THE CUT LOCUS OF A RANDERS ROTATIONAL SURFACE OF REVOLUTION HOMEOMORPHIC TO 2 DIMENSIONAL SPHERE

RATTANASAK HAMA

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY PROGRAM IN APPLIED MATHEMATICS

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

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Abstract

In this thesis, we study structure of the cut locus of a Randers rotational surface of revolution homeomorphic to a 2D sphere. The Randers rotational metric is obtained from Zermelo's navigation process with the wind blowing along the parallel. We show that in the case of flag curvature of the Randers surface is monotone along a meridian from pole to equator. The cut locus of a point $x \in M$ is a point on a subarc of the opposite half bending meridian or of the antipodal parallel. More generally, when the flag curvature is non-monotone along the meridian from pole to equator and the cut locus of a point x on the equator is a subarc of the same equator, then the cut locus of any point $\widetilde{x} \in M$ different from poles is a subarc of the antipodal parallel. Some examples are also given.

Keywords : Flag curvature, Cut locus, Meridian, Randers

หัวข้อวิทยานิพนธ์	คัทโลกัสของพื้นผิวที่เกิดจากการหมุนรอบสมานสัณฐานแบบทรงกลมสองมิติเชิงการหมุนแบบแรนเดอร์ส
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บทคัดย่อ

วิทยานิพนธ์เล่มนี้ ผู้วิจัยได้ศึกษาถึงโครงสร้างคัทโลกัสของพื้นผิวที่เกิดจากการหมุนรอบที่ สมานสัณฐานกับทรงกลมที่มีสองมิติเชิงการหมุนแบบแรนเดอร์ส เมตริกการหมุนแบบแรนเดอร์สได้มาจากกระบวนการเคลื่อนที่ของเซอร์เมโลเมื่อทิศทางของลมพัดตามแนวตั้งฉากกับเมอริเดียน ผู้วิจัยได้แสดงให้เห็นว่าในกรณีที่ค่าความโค้งแฟล็กบนพื้นผิวมีค่าทางเดียวตามเส้นเมอริเดียน คัทโลกัสของจุดบนพื้นผิวจะเป็นส่วนหนึ่งของเส้นเมอริเดียนแบบบิดที่อยู่ข้ามจุดนั้น หรือเป็นส่วนหนึ่งของเส้นที่ตั้งฉากกับเมอริเดียนที่อยู่ตรงข้ามจุดนั้น ในขณะที่ หากค่าความโค้งแฟล็กไม่ได้มีค่าทางเดียวตามเส้นเมอริเดียน และหากเราทราบว่าคัทโลกัสของจุดบนอีเควเตอร์นั้นเป็นส่วนหนึ่งของเส้นอีเควเตอร์ จะได้ว่าคัทโลกัสของจุดใดๆที่ไม่ใช่จุดขั้วจะเป็นส่วนหนึ่งของเส้นที่ตั้งฉากกับเมอริเดียนที่อยู่ตรงข้ามจุดนั้น ผู้วิจัยได้ทำการยกตัวอย่างกรณีต่างๆไว้ในงานวิจัย

คำสำคัญ : ค่าความโค้งแฟล็ก คัทโลกัส เมอริเดียน แรนเดอร์ส

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... Rattanasak HAMA ...

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Chapter 1

Introduction

The aim of this thesis is to study the geodesics and the cut locus of a Randers rotational surface of revolution homeomorphic to sphere. Before introducing the main theorems of this thesis, we would like to describe the history of the problem and our research motivation.

1.1 Research Motivation

B. Riemann introduced a new research area combining analysis with geometry in 1854, called today **Riemannian geometry**, that generalizes the flat **Euclidean space** to non-flat spaces called **Riemannian spaces** or **Riemannian manifolds** (see [8]).

A Riemannian manifold is a differentiable manifold with a given Riemannian metric, that is a norm in the tangent bundle whose associated distance function is symmetric. There are many researchers working in this field, studying the global behaviour of geodesics, conjugate points and cut locus amongst many other topics (see [1], [11], [12]).

In this work, we are interested in the more general case of the so-called **Finslerian manifold**, that is manifold endowed with a Finsler norm in the tangent bundle such that the distance function is not symmetric.

Z. Shen is the first person who pointed out the solutions of the Zermelo's navigation problem. Such that the solutions are the geodesics of a Finslerian metric which is called **Randers metric**, that is the deformation of Riemannian metric by adding a linear 1-form (see [13]).

For the sake of simplicity and motivation by examples in the real world, the Riemannian manifold that we use in this study is a **surface of revolution**, this surface is obtained by rotating a profile curve around an axis. The main mathematical reason that we use this manifold is because the geodesics equation can be integrated, therefore it is easy to study the global behaviour of geodesics, conjugate points and cut locus. In this thesis we consider a surface of revolution that is homeomorphic to a sphere, such a surface is called a **2 dimensional sphere of revolution** or **2D sphere of revolution** for convenience.

The global behaviour of geodesics, conjugate points and cut locus on Riemannian 2D sphere revolution with monotone curvature was studied by M. Tanaka and R. Sinclair in 2007 (see [12]).

R. Hama, P. Chitsakul and S. V. Sabau studied the geometry of Randers rotational surface of revolution homeomorphic to a plane in 2015 (see [9]).

Moreover, in 2015, B. Bonnard, B. Caillau, R. Sinclair and M. Tanaka have introduced interesting examples of 2D sphere of revolution whose sectional curvature is non-monotone and they show that if the cut locus of a point on the equator is a subarc of equator then the cut locus of any point, except the pair of poles, is a subarc of the antipodal parallel (see [5]). We will extend their result to find the cut locus on a 2D sphere of revolution endowed with a Finsler metric of Randers type whose curvature is non-monotone.

Therefore, this thesis will extend the results from [5], [9] and [12], in order to determine the behaviour of geodesics and cut locus on a Randers rotational 2D sphere of revolution.

1.2 Objectives of the study

1. To study the behavior of geodesics, cut points, and cut locus on a Randers rotational 2D sphere of revolution.
2. To construct concrete examples of Randers rotational 2D sphere of revolution that illustrate the behaviour of geodesics, cut points and cut locus.

1.3 Scopes of the study

1. The Randers metric is construct from Zermelo's navigation process.
2. The navigation data for Zermelo's navigation process is rotational along parallels.
3. The surfaces of revolution is homeomorphic to the sphere.
4. The 2D sphere of revolution is symmetric with respect to the equator.

1.4 Benefits of the study

1. To clarify the relation between the global behaviour of geodesics on a Riemannian and a Randers rotational 2D sphere of revolution.
2. To clarify the relation between the cut locus of a Riemannian and a Randers rotational 2D sphere of revolution.
3. To give concrete examples of Randers rotational 2D sphere of revolution.
4. To build the basic setting for a further study about conjugate points and conjugate locus.
5. To see the study of cut locus which is deeply related to the topology of the manifold.

Chapter 2

Preliminaries

In this chapter, we would like to introduce the collection of basic knowledge and notations on differential geometry, that will be used in this thesis.

2.1 Notations on differential geometry

2.1.1 Differential manifold and tangent bundle

Definition 2.1. [8] Let M be an n**-dimensional smooth manifold** and $(U;x)$ is local coordinates of M, that is for any point x,

$$x = (x^1, x^2, ..., x^n) \in U \subset M$$

there exists a homeomorphic mapping $f : x \in U \to \mathbb{R}^n$, where $\mathbb{R}^n$ is an n-dimensional Euclidean space.

Remark 2.1. The mapping f is called **homeomorphic mapping** or **homeomorphism** if f satisfies :

(i) f is bijective, i.e. f is surjective and injective,

(ii) f is continuous,

(iii) f^{-1} is continuous.

Definition 2.2. [8] The **tangent space** over a point x on M is a vector space with the basis $\left(\frac{\partial}{\partial x^i}\right)_x$, $i = 1, ..., n$ such that for every tangent vector $y \in T_xM$, we can write it in the form of the linear combination

$$y = \sum_{i=1}^{n} y^i(x) \left(\frac{\partial}{\partial x^i}\right) = y^i \frac{\partial}{\partial x^i}.$$

Remark 2.2. The set of all tangent vectors on M is called **tangent bundle** and denoted by

$$TM = \{(x, y) \mid x \in M, y \in T_xM\} \to M.$$

2.1.2 Riemannian manifold

Definition 2.3. [8] Suppose M is an n-dimensional smooth manifold, and H is a symmetric covariant tensor field of rank n on M. If $(U;x)$ is a local coordinates on M, then the tensor field H can be expressed as

$$H = h_{ij} dx^i \otimes dx^j,$$

on U, where $h_{ij} = h_{ji}$ is a smooth function on U.

The tensor H provides a bilinear function on T_xM at every point $x \in M$.

Suppose $Y = Y^i \frac{\partial}{\partial x^i}$ and $Z = Z^i \frac{\partial}{\partial x^i}$, then

$$H(Y, Z) = h_{ij} Y^i Z^j. \tag{2.1}$$

Definition 2.4. [8] The tensor H is **nondegenerate** at the point x, whenever $X \in T_xM$ and

$$H(X, Y) = 0 \tag{2.2}$$

for all $Y \in T_xM$, implies $X = 0$.

From definition 2.4, it follows that H is nondengenerate at x if and only if the system of linear equations

$$h_{ij}(x)X^i = 0, \qquad 1 \leq j \leq n,$$

has only the trivial solution, that is det$(h_{ij}(x)) \neq 0$.

Definition 2.5. [8] If for all $y \in T_xM$,

$$H(y, y) = h_{ij} y^i y^j \geq 0, \tag{2.3}$$

and $H(y, y) = 0$ for $y = 0$, then we say that H is **positive definite** at x.

Remark 2.3. From linear algebra, it is known that the condition for H to be positive definite is that the matrix (h_{ij}) is positive definite. Thus a positive definite tensor H is necessarily nondegenerate.

Definition 2.6. [8] Let M be an n-dimensional smooth manifold, if any point $x \in M$ is given a nondegenerate symmetric covariant tensor field H of rank n and H is positive definite then M is called a **Riemannian manifold**.

We define the inner product on the tangent space T_xM at every point $x \in M$. For any $y, z \in T_xM$ by

$$\langle y, z \rangle = H(y, z) = h_{ij}(x) y^i z^j. \tag{2.4}$$

Since H is positive definite, we can define the length of a tangent vector and the angle between two tangent vectors at the same point,

$$\begin{aligned} |y| &= \sqrt{h_{ij} y^i y^j} \\ \cos\angle(y, z) &= \frac{\langle y, z \rangle}{|y||z|}. \end{aligned} \tag{2.5}$$

The differential 2-form

$$ds^2 = h_{ij} dx^i dx^j \tag{2.6}$$

is independent of the local coordinates x and ds^2 is called the **Riemannian metric** or **metric form**. ds is the length of an infinitesimal tangent vector, called the **element of arclength**.

Remark 2.4. For the definition of a 2-form see [8].

Remark 2.5. The Riemannian norm $H : TM \to [0, \infty)$ satisfies

(i) H is positive and differentiable on $\widetilde{TM} := TM \setminus \{0\}$,

(ii) H is absolute homogeneous, that is $H(x, \lambda y) = \lambda^2 H(x, y)$ for any $\lambda \in \mathbb{R}$ and for all $(x, y) \in \widetilde{TM}$,

(iii) Hessian matrix $h_{ij} = \frac{1}{2}\frac{\partial^2 H}{\partial y^i \partial y^j}$ is positive definite on $\widetilde{TM}$.

Suppose $\gamma : x^i = x^i(t)$, $a \le t \le b$, is a continuous and smooth parametrized curve on M. Then the arclength of γ is defined by

$$\mathcal{L}(\gamma) = \int_a^b \sqrt{h_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}dt. \tag{2.7}$$

$\mathcal{L}(\gamma)$ is called the **integral length**.

Remark 2.6. If we consider the matrix $h_{ij} = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

then the metric form $ds^2 = \sum_{i=1}^n (dx^i)^2$ is called **Euclidean metric**.

Remark 2.7. The pair (M, h) is called **Riemannian manifold** where M is n-dimensional Riemannian manifold and h is Riemannian metric.

Theorem 2.8. [8] Suppose M is an n-dimensional Riemannian manifold. Then there exists a unique torsion-free and metric compatible connection on M, called the **Levi-Civita connnection** or **Riemannian connection** of M defined by

$$\Gamma^k_{ij} = \frac{1}{2}h^{kl}\left(\frac{\partial h_{il}}{\partial x^j} + \frac{\partial h_{jl}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^l}\right), \tag{2.8}$$

where h^{ij} is the inverse matrix of h_{ij}.

Remark 2.9. Γ^k_{ij} defined in (2.8) is called the **Christoffel symbol** of second kind while the first kind is defined by

$$\Gamma_{ikj} = \frac{1}{2}\left(\frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^k}\right) \qquad \text{or} \qquad \Gamma_{ilj} = h_{kl}\Gamma^k_{ij}.$$

Theorem 2.10. [8] The **curvature tensor** R_{ijkl} of a Riemannian manifold satisfies the following properties:

(i) $R_{ijkl} = -R_{jikl} = -R_{ijlk}$,

(ii) $R_{ijkl} + R_{iklj} + R_{iljk} = 0$,

(iii) $R_{ijkl} = R_{klij}$,

where R_{ijkl} is defined by

$$R_{ijkl} = \frac{\partial \Gamma_{ijl}}{\partial x^k} - \frac{\partial \Gamma_{ijk}}{\partial x^l} + \Gamma_{ik}^h \Gamma_{jhl} - \Gamma_{il}^h \Gamma_{jhk}. \tag{2.9}$$

Definition 2.7. [11] Let M be 2-dimensional Riemannian manifold or surface. The **Gaussian curvature** G at a point x on M is defined by

$$G(x) = -\frac{R_{1212}}{\det(h_{ij}(x))}. \tag{2.10}$$

2.1.3 Einstein manifold

Contracting the Riemannian curvature tensor R_{ijkl}, we get the **Ricci tensor**

$$R_{ij} = h^{km} R_{kijm}. \tag{2.11}$$

Definition 2.8. [8] The differential manifold (M, h) is called **Einstein manifold** if there exists a function $\lambda : \mathbb{R} \to \mathbb{R}$, such that

$$R_{ij} = \lambda h_{ij}$$

everywhere on M.

2.1.4 Geodesics on a Riemannian manifold

Definition 2.9. [8] Suppose (M, h) is the n-dimensional Riemannian manifold with the Riemannian metric defined in (2.6). Let $\gamma : x^i = x^i(t)$ be a parametrized curve on M, then $y(t)$ called a tangent vector field defined along γ if

$$y(t) := y^i(t) \left(\frac{\partial}{\partial x^i} \right)_{\gamma(t)}.$$

Lemma 2.11. If $\gamma(t)$ is a smooth curve then there exists a reparametrization $\gamma(s)$ such that $|\dot{\gamma}(s)| = 1$, where $|\cdot|$ is the Riemannian norm.

Proof. Consider the arclength function

$$s(t) = \int_a^t |\dot{\gamma}(u)| du.$$

We take derivative with respect to t, we have $\frac{ds}{dt} = |\dot{\gamma}(t)|$, it follows that

$$\frac{dt}{ds} = \frac{1}{|\dot{\gamma}(t)|}.$$

Let $\gamma(s) = \gamma(t(s))$, we take derivative with respect to s, therefore $\dot{\gamma}(s) = \dot{\gamma}(t(s)) \frac{dt}{ds}$, hence

$$|\dot{\gamma}(s)| = |\dot{\gamma}(t(s))| \frac{1}{|\dot{\gamma}(t)|} | = 1.$$

□

From [4] we know that if the tangent vector of a unit speed curve $\gamma(s)$ is parallel along $\gamma(s)$, then $\gamma(s)$ is a **geodesic** of the Riemannian manifold (M, h), that is $\gamma(s)$ satisfies

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{2.12}$$

where Γ^i_{jk} is the Christoffel symbols defined in (2.8).

Remark 2.12. The geodesic curve $\gamma : [0, t_0] \to M$ is the local length minimizer and we can defined the h-Riemannian distance $d_h(\cdot,\cdot)$ between $p = \gamma(0), q = \gamma(t_0)$ on M by the length of geodesic, i.e.

$$d_h(p, q) = \int_0^{t_0} \sqrt{h_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}dt.$$

Definition 2.10. [8] The Riemannian manifold (M, h) is called **complete Riemannian manifold** if and only if for any two points $x_1, x_2 \in M$, there exists a geodesic curve joining x_1 and x_2.

2.1.5 Jacobi fields and Conjugate points on a Riemannian manifold

Definition 2.11. [11] Let $\gamma : [0, r]$ be a smooth curve in manifold M. A variation of γ is a differentiable mapping $\sigma : (-\varepsilon, \varepsilon) \times [0, r] \to M$ such that $\sigma(0, t) = \gamma(t)$, $t \in [0, r]$. $\gamma(t)$ is called **base curve** of the variation. For each $u \in (-\varepsilon, \varepsilon)$, the parametrized curve $\sigma_u : [0, r] \to M$ defined by $\sigma_u(t) = \sigma(u, t)$ which is called a **curve in the variation** (t-curve); while the parametrized curves $\sigma_t(u) = \sigma(u, t)$, t fixed, are called **transversal curves** (u-curves) of the variation.

A variation $\sigma(u, t)$ gives us two vectors fields on M:

$$U(t) := \frac{\partial\sigma}{\partial u}, \qquad T(t) := \frac{\partial\sigma}{\partial t}.$$

In particular, the field $U(t, 0) = \frac{\partial\sigma}{\partial u}\big|_{u=0}$ is called the **variation field** of the variation σ along the base curve $\gamma(t)$ (see [7]).

Definition 2.12. [8] The vector field J along a geodesic is called Jacobi field if it satisfies the second order ordinary differential equation:

$$D_TD_TJ + R(J, T)T = 0,$$

where D_T is the covariant derivative along γ with respect to the tangent vector $T = \dot{\gamma}$.

Theorem 2.13. [7] Let $\sigma(u, t)$ be a variation of base curve $\gamma(t)$

(a) If any t-curves $\sigma_u(t)$ are geodesics then variation vector field $U(t)$ is a Jacobi field.

(b) If the base curve $\gamma(t)$ is a geodesic then there exists a variation field U , where U is Jacobi field.

Definition 2.13. [7] Let $x \in M$, $\gamma : [0, t_0] \to M$ is geodesic joining $x = \gamma(0)$ and $q = \gamma(t_0)$. A point q is called **conjugate point** to x along γ if and only if there exits a non-zero Jacobi field along γ that vanishes at x and q that is

$$J(0) = J(t_0) = 0 \text{ and } J(t) \neq 0, \qquad \forall t \in (0, t_0).$$

2.1.6 Cut points and Cut locus on a Riemannian manifold

Definition 2.14. [12] Let $\gamma : [0, t_0] \to M$ be a minimal geodesic segment on a complete Riemannian manifold (M, h). The endpoint $\gamma(t_0)$ of the geodesic segment is called a **cut point** of $p := \gamma(0)$ along γ if any extended geodesic segment $\widetilde{\gamma} : [0, t_1] \to M$ of γ, where $t_1 > t_0$, is not a minimizing arc joining p to $\widetilde{\gamma}(t_1)$ anymore. The **cut locus** C_p^h of the point p is defined by the set of the cut points along all geodesic segments emanating from p.

2.2 Two-dimensional sphere of revolution

In this section, we recall the details about a two-dimensional sphere of revolution.

Definition 2.15. [12] A compact Riemannian manifold (M, h) homeomorphic to a two-dimensional sphere is called a **two-dimensional sphere of revolution** or **2D sphere of revolution**, if M admits a point p, called **pole**, such that for any two points q_1, q_2 on M with $d_h(p, q_1) = d_h(p, q_2)$, there exists an h-isometry f on M satisfying $f(q_1) = q_2$, and $f(p) = p$, where $d_h(\cdot, \cdot)$ denote the h-Riemannian distance function on M.

Remark 2.14. The point p is also called the **vertex** of M, that is p is the intersection of M with rotating axis.

From Definition 2.15, we construct the 2D sphere of revolution by rotating the unit speed profile curve for any $a \in \mathbb{R}^+$, $m : (0, 2a) \to \mathbb{R}^+$, $m(0) = m(2a) = 0$ and $m'(0) = -m'(2a)$ in xz-plane around z-axis and obtain the following parametric equation.

Let (r, θ) be the geodesic polar coordinates around the pole $p = m(0)$, then we have

$$M := \{(m(r)\cos\theta, m(r)\sin\theta, z(r)), \qquad r \in [0, 2a], \qquad \theta \in [0, 2\pi)\}. \tag{2.13}$$

Lemma 2.15. The condition for unit speed profile curve is $z(r) = \int_0^r \sqrt{1 - m'(t)^2}dt$.

Proof. Since the condition for unit speed curve on the xz-plane is $m'(t)^2 + z'(t)^2 = 1$, the conclusion follows immediately. □

From lemma 2.1 in [12], $C_p^h = \{q\}$ and $C_q^h = \{p\}$ are **a pair of poles** $\{p, q\}$. We introduce the coordinate curves on M. Let r_0 and θ_0 be constants.

1. $\gamma(r(s), \theta_0)$ is called **meridian**. If $\theta_0 = 0$ then γ is called **profile curve**.

2. $\gamma(r_0, \theta(s))$ is called **parallel**. If $r_0 = a$ then γ is called **equator**.

Remark 2.16. In this work a 2D sphere of revolution is obtained by rotating an unit speed profile curve which is symmetric with respect to the equator $r = a$, that is $m(r) = m(2a - r)$.

2.2.1 The Riemannian metric on a 2D sphere of revolution

Using the parametric representation (2.13), we will compute the Riemannian metric on a 2D sphere of revolution

$$M : \begin{cases} x(r,\theta) = m(r)\cos\theta \\ y(r,\theta) = m(r)\sin\theta \\ z(r,\theta) = z(r). \end{cases}$$

It follows that

$$\begin{aligned} dx &= m'(r)\cos(\theta)dr - m(r)\sin(\theta)d\theta \\ dy &= m'(r)\sin(\theta)dr + m(r)\cos(\theta)d\theta \\ dz &= z'(r)dr. \end{aligned}$$

Since x, y, z are coordinates in $\mathbb{R}^3$, we use the Euclidean metric in Remark 2.6, therefore

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= dr^2 + m(r)^2 d\theta^2, \end{aligned}$$

where we use unit speed condition $m'(r)^2 + z'(r)^2 = 1$.

We obtain lemma 2.17.

Lemma 2.17. The Riemannian metric h of geodesic polar coordinates (r, θ) around pole of a 2D sphere of revolution (M, h) can be expressed by

$$ds^2 = dr^2 + m(r)^2 d\theta^2 \tag{2.14}$$

or, in the Hessian matrix form

$$(h_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & m(r)^2 \end{pmatrix}. \tag{2.15}$$

Remark 2.18. If s is the unit speed parameter then

$$|\dot{\gamma}(s)| = |\dot{\gamma}(r(s), \theta(s))| = \sqrt{\left(\frac{dr}{ds}\right)^2 + m(r)^2\left(\frac{d\theta}{ds}\right)^2} = 1.$$

2.2.2 Geodesics on a Riemannian 2D sphere of revolution

In the previous section, we have obtained the Riemannian metric h on M when using the geodesic polar coordinates $(x^1, x^2) = (r, \theta)$ on a 2D sphere of revolution. In this section we will show how to compute Gaussian curvature and Riemannian geodesic equations.

Recall the formula of Christoffel symbol (2.8),

$$\Gamma^k_{ij} = \frac{1}{2}h^{kl}\left(\frac{\partial h_{il}}{\partial x^j} + \frac{\partial h_{jl}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^l}\right).$$

Since

$$(h_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & m(r)^2 \end{pmatrix},$$

we obtain

$$(h^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{m(r)^2} \end{pmatrix},$$

by straightforward computation, we get

$$\begin{aligned} &\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^2_{11} = \Gamma^2_{22} = 0, \\ &\Gamma^1_{22} = -m(r)m'(r), \qquad \Gamma^2_{12} = \frac{m'(r)}{m(r)}. \end{aligned} \tag{2.16}$$

Lemma 2.19. The Gaussian curvature at a point x on a Riemannian 2D sphere of revolution is $G(x) = \frac{-m''(r(x))}{m(r(x))}$.

Proof. Consider the formula of curvature tensor (2.9)

$$R_{ijkl} = \frac{\partial \Gamma_{ijl}}{\partial x^k} - \frac{\partial \Gamma_{ijk}}{\partial x^l} + \Gamma^h_{ik}\Gamma_{jhl} - \Gamma^h_{il}\Gamma_{jhk},$$

with the relation between first and second kinds of Christoffel symbols

$$\Gamma_{ilj} = h_{kl}\Gamma^k_{ij},$$

it follows that

$$\begin{aligned} &\Gamma_{111} = \Gamma_{112} = \Gamma_{121} = \Gamma_{211} = \Gamma_{222} = 0, \\ &\Gamma_{122} = \Gamma_{221} = m(r)m'(r), \qquad \Gamma_{212} = -m(r)m'(r). \end{aligned}$$

From the Gaussian curvature G (2.10) at point x where x in on parallel $r = r(x)$ on a 2D sphere of revolution is

$$G(x) = -\frac{R_{1212}}{\det(h_{ij}(x))}.$$

The following computation show that

$$\begin{aligned} R_{1212} &= \frac{\partial \Gamma_{122}}{\partial x^1} - \frac{\partial \Gamma_{121}}{\partial x^2} + \Gamma^h_{11}\Gamma_{2h2} - \Gamma^h_{12}\Gamma_{2h1} \\ &= \frac{\partial \Gamma_{122}}{\partial r} + \Gamma^1_{11}\Gamma_{212} - \Gamma^1_{12}\Gamma_{211} + \Gamma^2_{11}\Gamma_{222} - \Gamma^2_{12}\Gamma_{221} \\ &= m(r)m''(r) + m'(r)^2 - \frac{m'(r)}{m(r)}m(r)m'(r) \\ &= m(r)m''(r). \end{aligned}$$

and det$(h_{ij}(x)) = m(r)^2$, therefore we obtain $G(x) = \frac{-m''(r(x))}{m(r(x))}$.

□

On the other hand, we consider the geodesics equation (2.12)

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0,$$

in the case of a 2D sphere of revolution, the indices $i, j, k \in \{1, 2\}$, and therefore it follows that

$$\frac{dx^1}{ds^2} + \Gamma^1_{11}\frac{dx^1}{ds}\frac{dx^1}{ds} + \Gamma^1_{12}\frac{dx^1}{ds}\frac{dx^2}{ds} + \Gamma^1_{21}\frac{dx^2}{ds}\frac{dx^1}{ds} + \Gamma^1_{22}\frac{dx^2}{ds}\frac{dx^2}{ds} = 0$$
$$\frac{dx^2}{ds^2} + \Gamma^2_{11}\frac{dx^1}{ds}\frac{dx^1}{ds} + \Gamma^2_{12}\frac{dx^1}{ds}\frac{dx^2}{ds} + \Gamma^2_{21}\frac{dx^2}{ds}\frac{dx^1}{ds} + \Gamma^2_{22}\frac{dx^2}{ds}\frac{dx^2}{ds} = 0.$$

From (2.16), we obtain the system of second order ordinary differential equations

$$\frac{d^2r}{ds^2} - m(r)m'(r)\left(\frac{d\theta}{ds}\right)^2 = 0, \qquad (2.17)$$

and

$$\frac{d^2\theta}{ds^2} + 2\frac{m'(r)}{m(r)}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right) = 0. \qquad (2.18)$$

The equations (2.17) and (2.18) are called **geodesic equations**. We obtain the lemma 2.20.

Lemma 2.20. Let $\gamma(s) = (r(s), \theta(s))$ be a smooth unit speed curve on a Riemannian 2D sphere of revolution, if $\gamma(s)$ is a geodesic then $\gamma(s)$ is satisfies (2.17) and (2.18).

According to lemma 2.20, we obtain the lemmas 2.21 and 2.22.

Lemma 2.21. Every unit speed meridian is geodesic.

Proof. Let $\gamma(s) = (r(s), \theta_0)$, where θ_0 is constant, be a meridian. We get

$$\frac{d\theta_0}{ds} = \frac{d^2\theta_0}{ds^2} = 0,$$

it follows that $\gamma(s)$ satisfies (2.18).

Since $\gamma(s)$ is unit speed, we obtain

$$\frac{dr}{ds} = 1.$$

We take the derivative with respect to s and we get

$$\frac{d^2r}{ds^2} = 0.$$

Hence $\gamma(s)$ satisfies (2.17).

□

Lemma 2.22. The unit speed parallel is geodesic if and only if $m'(r_0) = 0$.

Proof. First step, let $\gamma(s) = (r_0, \theta(s))$ where r_0 is constant be a unit parallel. We can see that

$$\frac{dr_0}{ds} = \frac{d^2 r_0}{ds^2} = 0,$$

from unit speed condition, we get

$$\frac{d\theta}{ds} = \frac{1}{m(r_0)^2} \text{ is constant.}$$

Therefore

$$\frac{d^2\theta}{ds^2} = 0.$$

Then $\gamma(s)$ satisfies (2.18), if $\gamma(s)$ is geodesic from (2.17), we obtain

$$m(r_0)m'(r_0)\left(\frac{d\theta}{ds}\right)^2 = 0.$$

Since $m(r_0) \neq 0$ and $\frac{d\theta}{ds} \neq 0$, hence $m'(r_0) = 0$.

Second step, let $m'(r_0) = 0$ and $\gamma(s) = (r_0, \theta(s))$ is unit speed parallel. It follows that $\gamma(s)$ satisfies (2.17). Recall the unit speed condition then $\gamma(s)$ satisfies (2.18). Hence $\gamma(s)$ is geodesic.

□

2.2.3 The Clairaut relation on a Riemannian 2D sphere of revolution

In this section, we consider on (2.18), and by taking the integral with respect to s we get

$$m(r)^2\left(\frac{d\theta}{ds}\right) = \nu, \tag{2.19}$$

where ν is constant called **Clairaut constant**.

We will use (2.19) to prove theorem 2.23.

Theorem 2.23 (Clairaut's relation)**.** If $\gamma(r(s), \theta(s))$ is a geodesic on a 2D sphere of revolution then the angle $\phi(s)$ between $\dot{\gamma}(s)$ and the meridian passing through $\gamma(s)$ satisfy

$$m(r(s))\,\text{sin}(\phi(s)) = \nu.$$

Proof. Let us denote the angle between the tangent vector $\dot{\gamma}(s)$ to the unit speed geodesic $\gamma(s)$ and the meridian by $\phi(s)$. If we take into account that parallels and meridians are perpendicular, then by the definition of inner product, we have

$$\left\langle \dot{\gamma}(s), \frac{\partial}{\partial\theta} \right\rangle = |\dot{\gamma}(s)| \left|\frac{\partial}{\partial\theta}\right| \cos\left(\frac{\pi}{2} - \phi(s)\right).$$

Using now the fact that $\gamma(s)$ is unit speed and $\dot{\gamma}(s) = \frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{ds}\frac{\partial}{\partial\theta}$, we obtain

$$\left\langle \dot{\gamma}(s), \frac{\partial}{\partial\theta} \right\rangle = \left\langle \frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{ds}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} \right\rangle = \left|\frac{\partial}{\partial\theta}\right| \text{sin}(\phi(s)).$$

From the following computation

$$\begin{aligned}\left\langle \frac{dr}{ds}\frac{\partial}{\partial r}+\frac{d\theta}{ds}\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\right\rangle &= \left\langle \frac{dr}{ds}\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right\rangle + \left\langle \frac{d\theta}{ds}\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\right\rangle \\ &= \frac{dr}{ds}\left\langle \frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right\rangle + \frac{d\theta}{ds}\left\langle \frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\right\rangle \\ &= \frac{d\theta}{ds}\left\langle \frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\right\rangle \\ &= \frac{d\theta}{ds}\left|\frac{\partial}{\partial\theta}\right|^2,\end{aligned} \tag{2.20}$$

taking into account that $\langle \frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\rangle = 0$, it results

$$\begin{aligned}\frac{d\theta}{ds}\left|\frac{\partial}{\partial\theta}\right|^2 &= \left|\frac{\partial}{\partial\theta}\right|\sin(\phi(s)) \\ \frac{d\theta}{ds}\left|\frac{\partial}{\partial\theta}\right| &= \sin(\phi(s)).\end{aligned}$$

Paying attention to $(\frac{\partial}{\partial\theta}) = (0,1)$, we get

$$\left|\frac{\partial}{\partial\theta}\right| = |(0,1)| = \sqrt{h_{11}(0)^2 + h_{22}(1)1^2} = m(r(s)).$$

Therefore

$$\begin{aligned}\frac{d\theta}{ds}m(r(s)) &= \sin(\phi(s)) \\ \frac{d\theta}{ds}m(r(s))^2 &= m(r(s))\sin\phi(s).\end{aligned}$$

From (2.19) we have

$$m(r(s))\sin(\phi(s)) = \nu.$$

□

From theorem 2.23, we obtain corollary 2.24.

Corollary 2.24. Let $\gamma(s) = (r(s),\theta(s))$ be an unit speed geodesic of a 2D sphere of revolution (M,h) with Clairaut constant ν.

1. The Clairaut constant ν vanishes if and only if $\gamma(s)$ is tangent to a meridian.
2. If the Clairaut constant ν is non-vanishing, then $\gamma(s)$ does not pass through the pole of M.

Proof. 1. If $\gamma(s)$ is meridian then the angle $\phi(s) = 0$ therefore $\nu = 0$.

2. We recall that the geodesic curve is passing through the pole is only the meridian then if $\nu \neq 0$, we get $\gamma(s)$ is not passing through the pole.

□

2.2.4 Half-period function on a Riemannian 2D sphere of revolution

In previous section, if we consider the geodesic $\gamma(s)$ emanating from equator $r = a$, we know that if Clairaut constant ν of $\gamma(s)$ is not vanishing. That is $\gamma(s)$ is not meridian and if we consider that the angle $\phi(s)$ between $\dot\gamma(s)$ and profile curve is not equal to $\frac{\pi}{2}$ then $\gamma(s)$ is not parallel, from theorem 2.23 that the geodesic $\gamma(s)$ emanating from equator $r = a$ with Clairaut constant $\nu \in (0, m(a))$ must tangent to the parallel $\xi(\nu) = m^{-1}(r)$ and then it must return to the equator. We will compute the distance function between the emanating point and the returning point.

Remark 2.25. The Riemannian universal covering manifold of the 2D sphere of revolution $(M \setminus \{p, q\}, h)$ is defined by (see Figure 2.1).

$$\widetilde{M} := ((0, 2a) \times \mathbb{R}, d\widetilde{r}^2 + m(\widetilde{r})^2 d\widetilde{\theta}^2).$$

Let $\Pi : \widetilde{M} \to M \setminus \{p, q\}$ be a covering projection.

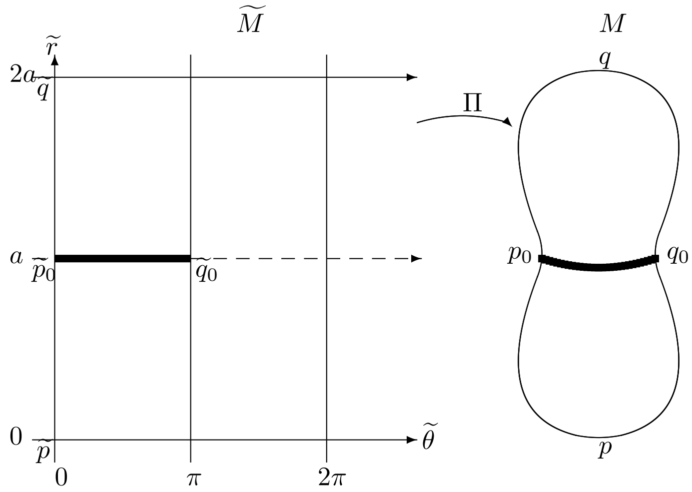

Figure 2.1*:* Universal covering manifold and manifold.

Lemma 2.26. Let $\widetilde{\gamma}_\nu^{\widetilde{p}_0}$ be a h-unit speed geodesic, where $\widetilde{p}_0 \in \widetilde{r} = a$ and $\nu \in (0, m(a))$, i.e. $\widetilde{p}_0$ is a point on equator and $\widetilde{\gamma}_\nu^{\widetilde{p}_0}$ is neither meridian nor parallel (equator) (see Figure 2.2). From Clairaut relation $\widetilde{\gamma}_\nu^{\widetilde{p}_0}$ must be tangent to the parallel $\xi(\nu)$ and return to the equator at $\widetilde{\gamma}_\nu^{\widetilde{p}_0}(t_0)$. The distance from $\widetilde{p}_0$ to $\widetilde{\gamma}_\nu^{\widetilde{p}_0}(t_0)$ can be computed by

$$\mathcal{H}(\nu) := 2\int_{\xi(\nu)}^{a} \frac{\nu}{m(t)\sqrt{m(t)^2 - \nu^2}} dt. \tag{2.21}$$

$\mathcal{H}(\nu)$ is called h-**half period function**.

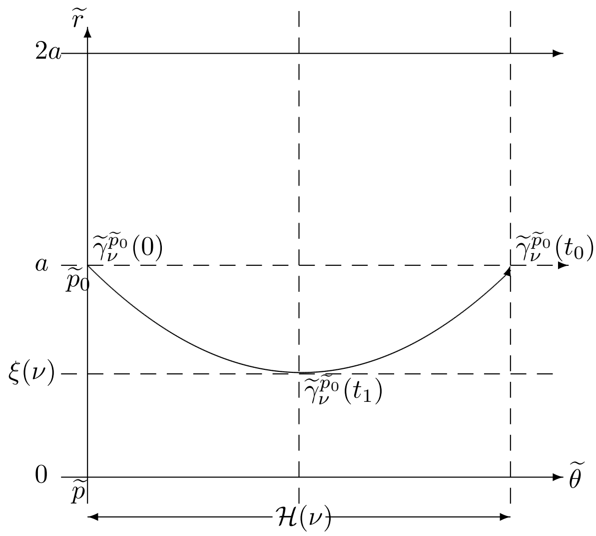

Figure 2.2*:* Half period function.

Proof. Let $\widetilde{\gamma}(s) = (\widetilde{r}(s), \widetilde{\theta}(s))$ be an h-unit speed geodesic emanating from equator, that is

$$\left(\frac{d\widetilde{r}}{ds}\right)^2 + m^2(\widetilde{r}(s))\left(\frac{d\widetilde{\theta}}{ds}\right)^2 = 1$$

multiply with $\left(\frac{ds}{d\widetilde{\theta}}\right)^2$, we have

$$\left(\frac{d\widetilde{r}}{d\widetilde{\theta}}\right)^2 + m^2(\widetilde{r}(s)) = \left(\frac{ds}{d\widetilde{\theta}}\right)^2,$$

from Clairaut relation it follows that

$$\frac{ds}{d\widetilde{\theta}} = \frac{m^2(\widetilde{r}(s))}{\nu}.$$

Therefore

$$\left(\frac{d\widetilde{r}}{d\widetilde{\theta}}\right)^2 = \frac{m^2(\widetilde{r}(s))(m^2(\widetilde{r}(s)) - \nu^2)}{\nu^2},$$

or

$$\frac{d\widetilde{\theta}}{d\widetilde{r}} = \frac{\nu}{m(\widetilde{r}(s))\sqrt{m^2(\widetilde{r}(s)) - \nu^2}}.$$

By integrating with respect to $\widetilde{r}$

$$\widetilde{\theta}(b) - \widetilde{\theta}(a) = \int_{\widetilde{r}(a)}^{\widetilde{r}(b)} \frac{\nu}{m(\widetilde{r}(s))\sqrt{m(\widetilde{r}(s))^2 - \nu^2}} d\widetilde{r}.$$

From the Clairaut relation, the geodesic $\widetilde{\gamma}(s)$ will be tangent to some parallel called $\widetilde{r} = \xi(\nu)$ at $\widetilde{\gamma}(t_1)$ and then $\widetilde{\gamma}$ will return to the equator at the point $\widetilde{\gamma}(t_0)$ and it follows that $\widetilde{\theta}(t_0) - \widetilde{\theta}(0) = 2(\widetilde{\theta}(t_0) - \widetilde{\theta}(t_1))$, we obtain

$$\mathcal{H}(\nu) := \widetilde{\theta}(t_0) - \widetilde{\theta}(0) = 2\int_{\xi(\nu)}^{a} \frac{\nu}{m(t)\sqrt{m(t)^2 - \nu^2}} dt. \tag{2.22}$$

□

2.2.5 Jacobi fields on a Riemannian 2D sphere of revolution

In this section, we construct the Jacobi field equation on a Riemannian 2D sphere of revolution in the similarly way as Riemannian surface of revolution case (see [11]).

Let $\widetilde{p}_0 \in \widetilde{M}$ and $\widetilde{\beta}_\nu(s)$ and $\widetilde{\gamma}_\nu(s)$ for any $\nu \in (0, m(\widetilde{r}(\widetilde{p}_0)))$ denote the geodesic emanating from $\widetilde{p}_0$ with $(\widetilde{r} \circ \widetilde{\beta_\nu})'(0) \geq 0$ and $(\widetilde{r} \circ \widetilde{\gamma_\nu})'(0) \leq 0$.

Since both geodesics $\widetilde{\beta}_\nu(s)$ and $\widetilde{\gamma}_\nu(s)$ depend smoothly on $\nu \in (0, m(a))$, we obtain the $\widetilde{h}$-Jacobi fields $X_\nu(t)$ and $Y_\nu(t)$:

$$X_\nu(t) := \frac{\partial}{\partial \nu}(\widetilde{\beta}_\nu(t)), \qquad Y_\nu(t) := \frac{\partial}{\partial \nu}(\widetilde{\gamma}_\nu(t)),$$

such that all curves in the variation are geodesics, along $\widetilde{\beta}_\nu(t)$ and $\widetilde{\gamma}_\nu(t)$. We can see that $X_\nu(0) = Y_\nu(0) = 0$.

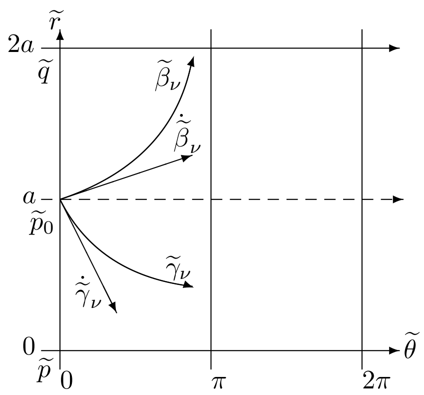

Figure 2.3: The $\widetilde{h}$-geodesic segments $\widetilde{\beta}_\nu(t)$ and $\widetilde{\gamma}_\nu(t)$.

Let us denote by $\widetilde{p}_u$ the point of coordinates $(\widetilde{r}(\widetilde{p}_u), \widetilde{\theta}(\widetilde{p}_u)) = (u, 0)$, where $u \in (0, 2a)$ and $\nu \in (0, m(a))$. Similarly with the case $u = a$, discussed above, for any $\nu \in (0, m(u))$, we consider the $\widetilde{h}$-geodesic where $\widetilde{\gamma}^u_\nu$ emanating from $\widetilde{p}_u$, with Clairaut constant ν and $(\widetilde{r} \circ \widetilde{\gamma}^u_\nu)'(0) \leq 0$.

The geodesic $\widetilde{\gamma}^u_\nu$ tangent to the parallel $\widetilde{r} = \xi(u)$ at a point $\widetilde{\gamma}^u_\nu(s_0)$, will intersect the equator and then will be tangent to the parallel $\widetilde{r} = 2a - \xi(u)$ at a point $\widetilde{\gamma}^u_\nu(s_1)$. Clearly, the parameter values s_0 and s_1 are solutions of the equation $(\widetilde{r} \circ \widetilde{\gamma}^u_\nu)'(s) = 0$. Then it is known from the proof of lemma 2.9 in [12], or proposition 7.2.3 in [11], that the Jacobi vector field Y_ν along $\widetilde{\gamma}^u_\nu$ is given by

$$Y_\nu(s) = \frac{\partial \widetilde{\theta}}{\partial \nu}(\widetilde{r}(s), u, \nu) \left[-\nu \frac{m(\widetilde{r}(s))}{\sqrt{m^2(\widetilde{r}(s)) - \nu^2}} \left(\frac{\partial}{\partial \widetilde{r}} \right)_{\widetilde{\gamma}^u_\nu(s)} + \left(\frac{\partial}{\partial \widetilde{\theta}} \right)_{\widetilde{\gamma}^u_\nu(s)} \right]. \tag{2.23}$$

Remark 2.27. We are interested in Jacobi field because the first zero solution of the Jacobi field, except emanating point, is the first conjugate point and this is related to the h-cut locus.

2.2.6 Cut locus on the Riemannian 2D sphere of revolution

In this section, we recall the theorem of the cut locus on the Riemannian 2D sphere of revolution.

Theorem 2.28. [12] Let $(M, dr^2 + m(r)^2 d\theta^2)$ be a 2-sphere of revolution with a pair of poles $\{p, q\}$ have the following properties

(i) M is symmetric with respect to the equator,

(ii) The Gaussian curvature is monotone along profile curve.

Then the h-cut locus of a point $x \in M \setminus \{p, q\}$ where $\theta(x) = 0$ is depends on the behaviour of the Gaussian curvature as follows. The cut point of x is:

1. A single point $\mathcal{C}_x^h = (2a - r(x), \pi)$, when $G(x)$ is a positive constant.
2. A subarc of the opposite half meridian $\mathcal{C}_x^h \subset \theta = \pi$, when $G(x)$ is monotone non-increasing along meridian from the pole p to the point on $r = a$. (see Figure 2.4).

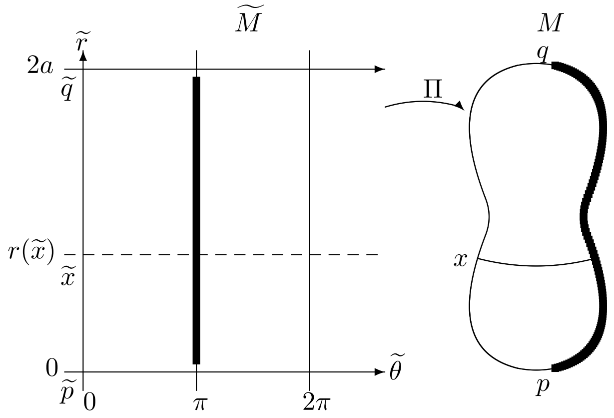

Figure 2.4*:* The cut locus when the Gaussian curvature G is monotone non-increasing.

3. A subarc of the antipodal parallel $r = 2a - r(x)$, that is $\mathcal{C}_x^h = r^{-1}(2a - r(x)) \cap \theta^{-1}(\mathcal{H}(m(r(x))), 2\pi - \mathcal{H}(m(r(x))))$, when $G(x)$ is monotone non-decreasing along the meridian from the pole p to the point on $r = a$. (see Figure 2.5).

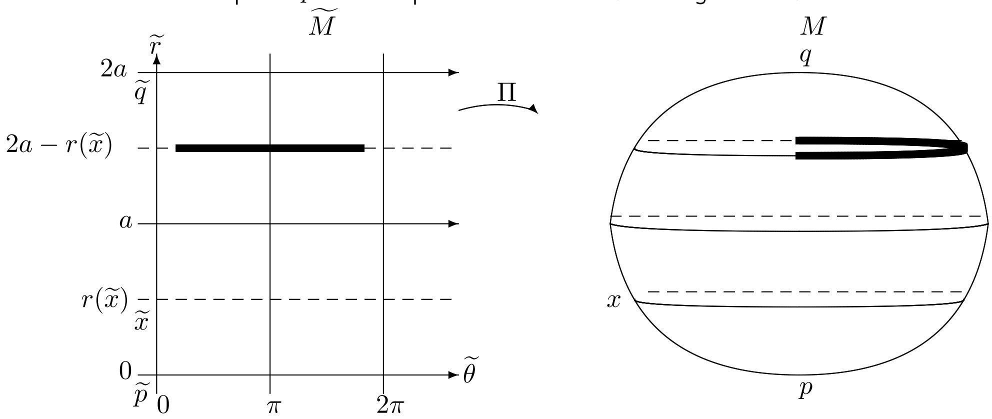

Figure 2.5*:* The cut locus when the Gaussian curvature G is monotone non-decreasing.

Here $r^{-1}(k)$ means the parallel $r = k$, $\theta^{-1}(a, b)$ means the region between $\theta = a$ to $\theta = b$.

On the other hand, if we consider the case that the Gaussian curvature is non-monotone along the profile curve, we can not find the general structure of the cut locus in this case.

But if we assume the condition, the cut locus of the point on equator is the subarc of the equator in [5], then we obtain theorem 2.29.

Theorem 2.29. [5] Let (M, h) denote a 2D sphere of revolution, where M is symmetric with respect to the equator. If the cut locus of a point on equator is a subset of equator then the cut locus of a point x with $r(x) \in (0, 2a) \setminus \{a\}$ is a subset of the antipodal parallel $r = 2a - r(x)$.

Therefore, we can find the cut locus of non-monotone Gaussian curvature on a 2D sphere of revolution, by adding only one assumption.

We can see that the h-cut locus of a point on the equator of the 2D sphere of revolution is the subarc of equator by using lemma 3.3 in [5], that is

Lemma 2.30. If $m' \neq 0$ on $(0, a)$ and the h-half period function is monotone non-increasing then the cut locus of each point on the equator is subset of the equator.

2.3 Finslerian metrics

In the previous section, we can see that the study about geodesics, conjugate points and cut locus is already done in Riemannian case, therefore we will go further by changing the metric on the 2D sphere of revolution from Riemannian metric to Finslerian metric and we use Randers metric to see that if there is a wind blowing along the parallel on the 2D sphere of revolution, what will happen to the properties of geodesics, conjugate points and cut locus.

We introduce the definition of Finslerian norm. Let (M, F) be a smooth n-dimensional differential manifold endowed with a Finslerian norm $F : TM \to [0, \infty)$ with properties

(i) F is positive and differentiable on $\widetilde{TM}$,

(ii) F is 1-positive homogeneous, that is $F(x, \lambda y) = \lambda F(x, y)$ for any $\lambda > 0$ and for all $(x, y) \in \widetilde{TM}$,

(iii) Hessian matrix $g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on $\widetilde{TM}$.

Then (M, F) is called a **Finsler manifold**, F is called the **fundamental function**, and g_{ij} the **fundamental tensor**.

Remark 2.31. We can see that the Finslerian norm does not imply symmetry, that is $F(x, y) \neq F(x, -y)$. If $F(x, y) = F(x, -y)$, then F is called an absolute homogeneous

Finsler metric. For instance, a Riemannian norm can be regarded as such a Finsler metric. Of course there are many non-Riemannian absolute homogeneous Finsler metrics (see [4]).

Next, we introduce a natural extension of the sectional curvature in Riemannian geometry, that is **flag curvature** on Finlserian manifold (M, F) (see [4] for the definition). We will use lemma 2.32 to see the behaviour of flag curvature.

Lemma 2.32. [9] Let (M, F) be a Finsler manifold of Randers type obtained by Zermelo's navigation method with navigation data (h, W), with flag curvature K. If M is Einstein manifold then $K = G$, where G is Gaussian curvature of Riemannian manifold (M, h).

A Finsler norm can be used for defining the integral length $\mathcal{L}_F$ of a smooth curve $\gamma : [a, b] \to M$ by

$$\mathcal{L}_F(\gamma) = \int_a^b F(\gamma(t), \dot{\gamma}(t))dt.$$

We can see that the length of curve on (M, F) depends on the direction, i.e. the distance is non-symmetric. We will introduce the example of deforming Riemannian metric $\alpha(x, y)$ by linear forms $\beta(x, y)$ defined by

$$\alpha(x, y) := \sqrt{a_{ij}(x)y^i y^j}, \qquad \beta(x, y) := b_i(x)y^i. \tag{2.24}$$

2.3.1 Randers metrics

In 1941, G. Randers studied a very interesting class of Finsler manifolds. Let M be an n-dimensional manifold. A **Randers metric** is a Finsler structure F that has the form

$$F(x, y) := \alpha(x, y) + \beta(x, y),$$

Lemma 2.33. The Randers metric is positive definite if and only if

$$|b| := \sqrt{b_i b^i} < 1$$

where $b^i := a^{ij} b_j$.

Proof. Fix $x \in M$. The positivity of the Randers metric F on $T_x M \setminus \{0\}$ means that

$$\begin{aligned} F = \alpha + \beta &> 0 \\ \alpha &> -\beta \\ \sqrt{a_{ij}y^i y^j} &> -b_i y^i \end{aligned}$$

for all $y \neq 0$.

We divide the proof in two steps (a) and (b)

(a) if $\sqrt{a_{ij}y^i y^j} > -b_i y^i$ then $\|b\| < 1$.

Proof of (a). Suppose

$$\sqrt{a_{ij}y^iy^j} > -b_iy^i. \tag{2.25}$$

If $b \neq 0$, substitute

$$y^i = -b^i := -a^{ij}b_j$$

into (2.25), therefore

$$\begin{aligned}\sqrt{a_{ij}(-a^{ij}b_j)(-b^j)} &> -b_i(-b^i)\\ \sqrt{b_jb^j} &> b_ib^i\\ \|b\| &> \|b\|^2\\ \|b\| &< 1.\end{aligned}$$

□

(b) if $\|b\| < 1$ then $\sqrt{a_{ij}y^iy^j} > -b_iy^i$.

Proof of (b). Recall the Cauchy-Schwarz inequality, we obtain

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle$$

therefore

$$\begin{aligned}|\alpha(b, y)|^2 &\leq \alpha(b, b)\alpha(y, y)\\ |\sqrt{a_{ij}b^iy^j}|^2 &\leq \sqrt{a_{ij}b^ib^j}\sqrt{a_{ij}y^iy^j}.\end{aligned}$$

Since

$$b^i := a^{ij}b_j$$

we can see that

$$\sqrt{a_{ij}b^ib^j} = \sqrt{a_{ij}(a^{ij}b_j)b^j} = \sqrt{b_jb^j} = \|b\|$$

and

$$|\sqrt{a_{ij}b^iy^j}|^2 = |a_{ij}b^iy^j| = |a_{ij}(a^{ij}b_j)y^j| = |b_jy^j| = |b_iy^i|.$$

We obtain

$$|b_iy^i| \leq \|b\|\sqrt{a_{ij}y^iy^j}.$$

Since $\|b\| < 1$, hence

$$|b_iy^i| < \sqrt{a_{ij}y^iy^j}.$$

□

From (a) and (b), we obtain the conclusion.

□

We consider that $\alpha(x,y)$ is Riemannian metric, it follows that $\alpha(x,y)$ is 1-positive homogeneous and from (2.24), we can see that

$$\beta(x,\lambda y) = b_i(x)\lambda y^i = \lambda b_i(x)y^i = \lambda\beta(x,y),$$

therefore $F = \alpha + \beta$ is 1-positive homogeneous.

The last condition about Hessian matrix (g_{ij}) is positive definite because the Hessian matrix (a_{ij}) of Riemannian metric α is positive definite because the definition of Riemannian norm and the following fact $\det(g_{ij}) = \left(\frac{F}{\alpha}\right)^{n+1}\det(a_{ij})$, (see [4] page 284).

2.3.2 Zermelo's navigation problem

In this section, we will introduce the method for constructing the Randers metric from Riemannian manifold with navigation data [6].

We start with the navigation problem called **Zermelo's navigation problem**. Zermelo considered the following problem in 1931.

"Find the paths of the shortest time travel between two points under the influence of a wind or a current when we travel by a boat with maximum speed."

It was proved by Z. Shen that the solutions of the Zermelo's navigation problem are the geodesics of a Randers metric.

If we consider that the sea is $\mathbb{R}^2$ (Euclidean case), we know that the straight line u is the shortest path between two points. However, in the presence of wind W with $\|W\| < 1$, the straight line does not give any more the shortest traveling time. We are going to compute a new norm (length function) F such that

$$F(v) = 1 \quad \textit{for all } v = u + W, \qquad \|u\| = 1.$$

We obtain

$$\|u\|^2 = \|v - W\|^2 = \langle v - W, v - W\rangle = \|v\|^2 - 2\langle v, W\rangle + \|W\|^2 = 1.$$

It follows that

$$\|v\|^2 - 2\|v\|\|W\|\cos\theta + \|W\|^2 - 1 = 0,$$

where we use the usual inner product in $\mathbb{R}^2$,

$$\langle v, W\rangle = \|v\|\|W\|\cos\theta, \qquad \theta = \measuredangle(v, W).$$

Then we get the equation

$$\|v\|^2 - 2(\|W\|\cos\theta)\|v\| - \lambda = 0, \qquad \textit{where} \qquad \lambda := 1 - \|W\|^2 > 0,$$

with the solution

$$\|v\| = \|W\|\cos\theta + \sqrt{\|W\|^2\cos^2\theta + \lambda}.$$

We denote $\|v\| = p + q$, *where*

$$p = \|W\| \cos\theta, \qquad q = \sqrt{\|W\|^2 \cos^2\theta + \lambda}$$

and compute a norm F *such that* $F(v) = 1$ *as follows :*

$$F(v) = 1 = \|v\| \frac{1}{p+q} = \|v\| \frac{q-p}{q^2 - p^2},$$

namely, we obtain

$$F(v) = \frac{\sqrt{\langle W, v\rangle^2 + \|v\|^2\lambda}}{\lambda} - \frac{\langle W, v\rangle}{\lambda},$$

where $\lambda = 1 - \|W\|^2 > 0$. *This is a Finsler norm written in terms of the Euclidean norm* $\|\cdot\|$ *and the wind* W.

Lemma 2.34. Consider Zermelo's navigation problem on the tangent space of (M, h) with navigation data (h, W), let $x \in M$ and $y \in T_xM$. We obtain that the time minimizing paths are the geodesics of the metric

$$F(x,y) = \alpha(x,y) + \beta(x,y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i,$$

where

$$a_{ij}(x) = \frac{W_i}{\lambda}\frac{W_j}{\lambda} + \frac{h_{ij}}{\lambda}, \qquad b_i(x) = -\frac{W_i}{\lambda}.$$

Proof. From Zermelo's navigation process, we obtain

$$F(x,y) = \frac{\sqrt{h(y,W)^2 + h(y,y)\lambda}}{\lambda} - \frac{h(y,W)}{\lambda}.$$

It follows that

$$\beta = -\frac{h(y,W)}{\lambda} = -\frac{h_{ij}y^iW^j}{\lambda} = -\frac{W_iy^i}{\lambda} = b_iy^i,$$

and

$$\begin{aligned}
\alpha^2 &= \frac{h(y,W)^2 + h(y,y)\lambda}{\lambda^2} \\
&= \frac{(h_{ij}y^iW^j)^2 + h_{ij}y^iy^j\lambda}{\lambda^2} \\
&= \frac{(h_{ij}y^iW^j)(h_{ij}y^jW^i)}{\lambda^2} + \frac{h_{ij}y^iy^j}{\lambda} \\
&= \left(\frac{h_{ij}W^j}{\lambda}\frac{h_{ij}W^i}{\lambda} + \frac{h_{ij}}{\lambda}\right) y^iy^j \\
&= a_{ij}y^iy^j,
\end{aligned}$$

where $W_i = h_{ij}W^j$.

□

Therefore, we obtain the corresponding Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ *and* 1*-form* $\beta = b_i(x)y^i$ *are given by*

$$a_{ij}(x) = \frac{\lambda h_{ij} + W_iW_j}{\lambda^2} \quad \textit{and} \quad b_i = -\frac{W_i}{\lambda},$$

where $W_i := h_{ij}W^j$.

2.3.3 Geodesics on Randers manifold

In this section, we let $\mathcal{P}(s)$ be an F-constant speed curve on Randers manifold $(M, F = \alpha + \beta)$.

Lemma 2.35. [4] The curve $\mathcal{P}(s)$ is called **Finslerian geodesic** on Randers manifold $(M, F = \alpha + \beta)$ if

$$\frac{d^2\mathcal{P}^i}{ds^2} + \left(\widetilde{\Gamma}^i_{jk} + l^i b_{j|k}\right)\frac{d\mathcal{P}^j}{ds}\frac{d\mathcal{P}^k}{ds} + \left(a^{ij} - l^i b^j\right)\left(b_{j|k} - b_{k|j}\right)\alpha(\mathcal{P}, \dot{\mathcal{P}})\frac{d\mathcal{P}^k}{ds} = 0,$$

where $l^i = \frac{a_{ij}y^j}{F}$, $b_{j|k} = \frac{\partial b^j}{\partial x^k} - b_s\widetilde{\Gamma}^s_{jk}$ and $\widetilde{\Gamma}^i_{jk}$ is a Christoffel symbols of α.

On the other hand, if the Randers manifold is obtained from Zermelo's navigation process, we have

Lemma 2.36. [10] The curve $\mathcal{P}(s)$ is called **F-unit speed geodesic** on Randers manifold $(M, F = \alpha + \beta)$, where F is obtained from Zermelo's navigation process if

$$\ddot{\mathcal{P}}^i + 2G^i(\mathcal{P}, \dot{\mathcal{P}}) = 0, \tag{2.26}$$

the geodesic coefficients of F are related to the those of h by

$$G^i = \mathcal{G}^i + \zeta^i,$$

where

$$\begin{aligned}\mathcal{G}^i &= \frac{1}{2}\Gamma^i_{jk}\dot{\mathcal{P}}^j\dot{\mathcal{P}}^k \\ \zeta^i &= \frac{1}{4}\left(\frac{1}{F}\dot{\mathcal{P}}^i - W^i\right)\left(2F\mathcal{S}_0 - \mathcal{L}_{00} - F^2\mathcal{L}_{WW}\right) - \frac{1}{4}F^2\left(\mathcal{S}^i + \mathcal{T}^i\right) - \frac{1}{2}F\mathcal{C}^i_0\end{aligned}$$

and

$$\begin{gathered}\mathcal{L}_{ij} = W_{i:j} + W_{j:i}, \quad \mathcal{C}_{ij} = W_{i:j} - W_{j:i}, \quad \mathcal{S}_0 = W^s\mathcal{L}_{si}\dot{\mathcal{P}}^i, \\ \mathcal{T}_i = W^s\mathcal{C}_{si}, \quad W_{i:j} = \frac{\partial W_i(x)}{\partial x^j} - W_s\Gamma^s_{ij}, \quad \mathcal{S}^i = h^{ij}\mathcal{S}_j, \\ C^i_0 = h^{ij}C_{jk}y^k, \quad \mathcal{L}_{WW} = W^iW^j\mathcal{L}_{ij}, \quad \mathcal{L}_{00} = \mathcal{L}_{ij}\dot{\mathcal{P}}^i\dot{\mathcal{P}}^j.\end{gathered}$$

Chapter 3

Auxiliary results

In this chapter, we will show how we construct our main theorems. We start from showing that a 2D sphere of revolution is Einstein manifold then the computation of the half period function in the Randers rotational case.

Lemma 3.1. Any Riemannian 2D sphere of revolution is an Einstein manifold.

Proof. From definition 2.8, the Riemannian manifold (M, h) is called Einstein manifold if

$$R_{ij} = \lambda h_{ij}$$

where $\lambda : M \to \mathbb{R}$ is smooth everywhere on M.

We start with the computation of curvature tensor, recall the formula (2.9)

$$R_{ijkl} = \frac{\partial \Gamma_{ijl}}{\partial x^k} - \frac{\partial \Gamma_{ijk}}{\partial x^l} + \Gamma^h_{ik}\Gamma_{jhl} - \Gamma^h_{il}\Gamma_{jhk}.$$

We obtain that

$$R_{1111}, R_{1112}, R_{1121}, R_{1211}, R_{2111}, R_{1122}, R_{2211}, R_{2221}, R_{2212}, R_{2122}, R_{1222}, R_{2222}$$

are vanish and

$$R_{1212} = R_{2121} = mm'', \qquad R_{2112} = R_{1221} = -mm''.$$

Next, we compute the Ricci tensor of (M, h)

$$R_{ij} := h^{km} R_{kijm},$$

we get

$$\begin{aligned}
R_{11} &= h^{km} R_{k11m} = h^{11} R_{1111} + h^{22} R_{2112} = -\frac{m''}{m} \\
R_{12} &= h^{km} R_{k12m} = h^{11} R_{1121} + h^{22} R_{2122} = 0 \\
R_{21} &= h^{km} R_{k21m} = h^{11} R_{1211} + h^{22} R_{2212} = 0 \\
R_{22} &= h^{km} R_{k22m} = h^{11} R_{1221} + h^{22} R_{2222} = -mm''.
\end{aligned}$$

We have to check

$$\begin{aligned}
R_{11} &= \lambda h_{11} \\
\lambda &= -\frac{m''}{m} \\
R_{22} &= \lambda h_{22} \\
\lambda &= -mm'' \left(\frac{1}{m^2}\right) = -\frac{m''}{m},
\end{aligned}$$

then we get $\lambda = -\frac{m''}{m}$, therefore the 2D sphere of revolution (M, h) is an Einstein manifold.

□

3.1 The Randers rotational metric on a 2D sphere of revolution

Recall the result in [9], where we have constructed a Randers rotational metric on a surface of revolution homeomorphic to $\mathbb{R}^2$*. We will construct the Randers rotational metric on a 2D sphere of revolution in a similar manner in the following.*

Let (M,h) *be the* 2*-sphere of revolution considered in the previous section. Observe that there exists a constant* $\mu < \{\frac{1}{\max\{m(r)\}} : r \in [0,2a]\}$*, such that* $\mu < \frac{1}{m(r)}$ *for any* $r \in [0,2a]$*.*

We construct a Randers rotational metric on M *by putting*

$$W := \mu\frac{\partial}{\partial\theta}$$

that is, in the polar coordinates system $(\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta})$ *of* T_xM *we have*

$$W = (W^1, W^2) = (0, \mu).$$

It follows

$$h(W,W) = h\left(\mu\frac{\partial}{\partial\theta}, \mu\frac{\partial}{\partial\theta}\right) = \mu^2 h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = (\mu m)^2 < 1.$$

We compute

$$a_{ij} = \frac{\lambda h_{ij} + W_iW_j}{\lambda^2},\ b_i = -\frac{W_i}{\lambda}$$

where $W_i = h_{ij}W^j$*,* $\lambda = 1 - h(W,W) = 1 - \mu^2m^2 > 0$*.*

Firstly, we have

$$W_1 = h_{11}W^1 + h_{12}W^2 = 0,\ W_2 = h_{21}W^1 + h_{22}W^2 = \mu m^2,$$

that is

$$(W_1, W_2) = (0, \mu m^2). \tag{3.1}$$

It follows

$$a_{11} = \frac{\lambda h_{11} + W_1W_1}{\lambda^2} = \frac{1}{1-\mu^2m^2}, \qquad a_{12} = \frac{\lambda h_{12} + W_1W_2}{\lambda^2} = 0$$
$$a_{21} = \frac{\lambda h_{21} + W_2W_1}{\lambda^2} = 0, \qquad a_{22} = \frac{\lambda h_{22} + W_2W_2}{\lambda^2} = \frac{m^2}{(1-\mu^2m^2)^2}$$

and

$$b_1 = -\frac{W_1}{\lambda} = 0,\ b_2 = -\frac{W_2}{\lambda} = -\frac{\mu m^2}{1-\mu^2m^2}.$$

In other words, we have

Proposition 3.2. If (M,h) is a Riemannian 2D sphere of revolution and $W = \mu\frac{\partial}{\partial\theta}$ is a breeze on M blowing along parallels, then the rotational Randers metric $(M, F = \alpha+\beta)$ obtained by the Zermelo's navigation process with data (h,W) which is a Finsler metric on M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$, $\beta = b_i(x)y^i$ are defined by

$$(a_{ij}) = \begin{pmatrix} \frac{1}{1-\mu^2m^2} & 0 \\ 0 & \frac{m^2}{(1-\mu^2m^2)^2} \end{pmatrix},\ b_i = \begin{pmatrix} 0 \\ -\frac{\mu m^2}{1-\mu^2m^2} \end{pmatrix}, \qquad i,j = 1,2. \tag{3.2}$$

Indeed, we have the inverse matrix of (a_{ij})

$$(a^{ij}) = \begin{pmatrix} 1-\mu^2m^2 & 0 \\ 0 & \frac{(1-\mu^2m^2)^2}{m^2} \end{pmatrix}$$

and therefore

$$a^{11}b_1b_1 = 0,\ a^{12}b_1b_2 = 0,\ a^{21}b_2b_1 = 0,$$
$$a^{22}b_2b_2 = \frac{(1-\mu^2m^2)^2}{m^2}\left(-\frac{\mu m^2}{1-\mu^2m^2}\right)^2 = \mu^2m^2.$$

It can be seen that

$$\alpha(b,b) = a^{ij}b_ib_j = h(W,W) = h_{ij}W^iW^j < 1.$$

Riemannian a*-norm of the covariant vector* $b = (b_1, b_2)$*. This condition guarantees the strong convexity of the Randers metric* $F = \alpha + \beta$ *(see [4]).*

Let us remark that the flow φ *of the vector field* W *is a rotation around the* z*-axis. We have*

Lemma 3.3. The flow of the vector field $W = \mu\frac{\partial}{\partial\theta}$ is given by

$$\varphi(s;r,\theta) = (r, \theta + \mu s).$$

Proof. The vector field W is tangent to the flow, that is

$$W = \mu\frac{\partial}{\partial\theta} = \frac{d\varphi^1}{ds}\Big|_{s=0}\left(\frac{\partial}{\partial r}\right) + \frac{d\varphi^2}{ds}\Big|_{s=0}\left(\frac{\partial}{\partial\theta}\right)$$

and $\varphi(0;r,\theta) = (r,\theta)$, that is

$$\begin{cases} \varphi^1(0;r,\theta) = r \\ \varphi^2(0;r,\theta) = \theta. \end{cases} \tag{3.3}$$

The flow is now given by the following system of differential equations

$$\begin{cases} \frac{d\varphi^1}{ds} = 0 \\ \frac{d\varphi^2}{ds} = \mu \end{cases} \Rightarrow \begin{cases} \varphi^1(s;r,\theta) = c_1 \\ \varphi^2(s;r,\theta) = \mu s + c_2 \end{cases}, \qquad c_1, c_2 \text{ are constants.}$$

Form (3.3) it follows $c_1 = r$, $c_2 = \theta$ and hence

$$\begin{cases} \varphi^1(s;r,\theta) = r \\ \varphi^2(s;r,\theta) = \mu s + \theta. \end{cases}$$

□

Lemma 3.4. W is Killing vector field of (M,h), that is

$$W_{i:j} + W_{j:i} = 0,$$

where $W_{i:j} = \frac{\partial W}{\partial x^j} - W_s\Gamma^s_{ij}$.

Proof. Recall the Christoffel symbols of Riemannian metric h from (2.16), we see that

$$\begin{aligned}
W_{1:1} &= \frac{\partial W_1}{\partial x^1} - W_s\Gamma^s_{11} = \frac{\partial W_1}{\partial r} - W_1\Gamma^1_{11} - W_2\Gamma^2_{11} = 0\\
W_{1:2} &= \frac{\partial W_1}{\partial x^2} - W_s\Gamma^s_{12} = \frac{\partial W_1}{\partial \theta} - W_1\Gamma^1_{12} - W_2\Gamma^2_{12} = -\mu m^2\frac{m'}{m} = -\mu m m'\\
W_{2:1} &= \frac{\partial W_2}{\partial x^1} - W_s\Gamma^s_{21} = \frac{\partial W_2}{\partial r} - W_1\Gamma^1_{21} - W_2\Gamma^2_{21} = 2\mu m m' - \mu m m' = \mu m m'\\
W_{2:2} &= \frac{\partial W_2}{\partial x^2} - W_s\Gamma^s_{22} = \frac{\partial W_2}{\partial \theta} - W_1\Gamma^1_{22} - W_2\Gamma^2_{22} = 0.
\end{aligned}$$

□

We obtain the global characterization of F-geodesics.

3.2 Geodesics on a Randers rotational 2D sphere of revolution

We compute the geodesic equations (2.26) *on a Randers rotational 2D sphere of revolution, then we have*

Lemma 3.5. Let (M, F) be a Randers rotational 2D sphere of revolution the F-unit speed geodesics $\mathcal{P} : (-\varepsilon, \varepsilon) \to M$ is satisfies

$$\begin{aligned}
\frac{d^2\mathcal{P}^1}{ds^2} - mm'\left(\frac{d\mathcal{P}^2}{ds}\right)^2 - \mu m m'\left(\mu - 2\frac{d\mathcal{P}^2}{ds}\right) &= 0\\
\frac{d^2\mathcal{P}^2}{ds^2} + \frac{2m'}{m}\left(\frac{d\mathcal{P}^1}{ds}\right)\left(\frac{d\mathcal{P}^2}{ds}\right) - \frac{2\mu m'}{m}\left(\frac{d\mathcal{P}^1}{ds}\right) &= 0.
\end{aligned} \tag{3.4}$$

Proof. We start with computation of $\mathcal{G}^i$, that is

$$\begin{aligned}
2\mathcal{G}^i &= \Gamma^i_{jk}\dot{\mathcal{P}}^j\dot{\mathcal{P}}^k\\
&= \begin{pmatrix} -mm'(\dot{\mathcal{P}}^2)^2 \\ 2\frac{m'}{m}\dot{\mathcal{P}}^1\dot{\mathcal{P}}^2 \end{pmatrix}.
\end{aligned}$$

Since W is Killing vector field, it follows that $W_{i:j} = -W_{j:i}$ therefore $\mathcal{L}_{ij} = 0$. Consequently,

$$\mathcal{S}_0 = \mathcal{S}^i = \mathcal{L}_{00} = \mathcal{L}_{WW} = 0,$$

thus

$$2\zeta^i = -\frac{1}{2}\mathcal{T}^i - \mathcal{C}^i_0.$$

Next, we compute

$$(\mathcal{C}_{ij}) = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 2W_{1:2} \\ -2W_{1:2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2\mu m m' \\ 2\mu m m' & 0 \end{pmatrix},$$

$$(\mathcal{T}^i) = \begin{pmatrix} W^s\mathcal{C}_{s1} \\ W^s\mathcal{C}_{s2} \end{pmatrix} = \begin{pmatrix} W^1\mathcal{C}_{11} + W^2\mathcal{C}_{12} \\ W^1\mathcal{C}_{12} + W^2\mathcal{C}_{22} \end{pmatrix} = \begin{pmatrix} -2\mu^2 m m' \\ 0 \end{pmatrix},$$

$$(\mathcal{T}^i) = (h^{ij})(\mathcal{T}_i) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{m^2} \end{pmatrix} \begin{pmatrix} -2\mu^2 mm' \end{pmatrix} = \begin{pmatrix} -2\mu^2 mm' \\ 0 \end{pmatrix},$$

$$(\mathcal{C}_0^i) = \begin{pmatrix} h^{1j}\mathcal{C}_{jk}\dot{\mathcal{P}}^k \\ h^{2j}\mathcal{C}_{jk}\dot{\mathcal{P}}^k \end{pmatrix} = \begin{pmatrix} h^{11}\mathcal{C}_{1k}\dot{\mathcal{P}}^k + h^{12}\mathcal{C}_{2k}\dot{\mathcal{P}}^k \\ h^{21}\mathcal{C}_{1k}\dot{\mathcal{P}}^k + h^{22}\mathcal{C}_{2k}\dot{\mathcal{P}}^k \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{11}\dot{\mathcal{P}}^1 + \mathcal{C}_{12}\dot{\mathcal{P}}^2 \\ h^{22}\mathcal{C}_{21}\dot{\mathcal{P}}^1 + h^{22}\mathcal{C}_{22}\dot{\mathcal{P}}^2 \end{pmatrix} = \begin{pmatrix} -2\mu mm'\dot{\mathcal{P}}^2 \\ \frac{2\mu m'}{m}\dot{\mathcal{P}}^1 \end{pmatrix}.$$

Hence

$$2\zeta^i = \begin{pmatrix} \mu^2 mm' \\ 0 \end{pmatrix} + \begin{pmatrix} 2\mu mm'\dot{\mathcal{P}}^2 \\ -\frac{2\mu m'}{m}\dot{\mathcal{P}}^1 \end{pmatrix}.$$

From (2.26), we get (3.4).

□

We obtain

Proposition 3.6. Let $(M, F = \alpha + \beta)$ be the Randers rotational metric constructed from the navigation data (h, W), where (M, h) is a Riemannian 2D sphere of revolution, and $W = \mu\frac{\partial}{\partial\theta}$, $\mu < \{\frac{1}{\max\{m(r)\}} : r \in [0, 2a]\}$, is the breeze on M blowing along parallels, then $\mathcal{P}(s) = \varphi(s; \gamma(s))$ is the F-unit speed geodesic, where $\gamma(s) = (r(s), \theta(s))$ is the h-unit speed on (M, h) and $\varphi(\cdot;\cdot)$ is the flow of vector field W.

Proof. Let $\gamma(s) = (r(s), \theta(s))$ be an h-unit speed, recall the flow of the vector field $W = \mu\frac{\partial}{\partial\theta}$ from lemma 3.3, that is

$$\mathcal{P}(s) = \varphi(s; \gamma(s)) = (r(s), \theta(s) + \mu s),$$

remark that from Zermelo's navigation process. Since $\gamma(s)$ is h-unit speed then $\mathcal{P}(s)$ is F-unit speed, that is

$$h(\dot{\gamma}(s), \dot{\gamma}(s)) = 1 \text{ if and only if } F(\mathcal{P}(s), \dot{\mathcal{P}}(s)) = 1. \tag{3.5}$$

We can see that

$$\begin{aligned} \frac{d\mathcal{P}^1}{ds} &= \frac{dr}{ds}, \qquad \frac{d\mathcal{P}^2}{ds} = \frac{d\theta}{ds} + \mu \\ \frac{d^2\mathcal{P}^1}{ds^2} &= \frac{d^2 r}{ds^2}, \qquad \frac{d^2\mathcal{P}^2}{ds^2} = \frac{d^2\theta}{ds^2}. \end{aligned} \tag{3.6}$$

We consider that $\mathcal{P}(s)$ is an F-unit speed geodesic therefore $\mathcal{P}(s)$ satisfies (3.4), we obtain

$$\frac{d^2\mathcal{P}^1}{ds^2} - mm'\left(\frac{d\mathcal{P}^2}{ds}\right)^2 - \mu mm'\left(\mu - 2\frac{d\mathcal{P}^2}{ds}\right) = 0$$

from (3.6) it follows that

$$\begin{aligned} &\frac{d^2 r}{ds^2} - mm'\left(\frac{d\theta}{ds} + \mu\right)^2 - \mu mm'\left(\mu - 2\left(\frac{d\theta}{ds} + \mu\right)\right) \\ &= \frac{d^2 r}{ds^2} - mm'\left(\left(\frac{d\theta}{ds}\right)^2 + 2\mu\frac{d\theta}{ds} + \mu^2\right) + \mu mm'\left(\mu + 2\frac{d\theta}{ds}\right) \\ &= \frac{d^2 r}{ds^2} - mm'\left(\frac{d\theta}{ds}\right)^2 = 0 \end{aligned} \tag{3.7}$$

and

$$\frac{d^2\mathcal{P}^2}{ds^2} + \frac{2m'}{m}\left(\frac{d\mathcal{P}^1}{ds}\right)\left(\frac{d\mathcal{P}^2}{ds}\right) - \frac{2\mu m'}{m}\left(\frac{d\mathcal{P}^1}{ds}\right) = 0 \tag{3.8}$$

substitute (3.6) in this equation, we get

$$\begin{aligned} &\frac{d^2\theta}{ds^2} + \frac{2m'}{m}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds} + \mu\right) - \frac{2\mu m'}{m}\left(\frac{dr}{ds}\right) \\ &= \frac{d^2\theta}{ds^2} + \frac{2m'}{m}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right) = 0 \end{aligned} \tag{3.9}$$

From (3.7) and (3.9), we can see that $r(s)$, $\theta(s)$ satisfy (2.17) and (2.18), hence $\gamma(s) = (r(s), \theta(s))$ is an h-unit speed geodesic.

□

3.3 The half period function in the Randers rotational case

In this section, we will compute the F-half period function of the F-unit speed geodesic emanating from a point on the equator. We will use a similar method for h-half period function, with the method used in the universal covering manifold in the Riemannian case. We obtain theorem 3.7.

Let $\widetilde{\mathcal{P}}_\nu^{\widetilde{p}_0}(s) = (\widetilde{r}(s), \widetilde{\theta}(s) + \mu s)$ be an $\widetilde{F}$-unit speed geodesic emanating from $\widetilde{p}_0$, $r(\widetilde{p}_0) = a$ and $\nu \in (0, m(a))$. We denote $\widetilde{\mathcal{P}}_\nu(s) := \widetilde{\mathcal{P}}_\nu^{\widetilde{p}_0}$, we can see that

$$\widetilde{\mathcal{P}}_\nu^2(r(b)) - \widetilde{\mathcal{P}}_\nu^2(r(a)) = \int_{r(a)}^{r(b)} \left(\frac{d\theta}{dr} + \mu\frac{ds}{dr}\right) dr.$$

We know that $\widetilde{\mathcal{P}}_\nu(s)$ must be tangent to the parallel $\xi(\nu)$ at $\widetilde{\mathcal{P}}_\nu(t_1)$ and then return to the equator at $\widetilde{\mathcal{P}}_\nu(t_0)$ (see Figure 3.1). Then, by a similar computation as in the Riemannian case, we obtain

Theorem 3.7. The F-distance from $\widetilde{p}_0$ to $\widetilde{\mathcal{P}}_\nu(t_0)$ in the wind direction is given by the following *F-half period function*

$$\mathcal{H}_F^+(\nu) = \mathcal{H}(\nu) + \psi(\nu), \tag{3.10}$$

where $\psi(\nu) := 2\mu(a - \xi(\nu))$, and $\mathcal{H}(\nu)$ is the h-half period function defined in (2.21). In the other hand for the direction against the wind, we obtain

$$\mathcal{H}_F^-(\nu) = \mathcal{H}(\nu) - \psi(\nu). \tag{3.11}$$

Proof. Recall that

$$b - a = \int_a^b ds = \int_{\widetilde{r}(a)}^{\widetilde{r}(b)} \frac{ds}{d\widetilde{r}} d\widetilde{r}. \tag{3.12}$$

From $\widetilde{\mathcal{P}}(s) = (\widetilde{r}(s), \widetilde{\theta}(s) + \mu s)$ obtaining from $\widetilde{\gamma}(s)$, we have

$$\frac{d\widetilde{\mathcal{P}}^1}{ds} = \frac{d\widetilde{r}}{ds}, \qquad \frac{d\widetilde{\mathcal{P}}^2}{ds} = \frac{d\widetilde{\theta}}{ds} + \mu$$

and therefore

$$\begin{aligned}\frac{d\widetilde{\mathcal{P}}^2}{d\widetilde{\mathcal{P}}^1} &= \frac{d\widetilde{\mathcal{P}}^2}{ds}\frac{ds}{d\widetilde{\mathcal{P}}^1} \\ \frac{d\widetilde{\mathcal{P}}^2}{d\widetilde{r}} &= \left(\frac{d\widetilde{\theta}}{ds} + \mu\right)\frac{ds}{d\widetilde{r}} \\ &= \frac{d\widetilde{\theta}}{d\widetilde{r}} + \mu\frac{ds}{d\widetilde{r}}.\end{aligned} \tag{3.13}$$

By integrating (3.13), we get

$$\widetilde{\mathcal{P}}^2(\widetilde{r}(b)) - \widetilde{\mathcal{P}}^2(\widetilde{r}(a)) = \int_{\widetilde{r}(a)}^{\widetilde{r}(a)} \left(\frac{d\widetilde{\theta}}{d\widetilde{r}} + \mu\frac{ds}{d\widetilde{r}}\right) d\widetilde{r}.$$

We will consider in the case that $\widetilde{\mathcal{P}}(s)$ will tangent to the parallel $\{\widetilde{r} = \xi(\nu)\}$ at $\widetilde{\mathcal{P}}(t_1)$ and return to equator at $\widetilde{\mathcal{P}}(t_0)$, from (2.22) and (3.12) therefore

$$\begin{aligned}\mathcal{H}_F^+(\nu) &= \widetilde{\mathcal{P}}^2(t_0) - \widetilde{\mathcal{P}}^2(0) \\ &= 2\int_{\xi(\nu)}^{a} \frac{d\widetilde{\theta}}{d\widetilde{r}} d\widetilde{r} + 2\int_{\xi(\nu)}^{a} \mu\frac{ds}{d\widetilde{r}} d\widetilde{r} \\ &= \mathcal{H}(\nu) + 2\mu(a - \xi(\nu)).\end{aligned} \tag{3.14}$$

If we consider the geodesic on backward $\widetilde{\mathcal{P}}^-(s) = (\widetilde{r}, \widetilde{\theta}(s) - \mu s)$, we get

$$\mathcal{H}_F^-(\nu) = \mathcal{H}(\nu) - 2\mu(a - \xi(\nu)).$$

□

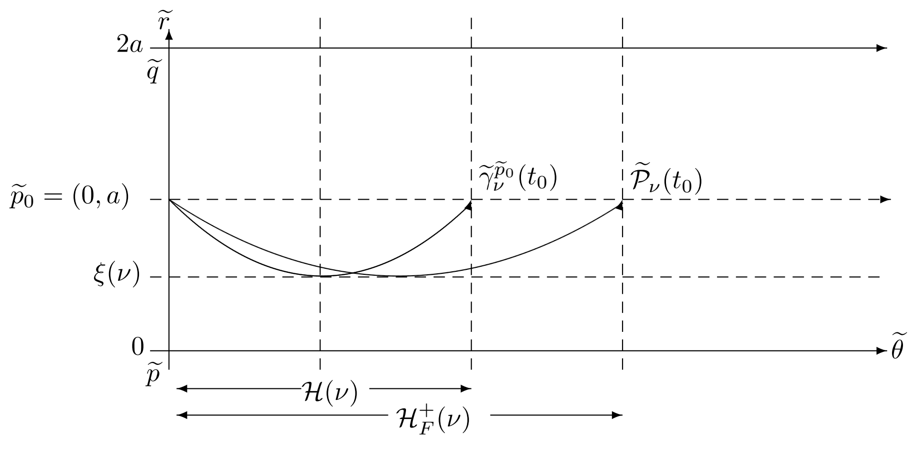

Figure 3.1: The h-half period function and F-half period function.

3.4 Jacobi fields of the Randers rotational 2D sphere of revolution

This section, we consider the property of Jacobi fields affected for any 2D sphere of revolution which is perturbed by the rotational wind.

Proposition 3.8. Let $(M, F = \alpha + \beta)$ be a Randers rotational 2D sphere of revolution with navigation data (h, W), where $W = \mu\frac{\partial}{\partial\theta}$ is the breeze on M blowing along parallels

$\mu < \frac{1}{m(r)}$ for any r. Suppose that $\gamma : [0,l] \to M$ is an h-geodesic and $\mathcal{P}(s) = \varphi(s;\gamma(s))$ is the corresponding F-geodesic, $t \in [0,l]$. Then $\mathcal{P}(l)$ is the first solution of $\mathcal{J}$ along $\mathcal{P}$ (with respect to the metric F) if and only if $\gamma(l)$ is first zero solution of J along γ (with respect to the metric h).

Proof. Let $\gamma : [0,l] \to M$ be an h-unit speed geodesic. Suppose $\Gamma(t,s) : (-\varepsilon,\varepsilon)\times[0,l] \to M$ be an h-geodesic variation of $\gamma(s) := \Gamma(0,s)$ with variation vector field

$$J(s) := \left.\frac{\partial\Gamma(t,s)}{\partial t}\right|_{t=0}.$$

Observe that this J is actually given by (2.23) for any $\nu \in (0, m(a))$. If we assume that $\gamma(l)$ is the first zero solution of J it follows that

$$J(0) = J(l) = 0 \quad \text{and} \quad J(s) \neq 0, \quad s \in (0,l).$$

By using the wind W, blowing up on M, with the flow φ, by deviating γ we obtain the corresponding F-geodesic $\mathcal{P}(s) = \varphi(s;\gamma(s))$.
Let us consider the F-geodesic variation

$$\overline{\mathcal{P}}(t,s) = \varphi(v(t)s;\Gamma(t,s)),$$

where $v(t)$ is the constant h-speed of the geodesic variation $\Gamma(t,s)$.

We compute the F-Jacobi field by

$$\begin{aligned}
\mathcal{J}(s) = \left.\frac{\partial\overline{\mathcal{P}}}{\partial t}\right|_{t=0} &= \left.\frac{\partial\varphi(v(t)s;\Gamma(t,s))}{\partial t}\right|_{t=0} \\
&= \left.\frac{\partial\varphi(v(t)s;\Gamma(t,s))}{\partial v(t)s}\right|_{t=0}\left.\frac{dv(t)s}{dt}\right|_{t=0} + \left.\frac{\partial\varphi(v(t)s;\Gamma(t,s))}{\partial\Gamma(t,s)}\right|_{t=0}\left.\frac{\partial\Gamma(t,s)}{\partial t}\right|_{t=0} \\
&= \frac{\partial\varphi(s;\gamma(s))}{\partial s}v'(0) + \frac{\partial\varphi(s;\gamma(s))}{\partial\gamma(s)}J(s) = d\varphi J(s).
\end{aligned}$$

If we consider the flow $\varphi = (\varphi_1,\varphi_2) = (r, \theta+\mu s)$, it follows that $d\varphi$ is the identity matrix.

$$d\varphi = \begin{pmatrix} \frac{\partial\varphi_1}{\partial r} & \frac{\partial\varphi_1}{\partial\theta} \\ \frac{\partial\varphi_2}{\partial r} & \frac{\partial\varphi_2}{\partial\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.15}$$

We obtain that $\mathcal{J}$ vanishes if and only if J does, hence

$$\mathcal{J}(0) = \mathcal{J}(l) = 0 \quad \text{and} \quad \mathcal{J}(s) \neq 0, \quad s \in (0,l),$$

that is $\mathcal{P}(l)$ is first zero solution of $\mathcal{J}$ along $\mathcal{P}$, whenever $\gamma(l)$ is first solution of J along γ.

□

Remark 3.9. From remark 2.27, we obtain that the first conjugate point of the F-geodesics is the displacement of the first conjugate point of h by the flow of the wind W.

3.5 Cut points of the Randers rotational 2D sphere of revolution

This section, we will show how to construct F-cut point on the Randers rotational 2D sphere of revolution $(M, F = \alpha + \beta)$.

Proposition 3.10. Let $x \in M \setminus \{p, q\}$ be an arbitrary point. Then q_0 is an F-cut point to x on $\mathcal{P}$ if and only if $\hat{q}_0$ is h-cut point to x on γ, where $\mathcal{P}(s) = \varphi(s; \gamma(s))$ is the corresponding F-geodesic obtained from γ, $\mathcal{P}(0) = \gamma(0) = x$.

Proof. Let $\gamma : [0, l] \to M$ be an h-unit minimizing geodesic from x to $\hat{q}_0 = \gamma(l)$ and $\hat{q}_0$ is a h-cut point of x, i.e. $\hat{q}_0 \in \mathcal{C}_x^h$.

Let $\mathcal{P}(s)$ be the F-unit geodesic obtained from $\gamma(s)$ and let $q_0 := \mathcal{P}(l)$.
Assume that q_0 is not F-cut point of x on $\mathcal{P}$, that is there exists a shorter minimizing F-geodesic $\mathcal{P}_0 : [0, l_0] \to M$ from $x = \mathcal{P}_0(0)$ to $q_0 = \mathcal{P}_0(l_0)$ where $d_F(x, q_0) := l_0 < l$.

From $\mathcal{P}_0$, we construct the corresponding h-geodesic

$$\gamma_0 : [0, l_0] \to M, \qquad \gamma_0(s) = \varphi(-s; \mathcal{P}(s)),$$

where $\gamma_0(0) = \mathcal{P}_0(0) = x$ and $\gamma_0(l_0) = \varphi(-l_0; \mathcal{P}_0(l_0)) = \varphi(-l_0; q_0) = \varphi_{q_0}(_0)$.

Let us denote by ζ the curve

$$\zeta : [-l_0, -l] \to M, \qquad \zeta(s) = \varphi(s; q_0),$$

(see Figure 3.2).

Then, from triangle inequality, we have

$$\mathcal{L}_h(\zeta) \geq \mathcal{L}_h(\gamma) - \mathcal{L}_h(\gamma_0). \tag{3.16}$$

On the other hand, we compute $\mathcal{L}_h(\zeta)$ as follows

$$\|\dot{\zeta}(s)\|_h^2 = \|W_{\varphi(s;q_0)}\|_h^2 = \|d\varphi(W_{q_0})\|_h^2 = \|W_{q_0}\|_h^2 = (\mu m(r(q_0)))^2 < 1,$$

where $d\varphi$ is identity from (3.15).

It follows that

$$\mathcal{L}_h(\zeta) = \int_{-l}^{-l_0} \|\dot{\zeta}(s)\|_h ds < \int_{-l}^{-l_0} ds = l - l_0 = \mathcal{L}_h(\gamma) - \mathcal{L}_h(\gamma_0). \tag{3.17}$$

From (3.16) and (3.17), we get a contradiction, hence q_0 is an F-cut point of x along $\mathcal{P}$.

□

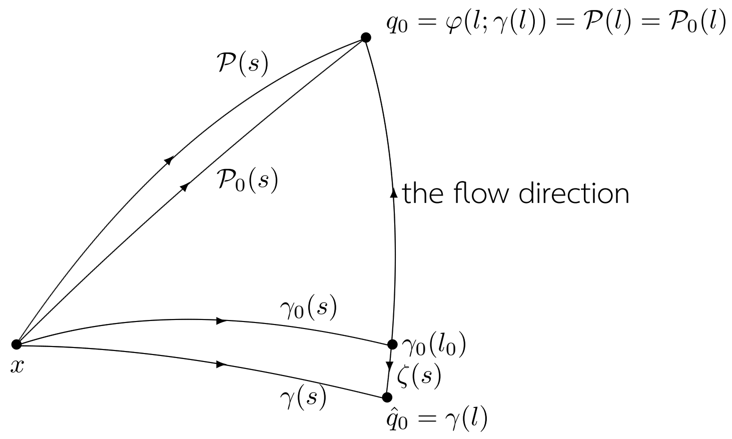

Figure 3.2*:* The proof of proposition 3.10.

Chapter 4

Main results

In this chapter, we will show the main theorems, we obtained from chapters 2 and 3. Let $x \in M \setminus \{p, q\}$. We recall that the flow for navigation data is $\varphi(s; r(s), \theta(s)) = (r(s), \theta(s) + \mu s)$.

Lemma 4.1. The flag curvature K at a point x on the Randers rotational 2D sphere of revolution $(M, F = \alpha + \beta)$ is monotone or non-monotone if and only if the Gaussian curvature G at the point x of the Riemannian 2D sphere of revolution (M, h) is monotone or non-monotone receptively.

Proof. From lemma 3.1, we obtain a Riemannian 2D sphere of revolution is Einstein manifold then lemma 2.32, tell us the flag curvature is equal to Gaussian curvature. □

Next, we will show the structure of the F-cut locus on the Randers rotational 2D sphere of revolution $(M, F = \alpha + \beta)$, when the flag curvature is monotone along the meridian.

We recall the result from [12] and we obtain theorem 4.2.

Theorem 4.2. Let (M, F) be a Randers rotational 2D sphere of revolution with navigation data (h, W), where $W = \mu\frac{\partial}{\partial\theta}$ is the wind blowing along parallels, $\mu < \{\frac{1}{\max\{m(r)\}} : r \in [0, 2a]\}$, with a pair of poles p, q, $d_h(p, q) = 2a$ and satisfies

(i) M is symmetric with respect to equator,

(ii) the flag curvature $\mathcal{K}$ is monotone along a meridian.

Then the F-cut locus $\mathcal{C}_x^F$ of a point $x \in M \setminus \{p, q\}$ with $\{\theta(x) = 0\}$ are

1. The subarc of the opposite half bending meridian,

$$\mathcal{C}_x^F = \varphi(d(x, \tau(t)), \tau(t)), \qquad t \in [c, 2a - c],$$

   where φ is the flow of the wind, when $\mathcal{K}$ is monotone non-increasing.

2. The following subarc of the antipodal parallel $r = 2a - r(x)$ to x,

$$\mathcal{C}_x^F = r^{-1}(2a - r(x)) \cap \theta^{-1}\{\mathcal{H}(m) + \psi(x), 2\pi - (\mathcal{H}(m) - \psi(x))\},$$

   where $\psi(x) = \mu d_h(x, \hat{q}_0)$, $\hat{q}_0$ is the h-first conjugate point of x with respect to h and $m := m(r(x))$, when $\mathcal{K}$ is monotone non-decreasing.

3. A single point on the antipodal parallel $\mathcal{C}_x^F = (2a - r(x), \pi(1 + \mu R))$, where R is radius of sphere, when $\mathcal{K} = \frac{1}{R^2}$ is constant.

4. If the cut locus of $x \in M \setminus \{p, q\}$ is a single point, then $\mathcal{K}$ is constant.

Proof. Propositions 3.8 and 3.10 imply that F-cut locus is corresponding to the h-cut locus.

1. The case of $\mathcal{K}$ is monotone non-increasing.

   In Riemannian case the h-cut locus $\mathcal{C}_x^h$ of x with the Gaussian curvature is monotone non-increasing, is a subarc of the opposite half meridian $\theta = \pi$, which is denote by $\tau_x|_{[c,2a-c]}$, where $\tau_x(c)$ is the h-first conjugate point of x along τ_x. Therefore by taking into account propositions 3.8 and 3.10, the F-cut locus is the following subarc of the opposite half bending meridian of x:

$$\mathcal{C}_x^F = \varphi(d(x, \tau(t)), \tau(t)), \qquad t \in [c, 2a - c].$$

2. The case of $\mathcal{K}$ is monotone non-decreasing.

   In the Riemannian case (see [12]), if the Gaussian curvature G is monotone non-decreasing then the h-cut locus $\mathcal{C}_x^h$ of x is a subarc of the antipodal parallel $r = 2a - r(x)$, that is

$$\mathcal{C}_x^h = r^{-1}(2a - r(x)) \cap \theta^{-1}\{\mathcal{H}(m), 2\pi - \mathcal{H}(m)\},$$

   where $\mathcal{H}$ is h-half period function defined in (2.21) and $m := m(r(x))$.
   Next, let $\hat{q}_0$ be the h-first conjugate point of x on front side, i.e.

$$\hat{q}_0 = (2a - r(x), \mathcal{H}(m)),$$

   and recall that our wind is blowing along the parallels, therefore the F-first conjugate point to x is

$$r^{-1}(2a - r(x)) \cap \theta^{-1}\{\mathcal{H}(m) + \psi(x)\},$$

   where $\psi(x) = \mu d(x, \hat{q}_0)$. On the other hand the F-first conjugate point to x on the back side is

$$r^{-1}(2a - r(x)) \cap \theta^{-1}\{2\pi - (\mathcal{H}(m) - \psi(x))\},$$

   hence we obtain

$$\mathcal{C}_x^F = r^{-1}(2a - r(x)) \cap \theta^{-1}\{\mathcal{H}(m) + \psi(x), 2\pi - (\mathcal{H}(m) - \psi(x))\}.$$

3. The case of $\mathcal{K}$ is constant.

   Let M be the round sphere of radius R. Recall that in the Riemannian case when $G = \frac{1}{R^2}$ is constant, the cut locus of any point $x \in M \setminus \{p, q\}$ is its antipodal point, i.e. $\mathcal{C}^h_x = \hat{q}_0 = (2a - r(x), \pi)$, where $\theta(x) = 0$. Since $d_h(x, \hat{q}_0)$ is equal to the half of circumference, i.e. $d_h(x, \hat{q}_0) = \pi R$, from proposition 3.10 we obtain that the F-cut locus of x is

$$\begin{aligned}\mathcal{C}^F_x &= \varphi(d_h(x, \hat{q}_0) = \varphi(\pi R, \hat{q}_0)\\ &= (2a - r(x), \pi(1 + \mu R),\end{aligned}$$

where R is radius of round sphere.

4. If the F-cut locus of $x \in M \setminus \{p, q\}$ is a single point, say $q \in M$, then $\hat{q} := \varphi(-l, q)$ is a h-cut point, where $d_F(x, q) = l$. Obviously $\hat{q}$ is the only h-cut point of h due to the proposition 3.10.

   Since the h-cut locus of $x \in M \setminus \{p, q\}$ is made of a single point $\hat{q}$, we know from [12] that $G = \frac{1}{R^2}$ must be a positive constant and hence (M, h) is the round sphere of radius R.

   Taking now into account that (M, h) is a constant Gaussian curvature Riemannian surface and W is a Killing field on (M, h), it follows from [6] that the corresponding Randers metric by the Zermelo navigation must be of constant flag curvature.

□

On the other hand, if we consider in the general case that is the flag curvature is non-monotone along the meridian, in this case is quite complicated to find the structure of the F-cut locus, even in the Riemannian case, we can not find the structure of the h-cut locus on the Riemannian 2D sphere of revolution (M, h) when the Gaussian curvature is non-monotone.

Therefore in [5] gives an extra assumption that if the h-cut locus of a point on the equator is the subarc of the equator then the structure of h-cut locus is explained in theorem 2.29.

We use the fact that when we move the point on the equator with the rotational wind, it is still on the equator because the equator is invariant under the rotational wind. It follows that if we assume the F-cut locus of the point on the equator is the subarc of the equator then we obtain theorem 4.3.

Theorem 4.3. Let $(M, F = \alpha + \beta)$ be the Randers rotational 2D sphere of revolution constructed from the navigation data (h, W) of the Riemannian 2D sphere of revolution (M, h).

If the F-cut locus of a point x on the equator $r = a$ is a subarc of the equator $r = a$, then the F-cut locus of any point $\widetilde{x}$ with $r(\widetilde{x}) \in (0, 2a) \setminus \{a\}$ is a subarc of the antipodal parallel $r = 2a - r(\widetilde{x})$.

Proof. If the cut locus of a point x on $r = a$ is a subarc of $r = a$, since the equator is invariant under the flow action, then by proposition 3.10 it follows that the h-cut locus of the point q is a subarc of $r = a$. Hence, by using theorem 3.5 in [5] it results that the h-cut locus of the point $\widetilde{x}$ is a subarc of the antipodal parallel $r = 2a - r(\widetilde{x})$.

Taking now into account that any parallel is flow-invariant by proposition 3.10, it follows that the F-cut locus of $\widetilde{x}$ must be a subarc in the antipodal parallel $r = 2a - r(\widetilde{x})$. Clearly, the F-cut locus is obtained by rotating the h-cut locus via flow action on the parallel $r = 2a - r(\widetilde{x})$.

□

Chapter 5

Examples

In this chapter, we will show some examples of the Randers rotational 2D sphere of revolution and find the F-half period function and the structure of the F-cut locus.

5.1 The example of the Randers rotational 2D sphere of revolution with monotone curvature

5.1.1 $m(r) = \sin r, \qquad r \in (0, \pi)$

We can see that the Gaussian curvature of this Riemannian 2D sphere of revolution is

$$G = -\frac{m''(r)}{m(r)} = 1, \qquad m(r) \neq 0.$$

From theorem 4.2, the F-cut locus is a single point.

5.2 The example of the Randers rotational 2D sphere of revolution with non-monotone curvature

5.2.1 The case $m_\lambda(r) = \frac{\sqrt{\lambda+1}\sin r}{\sqrt{1+\lambda\cos^2 r}}, \qquad r \in [0, \pi], \qquad \lambda \geq 0.$

Let us consider the Riemannian 2D sphere of revolution $M_\lambda := (\mathbb{S}^2, h_\lambda)$, introduced in [5], where

$$m_\lambda(r) = \frac{\sqrt{\lambda+1}\sin r}{\sqrt{1+\lambda\cos^2 r}}, \qquad r \in [0, \pi], \qquad \lambda \geq 0. \tag{5.1}$$

It is clear that the function $r \mapsto m_\lambda(r)$ is symmetric with respect to the equator $r = \frac{\pi}{2}$.

From (2.14)*, we get the Riemannian metric*

$$h_\lambda = dr^2 + m_\lambda^2(r)d\theta^2, \tag{5.2}$$

for $\lambda = 0$, we obtain the round sphere, that is

$$h_0 = dr^2 + \sin^2 r d\theta^2.$$

On the other hand, for $\lambda \to \infty$ the metric

$$h_\infty = dr^2 + \tan^2 r d\theta^2,$$

that is singular along the equator $r = \frac{\pi}{2}$.

A straightforward computation shows that the first derivative of $m_\lambda(r)$ is

$$
\begin{aligned}
m'_\lambda(r) &= \frac{\sqrt{1+\lambda\cos^2 r}(\sqrt{\lambda+1}\cos r) - \sqrt{\lambda+1}\sin r(\frac{1}{2}\frac{-2\lambda\sin r\cos r}{\sqrt{1+\lambda^2 r}})}{1+\lambda\cos^2 r} \\
&= \frac{(1+\lambda\cos^2 r)(\sqrt{\lambda+1}\cos r) + \lambda\sqrt{\lambda+1}\sin^2 r\cos r}{(1+\lambda\cos^2 r)^{3/2}} \\
&= \frac{\sqrt{\lambda+1}\cos r + \lambda\cos^2 r\sqrt{\lambda+1}\cos r + \lambda\sin^2 r\sqrt{\lambda+1}\cos r}{(1+\lambda\cos^2 r)^{3/2}} \\
&= \frac{\sqrt{\lambda+1}\cos r(1+\lambda)}{(1+\lambda\cos^2 r)^{3/2}} \\
&= \frac{(1+\lambda)\cos r}{1+\lambda\cos^2 r}\left(\frac{\sqrt{\lambda+1}\sin r}{\sqrt{1+\lambda\cos^2 r}}\right)\left(\frac{1}{\sin r}\right) \\
&= \frac{(1+\lambda)\cos r}{(1+\lambda\cos^2 r)\sin r} m_\lambda(r),
\end{aligned}
$$

and the second derivative $m_\lambda(r)$ is

$$
\begin{aligned}
m''_\lambda(r) &= \left(\frac{(1+\lambda)\cos r}{(1+\lambda\cos^2 r)\sin r}\right)^2 m_\lambda(r) \\
&+ m_\lambda(r)(1+\lambda)\left(\frac{-(1+\lambda\cos^2 r)\sin^2 r - (1+\lambda\cos^2 r)\cos^2 r + 2\lambda\cos^2 r\sin^2 r}{((1+\lambda\cos^2 r)\sin r)^2}\right) \\
&= \frac{(1+\lambda)m_\lambda(r)}{((1+\lambda\cos^2 r)\sin r)^2}\left((1+\lambda)\cos^2 r - (1+\lambda\cos^2 r) + 2\lambda\cos^2 r\sin^2 r\right) \\
&= \frac{(1+\lambda)m_\lambda(r)}{((1+\lambda\cos^2 r)\sin r)^2}\left(\cos^2 r + \lambda\cos^2 r - 1 - \lambda\cos^2 r + 2\lambda\cos^2 r\sin^2 r\right) \\
&= \frac{(1+\lambda)m_\lambda(r)}{((1+\lambda\cos^2 r)\sin r)^2}\left(-\sin^2 r + 2\lambda\cos^2 r\sin^2 r\right) \\
&= \frac{(1+\lambda)m_\lambda(r)}{(1+\lambda\cos^2 r)^2}\left(-1 + 2\lambda\cos^2 r\right).
\end{aligned}
$$

Therefore, from lemma 2.19 the Gaussian curvature of $(\mathbb{S}^2, h_\lambda)$ is

$$
\begin{aligned}
G_\lambda(r) &= \frac{-m''_\lambda(r)}{m_\lambda(r)} \\
&= \frac{(1+\lambda)(1-2\lambda\cos^2 r)}{(1+\lambda\cos^2 r)^2}.
\end{aligned}
$$

By taking the derivative of G_λ,

$$
\begin{aligned}
G'_\lambda(r) &= \frac{\lambda+1}{(1+\lambda\cos^2 r)^3}\left((1+\lambda\cos^2 r)(4\lambda\cos r\sin r) + (1-2\lambda\cos^2 r)(4\lambda\cos r\sin r)\right) \\
&= \frac{(\lambda+1)(4\lambda\cos r\sin r)}{(1+\lambda\cos^2 r)^3}\left(2-\lambda\cos^2 r\right) \\
&= \frac{(\lambda+1)(2\lambda\sin 2r)}{(1+\lambda\cos^2 r)^3}\left(2-\lambda\cos^2 r\right).
\end{aligned}
$$

Since

$$2-\lambda < 2-\lambda\cos^2 r < 2,$$

therefore G_λ is non-monotone along the meridian from a pole to the equator,when $\lambda > 2$.

Therefore $m_\lambda(r)$ is non-monotone Gaussian curvature, we will compute the h-half period function.

From

$$m_\lambda(r) = \frac{\sqrt{\lambda+1}\sin r}{\sqrt{1+\lambda\cos^2 r}}$$

and

$$m'_\lambda(r) = \frac{(\lambda+1)\cos r}{(1+\lambda\cos^2 r)\sin r} m_\lambda(r).$$

By putting $x = m_\lambda^2(r)$*, we have*

$$\begin{aligned}
\frac{dx}{dr} &= 2m_\lambda(r)m'_\lambda(r), \\
dr &= \frac{1}{2m_\lambda(r)m'_\lambda(r)}dx \\
&= \frac{(1+\lambda\cos^2 r)\tan r}{2(\lambda+1)x}dx.
\end{aligned}$$

We can see that

$$\begin{aligned}
x &= m_\lambda^2(r) \\
&= \frac{(\lambda+1)\sin^2 r}{1+\lambda\cos^2 r} \\
&= \frac{(\lambda+1)(1-\cos^2 r)}{1+\lambda\cos^2 r} \\
&= \frac{\lambda-\lambda\cos^2 r+1-\cos^2 r}{1+\lambda\cos^2 r} \\
x+\lambda x\cos^2 r &= \lambda-\lambda\cos^2 r+1-\cos^2 r \\
\lambda x\cos^2 r+\lambda\cos^2 r+\cos^2 r &= \lambda+1-x \\
\cos^2 r &= \frac{\lambda+1-x}{\lambda x+\lambda+1}.
\end{aligned}$$

Recall the fact that

$$\tan^2 r = \frac{\sin^2 r}{\cos^2 r} = \frac{1-\cos^2 r}{\cos^2 r} = \frac{1}{\cos^2 r} - 1,$$

we have

$$\begin{aligned}
\tan^2 r &= \frac{\lambda x+\lambda+1}{\lambda+1-x} - 1 \\
&= \frac{\lambda x+\lambda+1-\lambda-1+x}{\lambda+1-x} \\
&= \frac{(\lambda+1)x}{\lambda+1-x}
\end{aligned}$$

therefore

$$\begin{aligned}
dr &= \frac{\left(1+\lambda\left(\frac{\lambda+1-x}{\lambda x+\lambda+1}\right)\right)\left(\frac{(\lambda+1)x}{\lambda+1-x}\right)^{\frac{1}{2}}}{2(\lambda+1)x}dx \\
&= \frac{\frac{\lambda x+\lambda+1+\lambda^2+\lambda-\lambda x}{\lambda x+\lambda+1}}{2\sqrt{\lambda+1}\sqrt{x(\lambda+1-x)}}dx \\
&= \frac{2\lambda+1+\lambda^2}{2\sqrt{\lambda+1}\sqrt{x(\lambda+1-x)}(\lambda x+\lambda+1)}dx \\
&= \frac{(\lambda+1)^{\frac{3}{2}}}{2\sqrt{x(\lambda+1-x)}(\lambda x+\lambda+1)}dx,
\end{aligned}$$

we use that $r=\xi(\nu)=m_\lambda^{-1}(\nu)$ *implies* $x=m_\lambda^2(r)=m_\lambda^2(\xi(\nu))=\nu^2$ *and* $r=a=\frac{\pi}{2}$ *implies* $x=m_\lambda^2(\frac{\pi}{2})=\lambda+1$. *From lemma 2.26, we obtain*

$$\begin{aligned}\mathcal{H}(\nu)&=2\int_{\xi(\nu)}^{a}\frac{\nu}{m(r)\sqrt{m(r)^2-\nu^2}}dr\\&=2\int_{\nu^2}^{\lambda+1}\frac{\nu}{\sqrt{x}\sqrt{x-\nu^2}}\frac{(\lambda+1)^{\frac{3}{2}}}{2\sqrt{x(\lambda+1-x)(\lambda x+\lambda+1)}}dx\\&=(\lambda+1)^{\frac{3}{2}}\nu\int_{\nu^2}^{\lambda+1}\frac{1}{x\sqrt{(x-\nu^2)(\lambda+1-x)(\lambda x+\lambda+1)}}dx.\end{aligned}$$

Recall the lemma 4.2 in [5]

$$\int_b^c\frac{1}{x(x+a)\sqrt{(x-b)(c-x)}}=\frac{\pi}{a}\left(\frac{1}{\sqrt{bc}}-\frac{1}{\sqrt{(a+c)(a+b)}}\right).\tag{5.3}$$

We can see that, if we put

$$a=\frac{\lambda+1}{\lambda},\qquad b=\nu^2,\qquad c=\lambda+1,$$

therefore

$$\begin{aligned}\mathcal{H}(\nu)&=(\lambda+1)^{\frac{3}{2}}(\nu)\left(\frac{1}{\lambda}\right)\pi\frac{\lambda}{\lambda+1}\left(\frac{1}{\sqrt{\nu^2(\lambda+1)}}-\frac{1}{\sqrt{\left(\frac{\lambda+1}{\lambda}+\lambda+1\right)\left(\frac{\lambda+1}{\lambda}+\nu^2\right)}}\right)\\&=(\lambda+1)^{\frac{3}{2}}\frac{\pi\nu}{\lambda+1}\left(\frac{1}{\nu\sqrt{(\lambda+1)}}-\frac{1}{\sqrt{\left(\frac{2\lambda+1+\lambda^2}{\lambda}\right)\left(\frac{\lambda+1+\lambda\nu^2}{\lambda}\right)}}\right)\\&=(\lambda+1)^{\frac{3}{2}}\frac{\pi\nu}{\lambda+1}\left(\frac{1}{\nu\sqrt{(\lambda+1)}}-\frac{\lambda}{(\lambda+1)\sqrt{(\lambda+1+\lambda\nu^2)}}\right)\\&=(\lambda+1)^{\frac{3}{2}}\frac{\pi\nu}{\lambda+1}\left(\frac{(\lambda+1)\sqrt{(\lambda+1+\lambda\nu^2)}-\lambda(\nu\sqrt{(\lambda+1)})}{\nu(\lambda+1)^{\frac{3}{2}}\sqrt{(\lambda+1+\lambda\nu^2)}}\right)\\&=\frac{\pi}{\lambda+1}\left(\frac{(\lambda+1)\sqrt{(\lambda+1+\lambda\nu^2)}-\lambda(\nu\sqrt{(\lambda+1)})}{\sqrt{(\lambda+1+\lambda\nu^2)}}\right)\\&=\pi-\frac{\lambda\pi\nu}{\sqrt{\lambda+1}\sqrt{(\lambda+1+\lambda\nu^2)}}.\end{aligned}$$

We obtain proposition 5.1.

Proposition 5.1. For the Riemannian 2D sphere of revolution M_λ defined in (5.1), we get

$$\mathcal{H}(\nu)=\pi-\frac{\lambda\pi\nu}{\sqrt{\lambda+1}\sqrt{\lambda+1+\lambda\nu^2}},$$

for each $\nu\in[0,m_\lambda(\frac{\pi}{2})]$.

On the other hand, the first derivative of h-half period function is

$$\begin{aligned}\mathcal{H}'(\nu) &= \left(\frac{-\pi\lambda}{\sqrt{\lambda+1}}\right)\frac{\partial}{\partial r}\left(\frac{\nu}{\sqrt{(\lambda+1+\lambda\nu^2)}}\right)\\ &= \left(\frac{-\pi\lambda}{\sqrt{\lambda+1}}\right)\left(\frac{\sqrt{\lambda+1+\lambda\nu^2}-\nu\left(\frac{1}{2}\frac{1}{\sqrt{\lambda+1+\lambda\nu^2}}2\lambda\nu\right)}{(\lambda+1+\lambda\nu^2)}\right)\\ &= \left(\frac{-\pi\lambda}{\sqrt{\lambda+1}}\right)\left(\frac{\lambda+1+\lambda\nu^2-\lambda\nu^2}{(\lambda+1+\lambda\nu^2)^{\frac{3}{2}}}\right)\\ &= \frac{-\pi\lambda\sqrt{\lambda+1}}{(\lambda+1+\lambda\nu^2)^{\frac{3}{2}}} < 0.\end{aligned}$$

Therefore, $\mathcal{H}$ is monotone non-increasing from lemma 2.30, the h-cut locus of a point on equator is a subarc of equator then by theorem 2.29, we get

Proposition 5.2. If $\lambda > 0$, then, for each point of q of M_λ defined in (5.1) distinct from a pole, the cut locus of q is a subarc of the antipodal parallel to q.

Let us consider the associated Randers rotational metric $F = \alpha + \beta$ obtained by Zermelo's navigation method from the navigation data (h_λ, W), where $W = \mu\frac{\partial}{\partial\theta}$, $\mu < \left\{\frac{1}{\max m_\lambda(r)} : r \in [0,\pi]\right\} = \frac{1}{m_\lambda(\frac{\pi}{2})} = \frac{1}{\sqrt{\lambda+1}}$. From proposition 3.2 it follows

$$(a_{ij}) = \begin{pmatrix} \frac{1+\lambda\cos^2 r}{1+\lambda\cos^2 r-\mu^2(\lambda+1)\sin^2 r} & 0 \\ 0 & \frac{((\lambda+1)\sin^2 r)(1+\lambda\cos^2 r)}{(1+\lambda\cos^2 r-\mu^2(\lambda+1)\sin^2 r)^2}\end{pmatrix},\ b_i = \begin{pmatrix} 0 \\ \frac{-\mu(\lambda+1)\sin^2 r}{1+\lambda\cos^2 r-\mu^2(\lambda+1)\sin^2 r}\end{pmatrix}.$$

Observe that due to lemma 4.1 and the formula for G'_λ it follows that the Randers rotational metric constructed in this example is non-monotone flag curvature along meridian.

Moreover, observe that the F-cut locus of any point q in $r = \frac{\pi}{2}$ is a subarc of $r = \frac{\pi}{2}$, as well as, that the F-cut locus of any point $\tilde{q} \in M_\lambda$, such that $r(\tilde{q}) \in (0,\pi) \setminus \{\frac{\pi}{2}\}$ is a subarc of the antipodal parallel $r = \pi - r(\tilde{q})$. Indeed, taking into account the h-cut locus of the points q and $\tilde{q}$, respectively and the fact that the equator and parallels are invariant under the flow, the F-cut locus can be obtained from proposition 3.10.

Therefore, we obtain proposition 5.3.

Proposition 5.3. Let $(\mathbb{S}^2, F_\lambda = \alpha + \beta)$ be the Randers rotational metric induced from the navigation data (h_λ, W) on $\mathbb{S}^2$. If $\lambda > 0$, then

1. The cut locus of a point $q \in \mathbb{S}^2$ on the equator is a subarc of the equator.
2. The cut locus of a point $\tilde{q} \in \mathbb{S}^2$ which is distinct from the pair of poles, is a subarc of the antipodal parallel $r = \pi - r(\tilde{q})$.

This is the generalization of the theorem 2.29 to the Randers rotational case.

For the sake of simplicity, let us consider

$$\mu = \frac{1}{2}\frac{1}{\sqrt{\lambda+1}}.$$

Then (3.10) *implies*

$$\mathcal{H}_F^+(\nu) = \pi - \frac{\lambda\pi\nu}{\sqrt{\lambda+1}\sqrt{\lambda+1+\lambda\nu^2}} + \frac{1}{\sqrt{\lambda+1}}\left(\frac{\pi}{2} - \nu^2\right), \qquad \lambda > 0$$

and therefore

$$(\mathcal{H}_F^+)'(\nu) = \frac{-\lambda\pi\sqrt{\lambda+1}}{(\lambda+1+\lambda\nu^2)^{\frac{3}{2}}} - \frac{2\nu}{\sqrt{\lambda+1}}, \qquad \lambda > 0.$$

We observe that if $\mathcal{H}(\nu)$ is monotone non-increasing, then $\mathcal{H}_F^+(\nu)$ is decreasing on $\nu \in (0, \sqrt{\lambda+1})$.

5.2.2 The case $m_\lambda(r) = \frac{\sin r}{\sqrt{1-\lambda\sin^2 r}}, \quad r \in [0,\pi], \quad \lambda \in (0,1)$.

Another example is obtained from the Riemannian 2D sphere of revolution $(\mathbb{S}^2, h_\lambda)$ given in [3], where h_λ is declare in(5.2) *and*

$$m_\lambda(r) = \frac{\sin r}{\sqrt{1-\lambda\sin^2 r}}, \tag{5.4}$$

where $r \in [0,\pi]$, $\lambda \in (0,1)$.

The Riemannian metric (2.14)*, when $\lambda = 0$ and $\lambda = 1$ are*

$$h_0 = dr^2 + \sin^2 r d\theta^2, \qquad h_1 = dr^2 + \tan^2 r d\theta^2.$$

By straightforward computation it follows that the first derivative with respect to r is

$$\begin{aligned} m'_\lambda(r) &= \frac{\sqrt{1-\lambda\sin^2 r}(\cos r) - \sin r\frac{1}{2\sqrt{1-\lambda\sin^2 r}}(-2\lambda\sin r\cos r)}{1-\lambda\sin^2 r} \\ &= \frac{(1-\lambda\sin^2 r)(\cos r) + \lambda\sin^2 r\cos r}{(1-\lambda\sin^2 r)^{3/2}} \\ &= \frac{\cos r}{(1-\lambda\sin^2 r)^{3/2}}, \end{aligned}$$

and the second derivative is

$$\begin{aligned} m''_\lambda(r) &= \frac{(1-\lambda\sin^2 r)^{3/2}(-\sin r) - \cos r\frac{3}{2}(1-\lambda\sin^2 r)^{1/2} - 2\lambda\sin r\cos r}{(1-\lambda\sin^2 r)^3} \\ &= \frac{(-\sin r)(1-\lambda\sin^2 r - 3\lambda\cos^2 r)}{(1-\lambda\sin^2 r)^{5/2}} \\ &= \frac{(-\sin r)((1-\lambda) - 2\lambda\cos^2 r)}{(1-\lambda\sin^2 r)^{5/2}}. \end{aligned}$$

From lemma 2.19 the Gaussian curvature is

$$\begin{aligned} G_\lambda(r) &= \frac{-m''_\lambda(r)}{m_\lambda(r)} \\ &= \frac{(1-\lambda) - 2\lambda\cos^2 r}{(1-\lambda\sin^2 r)^2}. \end{aligned}$$

It follows that

$$G'_\lambda(r) = \frac{4\lambda \sin r \cos r(2(1-\lambda) - \lambda \cos^2 r)}{(1-\lambda \sin^2 r)^3}.$$

It is clear that for $\lambda \in (0,1)$, G'_λ vanishes at the pair of poles and the equator and the Gaussian curvature G_λ is non-monotone for $\lambda \in (0, \frac{2}{3})$ with a local extremum of $\lambda = \frac{2}{3}$.

Since, G_λ is non-monotone, the h-half period function can be obtained by the following computation as in previous example.

Let

$$m_\lambda(r) = \frac{\sin r}{\sqrt{1-\lambda \sin^2 r}}, \qquad r \in [0,\pi], \qquad \lambda \in (0,1)$$

and

$$m'_\lambda(r) = \frac{\cos r}{(1-\lambda \sin^2 r)^{3/2}}.$$

By putting $x = m^2_\lambda(r)$, we have

$$\begin{aligned} \frac{dx}{dr} &= 2m_\lambda(r)m'_\lambda(r) \\ &= \frac{2\sin r \cos r}{(1-\lambda \sin^2 r)^2} \\ \frac{dr}{dx} &= \frac{(1-\lambda \sin^2 r)^2}{2\sin r \cos r}. \end{aligned}$$

Since

$$\begin{aligned} x &= \frac{\sin^2 r}{1-\lambda \sin^2 r} \\ x - \lambda x \sin^2 r &= \sin r \\ \sin^2 r &= \frac{x}{1+\lambda x}, \end{aligned}$$

and

$$\begin{aligned} \cos^2 r &= 1 - \sin^2 r \\ &= 1 - \frac{x}{1+\lambda x} \\ &= \frac{1+\lambda x - x}{1+\lambda x}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{dr}{dx} &= \frac{\left(1 - \frac{\lambda x}{1+\lambda x}\right)^2}{2\sqrt{\frac{x}{1+\lambda x}}\sqrt{\frac{1+\lambda x - x}{1+\lambda x}}} \\ &= \frac{1+\lambda x - \lambda x}{(1+\lambda x)^2} \frac{1+\lambda x}{2\sqrt{x}\sqrt{1+\lambda x - x}}. \end{aligned}$$

Hence

$$dr = \frac{1}{2\sqrt{x}(1+\lambda x)\sqrt{1+\lambda x - x}} dx. \tag{5.5}$$

Recall the formula for h-half period function

$$\mathcal{H}(\nu) = 2\int_{\xi(\nu)}^{a} \frac{\nu}{m(r)\sqrt{m^2(r) - \nu^2}} dr$$

we use that $r = \xi(\nu) = m_\lambda^{-1}(\nu)$ *implies* $x = m_\lambda^2(r) = m_\lambda^2(\xi(\nu)) = \nu^2$ *and* $r = a = \frac{\pi}{2}$ *implies* $x = m_\lambda^2(\frac{\pi}{2}) = \frac{1}{1-\lambda}$, *we get*

$$\begin{aligned}
\mathcal{H}(\nu) &= 2\int_{\nu^2}^{\frac{1}{1-\lambda}} \frac{\nu}{\sqrt{x}\sqrt{x-\nu^2}} \frac{1}{2\sqrt{x}(1+\lambda x)\sqrt{1+\lambda x - x}} dx \\
&= \nu \int_{\nu^2}^{\frac{1}{1-\lambda}} \frac{1}{x\sqrt{x-\nu^2}(1+\lambda x)\sqrt{1+\lambda x - x}} dx \\
&= \nu \int_{\nu^2}^{\frac{1}{1-\lambda}} \frac{1}{x\lambda(x+\frac{1}{\lambda})\sqrt{x-\nu^2}\sqrt{\left(\frac{1}{1-\lambda} - x\right)(1-\lambda)}} dx \\
&= \frac{\nu}{\lambda\sqrt{1-\lambda}} \int_{\nu^2}^{\frac{1}{1-\lambda}} \frac{1}{x(x+\frac{1}{\lambda})\sqrt{x-\nu^2}\sqrt{\left(\frac{1}{1-\lambda} - x\right)}} dx.
\end{aligned}$$

By setting

$$a = \frac{1}{\lambda}, \qquad b = \nu^2, \qquad c = \frac{1}{1-\lambda},$$

we obtain

$$\mathcal{H}(\nu) = \frac{\nu}{\lambda\sqrt{1-\lambda}} \int_b^c \frac{1}{x(x+a)\sqrt{x-b}\sqrt{c-x}} dx. \tag{5.6}$$

From (5.3)*, we obtain*

$$\begin{aligned}
\mathcal{H}(\nu) &= \frac{\nu}{\lambda\sqrt{1-\lambda}} \lambda\pi \left(\frac{1}{\sqrt{\frac{\nu^2}{1-\lambda}}} - \frac{1}{\sqrt{\left(\frac{1}{\lambda} + \frac{1}{1-\lambda}\right)\left(\frac{1}{\lambda} + \nu^2\right)}} \right) \\
&= \frac{\nu\pi}{\sqrt{1-\lambda}} \left(\frac{\sqrt{1-\lambda}}{\nu} - \frac{1}{\sqrt{\left(\frac{1-\lambda+\lambda}{\lambda(1-\lambda)}\right)\left(\frac{1+\lambda\nu^2}{\lambda}\right)}} \right) \\
&= \frac{\nu\pi}{\sqrt{1-\lambda}} \left(\frac{\sqrt{1-\lambda}}{\nu} - \frac{\lambda\sqrt{1-\lambda}}{\sqrt{1+\lambda\nu^2}} \right) \\
&= \pi - \frac{\pi\nu\lambda}{\sqrt{1+\lambda\nu^2}}.
\end{aligned}$$

Next, we compute the first derivative with respect to ν *of* $\mathcal{H}(\nu)$

$$\begin{aligned}
\mathcal{H}'(\nu) &= -\pi\lambda \left(\frac{\sqrt{1+\lambda\nu^2} - \nu\frac{1}{2\sqrt{1+\lambda\nu^2}}(2\lambda\nu)}{1+\lambda\nu^2} \right) \\
&= -\pi\lambda \left(\frac{1+\lambda\nu^2 - \lambda\nu^2}{(1+\lambda\nu^2)^{\frac{3}{2}}} \right) \\
&= \frac{-\pi\lambda}{(1+\lambda\nu^2)^{\frac{3}{2}}}.
\end{aligned}$$

Therefore, $\mathcal{H}$ *is monotone non-increasing from lemma 2.30, the* h*-cut locus of a point on equator is a subarc of equator then by theorem 2.29, we get*

Proposition 5.4. If $\lambda \in (0,1)$, then, for each point of q of M_λ defined in (5.4) distinct from a pole, the cut locus of q is a subarc of the antipodal parallel to q.

If we consider again the Randers rotational metric $(\mathbb{S}^2, F_\lambda = \alpha + \beta)$ *obtained by Zermelo's navigation method from navigation data* (h_λ, W)*,* $W = \mu\frac{\partial}{\partial\theta}$*,* $\mu < \sqrt{1-\lambda}$*, then* (3.10) *gives*

$$\mathcal{H}_F^+(\nu) = \pi - \frac{\pi\nu\lambda}{\sqrt{1+\lambda\nu^2}} + \sqrt{1-\lambda}\left(\frac{\pi}{2} - \nu^2\right), \qquad \nu \in (0, \sqrt{1-\lambda}),$$

where we consider for simplicity $\mu = \frac{1}{2}\sqrt{1-\lambda}$*, and hence*

$$(\mathcal{H}_F^+)'(\nu) = \frac{-\pi\lambda}{(1+\lambda\nu^2)^{\frac{3}{2}}} - 2(\sqrt{1-\lambda})\nu.$$

By a similar argument with previous example it follows that proposition 5.3 is true for this example as well.

Remark 5.5. We can see that $\mathcal{H}$ and $\mathcal{H}_F^+$ have the same monotonicity.

Chapter 6

Conclusions and Suggestions

6.1 Conclusions

In this thesis, we have obtained the following conclusions:

1. *Lemma 3.1, Any Riemannian 2D sphere of revolution endowed with a Randers rotational metric is an Einstein manifold.*
2. *Lemma 3.4, the rotational wind is Killing vector field.*
3. *Proposition 3.6, we clarify the relation between any geodesic on a Riemannian and a Randers rotational 2D sphere of revolution, that is the F-geodesic obtain from moving the h-geodesic along rotational wind flow.*
4. *Proposition 3.8, when we move the h-Jacobi field with the rotational wind flow, we obtain the F-Jacobi field. This is the basic setting for study about conjugate points and conjugate locus on a Randers rotational 2D sphere of revolution.*
5. *Proposition 3.10, we clarify the relation between the cut locus of a Riemannian and a Randers rotational 2D sphere of revolution, that is if we move the h-cut point of Riemannian 2D sphere of revolution along the wind flow, we obtain the F-cut point on Randers 2D sphere of revolution.*
6. *Theorem 4.2, the F-cut locus on a Randers rotational 2D sphere of revolution with the flag curvature is monotone obtaining from moving the h-cut locus on Riemannian 2D sphere of revolution when the Gaussian curvature is monotone along the rotational wind flow.*
7. *Theorem 4.2, the fourth case told that, if the cut locus is a single point, then the flag curvature is constant, that is the topology of manifold is related to the cut locus.*
8. *Theorem 4.3, the F-cut locus on Randers rotational 2D sphere of revolution with non-monotone flag curvature of any point distinct from pole is the antipodal parallel, when the F-cut locus of a point on equator is the subarc of equator.*

6.2 Suggestions

There are many open problems following from this thesis, e.g.

1. *The case that the wind for navigation data is not Killing vector field.*
2. *The conjugate locus on Randers rotational 2D sphere of revolution.*
3. *The geometry of Randers rotational on cylinder of revolution, that is the surface of revolution is open two ends.*
4. *The Randers rotational Zoll surface of revolution, that is the sectional curvature is a function but it can not everywhere negative (see [2]).*

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Appendix

The Theory of Geodesics on Some Surface of Revolution

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Abstract

We study the properties of the geodesics on a Randers rotational surface of revolution by using Zermelo navigation data (h, W), where h is the induced Riemannian metric on the surface of revolution and W is the rotational wind. We are in special interested in the half-period function that can be computed by similar methods to the Riemannian case. Our result can be applied to find the structure of the cut locus of a Randers rotational 2-sphere of revolution.

Keywords: Randers rotational sphere, surface of revolution, Zermelo navigation

1. Introduction

The Riemannian geometry is one of the important research topics for differential geometry field. In general, Riemannian geometry has many interested topics to study, but they are almost well known study. So we are interested to do research in something more complicated or general (nearest the problem in real world) more than Riemannian geometry, that is Finsler geometry ([1], [2]). In this paper we will show that Riemannian case is the special case of Finsler case and we use the Randers metric as an examples for the Finsler case. In the case of a Riemannian surface of revolution, one can study the behaviour of geodesic by using Clairaut relation, we can see that if the geodesic is neither a profile curve nor s parallel then it will be tangent to the some parallel. The length between starting point and returning point can be calculate by using half period function.

The aims of studing the half-period function for Randers rotational case is to find the cut locus on Randers rotational surface of revolution. If we can find the exactly form of this function then we can see the behavior of the cut locus. In this paper, we will show how to construct the half period function for Randers rotational surface of revolution.

2. Materials and Methods

2.1 The geometry of Riemannian surface of revolution

We recall the definition of Riemannian geometry.

Definition 2.1 (Local surface) A subset S of $\mathbb{R}^3$ is called a local surface if there exists a C^∞ map φ of a domain D in $\mathbb{R}^2$ into $\mathbb{R}^3$, i.e. $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$, such that

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1. $S = \varphi(D)$
2. φ is injective.
3. the rank of the matrix $\begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix}$ is 2 at each point on D.

Definition 2.2 (2-sphere of revolution) A compact Riemannian manifold (M,h) homeomorphic to a 2-sphere is called a 2-sphere of revolution if M admits a point p such that for any two points q_1, q_2 on M with $d(p,q_1) = d(p,q_2)$, where $d(,)$ denoted the Riemannian distance function, there exists an isometry f on M satisfying $f(q_1) = q_2$ and $f(p) = p$. The point p is called a pole of M. Let (r,θ) denote geodesic polar coordinates around a pole p of (M,h). The Riemannian metric can be expressed as $h = dr^2 + m(r)^2 d\theta^2$ on $M \setminus \{p,q\}$, where q denotes the unique cut point of p, i.e. p, q are called a pair of poles.

From Definition 2.2 we can construct the classical Riemannian surface of revolution by rotating a unit speed smooth curve $x = f(z)$, where $f:[a,b] \to \mathbb{R}$, $a,b \in \mathbb{R}$ and $f(a) = f(b) = 0$, include in the xz plane around the z axis. We will consider the curve f in parametric form

$$f:\begin{cases} x = m(r) \\ z = z(r) \end{cases},$$

where $r \in [a,b]$, $m > 0$ and of the Euclidean unit speed condition, that is

$$(m'(r))^2 + (z'(r))^2 = 1.$$

Then we obtain the surface of revolution

$$M := \varphi(r,\theta) = \big(m(r)\cos\theta, m(r)\sin\theta, z(r)\big),\ r \in [a,b],\ \theta \in [0,2\pi).$$

One can see that the mapping φ is satisfied Definition 2.1.

We recall the Riemannian metric on surface of revolution is

$$ds^2 = dr^2 + m^2(r)d\theta^2,$$

and the geodesic equations of h -unit speed $\gamma(s) := (r(s),\theta(s))$ of (M,h) are

$$\begin{cases} \dfrac{d^2r}{ds^2} - mm'\left(\dfrac{d\theta}{ds}\right)^2 = 0 \\ \dfrac{d^2\theta}{ds^2} + 2\dfrac{m'}{m}\dfrac{dr}{ds}\dfrac{d\theta}{ds} = 0 \end{cases},$$

with the unit speed condition

$$\left(\frac{dr}{ds}\right)^2 + m^2\left(\frac{d\theta}{ds}\right)^2 = 1.$$

Remark 2.3 We can see that every profile curve, i.e. $\gamma(s) := (r(s),\theta_0)$, where θ_0 is constant, is an h -geodesic and parallel, i.e. $\gamma(s) := (r_0,\theta(s))$, where r_0 is constant and $m'(r_0) = 0$, is an h -geodesic.

Theorem 2.4 (Clairaut relation [6]) If $\gamma(s) := (r(s),\theta(s))$ is a geodesic on surface of revolution (M,h) then the angle $\phi(s)$ between tangent vector of $\gamma(s)$ and the profile curve passing through a point $\gamma(s)$ satisfy

$$m(r(s))\sin\phi(s) = \nu,$$

where ν is constant and called Clairaut constant.

Lemma 2.5 The Clairaut constant for any profile curve is vanishes, i.e. $\nu = 0$.

From Clairaut relation, we can see that if $\gamma(s)$ is neither a profile curve nor a parallel, i.e. $\nu \in (0, m(r))$, then for some $t_1 > 0$, $\gamma(t_1)$ will be tangent to the same parallel of $\gamma(0)$, where $\gamma(0)$ is the emanating point of geodesic.

Let us denoted the geodesic that emanating from a point p_0 with Clairaut constant ν by $\gamma_\nu^{p_0}$.

Remark 2.6 We always assume that our 2-sphere of revolution with a pair of poles p, q satisfying the following properties

1. (M,h) is symmetric with respect to the reflection fixing $r = a$, where $2a$ denotes the distance between p and q.

2. The Gaussian curvature G of M is monotone along a profile curve from the point p to the point on $r = a$.

We can find the length between $\gamma_\nu^{p_0}(0)$ and $\gamma_\nu^{p_0}(t_1)$ by using

Lemma 2.7 (Half period function of Riemannian surface of revolution) Let $\gamma_\nu^{p_0}$ be an h-unit speed geodesic, where $p_0 \in \{r = a\}$ and $\nu \in (0, m(a))$, i.e. p_0 is a point on equator and $\gamma_\nu^{p_0}$ is neither meridian nor equator. From Clairaut relation $\gamma_\nu^{p_0}$ must be tangent to the parallel $\xi(\nu)$ and return to the equator at $\gamma_\nu^{p_0}(t_1)$. The distance from p_0 to $\gamma_\nu^{p_0}(t_1)$ can be computed by

$$H(\nu) := 2\int_{\xi(\nu)}^{a} \frac{\nu}{m(t)\sqrt{m(t)^2 - \nu^2}}\, dt$$

where H is called half period function.

2.2 The geometry of Randers rotational surface of revolution

In this section, we will consider that if there is a wind blow up on our surface of revolution along the parallel, by using Zermelo navigation problem [3], therefore we obtained

Proposition 2.8 (Randers rotational metric [4]). If (M,h) is a surface of revolution whose profile curve is the bounded function $m(r) < \dfrac{1}{\mu}$ and $W = \mu \cdot \dfrac{\partial}{\partial \theta}$ is the breeze on M blowing along parallels, then the Randers metric $(M, F = \alpha + \beta)$ obtained by the Zermelo's navigation process on M is a Finsler metric on M, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$, $\beta = b_i(x) y^i$ are defined in

$$(a_{ij}) = \begin{pmatrix} \dfrac{1}{1-\mu^2 m^2} & 0 \\ 0 & \dfrac{m^2}{(1-\mu^2 m^2)^2} \end{pmatrix},\ b_i = \begin{pmatrix} 0 \\ \dfrac{\mu m^2}{1-\mu^2 m^2} \end{pmatrix}, i, j = 1, 2.$$

We obtained the flow of the wind $\varphi(s; r(s), \theta(s)) = (r(s), \theta(s) + \mu s)$.

From [5], the geodesic equation of F-unit geodesic $P(s) = (P^1(s), P^2(s))$ is

$$\begin{cases} \dfrac{d^2 P^1}{ds^2} - mm'\left(\dfrac{dP^2}{ds}\right)^2 - \mu mm'\left(\mu - 2\dfrac{dP^2}{ds}\right) = 0 \\ \dfrac{d^2 P^2}{ds^2} - 2\dfrac{m'}{m}\dfrac{dP^1}{ds}\dfrac{dP^2}{ds} - 2\mu\dfrac{m'}{m}\dfrac{dP^1}{ds} = 0 \end{cases}$$

Remark 2.9 If $\gamma(s) := (r(s), \theta(s))$ be a geodesic on (M,h) then we obtained geodesic $P(s)$ for (M,F) constructed as above by $P(s) = \varphi(s, \gamma(s)) = (r(s), \theta(s) + \mu s)$.

3. Results and Discussion

In this section we assume that there is a wind $W := \mu \frac{\partial}{\partial \theta}$ blowing along the parallels on 2-sphere of revolution (M,h), where $\mu \leq \frac{1}{\max\{m(r) : r \in [0, 2a]\}}$. Therefore we obtain the Randers rotational 2-sphere of revolution $(M, F = \alpha + \beta)$.

So, we can obtain our main result

Theorem 3.1 (Half period function of Randers rotational 2-sphere of revolution) Let $P_\nu^{p_0}(s) = (r(s), \theta(s) + \mu s)$ be an F-unit speed geodesic obtained by $\varphi(s; \gamma_\nu^{p_0}(s))$, where $\gamma_\nu^{p_0}(s)$ is an h-unit speed geodesic on (M,h), emanating from $p_0 \in \{r = a\}$ and $\nu \in (0, m(a))$ if the direction of $P_\nu^{p_0}(s)$ is along the wind then $P_\nu^{p_0}(s)$ will tangent to parallel $\xi(\nu)$ at $P_\nu^{p_0}(t_1)$ and return to the equator at $P_\nu^{p_0}(t_0)$. The distance from p_0 to $P_\nu^{p_0}(t_0)$ can be computed by

$$H_F^+(\nu) = H(\nu) + \psi(\nu), \tag{3.1}$$

where $\psi(\nu) := 2\mu(a - \xi(\nu))$.

In the others hand, if the direction of $P_\nu^{p_0}(s)$ is against the wind then the distance is

$$H_F^-(\nu) = H(\nu) - \psi(\nu). \tag{3.2}$$

Proof. In this proof we denoted $\gamma_\nu^{p_0}(s)$ by $\gamma(s)$ and $P_\nu^{p_0}(s)$ by $P(s)$.

Let $\gamma(s) = (r(s), \theta(s))$ be an h-unit speed geodesic, i.e.

$$\left(\frac{dr}{ds}\right)^2 + m^2(r(s))\left(\frac{d\theta}{ds}\right)^2 = 1. \tag{3.3}$$

Multiply (3.3) with $\left(\frac{ds}{d\theta}\right)^2$, we have

$$\left(\frac{dr}{d\theta}\right)^2 + m^2(r(s)) = \left(\frac{ds}{d\theta}\right)^2. \tag{3.4}$$

From Clairaut relation we have

$$\frac{ds}{d\theta} = \frac{m^2(r(s))}{\nu}. \tag{3.5}$$

Therefore (3.4) can be written as

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{m^2(r(s))(m^2(r(s)) - \nu^2)}{\nu^2}, \tag{3.6}$$

or

$$\frac{d\theta}{dr} = \frac{\nu}{m(r(s))\sqrt{m^2(r(s)) - \nu^2}}. \tag{3.7}$$

By integrating (3.7), we get

$$\theta(b)-\theta(a)=\int_{r(a)}^{r(b)}\frac{\nu}{m(r(s))\sqrt{m^2(r(s))-\nu^2}}dr\,. \tag{3.8}$$

From Clairaut relation we know that the geodesic $\gamma(s)$ emanating from the point $\gamma(0)$ on the parallel will tangent to other parallel called $\{r=\xi(\nu)\}$ at $\gamma(t_1)$ and after that it will return to the parallel again. We can see that $\theta(t_0)-\theta(0)=2(\theta(t_0)-\theta(t_1))$, i.e.

$$H(\nu):=\theta(t_0)-\theta(0)=2\int_{\xi(\nu)}^{a}\frac{\nu}{m(t)\sqrt{m(t)^2-\nu^2}}dt\,, \tag{3.9}$$

$H(\nu)$ is called h-half period function. Recall that

$$b-a=\int_a^b ds=\int_{r(a)}^{r(b)}\frac{ds}{dr}dr\,. \tag{3.10}$$

From $P(s)=(r(s),\theta(s)+\mu s)$ obtained from $\gamma(s)$, we have

$$\frac{dP^1}{ds}=\frac{dr}{ds}\,,\ \frac{dP^2}{ds}=\frac{d\theta}{ds}+\mu\,, \tag{3.11}$$

and therefore

$$\begin{aligned}\frac{dP^2}{dP^1}&=\frac{dP^2}{ds}\frac{ds}{dP^1}\\ \frac{dP^2}{dr}&=\left(\frac{d\theta}{ds}+\mu\right)\frac{ds}{dr}\,.\\ &=\frac{d\theta}{dr}+\mu\frac{ds}{dr}\end{aligned} \tag{3.12}$$

By integrating (3.12), we get

$$P^2(r(b))-P^2(r(a))=\int_{r(a)}^{r(b)}\left(\frac{d\theta}{dr}+\mu\frac{ds}{dr}\right)dr\,. \tag{3.13}$$

We will consider in the case that $P(s)$ will tangent to the parallel $\{r=\xi(\nu)\}$ at $P(t_1)$ and return to equator at $P(t_0)$, from (3.9) and (3.10) therefore we got (3.1)

$$\begin{aligned}H_F^{+}(\nu)&=P^2(t_0)-P^2(0)\\ &=2\int_{\xi(\nu)}^{a}\left(\frac{d\theta}{dr}+\mu\frac{ds}{dr}\right)dr\quad.\\ &=H(\nu)+2\mu(a-\xi(\nu))\end{aligned}$$

If we consider the geodesic that against the wind $P(s)=(r(s),\theta(s)-\mu s)$, we get (3.2)

$$H_F^{-}=H(\nu)-2\mu(a-\xi(\nu))\,.$$

Remark 3.2 The function $\psi(\nu)=2\mu(a-\xi(\nu))$ is decreasing function, where $\xi(\nu)\in(0,a)$, and it is increasing, where $\xi(\nu)\in(a,2a)$.

4. Conclusions

Finally, we can find the half period function for Randers rotational 2-sphere of revolution therefore we can see the structure of cut locus on Randers rotational 2-sphere of revolution as in [7].

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The cut locus of a Randers rotational 2-sphere of revolution

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Abstract. In the present paper we study the structure of the cut locus of a Randers rotational 2-sphere of revolution $(M, F = \alpha + \beta)$. We show that in the case when the Gaussian curvature of the Randers surface is monotone along a meridian, the cut locus of a point $q \in M$ is a point on a subarc of the opposite half bending meridian or of the antipodal parallel (Theorem 1.1). More generally, in the case when the Gaussian curvature is not monotone along the meridian, but the cut locus of a point q on the equator is a subarc of the same equator, the cut locus of any point $\widetilde{q} \in M$ different from poles is a subarc of the antipodal parallel (Theorem 1.2). Some examples are also given at the last section and some differences with the Riemannian case are pointed out.

1. Introduction

The study of the global behaviour of geodesics, conjugate points and cut locus is a fundamental problem in modern differential geometry. In the Riemannian case, an extensive literature is available (see [1], [10], [11]), but in the more general case of a Finsler manifold, the results are not so easily obtained. The main difficulty is that the dependence of the metric on the direction implies the non-symmetry of the distance function and the non-reversibility of the geodesics.

We recall (see [2] for details) that Finsler manifolds (M, F) generalize the Riemannian ones in the sense that they are defined by a norm $F : TM \to [0, \infty)$ with the properties

(i) F is positive and differentiable on $\widetilde{TM} := TM \setminus \{0\}$;

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(ii) F is 1-positive homogeneous, i.e. $F(x, \lambda y) = \lambda \cdot F(x, y)$ for any $\lambda > 0$ and for all $(x, y) \in \widetilde{TM}$;

(iii) the Hessian matrix $g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}$, $i, j \in \{1, ..., n\}$, is positive definite on $\widetilde{TM}$.

Here TM denotes the tangent bundle of an n-dimensional smooth manifold M and (x, y) the canonical coordinates on TM. The Finsler structure is called absolute homogeneous if the homogeneity condition (ii) is replaced by $F(x, \lambda y) = |\lambda| \cdot F(x, y)$ for any $\lambda \in \mathbb{R}$.

A Finsler norm F determines and it is determined by its indicatrix bundle $SM := \cup_{x \in M} S_x M$, where $S_x M := \{y \in T_x M : F(x, y) = 1\}$.

Obviously, the simplest Finsler manifolds are the Riemannian cases, but this is the trivial case for us. Less trivial examples are deformations of Riemannian metrics by linear forms $\beta = b_i(x)y^i$ defined on TM. This type of Finsler manifolds include Randers, Kropina and Matsumoto metrics ([13]).

A Finsler norm can be used for defining the integral length $\mathcal{L}_F$ of a C^∞ curve $\gamma : [a, b] \to M$ by

$$\mathcal{L}_F(\gamma\big|_{[a,b]}) = \int_a^b F(\gamma(t), \dot{\gamma}(t))dt,$$

where $\dot{\gamma}(t) = \frac{d\gamma}{dt}$ is the tangent vector of γ. This definition easily extends to the integral length of any piecewise C^∞ curve on M.

A smooth curve γ on a Finsler manifold that minimizes the integral length $\mathcal{L}_F$ over the set of all piecewise C^∞ curves with fixed end points is called an *F-geodesic*.

Any F-geodesic γ emanating from a point p in a compact Finslerian (or Riemannian) manifold is losing its global minimizing property of a point q on γ. Such point is called a *F-cut point of p* along γ. The *F-cut locus* of a point $p \in M$ is the set of all cut points along all geodesics emanating from p on a Finsler manifold. This is an important geometrical object related to the topology of the manifold and to the global geometrical properties of the Finsler manifold.

Even though in general the cut locus may have a very complicated structure, it is known that the F-cut locus $\mathcal{C}_p^F$ of a point p on a Finsler surface is a local tree and that any two points on the same connected component of $\mathcal{C}_p^F$ can be joined by a rectifiable Jordan arc in $\mathcal{C}_p^F$ (see [12] for details).

Based on this theoretical result, we have studied in [7] the actual structure of the cut locus of a point on a Randers rotational surface of revolution homeomorphic to $\mathbb{R}^2$.

The main aim of the present paper is to explicitly determine the structure of the cut locus of a point of a 2-sphere of revolution endowed with a Randers rotational metric.

Randers metrics are special Finsler metrics whose indicatrices are obtained by rigid translations of the Riemannian unit sphere. It was Shen [14] who pointed out for the first time that Randers metrics give solutions to the classical Zermelo's navigation problem, namely, *find the paths of shortest time travel between two points under the influence of a wind or a current when we travel by a boat capable of a certain maximum speed.*

Formally, if we consider the background landscape to be a Riemannian manifold (M, h), endowed with a vector field W on M, $\|W\|_h < 1$, then the shortest time travel paths are precisely the geodesics of a Finsler metric of Randers type

$$F(x, y) = \alpha(x, y) + \beta(x, y) = \frac{\sqrt{\lambda \cdot \|y\|_h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}$$

uniquely induced by the navigation data (h, W). Here $W = W^i \cdot \frac{\partial}{\partial x^i}$ is the velocity vector field of the wind, $\lambda = 1 - \|W\|_h^2$, $W_0 = h(W, y)$.

The corresponding Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and 1-form $\beta = b_i(x)y^i$ are given by

$$a_{ij}(x) = \frac{\lambda \cdot h_{ij} + W_i W_j}{\lambda^2} \quad \text{and} \quad b_i = -\frac{W_i}{\lambda},$$

where $W_i := h_{ij}W^j$.

This Randers metric satisfies all three conditions in the definition of a Finsler metric provided $\|W\|_h < 1$ (see [2], [5], [9], [14] for details).

Our main theorems on the structure of the F-cut locus of a surface of revolution endowed with a Randers rotational metric are the following.

Theorem 1.1. *Let (M, F) be a Randers rotational 2-sphere of revolution with navigation data (h, W), where $W = \mu \cdot \frac{\partial}{\partial \theta}$ is the wind blowing along parallels, $\mu < \{\frac{1}{\max\{m(r)\}} : r \in [0, 2a]\}$, with a pair of poles p, q, $d_h(p, q) = 2a$ and satisfying*

- *M is symmetric with respect to $\{r = a\}$,*
- *the flag curvature $\mathcal{K}$ is monotone along a meridian.*

Then the F-cut locus $\mathcal{C}_x^F$ of a point $x \in M \setminus \{p, q\}$ with $\{\theta(x) = 0\}$ is

(1) *The subarc of the opposite half bending meridian,*

$$\mathcal{C}_x^F = \varphi(d(x, \tau(t)), \tau(t)), \quad t \in [c, 2a - c],$$

where φ is the flow of the wind, when $\mathcal{K}$ is monotone non-increasing.

(2) *The following subarc of the antipodal parallel* $\{r = 2a - r(x)\}$ *to* x*:*

$$\mathcal{C}_x^F = r^{-1}(2a - r(x)) \cap \theta^{-1}\{\mathcal{H}(m) + \psi(x), 2\pi - (\mathcal{H}(m) - \psi(x))\}.$$

where $\psi(x) = \mu \cdot d_h(x, \hat{q}_0)$, $\hat{q}_0$ *is the* h*-first conjugate point of* x *with respect to* h, $m := m(r(x))$, *when* $\mathcal{K}$ *is monotone non-decreasing.*

(3) *A single point on the antipodal parallel* $\mathcal{C}_x^F = (2a - r(x), \pi(1 + \mu R))$, *where* R *is radius of sphere, when* $\mathcal{K} = \frac{1}{R^2}$ *is constant.*

(4) *If the cut locus of* $x \in M \setminus \{p, q\}$ *is a single point, then* $\mathcal{K}$ *is constant.*

More generally, if the Gaussian curvature of h, or of F, is not monotone, the following characterization of the cut locus is possible.

Theorem 1.2. *Let* $(M, F = \alpha + \beta)$ *be the Randers rotational 2-sphere of revolution constructed from the navigation data* (h, W) *of a 2-sphere of revolution* (M, h).

If the F*-cut locus of a point* x *on the equator* $\{r = a\}$ *is a subarc of the equator* $\{r = a\}$, *then the* F*-cut locus of any point* $\widetilde{x}$ *with* $r(\widetilde{x}) \in (0, 2a) \setminus \{a\}$ *is a subarc of the antipodal parallel* $\{r = 2a - r(\widetilde{x})\}$.

This is a generalization of Theorem 3.5 in [4] to the Randers case.

Here it is the structure of our paper.

We start by recalling the geometry of a Riemannian 2-sphere of revolution and the structure of its cut locus (Section 2.1). This section is an excerpt from [11].

By using the navigation data (h, W), where h is the induced Riemannian metric on the 2-sphere of revolution M, and $W := \mu \cdot \frac{\partial}{\partial \theta}$ a mild wind blowing along the parallels, we construct in Section 2.2 a Randers rotational metric $F = \alpha + \beta$ on the 2-sphere of revolution M. We determine the F-geodesic equations in Proposition 2 and extend the Clairaut relation to F-geodesics. The conjugate and cut points along F-geodesics are obtained by mapping the conjugate and cut points along h-geodesics by means of the flow φ, Propositions 3 and 4, respectively.

Moreover, we show here that the flag curvature of this Randers metric coincide with the Gaussian curvature of h (Lemma 2.4). Even though some of these results were proved already in [7], for a surface of revolution homeomorphic to $\mathbb{R}^2$, we show here how they extend to a 2-sphere of revolution.

Section 3 is where we prove Theorem 1.1 by using a certain number of lemmas. Finally, in Section 4, we prove Theorem 1.2 and give some examples of Randers rotational metrics whose Gaussian curvature is not monotone (Subsection 4.2).

We show that the convexity of the second derivative of the F-half period function different from the convexity of the h-half period function.

In a forthcoming research we will study the convexity of injectivity domain and other related topics of Randers rotational surface of revolution.

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2. The 2-sphere of revolution

2.1. The Riemannian 2-sphere of revolution. A compact Riemannian manifold (M, h) homeomorphic to a 2-sphere is called a 2-*sphere of revolution* if M admits a point p, called *pole*, such that for any two points q_1, q_2 on M with $d_h(p, q_1) = d_h(p, q_2)$, there exists an h-isometry f on M satisfying $f(q_1) = q_2$, and $f(p) = p$, where $d_h(\cdot, \cdot)$ denoted the h-Riemannian distance function on M.

Let (r, θ) denote geodesic polar coordinates around a pole p of (M, h). The Riemannian metric can be expressed as $h = dr^2 + m^2(r)d\theta^2$ on $M \setminus \{p, q\}$, where q denotes the unique h-cut point of p and

$$m(r(x)) := \sqrt{h\left(\left(\frac{\partial}{\partial\theta}\right)_x, \left(\frac{\partial}{\partial\theta}\right)_x\right)},$$

for any point $x \in M \setminus \{p, q\}$ with coordinates $(r(x), \theta(x))$ (see [11]).

It is known that each pole of a 2-sphere of revolution M has a unique cut point (see [11], Lemma 2.1.). A pole and its unique cut point are called *a pair of poles.*

From now, for the rest of the paper, we fix a pair of poles p, q and the geodesic polar coordinates (r, θ) around p.

Remark 2.1. We always assume about (M, h) the following conditions (as in [11]):

1. M is symmetric with respect to the equator, i.e. reflection fixing $\{r = a\}$, where $d_h(p, q) = 2a$. In other words, we assume

$$m(r) = m(2a - r), \quad \forall r \in (0, 2a).$$

2. The Gaussian curvature $G(x) = -\frac{m''(r(x))}{m(r(x))}$ of (M, h) is monotone along the meridian from pole to the equator.

We observe that both functions $m(r)$ and $m(2a-r)$ are extensible to a C^∞ odd function around $\{r=0\}$ and $m'(0)=1=-m'(2a)$.

Any periodic h-geodesic passing through a pair of poles is called a *meridian*, i.e. we have $\gamma(t)=\gamma(t+4a)$, for any $t\in\mathbb{R}$, and $p=\gamma(0)$.

Any curve $r=c\in(0,2a)$ is called a *parallel.* The parallel $\{r=a\}$ is called the *equator* of (M,h).

Remark 2.2. For the sake of simplicity we will often make use in the following of the Riemannian universal covering of $(M\setminus\{p,q\},dr^2+m(r)^2d\theta^2)$, namely

$$(\widetilde{M},\widetilde{h}):=((0,2a)\times\mathbb{R},d\widetilde{r}^2+m(\widetilde{r})^2d\widetilde{\theta}^2),$$

with the covering projection $\Pi:\widetilde{M}\to M\setminus\{p,q\}$.

Recall that the equations of an h-unit speed geodesic $\gamma(s):=(r(s),\theta(s))$ of (M,h) are

$$\begin{cases}\frac{d^2r}{ds^2}-mm'\left(\frac{d\theta}{ds}\right)^2=0\\ \frac{d^2\theta}{ds^2}+2\frac{m'}{m}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right)=0,\end{cases} \tag{1}$$

where s is the arclength parameter of γ with the h-unit speed parametrization condition

$$\left(\frac{dr}{ds}\right)^2+m^2\left(\frac{d\theta}{ds}\right)^2=1. \tag{2}$$

It follows that every profile curve, or *meridian*, is an h-geodesic, and that a parallel $\{r=r_0\}$ is geodesic, r_0 is constant, if and only if $m'(r_0)=0$.

We observe that (1) implies

$$\frac{d\theta(s)}{ds}m^2(r(s))=\nu,\quad \text{where } \nu \text{ is constant,} \tag{3}$$

that is, the quantity $\frac{d\theta}{ds}m^2$ is conserved along the h-geodesics.

Lemma 2.1 (The Clairaut relation)**.** *Let $\widetilde{\gamma}(s)=(\widetilde{r}(s),\widetilde{\theta}(s))$ be an $\widetilde{h}$-unit speed geodesic on $(\widetilde{M},\widetilde{h})$. There exists a constant ν such that*

$$m^2(\widetilde{r}(s))\widetilde{\theta}'(s)=m(\widetilde{r}(s))\cos\phi(s)=\nu \tag{4}$$

hold for any s, where $\phi(s)$ denotes the angle between tangent vector of $\widetilde{\gamma}(s)$ and $\frac{\partial}{\partial\widetilde{\theta}}|_{\widetilde{\gamma}(s)}$ (see Figure 1). The constant ν is called the Clairaut constant of $\widetilde{\gamma}$.

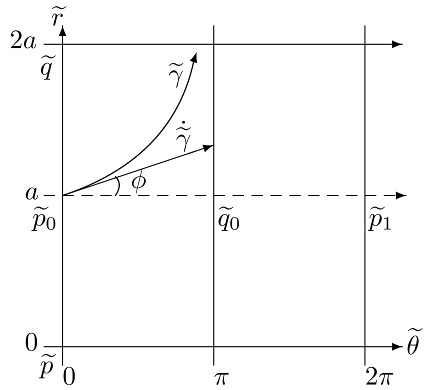

Figure 1. The angle ϕ between tangent vector of $\widetilde{\gamma}$ and $\frac{\partial}{\partial\widetilde{\theta}}|_{\widetilde{\gamma}(s)}$.

Remark 2.3. (1) Usually, a geodesic $\widetilde{\gamma}:[0,l]\to M$, $l>0$, is determined by its starting point $\widetilde{p}_0\in\widetilde{M}$ and initial velocity $v:=\widetilde{\gamma}(0)\in T_{\widetilde{p}_0}M$. However, from the Clairaut relation above one can see that this is equivalent to characterize geodesics by the initial point $\widetilde{p}_0$ and Clairaut constant ν. It is customary to use the notation $\widetilde{\gamma}_\nu^{\widetilde{p}_0}$.

(2) Let $\widetilde{p}_0\in\{\widetilde{r}=a\}$ be a point on the equator, and let $\widetilde{\gamma}_\nu^{\widetilde{p}_0}(s)=(\widetilde{r}(s),\widetilde{\theta}(s))$ be the $\widetilde{h}$-geodesic from $\widetilde{p}_0$ with Clairaut constant ν. Observe that

(a) if $\nu=0$, then $\phi=\pm\frac{\pi}{2}$ and $\widetilde{\gamma}_0^{\widetilde{p}_0}$ is a meridian, i.e. $\frac{d\widetilde{\theta}(s)}{ds}=0$;

(b) if $\nu=m(a)$, then $\phi=0$ and $\widetilde{\gamma}_{m(a)}^{\widetilde{p}_0}$ is a parallel, namely the equator in this case, i.e. $\frac{d\widetilde{r}(s)}{ds}=0$;

(c) if $\nu\in(0,m(a))$, then $\phi\in(-\frac{\pi}{2},\frac{\pi}{2})\backslash\{0\}$, and hence the geodesic $\widetilde{\gamma}_\nu^{\widetilde{p}_0}(s)=(\widetilde{r}(s),\widetilde{\theta}(s))$ is neither a meridian nor a parallel and $\frac{d\widetilde{\theta}(s)}{ds}>0$.

By combining the Clairaut relation with (2) it follows that the tangent vector along the unit $\widetilde{h}$-geodesic $\widetilde{\gamma}_\nu^{\widetilde{p}_0}$ has the components

$$\frac{d\widetilde{r}(s)}{ds}=\pm\sqrt{1-\frac{\nu^2}{m^2(\widetilde{r}(s))}},\quad \frac{d\widetilde{\theta}(s)}{ds}=\frac{\nu}{m^2(\widetilde{r}(s))}.$$

If we assume $\frac{d\widetilde{r}(s)}{ds}\neq 0$, for all s in some interval (s_1,s_2), i.e. our geodesic $\widetilde{\gamma}_\nu^{\widetilde{p}_0}$ is not tangent to a parallel, then it follows

$$\widetilde{\theta}(s_2)-\widetilde{\theta}(s_1)=\text{sign}\frac{d\widetilde{r}(s)}{ds}\int_{\widetilde{r}(s_1)}^{\widetilde{r}(s_2)}\frac{\nu}{m(\tau)\sqrt{m^2(\tau)-\nu^2}}d\tau, \tag{5}$$

where sign$\frac{d\widetilde{r}(s)}{ds}$ is the sign of the component $\frac{d\widetilde{r}(s)}{ds}$ of the tangent vector. Indeed, if the $\widetilde{h}$-geodesic is not a parallel, then the theorem of implicit functions allows us to write locally $\widetilde{\gamma}_\nu^{\widetilde{p}_0}$ as $\widetilde{\theta} = \widetilde{\theta}(\widetilde{r})$, for $\widetilde{r} \in (\widetilde{r}(s_1), \widetilde{r}(s_2))$, with the tangent vector

$$\frac{d\widetilde{\theta}}{d\widetilde{r}} = \text{sign}\frac{d\widetilde{r}(s)}{ds}\frac{\nu}{m(\widetilde{r})\sqrt{m^2(\widetilde{r}) - \nu^2}}. \tag{6}$$

Likewise, the $\widetilde{h}$-length of such an $\widetilde{\gamma}_\nu^{\widetilde{p}_0}|_{(s_1,s_2)}$ is given by

$$\mathcal{L}_{\widetilde{h}}(\widetilde{\gamma}_\nu^{\widetilde{p}_0}|_{(s_1,s_2)}) = \text{sign}\frac{d\widetilde{r}(s)}{ds}\int_{\widetilde{r}(s_1)}^{\widetilde{r}(s_2)} \frac{m(\tau)}{\sqrt{m^2(\tau) - \nu^2}}d\tau, \tag{7}$$

and taking into account the obvious identity

$$\frac{m(\tau)}{\sqrt{m^2(\tau) - \nu^2}} = \frac{\sqrt{m^2(\tau) - \nu^2}}{m(\tau)} + \frac{\nu^2}{m(\tau)\sqrt{m^2(\tau) - \nu^2}},$$

it follows

$$\mathcal{L}_{\widetilde{h}}(\widetilde{\gamma}_\nu^{\widetilde{p}_0}|_{(s_1,s_2)}) = \text{sign}\frac{d\widetilde{r}(s)}{ds}\int_{\widetilde{r}(s_1)}^{\widetilde{r}(s_2)} \frac{\sqrt{m^2(\tau) - \nu^2}}{m(\tau)}d\tau + \nu[\widetilde{\theta}(s_2) - \widetilde{\theta}(s_1)]. \tag{8}$$

Let us assume that $\widetilde{\gamma}_\nu^{\widetilde{p}_0}(s) = (\widetilde{r}(s), \widetilde{\theta}(s))$ is an $\widetilde{h}$-unit speed geodesic from $\widetilde{p}_0$, $\{\widetilde{r}(\widetilde{p}_0) = a\}$, $\{\widetilde{\theta}(\widetilde{p}_0) = 0\}$, such that $\nu \in (0, m(a))$, and $\frac{d\widetilde{r}(s)}{ds}|_{s=0} < 0$. From Clairaut relation it follows that $\widetilde{\gamma}_\nu^{\widetilde{p}_0}$ must be tangent to the parallel $\{\widetilde{r} = \xi(\nu)\}$ at a point $\widetilde{\gamma}_\nu^{\widetilde{p}_0}(t_1)$ and and return to the equator at $\widetilde{p}_1 = \widetilde{\gamma}_\nu^{\widetilde{p}_0}(t_0)$, where

$$t_0 = \min\{t > 0 \ : \ \widetilde{r}(t) = a\}.$$

Observe that here $\xi : (0, m(a)) \to \mathbb{R}$ is the inverse function of $m : [0, b) \to \mathbb{R}$, where b is the smallest value such that $m'|_{[0,b)]} > 0$.

On the universal covering, we can see that

$$\widetilde{\theta}(t_0) - \widetilde{\theta}(0) = 2(\widetilde{\theta}(t_0) - \widetilde{\theta}(t_1)) \tag{9}$$

By integrating (6) with condition (9), it follows (see [4]):

Lemma 2.2 (Half period function of Riemannian two-sphere of revolution)**.** *Let $\widetilde{\gamma}_\nu^{\widetilde{p}_0}$ be a $\widetilde{h}$-unit speed geodesic, where $\widetilde{p}_0 \in \{\widetilde{r} = a\}$ and $\nu \in (0, m(a))$, i.e. $\widetilde{p}_0$ is a point on equator and $\widetilde{\gamma}_\nu^{\widetilde{p}_0}$ is neither meridian nor parallel (equator) (see Figure 2). The $\widetilde{h}$-distance from $\widetilde{p}_0$ to $\widetilde{\gamma}_\nu^{\widetilde{p}_0}(t_0)$ is given by the $\widetilde{h}$-half period function*

$$\mathcal{H} : (0, m(a)) \to \mathbb{R}, \qquad \mathcal{H}(\nu) = 2\int_{\xi(\nu)}^{a} \frac{\nu}{m(\tau)\sqrt{m(\tau)^2 - \nu^2}}d\tau, \tag{10}$$

where $\widetilde{r} = \xi(\nu)$ is the parallel tangent to $\widetilde{\gamma}_\nu^{\widetilde{p}_0}(t_0)$.

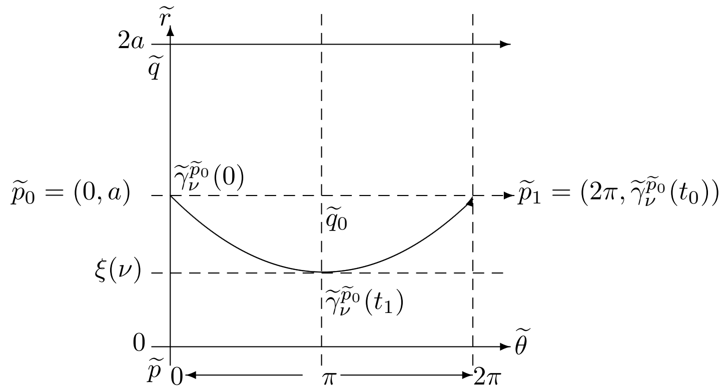

Figure 2. The Riemannian half period function $\mathcal{H}(\nu)$.

Let $\widetilde{p}_0 \in \widetilde{M}$ and $\widetilde{\beta}_\nu(s)$ and $\widetilde{\gamma}_\nu(s)$ for any $\nu \in (0, m(\widetilde{r}(\widetilde{p}_0)))$ denote the geodesic emanating from $\widetilde{p}_0$ with $(\widetilde{r} \circ \widetilde{\beta_\nu})'(0) \geq 0$ and $(\widetilde{r} \circ \widetilde{\gamma_\nu})'(0) \leq 0$.

Since both geodesics $\widetilde{\beta}_\nu(s)$ and $\widetilde{\gamma}_\nu(s)$ depend smoothly on $\nu \in (0, m(a))$ we obtain two geodesic variations such that all curves in the variation are geodesics.

We obtain the h-Jacobi fields $X_\nu(t)$ and $Y_\nu(t)$:

$$X_\nu(t) := \frac{\partial}{\partial \nu}(\widetilde{\beta}_\nu(t)), \quad Y_\nu(t) := \frac{\partial}{\partial \nu}(\widetilde{\gamma}_\nu(t)),$$

along $\widetilde{\beta}_\nu(t)$ and $\widetilde{\gamma}_\nu(t)$. We can see that $X_\nu(0) = Y_\nu(0) = 0$.

Let us denote by $\widetilde{p}_u$ the point of coordinates $(\widetilde{r}(\widetilde{p}_u), \widetilde{\theta}(\widetilde{p}_u)) = (u, 0)$, where $u \in (0, 2a)$ and $\nu \in (0, m(a))$. Similarly with the case $u = a$, discussed above, for any $\nu \in (0, m(u))$, we consider the $\widetilde{h}$-geodesic $\widetilde{\gamma}_\nu^u$ emanating from $\widetilde{p}_u$, with Clairaut constant ν and $(\widetilde{r} \circ \widetilde{\gamma}_\nu^u)'(0) \leq 0$.

The geodesic $\widetilde{\gamma}_\nu^u$ is tangent to the parallel $\{\widetilde{r} = \xi(u)\}$ at a point $\widetilde{\gamma}_\nu^u(s_0)$, will intersect the equator and then will be tangent to the parallel $\{\widetilde{r} = 2a - \xi(u)\}$ at a point $\widetilde{\gamma}_\nu^u(s_1)$. Clearly, the parameter values s_0 and s_1 are solutions of the equation $(\widetilde{r} \circ \widetilde{\gamma}_\nu^u)'(s) = 0$. Then it is known from the proof of Lemma 2.9 in [11], or Proposition 7.2.3 in [10], that the Jacobi vector field Y_ν along $\widetilde{\gamma}_\nu^u$ is given by

$$Y_\nu(s) = \frac{\partial \widetilde{\theta}}{\partial \nu}(\widetilde{r}(s), u, \nu) \left[-\nu \frac{m(\widetilde{r}(s))}{\sqrt{m^2(\widetilde{r}(s)) - \nu^2}} \left(\frac{\partial}{\partial \widetilde{r}} \right)_{\widetilde{\gamma}_\nu^u(s)} + \left(\frac{\partial}{\partial \widetilde{\theta}} \right)_{\widetilde{\gamma}_\nu^u(s)} \right]. \quad (11)$$

Pay attention to the fact that we are using here the parametrization $\widetilde{\theta} = \widetilde{\theta}(\widetilde{r}(s), u, \nu)$ explained above. Some straightforward computations show that $\widetilde{\theta}(\widetilde{r}, u, \nu)$

given by

$$\widetilde{\theta}(\widetilde{r},u,\nu)=\mathcal{H}(\nu)-\int_{\widetilde{r}}^{2a-u}\frac{\nu}{m(\tau)\sqrt{m^2(\tau)-\nu^2}}d\tau, \tag{12}$$

where $\mathcal{H}$ is the Riemannian half-period function in Lemma 2.2.

A point $q_0 := \widetilde{\gamma}_\nu(l)$, where $\nu \in (-m(\widetilde{r}(p_0)), m(\widetilde{r}(p_0)))$ and $l > 0$, is $\widetilde{h}$-*conjugate* to $\widetilde{p}_0$ along $\widetilde{\gamma}_\nu(t)$ if and only if $Y_\nu(l) = 0$, and taking into account that $\left(\frac{\partial}{\partial\widetilde{r}}\right)_{\widetilde{\gamma}^u_\nu(s)}, \left(\frac{\partial}{\partial\widetilde{\theta}}\right)_{\widetilde{\gamma}^u_\nu(s)}$ are linear independent vectors on $T_{\widetilde{\gamma}^u_\nu(s)}M$, one obtains the differential equation

$$\frac{\partial\widetilde{\theta}}{\partial\nu}(\widetilde{r},u,\nu)=0 \tag{13}$$

along $\widetilde{\gamma}^u_\nu(s)$.

It can be shown that this differential equation has a unique solution $\widetilde{r}(s_c,\nu,u)$ that is the $\widetilde{r}$ coordinate of the $\widetilde{h}$-first conjugate point of $\widetilde{p}_u$ on $\widetilde{\gamma}^u_\nu(s)$.

The $\widetilde{\theta}$ coordinate of the $\widetilde{h}$-first conjugate point is obtained by substitution $\widetilde{\theta}(s_c,u,\nu)=\widetilde{\theta}(\widetilde{r}(s_c,u,\nu),u,\nu)$.

Definition 2.1. Let $\gamma : [0,t_0] \to M$ be a minimal h-geodesic segment on a complete Riemannian manifold (M,h). The endpoint $\gamma(t_0)$ of the geodesic segment is called a *h-cut point* of $p := \gamma(0)$ along γ if any extended geodesic segment $\gamma^* : [0,t_1] \to M$ of γ, where $t_1 > t_0$, is not a minimizing arc joining p to $\gamma^*(t_1)$ anymore. The *h-cut locus* $\mathcal{C}^h_p$ of the point p is defined by the set of the cut points along all geodesics segments emanating from p.

The structure of the h-cut locus $\mathcal{C}^h_p$ of (M,h) was obtained in [11].

Theorem 2.3 ([11]). *Let $(M, dr^2 + m(r)^2 d\theta^2)$ be a 2-sphere of revolution with a pair of poles p,q and satisfying properties in Remark 2.1. Then the h-cut locus of a point $x \in M \setminus \{p,q\}$ with $\{\theta(x)=0\}$ is*

1. *the antipodal point $\mathcal{C}^h_x = (2a - r(x), \pi)$, when $G(x)$ is a positive constant;*
2. *a subarc of the opposite half meridian $\mathcal{C}^h_x \subset \{\theta = \pi\}$, when $G(x)$ is monotone non-increasing along meridian from the pole p to the point on $\{r = a\}$;*
3. *a subarc of the antipodal parallel $\{r = 2a - r(x)\}$, that is $\mathcal{C}^h_x = r^{-1}(2a - r(x)) \cap \theta^{-1}(\mathcal{H}(m(r(x))), 2\pi - \mathcal{H}(m(r(x))))$, when $G(x)$ is monotone non-decreasing along meridian from the pole p to the point on $\{r = a\}$.*

2.2. Randers rotational metrics. In a previous paper [7] we have constructed a Randers rotational metric on a surface of revolution homeomorphic to $\mathbb{R}^2$. We will construct a Randers rotational metric on a 2-sphere of revolution in a similar manner in the following.

Let (M, h) be the 2-sphere of revolution considered in the previous section. Observe that there exists a constant $\mu < \{\frac{1}{\max\{m(r)\}} : r \in [0, 2a]\}$, such that $\mu < \frac{1}{m(r)}$ for any $r \in [0, 2a]$.

Proposition 1. *If (M, h) is a surface of revolution and $W = \mu \cdot \frac{\partial}{\partial\theta}$ is a breeze on M blowing along parallels, then the Randers metric $(M, F = \alpha + \beta)$ obtained by the Zermelo's navigation process with data (h, W) is a Finsler metric on M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$, $\beta = b_i(x)y^i$ are defined by*

$$(a_{ij}) = \begin{pmatrix} \frac{1}{1-\mu^2 m^2} & 0 \\ 0 & \frac{m^2}{(1-\mu^2 m^2)^2} \end{pmatrix}, \ b_i = \begin{pmatrix} 0 \\ -\frac{\mu m^2}{1-\mu^2 m^2} \end{pmatrix}, \quad i, j = 1, 2. \tag{14}$$

Indeed, observe that due to our condition $\mu < \frac{1}{m(r)}$ for all $r \in [0, a]$, W in the canonical basis $(\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta})$ of $T_x M$, reads $W = (W^1, W^2) = (0, \mu)$, and hence $h(W, W) = b^2 = (\mu m)^2 < 1$, where $b^2 := a^{ij} b_i b_j$ is the Riemannian a-norm of the covariant vector $b = (b_1, b_2)$. This condition guarantees the strong convexity of the Randers metric $F = \alpha + \beta$ (see [2]).

It is trivial to see that W is a Killing vector field of (M, h), and taking into account that the flow of W is $\varphi(s; r(s), \theta(s)) = (r(s), \theta(s) + \mu \cdot s)$, we obtain the global characterization of F-geodesics.

Proposition 2. *Let $(M, F = \alpha + \beta)$ be the Randers rotational metric constructed from the navigation data (h, W), where (M, h) is a Riemannian 2-sphere of revolution, and $W = \mu \cdot \frac{\partial}{\partial\theta}$, $\mu < \{\frac{1}{\max\{m(r)\}} : r \in [0, 2a]\}$, is the breeze on M blowing along parallels, then the F-unit speed geodesics $\mathcal{P} : (-\epsilon, \epsilon) \to M$ are given by*

$$\mathcal{P}(s) = (r(s), \theta(s) + \mu s), \tag{15}$$

where $\gamma(s) = (r(s), \theta(s))$ is an h-unit speed geodesic.

Indeed, taking into account that Zermelo's navigation gives

$$h(\dot{\gamma}(s), \dot{\gamma}(s)) = 1 \text{ if and only if } F(\mathcal{P}(s), \dot{\mathcal{P}}(s)) = 1. \tag{16}$$

It follows that we can use the same arclength parameter s on both Riemannian and Randers geodesics, and since W is h-Killing vector field, the conclusion follows from [9], or can be verified by straightforward computation.

Corollary 1. The pair (M, F) is a forward complete Finsler surface of Randers type.

We recall from our previous work [7] the *Finsler version of Clairaut relation*. For an F-unit geodesic $\mathcal{P}(s) = \varphi(s, \gamma(s))$ obtained by deviating an h-geodesic $\gamma(s)$ with Clairaut constant ν by means of the W-flow φ, the following relation holds

$$\cos\psi(s) = \frac{\nu + \mu m^2(r(s))}{m(r(s))\sqrt{1 + 2\mu\nu + \mu^2 m^2(r(s))}}, \tag{17}$$

where $\psi(s)$ is the angle between the vectors $\dot{\mathcal{P}}(s)$ and $\frac{\partial}{\partial\theta}|_{\mathcal{P}(s)}$ (see Figure 3). We have proved this formula for a surface of revolution homeomorphic to $\mathbb{R}^2$ in [7], but the proof carries out on any kind of surface of revolution.

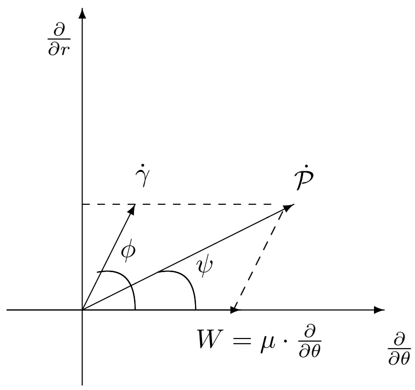

Figure 3. The angle ψ between $\dot{\mathcal{P}}$ and a parallel.

Remark 2.4. (1) We have seen in Remark 2.3 that an h-geodesic is characterised by its initial point and Clairaut constant (p_0, ν), that is equivalent to the usual initial conditions $(p_0, v) \in TM$. Since the corresponding Finsler geodesic is also determined by its starting point p_0 and initial velocity $y := v + \mu \cdot \frac{\partial}{\partial\theta} \in T_{p_0}M$, where μ is constant, we can see that this F-geodesic is uniquely determined by its initial point and Clairaut constant ν. We have to pay attention though that ν is the Clairaut constant of the original h-Riemannian geodesic.

(2) Let $p_0 \in \{r = a\}$ be a point on the equator, let $\gamma_\nu^{p_0}(s) = (r(s), \theta(s))$ be the h-geodesic from p_0 with Clairaut constant ν, and let $\mathcal{P}(s) = (r(s), \theta(s) + \mu s)$ be the corresponding F-geodesic. Observe that

(a) $\mathcal{P}$ is a meridian, that is $\psi = \pm\frac{\pi}{2}$, if and only if $\nu = -\mu m^2(a)$.
Indeed, $\psi = \pm\frac{\pi}{2}$ means $\cos\psi = 0$, and from Finslerian Clairaut relation (17) we obtain the desired value.

(b) $\mathcal{P}$ is a parallel, namely the equator in this case, that is $\psi = 0$, if and only if $\nu = m(a)$.

(c) if $\nu \in (-\mu m^2(a), 0) \cup (0, m(a))$, then $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\}$, and the geodesic $\mathcal{P}_\nu^{p_0}(s) = (r(s), \theta(s) + \mu s)$ is neither a meridian nor a parallel with $\frac{d\theta(s)}{ds} > 0$.

We have the following important result.

Lemma 2.4. *The flag curvature $\mathcal{K}$ of the Randers rotational metric $(M, F = \alpha + \beta)$ given by* (14) *lives on the base manifold M. Moreover, we have $\mathcal{K}(x, y) = \mathcal{K}(x) = G(x)$, for any $(x, y) \in TM$, where G is the Gaussian curvature of (M, h).*

Proof. Even though similar with the proof of Lemma 4.3 in [7] we sketch it here for the sake of completeness. We can see that our Randers rotational surface of revolution is Finsler-Einstein with Ricci scalar $Ric^{(F)} = \mathcal{K}(x)$, where $\mathcal{K}(x)$ is the sectional curvature of (M, F).

From [5], we know that (M, F) is Finsler-Einstein with Ricci scalar $Ric^{(F)} = \mathcal{K}(x)$ if and only if (M, h) is Einstein with Ricci scalar $Ric^{(h)} = \mathcal{K}(x)$ and W is Killing vector field for (M, h).

Next, we recall that any Riemannian surface (M, h) is an Einstein manifold with Ricci scalar $Ric^{(h)} = G(x)$ that completes the proof.

□

We turn now to the study of the conjugate points of F-geodesics.

Proposition 3. *Let $(M, F = \alpha + \beta)$ be a Randers rotational surface of revolution with navigation data (h, W), where $W = \mu \cdot \frac{\partial}{\partial \theta}$ is the breeze on M blowing along parallels $\mu < \frac{1}{m(r)}$ for any r. Suppose that $\gamma : [0, l] \to M$ is an h-geodesic and $\mathcal{P}(s) = \varphi(s, \gamma(s))$ is the corresponding F-geodesic, $t \in [0, l]$. Then $\mathcal{P}(l)$ is conjugate to $p = \mathcal{P}(0)$ along $\mathcal{P}$ (with respect to metric F) if and only if $\gamma(l)$ is conjugate to $p = \gamma(0)$ along γ (with respect to metric h).*

Proof. Let $\gamma : [0, l] \to M$ be an h-unit speed geodesic. Suppose $\Gamma(t, s) : (-\varepsilon, \varepsilon) \times [0, l] \to M$ be an h-geodesic variation of $\gamma(s) := \Gamma(0, s)$ with variation vector field

$$J(s) := \left. \frac{\partial \Gamma(t, s)}{\partial t} \right|_{t=0}.$$

Observe that this J is actually given by (11) for any $\nu \in (0, m(a))$. If we assume that $\gamma(l)$ is h-conjugate to $\gamma(0)$ it follows that

$$J(0) = J(l) = 0 \quad \text{and} \quad J(s) \neq 0, \quad s \in (0, l).$$

By using the wind W, blowing up on M, with the flow φ, by deviating γ we obtain the corresponding F-geodesic $\mathcal{P}(s) = \varphi(s, \gamma(s))$.
Let us consider the F-geodesic variation

$$\overline{\mathcal{P}}(t,s) = \varphi(v(t)s, \Gamma(t,s)),$$

where $v(t)$ is the constant h-speed of the geodesic variation $\Gamma(t,s)$.

We obtain the F-Jacobi field

$$\mathcal{J}(s)) = d\varphi \cdot J(s).$$

If we consider the flow $\varphi = (\varphi_1, \varphi_2) = (r, \theta + \mu s)$, it follows

$$d\varphi = \begin{pmatrix} \frac{\partial \varphi_1}{\partial r} & \frac{\partial \varphi_1}{\partial \theta} \\ \frac{\partial \varphi_2}{\partial r} & \frac{\partial \varphi_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{18}$$

that is the identity matrix.

We obtain that $\mathcal{J}$ vanishes if and only if J does, hence

$$\mathcal{J}(0) = \mathcal{J}(l) = 0 \quad \text{and} \quad \mathcal{J}(s) \neq 0, \quad s \in (0,l),$$

that is $\mathcal{P}(l)$ is conjugate to $p = \mathcal{P}(0)$ along $\mathcal{P}$, whenever $\gamma(l)$ is conjugate to $p = \gamma(0)$ along γ.

□

3. The proof of Theorem 1.1

Let $(M, F = \alpha+\beta)$ be a Randers rotational 2-sphere of revolution obtained from the navigation data (h, W), where h is the Riemannian metric of the 2-sphere of revolution M, and $W := \mu \cdot \frac{\partial}{\partial \theta}$ is the wind blowing along the parallels, where $\mu < \{\frac{1}{\max\{m(r)} : r \in [0, 2a]\}$.

Extending by analogy the definitions of poles from Riemannian setting (see Section 2.1), we obtain

Lemma 3.1. *The F-cut point of the pole p on (M,F) is the other pole q.*

Proof. Recall that a pair of poles p, q on (M,h) are invariant under the flow acting along parallels, i.e. $\varphi(s,p) = p$ and $\varphi(s,q) = q$, for any $s \in \mathbb{R}$.
Since the h-geodesics joining p and q are meridians and all of them have same h-length, it follows that the F-geodesics joining p and q are bending meridians with same F-length by (16). Hence we get that q is the cut point of p on (M,F).

□

Remark 3.1. In [7] the curve $\mathcal{P}(s) = \varphi(s, \gamma(s))$ is called a *twisted meridian*, where $\gamma(s)$ is a meridian. However since a pair of poles on 2-sphere of revolution M are invariant under the wind, we prefer to use the words *bending meridian* for $\mathcal{P}(s) = \varphi(s, \gamma(s))$, where $\gamma(s)$ is meridian on (M, h).

Corollary 2. The points p, q are a pair of poles on (M, F).

Remark 3.2. In the Finslerian universal covering manifold $(\widetilde{M}, \widetilde{F} = \widetilde{\alpha} + \widetilde{\beta})$, with the covering projection $\Pi : \widetilde{M} \to M \setminus \{p, q\}$ we use the notation $\widetilde{\mathcal{P}}^+(s) = (s, \widetilde{\gamma}(s))$ for an $\widetilde{F}$-geodesic obtained from $\widetilde{\gamma}(s)$ in the wind blowing direction and $\widetilde{\mathcal{P}}^-(s) = (-s, \widetilde{\gamma}(s))$ for an $\widetilde{F}$-geodesic advancing against the wind.

Lemma 3.2. *Let $\widetilde{\gamma}(s) = (\widetilde{r}(s), \widetilde{\theta}(s))$ be an $\widetilde{h}$-unit speed geodesic on $\widetilde{M}$ with Clairaut constant $\nu = m(a)$ joining the points $\widetilde{p}_0 := (a, 0)$ and $\widetilde{q}_0 := (a, \pi)$, i.e. $\widetilde{\gamma}(s)$ is an equator and $\widetilde{\theta}(\widetilde{p}_0) = 0$, $\widetilde{\theta}(\widetilde{q}_0) = \pi$ or $\widetilde{q}_0$ is antipodal point of $\widetilde{p}_0$ along $\widetilde{\gamma}$. Then the $\widetilde{F}$-unit speed geodesic $\widetilde{\mathcal{P}}^+(s) = \varphi(s, \widetilde{\gamma}(s))$ will join the point $\widetilde{p}_0 = \widetilde{\mathcal{P}}^+(0)$ with $\widetilde{q}_1 = \widetilde{\mathcal{P}}^+(\pi) = (a, \pi(1 + \mu))$. On the other hand, $\widetilde{\mathcal{P}}^-(s) = \varphi(-s, \widetilde{\gamma}(s))$ will join $\widetilde{p}_0 = \widetilde{\mathcal{P}}^-(0)$ to the point $\widetilde{q}_2 = \widetilde{\mathcal{P}}^-(\pi) = (a, \pi(1 - \mu))$.*

Remark 3.3. Observe that $\Pi(\widetilde{q}_1) = \Pi(\widetilde{q}_2) \in M$.

Proof. Let $\widetilde{p}_0$ be an arbitrary point on equator and $\widetilde{q}_0$ be an antipodal point to $\widetilde{p}_0$. Let $\widetilde{\gamma}(s) = (\widetilde{r}(s), \widetilde{\theta}(s))$, be an $\widetilde{h}$-unit speed geodesic joining $\widetilde{p}_0$ to $\widetilde{q}_0$, that is

$$\widetilde{p}_0 = \widetilde{\gamma}(0) = (a, 0), \quad \widetilde{q}_0 = \widetilde{\gamma}(\pi) = (a, \pi).$$

We recall that the wind is blowing along the parallels (see [7]). Since $\widetilde{d}_h(\widetilde{p}_0, \widetilde{q}_0) = \pi$, we know that the $\widetilde{F}$-unit speed geodesic $\widetilde{\mathcal{P}}^+(s) = \varphi(s, \widetilde{\gamma}(s))$ obtained by $\widetilde{\gamma}(s)$ is joining $\widetilde{p}_0$ to the point

$$\widetilde{\mathcal{P}}^+(\pi) = \varphi(\pi, \widetilde{\gamma}(\pi)) = (a, \pi(1 + \mu)),$$

therefore the point $\widetilde{q}_0$ will change the position to $\widetilde{q}_1 = (a, \pi(1 + \mu))$, hence $\widetilde{d}_F(\widetilde{p}_0, \widetilde{q}_1) = \pi$.

On the other hand $\widetilde{\mathcal{P}}^-(s)$ will join $\widetilde{p}_0$ to $\widetilde{q}_2 = (a, \pi(1 - \mu))$.

□

Remark 3.4. Let $\widetilde{\mathcal{P}}_\nu^{\widetilde{p}_0}(s) = (\widetilde{r}(s), \widetilde{\theta}(s) + \mu s)$ be an $\widetilde{F}$-unit speed geodesic emanating from $\widetilde{p}_0 \in \{r = a\}$ and $\nu \in (0, m(a))$. One can see that

$$(\widetilde{\mathcal{P}}_\nu^{\widetilde{p}_0})^2(r(b)) - (\widetilde{\mathcal{P}}_\nu^{\widetilde{p}_0})^2(r(a)) = \int_{r(a)}^{r(b)} \left(\frac{d\theta}{dr} + \mu \frac{ds}{dr} \right) dr.$$

We know that $\widetilde{\mathcal{P}}_\nu(s) := \widetilde{\mathcal{P}}_\nu^{\widetilde{p}_0}$ must be tangent to the parallel $\xi(\nu)$ at $\widetilde{\mathcal{P}}_\nu(t_1)$ and then return to the equator at $\widetilde{\mathcal{P}}_\nu(t_0)$ (see Figure 4). Then, by a similar computation as in the Riemannian case, the F-distance from $\widetilde{p}_0$ to $\widetilde{\mathcal{P}}_\nu(t_0)$ in the wind direction is given by the following *F-half period function*

$$\mathcal{H}_F^+(\nu) = \mathcal{H}(\nu) + \psi(\nu), \tag{19}$$

where $\psi(\nu) := 2\mu(a - \xi(\nu))$, and $\mathcal{H}$ is the h-half period function (see (10)). For the direction against the wind we obtain

$$\mathcal{H}_F^-(\nu) = \mathcal{H}(\nu) - \psi(\nu), \tag{20}$$

see [8] for computational details.

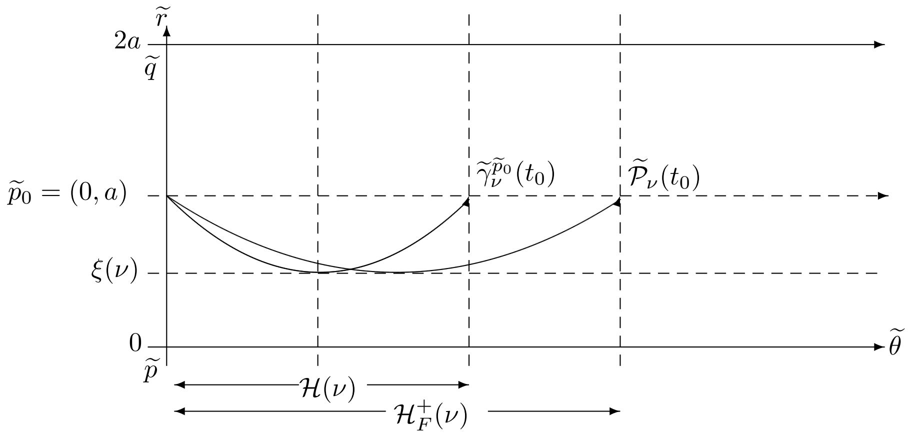

Figure 4. The h-half period function and F-half period function.

If $m'|_{[0,a)} \neq 0$ then we can assume $m' > 0$ on $(0, a)$, in this case, taking into account that $\xi(\nu) = (m|_{(0,a)})^{-1}$ observe that the function $\psi(\nu) = 2\mu(a - \xi(\nu))$ is decreasing function when $\xi(\nu) \in (0, a)$ and increasing when $\xi(\nu) \in (a, 2a)$.

Proposition 4. *Let $x \in M\backslash\{p, q\}$ be an arbitrary point. Then q_0 is an F-cut point to x on $\mathcal{P}$ if and only if $\hat{q}_0$ is h-cut point to x on γ, where $\mathcal{P}(s) = \varphi(s, \gamma(s))$ is the corresponding F-geodesic obtained from γ, $\mathcal{P}(0) = \gamma(0) = x$.*

Proof. Let $\gamma : [0, l] \to M$ be an h-unit minimizing geodesic from x to $\hat{q}_0 = \gamma(l)$ and $\hat{q}_0$ is a h-cut point of x, i.e. $\hat{q}_0 \in \mathcal{C}_x^h$.

Let $\mathcal{P}(s)$ be the F-unit geodesic obtain from $\gamma(s)$ and let $q_0 := \mathcal{P}(l)$. Assume q_0 is not F-cut point of x on $\mathcal{P}$, that is there exists a shorter minimizing F-geodesic $\mathcal{P}_0 : [0, l_0] \to M$ from $x = \mathcal{P}_0(0)$ to $q_0 = \mathcal{P}_0(l_0)$ where $d_F(x, q_0) := l_0 < l$.

From $\mathcal{P}_0$, we construct the corresponding h-geodesic

$$\gamma_0 : [0, l_0] \to M, \quad \gamma_0(s) = \varphi(-s, \mathcal{P}(s)),$$

where $\gamma_0(0) = \mathcal{P}_0(0) = x$ and $\gamma_0(l_0) = \varphi(-l_0, \mathcal{P}_0(l_0)) = \varphi(-l_0, q_0) = \varphi_{q_0}(_0)$.

Let us denote by ζ the curve

$$\zeta : [-l_0, -l] \to M, \quad \zeta(s) = \varphi(s, q_0),$$

(see Figure 5).

Then, from triangle inequality, we have

$$\mathcal{L}_h(\zeta) \geq \mathcal{L}_h(\gamma) - \mathcal{L}_h(\gamma_0). \tag{21}$$

On the other hand, we compute $\mathcal{L}_h(\zeta)$ as follows

$$\|\dot{\zeta}(s)\|_h^2 = \|W_{\varphi(s,q_0)}\|_h^2 = \|d\varphi(W_{q_0})\|_h^2 = \|W_{q_0}\|_h^2 = (\mu m(r(q_0)))^2 < 1,$$

where $d\varphi$ is identity from (18).

It follows that

$$\mathcal{L}_h(\zeta) = \int_{-l}^{-l_0} \|\dot{\zeta}(s)\|_h ds < \int_{-l}^{-l_0} ds = l - l_0 = \mathcal{L}_h(\gamma) - \mathcal{L}_h(\gamma_0). \tag{22}$$

From (21) and (22), we get a contradiction, hence q_0 is an F-cut point of x along $\mathcal{P}$.

□

Here is the proof of our Theorem 1.1

Proof of Theorem 1.1. Let $x \in M \setminus \{p, q\}$. We recall that the flow for navigation data is $\varphi(s; r(s), \theta(s)) = (r(s), \theta(s) + \mu s)$. Propositions 3 and 4 imply that F-cut locus is corresponding to the h-cut locus.

(1) The case when $\mathcal{K}$ is monotone non-increasing.

In Riemannian case the h-cut locus $\mathcal{C}_x^h$ of x, when the Gaussian curvature is monotone non-increasing, is a subarc of the opposite half meridian $\{\theta = \pi\}$, which are denote by $\tau_x|_{[c,2a-c]}$, where $\tau_x(c)$ is the h-first conjugate point of x along τ_x.

Therefore by taking into account Propositions 3 and 4 the F-cut locus is the following subarc of the opposite half bending meridian of x:

$$\mathcal{C}_x^F = \varphi(d(x, \tau(t)), \tau(t)), \quad t \in [c, 2a - c].$$

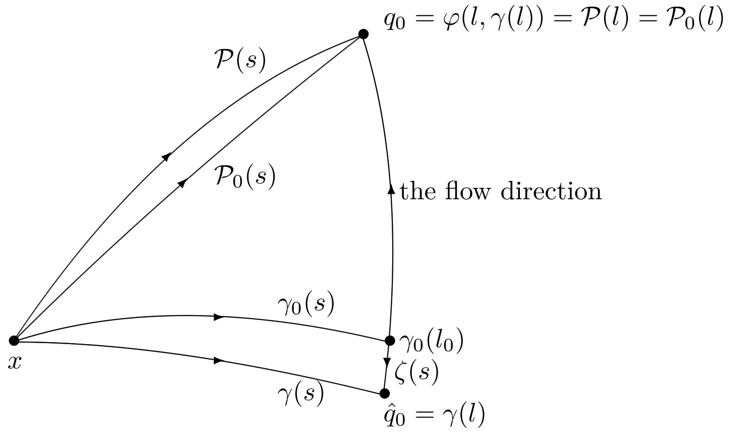

Figure 5. The proof of Proposition 4.

(2) The case when $\mathcal{K}$ is monotone non-decreasing.

In the Riemannian case (see [11]), if the Gaussian curvature G is monotone non-decreasing then the h-cut locus $\mathcal{C}_x^h$ of x is a subarc of the antipodal parallel $\{r = 2a - r(x)\}$, that is

$$\mathcal{C}_x^h = r^{-1}(2a - r(x)) \cap \theta^{-1}\{\mathcal{H}(m), 2\pi - \mathcal{H}(m)\},$$

where $\mathcal{H}$ is h-half period function defined in (10) and $m := m(r(x))$.
Next, let $\hat{q}_0$ be the h-first conjugate point of x on front side, i.e.

$$\hat{q}_0 = (2a - r(x), \mathcal{H}(m)),$$

and recall that our wind is blowing along the parallels, therefore the F-first conjugate point to x is

$$r^{-1}(2a - r(x)) \cap \theta^{-1}\{\mathcal{H}(m) + \psi(x)\},$$

where $\psi(x) = \mu \cdot d(x, \hat{q}_0)$. On the other hand the F-first conjugate point to x on the back side is

$$r^{-1}(2a - r(x)) \cap \theta^{-1}\{2\pi - (\mathcal{H}(m) - \psi(x))\},$$

hence we obtain

$$\mathcal{C}_x^F = r^{-1}(2a - r(x)) \cap \theta^{-1}\{\mathcal{H}(m) + \psi(x), 2\pi - (\mathcal{H}(m) - \psi(x))\}.$$

(3) The case when $\mathcal{K}$ is constant.

Let M be the round sphere of radius R. Recall that in the Riemannian case when $G = \frac{1}{R^2}$ is constant, the cut locus of any point $x \in M \setminus \{p, q\}$ is its antipodal point, i.e. $\mathcal{C}_x^h = \hat{q}_0 = (2a - r(x), \pi)$, where $\theta(x) = 0$. Since $d_h(x, \hat{q}_0)$ is equal to the half of circumference, i.e. $d_h(x, \hat{q}_0) = \pi \cdot R$, from Proposition 4 we obtain that the F-cut locus of x is

$$\begin{aligned}\mathcal{C}_x^F &= \varphi(d_h(x, \hat{q}_0) = \varphi(\pi R, \hat{q}_0) \\ &= (2a - r(x), \pi(1 + \mu R),\end{aligned}$$

where R is radius of round sphere.

(4) If the F-cut locus of $x \in M \setminus \{p, q\}$ is a single point, say $q \in M$, then $\hat{q} := \varphi(-l, q)$ is a h-cut point, where $d_F(x, q) = l$. Obviously $\hat{q}$ is the only h-cut point of h due to the Proposition 4.

Since the h-cut locus of $x \in M\{p, q\}$ is made of a single point $\hat{q}$, we know from [11] that $G = \frac{1}{R^2}$ must be a positive constant and hence (M, h) is actually the round sphere of radius R.

Taking now into account that (M, h) is a constant Gaussian curvature Riemannian surface and W a Killing field on (M, h), it follows from [6] that the corresponding Randers metric by the Zermelo navigation must be of constant flag curvature.

Remark 3.5. We recall that in order to obtain all F-geodesics $\mathcal{P}(s)$ with $\frac{d\mathcal{P}}{ds} > 0$ emanating from a point $p = \mathcal{P}(0)$, $\theta(p) = 0$, we need to consider the Riemannian geodesics γ_ν^p with Clairaut constant $\nu \in (-\mu \cdot m^2(a), m(a))$.

On the other hand, we have determined the F-cut point q of a point $p = \mathcal{P}(0)$ by mapping the h-cut point $\hat{q}$ of the same point $p = \gamma_\nu^p(0)$ along the corresponding h-geodesic $\gamma_\nu^p = \varphi(-s, \mathcal{P}(s))$, and using the structure of the h-cut locus for such h-geodesics determined in [11], for $\nu \in (0, m(a))$. Hence, strictly speaking we have determined only the F-cut locus of a point p along the F-geodesics corresponding to the h-geodesics having $\nu \in (0, m(a))$, having out the h-geodesics corresponding to $\nu \in (-\mu \cdot m^2(a), 0)$.

However, in the Riemannian case, due to the reversibility of h-geodesics and symmetry of the distance function d_h, it is easy to observe that the cut locus of a point $p \in M$ made of h-cut points along the h-geodesics γ_ν^p, $\nu \in (-\mu \cdot m^2(a), 0)$ is actually a subset of the h-cut locus of p made of h-cut points along all h-geodesics γ_ν^p, $\nu \in (0, m(a))$, hence we are not missing any points in $\mathcal{C}_p^F$.

4. The behaviour of cut locus when the cut locus of a point on equator is the subarc of the equator

4.1. The cut locus. From the previous section, we can see that, if the Gaussian curvature is monotone non-decreasing (increasing) then h-half period function is monotone non-increasing (decreasing), but the inverse is not true, i.e. if h-half period function is monotone non-increasing does not implies Gaussian curvature is monotone non-decreasing.

In this section, we will consider the more general case by extending the results in [4] to the Randers case.

Let (M, h) be the Riemannian 2-sphere of revolution considered in the previous sections, but in this section we do not assume the second condition in Remark 2.1, and let $W = \mu \cdot \frac{\partial}{\partial\theta}$ the wind blowing along the parallels, $\mu < \left\{\frac{1}{\max m(r)} : r \in [0, 2a]\right\}$. If we denote by $(M, F = \alpha + \beta)$ the Randers rotational constructed from navigation data (h, W) in Section 2.2, then we have

Proof of Theorem 1.2. If the cut locus of a point q on $\{r = a\}$ is a subarc of $\{r = a\}$, since the equator is invariant under the flow action, then by Proposition 4 it follows that the h-cut locus of the point q is a subarc of $\{r = a\}$. Hence, by using Theorem 3.5 in [4] it results that the h-cut locus of the point $\widetilde{q}$ is a subarc of the antipodal parallel $\{r = 2a - r(\widetilde{q})\}$.

Taking now into account that any parallel is flow-invariant by Proposition 4 it follows that the F-cut locus of $\widetilde{q}$ must be a subarc in the antipodal parallel $\{r = 2a - r(\widetilde{q})\}$. Clearly, the F-cut locus is obtained by rotating the h-cut locus by flow action on the parallel $\{r = 2a - r(\widetilde{q})\}$.

□

4.2. Examples.

Example 1. Let us consider the Riemannian 2-sphere of revolution $M_\lambda := (\mathbb{S}^2, h_\lambda)$, introduced in [4], where

$$h_\lambda = dr^2 + m_\lambda^2(r)d\theta^2 \tag{23}$$

and

$$m_\lambda(r) = \frac{\sqrt{\lambda+1} \cdot \sin r}{\sqrt{1 + \lambda \cos^2 r}}, \quad \lambda \geq 0. \tag{24}$$

It is clear that the function $r \mapsto m_\lambda(r)$ is symmetric with respect to the equator $\{r = \frac{\pi}{2}\}$, and a straightforward computation shows that the Gaussian curvature of $(\mathbb{S}^2, h_\lambda)$ is

$$G_\lambda(r) = \frac{(\lambda+1)(1-2\lambda\cos^2 r)}{(1+\lambda\cos^2 r)^2}.$$

For $\lambda = 0$ one obtains the the round sphere $\mathbb{S}^2$ with canonical Riemannian metric and for $\lambda \to \infty$ the metric

$$h_\infty = dr^2 + \tan^2 r d\theta^2,$$

that is singular along the equator $\{r = \frac{\pi}{2}\}$.

By taking the derivative of G_λ one can see that G_λ is not monotone along the meridian from a pole to the equator. Indeed, we have

$$G'_\lambda(r) = \frac{2\lambda(\lambda+1)\sin 2r}{(1+\lambda\cos^2 r)^3}(2-\lambda\cos^2 r).$$

On the other hand, more computations lead to

$$\mathcal{H}(\nu) = \pi - \frac{\lambda\pi\nu}{\sqrt{\lambda+1}\sqrt{(\lambda+1+\lambda\nu^2)}}, \quad \lambda > 0, \quad \nu \in (0, \sqrt{\lambda+1}),$$

where we use $\xi(\nu) = \nu^2$, and from here

$$\mathcal{H}'(\nu) = \frac{-\pi\lambda\sqrt{\lambda+1}}{(\lambda+1+\lambda\nu^2)^{\frac{3}{2}}}, \quad \lambda > 0,$$

moreover

$$\mathcal{H}''(\nu) = \frac{3\pi\lambda^2\nu\sqrt{\lambda+1}}{(\lambda+1+\lambda\nu^2)^{\frac{5}{2}}}, \quad \lambda > 0,$$

see [4] for detailed computations.

Then Lemma 3.3 in [4] implies that the h-cut locus of a point q on $\{r = \frac{\pi}{2}\}$ is a subarc in $\{r = \frac{\pi}{2}\}$ and hence by Theorem 3.5 in [4] it results that for this 2-sphere of revolution, the cut locus of any point $\widetilde{q} \in \mathbb{S}^2$, $r(\widetilde{q}) \in (0,\pi) \setminus \{\frac{\pi}{2}\}$ is a subarc of the antipodal parallel $\{r = 2a - r(\widetilde{q})\}$ (see [4] for details).

Let us consider the associated Randers rotational metric $F = \alpha + \beta$ obtained by Zermelo's navigation method ([7], [9]) from the navigation data (h_λ, W), where $W = \mu \cdot \frac{\partial}{\partial\theta}$, $\mu < \left\{\frac{1}{\max m_\lambda(r)} : r \in [0,\pi]\right\} = \frac{1}{m_\lambda(\frac{\pi}{2})} = \frac{1}{\sqrt{\lambda+1}}$. From Proposition 1 it follows

$$(a_{ij}) = \begin{pmatrix} \frac{1+\lambda\cos^2 r}{1+\lambda\cos^2 r - \mu^2(\lambda+1)\sin^2 r} & 0 \\ 0 & \frac{((\lambda+1)\sin^2 r)(1+\lambda\cos^2 r)}{(1+\lambda\cos^2 r-\mu^2(\lambda+1)\sin^2 r)^2} \end{pmatrix}, \; b_i = \begin{pmatrix} 0 \\ \frac{-\mu(\lambda+1)\sin^2 r}{1+\lambda\cos^2 r-\mu^2(\lambda+1)\sin^2 r} \end{pmatrix}.$$

For the sake of simplicity, let us consider

$$\mu = \frac{1}{2} \cdot \frac{1}{\sqrt{\lambda+1}}.$$

Then (19) implies

$$\mathcal{H}_F^+(\nu) = \pi - \frac{\lambda \pi \nu}{\sqrt{\lambda+1}\sqrt{\lambda+1+\lambda\nu^2}} + \frac{1}{\sqrt{\lambda+1}}\left(\frac{\pi}{2} - \nu^2\right), \quad \lambda > 0$$

and therefore

$$(\mathcal{H}_F^+)'(\nu) = \frac{-\lambda\pi\sqrt{\lambda+1}}{(\lambda+1+\lambda\nu^2)^{\frac{3}{2}}} - \frac{2\nu}{\sqrt{\lambda+1}}, \quad \lambda > 0,$$

and

$$(\mathcal{H}_F^+)''(\nu) = \frac{3\pi\lambda^2\nu\sqrt{\lambda+1}}{(\lambda+1+\lambda\nu^2)^{\frac{5}{2}}} - \frac{2}{\sqrt{\lambda+1}}, \quad \lambda > 0. \tag{25}$$

We observe that if $\mathcal{H}(\nu)$ is monotone non-increasing, then $\mathcal{H}_F^+(\nu)$ is decreasing on $\nu \in (0, \sqrt{\lambda+1})$.

Moreover, observe that the F-cut locus of any point q in $\{r = \frac{\pi}{2}\}$ is a subarc of $\{r = \frac{\pi}{2}\}$, as well as, that the F-cut locus of any point $\widetilde{q} \in M_\lambda$, such that $r(\widetilde{q}) \in (0, \pi) \setminus \{\frac{\pi}{2}\}$ is a subarc of the antipodal parallel $\{r = \pi - r(\widetilde{q})\}$. Indeed, taking into account the h-cut locus of the points q and $\widetilde{q}$, respectively and the fact that the equator and parallels are invariant under the flow, the F-cut locus can be obtained from Proposition 4.

Therefore, we obtain

Proposition 5. *Let $(\mathbb{S}^2, F_\lambda = \alpha + \beta)$ be the Randers rotational metric induced from the navigation data (h_λ, W) on $\mathbb{S}^2$ given by* (23), (24). *If $\lambda > 0$, then*

(i) *the cut locus of a point $q \in \mathbb{S}^2$ on the equator is a subarc of the equator.*

(ii) *the cut locus of a point $\widetilde{q} \in \mathbb{S}^2$, distinct from the pair of poles, is a subarc of the antipodal parallel $\{r = \pi - r(\widetilde{q})\}$.*

This is the generalization of the first part of Theorem 4.4 in [4] to the Randers case. Observe that due to Lemma 2.4 and the formula for G'_λ it follows that the Randers rotational metric constructed in this example is not of monotone flag curvature along meridian.

Remark 4.1. The Riemannian 2-sphere of revolution $(\mathbb{S}^2, h_\lambda)$ given by (23), (24), $\lambda \geq 0$ gives an example for Theorem 3.6 in [4] due to the fact that the h-half period function satisfies

$$\mathcal{H}'(\nu) < 0 < \mathcal{H}''(\nu) \text{ for any } \lambda > 0.$$

However, this type of relation is not true in the Randers case. Indeed, even though the h- and F-half period function $\mathcal{H}(\nu)$ and $\mathcal{H}_F^+(\nu)$ have the same monotonicity, respectively, they do not share the same convexity. Formula (25) implies that $(\mathcal{H}_F^+)''(\nu)$ is not always positive. For instance, numerical simulations show that $(\mathcal{H}_F^+)''(\nu) \leq 0$, for $\lambda \leq 1.5$, while for $\lambda > 1.5$ the function $(\mathcal{H}_F^+)''(\nu)$ can take both, positive and negative values, where $\nu \in (0, \sqrt{\lambda + 1})$, see Figure 6.

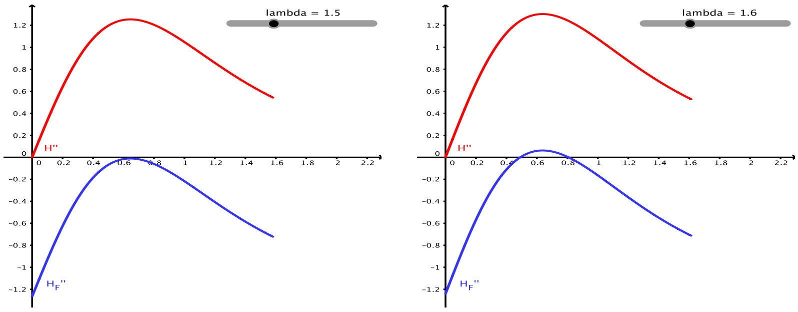

Figure 6. The graphs of $\mathcal{H}''(\nu)$ and $(\mathcal{H}_F^+)''(\nu)$, where $\lambda = 1.5$ and $\lambda = 1.6$ respectively.

Example 2. Another example is obtained from the Riemannian 2-sphere of revolution $(\mathbb{S}^2, h_\lambda)$ given in [3], where h_λ is given by (23) and

$$m_\lambda(r) = \frac{\sin r}{\sqrt{1 - \lambda \sin^2 r}}, \quad r \in [0, \pi], \quad \lambda \in (0, 1).$$

By straightforward computation one can see that

$$G_\lambda(r) = \frac{(1 - \lambda) - 2\lambda \cos^2 r}{(1 - \lambda \sin^2 r)^2}$$

and

$$G'_\lambda(r) = \frac{4\lambda \sin r \cos r (2(1 - \lambda) - \lambda \cos^2 r)}{(1 - \lambda \sin^2 r)^3}.$$

It is clear that for $\lambda \in (0,1)$, G'_λ vanishes at the pair of poles and the equator and the Gaussian curvature, G_λ is monotone for $\lambda \in (0, \frac{2}{3})$ with a local extremum of $\lambda = \frac{2}{3}$ (see [3] for details).

A similar computation with [4] shows that

$$\mathcal{H}(\nu) = \pi - \frac{\pi\nu\lambda}{\sqrt{1+\lambda\nu^2}}, \quad \nu \in \left(0, \frac{1}{\sqrt{1-\lambda}}\right)$$

and hence

$$\mathcal{H}'(\nu) = \frac{-\pi\lambda}{(1+\lambda\nu^2)^{\frac{3}{2}}}, \quad \mathcal{H}''(\nu) = \frac{3\pi\lambda^2\nu}{(1+\lambda\nu^2)^{\frac{5}{2}}}.$$

(compare with the form in [3] obtained in Hamiltonian formalism).

One can easily see that

$$\mathcal{H}'(\nu) < 0 < \mathcal{H}''(\nu) \tag{26}$$

and hence the h-cut locus of a point q on the equator is a subarc of the equator (Lemma 3.3 in [4]), and the h-cut locus of a point $\widetilde{q}$, distinct from equator of $(\mathbb{S}^2, h_\lambda)$ is a subarc of the opposite parallel (Lemma 3.4 in [4]).

If we consider again the Randers rotational metric $(\mathbb{S}^2, F_\lambda = \alpha + \beta)$ obtained by Zermelo's navigation method ([7], [9]) from navigation data (h_λ, W), $W = \mu \cdot \frac{\partial}{\partial\theta}$, $\mu < \sqrt{1-\lambda}$, then (19) gives

$$\mathcal{H}_F^+(\nu) = \pi - \frac{\pi\nu\lambda}{\sqrt{1+\lambda\nu^2}} + \sqrt{1-\lambda}\left(\frac{\pi}{2} - \nu^2\right), \quad \nu \in (0, \sqrt{1-\lambda}),$$

where we consider for simplicity $\mu = \frac{1}{2}\sqrt{1-\lambda}$, and hence

$$(\mathcal{H}_F^+)'(\nu) = \frac{-\pi\lambda}{(1+\lambda\nu^2)^{\frac{3}{2}}} - 2(\sqrt{1-\lambda})\nu,$$

$$(\mathcal{H}_F^+)''(\nu) = \frac{3\pi\lambda^2\nu}{(1+\lambda\nu^2)^{\frac{5}{2}}} - 2\sqrt{1-\lambda}.$$

By a similar argument with Example 1 it follows that Proposition 5 is true for this example as well.

Remark 4.2. When we consider the second derivative of the F-half period function $\mathcal{H}_F(\nu)$, we observe that, even through the Riemannian counter part satisfies (26), in the Finsler case we have $(\mathcal{H}_F^+)'(\nu) < 0$, however $(\mathcal{H}_F^+)''(\nu) \leq 0$, for $\lambda \leq 0.6$, while for $\lambda > 0.6$ the function $(\mathcal{H}_F^+)''(\nu)$ can take both, positive and negative values, where $\nu \in (0, \frac{1}{\sqrt{1-\lambda}})$, see Figure 7. that is, in this case also the convexity of the half period function in the Riemannian and Finsler case are quite different.

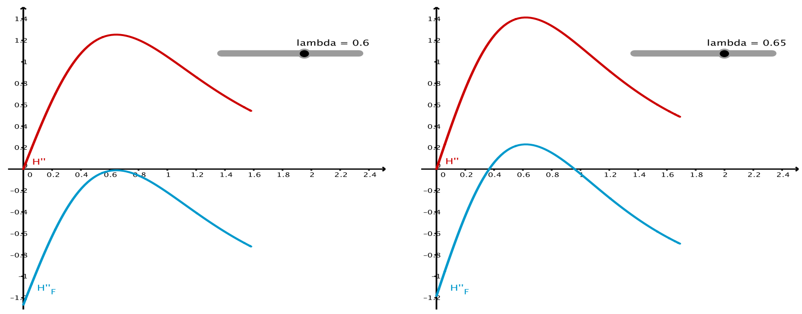

Figure 7. The graphs of $\mathcal{H}''(\nu)$ and $(\mathcal{H}_F^+)''(\nu)$, where $\lambda = 0.6$ and $\lambda = 0.65$ respectively.

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To
Professor Sorin V. Sabau

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23 April, 2018

Dear Professor Sabau,

It is my pleasure to inform you that your paper entitled

"The cut locus of a Randers rotational 2-sphere of revolution"
by Rattanasak Hama, Jaipong Kasemsuwan and Sorin V. Sabau

has been accepted for publication in the Publicationes Mathematicae Debrecen.

The paper will probably appear in the second half of 2018, in volume 93.

Sincerely yours

L. Tamassy

Prof. L. Tamassy
Managing Editor

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Academic Publications

1. with P. Chitsakul, S. V. Sabau, The Geometry of a Randers rotational surface, 2015, Publ. Math. Debrecen, 87/3-4 (2015), 473-502.

2. with J. Kasemsuwan, The Theory of Geodesics on Some Surface of Revolution, KMITL Science and Technology Journal, 17 (2017) no. 1, 42–47.

3. with J. Kasemsuwan, S. V. Sabau, The Cut locus of a Randers rotational 2-sphere of revolution, 2018, Publ. Math. Debrecen, volume 93.