 1$.

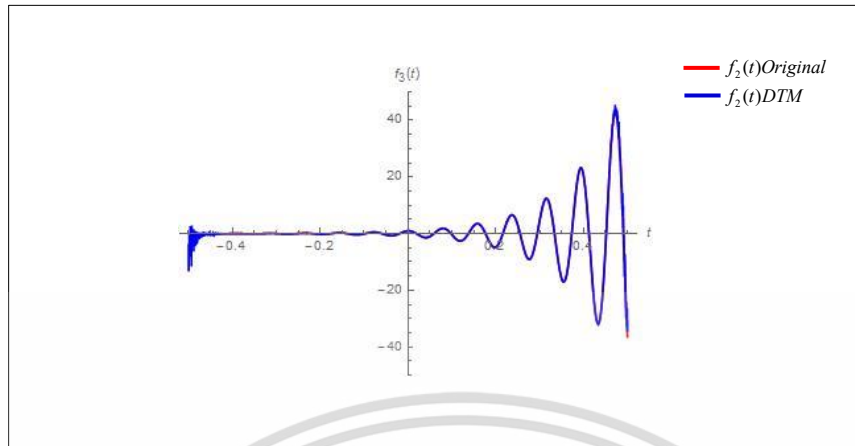


Figure 5.4 The graphical comparison results between $f_2(t) = e^{at} \cos \omega t$ and $f_2(t) = 1t^0 + 8t^1 - 3168t^2 - \frac{76544}{3}t^3 + \frac{4813312}{3}t^4 + \dots - 1.1 \times 10^{-265}t^{600}$ from DTM when and $t \in [-0.5, 0.5]$, $a = 8$, $k = 600$ and $\omega = 80$.

The graphical comparison results between the original functions and the approximate Taylor's series with the coefficients obtained from the modified differential transformation method as the recurrence relations of $f_1(t) = e^{at}$ in Equation (5.2) and the recurrence relations $f_2(t) = e^{at} \cos \omega t$ in Equation (5.6) are shown in Figures 5.5 - 5.6, respectively.

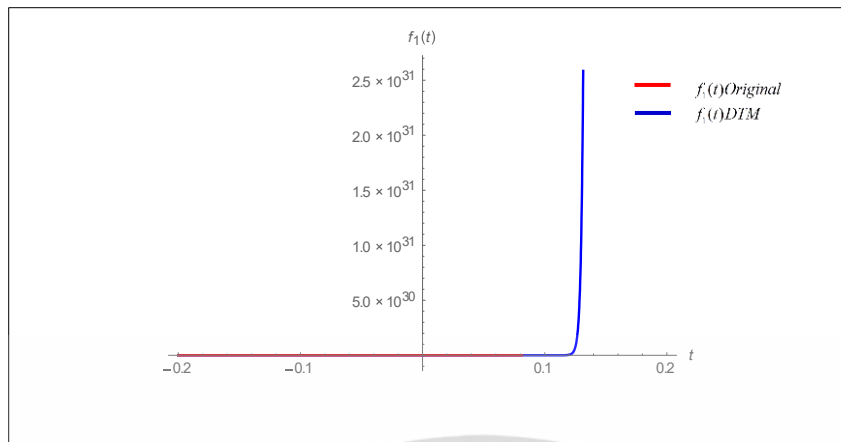


Figure 5.5 The graphical comparison results between $f_1(t) = e^{at}$ and $f_1(t) = 1t^0 + 550t^1 + 151250t^2 + \frac{83187500}{3}t^3 + \dots + 1.3 \times 10^{236}t^{600}$ using *DTM* when $t \in [-0.2, 0.2]$, $a = 550$, and $k = 600$.

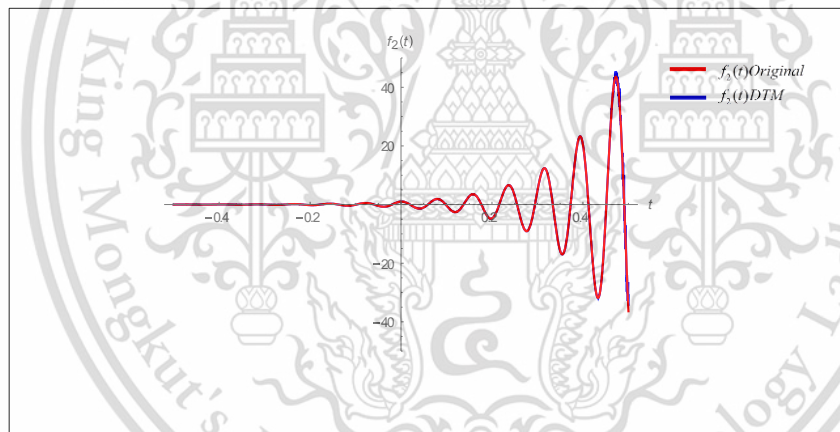


Figure 5.6 The graphical comparison results between $f_2(t) = e^{at} \sin \omega t$ and $f_2(t) = 1t^0 + 8t^1 - 3168t^2 - \frac{76544}{3}t^3 + \frac{4813312}{3}t^4 + \dots - 1.1 \times 10^{-265}t^{600}$ using *DTM* when $t \in [-0.5, 0.5]$, $a = 8$, $k = 600$, and $\omega = 80$.

Chapter 6

Conclusion

In first problem, the modified theorem of the two-dimensional differential transformation method for cubic term is manifested. The proof of the theorem is derived. The theorem can be applied in several applications. The cubic term of differential transformation can be applied efficiently to the nonlinear terms of suspended string equation. This work demonstrated that the differential transformation method can be applied to the initial and boundary value problem of the partial differential equation.

The second problem, the vibrating membrane equation is studied we found that the analytical solution and the approximate solution is very identical when the equation without a damping term and an external force that can depict the DTM (Differential Transformation Method) can efficiently detecting the vibration of membrane. Then we added a damping term, and an external force in the vibrating membrane equation to find the approximate solution by DTM, we found that as we added the damping term, and the external force can cause the amplitude of the vibrating membrane equation to decrease.

The third problem, the modified recursive formulas of differential transformation method is presented in this work. The obtained formulas is used to find the coefficients of the Taylor's series applied to the approximate the nonlinear functions. The comparison results between the approximate Taylor's series and the original function are the same. These obtained formulas can be used to solve many applications in the future works.

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Differential Transformation Method for Vibration of Membranes

Journal:	<i>Songklanakarin Journal of Science and Technology</i>
Manuscript ID	SJST-2017-0432.R1
Manuscript Type:	Original Article
Date Submitted by the Author:	14-Feb-2018
Complete List of Authors:	Mansilp, Kamonpad; King Mongkut's Institute of Technology Ladkrabang, mathematics Kasemsuwan, Jaipong; King Mongkut's Institute of Technology Ladkrabang, mathematics
Keyword:	Vibration of Membrane, Differential Transformation Method, Analytical Solution, Approximate Solution.



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DIFFERENTIAL TRANSFORMATION METHOD FOR VIBRATION OF MEMBRANES

Original Article

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Abstract

The purpose of this research is to apply the three-dimensional differential transform method to estimate solutions to the equation of motion for vibration in a membrane with certain boundary condition problems. The analytical solutions without an external force and damping term under the initial and boundary conditions are presented. We found that the comparison results between the analytical solution and the estimated solution are in good agreement in case of no damping and the external force. Furthermore, the differential transform method can be used to find approximated solutions taking into account both an external force and damping term. This cannot be achieved via an analytical solution.

2010 Mathematics Subject Classification: 35G20, 35L05, 35L10 , 35L70.

Keywords: Vibration of Membrane, Differential Transformation Method, Analytical Solution, Approximate Solution.

1. Introduction

Many researchers have attempted to understand several phenomena occurring in nature by applying knowledge from different field such as mechanical engineering, electrical engineering, industrial engineering, energy and medicine. Most of these problems have been studied by employing some form of mathematical modeling using both ordinary differential equation (ODE) and partial differential equation (PDE). These problems require an accurate solution, either as an analytical solution or approximate solution. The differential transform method (DTM) is among the most effective mathematical methods for finding solutions to these differential equations (Hatami, Ganji, & Sheikholeslami, 2017).

The differential transform method (DTM) is based on the high-order Taylor series expansion. This method is a powerful tool for solving linear and non-linear ordinary differential equations (Ayaz, 2004; Arikoglu, & Ozkol, 2006; Catal, 2008) and for solving two-and three-dimensionals partial differential equations in both linear and non-linear problems. DTM can be used to solve the differential equation subject to initial and boundary conditions having both linear and non-linear terms within an acceptable error range.

The two-dimensional differential transform method (2D-DTM) is used to find the solutions of both linear PDEs (Chen, & Ho, 1999; Othman, & Mahdy, 2010; Ayaz, 2003; Yang, Liu, & Bai, 2006) and nonlinear PDEs (Bildik, Konuralp, Bek, & Kucukarslan, 2006; Kangalgil, & Ayaz, 2009; Biazar, & Eslami, 2010; Biazar, Eslami, & Islam, 2012).

Additionally, the three-dimensional differential transform method (3D-DTM) is applied to find the solutions of linear and non-linear PDEs (Saravanan, & Magesh, 2013; Bagheri, & Manafianheris, 2012). It is noted that the differential transform method can be used to solve multidimensional PDEs such as the Westervelt equation (Jafari, Sadeghi, & Biswas, 2012), heat-like and wave-like equations (Tabaei, Celuk, & Tabaei, 2012), and fuzzy

partial differential equations (Mirzaee, & Yari, 2015) as well as the linear and nonlinear system of PDEs (Ayaz, 2004; Zedan, & AliAlghamdi, 2012).

Many researchers are concerned with the use of non-linear PDEs by applying several transform methods as follows. The Fitzhugh Nagumo (FN) equation is a mathematical model for solving scientific and engineering problems by using q-HATM and the fractional reduced differential transform method (FRDTM) which is based on DTM (Kumar, Singh, & Baleanu, 2017).

The numerical solutions to non-linear fractional dynamical model of interpersonal and romantic relationships are presented by applying q-homotopy analysis via the Sumudu transform method (q-HASTM) (Singh, Kumar, Qurashi, & Baleanu, 2017).

Jeffery-Hamel flow in non-parallel walls, which relates to non-linear PDE and occurs in fluid dynamics and other scientific applications, is solved by using an efficient hybrid computational technique, the homotopy analysis transform method (HATM) (Singh, Rashidi, Sushila, & Kumar, 2017).

The homotopy perturbation Sumudu transform method (HPSTM) and homotopy analysis Sumudu transform method (HASTM) are more convenient than the homotopy perturbation method (HPM) and the homotopy analysis method (HAM) since they produce a comparative analytical study for a system of time fractional non-linear differential equations (Choi, Kumar, Singh, & Swroop, 2016).

Further examples of the application of PDEs by Laplace transform method to various problem include Case I-application of drum head vibration solving by separation of variables method, and Case II and III-application of signal transmission and application of chemical communication in insects. All three cases give simulations on the solutions implemented by MATLAB software (Ojwando, 2016).

A modified He-Laplace method (MHLM) is applied to solve space and time nonlinear fractional differential-difference equations (NFDDEs) (Prakash, Kothandapan, & Bharathi, 2016).

In our previous work, we studied the suspended vibrating string equation using 2D – DTM. It was found that DTM can be applied to various problems of the suspended string equation.

In this proposed work, we study further the vibration equation in three dimensions of the motion of a membrane using the 3D-DTM. The oscillation of a membrane-like plate, which is determined by the tension and insignificant resistance to the bending. The differential transform method was applied to find solutions of the motion equation of a membrane with an external force and damping term. We compare the results with an analytical solution. We show in detail the derivation of the transformed formula of DTM which is of the n^{th} power form (to be shown in theorem 3.6). The obtained formula helps simplify the use of DTM in solving non-linear PDEs. It is noted that the formula requires high computation to find a complete solutions mainly due to the fact that the formula is in the recursive form.

The equation of motion of a membrane

The equation of motion for the forced transverse vibration of a membrane, after (Rao, 2011) is as follows:

$$P \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + f(x, y, t) = \rho(x, y) \frac{\partial^2 w}{\partial t^2}, \quad (1)$$

where $f(x, y, t)$ is the pressure acting in the z direction(external force), P is the intensity of tension at a point equal to production of the tensile stress and the thickness of the membrane,

and $\rho(x, y)$ is the mass per unit area. We assume that $f(x, y, t) = 0$, $P = 1$, and $\rho(x, y) = 1$.

Then Eq.(1) leads to:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial t^2}. \quad (2)$$

The initial conditions of (2) after (Rao, 2011) are:

$$w(x, y, 0) = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b,$$

$$\frac{\partial w}{\partial t}(x, y, 0) = 0, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b,$$

we assumed that $a = 1$ and $b = 1$. Therefore, the boundary conditions of the equation of motion of a membrane are defined thus:

$$\begin{aligned} w(x, 0, t) &= 0, \quad 0 \leq x \leq 1, \\ w(0, y, t) &= 0, \quad 0 \leq y \leq 1, \\ w(x, 1, t) &= 0, \quad 0 \leq x \leq 1, \\ w(1, y, t) &= 0, \quad 0 \leq y \leq 1, t \in \mathbb{R}. \end{aligned} \quad (3)$$

An analytical solution

The following is the derivation of the analytical solution of the problem (2).

Considering the initial conditions $w(x, y, 0) = \sin \pi x \sin \pi y$ and $\frac{\partial w}{\partial t}(x, y, 0) = 0$, the boundary conditions are shown in (3). The general solution can be derived by the separation of variables technique:

$$w(x, y, t) = X(x)Y(y)T(t), \quad (4)$$

Subject to the eigenvalue, $-\omega^2, \alpha^2$ and β^2 , can be written (2)

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{T''(t)}{T(t)} = -\omega^2,$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\omega^2,$$

and
$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \omega^2 = \alpha^2,$$

$$-\frac{X''(x)}{X(x)} = \alpha^2,$$

then
$$X''(x) + \alpha^2 X(x) = 0,$$

$$\frac{Y''(y)}{Y(y)} = \alpha^2 - \omega^2,$$

assumed
$$-\alpha^2 + \omega^2 = \beta^2, Y''(y) + (-\alpha^2 + \omega^2)Y(y) = 0,$$

then
$$Y''(y) + \beta^2 Y(y) = 0,$$

and
$$\frac{T''(t)}{T(t)} = \omega^2,$$

then
$$T''(t) + \omega^2 T(t) = 0,$$

$$X''(x) + \alpha^2 X(x) = 0, \tag{5}$$

$$Y''(y) + \beta^2 Y(y) = 0, \tag{6}$$

$$T''(t) + \omega^2 T(t) = 0. \tag{7}$$

We obtain a solution of (5) as $X(x) = C_1 \cos \alpha x + C_2 \sin \alpha x$, where C_1 and C_2 are arbitrary constants. Subject to the boundary conditions $X(0) = 0$, we have $C_1 = 0$. Then

$X(x) = C_2 \sin \alpha x$ full step the boundary condition $X(1) = 0$. Let $C_2 \neq 0$ give $\alpha = m\pi; m \in I$,

we have that

$$X_m(x) = C_m \sin m\pi x. \tag{8}$$

We obtain a solution of (6) as $Y(y) = C_3 \cos \beta y + C_4 \sin \beta y$. According to boundary conditions $Y(0) = 0$, we have $C_3 = 0$. Then the boundary conditions $Y(y) = 0$.

Letting $C_4 \neq 0$ gives $\beta = n\pi; n \in I$, then we see that:

$$Y_n(y) = C_n \sin n\pi y. \quad (9)$$

From

$$\begin{aligned} \omega^2 &= \beta^2 + \alpha^2, \\ \omega &= \pi\sqrt{m^2 + n^2}. \end{aligned}$$

We obtain a solution of (7) as

$$T_{mn}(t) = A_{mn} \cos \omega t + B_{mn} \sin \omega t.$$

Then

$$T_{mn}(t) = A_{mn} \cos \pi\sqrt{m^2 + n^2}t + B_{mn} \sin \pi\sqrt{m^2 + n^2}t. \quad (10)$$

According to (8), (9) and (10) can be written

$$\begin{aligned} W_{mn}(x, y, t) &= X(x)Y(y)T(t), \\ &= \left(A_{mn} C_m C_n \sin m\pi x \sin n\pi y \cos \pi\sqrt{m^2 + n^2}t \right) \\ &\quad + \left(B_{mn} C_m C_n \sin m\pi x \sin n\pi y \sin \pi\sqrt{m^2 + n^2}t \right), \end{aligned} \quad (11)$$

where $F_{mn} = A_{mn} C_m C_n$, $H_{mn} = B_{mn} C_m C_n$ and for all $m, n \in I$

By using the superposition principle (11) become

$$\begin{aligned} w(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}(x, y, t), \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(F_{mn} \sin m\pi x \sin n\pi y \cos \pi\sqrt{m^2 + n^2}t \right), \\ &\quad + \left(H_{mn} \sin m\pi x \sin n\pi y \sin \pi\sqrt{m^2 + n^2}t \right). \end{aligned} \quad (12)$$

According to the initial condition $w(x, y, 0) = \sin \pi x \sin \pi y$ and $\frac{\partial w}{\partial t}(x, y, 0) = 0$. By using

Fourier series, we have

$$\begin{aligned} F_{mn} &= 1 ; m=1 \text{ and } n=1, \\ H_{mn} &= 0; m, n \in I \\ \omega &= \sqrt{2}\pi. \end{aligned}$$

Substituting $F_{11} = 1$, $H_{mm} = 0$ and $\omega = \sqrt{2}\pi$ in (12), we obtain the analytical solution:

$$w(x, y, t) = \sin \pi x \sin \pi y \cos \sqrt{2}\pi t. \quad (13)$$

In the case of the problem (2) with damping term are represented by

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial t}.$$

By similar calculation, we obtain the solution of $T_{m,n}(t)$ as

$$T_{m,n}(t) = e^{-\frac{t}{2}} \left(A_{m,n} \cos \frac{\sqrt{1-4\omega^2}}{2} t + B_{m,n} \sin \frac{\sqrt{1-4\omega^2}}{2} t \right). \quad (14)$$

Considering (14), we can see that substituting $\omega = \sqrt{2}\pi$ to (14), it make $\sqrt{1-4\omega^2}$ be the complex number. Therefore $T(t)$ are not solvable.

2. Three-Dimensional Differential transform method

The basic definitions and fundamental operations of differential transform are defined below.

Definition 2.1 The three-dimensional differential transform of function $w(x, y, t)$ is defined as:

$$W(k, h, m) = \frac{1}{k!h!m!} \left. \frac{\partial^{k+h+m} w(x, y, t)}{\partial x^k \partial y^h \partial t^m} \right|_{(0,0,0)} \quad k \geq 0, \quad h \geq 0 \text{ and } m \geq 0.$$

Definition 2.2 The inverse three-dimensional differential transform of sequence

$\{W(k, h, m)\}_{k,h,m=0}^{\infty}$ is defined as:

$$w(x, y, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \sum_{m=0}^{\infty} W(k, h, m) x^k y^h t^m. \quad (14)$$

3. The fundamental operations of three-dimensional differential transform method .

From Table 1 theorem 3.1-3.5 shown in (Yang, Liu, & Bai, 2006).

The following is the derivation of theorem 3.6:

If $v(x, y, t) = w_1(x, y, t)w_2(x, y, t)...w_{n-1}(x, y, t)w_n(x, y, t)$ then,

$$V(k, h, m) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \sum_{h_3=0}^{h_2} \sum_{h_1=0}^{h_3} \sum_{m_3=0}^{m_2} \sum_{m_1=0}^{m_3} W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2}) W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).$$

For $V : I^+ \rightarrow \mathbb{R}$, $W_n : I^+ \rightarrow \mathbb{R}; n \in I^+$, $v : \mathbb{R} \rightarrow \mathbb{R}$, $w_n : \mathbb{R} \rightarrow \mathbb{R}; n \in I^+$, $x \in \mathbb{R}$

and $k, h, m = 0, 1, 2, 3, \dots$

By definition of the three-dimensional differential transform:

$$V(k, h, m) = \frac{1}{k!h!m!} \left[\frac{\partial^{k+h+m} v(x, y, t)}{\partial x^k \partial y^h \partial t^m} \right]_{(0,0,0)}$$

we obtain $v(x, y, t) = w_1(x, y, t)w_2(x, y, t)...w_{n-1}(x, y, t)w_n(x, y, t)$

Then by using definition (13), we have:

$$\begin{aligned} V(k, h, m) &= \frac{1}{k!h!m!} \left[\frac{\partial^{k+h+m}}{\partial x^k \partial y^h \partial t^m} w_1(x, y, t)w_2(x, y, t)...w_{n-1}(x, y, t)w_n(x, y, t) \right]_{(0,0,0)} \\ &= \frac{1}{k!h!m!} \sum_{k_{n-1}=0}^k \sum_{h_{n-1}=0}^h \sum_{m_{n-1}=0}^m \frac{k!}{(k_{n-1})!(k - k_{n-1})!} \frac{h!}{(h_{n-1})!(h - h_{n-1})!} \frac{m!}{(m_{n-1})!(m - m_{n-1})!} \\ &\quad \left[\frac{\partial^{k_{n-1}+h_{n-1}+m_{n-1}}}{\partial x^{k_{n-1}} \partial y^{h_{n-1}} \partial t^{m_{n-1}}} w_1(x, y, t)w_2(x, y, t)... \right. \\ &\quad \left. w_{n-1}(x, y, t) \frac{\partial^{k-k_{n-1}+h-h_{n-1}+m-m_{n-1}}}{\partial x^{k-k_{n-1}} \partial y^{h-h_{n-1}} \partial t^{m-m_{n-1}}} w_n(x, y, t) \right]_{(0,0,0)} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k_{n-1}=0}^k \sum_{h_{n-1}=0}^h \sum_{m_{n-1}=0}^m \\
&\left[\frac{1}{(k_{n-1})!(h_{n-1})!(m_{n-1})!} \frac{\partial^{k_{n-1}+h_{n-1}+m_{n-1}}}{\partial x^{k_{n-1}} \partial y^{h_{n-1}} \partial t^{m_{n-1}}} w_1(x, y, t) w_2(x, y, t) \dots w_{n-1}(x, y, t) \right]_{(0,0,0)} \\
&\left[\frac{1}{(k-k_{n-1})!(h-h_{n-1})!(m-m_{n-1})!} \frac{\partial^{k-k_{n-1}+h-h_{n-1}+m-m_{n-1}}}{\partial x^{k-k_{n-1}} \partial y^{h-h_{n-1}} \partial t^{m-m_{n-1}}} w_n(x, y, t) \right]_{(0,0,0)} \\
&= \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \\
&\left[\frac{1}{(k_{n-2})!(h_{n-2})!(m_{n-2})!} \frac{\partial^{k_{n-2}+h_{n-2}+m_{n-2}}}{\partial x^{k_{n-2}} \partial y^{h_{n-2}} \partial t^{m_{n-2}}} w_1(x, y, t) w_2(x, y, t) \dots w_{n-2}(x, y, t) \right]_{(0,0,0)} \\
&\left[\frac{1}{(k_{n-1}-k_{n-2})!(h_{n-1}-h_{n-2})!(m_{n-1}-m_{n-2})!} \right. \\
&\times \left. \frac{\partial^{k_{n-1}-k_{n-2}+h_{n-1}-h_{n-2}+m_{n-1}-m_{n-2}}}{\partial x^{k_{n-1}-k_{n-2}} \partial y^{h_{n-1}-h_{n-2}} \partial t^{m_{n-1}-m_{n-2}}} w_{n-1}(x, y, t) \right]_{(0,0,0)} \\
&\left[\frac{1}{(k-k_{n-1})!(h-h_{n-1})!(m-m_{n-1})!} \frac{\partial^{k-k_{n-1}+h-h_{n-1}+m-m_{n-1}}}{\partial x^{k-k_{n-1}} \partial y^{h-h_{n-1}} \partial t^{m-m_{n-1}}} w_n(x, y, t) \right]_{(0,0,0)} \\
&= \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{k_{n-3}=0}^{k_{n-2}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{h_{n-3}=0}^{h_{n-2}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \sum_{m_{n-3}=0}^{m_{n-2}} \\
&\left[\frac{1}{(k_{n-3})!(h_{n-3})!(m_{n-3})!} \frac{\partial^{k_{n-3}+h_{n-3}+m_{n-3}}}{\partial x^{k_{n-3}} \partial y^{h_{n-3}} \partial t^{m_{n-3}}} w_1(x, y, t) w_2(x, y, t) \dots w_{n-3}(x, y, t) \right]_{(0,0,0)} \\
&\left[\frac{1}{(k_{n-2}-k_{n-3})!(h_{n-2}-h_{n-3})!(m_{n-2}-m_{n-3})!} \right. \\
&\times \left. \frac{\partial^{k_{n-2}-k_{n-3}+h_{n-2}-h_{n-3}+m_{n-2}-m_{n-3}}}{\partial x^{k_{n-2}-k_{n-3}} \partial y^{h_{n-2}-h_{n-3}} \partial t^{m_{n-2}-m_{n-3}}} w_{n-2}(x, y, t) \right]_{(0,0,0)} \\
&\left[\frac{1}{(k_{n-1}-k_{n-2})!(h_{n-1}-h_{n-2})!(m_{n-1}-m_{n-2})!} \right. \\
&\times \left. \frac{\partial^{k_{n-1}-k_{n-2}+h_{n-1}-h_{n-2}+m_{n-1}-m_{n-2}}}{\partial x^{k_{n-1}-k_{n-2}} \partial y^{h_{n-1}-h_{n-2}} \partial t^{m_{n-1}-m_{n-2}}} w_{n-1}(x, y, t) \right]_{(0,0,0)} \\
&\left[\frac{1}{(1-k_{n-1})!(1-h_{n-1})!(1-m_{n-1})!} \frac{\partial^{1-k_{n-1}+1-h_{n-1}+1-m_{n-1}}}{\partial x^{1-k_{n-1}} \partial y^{1-h_{n-1}} \partial t^{1-m_{n-1}}} w_n(x, y, t) \right]_{(0,0,0)},
\end{aligned}$$

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$$\begin{aligned}
V(k, h, m) = & \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{k_{n-3}=0}^{k_{n-2}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{h_{n-3}=0}^{h_{n-2}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \sum_{m_{n-3}=0}^{m_{n-2}} \dots \\
& \sum_{k_3=0}^{k_4} \sum_{k_2=0}^{k_3} \sum_{h_3=0}^{h_4} \sum_{h_2=0}^{h_3} \sum_{m_3=0}^{m_4} \sum_{m_2=0}^{m_3} \\
& W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots \\
& W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2}) W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).
\end{aligned}$$

since $v(x, y, t) = w_1(x, y, t)w_2(x, y, t) \dots w_{n-1}(x, y, t)w_n(x, y, t)$ then,

$$\begin{aligned}
V(k, h, m) = & \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \sum_{h_2=0}^{h_3} \sum_{h_1=0}^{h_2} \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} \\
& W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots \\
& W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2}) W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).
\end{aligned}$$

4. Application

In this section, we apply the three-dimensional differential transform method to the vibration of a membrane. We demonstrate four examples of the problem under different conditions. The conditions are (i) without damping term, (ii) with damping term, (iii) with external force, and (iv) with damping term and external force. The initial and boundary conditions as defined as follows.

Example 4.1 Considering the equation of the motion for the vibration of a membrane

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \quad (15)$$

with the initial and boundary conditions,

$$w(x, y, 0) = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1,$$

$$\frac{\partial w}{\partial t}(x, y, 0) = 0, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1,$$

$$w(x, 0, t) = 0, \quad 0 \leq x \leq 1,$$

$$w(0, y, t) = 0, \quad 0 \leq y \leq 1,$$

$$w(x, 1, t) = 0, \quad 0 \leq x \leq 1,$$

$$w(1, y, t) = 0, \quad 0 \leq y \leq 1, t \in \mathbb{R}. \quad (16)$$

Comparing (15) to the general terms of PDEs in Table 1, we have:

$$a(x, y, t) = b(x, y, t) = c(x, y, t) = 1.$$

Then for all $i \geq 0$, $j \geq 0$ and $r \geq 0$, we have

$$A(i, j, r) = B(i, j, r) = C(i, j, r) = \delta(i, j, r).$$

The following Kronecker symbols can be used:

$$\delta(x, y, t) = \delta(x)\delta(y)\delta(t) = \begin{cases} 1; & x = y = t = 0 \\ 0; & \text{otherwise,} \end{cases} \quad \delta(x, y) = \delta(x)\delta(y).$$

By applying DTM and Kronecker symbols to the given equation of motion for the vibration of a membrane, we have

$$W(k, h, m+2) = \frac{1}{(m+2)(m+1)\delta(0,0,0)} \times \left[\sum_{i=0}^k \sum_{j=0}^{h-m} \sum_{r=0}^m (k-i+2)(k-i+1)A(i, j, r)W(k-i+2, h-j, m-r) + \sum_{i=0}^k \sum_{j=0}^{h-m} \sum_{r=0}^m (h-i+2)(h-i+1)B(i, j, r)U(k-i, h-j+2, m-r) \right],$$

$$\frac{1}{(m+2)(m+1)} (k+2)(k+1)(h+2)(h+1)W(k+2, h, m)W(k, h+2, m),$$

that is

$$W(k, h, m+2) = \left[(k+2)(k+1)(h+2)(h+1)W(k+2, h, m)W(k, h+2, m) \right] / (m+2)(m+1)(m+2)(m+1). \quad (17)$$

Comparing between the coefficient of Taylor's series of sine and the series in the definition

(14)

$$\sin(\pi x)\sin(\pi y) \cong \frac{2\pi^2}{2!}xt - \frac{4\pi^4}{4!}x^3t + \frac{6\pi^6}{6!}x^5t - \frac{8\pi^8}{8!}x^7t + \frac{10\pi^{10}}{10!}x^9t - \frac{12\pi^{12}}{12!}x^{11}t + \dots$$

$$\sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k, h, 0)x^k y^h = W(0, 0, 0)x^0 y^0 + W(1, 1, 0)xy + W(2, 1, 0)x^2 y + W(3, 1, 0)x^3 y + \dots + W(0, 1, 0)xy + W(1, 2, 0)xy^2 + W(2, 2, 0)x^2 y^2 + \dots \quad (18)$$

Then, we have

$$U(k, h, 0) = \begin{cases} 0 & \text{if } k = 0, 1, 2, \dots \text{ and } h = 0, 2, 4, 6, \dots \\ \frac{(k+h)\pi^{k+h}}{(k+h)!} & \text{if } k = 1, 5, 9, \dots \text{ and } h = 1, 3, 5, 7, \dots, \\ \frac{-(k+h)\pi^{k+h}}{(k+h)!} & \text{if } k = 3, 7, 11, \dots \text{ and } h = 1, 3, 5, 7, \dots \end{cases} \quad (19)$$

and from the initial condition (16)

$$W(k, h, 1) = 0, \quad k, h = 0, 1, 2, 3, \dots, \quad (20)$$

and from the boundary condition (16), we have

$$\begin{aligned} W(k, 0, m) &= 0, \quad k, m = 0, 1, 2, 3, \dots, \\ W(0, h, m) &= 0, \quad h, m = 0, 1, 2, 3, \dots, \\ W(k, 1, m) &= 0, \quad k, m = 0, 1, 2, 3, \dots, \\ W(1, h, m) &= 0, \quad h, m = 0, 1, 2, 3, \dots, \end{aligned} \quad (21)$$

for each k, h, m substituting (19)-(21) and by recursive method of (17), we obtain the coefficients $W(k, h, m)$ for the series solution.

That is

$$u(x, y, t) = \pi^2 xy - \pi^4 t^2 xy + \frac{1}{6} \pi^6 t^4 xy - \frac{1}{90} \pi^8 t^6 xy - \frac{1}{6} \pi^4 x^3 y + \frac{1}{6} \pi^6 t^2 x^3 y + \dots$$

As the result in (14), we can not apply the separation of variables technique to find the analytical solution to the problem with damping term as equation (22). In the following example, we apply the DTM to find the approximate solution to the problem.

Example 4.2 Considering the equation of motion for the vibration of a membrane with damping term.

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial t}, \quad (22)$$

comparing (22) to general terms of PDEs in table 1, we have

$$a(x, y, t) = b(x, y, t) = c(x, y, t) = 1, d(x, y, t) = -1.$$

Then for all $i \geq 0, j \geq 0$ and $r \geq 0$, we have

$$A(i, j, r) = B(i, j, r) = C(i, j, r) = \delta(i, j, r), D(i, j, r) = -\delta(i, j, r).$$

By applying DTM and Kronecker symbols to the given equation of motion for the vibration of a membrane, we have:

$$W(k, h, m + 2) = [(k + 2)(k + 1)(h + 2)(h + 1)W(k + 2, h, m)W(k, h + 2, m) - (h + 1)W(k, h + 1, m)] / (m + 2)(m + 1). \quad (23)$$

For each k, h, m substituting (19)-(21) and by recursive method of (23), we obtain the coefficients $W(k, h, m)$ for the series solution.

That is

$$w(x, y, t) = -\frac{5}{2}\pi^2 t^2 x + \frac{5}{6}\pi^6 t^4 x + \frac{5}{12}\pi^4 t^2 x^3 - \frac{5}{36}\pi^6 t^4 x^3 - \frac{1}{48}\pi^6 t^2 x^5 + \frac{1}{144}\pi^8 t^4 x^5 + \dots$$

Example 4.3 Considering the equation of motion for the vibration of a membrane with external force.

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - w^3. \quad (24)$$

Comparing (24) to general terms of PDEs in table 1, we have:

$$a(x, y, t) = b(x, y, t) = c(x, y, t) = 1, q(x, y, t) = -1.$$

Then for all $i \geq 0, j \geq 0$ and $r \geq 0$, we have:

$$A(i, j, r) = B(i, j, r) = C(i, j, r) = \delta(i, j, r), Q(i, j, r) = -\delta(i, j, r).$$

By applying DTM and Kronecker symbols to the given equation of motion for the vibration of a membrane, we have

$$W(k, h, m + 2) = [(k + 2)(k + 1)(h + 2)(h + 1)W(k + 2, h, m)W(k, h + 2, m) - \sum_{k_2=0}^k \sum_{h_1=0}^{h_2} \sum_{h_2=0}^h \sum_{m_1=0}^m \sum_{m_2=0}^{m_1} W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) W_3(k - k_2, h - h_2, m - m_2)] / (m + 2)(m + 1). \quad (25)$$

For each k, h, m substituting (19)-(21) and by recursive method of (25), we obtain the coefficients $W(k, h, m)$ for the series solution.

That is:

$$w(x, y, t) = \pi^2 xy - \pi^4 t^2 xy + \frac{1}{6} \pi^6 t^4 xy + \frac{1}{30} \left(-\frac{\pi^8}{6} + 3 \left(-\frac{\pi^8}{18} - \frac{\pi^{10}}{36} \right) \right) t^6 xy - \frac{1}{6} \pi^4 x^3 y + \frac{1}{6} \pi^6 t^2 x^3 y + \dots$$

Example 4.4 Considering the equation of the motion for the vibration of a membrane with external force and damping term.

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial t} - w^3. \quad (26)$$

Comparing (26) to general terms of PDEs in table 1, we have

$$a(x, y, t) = b(x, y, t) = c(x, y, t) = 1, d(x, y, t) = q(x, y, t) = -1.$$

Then for all $i \geq 0, j \geq 0$ and $r \geq 0$, we have

$$A(i, j, r) = B(i, j, r) = C(i, j, r) = \delta(i, j, r), D(i, j, r) = Q(i, j, r) = -\delta(i, j, r).$$

By applying DTM and Kronecker symbols to the given equation of motion for the vibration of a membrane, we have:

$$\begin{aligned} W(k, h, m+2) = & [(k+2)(k+1)(h+2)(h+1)W(k+2, h, m)W(k, h+2, m) \\ & - (h+1)W(k, h+1, m) \\ & - \sum_{k_2=0}^k \sum_{h_2=0}^{h-k_2} \sum_{m_2=0}^{m-h_2} \sum_{k_1=0}^{k_2} \sum_{h_1=0}^{h_2-k_1} \sum_{m_1=0}^{m_2-h_1} W_1(k_1, h_1, m_1) W_2(k_2-k_1, h_2-h_1, m_2-m_1) \\ & W_3(k-k_2, h-h_2, m-m_2)] / (m+2)(m+1). \end{aligned} \quad (27)$$

For each k, h, m substituting (19)-(21) and by recursive method of (27), we obtain the coefficients $W(k, h, m)$ for the series solution.

That is:

$$w(x, y, t) = -\frac{1}{2} \pi^2 t^2 x + 1/6 \pi^4 t^4 x + \frac{1}{30} \left(-(\pi^3)^6 + \frac{1}{12} (\pi^2 - 2\pi^6) \right) t^6 x + \frac{1}{12} \pi^4 t^2 x^3 - \frac{1}{36} \pi^6 t^4 x^3 + \dots$$

5. Results and Discussion

Next, we will show the results from examples 4.1 to– 4.4.

Figure 1 Graphical comparison between the amplitude of the analytical solution and the approximate solution from DTM of the vibration of a membrane.

In Figure 1, at the initial time, the approximate solution (dashed blue lines) obtained from DTM in Example 4.1 is close to the analytical solution (solid orange line) calculated from equation (13). With increasing time, both results continue to lie close to each other.

Figure 2 Graphical result of the amplitude of the approximate solution from DTM of the vibration of a membrane with a damping term.

The graphical results of the approximate solution from DTM of the vibration of a vibrating membrane with a damping term (Example 4.2) are shown in Fig.2. We found that the amplitude of vibration is less than that of the vibrating membrane without a damping term, as shown by the results in Table (2).

The values for the exact solution, approximate solution with a damping term and error for the vibration of membrane are shown in Tables 2 below.

Table 2 Data Value at $x = 0.5$, $y = 0.5$.

Figure 3 Graphical result of the amplitude of the approximate solution from DTM of the vibration of a membrane with an external force.

The graphical results of the approximate solution from DTM of the vibration of a membrane with the external force (Example 4.3) are shown in Fig.3. We found that the amplitude of vibration is less than that of the vibrating membrane without the external force, as shown by the results in Table (3).

Figure 4 Graphical result of the amplitude of the approximate solution from DTM of the vibration of a membrane with a damping term and external force.

The graphical results of the approximate solution from DTM of the vibration a membrane with a damping term and external force (Example 4.4) are shown in Fig.4. We found that the amplitude of vibration is less than that of the vibrating membrane without a damping term and external force as shown by the results in Table (3).

The values of the approximate solution for the vibration of a membrane with the external force, with the external force and a damping term, and error for the vibration of a membrane are shown in Table 3 below.

Table 3 Data Value at $x = 0.5$, $y = 0.5$.

6. Conclusions

We found that the analytical solutions and approximate solutions are very similar in case of the problem with no damping and external force. In the case of the problem with a damping term and external, the analytical solutions are not solvable. Therefore the DTM is used to find the approximate solutions. The obtained results show that the addition of a damping term, and addition of external force in the equation of motion of the membrane, can cause the amplitude of motion for the vibration of a membrane to decrease.

Acknowledgments

This research is supported by Rajamangala University of Technology Tawan-ok :Chantaburi Campus, Thailand.

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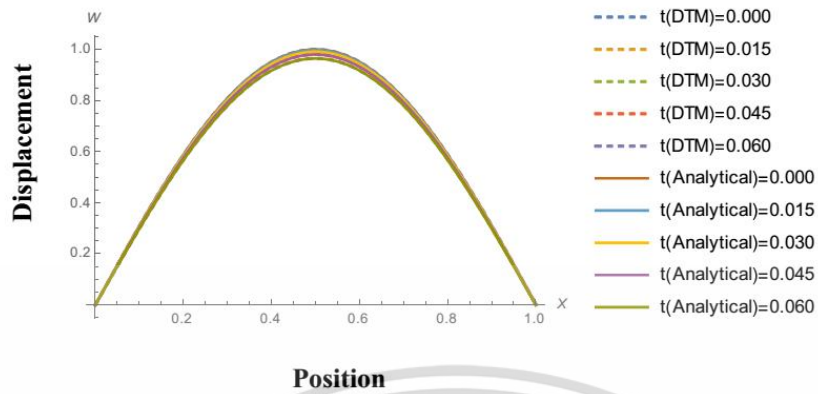


Figure 1 Graphical comparison between the amplitude of the analytical solution and the approximate solution from DTM of the vibration of a membrane.

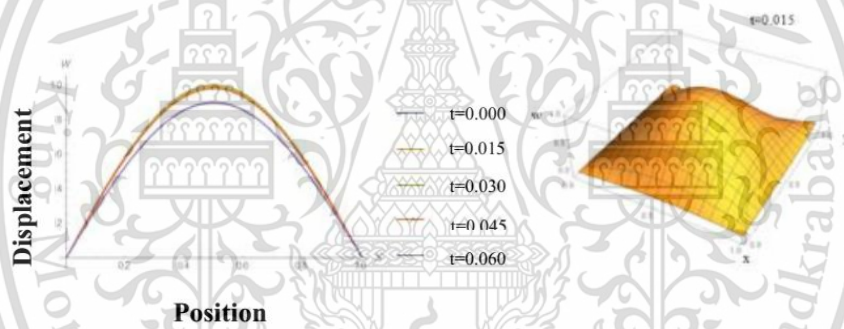


Figure 2 Graphical result of the amplitude of the approximate solution from DTM of the vibration of a membrane with a damping term.

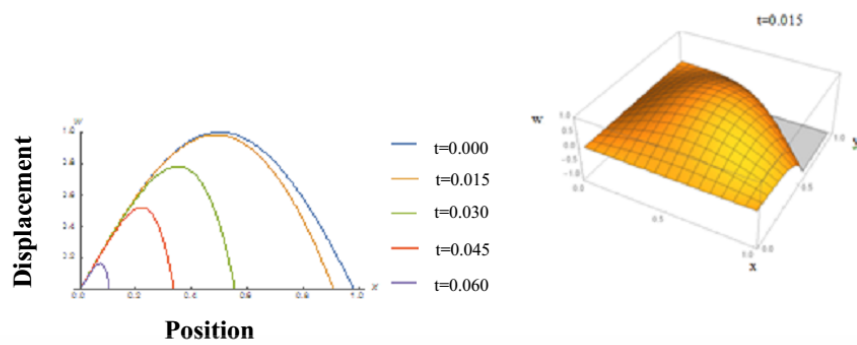


Figure 3 Graphical result of the amplitude of the approximate solution from DTM of the vibration of a membrane with an external force.

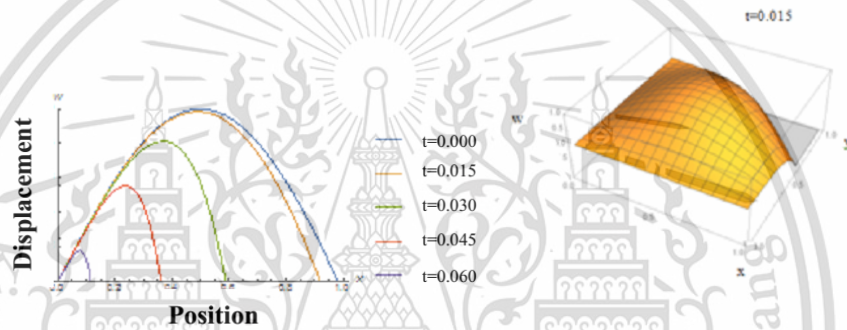


Figure 4 Graphical result of the amplitude of the approximate solution from DTM of the vibration of a membrane with a damping term and external force.

The fundamental operations of three-dimensional differential transform method

Original function	Transformed form
3.1 $v(x, y, t) = a(x, y, t) \frac{\partial^2 w(x, y, t)}{\partial x^2}$	$V(k, h, m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (k-i+2)(k-i+1) A(i, j, r) W(k-i+2, h-j, m-r).$
3.2 $v(x, y, t) = b(x, y, t) \frac{\partial^2 w(x, y, t)}{\partial y^2}$	$V(k, h, m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+2)(h-i+1) B(i, j, r) U(k-i, h-j+2, m-r).$
3.3 $v(x, y, t) = c(x, y, t) \frac{\partial^2 w(x, y, t)}{\partial t^2}$	$V(k, h, m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (m-i+2)(m-i+1) C(i, j, r) W(k-i, h-j, m-r+2).$
3.4 $v(x, y, t) = d(x, y, t) \frac{\partial w(x, y, t)}{\partial t}$	$V(k, h, m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+1) D(i, j, r) D(i, j, r) W(k-i, h-j+1, m-r).$
3.5 $v(x, y, t) = x^n y^l t^s$	$V(k, h, m) = \delta(k-n, h-l, m-s) = \delta((k-n)\delta(h-l)\delta(m-s)),$ where $\delta(k-n) = \begin{cases} 1 & k=n, \\ 0 & k \neq n, \end{cases} \delta(h-l) = \begin{cases} 1 & h=l, \\ 0 & h \neq l, \end{cases} \delta(m-p) = \begin{cases} 1 & m=s, \\ 0 & m \neq s, \end{cases}$
3.6 $v(x, y, t) = p(x, y, t) w^n(x, y, t)$	$V(k, h, m) = \sum_{k_1=0}^k \sum_{k_2=0}^{k-k_1} \sum_{h_1=0}^h \sum_{h_2=0}^{h-h_1} \sum_{m_1=0}^m \sum_{m_2=0}^{m-m_1} \dots \sum_{k_n=0}^{k-k_1-k_2-\dots-k_{n-1}} \sum_{h_n=0}^{h-h_1-h_2-\dots-h_{n-1}} \sum_{m_n=0}^{m-m_1-m_2-\dots-m_{n-1}} W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2}) W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).$
3.7 $v(x, y, t) = q(x, y, t) w^3(x, y, t)$	$V(k, h, m) = \sum_{k_2=0}^k \sum_{k_1=0}^{k-k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h-h_2} \sum_{m_2=0}^m \sum_{m_1=0}^{m-m_2} W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) W_3(k - k_2, h - h_2, m - m_2).$

Table 1 The fundamental operations of DTM

t(sec)	w(exact solution)	w(DTM)	w(damping)	w(exact)- w(DTM)	w(DTM)- w(damping)
$t = 0.000$	1.00000	1.000101	1.0001078	0.000101	0.0007806
$t = 0.015$	0.99113	0.997883	0.9970827	0.000103	0.0008003
$t = 0.030$	0.980081	0.991239	0.9780586	0.000109	0.0131804
$t = 0.045$	0.980081	0.980198	0.9133756	0.000117	0.0668224
$t = 0.060$	0.964679	0.96481	0.7544184	0.000149	0.2103916

Table 2 Data Value at $x = 0.5, y = 0.5$.

t(sec)	w(DTM)	w(external force)	w(damping+external force)	w(DTM)- w(external force)	w(DTM)- w(damping+external force)
$t = 0.000$	1.00000	0.999787	0.999787	0.000313	0.000313
$t = 0.015$	0.99113	0.9810904	0.9834975	0.0167926	0.0143855
$t = 0.030$	0.980081	0.4118883	0.5655184	0.5793507	0.4257206
$t = 0.045$	0.980081	-5.3055571	-3.556495	6.2857551	4.536693
$t = 0.060$	0.902917	-34.028812	-24.203159	34.9936223	25.1679693

Table 3 Data Value at $x = 0.5, y = 0.5$.

Differential Transform Method for the Suspended String Equations

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Abstract. The purpose of this paper is to show the derivation of theorem for the non-linear term transformation of the differential transformation method (DTM) and apply this method to find the approximated solution for the initial boundary value problem of the suspended string vibrating equation. The suspended string with non-uniform densities is solved easily by this method. The nonlinear external force with damping term and without damping term of this problem is also investigated.

Keywords: Suspended String Equation, Differential Transformation Method, Approximated Solution.

1 Introduction

The differential transformation method (DTM) is a very powerful and efficient tool applied in so many research works to solve a partial differential equation (PDE) such as done in [1]-[3]. [3] demonstrated the solving of the second-order PDE with the initial conditions but without boundary condition. The DTM applied to second order PDE under boundary conditions were studied in [4] without initial condition.

The suspended string equation was previously studied using numerical method in [5]-[10]. The numerical solution excluding an external force has been shown in [10] using the Crank-Nicolson method. [8] and [9] have shown the numerical simulation including both linear and nonlinear damping terms yet without an external force. The numerical simulation including an external force has been considered in [10]. However, all of those works considered suspended string only in the uniform density case and defined form of the nonlinear external force including the linear and nonlinear damping cases.

In this work, we apply the proposed the non-linear term transformation to find an approximated solution of the suspended string equation as shown in (1.1) where $u(x, t)$ denotes the horizontal displacement of the heavy and flexible string suspended with the upper end fixed and the lower end free.

$$\begin{aligned} \partial_t^2 u(x, t) - \frac{x}{m+1} \partial_x^2 u(x, t) - \partial_x u(x, t) + f(x, u(x, t)) &= 0, \quad (x, t) \in \Omega, \\ u(a, t) &= 0, \quad t \in [0, T], \\ u(x, 0) = \phi(x), \quad \partial_t u(x, 0) &= \psi(x), \quad x \in [0, a]. \end{aligned} \tag{1.1}$$

Where Ω is a cylindrical domain $(0, a) \times [0, T]$ and the power density are denoted by $\rho(x) = \alpha x^m$ where α and m are constants.

2. Two-Dimensional Differential transform method

The basic definitions and fundamental operations of differential transform method are defined in as followed.

Definition 2.1[3] The two-dimensional differential transform of function $w(x, y)$ is defined as

$$W(k, h) = \frac{1}{k!h!} \left. \frac{\partial^{k+h} w(x, y)}{\partial x^k \partial y^h} \right|_{(0,0)}, \quad k \geq 0 \quad \text{and} \quad h \geq 0, \tag{2.1}$$

Definition 2.2[3] The inverse two-dimensional differential transform of sequence $\{W(k, h)\}_{k, h=0}^{\infty}$ is defined as

$$w(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k, h) x^k y^h. \tag{2.2}$$

3. The fundamental operations on DTM [3]

a general terms of PDEs	Transformed form
3.1 $v(x, t) = a(x, t) \frac{\partial^2 u(x, t)}{\partial x^2}$.	$V(k, h) = \sum_{i=0}^k \sum_{j=0}^h (k-i+2)(k-i+1)A(i, j)U(k-i+2, h-j)$.

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Article History: Received: 15 Apr. 2018 Revised: 1 May.2018 Accepted:20 May 2018

3.2	$v(x,t) = b(x,t) \frac{\partial^2 u(x,t)}{\partial x \partial t}$	$V(k,h) = \sum_{i=0}^k \sum_{j=0}^h (k-i+1)(h-i+1)B(i,j)U(k-i+1,h-j+1)$
3.3	$v(x,t) = c(x,t) \frac{\partial^2 u(x,t)}{\partial t^2}$	$V(k,h) = \sum_{i=0}^k \sum_{j=0}^h (h-i+2)(h-i+1)C(i,j)U(k-i,h-j+2)$
3.4	$v(x,t) = d(x,t) \frac{\partial u(x,t)}{\partial x}$	$V(k,h) = \sum_{i=0}^k \sum_{j=0}^h (k-i+1)D(i,j)U(k-i+1,h-j)$
3.5	$v(x,t) = e(x,t) \frac{\partial u(x,t)}{\partial t}$	$V(k,h) = \sum_{i=0}^k \sum_{j=0}^h (h-i+1)E(i,j)U(k-i,h-j+1)$
3.6	$v(x,t) = x^m t^n$	$V(k,h) = \delta(k-m,h-n) = \delta(k-m)\delta(h-n)$, where $\delta(k-m) = \begin{cases} 1, & k=m \\ 0, & k \neq m \end{cases}$ and $\delta(h-n) = \begin{cases} 1, & h=n \\ 0, & h \neq n \end{cases}$
3.7	$v(x,t) = u^n(x,t)$	$V(k,h) = \sum_{k_1=0}^k \sum_{k_2=0}^{k-k_1} \sum_{h_1=0}^h \sum_{h_2=0}^{h-h_1} \dots \sum_{k_{n-1}=0}^{k-k_1-k_2-\dots-k_{n-2}} \sum_{h_{n-1}=0}^{h-h_1-h_2-\dots-h_{n-2}} U_1(k_1,h_1) \times U_2(k_2-h_2-h_1) \dots \times U_{n-1}(k_{n-1}-h_{n-1}-h_{n-2}-\dots-h_{n-2})U_n(k-k_{n-1},h-h_{n-1})$
3.8	$v(x,t) = u^2(x,t)$	$V(k,h) = \sum_{k_1=0}^k \sum_{k_2=0}^{k-k_1} \sum_{h_1=0}^h \sum_{h_2=0}^{h-h_1} U_1(k_1,h_1)U_2(k-k_1,h_2-h_1)U_3(k-k_2,h-h_2)$

Table 1: The fundamental operations on DTM

Theorem 3.1-3.5, 3.6 are derived and shown in [3] and [1], respectively.

The following is the derivation of the theorem 3.7.

If $v(x,t) = u_1(x,t)u_2(x,t) \dots u_{n-1}(x,t)u_n(x,t)$ then

$$V(k,h) = \sum_{k_1=0}^k \sum_{k_2=0}^{k-k_1} \sum_{k_3=0}^{k-k_1-k_2} \dots \sum_{k_{n-1}=0}^{k-k_1-k_2-\dots-k_{n-2}} \sum_{h_1=0}^h \sum_{h_2=0}^{h-h_1} \dots \sum_{h_{n-1}=0}^{h-h_1-h_2-\dots-h_{n-2}} U_1(k_1,h_1)U_2(k_2-h_2-h_1) \dots \times U_{n-1}(k_{n-1}-h_{n-1}-h_{n-2}-\dots-h_{n-2})U_n(k-k_{n-1},h-h_{n-1})$$

For $V: \mathbb{I}^+ \rightarrow \mathbb{R}$, $U_n: \mathbb{I}^+ \rightarrow \mathbb{R}$, $n \in \mathbb{I}^+$, $v: \mathbb{I} \rightarrow \mathbb{R}$, $u_n: \mathbb{I} \rightarrow \mathbb{R}$, $n \in \mathbb{I}^+$, $x \in \mathbb{R}$

and

By $k = 0, 1, 2, \dots$ definition of two-dimensional differential transform, we have

$$V(k,h) = \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} v(x,t) \right]_{(0,0)}$$

obtain

$$v(x,t) = u_1(x,t)u_2(x,t) \dots u_{n-1}(x,t)u_n(x,t)$$

using

definition(2.1), we have

For

$$\begin{aligned} V(k,h) &= \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} u_1(x,t)u_2(x,t) \dots u_{n-1}(x,t)u_n(x,t) \right]_{(0,0)} \quad \text{we have} \\ V(k,h) &= \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} u_1(x,t)u_2(x,t) \dots u_{n-1}(x,t)u_n(x,t) \right]_{(0,0)} \\ &= \frac{1}{k!h!} \sum_{k_1=0}^{k+1} \sum_{k_2=0}^{k-k_1} \frac{k!}{(k-k_1)!} \frac{h!}{(h-k_1)!} \\ &\quad \times \left[\frac{\partial^{k_1+h_1}}{\partial x^{k_1} \partial t^{h_1}} u_1(x,t)u_2(x,t) \dots u_{n-1}(x,t) \frac{\partial^{k-k_1+h-h_1}}{\partial x^{k-k_1} \partial t^{h-h_1}} u_n(x,t) \right]_{(0,0)} \\ &= \sum_{k_1=0}^k \sum_{k_2=0}^{k-k_1} \left[\frac{1}{(k_1-1)!(h_1-1)!} \frac{\partial^{k_1+h_1}}{\partial x^{k_1} \partial t^{h_1}} u_1(x,t)u_2(x,t) \dots u_{n-1}(x,t) \right]_{(0,0)} \\ &\quad \times \left[\frac{1}{(k-k_1)!(h-h_1)!} \frac{\partial^{k-k_1+h-h_1}}{\partial x^{k-k_1} \partial t^{h-h_1}} u_n(x,t) \right]_{(0,0)} \\ &= \sum_{k_1=0}^k \sum_{k_2=0}^{k-k_1} \sum_{k_3=0}^{k-k_1-k_2} \left[\frac{1}{(k_1-1)!(h_1-1)!} \frac{\partial^{k_1+h_1}}{\partial x^{k_1} \partial t^{h_1}} u_1(x,t)u_2(x,t) \dots u_{n-2}(x,t) \right]_{(0,0)} \\ &\quad \times \left[\frac{1}{(k_{n-1}-k_{n-2})!(h_{n-1}-h_{n-2})!} \frac{\partial^{k_{n-1}+h_{n-1}}}{\partial x^{k_{n-1}} \partial t^{h_{n-1}}} u_{n-1}(x,t) \right]_{(0,0)} \\ &\quad \times \left[\frac{1}{(k-k_{n-1})!(h-h_{n-1})!} \frac{\partial^{k-k_{n-1}+h-h_{n-1}}}{\partial x^{k-k_{n-1}} \partial t^{h-h_{n-1}}} u_n(x,t) \right]_{(0,0)} \end{aligned}$$

$$V(k, h) = \sum_{k_1=0}^k \sum_{k_2=0}^{k-k_1} \sum_{k_3=0}^{k-k_1-k_2} \dots \sum_{k_n=0}^{k-k_1-k_2-\dots-k_{n-1}} U_1(k_1, h_1) U_2(k_2, h_2 - h_1) \dots U_n(k - k_{n-1}, h - h_{n-1}).$$

Since $v(x, t) = u_1(x, t) u_2(x, t) \dots u_n(x, t)$, then $V(k, h) = \sum_{k_1=0}^k \sum_{k_2=0}^{k-k_1} \sum_{k_3=0}^{k-k_1-k_2} \dots \sum_{k_n=0}^{k-k_1-k_2-\dots-k_{n-1}} U_1(k_1, h_1) U_2(k_2, h_2 - h_1) \dots U_n(k - k_{n-1}, h - h_{n-1}).$

4. Application

In this section, two examples of the suspended string equations will be presented and used to demonstrate how DTM is applied to find approximated solutions.

Example 1 Consider Suspended string equation with external force

$$\frac{\partial^2 u(x, t)}{\partial t^2} - x \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{\partial u(x, t)}{\partial x} + u^3 = 0. \tag{4.1}$$

with the initial and boundary conditions,

$$u(x, 0) = \sin x, \quad x \in \mathbb{R},$$

$$u(1, t) = 0, \quad \text{and} \quad \frac{\partial u}{\partial x}(0, t) = 0, \quad t \in \mathbb{R}. \tag{4.2}$$

By applying DTM to the given suspended string equation, we have

$$(h+2)(h+1)U(k, h+2) = (k+1)^2 U(k+1, h) - \sum_{k_1=0}^k \sum_{k_2=0}^{k-k_1} \sum_{k_3=0}^{k-k_1-k_2} U_1(k_1, h_1) \times U_2(k_2, h_2 - h_1) U_3(k - k_3, h - h_2) \tag{4.3}$$

Comparing between the coefficient of the Taylor's series of sine and the series in the definition (2.2)

Then, we have

$$U(k, 0) = \begin{cases} 0 & , \text{if } k = 0, 2, 4, \dots \\ \frac{1}{k!} & , \text{if } k = 1, 5, 9, \dots \\ \frac{(-1)^k}{k!} & , \text{if } k = 3, 7, 11, \dots \end{cases} \tag{4.4}$$

and from the initial condition (4.2), we have

$$U(k, 1) = 0, \quad k = 0, 1, 2, \dots \tag{4.5}$$

From the boundary condition (4.2), we have

$$U(0, h) = 0, \quad h = 0, 1, 2, \dots \tag{4.6}$$

For each k and h using (4.4) - (4.6) and by recursive method of (4.3), we obtain the coefficients $U(k, h)$ for the series solution. That is

$$u(x, t) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + 0.5xt^2 - 0.75x^2t^2 + 0.0833x^3t^2 + 0.1042x^4t^2 + \dots$$

Example 2

Consider Suspended string equation non uniform $\left(m = \frac{1}{2}\right)$

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \frac{x}{1/2} \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{\partial u(x, t)}{\partial x} = 0. \tag{4.7}$$

By applying DTM to the given suspended string equation, we have

$$(h+2)(h+1)U(k, h+2) = (k+1)(2k+1)U(k+1, h). \tag{4.8}$$

For each k and h using (4.4) - (4.6) and by recursive method of (4.8), we obtain the coefficients $U(k, h)$ for the series solution. That is

$$u(x, t) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots - 1.25x^2t^2 + 0.1875x^4t^2 - 0.00903x^6t^2 + \dots$$

5. Results and Discussion

The derivation of the differential transform for non-linear term is presented. The proof of the theorem is also derived. The theorem can be applied in several applications. The non-linear term of differential transform can be applied efficiently to the nonlinear terms of suspended string equation. This work demonstrated that the

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Article History: Received: 15 Apr. 2018 Revised: 1 May.2018 Accepted:20 May 2018

differential transformation method can be applied to the initial and boundary conditions problem of partial differential equation.

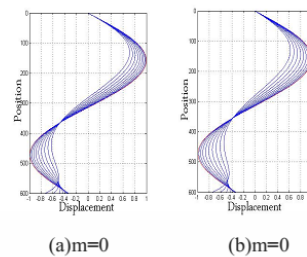


Figure 1 Graphical comparison between the vibration displacements between the without the external force (a) and with the external force(b).

The comparisons result of the suspended string vibration without external force (a) and with external force (b) are shown in Fig.1. We found that the amplitude of vibration with external force (b) is less than that of vibration without external force (a).

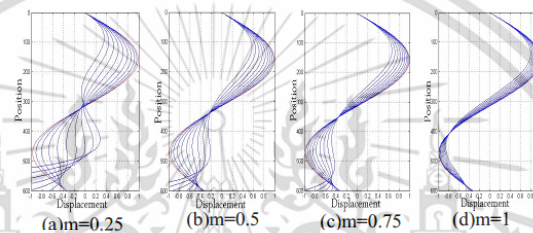


Figure 3 Graphical comparison between the vibration displacements with various non-uniform densities ($m=0.25, 0.5, 0.75, 1$).

The comparisons result of the suspended string vibration with the different values of m (i.e. $m=0.25, 0.5, 0.75, 1$) are shown in Fig.3. We found that the shape of vibration (blue line) approach to initial shape of vibration (red line) when the density and time increase.

6. Conclusions

We found that the amplitude of vibration with external force decreases when comparisons result of the suspended string vibration without the external force. The obtained results show that the vibration characteristics with non-uniform densities for different values of m , the shape of vibration approach to initial shape of vibration when the density and time increase.

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*Corresponding Author: K. Mansilp

Article History: Received: 15 Apr. 2018 Revised: 1 May.2018 Accepted:20 May 2018

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The Modified Differential Transformation Method of the Nonlinear Function

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Abstract. The propose of this work is to present the modified differential transformation method for the nonlinear functions based on Laplace table. The derivation of the differential transformed recurrence formulas are shown and applied to find the coefficients of the Taylor's series which are the approximant for the nonlinear functions. We found that the comparison results between the approximate Taylor's series and the nonlinear function are the same. The program are also implemented in this work.

Keywords: Differential Transform, Nonlinear Function, Laplace Transform Table.

Introduction

The differential transformation method (DTM) [1-2] is a very powerful and efficient tool applied in so many researches works to solve both linear and nonlinear ordinary differential equations (ODEs) [3-6], occurring in many areas of science and engineering such as vibration problems [8], prediction control problems [9-10] and steady nonlinear heat conduction [11] problems.

A new algorithm for calculating one-dimensional differential transform of nonlinear functions was first presented in [12] applying the Taylor's series about the origin (Maclaurin series) to approximate the nonlinear functions in the basic function forms such as exponential function, logarithm function, the trigonometric function and the hyperbolic trigonometric function. However we tried to apply the Maclaurin series to approximate the nonlinear functions based on Laplace Table, we found that the approximate Maclaurin series were not in good agreement with the nonlinear functions. In this work, we should study further from the previous work [12] to modify the differential transformation method to find the approximate Taylor's series about any points for the nonlinear functions based on Laplace Table.

The differential transform method is based on the coefficients of Taylor's series to find the transformation of the original function $u(t)$ at a point t_0 .

We recall that the Taylor series of $u(t)$ at a point t_0 is

$$u(t) = \sum_{k=0}^{\infty} \frac{u^{(k)}(t)}{k!} \Big|_{t=t_0} (t-t_0)^k, \text{ where}$$

$$\text{the transform function of } u(t) \text{ is defined by } U(k) = \frac{u^{(k)}(t)}{k!} \Big|_{t=t_0},$$

$$\text{Therefore, we can rewrite the Taylor series as } u(t) = \sum_{k=0}^{\infty} U(k) \cdot (t-t_0)^k.$$

Normally, we use $t_0 = 0$ for simplicity. However in this work, we need to find the differential transformation for any point t_0 .

In this paper, we study the modified differential transformation for the nonlinear functions base on Laplace table. The coefficients of the Taylor series are obtained by the recurrence differential transform formulas and calculated by computational work.

2. Differential transform method[1-2]

Definition 2.1 If $f(t)$ is analytic in time domain T , then it will be differential continuously with respect to time t .

For $t = t_0$ and k belong to the set of non-negative integers, denotes as K-domain.

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Article History: Received: 15 Apr. 2018 Revised: 1 May.2018 Accepted:20 May 2018

$$F(k) = \frac{1}{k!} \left[\frac{d^k}{dt^k} f(t) \right]_{t=t_0}, \forall k \in K, \tag{2.1}$$

where $f(t)$ is the original function and $F(k)$ is the transform function

Definition 2.2 The differential inverse transform of $F(k)$ is defined as

$$f(t) = \sum_{k=0}^{\infty} F(k)(t - t_0)^k. \tag{2.2}$$

In principal application, The function $f(t)$ is shown by finite numbers of terms and can be written as

$$f(t) = \sum_{k=0}^N F(k)(t - t_0)^k. \tag{2.3}$$

The fundamental mathematical operations are performed by differential transform method listed in Table 1.

3. The derivation of the modified differential transform method for the nonlinear function.

3.1 Considering $f_1(t) = e^{at}$ and from definition (2.1), when $k = 0$

differentiating $f_1(t_0) = e^{at_0}$ with respect to t ,

$$\frac{df_1(t_0)}{dt} = ae^{at_0} = af_1(t_0). \tag{3.1}$$

By the differential transform method, Eq. (3.1) became

$$(k + 1)F_1(k + 1) = aF_1(k).$$

Substituting $k + 1$ by k , we obtain

$$kF_1(k) = aF_1(k - 1) \\ F_1(k) = \frac{aF_1(k - 1)}{k}, \quad k \geq 1.$$

The recurrence differential formulas of $f_1(t_0) = e^{at_0}$ are obtained as follows

$$F_1(k) = \begin{cases} F_1[0] = e^{at_0}, & k = 0 \\ \frac{aF_1(0)}{k!}, & k \geq 1 \end{cases} \tag{3.2}$$

3.2. Considering $f_2(t) = e^{at} \sin \omega t$ and $h(t) = e^{at} \sin \omega t$ from definition (2.1), when $k = 0$ differentiating

$f_2(t) = e^{at} \cos \omega t$ and $h(t) = e^{at} \sin \omega t$ with respect to t ,

$$\frac{df_2(t)}{dt} = -\omega(e^{at} \sin \omega t) + a(e^{at} \cos \omega t) = -\omega h(t) + af_2(t) \\ \frac{dh(t)}{dt} = \omega(e^{at} \cos \omega t) + a(e^{at} \sin \omega t) = \omega f_2(t) + ah(t). \tag{3.3}$$

The recurrence differential formulas of $f_2(t) = e^{at} \cos \omega t$ are obtained as follows

$$F_2(k) = \begin{cases} e^{at_0} \cos \omega t_0, & k = 0 \\ \frac{-\omega H(0) + aF_2(0)}{k}, & k \geq 1 \end{cases} \tag{3.4}$$

and

$$H(k) = \begin{cases} e^{at_0} \sin \omega t_0, & k = 0 \\ \frac{\omega F_2(0) + aH(0)}{k}, & k \geq 1 \end{cases} \tag{3.5}$$

4. Results and Discussion.

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Article History: Received: 15 Apr. 2018 Revised: 1 May.2018 Accepted:20 May 2018

In this section, we will show the comparison results obtained from the recurrence relation of the differential transformation in the previous paper [12] and modified recurrence relation of the differential transformation in this work.

The following recursive formulas of the differential transformation of $f_1(t) = e^{at}$ is obtained by definition 2.1 when $t_0 = 0$ as applied in [12]

$$F_1(k) = \begin{cases} 1, & k = 0 \\ \frac{aF_1(k-1)}{k}, & k \geq 1 \end{cases} \quad (4.1)$$

The Eq. (4.1) is the recurrence relation for finding the coefficient of the approximate Maclaurin series of the original nonlinear function $f_1(t) = e^{at}$.

By the similar way to Eq. (4.1), the following recursive formulas of the differential transformation of $f_2(t) = e^{at} \cos \omega t$ is obtained by definition 2.1 when $t_0 = 0$ as applied in [12]

$$F_2(k) = \begin{cases} 1, & k = 1 \\ \frac{-\omega H(k-1) + aF_2(k-1)}{k}, & k \geq 1 \end{cases} \quad \text{and recursive}$$

method of T-function for $h(t) = e^{at} \sin \omega t$, we have

$$H(k) = \begin{cases} 0, & k = 0 \\ \frac{\omega F_2(k-1) + aH(k-1)}{k}, & k \geq 1 \end{cases} \quad (4.2)$$

The graphical comparison results between the original functions and the approximate Taylor's series with the coefficients obtained from the modified differential transformation method as the recurrence relations of $f_1(t) = e^{at}$ in Eq. (3.2) and the recurrence relations $f_2(t) = e^{at} \cos \omega t$ in Eq. (3.3) are shown in Fig 1 and 2, respectively.

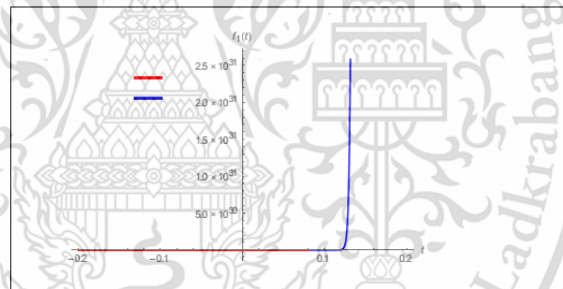


Fig. 1. The graphical comparison results between $f_1(t) = e^{at}$ and $f_1(t) = 1t^0 + 550t^1 + 151250t^2 + \frac{83187500}{3}t^3 + \dots + 1.3 \times 10^{236}t^{600}$ using *DTM* when $a = 550, k = 600$ and $t \in [-0.2, 0.2]$.

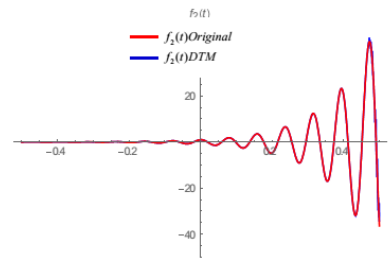


Fig. 2. The graphical comparison results between $f_2(t) = e^{at} \sin \omega t$ and

$$f_2(t) = 1t^0 + 8t^1 - 3168t^2 - \frac{76544}{3}t^3 + \frac{4813312}{3}t^4 + \dots - 1.1 \times 10^{-265}t^{600} \text{ using DTM when } a=8, \omega=80, k=600 \text{ and } t \in [-0.5, 0.5].$$

4. Conclusion

The modified recursive formulas of differential transformation method are presented in this work. The obtained formulas are used to find the coefficients of the Taylor's series applied to approximate the nonlinear functions based on the Laplace table. The comparison results between the approximate Taylor's series and the original function are the same. These obtained formulas can be used to solve differential transform problem in the future works.

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Article History: Received: 15 Apr. 2018 Revised: 1 May. 2018 Accepted: 20 May 2018

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Academic Publications	<ol style="list-style-type: none"> 1. Kamonpad Masilp and Jaipong Kasemsuwan. 2018 “Differential Transformation Method for Vibration of Membrane” <i>Songklanakarin Journal of Science and Technology</i> : 20 pages (accepted) 2. K. Mansilp and J. Kasemsuwan. 2018 “Differential Transform Method for the Suspended String Equations” <i>Journal of Advanced Research in Dynamical and Control Systems</i>, Vol.10, 06-Special Issue: 463 – 467 3. K. Mansilp and J. Kasemsuwan. 2018 “The Modified Differential Transformation Method of the Nonlinear Function” <i>Journal of Advanced Research in Dynamical and Control Systems</i>, Vol.10, 06-Special Issue: 468 – 471