

COMBINATION METHODS FOR SOLVING EQUILIBRIUM
PROBLEMS VARIATIONAL INEQUALITY PROBLEMS AND
FIXED POINT PROBLEMS

SARAWUT SUWANNAUT

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENT FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
(APPLIED MATHEMATICS)

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

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สำนักหอสมุดกลาง พระจอมเกล้าลาดกระบัง

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Abstract

For the purpose of this thesis, we first introduce the combination of equilibrium problem and the combination of variational inequality problem in a Hilbert space. Then, we prove strong convergence theorems for finding a common solution of these problems under appropriate conditions. Secondly, we obtain some properties of a nonspreading mapping. Consequently, we prove a strong convergence theorem for a proposed iterative scheme under some appropriate conditions. Thirdly, we modify the generalized equilibrium problem and introduce the K -mapping generated by a finite family of strictly pseudo-contractive mappings. Under certain control conditions, some strong convergence theorems of the proposed method are obtained. The results obtained in this thesis extend and improve many well-known results in the previous literature. Finally, we give numerical examples to support our main results in the space of real numbers.

Keywords : fixed point, equilibrium problem, variational inequality problem, nonlinear mapping, strong convergence theorem

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บทคัดย่อ

จุดประสงค์ของวิทยานิพนธ์นี้อันดับแรกคือเป็นการแนะนำปัญหาดุลยภาพรวมและปัญหาอสมการการแปรผันรวมในปริภูมิฮิลเบิร์ต จากนั้นทำการพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มสำหรับการหาค่าผลเฉลยร่วมของปัญหาดังกล่าว ขั้นที่สองเราได้ศึกษาถึงคุณสมบัติบางประการของการส่งแบบไม่กระจาย แล้วทำการพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มของกระบวนการทำซ้ำภายใต้เงื่อนไขที่เหมาะสม ขั้นที่สามคือการปรับปรุงปัญหาดุลยภาพทั่วไปและแนะนำการส่งแบบ K ที่สร้างจากวงศ์จำกัดของการส่งแบบหดเทียมโดยแท้ เราได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มของระเบียบการทำซ้ำภายใต้เงื่อนไขควบคุม ผลลัพธ์ที่ได้จากงานวิจัยนี้เป็นการขยายและปรับปรุงผลงานต่าง ๆ มากมายของการวิจัยในสาขานี้ ท้ายที่สุดเรายกตัวอย่างเชิงตัวเลขเพื่อยืนยันผลลัพธ์ที่ได้ในเซตของจำนวนจริง

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Chapter 1

Introduction

1.1 Iterative methods and nonexpansive mappings

Many problems in various fields of science can be formulated to the problems of finding a fixed point of nonlinear mappings. In the past few years, many mathematicians have been developed and widely studied fixed point theory. A fixed point of a mapping T is a point x such that $x = Tx$, where $T : X \rightarrow X$ is a nonlinear mapping. Throughout this thesis, $Fix(T)$ is used to denote the fixed point set of a mapping T . Therefore, fixed point theory is involved with finding conditions on the set X and the mapping $T : X \rightarrow X$ to guarantee the existence and uniqueness of fixed points. Furthermore, researchers have been studying about the structure of fixed point set and the approximation of fixed points.

Over the past decades, many others have constructed various types of iterative methods to approximate fixed points. Let C be a nonempty closed convex subset of a normed space and T be a mapping of C into itself.

The first one is the Mann iteration introduced by Mann [1] in 1953 and is defined as follows:

$$\begin{cases} x_0 \in H \text{ arbitrary chosen,} \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \forall n \geq 0, \end{cases} \quad (1.1)$$

where the sequence $\{\alpha_n\}$ is in the interval $(0,1)$. But this algorithm has only weak convergence. Thus, many mathematicians have been trying to modify Mann's iteration (1.1) and construct new iterative method to obtain the strong convergence theorem.

In 1967, Halpern [5] introduced the Halpern's iterative method as follows:

$$\begin{cases} x_0, u \in C \text{ arbitrary chosen,} \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \forall n \geq 0, \end{cases} \quad (1.2)$$

where $\{\alpha_n\} \subset (0,1)$ satisfying certain conditions. He proved that the sequence $\{x_n\}$ converges strongly to a fixed point of mapping T in a real Hilbert space, where T is a nonexpansive mapping.

By modification of Mann's iteration (1.1), the next iteration process is referred to as Ishikawa's iteration process [7] which is defined recursively as follows:

$$\begin{cases} x_0 \in H \text{ arbitrary chosen,} \\ y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \forall n \geq 0, \end{cases} \quad (1.3)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$. He also obtain the strong convergence theorem for the iterative method (1.3) converging to a fixed point of a nonexpansive mapping T . Observe that if $\beta_n = 1$, then the Ishikawa's iteration (1.3) reduces to the Mann's iteration (1.1).

In 2000, Moudafi [11] introduced the viscosity approximation method for non-expansive mapping S .

Let C be a closed convex subset of a real Hilbert space H and let $S : C \rightarrow C$ be a nonexpansive mapping such that $Fix(S)$ is nonempty. Let $f : C \rightarrow C$ be a contraction, that is, there exists $\alpha \in (0, 1)$ such that $\|fx - fy\| \leq \alpha \|x - y\|, \forall x, y \in C$, and let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_1 \in C \text{ arbitrary chosen,} \\ x_{n+1} = \frac{1}{1+\epsilon_n} Sx_n + \frac{\epsilon_n}{1+\epsilon_n} f(x_n), \forall n \in \mathbb{N}, \end{cases} \quad (1.4)$$

where $\{\epsilon_n\} \subset (0, 1)$ satisfies certain conditions. Then the sequence $\{x_n\}$ converges strongly to $z \in Fix(S)$, where $z = P_{Fix(S)}f(z)$ and $P_{Fix(S)}$ is the metric projection of H onto $Fix(S)$.

In 2006, using the concept of the viscosity approximation method (1.4), Marino and Xu [16] introduced the general iterative method and obtained the strong convergence theorem. Let $T : H \rightarrow H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $f : H \rightarrow H$ be a contractive mapping on H and let $\{x_n\}$ be generated by

$$\begin{cases} x_0 \in H \text{ arbitrary chosen,} \\ x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), n \geq 0, \end{cases} \quad (1.5)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the appropriate conditions. Then $\{x_n\}$ converges strongly to a fixed point \bar{x} of T which solves the variational inequality:

$$\langle (A - \gamma f)\bar{x}, \bar{x} - z \rangle \leq 0, z \in Fix(T).$$

For the past decades, many researcher are interested in studying the fixed point of a finite family of nonlinear mappings and its properties, see [18, 27, 28].

First, in 1999, Atsushiba and Takahashi [13] introduced the W -mapping for a finite family of nonexpansive mappings as follows. Let $W : C \rightarrow C$ be defined by

$$\begin{aligned} U_1 &= \beta_1 T_1 + (1 - \beta_1) I, \\ U_2 &= \beta_2 T_2 U_1 + (1 - \beta_2) I, \\ U_3 &= \beta_3 T_3 U_2 + (1 - \beta_3) I, \\ &\vdots \\ U_{N-1} &= \beta_{N-1} T_{N-1} U_{N-2} + (1 - \beta_{N-1}) I, \\ W = U_N &= \beta_N T_N U_{N-1} + (1 - \beta_N) I, \end{aligned}$$

where $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mapping of C into itself, I is the identity mapping and $\{\beta_i\}_{i=1}^N$ is a sequence in $[0, 1]$. Later, Takahashi and Shimoji [18] obtain that if X is strictly convex Banach space, then $Fix(W) = \bigcap_{i=1}^N Fix(T_i)$ and W is a nonexpansive mapping.

Later, in 2009, Kangtunyakarn and Suantai [24] proposed the K -mapping for a finite family of nonexpansive mappings as follows: Let a mapping $K : C \rightarrow C$ be defined by

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1 - \lambda_1) I, \\ U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2) U_1, \\ U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3) U_2, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1}) U_{N-2}, \\ K = U_N &= \lambda_N T_N U_{N-1} + (1 - \lambda_N) U_{N-1}, \end{aligned}$$

where $\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings of C into itself and let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers. Furthermore, they proved that $Fix(K) = \bigcap_{i=1}^N Fix(T_i)$ and K is a nonexpansive mapping under some control conditions.

In 2009, Kangtunyakarn and Suantai [23] introduced the S -mapping for a finite family of nonexpansive mappings as follows. Let $S : C \rightarrow C$ be defined by

$$\begin{aligned} U_1 &= \alpha_1^1 T_1 + \alpha_2^1 I + \alpha_3^1 I, \\ U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\ U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\ &\vdots \\ U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\ S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I, \end{aligned}$$

where $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mapping of C into itself and $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in [0, 1] \times [0, 1] \times [0, 1]$, where $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, for every $j = 1, 2, \dots, N$. Under some mild conditions, they also obtain that $Fix(S) = \bigcap_{i=1}^N Fix(T_i)$ and S is a nonexpansive mapping.

Remark 1.1. For the special cases of S -mapping, we have

- 1) If $\alpha_1^j = \beta_j$ and $\alpha_2^j = 0$, for $j = 1, 2, \dots, N$, then the S -mapping reduces to the W -mapping introduced by Atsushiba and Takahashi [13].
- 2) If we put $\alpha_1^j = \lambda_j$ and $\alpha_3^j = 0$, for $j = 1, 2, \dots, N$, then the S -mapping becomes the K -mapping defined by Kangtunyakarn and Suantai [24].

1.2 Equilibrium problems

In 1994, Blum [9] introduced the equilibrium problem which has had a great impact and influence in the development of several branches of pure and applied sciences. Many problems in physics, optimization, and economics are related to seeking the elements of $EP(F)$, see [9, 10]. Let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction. The equilibrium problem for F is to find a point x such that

$$F(x, y) \geq 0, \forall y \in C. \quad (1.6)$$

Moreover, the solution set of (1.6) is denoted by $EP(F)$.

Many authors have been studied iterative scheme for finding the solutions of the equilibrium problem, see, for example, [10].

In 2005, Combettes and Hirstoaga [10] introduced some iterative schemes for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved the strong convergence theorem.

In 2007, Takahashi and Takahashi [17] proved the following theorem:

Theorem 1.2. Let C be a nonempty closed convex subset of H . Let F be a bifunction from $C \times C$ to \mathbb{R} satisfying

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) For each $x, y, z \in C$,

$$\lim_{t \rightarrow 0^+} F(tz + (1-t)x, y) \leq F(x, y);$$

- (A4) For each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous;

and let S be a nonexpansive mapping of C into H such that $Fix(S) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset [0, 1]$ satisfy some control conditions. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in Fix(S) \cap EP(F)$, where $z = P_{Fix(S) \cap EP(F)} f(z)$.

For a specific case of equilibrium problem, if we take $F(x, y) = \langle y - x, Ax \rangle$, where $A : C \rightarrow H$ is a nonlinear mapping, then the equilibrium problem is equivalent to finding an element $x \in C$ such that

$$\langle y - x, Ax \rangle \geq 0, \forall y \in C, \quad (1.7)$$

which is well known as the variational inequality problem. The solution set of this problem (1.7) is denoted by $VI(C, A)$.

Variational inequalities were firstly introduced and studied by Stampacchia [2] in 1964. It is now well known that variational inequalities can be applied to various types of problems such as optimal control, optimization, mathematical programming, mechanics and finance, see [8, 12]. There are several techniques to analyze various iterative methods for solving variational inequality problem and the related optimization problems, see [21, 32] and the references therein.

Let C be a nonempty closed convex subset of H and $CB(H)$ be the family of all nonempty closed bounded subsets of H . Let $\varphi : C \rightarrow \mathbb{R}$ be a real-valued function, $T : C \rightarrow CB(H)$ a multivalued mapping and $\Phi : H \times C \times C \rightarrow \mathbb{R}$ an equilibrium-like function, that is, $\Phi(w, u, v) + \Phi(w, v, u) = 0$ for all $(w, u, v) \in H \times C \times C$ which satisfies the following conditions with respect to the multivalued mapping $T : C \rightarrow CB(H)$.

(H1) For each fixed $v \in C$, $(w, u) \mapsto \Phi(w, u, v)$ is an upper semicontinuous function from $H \times C \rightarrow \mathbb{R}$, that is, for $(w, u) \in H \times C$, whenever $w_n \rightarrow w$ and $u_n \rightarrow u$ as $n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \Phi(w_n, u_n, v) \leq \Phi(w, u, v);$$

(H2) For each fixed $(w, v) \in H \times C$, $u \mapsto \Phi(w, u, v)$ is a concave function;

(H3) For each fixed $(w, u) \in H \times C$, $v \mapsto \Phi(w, u, v)$ is a convex function.

In 2009, Ceng et al.[33] introduced the *generalized equilibrium problem (GEP)* as follows:

$$(GEP) \begin{cases} \text{Find } u \in C \text{ and } w \in T(u) \text{ such that} \\ \Phi(w, u, v) + \varphi(v) - \varphi(u) \geq 0, \forall v \in C. \end{cases} \quad (1.8)$$

The set $(GEP)_s(\Phi, \varphi) = \{(u, w) \in C \times T(u) : \Phi(w, u, v) + \varphi(v) - \varphi(u) \geq 0, \forall v \in C\}$ represents the solution set of the generalized equilibrium problem (GEP). In the case of $\varphi = 0$ and $\Phi(w, u, v) \equiv G(u, v)$, where the bifunction G satisfies (A1) – (A4), then the generalized equilibrium problem (1.8) reduces to the equilibrium problem (1.6) and $(GEP)_s(\Phi, \varphi)$ is denoted by $EP(G)$.

By using Nadler's theorem [6], they introduced the following algorithm:

Let $x_1 \in C$ and $w_1 \in T(x_1)$, there exists sequences $\{w_n\} \subseteq H$ and $\{x_n\}, \{u_n\} \subseteq C$ such that

$$\begin{cases} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n}) \mathcal{H}(T(x_n), T(x_{n+1})), \\ \Phi(w_n, u_n, v) + \varphi(v) - \varphi(u_n) + \frac{1}{r_n} \langle u_n - x_n, v - u_n \rangle \geq 0, \forall v \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n, n = 1, 2, \dots \end{cases} \quad (1.9)$$

They proved the strong convergence theorem of the sequence $\{x_n\}$ generated by (1.9) as follows:

Theorem 1.3 ([33]). Let C be a nonempty, bounded, closed and convex subset of a real Hilbert space H and let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : C \rightarrow CB(H)$ be \mathcal{H} -Lipschitz-continuous with constant μ , $\Phi : H \times C \times$

$C \rightarrow \mathbb{R}$ be an equilibrium-like function satisfying (H1)-(H3) and S be a nonexpansive mapping of C into itself such that $Fix(S) \cap (GEP)_s(\Phi, \varphi) \neq \emptyset$. Let f be a contraction of C into itself and let $\{x_n\}$, $\{w_n\}$, and $\{u_n\}$ be sequences generated by (1.9), where $\{\alpha_n\} \subseteq [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\liminf_{n \rightarrow \infty} r_n > 0 \text{ and } \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

If there exists a constant $\lambda > 0$ such that

$$\Phi(w_1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi(w_2, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\lambda \|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2,$$

for all $(r_1, r_2) \in \Xi \times \Xi$, $(x_1, x_2) \in C \times C$ and $w_i \in T(x_i)$, $i = 1, 2$, where $\Xi = \{r_n : n \geq 1\}$, then for $\hat{x} = P_{Fix(S) \cap (GEP)_s(\Phi, \varphi)} f(\hat{x})$, there exists $\hat{w} \in T(\hat{x})$ such that (\hat{x}, \hat{w}) is a solution of (GEP) and

$$x_n \rightarrow \hat{x}, w_n \rightarrow \hat{w} \text{ and } u_n \rightarrow \hat{x} \text{ as } n \rightarrow \infty.$$

In 2012, Kangtunyakarn [26] introduced the iterative algorithm as follows:

Algorithm 1.4 ([26]). Let $T_i : i = 1, 2, \dots, N$, be κ_i -pseudo contraction mappings of C into itself and $\kappa = \max \{\kappa_i : i = 1, 2, \dots, N\}$ and let S_n be the S -mappings generated by T_1, T_2, \dots, T_N and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1, 2, \dots, N-1$, $\kappa \leq \alpha_1^{n,N} \leq 1$, $\kappa \leq \alpha_3^{n,N} \leq d < 1$, $\kappa \leq \alpha_2^{n,N} \leq e < 1$ for all $j = 1, 2, \dots, N$. Let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1)$, $w_1^2 \in D(x_1)$, there exists sequence $\{w_n^1\}, \{w_n^2\} \in H$ and $\{x_n\}, \{u_n\}, \{v_n\} \subseteq C$ such that

$$\left\{ \begin{array}{l} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq (1 + \frac{1}{n}) \mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq (1 + \frac{1}{n}) \mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n} \langle u_n - x_n, u - u_n \rangle \geq 0, \forall u \in C, \\ \Phi(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n} \langle v_n - x_n, v - v_n \rangle \geq 0, \forall v \in C, \\ z_n = \delta_n P_C(I - \lambda A)u_n + (1 - \delta_n) P_C(I - \eta B)v_n, \\ y_n = \alpha_n z_n + (1 - \alpha_n) S_n z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{array} \right. \quad (1.10)$$

where $D, T : C \rightarrow CB(H)$ are \mathcal{H} -Lipschitz continuous with constants μ_1, μ_2 , respectively, $\Phi_1, \Phi_2 : H \times C \times C \rightarrow \mathbb{R}$ are equilibrium-like functions satisfying (H1) – (H3), $A : C \rightarrow H$ is an α -inverse strongly monotone mapping and $B : C \rightarrow H$ is a β -inverse strongly monotone mapping.

He proved under some control conditions on $\{\delta_n\}, \{\alpha_n\}, \{s_n\}$ and $\{r_n\}$ that the sequence $\{x_n\}$ generated by (1.10) converges strongly to $P_{\mathcal{F}} x_1$, where $\mathcal{F} = \bigcap_{i=1}^N Fix(T_i)$

$(GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \cap Fix(G_1) \cap Fix(G_2)$, $G_1, G_2 : C \rightarrow C$ are defined by $G_1(x) = P_C(x - \lambda Ax)$, $G_2(x) = P_C(x - \eta Bx)$, $\forall x \in C$ and $P_{\mathcal{F}x_1}$ is a solution of the following system of variational inequality:

$$\begin{cases} \langle Ax^*, x - x^* \rangle \geq 0, \\ \langle Bx^*, x - x^* \rangle \geq 0. \end{cases}$$

For the last decades, many mathematicians are interested in studying and trying to improve this problem, see [26, 34].

1.3 Objectives of the study

- 1) To propose some new mathematical tools and properties for the combination of equilibrium problem, the combination of variational inequality problem and fixed point problem in a Hilbert space.
- 2) To define the new iterative methods for approximating the solutions for the combination of equilibrium problem, the combination of variational inequality problem, the modified generalized equilibrium problem and fixed point problem in a Hilbert space.
- 3) To prove strong convergence theorems for equilibrium problems, variational inequality problems and fixed point problems in a Hilbert space under some control conditions.
- 4) To give numerical examples to support our main theorems.

1.4 Scopes of the study

- 1) Study variational inequality problems and equilibrium problems in a Hilbert space.
- 2) Investigate the fixed problems of nonlinear mappings including nonexpansive mapping, strictly pseudo-contractive mapping and nonspreading mapping in a Hilbert space.
- 3) All strong convergence theorems are considered and proved in Hilbert spaces.
- 4) Every numerical example for our main results are given in one- and two-dimensional space of real numbers.

1.5 Benefits of the study

- 1) Obtain new mathematical tools for the properties of variational inequality problems, equilibrium problems and fixed problems of nonlinear mappings in Hilbert spaces.

- 2) Obtain a strong convergence theorem for a common solution of equilibrium problems, variational inequality problems and fixed points of an infinite family of strictly pseudo-contractive mappings.
- 3) Obtain a strong convergence theorems for a common solution of equilibrium problems and fixed point of a nonspreading mapping.
- 4) Obtain a strong convergence theorem for a common solution of modified generalized equilibrium problems and fixed point of a finite family of strictly pseudo-contractions
- 5) Our new problems introduced in this thesis can be applied into the real-world problems such as economic problems and optimization problems.

This thesis consists of five chapters as follows:

In chapter 1, we show the background of this thesis, that is, iterative methods for fixed point theorems and the definitions and properties of a finite family of nonlinear mappings.

In chapter 2, we give the literature review and describe some definitions, notations and properties of fixed point problem, equilibrium problem and variational inequality problem that are necessary to prove our main theorems in the next chapter.

In chapter 3, we first introduce the combination of equilibrium problem and the combination of variational inequality problem for strongly positive linear bounded operators. Some Lemmas for these problems are obtained. Then we prove a strong convergence theorem for equilibrium problem, variational inequality problem and fixed point problem in Hilbert space. Secondly, we obtain a Lemma showing the properties of a nonspreading mapping. Consequently, we prove a strong convergence theorem for equilibrium problem and fixed point problem of a nonspreading mapping. Finally, the modified generalized equilibrium problem and K -mapping for a finite family of strictly pseudo-contractive mappings are introduced. Then we prove a strong convergence theorem for the modified generalized equilibrium problem and fixed point of strictly pseudo-contractions in Hilbert space.

In chapter 4, we obtain some additional results of our main results and give numerical examples for our theorems in the previous chapter.

In chapter 5, we describe the conclusion of the thesis.

Chapter 2

Preliminaries and Literature Reviews

2.1 Fundamental properties in Hilbert spaces

Lemma 2.1 ([38]). Let H be a real Hilbert space. Then the following results hold:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \text{ for each } x, y \in H. \quad (2.1)$$

Lemma 2.2 ([47]). Let H be a real Hilbert space. Then for all $x_i \in H$ and $\alpha_i \in [0, 1]$ for $i = 1, 2, \dots, n$ such that $\sum_{i=1}^n \alpha_i = 1$ the following equality holds:

$$\left\| \sum_{i=1}^n \alpha_i x_i \right\|^2 = \sum_{i=1}^n \alpha_i \|x_i\|^2 - \sum_{1 \leq i, j \leq n} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Definition 2.1 (Strong convergence [37]). A sequence $\{x_n\}$ of vectors in an inner product space K is called strongly convergent to a vector x in K if

$$\|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Definition 2.2 (Weak convergence [37]). A sequence $\{x_n\}$ of vectors in an inner product space K is called weakly convergent to a vector x in K if

$$\langle x_n, y \rangle \rightarrow \langle x, y \rangle \text{ as } n \rightarrow \infty \text{ for every } y \in K.$$

Theorem 2.3 ([37]). A strongly convergence sequence is weakly convergence (to the same limit), that is, $x_n \rightarrow x$ implies $x_n \rightharpoonup x$.

Remark 2.4 ([19]). If $x_n \rightarrow x$ and $x_n \rightarrow y$, then $x = y$.

Lemma 2.5 ([19]). Let $\{x_n\}$ be a Cauchy sequence of an inner product space C such that $x_n \rightharpoonup x$. Then $x_n \rightarrow x$.

Theorem 2.6 ([19]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Suppose that $\{x_n\} \subset C$ and $x_n \rightharpoonup x$. Then $x \in C$.

Definition 2.3 ([19]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called lower semicontinuous if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\} \text{ is closed.}$$

Moreover, f is called convex if for any $x_1, x_2 \in C$ and $t \in (0, 1)$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Similarly, f is said to be concave if for any $x_1, x_2 \in C$ and $t \in (0, 1)$,

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2).$$

Theorem 2.7 ([19]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightarrow x_0$. Then

$$f(x_0) \leq \liminf_{n \rightarrow \infty} f(x_n).$$

Lemma 2.8 ([4]). Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{u_n\} \subset H$ with $u_n \rightarrow u$, the inequality

$$\liminf_{n \rightarrow \infty} \|u_n - u\| < \liminf_{n \rightarrow \infty} \|u_n - v\|$$

holds for every $v \in H$ with $v \neq u$.

Theorem 2.9 ([19]). Let $\{a_n\}$ be a bounded of real numbers. Then, there exists subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n \rightarrow \infty} a_n = \lim_{i \rightarrow \infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n \rightarrow \infty} a_n = \lim_{j \rightarrow \infty} a_{n_j}.$$

Remark 2.10 ([19]). Let H be an inner product space. Then we know that the following (1) and (2) are equivalent:

- 1) H is complete,
- 2) each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Definition 2.4 (Metric projection [19]). The (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.11 ([20]). For a given $z \in H$ and $u \in C$,

$$u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \forall v \in C.$$

It is well-known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \forall x, y \in H.$$

Lemma 2.12 ([20]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H . Let $u \in C$. Then, for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C .

Lemma 2.13 ([14]). Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

$$(1) \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$(2) \limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then, $\lim_{n \rightarrow \infty} s_n = 0$.

2.2 Fixed point sets of nonexpansive mappings and strictly pseudo-contractive mappings

Let X be a nonempty set and $T : X \rightarrow X$ a self-mapping. We say that $x \in X$ is a fixed point of T if and only if $Tx = x$ and $Fix(T)$ represents the set of all fixed points of T .

Example 2.14 ([39]). 1) If $X = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$, then $Fix(T) = \{-2\}$;

2) If $X = \mathbb{R}$ and $T(x) = x^2 - x$, then $Fix(T) = \{0, 2\}$;

3) If $X = \mathbb{R}$ and $T(x) = x + 2$, then $Fix(T) = \emptyset$;

4) If $X = \mathbb{R}$ and $T(x) = x$, then $Fix(T) = \mathbb{R}$;

Definition 2.5. Let the mapping $T : C \rightarrow C$. Then T is called

1) a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

2) κ -strictly pseudo-contractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \forall x, y \in C.$$

Note that the class of κ -strictly pseudo-contractions strictly includes the class of nonexpansive mappings, that is, nonexpansive mapping is a 0-strictly pseudo-contractive mapping.

Example 2.15. Let $S, T : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $Sx = \frac{x}{2}$ and $Tx = -2x, \forall x \in \mathbb{R}$, respectively. Then S is nonexpansive and T is κ -strictly pseudo contractive, where $\kappa \in [\frac{1}{3}, 1)$.

Solution. Let $x, y \in \mathbb{R}$. Then we obtain

$$|Sx - Sy| = \left| \frac{x}{2} - \frac{y}{2} \right| = \frac{1}{2}|x - y| \leq |x - y|.$$

Thus S is nonexpansive.

Next, we compute

$$|Tx - Ty|^2 = |-2x + 2y|^2 = 4(x - y)^2$$

and

$$|(I - T)x - (I - T)y|^2 = |3x - 3y|^2 = 9(x - y)^2.$$

Choose $\kappa = \frac{1}{3}$. We derive

$$\begin{aligned} |x - y|^2 + \kappa|(I - T)x - (I - T)y|^2 &= (x - y)^2 + \frac{1}{3} \times 9(x - y)^2 \\ &= (x - y)^2 + 3(x - y)^2 = 4(x - y)^2 = |Tx - Ty|^2. \end{aligned}$$

Hence T is a $\frac{1}{3}$ -strictly pseudo contractive mapping.

Theorem 2.16 ([19]). Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H . Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C .

Theorem 2.17 ([19]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let T be a nonexpansive mapping of C into itself. Then $Fix(T)$ is closed and convex.

Lemma 2.18 (Demiclosedness principle [3]). Assume that T is a nonexpansive self-mapping of closed convex subset C of a Hilbert space H . If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y it follows that $(I - T)x = y$. Here, I is the identity mapping of H .

Lemma 2.19 ([40]). Let H be a Hilbert space, C be a closed convex subset of H . If T is a κ -strictly pseudocontractive mapping on C , then the fixed point set $Fix(T)$ is closed convex, so that the projection $P_{Fix(T)}$ is well defined.

Lemma 2.20 ([15]). Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \rightarrow C$ be a self-mapping of C . If S is a κ -strict pseudo-contractive mapping, then S satisfies the Lipschitz condition

$$\|Sx - Sy\| \leq \frac{1 + \kappa}{1 - \kappa} \|x - y\|, \forall x, y \in C.$$

Definition 2.6. Let the mapping $T : C \rightarrow C$. Then T is called an α -contractive mapping if there exists $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \forall x, y \in C.$$

Obviously, if T is contractive, then T is nonexpansive. That is, the class of nonexpansive mappings strictly includes the class of contractive mappings.

Example 2.21. Let the mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $fx = \frac{x}{4}$, for all $x \in \mathbb{R}$. Then f is a $\frac{1}{4}$ -contractive mapping.

Solution. Let $x, y \in \mathbb{R}$. Since $|fx - fy| = |\frac{x}{4} - \frac{y}{4}| = \frac{1}{4}|x - y|$, thus f is a contractive mapping with $\alpha = \frac{1}{4}$.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonlinear mappings with $\mathcal{F} = \bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. There are numerous mathematicians proposed the iterative methods and new techniques for finding an element of \mathcal{F} , see, for instance, [23, 26, 18, 13].

In 2009, Kangtunyakarn and Suantai [24] introduced the K -mapping for a finite family of nonexpansive mappings in a real Banach space. They obtain that $\text{Fix}(K) = \bigcap_{i=1}^N \text{Fix}(T_i)$ under some milds conditions.

Definition 2.7 ([24]). Let C be a nonempty closed convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers with $0 \leq \lambda_i \leq 1$ for every $i = 1, 2, \dots, N$. Define a mapping $K : C \rightarrow C$ as follows:

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1 - \lambda_1) I, \\ U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2) U_1, \\ U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3) U_2, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1}) U_{N-2}, \\ K = U_N &= \lambda_N T_N U_{N-1} + (1 - \lambda_N) U_{N-1}. \end{aligned} \tag{2.2}$$

Such a mapping K is called the K -mapping generated by T_1, T_2, \dots, T_N and $\lambda_1, \lambda_2, \dots, \lambda_N$.

Lemma 2.22 ([24]). Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \dots, N-1$ and $0 < \lambda_N \leq 1$. Let K be the K -mapping generated by T_1, T_2, \dots, T_N and $\lambda_1, \lambda_2, \dots, \lambda_N$. Then $\text{Fix}(K) = \bigcap_{i=1}^N \text{Fix}(T_i)$.

Later, in 2011, Kangtunyakarn [25] considered the fixed point sets of an infinite family of nonexpansive mappings and obtain the results as follows:

Definition 2.8 ([25]). Let C be a nonempty convex subset of a real Hilbert space. Let $T_i, i = 1, 2, \dots$ be mappings of C into itself. For each $j = 1, 2, \dots$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $n \in \mathbb{N}$, we define the mapping

$S_n : C \rightarrow C$ as follows:

$$\begin{aligned}
 U_{n,n+1} &= I \\
 U_{n,n} &= \alpha_1^n T_n U_{n,n+1} + \alpha_2^n U_{n,n+1} + \alpha_3^n I \\
 U_{n,n-1} &= \alpha_1^{n-1} T_{n-1} U_{n,n} + \alpha_2^{n-1} U_{n,n} + \alpha_3^{n-1} I \\
 &\vdots \\
 U_{n,k+1} &= \alpha_1^{k+1} T_{k+1} U_{n,k+2} + \alpha_2^{k+1} U_{n,k+2} + \alpha_3^{k+1} I \\
 U_{n,k} &= \alpha_1^k T_k U_{n,k+1} + \alpha_2^k U_{n,k+1} + \alpha_3^k I \\
 &\vdots \\
 U_{n,2} &= \alpha_1^2 T_2 U_{n,3} + \alpha_2^2 U_{n,3} + \alpha_3^2 I \\
 S_n = U_{n,1} &= \alpha_1^1 T_1 U_{n,2} + \alpha_2^1 U_{n,2} + \alpha_3^1 I.
 \end{aligned}$$

This mapping is called S -mapping generated by T_n, T_{n-1}, \dots, T_1 and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$.

Lemma 2.23 ([25]). Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^\infty$ be κ_i -strictly pseudo-contractive mappings of C into itself with $\bigcap_{i=1}^\infty \text{Fix}(T_i) \neq \emptyset$ and $\kappa = \sup_{i \in \mathbb{N}} \kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \dots$. For every $n \in \mathbb{N}$, let S_n be S -mapping generated by T_n, T_{n-1}, \dots, T_1 and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, $\lim_{n \rightarrow \infty} U_{n,k}x$ exists.

For every $k \in \mathbb{N}$ and $x \in C$, Kangtunyakarn [25] defined mapping $U_{\infty,k}$ and $S : C \rightarrow C$ as follows:

$$\lim_{n \rightarrow \infty} U_{n,k}x = U_{\infty,k}x$$

and

$$\lim_{n \rightarrow \infty} S_n x = \lim_{n \rightarrow \infty} U_{n,1}x = Sx.$$

Such a mapping S is called S -mapping generated by T_n, T_{n-1}, \dots and $\alpha_n, \alpha_{n-1}, \dots$

Remark 2.24 ([25]). For every $n \in \mathbb{N}$, S_n is nonexpansive and $\lim_{n \rightarrow \infty} \sup_{x \in D} \|S_n x - Sx\| = 0$ for every bounded subset D of C .

Lemma 2.25 ([25]). Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^\infty$ be κ_i -strictly pseudo-contractive mappings of C into itself with $\bigcap_{i=1}^\infty \text{Fix}(T_i) \neq \emptyset$ and $\kappa = \sup_{i \in \mathbb{N}} \kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \dots$. For every $n \in \mathbb{N}$, let S_n and S be S -mappings generated by T_n, T_{n-1}, \dots, T_1 and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$ and T_n, T_{n-1}, \dots and $\alpha_n, \alpha_{n-1}, \dots$, respectively. Then $\text{Fix}(S) = \bigcap_{i=1}^\infty \text{Fix}(T_i)$.

2.3 Fixed point sets of nonspreading mapping with some properties

Definition 2.9. Let $T : C \rightarrow C$ be a mapping. Then T is said to be

- 1) a quasi-nonexpansive mapping if

$$\|Tx - p\| \leq \|x - p\|, \text{ for every } x \in C \text{ and } p \in \text{Fix}(T).$$

- 2) a nonspreading mapping if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \forall x, y \in C. \quad (2.3)$$

In 2008, Kohsaka and Takahashi [41] introduced the nonspreading mapping T in Hilbert space H as defined in (2.3)

In 2009, it is shown by Iemoto and Takahashi [42] that (2.3) is equivalent to the following equation.

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle, \text{ for all } x, y \in C. \quad (2.4)$$

Many researcher proved the strong convergence theorem for nonspreading mapping and its generalized mappings in Hilbert spaces, see, for example, [43, 38, 44, 45].

Theorem 2.26 ([41]). Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a nonspreading mapping of C into itself. Then $\text{Fix}(T)$ is closed and convex.

Remark 2.27. From (2.4), if T is a nonspreading mapping with $\text{Fix}(T) \neq \emptyset$, then T is quasi-nonexpansive.

Example 2.28. Let $T : [1, 100] \rightarrow [1, 100]$ be defined by $Tx = \frac{2x+5}{7}$. Then T is a nonspreading mapping.

Solution. Let $x, y \in [1, 100]$. Thus we get

$$|Tx - Ty|^2 = \left| \frac{2x+5}{7} - \frac{2y+5}{7} \right|^2 = \left| \frac{2}{7}(x-y) \right|^2 = \frac{4}{49}|x-y|^2,$$

and

$$\begin{aligned} 2\langle x - Tx, y - Ty \rangle &= 2 \left(x - \frac{2x+5}{7} \right) \left(y - \frac{2y+5}{7} \right) \\ &= 2 \left(\frac{5x-5}{7} \right) \left(\frac{5y-5}{7} \right) \\ &= \frac{50}{49}(x-1)(y-1) \geq 0, \text{ (Since } x, y \geq 1). \end{aligned}$$

Therefore

$$\begin{aligned} |x-y|^2 + 2\langle x - Tx, y - Ty \rangle &\geq |x-y|^2 \\ &\geq \frac{4}{49}|x-y|^2 = |Tx - Ty|^2. \end{aligned}$$

Hence T is a nonspreading mapping.

Example 2.29. Let an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $\langle u, v \rangle = u \cdot v = u_1v_1 + u_2v_2$ and a usual norm $\| \cdot \| : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by $\|u\| = \sqrt{u_1^2 + u_2^2}$, for all $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$. Let $I = [1, 100]$ and let $T : I^2 \rightarrow I^2$ be defined by

$$Tu = \left(\frac{u_1 + 1}{2}, \frac{5u_2 + 1}{6} \right), \text{ for all } u = (u_1, u_2) \in I^2.$$

First, show that T is a nonspreading mapping.

For every $u, v \in I^2$, we obtain

$$\begin{aligned} \|Tu - Tv\|^2 &= \left\| \left(\frac{u_1 + 1}{2}, \frac{5u_2 + 1}{6} \right) - \left(\frac{v_1 + 1}{2}, \frac{5v_2 + 1}{6} \right) \right\|^2 \\ &= \left\| \left(\frac{1}{2}(u_1 - v_1), \frac{5}{6}(u_2 - v_2) \right) \right\|^2 \\ &= \frac{1}{4}(u_1 - v_1)^2 + \frac{25}{36}(u_2 - v_2)^2, \end{aligned}$$

and

$$\begin{aligned} &2\langle u - Tu, v - Tv \rangle \\ &= 2 \left\langle (u_1, u_2) - \left(\frac{u_1 + 1}{2}, \frac{5u_2 + 1}{6} \right), (v_1, v_2) - \left(\frac{v_1 + 1}{2}, \frac{5v_2 + 1}{6} \right) \right\rangle \\ &= 2 \left\langle \left(\frac{u_1 - 1}{2}, \frac{u_2 - 1}{6} \right), \left(\frac{v_1 - 1}{2}, \frac{v_2 - 1}{6} \right) \right\rangle \\ &= 2 \left(\frac{u_1 - 1}{2}, \frac{u_2 - 1}{6} \right) \cdot \left(\frac{v_1 - 1}{2}, \frac{v_2 - 1}{6} \right) \\ &= 2 \left[\left(\frac{u_1 - 1}{2} \right) \left(\frac{v_1 - 1}{2} \right) + \left(\frac{u_2 - 1}{6} \right) \left(\frac{v_2 - 1}{6} \right) \right] \\ &= \frac{(u_1 - 1)(v_1 - 1)}{2} + \frac{(u_2 - 1)(v_2 - 1)}{18} \\ &\geq 0. \end{aligned}$$

This yields that

$$\begin{aligned} \|u - v\|^2 + 2\langle u - Tu, v - Tv \rangle &\geq \|u - v\|^2 \\ &= \|(u_1 - v_1, u_2 - v_2)\|^2 \\ &= (u_1 - v_1)^2 + (u_2 - v_2)^2 \\ &\geq \frac{1}{4}(u_1 - v_1)^2 + \frac{25}{36}(u_2 - v_2)^2 \\ &= \|Tu - Tv\|^2. \end{aligned} \tag{2.5}$$

Then T is a nonspreading mapping and observe that $Fix(T) = \{\mathbf{1}\}$, where $\mathbf{1} = (1, 1)$.

For every $u \in I \times I$ and $\mathbf{1} \in Fix(T)$, from (2.5), we have

$$\begin{aligned} \|Tu - T\mathbf{1}\|^2 &\leq \|u - \mathbf{1}\|^2 + 2\langle u - Tu, \mathbf{1} - T\mathbf{1} \rangle \\ &= \|u - \mathbf{1}\|^2, \end{aligned}$$

Therefore T is a quasi-nonexpansive mapping.

The following example shows that the converse of Remark 2.27 does not hold.

Example 2.30. Let $I = [0, 2]$ and let $T : I^2 \rightarrow I^2$ be defined by

$$Tu = \begin{cases} \left(\frac{u_1+2}{2}, \frac{u_2+2}{2}\right) & \text{if } u \in (1, 2] \times (1, 2], \\ \left(\frac{u_1}{2}, \frac{u_2}{2}\right) & \text{if } u \in [0, 1] \times [0, 1]. \end{cases}$$

First, show that T is quasi-nonexpansive for all $u \in I^2$.

Observe that $Fix(T) = \{2\}$ if $x \in (1, 2] \times (1, 2]$ and $Fix(T) = \{0\}$ if $u \in [0, 1] \times [0, 1]$, where $2 = (2, 2)$ and $0 = (0, 0)$.

For any $u \in (1, 2] \times (1, 2]$, we have

$$\begin{aligned} \|Tu - T2\| &= \left\| \left(\frac{u_1+2}{2}, \frac{u_2+2}{2}\right) - (2, 2) \right\| = \left\| \left(\frac{u_1-2}{2}, \frac{u_2-2}{2}\right) \right\| \\ &= \frac{1}{2} \|(u_1 - 2, u_2 - 2)\| \\ &= \frac{1}{2} \|(u_1, u_2) - (2, 2)\| \\ &< \|u - 2\|. \end{aligned}$$

For every $u \in [0, 1] \times [0, 1]$, we obtain

$$\begin{aligned} \left\| \left(\frac{u_1}{2}, \frac{u_2}{2}\right) - (0, 0) \right\| &= \left\| \left(\frac{u_1}{2}, \frac{u_2}{2}\right) \right\| \\ &= \frac{1}{2} \|(u_1, u_2)\| \\ &< \|u\|. \end{aligned}$$

Therefore T is a quasi-nonexpansive for all $u \in I^2$.

Choose $u = \left(\frac{3}{2}, \frac{3}{2}\right)$ and $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, we have

$$\begin{aligned} \left\| T\left(\frac{3}{2}, \frac{3}{2}\right) - T\left(\frac{1}{2}, \frac{1}{2}\right) \right\|^2 &= \left\| \left(\frac{7}{4}, \frac{7}{4}\right) - \left(\frac{1}{4}, \frac{1}{4}\right) \right\|^2 \\ &= \left\| \left(\frac{6}{4}, \frac{6}{4}\right) \right\|^2 = \left(\frac{6}{4}\right)^2 + \left(\frac{6}{4}\right)^2 = \frac{9}{2}. \end{aligned}$$

Thus we get

$$\begin{aligned} &\|u - v\|^2 + 2\langle u - Tu, v - Tv \rangle \\ &= \left\| \left(\frac{3}{2}, \frac{3}{2}\right) - \left(\frac{1}{2}, \frac{1}{2}\right) \right\|^2 + 2 \left\langle \left(\frac{3}{2}, \frac{3}{2}\right) - T\left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right) - T\left(\frac{1}{2}, \frac{1}{2}\right) \right\rangle \\ &= \|(1, 1)\|^2 + 2 \left\langle \left(\frac{3}{2}, \frac{3}{2}\right) - \left(\frac{7}{4}, \frac{7}{4}\right), \left(\frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{4}, \frac{1}{4}\right) \right\rangle \\ &= 2 + 2 \left(\frac{3}{2} - \frac{7}{4}, \frac{3}{2} - \frac{7}{4} \right) \cdot \left(\frac{1}{2} - \frac{1}{4}, \frac{1}{2} - \frac{1}{4} \right) \\ &= 2 + 2 \left(-\frac{1}{4}, -\frac{1}{4} \right) \cdot \left(\frac{1}{4}, \frac{1}{4} \right) \\ &= 2 - \frac{2}{8} = \frac{14}{8}. \end{aligned}$$

Hence we have

$$\|Tu - Tv\|^2 > \|u - v\|^2 + 2\langle u - Tu, v - Tv \rangle.$$

That is, T is not a nonspreading mapping.

2.4 Properties of strongly positive bounded linear operators in Hilbert spaces

Definition 2.10 ([19]). Let E and F be linear spaces with the same scalars, and let T be a mapping of E into F . Then T is called *linear* if for any $x, y \in E$ and any scalar $\alpha \in \mathbb{R}$,

$$T(x + y) = T(x) + T(y) \text{ and } T(\alpha x) = \alpha T(x).$$

In particular, for the case of $\mathbb{F} = \mathbb{R}$, T is called a *linear functional*.

Note that if $T : E \rightarrow F$ is a linear mapping, then

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \forall x, y \in E \text{ and } \alpha, \beta \in \mathbb{R}.$$

Definition 2.11 ([19]). Let E and F be normed linear spaces with the same scalars, and let T be a linear mapping of E into F . Then T is called *bounded* if there exists $K \geq 0$ such that

$$\|T(x)\| \leq K\|x\| \text{ for all } x \in E.$$

Let T be a bounded linear mapping of E into F . So, we have that for $x \in E$ with $\|x\| \leq 1$,

$$\|T(x)\| \leq K, \tag{2.6}$$

where $T(x)$ is often denoted by Tx .

For a bounded linear mapping T of E into F , we define its norm by

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|. \tag{2.7}$$

For such $\|T\|$, we have the following results.

Definition 2.12 ([19]). Let E and F be normed linear spaces and let T be a bounded linear mapping of E into F . Then the following hold:

- 1) $\|Tx\| \leq \|T\|\|x\|$ for all $x \in E$,
- 2) $\|T\| = \sup_{\|x\|=1} \|Tx\|$.

Definition 2.13 ([37]). Let A be a bounded operator on a Hilbert space H . If $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x \in H$, then A is called the *self-adjoint* operator.

Theorem 2.31 ([37]). Let T be a bounded linear self-adjoint operator on a Hilbert space H . Then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x \rangle|.$$

Definition 2.14 ([37]). A self-adjoint operator A is called *positive* if $\langle Ax, x \rangle \geq 0$ for all $x \in H$.

Definition 2.15 ([16]). A self-adjoint operator A is a strongly positive operator on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \text{ for all } x \in H.$$

Example 2.32. Let an operator $A_i : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $A_i x = \frac{ix}{2}, \forall x \in \mathbb{R}$ and $i = 1, 2, \dots, N$. Then A_i is a strongly positive linear bounded on \mathbb{R} .

Solution. Let $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}$. Then, for every $i = 1, 2, \dots, N$, we have

$$A_i(\alpha x + \beta y) = \frac{i}{2}(\alpha x + \beta y) = \alpha \frac{ix}{2} + \beta \frac{iy}{2} = \alpha A_i x + \beta A_i y.$$

It follows that A_i is a linear operator, for all $i = 1, 2, \dots, N$.

Since

$$|A_i x| = \left| \frac{ix}{2} \right| = \frac{i}{2} |x|,$$

then, for any $i = 1, 2, \dots, N$, A_i is bounded.

Consider

$$(A_i x)y = \left(\frac{ix}{2} \right) y = x \left(\frac{iy}{2} \right) = x(A_i y),$$

then A_i is self-adjoint, for every $i = 1, 2, \dots, N$.

Observe that

$$(A_i x)x = \left(\frac{ix}{2} \right) x = \frac{ix^2}{2} \geq 0. \quad (2.8)$$

This yields that A_i is strongly positive, for $i = 1, 2, \dots, N$.

Example 2.33. Let \mathbb{R}^2 be the two dimensional space of real numbers with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $\langle u, v \rangle = u \cdot v = u_1 v_1 + u_2 v_2$ and a usual norm $\| \cdot \| : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $\|u\| = \sqrt{u_1^2 + u_2^2}$, for all $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$. Let an operator $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $Ax = \left(\frac{x_1}{2}, \frac{x_2}{2} \right), \forall x = (x_1, x_2) \in \mathbb{R}^2$. Then A is a strongly positive linear bounded on \mathbb{R}^2 .

Solution. Let $\alpha, \beta \in \mathbb{R}$ and $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Thus we derive

$$\begin{aligned} A(\alpha x + \beta y) &= A(\alpha(x_1, x_2), \beta(y_1, y_2)) \\ &= A(\alpha x_1 + \alpha x_2, \beta y_1 + \beta y_2) \\ &= \left(\frac{\alpha x_1 + \alpha x_2}{2}, \frac{\beta y_1 + \beta y_2}{2} \right) \\ &= \left(\frac{\alpha x_1}{2} + \frac{\beta y_1}{2} \right) + \left(\frac{\alpha x_2}{2} + \frac{\beta y_2}{2} \right) \\ &= \alpha \left(\frac{x_1}{2}, \frac{x_2}{2} \right) + \beta \left(\frac{y_1}{2}, \frac{y_2}{2} \right) \\ &= \alpha Ax + \beta Ay. \end{aligned}$$

This implies that A is linear. Observe that

$$\|Ax\| = \left| \left(\frac{x_1}{2}, \frac{x_2}{2} \right) \right| = \sqrt{\frac{x_1^2}{4} + \frac{x_2^2}{4}} = \sqrt{\frac{1}{4}(x_1^2 + x_2^2)} = \frac{1}{2} \sqrt{x_1^2 + x_2^2} = \frac{1}{2} \|x\|.$$

Then A is bounded. Next, we consider

$$\begin{aligned}\langle Ax, y \rangle &= \left\langle \left(\frac{x_1}{2}, \frac{x_2}{2} \right), (y_1, y_2) \right\rangle \\ &= \frac{x_1}{2}y_1 + \frac{x_2}{2}y_2 = \frac{y_1}{2}x_1 + \frac{y_2}{2}x_2 \\ &= \left\langle \left(\frac{y_1}{2}, \frac{y_2}{2} \right), (x_1, x_2) \right\rangle = \langle Ay, x \rangle.\end{aligned}$$

This yields that A is self-adjoint. Since

$$\langle Ax, x \rangle = \left\langle \left(\frac{x_1}{2}, \frac{x_2}{2} \right), (x_1, x_2) \right\rangle = \frac{1}{2}(x_1^2 + x_2^2) = \frac{1}{2}\|x\|^2.$$

Hence A is a $\frac{1}{2}$ -strongly positive operator.

Definition 2.16 ([22]). Let A be an operator of C into itself and $\alpha > 0$. Then, A is called α -strongly monotone if

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|x - y\|^2, \text{ and } x, y \in C.$$

Remark 2.34. If A is a strongly positive linear operator on H with coefficient $\alpha > 0$, then A is an α -strongly monotone mapping.

Proof. Let $x, y \in H$. Then $x - y \in H$. Since A is a strongly positive linear operator on H , we obtain

$$\langle x - y, Ax - Ay \rangle = \langle x - y, A(x - y) \rangle \geq \alpha \|x - y\|^2.$$

□

In 2006, under certain conditions, Marino and Xu [16] proved the property of a strongly positive linear bounded operator in a Hilbert space as shown in the following Lemma.

Lemma 2.35 ([16]). Assume A is a strongly positive linear bounded self-adjoint operator on a Hilbert space H with coefficient $\beta > 0$ and $0 < \delta < \|A\|^{-1}$. Then $\|I - \delta A\| \leq 1 - \delta\beta$.

Definition 2.17 ([22]). Let A be an operator of C into itself. Then, A is called an inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \forall x, y \in C.$$

Such an A is called α -inverse strongly monotone.

2.5 Equilibrium problems and generalized equilibrium problem in Hilbert spaces

The equilibrium problem covers monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems and certain fixed point problems.

Let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction. The classical equilibrium problem for F is to find $u \in C$ satisfying the following inequality

$$F(u, v) \geq 0, \forall v \in C. \quad (2.9)$$

We use $EP(F)$ to represent the set of solution of (2.9).

Let the bifunction F satisfy the following conditions for solving equilibrium problem.

(A1) $F(u, u) = 0$ for all $u \in C$;

(A2) F is monotone, i.e., $F(u, v) + F(v, u) \leq 0$ for all $u, v \in C$;

(A3) For each $u, v, w \in C$,

$$\lim_{t \rightarrow 0^+} F(tw + (1-t)u, v) \leq F(u, v);$$

(A4) For each $u \in C, v \mapsto F(u, v)$ is convex and lower semicontinuous.

Example 2.36. Let $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$F(x, y) = -3x^2 + xy + 2y^2, \forall x, y \in \mathbb{R}.$$

Then a bifunction F satisfies the condition (A1)-(A4) and $0 \in EP(F)$.

Solution. Let $x, y, z \in \mathbb{R}$. Since

$$F(x, x) = -3x^2 + x^2 + 2x^2 = 0,$$

thus we obtain (A1) holds. Next, observe that

$$F(x, y) + F(y, x) = (-3x^2 + xy + 2y^2) + (-3y^2 + xy + 2x^2) = -x^2 + 2xy - y^2 = -(x - y)^2 \leq 0.$$

This implies that F satisfies (A2). Let $t \in [0, 1]$. Consider

$$\lim_{t \rightarrow 0^+} F(tz + (1-t)x, y) = \lim_{t \rightarrow 0^+} -3(tz + (1-t)x)^2 + (tz + (1-t)x)y + 2y^2 = -3x^2 + xy + 2y^2 = F(x, y).$$

Therefore, (A3) is true. To show (A4), first let $\alpha \in (0, 1)$. Then we derive that

$$\begin{aligned} & F(x, \alpha z + (1-\alpha)y) \\ &= -3x^2 + x(\alpha z + (1-\alpha)y) + 2(\alpha z + (1-\alpha)y)^2 \\ &= -3x^2 + \alpha xz + (1-\alpha)xy + 2(\alpha^2 z^2 + 2\alpha(1-\alpha)zy + (1-\alpha)^2 y^2) \\ &\leq -3x^2 + \alpha xz + (1-\alpha)xy + 2(\alpha^2 z^2 + \alpha(1-\alpha)(z^2 + y^2) + (1-\alpha)^2 y^2) \\ &= \alpha(-3x^2 + xz + 2(\alpha z^2 + (1-\alpha)z^2)) + (1-\alpha)(-3x^2 + xy + 2(\alpha y^2 + (1-\alpha)y^2)) \\ &= \alpha(-3x^2 + xz + 2z^2) + (1-\alpha)(-3x^2 + xy + 2y^2) \\ &= \alpha F(x, z) + (1-\alpha)F(x, y). \end{aligned}$$

Hence F is a convex function. Let $\{y_n\} \subset \mathbb{R}$ with $y_n \rightarrow y$ as $n \rightarrow \infty$. Thus we get

$$\lim_{n \rightarrow \infty} F(x, y_n) = \lim_{n \rightarrow \infty} -3x^2 + xy_n + 2y_n^2 = -3x^2 + xy + 2y^2. \quad (2.10)$$

This yields that F is lower semicontinuous and (A4) holds. Since $F(0, y) = 2y^2 \geq 0, \forall y \in \mathbb{R}$, thus we have $0 \in EP(F)$.

Example 2.37. Let $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$F(x, y) = (y - x)(y + 2x - 3), \forall x, y \in \mathbb{R}.$$

Then a bifunction F satisfies the condition (A1)-(A4) and $1 \in EP(F)$.

Solution. Let $x, y, z \in \mathbb{R}$. Due to the fact that $F(x, x) = 0$, then (A1) holds. Observe that

$$\begin{aligned} F(x, y) + F(y, x) &= (y - x)(y + 2x - 3) + (x - y)(x + 2y - 3) \\ &= -(x - y)(y + 2x - 3 - x - 2y + 3) \\ &= -(x - y)^2. \end{aligned}$$

This yields that (A2) is true. Let $t \in [0, 1]$. We derive that

$$\begin{aligned} \lim_{t \rightarrow 0^+} F(tz + (1 - t)x, y) &= \lim_{t \rightarrow 0^+} (y - (tz + (1 - t)x))(y + 2(tz + (1 - t)x) - 3) \\ &= (y - x)(y + 2x - 3) = F(x, y). \end{aligned}$$

Then, we deduce that F satisfies (A3). By the definition of F , we obtain

$$F(x, y) = (y - x)(y + 2x - 3) = 3x - 2x^2 - 3y + xy + y^2. \quad (2.11)$$

Let $\alpha \in (0, 1)$. From (2.11), we have

$$\begin{aligned} &F(x, \alpha z + (1 - \alpha)y) \\ &= 3x - 2x^2 - 3(\alpha z + (1 - \alpha)y) + x(\alpha z + (1 - \alpha)y) + (\alpha z + (1 - \alpha)y)^2 \\ &= 3x - 2x^2 - 3(\alpha z + (1 - \alpha)y) + x(\alpha z + (1 - \alpha)y) + \alpha^2 z^2 + (1 - \alpha)^2 y^2 + 2\alpha(1 - \alpha)yz \\ &\leq 3x - 2x^2 - 3(\alpha z + (1 - \alpha)y) + x(\alpha z + (1 - \alpha)y) + \alpha^2 z^2 + (1 - \alpha)^2 y^2 + \alpha(1 - \alpha)(y^2 + z^2) \\ &= \alpha [3x - 2x^2 - 3z + xz + \alpha z^2 + (1 - \alpha)z^2] + (1 - \alpha) [3x - 2x^2 - 3y + xy + \alpha y^2 + (1 - \alpha)y^2] \\ &= \alpha [3x - 2x^2 - 3z + xz + z^2] + (1 - \alpha) [3x - 2x^2 - 3y + xy + y^2] \\ &= \alpha F(x, z) + (1 - \alpha)F(x, y). \end{aligned}$$

Thus F is convex. Next, we let $\{y_n\} \subset \mathbb{R}$ such that $y_n \rightarrow y$ as $n \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty} F(x, y_n) = \lim_{n \rightarrow \infty} (y_n - x)(y_n + 2x - 3) = (y - x)(y + 2x - 3) = F(x, y).$$

Hence F is lower semicontinuous. Therefore (A4) holds for a bifunction F .

Example 2.38. Let \mathbb{R}^2 be the two dimensional space of real numbers with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $\langle u, v \rangle = u \cdot v = u_1 v_1 + u_2 v_2$ and a usual norm $\| \cdot \| : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $\|u\| = \sqrt{u_1^2 + u_2^2}$, for all $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$. Let $F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$F(x, y) = (y - x) \cdot (y + 6x - (7, 7)), \forall x, y \in \mathbb{R}^2,$$

Then a bifunction F satisfies the condition (A1)-(A4) and $(1, 1) \in EP(F)$.

Solution. Let $x, y, z \in \mathbb{R}^2$. Consider

$$F(x, x) = (x - x) \cdot (x + 6x - (7, 7)) = 0.$$

Thus F satisfies (A1). Next, observe that

$$\begin{aligned}
& F(x, y) + F(y, x) \\
&= F((x_1, x_2), (y_1, y_2)) + F((y_1, y_2), (x_1, x_2)) \\
&= ((y_1, y_2) - (x_1, x_2)) \cdot ((y_1, y_2) + 6(x_1, x_2) - (7, 7)) \\
&\quad + ((x_1, x_2) - (y_1, y_2)) \cdot ((x_1, x_2) + 6(y_1, y_2) - (7, 7)) \\
&= (y_1 - x_1, y_2 - x_2) \cdot (y_1 + 6x_1 - 7, y_2 + 6x_2 - 7) \\
&\quad + (x_1 - y_1, x_2 - y_2) \cdot (x_1 + 6y_1 - 7, x_2 + 6y_2 - 7) \\
&= -(x_1 - y_1, x_2 - y_2) \cdot ((y_1 + 6x_1 - 7, y_2 + 6x_2 - 7) - (x_1 + 6y_1 - 7, x_2 + 6y_2 - 7)) \\
&= -(x_1 - y_1, x_2 - y_2) \cdot (5x_1 - 5y_1, 5x_2 - 5y_2) \\
&= -5 \left((x_1 - y_1)^2 + (x_2 - y_2)^2 \right) \leq 0.
\end{aligned}$$

Then (A2) is true. Let $t \in [0, 1]$. Derive that

$$\begin{aligned}
& \lim_{t \rightarrow 0^+} F(tz + (1-t)x, y) \\
&= \lim_{t \rightarrow 0^+} (y - (tz + (1-t)x)) \cdot (y + 6(tz + (1-t)x) - (7, 7)) \\
&= \lim_{t \rightarrow 0^+} ((y_1, y_2) - (t(z_1, z_2) + (1-t)(x_1, x_2))) \cdot ((y_1, y_2) + 6(t(z_1, z_2) + (1-t)(x_1, x_2)) - (7, 7)) \\
&= ((y_1, y_2) - (x_1, x_2)) \cdot ((y_1, y_2) + 6(x_1, x_2) - (7, 7)) \\
&= (y - x) \cdot (y + 6x - (7, 7)) = F(x, y).
\end{aligned}$$

Therefore (A3) holds. From the definition of F , we obtain

$$\begin{aligned}
F(x, y) &= F((x_1, x_2), (y_1, y_2)) \\
&= ((y_1, y_2) - (x_1, x_2)) \cdot ((y_1, y_2) + 6(x_1, x_2) - (7, 7)) \\
&= (y_1 - x_1, y_2 - x_2) \cdot (y_1 + 6x_1 - 7, y_2 + 6x_2 - 7) \\
&= (y_1 - x_1)(y_1 + 6x_1 - 7) + (y_2 - x_2)(y_2 + 6x_2 - 7) \\
&= (y_1^2 + 5x_1y_1 - 7y_1 - 6x_1^2 + 7x_1) + (y_2^2 + 5x_2y_2 - 7y_2 - 6x_2^2 + 7x_2). \tag{2.12}
\end{aligned}$$

Let $\alpha \in (0, 1)$. From (2.12), we deduce that

$$\begin{aligned}
& F(x, \alpha z + (1-\alpha)y) = F((x_1, x_2), \alpha(z_1, z_2) + (1-\alpha)(y_1, y_2)) \\
&= F((x_1, x_2), (\alpha z_1 + (1-\alpha)y_1, \alpha z_2 + (1-\alpha)y_2)) \\
&= \left[(\alpha z_1 + (1-\alpha)y_1)^2 + 5x_1(\alpha z_1 + (1-\alpha)y_1) - 7(\alpha z_1 + (1-\alpha)y_1) - 6x_1^2 + 7x_1 \right] \\
&\quad + \left[(\alpha z_2 + (1-\alpha)y_2)^2 + 5x_2(\alpha z_2 + (1-\alpha)y_2) - 7(\alpha z_2 + (1-\alpha)y_2) - 6x_2^2 + 7x_2 \right] \\
&= \left[\alpha^2 z_1^2 + (1-\alpha)^2 y_1^2 + 2\alpha(1-\alpha)z_1y_1 + 5x_1(\alpha z_1 + (1-\alpha)y_1) - 7(\alpha z_1 + (1-\alpha)y_1) - 6x_1^2 \right. \\
&\quad \left. + 7x_1 \right] + \left[\alpha^2 z_2^2 + (1-\alpha)^2 y_2^2 + 2\alpha(1-\alpha)z_2y_2 + 5x_2(\alpha z_2 + (1-\alpha)y_2) - 7(\alpha z_2 + (1-\alpha)y_2) \right. \\
&\quad \left. - 6x_2^2 + 7x_2 \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \left[\alpha^2 z_1^2 + (1-\alpha)^2 y_1^2 + \alpha(1-\alpha)(z_1^2 + y_1^2) + 5x_1(\alpha z_1 + (1-\alpha)y_1) - 7(\alpha z_1 + (1-\alpha)y_1) - 6x_1^2 \right. \\
&\quad \left. + 7x_1 \right] + \left[\alpha^2 z_2^2 + (1-\alpha)^2 y_2^2 + \alpha(1-\alpha)(z_2^2 + y_2^2) + 5x_2(\alpha z_2 + (1-\alpha)y_2) - 7(\alpha z_2 + (1-\alpha)y_2) \right. \\
&\quad \left. - 6x_2^2 + 7x_2 \right] \\
&= \left[\alpha z_1^2 + (1-\alpha)y_1^2 + 5x_1(\alpha z_1 + (1-\alpha)y_1) - 7(\alpha z_1 + (1-\alpha)y_1) - 6x_1^2 + 7x_1 \right] \\
&\quad + \left[\alpha z_2^2 + (1-\alpha)y_2^2 + 5x_2(\alpha z_2 + (1-\alpha)y_2) - 7(\alpha z_2 + (1-\alpha)y_2) - 6x_2^2 + 7x_2 \right] \\
&= \left[\alpha(z_1^2 + 5x_1 z_1 - 7z_1 - 6x_1^2 + 7x_1) + (1-\alpha)(y_1^2 + 5x_1 y_1 - 7y_1 - 6x_1^2 + 7x_1) \right] \\
&\quad + \left[\alpha(z_2^2 + 5x_2 z_2 - 7z_2 - 6x_2^2 + 7x_2) + (1-\alpha)(y_2^2 + 5x_2 y_2 - 7y_2 - 6x_2^2 + 7x_2) \right] \\
&= \alpha \left[(z_1^2 + 5x_1 z_1 - 7z_1 - 6x_1^2 + 7x_1) + (z_2^2 + 5x_2 z_2 - 7z_2 - 6x_2^2 + 7x_2) \right] \\
&\quad + (1-\alpha) \left[(y_1^2 + 5x_1 y_1 - 7y_1 - 6x_1^2 + 7x_1) + (y_2^2 + 5x_2 y_2 - 7y_2 - 6x_2^2 + 7x_2) \right] \\
&= \alpha F((x_1, x_2), (z_1, z_2)) + (1-\alpha) F((x_1, x_2), (y_1, y_2)) \\
&= \alpha F(x, z) + (1-\alpha) F(x, y).
\end{aligned}$$

This yields that F is convex. Suppose that $\{y_n\} \subset \mathbb{R}^k$ with $y_n = (y_n^1, y_n^2) \rightarrow (y_1, y_2)$ as $n \rightarrow \infty$. Thus we get

$$\lim_{n \rightarrow \infty} F(x, y_n) = \lim_{n \rightarrow \infty} (y_n - x) \cdot (y_n + 6x - (7, 7)) = (y - x) \cdot (y + 6x - (7, 7)) = F(x, y).$$

Hence F is lower semicontinuous. So the condition (A4) holds.

Because $F((1, 1), y) = (y - (1, 1)) \cdot (y + 6(1, 1) - (7, 7)) = (y - (1, 1)) \cdot (y - (1, 1)) = (y - (1, 1))^2 \geq 0$, then $(1, 1) \in EP(F)$.

In 1994, Blum and Oettli [9] proved the following existence result:

Lemma 2.39 ([9]). Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into \mathbb{R} satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C.$$

Lemma 2.40 ([10]). Assume that $F : C \times C \rightarrow \mathbb{R}$ satisfies (A1) - (A4). For $r > 0$, define a mapping $T_r : H \rightarrow C$ as follows:

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

- 1) T_r is single-valued;
- 2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y \rangle;$$

- 3) $Fix(T_r) = EP(F)$;
- 4) $EP(F)$ is closed and convex.

Let $\Psi = \{F_i\}_{i=1,2,\dots,N}$ be a finite family of bifunctions from $C \times C$ to \mathbb{R} . The system of equilibrium problem for Ψ is to determine common equilibrium points for $\Psi = \{F_i\}_{i=1,2,\dots,N}$, that is, the set

$$EP(\Psi) = \{u \in C : F_i(u, v) \geq 0, \forall v \in C, \forall i \in 1, 2, \dots, N\}. \quad (2.13)$$

Note that if $F_i = F$, for all $i = 1, 2, \dots, N$, then the system of equilibrium problem (2.13) becomes the equilibrium problem (2.9). Moreover, the problem (2.9) extends (2.13) to a system of such problems covering various forms of feasibility problems [10]. Several iterative algorithms are proposed to solve the equilibrium problems and a finite family of equilibrium problems, see, for instance, [31, 30, 10, 18, 17].

Let $CB(H)$ be the family of all nonempty closed bounded subsets of H and $\mathcal{H}(\cdot, \cdot)$ be a Pompeiu-Hausdorff metric on $CB(H)$ defined as

$$\mathcal{H}(U, V) = \max \left\{ \sup_{u \in U} d(u, V), \sup_{v \in V} d(U, v) \right\}, \forall U, V \in CB(H),$$

where $d(u, V) = \inf_{v \in V} d(u, v)$, $d(U, v) = \inf_{u \in U} d(u, v)$ and $d(u, v) = \|u - v\|$.

Example 2.41. Let $H = \mathbb{R}$, $U = [0, 2]$ and $V = [4, 7]$. Let $d : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by $d(x, y) = |x - y|$. Then $\mathcal{H}(U, V) = 5$.

Solution. Let the conditions hold. Then we have

$$\sup_{u \in U} d(u, V) = \sup_{u \in U} \inf_{v \in [4, 7]} (v - u) = \sup_{u \in [0, 2]} (4 - u) = 4,$$

and

$$\sup_{v \in V} d(U, v) = \sup_{v \in V} \inf_{u \in [0, 2]} (v - u) = \sup_{v \in [4, 7]} (v - 2) = 5.$$

This implies that

$$\mathcal{H}(U, V) = \max \left\{ \sup_{u \in U} d(u, V), \sup_{v \in V} d(U, v) \right\} = \max \{4, 5\} = 5.$$

Definition 2.18. A multivalued mapping $T : C \rightarrow CB(H)$ is said to be \mathcal{H} -Lipschitz continuous if there exists a constant $\mu > 0$ such that

$$\mathcal{H}(T(u), T(v)) \leq \mu \|u - v\|, \forall u, v \in C,$$

where $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$.

Lemma 2.42 (Nadler's theorem [6]). Let $(X, \|\cdot\|)$ be a normed vector space and $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$. If $U, V \in CB(H)$, then for every $\varepsilon > 0$ and $u \in U$, there exists $v \in V$ such that

$$\|u - v\| \leq (1 + \varepsilon)\mathcal{H}(U, V).$$

Let C be a nonempty closed convex subset of H . Let $\varphi : C \rightarrow \mathbb{R}$ be a real-valued function, $T : C \rightarrow CB(H)$ a multivalued mapping and $\Phi : H \times C \times C \rightarrow \mathbb{R}$ an

equilibrium-like function, that is, $\Phi(w, u, v) + \Phi(w, v, u) = 0$ for all $(w, u, v) \in H \times C \times C$ which satisfies the following conditions with respect to the multivalued mapping $T : C \rightarrow CB(H)$.

(H1) For each fixed $v \in C$, $(w, u) \mapsto \Phi(w, u, v)$ is an upper semicontinuous function from $H \times C \rightarrow \mathbb{R}$, that is, for $(w, u) \in H \times C$, whenever $w_n \rightarrow w$ and $u_n \rightarrow u$ as $n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \Phi(w_n, u_n, v) \leq \Phi(w, u, v);$$

(H2) For each fixed $(w, v) \in H \times C$, $u \mapsto \Phi(w, u, v)$ is a concave function;

(H3) For each fixed $(w, u) \in H \times C$, $v \mapsto \Phi(w, u, v)$ is a convex function.

In 2009, Ceng et al.[33] introduced *the generalized equilibrium problem (GEP)* as follows:

$$(GEP) \begin{cases} \text{Find } u \in C \text{ and } w \in T(u) \text{ such that} \\ \Phi(w, u, v) + \varphi(v) - \varphi(u) \geq 0, \forall v \in C. \end{cases} \quad (2.14)$$

The set of such solutions $u \in C$ of (GEP) is denoted by $(GEP)_s(\Phi, \varphi)$. In the case of $\varphi = 0$ and $\Phi(w, u, v) \equiv G(u, v)$, where the bifunction G satisfies (A1) – (A4), then $(GEP)_s(\Phi, \varphi)$ is denoted by $EP(G)$.

Example 2.43. Let $T : \mathbb{R} \rightarrow \mathbb{R}$ and $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$\Phi(w, u, v) = w(v - u), \forall (w, u, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

Then Φ satisfies the condition (H1)-(H3).

Solution. Let $w, u, v \in \mathbb{R}$. Then $\Phi(w, u, v) + \Phi(w, v, u) = w(v - u) + w(u - v) = 0$. Let $w_n \rightarrow \tilde{w}$ and $u_n \rightarrow \tilde{u}$ as $n \rightarrow \infty$. Since

$$\lim_{n \rightarrow \infty} \Phi(w_n, u_n, v) = \lim_{n \rightarrow \infty} w_n(v - u_n) = \tilde{w}(v - \tilde{u}) = \Phi(\tilde{w}, \tilde{u}, v), \text{ for all } y \in \mathbb{R},$$

then (H1) holds. Let $t \in (0, 1)$ and $\hat{u}, \hat{z} \in \mathbb{R}$. Because

$$\begin{aligned} \Phi(w, t\hat{u} + (1-t)\hat{z}, v) &= w(y - (t\hat{u} + (1-t)z)) \\ &= w(t(v - \hat{u}) + (1-t)(v - \hat{z})) \\ &= tw(v - \hat{u}) + (1-t)w(v - \hat{z}) = t\Phi(w, \hat{u}, v) + (1-t)\Phi(w, \hat{z}, v), \end{aligned}$$

for every $x, y \in \mathbb{R}$, thus Φ is a concave function and therefore (H2) is true. Let $t \in (0, 1)$ and $\bar{v}, \bar{z} \in \mathbb{R}$.

$$\begin{aligned} \Phi(w, u, t\bar{v} + (1-t)\bar{z}) &= w((t\bar{v} + (1-t)\bar{z}) - u) \\ &= w(t(\bar{v} - u) + (1-t)(\bar{z} - u)) \\ &= tw(\bar{v} - u) + (1-t)w(\bar{z} - u) = t\Phi(w, \bar{v}, u) + (1-t)\Phi(w, \bar{z}, u), \end{aligned}$$

for all $w, u \in \mathbb{R}$. Then Φ is a convex function and hence Φ satisfies (H3).

Theorem 2.44 ([33]). Let C be a nonempty, bounded, closed, and convex subset of a real Hilbert space H , and let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : H \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with constant μ , and $\Phi : H \times C \times C \rightarrow \mathbb{R}$ be an equilibrium-like function satisfying (H1) – (H3). Let $r > 0$ be a constant. For each $x \in H$, take $w_x \in T(x)$ arbitrarily and define a mapping $T_r : H \rightarrow C$ as follows:

$$T_r(x) = \left\{ u \in C : \Phi(w_x, u, v) + \varphi(v) - \varphi(u) + \frac{1}{r} \langle u - x, v - u \rangle \geq 0, \forall v \in C \right\}.$$

Then, there hold the following:

- 1) T_r is single-valued;
- 2) T_r is firmly nonexpansive (that is, for any $u, v \in H$, $\|T_r u - T_r v\|^2 \leq \langle T_r u - T_r v, u - v \rangle$) if

$$\Phi(w_1, T_r(x_1), T_r(x_2)) + \Phi(w_2, T_r(x_2), T_r(x_1)) \leq 0,$$

for all $(x_1, x_2) \in H \times H$ and all $w_i \in T(x_i)$, $i = 1, 2$;

- 3) $F(T_r) = (GEP)_s(\Phi, \varphi)$;
- 4) $(GEP)_s(\Phi, \varphi)$ is closed and convex.

Chapter 3

Strong Convergence Theorems

3.1 Convergence theorems for equilibrium problems, variational inequality problems and an infinite family of strictly pseudo-contractive mappings

In this section, we introduce *the combination of equilibrium problem* which is to find $u \in C$ such that

$$\left(\sum_{i=1}^N a_i F_i \right) (u, v) \geq 0, \forall v \in C, \quad (3.1)$$

where $F_i : C \times C \rightarrow \mathbb{R}$ be bifunctions and $a_i \in (0, 1)$ with $\sum_{i=1}^N a_i = 1$, for every $i = 1, 2, \dots, N$. The solution set of (3.1) is denoted by $EP \left(\sum_{i=1}^N a_i F_i \right)$.

If $F_i = F$, for all $i = 1, 2, \dots, N$, then the combination of equilibrium problem (3.1) reduces to the classical equilibrium problem (2.9).

Throughout this thesis, for solving the combination of equilibrium problem, let the bifunction $F_i, i = 1, 2, \dots, N$, satisfy the following conditions.

(A1)' $F_i(u, v) = 0$ if and only if $u = v$, for some $u, v \in C$;

(A2) F_i is monotone, i.e., $F_i(u, v) + F_i(v, u) \leq 0$ for all $u, v \in C$;

(A3) For each $u, v, w \in C$,

$$\lim_{t \rightarrow 0^+} F_i(tw + (1-t)u, v) \leq F_i(u, v);$$

(A4) For each $u \in C, v \mapsto F_i(u, v)$ is convex and lower semicontinuous.

Moreover, we obtain the following result.

Lemma 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H . For $i = 1, 2, \dots, N$, let $F_i : C \times C \rightarrow \mathbb{R}$ be bifunctions satisfying (A1)'–(A4) with $\bigcap_{i=1}^N EP(F_i) \neq \emptyset$. Then,

$$EP \left(\sum_{i=1}^N a_i F_i \right) = \bigcap_{i=1}^N EP(F_i),$$

where $a_i \in (0, 1)$ for every $i = 1, 2, \dots, N$ and $\sum_{i=1}^N a_i = 1$.

Proof. First, we will show that $\bigcap_{i=1}^N EP(F_i) \subseteq EP \left(\sum_{i=1}^N a_i F_i \right)$.

Let $x_0 \in \bigcap_{i=1}^N EP(F_i)$ and $a_i \in (0, 1)$ for every $i = 1, 2, \dots, N$. Thus we get

$$a_i F_i(x_0, y) \geq 0, \text{ for all } i = 1, 2, \dots, N, \text{ and } y \in C. \quad (3.2)$$

Combining the inequalities (3.2), therefore we have

$$\sum_{i=1}^N a_i F_i(x_0, y) \geq 0, \text{ for all } y \in C.$$

This follows that $x_0 \in EP\left(\sum_{i=1}^N a_i F_i\right)$. Hence we get $\bigcap_{i=1}^N EP(F_i) \subseteq EP\left(\sum_{i=1}^N a_i F_i\right)$. Next, let $x_0 \in EP\left(\sum_{i=1}^N a_i F_i\right)$ and $x^* \in \bigcap_{i=1}^N EP(F_i)$. Then, we have

$$\left(\sum_{i=1}^N a_i F_i\right)(x_0, y) \geq 0, \forall y \in C \quad (3.3)$$

and

$$F_k(x^*, y) \geq 0, \text{ for all } k = 1, 2, \dots, N, \text{ and } y \in C. \quad (3.4)$$

From (3.4) and $x_0 \in C$, we have

$$F_k(x^*, x_0) \geq 0, \quad (3.5)$$

for all $k = 1, 2, \dots, N$. From (3.5) and (A2), we obtain

$$F_k(x_0, x^*) \leq F_k(x^*, x_0) + F_k(x_0, x^*) \leq 0. \quad (3.6)$$

Since $x^* \in C$, it follows from (3.3) that

$$\sum_{i=1}^N a_i F_i(x_0, x^*) \geq 0. \quad (3.7)$$

Applying (3.7), for each $k = 1, 2, \dots, N$, we obtain

$$\begin{aligned} 0 &\leq \sum_{i=1}^N a_i F_i(x_0, x^*) \\ &= \sum_{i=1}^{k-1} a_i F_i(x_0, x^*) + a_k F_k(x_0, x^*) + \sum_{i=k+1}^N a_i F_i(x_0, x^*). \end{aligned} \quad (3.8)$$

From (3.5), (3.8) and (A2), it follows that

$$\begin{aligned} a_k F_k(x_0, x^*) &\geq -\sum_{i=1}^{k-1} a_i F_i(x_0, x^*) - \sum_{i=k+1}^N a_i F_i(x_0, x^*) \\ &\geq \sum_{i=1}^{k-1} a_i F_i(x^*, x_0) + \sum_{i=k+1}^N a_i F_i(x^*, x_0) \geq 0. \end{aligned} \quad (3.9)$$

Inequalities (3.9) and (3.6) guarantee that

$$F_k(x_0, x^*) = 0, \text{ for every } k = 1, 2, \dots, N. \quad (3.10)$$

By using (3.10) and (A1)', deduce that

$$x_0 = x^*.$$

It implies that

$$x_0 \in \bigcap_{i=1}^N EP(F_i).$$

Therefore,

$$EP\left(\sum_{i=1}^N a_i F_i\right) \subseteq \bigcap_{i=1}^N EP(F_i).$$

Hence, we have

$$EP \left(\sum_{i=1}^N a_i F_i \right) = \bigcap_{i=1}^N EP(F_i).$$

□

Remark 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H . For $i = 1, 2, \dots, N$, let $F_i : C \times C \rightarrow \mathbb{R}$ be bifunctions satisfying (A1)' - (A4). Then, $\sum_{i=1}^N a_i F_i$ satisfies (A1)'-(A4), where $a_i \in (0, 1)$ for every $i = 1, 2, \dots, N$ and $\sum_{i=1}^N a_i = 1$.

Proof. For every $i = 1, 2, \dots, N$, let $F_i : C \times C \rightarrow \mathbb{R}$ be bifunctions satisfying (A1)' - (A4) and let $x, y, z \in C$ and $a_i \in (0, 1)$ for all $i = 1, 2, \dots, N$ and $\sum_{i=1}^N a_i = 1$.

To prove (A1)', we get

$$\sum_{i=1}^N a_i F_i(x, y) = a_1 F_1(x, y) + a_2 F_2(x, y) + \dots + a_N F_N(x, y) = 0 \Leftrightarrow x = y.$$

Since

$$\sum_{i=1}^N a_i F_i(x, y) + \sum_{i=1}^N a_i F_i(y, x) = \sum_{i=1}^N a_i (F_i(x, y) + F_i(y, x)) \leq 0,$$

we have $\sum_{i=1}^N a_i F_i$ satisfies (A2).

Let $t \in [0, 1]$, then we have

$$\lim_{t \rightarrow 0^+} \sum_{i=1}^N a_i F_i(tz + (1-t)x, y) = \sum_{i=1}^N a_i \lim_{t \rightarrow 0^+} F_i(tz + (1-t)x, y) = \sum_{i=1}^N a_i F_i(x, y).$$

Thus (A3) holds.

To prove (A4), we first let $\alpha \in (0, 1)$. Therefore, we get

$$\begin{aligned} \sum_{i=1}^N a_i F_i(x, \alpha z + (1-\alpha)y) &\leq \sum_{i=1}^N a_i (\alpha F_i(x, z) + (1-\alpha)F_i(x, y)) \\ &= \alpha \sum_{i=1}^N a_i F_i(x, z) + (1-\alpha) \sum_{i=1}^N a_i F_i(x, y). \end{aligned}$$

It follows that $\sum_{i=1}^N a_i F_i$ is convex. Next, let $\{y_n\} \subset C$ with $y_n \rightarrow y$ as $n \rightarrow \infty$. Thus we obtain

$$\liminf_{n \rightarrow \infty} \sum_{i=1}^N a_i F_i(x, y_n) \geq \sum_{i=1}^N a_i \liminf_{n \rightarrow \infty} F_i(x, y_n) \geq \sum_{i=1}^N a_i F_i(x, y).$$

Then $\sum_{i=1}^N a_i F_i$ is lower semicontinuous. This implies that (A4) holds.

We can conclude that $\sum_{i=1}^N a_i F_i$ satisfies (A1)'-(A4). □

Remark 3.3. By Remark 3.2, we have $\sum_{i=1}^N a_i F_i$ satisfies (A1)'-(A4). By using Lemma 2.40 and Lemma 3.1, we obtain

$$Fix(T_r) = EP \left(\sum_{i=1}^N a_i F_i \right) = \bigcap_{i=1}^N EP(F_i),$$

where $a_i \in (0, 1)$, for each $i = 1, 2, \dots, N$, and $\sum_{i=1}^N a_i = 1$.

We now propose the combination of variational inequality problem as follows:
Find $x \in C$ such that

$$\left\langle y - x, \sum_{i=1}^N a_i A_i x \right\rangle \geq 0, \forall y \in C, \quad (3.11)$$

where $A_i : C \rightarrow C$ are nonlinear mappings and $a_i \in (0, 1)$ with $\sum_{i=1}^N a_i = 1$, for every $i = 1, 2, \dots, N$. The solution set of (3.11) is denoted by $VI\left(C, \sum_{i=1}^N a_i A_i\right)$.

If $A_i = A$, for all $i = 1, 2, \dots, N$, then the combination of variational inequality problem (3.11) reduces to the variational inequality problem (1.7).

For the generalization of Marino and Xu's result (Lemma 2.35), we now prove the following lemma to show the properties of a finite family of strongly positive linear bounded operators and the combination of variational inequality problem for these operators under mild conditions.

Lemma 3.4. Let C be a nonempty closed convex subset of a real Hilbert space H . For every $i = 1, 2, \dots, N$, let A_i be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\gamma_i > 0$, $\bar{\gamma} = \min_{i=1,2,\dots,N} \gamma_i$ and $\bigcap_{i=1}^N VI(C, A_i) \neq \emptyset$. Let $\{a_i\}_{i=1}^N \subseteq (0, 1)$ with $\sum_{i=1}^N a_i = 1$. Then, the following properties hold:

$$(i) \left\| I - \rho \sum_{i=1}^N a_i A_i \right\| \leq 1 - \rho \bar{\gamma} \text{ and } I - \rho \sum_{i=1}^N a_i A_i \text{ is a nonexpansive mapping, for every } 0 < \rho < \|A_i\|^{-1} (i = 1, 2, \dots, N).$$

$$(ii) VI\left(C, \sum_{i=1}^N a_i A_i\right) = \bigcap_{i=1}^N VI(C, A_i).$$

Proof. To show (i), we will show that $I - \rho \sum_{i=1}^N a_i A_i$ is a bounded linear self-adjoint operator.

Let $x, y \in H$. Since A_i is bounded, for each $i = 1, 2, \dots, N$, then there exists $k_i > 0$ such that

$$\|A_i x\| \leq k_i \|x\|, \text{ for all } i = 1, 2, \dots, N \text{ and } x \in H. \quad (3.12)$$

Choose $k = \max_{i=1,2,\dots,N} k_i$. From (3.12), we obtain

$$\begin{aligned} \left\| \left(I - \rho \sum_{i=1}^N a_i A_i \right) x \right\| &= \left\| x - \rho \sum_{i=1}^N a_i A_i x \right\| \\ &\leq \|x\| + \rho \left\| \sum_{i=1}^N a_i A_i x \right\| \\ &\leq \|x\| + \rho \sum_{i=1}^N a_i \|A_i x\| \\ &\leq \|x\| + \rho \sum_{i=1}^N a_i k_i \|x\| \\ &\leq \|x\| + \rho \sum_{i=1}^N a_i k \|x\| \\ &= (1 + \rho k) \|x\|. \end{aligned}$$

Then $I - \rho \sum_{i=1}^N a_i A_i$ is bounded.

Let $\omega_1, \omega_2 \in \mathbb{R}$. Since A_i is a linear operator for every $i = 1, 2, \dots, N$, thus we have

$$\sum_{i=1}^N a_i A_i (\omega_1 x + \omega_2 y) = \sum_{i=1}^N a_i (\omega_1 A_i x + \omega_2 A_i y) = \omega_1 \sum_{i=1}^N a_i A_i x + \omega_2 \sum_{i=1}^N a_i A_i y. \quad (3.13)$$

This implies that $\sum_{i=1}^N a_i A_i$ is linear. From (3.13), we have

$$\begin{aligned} \left(I - \rho \sum_{i=1}^N a_i A_i \right) (\omega_1 x + \omega_2 y) &= \omega_1 x + \omega_2 y - \rho \sum_{i=1}^N a_i A_i (\omega_1 x + \omega_2 y) \\ &= \omega_1 x + \omega_2 y - \rho \left(\omega_1 \sum_{i=1}^N a_i A_i x + \omega_2 \sum_{i=1}^N a_i A_i y \right) \\ &= \omega_1 \left(x - \rho \sum_{i=1}^N a_i A_i x \right) + \omega_2 \left(y - \rho \sum_{i=1}^N a_i A_i y \right) \\ &= \omega_1 \left(I - \rho \sum_{i=1}^N a_i A_i \right) x + \omega_2 \left(I - \rho \sum_{i=1}^N a_i A_i \right) y. \end{aligned}$$

Therefore, $I - \rho \sum_{i=1}^N a_i A_i$ is a linear operator.

Since A_i is strongly positive, $\forall i = 1, 2, \dots, N$, by Definition 2.14, then A_i is self-adjoint, $\forall i = 1, 2, \dots, N$. This yields that

$$\left\langle \sum_{i=1}^N a_i A_i x, y \right\rangle = \sum_{i=1}^N \langle a_i A_i x, y \rangle = \sum_{i=1}^N a_i \langle A_i x, y \rangle = \sum_{i=1}^N a_i \langle x, A_i y \rangle = \left\langle x, \sum_{i=1}^N a_i A_i y \right\rangle. \quad (3.14)$$

Thus, $\sum_{i=1}^N a_i A_i$ is self-adjoint.

From (3.14), we obtain

$$\begin{aligned} \left\langle \left(I - \rho \sum_{i=1}^N a_i A_i \right) x, y \right\rangle &= \left\langle x - \rho \sum_{i=1}^N a_i A_i x, y \right\rangle \\ &= \langle x, y \rangle - \rho \left\langle \sum_{i=1}^N a_i A_i x, y \right\rangle \\ &= \langle x, y \rangle - \rho \left\langle x, \sum_{i=1}^N a_i A_i y \right\rangle \\ &= \langle x, y \rangle - \left\langle x, \rho \sum_{i=1}^N a_i A_i y \right\rangle \\ &= \langle x, y - \rho \sum_{i=1}^N a_i A_i y \rangle \\ &= \left\langle x, \left(I - \rho \sum_{i=1}^N a_i A_i \right) y \right\rangle. \end{aligned}$$

This implies that $I - \rho \sum_{i=1}^N a_i A_i$ is self-adjoint. We can conclude that $I - \rho \sum_{i=1}^N a_i A_i$ is a bounded linear self-adjoint operator on H , which follows by Theorem 2.31 that

$$\left\| I - \rho \sum_{i=1}^N a_i A_i \right\| = \sup \left\{ \left| \left\langle I - \rho \sum_{i=1}^N a_i A_i x, x \right\rangle \right| : x \in H, \|x\| = 1 \right\}. \quad (3.15)$$

Since A_i is strongly positive operator for all $i = 1, 2, \dots, N$, we get

$$\left\langle \sum_{i=1}^N a_i A_i x, x \right\rangle = \sum_{i=1}^N a_i \langle A_i x, x \rangle \geq \sum_{i=1}^N a_i \gamma_i \|x\|^2 \geq \sum_{i=1}^N a_i \bar{\gamma} \|x\|^2 = \bar{\gamma} \|x\|^2, \quad (3.16)$$

which implies that $\sum_{i=1}^N a_i A_i$ is a $\bar{\gamma}$ -strongly positive operator.

Let $\|x\| = 1$. Then by using (3.16), we obtain

$$\begin{aligned} \left\langle \left(I - \rho \sum_{i=1}^N a_i A_i \right) x, x \right\rangle &= \left\langle x - \rho \sum_{i=1}^N a_i A_i x, x \right\rangle \\ &= \|x\|^2 - \rho \left\langle \sum_{i=1}^N a_i A_i x, x \right\rangle \\ &\leq (1 - \rho \bar{\gamma}) \|x\|^2 \\ &= 1 - \rho \bar{\gamma}. \end{aligned} \quad (3.17)$$

From (3.15) and (3.17), we have

$$\left\| I - \rho \sum_{i=1}^N a_i A_i \right\| \leq 1 - \rho \bar{\gamma}. \quad (3.18)$$

Next, we show that $I - \rho \sum_{i=1}^N a_i A_i$ is a nonexpansive mapping. Let $x, y \in C$. Then using (3.18), we obtain

$$\begin{aligned} \left\| \left(I - \rho \sum_{i=1}^N a_i A_i \right) x - \left(I - \rho \sum_{i=1}^N a_i A_i \right) y \right\| &= \left\| \left(I - \rho \sum_{i=1}^N a_i A_i \right) (x - y) \right\| \\ &\leq \left\| I - \rho \sum_{i=1}^N a_i A_i \right\| \|x - y\| \\ &\leq (1 - \rho \bar{\gamma}) \|x - y\| \\ &\leq \|x - y\|. \end{aligned}$$

Hence, $I - \rho \sum_{i=1}^N a_i A_i$ is a nonexpansive mapping.

To prove (ii), we will prove that

$$\bigcap_{i=1}^N VI(C, A_i) \subseteq VI\left(C, \sum_{i=1}^N a_i A_i\right). \quad (3.19)$$

Let $x_0 \in \bigcap_{i=1}^N VI(C, A_i)$ and $a_i \in (0, 1)$ for every $i = 1, 2, \dots, N$. Thus we get

$$\langle y - x_0, a_i A_i x_0 \rangle \geq 0, \forall i = 1, 2, \dots, N \text{ and } y \in C. \quad (3.20)$$

Combining the inequalities (3.20), hence we obtain

$$\sum_{i=1}^N \langle y - x_0, a_i A_i x_0 \rangle = \left\langle y - x_0, \sum_{i=1}^N a_i A_i x_0 \right\rangle \geq 0, \text{ for all } y \in C.$$

This follows that $x_0 \in VI\left(C, \sum_{i=1}^N a_i A_i\right)$.

Therefore we have $\bigcap_{i=1}^N VI(C, A_i) \subseteq VI\left(C, \sum_{i=1}^N a_i A_i\right)$.

Next, we let $x_0 \in VI\left(C, \sum_{i=1}^N a_i A_i\right)$ and $x^* \in \bigcap_{i=1}^N VI(C, A_i)$. Then, we have

$$\left\langle y - x_0, \sum_{i=1}^N a_i A_i x_0 \right\rangle \geq 0, \forall y \in C. \quad (3.21)$$

From (3.19), we have $x^* \in VI\left(C, \sum_{i=1}^N a_i A_i\right)$. It implies that

$$\left\langle y - x^*, \sum_{i=1}^N a_i A_i x^* \right\rangle \geq 0, \forall y \in C. \quad (3.22)$$

From (3.21), (3.22) and $x^*, x_0 \in C$, we obtain

$$\left\langle x^* - x_0, \sum_{i=1}^N a_i A_i x_0 \right\rangle \geq 0 \quad (3.23)$$

and

$$\left\langle x_0 - x^*, \sum_{i=1}^N a_i A_i x^* \right\rangle \geq 0. \quad (3.24)$$

By summing up (3.23), (3.24) and Remark 2.34, we have

$$\begin{aligned} 0 &\leq \left\langle x_0 - x^*, \sum_{i=1}^N a_i A_i x^* - \sum_{i=1}^N a_i A_i x_0 \right\rangle \\ &= \sum_{i=1}^N a_i \langle x_0 - x^*, A_i x^* - A_i x_0 \rangle \\ &\leq - \sum_{i=1}^N a_i \gamma_i \|x_0 - x^*\|^2 \\ &\leq - \sum_{i=1}^N a_i \bar{\gamma} \|x_0 - x^*\|^2 \\ &= -\bar{\gamma} \|x_0 - x^*\|^2. \end{aligned}$$

It implies that $x_0 = x^*$.

Then, we can conclude that $x_0 \in \bigcap_{i=1}^N VI(C, A_i)$. Therefore

$$VI\left(C, \sum_{i=1}^N a_i A_i\right) \subseteq \bigcap_{i=1}^N VI(C, A_i).$$

Hence, we have

$$VI\left(C, \sum_{i=1}^N a_i A_i\right) = \bigcap_{i=1}^N VI(C, A_i).$$

□

Theorem 3.5. Let C be a nonempty closed convex subset of a real Hilbert space H . For $i = 1, 2, \dots, N$, let $F_i : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)'–(A4) and let $A_i : C \rightarrow H$ be a strongly positive linear bounded operator on H with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\dots,N} \gamma_i$. Let $\{T_i\}_{i=1}^\infty$ be an infinite family of κ_i -strictly pseudocontractive mappings of C into itself and let $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq \eta < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p, q] \subset (\kappa, 1)$ for all $j \in \mathbb{N}$. For every $n \in \mathbb{N}$, let S_n and S be the S-mappings generated by T_n, T_{n-1}, \dots, T_1 and $\sigma_n, \sigma_{n-1}, \dots, \sigma_1$ and T_n, T_{n-1}, \dots and $\sigma_n, \sigma_{n-1}, \dots$, respectively. Assume that $\mathcal{F} = \bigcap_{i=1}^N EP(F_i) \cap \bigcap_{i=1}^N VI(C, A_i) \cap \bigcap_{i=1}^\infty Fix(T_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be

generated by $x_1, u \in C$ and

$$\begin{cases} \sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n (\alpha_n u + (1 - \alpha_n) S_n x_n) + (1 - \beta_n) P_C \left(I - \rho_n \sum_{i=1}^N b_i A_i \right) u_n, \forall n \geq 1, \end{cases} \quad (3.25)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0, 1)$ and $0 \leq a_i, b_i \leq 1$ for every $i = 1, 2, \dots, N$, satisfying the following conditions:

- (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (ii) $0 < a \leq \beta_n \leq b < 1, \forall n \in \mathbb{N}$,
- (iii) $\lim_{n \rightarrow \infty} \rho_n = 0$,
- (iv) $0 < c \leq r_n \leq d < 1, \forall n \in \mathbb{N}$,
- (v) $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$,
- (vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$
 $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$ and $\sum_{n=1}^{\infty} \alpha_n^q < \infty$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_{\mathcal{F}}u$.

Proof. Since $\rho_n \rightarrow 0$ as $n \rightarrow \infty$, without loss of generality we may assume that $\rho_n < \frac{1}{\|A_i\|}, \forall n \in \mathbb{N}$ and $i = 1, 2, \dots, N$. The proof will be divided into five steps.

Step 1. We will show that $\{x_n\}$ is bounded.

Since $\sum_{i=1}^N a_i F_i$ satisfies (A1)-(A4) and

$$\sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C,$$

by Lemma 2.40 and Remark 3.3, we have $u_n = T_{r_n} x_n$ and $\text{Fix}(T_{r_n}) = \bigcap_{i=1}^N EP(F_i)$.

Let $z \in \mathcal{F}$. Since $z \in \bigcap_{i=1}^N VI(C, A_i)$, by Lemma 2.12 and Lemma 3.4, we have

$$VI\left(C, \sum_{i=1}^N b_i A_i\right) = \text{Fix}\left(P_C\left(I - \rho_n \sum_{i=1}^N b_i A_i\right)\right).$$

From Lemma 3.4 and nonexpansiveness of T_{r_n} , we have

$$\begin{aligned} & \|x_{n+1} - z\| \\ &= \left\| \beta_n (\alpha_n u + (1 - \alpha_n) S_n x_n) + (1 - \beta_n) P_C \left(I - \rho_n \sum_{i=1}^N b_i A_i \right) u_n - z \right\| \\ &= \left\| \beta_n (\alpha_n u + (1 - \alpha_n) S_n x_n - z) + (1 - \beta_n) \left(P_C \left(I - \rho_n \sum_{i=1}^N b_i A_i \right) u_n - z \right) \right\| \\ &\leq \beta_n \|\alpha_n (u - z) + (1 - \alpha_n) (S_n x_n - z)\| + (1 - \beta_n) \left\| P_C \left(I - \rho_n \sum_{i=1}^N b_i A_i \right) u_n - z \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \beta_n (\alpha_n \|u - z\| + (1 - \alpha_n) \|x_n - z\|) + (1 - \beta_n) \|u_n - z\| \\
&= \beta_n (\alpha_n \|u - z\| + (1 - \alpha_n) \|x_n - z\|) + (1 - \beta_n) \|T_{r_n} x_n - z\| \\
&\leq \beta_n (\alpha_n \|u - z\| + (1 - \alpha_n) \|x_n - z\|) + (1 - \beta_n) \|x_n - z\| \\
&\leq \max\{\|u - z\|, \|x_1 - z\|\} = M.
\end{aligned}$$

By induction, we have $\|x_n - z\| \leq M, \forall n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded and so $\{u_n\}$ is a bounded sequence.

Step 2. We will show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

Putting $D = \sum_{i=1}^N b_i A_i$, from the definition of x_n , we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|\beta_n (\alpha_n u + (1 - \alpha_n) S_n x_n) + (1 - \beta_n) P_C (I - \rho_n D) u_n - \\
&\quad (\beta_{n-1} (\alpha_{n-1} u + (1 - \alpha_{n-1}) S_{n-1} x_{n-1}) + (1 - \beta_{n-1}) P_C (I - \rho_{n-1} D) u_{n-1})\| \\
&= \|\beta_n (\alpha_n u + (1 - \alpha_n) S_n x_n) - \beta_n (\alpha_{n-1} u + (1 - \alpha_{n-1}) S_{n-1} x_{n-1}) \\
&\quad + \beta_n (\alpha_{n-1} u + (1 - \alpha_{n-1}) S_{n-1} x_{n-1}) - \beta_{n-1} (\alpha_{n-1} u + (1 - \alpha_{n-1}) S_{n-1} x_{n-1}) \\
&\quad + (1 - \beta_n) P_C (I - \rho_n D) u_n - (1 - \beta_n) P_C (I - \rho_n D) u_{n-1} \\
&\quad + (1 - \beta_n) P_C (I - \rho_n D) u_{n-1} - (1 - \beta_n) P_C (I - \rho_{n-1} D) u_{n-1} \\
&\quad + (1 - \beta_n) P_C (I - \rho_{n-1} D) u_{n-1} - (1 - \beta_{n-1}) P_C (I - \rho_{n-1} D) u_{n-1}\| \\
&\leq \beta_n \left[|\alpha_n - \alpha_{n-1}| \|u\| + \|(1 - \alpha_n) S_n x_n - (1 - \alpha_n) S_n x_{n-1} + (1 - \alpha_n) S_n x_{n-1} \right. \\
&\quad \left. - (1 - \alpha_n) S_{n-1} x_{n-1} + (1 - \alpha_n) S_{n-1} x_{n-1} - (1 - \alpha_{n-1}) S_{n-1} x_{n-1}\| \right] \\
&\quad + |\beta_n - \beta_{n-1}| \left[\alpha_{n-1} \|u\| + (1 - \alpha_{n-1}) \|S_{n-1} x_{n-1}\| \right] \\
&\quad + (1 - \beta_n) \|P_C (I - \rho_n D) u_n - P_C (I - \rho_n D) u_{n-1}\| \\
&\quad + (1 - \beta_n) \|P_C (I - \rho_n D) u_{n-1} - P_C (I - \rho_{n-1} D) u_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|P_C (I - \rho_{n-1} D) u_{n-1}\| \\
&\leq \beta_n \left[|\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|S_n x_n - S_n x_{n-1}\| \right. \\
&\quad \left. + (1 - \alpha_n) \|S_n x_{n-1} - S_{n-1} x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|S_{n-1} x_{n-1}\| \right] \\
&\quad + |\beta_n - \beta_{n-1}| \left[\alpha_{n-1} \|u\| + (1 - \alpha_{n-1}) \|S_{n-1} x_{n-1}\| \right] \\
&\quad + (1 - \beta_n) \|u_n - u_{n-1}\| + (1 - \beta_n) \|(I - \rho_n D) u_{n-1} - (I - \rho_{n-1} D) u_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|P_C (I - \rho_{n-1} D) u_{n-1}\| \\
&\leq \beta_n \left[|\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|x_n - x_{n-1}\| \right. \\
&\quad \left. + (1 - \alpha_n) \|S_n x_{n-1} - S_{n-1} x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|S_{n-1} x_{n-1}\| \right] \\
&\quad + |\beta_n - \beta_{n-1}| \left[\alpha_{n-1} \|u\| + (1 - \alpha_{n-1}) \|S_{n-1} x_{n-1}\| \right] \\
&\quad + (1 - \beta_n) \|u_n - u_{n-1}\| + (1 - \beta_n) |\rho_n - \rho_{n-1}| \|D u_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|P_C (I - \rho_{n-1} D) u_{n-1}\|. \tag{3.26}
\end{aligned}$$

By using the same method as in [29], we have

$$\|S_n x_{n-1} - S_{n-1} x_{n-1}\| \leq \alpha_1^n \frac{2}{1-\kappa} \|x_{n-1} - z\|.$$

Indeed, fix $k \in \mathbb{N}$, then, for every $n \in \mathbb{N}$ with $n \geq k$, we get

$$\begin{aligned} & \|U_{n+1,k} x_n - U_{n,k} x_n\|^2 \\ &= \|\alpha_1^k T_k U_{n+1,k+1} x_n + \alpha_2^k U_{n+1,k+1} x_n + \alpha_3^k x_n - \alpha_1^k T_k U_{n,k+1} x_n - \alpha_2^k U_{n,k+1} x_n - \alpha_3^k x_n\|^2 \\ &= \|\alpha_1^k (T_k U_{n+1,k+1} x_n - T_k U_{n,k+1} x_n) + \alpha_2^k (U_{n+1,k+1} x_n - U_{n,k+1} x_n)\|^2 \\ &\leq \alpha_1^k \|T_k U_{n+1,k+1} x_n - T_k U_{n,k+1} x_n\|^2 + \alpha_2^k \|U_{n+1,k+1} x_n - U_{n,k+1} x_n\|^2 \\ &\quad - \alpha_1^k \alpha_2^k \|(I - T_k) U_{n,k+1} x_n - (I - T_k) U_{n+1,k+1} x_n\|^2 \\ &\leq \alpha_1^k \left(\|U_{n+1,k+1} x_n - U_{n,k+1} x_n\|^2 + \kappa \|(I - T_k) U_{n,k+1} x_n - (I - T_k) U_{n+1,k+1} x_n\|^2 \right) \\ &\quad + \alpha_2^k \|U_{n+1,k+1} x_n - U_{n,k+1} x_n\|^2 - \alpha_1^k \alpha_2^k \|(I - T_k) U_{n,k+1} x_n - (I - T_k) U_{n+1,k+1} x_n\|^2 \\ &\leq (1 - \alpha_3^k) \|U_{n+1,k+1} x_n - U_{n,k+1} x_n\|^2 \\ &\quad \vdots \\ &\leq \prod_{j=k}^n (1 - \alpha_3^j) \|U_{n+1,n+1} x_n - U_{n,n+1} x_n\|^2 \\ &\leq \|U_{n+1,n+1} x_n - x_n\|^2 \\ &= \|\alpha_1^{n+1} T_{n+1} U_{n+1,n+2} x_n + \alpha_2^{n+1} U_{n+1,n+2} x_n + \alpha_3^{n+1} x_n - x_n\|^2 \\ &= \|\alpha_1^{n+1} T_{n+1} x_n + (1 - \alpha_1^{n+1}) x_n - x_n\|^2 \\ &\leq (\alpha_1^{n+1})^2 \|T_{n+1} x_n - x_n\|^2. \end{aligned}$$

This implies that

$$\|U_{n+1,k} x_n - U_{n,k} x_n\| \leq \alpha_1^{n+1} \|T_{n+1} x_n - x_n\|. \quad (3.27)$$

From (3.27), we derive

$$\begin{aligned} \|S_n x_{n-1} - S_{n-1} x_{n-1}\| &= \|U_{n,1} x_{n-1} - U_{n-1,1} x_{n-1}\| \\ &\leq \alpha_1^{n+1} \|T_n x_{n-1} - x_{n-1}\| \\ &\leq \alpha_1^{n+1} (\|T_n x_{n-1} - z\| + \|x_{n-1} - z\|) \\ &\leq \alpha_1^{n+1} \left(\frac{1+\kappa}{1-\kappa} \|x_{n-1} - z\| + \frac{1-\kappa}{1-\kappa} \|x_{n-1} - z\| \right) \\ &\leq \alpha_1^n \frac{2}{1-\kappa} \|x_{n-1} - z\|. \end{aligned} \quad (3.28)$$

Since $u_n = T_{r_n} x_n$, by utilizing the definition of T_{r_n} , we obtain

$$\sum_{i=1}^N a_i F_i(T_{r_n} x_n, y) + \frac{1}{r_n} \langle y - T_{r_n} x_n, T_{r_n} x_n - x_n \rangle \geq 0, \forall y \in C, \quad (3.29)$$

and

$$\sum_{i=1}^N a_i F_i(T_{r_{n+1}} x_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - T_{r_{n+1}} x_{n+1}, T_{r_{n+1}} x_{n+1} - x_{n+1} \rangle \geq 0, \forall y \in C. \quad (3.30)$$

From (3.29) and (3.30), it follows that

$$\sum_{i=1}^N a_i F_i (T_{r_n} x_n, T_{r_{n+1}} x_{n+1}) + \frac{1}{r_n} \langle T_{r_{n+1}} x_{n+1} - T_{r_n} x_n, T_{r_n} x_n - x_n \rangle \geq 0, \quad (3.31)$$

and

$$\sum_{i=1}^N a_i F_i (T_{r_{n+1}} x_{n+1}, T_{r_n} x_n) + \frac{1}{r_{n+1}} \langle T_{r_n} x_n - T_{r_{n+1}} x_{n+1}, T_{r_{n+1}} x_{n+1} - x_{n+1} \rangle \geq 0. \quad (3.32)$$

From (3.31) and (3.32) and the fact that $\sum_{i=1}^N a_i F_i$ satisfies (A2), we have

$$\frac{1}{r_n} \langle T_{r_{n+1}} x_{n+1} - T_{r_n} x_n, T_{r_n} x_n - x_n \rangle + \frac{1}{r_{n+1}} \langle T_{r_n} x_n - T_{r_{n+1}} x_{n+1}, T_{r_{n+1}} x_{n+1} - x_{n+1} \rangle \geq 0,$$

which implies that

$$\left\langle T_{r_n} x_n - T_{r_{n+1}} x_{n+1}, \frac{T_{r_{n+1}} x_{n+1} - x_{n+1}}{r_{n+1}} - \frac{T_{r_n} x_n - x_n}{r_n} \right\rangle \geq 0.$$

It follows that

$$\left\langle T_{r_{n+1}} x_{n+1} - T_{r_n} x_n, T_{r_n} x_n - T_{r_{n+1}} x_{n+1} + T_{r_{n+1}} x_{n+1} - x_n - \frac{r_n}{r_{n+1}} (T_{r_{n+1}} x_{n+1} - x_{n+1}) \right\rangle \geq 0. \quad (3.33)$$

From (3.33), we obtain

$$\begin{aligned} & \|T_{r_{n+1}} x_{n+1} - T_{r_n} x_n\|^2 \\ & \leq \left\langle T_{r_{n+1}} x_{n+1} - T_{r_n} x_n, T_{r_{n+1}} x_{n+1} - x_n - \frac{r_n}{r_{n+1}} (T_{r_{n+1}} x_{n+1} - x_{n+1}) \right\rangle \\ & = \left\langle T_{r_{n+1}} x_{n+1} - T_{r_n} x_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right) (T_{r_{n+1}} x_{n+1} - x_{n+1}) \right\rangle \\ & \leq \|T_{r_{n+1}} x_{n+1} - T_{r_n} x_n\| \left\| x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right) (T_{r_{n+1}} x_{n+1} - x_{n+1}) \right\| \\ & \leq \|T_{r_{n+1}} x_{n+1} - T_{r_n} x_n\| \left[\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|T_{r_{n+1}} x_{n+1} - x_{n+1}\| \right] \\ & = \|T_{r_{n+1}} x_{n+1} - T_{r_n} x_n\| \left[\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|T_{r_{n+1}} x_{n+1} - x_{n+1}\| \right] \\ & \leq \|T_{r_{n+1}} x_{n+1} - T_{r_n} x_n\| \left[\|x_{n+1} - x_n\| + \frac{1}{d} |r_{n+1} - r_n| \|T_{r_{n+1}} x_{n+1} - x_{n+1}\| \right], \end{aligned}$$

which follows that

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{d} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \quad (3.34)$$

From (3.34), we have

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{d} |r_n - r_{n-1}| \|u_n - x_n\|. \quad (3.35)$$

By substituting (3.28) and (3.35) into (3.26), then we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \beta_n \left[|\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|x_n - x_{n-1}\| \right. \\
&\quad \left. + (1 - \alpha_n) \alpha_1^n \frac{2}{1 - \kappa} \|x_{n-1} - z\| + |\alpha_n - \alpha_{n-1}| \|S_{n-1}x_{n-1}\| \right] \\
&\quad + |\beta_n - \beta_{n-1}| \left[\alpha_{n-1} \|u\| + (1 - \alpha_{n-1}) \|S_{n-1}x_{n-1}\| \right] \\
&\quad + (1 - \beta_n) \left[\|x_n - x_{n-1}\| + \frac{1}{d} |r_n - r_{n-1}| \|u_n - x_n\| \right] \\
&\quad + (1 - \beta_n) |\rho_n - \rho_{n-1}| \|Du_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|P_C(I - \rho_{n-1}D)u_{n-1}\| \\
&\leq \beta_n (1 - \alpha_n) \|x_n - x_{n-1}\| + (1 - \beta_n) \|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|u\| + \alpha_1^n \frac{2}{1 - \kappa} \|x_{n-1} - z\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|S_{n-1}x_{n-1}\| + |\beta_n - \beta_{n-1}| \left[\|u\| \right. \\
&\quad \left. + \|S_{n-1}x_{n-1}\| \right] + \frac{1}{d} |r_n - r_{n-1}| \|u_n - x_n\| \\
&\quad + |\rho_n - \rho_{n-1}| \|Du_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|P_C(I - \rho_{n-1}D)u_{n-1}\| \\
&\leq (1 - \alpha_n \beta_n) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K + \alpha_1^n \frac{2}{1 - \kappa} K \\
&\quad + |\alpha_n - \alpha_{n-1}| K + |\beta_n - \beta_{n-1}| 2K + \frac{1}{d} |r_n - r_{n-1}| K \\
&\quad + |\rho_n - \rho_{n-1}| K + |\beta_n - \beta_{n-1}| K \\
&= (1 - \alpha_n \beta_n) \|x_n - x_{n-1}\| + 2K |\alpha_n - \alpha_{n-1}| + \frac{2K}{1 - \kappa} \alpha_1^n \\
&\quad + 3K |\beta_n - \beta_{n-1}| + \frac{K}{d} |r_n - r_{n-1}| + K |\rho_n - \rho_{n-1}|, \tag{3.36}
\end{aligned}$$

where $K = \max_{n \in \mathbb{N}} \{\|u\|, \|x_n - z\|, \|S_n x_n\|, \|u_n - x_n\|, \|Du_n\|, \|P_C(I - \rho_n D)u_n\|\}$. From (3.36), the conditions (i), (ii), (vi) and Lemma 2.13, we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.37}$$

Step 3. We will show that $\lim_{n \rightarrow \infty} \|u_n - x_n\| = \lim_{n \rightarrow \infty} \|P_C(I - \rho_n D)x_n - x_n\| = \lim_{n \rightarrow \infty} \|S_n x_n - x_n\| = 0$, where $D = \sum_{i=1}^N b_i A_i$.

To show this, let $z \in \mathcal{F}$. Since $u_n = T_{r_n} x_n$ and T_{r_n} is firmly nonexpansive mapping, then we obtain

$$\begin{aligned}
\|z - T_{r_n} x_n\|^2 &= \|T_{r_n} z - T_{r_n} x_n\|^2 \\
&\leq \langle T_{r_n} z - T_{r_n} x_n, z - x_n \rangle \\
&= \frac{1}{2} \left(\|T_{r_n} x_n - z\|^2 + \|x_n - z\|^2 - \|T_{r_n} x_n - x_n\|^2 \right),
\end{aligned}$$

which follows that

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{3.38}$$

By nonexpansiveness of $P_C(I - \rho_n D)$, (3.38) and the definition of x_n , then we have

$$\begin{aligned}
& \|x_{n+1} - z\|^2 \\
&= \|\beta_n(\alpha_n u + (1 - \alpha_n)S_n x_n) + (1 - \beta_n)P_C(I - \rho_n D)u_n - z\|^2 \\
&= \|\beta_n(\alpha_n(u - z) + (1 - \alpha_n)(S_n x_n - z)) + (1 - \beta_n)(P_C(I - \rho_n D)u_n - z)\|^2 \\
&\leq \beta_n \|\alpha_n(u - z) + (1 - \alpha_n)(S_n x_n - z)\|^2 + (1 - \beta_n) \|P_C(I - \rho_n D)u_n - z\|^2 \\
&\leq \beta_n \left(\alpha_n \|u - z\|^2 + (1 - \alpha_n) \|S_n x_n - z\|^2 \right) + (1 - \beta_n) \|u_n - z\|^2 \\
&\leq \beta_n \left(\alpha_n \|u - z\|^2 + (1 - \alpha_n) \|x_n - z\|^2 \right) + (1 - \beta_n) \left(\|x_n - z\|^2 - \|u_n - x_n\|^2 \right) \\
&\leq \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - (1 - \beta_n) \|u_n - x_n\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
(1 - \beta_n) \|u_n - x_n\|^2 &\leq \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\leq \alpha_n \|u - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\|. \quad (3.39)
\end{aligned}$$

By (3.37), (3.39), the conditions (i) and (ii), then we have

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \quad (3.40)$$

Put $w_n = \alpha_n u + (1 - \alpha_n)S_n x_n$. By the definition of x_n and $z \in \mathcal{F}$, we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\beta_n w_n + (1 - \beta_n)P_C(I - \rho_n D)u_n - z\|^2 \\
&= \|\beta_n(w_n - z) + (1 - \beta_n)(P_C(I - \rho_n D)u_n - z)\|^2 \\
&= \beta_n \|w_n - z\|^2 + (1 - \beta_n) \|P_C(I - \rho_n D)u_n - z\|^2 \\
&\quad - \beta_n(1 - \beta_n) \|w_n - P_C(I - \rho_n D)u_n\|^2 \\
&\leq \beta_n \|\alpha_n u + (1 - \alpha_n)S_n x_n - z\|^2 + (1 - \beta_n) \|u_n - z\|^2 \\
&\quad - \beta_n(1 - \beta_n) \|w_n - P_C(I - \rho_n D)u_n\|^2 \\
&\leq \beta_n \left(\alpha_n \|u - z\|^2 + (1 - \alpha_n) \|S_n x_n - z\|^2 \right) + (1 - \beta_n) \|x_n - z\|^2 \\
&\quad - \beta_n(1 - \beta_n) \|w_n - P_C(I - \rho_n D)u_n\|^2 \\
&\leq \beta_n \left(\alpha_n \|u - z\|^2 + (1 - \alpha_n) \|x_n - z\|^2 \right) + (1 - \beta_n) \|x_n - z\|^2 \\
&\quad - \beta_n(1 - \beta_n) \|w_n - P_C(I - \rho_n D)u_n\|^2 \\
&\leq \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \beta_n(1 - \beta_n) \|w_n - P_C(I - \rho_n D)u_n\|^2,
\end{aligned}$$

which follows that

$$\begin{aligned}
& \beta_n(1 - \beta_n) \|w_n - P_C(I - \rho_n D)u_n\|^2 \\
&\leq \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\leq \alpha_n \|u - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\|. \quad (3.41)
\end{aligned}$$

By (3.37) and the conditions (i) and (ii), we have

$$\lim_{n \rightarrow \infty} \|w_n - P_C(I - \rho_n D)u_n\| = 0. \quad (3.42)$$

By definition of x_n , we obtain

$$x_{n+1} - P_C(I - \rho_n D)u_n = \beta_n(w_n - P_C(I - \rho_n D)u_n). \quad (3.43)$$

By (3.43) and the definition of x_n , we have

$$\begin{aligned} \|x_n - P_C(I - \rho_n D)x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - P_C(I - \rho_n D)u_n\| \\ &\quad + \|P_C(I - \rho_n D)u_n - P_C(I - \rho_n D)x_n\| \\ &\leq \|x_n - x_{n+1}\| + \beta_n \|w_n - P_C(I - \rho_n D)u_n\| \\ &\quad + \|u_n - x_n\|. \end{aligned}$$

From (3.37), (3.40) and (3.42), we have

$$\lim_{n \rightarrow \infty} \|x_n - P_C(I - \rho_n D)x_n\| = 0. \quad (3.44)$$

Since

$$\begin{aligned} \|x_n - P_C(I - \rho_n D)u_n\| &\leq \|x_n - P_C(I - \rho_n D)x_n\| + \|P_C(I - \rho_n D)x_n - P_C(I - \rho_n D)u_n\| \\ &\leq \|x_n - P_C(I - \rho_n D)x_n\| + \|x_n - u_n\|, \end{aligned}$$

by using (3.40) and (3.44), we have

$$\lim_{n \rightarrow \infty} \|x_n - P_C(I - \rho_n D)u_n\| = 0. \quad (3.45)$$

By the definition of x_n , we obtain

$$\begin{aligned} x_{n+1} - x_n &= \beta_n(\alpha_n u + (1 - \alpha_n)S_n x_n) + (1 - \beta_n)P_C(I - \rho_n D)u_n - x_n \\ &= \beta_n(\alpha_n u + (1 - \alpha_n)S_n x_n - x_n) + (1 - \beta_n)(P_C(I - \rho_n D)u_n - x_n) \\ &= \beta_n(\alpha_n(u - x_n) + (1 - \alpha_n)(S_n x_n - x_n)) + (1 - \beta_n)(P_C(I - \rho_n D)u_n - x_n) \\ &= \alpha_n \beta_n(u - x_n) + \beta_n(1 - \alpha_n)(S_n x_n - x_n) + (1 - \beta_n)(P_C(I - \rho_n D)u_n - x_n), \end{aligned}$$

which follows that

$$\begin{aligned} &\beta_n(1 - \alpha_n)\|S_n x_n - x_n\| \\ &\leq \alpha_n \beta_n \|u - x_n\| + \|x_{n+1} - x_n\| + (1 - \beta_n)\|P_C(I - \rho_n D)u_n - x_n\|. \end{aligned}$$

From (3.37), (3.45), the conditions (i) and (ii), we have

$$\lim_{n \rightarrow \infty} \|S_n x_n - x_n\| = 0. \quad (3.46)$$

Step 4. We will show that $\limsup_{n \rightarrow \infty} \langle u - z, x_n - z \rangle \leq 0$, where $z = P_{\mathcal{F}}u$.

To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle u - z, x_n - z \rangle = \lim_{k \rightarrow \infty} \langle u - z, x_{n_k} - z \rangle. \quad (3.47)$$

Without loss of generality, we can assume that $x_{n_k} \rightarrow \omega$ as $k \rightarrow \infty$ where $\omega \in C$. From (3.40), we obtain $u_{n_k} \rightarrow \omega$ as $k \rightarrow \infty$.

Assume that $\omega \notin \bigcap_{i=1}^N VI(C, A_i)$. Since $\bigcap_{i=1}^N VI(C, A_i) = \text{Fix}(P_C(I - \rho_{n_k}D))$, we have $\omega \neq P_C(I - \rho_{n_k}D)\omega$, where $D = \sum_{i=1}^N b_i A_i$.

By nonexpansiveness of $P_C(I - \rho_{n_k}D)$, (3.44) and Opial's condition, we obtain

$$\begin{aligned} \liminf_{k \rightarrow \infty} \|x_{n_k} - \omega\| &< \liminf_{k \rightarrow \infty} \|x_{n_k} - P_C(I - \rho_{n_k}D)\omega\| \\ &\leq \liminf_{k \rightarrow \infty} \left(\|x_{n_k} - P_C(I - \rho_{n_k}D)x_{n_k}\| \right. \\ &\quad \left. + \|P_C(I - \rho_{n_k}D)x_{n_k} - P_C(I - \rho_{n_k}D)\omega\| \right) \\ &\leq \liminf_{k \rightarrow \infty} \|x_{n_k} - \omega\|. \end{aligned}$$

This is a contradiction. Then, we have

$$\omega \in \bigcap_{i=1}^N VI(C, A_i). \quad (3.48)$$

Next, we will show that $\omega \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$.

By Lemma 2.25, we have $\text{Fix}(S) = \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$. Assume that $\omega \neq S\omega$. Using Opial's condition, (3.46) and Remark 2.24, then we obtain

$$\begin{aligned} \liminf_{k \rightarrow \infty} \|x_{n_k} - \omega\| &< \liminf_{k \rightarrow \infty} \|x_{n_k} - S\omega\| \\ &\leq \liminf_{k \rightarrow \infty} \left(\|x_{n_k} - S_{n_k}x_{n_k}\| + \|S_{n_k}x_{n_k} - S_{n_k}\omega\| + \|S_{n_k}\omega - S\omega\| \right) \\ &\leq \liminf_{k \rightarrow \infty} \|x_{n_k} - \omega\|. \end{aligned}$$

This is a contradiction. Then we have

$$\omega \in \text{Fix}(S) = \bigcap_{i=1}^{\infty} \text{Fix}(T_i). \quad (3.49)$$

Next, we will show that $\omega \in \bigcap_{i=1}^N EP(F_i)$.

Since

$$\sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C,$$

and $\sum_{i=1}^N a_i F_i$ satisfies the conditions (A1)-(A4), we obtain

$$\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq \sum_{i=1}^N a_i F_i(y, u_n), \forall y \in C.$$

In particular, it follows that

$$\left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle \geq \sum_{i=1}^N a_i F_i(y, u_{n_k}), \forall y \in C. \quad (3.50)$$

From (3.40), (3.50) and (A4), we have

$$\sum_{i=1}^N a_i F_i(y, \omega) \leq 0, \forall y \in C. \quad (3.51)$$

Put $y_t := ty + (1-t)\omega$, $t \in (0, 1]$, we have $y_t \in C$. By using (A1)', (A4) and (3.51), we have

$$\begin{aligned} 0 &= \sum_{i=1}^N a_i F_i(y_t, y_t) \\ &= \sum_{i=1}^N a_i F_i(y_t, ty + (1-t)\omega) \\ &\leq t \sum_{i=1}^N a_i F_i(y_t, y) + (1-t) \sum_{i=1}^N a_i F_i(y_t, \omega) \\ &\leq t \sum_{i=1}^N a_i F_i(y_t, y). \end{aligned}$$

It implies that

$$\sum_{i=1}^N a_i F_i(ty + (1-t)\omega, y) \geq 0, \forall t \in (0, 1] \text{ and } \forall y \in C. \quad (3.52)$$

From (3.52), taking $t \rightarrow 0^+$ and using (A3), we can conclude that

$$0 \leq \sum_{i=1}^N a_i F_i(\omega, y), \forall y \in C.$$

Therefore, $\omega \in EP\left(\sum_{i=1}^N a_i F_i\right)$. By Lemma 3.1, we obtain $EP\left(\sum_{i=1}^N a_i F_i\right) = \bigcap_{i=1}^N EP(F_i)$.

It follows that

$$\omega \in \bigcap_{i=1}^N EP(F_i). \quad (3.53)$$

From (3.48), (3.49) and (3.53), we can deduce that $\omega \in \mathcal{F}$.

Since $x_{n_k} \rightarrow \omega$ and $\omega \in \mathcal{F}$, then, by Lemma 2.11, we can conclude that

$$\limsup_{n \rightarrow \infty} \langle u - z, x_n - z \rangle = \lim_{k \rightarrow \infty} \langle u - z, x_{n_k} - z \rangle = \langle u - z, \omega - z \rangle \leq 0. \quad (3.54)$$

Step 5. Finally, we will show that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}u$.

By nonexpansiveness of S_n and $P_C(I - \rho_n D)$, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\beta_n(\alpha_n u + (1 - \alpha_n)S_n x_n) + (1 - \beta_n)P_C(I - \rho_n D)u_n - z\|^2 \\ &= \|\beta_n(\alpha_n(u - z) + (1 - \alpha_n)(S_n x_n - z)) + (1 - \beta_n)(P_C(I - \rho_n D)u_n - z)\|^2 \\ &= \|\alpha_n \beta_n(u - z) + \beta_n(1 - \alpha_n)(S_n x_n - z) + (1 - \beta_n)(P_C(I - \rho_n D)u_n - z)\|^2 \\ &\leq \|\beta_n(1 - \alpha_n)(S_n x_n - z) + (1 - \beta_n)(P_C(I - \rho_n D)u_n - z)\|^2 \\ &\quad + 2\alpha_n \beta_n \langle u - z, x_{n+1} - z \rangle \\ &\leq (\beta_n(1 - \alpha_n)\|x_n - z\| + (1 - \beta_n)\|u_n - z\|)^2 + 2\alpha_n \beta_n \langle u - z, x_{n+1} - z \rangle \\ &\leq (\beta_n(1 - \alpha_n)\|x_n - z\| + (1 - \beta_n)\|x_n - z\|)^2 + 2\alpha_n \beta_n \langle u - z, x_{n+1} - z \rangle \\ &= (1 - \alpha_n \beta_n)^2 \|x_n - z\|^2 + 2\alpha_n \beta_n \langle u - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n \beta_n) \|x_n - z\|^2 + 2\alpha_n \beta_n \langle u - z, x_{n+1} - z \rangle. \end{aligned}$$

From the condition (ii), we have $\alpha_n \beta_n \leq \alpha_n \beta_n, \forall n \in \mathbb{R}$. Using the condition (i) and the comparison test, we deduce that $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$. From (3.54), the conditions (i), (ii)

and Lemma 2.13, we can conclude that $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}u$. By (3.40), we have $\{u_n\}$ converges strongly to $z = P_{\mathcal{F}}u$. This completes the proof. \square

Next, we apply our main theorem to prove strong convergence theorems for solving equilibrium problem, variational inequality problem and fixed point problem.

Corollary 3.6. Let C be a nonempty closed convex subset of a real Hilbert space H . For $i = 1, 2, \dots, N$, let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying $(A1)' - (A4)$ and let $A : C \rightarrow H$ be a strongly positive linear bounded operator on H with coefficient $\gamma > 0$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of κ_i -strictly pseudo-contractive mappings of C into itself and let $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq \eta < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p, q] \subset (\kappa, 1)$ for all $j \in \mathbb{N}$. For every $n \in \mathbb{N}$, let S_n and S be the S-mappings generated by T_n, T_{n-1}, \dots, T_1 and $\sigma_n, \sigma_{n-1}, \dots, \sigma_1$ and T_n, T_{n-1}, \dots and $\sigma_n, \sigma_{n-1}, \dots$, respectively. Assume that $\mathcal{F} = EP(F) \cap VI(C, A) \cap \bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n (\alpha_n u + (1 - \alpha_n) S_n x_n) + (1 - \beta_n) P_C (I - \rho_n A) u_n, \forall n \geq 1, \end{cases} \quad (3.55)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0, 1)$ and $0 \leq a_i, b_i \leq 1$ for every $i = 1, 2, \dots, N$, satisfying the following conditions:

- (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (ii) $0 < a \leq \beta_n \leq b < 1, \forall n \in \mathbb{N}$,
- (iii) $\lim_{n \rightarrow \infty} \rho_n = 0$,
- (iv) $0 < c \leq r_n \leq d < 1, \forall n \in \mathbb{N}$,
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$
 $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$ and $\sum_{n=1}^{\infty} \alpha_n^q < \infty$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_{\mathcal{F}}u$.

Proof. Taking $F = F_i$ and $A = A_i, \forall i = 1, 2, \dots, N$, in Theorem 3.5, then we obtain the desired conclusion. \square

Corollary 3.7. Let C be a nonempty closed convex subset of a real Hilbert space H . For every $i = 1, 2, \dots, N$, let $A_i : C \rightarrow H$ be a strongly positive linear bounded operator on H with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min \gamma_i$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of κ_i -strictly pseudo-contractive mappings of C into itself and let $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq \eta < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p, q] \subset (\kappa, 1)$ for all $j \in \mathbb{N}$. For every $n \in \mathbb{N}$, let S_n and S be the S-mappings generated by T_n, T_{n-1}, \dots, T_1 and $\sigma_n, \sigma_{n-1}, \dots, \sigma_1$ and T_n, T_{n-1}, \dots and $\sigma_n, \sigma_{n-1}, \dots$, respectively. Assume that $\mathcal{F} =$

$\bigcap_{i=1}^N VI(C, A_i) \cap \bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and

$$x_{n+1} = \beta_n (\alpha_n u + (1 - \alpha_n) S_n x_n) + (1 - \beta_n) P_C \left(I - \rho_n \sum_{i=1}^N b_i A_i \right) x_n, \forall n \geq 1, \quad (3.56)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0, 1)$ satisfying the following conditions:

- (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (ii) $0 < a \leq \beta_n \leq b < 1, \forall n \in \mathbb{N}$,
- (iii) $\lim_{n \rightarrow \infty} \rho_n = 0$,
- (iv) $0 < c \leq r_n \leq d < 1, \forall n \in \mathbb{N}$,
- (v) $\sum_{i=1}^N b_i = 1$,
- (vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$
 $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$ and $\sum_{n=1}^{\infty} \alpha_n^n < \infty$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_{\mathcal{F}} u$.

Proof. Putting $F_i = 0, \forall i = 1, 2, \dots, N$ in Theorem 3.5, then we have $u_n = P_C x_n = x_n, \forall n \in \mathbb{N}$ and the result of Corollary 3.7 can be obtained. \square

Corollary 3.8. Let C be a nonempty closed convex subset of a real Hilbert space H . For $i = 1, 2, \dots, N$, let $F_i : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)' - (A4). Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself and let $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j + \alpha_2^j \leq \eta < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p, q] \subset (0, 1)$ for all $j \in \mathbb{N}$. For every $n \in \mathbb{N}$, let S_n and S be the S -mappings generated by T_n, T_{n-1}, \dots, T_1 and $\sigma_n, \sigma_{n-1}, \dots, \sigma_1$ and T_n, T_{n-1}, \dots and $\sigma_n, \sigma_{n-1}, \dots$, respectively. Assume that $\mathcal{F} = \bigcap_{i=1}^N EP(F_i) \cap \bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and

$$\begin{cases} \sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n (\alpha_n u + (1 - \alpha_n) S_n x_n) + (1 - \beta_n) P_C u_n, \forall n \geq 1, \end{cases} \quad (3.57)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0, 1)$ and $0 \leq a_i \leq 1$ for every $i = 1, 2, \dots, N$, satisfying the following conditions:

- (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (ii) $0 < a \leq \beta_n \leq b < 1, \forall n \in \mathbb{N}$,
- (iii) $\lim_{n \rightarrow \infty} \rho_n = 0$,
- (iv) $0 < c \leq r_n \leq d < 1, \forall n \in \mathbb{N}$,
- (v) $\sum_{i=1}^N a_i = 1$,

$$(vi) \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \\ \sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty \text{ and } \sum_{n=1}^{\infty} \alpha_1^n < \infty.$$

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_{\mathcal{F}}u$.

Proof. Putting $A_i = 0, \forall i = 1, 2, \dots, N$ and $\kappa_i = 0$ in Theorem 3.5, then we obtain T_i is a nonexpansive mapping and Corollary 3.8. \square

3.2 Convergence analysis for a finite family of equilibrium problems and a nonspreading mapping

We now prove the lemma to show some properties of a nonspreading mapping T in a Hilbert space H as follows:

Lemma 3.9. Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \rightarrow C$ be a nonspreading mapping with $Fix(T) \neq \emptyset$. Then there hold the following statement:

- (i) $Fix(T) = VI(C, I - T)$;
- (ii) For every $u \in C$ and $v \in Fix(T)$,

$$\|P_C(I - \lambda(I - T))u - v\| \leq \|u - v\|, \text{ where } \lambda \in (0, 1).$$

Proof. To prove (i), let $x^* \in Fix(T)$. Then $x^* = Tx^*$. Since

$$\langle v - x^*, (I - T)x^* \rangle = 0, \forall v \in C,$$

then $x^* \in VI(C, I - T)$, which follows that $Fix(T) \subseteq VI(C, I - T)$.

Next, we show $VI(C, I - T) \subseteq Fix(T)$.

Let $\tilde{x} \in VI(C, I - T)$. This implies that

$$\langle v - \tilde{x}, (I - T)\tilde{x} \rangle \geq 0, \forall v \in C. \quad (3.58)$$

Let $x^* \in Fix(T)$. Then, by (2.4), we obtain

$$\|T\tilde{x} - Tx^*\|^2 \leq \|\tilde{x} - x^*\|^2 + 2\langle \tilde{x} - T\tilde{x}, x^* - Tx^* \rangle = \|\tilde{x} - x^*\|^2. \quad (3.59)$$

Observe that

$$\|T\tilde{x} - x^*\|^2 = \|\tilde{x} - x^* - (I - T)\tilde{x}\|^2 \\ = \|\tilde{x} - x^*\|^2 - 2\langle \tilde{x} - x^*, (I - T)\tilde{x} \rangle + \|(I - T)\tilde{x}\|^2. \quad (3.60)$$

From (3.58), (3.59) and (3.60), we get

$$\|(I - T)\tilde{x}\|^2 \leq 2\langle \tilde{x} - x^*, (I - T)\tilde{x} \rangle \leq 0,$$

which yields that $\tilde{x} \in Fix(T)$. Therefore $VI(C, I - T) \subseteq Fix(T)$.

To prove (ii), let $u \in C$ and $v \in \text{Fix}(T)$. Since T is a nonspreading mapping and (2.4), we get

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 + 2\langle u - Tu, v - Tv \rangle = \|u - v\|^2. \quad (3.61)$$

Thus we have

$$\begin{aligned} \|Tu - v\|^2 &= \|u - v - (I - T)u\|^2 \\ &= \|u - v\|^2 - 2\langle u - v, (I - T)u \rangle + \|(I - T)u\|^2. \end{aligned} \quad (3.62)$$

From (3.61) and (3.62), we obtain

$$\|(I - T)u\|^2 \leq 2\langle u - v, (I - T)u \rangle. \quad (3.63)$$

From (i) and Lemma 2.12, we have

$$v \in \text{Fix}(T) = VI(C, I - T) = \text{Fix}(P_C(I - \lambda(I - T))). \quad (3.64)$$

By the nonexpansiveness of P_C , (3.63) and (3.64), we get

$$\begin{aligned} \|P_C(I - \lambda(I - T))u - v\|^2 &= \|P_C(I - \lambda(I - T))u - P_C(I - \lambda(I - T))v\|^2 \\ &\leq \|(I - \lambda(I - T))u - (I - \lambda(I - T))v\|^2 \\ &= \|(u - v) - \lambda((I - T)u - (I - T)v)\|^2 \\ &= \|(u - v) - \lambda(I - T)u\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle u - v, (I - T)u \rangle + \lambda^2\|(I - T)u\|^2 \\ &\leq \|u - v\|^2 - \lambda\|(I - T)u\|^2 + \lambda^2\|(I - T)u\|^2 \\ &= \|u - v\|^2 - \lambda(1 - \lambda)\|(I - T)u\|^2 \\ &\leq \|u - v\|^2, \end{aligned}$$

which implies that $\|P_C(I - \lambda(I - T))u - v\| \leq \|u - v\|$. □

Remark 3.10. By Lemma 2.12 and Lemma 3.9, we can conclude that

$$\text{Fix}(T) = VI(C, I - T) = \text{Fix}(P_C(I - \lambda(I - T))), \forall \lambda > 0.$$

Theorem 3.11. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be an α -contractive mapping on H and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $T : C \rightarrow C$ be a nonspreading mapping. For every $i = 1, 2, \dots, N$, let $\Phi_i : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)'-(A4) with $\Omega := \text{Fix}(T) \cap \bigcap_{i=1}^N EP(\Phi_i) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} \sum_{i=1}^N a_i \Phi_i(v_n, y) + \frac{1}{\varphi_n} \langle y - v_n, v_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n = \theta_n P_C x_n + (1 - \theta_n) v_n, \\ x_{n+1} = \delta_n \gamma f(x_n) + (I - \delta_n A) P_C (I - \psi_n(I - T)) y_n, \forall n \in \mathbb{N}, \end{cases} \quad (3.65)$$

where $\{\delta_n\}, \{\theta_n\}, \{\varphi_n\}, \{\psi_n\} \subset (0, 1)$, $0 < a_i < 1$, for all $i = 1, 2, \dots, N$. Suppose the conditions (i)-(vi) hold.

- (i) $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$;
- (ii) $0 < \tau \leq \theta_n \leq \nu < 1$, for some $\tau, \nu > 0$;
- (iii) $\sum_{n=1}^{\infty} \psi_n < \infty$;
- (iv) $0 < \epsilon \leq \varphi_n \leq \eta < 1$, for some $\epsilon, \eta > 0$;
- (v) $\sum_{n=1}^N a_i = 1$;
- (vi) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$,
 $\sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$, $\sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ converge strongly to $q = P_{\Omega}(I - A + \gamma f)q$.

Proof. The proof of this theorem is separated into five steps.

Step 1. Claim that $\{x_n\}$ is a bounded sequence.

Since $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, without loss of generality, we assume $\delta_n < \frac{1}{\|A\|}$, for every $n \in \mathbb{N}$.

Since $\sum_{i=1}^N a_i \Phi_i$ satisfies (A1)-(A4) and

$$\sum_{i=1}^N a_i \Phi_i(v_n, y) + \frac{1}{\varphi_n} \langle y - v_n, v_n - x_n \rangle \geq 0, \forall y \in C,$$

by Lemma 2.40 and Remark 3.3, we have $v_n = T_{\varphi_n} x_n$ and $Fix(T_{\varphi_n}) = \bigcap_{i=1}^N EP(\Phi_i)$.

From Lemma 2.12 and Lemma 3.9(i), we obtain

$$Fix(T) = Fix(P_C(I - \psi_n(I - T))).$$

Let $z \in \Omega$. By nonexpansiveness of P_C and T_{φ_n} , we have

$$\|y_n - z\| \leq \theta_n \|P_C x_n - z\| + (1 - \theta_n) \|T_{\varphi_n} x_n - z\| \leq \|x_n - z\|. \quad (3.66)$$

From Lemma 2.35, Lemma 3.9(ii) and (3.66), we obtain

$$\begin{aligned} & \|x_{n+1} - z\| \\ & \leq \delta_n \|\gamma f(x_n) - Az\| + \|I - \delta_n A\| \|P_C(I - \psi_n(I - T))y_n - z\| \\ & \leq \delta_n \gamma \|f(x_n) - f(z)\| + \delta_n \|\gamma f(z) - Az\| + (1 - \delta_n \bar{\gamma}) \|y_n - z\| \\ & \leq \delta_n \gamma \alpha \|x_n - z\| + \delta_n \|\gamma f(z) - Az\| + (1 - \delta_n \bar{\gamma}) \|x_n - z\| \\ & = (1 - \delta_n(\bar{\gamma} - \gamma \alpha)) \|x_n - z\| + \delta_n \|\gamma f(z) - Az\| \\ & \leq \max \left\{ \|x_1 - z\|, \frac{\|\gamma f(z) - Az\|}{\bar{\gamma} - \gamma \alpha} \right\}. \end{aligned}$$

By induction, we obtain $\|x_n - z\| \leq \max \left\{ \|x_1 - z\|, \frac{\|\gamma f(z) - Az\|}{\bar{\gamma} - \gamma \alpha} \right\}$, $\forall n \in \mathbb{N}$. It yields that $\{x_n\}$ is bounded and so are $\{v_n\}$ and $\{y_n\}$.

Step 2. Show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

By the definition of x_n and Lemma 2.35, we obtain

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
& \leq \delta_n \gamma \|f(x_n) - f(x_{n-1})\| + \gamma |\delta_n - \delta_{n-1}| \|f(x_{n-1})\| \\
& \quad + \|(I - \delta_n A)\| \|P_C(I - \psi_n(I - T))y_n - P_C(I - \psi_{n-1}(I - T))y_{n-1}\| \\
& \quad + \|(I - \delta_n A)P_C(I - \psi_{n-1}(I - T))y_{n-1} - (I - \delta_{n-1}A)P_C(I - \psi_{n-1}(I - T))y_{n-1}\| \\
& \leq \delta_n \gamma \alpha \|x_n - x_{n-1}\| + \gamma |\delta_n - \delta_{n-1}| \|f(x_{n-1})\| \\
& \quad + (1 - \delta_n \bar{\gamma}) \|(I - \psi_n(I - T))y_n - (I - \psi_{n-1}(I - T))y_{n-1}\| \\
& \quad + |\delta_n - \delta_{n-1}| \|AP_C(I - \psi_{n-1}(I - T))y_{n-1}\| \\
& \leq \delta_n \gamma \alpha \|x_n - x_{n-1}\| + \gamma |\delta_n - \delta_{n-1}| \|f(x_{n-1})\| + (1 - \delta_n \bar{\gamma}) \left[\theta_n \|x_n - x_{n-1}\| \right. \\
& \quad + |\theta_n - \theta_{n-1}| \|P_C x_{n-1}\| + (1 - \theta_n) \|v_n - v_{n-1}\| + |\theta_n - \theta_{n-1}| \|v_{n-1}\| \\
& \quad \left. + \psi_n \|(I - T)y_n - (I - T)y_{n-1}\| + |\psi_n - \psi_{n-1}| \|(I - T)y_{n-1}\| \right] \\
& \quad + |\delta_n - \delta_{n-1}| \|AP_C(I - \psi_{n-1}(I - T))y_{n-1}\|. \tag{3.67}
\end{aligned}$$

Using the same method in (3.35), we have

$$\|v_n - v_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{\epsilon} |\varphi_n - \varphi_{n-1}| \|v_n - x_n\|. \tag{3.68}$$

Substituting (3.68) into (3.67), we get

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
& \leq \delta_n \gamma \alpha \|x_n - x_{n-1}\| + \gamma |\delta_n - \delta_{n-1}| \|f(x_{n-1})\| + (1 - \delta_n \bar{\gamma}) \left[\|x_n - x_{n-1}\| \right. \\
& \quad + |\theta_n - \theta_{n-1}| \|P_C x_{n-1}\| + \frac{1 - \theta_n}{\epsilon} |\varphi_n - \varphi_{n-1}| \|v_n - x_n\| + |\theta_n - \theta_{n-1}| \|v_{n-1}\| \\
& \quad \left. + \psi_n \|(I - T)y_n - (I - T)y_{n-1}\| + |\psi_n - \psi_{n-1}| \|(I - T)y_{n-1}\| \right] \\
& \quad + |\delta_n - \delta_{n-1}| \|AP_C(I - \psi_{n-1}(I - T))y_{n-1}\| \\
& \leq (1 - \delta_n(\bar{\gamma} - \gamma\alpha)) \|x_n - x_{n-1}\| + (1 + \gamma) |\delta_n - \delta_{n-1}| K + 2|\theta_n - \theta_{n-1}| K \\
& \quad + \frac{1}{\epsilon} |\varphi_n - \varphi_{n-1}| K + 2\psi_n K + |\psi_n - \psi_{n-1}| K, \tag{3.69}
\end{aligned}$$

where $K = \max_{n \in \mathbb{N}} \left\{ \|v_n\|, \|f(x_n)\|, \|v_n - x_n\|, \|P_C x_n\|, \|(I - T)y_n\|, \|AP_C(I - \psi_n(I - T))y_n\| \right\}$.

From (3.69), the condition (i), (iii), (v) and Lemma 2.13, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.70}$$

Step 3. Prove that $\lim_{n \rightarrow \infty} \|v_n - x_n\| = \lim_{n \rightarrow \infty} \|P_C(I - \psi_n(I - T))x_n - x_n\| = 0$.

To claim this, let $z \in \Omega$. Since $v_n = T_{\varphi_n} x_n$ and T_{φ_n} is a firmly nonexpansive mapping, we have

$$\begin{aligned}
\|z - T_{\varphi_n} x_n\|^2 &= \|T_{\varphi_n} z - T_{\varphi_n} x_n\|^2 \\
&\leq \langle T_{\varphi_n} z - T_{\varphi_n} x_n, z - x_n \rangle \\
&= \frac{1}{2} \left(\|T_{\varphi_n} x_n - z\|^2 + \|x_n - z\|^2 - \|T_{\varphi_n} x_n - x_n\|^2 \right),
\end{aligned}$$

which follows that

$$\|v_n - z\|^2 \leq \|x_n - z\|^2 - \|v_n - x_n\|^2. \quad (3.71)$$

By the definition of x_n , Lemma 2.35, Lemma 3.9(ii) and (3.71), we get

$$\begin{aligned} & \|x_{n+1} - z\|^2 \\ &= \|\delta_n (\gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n) + (P_C(I - \psi_n(I - T))y_n - z)\|^2 \\ &\leq \|P_C(I - \psi_n(I - T))y_n - z\|^2 \\ &\quad + 2\delta_n \langle \gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n, x_{n+1} - z \rangle \\ &\leq \|y_n - z\|^2 + 2\delta_n \|\gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n\| \|x_{n+1} - z\| \\ &\leq \theta_n \|P_C x_n - z\|^2 + (1 - \theta_n) \|v_n - z\|^2 \\ &\quad + 2\delta_n \|\gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n\| \|x_{n+1} - z\| \\ &\leq \theta_n \|x_n - z\|^2 + (1 - \theta_n) (\|x_n - z\|^2 - \|v_n - x_n\|^2) \\ &\quad + 2\delta_n \|\gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n\| \|x_{n+1} - z\| \\ &= \|x_n - z\|^2 - (1 - \theta_n) \|v_n - x_n\|^2 \\ &\quad + 2\delta_n \|\gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n\| \|x_{n+1} - z\|, \end{aligned}$$

which implies that

$$\begin{aligned} (1 - \theta_n) \|v_n - x_n\|^2 &\leq (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\| \\ &\quad + 2\delta_n \|\gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n\| \|x_{n+1} - z\|. \end{aligned}$$

From (3.70), the condition (i) and (ii), this yields that

$$\lim_{n \rightarrow \infty} \|v_n - x_n\| = 0. \quad (3.72)$$

By Lemma 2.35 and Lemma 3.9(ii), we get

$$\begin{aligned} & \|x_{n+1} - z\|^2 \\ &\leq \|P_C(I - \psi_n(I - T))y_n - z\|^2 \\ &\quad + 2\delta_n \langle \gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n, x_{n+1} - z \rangle \\ &\leq \|y_n - z\|^2 + 2\delta_n \|\gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n\| \|x_{n+1} - z\| \\ &= \theta_n \|P_C x_n - z\|^2 + (1 - \theta_n) \|v_n - z\|^2 - \theta_n (1 - \theta_n) \|P_C x_n - v_n\|^2 \\ &\quad + 2\delta_n \|\gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n\| \|x_{n+1} - z\| \\ &\leq \|x_n - z\|^2 - \theta_n (1 - \theta_n) \|P_C x_n - v_n\|^2 \\ &\quad + 2\delta_n \|\gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n\| \|x_{n+1} - z\|, \end{aligned}$$

which follows that

$$\begin{aligned} \theta_n (1 - \theta_n) \|P_C x_n - v_n\|^2 &\leq (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\| \\ &\quad + 2\delta_n \|\gamma f(x_n) - AP_C(I - \psi_n(I - T))y_n\| \|x_{n+1} - z\|. \end{aligned}$$

From (3.70), the condition (i) and (ii), this implies that

$$\lim_{n \rightarrow \infty} \|P_C x_n - v_n\| = 0. \quad (3.73)$$

Since

$$\|P_C x_n - x_n\| \leq \|P_C x_n - v_n\| + \|v_n - x_n\|,$$

using (3.72) and (3.73), we have

$$\lim_{n \rightarrow \infty} \|P_C x_n - x_n\| = 0. \quad (3.74)$$

Since

$$\|y_n - x_n\| \leq \theta_n \|P_C x_n - x_n\| + (1 - \theta_n) \|v_n - x_n\|,$$

by (3.72) and (3.74), thus we obtain

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \quad (3.75)$$

Observe that

$$\begin{aligned} & \|x_n - P_C (I - \psi_n(I - T)) y_n\| \\ & \leq \|x_n - x_{n+1}\| + \|x_{n+1} - P_C (I - \psi_n(I - T)) y_n\| \\ & = \|x_n - x_{n+1}\| + \delta_n \|\gamma f(x_n) - AP_C (I - \psi_n(I - T)) y_n\|, \end{aligned}$$

which implies by (3.70) and the condition (i) that

$$\lim_{n \rightarrow \infty} \|x_n - P_C (I - \psi_n(I - T)) y_n\| = 0. \quad (3.76)$$

Since

$$\begin{aligned} & \|x_n - P_C (I - \psi_n(I - T)) x_n\| \\ & \leq \|x_n - P_C (I - \psi_n(I - T)) y_n\| + \|P_C (I - \psi_n(I - T)) y_n - P_C (I - \psi_n(I - T)) x_n\| \\ & \leq \|x_n - P_C (I - \psi_n(I - T)) y_n\| + \|(I - \psi_n(I - T)) y_n - (I - \psi_n(I - T)) x_n\| \\ & \leq \|x_n - P_C (I - \psi_n(I - T)) y_n\| + \|y_n - x_n\| + \psi_n \|(I - T) y_n - (I - T) x_n\|, \end{aligned}$$

by (3.75), (3.76) and the condition (iii), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - P_C (I - \psi_n(I - T)) x_n\| = 0. \quad (3.77)$$

Step 4. Show that $\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \leq 0$, where $q = P_\Omega(I - A + \gamma f)q$.

First, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle = \lim_{k \rightarrow \infty} \langle \gamma f(q) - Aq, x_{n_k} - q \rangle.$$

Since $\{x_n\}$ is bounded, we can assume that $x_{n_k} \rightarrow \omega$ as $k \rightarrow \infty$. By (3.72), this follows that $v_{n_k} \rightarrow \omega$ as $k \rightarrow \infty$.

Assume $\omega \notin \text{Fix}(T)$. Since $\text{Fix}(T) = \text{Fix}(P_C(I - \psi_{n_k}(I - T)))$, we have $\omega \neq P_C(I -$

$\psi_{n_k}(I-T)\omega$. By nonexpansiveness of P_C , the condition (iii), (3.77) and the Opial's condition, we get

$$\begin{aligned}
\liminf_{k \rightarrow \infty} \|x_{n_k} - \omega\| &< \liminf_{k \rightarrow \infty} \|x_{n_k} - P_C(I - \psi_{n_k}(I-T))\omega\| \\
&\leq \liminf_{k \rightarrow \infty} \left(\|x_{n_k} - P_C(I - \psi_{n_k}(I-T))x_{n_k}\| \right. \\
&\quad \left. + \|P_C(I - \psi_{n_k}(I-T))x_{n_k} - P_C(I - \psi_{n_k}(I-T))\omega\| \right) \\
&\leq \liminf_{k \rightarrow \infty} \left(\|x_{n_k} - P_C(I - \psi_{n_k}(I-T))x_{n_k}\| \right. \\
&\quad \left. + \|(I - \psi_{n_k}(I-T))x_{n_k} - (I - \psi_{n_k}(I-T))\omega\| \right) \\
&\leq \liminf_{k \rightarrow \infty} \left(\|x_{n_k} - P_C(I - \psi_{n_k}(I-T))x_{n_k}\| \right. \\
&\quad \left. + \|x_{n_k} - \omega\| + \psi_{n_k} \|(I-T)x_{n_k} - (I-T)\omega\| \right) \\
&\leq \liminf_{k \rightarrow \infty} \|x_{n_k} - \omega\|.
\end{aligned}$$

This is a contradiction. Then, we have

$$\omega \in \text{Fix}(T). \quad (3.78)$$

By continuing the same argument as in (3.53), we obtain

$$\omega \in \bigcap_{i=1}^N EP(\Phi_i). \quad (3.79)$$

From (3.78) and (3.79), we get $\omega \in \Omega$. Since $x_{n_k} \rightarrow \omega$ as $k \rightarrow \infty$ and Lemma 2.11, we can conclude that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle = \lim_{k \rightarrow \infty} \langle \gamma f(q) - Aq, x_{n_k} - q \rangle = \langle \gamma f(q) - Aq, \omega - q \rangle \leq 0. \quad (3.80)$$

Step 5. Finally, claim that the sequence $\{x_n\}$ converges strongly to $q = P_\Omega(I - A + \gamma f)q$. By Lemma 2.1, Lemma 2.35 and Lemma 3.9(ii), we obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\delta_n(\gamma f(x_n) - Aq) + (I - \delta_n A)(P_C(I - \psi_n(I-T))y_n - q)\|^2 \\
&\leq \|(I - \delta_n A)(P_C(I - \psi_n(I-T))y_n - q)\|^2 + 2\delta_n \langle \gamma f(x_n) - Aq, x_{n+1} - q \rangle \\
&\leq (1 - \delta_n \bar{\gamma})^2 \|y_n - q\|^2 + 2\delta_n \gamma \|f(x_n) - f(q)\| \|x_{n+1} - q\| \\
&\quad + 2\delta_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
&\leq (1 - \delta_n \bar{\gamma})^2 \left(\theta_n \|P_C x_n - q\|^2 + (1 - \theta_n) \|v_n - q\|^2 \right) \\
&\quad + 2\delta_n \gamma \alpha \|x_n - q\| \|x_{n+1} - q\| + 2\delta_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
&\leq (1 - \delta_n \bar{\gamma})^2 \|x_n - q\|^2 + \delta_n \gamma \alpha \left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2 \right) \\
&\quad + 2\delta_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle,
\end{aligned}$$

which implies that

$$\begin{aligned} & \|x_{n+1} - q\|^2 \\ & \leq \frac{(1 - \delta_n \bar{\gamma})^2 + \delta_n \gamma \alpha}{1 - \delta_n \gamma \alpha} \|x_n - q\|^2 + \frac{2\delta_n}{1 - \delta_n \gamma \alpha} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\ & = \left(1 - \frac{2\delta_n(\bar{\gamma} - \gamma\alpha)}{1 - \delta_n \gamma \alpha}\right) \|x_n - q\|^2 + \frac{2\delta_n(\bar{\gamma} - \gamma\alpha)}{1 - \delta_n \gamma \alpha} \left(\frac{\delta_n \bar{\gamma}^2}{2(\bar{\gamma} - \gamma\alpha)} \|x_n - q\|^2\right. \\ & \quad \left. + \frac{1}{\bar{\gamma} - \gamma\alpha} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle\right). \end{aligned}$$

From (3.80), the condition (i) and Lemma 2.13, we can conclude that $\{x_n\}$ converges strongly to $q = P_\Omega(I - A + \gamma f)q$. By (3.72) and (3.75), we have $\{v_n\}$ and $\{y_n\}$ converge strongly to $q = P_\Omega(I - A + \gamma f)q$. This completes the proof. \square

The following corollaries are directed results from Theorem 3.11.

Corollary 3.12. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be an α -contractive mapping on H and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $T : C \rightarrow C$ be a nonspreading mapping. Let $\Phi : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) with $\Omega := \text{Fix}(T) \cap EP(\Phi) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} \Phi(v_n, y) + \frac{1}{\varphi_n} \langle y - v_n, v_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n = \theta_n P_C x_n + (1 - \theta_n) v_n, \\ x_{n+1} = \delta_n \gamma f(x_n) + (I - \delta_n A) P_C (I - \psi_n(I - T)) y_n, \forall n \in \mathbb{N}, \end{cases} \quad (3.81)$$

where $\{\delta_n\}$, $\{\theta_n\}$, $\{\varphi_n\}$, $\{\psi_n\} \subseteq (0, 1)$. Suppose the conditions (i)-(vi) hold.

- (i) $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$;
- (ii) $0 < \tau \leq \theta_n \leq v < 1$, for some $\tau, v > 0$;
- (iii) $\sum_{n=1}^{\infty} \psi_n < \infty$;
- (iv) $0 < \epsilon \leq \varphi_n \leq \eta < 1$, for some $\epsilon, \eta > 0$;
- (v) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$,
 $\sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$, $\sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ converge strongly to $q = P_\Omega(I - A + \gamma f)q$.

Proof. Putting $\Phi = \Phi_i$, for all $i = 1, 2, \dots, N$, in Theorem 3.11, then the desired result is obtained. \square

Corollary 3.13. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be an α -contractive mapping on H and let $A : H \rightarrow H$ be a strongly positive linear

bounded operator with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $T : C \rightarrow C$ be a nonspreading mapping with $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1 \in H$ and

$$x_{n+1} = \delta_n \gamma f(x_n) + (I - \delta_n A) P_C (I - \psi_n (I - T)) P_C x_n, \forall n \in \mathbb{N}, \quad (3.82)$$

where $\{\delta_n\}, \{\psi_n\} \subseteq (0, 1)$. Suppose the conditions (i)-(vi) hold.

- (i) $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$;
- (ii) $\sum_{n=1}^{\infty} \psi_n < \infty$;
- (iii) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $q = P_{Fix(T)}(I - A + \gamma f)q$.

Proof. Taking $\Phi_i = 0$, for every $i = 1, 2, \dots, N$, in Theorem 3.11, then we have $v_n = P_C x_n$, for every $n \in \mathbb{N}$ and the result of Corollary 3.13 can be obtained. \square

3.3 Viscosity approximation methods for the modified generalized equilibrium problem and a finite family of strictly pseudo-contractive mappings

In this section, modifying the generalized equilibrium problem (2.14), we introduce the modified generalized equilibrium problem (*MGEP*) as follows:

$$(MGEP) \begin{cases} \text{Find } u \in C \text{ and } w \in T(I - \lambda A)u, \forall \lambda > 0, \\ \Phi(w, u, v) + \varphi(v) - \varphi(u) + \langle v - u, Au \rangle \geq 0, \forall v \in C, \end{cases} \quad (3.83)$$

where $A : C \rightarrow C$ is a mapping. The solution set of (*MGEP*) is denoted by $(MGEP)_s(\Phi, \varphi, A)$. If $A = 0$, the modified generalized equilibrium problem (3.83) reduces to the generalized equilibrium problem (2.14).

Next, we prove the following Lemmas for the K -mapping of a finite family of strictly pseudo-contractive mapping that is necessary to prove our main results.

Lemma 3.14. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strictly pseudo-contractive mapping of C into itself with $\kappa_i \leq \gamma_1$, for all $i = 1, 2, \dots, N$, and $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers with $0 < \lambda_i < \gamma_2$, for all $i = 1, 2, \dots, N$ and $\gamma_1 + \gamma_2 < 1$. Let K be the K -mapping generated by T_1, T_2, \dots, T_N and $\lambda_1, \lambda_2, \dots, \lambda_N$. Then the following properties hold:

- (i) $Fix(K) = \bigcap_{i=1}^N Fix(T_i)$,
- (ii) K is a nonexpansive mapping.

Proof. To prove (i), first, we show that $\bigcap_{i=1}^N Fix(T_i) \subseteq Fix(K)$. Let $z \in \bigcap_{i=1}^N Fix(T_i)$. Then

$$z \in Fix(T_i), \forall i = 1, 2, \dots, N. \quad (3.84)$$

By the definition of U_1 and (3.84), we obtain

$$U_1 z = \lambda_1 T_1 z + (1 - \lambda_1) z = \lambda_1 z + (1 - \lambda_1) z = z,$$

which implies that

$$z \in \text{Fix}(U_1). \quad (3.85)$$

By the definition of U_2 , (3.84) and (3.85), we have

$$U_2 z = \lambda_2 T_2 U_1 z + (1 - \lambda_2) U_1 z = \lambda_2 T_2 z + (1 - \lambda_2) z = \lambda_2 z + (1 - \lambda_2) z = z,$$

which follows that $z \in \text{Fix}(U_2)$. By continuing this method above, we get

$$z \in \text{Fix}(U_i), \forall i = 1, 2, \dots, N-1. \quad (3.86)$$

By the definition of K -mapping, (3.84) and (3.86), we obtain

$$\begin{aligned} Kz &= U_N z = \lambda_N T_N U_{N-1} z + (1 - \lambda_N) U_{N-1} z \\ &= \lambda_N T_N U_{N-1} z + (1 - \lambda_N) U_{N-1} z \\ &= \lambda_N T_N z + (1 - \lambda_N) z \\ &= \lambda_N z + (1 - \lambda_N) z \\ &= z, \end{aligned}$$

that is $z \in \text{Fix}(K)$. Therefore $\bigcap_{i=1}^N \text{Fix}(T_i) \subseteq \text{Fix}(K)$.

Next, we claim that $\text{Fix}(K) \subseteq \bigcap_{i=1}^N \text{Fix}(T_i)$. Let $x \in \text{Fix}(K)$ and $y \in \bigcap_{i=1}^N \text{Fix}(T_i)$.

By the definition of K -mapping, we get

$$\begin{aligned} &\|x - y\|^2 \\ &= \|Kx - y\|^2 \\ &= \|\lambda_N T_N U_{N-1} x + (1 - \lambda_N) U_{N-1} x - y\|^2 \\ &= \|\lambda_N (T_N U_{N-1} x - y) + (1 - \lambda_N) (U_{N-1} x - y)\|^2 \\ &= \lambda_N^2 \|T_N U_{N-1} x - y\|^2 + (1 - \lambda_N)^2 \|U_{N-1} x - y\|^2 \\ &\quad + 2\lambda_N (1 - \lambda_N) \langle T_N U_{N-1} x - y, U_{N-1} x - y \rangle \\ &\leq \lambda_N^2 \left(\|U_{N-1} x - y\|^2 + \kappa_N \|T_N U_{N-1} x - U_{N-1} x\|^2 \right) + (1 - \lambda_N)^2 \|U_{N-1} x - y\|^2 \\ &\quad + 2\lambda_N (1 - \lambda_N) \left(\|U_{N-1} x - y\|^2 - \frac{1 - \kappa_N}{2} \|T_N U_{N-1} x - U_{N-1} x\|^2 \right) \\ &= \left(\lambda_N^2 + (1 - \lambda_N)^2 + 2\lambda_N (1 - \lambda_N) \right) \|U_{N-1} x - y\|^2 \\ &\quad + (\lambda_N^2 \kappa_N - \lambda_N (1 - \lambda_N) (1 - \kappa_N)) \|T_N U_{N-1} x - U_{N-1} x\|^2 \\ &= (\lambda_N + 1 - \lambda_N)^2 \|U_{N-1} x - y\|^2 \\ &\quad + \lambda_N (\lambda_N \kappa_N - (1 - \lambda_N) (1 - \kappa_N)) \|T_N U_{N-1} x - U_{N-1} x\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|U_{N-1}x - y\|^2 \\
&\quad + \lambda_N (\lambda_N \kappa_N - (1 - \kappa_N) + \lambda_N (1 - \kappa_N)) \|T_N U_{N-1}x - U_{N-1}x\|^2 \\
&= \|U_{N-1}x - y\|^2 + \lambda_N (\kappa_N + \lambda_N - 1) \|T_N U_{N-1}x - U_{N-1}x\|^2 \\
&\leq \|U_{N-1}x - y\|^2 + \lambda_N (\gamma_1 + \gamma_2 - 1) \|T_N U_{N-1}x - U_{N-1}x\|^2 \\
&\leq \|U_{N-1}x - y\|^2 \\
&\quad \vdots \\
&= \|U_2x - y\|^2 \\
&= \|\lambda_2 (T_2 U_1x - y) + (1 - \lambda_2) (U_1x - y)\|^2 \\
&= \lambda_2^2 \|T_2 U_1x - y\|^2 + (1 - \lambda_2)^2 \|U_1x - y\|^2 \\
&\quad + 2\lambda_2 (1 - \lambda_2) \langle T_2 U_1x - y, U_1x - y \rangle \\
&= \lambda_2^2 \left(\|U_1x - y\|^2 + \kappa_2 \|T_2 U_1x - U_1x\|^2 \right) + (1 - \lambda_2)^2 \|U_1x - y\|^2 \\
&\quad + 2\lambda_2 (1 - \lambda_2) \left(\|U_1x - y\|^2 - \frac{1 - \kappa_2}{2} \|T_2 U_1x - U_1x\|^2 \right) \\
&= \left(\lambda_2^2 + (1 - \lambda_2)^2 + 2\lambda_2 (1 - \lambda_2) \right) \|U_1x - y\|^2 \\
&\quad + (\lambda_2^2 \kappa_2 - \lambda_2 (1 - \lambda_2) (1 - \kappa_2)) \|T_2 U_1x - U_1x\|^2 \\
&= (\lambda_2 + 1 - \lambda_2)^2 \|U_1x - y\|^2 \\
&\quad + \lambda_2 (\lambda_2 \kappa_2 - (1 - \lambda_2) (1 - \kappa_2)) \|T_2 U_1x - U_1x\|^2 \\
&= \|U_1x - y\|^2 + \lambda_2 (\kappa_2 + \lambda_2 - 1) \|T_2 U_1x - U_1x\|^2 \\
&\leq \|U_1x - y\|^2 + \lambda_2 ((\gamma_1 + \gamma_2) - 1) \|T_2 U_1x - U_1x\|^2 \\
&\leq \|U_1x - y\|^2 \\
&= \|\lambda_1 (T_1x - y) + (1 - \lambda_1) (x - y)\|^2 \\
&= \lambda_1^2 \|T_1x - y\|^2 + (1 - \lambda_1)^2 \|x - y\|^2 \\
&\quad + 2\lambda_1 (1 - \lambda_1) \langle T_1x - y, x - y \rangle \\
&= \lambda_1^2 \left(\|x - y\|^2 + \kappa_1 \|T_1x - x\|^2 \right) + (1 - \lambda_1)^2 \|x - y\|^2 \\
&\quad + 2\lambda_1 (1 - \lambda_1) \left(\|x - y\|^2 - \frac{1 - \kappa_1}{2} \|T_1x - x\|^2 \right) \\
&= \left(\lambda_1^2 + (1 - \lambda_1)^2 + 2\lambda_1 (1 - \lambda_1) \right) \|x - y\|^2 \\
&\quad + (\lambda_1^2 \kappa_1 - \lambda_1 (1 - \lambda_1) (1 - \kappa_1)) \|T_1x - x\|^2 \\
&= (\lambda_1 + 1 - \lambda_1)^2 \|x - y\|^2 \\
&\quad + \lambda_1 (\lambda_1 \kappa_1 - (1 - \lambda_1) (1 - \kappa_1)) \|T_1x - x\|^2 \\
&= \|x - y\|^2 + \lambda_1 (\kappa_1 + \lambda_1 - 1) \|T_1x - x\|^2 \\
&\leq \|x - y\|^2 + \lambda_1 ((\gamma_1 + \gamma_2) - 1) \|T_1x - x\|^2. \tag{3.87}
\end{aligned}$$

From (3.87), it yields that $\lambda_1 (1 - (\gamma_1 + \gamma_2)) \|T_1x - x\|^2 \leq 0$. This implies that $\|T_1x - x\| =$

0. Therefore $x = T_1x$, that is,

$$x \in \text{Fix}(T_1). \quad (3.88)$$

By the definition of U_1 and (3.88), we have $U_1x = \lambda_1 T_1x + (1 - \lambda_1)x = x$, that is,

$$x \in \text{Fix}(U_1). \quad (3.89)$$

Again by (3.87) and (3.89), we obtain

$$\begin{aligned} \|x - y\|^2 &\leq \|U_1x - y\|^2 + \lambda_2((\gamma_1 + \gamma_2) - 1) \|T_2U_1x - U_1x\|^2 \\ &= \|x - y\|^2 + \lambda_2((\gamma_1 + \gamma_2) - 1) \|T_2x - x\|^2, \end{aligned}$$

which implies that $x = T_2x$, that is,

$$x \in \text{Fix}(T_2). \quad (3.90)$$

By the definition of U_2 , (3.89) and (3.90), we get $U_2x = \lambda_2 T_2U_1x + (1 - \lambda_2)U_1x = x$, which follows that $x \in \text{Fix}(U_2)$. Using the same argument, we can conclude that

$$x \in \text{Fix}(T_i) \text{ and } x \in \text{Fix}(U_i), \forall i = 1, 2, \dots, N-1.$$

Next, we show that $x \in \text{Fix}(T_N)$. Since

$$0 = Kx - x = \lambda_N T_N U_{N-1}x + (1 - \lambda_N)U_{N-1}x - x = \lambda_N (T_N x - x)$$

and $\lambda_N \in (0, 1]$, we obtain $x \in \text{Fix}(T_N)$, which follows that $x \in \bigcap_{i=1}^N \text{Fix}(T_i)$. Therefore $\text{Fix}(K) \subseteq \bigcap_{i=1}^N \text{Fix}(T_i)$ and the claim is proved. Hence, $\text{Fix}(K) = \bigcap_{i=1}^N \text{Fix}(T_i)$.

To prove (i), we claim that K is a nonexpansive mapping. Let $x, y \in C$. Then we obtain

$$\begin{aligned} &\|Kx - Ky\|^2 \\ &= \|(\lambda_N T_N U_{N-1}x + (1 - \lambda_N)U_{N-1}x) - (\lambda_N T_N U_{N-1}y + (1 - \lambda_N)U_{N-1}y)\|^2 \\ &= \|(U_{N-1}x - \lambda_N(U_{N-1}x - T_N U_{N-1}x)) - (U_{N-1}y - \lambda_N(U_{N-1}y - T_N U_{N-1}y))\|^2 \\ &= \|(U_{N-1}x - U_{N-1}y) - \lambda_N((I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y)\|^2 \\ &= \|U_{N-1}x - U_{N-1}y\|^2 + \lambda_N^2 \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\ &\quad - 2\lambda_N \langle U_{N-1}x - U_{N-1}y, (I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y \rangle \\ &\leq \|U_{N-1}x - U_{N-1}y\|^2 + \lambda_N^2 \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\ &\quad - 2\lambda_N \left(\frac{1 - \kappa_N}{2} \right) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\ &= \|U_{N-1}x - U_{N-1}y\|^2 \\ &\quad + \lambda_N(\lambda_N - (1 - \kappa_N)) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\ &\leq \|U_{N-1}x - U_{N-1}y\|^2 \\ &\quad + \lambda_N(\gamma_1 + \gamma_2 - 1) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|U_{N-1}x - U_{N-1}y\|^2 \\
&\quad - \lambda_N (1 - (\gamma_1 + \gamma_2)) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\
&= \|(\lambda_{N-1}T_{N-1}U_{N-2}x + (1 - \lambda_{N-1})U_{N-2}x) - (\lambda_{N-1}T_{N-1}U_{N-2}y + (1 - \lambda_{N-1})U_{N-2}y)\|^2 \\
&\quad - \lambda_N (1 - (\gamma_1 + \gamma_2)) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\
&= \|(U_{N-2}x - \lambda_{N-1}(I - T_{N-1})U_{N-2}x) - (U_{N-2}y - \lambda_{N-1}(I - T_{N-1})U_{N-2}y)\|^2 \\
&\quad - \lambda_N (1 - (\gamma_1 + \gamma_2)) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\
&= \|(U_{N-2}x - U_{N-2}y) - \lambda_{N-1}((I - T_{N-1})U_{N-2}x - (I - T_{N-1})U_{N-2}y)\|^2 \\
&\quad - \lambda_N (1 - (\gamma_1 + \gamma_2)) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\
&= \|U_{N-2}x - U_{N-2}y\|^2 + \lambda_{N-1}^2 \|(I - T_{N-1})U_{N-2}x - (I - T_{N-1})U_{N-2}y\|^2 \\
&\quad - 2\lambda_{N-1}(U_{N-2}x - U_{N-2}y, (I - T_{N-1})U_{N-2}x - (I - T_{N-1})U_{N-2}y) \\
&\quad - \lambda_N (1 - (\gamma_1 + \gamma_2)) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\
&\leq \|U_{N-2}x - U_{N-2}y\|^2 + \lambda_{N-1}^2 \|(I - T_{N-1})U_{N-2}x - (I - T_{N-1})U_{N-2}y\|^2 \\
&\quad - 2\lambda_{N-1} \left(\frac{1 - \kappa_{N-1}}{2} \right) \|(I - T_{N-1})U_{N-2}x - (I - T_{N-1})U_{N-2}y\|^2 \\
&\quad - \lambda_N (1 - (\gamma_1 + \gamma_2)) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\
&= \|U_{N-2}x - U_{N-2}y\|^2 \\
&\quad + \lambda_{N-1}(\lambda_{N-1} - (1 - \kappa_{N-1})) \|(I - T_{N-1})U_{N-2}x - (I - T_{N-1})U_{N-2}y\|^2 \\
&\quad - \lambda_N (1 - (\gamma_1 + \gamma_2)) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\
&\leq \|U_{N-2}x - U_{N-2}y\|^2 \\
&\quad + \lambda_{N-1}(\gamma_1 + \gamma_2 - 1) \|(I - T_{N-1})U_{N-2}x - (I - T_{N-1})U_{N-2}y\|^2 \\
&\quad - \lambda_N (1 - (\gamma_1 + \gamma_2)) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\
&= \|U_{N-2}x - U_{N-2}y\|^2 \\
&\quad - \lambda_{N-1}(1 - (\gamma_1 + \gamma_2)) \|(I - T_{N-1})U_{N-2}x - (I - T_{N-1})U_{N-2}y\|^2 \\
&\quad - \lambda_N (1 - (\gamma_1 + \gamma_2)) \|(I - T_N)U_{N-1}x - (I - T_N)U_{N-1}y\|^2 \\
&= \|U_{N-2}x - U_{N-2}y\|^2 \\
&\quad - (1 - (\gamma_1 + \gamma_2)) \sum_{i=N-1}^N \lambda_i \|(I - T_i)U_{i-1}x - (I - T_i)U_{i-1}y\|^2 \\
&\quad \vdots \\
&\leq \|x - y\|^2 - (1 - (\gamma_1 + \gamma_2)) \sum_{i=1}^N \lambda_i \|(I - T_i)U_{i-1}x - (I - T_i)U_{i-1}y\|^2,
\end{aligned}$$

which implies that

$$\|Kx - Ky\|^2 \leq \|x - y\|^2 - (1 - (\gamma_1 + \gamma_2)) \sum_{i=1}^N \lambda_i \|(I - T_i)U_{i-1}x - (I - T_i)U_{i-1}y\|^2. \quad (3.91)$$

From (3.91) and $\gamma_1 + \gamma_2 < 1$, we obtain $\|Kx - Ky\| \leq \|x - y\|, \forall x, y \in C$, that is, K is a

nonexpansive mapping. □

Next, we give an example for Lemma 3.14.

Example 3.15. Let \mathbb{R} be the set of real numbers and let $T_i : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$T_i x = -(i+1)x, \text{ for all } x \in \mathbb{R}$$

and $\lambda_i = \frac{i}{5i+1}$, for all $i = 1, 2, \dots, 5$. Obviously, for all $i = 1, 2, \dots, 5$, T_i is κ_i -strictly pseudo-contractions with $\kappa_i = \frac{i}{2+i}$. Let K be the K -mapping generated by T_1, T_2, \dots, T_5 and $\lambda_1, \lambda_2, \dots, \lambda_5$. Choose $\gamma_1 = \frac{11}{14}$ and $\gamma_2 = \frac{11}{52}$, which follows that $\gamma_1 + \gamma_2 = \frac{11}{14} + \frac{11}{52} = \frac{726}{728} < 1$. Then, by Lemma 3.14, we obtain $Fix(K) = \bigcap_{i=1}^5 Fix(T_i) = \{0\}$.

Lemma 3.16. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strictly pseudo contractive mappings of C into itself with $\kappa_i \leq \gamma_1$ and $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. For every $i = 1, 2, \dots, N$ and $n \in \mathbb{N}$, let $\lambda_1, \lambda_2, \dots, \lambda_N$ and $\lambda_1^n, \lambda_2^n, \dots, \lambda_N^n$ be real numbers with $0 < \lambda_i, \lambda_i^n < \gamma_2$ and $\gamma_1 + \gamma_2 < 1$ such that $\lambda_i^n \rightarrow \lambda_i$ as $n \rightarrow \infty$ and $\sum_{n=1}^{\infty} |\lambda_i^{n+1} - \lambda_i^n| < \infty$. For every $n \in \mathbb{N}$, let K and K_n be the K -mappings generated by T_1, T_1, \dots, T_N and $\lambda_1, \lambda_2, \dots, \lambda_N$ and T_1, T_2, \dots, T_N and $\lambda_1^n, \lambda_2^n, \dots, \lambda_N^n$, respectively. Then, for every bounded sequences $\{x_n\}$ in C , the following properties hold:

- (i) $\lim_{n \rightarrow \infty} \|K_n x_n - K x_n\| = 0$;
- (ii) $\sum_{n=1}^{\infty} \|K_n x_{n-1} - K_{n-1} x_{n-1}\| < \infty$.

Proof. Let $\{x_n\}$ be a bounded sequence in C and let U_k and $U_{n,k}$ be generated by T_1, T_1, \dots, T_N and $\lambda_1, \lambda_2, \dots, \lambda_N$ and T_1, T_1, \dots, T_N and $\lambda_1^n, \lambda_2^n, \dots, \lambda_N^n$, respectively.

First, we shall prove that (i) holds. For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \|U_{n,1} x_n - U_1 x_n\| &= \|\lambda_1^n T_1 x_n + (1 - \lambda_1^n) x_n - (\lambda_1 T_1 x_n + (1 - \lambda_1) x_n)\| \\ &= \|\lambda_1^n T_1 x_n - \lambda_1^n x_n - \lambda_1 T_1 x_n + \lambda_1 x_n\| \\ &= \|(\lambda_1^n - \lambda_1) T_1 x_n - (\lambda_1^n - \lambda_1) x_n\| \\ &= |\lambda_1^n - \lambda_1| \|T_1 x_n - x_n\|. \end{aligned} \tag{3.92}$$

For $k \in \{2, 3, \dots, N\}$, we have

$$\begin{aligned} &\|U_{n,k} x_n - U_k x_n\| \\ &= \|\lambda_k^n T_k U_{n,k-1} x_n + (1 - \lambda_k^n) U_{n,k-1} x_n - (\lambda_k T_k U_{k-1} x_n + (1 - \lambda_k) U_{k-1} x_n)\| \\ &= \|\lambda_k^n T_k U_{n,k-1} x_n - \lambda_k T_k U_{k-1} x_n + (1 - \lambda_k^n) U_{n,k-1} x_n - (1 - \lambda_k) U_{k-1} x_n\| \\ &= \left\| \lambda_k^n T_k U_{n,k-1} x_n - \lambda_k^n T_k U_{k-1} x_n + \lambda_k^n T_k U_{k-1} x_n - \lambda_k T_k U_{k-1} x_n + (1 - \lambda_k^n) U_{n,k-1} x_n \right. \\ &\quad \left. - (1 - \lambda_k^n) U_{k-1} x_n + (1 - \lambda_k^n) U_{k-1} x_n - (1 - \lambda_k) U_{k-1} x_n \right\| \\ &= \left\| \lambda_k^n (T_k U_{n,k-1} x_n - T_k U_{k-1} x_n) + (\lambda_k^n - \lambda_k) T_k U_{k-1} x_n \right. \\ &\quad \left. + (1 - \lambda_k^n) (U_{n,k-1} x_n - U_{k-1} x_n) + (1 - \lambda_k^n - (1 - \lambda_k)) U_{k-1} x_n \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \lambda_k^n \|T_k U_{n,k-1} x_n - T_k U_{k-1} x_n\| + |\lambda_k^n - \lambda_k| \|T_k U_{k-1} x_n\| \\
&\quad + (1 - \lambda_k^n) \|U_{n,k-1} x_n - U_{k-1} x_n\| + |\lambda_k - \lambda_k^n| \|U_{k-1} x_n\| \\
&\leq \lambda_k^n \frac{1 + \kappa_k}{1 - \kappa_k} \|U_{n,k-1} x_n - U_{k-1} x_n\| + |\lambda_k^n - \lambda_k| \|T_k U_{k-1} x_n\| \\
&\quad + (1 - \lambda_k^n) \|U_{n,k-1} x_n - U_{k-1} x_n\| + |\lambda_k - \lambda_k^n| \|U_{k-1} x_n\| \\
&\leq \frac{1 + \kappa_k}{1 - \kappa_k} \|U_{n,k-1} x_n - U_{k-1} x_n\| + \frac{1 - \kappa_k}{1 - \kappa_k} \|U_{n,k-1} x_n - U_{k-1} x_n\| \\
&\quad + |\lambda_k^n - \lambda_k| (\|T_k U_{k-1} x_n\| + \|U_{k-1} x_n\|) \\
&= \frac{2}{1 - \kappa_k} \|U_{n,k-1} x_n - U_{k-1} x_n\| + |\lambda_k^n - \lambda_k| (\|T_k U_{k-1} x_n\| + \|U_{k-1} x_n\|). \tag{3.93}
\end{aligned}$$

By (3.92) and (3.93), we get

$$\begin{aligned}
&\|K_n x_n - K x_n\| \\
&= \|U_{n,N} x_n - U_N x_n\| \\
&\leq \frac{2}{1 - \kappa_N} \|U_{n,N-1} x_n - U_{N-1} x_n\| + |\lambda_N^n - \lambda_N| (\|T_N U_{N-1} x_n\| + \|U_{N-1} x_n\|) \\
&\leq \frac{2}{1 - \kappa_N} \left(\frac{2}{1 - \kappa_{N-1}} \|U_{n,N-2} x_n - U_{N-2} x_n\| + |\lambda_{N-1}^n - \lambda_{N-1}| \right. \\
&\quad \left. (\|T_{N-1} U_{N-2} x_n\| + \|U_{N-2} x_n\|) \right) + |\lambda_N^n - \lambda_N| (\|T_N U_{N-1} x_n\| + \|U_{N-1} x_n\|) \\
&= \left(\frac{2}{1 - \kappa_N} \right) \left(\frac{2}{1 - \kappa_{N-1}} \right) \|U_{n,N-2} x_n - U_{N-2} x_n\| \\
&\quad + \frac{2}{1 - \kappa_N} |\lambda_{N-1}^n - \lambda_{N-1}| (\|T_{N-1} U_{N-2} x_n\| + \|U_{N-2} x_n\|) \\
&\quad + |\lambda_N^n - \lambda_N| (\|T_N U_{N-1} x_n\| + \|U_{N-1} x_n\|) \\
&= \prod_{j=N-1}^N \left(\frac{2}{1 - \kappa_j} \right) \|U_{n,N-2} x_n - U_{N-2} x_n\| \\
&\quad + \sum_{j=N-1}^N \left(\frac{2}{1 - \kappa_{j+1}} \right)^{N-j} |\lambda_j^n - \lambda_j| (\|T_j U_{j-1} x_n\| + \|U_{j-1} x_n\|) \\
&\quad \vdots \\
&\leq \prod_{j=2}^N \left(\frac{2}{1 - \kappa_j} \right) \|U_{n,1} x_n - U_1 x_n\| \\
&\quad + \sum_{j=2}^N \left(\frac{2}{1 - \kappa_{j+1}} \right)^{N-j} |\lambda_j^n - \lambda_j| (\|T_j U_{j-1} x_n\| + \|U_{j-1} x_n\|) \\
&= \prod_{j=2}^N \left(\frac{2}{1 - \kappa_j} \right) |\lambda_1^n - \lambda_1| \|T_1 x_n - x_n\| \\
&\quad + \sum_{j=2}^N \left(\frac{2}{1 - \kappa_{j+1}} \right)^{N-j} |\lambda_j^n - \lambda_j| (\|T_j U_{j-1} x_n\| + \|U_{j-1} x_n\|). \tag{3.94}
\end{aligned}$$

By (3.94) and the fact that $\lambda_i^n \rightarrow \lambda_i$ as $n \rightarrow \infty$ for all $i = 1, 2, \dots, N$, we can deduce that $\lim_{n \rightarrow \infty} \|K_n x_n - K x_n\| = 0$.

Next, we will claim that (ii) holds. For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\|U_{n,1}x_{n-1} - U_{n-1,1}x_{n-1}\| &= \|\lambda_1^n T_1 x_{n-1} + (1 - \lambda_1^n) x_{n-1} - (\lambda_1^{n-1} T_1 x_{n-1} + (1 - \lambda_1^{n-1}) x_{n-1})\| \\
&= \|\lambda_1^n T_1 x_{n-1} - \lambda_1^n x_{n-1} - \lambda_1^{n-1} T_1 x_{n-1} + \lambda_1^{n-1} x_{n-1}\| \\
&= \|(\lambda_1^n - \lambda_1^{n-1}) T_1 x_{n-1} - (\lambda_1^n - \lambda_1^{n-1}) x_{n-1}\| \\
&= |\lambda_1^n - \lambda_1^{n-1}| \|T_1 x_{n-1} - x_{n-1}\|. \tag{3.95}
\end{aligned}$$

For $k \in \{2, 3, \dots, N\}$, we have

$$\begin{aligned}
&\|U_{n,k}x_{n-1} - U_{n-1,k}x_{n-1}\| \\
&= \|\lambda_k^n T_k U_{n,k-1}x_{n-1} + (1 - \lambda_k^n) U_{n,k-1}x_{n-1} - (\lambda_k^{n-1} T_k U_{n-1,k-1}x_{n-1} \\
&\quad + (1 - \lambda_k^{n-1}) U_{n-1,k-1}x_{n-1})\| \\
&= \|\lambda_k^n T_k U_{n,k-1}x_{n-1} - \lambda_k^{n-1} T_k U_{n-1,k-1}x_{n-1} + (1 - \lambda_k^n) U_{n,k-1}x_{n-1} \\
&\quad - (1 - \lambda_k^{n-1}) U_{n-1,k-1}x_{n-1}\| \\
&= \|\lambda_k^n T_k U_{n,k-1}x_{n-1} - \lambda_k^n T_k U_{n-1,k-1}x_{n-1} + \lambda_k^n T_k U_{n-1,k-1}x_{n-1} \\
&\quad - \lambda_k^{n-1} T_k U_{n-1,k-1}x_{n-1} + (1 - \lambda_k^n) U_{n,k-1}x_{n-1} - (1 - \lambda_k^n) U_{n-1,k-1}x_{n-1} \\
&\quad + (1 - \lambda_k^n) U_{n-1,k-1}x_{n-1} - (1 - \lambda_k^{n-1}) U_{n-1,k-1}x_{n-1}\| \\
&= \|\lambda_k^n (T_k U_{n,k-1}x_{n-1} - T_k U_{n-1,k-1}x_{n-1}) + (\lambda_k^n - \lambda_k^{n-1}) T_k U_{n-1,k-1}x_{n-1} \\
&\quad + (1 - \lambda_k^n) (U_{n,k-1}x_{n-1} - U_{n-1,k-1}x_{n-1}) + (1 - \lambda_k^n - (1 - \lambda_k^{n-1})) U_{n-1,k-1}x_{n-1}\| \\
&\leq \lambda_k^n \|T_k U_{n,k-1}x_{n-1} - T_k U_{n-1,k-1}x_{n-1}\| + |\lambda_k^n - \lambda_k^{n-1}| \|T_k U_{n-1,k-1}x_{n-1}\| \\
&\quad + (1 - \lambda_k^n) \|U_{n,k-1}x_{n-1} - U_{n-1,k-1}x_{n-1}\| + |\lambda_k^n - \lambda_k^{n-1}| \|U_{n-1,k-1}x_{n-1}\| \\
&\leq \lambda_k^n \frac{1 + \kappa_k}{1 - \kappa_k} \|U_{n,k-1}x_{n-1} - U_{n-1,k-1}x_{n-1}\| + |\lambda_k^n - \lambda_k^{n-1}| \|T_k U_{n-1,k-1}x_{n-1}\| \\
&\quad + (1 - \lambda_k^n) \|U_{n,k-1}x_{n-1} - U_{n-1,k-1}x_{n-1}\| + |\lambda_k^n - \lambda_k^{n-1}| \|U_{n-1,k-1}x_{n-1}\| \\
&\leq \frac{1 + \kappa_k}{1 - \kappa_k} \|U_{n,k-1}x_{n-1} - U_{n-1,k-1}x_{n-1}\| + \frac{1 - \kappa_k}{1 - \kappa_k} \|U_{n,k-1}x_{n-1} - U_{n-1,k-1}x_{n-1}\| \\
&\quad + |\lambda_k^n - \lambda_k^{n-1}| (\|T_k U_{n-1,k-1}x_{n-1}\| + \|U_{n-1,k-1}x_{n-1}\|) \\
&= \frac{2}{1 - \kappa_k} \|U_{n,k-1}x_{n-1} - U_{n-1,k-1}x_{n-1}\| \\
&\quad + |\lambda_k^n - \lambda_k^{n-1}| (\|T_k U_{n-1,k-1}x_{n-1}\| + \|U_{n-1,k-1}x_{n-1}\|). \tag{3.96}
\end{aligned}$$

From (3.95) and (3.96), we obtain

$$\begin{aligned}
&\|K_n x_{n-1} - K_{n-1} x_{n-1}\| \\
&= \|U_{n,N} x_{n-1} - U_{n-1,N} x_{n-1}\| \\
&\leq \frac{2}{1 - \kappa_N} \|U_{n,N-1} x_{n-1} - U_{n-1,N-1} x_{n-1}\| \\
&\quad + |\lambda_N^n - \lambda_N^{n-1}| (\|T_N U_{n-1,N-1} x_{n-1}\| + \|U_{n-1,N-1} x_{n-1}\|) \\
&\leq \frac{2}{1 - \kappa_N} \left(\frac{2}{1 - \kappa_{N-1}} \|U_{n,N-2} x_{n-1} - U_{n-1,N-2} x_{n-1}\| + |\lambda_{N-1}^n - \lambda_{N-1}^{n-1}| \right)
\end{aligned}$$

$$\begin{aligned}
& (\|T_{N-1}U_{n-1,N-2}x_{n-1}\| + \|U_{n-1,N-2}x_{n-1}\|) \\
& + |\lambda_N^n - \lambda_N^{n-1}| (\|T_N U_{n-1,N-1}x_{n-1}\| + \|U_{n-1,N-1}x_{n-1}\|) \\
= & \left(\frac{2}{1-\kappa_N}\right) \left(\frac{2}{1-\kappa_{N-1}}\right) \|U_{n,N-2}x_{n-1} - U_{n-1,N-2}x_{n-1}\| \\
& + \frac{2}{1-\kappa_N} |\lambda_{N-1}^n - \lambda_{N-1}^{n-1}| (\|T_{N-1}U_{n-1,N-2}x_{n-1}\| + \|U_{n-1,N-2}x_{n-1}\|) \\
& + |\lambda_N^n - \lambda_N^{n-1}| (\|T_N U_{n-1,N-1}x_{n-1}\| + \|U_{n-1,N-1}x_{n-1}\|) \\
= & \prod_{j=N-1}^N \left(\frac{2}{1-\kappa_j}\right) \|U_{n,N-2}x_{n-1} - U_{n-1,N-2}x_{n-1}\| \\
& + \sum_{j=N-1}^N \left(\frac{2}{1-\kappa_{j+1}}\right)^{N-j} |\lambda_j^n - \lambda_j^{n-1}| (\|T_j U_{n-1,j-1}x_{n-1}\| + \|U_{n-1,j-1}x_{n-1}\|) \\
& \vdots \\
\leq & \prod_{j=2}^N \left(\frac{2}{1-\kappa_j}\right) \|U_{n,1}x_{n-1} - U_{n-1,1}x_{n-1}\| \\
& + \sum_{j=2}^N \left(\frac{2}{1-\kappa_{j+1}}\right)^{N-j} |\lambda_j^n - \lambda_j^{n-1}| (\|T_j U_{n-1,j-1}x_{n-1}\| + \|U_{n-1,j-1}x_{n-1}\|) \\
= & \prod_{j=2}^N \left(\frac{2}{1-\kappa_j}\right) |\lambda_1^n - \lambda_1^{n-1}| \|T_1 x_{n-1} - x_{n-1}\| \\
& + \sum_{j=2}^N \left(\frac{2}{1-\kappa_{j+1}}\right)^{N-j} |\lambda_j^n - \lambda_j^{n-1}| (\|T_j U_{n-1,j-1}x_{n-1}\| + \|U_{n-1,j-1}x_{n-1}\|) \\
\leq & \prod_{j=2}^N \left(\frac{2}{1-\kappa_j}\right) |\lambda_1^n - \lambda_1^{n-1}| M + 2 \sum_{j=2}^N \left(\frac{2}{1-\kappa_{j+1}}\right)^{N-j} |\lambda_j^n - \lambda_j^{n-1}| M, \quad (3.97)
\end{aligned}$$

where $M = \max_{n \in \mathbb{N}} \{\|T_1 x_{n-1} - x_{n-1}\|, \|T_j U_{n-1,j-1}x_{n-1}\|, \|U_{n-1,j-1}x_{n-1}\|\}$, for all $j = 2, 3, \dots, N$. Hence, by (3.97) and $\sum_{n=1}^{\infty} |\lambda_i^{n+1} - \lambda_i^n| < \infty$ for all $i = 1, 2, \dots, N$, we have

$$\sum_{n=1}^{\infty} \|K_n x_{n-1} - K_{n-1} x_{n-1}\| < \infty.$$

□

Theorem 3.17. Let C be a nonempty closed convex subset of a real Hilbert space H . For every $i = 1, 2, \dots, N$, $S_i : H \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with coefficients μ_i , $\Phi_i : H \times C \times C \rightarrow \mathbb{R}$ be equilibrium-like function satisfying (H1)-(H3). Let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function and $A : C \rightarrow C$ be an α -inverse strongly monotone mapping. Let $\{T_i\}_{i=1}^{\bar{N}}$ be a finite family of κ_i -strictly pseudo-contractive mappings and $\kappa_i \leq \gamma_1$ with $\mathcal{F} := \bigcap_{i=1}^{\bar{N}} \text{Fix}(T_i) \cap \bigcap_{i=1}^N (MGEP)_s(\Phi_i, \varphi, A) \neq \emptyset$. For every $n \in \mathbb{N}$, let K_n be the K-mapping generated by $T_1, T_2, \dots, T_{\bar{N}}$ and $\lambda_1^n, \lambda_2^n, \dots, \lambda_{\bar{N}}^n$ where $0 < \phi \leq \lambda_i^n \leq \psi < \gamma_2$, for all $i = 1, 2, \dots, \bar{N}$ and $\gamma_1 + \gamma_2 < 1$. For every $i = 1, 2, \dots, N$, let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and $w_i^1 \in S_i(I - r_1^i A)x_1$, there exists

sequences $\{w_n^i\} \in H$ and $\{x_n\}, \{u_n^i\} \subseteq C$ such that

$$\begin{cases} w_n^i \in S_i(I - r_n^i A)x_n, \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n}) \mathcal{H}(S_i(I - r_n^i A)x_n, S_i(I - r_{n+1}^i A)x_{n+1}), \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - x_n, y - u_n^i \rangle + \langle Ax_n, y - u_n^i \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n \left(\sum_{i=1}^N a_n^i u_n^i \right) + \delta_n K_n x_n, \forall n \geq 1, \end{cases} \quad (3.98)$$

where $f : C \rightarrow C$ be a contraction mapping with a constant ξ and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subseteq (0, 1)$ with $\alpha_n + \beta_n + \delta_n = 1, \forall n \geq 1$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < \tau \leq \beta_n, \delta_n \leq \nu < 1$;
- (iii) $0 \leq \eta \leq a_n^i \leq \sigma < 1$, for all $i = 1, 2, \dots, N-1$ and $0 < \eta \leq a_n^N \leq \sigma \leq 1$ with $\sum_{n=1}^N a_n^i = 1$;
- (iv) $0 < \epsilon \leq r_n^i \leq \omega < 2\alpha$, for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, N$;
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty,$
 $\sum_{n=1}^{\infty} |r_{n+1}^i - r_n^i| < \infty, \sum_{n=1}^{\infty} |a_{n+1}^i - a_n^i| < \infty, \sum_{n=1}^{\infty} |\lambda_k^{n+1} - \lambda_k^n| < \infty$, for all $i = 1, 2, \dots, N$ and $k = 1, 2, \dots, \bar{N}$;
- (vi) For each $i = 1, 2, \dots, N$, there exists $\rho_i > 0$ such that

$$\Phi_i(w_{r_1^i}^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)) + \Phi_i(w_{r_2^i}^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)) \leq -\rho_i \|T_{r_1^i}(x_1) - T_{r_2^i}(x_2)\|^2, \quad (3.99)$$

for all $(r_1^i, r_2^i) \in \Theta_i \times \Theta_i, (x_1, x_2) \in H \times H$ and $w_j^i \in S_i(x_j)$, for $j = 1, 2$, where $\Theta_i = \{r_n^i : n \geq 1\}$.

Then $\{x_n\}$ and $\{u_n^i\}$ converge strongly to $q = P_{\mathcal{F}}f(q)$, for every $i = 1, 2, \dots, N$.

Proof. Let $z \in \mathcal{F}$. From (3.98), we have

$$\Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - (I - r_n^i A)x_n, y - u_n^i \rangle \geq 0, \quad (3.100)$$

for every $y \in C$. From (3.100) and Theorem 2.44, we obtain

$$u_n^i = T_{r_n^i}(I - r_n^i A)x_n, \forall i = 1, 2, \dots, N.$$

Since $z \in \mathcal{F} = \bigcap_{i=1}^{\bar{N}} \text{Fix}(T_i) \cap \bigcap_{i=1}^N (MGE P)_s(\Phi_i, \varphi, A)$, then, for every $i = 1, 2, \dots, N$, we derive

$$\begin{aligned} & \Phi_i(w, z, v) + \varphi(y) - \varphi(z) + \langle y - z, Az \rangle \geq 0, \forall y \in C, \\ & \Rightarrow \Phi_i(w, z, v) + \varphi(y) - \varphi(z) + \frac{1}{r_n^i} \langle y - z, z - z + r_n^i Az \rangle \geq 0, \forall y \in C, \\ & \Rightarrow \Phi_i(w, z, v) + \varphi(y) - \varphi(z) + \frac{1}{r_n^i} \langle y - z, z - (I - r_n^i A)z \rangle \geq 0, \forall y \in C. \end{aligned}$$

From Theorem 2.44, we obtain

$$z = T_{r_n^i} (I - r_n^i A) z, \forall i = 1, 2, \dots, N.$$

Put $r^i \in \Theta_i$ for all $i = 1, 2, \dots, N$. From (3.99), we have

$$\Phi_i (w_1^i, T_{r^i} (x_1), T_{r^i} (x_2)) + \Phi_i (w_2^i, T_{r^i} (x_2), T_{r^i} (x_1)) \leq -\rho_i \|T_{r^i} (x_1) - T_{r^i} (x_2)\|^2 \leq 0, \quad (3.101)$$

for all $(x_1, x_2) \in H \times H$ and $w_j^i \in S_i (x_j), j = 1, 2$.

From (3.101), it implies that Theorem 2.44 holds.

The proof shall be divided into seven steps.

Step 1. We will prove that $I - r_n^i A$ is nonexpansive, for all $i = 1, 2, \dots, N$.

It is obvious to see that $I - r_n^i A$ is a nonexpansive mapping, for every $i = 1, 2, \dots, N$.

Indeed, A is α -inverse strongly monotone with $r_n^i \in (0, 2\alpha)$, we get

$$\begin{aligned} \|(I - r_n^i A)x - (I - r_n^i A)y\|^2 &= \|x - y - r_n^i (Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2r_n^i \langle x - y, Ax - Ay \rangle + (r_n^i)^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\alpha r_n^i \|Ax - Ay\|^2 + (r_n^i)^2 \|Ax - Ay\|^2 \\ &= \|x - y\|^2 + r_n^i (r_n^i - 2\alpha) \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Thus $I - r_n^i A$ is a nonexpansive mapping, for all $i = 1, 2, \dots, N$.

Step 2. We will show that $\{x_n\}$ is bounded.

By nonexpansiveness of K_n , we have

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n \|f(x_n) - z\| + \beta_n \left\| \sum_{i=1}^N a_n^i (u_n^i - z) \right\| + \delta_n \|K_n x_n - z\| \\ &\leq \alpha_n \|f(x_n) - f(z) + f(z) - z\| + \beta_n \sum_{i=1}^N a_n^i \|u_n^i - z\| + \delta_n \|x_n - z\| \\ &\leq \alpha_n (\|f(x_n) - f(z)\| + \|f(z) - z\|) + \beta_n \sum_{i=1}^N a_n^i \|T_{r_n^i} (I - r_n^i A) x_n - z\| \\ &\quad + \delta_n \|x_n - z\| \\ &\leq \alpha_n (\xi \|x_n - z\| + \|f(z) - z\|) + \beta_n \sum_{i=1}^N a_n^i \|x_n - z\| + \delta_n \|x_n - z\| \\ &= (1 - \alpha_n (1 - \xi)) \|x_n - z\| + \alpha_n \|f(z) - z\| \\ &\leq \max \left\{ \|x_1 - z\|, \frac{\|f(z) - z\|}{1 - \xi} \right\}. \end{aligned}$$

By induction, we have $\|x_n - z\| \leq \max \left\{ \|x_1 - z\|, \frac{\|f(z) - z\|}{1 - \xi} \right\}, \forall n \in \mathbb{N}$. It follows that $\{x_n\}$ is bounded so is $\{u_n^i\}, \forall i = 1, 2, \dots, N$.

Step 3. We will show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

By the definition of x_n , we obtain

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
&= \left\| \alpha_n f(x_n) + \beta_n \left(\sum_{i=1}^N a_n^i u_n^i \right) + \delta_n K_n x_n \right. \\
&\quad \left. - \left(\alpha_{n-1} f(x_{n-1}) + \beta_{n-1} \left(\sum_{i=1}^N a_{n-1}^i u_{n-1}^i \right) + \delta_{n-1} K_{n-1} x_{n-1} \right) \right\| \\
&\leq \alpha_n \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
&\quad + \beta_n \left\| \sum_{i=1}^N a_n^i u_n^i - \sum_{i=1}^N a_{n-1}^i u_{n-1}^i \right\| + |\beta_n - \beta_{n-1}| \left\| \sum_{i=1}^N a_{n-1}^i u_{n-1}^i \right\| \\
&\quad + \delta_n \|K_n x_n - K_n x_{n-1}\| + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\| \\
&\leq \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
&\quad + \beta_n \left\| \sum_{i=1}^N a_n^i u_n^i - \sum_{i=1}^N a_{n-1}^i u_{n-1}^i + \sum_{i=1}^N a_n^i u_{n-1}^i - \sum_{i=1}^N a_{n-1}^i u_{n-1}^i \right\| \\
&\quad + |\beta_n - \beta_{n-1}| \sum_{i=1}^N a_{n-1}^i \|u_{n-1}^i\| + \delta_n \|x_n - x_{n-1}\| \\
&\quad + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\| \\
&\leq \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \left\| \sum_{i=1}^N a_n^i (u_n^i - u_{n-1}^i) \right\| \\
&\quad + \beta_n \left\| \sum_{i=1}^N (a_n^i - a_{n-1}^i) u_{n-1}^i \right\| + |\beta_n - \beta_{n-1}| \sum_{i=1}^N a_{n-1}^i \|u_{n-1}^i\| \\
&\quad + \delta_n \|x_n - x_{n-1}\| + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\| \\
&\leq \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \sum_{i=1}^N a_n^i \|u_n^i - u_{n-1}^i\| \\
&\quad + \beta_n \sum_{i=1}^N |a_n^i - a_{n-1}^i| \|u_{n-1}^i\| + |\beta_n - \beta_{n-1}| \sum_{i=1}^N a_{n-1}^i \|u_{n-1}^i\| \\
&\quad + \delta_n \|x_n - x_{n-1}\| + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\|. \quad (3.102)
\end{aligned}$$

From $u_n^i = T_{r_n^i}(I - r_n^i A)x_n$, for all $i = 1, 2, \dots, N$, we have

$$\Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - (I - r_n^i A)x_n, y - u_n^i \rangle \geq 0, \forall y \in C,$$

and

$$\Phi_i(w_{n+1}^i, u_{n+1}^i, y) + \varphi(y) - \varphi(u_{n+1}^i) + \frac{1}{r_{n+1}^i} \langle u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}, y - u_{n+1}^i \rangle \geq 0, \forall y \in C.$$

In particular, we obtain

$$\Phi_i(w_n^i, u_n^i, u_{n+1}^i) + \varphi(u_{n+1}^i) - \varphi(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - (I - r_n^i A)x_n, u_{n+1}^i - u_n^i \rangle \geq 0, \quad (3.103)$$

and

$$\Phi_i(w_{n+1}^i, u_{n+1}^i, u_n^i) + \varphi(u_n^i) - \varphi(u_{n+1}^i) + \frac{1}{r_{n+1}^i} \langle u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}, u_n^i - u_{n+1}^i \rangle \geq 0. \quad (3.104)$$

Summing up (3.103) with (3.104) and applying (3.101), we get

$$\frac{1}{r_n^i} \langle u_n^i - (I - r_n^i A)x_n, u_{n+1}^i - u_n^i \rangle + \frac{1}{r_{n+1}^i} \langle u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}, u_n^i - u_{n+1}^i \rangle \geq 0,$$

which implies that

$$\left\langle u_{n+1}^i - u_n^i, \frac{u_n^i - (I - r_n^i A)x_n}{r_n^i} - \frac{u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}}{r_{n+1}^i} \right\rangle \geq 0.$$

It follows that

$$\left\langle u_{n+1}^i - u_n^i, u_n^i - u_{n+1}^i + u_{n+1}^i - (I - r_n^i A)x_n - \frac{r_n^i}{r_{n+1}^i} (u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}) \right\rangle \geq 0. \quad (3.105)$$

From (3.105), we obtain

$$\begin{aligned} & \|u_{n+1}^i - u_n^i\|^2 \\ & \leq \left\langle u_{n+1}^i - u_n^i, u_{n+1}^i - (I - r_n^i A)x_n - \frac{r_n^i}{r_{n+1}^i} (u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}) \right\rangle \\ & = \left\langle u_{n+1}^i - u_n^i, (I - r_{n+1}^i A)x_{n+1} - (I - r_n^i A)x_n \right. \\ & \quad \left. + \left(1 - \frac{r_n^i}{r_{n+1}^i}\right) (u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}) \right\rangle \\ & \leq \|u_{n+1}^i - u_n^i\| \left\| (I - r_{n+1}^i A)x_{n+1} - (I - r_n^i A)x_n \right. \\ & \quad \left. + \left(1 - \frac{r_n^i}{r_{n+1}^i}\right) (u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}) \right\| \\ & \leq \|u_{n+1}^i - u_n^i\| \left[\left\| (I - r_{n+1}^i A)x_{n+1} - (I - r_{n+1}^i A)x_n + (I - r_{n+1}^i A)x_n \right. \right. \\ & \quad \left. \left. - (I - r_n^i A)x_n \right\| + \left|1 - \frac{r_n^i}{r_{n+1}^i}\right| \|u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}\| \right] \\ & \leq \|u_{n+1}^i - u_n^i\| \left[\left\| (I - r_{n+1}^i A)x_{n+1} - (I - r_{n+1}^i A)x_n \right\| \right. \\ & \quad \left. + \left\| (I - r_{n+1}^i A)x_n - (I - r_n^i A)x_n \right\| \right. \\ & \quad \left. + \frac{1}{r_{n+1}^i} |r_{n+1}^i - r_n^i| \|u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}\| \right] \\ & \leq \|u_{n+1}^i - u_n^i\| \left[\|x_{n+1} - x_n\| + |r_{n+1}^i - r_n^i| \|Ax_n\| \right. \\ & \quad \left. + \frac{1}{\epsilon} |r_{n+1}^i - r_n^i| \|u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}\| \right], \end{aligned}$$

which follows that

$$\|u_{n+1}^i - u_n^i\| \leq \|x_{n+1} - x_n\| + |r_{n+1}^i - r_n^i| \|Ax_n\| + \frac{1}{\epsilon} |r_{n+1}^i - r_n^i| \|u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}\|. \quad (3.106)$$

From (3.106), we have

$$\|u_n^i - u_{n-1}^i\| \leq \|x_n - x_{n-1}\| + |r_n^i - r_{n-1}^i| \|Ax_{n-1}\| + \frac{1}{\epsilon} |r_n^i - r_{n-1}^i| \|u_n^i - (I - r_n^i A)x_n\|. \quad (3.107)$$

From (3.102) and (3.107), we obtain

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
& \leq \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \sum_{i=1}^N a_n^i \left[\|x_n - x_{n-1}\| \right. \\
& \quad \left. + |r_n^i - r_{n-1}^i| \|Ax_{n-1}\| + \frac{1}{\epsilon} |r_n^i - r_{n-1}^i| \|u_n^i - (I - r_n^i A)x_n\| \right] \\
& \quad + \beta_n \sum_{i=1}^N |a_n^i - a_{n-1}^i| \|u_{n-1}^i\| + |\beta_n - \beta_{n-1}| \sum_{i=1}^N a_{n-1}^i \|u_{n-1}^i\| + \delta_n \|x_n - x_{n-1}\| \\
& \quad + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\| \\
& = \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \sum_{i=1}^N a_n^i \|x_n - x_{n-1}\| \\
& \quad + \beta_n \sum_{i=1}^N a_n^i |r_n^i - r_{n-1}^i| \|Ax_{n-1}\| + \frac{\beta_n}{\epsilon} \sum_{i=1}^N a_n^i |r_n^i - r_{n-1}^i| \|u_n^i - (I - r_n^i A)x_n\| \\
& \quad + \beta_n \sum_{i=1}^N |a_n^i - a_{n-1}^i| \|u_{n-1}^i\| + |\beta_n - \beta_{n-1}| \sum_{i=1}^N a_{n-1}^i \|u_{n-1}^i\| + \delta_n \|x_n - x_{n-1}\| \\
& \quad + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\| \\
& \leq (1 - \alpha_n(1 - \xi)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
& \quad + \sum_{i=1}^N |r_n^i - r_{n-1}^i| \|Ax_{n-1}\| + \frac{1}{\epsilon} \sum_{i=1}^N |r_n^i - r_{n-1}^i| \|u_n^i - (I - r_n^i A)x_n\| \\
& \quad + \sum_{i=1}^N |a_n^i - a_{n-1}^i| \|u_{n-1}^i\| + |\beta_n - \beta_{n-1}| \sum_{i=1}^N a_{n-1}^i \|u_{n-1}^i\| \\
& \quad + \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\| \\
& \leq (1 - \alpha_n(1 - \xi)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M + \sum_{i=1}^N |r_n^i - r_{n-1}^i| M + \frac{1}{\epsilon} \sum_{i=1}^N |r_n^i - r_{n-1}^i| M \\
& \quad + \sum_{i=1}^N |a_n^i - a_{n-1}^i| M + |\beta_n - \beta_{n-1}| M + \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| M.
\end{aligned}$$

Choose $M = \max_{n \in \mathbb{N} \text{ and } i=1,2,\dots,N} \{\|f(x_n)\|, \|Ax_n\|, \|u_n^i - (I - r_n^i A)x_n\|, \|u_n^i\|, \|K_n x_n\|\}$. Applying the conditions (i), (v), Lemma 2.13 and Lemma 3.16(i), we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.108)$$

Step 4. We will show that $\lim_{n \rightarrow \infty} \|u_n^i - x_n\| = \lim_{n \rightarrow \infty} \|K_n x_n - x_n\| = 0, \forall i = 1, 2, \dots, N$. Since $T_{r_n^i}$ is a firmly nonexpansive mapping, for every $i = 1, 2, \dots, N$, we obtain

$$\begin{aligned}
& \|T_{r_n^i}(I - r_n^i A)x_n - z\|^2 = \|T_{r_n^i}(I - r_n^i A)x_n - T_{r_n^i}(I - r_n^i A)z\|^2 \\
& \leq \langle (I - r_n^i A)x_n - (I - r_n^i A)z, u_n^i - z \rangle \\
& = \frac{1}{2} \left(\|(I - r_n^i A)x_n - (I - r_n^i A)z\|^2 + \|u_n^i - z\|^2 \right. \\
& \quad \left. - \|(I - r_n^i A)x_n - (I - r_n^i A)z - (u_n^i - z)\|^2 \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \left(\|x_n - z\|^2 + \|u_n^i - z\|^2 - \|(x_n - u_n^i) - r_n^i (Ax_n - Az)\|^2 \right) \\
&= \frac{1}{2} \left(\|x_n - z\|^2 + \|u_n^i - z\|^2 - \|x_n - u_n^i\|^2 - (r_n^i)^2 \|Ax_n - Az\|^2 \right. \\
&\quad \left. + 2r_n^i \langle x_n - u_n^i, Ax_n - Az \rangle \right) \\
&\leq \frac{1}{2} \left(\|x_n - z\|^2 + \|u_n^i - z\|^2 - \|x_n - u_n^i\|^2 + 2r_n^i \|x_n - u_n^i\| \|Ax_n - Az\| \right),
\end{aligned}$$

which implies that

$$\|u_n^i - z\|^2 \leq \|x_n - z\|^2 - \|x_n - u_n^i\|^2 + 2r_n^i \|x_n - u_n^i\| \|Ax_n - Az\|. \quad (3.109)$$

From the nonexpansiveness of $T_{r_n^i}$ and $u_n^i = T_{r_n^i} (I - r_n^i A) x_n$, for every $i = 1, 2, \dots, N$, we have

$$\begin{aligned}
\|u_n^i - z\|^2 &= \|T_{r_n^i} (I - r_n^i A) x_n - T_{r_n^i} (I - r_n^i A) z\|^2 \\
&\leq \|(x_n - z) - r_n^i (Ax_n - Az)\|^2 \\
&= \|x_n - z\|^2 - 2r_n^i \langle x_n - z, Ax_n - Az \rangle + (r_n^i)^2 \|Ax_n - Az\|^2 \\
&\leq \|x_n - z\|^2 - 2\alpha r_n^i \|Ax_n - Az\|^2 + (r_n^i)^2 \|Ax_n - Az\|^2 \\
&= \|x_n - z\|^2 - r_n^i (2\alpha - r_n^i) \|Ax_n - Az\|^2.
\end{aligned} \quad (3.110)$$

From the definition of x_n and (3.110), we get

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\leq \alpha_n \|f(x_n) - z\|^2 + \beta_n \sum_{i=1}^N a_n^i \|u_n^i - z\|^2 + \delta_n \|x_n - z\|^2 \\
&\leq \alpha_n \|f(x_n) - z\|^2 + \beta_n \sum_{i=1}^N a_n^i \left(\|x_n - z\|^2 - r_n^i (2\alpha - r_n^i) \|Ax_n - Az\|^2 \right) \\
&\quad + \delta_n \|x_n - z\|^2 \\
&= \alpha_n \|f(x_n) - z\|^2 + \beta_n \sum_{i=1}^N a_n^i \|x_n - z\|^2 - \beta_n \sum_{i=1}^N a_n^i r_n^i (2\alpha - r_n^i) \|Ax_n - Az\|^2 \\
&\quad + \delta_n \|x_n - z\|^2 \\
&\leq \|x_n - z\|^2 + \alpha_n \|f(x_n) - z\|^2 - \beta_n \sum_{i=1}^N a_n^i r_n^i (2\alpha - r_n^i) \|Ax_n - Az\|^2,
\end{aligned}$$

which follows that

$$\begin{aligned}
\beta_n \sum_{i=1}^N a_n^i r_n^i (2\alpha - r_n^i) \|Ax_n - Az\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n \|f(x_n) - z\|^2 \\
&\leq (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - z\|^2.
\end{aligned} \quad (3.111)$$

From (3.108), (3.111), the conditions (i), (ii), (iii) and (iv), we obtain

$$\lim_{n \rightarrow \infty} \|Ax_n - Az\| = 0. \quad (3.112)$$

From the definition of x_n and (3.109), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \alpha_n \|f(x_n) - z\|^2 + \beta_n \sum_{i=1}^N a_n^i \|u_n^i - z\|^2 + \delta_n \|x_n - z\|^2 \\
&\leq \alpha_n \|f(x_n) - z\|^2 + \beta_n \sum_{i=1}^N a_n^i \left(\|x_n - z\|^2 - \|x_n - u_n^i\|^2 \right. \\
&\quad \left. + 2r_n^i \|x_n - u_n^i\| \|Ax_n - Az\| \right) + \delta_n \|x_n - z\|^2 \\
&\leq \alpha_n \|f(x_n) - z\|^2 + \beta_n \sum_{i=1}^N a_n^i \|x_n - z\|^2 - \beta_n \sum_{i=1}^N a_n^i \|x_n - u_n^i\|^2 \\
&\quad + 2\beta_n \sum_{i=1}^N a_n^i r_n^i \|x_n - u_n^i\| \|Ax_n - Az\| + \delta_n \|x_n - z\|^2 \\
&\leq \|x_n - z\|^2 + \alpha_n \|f(x_n) - z\|^2 - \beta_n \sum_{i=1}^N a_n^i \|x_n - u_n^i\|^2 \\
&\quad + 2\beta_n \sum_{i=1}^N a_n^i r_n^i \|x_n - u_n^i\| \|Ax_n - Az\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\beta_n \sum_{i=1}^N a_n^i \|x_n - u_n^i\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n \|f(x_n) - z\|^2 \\
&\quad + 2\beta_n \sum_{i=1}^N a_n^i r_n^i \|x_n - u_n^i\| \|Ax_n - Az\| \\
&\leq (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - z\|^2 \\
&\quad + 2\beta_n \sum_{i=1}^N a_n^i r_n^i \|x_n - u_n^i\| \|Ax_n - Az\|. \tag{3.113}
\end{aligned}$$

From (3.108), (3.112), (3.113) and the conditions (i), (ii), (iii), we get

$$\lim_{n \rightarrow \infty} \|x_n - u_n^i\| = 0, \text{ for all } i = 1, 2, \dots, N. \tag{3.114}$$

By the definition of x_n , we obtain

$$\begin{aligned}
x_{n+1} - x_n &= \alpha_n f(x_n) + \beta_n \left(\sum_{i=1}^N a_n^i u_n^i \right) + \delta_n K_n x_n - x_n \\
&= \alpha_n (f(x_n) - x_n) + \beta_n \sum_{i=1}^N a_n^i (u_n^i - x_n) + \delta_n (K_n x_n - x_n).
\end{aligned}$$

From (3.108), (3.114) and the conditions (i) and (ii), we get

$$\lim_{n \rightarrow \infty} \|K_n x_n - x_n\| = 0. \tag{3.115}$$

Step 5. We show that $\{x_n\}$, $\{w_n^i\}$ and $\{r_n^i\}$ are Cauchy sequences, for every $i = 1, 2, \dots, N$.

Let $a \in (0, 1)$, by (3.108), there exists $N \in \mathbb{N}$ such that

$$\|x_{n+1} - x_n\| < a^n, \forall n \geq N. \tag{3.116}$$

Thus, for any $n \geq N \in \mathbb{N}$ and $p \in \mathbb{N}$, we have

$$\|x_{n+p} - x_n\| \leq \sum_{k=n}^{n+p-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{n+p-1} a^k < \sum_{k=n}^{\infty} a^k = \frac{a^n}{1-a}. \quad (3.117)$$

Since $a \in (0, 1)$, we get $\lim_{n \rightarrow \infty} a^n = 0$. From (3.117), taking $n \rightarrow \infty$, we obtain $\{x_n\}$ is a Cauchy sequence in a Hilbert space H . Let $\lim_{n \rightarrow \infty} x_n = x^*$. Since $S_i : C \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous on H with coefficients μ_i , for every $i = 1, 2, \dots, N$, and (3.98), we have

$$\begin{aligned} \|w_n^i - w_{n+1}^i\| &\leq \left(1 + \frac{1}{n}\right) \mathcal{H}(S_i(I - r_n^i A)x_n, S_i(I - r_{n+1}^i A)x_{n+1}) \\ &\leq \left(1 + \frac{1}{n}\right) \mu_i \|(I - r_n^i A)x_n - (I - r_{n+1}^i A)x_{n+1}\| \\ &\leq \left(1 + \frac{1}{n}\right) \mu_i \left(\|(I - r_n^i A)x_n - (I - r_n^i A)x_{n+1}\| \right. \\ &\quad \left. + \|(I - r_n^i A)x_{n+1} - (I - r_{n+1}^i A)x_{n+1}\|\right) \\ &\leq \left(1 + \frac{1}{n}\right) \mu_i \left(\|x_n - x_{n+1}\| + |r_{n+1}^i - r_n^i| \|Ax_{n+1}\|\right) \\ &\leq \left(1 + \frac{1}{n}\right) \mu_i \left(\|x_n - x_{n+1}\| + |r_{n+1}^i - r_n^i| M\right), \end{aligned} \quad (3.118)$$

where $M = \max_{n \in \mathbb{N}} \{\|Ax_n\|\}$. From (3.108), (3.118) and the condition (vi), we obtain

$$\lim_{n \rightarrow \infty} \|w_n^i - w_{n+1}^i\| = 0, \text{ for every } i = 1, 2, \dots, N.$$

By continuing the same argument as (3.116) and (3.117), we have $\{w_n^i\}$ is a Cauchy sequence in a Hilbert space H , for all $i = 1, 2, \dots, N$. Let $\lim_{n \rightarrow \infty} w_n^i = w_i^*$, for every $i = 1, 2, \dots, N$. Using the same method as above and the condition (vi), we have $\{r_n^i\}$ is a Cauchy sequence, for all $i = 1, 2, \dots, N$. Put $\lim_{n \rightarrow \infty} r_n^i = r_i^*$, for every $i = 1, 2, \dots, N$. Next, we will prove that $w_i^* \in S_i(I - r_i^* A)x^*$, for all $i = 1, 2, \dots, N$.

Since $w_n^i \in S_i(I - r_n^i A)x_n$, for every $i = 1, 2, \dots, N$, we obtain

$$d(w_n^i, S_i(I - r_i^* A)x^*) = \inf_{\tilde{w}_i \in S_i(I - r_i^* A)x^*} d(w_n^i, \tilde{w}_i) \leq d(w_n^i, \tilde{w}_i), \forall \tilde{w}_i \in S_i(I - r_i^* A)x^*. \quad (3.119)$$

and

$$d(S_i(I - r_i^* A)x_n, \tilde{w}_i) = \inf_{\tilde{S}_i \in S_i(I - r_i^* A)x_n} d(\tilde{S}_i, \tilde{w}_i) \leq d(\tilde{S}_i, \tilde{w}_i), \forall \tilde{S}_i \in S_i(I - r_i^* A)x_n. \quad (3.120)$$

From (3.119) and (3.120), we get

$$\begin{aligned} d(w_n^i, S_i(I - r_i^* A)x^*) &\leq d(S_i(I - r_i^* A)x_n, \tilde{w}_i), \forall \tilde{w}_i \in S_i(I - r_i^* A)x^* \\ &\leq \sup_{\tilde{w}_i \in S_i(I - r_i^* A)x^*} d(S_i(I - r_i^* A)x_n, \tilde{w}_i). \end{aligned} \quad (3.121)$$

From (3.121), we can assume that

$$\begin{aligned}
& d(w_n^i, S_i(I - r_i^* A)x^*) \\
& \leq \max \left\{ d(w_n^i, S_i(I - r_i^* A)x^*), \sup_{\tilde{w}_i \in S_i(I - r_i^* A)x^*} d(S_i(I - r_n^i A)x_n, \tilde{w}_i) \right\} \\
& \leq \max \left\{ \sup_{\tilde{w}_i \in S_i(I - r_n^i A)x_n} d(\tilde{w}_i, S_i(I - r_i^* A)x^*), \sup_{\tilde{w}_i \in S_i(I - r_i^* A)x^*} d(S_i(I - r_n^i A)x_n, \tilde{w}_i) \right\} \\
& = \mathcal{H}(S_i(I - r_n^i A)x_n, S_i(I - r_i^* A)x^*), \text{ for every } i = 1, 2, \dots, N. \tag{3.122}
\end{aligned}$$

Observe that

$$\begin{aligned}
d(w_i^*, S_i(I - r_i^* A)x^*) &= \inf_{\tilde{S}_i \in S_i(I - r_i^* A)x^*} d(w_i^*, \tilde{S}_i) \leq d(w_i^*, \tilde{S}_i), \forall \tilde{S}_i \in S_i(I - r_i^* A)x^* \\
&= \|w_i^* - \tilde{S}_i\|, \forall \tilde{S}_i \in S_i(I - r_i^* A)x^* \\
&\leq \|w_i^* - w_n^i\| + \|w_n^i - \tilde{S}_i\|, \forall \tilde{S}_i \in S_i(I - r_i^* A)x^*.
\end{aligned}$$

This follows that

$$d(w_i^*, S_i(I - r_i^* A)x^*) - \|w_i^* - w_n^i\| \leq \|w_n^i - \tilde{S}_i\|, \forall \tilde{S}_i \in S_i(I - r_i^* A)x^*.$$

Since $d(w_i^*, S_i(I - r_i^* A)x^*) - \|w_i^* - w_n^i\|$ is a lower bound of $\|w_n^i - \tilde{S}_i\|$ and $d(w_n^i, S_i(I - r_i^* A)x^*)$ is the greatest lower bound of $\|w_n^i - \tilde{S}_i\|$, for every $\tilde{S}_i \in S_i(I - r_i^* A)x^*$, then it implies that

$$d(w_i^*, S_i(I - r_i^* A)x^*) - \|w_i^* - w_n^i\| \leq d(w_n^i, S_i(I - r_i^* A)x^*).$$

That is, by (3.122), we derive

$$\begin{aligned}
d(w_i^*, S_i(I - r_i^* A)x^*) &\leq \|w_i^* - w_n^i\| + d(w_n^i, S_i(I - r_i^* A)x^*) \\
&\leq \|w_i^* - w_n^i\| + \mathcal{H}(S_i(I - r_n^i A)x_n, S_i(I - r_i^* A)x^*) \\
&\leq \|w_i^* - w_n^i\| + \mu_i \|(I - r_n^i A)x_n - (I - r_i^* A)x^*\| \\
&= \|w_i^* - w_n^i\| + \mu_i \|(x_n - x^*) - (r_n^i A x_n - r_i^* A x^*)\|,
\end{aligned}$$

taking $n \rightarrow \infty$, we have

$$d(w_i^*, S_i(I - r_i^* A)x^*) = 0,$$

Let $\varepsilon > 0$. Since $d(w_i^*, S_i(I - r_i^* A)x^*) = 0$, we obtain

$$d(w_i^*, S_i(I - r_i^* A)x^*) = \inf_{\tilde{S}_i \in S_i(I - r_i^* A)x^*} d(w_i^*, \tilde{S}_i) < \varepsilon.$$

Since ε is not a lower bound of $d(w_i^*, \tilde{S}_i)$, where $\tilde{S}_i \in S_i(I - r_i^* A)x^*$, then there exists $S_i^* \in S_i(I - r_i^* A)x^*$ such that

$$\begin{aligned}
d(w_i^*, S_i^*) < \varepsilon &\Rightarrow d(w_i^*, S_i^*) \leq 0 \Rightarrow d(w_i^*, S_i^*) = 0 \\
&\Rightarrow \|w_i^* - S_i^*\| = 0 \\
&\Rightarrow w_i^* - S_i^* = 0 \\
&\Rightarrow w_i^* = S_i^* \in S_i(I - r_i^* A)x^*,
\end{aligned}$$

which implies that

$$w_i^* \in S_i(I - r_i^* A)x^*, \text{ for all } i = 1, 2, \dots, N. \quad (3.123)$$

Step 6. We will show that $\limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle \leq 0$, where $q = P_{\mathcal{F}}f(q)$.

To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle = \lim_{k \rightarrow \infty} \langle f(q) - q, x_{n_k} - q \rangle.$$

Without loss of generality, we can assume that $x_{n_k} \rightarrow \bar{x}$ as $k \rightarrow \infty$.

For every $i = 1, 2, \dots, \bar{N}$, $0 < \phi \leq \lambda_i^n \leq \psi < \gamma_2 < 1$, without loss of generality, we may assume that

$$\lambda_i^{n_k} \rightarrow \lambda_i \in (0, 1) \text{ as } k \rightarrow \infty, \text{ for every } i = 1, 2, \dots, \bar{N}.$$

Let K be the K -mapping generated by $T_1, T_2, \dots, T_{\bar{N}}$ and $\lambda_1, \lambda_2, \dots, \lambda_{\bar{N}}$. By Lemma 2.12, we have K is nonexpansive and $\text{Fix}(K) = \bigcap_{i=1}^{\bar{N}} \text{Fix}(T_i)$.

From Lemma 3.16 (i), we obtain

$$\lim_{k \rightarrow \infty} \|K_{n_k} x_{n_k} - Kx_{n_k}\| = 0. \quad (3.124)$$

Since

$$\|x_{n_k} - Kx_{n_k}\| \leq \|x_{n_k} - K_{n_k} x_{n_k}\| + \|K_{n_k} x_{n_k} - Kx_{n_k}\|,$$

by (3.115) and (3.124), we have

$$\lim_{k \rightarrow \infty} \|x_{n_k} - Kx_{n_k}\| = 0. \quad (3.125)$$

Since $x_{n_k} \rightarrow \bar{x}$ as $n \rightarrow \infty$, (3.125) and Lemma 3.14(i), we have

$$\bar{x} \in \text{Fix}(K) = \bigcap_{i=1}^{\bar{N}} \text{Fix}(T_i). \quad (3.126)$$

Next, we show that $x^* \in \bigcap_{i=1}^N (\text{GEP})_s(\Phi_i, \varphi, A)$.

Since $x_{n_k} \rightarrow x^*$ as $k \rightarrow \infty$ and (3.114), we have

$$u_{n_k}^i \rightarrow x^* \text{ as } k \rightarrow \infty, \text{ for all } i = 1, 2, \dots, N. \quad (3.127)$$

From (3.98), we obtain

$$\Phi_i(w_{n_k}^i, u_{n_k}^i, y) + \varphi(y) - \varphi(u_{n_k}^i) + \frac{1}{r_{n_k}^i} \langle u_{n_k}^i - x_{n_k}, y - u_{n_k}^i \rangle + \langle Ax_{n_k}, y - u_{n_k}^i \rangle \geq 0,$$

for every $y \in C$ and $i = 1, 2, \dots, N$. From (3.114), (3.127), the condition (H1) and the lower semicontinuity of φ , we get

$$\begin{aligned} \Phi_i(w_i^*, x^*, y) + \varphi(y) + \langle Ax^*, y - x^* \rangle &\geq \limsup_{k \rightarrow \infty} \varphi(u_{n_k}^i) \geq \liminf_{k \rightarrow \infty} \varphi(u_{n_k}^i) \geq \varphi(x^*) \\ &\Rightarrow \Phi_i(w_i^*, x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^* \rangle \geq 0, \end{aligned}$$

for every $y \in C$ and $i = 1, 2, \dots, N$, which follows by (3.123) that

$$x^* \in (GEP)_s(\Phi_i, \varphi, A), \text{ for every } i = 1, 2, \dots, N.$$

It implies that

$$x^* \in \bigcap_{i=1}^N (GEP)_s(\Phi_i, \varphi, A). \quad (3.128)$$

Since $x_{n_k} \rightarrow \bar{x}$ and $x_{n_k} \rightarrow x^*$ as $n \rightarrow \infty$, then $\bar{x} = x^*$. From (3.126) and (3.128), we have

$$x^* \in \mathcal{F}. \quad (3.129)$$

Indeed, since $x_{n_k} \rightarrow x^*$ as $k \rightarrow \infty$, (3.129) and Lemma 2.11, therefore we obtain

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle = \lim_{k \rightarrow \infty} \langle f(q) - q, x_{n_k} - q \rangle = \langle f(q) - q, x^* - q \rangle \leq 0. \quad (3.130)$$

Step 7. Finally, we will prove that $\{x_n\}$ and $\{u_n^i\}$ converges strongly to $q = P_{\mathcal{F}}f(q)$, for every $i = 1, 2, \dots, N$.

By Lemma 2.1, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \left\| \alpha_n (f(x_n) - q) + \beta_n \sum_{i=1}^N a_n^i (u_n^i - q) + \delta_n (K_n x_n - q) \right\|^2 \\ &\leq \left\| \beta_n \sum_{i=1}^N a_n^i (u_n^i - q) + \delta_n (K_n x_n - q) \right\|^2 + 2\alpha_n \langle f(x_n) - q, x_{n+1} - q \rangle \\ &\leq \left(\beta_n \left\| \sum_{i=1}^N a_n^i (u_n^i - q) \right\| + \delta_n \|K_n x_n - q\| \right)^2 \\ &\quad + 2\alpha_n \langle f(x_n) - f(q), x_{n+1} - q \rangle + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\ &\leq \left(\beta_n \sum_{i=1}^N a_n^i \|x_n - q\| + \delta_n \|x_n - q\| \right)^2 + 2\alpha_n \|f(x_n) - f(q)\| \|x_{n+1} - q\| \\ &\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\ &\leq ((1 - \alpha_n) \|x_n - q\|)^2 + 2\alpha_n \xi \|x_n - q\| \|x_{n+1} - q\| \\ &\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + \alpha_n \xi (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\ &\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \frac{(1 - \alpha_n)^2 + \alpha_n \xi}{1 - \alpha_n \xi} \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n \xi} \langle f(q) - q, x_{n+1} - q \rangle \\ &= \frac{1 - \alpha_n \xi - 2\alpha_n(1 - \xi)}{1 - \alpha_n \xi} \|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n \xi} \|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n \xi} \langle f(q) - q, x_{n+1} - q \rangle \\ &= \left(1 - \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi} \right) \|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n \xi} \|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n \xi} \langle f(q) - q, x_{n+1} - q \rangle \end{aligned}$$

$$= \left(1 - \frac{2\alpha_n(1-\xi)}{1-\alpha_n\xi} \right) \|x_n - q\|^2 + \frac{2\alpha_n(1-\xi)}{1-\alpha_n\xi} \left(\frac{\alpha_n}{2(1-\xi)} \|x_n - q\|^2 + \frac{1}{1-\xi} \langle f(q) - q, x_{n+1} - q \rangle \right).$$

Applying the condition (i), (3.130) and Lemma 2.13, we have the sequence $\{x_n\}$ converges strongly to $q = P_{\mathcal{F}}f(q)$. From (3.114), we also obtain $\{u_n^i\}$ converges strongly to $q = P_{\mathcal{F}}f(q)$, for every $i = 1, 2, \dots, N$. This completes the proof. \square

The following corollaries are consequences which are applied by Theorem 3.17.

Corollary 3.18. Let C be a nonempty closed convex subset of a real Hilbert space H . For every $i = 1, 2, \dots, N$, $S_i : H \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with coefficients μ_i , $\Phi_i : H \times C \times C \rightarrow \mathbb{R}$ be equilibrium-like function satisfying (H1)-(H3). Let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function and $A : C \rightarrow C$ be an α -inverse strongly monotone mapping. Let $T : C \rightarrow C$ be κ -strictly pseudo-contractive mapping with $\kappa \leq \gamma_1$ and $\mathcal{F} := F(T) \cap \bigcap_{i=1}^N (MGEP)_s(\Phi_i, \varphi, A) \neq \emptyset$. For every $n \in \mathbb{N}$, let $\{\lambda_n\}$ be a sequence of real numbers where $0 < \lambda_n < \gamma_2$ and $\gamma_1 + \gamma_2 < 1$. For every $i = 1, 2, \dots, N$, let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and $w_1^i \in S_i(I - r_1^i A)x_1$, there exists sequences $\{w_n^i\} \in H$ and $\{x_n\}, \{u_n^i\} \subseteq C$ such that

$$\begin{cases} w_n^i \in S_i(I - r_n^i A)x_n, \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n}) \mathcal{H}(S_i(I - r_n^i A)x_n, S_i(I - r_{n+1}^i A)x_{n+1}), \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - x_n, y - u_n^i \rangle + \langle Ax_n, y - u_n^i \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n \left(\sum_{i=1}^N a_n^i u_n^i \right) + \delta_n (\lambda_n T + (1 - \lambda_n) I)x_n, \forall n \geq 1, \end{cases} \quad (3.131)$$

where $f : C \rightarrow C$ be a contraction mapping with a constant ξ and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subseteq (0, 1)$ with $\alpha_n + \beta_n + \delta_n = 1, \forall n \geq 1$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < \tau \leq \beta_n, \delta_n \leq \nu < 1$;
- (iii) $0 \leq \eta \leq a_n^i \leq \sigma < 1$, for all $i = 1, 2, \dots, N - 1$ and $0 < \eta \leq a_n^N \leq \sigma \leq 1$ with $\sum_{n=1}^N a_n^i = 1$;
- (iv) $0 < \epsilon \leq r_n^i \leq \omega < 2\alpha$, for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, N$;
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$
 $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1}^i - r_n^i| < \infty,$
 $\sum_{n=1}^{\infty} |a_{n+1}^i - a_n^i| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, for all $i = 1, 2, \dots, N$;
- (vi) For each $i = 1, 2, \dots, N$, there exists $\rho_i > 0$ such that

$$\Phi_i(w_1^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)) + \Phi_i(w_2^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)) \leq -\rho_i \|T_{r_1^i}(x_1) - T_{r_2^i}(x_2)\|^2, \quad (3.132)$$

for all $(r_1^i, r_2^i) \in \Theta_i \times \Theta_i$, $(x_1, x_2) \in H \times H$ and $w_j^i \in S_i(x_j)$, for $j = 1, 2$, where $\Theta_i = \{r_n^i : n \geq 1\}$.

Then $\{x_n\}$ and $\{u_n^i\}$, for every $i = 1, 2, \dots, N$, converges strongly to $q = P_{\mathcal{F}}f(q)$.

Proof. Taking $\bar{N} = 1$ in Theorem 3.17. Then, we obtain the desired conclusion. \square

Corollary 3.19. Let C be a nonempty closed convex subset of a real Hilbert space H . For every $i = 1, 2, \dots, N$, $S_i : H \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with coefficients μ_i , $\Phi_i : H \times C \times C \rightarrow \mathbb{R}$ be equilibrium-like function satisfying (H1)-(H3). Let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function. Let $\{T_i\}_{i=1}^{\bar{N}}$ be a finite family of κ_i -strictly pseudo-contractive mappings and $\kappa_i \leq \gamma_1$ with $\mathcal{F} := \bigcap_{i=1}^{\bar{N}} F(T_i) \cap \bigcap_{i=1}^{\bar{N}} (GEP)_s(\Phi_i, \varphi) \neq \emptyset$. For every $n \in \mathbb{N}$, let K_n be the K-mapping generated by $T_1, T_2, \dots, T_{\bar{N}}$ and $\lambda_1^n, \lambda_2^n, \dots, \lambda_{\bar{N}}^n$ where $0 < \phi \leq \lambda_i^n \leq \psi < \gamma_2 < 1$, for all $i = 1, 2, \dots, \bar{N}$ and $\gamma_1 + \gamma_2 < 1$. For every $i = 1, 2, \dots, N$, let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and $w_1^i \in S_i(x_1)$, there exists sequences $\{w_n^i\} \in H$ and $\{x_n\}, \{u_n^i\} \subseteq C$ such that

$$\begin{cases} w_n^i \in S_i(x_n), \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n}) \mathcal{H}(S_i(x_n), S_i(x_{n+1})), \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - x_n, y - u_n^i \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n \left(\sum_{i=1}^N a_n^i u_n^i \right) + \delta_n K_n x_n, \forall n \geq 1, \end{cases} \quad (3.133)$$

where $f : C \rightarrow C$ be a contraction mapping with a constant ξ and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subseteq (0, 1)$ with $\alpha_n + \beta_n + \delta_n = 1, \forall n \geq 1$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < \tau \leq \beta_n, \delta_n \leq \nu < 1$;
- (iii) $0 \leq \eta \leq a_n^i \leq \sigma < 1$, for all $i = 1, 2, \dots, N-1$ and $0 < \eta \leq a_n^N \leq \sigma \leq 1$ with $\sum_{n=1}^N a_n^i = 1$;
- (iv) $0 < \epsilon \leq r_n^i \leq \omega < 1$, for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, N$;
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$
 $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1}^i - r_n^i| < \infty,$
 $\sum_{n=1}^{\infty} |a_{n+1}^i - a_n^i| < \infty, \sum_{n=1}^{\infty} |\lambda_k^{n+1} - \lambda_k^n| < \infty$, for all $i = 1, 2, \dots, N$ and $k = 1, 2, \dots, \bar{N}$;
- (vi) For each $i = 1, 2, \dots, N$, there exists $\rho_i > 0$ such that

$$\Phi_i \left(w_{r_1^i}^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2) \right) + \Phi_i \left(w_{r_2^i}^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1) \right) \leq -\rho_i \left\| T_{r_1^i}(x_1) - T_{r_2^i}(x_2) \right\|^2, \quad (3.134)$$

for all $(r_1^i, r_2^i) \in \Theta_i \times \Theta_i$, $(x_1, x_2) \in H \times H$ and $w_j^i \in S_i(x_j)$, for $j = 1, 2$, where $\Theta_i = \{r_n^i : n \geq 1\}$.

Then $\{x_n\}$ and $\{u_n^i\}$ converges strongly to $q = P_{\mathcal{F}}f(q)$, for every $i = 1, 2, \dots, N$.

Proof. Putting $A = 0$ in Theorem 3.17, hence, Corollary 3.19 holds. \square

Remark 3.20. From Corollary 3.19, put $\bar{N} = 1$, then iterative scheme (3.133) reduces to

$$\begin{cases} w_n^1 \in S_1(x_n), \|w_n^1 - w_{n+1}^1\| \leq (1 + \frac{1}{n}) \mathcal{H}(S_1(x_n), S_1(x_{n+1})), \\ \Phi_1(w_n^1, u_n^1, y) + \varphi(y) - \varphi(u_n^1) + \frac{1}{r_n^1} \langle u_n^1 - x_n, y - u_n^1 \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n u_n^1 + \delta_n (\lambda_1^n T_1 + (1 - \lambda_1^n) I) x_n, \forall n \geq 1, \end{cases}$$

which is a modification of iterative scheme (1.8) in the results of Ceng, Ansari and Yao [33]. By assuming the initial condition $x_1 \in C$, $w_1^1 \in S_1(x_1)$ and the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < \tau \leq \beta_n, \delta_n \leq v < 1$;
- (iii) $0 < \epsilon \leq r_n^1 \leq \omega < 1$, for all $n \in \mathbb{N}$;
- (iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
 $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1}^1 - r_n^1| < \infty$,
 $\sum_{n=1}^{\infty} |\lambda_1^{n+1} - \lambda_1^n| < \infty$;

(v) There exists $\rho_1 > 0$ such that

$$\Phi_1(w_1^1, T_{r_1^1}(x_1), T_{r_2^1}(x_2)) + \Phi_1(w_2^1, T_{r_2^1}(x_2), T_{r_1^1}(x_1)) \leq -\rho_1 \|T_{r_1^1}(x_1) - T_{r_2^1}(x_2)\|^2,$$

for all $(r_1^1, r_2^1) \in \Theta_1 \times \Theta_1$, $(x_1, x_2) \in H \times H$ and $w_j^1 \in S_1(x_j)$, for $j = 1, 2$, where $\Theta_1 = \{r_n^1 : n \geq 1\}$.

Then $\{x_n\}$ and $\{u_n^1\}$ converge strongly to $q = P_{\mathcal{F}}f(q)$.

Proof. Taking $\bar{N} = 1$ in Corollary 3.19, we obtain the desired result. \square

Chapter 4

Applications and Numerical Examples

4.1 General iterative methods for equilibrium problems, nonspreading mapping and quasi-nonexpansive mapping

By changing T from a nonspreading mapping to a quasi-nonexpansive mapping with $Fix(T) \neq \emptyset$, we obtain the same result as shown in Lemma 3.9.

Lemma 4.1. Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \rightarrow C$ be a quasi-nonexpansive mapping with $Fix(T) \neq \emptyset$. Then there hold the following statement:

- (i) $Fix(T) = VI(C, I - T)$;
- (ii) For every $u \in C$ and $v \in Fix(T)$,

$$\|P_C(I - \lambda(I - T))u - v\| \leq \|u - v\|, \text{ where } \lambda \in (0, 1).$$

Definition 4.1 ([23]). Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, \dots$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \rightarrow C$ as follows:

$$\begin{aligned} U_0 &= I, \\ U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\ U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\ U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\ &\vdots \\ U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\ S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I. \end{aligned}$$

This mapping is called the S -mapping generated by T_1, T_2, \dots, T_N and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 4.2 ([27]). Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \dots, N-1$ and $\alpha_1^N \in (0, 1)$, $\alpha_3^N \in [0, 1)$, $\alpha_2^j \in (0, 1)$ for all $j = 1, 2, \dots, N$. Let S be the S -mapping generated by T_1, T_2, \dots, T_N and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $Fix(S) = \bigcap_{i=1}^N Fix(T_i)$ and S is a quasi-nonexpansive mapping.

Theorem 4.3. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $f : C \rightarrow C$ be an α -contractive mapping, let $A : C \rightarrow C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. For $i = 1, 2, \dots, \bar{N}$, let $\Phi_i : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)'-(A4). Let $T_i : C \rightarrow C$, for $i = 1, 2, \dots, N$ be a finite family of nonspreading mappings with $\Omega := \bigcap_{i=1}^N \text{Fix}(T_i) \cap \bigcap_{i=1}^{\bar{N}} EP(\Phi_i) \neq \emptyset$. Let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \dots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \dots, N-1$ and $\alpha_1^N \in (0, 1]$, $\alpha_3^N \in [0, 1)$, $\alpha_2^j \in (0, 1)$ for all $j = 1, 2, \dots, N$, and let S be the S -mapping generated by T_1, T_2, \dots, T_N and $\rho_1, \rho_2, \dots, \rho_N$. Let $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} \sum_{i=1}^{\bar{N}} a_i \Phi_i(v_n, y) + \frac{1}{\varphi_n} \langle y - v_n, v_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n = \theta_n P_C x_n + (1 - \theta_n) v_n, \\ x_{n+1} = \delta_n \gamma f(x_n) + (I - \delta_n A) P_C (I - \psi_n (I - S)) y_n, \forall n \in \mathbb{N}, \end{cases} \quad (4.1)$$

where $\{\delta_n\}$, $\{\theta_n\}$, $\{\varphi_n\}$, $\{\psi_n\} \subseteq (0, 1)$, $0 < a_i < 1$, for all $i = 1, 2, \dots, \bar{N}$. Suppose the conditions (i)-(vi) hold.

- (i) $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$;
- (ii) $0 < \tau \leq \theta_n \leq \nu < 1$, for some $\tau, \nu > 0$;
- (iii) $\sum_{n=1}^{\infty} \psi_n < \infty$;
- (iv) $0 < \epsilon \leq \varphi_n \leq \eta < 1$, for some $\epsilon, \eta > 0$;
- (v) $\sum_{n=1}^{\bar{N}} a_i = 1$;
- (vi) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$,
 $\sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$, $\sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ converge strongly to $q = P_{\Omega}(I - A + \gamma f)q$.

Proof. Using Lemma 4.1, Lemma 4.2 and the same method in Theorem 3.11, we have the desired conclusion. \square

Remark 4.4. Theorem 4.3 can be considered as an improvement of Theorem 3.1 in the result of Tien and Jin [48] in the sense that some conditions are not assumed.

- (i) $T_{\omega} := (1 - \omega)I + \omega T$, $\omega \in (0, \frac{1}{2})$,
- (ii) T is demi-closed on H ,

where T is quasi-nonexpansive mapping on H .

4.2 Examples and numerical results

In this section, an example is given for supporting Theorem 3.5.

Example 4.5. Let \mathbb{R} be the set of real numbers and let the mapping $A_i : \mathbb{R} \rightarrow \mathbb{R}$ defined by $A_i x = \frac{ix}{2}, \forall x \in \mathbb{R}$ and $b_i = \frac{7}{8^i} + \frac{1}{N8^N}$ for every $i = 1, 2, \dots, N$. For $n \in \mathbb{N}$, let the mapping $T_n : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$T_n x = \frac{7n}{7n+1}x, \forall x \in \mathbb{R},$$

and let $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$F_i(x, y) = i(-3x^2 + xy + 2y^2), \forall x, y \in \mathbb{R}.$$

Furthermore, let $a_i = \frac{2}{3^i} + \frac{1}{N3^N}$ for every $i = 1, 2, \dots, N$. Then, we have

$$\begin{aligned} \sum_{i=1}^N a_i F_i(x, y) &= \sum_{i=1}^N \left(\frac{2}{3^i} + \frac{1}{N3^N} \right) i(-3x^2 + xy + 2y^2) \\ &= E(-3x^2 + xy + 2y^2), \end{aligned}$$

where $E = \sum_{i=1}^N \left(\frac{2}{3^i} + \frac{1}{N3^N} \right) i$. From Example 2.36, we have F_i satisfies the conditions (A1)-(A4) for each $i = 1, 2, \dots, N$. Then, by Remark 3.2 and Lemma 3.1, we have $\sum_{i=1}^N a_i F_i$ satisfies the conditions (A1)-(A4) and $0 \in EP(\sum_{i=1}^N a_i F_i) = \bigcap_{i=1}^N EP(F_i)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by (3.25). By the definition of F , we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \\ &= E(-3u_n^2 + u_n y + 2y^2) + \frac{1}{r_n} (y - u_n)(u_n - x_n) \\ &= E(-3u_n^2 + u_n y + 2y^2) + \frac{1}{r_n} (yu_n - yx_n - u_n^2 + u_n x_n) \\ &\Leftrightarrow \\ 0 &\leq Er_n(-3u_n^2 + u_n y + 2y^2) + (yu_n - yx_n - u_n^2 + u_n x_n) \\ &= -3Eu_n^2 r_n + Eu_n r_n y + 2Ey^2 r_n + yu_n - yx_n - u_n^2 + u_n x_n \\ &= 2Er_n y^2 + (Eu_n r_n + u_n - x_n)y + u_n x_n - u_n^2 - 3Eu_n^2 r_n. \end{aligned}$$

Let $G(y) = 2Er_n y^2 + (u_n(Er_n + 1) - x_n)y + u_n x_n - u_n^2 - 3Eu_n^2 r_n$. $G(y)$ is a quadratic function of y with coefficient $a = 2Er_n$, $b = u_n(Er_n + 1) - x_n$, and $c = u_n x_n - u_n^2 - 3Eu_n^2 r_n$. Determine the discriminant Δ of G as follows

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (u_n(Er_n + 1) - x_n)^2 - 4(2Er_n)(u_n x_n - u_n^2 - 3Eu_n^2 r_n) \\ &= u_n^2(Er_n + 1)^2 - 2x_n u_n(Er_n + 1) + x_n^2 - 8Er_n u_n x_n + 24E^2 u_n^2 r_n^2 \\ &\quad + 8Eu_n^2 r_n \end{aligned}$$

$$\begin{aligned}
&= E^2 u_n^2 r_n^2 + 2Er_n u_n^2 + u_n^2 - 2Ex_n u_n r_n - 2x_n u_n + x_n^2 - 8Er_n u_n x_n \\
&\quad + 24E^2 u_n^2 r_n^2 + 8Eu_n^2 r_n \\
&= u_n^2 + 10Er_n u_n^2 + 25E^2 u_n^2 r_n^2 - 2x_n u_n - 10Ex_n u_n r_n + x_n^2 \\
&= (u_n + 5Eu_n r_n)^2 - 2x_n (u_n + 5Eu_n r_n) + x_n^2 \\
&= (u_n + 5Eu_n r_n - x_n)^2.
\end{aligned}$$

We know that $G(y) \geq 0, \forall y \in \mathbb{R}$. If it has most one solution in \mathbb{R} , then $\Delta \leq 0$, so we obtain

$$u_n = \frac{x_n}{1 + 5 \sum_{i=1}^N \left(\frac{2}{3^i} + \frac{1}{N3^N} \right) i r_n}. \quad (4.2)$$

For every $n \in \mathbb{N}$, $x, y \in \mathbb{R}$ and $T_n x = \frac{7n}{7n+1}x$, then we have

$$|T_n x - T_n y| = \left| \frac{7n}{7n+1}x - \frac{7n}{7n+1}y \right| = \frac{7n}{7n+1} |x - y| \leq |x - y|.$$

This implies that T_n is a nonexpansive mapping, that is, T is 0-strictly pseudo contractive for every $n \in \mathbb{N}$. For every $j = 1, 2, \dots$ and let $\alpha_1^j = \frac{3}{5j^2+3}$, $\alpha_2^j = \frac{3j^2}{5j^2+3}$, $\alpha_3^j = \frac{2j^2}{5j^2+3}$. Then $\sigma_j = \left(\frac{3}{5j^2+3}, \frac{3j^2}{5j^2+3}, \frac{2j^2}{5j^2+3} \right)$ for all $j = 1, 2, \dots$. Since S_n is S -mapping generated by T_n, T_{n-1}, \dots, T_1 and $\sigma_n, \sigma_{n-1}, \dots, \sigma_1$, we obtain

$$\begin{aligned}
U_{n,n+1} x_n &= x_n \\
U_{n,n} x_n &= \left(\frac{3}{5n^2+3} \left(\frac{7n}{7n+1} \right) U_{n,n+1} + \left(\frac{3n^2}{5n^2+3} \right) U_{n,n+1} + \frac{2n^2}{5n^2+3} \right) x_n \\
U_{n,n-1} x_n &= \left(\frac{3}{5(n-1)^2+3} \left(\frac{7(n-1)}{7(n-1)+1} \right) U_{n,n} + \left(\frac{3(n-1)^2}{5(n-1)^2+3} \right) U_{n,n} + \frac{2(n-1)^2}{5(n-1)^2+3} \right) x_n \\
&\vdots \\
U_{n,k+1} x_n &= \left(\frac{3}{5(k+1)^2+3} \left(\frac{7(k+1)}{7(k+1)+1} \right) U_{n,k+2} + \left(\frac{3(k+1)^2}{5(k+1)^2+3} \right) U_{n,k+2} + \frac{2(k+1)^2}{5(k+1)^2+3} \right) x_n \\
U_{n,k} x_n &= \left(\frac{3}{5k^2+3} \left(\frac{7k}{7k+1} \right) U_{n,k+1} + \left(\frac{3k^2}{5k^2+3} \right) U_{n,k+1} + \frac{2k^2}{5k^2+3} \right) x_n \\
&\vdots \\
U_{n,2} x_n &= \left(\frac{3}{23} \left(\frac{14}{15} \right) U_{n,3} + \left(\frac{12}{23} \right) U_{n,2} + \frac{8}{23} \right) x_n \\
S_n x_n &= U_{n,1} x_n = \left(\frac{3}{8} \left(\frac{7}{8} \right) U_{n,2} + \left(\frac{3}{8} \right) U_{n,2} + \frac{2}{8} \right) x_n.
\end{aligned}$$

From the definition of T_n , we obtain

$$\{0\} = \bigcap_{i=1}^{\infty} \text{Fix}(T_i). \quad (4.3)$$

Since

$$\left\langle y - x, \sum_{i=1}^N a_i A_i x \right\rangle = (y - x) \sum_{i=1}^N \left(\frac{7}{8^i} + \frac{1}{N8^N} \right) \frac{ix}{2},$$

then $0 \in VI \left(C, \sum_{i=1}^N a_i A_i \right) = \bigcap_{i=1}^N VI(C, A_i)$. From (4.3), it follows that

$$\{0\} = \bigcap_{i=1}^N EP(F_i) \cap \bigcap_{i=1}^N VI(C, A_i) \cap \bigcap_{i=1}^{\infty} \text{Fix}(T_i).$$

Put $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n+1}{7n+2}$, $r_n = \frac{n}{2n+1}$, $\rho_n = \frac{1}{n^3}$, $\forall n \in \mathbb{N}$. Since

$$\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0,$$

and

$$\sum_{n=1}^{\infty} \alpha_n = \sum_{i=1}^{\infty} \frac{1}{n} = \infty,$$

then the sequence $\{\alpha_n\}$ satisfies the condition (i). Since $\lim_{n \rightarrow \infty} \beta_n = \frac{1}{7}$ and $\lim_{n \rightarrow \infty} r_n = \frac{1}{2}$, then the sequences $\{\beta_n\}$ and $\{r_n\}$ are bounded. This follows that the conditions (ii) and (iv) hold. Because $\lim_{n \rightarrow \infty} \rho_n = \lim_{n \rightarrow \infty} \frac{1}{n^3} = 0$, then the condition (iii) holds. By Example 2.32, we get A_i is $\frac{1}{2}$ -strongly positive, for $i = 1, 2, \dots, N$. Since $a_i = \frac{2}{3^i} + \frac{1}{N3^N}$ for every $i = 1, 2, \dots, N$, we have

$$\sum_{i=1}^N a_i = \sum_{i=1}^N \left(\frac{2}{3^i} + \frac{1}{N3^N} \right) = 2 \sum_{i=1}^N \frac{1}{3^i} + \sum_{i=1}^N \frac{1}{N3^N} = 2 \left[\frac{\frac{1}{3} \left(1 - \left(\frac{1}{3} \right)^N \right)}{1 - \frac{1}{3}} \right] + \frac{1}{3^N} = 1$$

Using the same method as above, we get $\sum_{i=1}^N b_i = 1$. This deduces that the condition (iv) is true. We compute

$$|\alpha_{n+1} - \alpha_n| = \left| \frac{1}{n+1} - \frac{1}{n} \right| = \frac{1}{n^2 + n}.$$

By using the comparison test, we obtain $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. We derive

$$|\beta_{n+1} - \beta_n| = \left| \frac{n+2}{7n+9} - \frac{n+1}{7n+2} \right| = \frac{5}{49n^2 + 77n + 18}.$$

This follows by the comparison test that $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$. Similarly, we obtain the same result for the sequences $\{r_n\}$, $\{\rho_n\}$ and $\{\alpha_n^1\}$. Thus, the condition (vi) is true. For every $n \in \mathbb{N}$, from (4.2) we rewrite (3.25) as follows:

$$x_{n+1} = \frac{n+1}{7n+2} \left(\frac{1}{n} u + \left(1 - \frac{1}{n} \right) S_n x_n \right) + \left(1 - \frac{n+1}{7n+2} \right) \left(I - \frac{1}{n^3} \sum_{i=1}^N \left(\frac{7}{8^i} + \frac{1}{N8^N} \right) A_i \right) \frac{x_n}{1 + 5 \sum_{i=1}^N \left(\frac{2}{3^i} + \frac{1}{N3^N} \right) i r_n}, \forall n \geq 1. \quad (4.4)$$

Then, we can conclude that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to 0.

The following table shows the values of sequences $\{x_n\}$ and $\{u_n\}$ where $u = x_1 = -5$ and $u = x_1 = 5$ and $n = N = 30$.

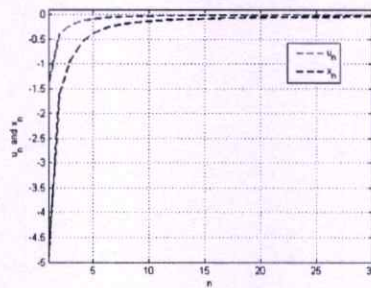
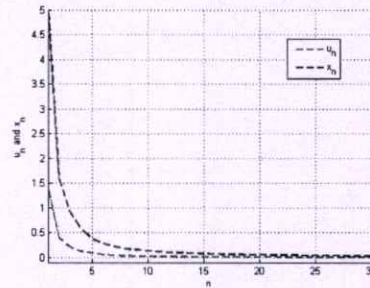
Remark 4.6. (i) Table 4.1 and Figure 4.1 show that the sequences $\{x_n\}$ and $\{u_n\}$ converge to 0, where $\{0\} = \bigcap_{i=1}^N EP(F_i) \cap \bigcap_{i=1}^N VI(C, A_i) \cap \bigcap_{i=1}^{\infty} Fix(T_i)$.

(ii) Theorem 3.5 guarantees the convergence of $\{x_n\}$ and $\{u_n\}$ in Example 4.5.

Next, the following numerical examples are given for supporting Theorem 3.11. Using these examples, we see that our iteration for the combination of equilibrium problem converges faster than our iteration for the classical equilibrium problem.

Table 4.1: The values of $\{u_n\}$ and $\{x_n\}$ with $n = N = 30$.

n	$u = x_1 = -5$		$u = x_1 = 5$	
	u_n	x_n	u_n	x_n
1	-1.428571	-5.000000	1.428571	5.000000
2	-0.396825	-1.587302	0.396825	1.587302
3	-0.215646	-0.908795	0.215646	0.908795
4	-0.130112	-0.563820	0.130112	0.563820
5	-0.086732	-0.382407	0.086732	0.382407
\vdots	\vdots	\vdots	\vdots	\vdots
15	-0.017597	-0.081457	0.017597	0.081457
\vdots	\vdots	\vdots	\vdots	\vdots
26	-0.009365	-0.043819	0.009365	0.043819
27	-0.008982	-0.042053	0.008982	0.042053
28	-0.008630	-0.040424	0.008630	0.040424
29	-0.008304	-0.038916	0.008304	0.038916
30	-0.008002	-0.037517	0.008002	0.037517

(a) $u = x_1 = -5$ (b) $u = x_1 = 5$ Figure 4.1: The convergence numbers of iterative calculations with different initial values of u and x_1 .

Example 4.7. Let the mappings $A : \mathbb{R} \rightarrow \mathbb{R}$, $f : \mathbb{R} \rightarrow \mathbb{R}$, be defined by

$$Ax = \frac{x}{2},$$

$$fx = \frac{x}{4}, \text{ for all } x \in \mathbb{R}.$$

For every $i = 1, 2, \dots, N$, let $\Phi_i : [1, 100] \times [1, 100] \rightarrow \mathbb{R}$ and $T : [1, 100] \rightarrow [1, 100]$ be defined by

$$Tx = \frac{2x + 5}{7},$$

$$\Phi_i(x, y) = i(y - x)(y + 2x - 3), \text{ for all } x, y \in [1, 100].$$

Put $a_i = \frac{4}{5^i} + \frac{1}{N5^N}$, for every $i = 1, 2, \dots, N$. Let $\gamma = \frac{1}{3}$, $\delta_n = \frac{1}{3n}$, $\theta_n = \frac{n}{2n+3}$, $\varphi_n = \frac{2n}{3n+2}$ and $\psi_n = \frac{1}{n^2}$ for every $n \in \mathbb{N}$. Let the initial values be defined as in the following cases:

(i) $x_1 = 50$, $N = 1$ and $n = 10$,

(ii) $x_1 = 50$ and $n = N = 10$.

Then, for both cases, the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ converge strongly to 1.

Solution. From Example 2.28, we get T is a nonspreading mapping and $Fix(T) = \{1\}$.

Since $a_i = \frac{4}{5^i} + \frac{1}{N5^N}$, we obtain

$$\begin{aligned} \sum_{i=1}^N a_i \Phi_i(x, y) &= \sum_{i=1}^N \left(\frac{4}{5^i} + \frac{1}{N5^N} \right) i(y-x)(y+2x-3) \\ &= \mu(y-x)(y+2x-3), \end{aligned}$$

where $\mu = \sum_{i=1}^N \left(\frac{4}{5^i} + \frac{1}{N5^N} \right) i$. By Example 2.37, we get Φ_i satisfies the conditions (A1)'-(A4). It follows by Remark 3.2 and Lemma 3.1 that $\sum_{i=1}^N a_i \Phi_i$ satisfies the conditions (A1)'-(A4) and $1 \in EP(\sum_{i=1}^N a_i \Phi_i) = \bigcap_{i=1}^N EP(\Phi_i)$. Then we have

$$Fix(T) \cap \bigcap_{i=1}^N EP(\Phi_i) = \{1\}.$$

Observe that

$$\begin{aligned} 0 &\leq \sum_{i=1}^N a_i \Phi_i(v_n, y) + \frac{1}{\varphi_n} \langle y - v_n, v_n - x_n \rangle \\ &= \mu(y - v_n)(y + 2v_n - 3) + \frac{1}{\varphi_n} (y - v_n)(v_n - x_n) \\ &\Leftrightarrow \\ 0 &\leq \mu\varphi_n(y - v_n)(y + 2v_n - 3) + (y - v_n)(v_n - x_n) \\ &= \mu\varphi_n y^2 + (\mu v_n \varphi_n + v_n - x_n - 3\mu\varphi_n)y + 3\mu\varphi_n v_n - v_n^2 - 2\mu\varphi_n v_n^2 + v_n x_n. \end{aligned} \quad (4.5)$$

Let $G(y) = \mu\varphi_n y^2 + (\mu v_n \varphi_n + v_n - x_n - 3\mu\varphi_n)y + 3\mu\varphi_n v_n - v_n^2 - 2\mu\varphi_n v_n^2 + v_n x_n$. $G(y)$ is a quadratic function of y with coefficients $a = \mu\varphi_n$, $b = \mu v_n \varphi_n + v_n - x_n - 3\mu\varphi_n$, and $c = 3\mu\varphi_n v_n - v_n^2 - 2\mu\varphi_n v_n^2 + v_n x_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (\mu v_n \varphi_n + v_n - x_n - 3\mu\varphi_n)^2 - 4(\mu\varphi_n)(3\mu\varphi_n v_n - v_n^2 - 2\mu\varphi_n v_n^2 + v_n x_n) \\ &= 9\mu^2 \varphi_n^2 - 6\mu\varphi_n v_n - 18\mu^2 \varphi_n^2 v_n + v_n^2 + 6\mu\varphi_n v_n^2 + 9\mu^2 \varphi_n^2 v_n^2 + 6\mu\varphi_n x_n - 2v_n x_n \\ &\quad - 6\mu\varphi_n v_n x_n + x_n^2 \\ &= (v_n - 3\mu\varphi_n + 3\mu\varphi_n v_n - x_n)^2. \end{aligned}$$

From (4.5), we have $G(y) \geq 0$, for every $y \in \mathbb{R}$. If $G(y)$ has most one solution in \mathbb{R} , thus we have $\Delta \leq 0$. This implies that

$$v_n = \frac{x_n + 3\mu\varphi_n}{1 + 3\mu\varphi_n}, \quad (4.6)$$

where $\mu = \sum_{i=1}^N \left(\frac{4}{5^i} + \frac{1}{N5^N} \right) i$. Put $\delta_n = \frac{1}{3n}$, $\theta_n = \frac{n}{2n+3}$, $\varphi_n = \frac{2n}{3n+2}$, $\psi_n = \frac{1}{n^2}$, $\forall n \in \mathbb{N}$.

Since $\lim_{n \rightarrow \infty} \delta_n = \lim_{n \rightarrow \infty} \frac{1}{3n} = 0$, and $\sum_{n=1}^{\infty} \delta_n = \sum_{i=1}^{\infty} \frac{1}{3n} = \infty$, then the sequence $\{\delta_n\}$

satisfies the condition (i). Since $\lim_{n \rightarrow \infty} \theta_n = \frac{1}{2}$ and $\lim_{n \rightarrow \infty} \varphi_n = \frac{2}{3}$, then the sequences $\{\theta_n\}$ and $\{\varphi_n\}$ are bounded. This follows that the condition (ii) and (iv) hold. Because $\lim_{n \rightarrow \infty} \psi_n = \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$, then the condition (iii) holds. Since $a_i = \frac{4}{5^i} + \frac{1}{N5^N}$, for every $i = 1, 2, \dots, N$, we have

$$\sum_{i=1}^N a_i = \sum_{i=1}^N \left(\frac{4}{5^i} + \frac{1}{N5^N} \right) = 4 \sum_{i=1}^N \frac{1}{5^i} + \sum_{i=1}^N \frac{1}{N5^N} = 4 \left[\frac{\frac{1}{5} \left(1 - \left(\frac{1}{5} \right)^N \right)}{1 - \frac{1}{5}} \right] + \frac{1}{5^N} = 1.$$

This deduces that the condition (v) is true. We compute

$$|\delta_{n+1} - \delta_n| = \left| \frac{1}{3(n+1)} - \frac{1}{3n} \right| = \frac{1}{3n^2 + 3n}.$$

By using the comparison test, we obtain $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$. We derive

$$|\theta_{n+1} - \theta_n| = \left| \frac{n+1}{2n+5} - \frac{n}{2n+3} \right| = \frac{3}{4n^2 + 11n + 15}.$$

This follows by the comparison test that $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$. Similarly, we obtain the same result for the sequences $\{\varphi_n\}$ and $\{\psi_n\}$. Thus, the condition (vi) is true.

For every $n \in \mathbb{N}$, from (4.6), we rewrite (3.65) as follows:

$$\begin{cases} y_n = \frac{n}{2n+3} P_{[1,100]} x_n + \left(1 - \frac{n}{2n+3} \right) \frac{1}{1+3\mu \frac{2n}{3n+2}} \left(x_n + 3\mu \frac{2n}{3n+2} \right), \\ x_{n+1} = \frac{1}{36n} x_n + \left(I - \frac{1}{3n} A \right) P_{[1,100]} \left(I - \frac{1}{n^2} (I - T) \right) y_n, \forall n \in \mathbb{N}. \end{cases} \quad (4.7)$$

From Theorem 3.11, we can conclude that the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ generated by (4.7) converge strongly to 1.

For case (i), with $N = 1$, we have $\mu = 1$. Then (4.6) becomes

$$v_n = \frac{x_n + 3\varphi_n}{1 + 3\varphi_n}. \quad (4.8)$$

Then we have

$$\begin{cases} y_n = \frac{n}{2n+3} P_{[1,100]} x_n + \left(1 - \frac{n}{2n+3} \right) \frac{1}{1+3 \frac{2n}{3n+2}} \left(x_n + 3 \frac{2n}{3n+2} \right), \\ x_{n+1} = \frac{1}{36n} x_n + \left(I - \frac{1}{3n} A \right) P_{[1,100]} \left(I - \frac{1}{n^2} (I - T) \right) y_n, \forall n \in \mathbb{N}. \end{cases} \quad (4.9)$$

From Corollary 3.12, we can conclude that the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ generated by (4.9) converge strongly to 1.

The following table and figure show the values of sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ in two cases.

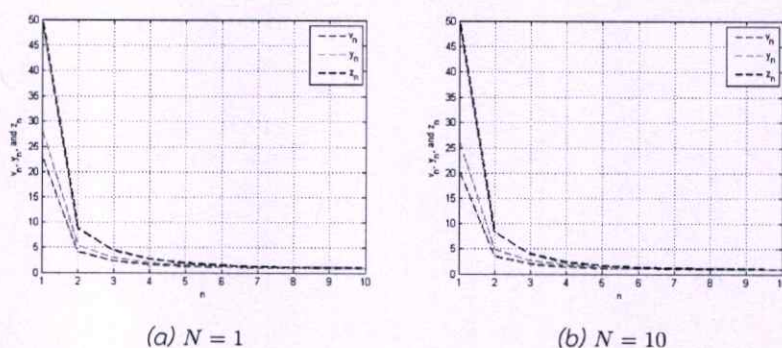
Remark 4.8. (i) From Table 4.2 and Figure 4.2, the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ converge to 1, where $\{1\} = \text{Fix}(T) \cap \bigcap_{i=1}^N EP(\Phi_i)$.

(ii) For case (i), Corollary 3.12 guarantees the convergence of $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$.

(iii) For case (ii), the convergence of $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ can be guaranteed by Theorem 3.11.

Table 4.2: The values of $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ with $x_1 = 50$.

n	$N = 1$			$N = 10$		
	v_n	y_n	x_n	v_n	y_n	x_n
1	23.272727	28.618182	50.000000	20.600006	26.480004	50.000000
2	4.119192	5.455988	8.797980	3.535267	4.893445	8.288890
3	2.287423	2.989653	4.394114	1.973077	2.636539	3.963461
4	1.631883	2.025784	2.715111	1.446758	1.794880	2.404095
5	1.331444	1.556406	1.916346	1.219216	1.405203	1.702781
6	1.178502	1.307023	1.499805	1.109733	1.208492	1.356631
7	1.096153	1.168453	1.271738	1.053976	1.104708	1.177182
8	1.050342	1.089475	1.143282	1.024666	1.048632	1.081587
9	1.024377	1.043831	1.069769	1.009019	1.018015	1.030010
10	1.009539	1.017314	1.027423	1.000654	1.001320	1.002186

Figure 4.2: The convergence numbers of iterative calculations with an initial value $x_1 = 50$.

- (iv) The iteration (4.7) for the combination of equilibrium problem converges faster than the iteration (4.9) for the classical equilibrium problem.

Finally, we give the numerical example for our main theorem in two dimensional space of real numbers.

Example 4.9. Let \mathbb{R}^2 be the two dimensional space of real numbers with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $\langle u, v \rangle = u \cdot v = u_1v_1 + u_2v_2$ and a usual norm $\| \cdot \| : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $\|u\| = \sqrt{u_1^2 + u_2^2}$, for all $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$. Let the mappings $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2, f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$Au = \left(\frac{u_1}{2}, \frac{u_2}{2} \right),$$

$$fu = \left(\frac{u_1}{4}, \frac{u_2}{4} \right), \text{ for all } u = (u_1, u_2) \in \mathbb{R}^2.$$

For every $i = 1, 2, \dots, N$ and $I = [0, 1]$, let $\Phi_i : I^2 \times I^2 \rightarrow \mathbb{R}$ and $T : I^2 \rightarrow I^2$ be defined by

$$Tu = \left(\frac{u_1 + 1}{2}, \frac{5u_2 + 1}{6} \right),$$

$$\Phi_i(u, v) = i(v - u) \cdot (v + 6u - 7), \text{ for all } u = (u_1, u_2), v = (v_1, v_2) \in I^2,$$

where $7 = (7, 7)$. Let $\gamma = \frac{1}{3}$, $\delta_n = \frac{1}{3n}$, $\theta_n = \frac{n}{2n+3}$, $\varphi_n = \frac{2n}{3n+2}$ and $\psi_n = \frac{1}{n^2}$ for every $n \in \mathbb{N}$. By Example 2.29, we obtain T is a nonspreading mapping and $Fix(T) = \{1\}$, where $1 = (1, 1)$.

Put $a_i = \frac{8}{9^i} + \frac{1}{N9^N}$, for every $i = 1, 2, \dots, N$. From Example 2.38, we can conclude that $\Phi_{[i]}$ satisfies the conditions (A1)'-(A4). This implies by Remark 3.2 and Lemma 3.1 that $\sum_{i=1}^N a_i \Phi_i$ satisfies the conditions (A1)'-(A4) and $1 \in EP(\sum_{i=1}^N a_i \Phi_i) = \bigcap_{i=1}^N EP(\Phi_i)$, where $1 = (1, 1)$. Then we have

$$Fix(T) \cap \bigcap_{i=1}^N EP(\Phi_i) = \{1\}.$$

Then, by Theorem 3.11, the sequences $x_n = (x_n^1, x_n^2)$, $y_n = (y_n^1, y_n^2)$ and $v_n = (v_n^1, v_n^2)$ converge strongly to $\{1\}$.

Putting $\rho = \sum_{i=1}^N (\frac{8}{9^i} + \frac{1}{N9^N}) i$, we obtain

$$\begin{aligned} 0 &\leq \sum_{i=1}^N a_i \Phi_i(v_n, y) + \frac{1}{\varphi_n} \langle y - v_n, v_n - x_n \rangle \\ &= \rho(y - v_n) \cdot (y + 6v_n - (7, 7)) + \frac{1}{\varphi_n} (y - v_n) \cdot (v_n - x_n) \\ &= \rho(y_1 - v_n^1, y_2 - v_n^2) \cdot (y_1 + 6v_n^1 - 7, y_2 + 6v_n^2 - 7) \\ &\quad + \frac{1}{\varphi_n} (y_1 - v_n^1, y_2 - v_n^2) \cdot (v_n^1 - x_n^1, v_n^2 - x_n^2) \\ &= \rho((y_1 - v_n^1)(y_1 + 6v_n^1 - 7) + (y_2 - v_n^2)(y_2 + 6v_n^2 - 7)) \\ &\quad + \frac{1}{\varphi_n} ((y_1 - v_n^1)(v_n^1 - x_n^1) + (y_2 - v_n^2)(v_n^2 - x_n^2)) \\ &= \left(\rho(y_1 - v_n^1)(y_1 + 6v_n^1 - 7) + \frac{1}{\varphi_n} (y_1 - v_n^1)(v_n^1 - x_n^1) \right) \\ &\quad + \left(\rho(y_2 - v_n^2)(y_2 + 6v_n^2 - 7) + \frac{1}{\varphi_n} (y_2 - v_n^2)(v_n^2 - x_n^2) \right) \\ &\Leftrightarrow \\ 0 &\leq (\rho\varphi_n(y_1 - v_n^1)(y_1 + 6v_n^1 - 7) + (y_1 - v_n^1)(v_n^1 - x_n^1)) \\ &\quad + (\rho\varphi_n(y_2 - v_n^2)(y_2 + 6v_n^2 - 7) + (y_2 - v_n^2)(v_n^2 - x_n^2)) \\ &= (\rho\varphi_n(y_1)^2 + (5\rho v_n^1\varphi_n + v_n^1 - x_n^1 - 7\rho\varphi_n)y_1 + 7\rho\varphi_n v_n^1 - (v_n^1)^2 - 6\rho\varphi_n(v_n^1)^2 + v_n^1 x_n^1) \\ &\quad + (\rho\varphi_n(y_2)^2 + (5\rho v_n^2\varphi_n + v_n^2 - x_n^2 - 7\rho\varphi_n)y_2 + 7\rho\varphi_n v_n^2 - (v_n^2)^2 - 6\rho\varphi_n(v_n^2)^2 + v_n^2 x_n^2) \\ &= G_1(y_1) + G_2(y_2), \end{aligned} \tag{4.10}$$

where $G_1(y_1) = \rho\varphi_n(y_1)^2 + (5\rho v_n^1\varphi_n + v_n^1 - x_n^1 - 7\rho\varphi_n)y_1 + 7\rho\varphi_n v_n^1 - (v_n^1)^2 - 6\rho\varphi_n(v_n^1)^2 + v_n^1 x_n^1$ and $G_2(y_2) = \rho\varphi_n(y_2)^2 + (5\rho v_n^2\varphi_n + v_n^2 - x_n^2 - 7\rho\varphi_n)y_2 + 7\rho\varphi_n v_n^2 - (v_n^2)^2 - 6\rho\varphi_n(v_n^2)^2 + v_n^2 x_n^2$. Then $G_1(y_1)$ and $G_2(y_2)$ are quadratic functions with coefficients $a_1 = \rho\varphi_n$, $b_1 = 5\rho v_n^1\varphi_n + v_n^1 - x_n^1 - 7\rho\varphi_n$ and $c_1 = 7\rho\varphi_n v_n^1 - (v_n^1)^2 - 6\rho\varphi_n(v_n^1)^2 + v_n^1 x_n^1$ and $a_2 = \rho\varphi_n$, $b_2 = 5\rho v_n^2\varphi_n + v_n^2 - x_n^2 - 7\rho\varphi_n$ and $c_2 = 7\rho\varphi_n v_n^2 - (v_n^2)^2 - 6\rho\varphi_n(v_n^2)^2 + v_n^2 x_n^2$, respectively.

Determine the discriminant Δ_1 of G_1 as follows:

$$\begin{aligned}\Delta_1 &= (b_1)^2 - 4a_1c_1 \\ &= (5\rho v_n^1 \varphi_n + v_n^1 - x_n^1 - 7\rho\varphi_n)^2 - 4\rho\varphi_n \left(7\rho\varphi_n v_n^1 - (v_n^1)^2 - 6\rho\varphi_n (v_n^1)^2 + v_n^1 x_n^1 \right) \\ &= (v_n^1 - 7\rho\varphi_n + 7\rho\varphi_n v_n^1 - x_n^1)^2.\end{aligned}$$

From (4.10), if $G_1(y_1) \geq 0, \forall y_1 \in \mathbb{R}$ and it has most one solution in \mathbb{R} , thus $\Delta_1 \leq 0$. This follows that

$$v_n^1 = \frac{x_n^1 + 7\rho\varphi_n}{1 + 7\rho\varphi_n}. \quad (4.11)$$

Next, we determine the discriminant Δ_2 of G_2 as follows:

$$\begin{aligned}\Delta_2 &= (b_2)^2 - 4a_2c_2 \\ &= (5\rho v_n^2 \varphi_n + v_n^2 - x_n^2 - 7\rho\varphi_n)^2 - 4\rho\varphi_n \left(7\rho\varphi_n v_n^2 - (v_n^2)^2 - 6\rho\varphi_n (v_n^2)^2 + v_n^2 x_n^2 \right) \\ &= (v_n^2 - 7\rho\varphi_n + 7\rho\varphi_n v_n^2 - x_n^2)^2.\end{aligned}$$

From (4.10), if $G_2(y_2) \geq 0, \forall y_2 \in \mathbb{R}$ and it has most one solution in \mathbb{R} , then $\Delta_2 \leq 0$. This yields that

$$v_n^2 = \frac{x_n^2 + 7\rho\varphi_n}{1 + 7\rho\varphi_n}. \quad (4.12)$$

Put $\delta_n = \frac{1}{3n}$, $\theta_n = \frac{n}{2n+3}$, $\varphi_n = \frac{2n}{3n+2}$, $\psi_n = \frac{1}{n^2}$, for all $n \in \mathbb{N}$. From Example 4.7, it is obvious to prove that the sequences $\{\delta_n\}, \{\theta_n\}, \{\varphi_n\}$ and $\{\psi_n\}$ satisfy the conditions (i)-(vi) in Theorem 3.11. For every $n \in \mathbb{N}$, from (4.11) and (4.12), the iterative scheme (3.65) becomes

$$\begin{cases} y_n = \frac{n}{2n+3} P_C x_n + \left(1 - \frac{n}{2n+3}\right) v_n, \\ x_{n+1} = \frac{1}{36n} x_n + \left(I - \frac{1}{3n} A\right) P_C \left(I - \frac{1}{n^2} (I - T)\right) y_n, \forall n \in \mathbb{N}, \end{cases} \quad (4.13)$$

where $x_n = (x_n^1, x_n^2)$, $y_n = (y_n^1, y_n^2)$ and $v_n = (v_n^1, v_n^2) = \left(\frac{x_n^1 + 7\rho\varphi_n}{1 + 7\rho\varphi_n}, \frac{x_n^2 + 7\rho\varphi_n}{1 + 7\rho\varphi_n}\right)$.

Let the initial values be defined as in the following cases.

(i) $x_1 = (x_1^1, x_1^2) = (-1, 0)$, $N = 1$ and $n = 20$,

(ii) $x_1 = (x_1^1, x_1^2) = (-1, 0)$ and $n = N = 20$.

For case (i), with $N = 1$, we have $\rho = 1$. Then, from (4.11) and (4.12), we obtain $v_n^1 = \frac{x_n^1 + 3\varphi_n}{1 + 7\varphi_n}$, and $v_n^2 = \frac{x_n^2 + 3\varphi_n}{1 + 7\varphi_n}$. Then we have

$$\begin{cases} y_n = \frac{n}{2n+3} P_C x_n + \left(1 - \frac{n}{2n+3}\right) v_n, \\ x_{n+1} = \frac{1}{36n} x_n + \left(I - \frac{1}{3n} A\right) P_C \left(I - \frac{1}{n^2} (I - T)\right) y_n, \forall n \in \mathbb{N}, \end{cases} \quad (4.14)$$

where $x_n = (x_n^1, x_n^2)$, $y_n = (y_n^1, y_n^2)$ and $v_n = (v_n^1, v_n^2) = \left(\frac{x_n^1 + 3\varphi_n}{1 + 7\varphi_n}, \frac{x_n^2 + 3\varphi_n}{1 + 7\varphi_n}\right)$.

The following table and figure show the values of sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ in two cases.

Table 4.3: The values of $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ with $x_1 = (-1, 0)$ and $N = 1$.

n	$v_n = (v_n^1, v_n^2)$	$y_n = (y_n^1, y_n^2)$	$x_n = (x_n^1, x_n^2)$
1	(0.473684, 0.736842)	(0.178947, 0.589474)	(-1.000000, 0.000000)
2	(0.880767, 0.899610)	(0.761533, 0.799220)	(0.463450, 0.548246)
3	(0.944343, 0.947678)	(0.873506, 0.881086)	(0.731833, 0.747902)
4	(0.967678, 0.968228)	(0.920665, 0.922015)	(0.838391, 0.841141)
5	(0.978604, 0.978548)	(0.944718, 0.944575)	(0.890502, 0.890217)
\vdots	\vdots	\vdots	\vdots
10	(0.992787, 0.992723)	(0.979067, 0.978881)	(0.961231, 0.960887)
\vdots	\vdots	\vdots	\vdots
16	(0.995890, 0.995878)	(0.987473, 0.987437)	(0.977477, 0.977412)
17	(0.996158, 0.996149)	(0.988232, 0.988203)	(0.978908, 0.978856)
18	(0.996393, 0.996386)	(0.988903, 0.988879)	(0.980164, 0.980121)
19	(0.996601, 0.996595)	(0.989499, 0.989479)	(0.981276, 0.981241)
20	(0.996785, 0.996780)	(0.990033, 0.990017)	(0.982268, 0.982239)

Table 4.4: The values of $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ with $x_1 = (-1, 0)$, $N = 20$.

n	$v_n = (v_n^1, v_n^2)$	$y_n = (y_n^1, y_n^2)$	$x_n = (x_n^1, x_n^2)$
1	(0.518072, 0.759036)	(0.214458, 0.607229)	(-1.000000, 0.000000)
2	(0.894328, 0.911003)	(0.775448, 0.810881)	(0.478246, 0.560576)
3	(0.951505, 0.954360)	(0.882070, 0.889012)	(0.743199, 0.758316)
4	(0.972025, 0.972470)	(0.926247, 0.927421)	(0.846135, 0.848585)
5	(0.981489, 0.981428)	(0.948508, 0.948339)	(0.895738, 0.895396)
\vdots	\vdots	\vdots	\vdots
10	(0.993655, 0.993602)	(0.980077, 0.979909)	(0.962426, 0.962110)
\vdots	\vdots	\vdots	\vdots
16	(0.996360, 0.996351)	(0.987975, 0.987942)	(0.978017, 0.977957)
17	(0.996597, 0.996589)	(0.988698, 0.988671)	(0.979405, 0.979357)
18	(0.996804, 0.996798)	(0.989336, 0.989315)	(0.980624, 0.980585)
19	(0.996987, 0.996982)	(0.989905, 0.989888)	(0.981706, 0.981673)
20	(0.997150, 0.997146)	(0.990416, 0.990401)	(0.982671, 0.982644)

Remark 4.10. (i) Table 4.3, 4.4 and Figure 4.3 show that the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ converge to $\mathbf{1}$, where $\{\mathbf{1}\} = \{(1, 1)\} = \text{Fix}(T) \cap \bigcap_{i=1}^N EP(\Phi_i)$.

(ii) For case (i), Corollary 3.12 guarantees the convergence of $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$.

(iii) For case (ii), the convergence of $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ can be guaranteed by Theorem 3.11.

(iv) The iteration (4.13) for the combination of equilibrium problem converges faster than the iteration (4.14) for the classical equilibrium problem.

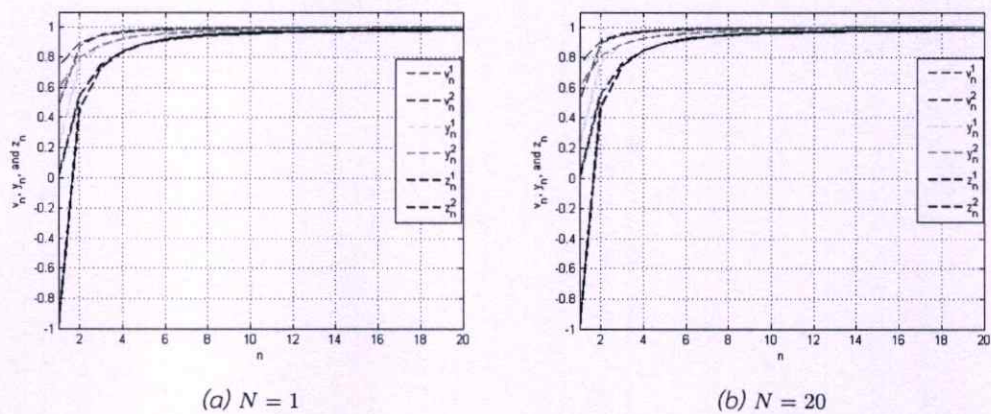


Figure 4.3: The convergence numbers of iterative calculations with an initial value $x_1 = (-1, 0)$ for both cases.

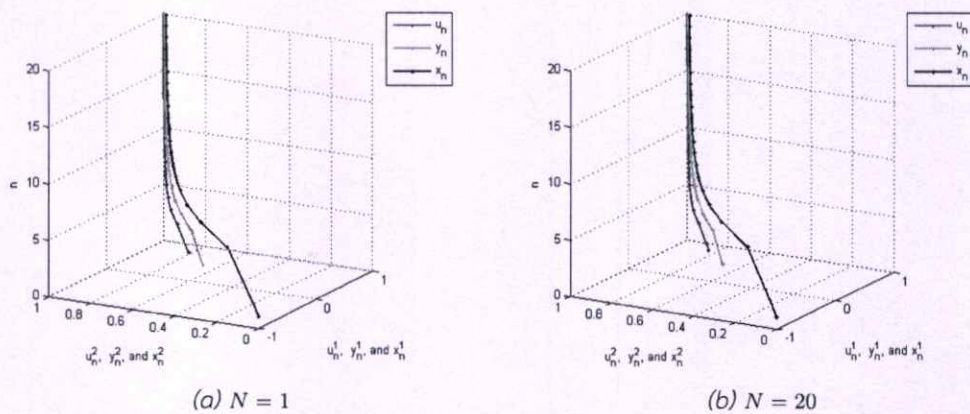


Figure 4.4: The convergence numbers of iterative calculations with an initial value $x_1 = (-1, 0)$ for both cases in three-dimensional spaces.

Chapter 5

Conclusions and Suggestions

In this section, we conclude all main results obtained in this thesis.

5.1 Convergence theorems for equilibrium problems, variational inequality problems and an infinite family of strictly pseudo-contractive mappings

- (1) Let C be a nonempty closed convex subset of a real Hilbert space H . For $i = 1, 2, \dots, N$, let $F_i : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)' – (A4) and let $A_i : C \rightarrow H$ be a strongly positive linear bounded operator on H with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\dots,N} \gamma_i$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of κ_i -strictly pseudo-contractive mappings of C into itself and let $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq \eta < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p, q] \subset (\kappa, 1)$ for all $j \in \mathbb{N}$. For every $n \in \mathbb{N}$, let S_n and S be the S-mappings generated by T_n, T_{n-1}, \dots, T_1 and $\sigma_n, \sigma_{n-1}, \dots, \sigma_1$ and T_n, T_{n-1}, \dots and $\sigma_n, \sigma_{n-1}, \dots$, respectively. Assume that $\mathcal{F} = \bigcap_{i=1}^N EP(F_i) \cap \bigcap_{i=1}^N VI(C, A_i) \cap \bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and

$$\begin{cases} \sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n (\alpha_n u + (1 - \alpha_n) S_n x_n) + (1 - \beta_n) P_C \left(I - \rho_n \sum_{i=1}^N b_i A_i \right) u_n, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0, 1)$ and $0 \leq a_i, b_i \leq 1$ for every $i = 1, 2, \dots, N$, satisfying the following conditions:

- (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (ii) $0 < a \leq \beta_n \leq b < 1, \forall n \in \mathbb{N}$,
- (iii) $\lim_{n \rightarrow \infty} \rho_n = 0$,
- (iv) $0 < c \leq r_n \leq d < 1, \forall n \in \mathbb{N}$,
- (v) $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$,
- (vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$
 $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$ and $\sum_{n=1}^{\infty} \alpha_1^n < \infty$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_{\mathcal{F}} u$.

- (2) Let C be a nonempty closed convex subset of a real Hilbert space H . For $i = 1, 2, \dots, N$, let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)' – (A4) and let $A : C \rightarrow H$ be a strongly positive linear bounded operator on H with coefficient $\gamma > 0$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of κ_i -strictly pseudo-contractive mappings of C

into itself and let $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq \eta < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p, q] \subset (\kappa, 1)$ for all $j \in \mathbb{N}$. For every $n \in \mathbb{N}$, let S_n and S be the S-mappings generated by T_n, T_{n-1}, \dots, T_1 and $\sigma_n, \sigma_{n-1}, \dots, \sigma_1$ and T_n, T_{n-1}, \dots and $\sigma_n, \sigma_{n-1}, \dots$, respectively. Assume that $\mathcal{F} = EP(F) \cap VI(C, A) \cap \bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n(\alpha_n u + (1 - \alpha_n)S_n x_n) + (1 - \beta_n)P_C(I - \rho_n A)u_n, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0, 1)$ and $0 \leq a_i, b_i \leq 1$ for every $i = 1, 2, \dots, N$, satisfying the following conditions:

- (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (ii) $0 < a \leq \beta_n \leq b < 1, \forall n \in \mathbb{N}$,
- (iii) $\lim_{n \rightarrow \infty} \rho_n = 0$,
- (iv) $0 < c \leq r_n \leq d < 1, \forall n \in \mathbb{N}$,
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$
 $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$ and $\sum_{n=1}^{\infty} \alpha_n^n < \infty$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_{\mathcal{F}}u$.

- (3) Let C be a nonempty closed convex subset of a real Hilbert space H . For every $i = 1, 2, \dots, N$, let $A_i : C \rightarrow H$ be a strongly positive linear bounded operator on H with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min \gamma_i$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of κ_i -strictly pseudo-contractive mappings of C into itself and let $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq \eta < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p, q] \subset (\kappa, 1)$ for all $j \in \mathbb{N}$. For every $n \in \mathbb{N}$, let S_n and S be the S-mappings generated by T_n, T_{n-1}, \dots, T_1 and $\sigma_n, \sigma_{n-1}, \dots, \sigma_1$ and T_n, T_{n-1}, \dots and $\sigma_n, \sigma_{n-1}, \dots$, respectively. Assume that $\mathcal{F} = \bigcap_{i=1}^N VI(C, A_i) \cap \bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and

$$x_{n+1} = \beta_n(\alpha_n u + (1 - \alpha_n)S_n x_n) + (1 - \beta_n)P_C\left(I - \rho_n \sum_{i=1}^N b_i A_i\right)x_n, \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0, 1)$ satisfying the following conditions:

- (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (ii) $0 < a \leq \beta_n \leq b < 1, \forall n \in \mathbb{N}$,
- (iii) $\lim_{n \rightarrow \infty} \rho_n = 0$,
- (iv) $0 < c \leq r_n \leq d < 1, \forall n \in \mathbb{N}$,
- (v) $\sum_{i=1}^N b_i = 1$,

$$(vi) \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \\ \sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty \text{ and } \sum_{n=1}^{\infty} \alpha_1^n < \infty.$$

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_{\mathcal{F}}u$.

- (4) Let C be a nonempty closed convex subset of a real Hilbert space H . For $i = 1, 2, \dots, N$, let $F_i : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)' - (A4). Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself and let $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq \eta < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p, q] \subset (0, 1)$ for all $j \in \mathbb{N}$. For every $n \in \mathbb{N}$, let S_n and S be the S -mappings generated by T_n, T_{n-1}, \dots, T_1 and $\sigma_n, \sigma_{n-1}, \dots, \sigma_1$ and T_n, T_{n-1}, \dots and $\sigma_n, \sigma_{n-1}, \dots$, respectively. Assume that $\mathcal{F} = \bigcap_{i=1}^N EP(F_i) \cap \bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and

$$\begin{cases} \sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n (\alpha_n u + (1 - \alpha_n) S_n x_n) + (1 - \beta_n) P_C u_n, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0, 1)$ and $0 \leq a_i \leq 1$ for every $i = 1, 2, \dots, N$, satisfying the following conditions:

- (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (ii) $0 < a \leq \beta_n \leq b < 1, \forall n \in \mathbb{N}$,
- (iii) $\lim_{n \rightarrow \infty} \rho_n = 0$,
- (iv) $0 < c \leq r_n \leq d < 1, \forall n \in \mathbb{N}$,
- (v) $\sum_{i=1}^N a_i = 1$,
- (vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \\ \sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty \text{ and } \sum_{n=1}^{\infty} \alpha_1^n < \infty.$

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_{\mathcal{F}}u$.

5.2 Convergence analysis for a finite family of equilibrium problems and a nonspreading mapping

- (1) Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be an α -contractive mapping on H and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $T : C \rightarrow C$ be a nonspreading mapping. For every $i = 1, 2, \dots, N$, let $\Phi_i : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) with $\Omega := Fix(T) \cap \bigcap_{i=1}^N EP(\Phi_i) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ and $\{v_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} \sum_{i=1}^N a_i \Phi_i(v_n, y) + \frac{1}{\varphi_n} \langle y - v_n, v_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n = \theta_n P_C x_n + (1 - \theta_n) v_n, \\ x_{n+1} = \delta_n \gamma f(x_n) + (I - \delta_n A) P_C (I - \psi_n (I - T)) y_n, \forall n \in \mathbb{N}, \end{cases}$$

where $\{\delta_n\}, \{\theta_n\}, \{\varphi_n\}, \{\psi_n\} \subset (0, 1)$, $0 < a_i < 1$, for all $i = 1, 2, \dots, N$. Suppose the conditions (i)-(vi) hold.

- (i) $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$;
- (ii) $0 < \tau \leq \theta_n \leq v < 1$, for some $\tau, v > 0$;
- (iii) $\sum_{n=1}^{\infty} \psi_n < \infty$;
- (iv) $0 < \epsilon \leq \varphi_n \leq \eta < 1$, for some $\epsilon, \eta > 0$;
- (v) $\sum_{n=1}^N a_i = 1$;
- (vi) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$,
 $\sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$, $\sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{v_n\}$ converge strongly to $q = P_{\Omega}(I - A + \gamma f)q$.

- (2) Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be an α -contractive mapping on H and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $T : C \rightarrow C$ be a nonspreading mapping. Let $\Phi : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)'-(A4) with $\Omega := \text{Fix}(T) \cap EP(\Phi) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ and $\{v_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} \Phi(v_n, y) + \frac{1}{\varphi_n} \langle y - v_n, v_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n = \theta_n P_C x_n + (1 - \theta_n) v_n, \\ x_{n+1} = \delta_n \gamma f(x_n) + (I - \delta_n A) P_C (I - \psi_n (I - T)) y_n, \forall n \in \mathbb{N}, \end{cases}$$

where $\{\delta_n\}, \{\theta_n\}, \{\varphi_n\}, \{\psi_n\} \subseteq (0, 1)$. Suppose the conditions (i)-(vi) hold.

- (i) $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$;
- (ii) $0 < \tau \leq \theta_n \leq v < 1$, for some $\tau, v > 0$;
- (iii) $\sum_{n=1}^{\infty} \psi_n < \infty$;
- (iv) $0 < \epsilon \leq \varphi_n \leq \eta < 1$, for some $\epsilon, \eta > 0$;
- (v) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$,
 $\sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$, $\sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{v_n\}$ converge strongly to $q = P_{\Omega}(I - A + \gamma f)q$.

- (3) Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be an α -contractive mapping on H and let $A : H \rightarrow H$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $T : C \rightarrow C$ be a nonspreading mapping with $\text{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1 \in H$ and

$$x_{n+1} = \delta_n \gamma f(x_n) + (I - \delta_n A) P_C (I - \psi_n (I - T)) P_C x_n, \forall n \in \mathbb{N},$$

where $\{\delta_n\}, \{\psi_n\} \subseteq (0, 1)$. Suppose the conditions (i)-(vi) hold.

- (i) $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$;
- (ii) $\sum_{n=1}^{\infty} \psi_n < \infty$;
- (iii) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $q = P_{\text{Fix}(T)}(I - A + \gamma f)q$.

- (4) Let C be a nonempty closed convex subset of a real Hilbert space H . Let $f : C \rightarrow C$ be an α -contractive mapping, let $A : C \rightarrow C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. For $i = 1, 2, \dots, \bar{N}$, let $\Phi_i : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $T_i : C \rightarrow C$, for $i = 1, 2, \dots, N$ be a finite family of nonspreading mappings with $\Omega := \bigcap_{i=1}^N \text{Fix}(T_i) \cap \bigcap_{i=1}^{\bar{N}} EP(\Phi_i) \neq \emptyset$. Let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \dots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \dots, N - 1$ and $\alpha_1^N \in (0, 1]$, $\alpha_3^N \in [0, 1)$, $\alpha_2^j \in (0, 1)$ for all $j = 1, 2, \dots, N$, and let S be the S -mapping generated by T_1, T_2, \dots, T_N and $\rho_1, \rho_2, \dots, \rho_N$. Let $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} \sum_{i=1}^{\bar{N}} a_i \Phi_i(v_n, y) + \frac{1}{\varphi_n} \langle y - v_n, v_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n = \theta_n P_C x_n + (1 - \theta_n) v_n, \\ x_{n+1} = \delta_n \gamma f(x_n) + (I - \delta_n A) P_C (I - \psi_n (I - S)) y_n, \forall n \in \mathbb{N}, \end{cases}$$

where $\{\delta_n\}$, $\{\theta_n\}$, $\{\varphi_n\}$, $\{\psi_n\} \subseteq (0, 1)$, $0 < a_i < 1$, for all $i = 1, 2, \dots, \bar{N}$. Suppose the conditions (i)-(vi) hold.

- (i) $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$;
- (ii) $0 < \tau \leq \theta_n \leq \nu < 1$, for some $\tau, \nu > 0$;
- (iii) $\sum_{n=1}^{\infty} \psi_n < \infty$;
- (iv) $0 < \epsilon \leq \varphi_n \leq \eta < 1$, for some $\epsilon, \eta > 0$;
- (v) $\sum_{n=1}^{\bar{N}} a_i = 1$;
- (vi) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$,
 $\sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty$, $\sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ converge strongly to $q = P_{\Omega}(I - A + \gamma f)q$.

5.3 Viscosity approximation methods for the modified generalized equilibrium problem and a finite family of strictly pseudo-contractive mappings

- (1) Let C be a nonempty closed convex subset of a real Hilbert space H . For every $i = 1, 2, \dots, N$, $S_i : C \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with coefficients μ_i , $\Phi_i : H \times C \times C \rightarrow \mathbb{R}$ be equilibrium-like function satisfying (H1)-(H3). Let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function and

$A : C \rightarrow C$ be an α -inverse strongly monotone mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strictly pseudo-contractive mappings and $\kappa_i \leq \gamma_1$ with $\mathcal{F} := \bigcap_{i=1}^N \text{Fix}(T_i) \cap \bigcap_{i=1}^N (\text{MGEP})_s(\Phi_i, \varphi, A) \neq \emptyset$. For every $n \in \mathbb{N}$, let K_n be the K -mapping generated by T_1, T_2, \dots, T_N and $\lambda_1^n, \lambda_2^n, \dots, \lambda_N^n$ where $0 < \phi \leq \lambda_i^n \leq \psi < \gamma_2$, for all $i = 1, 2, \dots, N$ and $\gamma_1 + \gamma_2 < 1$. For every $i = 1, 2, \dots, N$, let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and $w_1^i \in S_i(I - r_1^i A)x_1$, there exists sequences $\{w_n^i\} \in H$ and $\{x_n\}, \{u_n^i\} \subseteq C$ such that

$$\begin{cases} w_n^i \in S_i(I - r_n^i A)x_n, \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n}) \mathcal{H}(S_i(I - r_n^i A)x_n, S_i(I - r_{n+1}^i A)x_{n+1}), \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - x_n, y - u_n^i \rangle + \langle Ax_n, y - u_n^i \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n \left(\sum_{i=1}^N a_n^i u_n^i \right) + \delta_n K_n x_n, \forall n \geq 1, \end{cases}$$

where $f : C \rightarrow C$ be a contraction mapping with a constant ξ and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subseteq (0, 1)$ with $\alpha_n + \beta_n + \delta_n = 1, \forall n \geq 1$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < \tau \leq \beta_n, \delta_n \leq v < 1$;
- (iii) $0 \leq \eta \leq a_n^i \leq \sigma < 1$, for all $i = 1, 2, \dots, N - 1$ and $0 < \eta \leq a_n^N \leq \sigma \leq 1$ with $\sum_{n=1}^N a_n^i = 1$;
- (iv) $0 < \epsilon \leq r_n^i \leq \omega < 2\alpha$, for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, N$;
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty,$
 $\sum_{n=1}^{\infty} |r_{n+1}^i - r_n^i| < \infty, \sum_{n=1}^{\infty} |a_{n+1}^i - a_n^i| < \infty, \sum_{n=1}^{\infty} |\lambda_i^{n+1} - \lambda_i^n| < \infty$, for all $i = 1, 2, \dots, N$;
- (vi) For each $i = 1, 2, \dots, N$, there exists $\rho_i > 0$ such that

$$\Phi_i(w_{r_1^i}^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)) + \Phi_i(w_{r_2^i}^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)) \leq -\rho_i \|T_{r_1^i}(x_1) - T_{r_2^i}(x_2)\|^2,$$

for all $(r_1^i, r_2^i) \in \Theta_i \times \Theta_i, (x_1, x_2) \in C \times C$ and $w_j^i \in S_i(x_j)$, for $j = 1, 2$, where $\Theta_i = \{r_n^i : n \geq 1\}$.

Then $\{x_n\}$ and $\{u_n^i\}$ converge strongly to $q = P_{\mathcal{F}}f(q)$, for every $i = 1, 2, \dots, N$.

- (2) Let C be a nonempty closed convex subset of a real Hilbert space H . For every $i = 1, 2, \dots, N, S_i : C \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with coefficients $\mu_i, \Phi_i : H \times C \times C \rightarrow \mathbb{R}$ be equilibrium-like function satisfying (H1)-(H3). Let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function and $A : C \rightarrow C$ be an α -inverse strongly monotone mapping. Let $T : C \rightarrow C$ be κ -strictly pseudo-contractive mapping with $\kappa \leq \gamma_1$ and $\mathcal{F} := F(T) \cap \bigcap_{i=1}^N (\text{MGEP})_s(\Phi_i, \varphi, A) \neq \emptyset$. For every $n \in \mathbb{N}$, let $\{\lambda_n\}$ be a sequence of real numbers where $0 < \lambda_n < \gamma_2$ and $\gamma_1 + \gamma_2 < 1$. For every $i = 1, 2, \dots, N$, let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and $w_1^i \in S_i(I - r_1^i A)x_1$, there exists sequences $\{w_n^i\} \in H$ and $\{x_n\}, \{u_n^i\} \subseteq C$ such

that

$$\begin{cases} w_n^i \in S_i(I - r_n^i A)x_n, \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n}) \mathcal{H}(S_i(I - r_n^i A)x_n, S_i(I - r_{n+1}^i A)x_{n+1}), \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - x_n, y - u_n^i \rangle + \langle Ax_n, y - u_n^i \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n \left(\sum_{i=1}^N \alpha_n^i u_n^i \right) + \delta_n (\lambda_n T + (1 - \lambda_n) I)x_n, \forall n \geq 1, \end{cases}$$

where $f : C \rightarrow C$ be a contraction mapping with a constant ξ and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subseteq (0, 1)$ with $\alpha_n + \beta_n + \delta_n = 1, \forall n \geq 1$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < \tau \leq \beta_n, \delta_n \leq v < 1$;
- (iii) $0 \leq \eta \leq a_n^i \leq \sigma < 1$, for all $i = 1, 2, \dots, N - 1$ and $0 < \eta \leq a_n^N \leq \sigma \leq 1$ with $\sum_{n=1}^N a_n^i = 1$;
- (iv) $0 < \epsilon \leq r_n^i \leq \omega < 2\alpha$, for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, N$;
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$
 $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1}^i - r_n^i| < \infty,$
 $\sum_{n=1}^{\infty} |a_{n+1}^i - a_n^i| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, for all $i = 1, 2, \dots, N$;
- (vi) For each $i = 1, 2, \dots, N$, there exists $\rho_i > 0$ such that

$$\Phi_i(w_{r_1^i}^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)) + \Phi_i(w_{r_2^i}^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)) \leq -\rho_i \|T_{r_1^i}(x_1) - T_{r_2^i}(x_2)\|^2,$$

for all $(r_1^i, r_2^i) \in \Theta_i \times \Theta_i, (x_1, x_2) \in C \times C$ and $w_j^i \in S_i(x_j)$, for $j = 1, 2$, where $\Theta_i = \{r_n^i : n \geq 1\}$.

Then $\{x_n\}$ and $\{u_n^i\}$ converges strongly to $q = P_{\mathcal{F}}f(q)$, for every $i = 1, 2, \dots, N$.

- (3) Let C be a nonempty closed convex subset of a real Hilbert space H . For every $i = 1, 2, \dots, N, S_i : C \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with coefficients $\mu_i, \Phi_i : H \times C \times C \rightarrow \mathbb{R}$ be equilibrium-like function satisfying (H1)-(H3). Let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strictly pseudo-contractive mappings and $\kappa_i \leq \gamma_1$ with $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N (GEP)_s(\Phi_i, \varphi) \neq \emptyset$. For every $n \in \mathbb{N}$, let K_n be the K-mapping generated by T_1, T_2, \dots, T_N and $\lambda_1^n, \lambda_2^n, \dots, \lambda_N^n$ where $0 < \phi \leq \lambda_i^n \leq \psi < \gamma_2 < 1$, for all $i = 1, 2, \dots, N$ and $\gamma_1 + \gamma_2 < 1$. For every $i = 1, 2, \dots, N$, let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and $w_1^i \in S_i(x_1)$, there exists sequences $\{w_n^i\} \in H$ and $\{x_n\}, \{u_n^i\} \subseteq C$ such that

$$\begin{cases} w_n^i \in S_i(x_n), \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n}) \mathcal{H}(S_i(x_n), S_i(x_{n+1})), \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - x_n, y - u_n^i \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n \left(\sum_{i=1}^N \alpha_n^i u_n^i \right) + \delta_n K_n x_n, \forall n \geq 1, \end{cases}$$

where $f : C \rightarrow C$ be a contraction mapping with a constant ξ and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subseteq (0, 1)$ with $\alpha_n + \beta_n + \delta_n = 1, \forall n \geq 1$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < \tau \leq \beta_n, \delta_n \leq v < 1$;
- (iii) $0 \leq \eta \leq a_n^i \leq \sigma < 1$, for all $i = 1, 2, \dots, N-1$ and $0 < \eta \leq a_n^N \leq \sigma \leq 1$ with $\sum_{n=1}^N a_n^i = 1$;
- (iv) $0 < \epsilon \leq r_n^i \leq \omega < 1$, for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, N$;
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
 $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1}^i - r_n^i| < \infty$,
 $\sum_{n=1}^{\infty} |a_{n+1}^i - a_n^i| < \infty$, $\sum_{n=1}^{\infty} |\lambda_i^{n+1} - \lambda_i^n| < \infty$, for all $i = 1, 2, \dots, N$;
- (vi) For each $i = 1, 2, \dots, N$, there exists $\rho_i > 0$ such that

$$\Phi_i(w_1^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)) + \Phi_i(w_2^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)) \leq -\rho_i \|T_{r_1^i}(x_1) - T_{r_2^i}(x_2)\|^2,$$

for all $(r_1^i, r_2^i) \in \Theta_i \times \Theta_i$, $(x_1, x_2) \in C \times C$ and $w_j^i \in S_i(x_j)$, for $j = 1, 2$, where $\Theta_i = \{r_n^i : n \geq 1\}$.

Then $\{x_n\}$ and $\{u_n^i\}$ converges strongly to $q = P_{\mathcal{F}}f(q)$, for every $i = 1, 2, \dots, N$.

5.4 General iterative methods for equilibrium problems, nonspreading mapping and quasi-nonexpansive mapping

- (1) Let C be a nonempty closed convex subset of a real Hilbert space H . Let $f : C \rightarrow C$ be an α -contractive mapping, let $A : C \rightarrow C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. For $i = 1, 2, \dots, \bar{N}$, let $\Phi_i : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $T_i : C \rightarrow C$, for $i = 1, 2, \dots, N$ be a finite family of nonspreading mappings with $\Omega := \bigcap_{i=1}^N \text{Fix}(T_i) \cap \bigcap_{i=1}^{\bar{N}} EP(\Phi_i) \neq \emptyset$. Let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \dots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \dots, N-1$ and $\alpha_1^N \in (0, 1)$, $\alpha_3^N \in [0, 1)$, $\alpha_2^j \in (0, 1)$ for all $j = 1, 2, \dots, N$, and let S be the S -mapping generated by T_1, T_2, \dots, T_N and $\rho_1, \rho_2, \dots, \rho_N$. Let $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} \sum_{i=1}^{\bar{N}} a_i \Phi_i(v_n, y) + \frac{1}{\varphi_n} \langle y - v_n, v_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n = \theta_n P_C x_n + (1 - \theta_n) v_n, \\ x_{n+1} = \delta_n \gamma f(x_n) + (I - \delta_n A) P_C (I - \psi_n (I - S)) y_n, \forall n \in \mathbb{N}, \end{cases}$$

where $\{\delta_n\}$, $\{\theta_n\}$, $\{\varphi_n\}$, $\{\psi_n\} \subseteq (0, 1)$, $0 < a_i < 1$, for all $i = 1, 2, \dots, \bar{N}$. Suppose the conditions (i)-(vi) hold.

- (i) $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$;
- (ii) $0 < \tau \leq \theta_n \leq v < 1$, for some $\tau, v > 0$;
- (iii) $\sum_{n=1}^{\infty} \psi_n < \infty$;
- (iv) $0 < \epsilon \leq \varphi_n \leq \eta < 1$, for some $\epsilon, \eta > 0$;

$$(v) \sum_{n=1}^{\tilde{N}} a_i = 1;$$

$$(vi) \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty, \\ \sum_{n=1}^{\infty} |\psi_{n+1} - \psi_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ converge strongly to $q = P_{\Omega}(I - A + \gamma f)q$.

(2) Theorem 4.3 can be considered as an improvement of Theorem 3.1 in the result of Tien and Jin [48] in the sense that some conditions are not assumed.

$$(i) T_{\omega} := (1 - \omega)I + \omega T, \omega \in (0, \frac{1}{2}),$$

(ii) T is demi-closed on H ,

where T is quasi-nonexpansive mapping on H .

5.5 Examples and numerical results

5.5.1 Conclusions for Example 4.5

(1) Table 4.1 and Figure 4.1 show that the sequences $\{x_n\}$ and $\{u_n\}$ converge to 0, where $\{0\} = \bigcap_{i=1}^N EP(F_i) \cap \bigcap_{i=1}^N VI(C, A_i) \cap \bigcap_{i=1}^{\infty} Fix(T_i)$.

(2) Theorem 3.5 guarantees the convergence of $\{x_n\}$ and $\{u_n\}$ in Example 4.5.

5.5.2 Conclusions for Example 4.7

(1) From Table 4.2 and Figure 4.2, the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ converge to 1, where $\{1\} = Fix(T) \cap \bigcap_{i=1}^N EP(\Phi_i)$.

(2) For case (i), Corollary 3.12 guarantees the convergence of $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$.

(3) For case (ii), the convergence of $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ can be guaranteed by Theorem 3.11.

(4) The iteration (4.7) for the combination of equilibrium problem converges faster than the iteration (4.9) for the classical equilibrium problem.

5.5.3 Conclusions for Example 4.9

(1) Table 4.3, 4.4 and Figure 4.3 show that the sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ converge to $\mathbf{1}$, where $\{\mathbf{1}\} = \{(1, 1)\} = Fix(T) \cap \bigcap_{i=1}^N EP(\Phi_i)$.

(2) For case (i), Corollary 3.12 guarantees the convergence of $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$.

(3) For case (ii), the convergence of $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ can be guaranteed by Theorem 3.11.

(4) The iteration (4.13) for the combination of equilibrium problem converges faster than the iteration (4.14) for the classical equilibrium problem.

5.6 Suggestions

In our thesis, we obtain some results of equilibrium problem, variational inequality problem and fixed point problem in a Hilbert space. For those who would like to extend these results, we suggest that proving our results in a Banach space would be better. Furthermore, if some conditions of our Lemmas and Theorems are omitted, the proof would be more complicated.

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