CONVERGENCE THEOREMS FOR A NEW TYPE OF VARIATIONAL INCLUSION PROBLEMS IN A HILBERT SPACE

WONGVISARUT KHUANGSATUNG

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
(APPLIED MATHEMATICS)

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

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Thesis Title	Convergence Theorems for a New Type of Variational Inclusion Problems in a Hilbert Space
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Abstract

The purposes of this thesis is first to modify variational inclusion problems and prove a strong convergence theorem for approximating a common element of two solution sets of variational inclusion problems, the solution set of variational inequality problem and the set of fixed points of a nonexpansive mapping with some new sufficient conditions in a framework of Hilbert space. Secondly, we introduce a new iterative method for finding a common element of the set of fixed points of a κ-strictly pseudononspreading mapping, the set of solutions of a finite family of variational inclusion problems and the set of solutions of a finite family of equilibrium problems in Hilbert space. Then, we prove a strong convergence theorem of the proposed iterative scheme. Thirdly, we prove a strong convergence theorem of a new iterative algorithm for finding a common element of the set of fixed points of a finite family of nonspreading mappings, the set of solutions of a finite family of equilibrium problems and the two sets of solutions of variational inequality problems in a real Hilbert space. The results obtained in this thesis extend and improve the corresponding results existed in the literature. Finally, we give the numerical examples to support some of our results.

Keywords : Variational inclusion problem, Strictly pseudononspreading mapping, Equilibrium problem, Fixed point problem

หัวข้อวิทยานิพนธ์	ทฤษฎีบทการลู่เข้าสำหรับปัญหาการแปรผันอินคลูชั่นแบบใหม่ในปริภูมิฮิลเบิร์ต
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บทคัดย่อ

จุดประสงค์ของวิทยานิพนธ์นี้คือ อย่างแรกเป็นการแนะนำปัญหาการแปรผันอินคลูชั่นแบบใหม่ และทำการพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มสำหรับการประมาณค่าสมาชิกร่วมของผลเฉลยของปัญหาการแปรผันอินคลูชั่นสองเซต เซตผลเฉลยของปัญหาอสมการการแปรผัน และเซตจุดตรึงของการส่งแบบไม่ขยายซึ่งใช้เงื่อนไขใหม่ที่พอเพียงในปริภูมิฮิลเบิร์ต อย่างที่สอง ผู้วิจัยได้สร้างกระบวนการทำซ้ำแบบใหม่สำหรับการหาค่าสมาชิกร่วมของเซตจุดตรึงของการส่งแบบไม่กระจายเทียมโดยแท้ เซตผลเฉลยของวงศ์จำกัดของปัญหาการแปรผันอินคลูชั่นและเซตผลเฉลยของวงศ์จำกัดของปัญหาดุลยภาพในปริภูมิฮิลเบิร์ต จากนั้น ผู้วิจัยได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มของกระบวนการทำซ้ำดังกล่าว อย่างที่สาม ผู้วิจัยได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้ม สำหรับการหาค่าสมาชิกร่วมของเซตจุงตรึงของวงศ์จำกัดของการส่งแบบไม่กระจาย เซตผลเฉลยของวงศ์จำกัดของปัญหาดุลยภาพ และเซตผลเฉลยของปัญหาอสมการการแปรผันสองเซตในปริภูมิฮิลเบิร์ต ผลลัพธ์ที่ได้ในวิทยานิพนธ์นี้ได้ขยายและพัฒนาผลลัพธ์ที่เกี่ยวข้องที่มีอยู่ในงานวิจัยต่างๆ ท้ายที่สุด ผู้วิจัยยังได้ให้ตัวอย่างเชิงตัวเลขเพื่อสนับสนุนทฤษฎีบทที่ได้

คำสำคัญ : ปัญหาการแปรผันอินคลูชั่น การส่งแบบไม่กระจายเทียมโดยแท้ ปัญหาดุลยภาพ ปัญหาจุดตรึง

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Wongvisarut Khuangsatung

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Chapter 1

Introduction

The fixed point theory plays an important role in nonlinear functional analysis and becomes a very useful tool in various fields. In the past few decades, fixed point theorem has been applied in many branches such as mathematics, engineering, economics, optimization, physics, and computer sciences, etc. For an interesting example, fixed point theorems are utilized to the split feasibility problem, which was introduced by Censor and Elfving [1]. Split feasibility problem has been used to model important real world problems in signal processing and image reconstruction, and especially applied in medical fields such as intensity-modulated radiation therapy (IMRT) (see, *e.g.* [2, 3]). Fixed point Theory are also applied to the optical effects in multi-layer structures in order to ensure the convergence of the iterative algorithm [4]. In applications to neural networks, fixed point theorems can be used to design a dynamic neural network in order to solve steady state solutions [6].

Let X be a nonempty set and let $T : X \to X$ be a mapping. We say that $x \in X$ is a fixed point of T if $Tx = x$. The set of all fixed points of T is denoted by $F(T)$. The study of fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the mapping $T : X \to X$, in order to obtain results on:

1) the existence and the uniqueness of fixed points;

2) the structure of fixed point sets;

3) the approximation of fixed points.

For proving the existence and uniqueness of fixed point theory, the Banach contraction mapping principle, which was firstly introduced by Banach [5] in 1992, has been an important source in fixed point theory as follows:

Theorem 1.1. (**The Banach Contraction Mapping Principle**). Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction, that is, there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in X.$$

Then T has a unique fixed point $z \in X$. Moreover, for each $x \in X$, the sequence $\{T^n x\}$ converges strongly to z

It is well-known that the Banach's contraction mapping principle is the basis theorem of the research related to fixed point problem. This theorem is simple and useful, so it has become the most significant tool in solving many existence problems in various fields of mathematical analysis. Banach's contraction mapping principle has been applied in many branches such as digital images [7].

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

Many mathematicians have been studying to construct iterative schemes and conditions to guarantee the existence and the uniqueness of fixed point theory for nonexpansive mappings, see, for instance [9, 10], and references therein.

Let $A : C \to H$ be a nonlinear mapping. The *variational inequality problem* is to find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \tag{1.1}$$

for all $y \in C$. The set of solutions of the (1.1) is denoted by $VI(C, A) = \{x \in C : \langle Ax, y - x\rangle \geq 0, \forall y \in C\}$. Variational inequality problems were introduced and investigated by Stampacchia [8] in 1964. The applications of this problem have been expanded to problems from economics, finance, optimization and game theory, etc. Some methods have been proposed to solve the variational inequality problem ; see, for example, [11, 12, 13] and the references therein.

In 2003, Takahashi and Toyoda [14] introduced an iterative scheme for finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequalities problem in a Hilbert space as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)TP_C(I - \lambda_n A)x_n, \forall n \geq 0,$$

and proved weak convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$.

In 2005, Iiduka and Takahashi [15] introduced an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem in a Hilbert space as follows:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)TP_C(I - \lambda_n A)x_n, \forall n \geq 0,$$

and proved strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$.

In 2006, Marino and Xu [16] modified the viscosity approximation method which was first introduced by Moudafi [17]. They introduced the new algorithm called the general iterative method as follows:

$$x_{n+1} = (I - \alpha_n A)\, Tx_n + \alpha_n \gamma f(x_n), n \geq 0, \tag{1.2}$$

where T is a nonexpansive mapping, f is a contractive mapping on H, and $\{\alpha_n\}$ is a sequence in (0,1). Then, they proved that the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, x - \tilde{x} \rangle \geq 0, x \in F(T).$$

By using the concept of Banach's contraction mapping principle, Kangtunyakarn [18] proved a strong convergence theorem of the sequence $\{x_n\}$ for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problem without assumptions on the set of fixed points of a nonexpansive mapping and the set of variational inequality as the following iteration.

$$x_{n+1} = \alpha T x_n + (1 - \alpha) P_C (I - \rho A) x_n, \forall n \geq 0.$$

Now, we will introduce one of the famous problems in many fields of pure and applied sciences, that is the Equilibrium problem. Equilibrium problem can be applied to many problems in physics, finance, network, optimization problems, variational inequality problems and Nash equilibrium problem as special cases, see [22, 23]. Let $F : C \times C \to \mathbb{R}$ be a bifunction where $\mathbb{R}$ is the set of real numbers. The equilibrium problem of F is to find a point x such that

$$F(x, y) \geq 0, \forall y \in C. \tag{1.3}$$

The set of solution of equilibrium problem is denoted by $EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}$. Given a mapping $A : C \to H$, let $F(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Az, y - z \rangle \geq 0$ for all $y \in C$, That is, z is a solution of the variational inequality. Many authors have studied fixed point theorem for the equilibrium problem; see, for example [24, 25, 26] and the references therein.

In 2005, Combettes and Hirstoaga [23] introduced an iterative scheme for finding the best approximation to the initial data when the set of solution of equilibrium problem is nonempty and proved a strong convergence theorem. By using the viscosity approximation method, Takahashi and Takahashi [25] proved a strong convergence theorem for finding a common element of the set of solution of equilibrium problem and and the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x \rangle \geq 0, \forall y \in C \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset [0, 1]$ satisfy some suitable conditions.

Over the past few decades, many authors have been trying to construct new iterations to prove strong convergence theorems for finding a common element of the set of fixed point problem and the solution set of equilibrium problem and the solution set of variational inequality problem; see, for example, [19, 20], and the references therein.

In 2013, by using the concept of variational inequality problems, Kangtunyakarn [21] modified variational inequality problems and proved a strong convergence theorem for finding a common element of the fixed point sets of nonlinear mappings and the set of equilibrium problem and two sets of variational inequality problems as follows:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) S_n P_C(I - \gamma(aA + (1-a)B)), \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, γ and $\{r_n\}$ satisfy some suitable conditions.

The problem for finding a common element of a finite family of the set of solutions of equilibrium problem and the solution sets of various nonlinear operator problems in Hilbert spaces has been investigated by many authors; see, for example, [27, 28] and the references therein. In 2013, Suwannaut and Kangtunyakarn [29] modified the set $EP(F)$ as follows:

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \left\{x \in C : \left(\sum_{i=1}^{N} a_i F_i\right)(x, y) \geq 0, \forall y \in C\right\}, \tag{1.4}$$

where $F_i : C \times C \to \mathbb{R}$ is a bifunction and $a_i > 0$ with $\sum_{i=1}^{N} a_i = 1$, for every $i = 1, 2, \ldots, N$. It is obvious that (1.4) reduces to (1.3), if $F_i = F$, for all $i = 1, 2, \ldots, N$.

They also introduced an iterative method for finding a common element of the set of fixed points of an infinite family of nonlinear mappings and the solution sets of a finite family of equilibrium problems and variational inequalities problems as follows:

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n) S_n x_n) + (1-\beta_n) P_C\left(I - \rho_n \sum_{i=1}^{N} b_i A_i\right) u_n, \forall n \geq 1. \end{cases} \tag{1.5}$$

Under some appropriate conditions, they proved a strong convergence theorem of the sequence $\{x_n\}$ generated by (1.5).

Variational inclusion problem, which was introduced and studied by Noor and Noor [30], has a great impact and influence in the classes of mathematical problems and widely studied in many fields of pure and applied sciences such as mathematical programming, complementarity problems, optimal control, mathematical economics and game theory, etc. variational inclusion is a useful and significant generalization of the classical variational principles that include variational, quasi-variational, variational-like inequalities as special cases. Let $B : H \to H$ be a mapping and $M : H \to 2^H$ be a multi-valued mapping. The *variational inclusion problem* is to find $u \in H$ such that

$$\theta \in Bu + Mu, \tag{1.6}$$

where θ is a zero vector in H. The set of the solution of (1.6) is denoted by $VI(H, B, M)$. Many research papers have increasingly investigated this problem, see, for instance, [31, 32, 33, 34] and references therein.

In 2008, Zhang *et al.*[34] introduced an iterative scheme for finding a common element of the set of solutions of the variational inclusion problem with multi-valued maximal monotone mapping and inverse strongly monotone mappings and the set of fixed points of nonexpansive mappings in Hilbert space. They introduced the iterative scheme as follows:

$$\begin{cases} y_n = J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) S y_n, \forall n \geq 0, \end{cases}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable condition of parameters $\{\alpha_n\}$ and λ.

In 2008, Peng *et al.*[35] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the variational inclusion with a multi-valued maximal monotone mapping and an inverse strongly monotone mapping, the set of solutions of equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x \rangle \geq 0, \forall y \in C \\ y_n = J_{M,\lambda}(u_n - \lambda A u_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S y_n, \forall n \geq 0, \end{cases}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under control condition of parameters $\{\alpha_n\}$, $\{r_n\}$ and λ.

In the past few decades, approximation method for finding fixed points of nonexpansive mappings in Hilbert space have been widely studied by many researchers. Moreover, the fixed point theory for various mappings such as nonspreading mapping and κ-strictly pseudononspreading mapping is also interesting. In 2008, Kohsaka and Takahashi [37] introduced the *nonspreading* mapping in Hilbert space H. The classes of nonexpansive mappings and nonspreading mappings in a Hilbert space are deduced from the class of firmly nonexpansive mappings (see, [36]). In 2010, Plubtieng and Chornphrom [39] proved a strong convergence theorem for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of a nonspreading mapping in a Hilbert space. In the recent years, many mathematicians have been studying and concerning the problem of approximating a common fixed point of the set of solutions of a finite family of nonspreading mappings by using the W-mapping [41], the K-mapping[43] and the S-mapping [42]; see, for example [44, 45, 46].

For the generalization of the nonspreading mapping, in 2011 Osilike and Isiogugu

[40] introduced κ-strictly pseudononspreading mapping. It is well-known that every nonspreading mapping is a κ-strictly pseudononspreading mapping. In the past few years, many authors have been trying to construct new iterative schemes to prove strong convergence theorems for finding a common element of κ-strictly pseudononspreading mapping and the solution sets of various nonlinear operator problems in Hilbert spaces; see, for example, [47, 48] and the references therein.

In 2013, Kangtunyakarn [49] introduced an iterative algorithm for finding a common element of the set of fixed points of a κ-strictly pseudononspreading mapping and the set of solutions of a finite family of variational inequality problems as follows:

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n S x_n, \forall n \in \mathbb{N},$$

and they proved a strong convergence theorem of the sequence $\{x_n\}$ under appropriate conditions of the parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$.

1.1 Objectives of the study

1) To study some new tools and properties for variational inclusion problems in Hilbert spaces.

2) To study some new knowledge about fixed point theorem used in strictly pseudononspreading mapping and nonspreading mapping in Hilbert spaces.

3) To investigate a strong convergence theorem for variational inclusion problems, variational inequality problems, and fixed point problems of nonexpansive mapping in Hilbert spaces.

4) To investigate a strong convergence theorem for variational inclusion problems, equilibrium problems, and fixed point problems of strictly pseudononspreading mapping in Hilbert spaces.

5) To investigate a strong convergence theorem for variational inequality problems, equilibrium problems, and fixed point problems of nonspreading mapping in Hilbert spaces.

6) To give numerical examples to support our theorems in one- and three-dimensional space of real numbers.

1.2 Scopes of the study

1) Study the definitions and properties of variational inclusion problems, variational inequality problems, equilibrium problems and fixed point problems in Hilbert spaces.

2) Prove a strong convergence theorem for variational inclusion problems, variational inequality problems, and fixed point problems of nonexpansive mapping with some new sufficient conditions in a framework of Hilbert spaces.

3) Prove a strong convergence theorem for variational inclusion problems, equilibrium problems, and fixed point problems of strictly pseudononspreading mapping in a framework of Hilbert spaces.

4) Prove a strong convergence theorem for variational inequality problems, equilibrium problems, and fixed point problems of nonspreading mapping in a framework of Hilbert spaces.

5) Give numerical examples in one- and three-dimensional space of real numbers for supporting our main theorems.

1.3 Benefits of the study

1) Obtain new mathematical tools for the properties of variational inclusion problems and fixed point problems of nonlinear mappings in Hilbert spaces.

2) Obtain a strong convergence theorem for approximating a common element of two sets of solutions of variational inclusion problems, the solution set of variational inequality problem and the set of fixed points of a nonexpansive mapping with some new sufficient conditions in a framework of Hilbert space.

3) Obtain a strong convergence theorem for finding a common element of the set of fixed points of a κ-strictly pseudononspreading mapping and the set of solutions of a finite family of variational inclusion problems and the set of solutions of a finite family of equilibrium problems in Hilbert space.

4) Obtain a strong convergence theorem for finding a common element of the set of fixed points of a finite family of nonspreading mappings, the set of solutions of a finite family of equilibrium problems and the set of solutions of two variational inequality problems in a real Hilbert space.

This thesis is divided into 5 chapters as follows:

In Chapter 1 is an introduction of this thesis such as the history of the fixed point theory of nonexpansive mapping, nonspreading mapping, κ-strictly pseudononspreading mapping, variational inequality problem, equilibrium problem, variational inclusion problem.

In chapter 2, basic definitions, some lemmas and properties to prove our main theorem in the next chapter are presented.

In chapter 3, the main result of this thesis is organized as follows:

In section 3.1, we introduce the modified variational inclusion and prove a strong convergence theorem for approximating a common element of two sets of solutions of variational inclusion problems, variational inequality problem and the set of fixed points of a nonexpansive mapping without the conditions $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ in a Hilbert space.

In section 3.2, we introduce new lemmas for the combination of equilibrium problem and fixed point problem of a κ-strictly pseudononspreading mapping. Then, we prove a strong convergence theorem for finding a common element of the set of fixed points of a κ-strictly pseudononspreading mapping and the set of solutions of a finite family of variational inclusion problems and the set of solutions of a finite family of equilibrium problems in Hilbert space.

In section 3.3, we introduce a new lemmas for fixed point problem of a nonspreading mapping. Then, we prove a strong convergence of a new iterative algorithm for finding a common element of the set of fixed points of a finite family of nonspreading mappings, the set of solutions of a finite family of equilibrium problems and the set of solutions of two variational inequality problems in a real Hilbert space.

In chapter 4, we study some applications of our main results and give the numerical examples to support our main results.

In chapter 5, we describe the conclusion of the thesis.

Chapter 2

Basic Concepts and Preliminaries

The purpose of this chapter is to collect notations, terminologies and elementary results used throughout the thesis.

2.1 Basic Concepts

In this section, we will give some definitions and lemmas that will be used in our main theorems. For more detail, basic definitions in this chapter can be found in most standard books on functional analysis for example [50], [51], and [52].

Definition 2.1 (**Cauchy sequence** [53]). A sequence of vectors $\{x_n\}$ in a normed space X is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n > N$.

Definition 2.2 (**complete** [53]). A normed space X is called *complete* if every Cauchy sequence in X converges to an element of X.

Definition 2.3 (**Convex set** [59]). Let X be a normed space and let C be a subset of X. Then the set C is called *convex* if

$$\alpha x + (1 - \alpha) y \in C,$$

for all $x, y \in C$ and $\alpha \in [0, 1]$.

Remark 2.1 ([59]). An inner product space is called a *real inner product space* for the case when the scalars are the real numbers and $\langle x, y\rangle$ is a real number. For the case, (3) means

$$\langle x, y\rangle = \langle y, x\rangle.$$

Theorem 2.2 (**Schwarz inequality**[59]). Let X be an inner product space and let x and y be element in X. Then the following holds:

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Theorem 2.3 (**Parallelogram Law**[53]). For any two elements x and y of inner product space X we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Definition 2.4 (**Strong convergence** [53]). A sequence $\{x_n\}$ of vectors in an inner product space X is called *strongly convergent* to x in X if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.5 (**Weak convergence** [53]). A sequence $\{x_n\}$ of vectors in an inner product space X is called *weakly convergent* to x in X if

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ as } n \to \infty \text{ for all } y \in X.$$

Theorem 2.4 ([59]). The inner product of an inner product space X is jointly continuous:

$$x_n \to x \text{ and } y_n \to y \Rightarrow \langle x_n, y_n\rangle \to \langle x, y\rangle.$$

In this thesis, we denote weak and strong convergence by the notations " $\rightharpoonup$" and " $\to$", respectively.

Remark 2.5 ([59]). We of course obtain from Theorem 2.4 that if $x_n \to x$, then for a fixed $y \in X$

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ and } \langle y, x_n\rangle \to \langle x, y\rangle$$

Theorem 2.6 ([53]). A strongly convergence sequence is weakly convergence (to the same limit), that is, $x_n \to x$ implies $x_n \rightharpoonup x$.

Remark 2.7 ([59]). If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Lemma 2.8 ([59]). Let $\{x_n\}$ be a Cauchy sequence of an inner product space C such that $x_n \rightharpoonup x$. Then $x_n \to x$.

Lemma 2.9 ([58]). Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \le (1 - \alpha_n)s_n + \delta_n, \forall n \ge 0,$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup\limits_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ or $\sum\limits_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim\limits_{n\to\infty} s_n = 0$.

2.2 Some Properties in Hilbert space

In this section, we indicate the following theorems in Hilbert space which are very useful for our results.

Definition 2.6 (**Hilbert space** [53]). Let X be an inner product space and X is called *Hilbert space* if X is complete inner product space.

Example 2.10 ([56]). The n-dimensional Euclidean space $\mathbb{R}^n$ is Hilbert spaces with inner product defined by

$$\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + x_2y_2 + x_3y_3 + \cdots + x_ny_n \text{ and } \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$$

where $\mathbf{x} = (x_1, x_2, x_3, ..., x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, y_2, y_3, ..., y_n) \in \mathbb{R}^n$.

Lemma 2.11 ([59]). Let H be a real Hilbert space. Then the following results hold:

1) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$,

2) $\|x+y\|^2 \leq \|y\|^2 + 2\langle y, x+y\rangle$,

for all $x, y \in H$ and $\lambda \in \mathbb{R}$

Lemma 2.12 ([56]). Let H be a Hilbert space. Then for all $x, y \in H$ and $\alpha_i \in [0,1]$ for $i = 1, 2, ..., n$ such that $\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1$ the following equality holds:

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0 \leq i,j \leq n} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Remark 2.13 ([59]). Let H be an inner product space. Then we know that the following (1) and (2) are equivalent:

1) H is complete,

2) each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Theorem 2.14 ([59]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H with $\{x_n\} \subset C$ and $x_n \rightharpoonup x$, then $x \in C$.

Definition 2.7 (**Metric projection** [54]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H. The metric projection or nearest point projection of H on to C, denoted by P_C, is defined, for any $x \in H$, as the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x-y\| : y \in C\}.$$

Lemma 2.15 ([54]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Given $x \in H$ and $y \in C$, then following holds

$$P_C x = y \Leftrightarrow \langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Lemma 2.16 ([60]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Then the following hods:

1) $\|P_C x - P_C y\| \leq \|x - y\|$, for all $x, y \in H$,

2) $\langle y - x, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$, for all $x, y \in H$.

Lemma 2.17 (**Opial's theorem** [59]). Let H be a Hilbert space and suppose $x_n \rightharpoonup x$. Then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

Definition 2.8 (Lower semicontinuous [59]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\} \text{ is closed.}$$

Moreover, f is called convex if for any $x_1, x_2 \in C$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Theorem 2.18 ([59]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightharpoonup x_0$. Then

$$f(x_0) \leq \liminf_{n \to \infty} f(x_n).$$

Definition 2.9 ([53]). Let C be a nonempty closed convex subset of a real Hilbert space H. Then

1) A is called a *linear operator* if

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y), \text{ for all } x, y \in H \text{ and all scalars } \alpha, \beta.$$

2) A is called *bounded* if there is a number K such that

$$\|Ax\| \leq K\|x\|, \text{ for all } x \text{ in } H.$$

Definition 2.10 ([53]). Let A be a bounded operator on a Hilbert space H. If $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x \in H$, then A is called the *self-adjoint* operator.

Theorem 2.19 ([53]). Let T be a bounded linear self-adjoint operator on a Hilbert space H. Then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Definition 2.11 ([53]). A self-adjoint operator A is called *positive* if $\langle Ax, x\rangle \geq 0$ for all $x \in H$.

Definition 2.12 ([16]). A self-adjoint operator A is a strongly positive operator on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma} \|x\|^2, \text{ for all } x \in H.$$

Lemma 2.20 ([16]). Let A be a strongly positive linear bounded self-adjoint operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Definition 2.13 (Inverse strongly monotone [59]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H. The mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$.

2.3 Fixed Point Theorems

Let X be a nonempty set and let $T : X \to X$. We say that $x \in X$ is a fixed point of T if and only if $Tx = x$ and denote by $F(T)$ the set of all fixed points of T, i.e.,

$$F(T) = \{x \in C : Tx = x\}.$$

Example 2.21. [54] Let $X = \mathbb{R}$.

1) If $T(x) = x^2 + 5x + 4$, then $F(T) = \{-2\}$;

2) If $T(x) = x^2 - x$, then $F(T) = \{0, 2\}$;

3) If $T(x) = x + 2$, then $F(T) = \emptyset$;

4) If $T(x) = x$, then $F(T) = \mathbb{R}$.

2.3.1 Fixed Points of nonexpansive mapping

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H.

Definition 2.14 (Contraction mapping [59]**).** A mapping $f : C \to H$ is called *contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \forall x, y \in C.$$

Theorem 2.22 ([59]). Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Theorem 2.23 ([59]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.

The following classical demiclosedness principle was introduced by Browder in 1967.

Lemma 2.24 (Demiclosedness principle [9]**).** Assume that T is a nonexpansive self-mapping on closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y it follows that $(I - T)x = y$. Here, I is the identity mapping of H.

Lemma 2.25 ([59]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let $A : C \to H$ be a α-inverse strongly monotone. If $0 < \lambda < 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.

2.3.2 Fixed Points of a nonspreading mapping and a κ-strictly pseudononspreading mapping

In 2008, Kohsaka and Takahashi [37] introduced the nonspreading mapping in Hilbert spaces H which is defined as follows:

Definition 2.15 (Nonspreading mapping). The mapping $T : C \to C$ is called *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \forall x, y \in C. \tag{2.1}$$

It is shown in [38] that (2.1) is equivalent to the following equation.

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \forall x, y \in C. \tag{2.2}$$

Proposition 2.26 ([37]). Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonspreading mapping from C into itself. Then $F(T)$ is closed and convex.

Example 2.27. Let $T : [0, \infty) \to [0, \infty)$ be defined by

$$Tx = \frac{x}{x+1}, \forall x \in [0, \infty). \tag{2.3}$$

To see that T is a nonspreading mapping. For every $x, y \in [0, \infty)$, we have $Tx = \dfrac{x}{x+1}$ and $Ty = \dfrac{y}{y+1}$. From (2.3), we have

$$\begin{aligned} |Tx - Ty|^2 =& \left|\frac{x}{x+1} - \frac{y}{y+1}\right|^2 \\ =& \frac{1}{(x+1)^2(y+1)^2}|x - y|^2 \\ \leq& |x - y|^2 \end{aligned}$$

and

$$\begin{aligned} 2\langle x - Tx, y - Ty\rangle =& 2\langle x - \frac{x}{x+1}, y - \frac{y}{y+1}\rangle \\ =& 2\left(\frac{x^2 + x - x}{x+1}\right)\left(\frac{y^2 + y - y}{y+1}\right) \\ =& \frac{2}{(x+1)(y+1)}x^2y^2 \geq 0. \ (Since \ \ 0 \leq x, y < \infty). \end{aligned}$$

From the above, we have

$$|x - y|^2 + 2\langle x - Tx, y - Ty\rangle \geq |Tx - Ty|^2.$$

In 2011, Osilike and Isiogugu [40] introduced the κ-strictly pseudononspreading mapping in Hilbert spaces H which is defined as follows:

Definition 2.16 (κ-strictly pseudononspreading mapping). A mapping $T : C \to C$ is called a *κ-strictly pseudononspreading mapping* if there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 + 2\langle x - Tx, y - Ty\rangle, \forall x, y \in C.$$

Clearly every nonspreading mapping is a κ-strictly pseudononspreading mapping.

Example 2.28. Let $T : [1,\infty) \to [1,\infty)$ be defined by

$$Tx = \sin x, \forall x \in [1,\infty).$$

To see that T is a κ-strictly pseudononspreading mapping. For every $x, y \in [1,\infty)$, we have $Tx = \sin x$ and $Ty = \sin y$. From the definition of T, we have

$$|Tx - Ty|^2 = |\sin x - \sin y|^2 \leq |x - y|^2,$$

$$\begin{aligned} |(I-T)x - (I-T)y|^2 =& |x - \sin x - (y - \sin y)|^2 \\ =& |(x-y) - (\sin x - \sin y)|^2 \\ \leq& (|x-y| + |\sin x - \sin y|)^2 \\ \leq& (|x-y| + |x-y|)^2 \\ =& 4|x-y|^2, \end{aligned}$$

and

$$\begin{aligned} 2\langle x - Tx, y - Ty\rangle =& 2\langle x - \sin x, y - \sin y\rangle \\ =& 2(x - \sin x)(y - \sin y) \geq 0. \quad (\textit{Since} \ \ 1 \leq x, y < \infty, (x - \sin x)(y - \sin y) \geq 0). \end{aligned}$$

From the above, then there exists $\kappa \in [0,1)$ such that

$$\begin{aligned} |x-y|^2 + \kappa|(I-T)x - (I-T)y|^2 + 2\langle x - Tx, y - Ty\rangle \geq& |x-y|^2 \\ \geq& |\sin x - \sin y|^2 \\ =& |Tx - Ty|^2. \end{aligned}$$

Proposition 2.29 ([40]). Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a κ-strictly pseudononspreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.

Proposition 2.30 ([40]). Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a κ-strictly pseudononspreading mapping. Then $(I-T)$ is demiclosed at 0.

Lemma 2.31 ([49]). Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a κ-strictly pseudononspreading mapping with $F(T) \neq \emptyset$. Then $F(T) = VI(C, I-T)$.

2.4 Variational inequality Problems and Equilibrium Problems

In this section, we give some useful lemmas, propositions and theorems for variational inequality problems and equilibrium problems to prove the main results.

Lemma 2.32 ([59]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.33 ([59]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let $A : C \to H$ be a α-inverse strongly monotone. Then $VI(C, A) \neq \emptyset$.

In 2013, Kangtunyakarn [21] modified $VI(C, A)$ as follows:

$$VI(C, aA + (1 - a)B) = \{x \in C : \langle y - x, (aA + (1 - a)B)x\rangle \geq 0, \forall y \in C, a \in (0, 1)\}, \quad (2.4)$$

where $A, B : C \to H$ are nonlinear mapping. It obvious that $VI(C, aA+(1-a)B)$ reduces to $VI(C, A)$, if $A = B$. He also introduced a new mathematical tool for variational inequality problems as follows:

Lemma 2.34 ([21]). Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α and β-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C, A) \bigcap VI(C, B) \neq \emptyset$. Then

$$VI(C, aA + (1 - a)B) = VI(C, A) \bigcap VI(C, B), \forall a \in (0, 1).$$

Furthermore, if $0 < \gamma < min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1 - a)B)$ is a nonexpansive mapping.

Definition 2.17 ([22]). For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfy the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) For each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) For each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.35 ([22]). Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.36 ([23]). Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:

1) T_r is a single valued;

2) T_r is a firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle ;$$

3) $F(T_r) = EP(F)$;

4) $EP(F)$ is closed and convex.

Lemma 2.37 ([29]). Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying $(A1) - (A4)$ with $\bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Then,

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0, 1)$ for every $i = 1, 2, \ldots, N$ and $\sum_{i=1}^{N} a_i = 1$.

Remark 2.38 ([29]). From Lemma 2.37

$$F(T_r) = EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0, 1)$, for each $i = 1, 2, \ldots, N$, and $\sum_{i=1}^{N} a_i = 1$.

2.5 Variational inclusion problems

In this section, we give some useful lemmas and definitions for variational inclusion problems to prove the main results.

Definition 2.18 ([57]). The graph of a multi-valued mapping $M : H \to 2^H$ is defined by

$$\text{Graph}(M) = \{(x, y) \in H \times H : x \in H, y \in Mx\}.$$

Definition 2.19 (Monotone[57]**).** A multi-valued mapping $M : H \to 2^H$ is called *monotone*, if for all $x, y \in H$, $u \in Mx$ and $v \in My$ implies that $\langle u - v, x - y\rangle \geq 0$.

Definition 2.20 (Maximal monotone[57]**).** A multi-valued mapping $M : H \to 2^H$ is called *maximal monotone*, if it is monotone and if for any $(x, u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0$ for every $(y, v) \in \text{Graph}(M)$ (the graph of mapping M) implies that $u \in Mx$.

Definition 2.21 (Resolvent operator[34]). Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(z) = (I + \lambda M)^{-1}(z), \forall z \in H,$$

is called the *resolvent operator* associated with M, where λ is any positive number and I is an identity mapping.

Lemma 2.39 ([34]). $u \in H$ is a solution of variational inclusion (1.6) if and only if $u = J_{M,\lambda}(u - \lambda Bu), \forall \lambda > 0$, i.e.,

$$VI(H, A, M) = F(J_{M,\lambda}(I - \lambda A)), \forall \lambda > 0.$$

Further, if $\lambda \in (0, 2\alpha]$, then $VI(H, A, M)$ is closed convex subset in H.

Lemma 2.40 ([34]). The resolvent operator $J_{M,\lambda}$ associated with M is single-valued, nonexpansive for all $\lambda > 0$ and 1-inverse strongly monotone.

Chapter 3

Strong Convergence Theorems in Hilbert space

In this chapter, we introduce and study the new iterative schemes for fixed point theorem of nonlinear mappings in Hilbert space.

3.1 A Strong Convergence Theorem for Nonexpansive Mapping with Some New Sufficient Conditions

In this section is to modify the variational inclusion problems and introduce a strong convergence theorem for approximating a common element of two sets of solutions of variational inclusion problems, variational inequality problem and the set of fixed points of a nonexpansive mapping without the conditions $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ in a Hilbert space.

We first modify the variational inclusion problems in a Hilbert space as follows.

For $i = 1, 2, ..., N$, let $A_i : H \to H$ be a single valued mapping and let $M : H \to 2^H$ be a multi-valued mapping. From the concept of (1.6), we introduce a new problem for finding $u \in H$ such that

$$\theta \in \sum_{i=1}^{N} a_i A_i u + Mu, \tag{3.1}$$

for all $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$ and θ is a zero vector. This problem is called *the modified variational inclusion*. The set of solutions (3.1) is denoted by $VI\left(H, \sum_{i=1}^{N} a_i A_i, M\right)$. If $A_i \equiv A$ for all $i = 1, 2, ..., N$, then (3.1) reduce to (1.6).

Now, we introduce and prove the following lemma and remark which will be needed for the proof of our main theorems.

Lemma 3.1. Let H be a real Hilbert space and let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, ..., N$, let $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,...,N}\{\alpha_i\}$ and $\bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. Then

$$VI(H, \sum_{i=1}^{N} a_i A_i, M) = \bigcap_{i=1}^{N} VI(H, A_i, M),$$

where $\sum_{i=1}^{N} a_i = 1$, and $0 < a_i < 1$ for every $i = 1, 2, \ldots, N$. Moreover, we then have $J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} a_i A_i)$ is a nonexpansive mapping, for all $0 < \lambda < 2\eta$.

Proof. We will first show that $\bigcap_{i=1}^{N} VI(H, A_i, M) \subseteq VI(H, \sum_{i=1}^{N} a_i A_i, M)$. Let $z \in \bigcap_{i=1}^{N} VI(H, A_i, M)$, then $z \in VI(H, A_i, M)$, for all $i = 1, 2, 3, ..., N$. Since $z \in VI(H, A_i, M)$, for all $i = 1, 2, 3, ..., N$, then

$$\theta \in A_i z + Mz, \forall i = 1, 2, 3, ..., N.$$

Since $\sum_{i=1}^{N} a_i = 1$, and $0 < a_i < 1$, for all $i = 1, 2, \ldots, N$, then

$$\theta \in \sum_{i=1}^{N} a_i A_i z + Mz.$$

It implies that $z \in VI(H, \sum_{i=1}^{N} a_i A_i, M)$. Next, we will show that $VI(H, \sum_{i=1}^{N} a_i A_i, M) \subseteq \bigcap_{i=1}^{N} VI(H, A_i, M)$. Let $x_0 \in VI(H, \sum_{i=1}^{N} a_i A_i, M)$ and let $x^* \in \bigcap_{i=1}^{N} VI(H, A_i, M)$. From Lemma 2.39, we have

$$x_0 \in F(J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} a_i A_i)).$$

Since $\bigcap_{i=1}^{N} VI(H, A_i, M) \subseteq VI(H, \sum_{i=1}^{N} a_i A_i, M)$, we have $x^* \in VI(H, \sum_{i=1}^{N} a_i A_i, M)$. From Lemma 2.39, we have

$$x^* \in F(J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} a_i A_i)).$$

From the nonexpansiveness of $J_{M,\lambda}$, we have

$$\begin{aligned}
\|x^* - x_0\|^2 &= \|J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} a_i A_i)x^* - J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} a_i A_i)x_0\|^2 \\
&\leq \|(I - \lambda \sum_{i=1}^{N} a_i A_i)x^* - (I - \lambda \sum_{i=1}^{N} a_i A_i)x_0\|^2 \\
&= \|(x^* - x_0) - \lambda(\sum_{i=1}^{N} a_i A_i x^* - \sum_{i=1}^{N} a_i A_i x_0)\|^2 \\
&\leq \|x^* - x_0\|^2 - 2\lambda \sum_{i=1}^{N} a_i \langle x^* - x_0, A_i x^* - A_i x_0 \rangle + \lambda^2 \sum_{i=1}^{N} a_i \|A_i x^* - A_i x_0\|^2 \\
&\leq \|x^* - x_0\|^2 - 2\lambda \sum_{i=1}^{N} a_i \alpha_i \|A_i x^* - A_i x_0\|^2 + \lambda^2 \sum_{i=1}^{N} a_i \|A_i x^* - A_i x_0\|^2 \\
&\leq \|x^* - x_0\|^2 - 2\lambda\eta \sum_{i=1}^{N} a_i \|A_i x^* - A_i x_0\|^2 + \lambda^2 \sum_{i=1}^{N} a_i \|A_i x^* - A_i x_0\|^2 \\
&= \|x^* - x_0\|^2 + \lambda \sum_{i=1}^{N} a_i (\lambda - 2\eta) \|A_i x^* - A_i x_0\|^2. \qquad (3.2)
\end{aligned}$$

This implies that

$$\lambda \sum_{i=1}^{N} a_i (2\eta - \lambda) \|A_i x^* - A_i x_0\|^2 \leq 0.$$

Then

$$A_i x^* = A_i x_0, \forall i = 1, 2, ..., N. \qquad (3.3)$$

Since $x_0 \in VI(H, \sum_{i=1}^{N} a_i A_i, M)$, we have

$$\theta \in Mx_0 + \sum_{i=1}^{N} a_i A_i x_0. \qquad (3.4)$$

From $x^* \in VI(H, \sum_{i=1}^{N} a_i A_i, M)$, we have

$$\theta \in Mx^* + \sum_{i=1}^{N} a_i A_i x^*. \qquad (3.5)$$

From (3.4) and (3.5), we have

$$\theta \in Mx_0 + \sum_{i=1}^{N} a_i A_i x_0 - Mx^* - \sum_{i=1}^{N} a_i A_i x^*. \tag{3.6}$$

From (3.3) and (3.6), we have

$$\theta \in Mx_0 - Mx^*. \tag{3.7}$$

Since $x^* \in \bigcap_{i=1}^{N} VI(H, A_i, M)$ and we have (3.3) and (3.7),

$$\begin{aligned} \theta &\in Mx_0 - Mx^* + Mx^* + A_i x^* \\ &\subseteq Mx_0 + A_i x_0, \end{aligned}$$

for all $i = 1, 2, ..., N$. It implies that $x_0 \in \bigcap_{i=1}^{N} VI(H, A_i, M)$. Hence

$$VI(H, \sum_{i=1}^{N} a_i A_i, M) \subseteq \bigcap_{i=1}^{N} VI(H, A_i, M).$$

Applying (3.2), we can conclude that $J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} a_i A_i)$ is a nonexpansive mapping for all i=1,2,...,N.

□

Next, we will prove a strong convergence theorem for approximating a common element of two sets of solutions of variational inclusion problems, variational inequality problem and the set of fixed points of a nonexpansive mapping without the conditions $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ in a Hilbert.

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, and let $D : C \to H$ be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Let $T : C \to C$ be a nonexpansive mapping with $\mathcal{F} := F(T) \cap VI(H, A, M) \cap VI(H, B, M) \cap VI(C, D) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} y_n = J_{M,\lambda}(I - \lambda(aA + (1-a)B))x_n, \\ x_{n+1} = \alpha_n P_C(I - \rho D)y_n + (1 - \alpha_n)Tx_n, \forall n \geq 1, \end{cases} \tag{3.8}$$

where $a \in (0, 1)$, $\{\alpha_n\} \subseteq [c, d] \subset [0, 1]$, for all $n \in \mathbb{N}$, $0 < \rho \leq \|D\|^{-1}$ and $0 < \lambda < 2\eta$ with $\eta = min\{\alpha, \beta\}$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$.

Proof. Let $x, y \in C$. From Lemma 2.20, we have

$$\begin{aligned} \|(I - \rho D)x - (I - \rho D)y\| =& \|(I - \rho D)(x - y)\| \\ \leq& (1 - \rho\bar{\gamma})\|x - y\|. \end{aligned} \tag{3.9}$$

Let $x^* \in \mathcal{F}$. From the definition of y_n, Lemma 3.1, we have

$$\begin{aligned}\|y_n - x^*\| =& \|J_{M,\lambda}(I - \lambda(aA + (1-a)B))x_n - x^*\| \\ \leq& \|x_n - x^*\|. \end{aligned} \tag{3.10}$$

From the definition of x_n, (3.9) and (3.10), we have

$$\begin{aligned}\|x_{n+1} - x^*\| =& \|\alpha_n(P_C(I - \rho D)y_n - x^*) + (1-\alpha_n)(Tx_n - x^*)\| \\ \leq& \alpha_n\|P_C(I - \rho D)y_n - x^*\| + (1-\alpha_n)\|Tx_n - x^*\| \\ \leq& \alpha_n\|(I - \rho D)y_n - (I - \rho D)x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ \leq& \alpha_n(1 - \rho\bar{\gamma})\|y_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ \leq& \alpha_n(1 - \rho\bar{\gamma})\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ \leq& (1 - \rho\bar{\gamma}\alpha_n)\|x_n - x^*\| \\ \leq& (1 - \rho\bar{\gamma}c)\|x_n - x^*\| \\ =& p\|x_n - x^*\| \\ \leq& p(p\|x_{n-1} - x^*\|) \\ =& p^2\|x_{n-1} - x^*\| \\ & \vdots \\ \leq& p^n\|x_1 - x^*\|, \end{aligned} \tag{3.11}$$

where $p = (1 - \rho\bar{\gamma}c) \in (0, 1)$.
Since $p^n \to 0$ as $n \to \infty$ and (3.11), we can conclude that the sequence $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$. □

3.2 A Strong Convergence Theorem for a κ-Strictly Pseudononspreading Mapping in Hilbert space

In this section, we introduce a strong convergence theorem for finding a common element of the set of fixed points of a κ-strictly pseudononspreading mapping and the set of solutions of a finite family of variational inclusion problems and the set of solutions of a finite family of equilibrium problems in Hilbert space.

Now, we prove the following results which will be needed for the proof of our main theorems.

Remark 3.3. Let C be a nonempty closed convex subset of H. Then following statements are equivalent:

(i) $T : C \to C$ is a κ-strictly pseudononspreading mapping,

(ii) $\frac{1-\kappa}{2}\|(I-T)x - (I-T)y\|^2 \leq \langle (I-T)x - (I-T)y, x - y\rangle + \langle (I-T)x, (I-T)y\rangle$,

for all $x, y \in C$.

Proof. Let $x, y \in C$. *Case* (i)⇒(ii), let T be a κ-strictly pseudononspreading mapping, then there exists $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2 + 2\langle x - Tx, y - Ty\rangle. \tag{3.12}$$

Since

$$\|(I - T)x - (I - T)y\|^2 = \|Tx - Ty\|^2 - 2\langle Tx - Ty, x - y\rangle + \|x - y\|^2, \tag{3.13}$$

we have

$$\|Tx - Ty\|^2 = \|(I - T)x - (I - T)y\|^2 + 2\langle Tx - Ty, x - y\rangle - \|x - y\|^2. \tag{3.14}$$

From (3.12) and (3.14), we have

$$\begin{aligned}&\|(I - T)x - (I - T)y\|^2 + 2\langle Tx - Ty, x - y\rangle - \|x - y\|^2\\ &\quad\leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2 + 2\langle x - Tx, y - Ty\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}(1 - \kappa)\|(I - T)x - (I - T)y\|^2 &\leq 2\|x - y\|^2 - 2\langle Tx - Ty, x - y\rangle + 2\langle x - Tx, y - Ty\rangle\\ &= 2\langle (I - T)x - (I - T)y, x - y\rangle + 2\langle x - Tx, y - Ty\rangle.\end{aligned}$$

Then

$$\frac{1 - \kappa}{2}\|(I - T)x - (I - T)y\|^2 \leq \langle (I - T)x - (I - T)y, x - y\rangle + \langle (I - T)x, (I - T)y\rangle.$$

Case (ii)⇒(i), let $x, y \in C$ and

$$\begin{aligned}\frac{1 - \kappa}{2}\|(I - T)x - (I - T)y\|^2 &\leq \langle (I - T)x - (I - T)y, x - y\rangle + \langle x - Tx, y - Ty\rangle\\ &= \|x - y\|^2 - \langle Tx - Ty, x - y\rangle + \langle x - Tx, y - Ty\rangle.\end{aligned}$$

Then

$$(1 - \kappa)\|(I - T)x - (I - T)y\|^2 \leq 2\|x - y\|^2 - 2\langle Tx - Ty, x - y\rangle + 2\langle x - Tx, y - Ty\rangle.$$

It follows that

$$2\langle Tx - Ty, x - y\rangle \leq 2\|x - y\|^2 - (1 - \kappa)\|(I - T)x - (I - T)y\|^2 + 2\langle x - Tx, y - Ty\rangle. \tag{3.15}$$

From (3.13), we have

$$2\langle Tx - Ty, x - y\rangle = \|Tx - Ty\|^2 + \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2. \tag{3.16}$$

From (3.15) and (3.16), we have

$$\begin{aligned}\|Tx - Ty\|^2 + \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2 \leq & 2\|x - y\|^2 - (1 - \kappa)\|(I - T)x - (I - T)y\|^2\\ &+ 2\langle x - Tx, y - Ty\rangle.\end{aligned}$$

Then

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2 + 2\langle (I - T)x, (I - T)y\rangle.$$

Therefore, $T : C \to C$ is a κ-strictly pseudononspreading mapping. □

Example 3.4. Let $T : [0, \infty) \to [0, \infty)$ be defined by

$$Tx = \frac{4x^2}{5+4x}, \forall x \in C.$$

To see that T is a κ-strictly pseudononspreading mapping. For every $x, y \in [1, \infty)$, we have

$$\begin{aligned}
|(I-T)x-(I-T)y|^2 &= |x-Tx-(y-Ty)|^2 \\
&= \left|x - \frac{4x^2}{5+4x} - \left(y - \frac{4y^2}{5+4y}\right)\right|^2 \\
&= \left|\frac{5x+4x^2-4x^2}{5+4x} - \left(\frac{5y+4y^2-4y^2}{5+4y}\right)\right|^2 \\
&= \left|\frac{5x}{5+4x} - \left(\frac{5y}{5+4y}\right)\right|^2 \\
&= \left|\frac{5x(5+4y)-5y(5+4x)}{(5+4x)(5+4y)}\right|^2 \\
&= \left|\frac{25x+20xy-25y-20xy}{(5+4x)(5+4y)}\right|^2 \\
&= \left|\frac{25x-25y}{(5+4x)(5+4y)}\right|^2 \\
&= \frac{625}{(5+4x)^2(5+4y)^2}|x-y|^2 \geq 0,
\end{aligned}$$

$$\begin{aligned}
\langle x-Tx-(y-Ty), x-y\rangle &= \left\langle \frac{25x-25y}{(5+4x)(5+4y)}, x-y \right\rangle \\
&= \frac{25}{(5+4x)(5+4y)}(x-y)^2 \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
\langle x-Tx, y-Ty\rangle &= \left\langle \frac{5x}{5+4x}, \frac{5y}{5+4y} \right\rangle \\
&= \frac{25}{(5+4x)(5+4y)}xy \geq 0.
\end{aligned}$$

From the above inequalities, we have

$$\begin{aligned}
\langle (I-T)x-(I-T)y, x-y\rangle + \langle x-Tx, y-Ty\rangle &= \frac{25}{(5+4x)(5+4y)}(x-y)^2 + \frac{25}{(5+4x)(5+4y)}xy \\
&\geq \frac{25}{(5+4x)(5+4y)}(x-y)^2 \\
&\geq \frac{25}{(5+4x)^2(5+4y)^2}(x-y)^2 \\
&= \frac{25}{25}\left(\frac{25}{(5+4x)^2(5+4y)^2}(x-y)^2\right) \\
&= \frac{1}{25}\left(\frac{625}{(5+4x)^2(5+4y)^2}(x-y)^2\right) \\
&= \frac{1}{25}|(I-T)x-(I-T)y|^2.
\end{aligned}$$

From Remark 3.3, we have T is a $\frac{23}{25}$-strictly pseudononspreading mapping.

Remark 3.5. Let $T : H \to H$ be a κ-strictly pseudononspreading mapping with $F(T) \neq \emptyset$. Define $S : H \to H$ by $Sx := ((1-\lambda)I + \lambda T)x$, where $\lambda \in (0, 1-\kappa)$. Then the following hold:

1) $F(T) = F(S) = F(I - \lambda(I - T))$;

2) for every $x \in H$ and $y \in F(T)$,

$$\|Sx - y\| \leq \|x - y\|.$$

Proof. From the definition of the mapping S, we have

$$\begin{aligned} x \in F(I - \lambda(I-T)), \forall \lambda \in (0, 1-\kappa) &\Leftrightarrow x = (I - \lambda(I-T))x, \forall \lambda \in (0, 1-\kappa) \\ &\Leftrightarrow x = x - \lambda x + \lambda Tx, \forall \lambda \in (0, 1-\kappa) \\ &\Leftrightarrow x = Tx \\ &\Leftrightarrow x \in F(T). \end{aligned}$$

Hence $F(T) = F(S) = F(I - \lambda(I - T))$.
(ii) Next, we show that $\|Sx - y\| \leq \|x - y\|$. For every $x \in H$ and $y \in F(T)$, we have

$$\begin{aligned} \|Sx - y\|^2 =\ & \|(1-\lambda)(x-y) + \lambda(Tx - y)\|^2 \\ =\ & (1-\lambda)\|x-y\|^2 + \lambda\|Tx - y\|^2 - \lambda(1-\lambda)\|Tx - x\|^2 \\ \leq\ & (1-\lambda)\|x-y\|^2 + \lambda(\|x-y\|^2 + \kappa\|(I-T)x\|^2) \\ & -\ \lambda(1-\lambda)\|Tx - x\|^2 \\ =\ & \|x-y\|^2 + \kappa\lambda\|Tx - x\|^2 - \lambda(1-\lambda)\|Tx - x\|^2 \\ =\ & \|x-y\|^2 + \lambda(\lambda - (1-\kappa))\|Tx - x\|^2 \\ \leq\ & \|x-y\|^2. \end{aligned}$$

□

Lemma 3.6. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\sum_{i=1}^{N} a_i F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying $(A1) - (A4)$ where $a_i \in (0,1)$, for each $i = 1, 2, \ldots, N$ and $\sum_{i=1}^{N} a_i = 1$. For every $n \in \mathbb{N}$, let $0 < c \leq r_n \leq d$ with $r_n \to r$ as $n \to \infty$. Then $\|T_{r_n}x - T_r x\| \to 0$ as $n \to \infty$ for all $x \in H$.

Proof. For every $n \in \mathbb{N}$, let $0 < c \leq r_n \leq d$ with $r_n \to r$ as $n \to \infty$, which it follows that $0 < c \leq r \leq d$. For every $x \in H$ and Lemma 2.36 , we then have

$$\sum_{i=1}^{N} a_i F_i(T_{r_n}x, y) + \frac{1}{r_n}\langle y - T_{r_n}x, T_{r_n}x - x\rangle \geq 0, \forall y \in C.$$

and

$$\sum_{i=1}^{N} a_i F_i(T_r x, y) + \frac{1}{r}\langle y - T_r x, T_{r_n}x - x\rangle \geq 0, \forall y \in C.$$

In particular, we have

$$\sum_{i=1}^{N} a_i F_i(T_{r_n}x, T_r x) + \frac{1}{r_n}\langle T_r x - T_{r_n}x, T_{r_n}x - x\rangle \geq 0, \tag{3.17}$$

and

$$\sum_{i=1}^{N} a_i F_i(T_r x, T_{r_n}x) + \frac{1}{r}\langle T_{r_n}x - T_r x, T_r x - x\rangle \geq 0. \tag{3.18}$$

Summing up (3.17) and (3.18) and using $(A2)$, we have

$$\frac{1}{r}\langle T_{r_n}x - T_r x, T_r x - x\rangle + \frac{1}{r_n}\langle T_r x - T_{r_n}x, T_{r_n}x - x\rangle \geq 0.$$

It follows that

$$\left\langle T_{r_n}x - T_r x, \frac{T_r x - x}{r} - \frac{T_{r_n}x - x}{r_n}\right\rangle \geq 0.$$

This implies that

$$\begin{aligned} 0 &\leq \left\langle T_r x - T_{r_n}x, T_{r_n}x - x - \frac{r_n}{r}(T_r x - x)\right\rangle \\ &= \left\langle T_r x - T_{r_n}x, T_{r_n}x - T_r x + \left(1 - \frac{r_n}{r}\right)(T_r x - x)\right\rangle. \end{aligned}$$

It follows that

$$\|T_r x - T_{r_n}x\|^2 \leq \left|1 - \frac{r_n}{r}\right| \|T_r x - T_{r_n}x\| \left(\|T_r x\| + \|x\|\right).$$

Then we have

$$\|T_r x - T_{r_n}x\| \leq \frac{1}{r}|r - r_n| \left(\|T_r x\| + \|x\|\right).$$

Since $r_n \to r$ as $n \to \infty$, we have

$$\|T_{r_n}x - T_r x\| \to 0 \text{ as } n \to \infty.$$

□

Next, we prove that a strong convergence theorem for finding a common element of the set of fixed points of a κ-strictly pseudononspreading mapping and the set of solutions of a finite family of variational inclusion problems and the set of solutions of a finite family of equilibrium problems in Hilbert space.

Theorem 3.7. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$ and $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,\ldots,N}\{\alpha_i\}$. Let $T : H \to H$ be a κ-strictly pseudononspreading mapping. Assume $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} \displaystyle\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ \displaystyle x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda\sum_{i=1}^{N} b_i A_i)x_n \\ \qquad + \eta_n(I - \rho_n(I - T))x_n + \delta_n u_n, \forall n \geq 1, \end{cases} \tag{3.19}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq (0,1)$ and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n + \eta_n + \delta_n = 1$, $0 < \alpha < 1$, and $0 \leq a_i, b_i \leq 1$, for every $i = 1, 2, ..., N$, $r_n \in [c,d] \subset (0,1)$, $0 < p \leq \beta_n, \gamma_n, \eta_n, \delta_n \leq q < 1$, $\rho_n \in (0, 1-\kappa)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \rho_n < \infty$;

(iii) $0 < \lambda < 2\eta$;

(iv) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = 1$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathcal{F}} u$.

Proof. The proof of Theorem 3.7 will be divided into five steps:

Step 1. We show that the sequence $\{x_n\}$ is bounded.

Since $\sum_{i=1}^{N} a_i F_i$ satisfies $(A1)$-$(A4)$, and

$$\sum_{i=1}^{N} a_i F_i (u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C,$$

by Lemma 2.36 and Remark 2.38, we have $u_n = T_{r_n} x_n$ and $F(T_{r_n}) = \bigcap_{i=1}^{N} EP(F_i)$. Let $z \in \mathcal{F}$. From Lemma 2.39 and Lemma 3.1, we have

$$z = J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} b_i A_i) z.$$

From the nonexpansiveness of $J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} b_i A_i)$, we have

$$\|J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} b_i A_i) x_n - z\| \leq \|x_n - z\|. \tag{3.20}$$

From Remark 3.5, we have

$$\begin{aligned} \|(I - \rho_n(I - T)) x_n - z\|^2 &= \|(1 - \rho_n) x_n + \rho_n T x_n - z\|^2 \\ &\leq \|x_n - z\|^2. \end{aligned} \tag{3.21}$$

From the definition of x_n, (3.20), and (3.21), we have

$$\begin{aligned}\|x_{n+1}-z\| &= \|\alpha_n u+\beta_n x_n+\gamma_n J_{M,\lambda}(I-\lambda\sum_{i=1}^{N} b_iA_i)x_n \\ &\quad + \eta_n(I-\rho_n(I-T))x_n+\delta_n u_n - z\| \\ &\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\gamma_n\|J_{M,\lambda}(I-\lambda\sum_{i=1}^{N} b_iA_i)x_n-z\| \\ &\quad + \eta_n\|(I-\rho_n(I-T))x_n-z\|+\delta_n\|u_n-z\| \\ &\le \alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\| \\ &\le \max\{\|u-z\|,\|x_1-z\|\} = K.\end{aligned}$$

By using the mathematical induction, we have $\|x_n-z\|\le K, \forall n\in\mathbb{N}$. It implies that $\{x_n\}$ is bounded and so is $\{u_n\}$.
By continuing in the same direction as in step 1 of Theorem 3.1 in [49], we have

$$\|Tx_n-z\|\le\frac{1+\kappa}{1-\kappa}\|x_n-z\|. \tag{3.22}$$

Indeed, since T is a κ-strictly pseudononspreading mapping, we have

$$\begin{aligned}\|Tx_n-z\|^2 &\le \|x_n-z\|^2+\kappa\|(I-T)x_n-(I-T)z\|^2+2\langle x_n-Tx_n, z-Tz\rangle \\ &= \|x_n-z\|^2+\kappa\|(x_n-z)-(Tx_n-z)\|^2 \\ &= \|x_n-z\|^2+\kappa\left(\|x_n-z\|^2+\|Tx_n-z\|^2-2\langle x_n-z,Tx_n-z\rangle\right) \\ &= \|x_n-z\|^2+\kappa\|x_n-z\|^2+\kappa\|Tx_n-z\|^2+2\kappa\langle z-x_n,Tx_n-z\rangle.\end{aligned}$$

It implies that

$$(1-\kappa)\|Tx_n-z\|^2\le(1+\kappa)\|x_n-z\|^2+2\kappa\|x_n-z\|\|Tx_n-z\|.$$

From the above inequality, we have

$$\begin{aligned}0 &\ge (1-\kappa)\|Tx_n-z\|^2-(1+\kappa)\|x_n-z\|^2-2\kappa\|x_n-z\|\|Tx_n-z\| \\ &= (1-\kappa)\|Tx_n-z\|^2-\kappa\|x_n-z\|\|Tx_n-z\|-\left((1+\kappa)\|x_n-z\|^2+\kappa\|x_n-z\|\|Tx_n-z\|\right) \\ &= (1-\kappa)\|Tx_n-z\|^2-\kappa\|x_n-z\|\|Tx_n-z\|+\|x_n-z\|\|Tx_n-z\| \\ &\quad -\left((1+\kappa)\|x_n-z\|^2+\kappa\|x_n-z\|\|Tx_n-z\|+\|x_n-z\|\|Tx_n-z\|\right) \\ &= (1-\kappa)\left(\|Tx_n-z\|^2+\|x_n-z\|\|Tx_n-z\|\right)-\left((1+\kappa)\left(\|x_n-z\|^2+\|x_n-z\|\|Tx_n-z\|\right)\right) \\ &= (1-\kappa)\|Tx_n-z\|\left(\|Tx_n-z\|+\|x_n-z\|\right)-\left((1+\kappa)\|x_n-z\|\left(\|x_n-z\|+\|Tx_n-z\|\right)\right).\end{aligned}$$

It implies that

$$\|Tx_n-z\|\le\frac{1+\kappa}{1-\kappa}\|x_n-z\|.$$

From (3.22), we can conclude that $\{Tx_n\}$ is bounded.

Step 2. Put $G=\sum_{i=1}^{N} b_iA_i$ and $P=I-T$. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n u+\beta_n x_n+\gamma_n J_{M,\lambda}(I-\lambda G)x_n+\eta_n(I-\rho_n P)x_n \\
&+\ \delta_n u_n-\alpha_{n-1}u-\beta_{n-1}x_{n-1}-\gamma_{n-1}J_{M,\lambda}(I-\lambda G)x_{n-1} \\
&-\ \eta_{n-1}(I-\rho_{n-1}P)x_{n-1}-\delta_{n-1}u_{n-1}\| \\
&\le |\alpha_n-\alpha_{n-1}|\,\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\,\|x_{n-1}\| \\
&+\ \gamma_n\|J_{M,\lambda}(I-\lambda G)x_n-J_{M,\lambda}(I-\lambda G)x_{n-1}\| \\
&+\ |\gamma_n-\gamma_{n-1}|\,\|J_{M,\lambda}(I-\lambda G)x_{n-1}\| \\
&+\ \eta_n\|(I-\rho_n P)x_n-(I-\rho_{n-1}P)x_{n-1}\|+|\eta_n-\eta_{n-1}|\,\|(1-\rho_{n-1}P)x_{n-1}\| \\
&+\ \delta_n\|u_n-u_{n-1}\|+|\delta_n-\delta_{n-1}|\,\|u_{n-1}\| \\
&\le |\alpha_n-\alpha_{n-1}|\,\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\,\|x_{n-1}\|+\gamma_n\|x_n-x_{n-1}\| \\
&+\ |\gamma_n-\gamma_{n-1}|\,\|J_{M,\lambda}(I-\lambda G)x_{n-1}\|+|\eta_n-\eta_{n-1}|\,\|(I-\rho_{n-1}P)x_{n-1}\| \\
&+\ \eta_n\left(\|x_n-x_{n-1}\|+\rho_n\|Px_n-Px_{n-1}\|+|\rho_n-\rho_{n-1}|\,\|Px_{n-1}\|\right) \\
&+\ \delta_n\|u_n-u_{n-1}\|+|\delta_n-\delta_{n-1}|\,\|u_{n-1}\| \\
&\le |\alpha_n-\alpha_{n-1}|\,\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\,\|x_{n-1}\|+\gamma_n\|x_n-x_{n-1}\| \\
&+\ |\gamma_n-\gamma_{n-1}|\,\|J_{M,\lambda}(I-\lambda G)x_{n-1}\|+|\eta_n-\eta_{n-1}|\,\|(I-\rho_{n-1}P)x_{n-1}\| \\
&+\ \eta_n\|x_n-x_{n-1}\|+\rho_n\|Px_n-Px_{n-1}\|+|\rho_n-\rho_{n-1}|\,\|Px_{n-1}\| \\
&+\ \delta_n\|u_n-u_{n-1}\|+|\delta_n-\delta_{n-1}|\,\|u_{n-1}\|. \qquad (3.23)
\end{aligned}$$

By continuing in the same direction as in step 2 of Theorem 3.1 in [29], we have

$$\|u_n-u_{n-1}\|\le\|x_n-x_{n-1}\|+\frac{1}{d}|r_n-r_{n-1}|\,\|u_n-x_n\|.$$

Indeed, since $u_n=T_{r_n}x_n$, by utilizing the definition of T_{r_n}, we obtain

$$\sum_{i=1}^{N} a_iF_i\left(T_{r_n}x_n,y\right)+\frac{1}{r_n}\langle y-T_{r_n}x_n,T_{r_n}x_n-x_n\rangle\ge 0,\forall y\in C, \qquad (3.24)$$

and

$$\sum_{i=1}^{N} a_iF_i\left(T_{r_{n+1}}x_{n+1},y\right)+\frac{1}{r_{n+1}}\left\langle y-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\right\rangle\ge 0,\forall y\in C. \qquad (3.25)$$

From (3.24) and (3.25), it follows that

$$\sum_{i=1}^{N} a_iF_i\left(T_{r_n}x_n,T_{r_{n+1}}x_{n+1}\right)+\frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n,T_{r_n}x_n-x_n\right\rangle\ge 0, \qquad (3.26)$$

and

$$\sum_{i=1}^{N} a_iF_i\left(T_{r_{n+1}}x_{n+1},T_{r_n}x_n\right)+\frac{1}{r_{n+1}}\left\langle T_{r_n}x_n-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\right\rangle\ge 0. \qquad (3.27)$$

From (3.26) and (3.27) and the fact that $\sum_{i=1}^{N} a_i F_i$ satisfies (A2), we have

$$\frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\right\rangle + \frac{1}{r_{n+1}}\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \geq 0,$$

which implies that

$$\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, \frac{T_{r_{n+1}}x_{n+1} - x_{n+1}}{r_{n+1}} - \frac{T_{r_n}x_n - x_n}{r_n}\right\rangle \geq 0.$$

It follows that

$$\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - T_{r_{n+1}}x_{n+1} + T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\rangle \geq 0. \tag{3.28}$$

From (3.28), we obtain

$$\begin{aligned}
&\left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\|^2 \\
&\leq \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\rangle \\
&\leq \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right] \\
&= \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right] \\
&\leq \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right],
\end{aligned}$$

which follows that

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \tag{3.29}$$

By applying (3.29) and (3.23), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| \leq\ & |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| \\
&+ |\gamma_n - \gamma_{n-1}| \|J_{M,\lambda}(I - \lambda G)x_{n-1}\| + |\eta_n - \eta_{n-1}| \|(I - \rho_{n-1}P)x_{n-1}\| \\
&+ \eta_n \|x_n - x_{n-1}\| + \rho_n \|Px_n - Px_{n-1}\| + |\rho_n - \rho_{n-1}| \|Px_{n-1}\| \\
&+ \delta_n \|u_n - u_{n-1}\| + |\delta_n - \delta_{n-1}| \|u_{n-1}\| \\
\leq\ & |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| \\
&+ |\gamma_n - \gamma_{n-1}| \|J_{M,\lambda}(I - \lambda G)x_{n-1}\| + |\eta_n - \eta_{n-1}| \|(I - \rho_{n-1}P)x_{n-1}\| \\
&+ \eta_n \|x_n - x_{n-1}\| + \rho_n \|Px_n - Px_{n-1}\| + |\rho_n - \rho_{n-1}| \|Px_{n-1}\| \\
&+ \delta_n \left(\|x_n - x_{n-1}\| + \frac{1}{d}|r_n - r_{n-1}| \|u_n - x_n\|\right) + |\delta_n - \delta_{n-1}| \|u_{n-1}\| \\
\leq\ & |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| \\
&+ |\gamma_n - \gamma_{n-1}| \|J_{M,\lambda}(I - \lambda G)x_{n-1}\| + |\eta_n - \eta_{n-1}| \|(I - \rho_{n-1}P)x_{n-1}\| \\
&+ \rho_n \|Px_n - Px_{n-1}\| + |\rho_n - \rho_{n-1}| \|Px_{n-1}\| \\
&+ \frac{1}{d}|r_n - r_{n-1}| \|u_n - x_n\| + |\delta_n - \delta_{n-1}| \|u_{n-1}\|.
\end{aligned} \tag{3.30}$$

Applying Lemma 2.9, (3.30), and the conditions (i), (ii), (v), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.31}$$

Step 3. We show that $\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|(I - \rho_n P)x_n - x_n\|$ $= \lim_{n\to\infty} \|J_{M,\lambda}(I - \lambda G)x_n - x_n\| = 0$. By the definition of x_n, (3.20), and (3.21), we have

$$\|x_{n+1} - z\|^2 = \|\alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda G)x_n + \eta_n(I - \rho_n P)x_n + \delta_n u_n - z\|^2$$

$$\begin{aligned}
&\le \alpha_n \|u - z\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \|J_{M,\lambda}(I - \lambda G)x_n - z\|^2 \\
&\quad + \eta_n \|(I - \rho_n P)x_n - z\|^2 + \delta_n \|u_n - z\|^2 - \beta_n\delta_n\|x_n - u_n\|^2 \\
&\quad - \beta_n\gamma_n \|J_{M,\lambda}(I - \lambda G)x_n - x_n\|^2 \\
&= \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|x_n - z\|^2 - \beta_n\delta_n\|x_n - u_n\|^2 \\
&\quad - \beta_n\gamma_n \|J_{M,\lambda}(I - \lambda G)x_n - x_n\|^2 \\
&\le \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \beta_n\delta_n\|x_n - u_n\|^2 - \beta_n\gamma_n \|J_{M,\lambda}(I - \lambda G)x_n - x_n\|^2 .
\end{aligned}$$

It implies that

$$\begin{aligned}
\beta_n\delta_n \|u_n - x_n\|^2 &\le \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 - \beta_n\gamma_n \|J_{M,\lambda}(I - \lambda G)x_n - x_n\|^2 \\
&\le \alpha_n \|u - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\| .
\end{aligned}$$

From the condition (i) and (3.31), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.32}$$

By continuing in the same direction as (3.32), we have

$$\lim_{n\to\infty} \|J_{M,\lambda}(I - \lambda G)x_n - x_n\| = 0. \tag{3.33}$$

From the definition of x_n, we have

$$\begin{aligned}
x_{n+1} - x_n &= \alpha_n (u - x_n) + \gamma_n (J_{M,\lambda}(I - \lambda G)x_n - x_n) \\
&\quad + \eta_n ((I - \rho_n P)x_n - x_n) + \delta_n(u_n - x_n).
\end{aligned}$$

From the condition (i), (3.31), (3.32), and (3.33), we have

$$\lim_{n\to\infty} \|(I - \rho_n(I - T))x_n - x_n\| = 0. \tag{3.34}$$

Step 4. We will show that $\limsup_{n\to\infty} \langle u - z, x_n - z\rangle \le 0$, where $z = P_{\mathcal{F}}u$.
To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - z, x_n - z\rangle = \lim_{k\to\infty} \langle u - z, x_{n_k} - z\rangle . \tag{3.35}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$. From (3.32), we obtain $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$.

First, we will show that $\omega \in \bigcap_{i=1}^{N} VI(H, A_i, M)$. Assume that $\omega \notin \bigcap_{i=1}^{N} VI(H, A_i, M)$. By Lemmas 2.39 and 3.1, $\bigcap_{i=1}^{N} VI(H, A_i, M) = F(J_{M,\lambda}((I-\lambda G)))$. Then $\omega \neq J_{M,\lambda}(I-\lambda G)\omega$, where $G = \sum_{i=1}^{N} b_i A_i$. By taking the nonexpansiveness of $J_{M,\lambda}((I-\lambda G))$, (3.33), and Opial's condition, we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - J_{M,\lambda}((I-\lambda G))\omega\| \\
&\leq \liminf_{k\to\infty} (\|x_{n_k} - J_{M,\lambda}((I-\lambda G))x_{n_k}\| \\
&\quad + \|J_{M,\lambda}((I-\lambda G))x_{n_k} - J_{M,\lambda}((I-\lambda G))\omega\|) \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - \omega\|.
\end{aligned}$$

This is a contradiction. Then we have

$$\omega \in \bigcap_{i=1}^{N} VI(H, A_i, M). \tag{3.36}$$

Next, we will show that $\omega \in F(T)$. Assume that $\omega \notin F(T)$. From Remark 3.5 (i), $F(T) = F(I - \rho_{n_k}(I-T))$. Then $\omega \neq (I - \rho_{n_k}(I-T))\omega$. From the condition (ii), (3.34), and Opial's condition, we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - (I - \rho_{n_k}(I-T))\omega\| \\
&\leq \liminf_{k\to\infty} (\|x_{n_k} - (I - \rho_{n_k}(I-T))x_{n_k}\| \\
&\quad + \|(I - \rho_{n_k}(I-T))x_{n_k} - (I - \rho_{n_k}(I-T))\omega\|) \\
&\leq \liminf_{k\to\infty} (\|x_{n_k} - (I - \rho_{n_k}(I-T))x_{n_k}\| \\
&\quad + \|x_{n_k} - \omega\| + \rho_{n_k}\|(I-T)x_{n_k} - (I-T)\omega\|) \\
&= \liminf_{k\to\infty} \|x_{n_k} - \omega\|.
\end{aligned}$$

This is a contradiction. Then we have

$$\omega \in F(T). \tag{3.37}$$

Since $0 < c \leq r_n \leq d, \forall n \in \mathbb{N}$, then we have $r_{n_k} \to r$ as $k \to \infty$ with $0 < c \leq r \leq d$. Applying Lemma 3.6, we have $\|T_{r_{n_k}} x_{n_k} - T_r x_{n_k}\| \to 0$ as $k \to \infty$. Next, we will show that $\omega \in \bigcap_{i=1}^{N} EP(F_i)$. Assume that $\omega \notin \bigcap_{i=1}^{N} EP(F_i)$. From Remark 2.38, we have $\omega \notin F(T_r)$. By Opial's condition and (3.32), we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - T_r\omega\| \\
&\leq \liminf_{k\to\infty} (\left\|x_{n_k} - T_{r_{n_k}} x_{n_k}\right\| \\
&\quad + \left\|T_{r_{n_k}} x_{n_k} - T_r x_{n_k}\right\| + \|T_r x_{n_k} - T_r\omega\|) \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - \omega\|.
\end{aligned}$$

This is a contradiction. Then, we have

$$\omega \in \bigcap_{i=1}^{N} EP(F_i). \tag{3.38}$$

From (3.36), (3.37), and (3.38), we can conclude that $\omega \in \mathcal{F}$.
Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in \mathcal{F}$. By (3.35) and Lemma 2.15, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle u - z, x_n - z\rangle &= \lim_{k\to\infty} \langle u - z, x_{n_k} - z\rangle \\ &= \langle u - z, \omega - z\rangle \\ &\leq 0. \end{aligned} \tag{3.39}$$

Step 5. Finally, we will show that $\lim_{n\to\infty} x_n = z$, where $z = P_{\mathcal{F}}u$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda G)x_n + \eta_n(I - \rho_n P)x_n + \delta_n u_n - z\|^2 \\ &\leq \|\alpha_n (u - z) + \beta_n (x_n - z) + \gamma_n (J_{M,\lambda}(I - \lambda G)x_n - z) \\ &\quad + \eta_n ((I - \rho_n P)x_n - z) + \delta_n (u_n - z) \|^2 \\ &\leq (\beta_n \|x_n - z\| + \gamma_n \|J_{M,\lambda}(I - \lambda G)x_n - z\| + \eta_n \|(I - \rho_n P)x_n - z\| \\ &\quad + \delta_n \|u_n - z\|)^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\leq (1 - \alpha_n) \|x_n - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle.\end{aligned}$$

From the condition (i), (3.39), and Lemma 2.9, we can conclude that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}u$. By (3.32), we find that $\{u_n\}$ converges strongly to $z = P_{\mathcal{F}}u$. This completes the proof. □

As a direct proof of Theorem 3.7, we obtain the following result.

Corollary 3.8. Let C be a nonempty closed convex subset of a real Hilbert space H and let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$ and let $A : H \to H$ be an α-inverse strongly monotone mapping. Let $T : H \to H$ be a κ-strictly pseudononspreading mapping. Assume $\mathcal{F} := F(T) \cap \bigcap_{i=1}^N EP(F_i) \cap VI(H, A, M) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} \displaystyle\sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda A)x_n \\ \qquad + \eta_n(I - \rho_n(I - T))x_n + \delta_n u_n, \forall n \geq 1, \end{cases} \tag{3.40}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq (0, 1)$ and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n + \eta_n + \delta_n = 1, 0 < \alpha < 1$, and $0 \leq a_i \leq 1$, for every $i = 1, 2, ..., N$, $r_n \in [c, d] \subset (0, 1)$, $0 < p \leq \beta_n, \gamma_n, \eta_n, \delta_n \leq q < 1$, $\rho_n \in (0, 1 - \kappa)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \rho_n < \infty$;

(iii) $0 < \lambda < 2\alpha$;

(iv) $\sum_{i=1}^{N} a_i = 1$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
$\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.

Proof. Put $A_i \equiv A$ for all $i = 1, 2, ..., N$ in Theorem 3.7. So, from Theorem 3.7, we obtain the desired result. □

Corollary 3.9. Let C be a nonempty closed convex subset of a real Hilbert space H and let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$. For every $i = 1, 2, \dots, N$, $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,...,N}\{\alpha_i\}$. Let $T : H \to H$ be a κ-strictly pseudononspreading mapping. Assume $\mathcal{F} := F(T) \cap EP(F) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. Let the sequences $\{x_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} b_i A_i) x_n \\ \qquad + \eta_n (I - \rho_n (I - T)) x_n + \delta_n u_n, \forall n \geq 1, \end{cases} \tag{3.41}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq (0, 1)$ and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n + \eta_n + \delta_n = 1$, $0 < \alpha < 1$, and $0 \leq b_i \leq 1$, for every $i = 1, 2, ..., N$, $r_n \in [c, d] \subset (0, 1)$, $0 < p \leq \beta_n, \gamma_n, \eta_n, \delta_n \leq q < 1$, $\rho_n \in (0, 1 - \kappa)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \rho_n < \infty$;

(iii) $0 < \lambda < 2\eta$;

(iv) $\sum_{i=1}^{N} b_i = 1$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
$\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.

Proof. Take $F = F_i$ and $N = 1$, $\forall i = 1, 2, \dots, N$ in Theorem 3.7, we obtain the desired conclusion. □

3.3 A Strong Convergence Theorem for Nonspreading Mappings in Hilbert space

In this section, we prove a strong convergence therem of a new iterative algorithm for finding a common element of the set of fixed points of a finite family of nonspreading mappings, the set of solutions of a finite family of equilibrium problems and the set of solutions of two variational inequality problems in a real Hilbert space.

Now, we introduce and prove the following remark which will be needed for the proof of our main theorems.

Remark 3.10. If we replace the mapping T in Lemma 2.31 by T is a nonspreading mapping, then Lemma 2.31 is still holds. Applying the same method in [49], we have

$$\|P_C(I-\lambda(I-T))x-y\| \leq \|x-y\|,$$

for all $x \in C, y \in F(T)$ and $\lambda \in (0,1)$.

Proof. Indeed, let $x \in C$ and $y \in F(T)$. Since T is a nonspreading mapping, we have

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + 2\langle x-Tx, y-Ty\rangle = \|x-y\|^2. \tag{3.42}$$

From (3.42), we obtain

$$\begin{aligned}
\|Tx-y\|^2 &= \|Tx-y-x+x\|^2 \\
&= \|x-y-(I-T)x\|^2 \\
&= \|x-y\|^2 - 2\langle x-y,(I-T)x\rangle + \|(I-T)x\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

It implies that

$$\|(I-T)x\|^2 \leq 2\langle x-y,(I-T)x\rangle. \tag{3.43}$$

From $F(T)=VI(C,I-T)$ and Lemma 2.32, we have

$$y \in F(T) = VI(C,I-T) = F\left(P_C(I-\lambda(I-T))\right). \tag{3.44}$$

By the nonexpansiveness of P_C, (3.43) and (3.44), we get

$$\begin{aligned}
\|P_C(I-\lambda(I-T))x-y\|^2 &= \|P_C(I-\lambda(I-T))x-P_C(I-\lambda(I-T))y\|^2 \\
&\leq \|(I-\lambda(I-T))x-(I-\lambda(I-T))y\|^2 \\
&= \|(x-y)-\lambda(I-T)x\|^2 \\
&= \|x-y\|^2 - 2\lambda\langle x-y,(I-T)x\rangle + \lambda^2\|(I-T)x\|^2 \\
&\leq \|x-y\|^2 - \lambda\|(I-T)x\|^2 + \lambda^2\|(I-T)x\|^2 \\
&= \|x-y\|^2 - \lambda(1-\lambda)\|(I-T)x\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

□

Next, we show that a strong convergence therem of a new iterative algorithm for finding a common element of the set of fixed points of a finite family of nonspreading mappings, the set of solutions of a finite family of equilibrium problems and the set of solutions of two variational inequality problems in a real Hilbert space.

Theorem 3.11. Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, with $\eta = min\{\alpha, \beta\}$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0, 1)$. Suppose that $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N EP(F_i) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let the sequences $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n^i = b_n x_n + (1 - b_n) P_C(I - \lambda_n(I - T_i)) u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \rho_n(aA + (1-a)B)) x_n + \gamma_n \sum_{i=1}^N c_i y_n^i, \forall n \geq 1, \end{cases} \tag{3.45}$$

where $a \in (0, 1)$, $\{b_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\rho_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $0 \leq a_i, c_i \leq 1$, for every $i = 1, 2, ..., N$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} b_n = b \in (0, 1)$;

(iii) $0 < \rho_n < 2\eta$;

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(v) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} c_i = 1$ and $0 < g \leq a_i, c_i \leq h < 1$ for all $i = 1, 2, ..., N$;

(vi) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(vii) $0 < c \leq r_n, \beta_n, \gamma_n \leq d < 1, \forall n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$.

Proof. We divide the proof of Theorem 3.11 into five steps:

Step 1. We show that the sequence $\{x_n\}$ is bounded. Since

$$\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C,$$

by Lemma 2.36 and Remark 2.38, we have $u_n = T_{r_n} x_n$ and $F(T_{r_n}) = \bigcap_{i=1}^{N} EP(F_i)$.

Let $z \in \mathcal{F}$. From Lemma 2.34 and Lemma 2.32, we have

$$z \in VI(C, aA + (1-a)B) = F(P_C(I - \rho_n(aA + (1-a)B))).$$

Since $z \in \bigcap_{i=1}^{N} F(T_i)$. By Remark 3.10, we have

$$\begin{aligned} \|P_C(I - \lambda_n(I - T_i))u_n - z\|^2 &\leq \|u_n - z\|^2 \\ &= \|T_{r_n} x_n - z\|^2 \\ &\leq \|x_n - z\|^2. \end{aligned} \tag{3.46}$$

Put $M_n = \sum_{i=1}^{N} c_i y_n^i$ and (3.46), we have

$$\begin{aligned} \|M_n - z\| &= \left\| \sum_{i=1}^{N} c_i \left(y_n^i - z\right) \right\| \\ &\leq \sum_{i=1}^{N} c_i \left\|y_n^i - z\right\| \\ &= \sum_{i=1}^{N} c_i \left\|b_n x_n + (1-b_n) P_C(I - \lambda_n(I - T_i))u_n - z\right\| \\ &\leq \sum_{i=1}^{N} c_i \left(b_n \|x_n - z\| + (1-b_n) \|P_C(I - \lambda_n(I - T_i))u_n - z\|\right) \\ &\leq \left(b_n \|x_n - z\| + (1-b_n) \|x_n - z\|\right) \\ &= \|x_n - z\|. \end{aligned} \tag{3.47}$$

From the definition of x_n, Lemma 2.34 and (3.47), we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n f(x_n) + \beta_n P_C(I - \rho_n(aA + (1-a)B))x_n + \gamma_n M_n - z\| \\ &\leq \alpha_n \|f(x_n) - z\| + \beta_n \|P_C(I - \rho_n(aA + (1-a)B))x_n - z\| + \gamma_n \|M_n - z\| \\ &\leq \alpha_n (\|f(x_n) - f(z)\| + \|f(z) - z\|) + \beta_n \|x_n - z\| + \gamma_n \|x_n - z\| \\ &\leq (1 - \alpha_n(1-\alpha)) \|x_n - z\| + \alpha_n \|f(z) - z\| \\ &\leq \max\left\{ \|x_1 - z\|, \frac{\|f(z) - z\|}{1-\alpha} \right\} \end{aligned}$$

Put $K = \max\left\{ \|x_1 - z\|, \frac{\|f(z) - z\|}{1-\alpha} \right\}$. By employing the mathematical induction, we have $\|x_n - z\| \leq K, \forall n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{u_n\}$.

Step 2. For every $i = 1, 2, ..., N$, put $P = aA + (1-a)B$ and $D_i = I - T_i$. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. By the definition of x_n and Lemma 2.34, we obtain

$$\begin{aligned}
\|x_{n+1}-x_n\| =&\|\alpha_n f(x_n)+\beta_n P_C(I-\rho_n P)x_n+\gamma_n M_n \\
&-\alpha_{n-1}f(x_{n-1})-\beta_{n-1}P_C(I-\rho_{n-1}P)x_{n-1}-\gamma_{n-1}M_{n-1}\| \\
\leq&\alpha_n\|f(x_n)-f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| \\
&+\beta_n\|P_C(I-\rho_n P)x_n-P_C(I-\rho_n P)x_{n-1}\| \\
&+\beta_n\|P_C(I-\rho_n P)x_{n-1}-P_C(I-\rho_{n-1}P)x_{n-1}\| \\
&+|\beta_n-\beta_{n-1}|\|P_C(I-\rho_{n-1}P)x_{n-1}\| \\
&+\gamma_n\|M_n-M_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|M_{n-1}\| \\
\leq&\alpha_n\alpha\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|+\beta_n\|x_n-x_{n-1}\| \\
&+\beta_n\|(I-\rho_n P)x_{n-1}-(I-\rho_{n-1}P)x_{n-1}\|+|\beta_n-\beta_{n-1}|\|P_C(I-\rho_{n-1}P)x_{n-1}\| \\
&+\gamma_n\|M_n-M_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|M_{n-1}\| \\
\leq&\alpha_n\alpha\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|+\beta_n\|x_n-x_{n-1}\| \\
&+\beta_n|\rho_n-\rho_{n-1}|\|Px_{n-1}\|+|\beta_n-\beta_{n-1}|\|P_C(I-\rho_{n-1}P)x_{n-1}\| \\
&+\gamma_n\|M_n-M_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|M_{n-1}\|. \qquad (3.48)
\end{aligned}$$

By continuing in the same direction as in step 2 of Theorem 3.1 in [29], we have

$$\|u_n-u_{n-1}\|\leq\|x_n-x_{n-1}\|+\frac{1}{d}|r_n-r_{n-1}|\,\|u_n-x_n\|.$$

Indeed, since $u_n=T_{r_n}x_n$, by utilizing the definition of T_{r_n}, we obtain

$$\sum_{i=1}^{N}a_iF_i\left(T_{r_n}x_n,y\right)+\frac{1}{r_n}\left\langle y-T_{r_n}x_n,T_{r_n}x_n-x_n\right\rangle\geq 0,\forall y\in C, \qquad (3.49)$$

and

$$\sum_{i=1}^{N}a_iF_i\left(T_{r_{n+1}}x_{n+1},y\right)+\frac{1}{r_{n+1}}\left\langle y-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\right\rangle\geq 0,\forall y\in C. \qquad (3.50)$$

From (3.49) and (3.50), it follows that

$$\sum_{i=1}^{N}a_iF_i\left(T_{r_n}x_n,T_{r_{n+1}}x_{n+1}\right)+\frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n,T_{r_n}x_n-x_n\right\rangle\geq 0, \qquad (3.51)$$

and

$$\sum_{i=1}^{N}a_iF_i\left(T_{r_{n+1}}x_{n+1},T_{r_n}x_n\right)+\frac{1}{r_{n+1}}\left\langle T_{r_n}x_n-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\right\rangle\geq 0. \qquad (3.52)$$

From (3.51) and (3.52) and the fact that $\sum_{i=1}^{N}a_iF_i$ satisfies (A2), we have

$$\frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n,T_{r_n}x_n-x_n\right\rangle+\frac{1}{r_{n+1}}\left\langle T_{r_n}x_n-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\right\rangle\geq 0,$$

which implies that

$$\left\langle T_{r_n}x_n-T_{r_{n+1}}x_{n+1},\frac{T_{r_{n+1}}x_{n+1}-x_{n+1}}{r_{n+1}}-\frac{T_{r_n}x_n-x_n}{r_n}\right\rangle\geq 0.$$

It follows that

$$\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - T_{r_{n+1}}x_{n+1} + T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\rangle \geq 0. \tag{3.53}$$

From (3.53), we obtain

$$\begin{aligned}
&\left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\|^2 \\
\leq &\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\rangle \\
\leq &\left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right] \\
= &\left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right] \\
\leq &\left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right],
\end{aligned}$$

which follows that

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \tag{3.54}$$

By the definition of M_n and applying (3.54), we have

$$\begin{aligned}
\|M_n - M_{n-1}\| =&\| \sum_{i=1}^{N} c_i y_n^i - \sum_{i=1}^{N} c_i y_{n-1}^i \| \\
\leq &\sum_{i=1}^{N} c_i \|y_n^i - y_{n-1}^i\| \\
= &\sum_{i=1}^{N} c_i \|b_n x_n + (1 - b_n) P_C(I - \lambda_n D_i) u_n - b_{n-1} x_{n-1} \\
&- (1 - b_{n-1}) P_C(I - \lambda_{n-1} D_i) u_{n-1}\| \\
\leq &\sum_{i=1}^{N} c_i \big(b_n \|x_n - x_{n-1}\| + |b_n - b_{n-1}| \|x_{n-1}\| \\
&+ (1 - b_n) \|(I - \lambda_n D_i) u_n - (I - \lambda_{n-1} D_i) u_{n-1}\| \\
&+ |b_n - b_{n-1}| \|P_C(I - \lambda_{n-1} D_i) u_{n-1}\|\big) \\
\leq &\sum_{i=1}^{N} c_i \big(b_n \|x_n - x_{n-1}\| + |b_n - b_{n-1}| \|x_{n-1}\| \\
&+ (1 - b_n) \left(\|u_n - u_{n-1}\| + \lambda_n \|D_i u_n - D_i u_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|D_i u_{n-1}\|\right) \\
&+ |b_n - b_{n-1}| \|P_C(I - \lambda_{n-1} D_i) u_{n-1}\|\big) \\
\leq &b_n \|x_n - x_{n-1}\| + |b_n - b_{n-1}| \|x_{n-1}\| + (1 - b_n) \|u_n - u_{n-1}\| \\
&+ \sum_{i=1}^{N} c_i \lambda_n (1 - b_n) \|D_i u_n - D_i u_{n-1}\| + \sum_{i=1}^{N} c_i (1 - b_n) |\lambda_n - \lambda_{n-1}| \|D_i u_{n-1}\| \\
&+ \sum_{i=1}^{N} c_i |b_n - b_{n-1}| \|P_C(I - \lambda_{n-1} D_i) u_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&\leq b_n\|x_n - x_{n-1}\| + |b_n - b_{n-1}|\|x_{n-1}\| \\
&\quad + (1-b_n)\left(\|x_n - x_{n-1}\| + \frac{1}{d}\,|r_n - r_{n-1}|\,\|u_n - x_n\|\right) \\
&\quad + \sum_{i=1}^{N} c_i\lambda_n(1-b_n)\|D_iu_n - D_iu_{n-1}\| + \sum_{i=1}^{N} c_i(1-b_n)|\lambda_n - \lambda_{n-1}|\|D_iu_{n-1}\| \\
&\quad + \sum_{i=1}^{N} c_i|b_n - b_{n-1}|\|P_C(I - \lambda_{n-1}D_i)u_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + |b_n - b_{n-1}|\|x_{n-1}\| + \frac{1}{d}\,|r_n - r_{n-1}|\,\|u_n - x_n\| \\
&\quad + \sum_{i=1}^{N} c_i\lambda_n(1-b_n)\|D_iu_n - D_iu_{n-1}\| + \sum_{i=1}^{N} c_i(1-b_n)|\lambda_n - \lambda_{n-1}|\|D_iu_{n-1}\| \\
&\quad + \sum_{i=1}^{N} c_i|b_n - b_{n-1}|\|P_C(I - \lambda_{n-1}D_i)u_{n-1}\|. \qquad (3.55)
\end{aligned}
$$

Substitute (3.55) into (3.48), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq \alpha_n\alpha\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \beta_n\|x_n - x_{n-1}\| \\
&\quad + \beta_n|\rho_n - \rho_{n-1}|\|Px_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I - \rho_{n-1}P)x_{n-1}\| \\
&\quad + \gamma_n\|M_n - M_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|M_{n-1}\| \\
&\leq \alpha_n\alpha\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \beta_n\|x_n - x_{n-1}\| \\
&\quad + \beta_n|\rho_n - \rho_{n-1}|\|Px_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I - \rho_{n-1}P)x_{n-1}\| \\
&\quad + \gamma_n\Big(\|x_n - x_{n-1}\| + |b_n - b_{n-1}|\|x_{n-1}\| + \frac{1}{d}\,|r_n - r_{n-1}|\,\|u_n - x_n\| \\
&\quad + \sum_{i=1}^{N} c_i\lambda_n(1-b_n)\|D_iu_n - D_iu_{n-1}\| + \sum_{i=1}^{N} c_i(1-b_n)|\lambda_n - \lambda_{n-1}|\|D_iu_{n-1}\| \\
&\quad + \sum_{i=1}^{N} c_i|b_n - b_{n-1}|\|P_C(I - \lambda_{n-1}D_i)u_{n-1}\|\Big) \\
&\quad + |\gamma_n - \gamma_{n-1}|\|M_{n-1}\| \\
&\leq \alpha_n\alpha\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\
&\quad + \beta_n|\rho_n - \rho_{n-1}|\|Px_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I - \rho_{n-1}P)x_{n-1}\| \\
&\quad + |b_n - b_{n-1}|\|x_{n-1}\| + \frac{1}{d}\,|r_n - r_{n-1}|\,\|u_n - x_n\| \\
&\quad + \sum_{i=1}^{N} c_i\lambda_n(1-b_n)\|D_iu_n - D_iu_{n-1}\| + \sum_{i=1}^{N} c_i(1-b_n)|\lambda_n - \lambda_{n-1}|\|D_iu_{n-1}\| \\
&\quad + \sum_{i=1}^{N} c_i|b_n - b_{n-1}|\|P_C(I - \lambda_{n-1}D_i)u_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\|M_{n-1}\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\alpha_n(1-\alpha))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad + |\rho_n - \rho_{n-1}|\|Px_{n-1}\| + |\beta_n - \beta_{n-1}|\|P_C(I-\rho_{n-1}P)x_{n-1}\| \\
&\quad + |b_n - b_{n-1}|\|x_{n-1}\| + \frac{1}{d}|r_n - r_{n-1}|\|u_n - x_n\| \\
&\quad + \lambda_n \sum_{i=1}^{N} c_i\|D_i u_n - D_i u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\sum_{i=1}^{N} c_i\|D_i u_{n-1}\| \\
&\quad + |b_n - b_{n-1}|\sum_{i=1}^{N} c_i\|P_C(I-\lambda_{n-1}D_i)u_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\|M_{n-1}\|.
\end{aligned} \tag{3.56}
$$

Applying Lemma 2.9, (3.56), and the conditions (i), (iv), (vi), we have

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.57}
$$

Step 3. We show that $\lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\|P_C(I-\rho_n P)x_n - x_n\| = \lim_{n\to\infty}\|P_C(I-\lambda_n D_i)x_n - x_n\| = 0$, for all $i = 1, 2, 3, ..., N$.
To show this, let $z \in \mathcal{F}$. Since $u_n = T_{r_n}x_n$ and T_{r_n} is firmly nonexpansive mapping, then we obtain

$$
\begin{aligned}
\|z - T_{r_n}x_n\|^2 &= \|T_{r_n}z - T_{r_n}x_n\|^2 \\
&\leq \langle T_{r_n}z - T_{r_n}x_n, z - x_n\rangle \\
&= \frac{1}{2}\left(\|T_{r_n}x_n - z\|^2 + \|x_n - z\|^2 - \|T_{r_n}x_n - x_n\|^2\right),
\end{aligned}
$$

which follows that

$$
\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{3.58}
$$

From the definition of x_n and (3.58), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n f(x_n) + \beta_n P_C(I-\rho_n P)x_n + \gamma_n M_n - z\|^2 \\
&\leq \alpha_n\|f(x_n) - z\|^2 + \beta_n\|P_C(I-\rho_n P)x_n - z\|^2 + \gamma_n\|M_n - z\|^2 \\
&\leq \alpha_n\|f(x_n) - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left\|\sum_{i=1}^{N} c_i\left(y_n^i - z\right)\right\|^2 \\
&\leq \alpha_n\|f(x_n) - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + \gamma_n\sum_{i=1}^{N} c_i\|(b_n x_n + (1-b_n)P_C(I-\lambda_n D_i)u_n - z)\|^2 \\
&\leq \alpha_n\|f(x_n) - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + \gamma_n\sum_{i=1}^{N} c_i\left(b_n\|x_n - z\|^2 + (1-b_n)\|P_C(I-\lambda_n D_i)u_n - z\|^2\right) \\
&= \alpha_n\|f(x_n) - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + \gamma_n\left(b_n\|x_n - z\|^2 + (1-b_n)\|u_n - z\|^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha_n \|f(x_n) - z\|^2 + \beta_n \|x_n - z\|^2 \\
&\quad + \gamma_n \left(b_n \|x_n - z\|^2 + (1-b_n)\|x_n - z\|^2 - (1-b_n)\|u_n - x_n\|^2 \right) \\
&\le \alpha_n \|f(x_n) - z\|^2 + \|x_n - z\|^2 - (1-b_n)\gamma_n \|u_n - x_n\|^2 .
\end{aligned}
$$

It implies that

$$
\begin{aligned}
(1-b_n)\gamma_n \|u_n - x_n\|^2 &\le \alpha_n \|f(x_n) - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\le \alpha_n \|f(x_n) - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\|.
\end{aligned}
$$

From the condition (i) and (3.57), we have

$$
\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.59}
$$

From the nonexpansive of P_C, we have

$$
\begin{aligned}
\|P_C(I-\rho_n P)x_n - z\|^2 &= \|P_C(I-\rho_n P)x_n - P_C(I-\rho_n P)z\|^2 \\
&\le \|(I-\rho_n P)x_n - (I-\rho_n P)z\|^2 \\
&= \|(x_n - z) - \rho_n(Px_n - Pz)\|^2 \\
&\le \|x_n - z\|^2 + \rho_n^2\|Px_n - Pz\|^2 - 2\rho_n\langle x_n - z, Px_n - Pz\rangle. \qquad (3.60)
\end{aligned}
$$

For every $x, y \in C$ and $P = aA + (1-a)B$, we have

$$
\begin{aligned}
\langle Px - Py, x - y\rangle &= \langle aAx + (1-a)Bx - (aAy + (1-a)By), x - y\rangle \\
&= \langle a(Ax - Ay) + (1-a)(Bx - By), x - y\rangle \\
&= a\langle Ax - Ay, x - y\rangle + (1-a)\langle Bx - By, x - y\rangle \\
&\ge a\alpha\|Ax - Ay\|^2 + (1-a)\beta\|Bx - By\|^2 \\
&\ge \eta\left(a\|Ax - Ay\|^2 + (1-a)\|Bx - By\|^2\right) \\
&\ge \eta\|a(Ax - Ay) + (1-a)(Bx - By)\|^2 \\
&= \eta\|aAx + (1-a)Bx - (aAy + (1-a)By)\|^2 \\
&= \eta\|Px - Py\|^2 . \qquad (3.61)
\end{aligned}
$$

From (3.60) and (3.61), we have

$$
\begin{aligned}
\|P_C(I-\rho_n P)x_n - z\|^2 &\le \|x_n - z\|^2 + \rho_n^2\|Px_n - Pz\|^2 - 2\rho_n\langle x_n - z, Px_n - Pz\rangle \\
&\le \|x_n - z\|^2 + \rho_n^2\|Px_n - Pz\|^2 - 2\rho_n\eta\|Px_n - Pz\|^2 \\
&= \|x_n - z\|^2 + \rho_n(\rho_n - 2\eta)\|Px_n - Pz\|^2. \qquad (3.62)
\end{aligned}
$$

From the definition of x_n and (3.62), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n f(x_n) + \beta_n P_C(I-\rho_n P)x_n + \gamma_n M_n - z\|^2 \\
&\le \alpha_n \|f(x_n) - z\|^2 + \beta_n \|P_C(I-\rho_n P)x_n - z\|^2 + \gamma_n \|M_n - z\|^2 \\
&\le \alpha_n \|f(x_n) - z\|^2 + \beta_n\left(\|x_n - z\|^2 + \rho_n(\rho_n - 2\eta)\|Px_n - Pz\|^2\right) + \gamma_n \|x_n - z\|^2 \\
&\le \alpha_n \|f(x_n) - z\|^2 + \|x_n - z\|^2 + \rho_n\beta_n(\rho_n - 2\eta)\|Px_n - Pz\|^2.
\end{aligned}
$$

It follows that

$$\begin{aligned}\rho_n\beta_n(2\eta-\rho_n)\|Px_n-Pz\|^2 \leq&\alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2\\ \leq&\alpha_n\|f(x_n)-z\|^2+(\|x_n-z\|+\|x_{n+1}-z\|)\|x_{n+1}-x_n\|.\end{aligned}$$

From the condition (i) and (3.57), we have

$$\lim_{n\to\infty}\|Px_n-Pz\|=0. \tag{3.63}$$

From Lemma 2.34, we have

$$\begin{aligned}\|P_C(I-\rho_nP)x_n-z\|^2=&\|P_C(I-\rho_nP)x_n-P_C(I-\rho_nP)z\|^2\\ \leq&\langle(I-\rho_nP)x_n-(I-\rho_nP)z,P_C(I-\rho_nP)x_n-z\rangle\\ =&\frac{1}{2}\big(\|(I-\rho_nP)x_n-(I-\rho_nP)z\|^2+\|P_C(I-\rho_nP)x_n-z\|^2\\ &-\|(I-\rho_nP)x_n-(I-\rho_nP)z-(P_C(I-\rho_nP)x_n-z)\|^2\big)\\ \leq&\frac{1}{2}\big(\|x_n-z\|^2+\|P_C(I-\rho_nP)x_n-z\|^2\\ &-\|x_n-P_C(I-\rho_nP)x_n-\rho_n(Px_n-Pz)\|^2\big)\\ =&\frac{1}{2}\big(\|x_n-z\|^2+\|P_C(I-\rho_nP)x_n-z\|^2-\|x_n-P_C(I-\rho_nP)x_n\|^2\\ &-\rho_n^2\|Px_n-Pz\|^2+2\rho_n\langle x_n-P_C(I-\rho_nP)x_n,Px_n-Pz\rangle\big)\\ \leq&\frac{1}{2}\big(\|x_n-z\|^2+\|P_C(I-\rho_nP)x_n-z\|^2-\|x_n-P_C(I-\rho_nP)x_n\|^2\\ &-\rho_n^2\|Px_n-Pz\|^2+2\rho_n\|x_n-P_C(I-\rho_nP)x_n\|\|Px_n-Pz\|\big).\end{aligned}$$

It follows that

$$\begin{aligned}\|P_C(I-\rho_nP)x_n-z\|^2\leq&\|x_n-z\|^2-\|x_n-P_C(I-\rho_nP)x_n\|^2\\ &+2\rho_n\|x_n-P_C(I-\rho_nP)x_n\|\|Px_n-Pz\|.\end{aligned} \tag{3.64}$$

From the definition of x_n and (3.64), we have

$$\begin{aligned}\|x_{n+1}-z\|^2=&\|\alpha_nf(x_n)+\beta_nP_C(I-\rho_nP)x_n+\gamma_nM_n-z\|^2\\ \leq&\alpha_n\|f(x_n)-z\|^2+\beta_n\|P_C(I-\rho_nP)x_n-z\|^2+\gamma_n\|M_n-z\|^2\\ \leq&\alpha_n\|f(x_n)-z\|^2+\beta_n\big(\|x_n-z\|^2-\|x_n-P_C(I-\rho_nP)x_n\|^2\\ &+2\rho_n\|x_n-P_C(I-\rho_nP)x_n\|\|Px_n-Pz\|\big)+\gamma_n\|x_n-z\|^2\\ \leq&\alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2-\beta_n\|x_n-P_C(I-\rho_nP)x_n\|^2\\ &+2\rho_n\|x_n-P_C(I-\rho_nP)x_n\|\|Px_n-Pz\|.\end{aligned}$$

Which implies that

$$\begin{aligned}\beta_n\|x_n-P_C(I-\rho_nP)x_n\|^2\leq&\alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2\\ &+2\rho_n\|x_n-P_C(I-\rho_nP)x_n\|\|Px_n-Pz\|\\ \leq&\alpha_n\|f(x_n)-z\|^2+(\|x_n-z\|+\|x_{n+1}-z\|)\|x_{n+1}-x_n\|\\ &+4\eta\|x_n-P_C(I-\rho_nP)x_n\|\|Px_n-Pz\|.\end{aligned}$$

From the condition (i), (3.57), and (3.63), we have

$$\lim_{n\to\infty} \|P_C(I-\rho_n P)x_n - x_n\| = 0. \tag{3.65}$$

From the definition of x_n, we have

$$x_{n+1} - x_n = \alpha_n(f(x_n) - x_n) + \beta_n(P_C(I-\rho_n P)x_n - x_n) + \gamma_n(M_n - x_n).$$

From the condition (i), (3.57), (3.63) and (3.65), we have

$$\lim_{n\to\infty} \|M_n - x_n\| = 0. \tag{3.66}$$

From the definition of M_n and (3.46), we have

$$\begin{aligned}
\|M_n - z\|^2 &\le \sum_{i=1}^{N} c_i \left\|y_n^i - z\right\|^2 \\
&= \sum_{i=1}^{N} c_i \|b_n(x_n - z) + (1-b_n)(P_C(I-\lambda_n D_i)u_n - z)\|^2 \\
&= \sum_{i=1}^{N} c_i \big(b_n \|x_n - z\|^2 + (1-b_n)\|P_C(I-\lambda_n D_i)u_n - z\|^2 \\
&\quad - b_n(1-b_n)\|x_n - P_C(I-\lambda_n D_i)u_n\|^2\big) \\
&\le \|x_n - z\|^2 - b_n(1-b_n)\sum_{i=1}^{N} c_i \|x_n - P_C(I-\lambda_n D_i)u_n\|^2 .
\end{aligned}$$

It implies that

$$\begin{aligned}
b_n(1-b_n)\sum_{i=1}^{N} c_i \|x_n - P_C(I-\lambda_n D_i)u_n\|^2 &\le \|x_n - z\|^2 - \|M_n - z\|^2 \\
&\le (\|x_n - z\| + \|M_n - z\|)\|M_n - x_n\| .
\end{aligned}$$

From the condition (ii) and (3.66), we have

$$\lim_{n\to\infty} \|P_C(I-\lambda_n D_i)u_n - x_n\| = 0, \tag{3.67}$$

for all $i = 1, 2, ..., N$.

Step 4. We show that $\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle \le 0$ where $z_0 = P_{\mathcal{F}} f(z_0)$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle = \lim_{k\to\infty} \langle f(z_0) - z_0, x_{n_k} - z_0\rangle . \tag{3.68}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$. From (3.59), we obtain $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$.

First, we will show that $\omega \in \bigcap_{i=1}^{N} F(T_i)$. Assume $\omega \notin \bigcap_{i=1}^{N} F(T_i)$, then we have $\omega \notin F(T_{i_0})$, for some $i_0 = 1, 2, ..., N$. From Lemma 2.32 and Remark 3.10, we have $\omega \neq P_C(I - \lambda_{n_k}(I - T_{i_0}))\omega$. From the nonexpansiveness of P_C, the condition (iv), (3.67) and the Opial's condition, we have

$$
\begin{aligned}
\liminf_{k\to\infty}\|u_{n_k}-\omega\| < & \liminf_{k\to\infty}\|u_{n_k}-P_C(I-\lambda_{n_k}(I-T_{i_0}))\omega\| \\
\le & \liminf_{k\to\infty}\big(\|u_{n_k}-x_{n_k}\|+\|x_{n_k}-P_C(I-\lambda_{n_k}(I-T_{i_0}))u_{n_k}\| \\
& + \|P_C(I-\lambda_{n_k}(I-T_{i_0}))u_{n_k}-P_C(I-\lambda_{n_k}(I-T_{i_0}))\omega\|\big) \\
\le & \liminf_{k\to\infty}\big(\|u_{n_k}-x_{n_k}\|+\|x_{n_k}-P_C(I-\lambda_{n_k}(I-T_{i_0}))u_{n_k}\| \\
& + \|u_{n_k}-\omega\|+\lambda_{n_k}\|(I-T_{i_0})u_{n_k}-(I-T_{i_0})\omega\|\big) \\
\le & \liminf_{k\to\infty}\|u_{n_k}-\omega\|.
\end{aligned}
$$

This is a contradiction. Then

$$\omega \in \bigcap_{i=1}^{N} F(T_i). \tag{3.69}$$

By continuing the same method of proof as Step 4 of Theorem 3.1 in [29], we obtain

$$\omega \in \bigcap_{i=1}^{N} EP(F_i). \tag{3.70}$$

Indeed, Next, we will show that $\omega \in \bigcap_{i=1}^{N} EP(F_i)$.

Since

$$\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \forall y \in C,$$

and $\sum_{i=1}^{N} a_i F_i$ satisfies the conditions (A1)-(A4), we obtain

$$\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge \sum_{i=1}^{N} a_i F_i(y, u_n), \forall y \in C.$$

In particular, it follows that

$$\left\langle y-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle \ge \sum_{i=1}^{N} a_i F_i(y, u_{n_k}), \forall y \in C. \tag{3.71}$$

From (3.59), (3.71) and (A4), we have

$$\sum_{i=1}^{N} a_i F_i(y, \omega) \le 0, \ \forall y \in C. \tag{3.72}$$

Put $y_t := ty + (1-t)\omega$, $t \in (0,1]$, we have $y_t \in C$. By using (A1), (A4) and (3.72), we have

$$
\begin{aligned}
0 &= \sum_{i=1}^{N} a_i F_i(y_t, y_t) \\
&= \sum_{i=1}^{N} a_i F_i(y_t, ty+(1-t)\omega) \\
&\le t\sum_{i=1}^{N} a_i F_i(y_t, y) + (1-t)\sum_{i=1}^{N} a_i F_i(y_t, \omega) \\
&\le t\sum_{i=1}^{N} a_i F_i(y_t, y).
\end{aligned}
$$

It implies that

$$\sum_{i=1}^{N} a_i F_i \left(ty + (1-t)\omega, y\right) \geq 0, \forall t \in (0,1] \text{ and } \forall y \in C. \tag{3.73}$$

From (3.73), taking $t \to 0^+$ and using (A3), we can conclude that

$$0 \leq \sum_{i=1}^{N} a_i F_i \left(\omega, y\right), \ \forall y \in C.$$

Therefore, $\omega \in EP\left(\sum_{i=1}^{N} a_i F_i\right)$. By Lemma 2.37, we obtain $EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP\left(F_i\right)$. It follows that

$$\omega \in \bigcap_{i=1}^{N} EP\left(F_i\right).$$

Next, we will show that $\omega \in VI(C,A) \cap VI(C,B)$. Assume that $\omega \notin VI(C,A) \cap VI(C,B)$. Since Lemma 2.32 and Lemma 2.34, we have $\omega \neq P_C(I - \rho_{n_k}(aA + (1-a)B))\omega$. By the nonexpansiveness of $P_C(I - \rho_{n_k}(aA + (1-a)B))$, (3.65) and the Opial's condition, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - P_C(I - \rho_{n_k}(aA + (1-a)B))x_{n_k} \\
&\quad + P_C(I - \rho_{n_k}(aA + (1-a)B))x_{n_k} - P_C(I - \rho_{n_k}(aA + (1-a)B))\omega\| \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - \omega\|.
\end{aligned}$$

This is a contradiction. Then

$$\omega \in VI(C,A) \cap VI(C,B). \tag{3.74}$$

From (3.69), (3.70), and (3.74), we get $\omega \in \mathcal{F}$. Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in \mathcal{F}$. By (3.68) and Lemma 2.32, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0 \rangle &= \lim_{k\to\infty} \langle f(z_0) - z_0, x_{n_k} - z_0 \rangle \\
&= \langle f(z_0) - z_0, \omega - z_0 \rangle \\
&\leq 0. \qquad (3.75)
\end{aligned}$$

Step 5. Finally, we show that $\lim_{n\to\infty} x_n = z_0$, where $z_0 = P_{\mathcal{F}} f(z_0)$. From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n(f(x_n) - z_0) + \beta_n(P_C(I - \rho_n P)x_n - z_0) + \gamma_n(M_n - z_0)\|^2 \\
&\leq \|\beta_n(P_C(I - \rho_n P)x_n - z_0) + \gamma_n(M_n - z_0)\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\|f(x_n) - f(z_0)\|\|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\alpha\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + \alpha_n\alpha(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\quad + 2\alpha_n\langle f(z) - z_0, x_{n+1} - z_0\rangle \\
&= (1-\alpha_n)^2\|x_n - z_0\|^2 + \alpha_n\alpha\|x_n - z_0\|^2 + \alpha_n\alpha\|x_{n+1} - z_0\|^2 \\
&\quad + 2\alpha_n\langle f(z) - z_0, x_{n+1} - z_0\rangle.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}\|x_n - z_0\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= \frac{1 - 2\alpha_n + \alpha_n^2 + \alpha_n\alpha}{1-\alpha_n\alpha}\|x_n - z_0\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= \frac{1 - 2\alpha_n + \alpha_n\alpha}{1-\alpha_n\alpha}\|x_n - z_0\|^2 + \frac{\alpha_n^2}{1-\alpha_n\alpha}\|x_n - z_0\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= \left(1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\right)\|x_n - z_0\|^2 + \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\left(\frac{\alpha_n}{2(1-\alpha)}\|x_n - z_0\|^2\right. \\
&\quad \left. + \frac{1}{1-\alpha}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle\right).
\end{aligned}
$$

From the condition (i), (3.75) and Lemma 2.9, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}f(z_0)$. This completes the proof. □

As a direct consequence of Theorem 3.11, we obtain the following result.

Corollary 3.12. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, with $\eta = min\{\alpha, \beta\}$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself with $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap EP(F) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0,1)$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$
\begin{cases}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\
y_n^i = b_n x_n + (1-b_n)P_C(I - \lambda_n(I - T_i))u_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \rho_n(aA + (1-a)B))x_n + \gamma_n \sum_{i=1}^N c_i y_n^i, \forall n \geq 1,
\end{cases}
\tag{3.76}
$$

where $a \in (0,1)$, $\{b_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\rho_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$. Assume the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} b_n = b \in (0,1)$;

(iii) $0 < \rho_n < 2\eta$;

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(v) $\sum_{i=1}^{N} c_i = 1$ and $0 < g \le c_i \le h < 1$ for all $i = 1, 2, ..., N$;

(vi) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$; $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(vii) $0 < c \le r_n, \beta_n, \gamma_n \le d < 1, \forall n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$.

Proof. For every $i = 1, 2, \ldots, N$, put $F = F_i$ in Theorem 3.11, we obtain the desired result. □

Corollary 3.13. Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)-(A4)$ and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, with $\eta = min\{\alpha, \beta\}$. Let T be a nonspreading mappings of C into itself with $\mathcal{F} := F(T) \bigcap_{i=1}^{N} EP(F_i) \cap VI(C, A)$ $\cap VI(C, B) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0,1)$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \forall y \in C, \\ y_n = b_n x_n + (1 - b_n) P_C(I - \lambda_n(I - T)) u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \rho_n(aA + (1-a)B)) x_n + \gamma_n y_n, \forall n \ge 1, \end{cases} \tag{3.77}$$

where $a \in (0,1)$, $\{b_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\rho_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$. Assume the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} b_n = b \in (0,1)$;

(iii) $0 < \rho_n < 2\eta$;

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(v) $\sum_{i=1}^{N} a_i = 1$ and $0 < g \le a_i \le h < 1$ for all $i = 1, 2, ..., N$;

(vi) $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty$, $\sum_{n=1}^{\infty}|\rho_{n+1}-\rho_n|<\infty$,
$\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|b_{n+1}-b_n|<\infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$;

(vii) $0<c\leq r_n,\beta_n,\gamma_n\leq d<1,\forall n\in\mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $z_0=P_{\mathcal{F}}f(z_0)$.

Proof. For every $i=1,2,\ldots,N$, put $T=T_i$ in Theorem 3.11, we obtain the desired result. □

Corollary 3.14. Let C be a nonempty closed convex subset of a real Hilbert space H. For $i=1,2,\ldots,N$, let $F_i:C\times C\to\mathbb{R}$ be a bifunction satisfying $(A1)-(A4)$ and $A:C\to H$ be α -inverse strongly monotone mapping. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into itself with $\mathcal{F}:=\bigcap_{i=1}^{N}F(T_i)\cap\bigcap_{i=1}^{N}EP(F_i)\cap VI(C,A)$ $\neq\emptyset$. Let $f:C\to C$ be a contractive mapping with $\alpha\in(0,1)$. Suppose that the sequence $\{x_n\}$ is generated by $x_1\in C$ and

$$\begin{cases}\sum_{i=1}^{N}a_iF_i(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\forall y\in C,\\ y_n^i=b_nx_n+(1-b_n)P_C(I-\lambda_n(I-T_i))u_n,\\ x_{n+1}=\alpha_nf(x_n)+\beta_nP_C(I-\rho_nA)x_n+\gamma_n\sum_{i=1}^{N}c_iy_n^i,\forall n\geq 1\end{cases}\tag{3.78}$$

where $\{b_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\rho_n\}\subset[0,1]$ with $\alpha_n+\beta_n+\gamma_n=1$ and $0\leq a_i,c_i\leq 1$, for every $i=1,2,...,N$. Assume the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(ii) $\lim_{n\to\infty}b_n=b\in(0,1)$;

(iii) $0<\rho_n<2\alpha$;

(iv) $\sum_{n=1}^{\infty}\lambda_n<\infty$;

(v) $\sum_{i=1}^{N}a_i=\sum_{i=1}^{N}c_i=1$ and $0<g\leq a_i,c_i\leq h<1$ for all $i=1,2,...,N$;

(vi) $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty$, $\sum_{n=1}^{\infty}|\rho_{n+1}-\rho_n|<\infty$,
$\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|b_{n+1}-b_n|<\infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$;

(vii) $0<c\leq r_n,\beta_n,\gamma_n\leq d<1,\forall n\in\mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $z_0=P_{\mathcal{F}}f(z_0)$.

Proof. Put $A=B$ in Theorem 3.11, we obtain the desired result. □

Chapter 4

Applications and Numerical Examples

4.1 A Strong Convergence Theorem for a Finite Family of Nonexpansive Mappings in Hilbert Space

To prove a strong convergence theorem in this section, we need the definition and lemma as follows:

Definition 4.1 ([26]). Let C be a nonempty convex subset of a real Banach space X. Let $\{T_i\}_{n=1}^{N}$ be a finite family of nonexpansive mappings of C into itself and let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, ..., N$. Define a mapping $K : C \to C$ as follows:

$$
\begin{aligned}
U_1 &= \lambda_1 T_1 + (1 - \lambda_1) I, \\
U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2) U_1, \\
U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3) U_2, \\
&\vdots \\
U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1}) U_{N-2}, \\
K = U_N &= \lambda_N T_N U_{N-1} + (1 - \lambda_N) U_{N-1}.
\end{aligned}
$$

Such a mapping is called the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1,\ \lambda_2, ...\lambda_N$.

Lemma 4.1 ([26]). Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ...\lambda_N$. Then $F(K) = \bigcap_{i=1}^{N} F(T_i)$.

Remark 4.2. From the definition of K, it is obvious that K is a nonexpansive mapping.

Theorem 4.3. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, and let $D : C \to H$ be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself and let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_1, ..., T_N$ and $\lambda_1, \lambda_1, ...\lambda_N$. Assume $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i)$

$\cap VI(H, A, M) \cap VI(H, B, M) \cap VI(C, D) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} y_n = J_{M,\lambda}(I - \lambda(aA + (1-a)B))x_n, \\ x_{n+1} = \alpha_n P_C(I - \rho D)y_n + (1-\alpha_n)Kx_n, \forall n \geq 1, \end{cases} \tag{4.1}$$

where $a \in (0, 1)$, $\{\alpha_n\} \subseteq [c, d] \subset [0, 1]$, for all $n \in \mathbb{N}$, $0 < \rho \leq \|D\|^{-1}$ and $0 < \lambda < 2\eta$ with $\eta = min\{\alpha, \beta\}$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$.

Proof. From Theorem 3.2 and Lemma 4.1, we obtain the desired conclusion. □

4.2 A Strong Convergence Theorem for a κ-Quasi-Strictly Pseudo-Contractive Mapping in Hilbert space

In this section, we utilize our main theorem to prove a strong convergence theorem for finding a common element of the set of fixed points of a κ-quasi-strictly pseudo-contractive mapping and the set of solutions of a finite family of variational inclusion problems and the set of solutions of a finite family of equilibrium problem in Hilbert space.

To obtain this result, we recall some definitions, lemmas, and remarks as follows

Definition 4.2. Let C be a subset of a real Hilbert space H and let $T : C \to C$ be a mapping. Then T is said to be *κ-quasi-strictly pseudo-contractive* or *demicontractive mappings* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \kappa\|x - Tx\|^2, \forall x \in C \text{ and } \forall p \in F(T).$$

T is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|, \forall x \in C \text{ and } \forall p \in F(T).$$

The class of κ-quasi-strictly pseudo-contractions includes the class of quasi-nonexpansive mappings.

Remark 4.4. If $T : C \to C$ be a κ-strictly pseudononspreading mapping with $F(T) \neq \emptyset$, then T is a κ-quasi-strictly pseudo-contractive mapping.

Example 4.5. Let $T : [0, 1] \to [0, 1]$ be defined by

$$Tx = \frac{2x+1}{3}, \text{ for all } x \in [0, 1].$$

Then T is a κ-strictly pseudononspreading mapping where $\kappa \in [0, 1)$. Since $1 \in F(T)$, T is also κ-quasi-strictly pseudo-contractive mapping.

Next, we give the example to shows that the converse of Remark 4.4 is not true.

Example 4.6. Let $T : [-2,2] \to [-2,2]$ be defined by

$$Tx = -\frac{5}{3}x, \forall x \in [-2,2]$$

First, show that T is a κ-quasi-strictly pseudo-contractive mapping for all $x \in [-2,2]$. Observe that $F(T) = \{0\}$. Let $x \in [-2,2]$, we have

$$|Tx - T0|^2 = \left|-\frac{5}{3}x - 0\right|^2 = \frac{25}{9}|x|^2$$

and

$$\begin{aligned}|x-0|^2 + \frac{1}{4}|(I-T)x|^2 =& |x|^2 + \frac{1}{4}\left|x + \frac{5}{3}x\right|^2 \\ =& |x|^2 + \frac{1}{4}\left|\frac{8}{3}x\right|^2 \\ =& |x|^2 + \frac{64}{9}\left(\frac{1}{4}\right)|x|^2 \\ =& \left(\frac{25}{9}\right)|x|^2.\end{aligned}$$

Then T is a $\frac{1}{4}$-quasi-strictly pseudo-contractive mapping. Next, we show that T is not a $\frac{1}{4}$-strictly pseudononspreading mapping .
Choose $x = \frac{3}{2}$ and $y = \frac{-3}{2}$, we have

$$\begin{aligned}\left|T\left(\frac{3}{2}\right) - T\left(-\frac{3}{2}\right)\right|^2 =& \left|-\frac{5}{3}\left(\frac{3}{2}\right) + \frac{5}{3}\left(-\frac{3}{2}\right)\right|^2 \\ =& \left|-\frac{10}{2}\right|^2 \\ =& 25,\end{aligned}$$

$$|x-y|^2 = \left|\frac{3}{2} + \frac{3}{2}\right|^2 = 9,$$

$$\begin{aligned}\frac{1}{4}\left|(I-T)\left(\frac{3}{2}\right) - (I-T)\left(\frac{-3}{2}\right)\right|^2 =& \frac{1}{4}\left|\left(\frac{3}{2}\right) + \frac{5}{3}\left(\frac{3}{2}\right) - \left(\left(-\frac{3}{2}\right) + \frac{5}{3}\left(-\frac{3}{2}\right)\right)\right|^2 \\ =& \frac{1}{4}|8|^2 \\ =& 16\end{aligned}$$

and

$$\begin{aligned}2\langle(I-T)\left(\frac{3}{2}\right), (I-T)\left(\frac{-3}{2}\right)\rangle =& 2\langle\left(\frac{3}{2}\right) + \frac{5}{3}\left(\frac{3}{2}\right), \left(\left(-\frac{3}{2}\right) + \frac{5}{3}\left(-\frac{3}{2}\right)\right)\rangle \\ =& 2(4)(-4) = -32.\end{aligned}$$

Then we have

$$|Tx - Ty|^2 > |x-y|^2 + \frac{1}{4}|(I-T)x - (I-T)y|^2 + 2\langle x - Tx, y - Ty\rangle.$$

By replacing κ-strictly pseudononspreading mapping with κ-quasi-strictly pseudo-contractive mapping, we obtain the same result as in Remark 3.5 as follows:

Remark 4.7. Let $T : H \to H$ be a κ-quasi-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Define $S : H \to H$ by $Sx := ((1-\lambda)I + \lambda T)x$, where $\lambda \in (0, 1-\kappa)$. Then the following hold:

1) $F(T) = F(S) = F(I - \lambda(I - T))$;

2) for every $x \in H$ and $y \in F(T)$,

$$\|Sx - y\| \leq \|x - y\|.$$

In 2009, Kangtunyakarn and Suantai [42] introduced the S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$ as follows:

Definition 4.3 ([42]). Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, ...,$ let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\
&\vdots \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called the *S-mapping* generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 4.8 ([46]). Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \ldots, N-1$ and $\alpha_1^N \in (0, 1]$, $\alpha_3^N \in [0, 1)$, $\alpha_2^j \in (0, 1)$ for all $j = 1, 2, \ldots, N$. Let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a quasi-nonexpansive mapping.

Remark 4.9. From Lemma 4.8 is still holds if $C \equiv H$.

Theorem 4.10. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$ and $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,...,N}\{\alpha_i\}$. Let $T : H \to H$ be a κ-quasi-strictly pseudo-contractive mapping. Assume $\mathcal{F} := F(T) \cap \bigcap_{i=1}^N EP(F_i) \cap$

$\bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} b_i A_i)x_n \\ \qquad + \eta_n(I - \rho_n(I - T))x_n + \delta_n u_n, \forall n \geq 1, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq (0,1)$, and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n + \eta_n + \delta_n = 1$, $0 < \alpha < 1$ and $0 \leq a_i, b_i \leq 1$, for every $i = 1, 2, ..., N$, $r_n \in [c, d] \subset (0,1)$, $0 < p \leq \beta_n, \gamma_n, \eta_n, \delta_n \leq q < 1$, $\rho_n \in (0, 1-\kappa)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \rho_n < \infty$;

(iii) $0 < \lambda < 2\eta$;

(iv) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = 1$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
$\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.

Proof. Using Remark 4.7 and the same method of proof in Theorem 3.7, we have the desired conclusion. □

Theorem 4.11. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, \dots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$ and let $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,\dots,N}\{\alpha_i\}$. Let $T_i : H \to H$, for $i = 1, 2, \dots, N$ be a finite family of nonspreading mappings with $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. Let $\theta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \dots, N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \dots, N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in (0,1)$ for all $j = 1, 2, \dots, N$, and let S be the S-mapping generated by $T_1, T_2, \dots, T_N$ and $\theta_1, \theta_2, \dots, \theta_N$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} b_i A_i)x_n \\ \qquad + \eta_n(I - \rho_n(I - S))x_n + \delta_n u_n, \forall n \geq 1, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq (0,1)$, and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n + \eta_n + \delta_n = 1$, $0 < \alpha < 1$, and $0 \leq a_i, b_i \leq 1$, for every $i = 1, 2, ..., N$, $r_n \in [c, d] \subset (0,1)$, $0 < p \leq \beta_n, \gamma_n, \eta_n, \delta_n \leq q < 1$, $\rho_n \in (0,1)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \rho_n < \infty$;

(iii) $0 < \lambda < 2\eta$;

(iv) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = 1$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
$\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.

Proof. From Theorem 4.10 and Remark 4.9, we obtain the desired conclusion. □

4.3 A Strong Convergence Theorem for a Quasi-Nonexpansive Mapping in Hilbert space

In this section, we utilize our main theorem to prove a strong convergence theorem for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a finite family of equilibrium problem and two set of solutions of of variational inequality problems in Hilbert space.

To obtain this result, we recall some definitions, lemmas, and remark as follows:

Remark 4.12. If $T : C \to C$ be a nonspreading mapping with $F(T) \neq \emptyset$, then T is a quasi-nonexpansive mapping.

Next, we give two examples of quasi-nonexpansive mapping.

Example 4.13. For every $i = 1, 2, ..., N$, let $T_i : [0, \infty) \to [0, \infty)$ be defined by

$$T_i x = \frac{2i}{2i+1}x, \text{ for all } x \in [0, \infty).$$

For every $x, y \in [0, \infty)$, we obtain

$$|T_i x - T_i y|^2 = \left|\frac{2i}{2i+1}x - \frac{2i}{2i+1}y\right|^2 = \frac{4i^2}{4i^2 + 4i + 1}|x - y|^2$$

and

$$\begin{aligned} 2\langle x - T_i x, y - T_i y\rangle &= 2\left\langle x - \frac{2i}{2i+1}x, y - \frac{2i}{2i+1}y\right\rangle \\ &= \frac{2}{(2i+1)^2}(x)(y) \geq 0, (\text{ since } x, y \geq 0). \end{aligned}$$

It follows that

$$|x-y|^2+2\langle x-T_ix,y-T_iy\rangle \geq |x-y|^2 \geq \frac{(2i)^2}{(2i+1)^2}|x-y|^2=|T_ix-T_iy|^2.$$

Then T_i is a nonspreading mapping for all $i=1,2,...,N$ and observe that $\bigcap_{i=1}^{N}F(T_i)=\{0\}$. For every $x\in[0,\infty)$ and $0\in F(T)$, we have

$$|T_ix-0|=\left|\frac{2i}{2i+1}x-0\right|=\frac{2i}{2i+1}|x|\leq |x|=|x-0|.$$

Therefore T_i is a quasi-nonexpansive mapping for all $i=1,2,...,N$.

Example 4.14. For every $i=1,2,...,N$, let $T_i:[0,\infty)\to[0,\infty)$ be defined by

$$T_ix=\frac{\sin(ix)}{i},\ \text{for all } x\in[0,\infty).$$

For every $x,y\in[0,\infty)$, we obtain

$$|T_ix-T_iy|^2=\left|\frac{\sin(ix)}{i}-\frac{\sin(iy)}{i}\right|^2=\frac{1}{i^2}|\sin(ix)-\sin(iy)|^2\leq |x-y|^2$$

and

$$\begin{aligned}2\langle x-T_ix,y-T_iy\rangle &= 2\left\langle x-\frac{\sin(ix)}{i},y-\frac{\sin(iy)}{i}\right\rangle\\ &=\frac{2}{i^2}(ix-\sin(ix))(iy-\sin(iy))\geq 0,(\text{ since } x,y\geq 0).\end{aligned}$$

It follows that

$$|x-y|^2+2\langle x-T_ix,y-T_iy\rangle \geq |x-y|^2\geq \frac{1}{i^2}|\sin(ix)-\sin(iy)|^2=|T_ix-T_iy|^2.$$

Then T_i is a nonspreading mapping for all $i=1,2,...,N$ and observe that $\bigcap_{i=1}^{N}F(T_i)=\{0\}$. For every $x\in[0,\infty)$ and $0\in F(T_i)$ for all $i=1,2,...,N$, we have

$$|T_ix-0|=\left|\frac{\sin(ix)}{i}-0\right|=\frac{1}{i}|\sin(ix)|\leq\frac{1}{i}|ix|=|x-0|.$$

Therefore T_i is a quasi-nonexpansive mapping for all $i=1,2,...,N$.

Remark 4.15. Let C be a nonempty closed convex subset of a real Hilbert space H and let $T:C\to C$ be a quasi-nonexpansive mapping with $F(T)\neq\emptyset$. Then there hold the following statement:

1) $F(T)=VI(C,I-T)$;

2) For every $x\in C$ and $y\in F(T)$,

$$\|P_C(I-\lambda(I-T))x-y\|\leq\|x-y\|,\ \text{for all } \lambda\in(0,1).$$

From remark 4.15 and Theorem 3.11, we obtain the following theorem.

Theorem 4.16. Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, with $\eta = min\{\alpha, \beta\}$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mapping of C into itself with $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N EP(F_i) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0, 1)$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n^i = b_n x_n + (1 - b_n) P_C(I - \lambda_n(I - T_i)) u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \rho_n(aA + (1 - a)B)) x_n + \gamma_n \sum_{i=1}^N c_i y_n^i, \forall n \geq 1, \end{cases} \tag{4.4}$$

where $a \in (0, 1)$, $\{b_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\rho_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $0 \leq a_i, c_i \leq 1$, for every $i = 1, 2, ..., N$. Assume the following conditions hold:

1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

2) $\lim_{n\to\infty} b_n = b \in (0, 1)$;

3) $0 < \rho_n < 2\eta$, where $\eta = min\{\alpha, \beta\}$;

4) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

5) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} c_i = 1$ and $0 < g \leq a_i, c_i \leq h < 1$ for all $i = 1, 2, ..., N$;

6) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$,
$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

7) $0 < c \leq r_n, \beta_n, \gamma_n \leq d < 1, \forall n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$.

4.4 Examples and Numerical Results

For the purpose of this section we give a numerical example to support our some result. Theorem 3.7 is supported by the following example.

Example 4.17. Let $\mathbb{R}$ be the set of real numbers. For every $i = 1, 2, \dots, N$, let $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $A_i : \mathbb{R} \to \mathbb{R}$ be defined by

$$F_i(x, y) = i(y - x)(3x + y),$$
$$A_i x = \frac{ix}{10},$$

for all $x, y \in \mathbb{R}$ and let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \begin{cases} \frac{-3x}{5} & \text{if } x \in [0, \infty), \\ x & \text{if } x \in (-\infty, 0). \end{cases}$$

For every $i = 1, 2, \dots, N$, suppose that $J_{M,\lambda} = I$, $\lambda = \frac{1}{N}$, $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$, $b_i = \frac{8}{9^i} + \frac{1}{N9^N}$. Let $\{x_n\}$ and $\{u_n\}$ be generated by (3.19), where $\alpha_n = \frac{1}{20n}$, $\beta_n = \frac{3(20n-1)}{220n}$, $\gamma_n = \frac{2(20n-1)}{220n}$, $\eta_n = \frac{5(20n-1)}{220n}$, $\delta_n = \frac{20n-1}{220n}$, $r_n = \frac{3n}{5n+6}$, and $\rho_n = \frac{1}{9n^2}$ for every $n \in \mathbb{N}$. Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to 0.

Solution. To show that T is a $\frac{3}{4}$-strictly pseudononspreading mapping, we consider three cases as follows.

Case 1. Let $x, y \in [0, \infty)$. From the definition of T, we have

$$|Tx - Ty|^2 = \left| \frac{-3x}{5} + \frac{3y}{5} \right|^2 = \frac{9}{25}|x - y|^2,$$

$$\begin{aligned} |(I - T)x - (I - T)y|^2 &= \left| x + \frac{3x}{5} - (y + \frac{3y}{5}) \right|^2 \\ &= \left| \frac{8x}{5} - \frac{8y}{5} \right|^2 \\ &= \frac{64}{25}|x - y|^2, \end{aligned}$$

and

$$\begin{aligned} 2\langle x - Tx, y - Ty\rangle &= 2\left\langle x + \frac{3x}{5}, y + \frac{3y}{5} \right\rangle \\ &= 2\left(\frac{64}{25}\right) xy \geq 0. \quad (Since \ \ 0 \leq x, y < \infty). \end{aligned}$$

Choose $\kappa = \frac{3}{4}$, we have

$$\begin{aligned} |x - y|^2 + \frac{3}{4}|(I - T)x - (I - T)y|^2 + 2\langle x - Tx, y - Ty\rangle &\geq |x - y|^2 \\ &\geq \frac{9}{25}|x - y|^2 \\ &= |Tx - Ty|^2. \end{aligned}$$

Case 2. Let $x, y \in (-\infty, 0)$. From the definition of T, we have

$$|Tx - Ty|^2 = |x - y|^2,$$

$$|(I-T)x - (I-T)y|^2 = |x - x - (y - y)|^2 = 0,$$

and

$$2\langle x - Tx, y - Ty\rangle = 2\langle x - x, y - y\rangle = 0.$$

Choose $\kappa = \frac{3}{4}$, we have

$$\begin{aligned}|x-y|^2 + \kappa|(I-T)x - (I-T)y|^2 + 2\langle x - Tx, y - Ty\rangle \geq& |x-y|^2 \\ =& |Tx - Ty|^2.\end{aligned}$$

Case 3. Let $x \in [0, \infty)$ and $y \in (-\infty, 0)$. From the definition of T, we have

$$|Tx - Ty|^2 = \left|\frac{-3x}{5} - y\right|^2 = \left|\frac{3x}{5} + y\right|^2 = \frac{9}{25}x^2 + \frac{6}{5}xy + y^2,$$

$$\begin{aligned}|(I-T)x - (I-T)y|^2 &= \left|x + \frac{3x}{5} - (y - y)\right|^2 \\ &= \left(\frac{8x}{5}\right)^2,\end{aligned}$$

and

$$2\langle x - Tx, y - Ty\rangle = 2\left\langle x + \frac{3x}{5}, y - y\right\rangle = 0.$$

Choose $\kappa = \frac{3}{4}$, we have

$$\begin{aligned}|x-y|^2 + \kappa|(I-T)x - (I-T)y|^2 + 2\langle x - Tx, y - Ty\rangle \geq& |x-y|^2 \\ =& x^2 - 2xy + y^2 + \frac{9}{25}x^2 + \frac{6}{5}xy - \frac{9}{25}x^2 - \frac{6}{5}xy \\ =& \frac{9}{25}x^2 + \frac{6}{5}xy + y^2 + x^2 - 2xy - \frac{9}{25}x^2 - \frac{6}{5}xy \\ =& \left(\frac{3x}{5} + y\right)^2 + \frac{16}{25}x^2 - \frac{16}{5}xy \\ \geq& \left(\frac{3x}{5} + y\right)^2 \\ =& |Tx - Ty|^2.\end{aligned}$$

Thus, T is a κ-strictly pseudononspreading mapping. From the definition of T, then $F(T) = \{0\}$.

Next, we will show that for every $i = 1, 2, 3, ..., N$, A_i is $\frac{1}{i}$-inverse strongly monotone mapping. Let $x, y \in \mathbb{R}$. From the definition of A_i, we have

$$|A_i x - A_i y|^2 = \left|\frac{ix}{10} - \frac{iy}{10}\right|^2 = \frac{i^2}{100}|x - y|^2.$$

From the above equation, we have

$$\begin{aligned}\langle A_ix - A_iy, x-y\rangle =& \left\langle \frac{ix}{10} - \frac{iy}{10}, x-y \right\rangle\\ =&\frac{i}{10}|x-y|^2\\ \geq&\frac{i}{100}|x-y|^2\\ =&\frac{1}{i}\left(\frac{i^2}{100}|x-y|^2\right)\\ =&\frac{1}{i}|A_ix - A_iy|^2.\end{aligned}$$

Then, for every $i = 1, 2, 3, ..., N$, A_i is a $\frac{1}{i}$-inverse strongly monotone mapping. Moreover, we have $\eta = \min_{i=1,2,...,N}\{\frac{1}{i}\} = \frac{1}{N}$.

Next, we show that F_i satisfies $(A1) - (A4)$, for all $i = 1, 2, ..., N$. For every $i = 1, 2, ..., N$, from the definition of F_i, we have

$$F_i(x, x) = i(x - x)(3x + x) = 0.$$

Then F_i satisfies (A1), for all $i = 1, 2, ..., N$.
From the definition of F_i, we have,

$$\begin{aligned}F_i(x,y) + F_i(y,x) =& i(y-x)(3x+y) + i(x-y)(3y+x)\\ =&i[2xy + y^2 - 3x^2 + 2xy + x^2 - 3y^2]\\ =&i[-2x^2 + 4xy - 2y^2]\\ =&-2i[x^2 - 2xy + y^2]\\ =&-2i(x-y)^2\\ \leq&0.\end{aligned}$$

Then F_i satisfies (A2), for all $i = 1, 2, ..., N$.
Let $t \in [0, 1]$, we then have

$$\begin{aligned}\lim_{t\to 0^+} F_i(tz + (1-t)x, y) =& \lim_{t\to 0^+} i(y - (tz + (1-t)x))(3(tz + (1-t)x) + y)\\ =&i(y-x)(3x+y)\\ =&F_i(x,y).\end{aligned}$$

Then F_i satisfies (A3), for all $i = 1, 2, ..., N$.
Finally, we will prove (A4). Since

$$F_i(x, y) = i(y - x)(3x + y) = 2xy + y^2 - 3x^2.$$

Let $\alpha \in (0,1)$ and above the equation, we get

$$\begin{aligned}
F_i(x,\alpha z+(1-\alpha)y) =& i\left(2x(\alpha z+(1-\alpha)y)+(\alpha z+(1-\alpha)y)^2-3x^2\right)\\
=& i\left(2\alpha xz+2(1-\alpha)xy+\alpha^2z^2+2\alpha(1-\alpha)zy+(1-\alpha)^2y^2-3x^2\right)\\
\leq& i\left(2\alpha xz+2(1-\alpha)xy+\alpha^2z^2+\alpha(1-\alpha)(z^2+y^2)+(1-\alpha)^2y^2-(\alpha+(1-\alpha))3x^2\right)\\
=& i\big(\alpha\left(2xz+\alpha z^2+(1-\alpha)z^2-3x^2\right)\\
&+(1-\alpha)\left(2xy+\alpha y^2+(1-\alpha)y^2-3x^2\right)\big)\\
=& i\left(\alpha\left(2xz+z^2-3x^2\right)+(1-\alpha)\left(2xy+y^2-3x^2\right)\right)\\
=& \alpha i\left(2xz+z^2-3x^2\right)+(1-\alpha)i\left(2xy+y^2-3x^2\right)\\
=& \alpha F_i(x,z)+(1-\alpha)F_i(x,y).
\end{aligned}$$

Then F_i is convex function, for all $i=1,2,...,N$. Let $\{y_n\}\subset\mathbb{R}$ with $y_n\to y$ as $n\to\infty$. Thus we obtain

$$\begin{aligned}
\liminf_{n\to\infty} F_i(x,y_n) =& \liminf_{n\to\infty}(i(y_n-x)(3x+y_n))\\
=& (i(y-x)(3x+y))\\
=& F_i(x,y).
\end{aligned}$$

Then F_i is lower semicontinuous, for all $i=1,2,...,N$. Hence F_i satisfies $(A1)-(A4)$, for all $i=1,2,...,N$. Since $a_i=\frac{3}{4^i}+\frac{1}{N4^N}$, then

$$\begin{aligned}
\sum_{i=1}^{N}a_i =& \sum_{i=1}^{N}\left(\frac{3}{4^i}+\frac{1}{N4^N}\right)\\
=& \left(\frac{3}{4^1}+\frac{1}{N4^N}\right)+\left(\frac{3}{4^2}+\frac{1}{N4^N}\right)+\left(\frac{3}{4^3}+\frac{1}{N4^N}\right)+\cdots+\left(\frac{3}{4^N}+\frac{1}{N4^N}\right)\\
=& \left(\frac{3}{4^1}+\frac{3}{4^2}+\frac{3}{4^3}+\cdots+\frac{3}{4^N}\right)+\left(\frac{1}{N4^N}+\frac{1}{N4^N}+\frac{1}{N4^N}+\cdots+\frac{1}{N4^N}\right)\\
=& 3\left(\frac{1}{4^1}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots+\frac{1}{4^N}\right)+\frac{N}{N4^N}\\
=& 3\left(\frac{\frac{1}{4}\left(1-\frac{1}{4^N}\right)}{1-\frac{1}{4}}\right)+\frac{1}{4^N}\\
=& \left(1-\frac{1}{4^N}\right)+\frac{1}{4^N}\\
=& 1.
\end{aligned}$$

Then $\sum_{i=1}^{N}a_i=1$ and so is $\sum_{i=1}^{N}b_i=1$, where $b_i=\frac{8}{9^i}+\frac{1}{N9^N}$. Using the same method as the proof of F_i satisfying $(A1)-(A4)$, for all $i=1,2,...,N$, then we can conclude that $\sum_{i=1}^{N}a_iF_i$ satisfies (A1)-(A4), where $a_i=\frac{3}{4^i}+\frac{1}{N4^N}$.
Since $a_i=\frac{3}{4^i}+\frac{1}{N4^N}$, we obtain

$$\sum_{i=1}^{N}a_iF_i(x,y)=\sum_{i=1}^{N}\left(\frac{3}{4^i}+\frac{1}{N4^N}\right)i(y-x)(y+3x).$$

Since $\sum_{i=1}^{N} a_iF_i(0,y) = \sum_{i=1}^{N} a_i i\,(y-0)\cdot(3(0)+y) = \sum_{i=1}^{N} a_i i(y^2) \geq 0$, then $0 \in EP(\sum_{i=1}^{N} a_iF_i) = \bigcap_{i=1}^{N} EP\,(F_i)$. It implies that

$$F(T) \cap \bigcap_{i=1}^{N} EP(F_i) = \{0\}. \tag{4.5}$$

Put $S_1 = \sum_{i=1}^{N} \left(\frac{3}{4^i} + \frac{1}{N4^N}\right) i$, then we have

$$\begin{aligned}
0 \leq & \sum_{i=1}^{N} a_iF_i\,(u_n, y) + \frac{1}{r_n}\,\langle y-u_n, u_n - x_n\rangle \\
&= S_1\,(y-u_n)\,(y+3u_n) + \frac{1}{r_n}\,(y-u_n)\,(u_n - x_n) \\
&\Leftrightarrow \\
0 \leq & S_1r_n\,(y-u_n)\,(y+3u_n) + (y-u_n)\,(u_n-x_n) \\
&= S_1r_ny^2 + (u_n + 2r_nS_1u_n - x_n)\,y + u_nx_n - 3r_nS_1u_n^2 - u_n^2.
\end{aligned}$$

Let $G(y) = S_1r_ny^2 + (u_n + 2r_nS_1u_n - x_n)\,y + u_nx_n - 3r_nS_1u_n^2 - u_n^2$. $G(y)$ is a quadratic function of y with coefficient $a = S_1r_n$, $b = u_n + 2r_nS_1u_n - x_n$, and $c = u_nx_n - 3r_nS_1u_n^2 - u_n^2$. We will determine the discriminant Δ of G as follows

$$\begin{aligned}
\Delta =& b^2 - 4ac \\
&= (u_n + 2r_nS_1u_n - x_n)^2 - 4\,(S_1r_n)\,(u_nx_n - 3r_nS_1u_n^2 - u_n^2) \\
=& u_n^2 + 8r_nS_1u_n^2 + 16r_n^2S_1^2u_n^2 - 2u_nx_n - 8r_nS_1u_nx_n + x_n^2 \\
&= (u_n + 4S_1r_nu_n - x_n)^2.
\end{aligned}$$

We know that $G(y) \geq 0, \forall y \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so we obtain

$$u_n = \frac{x_n}{1 + 4S_1r_n}, \tag{4.6}$$

where $S_1 = \sum_{i=1}^{N} \left(\frac{3}{4^i} + \frac{1}{N4^N}\right) i$.
Since $A_ix = \frac{ix}{10}$ and $b_i = \frac{8}{9^i} + \frac{1}{N9^N}$, then

$$\sum_{i=1}^{N} b_iA_ix = \sum_{i=1}^{N} \left(\frac{8}{9^i} + \frac{1}{N9^N}\right) \frac{ix}{10}.$$

From the definition of the modified variational inclusion, it follows that

$$\bigcap_{i=1}^{N} VI(H, A_i, M) = \{0\}. \tag{4.7}$$

From (4.5) and (4.7), we have

$$F(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) = \{0\}. \tag{4.8}$$

For every $n \in \mathbb{N}$, let $\alpha_n = \frac{1}{20n}$, $\beta_n = \frac{3(20n-1)}{220n}$, $\gamma_n = \frac{2(20n-1)}{220n}$, $\eta_n = \frac{20n-1}{220n}$, $\delta_n = \frac{5(20n-1)}{220n}$, we have

$$\begin{aligned}\alpha_n+\beta_n+\gamma_n+\eta_n+\delta_n =& \frac{1}{20n}+\frac{3(20n-1)}{220n}+\frac{2(20n-1)}{220n}+\frac{20n-1}{220n}+\frac{5(20n-1)}{220n}\\ =&\frac{1}{20n}+\frac{11(20n-1)}{220n}\\ =&\frac{1}{20n}+1-\frac{1}{20n}\\ =&1.\end{aligned}$$

Since $\alpha_n = \frac{1}{20n}$, for all $n \in \mathbb{N}$, then $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\frac{1}{20n} = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \sum_{n=1}^{\infty}\frac{1}{20n} = \infty$. It implies that the condition (i) holds.

Since $\rho_n = \frac{1}{9n^2}$, for all $n \in \mathbb{N}$, by the p-series, then $\sum_{n=1}^{\infty}\frac{1}{9n^2}$ converges. That is the condition (ii) holds. Moreover, since T is a $\frac{3}{4}$-strictly pseudononspreading mapping, then $\rho_n \in (0, \frac{1}{4})$. From $\lambda = \frac{1}{N}$ and $\eta = \frac{1}{N}$, then $0 < \lambda < 2\eta$. It implies that λ satisfy the condition (iii). Observe that

$$|\alpha_{n+1}-\alpha_n| = |\frac{1}{20(n+1)}-\frac{1}{20n}| = \frac{1}{20n^2+20n}$$

By using the Limit comparison test, putting $p_n = \frac{1}{20n^2+20n}$ and $q_n = \frac{1}{n^2}$, then

$$\lim_{n\to\infty}\frac{p_n}{q_n} = \lim_{n\to\infty}\frac{n^2}{20n^2+20n} = \frac{1}{20}.$$

Since $\sum_{n=1}^{\infty}q_n$ converges, then $\sum_{n=1}^{\infty}p_n$ converges. It implies that $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ and so are $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$, $\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n| < \infty$, $\sum_{n=1}^{\infty}|\rho_{n+1}-\rho_n| < \infty$, $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n| < \infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$. Then the condition (v) holds. Thai implies that the sequences $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\eta_n\},\{\delta_n\},\{r_n\}$, and $\{\rho_n\}$ satisfy all the conditions of Theorem 3.7.

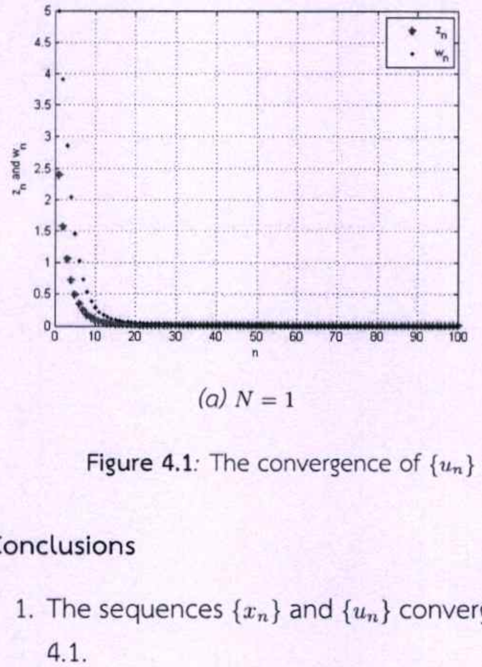
For every $n \in \mathbb{N}$, from (4.6), we rewrite (3.19) as follows:

$$\begin{aligned}x_{n+1} =& \frac{1}{20n}u+\frac{3(20n-1)}{220n}x_n+\frac{2(20n-1)}{220n}\left(x_n-\frac{1}{N}\sum_{i=1}^{N}\left(\frac{8}{9^i}+\frac{1}{N9^N}\right)\frac{ix_n}{10}\right)\\ &+\frac{5(20n-1)}{220n}\left(I-\frac{1}{9n^2}(I-T)\right)x_n+\frac{20n-1}{220n}\left(\frac{x_n}{1+4\left(\sum_{i=1}^{N}\left(\frac{3}{4^i}+\frac{1}{N4^N}\right)i\right)\frac{3n}{5n+6}}\right).\end{aligned} \tag{4.9}$$

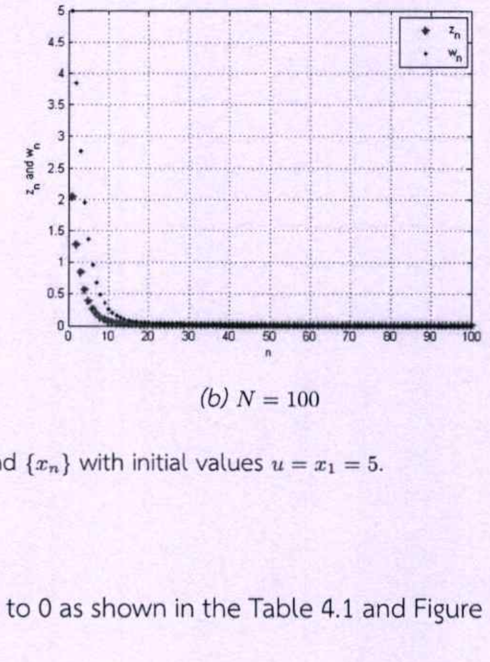
Using the algorithm (4.9) and choosing $u = x_1 = 5$ with $N = 1$ and $N = 100$, we have the numerical result as follows:

Table 4.1: The values of the sequences $\{u_n\}$ and $\{x_n\}$ with initial values $u = x_1 = 5$.

	$N=1$		$N=100$	
n	u_n	x_n	u_n	x_n
1	2.391304	5.000000	2.037037	5.000000
2	1.480316	3.700791	1.211068	3.633203
3	0.982649	2.667191	0.784335	2.577101
4	0.667619	1.900147	0.523043	1.810533
5	0.459689	1.349410	0.354712	1.270096
⋮	⋮	⋮	⋮	⋮
50	0.004726	0.015803	0.003739	0.015422
⋮	⋮	⋮	⋮	⋮
96	0.002359	0.007951	0.001867	0.007854
97	0.002334	0.007866	0.001847	0.007687
98	0.002309	0.007783	0.001828	0.007606
99	0.002284	0.007702	0.001808	0.007526
100	0.002261	0.007622	0.001790	0.007448



(a) $N=1$



(b) $N=100$

Figure 4.1: The convergence of $\{u_n\}$ and $\{x_n\}$ with initial values $u = x_1 = 5$.

Conclusions

1. The sequences $\{x_n\}$ and $\{u_n\}$ converge to 0 as shown in the Table 4.1 and Figure 4.1.
2. From Theorem 3.7, we can conclude that the sequence $\{x_n\}$ and $\{u_n\}$, in Example 4.17, converge to 0.

Next, we give the numerical example to support Theorem 3.7 in a three dimensional space of real numbers.

Example 4.18. Let $\mathbb{R}^3$ be the three dimensional space of real numbers with an inner product $\langle\cdot,\cdot\rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $\langle \mathbf{x},\mathbf{y}\rangle = \mathbf{x}\cdot\mathbf{y} = x_1\cdot y_1 + x_2\cdot y_2 + x_3\cdot y_3$ and a usual norm $\|\cdot\| : \mathbb{R}^3 \to \mathbb{R}$ defined by $\|\mathbf{x}\| = \sqrt{x_1^2+x_2^2+x_3^2}$ for all $\mathbf{x} = (x_1,x_2,x_3)$, $\mathbf{y} = (y_1,y_2,y_3)$. For every $i = 1,2,\ldots,N$, let $F_i : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $A_i : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F_i(\mathbf{x},\mathbf{y}) = i(\mathbf{y}-\mathbf{x})\cdot(9\mathbf{x}+\mathbf{y}),$$
$$A_i\mathbf{x} = \left(\frac{ix_1}{6},\frac{ix_2}{6},\frac{ix_3}{6}\right),$$

and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\mathbf{x} = \begin{cases} \left(\frac{-3x_1}{5},\frac{-5x_2}{7},\frac{-7x_3}{9}\right) & \text{if } \mathbf{x} \in [0,\infty)\times[0,\infty)\times[0,\infty), \\ (x_1,x_2,x_3) & \text{if } \mathbf{x} \in (-\infty,0)\times(-\infty,0)\times(-\infty,0), \end{cases}$$

for all $\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$, $\mathbf{y} = (y_1,y_2,y_3) \in \mathbb{R}^3$. For every $i = 1,2,\ldots,N$, suppose that $J_{M,\lambda} = I$, $\lambda = \frac{1}{N}$, $a_i = \frac{4}{5^i} + \frac{1}{N5^N}$, $b_i = \frac{7}{8^i} + \frac{1}{N8^N}$. Let $\mathbf{x}_n = (x_n^1,x_n^2,x_n^3)$ and $\mathbf{u}_n = (u_n^1,u_n^2,u_n^3)$ be generated by (3.19), where $\alpha_n = \frac{1}{15n}$, $\beta_n = \frac{3(15n-1)}{165n}$, $\gamma_n = \frac{2(15n-1)}{165n}$, $\eta_n = \frac{5(15n-1)}{165n}$, $\delta_n = \frac{15n-1}{165n}$, $r_n = \frac{2n}{7n+6}$ and $\rho_n = \frac{1}{9n^2}$ for every $n \in \mathbb{N}$. Then the sequences $\mathbf{x}_n = (x_n^1,x_n^2,x_n^3)$, and $\mathbf{u}_n = (u_n^1,u_n^2,u_n^3)$ converge strongly to $\mathbf{0}$ where $\mathbf{0} = (0,0,0)$.

Solution. To show that T is a $\frac{7}{8}$-strictly pseudononspreading mapping , we consider three cases as follows.

Case 1. Let $\mathbf{x},\mathbf{y} \in [0,\infty)\times[0,\infty)\times[0,\infty)$. From the definition of T, we have

$$\begin{aligned}\|T\mathbf{x}-T\mathbf{y}\|^2 &= \left\|\left(\frac{-3x_1}{5},\frac{-5x_2}{7},\frac{-7x_3}{9}\right)-\left(\frac{-3y_1}{5},\frac{-5y_2}{7},\frac{-7y_3}{9}\right)\right\|^2 \\ &= \left\|\left(\frac{-3x_1}{5}+\frac{3y_1}{5}\right),\left(\frac{-5x_2}{7}+\frac{5y_2}{7}\right),\left(\frac{-7x_3}{9}+\frac{7y_3}{9}\right)\right\|^2 \\ &= \frac{9}{25}(x_1-y_1)^2+\frac{25}{49}(x_2-y_2)^2+\frac{49}{81}(x_3-y_3)^2,\end{aligned}$$

$$\begin{aligned}\|(I-T)\mathbf{x}-(I-T)\mathbf{x}\|^2 &= \Bigg\|\left((x_1,x_2,x_3)-\left(\frac{-3x_1}{5},\frac{-5x_2}{7},\frac{-7x_3}{9}\right)\right) \\ &\quad -\left((y_1,y_2,y_3)-\left(\frac{-3y_1}{5},\frac{-5y_2}{7},\frac{-7y_3}{9}\right)\right)\Bigg\|^2 \\ &= \left\|\left(\frac{8x_1}{5},\frac{12x_2}{7},\frac{16x_3}{9}\right)-\left(\frac{8y_1}{5},\frac{12y_2}{7},\frac{16y_3}{9}\right)\right\|^2 \\ &= \|\frac{8}{5}(x_1-y_1)^2,\frac{12}{7}(x_2-y_2)^2,\frac{16}{9}(x_3-y_3)^2\|^2 \\ &= \frac{64}{25}(x_1-y_1)^2+\frac{144}{49}(x_2-y_2)^2+\frac{256}{81}(x_3-y_3)^2,\end{aligned}$$

and

$$2\langle \mathbf{x}-T\mathbf{x},\mathbf{y}-T\mathbf{y}\rangle =2\left\langle (x_1,x_2,x_3)-\left(\frac{-3x_1}{5},\frac{-5x_2}{7},\frac{-7x_3}{9}\right),(y_1,y_2,y_3)-\left(\frac{-3y_1}{5},\frac{-5y_2}{7},\frac{-7y_3}{9}\right)\right\rangle$$
$$=2\left\langle \left(\frac{8x_1}{5},\frac{12x_2}{7},\frac{16x_3}{9}\right),\left(\frac{8y_1}{5},\frac{12y_2}{7},\frac{16y_3}{9}\right)\right\rangle$$
$$=2\left(\left(\frac{8x_1}{5}\right)\left(\frac{8y_1}{5}\right)+\left(\frac{12x_2}{7}\right)\left(\frac{12y_2}{7}\right)+\left(\frac{16x_3}{9}\right)\left(\frac{16y_3}{9}\right)\right)$$
$$=2\left(\left(\frac{64}{25}\right)x_1y_1+\left(\frac{144}{49}\right)x_2y_2+\left(\frac{256}{81}\right)x_3y_3\right)\geq 0.\ \ (Since\ \ 0\leq \mathbf{x},\mathbf{y}<\infty).$$

From the above equation, choose $\kappa=\frac{7}{8}$ such that

$$\|\mathbf{x}-\mathbf{y}\|^2+\frac{7}{8}\|(I-T)\mathbf{x}-(I-T)\mathbf{y}\|^2+2\langle \mathbf{x}-T\mathbf{x},\mathbf{y}-T\mathbf{y}\rangle \geq \|\mathbf{x}-\mathbf{y}\|^2$$
$$\geq\|\,(x_1,x_2,x_3)-(y_1,y_2,y_3)\,\|^2$$
$$=\|(x_1-y_1),(x_2-y_2),(x_3-y_3)\|^2$$
$$=(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2$$
$$\geq\frac{9}{25}(x_1-y_1)^2+\frac{25}{49}(x_2-y_2)^2+\frac{49}{81}(x_3-y_3)^2$$
$$=\|T\mathbf{x}-T\mathbf{y}\|^2.$$

Case 2. Let $\mathbf{x},\mathbf{y}\in(-\infty,0)\times(-\infty,0)\times(-\infty,0)$. From the definition of T, we have

$$\|T\mathbf{x}-T\mathbf{y}\|^2=\|\mathbf{x}-\mathbf{y}\|^2,$$

$$\|(I-T)\mathbf{x}-(I-T)\mathbf{y}\|^2=\|\mathbf{x}-\mathbf{x}-(\mathbf{y}-\mathbf{y})\|^2=0,$$

and

$$2\langle \mathbf{x}-T\mathbf{x},\mathbf{y}-T\mathbf{y}\rangle =2\langle \mathbf{x}-\mathbf{x},\mathbf{y}-\mathbf{y}\rangle=0.$$

From the above equation, choose $\kappa=\frac{7}{8}$ such that

$$\|\mathbf{x}-\mathbf{y}\|^2+\frac{7}{8}\|(I-T)\mathbf{x}-(I-T)\mathbf{y}\|^2+2\langle \mathbf{x}-T\mathbf{x},\mathbf{y}-T\mathbf{y}\rangle \geq\|\mathbf{x}-\mathbf{y}\|^2$$
$$=\|T\mathbf{x}-T\mathbf{y}\|^2.$$

Case 3. Let $\mathbf{x}\in[0,\infty)\times[0,\infty)\times[0,\infty)$ and $\mathbf{y}\in(-\infty,0)\times(-\infty,0)\times(-\infty,0)$. From the definition of T, we have

$$\|T\mathbf{x}-T\mathbf{y}\|^2=\left\|\left(\frac{-3x_1}{5},\frac{-5x_2}{7},\frac{-7x_3}{9}\right)-(y_1,y_2,y_3)\right\|^2$$
$$=\left\|\left(\frac{-3x_1}{5}-y_1\right),\left(\frac{-5x_2}{7}-y_2\right),\left(\frac{-7x_3}{9}-y_3\right)\right\|^2$$
$$=\left(\frac{-3x_1}{5}-y_1\right)^2+\left(\frac{-5x_1}{7}-y_2\right)^2+\left(\frac{-7x_3}{9}-y_3\right)^2$$
$$=\left(\frac{3x_1}{5}+y_1\right)^2+\left(\frac{5x_1}{7}+y_2\right)^2+\left(\frac{7x_3}{9}+y_3\right)^2$$
$$=\frac{9}{25}x_1^2+\frac{6}{5}x_1y_1+y_1^2+\frac{25}{49}x_2^2+\frac{10}{7}x_2y_2+y_2^2+\frac{49}{81}x_3^2+\frac{14}{9}x_3y_3+y_3^2,$$

$$\|(I-T)\mathbf{x}-(I-T)\mathbf{y}\|^2 = \left\| \left((x_1,x_2,x_3) - \left(\frac{-3x_1}{5}, \frac{-5x_2}{7}, \frac{-7x_3}{9} \right) \right) - ((y_1,y_2,y_3)-(y_1,y_2,y_3)) \right\|^2$$
$$= \left\| \left((x_1,x_2,x_3) - \left(\frac{-3x_1}{5}, \frac{-5x_2}{7}, \frac{-7x_3}{9} \right) \right) \right\|^2$$
$$= \left\| \frac{8x_1}{5}, \frac{12x_2}{7}, \frac{16x_3}{9} \right\|^2$$
$$= \left(\frac{8x_1}{5} \right)^2 + \left(\frac{12x_2}{7} \right)^2 + \left(\frac{16x_3}{9} \right)^2$$
$$\geq 0,$$

and

$$2\langle \mathbf{x}-T\mathbf{x}, \mathbf{y}-T\mathbf{y} \rangle = 2\left\langle \left((x_1,x_2,x_3) - \left(\frac{-3x_1}{5}, \frac{-5x_2}{7}, \frac{-7x_3}{9} \right) \right), (y_1,y_2,y_3)-(y_1,y_2,y_3) \right\rangle$$
$$=0.$$

From the above equation, choose $\kappa = \frac{7}{8}$ such that

$$\|\mathbf{x}-\mathbf{y}\|^2 + \frac{7}{8}\|(I-T)\mathbf{x}-(I-T)\mathbf{y}\|^2 + 2\langle \mathbf{x}-T\mathbf{x}, \mathbf{y}-T\mathbf{y} \rangle \geq \|\mathbf{x}-\mathbf{y}\|^2$$
$$= \| (x_1,x_2,x_3) - (y_1,y_2,y_3) \|^2$$
$$= \|(x_1-y_1),(x_2-y_2),(x_3-y_3)\|^2$$
$$= (x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2$$
$$= x_1^2 - 2x_1y_1 + y_1^2 + x_2^2 - 2x_2y_2 + y_2^2 + x_3^2 - 2x_3y_3 + y_3^2$$
$$= x_1^2 - 2x_1y_1 + y_1^2 + \frac{9}{25}x_1^2 + \frac{6}{5}x_1y_1 - \frac{9}{25}x_1^2 - \frac{6}{5}x_1y_1$$
$$+ x_2^2 - 2x_2y_2 + y_2^2 + \frac{25}{49}x_2^2 + \frac{10}{7}x_2y_2 - \frac{25}{49}x_2^2 - \frac{10}{7}x_2y_2$$
$$+ x_3^2 - 2x_3y_3 + y_3^2 + \frac{49}{81}x_3^2 + \frac{14}{9}x_3y_3 - \frac{49}{81}x_3^2 - \frac{14}{9}x_3y_3$$
$$= \frac{9}{25}x_1^2 + \frac{6}{5}x_1y_1 + y_1^2 + x_1^2 - 2x_1y_1 - \frac{9}{25}x_1^2 - \frac{6}{5}x_1y_1$$
$$+ \frac{25}{49}x_2^2 + \frac{10}{7}x_2y_2 + y_2^2 + x_2^2 - 2x_2y_2 - \frac{25}{49}x_2^2 - \frac{10}{7}x_2y_2$$
$$+ \frac{49}{81}x_3^2 + \frac{14}{9}x_3y_3 + y_3^2 + x_3^2 - 2x_3y_3 - \frac{49}{81}x_3^2 - \frac{14}{9}x_3y_3$$
$$= \left(\frac{3x_1}{5} + y_1 \right)^2 + \frac{16}{25}x_1^2 - \frac{16}{5}x_1y_1$$
$$+ \left(\frac{3x_1}{5} + y_1 \right)^2 + \frac{24}{49}x_2^2 - \frac{24}{7}x_2y_2$$
$$+ \left(\frac{7x_3}{9} + y_3 \right)^2 + \frac{32}{81}x_3^2 - \frac{32}{9}x_3y_3$$
$$\geq \left(\frac{3x_1}{5} + y_1 \right)^2 + \left(\frac{3x_1}{5} + y_1 \right)^2 + \left(\frac{7x_3}{9} + y_3 \right)^2$$
$$= \|T\mathbf{x} - T\mathbf{y}\|^2.$$

Thus, T is a $\frac{7}{8}$-strictly pseudononspreading mapping. From the definition of T, we have $F(T) = \{\mathbf{0}\}$, where $\mathbf{0} = (0,0,0)$.

Next, we will show that for every $i = 1,2,3,...,N$, A_i is a $\frac{1}{i}$-inverse strongly monotone mapping. Let $\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$, $\mathbf{y} = (y_1,y_2,y_3) \in \mathbb{R}^3$. For every $i = 1,2,3,...,N$, from the definition of A_i, we have

$$\begin{aligned}
\|A_i\mathbf{x} - A_i\mathbf{y}\|^2 &= \left\|\left(\frac{ix_1}{6},\frac{ix_2}{6},\frac{ix_3}{6}\right) - \left(\frac{iy_1}{6},\frac{iy_2}{6},\frac{iy_3}{6}\right)\right\|^2 \\
&= \left\|\left(\frac{ix_1}{6} - \frac{iy_1}{6}\right),\left(\frac{ix_2}{6} - \frac{iy_2}{6}\right),\left(\frac{ix_3}{6} - \frac{iy_3}{6}\right)\right\|^2 \\
&= \left(\frac{ix_1}{6} - \frac{iy_1}{6}\right)^2 + \left(\frac{ix_2}{6} - \frac{iy_2}{6}\right)^2 + \left(\frac{ix_3}{6} - \frac{iy_3}{6}\right)^2 \\
&= \frac{i^2}{36}\left((x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2\right) \\
&= \frac{i^2}{36}\|\mathbf{x} - \mathbf{y}\|^2.
\end{aligned}$$

From the above equation, we have

$$\begin{aligned}
\langle A_i\mathbf{x} - A_i\mathbf{y}, \mathbf{x} - \mathbf{y}\rangle &= \left\langle \left(\frac{ix_1}{6},\frac{ix_2}{6},\frac{ix_3}{6}\right) - \left(\frac{iy_1}{6},\frac{iy_2}{6},\frac{iy_3}{6}\right), (x_1,x_2,x_3) - (y_1,y_2,y_3)\right\rangle \\
&= \Big\langle \left(\frac{ix_1}{6} - \frac{iy_1}{6}\right),\left(\frac{ix_2}{6} - \frac{iy_2}{6}\right),\left(\frac{ix_3}{6} - \frac{iy_3}{6}\right), (x_1-y_1) + (x_2-y_2) \\
&\quad + (x_3-y_3)\Big\rangle \\
&= \frac{i}{6}(x_1-y_1)^2 + \frac{i}{6}(x_2-y_2)^2 + \frac{i}{6}(x_3-y_3)^2 \\
&= \frac{i}{6}\|\mathbf{x} - \mathbf{y}\|^2 \\
&\geq \frac{i}{36}\|\mathbf{x} - \mathbf{y}\|^2 \\
&= \frac{1}{i}\left((\frac{i^2}{36}\|\mathbf{x} - \mathbf{y}\|^2\right) \\
&= \frac{1}{i}\|A_i\mathbf{x} - A_i\mathbf{y}\|^2.
\end{aligned}$$

Then, for every $i = 1,2,3,...,N$, A_i is a $\frac{1}{i}$-inverse strongly monotone mapping. Moreover,we have $\eta = min_{i=1,2,...,N}\{\frac{1}{i}\} = \frac{1}{N}$. Since $\lambda = \frac{1}{N}$ and $\eta = \frac{1}{N}$, then $0 < \lambda < 2\eta$. It implies that λ satisfies the condition (iii).

Next, we show that F_i satisfying $(A1) - (A4)$, for all $i = 1,2,...,N$. To prove (A1), let $\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$, $\mathbf{y} = (y_1,y_2,y_3) \in \mathbb{R}^3$, then we have

$$\begin{aligned}
F_i(\mathbf{x},\mathbf{x}) &= i(\mathbf{x} - \mathbf{x}) \cdot (9\mathbf{x} + \mathbf{x}) \\
&= i((x_1,x_2,x_3) - (x_1,x_2,x_3)) \cdot (9(x_1,x_2,x_3) + (x_1,x_2,x_3)) \\
&= 0.
\end{aligned}$$

Then F_i satisfies (A1), for all $i = 1,2,...,N$.

To prove (A2), we have

$$\begin{aligned}
F_i(\mathbf{x},\mathbf{y}) + F_i(\mathbf{y},\mathbf{x}) =& i(\mathbf{y}-\mathbf{x})\cdot(9\mathbf{x}+\mathbf{y}) + i(\mathbf{x}-\mathbf{y})\cdot(9\mathbf{y}+\mathbf{x}) \\
=& i\left((y_1,y_2,y_3)-(x_1,x_2,x_3)\right)\cdot\left(9(x_1,x_2,x_3)+(y_1,y_2,y_3)\right) \\
&+ i\left((x_1,x_2,x_3)-(y_1,y_2,y_3)\right)\cdot\left(9(y_1,y_2,y_3)+((x_1,x_2,x_3)\right) \\
=& i\big(8(x_1,x_2,x_3)(y_1,y_2,y_3)+(y_1,y_2,y_3)^2-9(x_1,x_2,x_3)^2 \\
&+ 8(x_1,x_2,x_3)(y_1,y_2,y_3)+(x_1,x_2,x_3)^2-9(y_1,y_2,y_3)^2\big) \\
=& i\left(16(x_1,x_2,x_3)(y_1,y_2,y_3)-8(y_1,y_2,y_3)^2-8(x_1,x_2,x_3)^2\right) \\
=& -8i\left((x_1,x_2,x_3)^2-2(x_1,x_2,x_3)(y_1,y_2,y_3)+(y_1,y_2,y_3)^2\right) \\
=& -8i\left((x_1,x_2,x_3)-(y_1,y_2,y_3)\right)^2 \\
\leq& 0.
\end{aligned}$$

Then F_i satisfies (A2), for all $i = 1,2,...,N$.

To prove (A3), let $\mathbf{z} = (z_1,z_2,z_3)$ and $t \in [0,1]$, then we have

$$\begin{aligned}
\lim_{t\to0^+} F_i(t\mathbf{z}+(1-t)\mathbf{x},\mathbf{y}) =& \lim_{t\to0^+} i(\mathbf{y}-(t\mathbf{z}+(1-t)\mathbf{x}))(9(t\mathbf{z}+(1-t)\mathbf{x})+\mathbf{y}) \\
=& \lim_{t\to0^+} i\big((y_1,y_2,y_3)-(t(z_1,z_2,z_3)+(1-t)(x_1,x_2,x_3))\big)\big(9(t(z_1,z_2,z_3) \\
&+(1-t)(x_1,x_2,x_3))+(y_1,y_2,y_3)\big) \\
=& i\big((y_1,y_2,y_3)-(x_1,x_2,x_3)\big)\big(9((x_1,x_2,x_3))+(y_1,y_2,y_3)\big) \\
=& F_i(\mathbf{x},\mathbf{y}).
\end{aligned}$$

Then F_i satisfies (A3), for all $i = 1,2,...,N$.

To prove (A4), we first show that F_i is convex. Since $F_i(\mathbf{x},\mathbf{y}) = i(\mathbf{y}-\mathbf{x})\cdot(9\mathbf{x}+\mathbf{y})$, we get

$$\begin{aligned}
F_i(\mathbf{x},\mathbf{y}) =& i\left((y_1,y_2,y_3)-(x_1,x_2,x_3)\right)\cdot\left(9(x_1,x_2,x_3)+(y_1,y_2,y_3)\right) \\
=& i\left(y_1-x_1,y_2-x_2,y_3-x_3\right)\cdot\left(9x_1+y_1,9x_2+y_2,9x_3+y_3\right) \\
=& i\left((y_1-x_1)(9x_1+y_1)+(y_2-x_2)(9x_2+y_2)+(y_3-x_3)(9x_3+y_3)\right) \\
=& i\left(\left(8y_1x_1+y_1^2-9x_1^2\right)+\left(8y_2x_2+y_2^2-9x_2^2\right)+\left(8y_3x_3+y_3^2-9x_3^2\right)\right).
\end{aligned}$$

Let $\alpha \in (0,1)$ and the above equation, we get

$$\begin{aligned}
F_i(\mathbf{x},\alpha z+(1-\alpha)y) =& F_i\left((x_1,x_2,x_3),\alpha(z_1,z_2,z_3)+(1-\alpha)(y_1,y_2,y_3)\right) \\
=& F_i\left((x_1,x_2,x_3),(\alpha z_1+(1-\alpha)y_1,\alpha z_2+(1-\alpha)y_2,\alpha z_3+(1-\alpha)y_3)\right) \\
=& i\big(\left(8(\alpha z_1+(1-\alpha)y_1)x_1+(\alpha z_1+(1-\alpha)y_1)^2-9x_1^2\right) \\
&+\left(8(\alpha z_2+(1-\alpha)y_2)x_2+(\alpha z_2+(1-\alpha)y_2)^2-9x_2^2\right) \\
&+\left(8(\alpha z_3+(1-\alpha)y_3)x_3+(\alpha z_3+(1-\alpha)y_3)^2-9x_3^2\right)\big)
\end{aligned}$$

$$
\begin{aligned}
=&i\big(\left(\alpha^2 z_1^2 + (1-\alpha)^2 y_1^2 + 2\alpha(1-\alpha) z_1 y_1 + 8x_1(\alpha z_1 + (1-\alpha) y_1) - 9x_1^2\right) \\
&+ \left(\alpha^2 z_2^2 + (1-\alpha)^2 y_2^2 + 2\alpha(1-\alpha) z_2 y_2 + 8x_2(\alpha z_2 + (1-\alpha) y_2) - 9x_2^2\right) \\
&+ \left(\alpha^2 z_3^2 + (1-\alpha)^2 y_3^2 + 2\alpha(1-\alpha) z_3 y_3 + 8x_3(\alpha z_3 + (1-\alpha) y_3) - 9x_3^2\right) \big) \\
\leq&i\big(\left(\alpha^2 z_1^2 + (1-\alpha)^2 y_1^2 + \alpha(1-\alpha)(z_1^2 + y_1^2) + 8x_1(\alpha z_1 + (1-\alpha) y_1) - 9x_1^2\right) \\
&+ \left(\alpha^2 z_2^2 + (1-\alpha)^2 y_2^2 + \alpha(1-\alpha)(z_2^2 + y_2^2) + 8x_2(\alpha z_2 + (1-\alpha) y_2) - 9x_2^2\right) \\
&+ \left(\alpha^2 z_3^2 + (1-\alpha)^2 y_3^2 + \alpha(1-\alpha)(z_3^2 + y_3^2) + 8x_3(\alpha z_3 + (1-\alpha) y_3) - 9x_3^2\right) \big) \\
=&i\big(\alpha\left((z_1^2 + 8x_1 z_1 - 9x_1^2) + (z_2^2 + 8x_2 z_2 - 9x_2^2) + (z_3^2 + 8x_3 z_3 - 9x_3^2)\right) \\
&+ (1-\alpha)\left((y_1^2 + 8x_1 y_1 - 9x_1^2) + (y_2^2 + 8x_2 y_2 - 9x_2^2) + (y_2^2 + 8x_2 y_2 - 9x_2^2)\right)\big) \\
=&i\big(\alpha\big((z_1^2 + 9x_1 z_1 - x_1 z_1 - 9x_1^2) + (z_2^2 + 9x_2 z_2 - x_2 z_2 - 9x_2^2) \\
&+ (z_3^2 + 9x_3 z_3 - x_3 z_3 - 9x_3^2) \big) + (1-\alpha)\big((y_1^2 + 9x_1 z_1 - x_1 z_1 - 9x_1^2) \\
&+ (y_2^2 + 9x_2 z_2 - x_2 z_2 - 9x_2^2) + (y_2^2 + 9x_3 z_3 - x_3 z_3 - 9x_2^2) \big)\big) \\
=&i\big(\alpha\big((z_1 - x_1)(9x_1 + z_1) + (z_2 - x_2)(9x_2 + z_2) + (z_3 - x_3)(9x_3 + z_3)\big) \\
&+ (1-\alpha)\big((y_1 - x_1)(9x_1 + y_1) + (y_2 - x_2)(9x_2 + y_2) + (y_3 - x_3)(9x_3 + y_3)\big)\big) \\
=&\alpha\left(i\,(z_1 - x_1, z_2 - x_2, z_3 - x_3) \cdot (9x_1 + z_1, 9x_2 + z_2, 9x_3 + z_3)\right) \\
&+ (1-\alpha)\left(i\,(y_1 - x_1, y_2 - x_2, y_3 - x_3) \cdot (9x_1 + y_1, 9x_2 + y_2, 9x_3 + y_3)\right) \\
=&\alpha\left(i\left((z_1, z_2, z_3) - (x_1, x_2, x_3)\right) \cdot \left(9(x_1, x_2, x_3) + (z_1, z_2, z_3)\right)\right) \\
&+ ((1-\alpha))\left(i\left((y_1, y_2, y_3) - (x_1, x_2, x_3)\right) \cdot \left(9(x_1, x_2, x_3) + (y_1, y_2, y_3)\right)\right) \\
=&\alpha F_i(\mathbf{x}, \mathbf{z}) + (1-\alpha) F_i(\mathbf{x}, \mathbf{y}).
\end{aligned}
$$

Thus F_i is convex function, for all $i = 1, 2, 3, ..., N$. Suppose that $\{y_n\} \subset \mathbb{R}^3$ with $\mathbf{y}_n = (y_n^1, y_n^2, y_n^3) \to \mathbf{y} = (y_1, y_2, y_3)$ as $n \to \infty$. Thus we have

$$\lim_{n\to\infty} F_i(\mathbf{x}, \mathbf{y}_n) = \lim_{n\to\infty} i\,(\mathbf{y}_n - \mathbf{x}) \cdot (9\mathbf{x} + \mathbf{y}_n) = i\,(\mathbf{y} - \mathbf{x}) \cdot (9\mathbf{x} + \mathbf{y}) = F(\mathbf{x}, \mathbf{y}).$$

Then F_i is lower semicontinuous , for all $i = 1, 2, ..., N$. Hence the condition (A4) holds. Therefore F_i satisfies $(A1) - (A4)$, for all $i = 1, 2, ..., N$.

Since $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$ and by using the same method as Example 4.17, we have $\sum_{i=1}^N a_i = 1$, which implies that

$$\sum_{i=1}^N a_i F_i(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^N \left(\frac{4}{5^i} + \frac{1}{N5^N}\right) i(\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} + 9\mathbf{x}),$$

for all $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. It implies that $\sum_{i=1}^N a_i F_i$ satisfies $(A1) - (A4)$.

Since $\sum_{i=1}^N a_i F_i(\mathbf{0}, \mathbf{y}) = \sum_{i=1}^N a_i F_i((0,0,0), (y_1, y_2, y_3)) = \sum_{i=1}^N a_i i\,((y_1, y_2, y_3) - (0,0,0)) \cdot (9(0,0,0) + (y_1, y_2, y_3)) = \sum_{i=1}^N a_i (\mathbf{y})^2 \geq 0$, then $\mathbf{0} \in EP(\sum_{i=1}^N a_i F_i) = \bigcap_{i=1}^N EP\,(F_i)$. It implies that

$$F(T) \cap \bigcap_{i=1}^N EP(F_i) = \{\mathbf{0}\}. \tag{4.10}$$

Put $S_2 = \sum_{i=1}^{N}\left(\frac{4}{5^i} + \frac{1}{N5^N}\right) i$, then we have

$$
\begin{aligned}
0 \leq & \sum_{i=1}^{N} a_i F_i(u_n, \mathbf{y}) + \frac{1}{r_n}\langle \mathbf{y} - u_n, u_n - x_n\rangle \\
= & S_2(\mathbf{y} - u_n)\cdot(\mathbf{y} + 9u_n) + \frac{1}{r_n}(\mathbf{y} - u_n)(u_n - x_n) \\
= & S_2\left(y_1 - u_n^1, y_2 - u_n^2, y_3 - u_n^3\right)\cdot\left(y_1 + 9u_n^1, y_2 + 9u_n^2, y_3 + 9u_n^3\right) \\
& + \frac{1}{r_n}\left(y_1 - u_n^1, y_2 - u_n^2, y_3 - u_n^3\right)\cdot\left(u_n^1 - x_n^1, u_n^2 - x_n^2, u_n^3 - x_n^3\right) \\
= & S_2\left(\left(y_1 - u_n^1\right)\left(y_1 + 9u_n^1\right) + \left(y_2 - u_n^2\right)\left(y_2 + 9u_n^2\right) + \left(y_3 - u_n^3\right)\left(y_3 + 9u_n^3\right)\right) \\
& + \frac{1}{r_n}\left(\left(y_1 - u_n^1\right)\left(u_n^1 - x_n^1\right) + \left(y_2 - u_n^2\right)\left(u_n^2 - x_n^2\right) + \left(y_3 - u_n^3\right)\left(u_n^3 - x_n^3\right)\right) \\
= & \left(S_2\left(y_1 - u_n^1\right)\left(y_1 + 9u_n^1\right) + \frac{1}{r_n}\left(y_1 - u_n^1\right)\left(u_n^1 - x_n^1\right)\right) \\
& + \left(S_2\left(y_2 - u_n^2\right)\left(y_2 + 9u_n^2\right) + \frac{1}{r_n}\left(y_2 - u_n^2\right)\left(u_n^2 - x_n^2\right)\right) \\
& + \left(S_2\left(y_3 - u_n^3\right)\left(y_3 + 9u_n^3\right) + \frac{1}{r_n}\left(y_3 - u_n^3\right)\left(u_n^3 - x_n^3\right)\right) \\
& \Leftrightarrow
\end{aligned}
$$

$$
\begin{aligned}
0 \leq & \left(S_2 r_n\left(y_1 - u_n^1\right)\left(y_1 + 9u_n^1\right) + \left(y_1 - u_n^1\right)\left(u_n^1 - x_n^1\right)\right) \\
& + \left(S_2 r_n\left(y_2 - u_n^2\right)\left(y_2 + 9u_n^2\right) + \left(y_2 - u_n^2\right)\left(u_n^2 - x_n^2\right)\right) \\
& + \left(S_2 r_n\left(y_3 - u_n^3\right)\left(y_3 + 9u_n^3\right) + \left(y_3 - u_n^3\right)\left(u_n^3 - x_n^3\right)\right) \\
= & \left(S_2 r_n (y_1)^2 + \left(u_n^1 + 8r_n S_2 u_n^1 - x_n^1\right) y_1 + u_n^1 x_n^1 - 9r_n S_2\left(u_n^1\right)^2 - \left(u_n^1\right)^2\right) \\
& + \left(S_2 r_n (y_2)^2 + \left(u_n^2 + 8r_n S_2 u_n^2 - x_n^2\right) y_2 + u_n^2 x_n^2 - 9r_n S_2\left(u_n^2\right)^2 - \left(u_n^2\right)^2\right) \\
& + \left(S_2 r_n (y_3)^2 + \left(u_n^3 + 8r_n S_2 u_n^3 - x_n^3\right) y_3 + u_n^3 x_n^3 - 9r_n S_2\left(u_n^3\right)^2 - \left(u_n^3\right)^2\right) \\
= & G(y_1) + G(y_2) + G(y_3), \qquad (4.11)
\end{aligned}
$$

where $G(y_1) = S_2 r_n (y_1)^2 + \left(u_n^1 + 8r_n S_2 u_n^1 - x_n^1\right) y_1 + u_n^1 x_n^1 - 9r_n S_2\left(u_n^1\right)^2 - \left(u_n^1\right)^2$, $G(y_2) = S_2 r_n (y_2)^2 + \left(u_n^2 + 8r_n S_2 u_n^2 - x_n^2\right) y_2 + u_n^2 x_n^2 - 9r_n S_2\left(u_n^2\right)^2 - \left(u_n^2\right)^2$ and $G(y_3) = S_2 r_n (y_3)^2 + \left(u_n^3 + 8r_n S_2 u_n^3 - x_n^3\right) y_3 + u_n^3 x_n^3 - 9r_n S_2\left(u_n^3\right)^2 - \left(u_n^3\right)^2$. Then $G(y_1)$, $G(y_2)$ and $G(y_3)$ is a quadratic function of $\mathbf{y}$ with coefficient $a_1 = S_2 r_n$, $b_1 = u_n^1 + 8r_n S_2 u_n^1 - x_n^1$, $c_1 = u_n^1 x_n^1 - 9r_n S_2\left(u_n^1\right)^2 - \left(u_n^1\right)^2$, $a_2 = S_2 r_n$, $b_2 = u_n^2 + 8r_n S_2 u_n^2 - x_n^2$, $c_2 = u_n^2 x_n^2 - 9r_n S_2\left(u_n^2\right)^2 - \left(u_n^2\right)^2$, $a_3 = S_1 r_n$, $b_3 = u_n^3 + 8r_n S_2 u_n^3 - x_n^3$ and $c_3 = u_n^3 x_n^3 - 9r_n S_2\left(u_n^3\right)^2 - \left(u_n^3\right)^2$, respectively. Determine the discriminant Δ_1 of G_1 as follows:

$$
\begin{aligned}
\Delta_1 = & b_1^2 - 4a_1 c_1 \\
= & \left(u_n^1 + 8r_n S_2 u_n^1 - x_n^1\right)^2 - 4\left(S_2 r_n\right)\left(u_n^1 x_n^1 - 9r_n S_2\left(u_n^1\right)^2 - \left(u_n^1\right)^2\right) \\
= & \left(u_n^1\right)^2 + 20r_n S_2\left(u_n^1\right)^2 + 100r_n^2 S_2^2\left(u_n^1\right)^2 - 2u_n^1 x_n^1 - 20r_n S_2 u_n^1 x_n^1 + \left(x_n^1\right)^2 \\
= & \left(u_n^1 + 10S_2 r_n u_n^1 - x_n^1\right)^2.
\end{aligned}
$$

From (4.11), if $G(y_1) \geq 0, \forall y_1 \in \mathbb{R}$ and it has most one solution in $\mathbb{R}$, then $\Delta_1 \leq 0$, so we obtain

$$u_n^1 = \frac{x_n^1}{1 + 10S_2 r_n}. \tag{4.12}$$

Next, determine the discriminant Δ_2 of G_2 as follows:

$$\begin{aligned}
\Delta_2 =& b_2^2 - 4a_2c_2 \\
=& \left(u_n^2 + 8r_nS_2u_n^2 - x_n^2\right)^2 - 4\left(S_2r_n\right)\left(u_n^2x_n^2 - 9r_nS_2\left(u_n^2\right)^2 - \left(u_n^2\right)^2\right) \\
=& \left(u_n^2\right)^2 + 20r_nS_2\left(u_n^2\right)^2 + 100r_n^2S_2^2\left(u_n^2\right)^2 - 2u_n^2x_n^2 - 20r_nS_2u_n^2x_n^2 + \left(x_n^2\right)^2 \\
=& \left(u_n^2 + 10S_2r_nu_n^2 - x_n^2\right)^2.
\end{aligned}$$

From (4.11), if $G(y_2) \geq 0, \forall y_2 \in \mathbb{R}$ and it has at most one solution in $\mathbb{R}$, then $\Delta_2 \leq 0$, so we obtain

$$u_n^2 = \frac{x_n^2}{1 + 10S_2 r_n}. \tag{4.13}$$

Next, determine the discriminant Δ_3 of G_3 as follows:

$$\begin{aligned}
\Delta_3 =& b_3^2 - 4a_3c_3 \\
=& \left(u_n^3 + 8r_nS_2u_n^3 - x_n^3\right)^2 - 4\left(S_2r_n\right)\left(u_n^3x_n^3 - 9r_nS_2\left(u_n^3\right)^2 - \left(u_n^3\right)^2\right) \\
=& \left(u_n^3\right)^2 + 20r_nS_2\left(u_n^3\right)^2 + 100r_n^2S_2^2\left(u_n^3\right)^2 - 2u_n^3x_n^3 - 20r_nS_2u_n^3x_n^3 + \left(x_n^3\right)^2 \\
=& \left(u_n^3 + 10S_2r_nu_n^3 - x_n^3\right)^2.
\end{aligned}$$

From (4.11), if $G(y_3) \geq 0, \forall y_3 \in \mathbb{R}$ and it has at most one solution in $\mathbb{R}$, then $\Delta_3 \leq 0$, so we obtain

$$u_n^3 = \frac{x_n^3}{1 + 10S_2 r_n}. \tag{4.14}$$

Since $A_i\mathbf{x} = \left(\frac{ix_1}{6}, \frac{ix_2}{6}, \frac{ix_3}{6}\right)$ and $b_i = \frac{7}{8^i} + \frac{1}{N8^N}$, then

$$\sum_{i=1}^{N} b_iA_i\mathbf{x} = \sum_{i=1}^{N}\left(\frac{7}{8^i} + \frac{1}{N8^N}\right)A_i\mathbf{x}.$$

From above and the definition of the modified variational inclusion, we obtain

$$\bigcap_{i=1}^{N} VI(H, A_i, M) = \{\mathbf{0}\}. \tag{4.15}$$

From (4.10) and (4.15), we have

$$F(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) = \{\mathbf{0}\}. \tag{4.16}$$

For every $n \in \mathbb{N}$, $\alpha_n = \frac{1}{15n}$, $\beta_n = \frac{3(15n-1)}{165n}$, $\gamma_n = \frac{2(15n-1)}{165n}$, $\eta_n = \frac{5(15n-1)}{165n}$, $\delta_n = \frac{15n-1}{165n}$, $r_n = \frac{2n}{7n+6}$ and $\rho_n = \frac{1}{9n^2}$. By using the same method as Example 4.17, then the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\}, \{r_n\}$ and $\{\rho_n\}$ satisfy all the conditions of Theorem 3.7. For every $n \in \mathbb{N}$, from (4.12), (4.13), and (4.14), we rewrite (3.19) as follows:

$$\begin{aligned}
\mathbf{x}_{n+1} =& \frac{1}{15n}\mathbf{u} + \frac{3(15n-1)}{165n}\mathbf{x}_n + \frac{2(15n-1)}{165n}\left(\mathbf{x}_n - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{7}{8^i} + \frac{1}{N8^N}\right)A_i\mathbf{x}\right) \\
&+ \frac{5(15n-1)}{165n}\left(I - \frac{1}{9n^2}(I - T)\right)\mathbf{x}_n + \frac{15n-1}{165n}\mathbf{u}_n,
\end{aligned} \tag{4.17}$$

where $\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$ and $\mathbf{u}_n = (u_n^1, u_n^2, u_n^3) = \left(\frac{x_n^1}{1+10S_2r_n}, \frac{x_n^2}{1+10S_2r_n}, \frac{x_n^3}{1+10S_2r_n}\right)$.

Using the algorithm (4.17) and choosing $\mathbf{u} = \mathbf{x}_1 = (5, 10, 15)$, we consider the parameters for two cases as follows: (i) $n = 100$ and $N = 1$. (ii) $n = 100$ and $N = 100$. The numerical results for the sequences x_n and u_n in two cases are show in the following table and figure.

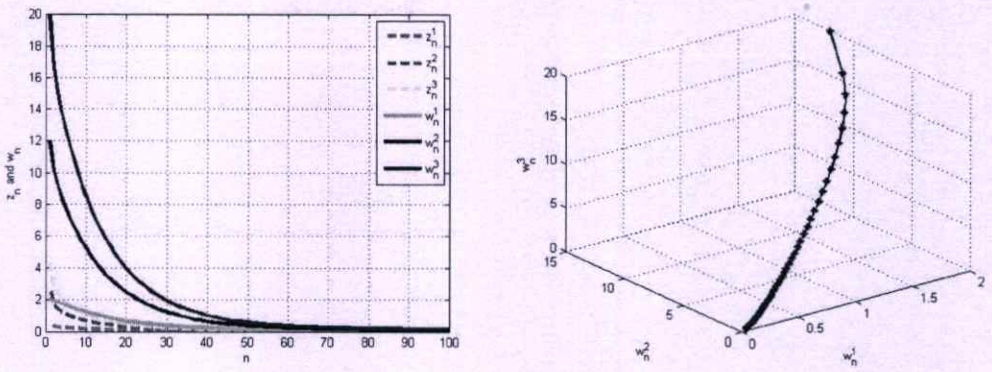
Table 4.2: The values of the sequences $\{\mathbf{u}_n\}$ and $\{\mathbf{x}_n\}$ with initial values $\mathbf{u} = (5, 10, 15)$, $\mathbf{x}_1 = (2, 12, 20)$, and $n = N = 100$.

n	u_n	x_n
1	(0.453608, 2.721649, 4.536082)	(2.000000, 12.000000, 20.000000)
2	(0.336995, 1.776168, 2.930220)	(1.916656, 10.101958, 16.665628)
3	(0.289738, 1.439132, 2.364314)	(1.841903, 9.148767, 15.030279)
4	(0.259512, 1.239451, 2.030799)	(1.756695, 8.390131, 13.746949)
5	(0.236507, 1.096332, 1.792646)	(1.666990, 7.727376, 12.635260)
⋮	⋮	⋮
50	(0.017724, 0.045633, 0.070850)	(0.147537, 0.379859, 0.589773)
⋮	⋮	⋮
96	(0.006069, 0.012338, 0.018554)	(0.051022, 0.103729, 0.155990)
97	(0.005987, 0.012158, 0.018280)	(0.050339, 0.102226, 0.153703)
98	(0.005907, 0.011984, 0.018015)	(0.049677, 0.100775, 0.151496)
99	(0.005830, 0.011816, 0.017760)	(0.049033, 0.099372, 0.149365)
100	(0.005755, 0.011653, 0.017513)	(0.048408, 0.098015, 0.147304)

Conclusions

1. The sequence $\{\mathbf{x}_n\}$ and $\{\mathbf{u}_n\}$ converge to $\mathbf{0}$ as shown in the Table 4.2 and Figure 4.2, where $\mathbf{0} = (0, 0, 0)$.

2. From Theorem 3.7, we can conclude that the sequence $\{\mathbf{x}_n\}$ and $\{\mathbf{u}_n\}$, in Example 4.18, converge to $\mathbf{0}$, where $\mathbf{0} = (0, 0, 0)$.



(a) 2D *(b)* 3D

Figure 4.2: The convergence of $\{\mathbf{u}_n\}$ and $\{\mathbf{x}_n\}$ with initial values $\mathbf{u} = (5, 10, 15)$, $\mathbf{x}_1 = (2, 12, 20)$ and $n = N = 100$.

The following example is given to support Theorem 3.11.

Example 4.19. Let $\mathbb{R}$ be the set of real numbers and let the mapping $A, B : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, be defined by

$$Ax = \frac{x}{5}, Bx = \frac{x}{10}, fx = \frac{x}{6}, \text{ for all } x, y \in \mathbb{R}.$$

For every $i = 1, 2, \ldots, N$, let $F_i : [0, 10^3] \times [0, 10^3] \to \mathbb{R}$ and $T_i : [0, 10^3] \to [0, 10^3]$ be defined by

$$T_i x = \frac{\sin(ix)}{i}, F_i(x, y) = i(y - x)(y + 5x), \text{ for all } x, y \in [0, 10^3].$$

For every $i = 1, 2, \ldots, N$, suppose that $a_i = \frac{5}{6^i} + \frac{1}{N6^N}$, $c_i = \frac{4}{5^i} + \frac{1}{N5^N}$ and $a = \frac{1}{3}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be generated by (3.45), where $\alpha_n = \frac{1}{6n}$, $\beta_n = \frac{3(6n-1)}{30n}$, $\gamma_n = \frac{2(6n-1)}{30n}$, $r_n = \frac{2n}{6n+9}$, $b_n = \frac{n}{5n+4}$, $\lambda_n = \frac{1}{2n^2}$ and $\rho = \frac{n}{5n+1}$ for every $n \in \mathbb{N}$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to 0.

Solution. For every $i = 1, 2, ..., N$. From Example 4.14, T_i is a nonspreading mappings and $F(T_i) = \{0\}$, for all $i = 1, 2, ..., N$. It implies that

$$\bigcap_{i=1}^{N} F(T_i) = \{0\}. \tag{4.18}$$

Next, we will show that A isa 1-inverse strongly monotone mapping. Let $x, y \in \mathbb{R}$. From the definition of A, we have

$$|Ax - Ay|^2 = |\frac{x}{5} - \frac{y}{5}|^2 = \frac{1}{25}|x - y|^2.$$

From the above equation, we have

$$\begin{aligned}\langle Ax - Ay, x - y\rangle =& \langle \frac{x}{5} - \frac{y}{5}, x - y\rangle \\ =& \frac{1}{5}|x - y|^2 \\ \geq& \frac{1}{25}|x - y|^2 \\ =& |A_i x - A_i y|^2.\end{aligned}$$

Then, for every $i = 1, 2, 3, ..., N$, A_i is a 1-inverse strongly monotone mapping. Similarly, B is a 1-inverse strongly monotone mapping.

From the definition of f, then

$$|f(x) - f(y)| = |\frac{x}{6} - \frac{y}{6}| = \frac{1}{6}|x - y|.$$

Then f is a $\frac{1}{6}$-contractive mapping. From the definition of A and B, we have

$$\langle y - x, (a\frac{x}{5} + (1 - a)\frac{x}{10})\rangle = (y - x)(2a\frac{x}{10} + (1 - a)\frac{x}{10}) = (1 + a)(y - x)(\frac{x}{10}).$$

From the above equation, we have

$$0 \in VI(C, aA + (1 - a)B) = VI(C, A) \cap VI(C, B). \tag{4.19}$$

Similarly, from Example 4.17 and the definition of F_i, we have F_i is satisfies (A1)-(A4) for all $i = 1, 2, ..., N$.

Since $a_i = \frac{5}{6^i} + \frac{1}{N6^N}$, then

$$\begin{aligned}\sum_{i=1}^{N} a_i =& \sum_{i=1}^{N}\left(\frac{5}{6^i} + \frac{1}{N6^N}\right) \\ =& \left(\frac{5}{6^1} + \frac{1}{N6^N}\right) + \left(\frac{5}{6^2} + \frac{1}{N6^N}\right) + \left(\frac{5}{6^3} + \frac{1}{N6^N}\right) + \cdots + \left(\frac{5}{6^N} + \frac{1}{N6^N}\right) \\ =& \left(\frac{5}{6^1} + \frac{5}{6^2} + \frac{5}{6^3} + \cdots + \frac{5}{6^N}\right) + \left(\frac{1}{N6^N} + \frac{1}{N6^N} + \frac{1}{N6^N} + \cdots + \frac{1}{N6^N}\right) \\ =& 5\left(\frac{1}{6^1} + \frac{1}{6^2} + \frac{1}{6^3} + \cdots + \frac{1}{6^N}\right) + \frac{N}{N6^N} \\ =& 5\left(\frac{\frac{1}{6}\left(1 - \frac{1}{6^N}\right)}{1 - \frac{1}{6}}\right) + \frac{1}{6^N} \\ =& \left(1 - \frac{1}{6^N}\right) + \frac{1}{6^N} \\ =& 1.\end{aligned}$$

Then $\sum_{i=1}^{N} a_i = 1$ and so is $\sum_{i=1}^{N} b_i = 1$, where $c_i = \frac{4}{5^i} + \frac{1}{N5^N}$. It implies that the condition (v) holds. Using the same mathod as the proof of F_i satisfying $(A1) - (A4)$, for all $i = 1, 2, ..., N$, then we can conclude that $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4), where $a_i = \frac{5}{6^i} + \frac{1}{N6^N}$.

From the definition of a_i and F_i for all $i = 1, 2, 3, ..., N$, we obtain $\sum_{i=1}^{N} a_iF_i(0, y) = \sum_{i=1}^{N} a_iF_i((0, y) = \sum_{i=1}^{N} a_ii\,(y-0)\cdot(5(0)+y) = \sum_{i=1}^{N} a_ii(y^2) \geq 0, \forall y \in R$. It implies that

$$0 \in EP\left(\sum_{i=1}^{N} a_iF_i\right) = \bigcap_{i=1}^{N} EP\left(F_i\right). \tag{4.20}$$

From (4.18), (4.19), and (4.21), we have

$$\bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} EP\left(F_i\right) \cap VI(C, A) \cap VI(C, B) = \{0\}. \tag{4.21}$$

Put $S_3 = \sum_{i=1}^{N}\left(\frac{5}{6^i} + \frac{1}{N6^N}\right) i$, then we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^{N} a_iF_i\left(u_n, y\right) + \frac{1}{r_n}\left\langle y - u_n, u_n - x_n\right\rangle \\ &= \sum_{i=1}^{N}\left(\frac{5}{6^i} + \frac{1}{N6^N}\right) i\left(y-u_n\right)\left(y+5u_n\right) + \frac{1}{r_n}\left(y-u_n\right)\left(u_n - x_n\right) \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} 0 &\leq S_3r_n\left(y-u_n\right)\left(y+5u_n\right) + \left(y-u_n\right)\left(u_n-x_n\right) \\ &= S_3r_ny^2 + \left(u_n + 4r_nS_1u_n - x_n\right)y + u_nx_n - 5r_nS_1u_n^2 - u_n^2. \end{aligned}$$

Let $G(y) = S_3r_ny^2 + \left(u_n + 4r_nS_3u_n - x_n\right)y + u_nx_n - 5r_nS_3u_n^2 - u_n^2$. $G(y)$ is a quadratic function of y with coefficient $a = S_3r_n$, $b = u_n + 4r_nS_3u_n - x_n$, and $c = u_nx_n - 5r_nS_3u_n^2 - u_n^2$. Determine the discriminant Δ of G as follows

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= \left(u_n + 4r_nS_3u_n - x_n\right)^2 - 4\left(S_3r_n\right)\left(u_nx_n - 5r_nS_3u_n^2 - u_n^2\right) \\ &= u_n^2 + 12r_nS_3u_n^2 + 36r_n^2S_3^2u_n^2 - 2u_nx_n - 12r_nS_3u_nx_n + x_n^2 \\ &= \left(u_n + 6S_3r_nu_n - x_n\right)^2. \end{aligned}$$

We know that $G(y) \geq 0, \forall y \in \mathbb{R}$. If it has most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so we obtain

$$u_n = \frac{x_n}{1 + 6S_3r_n}, \tag{4.22}$$

where $S_3 = \sum_{i=1}^{N}\left(\frac{5}{6^i} + \frac{1}{N6^N}\right) i$.

For every $n \in \mathbb{N}$, let $\alpha_n = \frac{1}{6n}$, $\beta_n = \frac{3(6n-1)}{30n}$, $\gamma_n = \frac{2(6n-1)}{30n}$, $r_n = \frac{2n}{6n+9}$, $b_n = \frac{n}{5n+4}$, $\lambda_n = \frac{1}{2n^2}$ and $\rho = \frac{n}{5n+1}$, we have

$$\alpha_n + \beta_n + \gamma_n = \frac{1}{6n} + \frac{3(6n-1)}{30n} + \frac{2(6n-1)}{30n} = \frac{1}{6n} + 1 - \frac{1}{6n} = 1.$$

Since $\alpha_n = \frac{1}{6n}$, for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{1}{6n} = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \frac{1}{6n} = \infty$. It implies that the condition (i) holds.

Since $b_n = \frac{n}{5n+4}$, for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{5n+4} = \frac{1}{5} \in (0,1)$. It implies that the condition (ii) holds.

Since A_i , B are 1-inverse strongly monotone mappings and $\rho = \frac{n}{5n+1}$, for all $n \in \mathbb{N}$, then $0 < \rho_n < 2$. Then we have the condition (iii) holds

Since $\lambda_n = \frac{1}{2n^2}$, for all $n \in \mathbb{N}$, by the p-series, then $\sum_{n=1}^{\infty} \frac{1}{2n^2}$ converges. That is the condition (iv) holds. Observe that

$$|\alpha_{n+1} - \alpha_n| = |\frac{1}{20(n+1)} - \frac{1}{20n}| = \frac{1}{6n^2+6n}.$$

By using the Limit comparison test, putting $p_n = \frac{1}{20n^2+20n}$ and $q_n = \frac{1}{n^2}$, then

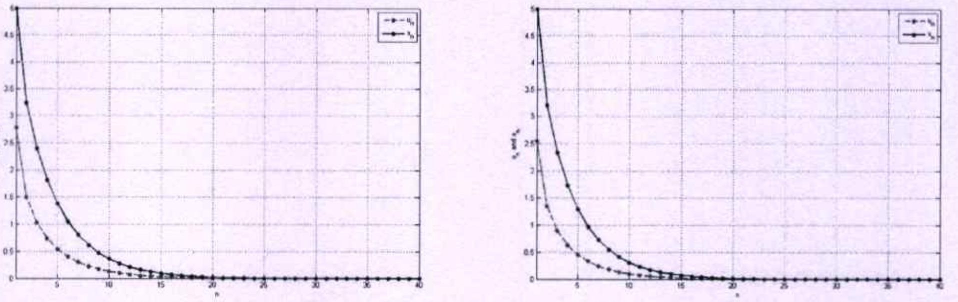
$$\lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} \frac{n^2}{6n^2+6n} = \frac{1}{6}.$$

Since $\sum_{n=1}^{\infty} q_n$ converges, then $\sum_{n=1}^{\infty} p_n$ also converges . It implies that $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and so are $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then the condition (vi) holds. Moreover, $\lim_{n\to\infty} r_n = \lim_{n\to\infty} \frac{2n}{6n+9} = \frac{1}{3}$. Then $0 < c \le r_n \le d < 1$, $\forall n \in \mathbb{N}$ and so are $\{\beta_n\}$, $\{\gamma_n\}$. It implies that the condition (vii) holds.

From the above description, we have the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{r_n\}$, $\{b_n\}$, $\{\rho_n\}$ and $\{\lambda_n\}$ satisfy all the conditions of Theorem 3.11. From (4.22) and (3.45), we have

$$\begin{cases} y_n^i = \left(\frac{n}{5n+4}\right) x_n + \left(1 - \frac{n}{5n+4}\right) P_{[0,10^3]} \left(I - \frac{1}{2n^2}(I - T_i)\right) \left(\frac{x_n}{1+6S_3 r_n}\right), \\ x_{n+1} = \left(\frac{1}{6n}\right) \frac{x_n}{6} + \left(\frac{3(6n-1)}{30n}\right) P_{[0,10^3]} \left(I - \frac{n}{5n+1}\left(\frac{1}{3}A + \left(1 - \frac{1}{3}\right) B\right)\right) x_n \\ \quad + \left(\frac{2(6n-1)}{30n}\right) \sum_{i=1}^{N} \left(\frac{4}{5^i} + \frac{1}{N5^N}\right) y_n^i, \end{cases} \tag{4.23}$$

for all $n \ge 1$ and $i = 1, 2, ..., N$. Using the algorithm (4.23) and choosing $u = x_1 = 5$ with $N = 1$ and $N = 100$, we have the numerical result as follows:



(a) $N = 1$ (b) $N = 100$

Figure 4.3: The convergence of $\{u_n\}$ and $\{x_n\}$ with initial values $u = x_1 = 5$.

Table 4.3: The values of the sequences $\{u_n\}$ and $\{x_n\}$ with initial values $u = x_1 = 5$.

	$N = 1$		$N = 100$	
n	u_n	x_n	u_n	x_n
1	2.777778	5.000000	2.551020	5.000000
2	1.508621	3.232758	1.350608	3.202870
3	1.029955	2.403229	0.897762	2.334181
4	0.740664	1.817993	0.631121	1.732715
5	0.545021	1.383515	0.454871	1.294633
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	0.009089	0.025998	0.005806	0.018768
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
36	0.000134	0.000392	0.000065	0.000216
37	0.000103	0.000302	0.000050	0.000164
38	0.000079	0.000232	0.000037	0.000124
39	0.000061	0.000179	0.000028	0.000094
40	0.000047	0.000138	0.000021	0.000071

Conclusions

1. Table 4.3 shows that the sequence $\{x_n\}$ and $\{u_n\}$ converge to 0.

2. Theorem 3.11 guarantees the convergence of $\{x_n\}$ and $\{u_n\}$ to 0 in Example 4.19.

Chapter 5

Conclusions

In this chapter, we conclude all main theorems obtained in this thesis.

5.1 Strong Convergence Theorems for Nonexpansive Mapping with Some New Sufficient Conditions

(1) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, and let $D : C \to H$ be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Let $T : C \to C$ be a nonexpansive mapping with $\mathcal{F} := F(T) \cap VI(H, A, M) \cap VI(H, B, M) \cap VI(C, D) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} y_n = J_{M,\lambda}(I - \lambda(aA + (1-a)B))x_n, \\ x_{n+1} = \alpha_n P_C(I - \rho D)y_n + (1 - \alpha_n)Tx_n, \forall n \geq 1, \end{cases}$$

where $a \in (0, 1)$, $\{\alpha_n\} \subseteq [c, d] \subset [0, 1]$, for all $n \in \mathbb{N}$, $0 < \rho \leq \|D\|^{-1}$ and $0 < \lambda < 2\eta$ with $\eta = min\{\alpha, \beta\}$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, and let $D : C \to H$ be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_1, ..., T_N$ and $\lambda_1, \lambda_1, ...\lambda_N$. Assume $\mathcal{F} := \bigcap_{i=1}^N F(T_i)$ $\cap VI(H, A, M)$ $\cap VI(H, B, M)$ $\cap VI(C, D) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} y_n = J_{M,\lambda}(I - \lambda(aA + (1-a)B))x_n, \\ x_{n+1} = \alpha_n P_C(I - \rho D)y_n + (1 - \alpha_n)Kx_n, \forall n \geq 1, \end{cases}$$

where $a \in (0, 1)$, $\{\alpha_n\} \subseteq [c, d] \subset [0, 1]$, for all $n \in \mathbb{N}$, $0 < \rho \leq \|D\|^{-1}$ and $0 < \lambda < 2\eta$ with $\eta = min\{\alpha, \beta\}$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$.

5.2 Strong Convergence Theorems for a κ-Strictly Pseudononspreading Mapping in Hilbert space

(1) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, \ldots, N$,

let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$ and $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,...,N}\{\alpha_i\}$. Let $T : H \to H$ be a κ-strictly pseudononspreading mapping. Assume $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} b_i A_i)x_n \\ \qquad + \eta_n(I - \rho_n(I - T))x_n + \delta_n u_n, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq (0,1)$ and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n + \eta_n + \delta_n = 1$, $0 < \alpha < 1$, and $0 \leq a_i, b_i \leq 1$, for every $i = 1, 2, ..., N$, $r_n \in [c, d] \subset (0,1)$, $0 < p \leq \beta_n, \gamma_n, \eta_n, \delta_n \leq q < 1$, $\rho_n \in (0, 1 - \kappa)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \rho_n < \infty$;

(iii) $0 < \lambda < 2\eta$;

(iv) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = 1$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
$\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathcal{F}} u$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H and let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$ and let $A : H \to H$ be an α-inverse strongly monotone mapping. Let $T : H \to H$ be a κ-strictly pseudononspreading mapping. Assume $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap VI(H, A, M) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda A)x_n \\ \qquad + \eta_n(I - \rho_n(I - T))x_n + \delta_n u_n, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq (0,1)$ and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n + \eta_n + \delta_n = 1$, $0 < \alpha < 1$, and $0 \leq a_i \leq 1$, for every $i = 1, 2, ..., N$, $r_n \in [c, d] \subset (0,1)$, $0 < p \leq \beta_n, \gamma_n, \eta_n, \delta_n \leq q < 1$, $\rho_n \in (0, 1 - \kappa)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \rho_n < \infty$;

(iii) $0 < \lambda < 2\alpha$;

(iv) $\sum_{i=1}^{N} a_i = 1$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.

(3) Let C be a nonempty closed convex subset of a real Hilbert space H and let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$. For every $i = 1, 2, \ldots, N$, $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,\ldots,N}\{\alpha_i\}$. Let $T : H \to H$ be a κ-strictly pseudononspreading mapping. Assume $\mathcal{F} := F(T) \cap EP(F) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. Let the sequences $\{x_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} b_i A_i)x_n \\ \qquad + \eta_n(I - \rho_n(I - T))x_n + \delta_n u_n, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq (0, 1)$ and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n + \eta_n + \delta_n = 1$, $0 < \alpha < 1$, and $0 \leq b_i \leq 1$, for every $i = 1, 2, ..., N$, $r_n \in [c, d] \subset (0, 1)$, $0 < p \leq \beta_n, \gamma_n, \eta_n, \delta_n \leq q < 1$, $\rho_n \in (0, 1 - \kappa)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \rho_n < \infty$;

(iii) $0 < \lambda < 2\eta$;

(iv) $\sum_{i=1}^{N} b_i = 1$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.

(4) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, \ldots, N$,

let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$ and $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,...,N}\{\alpha_i\}$. Let $T : H \to H$ be a κ-quasi-strictly pseudo-contractive mapping. Assume $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda\sum_{i=1}^{N} b_i A_i)x_n \\ \qquad + \eta_n(I - \rho_n(I - T))x_n + \delta_n u_n, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq (0, 1)$, and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n + \eta_n + \delta_n = 1$, $0 < \alpha < 1$ and $0 \leq a_i, b_i \leq 1$, for every $i = 1, 2, ..., N$, $r_n \in [c, d] \subset (0, 1)$, $0 < p \leq \beta_n, \gamma_n, \eta_n, \delta_n \leq q < 1$, $\rho_n \in (0, 1 - \kappa)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \rho_n < \infty$;

(iii) $0 < \lambda < 2\eta$;

(iv) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = 1$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
$\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.

(5) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$ and let $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,...,N}\{\alpha_i\}$. Let $T_i : H \to H$, for $i = 1, 2, \ldots, N$ be a finite family of nonspreading mappings with $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. Let $\theta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \ldots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \ldots, N - 1$ and $\alpha_1^N \in (0, 1]$, $\alpha_3^N \in [0, 1)$, $\alpha_2^j \in (0, 1)$ for all $j = 1, 2, \ldots, N$, and let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\theta_1, \theta_2, \ldots, \theta_N$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda\sum_{i=1}^{N} b_i A_i)x_n \\ \qquad + \eta_n(I - \rho_n(I - S))x_n + \delta_n u_n, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq (0,1)$, and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n + \eta_n + \delta_n = 1$, $0 < \alpha < 1$, and $0 \leq a_i, b_i \leq 1$, for every $i = 1, 2, ..., N$, $r_n \in [c, d] \subset (0,1)$, $0 < p \leq \beta_n, \gamma_n, \eta_n, \delta_n \leq q < 1$, $\rho_n \in (0,1)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \rho_n < \infty$;

(iii) $0 < \lambda < 2\eta$;

(iv) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = 1$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
$\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.

5.3 Strong Convergence Theorems for Nonspreading Mappings in Hilbert space

(1) Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, with $\eta = min\{\alpha, \beta\}$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into itself. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0,1)$. Suppose that $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} EP(F_i) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let the sequences $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n^i = b_n x_n + (1 - b_n) P_C(I - \lambda_n(I - T_i))u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \rho_n(aA + (1-a)B))x_n + \gamma_n \sum_{i=1}^{N} c_i y_n^i, \forall n \geq 1, \end{cases}$$

where $a \in (0,1)$, $\{b_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\rho_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $0 \leq a_i, c_i \leq 1$, for every $i = 1, 2, ..., N$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} b_n = b \in (0,1)$;

(iii) $0 < \rho_n < 2\eta$;

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(v) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} c_i = 1$ and $0 < g \leq a_i, c_i \leq h < 1$ for all $i = 1, 2, ..., N$;

(vi) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty, \sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty,$
$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty;$

(vii) $0 < c \leq r_n, \beta_n, \gamma_n \leq d < 1, \forall n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, with $\eta = min\{\alpha, \beta\}$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into itself with $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap EP(F) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0, 1)$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n^i = b_n x_n + (1 - b_n) P_C (I - \lambda_n (I - T_i)) u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C (I - \rho_n (aA + (1 - a)B)) x_n + \gamma_n \sum_{i=1}^{N} c_i y_n^i, \forall n \geq 1, \end{cases}$$

where $a \in (0, 1)$, $\{b_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\rho_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$. Assume the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} b_n = b \in (0, 1)$;

(iii) $0 < \rho_n < 2\eta$;

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(v) $\sum_{i=1}^{N} c_i = 1$ and $0 < g \leq c_i \leq h < 1$ for all $i = 1, 2, ..., N$;

(vi) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty; \sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty,$
$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty; \sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty;$

(vii) $0 < c \leq r_n, \beta_n, \gamma_n \leq d < 1, \forall n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$.

(3) Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, with $\eta = min\{\alpha, \beta\}$. Let T be a nonspreading mappings of C into itself with $\mathcal{F} := F(T) \bigcap_{i=1}^{N} EP(F_i) \cap$

$VI(C,A) \cap VI(C,B) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0,1)$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = b_n x_n + (1-b_n)P_C(I - \lambda_n(I-T))u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \rho_n(aA + (1-a)B))x_n + \gamma_n y_n, \forall n \geq 1, \end{cases}$$

where $a \in (0,1)$, $\{b_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\rho_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$. Assume the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} b_n = b \in (0,1)$;

(iii) $0 < \rho_n < 2\eta$;

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(v) $\sum_{i=1}^{N} a_i = 1$ and $0 < g \leq a_i \leq h < 1$ for all $i = 1, 2, ..., N$;

(vi) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(vii) $0 < c \leq r_n, \beta_n, \gamma_n \leq d < 1, \forall n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$.

(4) Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and $A : C \to H$ be α -inverse strongly monotone mapping. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into itself with $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} EP(F_i) \cap VI(C,A) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0,1)$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n^i = b_n x_n + (1-b_n)P_C(I - \lambda_n(I-T_i))u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \rho_n A)x_n + \gamma_n \sum_{i=1}^{N} c_i y_n^i, \forall n \geq 1 \end{cases}$$

where $\{b_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\rho_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $0 \leq a_i, c_i \leq 1$, for every $i = 1, 2, ..., N$. Assume the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} b_n = b \in (0,1)$;

(iii) $0 < \rho_n < 2\alpha$;

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(v) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} c_i = 1$ and $0 < g \leq a_i, c_i \leq h < 1$ for all $i = 1, 2, ..., N$;

(vi) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$,
$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(vii) $0 < c \leq r_n, \beta_n, \gamma_n \leq d < 1, \forall n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$.

(5) Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)-(A4)$ and $A, B : C \to H$ be α and β-inverse strongly monotone mapping, respectively, with $\eta = min\{\alpha, \beta\}$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of quasi-nonexpansive mapping of C into itself with $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} EP(F_i) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0, 1)$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n^i = b_n x_n + (1 - b_n) P_C(I - \lambda_n(I - T_i)) u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \rho_n(aA + (1 - a)B)) x_n + \gamma_n \sum_{i=1}^{N} c_i y_n^i, \forall n \geq 1, \end{cases}$$

where $a \in (0, 1)$, $\{b_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\rho_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $0 \leq a_i, c_i \leq 1$, for every $i = 1, 2, ..., N$. Assume the following conditions hold:

1) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

2) $\lim_{n \to \infty} b_n = b \in (0, 1)$;

3) $0 < \rho_n < 2\eta$;

4) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

5) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} c_i = 1$ and $0 < g \leq a_i, c_i \leq h < 1$ for all $i = 1, 2, ..., N$;

6) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$,
$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

7) $0 < c \leq r_n, \beta_n, \gamma_n \leq d < 1, \forall n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$.

5.4 Examples and numerical results

5.4.1 Conclusions for Example 4.17

(1) The sequences $\{x_n\}$ and $\{u_n\}$ converge to 0 as shown in the Table 4.1 and Figure 4.1.

(2) From Theorem 3.7, we can conclude that the sequence $\{x_n\}$ and $\{u_n\}$, in Example 4.17, converge to 0.

5.4.2 Conclusions for Example 4.18

(1) The sequence $\{\mathbf{x}_n\}$ and $\{\mathbf{u}_n\}$ converge to **0** as shown in the Table 4.2 and Figure 4.2, where $\mathbf{0} = (0, 0, 0)$.

(2) From Theorem 3.7, we can conclude that the sequence $\{\mathbf{x}_n\}$ and $\{\mathbf{u}_n\}$, in Example 4.18, converge to **0**, where $\mathbf{0} = (0, 0, 0)$.

5.4.3 Conclusions for Example 4.19

(1) Table 4.3 shows that the sequence $\{x_n\}$ and $\{u_n\}$ converge to 0.

(2) Theorem 3.11 guarantees the convergence of $\{x_n\}$ and $\{u_n\}$ to 0 in Example 4.19.

5.5 Suggestions

In our thesis, we obtain some results of variational inclusion problems, equilibrium problem, variational inequality problem and fixed point problem in a Hilbert space. For those who would like to extend these results, we suggest that they should prove our results of fixed point problem for demicontractive mappings which is more general than quasi-nonexpansive mapping in a Hilbert space and investigate our results in a Banach space.

References

[1] Censor, Y. and Elfving, T. 1994. "A multiprojection algorithm using Bregman projections in a product space." *Numer. Algorithms.* 8:221-239

[2] Censor, Y. Bortfeld, T. Martin, B. and Trofimov, A. 2006. "A unified approach for inversion problems in intensity-modulated radiation therapy." *Phys. Med. Biol.* 51: 2353–2365.

[3] Censor, Y. Elfving, T. Kopf, N. and Bortfeld, T. 2005. "The multiple-sets split feasibility problem and its applications for inverse problems." *Inverse Probl.* 21: 2071–2084.

[4] Yuan, J. Yang, J. Shi, D. and Ai, W. 2015. " A continuation fixed-point iterative method on harmonic generations with strong nonlinear optical effects in multilayer structures." *Comp. Appl. Math.* DOI 10.1007/s40314-015-0267-7.

[5] Banach, S. 1992. "Sur les operations dans les ensembles abstraits et leur applications aux equations integrales." *Fund. Math.* 3: 133-181.

[6] Wang, G.Q. and Cheng, S.S. 2009. "Fixed point theorems arising from seeking steady states of neural networks." *Appl. Math. Model.* 33: 499-506.

[7] Ege, O. and Karaca, I. 2015."Banach fixed point theorem for digital images." *J. Nonlinear. Sci. Appl.* 8: 237–245.

[8] Stampacchia, G. 1964. "Formes bilineaires coercitives sur les ensembles convexes." *Comptes rendus hebdomadaires des s´eances de l'Acad´emie des sciences.* 258: 4413-4416.

[9] Browder, F.E. 1967. "Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces." *Arch. Ration. Mech. Anal.* 24: 82-90.

[10] Reich, S. 1980. "Strong convergence theorems for resolvents of accretive operators in Banach spaces." *J. Math. Anal. Appl.* 75: 287-292.

[11] Nadezhkina, N. and Takahashi, W. 2006. "Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings." *J. Optim. Theory Appl.* 128: 191-201.

[12] Chang, S.S. Joseph Lee, H.W. and Chan, C.K. 2009. " A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization." *Nonlinear Anal.* 70: 3307-3319.

[13] Zeng, L.C. and Yao, J.C. 2006. "Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems." *Taiwan. J. Math.*. 10 (5): 1293-1303.

[14] Takahashi, W. and Toyoda, M. 2003. "Weak convergence theorems for nonexpansive mappings and monotone mappings." *J. Optim. Theory Appl.* 118: 417-428.

[15] Iiduka, H. and Takahashi, W. 2005. "Strong convergence theorems for nonexpansive mappings and inversestrongly monotone mappings." *Nonlinear Anal TMA.* 61(3):341–350.

[16] Marino, G. and Xu, H.K. 2006. "A general iterative method for nonexpansive mappings in Hilbert spaces." *J. Math. Anal. Appl,* 318: 43-52.

[17] Moudafi, A. 2000. "Viscosity approximation methods for fixed-points problems." *J. Math. Anal. Appl*, 241: 46-55.

[18] Kangtunyakarn, A. 2013. "Approximation method for fixed point problem and variational inequality problem without assumption on the set of fixed point and the set of variational inequality." *Thai J. Math.* 11(2) : 313-317.

[19] Chang, S.S. Lee, H.W.J. and Chan, C.K. 2009. " A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization." *Nonlinear Anal.* 70: 3307–3319.

[20] Qin, X. Shang, M. and Suc, Y. 2008. "Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems." *Math. Comput. Model.* 48: 1033–1046

[21] Kangtunyakarn, A. 2013. "A new iterative scheme for fixed point problems of infinite family of κ_i-pseudo contractive mappings, equilibrium problem, variational inequality problems." *J. Glob. Optim.* 56(4): 1543-1562.

[22] Blum, E. and Oettli, W. 1994. " From optimization and variational inequilities to equilibrium problems." *Math. Student.* 63(14): 123-145.

[23] Combettes, P.L. and Hirstoaga, S.A. 2005. "Equilibrium programming in hilbert spaces." *J. Nonlinear Convex Anal.* 6(1): 117-136.

[24] Plubtieng, S. and Punpaeng, R. 2007. "A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces." *J. Math. Anal. Appl.* 336(1): 455-469.

[25] Takahashi, S. and Takahashi, W. 2007. "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces." *J. Math.Anal. Appl.* 331: 506 - 515.

[26] Kangtunyakarn, A. and Suantai, S. 2009. "A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings." *Nonlinear Anal.* 71:4448-4460.

[27] Kangtunyakarn, A. 2013. "Iterative approximation of common element of solution sets of various nonlinear operator problems." *Fix Point Theory Appl.* 2013:295

[28] Peng, J.W. and Yao,J.C. 2009. "A viscosity approximation scheme for system of equilibrium problems, nonexpansive mappings and monotone mappings." *Nonlinear Anal.* 71: 6001--6010.

[29] Suwannaut, S. and Kangtunyakarn, A. 2013. "The combination of the set of solutions of equilibrium problem for convergence theorem of the set of fixed points of strictly pseudo-contractive mappings and variational inequalities problem." *Fix Point Theory Appl.* 2013:291.

[30] Noor, M.A. and Noor, K.I. 1999. "Sensitivity analysis of quasi variational inclusions." *J. Math. Anal. Appl.* 236:290–299.

[31] Kangtunyakarn, A. 2011. "Iterative algorithms for finding a common solution of system of the set of variational inclusion problems and the set of fixed point problems." *Fix Point Theory Appl.* 38: 1--16.

[32] Chang, S.S. 2000. "Set-valued variational inclusions in Banach spaces." *J. Math. Anal. Appl.* 248:438-454.

[33] Hao, Y. "On variational inclusion and common fixed point problems in Hilbert spaces with applications." *Appl. Math.Comput.* 217(7): 3000-3010.

[34] Zhang, S.S Lee, J.H.W. and Chan, C.K. 2008. "Algorithms of common solutions for quasi variational inclusion and fixed point problems." *Appl. Math. Mech.* 29:571–581.

[35] Peng J.W. Wang Y. Shyu D.S. and Yao J.C. 2008. "Common solutions of an iterative scheme for variational inclusions, equilibrium problems and fixed point problems" *J. Inequal. Appl.* ID 720371.

[36] Takahashi, W. 2010. "Fixed point theorems for new nonlinear mappings in a Hilbert space." *J. Nonlinear Convex Anal.* 11:79–88

[37] Kohsaka, F. and Takahashi, W. 2008. "fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces." *Arch. Math.* 91:166-177.

[38] Iemoto. S. and Takahashi, W. 2009. "Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space." *Nonlinear Anal.* 71:2080–2089.

[39] Plubtieng, S. and Chornphrom, S. 2010. "Strong Convergence Theorem for Equilibrium Problems and Fixed Points of a Nonspreading Mapping in Hilbert Spaces." *Fix Point Theory Appl.* 2010: ID 296759

[40] Osilike, M.O. and Isiogugu, F.O. 2011. "Weak and strong convergence theorems for nonspreading-type mappings in Hilbert spaces." *Nonlinear Anal.* 74: 1814-1822.

[41] Atsushiba, S. and Takahashi, W. 1999. "Strong convergence theorems for a finite family of nonexpansive mappings and applications." *B. N. Prasad birth centenary commemoration volume. Indian J. Math.* 41: 435-453.

[42] Kangtunyakarn, A. and Suantai, S. 2009. "Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings." *Nonlinear Anal. Hybrid Syst.* 3:296-309.

[43] Kangtunyakarn, A. 2009. "A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings." *Nonlinear Anal.* 71: 4448-4460.

[44] Xiaoyan, H. Haiyun, Z. and Jiangyan, H.E. 2015. "Viscosity approximation methods for equilibrium problems and a finite family of nonspreading mappings in a Hilbert space." *Advances In Mathematics(China).* 44(2):284-298.

[45] Kangtunyakarn A. 2012. "Convergence theorem of common fixed points for a family of nonspreading mappings in Hilbert space." *Optimization Letters.* 6(5): 957-961.

[46] Kangtunyakarn A. 2012. "Strong convergence of the hybrid method for a finite family of nonspreading mappings and variational inequality problems." *Fix Point Theory Appl.* 2012:188.

[47] Wen D.J. Chen Y.A. and Tang, Y. 2014. "Strong Convergence of a Unified General Iteration for k-Strictly Pseudononspreading Mapping in Hilbert Spaces." *Abstr. Appl. Anal.* 2014 :Article ID 219695.

[48] Onjai-uea, N. and Phuengrattana, W. 2015. "A hybrid iterative method for common solutions of variational inequality problems and fixed point problems for single-valued and multi-valued mappings with applications." *Fixed Point Theory Appl.* 2015:16

[49] Kangtunyakarn, A. 2013. "The methods for variational inequality problems and fixed point of κ_i-strictly pseudononspreading mapping." *Fixed Point Theory Appl.* 2013:171.

[50] Kreyszig, E. **Introductory functional analysis with applications**. New York:John Wiley and Sons. 1978.

[51] Agarwal, R.P. O'Regan, D. and Sahu, D.R. **Fixed Point Theory for Lipschitziantype Mappings with Applications**. Topological Fixed Point Theory and Its Applications 6. (2009).

[52] Megginson, R.E. **An Introduction to Banach Space**.New York: JSpringers. 1998.

[53] Debnath, L. and Mikusinski, P. **Introduction to Hilbert Spaces with Applications**. London:Elsevier Academic Press. 2005.

[54] Berinde, V. **Iterative Approximation of Fixed Points.** New York:Springer Berlin Heidelberg. 2007.

[55] Opial, Z. 1967. "Weak convergence of the sequence of successive approximation of nonexpansive mappings." *Bull. Amer. Math. Soc.* 73:591-597.

[56] Zegeye, H. and Shahzadb, N. 2011. "Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings." *Comput. Math. Appl.* 62:4007–4014.

[57] Bauschke, H.H. and CombettesP. L. **Convex Analysis and Monotone Operator Theory in Hilbert Spaces.** New York:Springer. 2011.

[58] Xu, H.K. 2003. "An iterative approach to quadric optimization." *J. Optim. Theory Appl.* 116:659-678.

[59] Takahashi, W. **Introduction to Nonlinear and Convex Analysis.** Yokohama: Yokohama Publishers. 2009.

[60] Takahashi, W. **Nonlinear Functional Analysis**. Yokohama: Yokohama Publishers. 2000.

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Academic Publications

1. Khuangsatung, W., Kangtunyakarn, A. 2014. "Algorithm of a new variational inclusion problem and strictly pseudononspreading mapping with application ." *Fixed Point Theory and Applications.* 2014:209.
2. Khuangsatung, W., Kangtunyakarn, A. 2015. "The new variational inclusion problems for fixed point theorem." *Journal of Nonlinear and Convex Analysis.* Volume 16, Number 3, 2015 pp. 459-466
3. Khuangsatung, W., Kangtunyakarn, A. 2016. "Fixed point theory for various nonlinear mappings and equilibrium problems with numerical example ." *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matemáticas.* 2016. DOI: 10.1007/s13398-016-0276-4.