

NUMERICAL SIMULATIONS OF HYDRODYNAMIC AND WATER  
POLLUTION MEASUREMENT MODELS IN OPENED-CLOSED  
RESERVOIR WITH CONTAMINANT REMOVAL MECHANISMS

KABOON THONGTHA

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE  
DEGREE OF DOCTOR OF PHILOSOPHY IN APPLIED MATHEMATICS  
DEPARTMENT OF MATHEMATICS  
FACULTY OF SCIENCE  
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG  
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<b>Thesis Title</b>	Numerical Simulations of Hydrodynamic and Water Pollution Measurement Models in Opened-Closed Reservoir with Contaminant Removal Mechanisms
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### **Abstract**

In this research, a special type opened-closed reservoir with the contaminant removal mechanisms within the reservoir is assigned. The water pollution measurement model consist the hydrodynamic and the transient advection-diffusion models. The hydrodynamic model is used to describe the water current in the opened-closed reservoir. The transient advection-diffusion equation with contaminant removal mechanism provides the water pollutant concentration. A new method in finding the analytical solution of the hydrodynamic model in two spatial spaces XY domain with time t by using the separation of variables method with mathematical induction is introduced. The numerical simulations are performed with respect to contaminant activated at the different rate of flows and assigned opened-closed gates of the reservoir. The models were solved numerically using finite difference methods via, Lax-Wendroff and the forward time centered space methods. The proposed numerical solutions give the water quality levels at assigned removal mechanism without using field monitoring measurements. The proposed numerical simulations give suitable area and period of time for the removal mechanism management. This model can be applied to the similar reservoir in another real problem of the mechanical treatment of wasted water control.

**Keywords** : Advection-Diffusion Equation, Contaminant Removal Mechanisms, Hydrodynamic Model, Opened-Closed Reservoir, Separation of Variables Method, Water Pollution Measurement Model.

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Miss Kaboon Thongtha

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## Lists of Symbols

$C$	Pollutant concentration in the water area
$C(x,y,t)$	Particle mass concentration at point $(x,y)$ and time $t$
$C_{i,j}^n$	Particle mass concentration at point $(x_i,y_j)$ and time $n\Delta t$
$D$	The pollutant dispersion coefficient
$g$	Acceleration due to gravity
$h$	The height of water surface
$H$	The total depth of the water area
$i, j, k$	Unit vector along the Rectangular coordinate axes
$L$	The length of the reservoir
$Q$	The rate of change of substance concentration due to a source
$R$	The water pollutant decaying rate or removal term
$t$	Time
$u$	Velocity component in $x$ -direction
$v$	Velocity component in $y$ -direction
$W$	Velocity component in $z$ -direction
$x$	$x$ -coordinate
$y$	$y$ -coordinate
$z$	$z$ -coordinate
$\mu$	The dynamic viscosity coefficient
$\kappa$	The rate of change between inner and outer water level through the eastern exit gate
$\zeta$	The water elevation
$\Omega$	Domain
$\mathfrak{R}$	Set of the real number
$\Delta x$	Mesh width in $x$ -direction
$\Delta y$	Mesh width in $y$ -direction
$\Delta t$	Time increment

## Lists of Technical Terms and Abbreviations

FDM	Finite Difference Method
PDE	Partial differential equation
ODE	Ordinary differential equation
Kg	Kilogram
l	Litre
m	Meter
m/s	Meter per second
mg	Milligram
mg/l	Milligram per Litre
sec, s	Second

# Chapter 1

## Introduction

### 1.1 The water pollution problem

Water is important for the life of all existence forms. In addition to household uses, water is important for agriculture, industry, fishery and tourism and many others. The quality of water used is also being deteriorated as it's far getting an increasing number of polluted. The increases in an industrial occupation and urbanization are the main reasons for the growth of pollutants. Water quality need to be protected and maintained for several makes use of, the essential ones being domestic water deliver, energy production, industry, agriculture, aquatic animals and natural world.

The water pollution can be defined because the contamination of streams, lakes, seas, underground water or oceans by way of substances, which can be dangerous for living beings. If the concentration of substances naturally found in water will increase then additionally the water is stated to be polluted. Control of pollution is important for the protection of the water environment and the maintenance of perfect quality in rivers, lakes, reservoir, seas, oceans and underground water.

The water pollution consists of the following parameters stated below reach past a specified concentration in water.

1) Physical parameters are color, odor, turbidity, flavor, temperature and electric conductivity represent the physical parameters and are right signs of contamination. For example, coloration and turbidity are seen evidences of polluted water and difference than normal taste additionally makes water not worthy for drinking.

2) Chemical parameters are the amount of carbonates, sulphates, chlorides, fluorides, nitrates, and metal ions. These chemical compounds form the total dissolved solids, found in water.

3) Biological parameters are included matter like algae, fungi, viruses, protozoa and bacteria. The life forms found in water are affected to a terrific volume by using

the presence of pollutants. The pollutants in water may cause a reduction in the population of both lower and higher plant and animal lives. Therefore, the biological parameters supply an indirect indication of the quantity of pollution in water.

Wastewater treatment is the removal of contaminants from the water in order to decrease the possibility of detrimental impacts on humans and the rest of the ecosystem. The “contaminants” is used here to refer to an undesirable constituent in the water or wastewater that may directly or indirectly affect human or environmental. Many contaminants, including a wide variety of organic compounds and metals, are toxic to humans and other organisms. For example, loading of nutrients to waterways can result in excessive growth of algae and unwanted vegetation, diminishing the recreational, economic and aesthetic values of lakes, bays, streams and reservoirs. Its treatment aims at reducing the contaminants to acceptable levels to make the water safe for discharge back into the environment. There are two wastewater treatment plants namely chemical or physical treatment plant, and biological wastewater treatment plant. Biological waste treatment plants use biological matter and bacteria to break down waste matter. Physical waste treatment plants use chemical reactions as well as physical processes to treat wastewater. Biological treatment systems are ideal for treating wastewater from households and business premises. Physical wastewater treatment plants are mostly used to treat wastewater from industries, factories and manufacturing firms. This is because most of the wastewater from these industries contains chemicals and other toxins that can largely harm the environment.

In this research, the mathematical models for water quality measurement which consist of the hydrodynamic model and the dispersion model, used to simulate water quality in a water flow systems, were considered. The first is a hydrodynamic model that provides the water current and the elevation of water in an opened-closed reservoir. The second is a dispersion model that gives the concentration of pollutant in an opened-closed reservoir with the contaminant removal mechanism. For numerical techniques, we used the Lax-Wendroff method to the system of the Hydrodynamic model and the forward in time central in space (FTCS) to the dispersion model. The results from the shallow water equation of the hydrodynamic model are the water flow velocity which are input data for advection-

diffusion reaction equation which provides the level of pollutant concentration field. Averaging the equation over the depth with bottom topography and discarding the term due to the Coriolis force, surface wind effect, and external forces, it follows that the two-dimensional shallow water and advection-diffusion-reaction equations are applicable.

## 1.2 Literature Reviews

The several numerical techniques for solving such models were available. In [1–3], they used the hydrodynamic model and the dispersion model with the finite element method to approximate the velocity of the water current in bay, estuaries, and open reservoir, respectively. In [4], the finite element method was used for solving the water pollution levels to the optimal control of the water treatment plants to achieve minimum cost. In [5], the method of the characteristic technique combined with Galerkin finite element method is used to solve the shallow water mass transport problems. In [22], used a weighted discretisation with the modified equivalent partial differential equation approach for solving one-dimension convection-diffusion equation. In [21], presents the mathematical modeling of the transport salinity, pollutants and suspended matter in shallow water involves the numerical solution of a convection-diffusion equation. The technique of upwind scheme of finite difference methods is described. In [27] explains open boundaries for three-dimensional primitive equation models, and received little attention. Many open boundary conditions used are based on the Somerfield radiation condition. In [8], the Crank-Nicolson method is used to solve the hydrodynamic model and the backward time central space (BTCS) for the dispersion model. In [9], the approximated solutions of the hydrodynamic model and advection-diffusion-reaction equation in a uniform reservoir are proposed. In [23], developed the three-dimensional transport models. They used an algorithm splitting the horizontal convection and diffusion by an implicit finite element method, treating the horizontal convection-diffusion explicitly and vertical convection-diffusion by an implicit finite element method. They also used a first-order upwind scheme for the convection terms of the convection-diffusion equations. In [24], gives a third-order upwind difference that is used for the convection terms of convection-diffusion

equation. In [25], used a third-order upwind scheme for the convective terms of shallow water momentum equations. In [26], used the finite element method to simulate the water pollution in the reservoir for the case of steady flow. They used the optimization to control the water pollution at least cost of waste water treatment process. In [6, 7], the numerical techniques are used to solve the nonuniform flow of stream water quality model with the advection-dispersion-reaction equations. In [10], a nondimensional form of the hydrodynamic model with variable coefficients using Lax-Wendroff method is presented. In [11], the Lax-Wendroff finite difference method is also proposed to approximate the water elevation and water flow velocity with a rectangular domain. In [12], the mathematical models and numerical methods for approximating water current and pollutant concentration level in Rama-nine reservoirs are presented. In [13], the Lax-Wendroff method for solving the dimensional form of shallow water equation in spherical model with Matlab program is proposed. In [14], an analytical solution to a hydrodynamic model in an open uniform reservoir with the specified tidal wave functions is proposed. In this research, the mathematical models for water quality measurement which consist of the hydrodynamic model and the dispersion model, used to simulate water quality in a water flow systems, were considered. The first is a hydrodynamic model that provides the water current and the elevation of water in an opened-closed reservoir. The second is a dispersion model that gives the concentration of pollutant in an opened-closed reservoir with the contaminant removal mechanism. For numerical techniques, we used the Lax-Wendroff method to the system of the hydrodynamic model and the forward in time central in space (FTCS) to the dispersion model. The results from the shallow water equation of the hydrodynamic model are the water flow velocity which are input data for advection-diffusion-reaction equation which provides the level of pollutant concentration field. Averaging the equation over the depth with anisotropic bottom topography and discarding the term due to the Coriolis force, surface wind effect, and external forces, it follows that the two-dimensional shallow water and advection-diffusion-reaction equations are applicable.

### 1.3 Objectives of the thesis

- 1) To study a simple form of a two-dimensional hydrodynamic model in an open reservoir.
- 2) To propose an analytical solution: the water elevation of a simple form of a two-dimensional hydrodynamic model in an open reservoir.
- 3) To study a general two-dimensional hydrodynamic model in opened-closed reservoir.
- 4) To propose numerical solutions such as the water elevation and the water flow velocity of a two-dimensional hydrodynamic model in opened-closed reservoir.
- 5) To propose numerical techniques for water pollution measurements in an opened-closed reservoir.
- 6) To propose numerical simulations for water pollution control in an opened-closed reservoir using contaminant removal mechanism.

### 1.4 Scope of the thesis

The scope of the thesis is restricted to the application of the analytical method and the numerical method to the water quality problem in opened-closed reservoir. In the first part, an open uniform reservoir is supplied by the outer water wave. In the former studies, the elevation of water wave (tidal elevation) can be found only by numerical approximations. The purpose of this research is to derive an analytical solution to a model. In the second part, the mathematical model of the approximated water quality in an opened-closed reservoir, measurement and control with of three contaminant removal mechanisms in reservoir is considered.

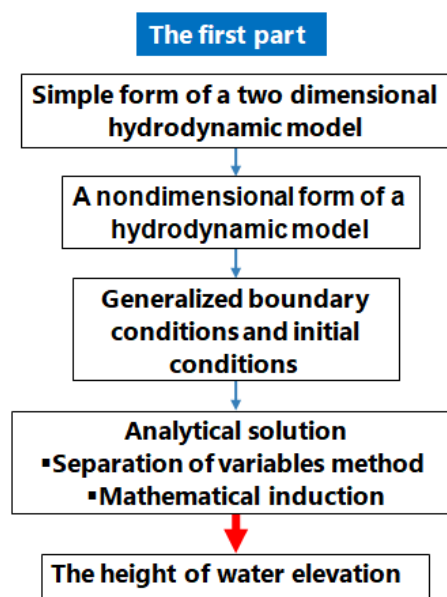
### 1.5 Steps in the thesis conducting

In the thesis describes the mathematical modeling of water pollution measurement and water quality control in reservoirs. We have to establish a simulation process by means of which water pollution levels can be reduced. The first part gives the detail of the basic knowledge of the flow analysis for the simple form of two-dimensional hydrodynamic model in open reservoir by defining the domain of problem. The simple form of hydrodynamic model is used to determine

the water wave flow. A nondimensional form of a two-dimensional hydrodynamic model with the generalized boundary condition  $g(x, t)$  and initial conditions for describing the elevation of water wave in an open uniform reservoir is proposed. The separation of variables method with mathematical induction is employed to find an analytical solution to the model.

The second part will propose the numerical method for solving a general two-dimensional hydrodynamic model and water pollution dispersion model with removal mechanism. Two mathematical models are used to simulate pollution in opened-closed reservoirs with varied current velocities. The first is a general two-dimensional hydrodynamic model that gives the velocity component and elevation of water. The second is the dispersion with removal mechanism model that gives the concentration of pollution. In the simulating processes, we used the Lax-Wendroff method to system of a general two-dimension hydrodynamic model and the forward in time and central in space technique to the dispersion with removal mechanism model.

Finally, the numerical simulations give a suitable area of zonal removal mechanism placement. The proposed simulations also give the overall and specified approximated water quality for each point and time when the exit gate is opened on the different periods of time. In addition, the proposed techniques can give a suitable period of time to open the exit gate to achieve a good agreement water quality by using contaminant removal mechanism.



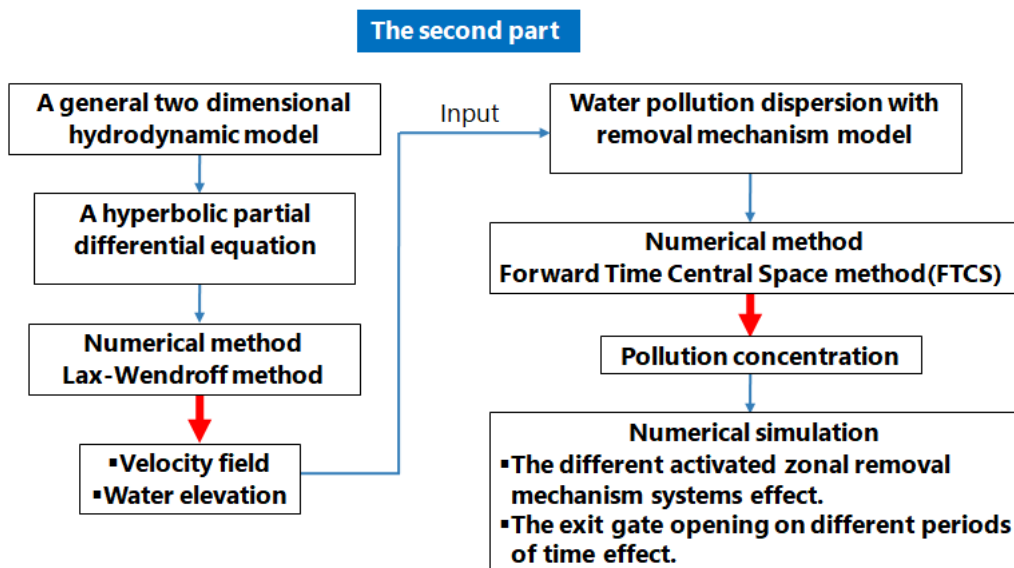


Figure 1.1 Steps of the numerical simulations of hydrodynamic and water pollution models in opened-closed reservoir with contaminant removal mechanism

## 1.6 Expected results

The expected results of the thesis are the analytical solution of the water elevation to a simple form of a two-dimensional hydrodynamic model in an open reservoir in the first part and the approximate solutions for model of water pollution assessment and control of water quality with removal mechanism system in an opened-closed reservoir in the second part.

## Chapter 2

# Mathematical Models for Water Quality Measurement

Field measurement and mathematical simulation are methods to detect the amount of the level of pollutants in water area. In water quality modeling for reservoir, the general governing equations used are the hydrodynamic model and the dispersion model. The two-dimensional shallow water equation and the advection-diffusion-reaction equation govern the first and the second models, respectively.

### 2.1 Reservoir

In this section, we will explain the description of the domain used in the research include opened-closed reservoir as shown in Figure 2.1. A reservoir is an artificial lake where water is saved. Most reservoirs are formed by constructing dams across rivers. The dam controls the quantity of water that flows out of the reservoir. People construct reservoirs because the quantity of water in a river varies over time. All through very rainy instances, the water in a river rises and sometimes overflows its banks, reservoirs assist manipulate flooding.

Reservoirs are usually found in areas of water scarcity or excess, or where there are agricultural or technological reasons to have a controlled water facility. Where water is scarce, for example, reservoirs are mainly used to conserve available water for use during those periods in which it is most needed for irrigation or drinking water supply. When excess water may be the problem, then a reservoir can be used for flood control to prevent downstream areas from being inundated during periods of upstream rainfall or snow-melt. Particular activities such as power generation, fish-farming, paddy-field management or general wet-land formation, for example, are also met by constructing reservoirs. By implication, they are also water bodies which are potentially subject to significant human control, in addition to any other impact. Reservoirs are, nonetheless, a considerable, frequently undervalued, water resource: approximately 25 percent of all waters flowing to the oceans have previously been impounded in reservoirs [32].

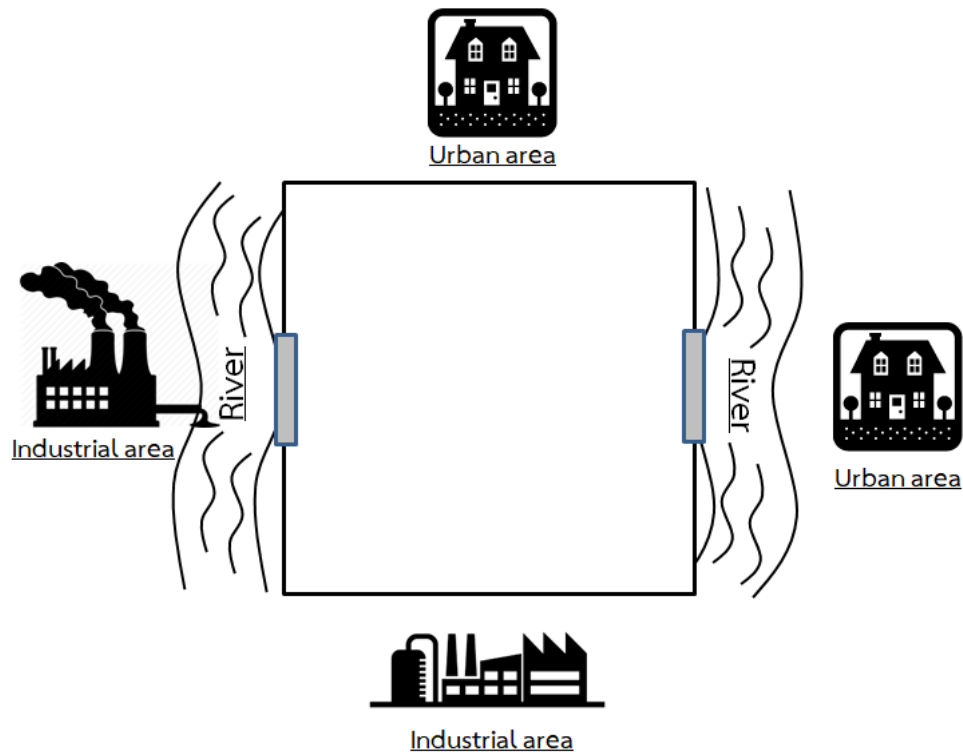


Figure 2.1 The two-dimensional water area.

### 2.1.1 Types of reservoirs

Types of reservoirs classified depending upon purpose served [33]

1) Storage or Conservation Reservoir : A storage or a conservation reservoir can retain such excess supplies during periods of peak flows, and can release them gradually during low flows as and when the need arises such as irrigation, power generation, recreation, domestic, industrial and municipal supply etc.

2) Flood control Reservoir : A flood control reservoir, generally called a flood-mitigation reservoir, stores a portions if the flood flows in such a way as to minimize the flood peaks at the areas to be protected downstream. A reservoir where in water is stored for a relatively brief period of time, past of it being retained until the stream can safely carry the ordinary flow plus the released water. Such reservoirs usually have outlets without control gates and are used for flood regulation.

3) Multipurpose Reservoir : A reservoir planned and constructed to serve not only one purpose but various purpose together such as irrigation, flood control, power generation, navigation, pollution abatement, domestic and industrial water supply, fish culture, recreation, etc.

4) Distribution Reservoir : A distribution reservoir connected with the conduits of a primary water supply; used to supply water to consumers according to fluctuations in demand over short time periods and serves for local storage in case of emergency such as a break in a main supply line failure of a pumping plant.

### **2.1.2 Fully closed water reservoir**

Fully closed water reservoir is a closed water storage system. Fully closed water reservoir is a solution for preventing algae growth and contamination of the water because external influences on the quality of the water are no longer possible. The plastic foil to be used is made depending on the location and the water quality.

### **2.1.3 Opened-closed reservoir**

An opened-closed reservoir or basin is any area of land where precipitation collects and drains off into a common inlet or outlet, such as into a river, bay, or other body of water. The opened-closed reservoir includes all the surface water from rain runoff, snowmelt, and nearby streams that there is water flow inlet towards the downstream outlet.

Opened-closed reservoirs connect into other basins or rivers at lower elevations in a hierarchical pattern, with smaller sub-drainage basins, which in turn drain into another common outlet.

Opened-closed reservoirs are important in environmental system. As water flows over the ground and along rivers it can carries nutrients, sediment, and water pollutants. Modern use of artificial fertilizers, containing nitrogen, phosphorus, and potassium, has affected the environmental system. The water pollutants are carried by the opened-closed reservoir to connected rivers, and may accumulate there, disturbing the natural mineral balance. This can cause eutrophication where plant growth is accelerated by the additional material. With the water body, they are transported towards the outlet of the reservoir, and can affect the ecological processes along the way as well as in the receiving water source.

## 2.2 A two-dimensional hydrodynamic model

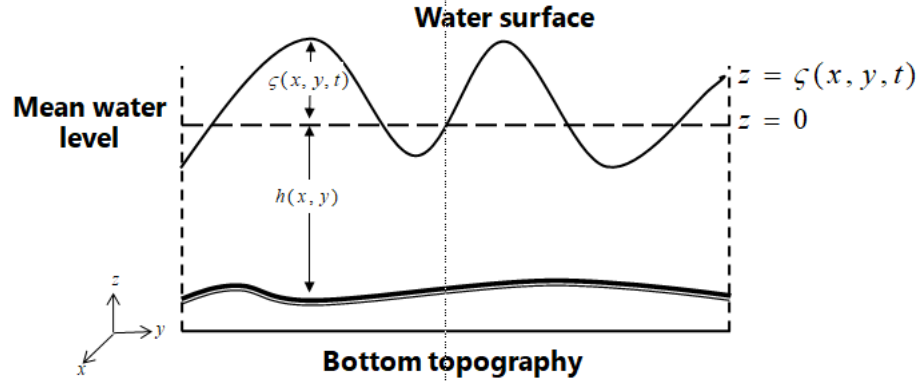


Figure 2.2 The shallow water system.

### 2.2.1 A general two-dimensional hydrodynamic model

The reservoir has a simple two-space dimension as shown in Figure 2.2. The two-dimensional unsteady water current into and out of the reservoir can be determined by using the system of shallow water equations as the conservation of mass and conservation of momentum. It is taken into account that the equations of the system of shallow water can be derived from depth-averaging Navier-Stokes equations in the vertical direction, neglecting the diffusion of momentum due to turbulence and discarding the terms expressing the effects of friction, surface wind, Coriolis factor, and shearing stresses. The continuity and momentum equation governs the hydrodynamic behavior of the reservoir [15]. The well known two-dimensional shallow water equations are [1, 2, 13, 16]

$$\frac{\partial(h+\zeta)}{\partial t} + \frac{\partial}{\partial x}(u(h+\zeta)) + \frac{\partial}{\partial y}(v(h+\zeta)) = 0, \quad (2.1)$$

$$\frac{\partial(u(h+\zeta))}{\partial t} + \frac{\partial(u^2(h+\zeta) + \frac{1}{2}g(h+\zeta)^2)}{\partial x} + \frac{\partial(uv(h+\zeta))}{\partial y} = 0, \quad (2.2)$$

$$\frac{\partial(v(h+\zeta))}{\partial t} + \frac{\partial(uv(h+\zeta))}{\partial x} + \frac{\partial(v^2(h+\zeta) + \frac{1}{2}g(h+\zeta)^2)}{\partial y} = 0, \quad (2.3)$$

where  $x$  is the longitudinal distance along the reservoir (m),  $y$  is the transverse distance along the reservoir (m),  $t$  is the time (s),  $h(x, y)$  is the depth measured from the mean water level to the reservoir bed (m),  $\zeta(x, y, t)$  is the elevation of water surface from the mean water level in reservoir (m),  $u(x, y, t)$  is velocity in x-direction (m/s),  $v(x, y, t)$  is velocity in y-direction (m/s),  $H(x, y, t) = h(x, y) + \zeta(x, y, t)$  is the elevation of water surface measured from the mean water level to the bed of the reservoir and  $g$  is gravitational constant ( $9.81 \text{ m/s}^2$ ).

Such independent variables  $x$ ,  $y$  and  $t$  make up the special dimensions and time. The dependent variables are the depth  $h$  with respect to the surface and the two-dimensional velocities  $u$  and  $v$ . The partial derivatives taken with respect to the same term  $(\partial t, \partial x, \partial y)$  are grouped into vectors and rewritten as a single hyperbolic partial differential [13, 15],

$$\frac{\partial H}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \quad (2.4)$$

where

$$H = \begin{pmatrix} h + \zeta \\ u(h + \zeta) \\ v(h + \zeta) \end{pmatrix},$$

$$U = \begin{pmatrix} u(h + \zeta) \\ u^2(h + \zeta) + \frac{1}{2}g(h + \zeta)^2 \\ uv(h + \zeta) \end{pmatrix},$$

$$V = \begin{pmatrix} v(h + \zeta) \\ uv(h + \zeta) \\ v^2(h + \zeta) + \frac{1}{2}g(h + \zeta)^2 \end{pmatrix}.$$

### 2.2.2 A simple form of two-dimensional hydrodynamic model

The governing equation of the hydrodynamic describes the behavior of the reservoir. We derive a general two-dimensional hydrodynamic model into a simple form of two-dimensional hydrodynamic model for describing water current and elevation. We make the usual assumption in the continuity and momentum balance; that is, we assume that the Coriolis, shearing stresses, and the surface wind are small

[7, 9, 10, 20] and  $h$  is a flat bottom topography [3]. We obtain the two-dimensional shallow water equations [2, 5] as follows

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}[(h + \zeta)u] + \frac{\partial}{\partial y}[(h + \zeta)v] = 0, \quad (2.5)$$

$$\frac{\partial u}{\partial t} + g \frac{\partial \zeta}{\partial x} = 0, \quad (2.6)$$

$$\frac{\partial v}{\partial t} + g \frac{\partial \zeta}{\partial y} = 0, \quad (2.7)$$

where  $h(x, y)$  is the depth measured from the mean water level to the bed of the reservoir,  $\zeta(x, y, t)$  is the elevation from the mean water level to the temporary water surface or the tidal elevation,  $g$  is the acceleration due to gravity, and  $u(x, y, t)$  and  $v(x, y, t)$  are the velocity components in  $x$  and  $y$  directions, respectively, for all  $(x, y, t) \in [0, l] \times [0, l] \times [0, T]$ . We assume that  $h$  is a constant and  $\zeta \ll h$ . Then Eqs. (2.5)-(2.7) lead to

$$\frac{\partial \zeta}{\partial t} + h \frac{\partial u}{\partial x} + h \frac{\partial v}{\partial y} = 0, \quad (2.8)$$

$$\frac{\partial u}{\partial t} + g \frac{\partial \zeta}{\partial x} = 0, \quad (2.9)$$

$$\frac{\partial v}{\partial t} + g \frac{\partial \zeta}{\partial y} = 0. \quad (2.10)$$

We will consider the equation in a dimensionless problem by letting  $U = u/\sqrt{gh}$ ,  $V = v/\sqrt{gh}$ ,  $X = x/l$ ,  $Y = y/l$ ,  $Z = \zeta/h$  and  $T = t\sqrt{gh}/l$ . Substituting them into Eqs. (2.8)-(2.10) leads to

$$\frac{\partial Z}{\partial T} + \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \quad (2.11)$$

$$\frac{\partial U}{\partial T} + \frac{\partial Z}{\partial X} = 0, \quad (2.12)$$

$$\frac{\partial V}{\partial T} + \frac{\partial Z}{\partial Y} = 0. \quad (2.13)$$

In order to solve Eqs. (2.11)-(2.13) in  $\Omega \times [0, T]$ , where  $\Omega = (0, 1) \times (0, 1)$ , for convenience using  $u, v, d$  and  $t$  for  $U, V, Z$  and  $T$ , respectively [5], we get

$$\frac{\partial d}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2.14)$$

$$\frac{\partial u}{\partial t} + \frac{\partial z}{\partial x} = 0, \quad (2.15)$$

$$\frac{\partial v}{\partial t} + \frac{\partial z}{\partial y} = 0. \quad (2.16)$$

Eqs. (2.14)-(2.16) can be written as a non-conservation form of equation, as a single equation,

$$\frac{\partial^2 d}{\partial t^2} = \frac{\partial^2 d}{\partial x^2} + \frac{\partial^2 d}{\partial y^2}, \quad (2.17)$$

subject to the initial conditions  $d(x, y, 0) = f_1(x, y)$  and  $\frac{\partial d}{\partial t} = f_2(x, y)$ . The boundary conditions are  $d = 0$  at  $\partial\Omega \setminus \{(x, y) \in \partial\Omega : y = 1\}$ ,  $d(x, 1, t) = g(x, t)$  and  $\frac{\partial d}{\partial y} = 0$  at the plane  $y = 1$ .

## 2.3 A two-dimensional dispersion model

### 2.3.1 Water pollutant dispersion model

Applying the distributed pollutant process, including the transportation and diffusion, we have the mass transfer equation. There is a representation simplified by averaging the equation over the depth, generating the advection-diffusion equation as follows [4]:

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right), \quad (2.18)$$

where  $C(x, y, t)$  is the depth averaged water pollutant concentration at the point  $(x, y)$  and at the time  $t$  ( $\text{kg/m}^3$ ),  $D$  is the pollutant dispersion coefficient ( $\text{m}^2/\text{s}$ ) and  $u(x, y, t)$ ,  $v(x, y, t)$  are the velocity component.

### 2.3.2 Water pollutant dispersion model with removal mechanism

The mechanisms of pollutant removal are introduced by decaying chemical reaction and absorptive reduction. A representation is modified by generating the advection-diffusion- reaction with sink term

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) - RC - Q(x, y, t), \quad (2.19)$$

where  $C(x, y, t)$  is the depth averaged water pollutant concentration at the point  $(x, y)$  and at the time  $t$  ( $\text{kg}/\text{m}^3$ ),  $D$  is the pollutant dispersion coefficient ( $\text{m}^2/\text{s}$ ),  $R \geq 0$  is the water pollutant decaying rate ( $\text{s}^{-1}$ ), and  $Q(x, y, t)$  is the decreasing rate of water pollutant concentration due to a water pollutant sink ( $\text{kg}/\text{m}^3 \text{ s}$ ). The initial condition  $C(x, y, 0) = c_0$  at  $t = 0$  and the boundary condition  $\frac{\partial C}{\partial n} = 0$  on  $\partial\Omega$ .

## Chapter 3

### Analytical Solution to a Simple form of Two-Dimensional Hydrodynamic Model in an Open Uniform Reservoir

A simple form of two-dimensional hydrodynamic model is used to determine the water wave flow. In this chapter, a nondimensional form of a two-dimensional hydrodynamic model in an open uniform reservoir with generalized boundary condition  $g(x, t)$  and initial conditions for describing the elevation of water wave in an open uniform reservoir is proposed. The separation of variables method with mathematical induction is employed to find an analytical solution to the model. In the final part of this section, an example of flow calculations in an open uniform reservoir is also demonstrated.

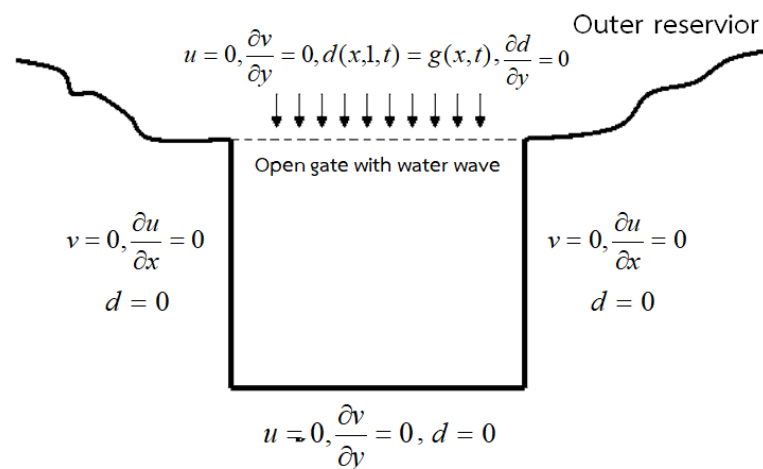


Figure 3.1 An open uniform reservoir is connected to an outer reservoir.

#### 3.1 The initial and boundary conditions of a simple form of two-dimensional hydrodynamic model in an open uniform reservoir

An open uniform reservoir is supplied by the outer water wave as shown in Figure 3.1. The initial conditions of Eqs. (2.14)-(2.16) are assumed to be  $u = 0, v = 0, d = f_1(x, y)$  and  $\frac{\partial d}{\partial t} = f_2(x, y)$ . The boundary conditions of the model in

an open uniform reservoir are assumed to be  $v = 0, \frac{\partial u}{\partial x} = 0$  at the planes  $x = 0$  and  $x = 1$ .  $u = 0, \frac{\partial v}{\partial y} = 0$  at the planes  $y = 0$ .  $u = 0, \frac{\partial v}{\partial y} = 0, d(x,1,t) = g(x,t)$  and  $\frac{\partial d}{\partial y} = 0$  at the planes  $y = 1$ .  $d = 0$  on  $\partial\Omega \setminus \{(x,y) \in \partial\Omega : y = 1\}$ .

### 3.2 An analytical solution

From Eq. (2.17) is to derive a formal solution of the following problem

$$d_t = d_{xx} + d_{yy} \quad \text{for} \quad x, y \in (0,1), t > 0. \quad (3.1)$$

Hence, in addition to the differential equation (3.1) we specify initial and boundary conditions of the form

$$d(x, y, 0) = f_1(x, y), \quad x, y \in (0,1), t > 0, \quad (3.2)$$

$$d_t(x, y, 0) = f_2(x, y), \quad x, y \in (0,1), t > 0, \quad (3.3)$$

$$d(x, 1, t) = g(x, t), \quad x \in (0,1), t > 0, \quad (3.4)$$

$$d_t(x, 1, t) = 0, \quad x \in (0,1), t > 0, \quad (3.5)$$

$$d(x, 0, t) = 0, \quad x \in (0,1), t > 0, \quad (3.6)$$

$$d(0, y, t) = 0, \quad y \in (0,1), t > 0, \quad (3.7)$$

$$d(1, y, t) = 0, \quad y \in (0,1), t > 0. \quad (3.8)$$

It is readily verified that the general solution can be written by the separation of variables technique

$$d(x, y, t) = X(x)Y(y)T(t), \quad (3.9)$$

where  $X$  is a function of  $x$  alone,  $Y$  is a function of  $y$  alone and  $T$  is a function of  $t$  alone.

Then, its partial derivatives can also be expressed simply by

$$d(x, y, t) = X(x)Y(y)T(t), \quad (3.11)$$

$$d_x(x, y, t) = X'(x)Y(y)T(t), \quad (3.12)$$

$$d_{xx}(x, y, t) = X''(x)Y(y)T(t), \quad (3.13)$$

$$d_y(x, y, t) = X(x)Y'(y)T(t), \quad (3.14)$$

$$d_{yy}(x, y, t) = X(x)Y''(y)T(t), \quad (3.15)$$

$$d_t(x, y, t) = X(x)Y(y)T'(t), \quad (3.16)$$

$$d_{tt}(x, y, t) = X(x)Y(y)T''(t). \quad (3.17)$$

Hence, a single equation of a dimensionless problem Eq. (3.1) can be rewritten as

$$X(x)Y(y)T''(t) = X''(x)Y(y)T(t) + X(x)Y''(y)T(t). \quad (3.18)$$

Dividing the both sides by  $X(x)Y(y)T(t)$ , we have separated the variables

$$\frac{X(x)Y(y)T''(t)}{X(x)Y(y)T(t)} = \frac{X''(x)Y(y)T(t)}{X(x)Y(y)T(t)} + \frac{X(x)Y''(y)T(t)}{X(x)Y(y)T(t)}, \quad (3.19)$$

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}, \quad (3.20)$$

or

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} - \frac{Y''(y)}{Y(y)}. \quad (3.21)$$

Since  $X$  only depends on  $x$ ,  $Y$  only depends on  $y$  and  $T$  only depends on  $t$ .

Let Eq. (3.21) is equal to the constant  $-\lambda^2$  to be negative real number; it is called the constant of separation.

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} - \frac{Y''(y)}{Y(y)} = -\lambda^2, \quad (3.22)$$

The first equality Eq. (3.22) becomes

$$\frac{X''(x)}{X(x)} = -\lambda^2, \quad (3.23)$$

$$X''(x) = -\lambda^2 X(x). \quad (3.24)$$

Hence, this gives the ordinary differential equation of function  $X(x)$

$$X''(x) + \lambda^2 X(x) = 0. \quad (3.25)$$

The second equality Eq. (3.22) can be rewritten as

$$\frac{T''(t)}{T(t)} - \frac{Y''(y)}{Y(y)} = -\lambda^2, \quad (3.26)$$

so 
$$\frac{Y''(y)}{Y(y)} = \frac{T''(t)}{T(t)} + \lambda^2. \quad (3.27)$$

Let Eq. (3.27) is equal to the constant, say  $-\mu^2$  to be negative real number; it is called the constant of separation. Thus,

$$\frac{Y''(y)}{Y(y)} = \frac{T''(t)}{T(t)} + \lambda^2 = -\mu^2. \quad (3.28)$$

The first equality Eq. (3.28) becomes

$$\begin{aligned} \frac{Y''(y)}{Y(y)} &= -\mu^2, \quad (3.29) \\ Y''(y) &= -\mu^2 Y(y). \end{aligned}$$

Hence, this gives the ordinary differential equation of function  $Y(y)$

$$Y''(y) + \mu^2 Y(y) = 0. \quad (3.30)$$

The second equality Eq. (3.28) can be rewritten as

$$\frac{T''(t)}{T(t)} + \lambda^2 = -\mu^2, \quad (3.31)$$

$$\frac{T''(t)}{T(t)} = -(\lambda^2 + \mu^2), \quad (3.32)$$

$$T''(t) = -(\lambda^2 + \mu^2)T(t). \quad (3.33)$$

Hence, this gives the ordinary differential equation of function  $T(t)$

$$T''(t) + (\lambda^2 + \mu^2)T(t) = 0. \quad (3.34)$$

Hence, the functions  $X(x)$ ,  $Y(y)$  and  $T(t)$  are eigenfunctions of the problem Eq. (3.1), with corresponding eigenvalues  $\lambda$  and  $\mu$ .

We then have the general solutions of the three ODEs Eqs. (3.25), (3.30) (3.34) are

$$X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x), \quad (3.35)$$

$$Y(y) = C_3 \cos(\mu y) + C_4 \sin(\mu y), \quad (3.36)$$

$$T(t) = C_5 \cos(\sqrt{\lambda^2 + \mu^2} t) + C_6 \sin(\sqrt{\lambda^2 + \mu^2} t), \quad (3.37)$$

where  $C_1, C_2, C_3, C_4, C_5, C_6$  are arbitrary constants.

The boundary conditions can be rewritten as

$$d(0, y, t) = 0 \rightarrow X(0) = 0, \quad (3.38)$$

$$d(1, y, t) = 0 \rightarrow X(1) = 0, \quad (3.39)$$

$$d(x, 1, t) = g(x, t) \rightarrow Y(1) = g(x, t), \quad (3.40)$$

$$d_y(x, 1, t) = 0 \rightarrow Y'(1) = 0, \quad (3.41)$$

$$d(x, 0, t) = 0 \rightarrow Y(0) = 0. \quad (3.42)$$

We first consider Eq. (3.35) and the boundary conditions from Eqs. (3.38-3.39).

According to the boundary condition  $X(0) = 0$ , we get  $C_1 = 0$ . Then

$$X(x) = C_2 \sin(\lambda x). \quad (3.43)$$

Now using the boundary condition  $X(1) = 0$ , we then have

$$C_2 \sin(\lambda) = 0. \quad (3.44)$$

Thus,  $C_2 = 0$  and  $X(x) = 0$ , we have seen this cannot be a solution.

Hence  $C_2 \neq 0$  and  $\sin(\lambda) = 0$ ,

we conclude that

$$\lambda = n\pi \text{ for } n = 1, 2, 3, \dots, \quad (3.45)$$

with corresponding eigenfunctions

$$X_n(x) = C_2 \sin(n\pi x), \text{ for } n = 1, 2, 3, \dots \quad (3.46)$$

We consider Eq. (3.36) and the boundary conditions from Eqs. (3.40-2.42). According to the boundary condition  $Y(0) = 0$ , we get  $C_3 = 0$ . Then

$$Y(y) = C_4 \sin(\mu y). \quad (3.47)$$

Using the boundary condition  $Y'(1) = 0$ , we then have

$$C_4 \mu \cos(\mu) = 0. \quad (3.48)$$

Thus,  $C_4 = 0$  and  $Y(y) = 0$ , we have seen this cannot be a solution.

Hence  $C_4 \neq 0$ ,  $\mu \neq 0$  and  $\cos(\mu) = 0$ , we conclude that

$$\mu = (2m - 1) \frac{\pi}{2} \text{ for } m = 1, 2, 3, \dots \quad (3.49)$$

Thus, 
$$Y_m(y) = C_4 \sin\left((2m - 1) \frac{\pi}{2} y\right) \text{ for } m = 1, 2, 3, \dots \quad (3.50)$$

Now using the boundary condition  $Y(1) = g(x, t)$ , we have

$$C_4 \sin\left((2m - 1) \frac{\pi}{2}\right) = g(x, t), \quad (3.51)$$

$$C_4 = \frac{g(x, t)}{\sin\left((2m - 1) \frac{\pi}{2}\right)}, \quad (3.52)$$

$$C_4 = \frac{g(x, t)}{(-1)^{m+1}}, \quad (3.53)$$

$$\text{so} \quad C_4 = (-1)^{-m-1} g(x, t). \quad (3.54)$$

We get

$$Y_m(y) = (-1)^{-m-1} g(x, t) \sin\left((2m-1)\frac{\pi}{2}y\right) \text{ for } m = 1, 2, 3, \dots \quad (3.55)$$

We consider Eq. (3.37), since  $\lambda = n\pi$  and  $\mu = (2m-1)\frac{\pi}{2}$ , we then have

$$T_{mn}(t) = C_5 \cos\left(\sqrt{(n\pi)^2 + \left((2m-1)\frac{\pi}{2}\right)^2}t\right) + C_6 \sin\left(\sqrt{(n\pi)^2 + \left((2m-1)\frac{\pi}{2}\right)^2}t\right), \quad (3.56)$$

$$T_{mn}(t) = C_5 \cos\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) + C_6 \sin\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right). \quad (3.57)$$

According to Eqs. (3.46), (3.55) and (3.57), the solution of our problem can be written in the form

$$d_{mn}(x, y, t) = X_n(x)Y_m(y)T_{mn}(t), \quad (3.58)$$

$$\begin{aligned} &= \sin(n\pi x)(-1)^{-m-1} g(x, t) \sin\left((2m-1)\frac{\pi}{2}y\right) \\ &\quad \times \left[ A_{mn} \cos\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) + B_{mn} \sin\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right], \end{aligned} \quad (3.59)$$

satisfy the differential equation and the boundary values prescribed by the initial-boundary value problem, where  $A_{mn} = C_2C_5$  and  $B_{mn} = C_2C_6$  are arbitrary constants and for all  $m, n = 1, 2, 3, \dots$ . By using the superposition principle, Eq. (3.40) becomes

$$d(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} d_{mn}(x, y, t), \quad (3.60)\#$$

$$\begin{aligned} d(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(n\pi x)(-1)^{-m-1} g(x, t) \sin\left((2m-1)\frac{\pi}{2}y\right) \\ &\quad \times \left[ A_{mn} \cos\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) + B_{mn} \sin\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right], \end{aligned} \quad (3.61)$$

where  $A_{mn}$  and  $B_{mn}$  are fixed by the value of  $d(x, y, 0) = f_1(x, y)$  and  $\frac{\partial d}{\partial t}(x, y, 0) = f_2(x, y)$ , respectively.

### 3.3 Application of a simple form of two-dimensional hydrodynamic model in an open uniform reservoir

The equations (2.14)-(2.16) can be written to be the single variable equation (2.17). The elevation of water along the northern boundary is distributed by  $\zeta(x, l, t) = 2 \sin(\pi x) \cos(t)$ . The open boundary condition is assumed to be  $d(x, l, t) = g(x, t) = 2 \sin(\pi x) \cos(t)$ . It is readily verified that the general solution is the form

$$d(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \cos(t) \sin\left((2m-1)\frac{\pi}{2}y\right) \times \left[ A_{mn} \cos\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) + B_{mn} \sin\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right]. \quad (3.62)$$

The initial conditions are assumed to be

$$d(x, y, 0) = f_1(x, y) = 2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right), \quad (3.63)$$

$$\frac{\partial d}{\partial t}(x, y, 0) = f_2(x, y) = -\sin(2\pi x) \sin\left(\frac{\pi}{2}y\right). \quad (3.64)$$

Substituting Eq. (3.63) into Eq. (3.62) we get

$$d(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \cos(0) \sin\left((2m-1)\frac{\pi}{2}y\right) \times [A_{mn} \cos(0) + B_{mn} \sin(0)] \quad (3.65)$$

Furthermore, the solution in Eq. (3.65) satisfy the initial condition Eq. (3.63) becomes

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2}y\right) = 2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right). \quad (3.66)$$

Then, it follows that Eq. (3.62) is given by

$$d(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(n\pi x) (-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2}y\right) \\ \times \left[ A_{mn} 2 \cos(t) \cos\left(\pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) + B_{mn} 2 \cos(t) \sin\left(\pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right], \quad (3.67)$$

$$d(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(n\pi x) (-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2}y\right) \\ \times \left[ A_{mn} \cos\left(t + \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) + A_{mn} \cos\left(t - \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right. \\ \left. + B_{mn} \sin\left(t + \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) - B_{mn} \sin\left(t - \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right], \quad (3.68)$$

$$d(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(n\pi x) (-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2}y\right) \\ \times \left[ A_{mn} \cos\left(t \left(1 + \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right)\right) + A_{mn} \cos\left(t \left(1 - \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right)\right) \right. \\ \left. + B_{mn} \sin\left(t \left(1 + \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right)\right) - B_{mn} \sin\left(t \left(1 - \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right)\right) \right], \quad (3.69)$$

considering Eq. (3.69), we evaluate  $\frac{\partial d}{\partial t}(x, y, t)$  as follows

$$\frac{\partial d}{\partial t}(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(n\pi x) (-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2}y\right)$$

$$\begin{aligned}
& \times \left[ -A_{mn} \sin \left( t \left( 1 + \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \right) \left( 1 + \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \right. \\
& - A_{mn} \sin \left( t \left( 1 - \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \right) \left( 1 - \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \\
& + B_{mn} \cos \left( t \left( 1 + \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \right) \left( 1 + \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \\
& \left. - B_{mn} \cos \left( t \left( 1 - \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \right) \left( 1 - \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \right]. \quad (3.70)
\end{aligned}$$

Substituting Eq. (3.64) into Eq. (3.70) we get

$$\begin{aligned}
\frac{\partial d}{\partial t}(x, y, 0) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(n\pi x) (-1)^{-m-1} \sin(\pi x) \sin \left( (2m-1) \frac{\pi}{2} y \right) \\
& \times \left[ -A_{mn} \sin(0) \left( 1 + \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) - A_{mn} \sin(0) \left( 1 - \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \right. \\
& \left. + B_{mn} \cos(0) \left( 1 + \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) - B_{mn} \cos(0) \left( 1 - \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \right], \quad (3.71)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial d}{\partial t}(x, y, 0) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(n\pi x) (-1)^{-m-1} \sin(\pi x) \sin \left( (2m-1) \frac{\pi}{2} y \right) \\
& \times B_{mn} \left[ 2\pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right], \quad (3.72)
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial d}{\partial t}(x, y, 0) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \left( \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \\
& \times \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin \left( (2m-1) \frac{\pi}{2} y \right). \quad (3.73)\#
\end{aligned}$$

Furthermore, the solution in Eq. (3.73) satisfy the initial condition Eq. (3.64) becomes

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \left( \pi \sqrt{n^2 + \left( \frac{2m-1}{2} \right)^2} \right) \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin\left( (2m-1) \frac{\pi}{2} y \right) \\ = -\sin(2\pi x) \sin\left( \frac{\pi}{2} y \right). \end{aligned} \quad (3.74)$$

We will consider the coefficients  $A_{mn}$  by comparing coefficients and using the mathematical induction on the finite sum of Eq. (3.66) as follows

$$2 \sin(\pi x) \sin\left( \frac{\pi}{2} y \right) \approx \sum_{m=1}^M \sum_{n=1}^N A_{mn} \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin\left( (2m-1) \frac{\pi}{2} y \right). \quad (3.75)$$

By comparing their coefficients in Eq. (3.75), we can see that if  $M=1$  and  $N=1$ ,

$$2 \sin(\pi x) \sin\left( \frac{\pi}{2} y \right) \approx \sum_{m=1}^1 \sum_{n=1}^1 A_{mn} \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin\left( (2m-1) \frac{\pi}{2} y \right), \quad (3.76)$$

$$= A_{11} \sin(\pi x) 2(-1)^{-2} \sin(\pi x) \sin\left( \frac{\pi}{2} y \right), \quad (3.77)$$

$$= A_{11} 2 \sin^2(\pi x) \sin\left( \frac{\pi}{2} y \right), \quad (3.78)$$

we get that the coefficients in Eq. (3.78) are

$$A_{11} = \frac{1}{\sin(\pi x)}, \quad (3.79)$$

where  $\sin(\pi x) \neq 0$ . Assume that  $A_{11} = \frac{1}{\sin(\pi x)}$ , where  $M, N = k$  and  $\sin(\pi x) \neq 0$ .

if  $M = k + 1$  and  $N = k + 1$ , then

$$\begin{aligned}
2 \sin(\pi x) \sin\left(\frac{\pi}{2} y\right) &= \sum_{m=1}^{k+1} \sum_{n=1}^{k+1} A_{mn} \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2} y\right), \quad (3.80) \\
&= A_{11} \sin(\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2} y\right) + A_{12} \sin(2\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2} y\right) \\
&\quad + \dots + A_{1k} \sin(k\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2} y\right) + A_{1(k+1)} \sin((k+1)\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2} y\right) \\
&\quad + A_{21} \sin(\pi x) 2(-1) \sin(\pi x) \sin\left(\frac{3\pi}{2} y\right) + A_{22} \sin(2\pi x) 2(-1) \sin(\pi x) \sin\left(\frac{3\pi}{2} y\right) \\
&\quad + \dots + A_{2k} \sin(k\pi x) 2(-1) \sin(\pi x) \sin\left(\frac{3\pi}{2} y\right) \\
&\quad + A_{2(k+1)} \sin((k+1)\pi x) 2(-1) \sin(\pi x) \sin\left(\frac{3\pi}{2} y\right) \\
&\quad + \dots + A_{(k+1)1} \sin(\pi x) 2(-1)^{-k-2} \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2} y\right) \\
&\quad + A_{(k+1)2} \sin(2\pi x) 2(-1)^{-k-2} \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2} y\right) \\
&\quad + \dots + A_{(k+1)k} \sin(k\pi x) 2(-1)^{-k-2} \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2} y\right) \\
&\quad + A_{(k+1)(k+1)} \sin((k+1)\pi x) 2(-1)^{-k-2} \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2} y\right). \quad (3.81)
\end{aligned}$$

Since  $\sin(\pi x) \neq 0$  and  $\sin\left((2m-1)\frac{\pi}{2} y\right) \neq 0$  for all  $x, y \in (0, 1)$ , substituting Eq. (3.79) into Eq. (3.81) and comparing the coefficients in Eq. (3.81), we can see that  $A_{12}, A_{13}, \dots, A_{1(k+1)} = 0, A_{21}, A_{22}, \dots, A_{2(k+1)} = 0, \dots, A_{(k+1)1}, A_{(k+1)2}, \dots, A_{(k+1)(k+1)} = 0$ , except  $A_{11} \neq 0$ . we get

$$A_{11} = \frac{1}{\sin(\pi x)}, \quad (3.82)$$

Thus, Eq. (3.75) holds for  $M = k + 1$  and  $N = k + 1$ , the proof of the induction step is completed.

Similarly, we also consider the coefficients  $B_{mn}$  by comparing coefficients and the mathematical induction on the finite sum of Eq. (3.74) as follows

$$\begin{aligned}
-\sin(2\pi x) \sin\left(\frac{\pi}{2} y\right) &\approx \sum_{m=1}^M \sum_{n=1}^N B_{mn} \left( \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2} \right) \\
&\quad \times \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2} y\right). \quad (3.83)
\end{aligned}$$

If  $M=1$  and  $N=2$ , then

$$\begin{aligned}
-\sin(2\pi x) \sin\left(\frac{\pi}{2} y\right) &\approx \sum_{m=1}^1 \sum_{n=1}^2 B_{mn} \left( \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2} \right) \\
&\quad \times \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2} y\right), \quad (3.84)
\end{aligned}$$

$$\begin{aligned}
&= B_{11} \left( \pi \sqrt{1^2 + \left(\frac{1}{2}\right)^2} \right) \sin(\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2} y\right) \\
&\quad + B_{12} \left( \pi \sqrt{2^2 + \left(\frac{1}{2}\right)^2} \right) \sin(2\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2} y\right). \quad (3.85)
\end{aligned}$$

By comparing their coefficients in Eq. (3.85), we get

$$B_{12} = -\frac{1}{2\pi\sqrt{4.25} \sin(\pi x)}, \quad (3.86)$$

where  $B_{11} = 0$  and  $\sin(\pi x) \neq 0$ . Assume that  $B_{12} = -\frac{1}{2\pi\sqrt{4.25} \sin(\pi x)}$ , where  $M, N = k$  and  $\sin(\pi x) \neq 0$ . If  $M = k+1$  and  $N = k+1$ , then

$$\begin{aligned}
-\sin(2\pi x) \sin\left(\frac{\pi}{2} y\right) &= \sum_{m=1}^{k+1} \sum_{n=1}^{k+1} B_{mn} \left( \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2} \right) \\
&\quad \times \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2} y\right) \quad (3.87)
\end{aligned}$$

$$\begin{aligned}
& -\sin(2\pi x) \sin\left(\frac{\pi}{2} y\right) = B_{11}(\pi\sqrt{1.25})\sin(\pi x)2\sin(\pi x) \sin\left(\frac{\pi}{2} y\right) \\
& + B_{12}(\pi\sqrt{4.25})\sin(4\pi x)2\sin(\pi x) \sin\left(\frac{\pi}{2} y\right) \\
& + \dots + B_{1k}(\pi\sqrt{k^2 + 0.25})\sin(k\pi x)2\sin(\pi x) \sin\left(\frac{\pi}{2} y\right) \\
& + \dots + B_{1(k+1)}(\pi\sqrt{(k+1)^2 + 0.25})\sin((k+1)\pi x)2\sin(\pi x) \sin\left(\frac{\pi}{2} y\right) \\
& + B_{21}(\pi\sqrt{1 + 2.25})\sin(\pi x)2(-1)\sin(\pi x) \sin\left(\frac{3\pi}{2} y\right) \\
& + B_{22}(\pi\sqrt{2^2 + 2.25})\sin(2\pi x)2(-1)\sin(\pi x) \sin\left(\frac{3\pi}{2} y\right) \\
& + \dots + B_{2k}(\pi\sqrt{k^2 + 2.25})\sin(k\pi x)2(-1)\sin(\pi x) \sin\left(\frac{3\pi}{2} y\right) \\
& + B_{2(k+1)}(\pi\sqrt{(k+1)^2 + 2.25})\sin((k+1)\pi x)2(-1)\sin(\pi x) \sin\left(\frac{3\pi}{2} y\right) \\
& + \dots + B_{(k+1)1} \left( \pi\sqrt{1^2 + \left(\frac{2(k+1)-1}{2}\right)^2} \right) \sin(\pi x)2(-1)^{-k-2} \\
& \quad \times \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2} y\right) \\
& + \dots + B_{(k+1)2} \left( \pi\sqrt{2^2 + \left(\frac{2(k+1)-1}{2}\right)^2} \right) \sin(2\pi x)2(-1)^{-k-2} \\
& \quad \times \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2} y\right) \\
& + \dots + B_{(k+1)k} \left( \pi\sqrt{k^2 + \left(\frac{2(k+1)-1}{2}\right)^2} \right) \sin(k\pi x)2(-1)^{-k-2} \\
& \quad \times \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2} y\right) \\
& + B_{(k+1)(k+1)} \left( \pi\sqrt{(k+1)^2 + \left(\frac{2(k+1)-1}{2}\right)^2} \right) \sin((k+1)\pi x)2(-1)^{-k-2} \\
& \quad \times \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2} y\right). \tag{3.88}
\end{aligned}$$

Since  $\sin(\pi x) \neq 0$  and  $\sin\left((2m-1)\frac{\pi}{2} y\right) \neq 0$  for all  $x, y \in (0,1)$ , substituting Eq. (3.86) into Eq. (3.88) and comparing the coefficients in Eq. (3.88), we can see that

$B_{11}, B_{13}, B_{14}, \dots, B_{1(k+1)} = 0, B_{21}, B_{22}, \dots, B_{2(k+1)} = 0, \dots, B_{(k+1)1}, B_{(k+1)2}, \dots, B_{(k+1)(k+1)} = 0,$   
except  $B_{12} \neq 0$ . We get

$$B_{12} = -\frac{1}{2\pi\sqrt{4.25}\sin(\pi x)}. \quad (3.89)$$

Thus, Eq. (3.83) holds for  $M = k + 1$  and  $N = k + 1$ , the proof of the induction step is completed.

Therefore, an analytical solution  $d(x, y, t)$  of Eq. (3.62) with given initial and boundary conditions becomes

$$d(x, y, t) = 2\sin(\pi x)\cos(t)\sin\left(\frac{\pi}{2}y\right)\cos(\pi\sqrt{1.25}) \\ - \frac{1}{\pi\sqrt{4.25}}\sin(2\pi x)\cos(t)\sin\left(\frac{\pi}{2}y\right)\sin(\pi\sqrt{4.25}), \quad (3.90)$$

for all  $(x, y, t) \in \Omega \times [0, T]$ .

### 3.4 Water elevation evaluation in an open uniform reservoir

We consider an open uniform reservoir with dimension  $3.2 \times 3.2$  km ( $l = 3.2$ ) km and the constant depth  $h = 0$  m, as shown in Figure 3.2. The reservoir is meshed with 6,400 grids points with  $\Delta x = \Delta y = 40$  m and taking time interval at  $\Delta t = 2.5$  sec. Initially, the water in the reservoir is assumed to be motionless  $u = 0, v = 0$  and the water elevation is specified by  $\zeta(x, y, 0) = 2\sin(\pi x)\sin\left(\frac{\pi}{2}y\right)$  and  $\frac{\partial \zeta}{\partial t}(x, y, 0) = -\sin(2\pi x)\sin\left(\frac{\pi}{2}y\right)$ . The elevation of water along the northern boundary is distributed by  $\zeta(x, l, t) = 2\sin(\pi x)\cos(t)$ . The open boundary condition is assumed to be  $d(x, l, t) = g(x, t) = 2\sin(\pi x)\cos(t)$ .

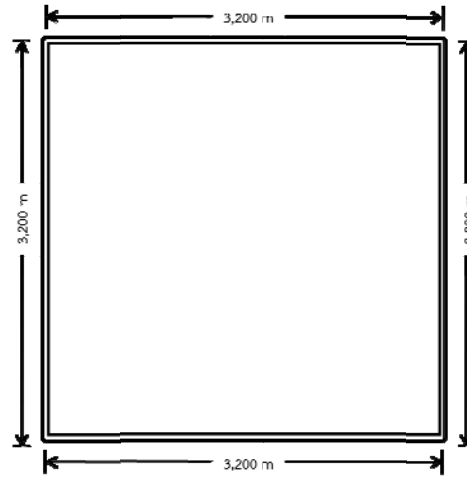


Figure 3.2 Rectangular reservoir.

The solution of water wave elevations (tidal elevation) by using Eq. (3.90) is shown in Table 3.1.

Table 3.1 Water elevation  $\zeta(x, y, t)$  (m) at  $t = 2$  hrs 50 min

$y \setminus x$	320	640	960	1,280	1,600	1,920	2,240	2,560	2,880
320	0.1729	0.2620	0.3174	0.3352	0.3177	0.2716	0.2055	0.1272	0.0430
640	0.2835	0.4295	0.5203	0.5495	0.5209	0.4453	0.3368	0.2085	0.0704
960	0.3871	0.5864	0.7104	0.7503	0.7112	0.6080	0.4599	0.2847	0.0962
1,280	0.4811	0.7289	0.8830	0.9326	0.8840	0.7557	0.5716	0.3538	0.1195
1,600	0.5633	0.8534	1.0338	1.0919	1.0350	0.8848	0.6692	0.4143	0.1400
1,920	0.6316	0.9570	1.1592	1.2243	1.1605	0.9922	0.7504	0.4646	0.1570
2,240	0.6844	1.0369	1.2561	1.3266	1.2575	1.0751	0.8131	0.5034	0.1701
2,560	0.7203	1.0913	1.3220	1.3963	1.3235	1.1315	0.8558	0.5298	0.1790
2,880	0.7385	1.1189	1.3554	1.4315	1.3569	1.1600	0.8774	0.5432	0.1835

The analytical solution  $d(x, y, t)$  at  $x = y = \frac{1}{2}$  of the system Eqs. (2.14) - (2.16) with given initial and boundary conditions,

$$\begin{aligned}
 d\left(\frac{1}{2}, \frac{1}{2}, t\right) &= 2 \sin\left(\frac{\pi}{2}\right) \cos(t) \sin\left(\frac{\pi}{4}\right) \cos\left(\pi\sqrt{1.25}\right) \\
 &\quad - \frac{1}{\pi\sqrt{4.25}} \sin(\pi) \cos(t) \sin\left(\frac{\pi}{4}\right) \sin\left(\pi\sqrt{4.25}\right), \quad (3.91)
 \end{aligned}$$

The solution of water wave elevations (tidal elevation) at the center point of an open uniform reservoir by using Eq. (3.91) is shown in Figures 3.3-3.4.

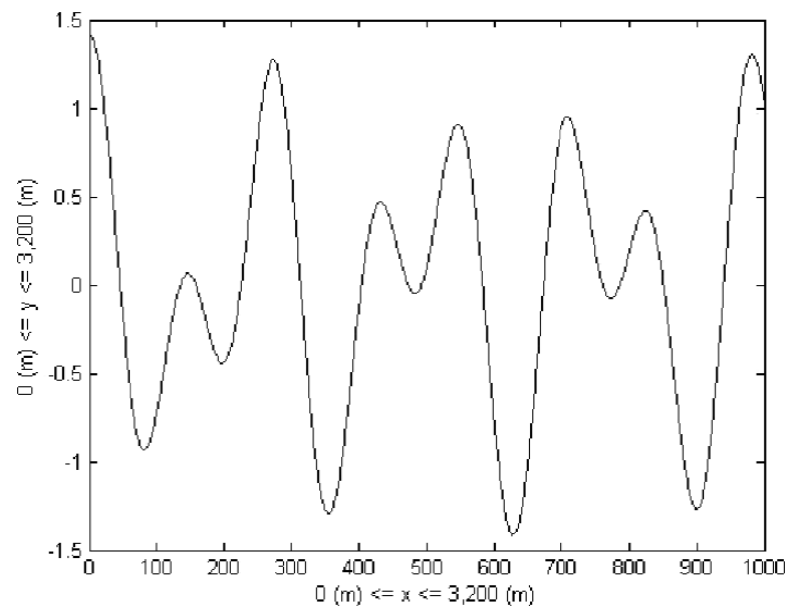


Figure 3.3 Analytical solution for height of water elevation  $\zeta(x, y, t)$  where  $0 \leq t \leq 170$  (min) at the center point of an open uniform reservoir.

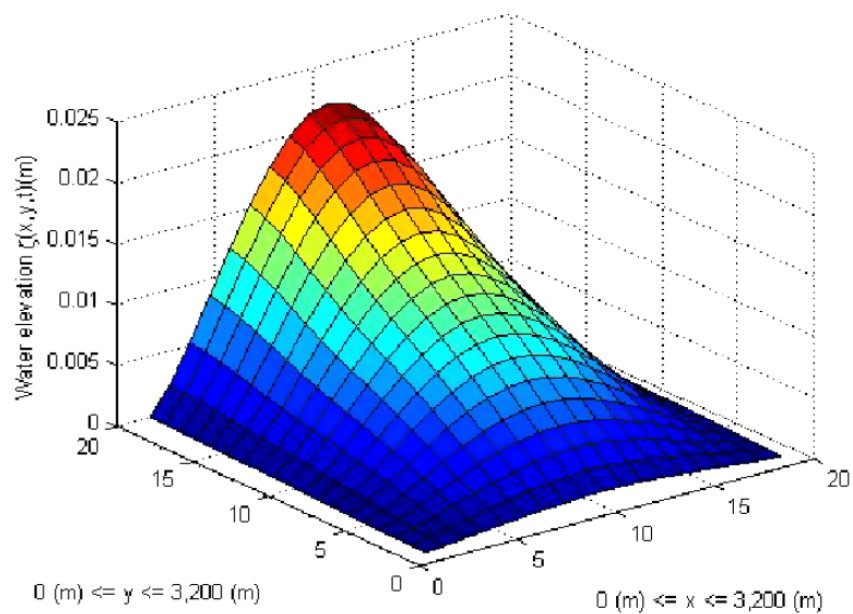


Figure 3.4 Surface plot of water elevation  $\zeta(x, y, t)$  where  $0 \leq t \leq 170$  (min) over an open uniform reservoir.

## Chapter 4

# The Numerical Simulations of Water Quality Measurement Model with Contaminant Removal Mechanisms

In this chapter, the finite difference techniques are used to approximate the solution of the water quality model. We used the Lax-Wendroff method for solving the hydrodynamic model to describe the velocity field and elevation of the water in the opened-closed reservoir. The forward differences in time and central difference in space technique for solving the transient advection-diffusion equation with removal mechanism provides the water pollutant concentration. The calculated results of the model are input to the dispersion model with removal mechanisms, which provide the pollutant concentration field. To obtain an approximate solution of the initial boundary value problem.

### 4.1 The initial and boundary conditions

#### 4.1.1 The initial and boundary conditions of a general two-dimensional hydrodynamic model in opened-closed reservoir

An opened-closed reservoir supplied by the outer water wave is going to flow into the reservoir such that the water is drained. The initial conditions of Eq. (2.4) are assumed to be motionless,

$$u(x, y, 0) = 0, \quad (4.1)$$

$$v(x, y, 0) = 0, \quad (4.2)$$

$$h(x, y, 0) = 0. \quad (4.3)$$

The elevation of water on the opened gate is assumed to be a wave maker function, say  $f$ . The elevation of water on the exit gates is assumed to be  $\frac{\partial h}{\partial x} = \kappa$ , where  $\kappa < 0$  is the rate of change between inner and outer water level through the eastern exit gate. The water flow acceleration in  $x$ -direction is assumed to be  $\frac{\partial u}{\partial x} = u_e$ ,

where  $u_e > 0$  is the rate of change between inner and outer water level through the eastern exit gate.

The boundary conditions of the model in an opened-closed reservoir are assumed as follows:

(i)  $u = 0, \frac{\partial v}{\partial y} = 0$  and  $\frac{\partial h}{\partial y} = 0$  for all  $(x, y)$  lie on the horizontal edges of the reservoir.

(ii)  $v = 0, \frac{\partial u}{\partial x} = 0$  and  $\frac{\partial h}{\partial x} = 0$  for all  $(x, y)$  lie on the vertical edges of the reservoir.

(iii)  $v = 0, \frac{\partial u}{\partial x} = 0$  and  $h = f$  for all  $(x, y)$  lie on the flowing water into the opened entrance gate.

(iv)  $v = 0, \frac{\partial u}{\partial x} = u_e$  and  $\frac{\partial h}{\partial x} = \kappa$  for all  $(x, y)$  lie on the flowing water at the opened-closed exit gate. The initial and boundary conditions are also shown in Figure 4.1.

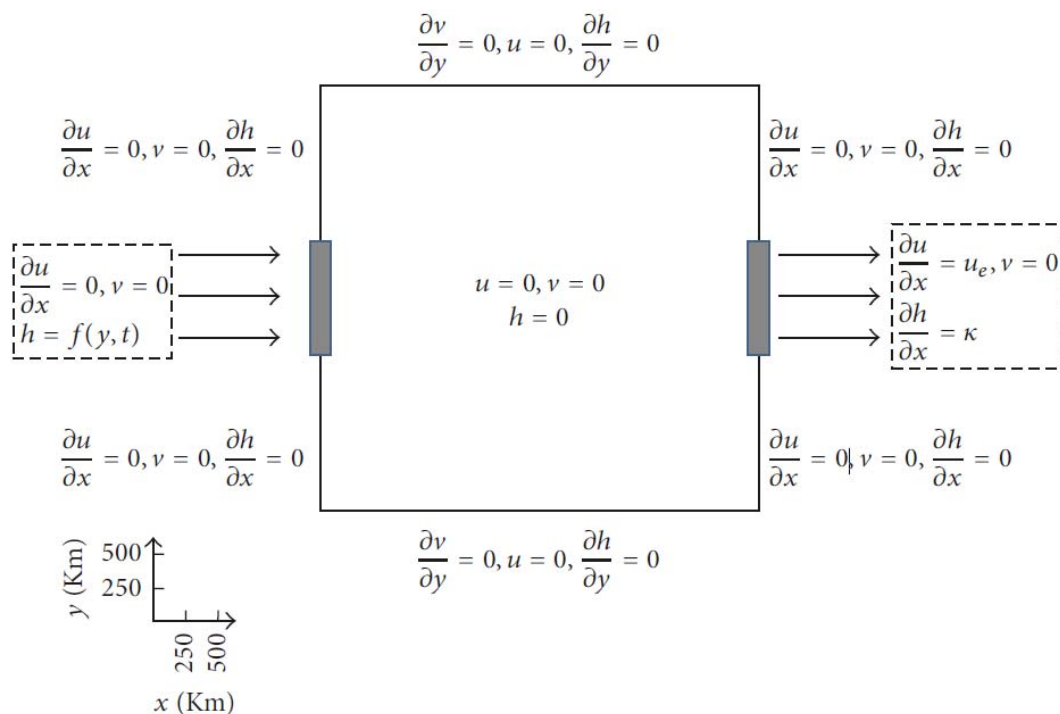


Figure 4.1 Initial and boundary conditions of hydrodynamic model in opened-closed reservoir.

#### 4.1.2 The initial and boundary conditions of water pollutant dispersion model in opened-closed reservoir.

The initial pollutant concentration in reservoir is  $c_0$  ( $\text{kg}/\text{m}^3$ ). The water pollutant is discharged from the open gate into the opened-closed reservoir which are assumed to be  $C(x, y, t) = c_1$ , where  $c_1$  ( $\text{kg}/\text{m}^3$ ) is the averaged discharged pollutant concentration along the entrance gate. The opened-closed reservoir drain water at the exit gate by assuming rate of water drain as  $\frac{\partial C}{\partial x} = -c_2$ , where  $c_2$  is a nonnegative rate of change of pollutant concentration across the exit gate. The embankment of reservoir is a nonabsorbing boundary bank. Consequently, there is no rate of change of pollutant concentration at the boundary of opened-closed reservoir,  $\frac{\partial C}{\partial n} = 0$ , where  $n$  is a normal vector, as shown in Figure 4.2.

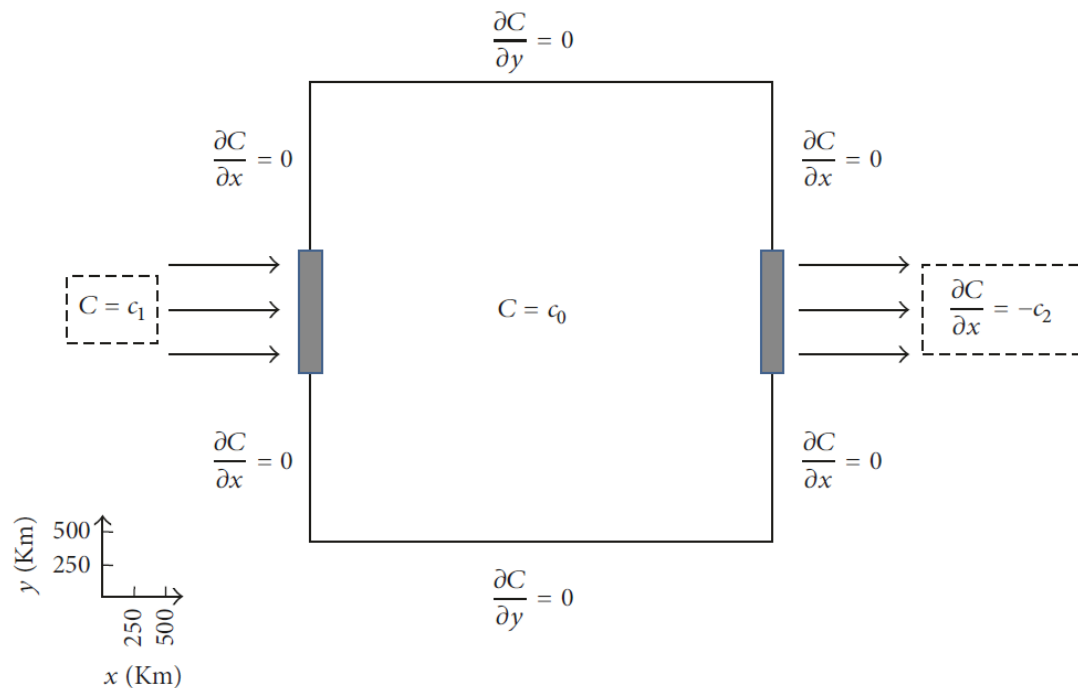


Figure 4.2 The initial and boundary conditions of water pollutant dispersion model in opened-closed reservoir.

### 4.1.3 Removal pollutant concentration mechanism in water pollutant dispersion model

The removal terms can model a variety of different phenomena, that is, the removal of pollutant concentration in (5) with absorptive rate function  $Q = Q(x, y, t)$  ( $\text{kg}/\text{m}^3 \text{ s}$ ), where  $Q$  is a nonnegative real value function.

The term  $R$  in Eq. (2.19) arises when there is pollutant concentration loss from a water surface; the reaction term is used to describe the decaying rate of pollutant concentration, where  $R$  is a nonnegative constant.

## 4.2 The numerical solutions

The hydrodynamic model provides the velocity field and elevation of the water. Then the calculated results will be input into the dispersion model that provides the pollutant concentration results. The reservoir has a simple two space dimensional as shown in Figure 2.1. Averaging the equation over the depth, discarding the term due to Coriolis force, shearing stresses and surface wind, it follows that the two-dimensional shallow water and advection-diffusion-reaction equation. In this study, we use Lax-Wendroff method for solving a general two-dimensional hydrodynamic model and the forward differences in time and central difference in space technique to approximate the velocity, the tidal elevation and the concentration, respectively.

### 4.2.1 The numerical solution for a general two-dimensional hydrodynamic model in opened-closed reservoir

We apply the numerical method for solving the single hyperbolic partial differential equations known as the Lax-Wendroff method. The Lax-Wendroff method involves starting to calculate a first half step and then using the result from the half step to calculate the full step [7, 8]. To find the water velocity and water elevation, we now discretize Eq. (2.4) on a rectangle reservoir with length  $l$  and width  $m$ . The domain  $[0, l] \times [0, m]$  is divided into  $L$  and  $M$  subintervals such that  $L\Delta x = l$  and  $M\Delta y = m$ , respectively. The time intervals  $[0, k]$  are also divided into  $T$  subintervals such that  $T\Delta t = k$ . We can then approximate  $H(x_i, y_j, t_n)$  by  $H_{i,j}^n$  and the value of the difference approximation of  $H(x, y, t)$  at points  $x = i\Delta x$  and  $y = j\Delta y$  and time

$t = n\Delta t$  where  $0 \leq i \leq L, 0 \leq j \leq M$  and  $0 \leq n \leq T$ , similarly defined for  $U_{i,j}^n$  and  $V_{i,j}^n$ . The grid points are defined by  $x_i = i\Delta x$  for all  $i = 0, 1, 2, \dots, L$ ,  $y_j = j\Delta y$  for all  $j = 0, 1, 2, \dots, M$  and  $t_n = n\Delta t$  for all  $n = 0, 1, 2, \dots, T$  in which  $L$ ,  $M$  and  $T$  are positive integers. Using the Lax-Wendroff method [7,8] on Eq. (2.4), we can obtain the following finite difference equation of hyperbolic partial differential equations.

The first half step defines values of  $H$  at time step  $n + \frac{1}{2}$  and the midpoint of the grid cell; the first half steps are stored in separate matrices  $H_x$  and  $H_y$  to be reintroduced during the second half step as shown in Figure 4.3.

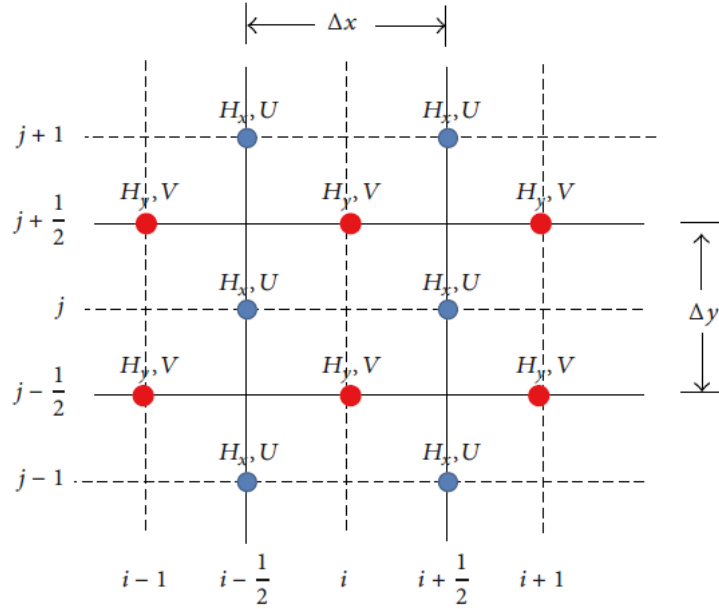


Figure 4.3 The values of vector  $H_x, H_y, U$  and  $V$  represented the solution at the midpoints of the grid cell.

The first half steps for  $U$  and  $V$  are approximated in a similar process and they are stored in  $U_x, U_y$  and  $V_x, V_y$  as follows [13, 15]

$$H_{x, i+\frac{1}{2}, j}^{n+\frac{1}{2}} = \frac{1}{2}(H_{i+1, j}^n + H_{i, j}^n) - \frac{\Delta t}{2\Delta x}(U_{i+1, j}^n + U_{i, j}^n), \quad (4.4)$$

$$H_{y, i, j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(H_{i, j+1}^n + H_{i, j}^n) - \frac{\Delta t}{2\Delta y}(V_{i, j+1}^n + V_{i, j}^n), \quad (4.5)$$

$$U_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} = \frac{1}{2}(U_{i+1,j}^n + U_{i,j}^n) - \frac{\Delta t}{2\Delta x} \left[ \left( (U_{i+1,j}^n)^2 H_{i+1,j}^n - (U_{i,j}^n)^2 H_{i,j}^n \right) + \left( \frac{g}{2} (H_{i+1,j}^n)^2 - \frac{g}{2} (H_{i,j}^n)^2 \right) \right], \quad (4.6)$$

$$U_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} = \frac{1}{2}(U_{i,j+1}^n + U_{i,j}^n) - \frac{\Delta t}{2\Delta y} \left[ (U_{i,j+1}^n V_{i,j+1}^n H_{i,j+1}^n) - (U_{i,j}^n V_{i,j}^n H_{i,j}^n) \right], \quad (4.7)$$

$$V_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} = \frac{1}{2}(V_{i+1,j}^n + V_{i,j}^n) - \frac{\Delta t}{2\Delta x} \left[ (U_{i+1,j}^n V_{i+1,j}^n H_{i+1,j}^n) - (U_{i,j}^n V_{i,j}^n H_{i,j}^n) \right], \quad (4.8)$$

$$V_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} = \frac{1}{2}(V_{i,j+1}^n + V_{i,j}^n) - \frac{\Delta t}{2\Delta y} \left[ \left( (U_{i,j+1}^n)^2 H_{i,j+1}^n - (U_{i,j}^n)^2 H_{i,j}^n \right) + \left( \frac{g}{2} (H_{i,j+1}^n)^2 - \frac{g}{2} (H_{i,j}^n)^2 \right) \right]. \quad (4.9)$$

The second half step completes the time step  $n+1$  by using the values calculated in the first half step to calculate new values at the centre of the grid cell [13, 15]

$$H_{i,j}^{n+1} = H_{i,j}^n - \frac{\Delta t}{\Delta x} \left( U_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} + U_{x_{i-\frac{1}{2},j}}^{n+\frac{1}{2}} \right) - \frac{\Delta t}{\Delta x} \left( V_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} + V_{y_{i,j-\frac{1}{2}}}^{n+\frac{1}{2}} \right), \quad (4.10)$$

$$U_{i,j}^{n+1} = U_{i,j}^n - \frac{\Delta t}{\Delta x} \left[ \left( \left( U_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} \right)^2 \left( H_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} \right) + \frac{g}{2} \left( H_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} \right)^2 \right) - \left( \left( U_{x_{i-\frac{1}{2},j}}^{n+\frac{1}{2}} \right)^2 \left( H_{x_{i-\frac{1}{2},j}}^{n+\frac{1}{2}} \right) + \frac{g}{2} \left( H_{x_{i-\frac{1}{2},j}}^{n+\frac{1}{2}} \right)^2 \right) \right] - \frac{\Delta t}{\Delta y} \left[ \left( \left( U_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} \right)^2 \left( H_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} \right) + \frac{g}{2} \left( H_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} \right)^2 \right) - \left( \left( U_{y_{i,j-\frac{1}{2}}}^{n+\frac{1}{2}} \right)^2 \left( H_{y_{i,j-\frac{1}{2}}}^{n+\frac{1}{2}} \right) + \frac{g}{2} \left( H_{y_{i,j-\frac{1}{2}}}^{n+\frac{1}{2}} \right)^2 \right) \right], \quad (4.11)$$

$$\begin{aligned}
V_{i,j}^{n+1} = & V_{i,j}^n - \frac{\Delta t}{\Delta x} \left[ \left( \left( U_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} \right) \left( V_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} \right) \left( H_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} \right) \right) \right. \\
& - \left. \left( \left( U_{x_{i-\frac{1}{2},j}}^{n+\frac{1}{2}} \right) \left( V_{x_{i-\frac{1}{2},j}}^{n+\frac{1}{2}} \right) \left( H_{x_{i-\frac{1}{2},j}}^{n+\frac{1}{2}} \right) \right) \right] \\
& - \frac{\Delta t}{\Delta y} \left[ \left( \left( V_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} \right) \left( H_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} \right) + \frac{\mathcal{G}}{2} \left( H_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} \right)^2 \right) \right. \\
& - \left. \left( \left( V_{y_{i,j-\frac{1}{2}}}^{n+\frac{1}{2}} \right) \left( H_{y_{i,j-\frac{1}{2}}}^{n+\frac{1}{2}} \right) + \frac{\mathcal{G}}{2} \left( H_{y_{i,j-\frac{1}{2}}}^{n+\frac{1}{2}} \right)^2 \right) \right]. \tag{4.12}
\end{aligned}$$

The finite difference method computes a numerical approximation to the boundary conditions of the reservoir. For the left boundary condition, we have  $i = 0$  and  $1 \leq j \leq M$ , letting  $H_{0,j}^n = H_{1,j}^n, U_{0,j}^n = U_{1,j}^n$  and  $V_{0,j}^n = V_{1,j}^n$ . Substituting the approximate unknown vector nodes  $H_{0,j}^n, U_{0,j}^n$  and  $V_{0,j}^n$  of the left boundary into Eqs. (4.4), (4.6) and (4.8), respectively,

$$H_{x_{\frac{1}{2},j}}^{n+\frac{1}{2}} = \frac{1}{2} (H_{1,j}^n + H_{0,j}^n) - \frac{\Delta t}{2\Delta x} (U_{1,j}^n + U_{0,j}^n) = H_{1,j}^n, \tag{4.13}$$

$$\begin{aligned}
U_{x_{\frac{1}{2},j}}^{n+\frac{1}{2}} = & \frac{1}{2} (U_{1,j}^n + U_{0,j}^n) \\
& - \frac{\Delta t}{2\Delta x} \left[ \left( (U_{1,j}^n)^2 H_{1,j}^n - (U_{0,j}^n)^2 H_{0,j}^n \right) + \left( \frac{\mathcal{G}}{2} (H_{1,j}^n)^2 - \frac{\mathcal{G}}{2} (H_{0,j}^n)^2 \right) \right], \\
= & U_{1,j}^n, \tag{4.14}
\end{aligned}$$

$$V_{x_{\frac{1}{2},j}}^{n+\frac{1}{2}} = \frac{1}{2} (V_{1,j}^n + V_{0,j}^n) - \frac{\Delta t}{2\Delta x} \left[ (U_{1,j}^n V_{1,j}^n H_{1,j}^n) - (U_{0,j}^n V_{0,j}^n H_{0,j}^n) \right] \tag{4.15}$$

For the right boundary condition, we have  $i = L$  and  $1 \leq j \leq M$ , letting  $H_{L+1,j}^n = H_{L,j}^n, U_{L+1,j}^n = U_{L,j}^n$  and  $V_{L+1,j}^n = V_{L,j}^n$ . Substituting the approximate unknown vector nodes  $H_{L+1,j}^n, U_{L+1,j}^n$  and  $V_{L+1,j}^n$  of the right boundary into Eqs. (4.4), (4.6) and (4.8), respectively,

$$H_{x_{L+\frac{1}{2},j}}^{n+\frac{1}{2}} = \frac{1}{2}(H_{L+1,j}^n + H_{L,j}^n) - \frac{\Delta t}{2\Delta x}(U_{L+1,j}^n + U_{L,j}^n) = H_{L,j}^n, \quad (4.16)$$

$$\begin{aligned} U_{x_{L+\frac{1}{2},j}}^{n+\frac{1}{2}} &= \frac{1}{2}(U_{L+1,j}^n + U_{L,j}^n) \\ &\quad - \frac{\Delta t}{2\Delta x} \left[ \left( (U_{L+1,j}^n)^2 H_{L+1,j}^n - (U_{L,j}^n)^2 H_{L,j}^n \right) + \left( \frac{g}{2} (H_{L+1,j}^n)^2 - \frac{g}{2} (H_{L,j}^n)^2 \right) \right], \\ &= U_{L,j}^n, \end{aligned} \quad (4.17)$$

$$V_{x_{L+\frac{1}{2},j}}^{n+\frac{1}{2}} = \frac{1}{2}(V_{L+1,j}^n + V_{L,j}^n) - \frac{\Delta t}{2\Delta x} \left[ (U_{L+1,j}^n V_{L+1,j}^n H_{L+1,j}^n) - (U_{L,j}^n V_{L,j}^n H_{L,j}^n) \right] = V_{L,j}^n. \quad (4.18)$$

For the lower boundary condition, we have  $1 \leq i \leq L$  and  $j = 0$ , letting  $H_{i,0}^n = H_{i,1}^n, U_{i,0}^n = U_{i,1}^n$  and  $V_{i,0}^n = V_{i,1}^n$ . Substituting the approximate unknown vector nodes  $H_{i,0}^n, U_{i,0}^n$  and  $V_{i,0}^n$  of the lower boundary into Eqs. (4.5), (4.7) and (4.9), respectively,

$$H_{y_{i,\frac{1}{2}}}^{n+\frac{1}{2}} = \frac{1}{2}(H_{i,1}^n + H_{i,0}^n) - \frac{\Delta t}{2\Delta y}(V_{i,1}^n + V_{i,0}^n) = H_{i,1}^n, \quad (4.19)$$

$$U_{y_{i,\frac{1}{2}}}^{n+\frac{1}{2}} = \frac{1}{2}(U_{i,1}^n + U_{i,0}^n) - \frac{\Delta t}{2\Delta y} \left[ (U_{i,1}^n V_{i,1}^n H_{i,1}^n) - (U_{i,0}^n V_{i,0}^n H_{i,0}^n) \right] = U_{i,1}^n, \quad (4.20)$$

$$\begin{aligned} V_{y_{i,\frac{1}{2}}}^{n+\frac{1}{2}} &= \frac{1}{2}(V_{i,1}^n + V_{i,0}^n) \\ &\quad - \frac{\Delta t}{2\Delta y} \left[ \left( (U_{i,1}^n)^2 H_{i,1}^n - (U_{i,0}^n)^2 H_{i,0}^n \right) + \left( \frac{g}{2} (H_{i,1}^n)^2 - \frac{g}{2} (H_{i,0}^n)^2 \right) \right] = V_{i,1}^n. \end{aligned} \quad (4.21)$$

For the upper boundary condition, we have  $1 \leq i \leq L$  and  $j = M$ , letting  $H_{i,M+1}^n = H_{i,M}^n, U_{i,M+1}^n = U_{i,M}^n$  and  $V_{i,M+1}^n = V_{i,M}^n$ . Substituting the approximate unknown vector nodes  $H_{i,M+1}^n, U_{i,M+1}^n$  and  $V_{i,M+1}^n$  of the upper boundary into Eqs. (4.5), (4.7) and (4.9), respectively,

$$H_{y_{i,M+\frac{1}{2}}}^{n+\frac{1}{2}} = \frac{1}{2}(H_{i,M+1}^n + H_{i,M}^n) - \frac{\Delta t}{2\Delta y}(V_{i,M+1}^n + V_{i,M}^n) = H_{i,M}^n, \quad (4.22)$$

$$\begin{aligned}
U_{y_{i,M+\frac{1}{2}}}^{n+\frac{1}{2}} &= \frac{1}{2} (U_{i,M+1}^n + U_{i,M}^n) - \frac{\Delta t}{2\Delta y} \left[ (U_{i,M+1}^n V_{i,M+1}^n H_{i,M+1}^n) - (U_{i,M}^n V_{i,M}^n H_{i,M}^n) \right] \\
&= U_{i,M}^n,
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
V_{y_{i,M+\frac{1}{2}}}^{n+\frac{1}{2}} &= \frac{1}{2} (V_{i,M+1}^n + V_{i,M}^n) \\
&\quad - \frac{\Delta t}{2\Delta y} \left[ \left( (U_{i,M+1}^n)^2 H_{i,M+1}^n - (U_{i,M}^n)^2 H_{i,M}^n \right) + \left( \frac{g}{2} (H_{i,M+1}^n)^2 - \frac{g}{2} (H_{i,M}^n)^2 \right) \right], \\
&= V_{i,M}^n.
\end{aligned} \tag{4.24}$$

#### 4.2.2 The numerical solution for water pollutant dispersion model with removal mechanism

To obtain an approximate solution of the dispersion model with removal mechanism Eq. (2.19), the forward differences in time and central difference in space techniques are used. We can then approximate  $C(x_i, y_j, t_n)$  by  $C_{i,j}^n$ ; the value of the difference approximation of  $C(x, y, t)$  at points the grid point  $(x_i, y_j, t_n)$  is defined by  $x_i = i\Delta x$  for all  $i = 0, 1, 2, \dots, L$ ,  $y_j = j\Delta y$  for all  $j = 0, 1, 2, \dots, M$  and  $t_n = n\Delta t$  for all  $n = 0, 1, 2, \dots, T$  in which  $L$ ,  $M$  and  $T$  are positive integers. We use the forward differences in time and central difference in space in advection-diffusion equation; we get the finite difference equations as follows [15]:

$$C \approx C_{i,j}^n, \tag{4.25}$$

$$\frac{\partial C}{\partial t} \approx \frac{C_{i,j}^{n+1} - C_{i,j}^n}{\Delta t}, \tag{4.26}$$

$$\frac{\partial C}{\partial x} \approx \frac{C_{i+1,j}^n - C_{i-1,j}^n}{2\Delta x}, \tag{4.27}$$

$$\frac{\partial C}{\partial y} \approx \frac{C_{i,j+1}^n - C_{i,j-1}^n}{2\Delta y}, \tag{4.28}$$

$$\frac{\partial^2 C}{\partial x^2} \approx \frac{C_{i+1,j}^n - 2C_{i,j}^n + C_{i-1,j}^n}{(\Delta x)^2}, \tag{4.29}$$

$$\frac{\partial^2 C}{\partial y^2} \approx \frac{C_{i,j+1}^n - 2C_{i,j}^n + C_{i,j-1}^n}{(\Delta y)^2}, \tag{4.30}$$

$$u \approx U_{i,j}^n, \tag{4.31}$$

$$v \approx V_{i,j}^n, \quad (4.32)$$

$$Q \approx Q_{i,j}^n. \quad (4.33)$$

Taking the forward time central space technique [15] into Eq. (2.19), we get the following finite difference equation

$$\begin{aligned} \frac{C_{i,j}^{n+1} - C_{i,j}^n}{\Delta t} + u_{i,j}^n \left( \frac{C_{i+1,j}^n - C_{i-1,j}^n}{2\Delta x} \right) + v_{i,j}^n \left( \frac{C_{i,j+1}^n - C_{i,j-1}^n}{2\Delta y} \right) = \\ D \left( \frac{C_{i+1,j}^n - 2C_{i,j}^n + C_{i-1,j}^n}{(\Delta x)^2} + \frac{C_{i,j+1}^n - 2C_{i,j}^n + C_{i,j-1}^n}{(\Delta y)^2} \right) - RC_{i,j}^n - Q_{i,j}^n. \end{aligned} \quad (4.34)$$

Rearrangement of Eq. (4.34) gives

$$\begin{aligned} C_{i,j}^{n+1} = & (\gamma_x - \lambda_x u_{i,j}^n) C_{i+1,j}^n + (\gamma_y - \lambda_y v_{i,j}^n) C_{i,j+1}^n \\ & + (\gamma_x + \lambda_x u_{i,j}^n) C_{i-1,j}^n + (\gamma_y + \lambda_y v_{i,j}^n) C_{i,j-1}^n \\ & + (1 - 2\gamma_x - 2\gamma_y - R\Delta t) C_{i,j}^n - Q_{i,j}^n \Delta t, \end{aligned} \quad (4.35)$$

where  $\lambda_x = \frac{\Delta t}{2\Delta x}$ ,  $\lambda_y = \frac{\Delta t}{2\Delta y}$ ,  $\gamma_x = \frac{D\Delta t}{(\Delta x)^2}$ ,  $\gamma_y = \frac{D\Delta t}{(\Delta y)^2}$ ,  $D$  is the diffusion coefficient ( $\text{m}^2/\text{s}$ ),  $R$  is the substance decaying rate ( $\text{s}^{-1}$ ) and  $Q(x, y, t)$  is the decreasing rate of water pollutant concentration due to a water pollutant sink ( $\text{kg}/\text{m}^3\text{s}$ ).

If  $C_{i,j}^n$  is on the boundary of the opened-closed reservoir, it is calculated by applying the backward difference scheme to the right boundary condition and the upper boundary conditions. We also take the forward difference scheme to the left boundary condition and the lower boundary condition.

For the left boundary condition, we have  $i=1$  and  $1 \leq j \leq M$ , letting  $C_{0,j}^n = C_{1,j}^n$ . Substituting the approximate unknown vector nodes  $C_{0,j}^n$  of the left boundary into Eq. (4.34) as follows

$$\begin{aligned} \frac{C_{1,j}^{n+1} - C_{1,j}^n}{\Delta t} + u_{1,j}^n \left( \frac{C_{2,j}^n - C_{0,j}^n}{2\Delta x} \right) + v_{1,j}^n \left( \frac{C_{1,j+1}^n - C_{1,j-1}^n}{2\Delta y} \right) = \\ D \left( \frac{C_{2,j}^n - 2C_{1,j}^n + C_{0,j}^n}{(\Delta x)^2} + \frac{C_{1,j+1}^n - 2C_{1,j}^n + C_{1,j-1}^n}{(\Delta y)^2} \right) - RC_{1,j}^n - Q_{1,j}^n, \end{aligned} \quad (4.36)$$

then we have

$$\begin{aligned} \frac{C_{1,j}^{n+1} - C_{1,j}^n}{\Delta t} + u_{1,j}^n \left( \frac{C_{2,j}^n - C_{0,j}^n}{2\Delta x} \right) + v_{1,j}^n \left( \frac{C_{1,j+1}^n - C_{1,j-1}^n}{2\Delta y} \right) = \\ D \left( \frac{C_{2,j}^n - C_{1,j}^n}{(\Delta x)^2} + \frac{C_{1,j+1}^n - 2C_{1,j}^n + C_{1,j-1}^n}{(\Delta y)^2} \right) - RC_{1,j}^n - Q_{1,j}^n. \end{aligned} \quad (4.37)$$

For the right boundary condition, we have  $i=L$  and  $1 \leq j \leq M$ , letting  $C_{L+1,j}^n = C_{L,j}^n$ . Substituting the approximate unknown vector nodes  $C_{L+1,j}^n$  of the right boundary into Eq. (4.34) as follows

$$\begin{aligned} \frac{C_{L,j}^{n+1} - C_{L,j}^n}{\Delta t} + u_{L,j}^n \left( \frac{C_{L+1,j}^n - C_{L-1,j}^n}{2\Delta x} \right) + v_{L,j}^n \left( \frac{C_{L,j+1}^n - C_{L,j-1}^n}{2\Delta y} \right) = \\ D \left( \frac{C_{L+1,j}^n - 2C_{L,j}^n + C_{L-1,j}^n}{(\Delta x)^2} + \frac{C_{L,j+1}^n - 2C_{L,j}^n + C_{L,j-1}^n}{(\Delta y)^2} \right) - RC_{L,j}^n - Q_{L,j}^n. \end{aligned} \quad (4.38)$$

then we have

$$\begin{aligned} \frac{C_{L,j}^{n+1} - C_{L,j}^n}{\Delta t} + u_{L,j}^n \left( \frac{C_{L+1,j}^n - C_{L-1,j}^n}{2\Delta x} \right) + v_{L,j}^n \left( \frac{C_{L,j+1}^n - C_{L,j-1}^n}{2\Delta y} \right) = \\ D \left( \frac{-C_{L,j}^n + C_{L-1,j}^n}{(\Delta x)^2} + \frac{C_{L,j+1}^n - 2C_{L,j}^n + C_{L,j-1}^n}{(\Delta y)^2} \right) - RC_{L,j}^n - Q_{L,j}^n. \end{aligned} \quad (4.39)$$

For the lower boundary condition, we have  $1 \leq i \leq L$  and  $j = 1$ , letting  $C_{i,0}^n = C_{i,1}^n$ . Substituting the approximate unknown vector nodes  $C_{i,0}^n$  of the lower boundary into Eq. (4.34) as follows

$$\begin{aligned} \frac{C_{i,1}^{n+1} - C_{i,1}^n}{\Delta t} + u_{i,1}^n \left( \frac{C_{i+1,1}^n - C_{i-1,1}^n}{2\Delta x} \right) + v_{i,1}^n \left( \frac{C_{i,2}^n - C_{i,0}^n}{2\Delta y} \right) = \\ D \left( \frac{C_{i+1,1}^n - 2C_{i,1}^n + C_{i-1,1}^n}{(\Delta x)^2} + \frac{C_{i,2}^n - 2C_{i,1}^n + C_{i,0}^n}{(\Delta y)^2} \right) - RC_{i,1}^n - Q_{i,1}^n, \end{aligned} \quad (4.40)$$

then we have

$$\begin{aligned} \frac{C_{i,1}^{n+1} - C_{i,1}^n}{\Delta t} + u_{i,1}^n \left( \frac{C_{i+1,1}^n - C_{i-1,1}^n}{2\Delta x} \right) + v_{i,1}^n \left( \frac{C_{i,2}^n - C_{i,0}^n}{2\Delta y} \right) = \\ D \left( \frac{C_{i+1,1}^n - 2C_{i,1}^n + C_{i-1,1}^n}{(\Delta x)^2} + \frac{C_{i,2}^n - C_{i,1}^n}{(\Delta y)^2} \right) - RC_{i,1}^n - Q_{i,1}^n. \end{aligned} \quad (4.41)$$

For the upper boundary condition, we have  $1 \leq i \leq L$  and  $j = M$ , letting  $C_{i,M+1}^n = C_{i,M}^n$ . Substituting the approximate unknown vector nodes  $C_{i,M+1}^n$  of the upper boundary into Eq. (4.34) as follows

$$\begin{aligned} \frac{C_{i,M}^{n+1} - C_{i,M}^n}{\Delta t} + u_{i,M}^n \left( \frac{C_{i+1,M}^n - C_{i-1,M}^n}{2\Delta x} \right) + v_{i,M}^n \left( \frac{C_{i,M+1}^n - C_{i,M-1}^n}{2\Delta y} \right) = \\ D \left( \frac{C_{i+1,M}^n - 2C_{i,M}^n + C_{i-1,M}^n}{(\Delta x)^2} + \frac{C_{i,M+1}^n - 2C_{i,M}^n + C_{i,M-1}^n}{(\Delta y)^2} \right) - RC_{i,M}^n - Q_{i,M}^n, \end{aligned} \quad (4.42)$$

then we have

$$\begin{aligned} \frac{C_{i,M}^{n+1} - C_{i,M}^n}{\Delta t} + u_{i,M}^n \left( \frac{C_{i+1,M}^n - C_{i-1,M}^n}{2\Delta x} \right) + v_{i,M}^n \left( \frac{C_{i,M+1}^n - C_{i,M-1}^n}{2\Delta y} \right) = \\ D \left( \frac{C_{i+1,M}^n - 2C_{i,M}^n + C_{i-1,M}^n}{(\Delta x)^2} + \frac{-C_{i,M}^n + C_{i,M-1}^n}{(\Delta y)^2} \right) - RC_{i,M}^n - Q_{i,M}^n. \end{aligned} \quad (4.43)$$

### 4.3 Numerical simulations of water quality measurement model in an opened-closed reservoir with contaminant removal mechanism

The mathematical simulation of water contaminant measurement is often used to assess the water quality. The monitoring point placement for water quality measurement in an opened-closed reservoir can give accurate or inaccurate assessment. The mathematical model of the approximated water quality in an opened-closed reservoir with removal mechanism system is proposed. The water quality model consists of the hydrodynamic model and the dispersion model. The numerical simulation of this section shows that the proposed water quality measurement model give a suitable area of zonal removal mechanism placement. The proposed simulations also give the overall and specified approximated water quality for each point and time when the exit gate is opened on the different periods of time. In addition, the proposed techniques can give a suitable period of time to open the exit gate to achieve a good agreement water quality by using contaminant removal mechanism.

Assume that the western gate is opened and the water elevation along the gate is described as a function  $f(y,t) = 1 + (0.25 \sin(0.1t))$ . The elevation of water on the eastern exit gates is assumed to be  $\frac{\partial h}{\partial x} = \kappa = -0.05$ , where  $\kappa$  is the rate of change between inner and outer water level through the eastern exit gate. The water flow acceleration in  $x$ -direction is assumed to be  $\frac{\partial u}{\partial x} = u_e = 0.01$ , where  $u_e$  is the rate of change between inner and outer water flow velocity through the eastern exit gate. The initial pollutant concentration in reservoir is  $c_0 = 1.50(\text{kg/m}^3)$ . The water pollutant is released from the open western gate into the reservoir, which is the averaged pollutant concentration along the entrance gate defined by  $c_1 = 10.00(\text{kg/m}^3)$ . The reservoir has drained water off through the exit eastern gate with the rate of change of pollutant concentration across the gate defined by  $c_2 = 0.001$ . The embankment of reservoir is a nonabsorbing boundary bank. Consequently, there is no rate of change of pollutant concentration at the boundary of opened-closed reservoir,  $\frac{\partial C}{\partial n} = 0$ , where  $n$  is a normal vector. There are three contaminant removal mechanisms in the reservoir as shown in Figure 4.4. There are 3 mechanisms such as northern, center, and southern mechanisms which are contained in each zone. The

removal mechanisms for each zone are located as shown in Table 4.1. Five monitoring nodes are located as shown in Table 4.2. If the contaminant removal mechanism is added, assume that the removal mechanisms have the same absorptive rate that can be described as a function  $Q(x, y, t) = 0.10(\text{kg}/\text{m}^3)$ . There is a pollutant concentration which is loss from the water surface, the decaying rate  $R = 0.1 \times 10^{-6}(\text{s}^{-1})$ .

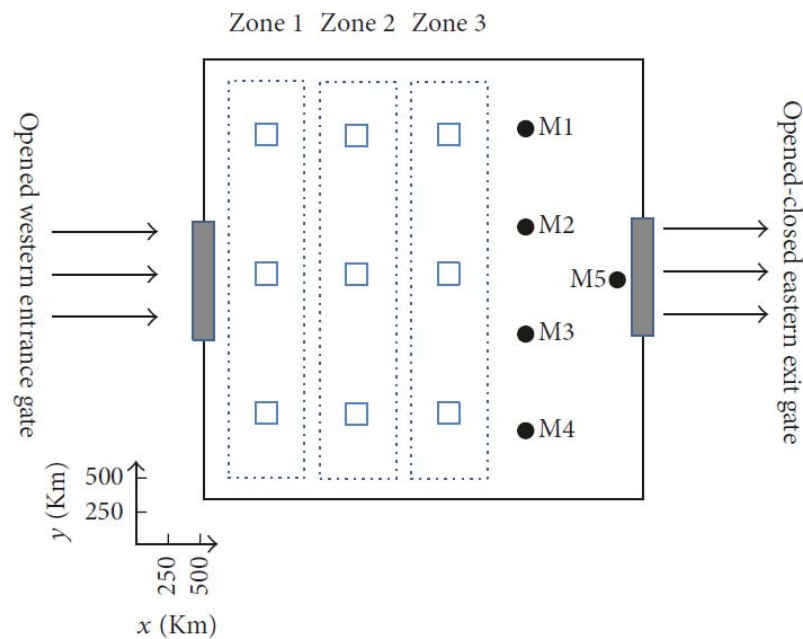


Figure 4.4 An Opened-closed reservoir with contaminant removal mechanism zones and monitoring points M1, M2, M3, M4 and M5.

Table 4.1 Removal mechanism locations  $(x, y)$  (meter, meter)

Removal mechanism	Northern	Center	Southern
Zone 1	(200, 1600)	(200, 1080)	(200, 400)
Zone 2	(800, 1600)	(800, 1080)	(800, 400)
Zone 3	(1200, 1600)	(1200, 1080)	(1200, 400)

Table 4.2 Monitoring point locations  $(x, y)$  (meter, meter)

Monitoring points	M1	M2	M3	M4	M5
Locations	(1600,1600)	(1600,1200)	(1600,800)	(1600,400)	(1200,1000)

The water flow velocity in  $x$ - and  $y$ -direction and the water elevations are approximated by using Eqs. (4.10)-(4.12) as shown in Tables 4.3 and 4.4 and Figure 4.5. The water pollutant concentrations are approximated by using Eq. (4.35). The approximated pollutant concentrations in an opened-closed reservoir are shown in Table 4.5.

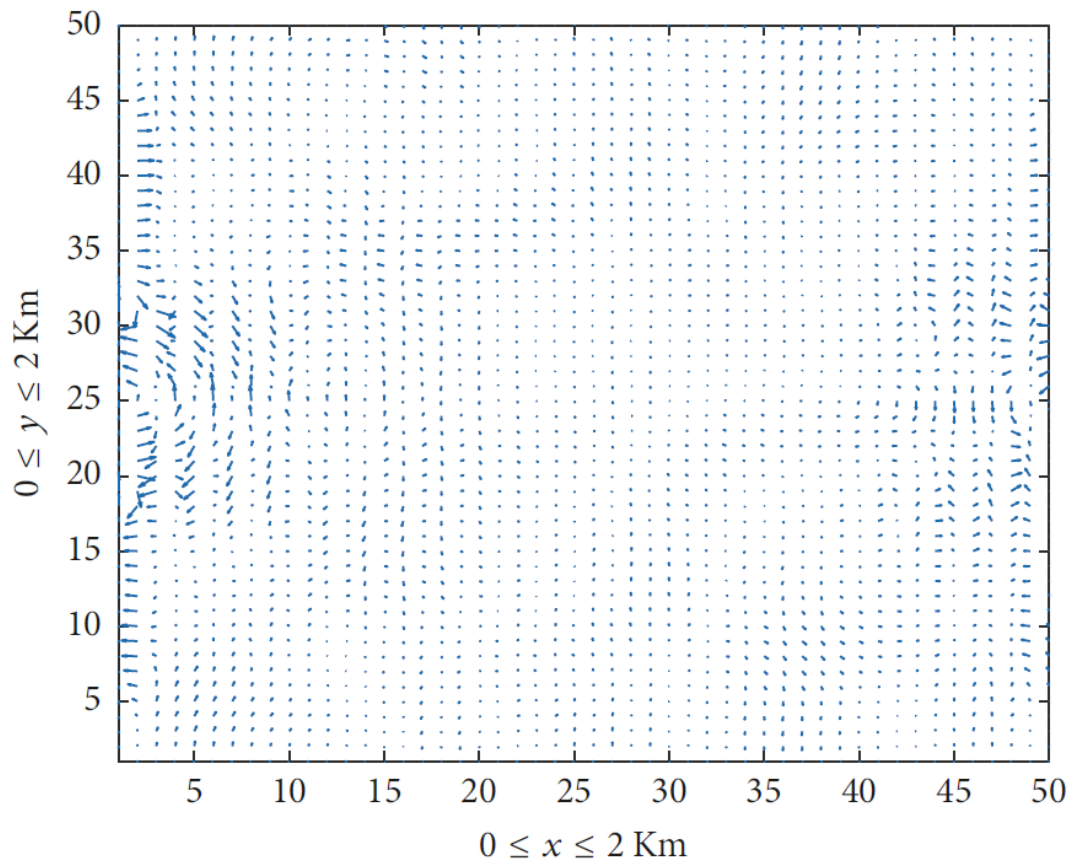


Figure 4.5 Approximated water velocities in an opened-closed reservoir

Table 4.3 Approximated water flow velocity in  $x$ -direction  $u(x, y, t)$  (m/s).

$y/x$ (m)	0	400	800	1200	1600	2000
800	0.0000	0.0870	-0.0356	0.0411	0.0219	-0.1939
1000	0.0000	-0.0344	-0.0021	0.0103	-0.0487	0.0305
1200	0.0000	-0.1260	0.0491	-0.0584	0.0283	0.1866

Table 4.4 Approximated water flow velocity in  $y$ -direction  $v(x, y, t)$  (m/s).

$y/x$ (m)	0	400	800	1200	1600	2000
800	0.6213	0.0256	-0.1082	0.0412	0.0323	0.1251
1000	-0.2959	0.4853	-0.1423	-0.0040	-0.1365	-0.5382
1200	0.5779	-0.0048	-0.1376	0.1127	0.0168	0.1442

Table 4.5 Approximated pollutant concentration  $C(x,y,t)$  ( $\text{kg/m}^3$ ).

$y/x$ (m)	0	400	800	1200	1600	2000
800	5.7854	2.9295	1.5580	0.7936	1.0637	1.3736
1000	10.0000	4.0758	1.6448	0.5297	1.0873	1.3920
1200	5.8178	2.9421	1.5798	0.9594	1.1355	1.3860

#### 4.3.1 Simulation 1: The different activated zonal removal mechanism systems effect

The approximated pollutant concentration levels in removal mechanism system and nonremoval mechanism system are compared in Figure 4.6. The approximated pollutant concentrations, when the first zonal removal mechanism systems are activated, are shown as a surface plot in Figure 4.7. The approximated pollutant concentrations, when the second zonal removal mechanism systems are activated, are shown as a surface plot in Figure 4.8. The approximated pollutant concentrations, when the third zonal removal mechanism systems are activated, are shown as a surface plot in Figure 4.9.

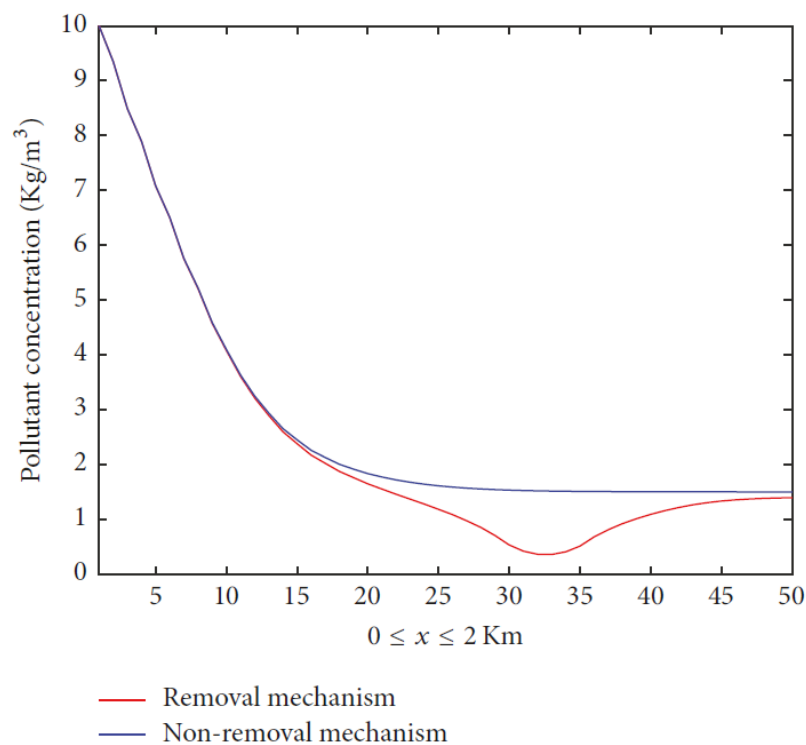


Figure 4.6 Comparison of approximated pollutant concentrations in removal mechanism system and nonremoval mechanism system.

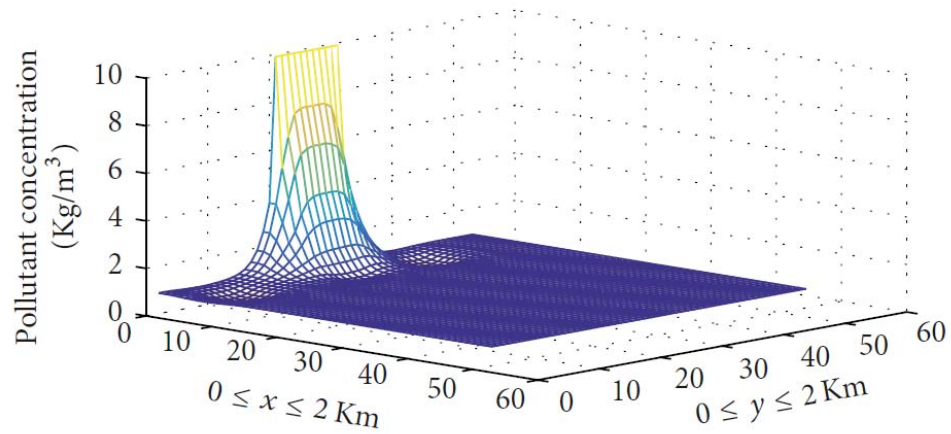


Figure 4.7 Approximated pollutant concentrations when the first zonal removal mechanism system is activated.

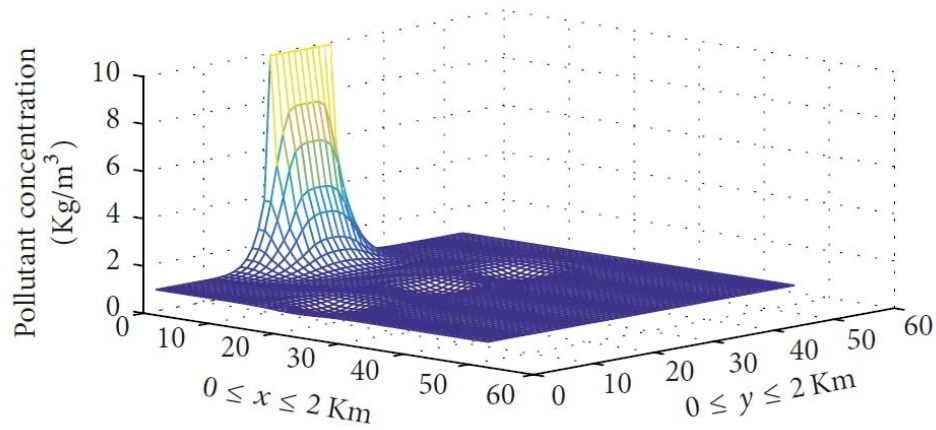


Figure 4.8 Approximated pollutant concentrations when the second zonal removal mechanism system is activated.

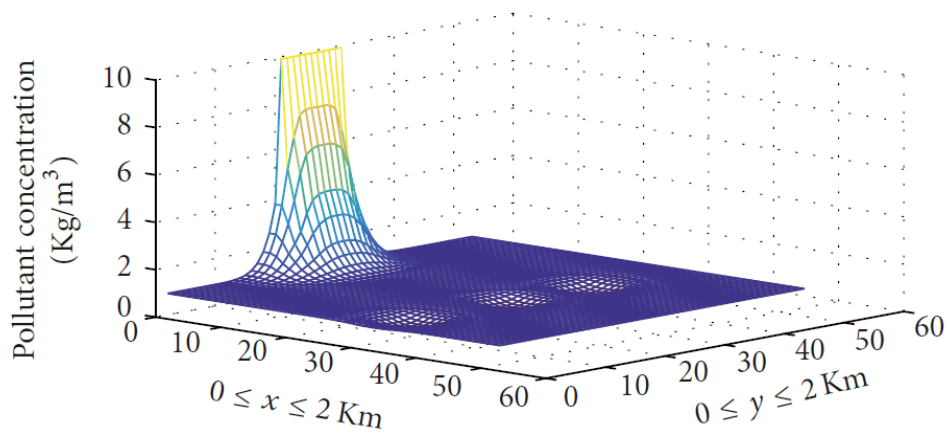


Figure 4.9 Approximated pollutant concentrations when the third zonal removal mechanism system is activated.

The approximated pollutant concentrations along the northern part,  $C(x,y,t)$ , for all  $0 \leq x \leq 2000$  m,  $y=1200$  m and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated, are compared in Figure 4.10. The approximated pollutant concentrations along the middle part,  $C(x,y,t)$ , for all  $0 \leq x \leq 2000$  m,  $y=1000$  m and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated, are compared in Figure 4.11. The approximated pollutant concentrations along the southern part,  $C(x,y,t)$ , for all  $0 \leq x \leq 2000$  m,  $y=800$  m and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated, are compared in Figure 4.12.

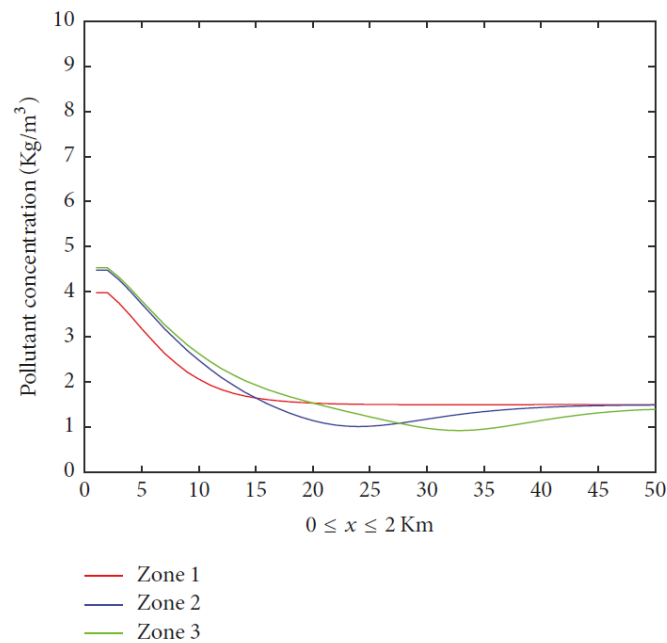


Figure 4.10 Comparison of approximated pollutant concentrations along the northern part,  $C(x,y,t)$ , for all  $0 \leq x \leq 2000$  m,  $y=1200$  m and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated.

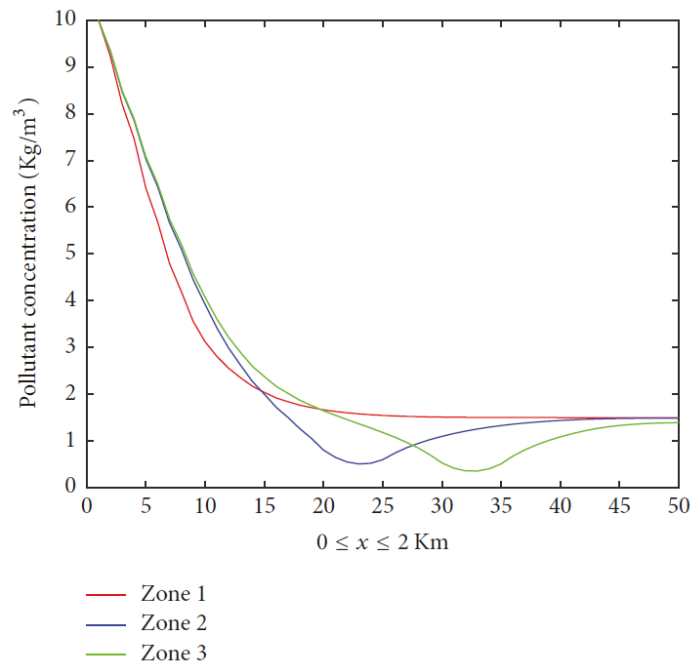


Figure 4.11 Comparison of approximated pollutant concentrations along the northern part,  $C(x,y,t)$ , for all  $0 \leq x \leq 2000$  m,  $y = 1000$  m and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated.

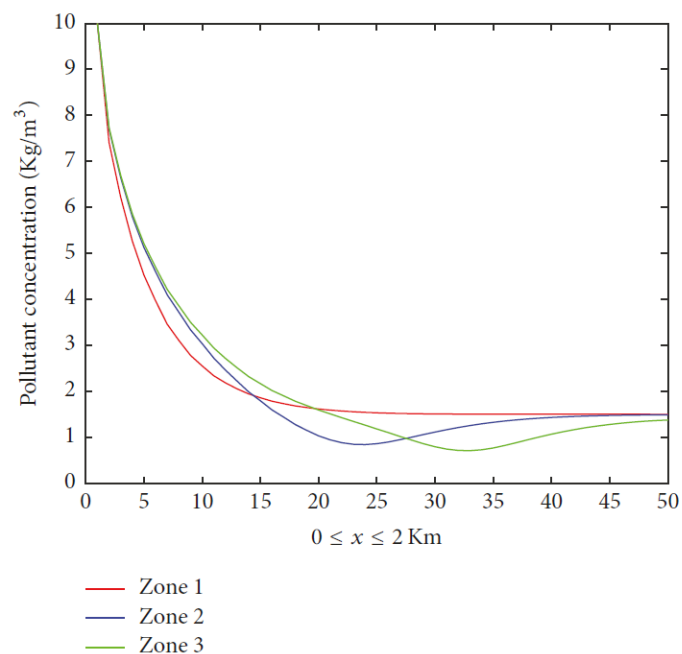


Figure 4.12 Comparison of approximated pollutant concentrations along the northern part,  $C(x,y,t)$ , for all  $0 \leq x \leq 2000$  m,  $y = 800$  m and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated.

### 4.3.2 Simulation 2: The exit gate opening on different periods of time effect

Assuming that the third zonal removal mechanism systems are activated. The approximated pollutant concentrations over 5 monitoring nodes, when the third zonal removal mechanism systems are activated and the eastern exit gate is opened, are shown in Figure 4.13. Five monitoring nodes are located as shown in Table 4.2. If the eastern exit gate is opened or shut on different periods of time such as 0–100 min, 33–100 min, and 66–100 min, the approximated pollutant concentrations over 5 monitoring nodes are also compared in Figure 4.13. The comparison of approximated pollutant concentrations along the eastern exit gate,  $C(x,y,t)$ , for all  $800 \leq y \leq 1200$  m,  $x = 2000$  m and  $0 \leq t \leq 100$  min, when the eastern exit gate is opened or shut on the difference periods of time such as 0–100min, 33–100min, and 66–100min, is shown in Figure 4.14. The overall approximated pollutant concentrations, when the eastern exit gate is opened on the period 0–100min, are shown in Figure 4.15. The overall approximated pollutant concentrations, when the eastern exit gate is opened on the period 33–100min, are shown in Figure 4.16. The overall approximated pollutant concentrations, when the eastern exit gate is opened on the period 66–100min, are also shown in Figure 4.17.

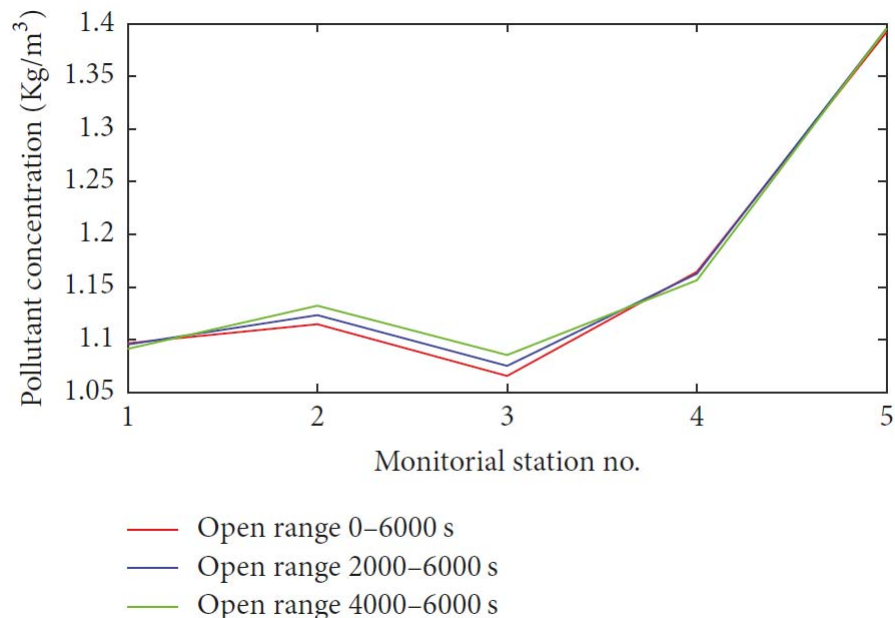


Figure 4.13 The approximated pollutant concentrations over 5 monitoring nodes, when the eastern exit gate is opened on 0–100 min, 33–100 min, and 66–100 min.

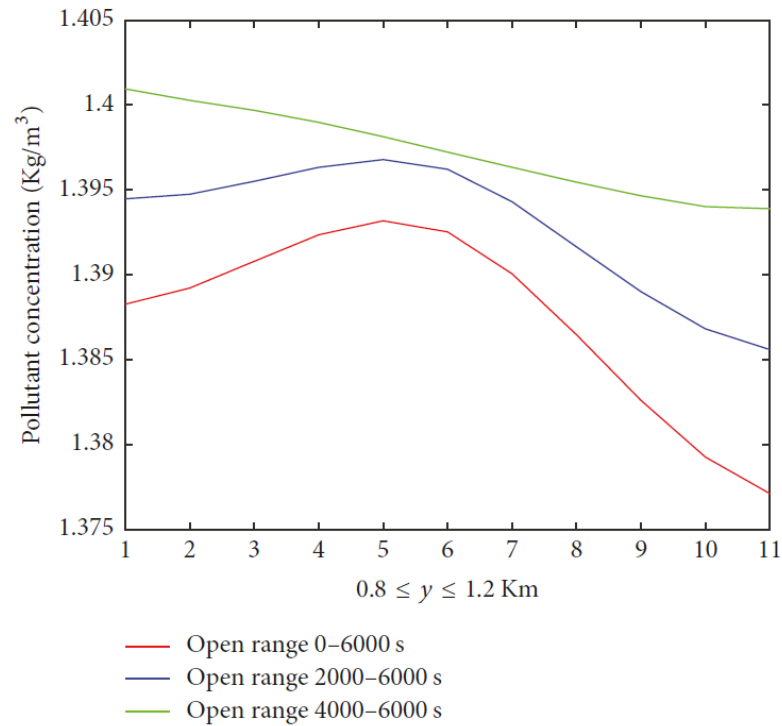


Figure 4.14 The approximated pollutant concentrations over the eastern exit gate is opened on 0–100 min, 33–100 min and 66–100 min, when  $C(x,y,t)$ , for all  $800 \leq y \leq 1200$  m,  $x = 2000$  m.

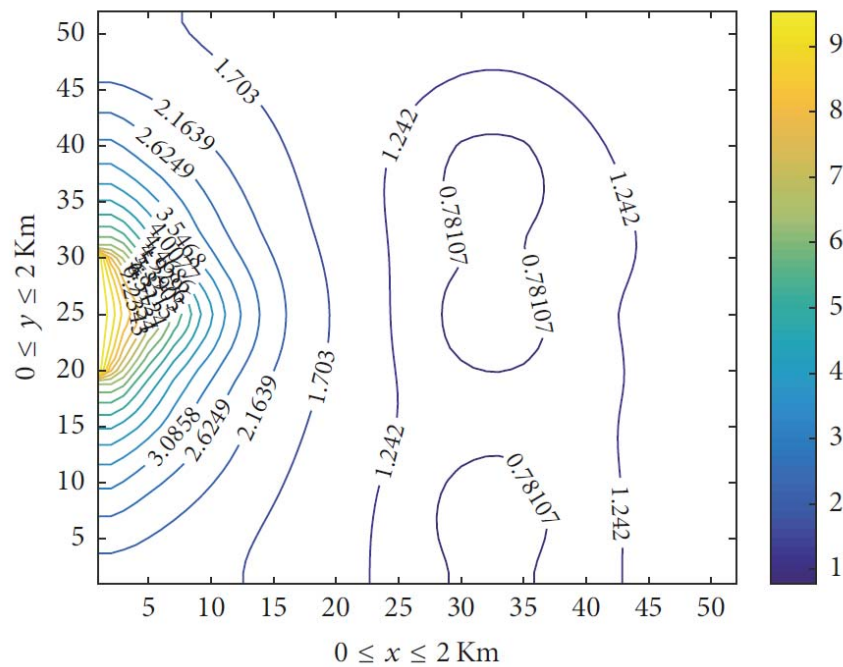


Figure 4.15 The overall approximated pollutant concentrations when the eastern exit gate is opened on the period 0–100 min.

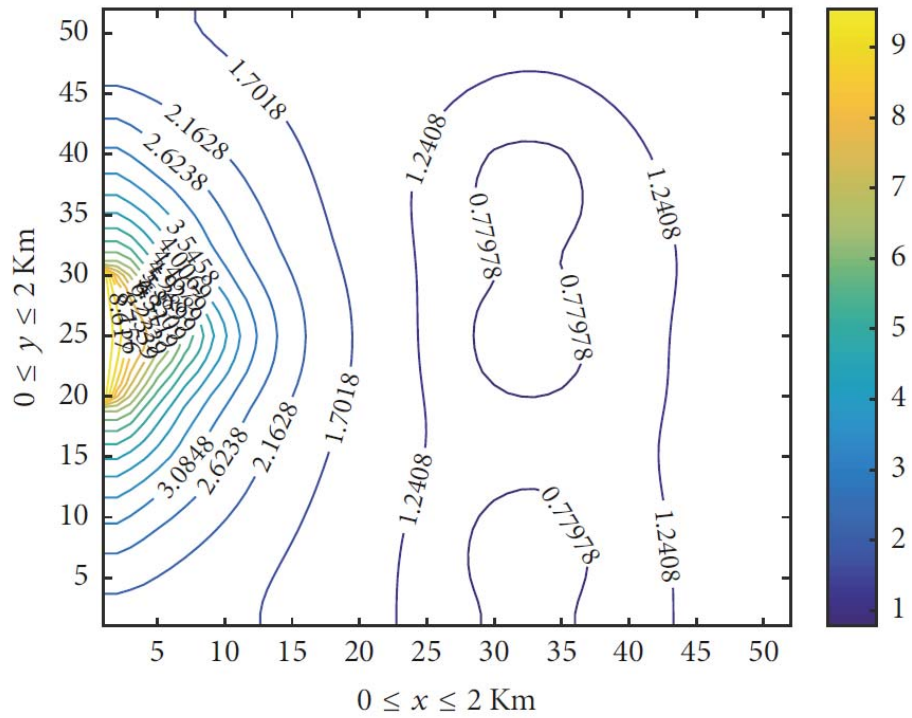


Figure 4.16 The overall approximated pollutant concentrations when the eastern exit gate is opened on the period 33–100 min.

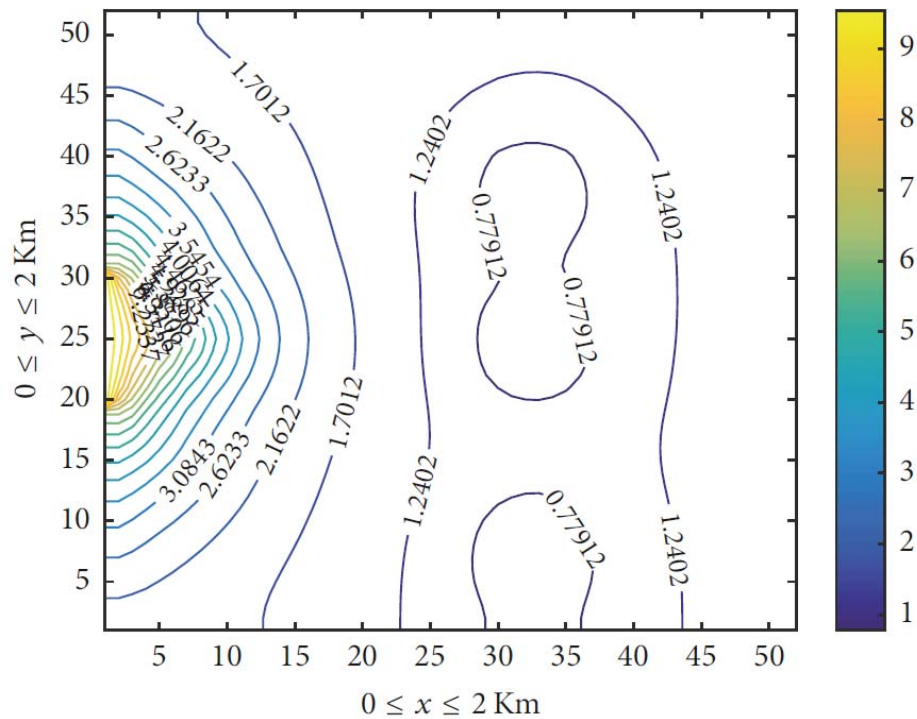


Figure 4.17 The overall approximated pollutant concentrations when the eastern exit gate is opened on the period 66–100 min.

## Chapter 5

# Discussions and Conclusions

In this thesis, the mathematical model of the approximated water quality in an opened-closed reservoir with contaminant removal mechanism system is considered. The water quality model consists of the hydrodynamic model and the dispersion model. We have introduced the hydrodynamic model with wave maker functions to compute the velocity and the elevation of water current for the transient advection-diffusion-reaction of water pollution model in two-dimensional spaces.

We applied the method to assess the water pollution in an opened-closed reservoir. We claim that our methods introducing the hydrodynamic model with water quality measurement model in opened-closed reservoir with contaminant removal mechanism, developed in this thesis is a new approach for water pollution assessment and control.

### 5.1 Discussions

#### 5.1.1 Discussion for simulation 1: The Different Activated Zonal Removal Mechanism Systems Effect

In simulation 1, the different activated zonal removal mechanism systems are considered. The approximated pollutant concentration levels with the contaminant removal mechanism system are shown in Figure 4.6. We can see that the removal mechanism can be used to reduce the pollutant concentration in the opened-closed reservoir. The approximated pollutant concentrations, when the first/second/third zonal removal mechanism systems are activated, are shown as a surface plot in Figures 4.7, 4.8 and 4.9. The approximated pollutant concentrations along the northern/middle/southern parts, when the first, second, and third zonal removal mechanism systems are activated, are compared in Figures 4.10, 4.11 and 4.12. We can see that the third zonal removal mechanism systems give better water quality than another activated zones.

### 5.1.2 Discussion In simulation 2: The Exit Gate Opening on Different Periods of Time Effect.

In simulation 2, the eastern exit gate opening on different periods of time is considered. The suitable third zonal removal mechanism system has been chosen to be activated. If the eastern exit gate is opened or shut on different periods of time such as 0–100 min, 33–100 min, and 66–100 min, the approximated pollutant concentrations over 5 monitoring nodes are shown in Figure 4.13. We can see that the full periods (0–100 min) of opened exit gate give better water quality for all 5 monitoring nodes.

If we consider the approximated pollutant concentrations along the eastern exit gate, when the eastern exit gate is opened or shut on the difference periods of time as shown in Figure 4.14, we can also see that the long opened exit gate period (0–100 min) gives the better monitored water quality than another periods as well.

However, the overall approximated pollutant concentrations, when the eastern exit gate is opened on the different period of times, are shown in Figures 4.15, 4.16, and 4.17. We can see that the period (66–100min) of the opened exit gate, which is shortest, gives better overall water quality than other periods. These mean that the monitoring point placement for water quality measurement in an opened-closed reservoir affect to water quality assessment. These mean that the approximated water quality measurement on monitoring points in an opened-closed reservoir affect the accuracy assessment. The overall simulated pollutant concentration method gives more precise water quality assessment than the monitoring assessment method.

## 5.2 Conclusions

The first part, a simple form of two-dimensional hydrodynamic model for describing the elevation of water wave in an open uniform reservoir is derived and presented. The separation of variables method is employed to find an analytical solution to the model. The second part, the mathematical simulation of water contaminant measurement is often used to assess the water quality. The monitoring point placement for water quality measurement in an opened-closed reservoir can give accurate or inaccurate assessment. However, the proposed numerical simulations have given the overall approximated water quality when the removal

mechanism systems are activated in different zones. The proposed techniques give the zonal removal mechanism placement to specialists who want to control the water quality. Moreover, we obtain the overall and specified approximated water quality for each points and time when the exit gate is opened on the different periods of time. The techniques can give a suitable period of time to open the exit gate to specialists who want to get a good agreement water quality by using contaminant removal mechanism.

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Appendix A  
Research Journals

## RESEARCH

## Open Access



# Analytical solution to a hydrodynamic model in an open uniform reservoir

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## Abstract

The hydrodynamic model is used to determine the water wave flow. In this research, a nondimensional form of a two-dimensional hydrodynamic model with generalized boundary condition  $g(x, t)$  and initial conditions for describing the elevation of water wave in an open uniform reservoir is proposed. The separation of variables method with mathematical induction is employed to find an analytical solution to the model. An example of flow calculations in an open uniform reservoir is also demonstrated.

**MSC:** 35L05; 35L04; 35L40

**Keywords:** separation of variables method; hydrodynamic model; open uniform reservoir; water elevation

## 1 Introduction

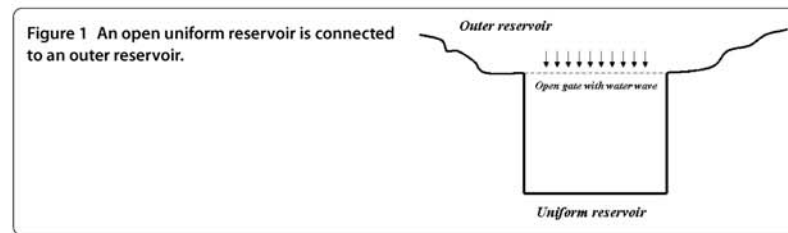
In [1] and [2], the finite element method was used to solve the water pollution models. In literature, several mathematical models need the data of water flow, while the velocity and elevation of water flow are provided by the hydrodynamic model. In [3], the finite difference method was used to solve the hydrodynamic model with constant coefficients in the closed uniform reservoir.

In [4], an analytical solution to the hydrodynamic model in a closed uniform reservoir was proposed. In [5], the Lax-Wendroff finite difference method was also proposed to approximate the water elevation and water flow velocity. However, the analytical solution to the hydrodynamic model in an open reservoir has not been considered.

An open uniform reservoir is supplied by the outer water wave as shown in Figure 1. In the former studies, the elevation of water wave (tidal elevation) can be found only by numerical approximations. The purpose of this research is to derive an analytical solution to a model. An analytical solution is needed for comparison with the approximated solutions of the flow. Hence, the analytical solution becomes a benchmark for any related numerical approximations.

## 2 A nondimensional form of a hydrodynamic model

The continuity and momentum equations are governed by the hydrodynamic behavior on the reservoir. We average the equations over the depth, discarding the term due to Coriolis parameter, shearing stresses and surface wind. We introduce the well-known two-



dimensional shallow water equations [2, 5] as follows:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}[(h + \zeta)u] + \frac{\partial}{\partial y}[(h + \zeta)v] = 0, \quad (1)$$

$$\frac{\partial u}{\partial t} + g \frac{\partial \zeta}{\partial x} = 0, \quad (2)$$

$$\frac{\partial v}{\partial t} + g \frac{\partial \zeta}{\partial y} = 0, \quad (3)$$

where  $h(x, y)$  is the depth measured from the mean water level to the bed of the reservoir,  $\zeta(x, y, t)$  is the elevation from the mean water level to the temporary water surface or the tidal elevation,  $g$  is the acceleration due to gravity, and  $u(x, y, t)$  and  $v(x, y, t)$  are the velocity components in  $x$  and  $y$  directions, respectively, for all  $(x, y, t) \in [0, l] \times [0, l] \times [0, T]$ . We assume that  $h$  is a constant and  $\zeta \ll h$ . Then Eqs. (1)-(3) lead to

$$\frac{\partial \zeta}{\partial t} + h \frac{\partial u}{\partial x} + h \frac{\partial v}{\partial y} = 0, \quad (4)$$

$$\frac{\partial u}{\partial t} + g \frac{\partial \zeta}{\partial x} = 0, \quad (5)$$

$$\frac{\partial v}{\partial t} + g \frac{\partial \zeta}{\partial y} = 0. \quad (6)$$

We will consider the equation in a dimensionless problem by letting  $U = u/\sqrt{gh}$ ,  $V = v/\sqrt{gh}$ ,  $X = x/l$ ,  $Y = y/l$ ,  $Z = \zeta/h$  and  $T = t\sqrt{gh}/l$ . Substituting them into Eqs. (4)-(6) leads to

$$\frac{\partial Z}{\partial T} + \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \quad (7)$$

$$\frac{\partial U}{\partial T} + \frac{\partial Z}{\partial X} = 0, \quad (8)$$

$$\frac{\partial V}{\partial T} + \frac{\partial Z}{\partial Y} = 0. \quad (9)$$

In order to solve Eqs. (7)-(9) in  $\Omega \times [0, T]$ , where  $\Omega = (0, 1) \times (0, 1)$ , for convenience using  $u, v, d$  for  $U, V$  and  $Z$ , respectively [5], we get

$$\frac{\partial d}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (10)$$

$$\frac{\partial u}{\partial t} + \frac{\partial d}{\partial x} = 0, \quad (11)$$

$$\frac{\partial v}{\partial t} + \frac{\partial d}{\partial y} = 0. \quad (12)$$

### 2.1 Initial and boundary conditions for an open uniform reservoir

The initial conditions of Eqs. (10)-(12) are assumed to be  $u = 0$ ,  $v = 0$ ,  $d = f_1(x, y)$  and  $\frac{\partial d}{\partial t} = f_2(x, y)$ . The boundary conditions of the model in an open uniform reservoir are assumed to be  $v = 0$ ,  $\frac{\partial u}{\partial x} = 0$  at the planes  $x = 0$  and  $x = 1$ .  $u = 0$ ,  $\frac{\partial v}{\partial y} = 0$  at the planes  $y = 0$ .  $u = 0$ ,  $\frac{\partial v}{\partial y} = 0$ ,  $d(x, 1, t) = g(x, t)$  and  $\frac{\partial d}{\partial y} = 0$  at the planes  $y = 1$ .  $d = 0$  on  $\partial\Omega \setminus \{(x, y) \in \partial\Omega : y = 1\}$ .

### 3 An analytical solution

Eqs. (10)-(12) can be written as a non-conservation form of equation, as a single equation,

$$\frac{\partial^2 d}{\partial t^2} = \frac{\partial^2 d}{\partial x^2} + \frac{\partial^2 d}{\partial y^2}, \quad (13)$$

subject to the initial conditions  $d(x, y, 0) = f_1(x, y)$  and  $\frac{\partial d}{\partial t} = f_2(x, y)$ . The boundary conditions are  $d = 0$  at  $\partial\Omega \setminus \{(x, y) \in \partial\Omega : y = 1\}$ ,  $d(x, 1, t) = g(x, t)$  and  $\frac{\partial d}{\partial y} = 0$  at the plane  $y = 1$ . It is readily verified that the general solution can be written by the separation of variables technique

$$d(x, y, t) = X(x)Y(y)T(t) \quad (14)$$

subject to the boundary conditions  $X(0) = X(1) = 0$ ,  $Y(0) = Y'(1) = 0$  and  $Y(1) = g(x, t)$ . Due to the eigenvalue needing to be a negative real number, say  $-\lambda^2$ , we can now obtain that Eq. (13) becomes

$$\frac{X''(x)}{X(x)} = -\lambda^2, \quad (15)$$

$$-\frac{Y''(y)}{Y(y)} + \frac{T''(t)}{T(t)} = -\lambda^2. \quad (16)$$

We get the solution of Eq. (15) as  $X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$ , where  $C_1, C_2$  are arbitrary constants. According to the boundary condition  $X(0) = 0$ , we have  $C_1 = 0$ . Then  $X(x) = C_2 \sin(\lambda x)$ . The boundary condition  $X(1) = 0$  gives  $\lambda = n\pi$ , we then have

$$X_n(x) = C_2 \sin(n\pi x), \quad (17)$$

where  $n = 1, 2, 3, \dots$ . Since the eigenvalue of Eq. (16) is a negative real number, say  $-\mu^2$ , we have

$$\frac{Y''(y)}{Y(y)} = -\mu^2, \quad (18)$$

$$\frac{T''(t)}{T(t)} + \lambda^2 = -\mu^2. \quad (19)$$

We then have the solutions  $Y(y) = C_3 \cos(\mu y) + C_4 \sin(\mu y)$  and  $T(t) = C_5 \cos(t\sqrt{\lambda^2 + \mu^2}) + C_6 \sin(t\sqrt{\lambda^2 + \mu^2})$ , where  $C_3, C_4, C_5, C_6$  are arbitrary constants. According to the boundary conditions  $Y(0) = 0$  and  $Y'(1) = 0$ , we have  $C_3 = 0$  and  $\mu = (2m-1)\frac{\pi}{2}$ . Thus,

$$Y_m(y) = C_4 \sin\left((2m-1)\frac{\pi}{2}y\right), \quad (20)$$

where  $m = 1, 2, 3, \dots$ . Then  $Y(1) = g(x, t) = C_4 \sin((2m-1)\frac{\pi}{2})$ , we obtain  $C_4 = (-1)^{-m-1}g(x, t)$ . We get

$$Y_m(y) = (-1)^{-m-1}g(x, t) \sin\left((2m-1)\frac{\pi}{2}y\right). \quad (21)$$

Since  $\lambda = n\pi$  and  $\mu = (2m-1)\frac{\pi}{2}$ , we then have

$$T_{mn}(t) = C_5 \cos\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) + C_6 \sin\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right). \quad (22)$$

According to Eqs. (17), (21) and (22), the solution of our problem can be written in the form

$$\begin{aligned} d_{mn}(x, y, t) &= X_n(x)Y_m(y)T_{mn}(t) \quad (23) \\ &= \sin(n\pi x)(-1)^{-m-1}g(x, t) \sin\left((2m-1)\frac{\pi}{2}y\right) \\ &\quad \times \left( A_{mn} \cos\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right. \\ &\quad \left. + B_{mn} \sin\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right), \quad (24) \end{aligned}$$

where  $A_{mn} = C_2C_5$  and  $B_{mn} = C_2C_6$  are arbitrary constants and for all  $m, n = 1, 2, 3, \dots$ . By using the superposition principle, Eq. (24) becomes

$$\begin{aligned} d(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} d_{mn}(x, y, t) \quad (25) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(n\pi x)(-1)^{-m-1}g(x, t) \sin\left((2m-1)\frac{\pi}{2}y\right) \\ &\quad \times \left( A_{mn} \cos\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right. \\ &\quad \left. + B_{mn} \sin\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right), \quad (26) \end{aligned}$$

where  $A_{mn}$  and  $B_{mn}$  are fixed by the value of  $d(x, y, 0) = f_1(x, y)$  and  $\frac{\partial d}{\partial t}(x, y, 0) = f_2(x, y)$ , respectively.

#### 4 Application to open uniform reservoir

We consider an open uniform reservoir with dimension  $3.2 \times 3.2$  km ( $l = 3.2$  km) and the constant depth  $h = 1$  m. Initially, the water in the reservoir is assumed to be motionless  $u = 0$ ,  $v = 0$  and the water elevation is specified by  $\zeta(x, y, 0) = 2 \sin(\pi x) \sin(\frac{\pi}{2}y)$  and  $\frac{\partial \zeta}{\partial t}(x, y, 0) = -\sin(2\pi x) \sin(\frac{\pi}{2}y)$ . The elevation of water along the northern boundary is distributed by  $\zeta(x, l, t) = 2 \sin(\pi x) \cos(t)$ . The open boundary condition is assumed to be

$d(x, 1, t) = g(x, t) = 2 \sin(\pi x) \cos(t)$ . Eq. (26) becomes

$$\begin{aligned} d(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \cos(t) \sin\left((2m-1)\frac{\pi}{2}y\right) \\ &\quad \times \left( A_{mn} \cos\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right. \\ &\quad \left. + B_{mn} \sin\left(\pi t \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \right). \end{aligned} \quad (27)$$

The initial conditions are assumed to be  $d(x, y, 0) = f_1(x, y) = 2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right)$ ,  $\frac{\partial d}{\partial t} = f_2(x, y) = -\sin(2\pi x) \sin\left(\frac{\pi}{2}y\right)$ . Thus Eq. (27) becomes

$$\begin{aligned} &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2}y\right) \\ &= 2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right), \end{aligned} \quad (28)$$

and

$$\begin{aligned} &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \left(\pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2}\right) \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2}y\right) \\ &= -\sin(2\pi x) \sin\left(\frac{\pi}{2}y\right). \end{aligned} \quad (29)$$

We will consider the coefficients  $A_{mn}$  by comparing coefficients and using the mathematical induction on the finite sum of Eq. (28) as follows:

$$2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right) \simeq \sum_{m=1}^M \sum_{n=1}^N A_{mn} \sin(n\pi x) 2(-1)^{-m-1} \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2}y\right). \quad (30)$$

By comparing their coefficients in Eq. (30), we can see that if  $M = 1$  and  $N = 1$ ,

$$A_{11} = \frac{1}{\sin(\pi x)}, \quad (31)$$

where  $\sin(\pi x) \neq 0$ . Assume that  $A_{11} = \frac{1}{\sin(\pi x)}$ , where  $M, N = k$  and  $\sin(\pi x) \neq 0$ . If  $M = k + 1$  and  $N = k + 1$ , then

$$\begin{aligned} &2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right) \\ &= A_{11} \sin(\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right) + A_{12} \sin(2\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right) \\ &\quad + \cdots + A_{1k} \sin(k\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right) \\ &\quad + A_{1(k+1)} \sin((k+1)\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right) \end{aligned}$$

$$\begin{aligned}
& + A_{21} \sin(\pi x) 2(-1) \sin(\pi x) \sin\left(\frac{3\pi}{2}y\right) \\
& + A_{22} \sin(2\pi x) 2(-1) \sin(\pi x) \sin\left(\frac{3\pi}{2}y\right) \\
& + \cdots + A_{2k} \sin(k\pi x) 2(-1) \sin(\pi x) \sin\left(\frac{3\pi}{2}y\right) \\
& + A_{2(k+1)} \sin((k+1)\pi x) 2(-1) \sin(\pi x) \sin\left(\frac{3\pi}{2}y\right) \\
& + \cdots + A_{(k+1)1} \sin(\pi x) 2(-1)^{-k-2} \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2}y\right) \\
& + A_{(k+1)2} \sin(2\pi x) 2(-1)^{-k-2} \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2}y\right) \\
& + \cdots + A_{(k+1)k} \sin(k\pi x) 2(-1)^{-k-2} \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2}y\right) \\
& + A_{(k+1)(k+1)} \sin((k+1)\pi x) 2(-1)^{-k-2} \sin(\pi x) \sin\left(\frac{(2(k+1)-1)\pi}{2}y\right). \quad (32)
\end{aligned}$$

Since  $\sin(n\pi x) \neq 0$  and  $\sin\left(\frac{(2m-1)\pi}{2}y\right) \neq 0$  for all  $x, y \in (0, 1)$ , substituting Eq. (31) into Eq. (32) and comparing the coefficients in Eq. (32), we can see that  $A_{12}, A_{13}, \dots, A_{1(k+1)} = 0$ ,  $A_{21}, A_{22}, \dots, A_{2(k+1)} = 0, \dots, A_{(k+1)1}, A_{(k+1)2}, \dots, A_{(k+1)(k+1)} = 0$ , except  $A_{11} \neq 0$ . We get

$$A_{11} = \frac{1}{\sin(\pi x)}. \quad (33)$$

Thus, Eq. (30) holds for  $M = k + 1$  and  $N = k + 1$ , the proof of the induction step is completed. Similarly, we also consider the coefficients  $B_{mn}$  by comparing coefficients and the mathematical induction on the finite sum of Eq. (29) as follows:

$$\begin{aligned}
-\sin(2\pi x) \sin\left(\frac{\pi}{2}y\right) & \simeq \sum_{m=1}^M \sum_{n=1}^N B_{mn} \left( \pi \sqrt{n^2 + \left(\frac{2m-1}{2}\right)^2} \right) \sin(n\pi x) 2(-1)^{-m-1} \\
& \times \sin(\pi x) \sin\left((2m-1)\frac{\pi}{2}y\right). \quad (34)
\end{aligned}$$

If  $M = 1$  and  $N = 2$ , then

$$\begin{aligned}
-\sin(2\pi x) \sin\left(\frac{\pi}{2}y\right) & = B_{11} \left( \pi \sqrt{1^2 + \left(\frac{1}{2}\right)^2} \right) \sin(\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right) \\
& + B_{12} \left( \pi \sqrt{2^2 + \left(\frac{1}{2}\right)^2} \right) \sin(2\pi x) 2 \sin(\pi x) \sin\left(\frac{\pi}{2}y\right). \quad (35)
\end{aligned}$$

By comparing their coefficients in Eq. (35), we get

$$B_{12} = -\frac{1}{2\pi \sqrt{4.25} \sin(\pi x)}, \quad (36)$$

where  $B_{11} = 0$  and  $\sin(\pi x) \neq 0$ . Assume that  $B_{12} = -\frac{1}{2\pi\sqrt{4.25}\sin(\pi x)}$ , where  $M, N = k$  and  $\sin(\pi x) \neq 0$ . If  $M, N = k + 1$ , then

$$\begin{aligned}
& -\sin(2\pi x)\sin\left(\frac{\pi}{2}y\right) \\
& = B_{11}(\pi\sqrt{1.25})\sin(\pi x)2\sin(\pi x)\sin\left(\frac{\pi}{2}y\right) \\
& \quad + B_{12}(\pi\sqrt{4.25})\sin(2\pi x)2\sin(\pi x)\sin\left(\frac{\pi}{2}y\right) \\
& \quad + \cdots + B_{1k}(\pi\sqrt{k^2 + 0.25})\sin(k\pi x)2\sin(\pi x)\sin\left(\frac{\pi}{2}y\right) \\
& \quad + B_{1(k+1)}(\pi\sqrt{(k+1)^2 + 0.25})\sin((k+1)\pi x)2\sin(\pi x)\sin\left(\frac{\pi}{2}y\right) \\
& \quad + B_{21}(\pi\sqrt{1 + 2.25})\sin(\pi x)2(-1)\sin(\pi x)\sin\left(\frac{3\pi}{2}y\right) \\
& \quad + B_{22}(\pi\sqrt{2^2 + 2.25})\sin(2\pi x)2(-1)\sin(\pi x)\sin\left(\frac{3\pi}{2}y\right) \\
& \quad + \cdots + B_{2k}(\pi\sqrt{k^2 + 2.25})\sin(k\pi x)2(-1)\sin(\pi x)\sin\left(\frac{3\pi}{2}y\right) \\
& \quad + B_{2(k+1)}(\pi\sqrt{(k+1)^2 + 2.25})\sin((k+1)\pi x)2(-1)\sin(\pi x)\sin\left(\frac{3\pi}{2}y\right) \\
& \quad + \cdots + B_{(k+1)1}\left(\pi\sqrt{1 + \left(\frac{2(k+1)-1}{2}\right)^2}\right)\sin(\pi x)2(-1)^{-k-2} \\
& \quad \times \sin(\pi x)\sin\left(\frac{(2(k+1)-1)\pi}{2}y\right) \\
& \quad + \cdots + B_{(k+1)1}\left(\pi\sqrt{1 + \left(\frac{2(k+1)-1}{2}\right)^2}\right)\sin(\pi x)2(-1)^{-k-2} \\
& \quad \times \sin(\pi x)\sin\left(\frac{(2(k+1)-1)\pi}{2}y\right) \\
& \quad + B_{(k+1)2}\left(\pi\sqrt{2^2 + \left(\frac{2(k+1)-1}{2}\right)^2}\right)\sin(2\pi x)2(-1)^{-k-2} \\
& \quad \times \sin(\pi x)\sin\left(\frac{(2(k+1)-1)\pi}{2}y\right) \\
& \quad + \cdots + B_{(k+1)k}\left(\pi\sqrt{k^2 + \left(\frac{2(k+1)-1}{2}\right)^2}\right)\sin(k\pi x)2(-1)^{-k-2} \\
& \quad \times \sin(\pi x)\sin\left(\frac{(2(k+1)-1)\pi}{2}y\right) \\
& \quad + B_{(k+1)(k+1)}\left(\pi\sqrt{(k+1)^2 + \left(\frac{2(k+1)-1}{2}\right)^2}\right)\sin((k+1)\pi x)2(-1)^{-k-2} \\
& \quad \times \sin(\pi x)\sin\left(\frac{(2(k+1)-1)\pi}{2}y\right). \tag{37}
\end{aligned}$$

Since  $\sin(n\pi x) \neq 0$  and  $\sin(\frac{(2m-1)\pi}{2}y) \neq 0$  for all  $x, y \in (0, 1)$ , we can see that  $B_{11}, B_{13}, B_{14}, \dots, B_{1(k+1)} = 0, B_{21}, B_{22}, \dots, B_{2(k+1)} = 0, \dots, B_{(k+1)1}, B_{(k+1)2}, \dots, B_{(k+1)(k+1)} = 0$ , except  $B_{12} \neq 0$ . We get

$$B_{12} = \frac{1}{2\pi\sqrt{4.25}\sin(\pi x)}. \tag{38}$$

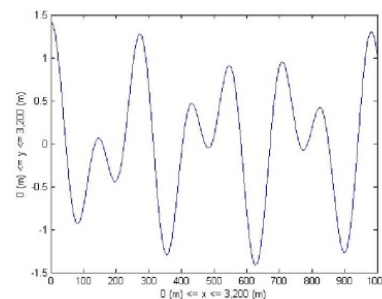
Thus, Eq. (34) holds for  $M = k + 1$  and  $N = k + 1$ , the proof of the induction step is completed. Therefore, an analytical solution  $d(x, y, t)$  of Eq. (27) with given initial and boundary conditions becomes

$$d(x, y, t) = 2\sin(\pi x)\cos(t)\sin\left(\frac{\pi}{2}y\right)\cos(\pi t\sqrt{1.25}) - \frac{1}{\pi\sqrt{4.25}}\sin(2\pi x)\cos(t)\sin\left(\frac{\pi}{2}y\right)\sin(\pi t\sqrt{4.25}) \tag{39}$$

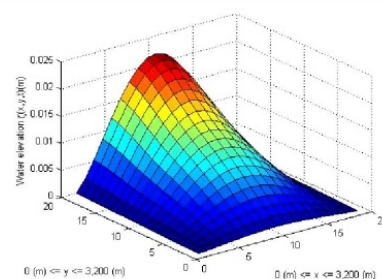
**Table 1** Water elevation  $\zeta(x, y, t)$  (m) at  $t = 2$  hrs 50 min

$y \setminus x$	320	640	960	1,280	1,600	1,920	2,240	2,560	2,880
320	0.1729	0.2620	0.3174	0.3352	0.3177	0.2716	0.2055	0.1272	0.0430
640	0.2835	0.4295	0.5203	0.5495	0.5209	0.4453	0.3368	0.2085	0.0704
960	0.3871	0.5864	0.7104	0.7503	0.7112	0.6080	0.4599	0.2847	0.0962
1,280	0.4811	0.7289	0.8830	0.9326	0.8840	0.7557	0.5716	0.3538	0.1195
1,600	0.5633	0.8534	1.0338	1.0919	1.0350	0.8848	0.6692	0.4143	0.1400
1,920	0.6316	0.9570	1.1592	1.2243	1.1605	0.9922	0.7504	0.4646	0.1570
2,240	0.6844	1.0369	1.2561	1.3266	1.2575	1.0751	0.8131	0.5034	0.1701
2,560	0.7203	1.0913	1.3220	1.3963	1.3235	1.1315	0.8558	0.5298	0.1790
2,880	0.7385	1.1189	1.3554	1.4315	1.3569	1.1600	0.8774	0.5432	0.1835

**Figure 2** Water elevation  $\zeta(x, y, t)$ , where  $0 \leq t \leq 170$  (min) at the center point of an open uniform reservoir.



**Figure 3** Surface plot of water elevation  $\zeta(x, y, t)$  at  $t = 170$  (min) over an open uniform reservoir.



for all  $(x, y, t) \in \Omega \times [0, T]$ . The solution of water wave elevations (tidal elevation) by using Eq. (39) is shown in Table 1 and Figures 2-3.

## 5 Conclusions

A two-dimensional hydrodynamic model for describing the elevation of water wave in an open uniform reservoir is derived and presented. The separation of variables method is employed to find an analytical solution to the model.

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### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

KT wrote the computer codes for the theoretical analysis, data acquisition, constructed the software and calibrated it and wrote the paper. JK contributed equally to the theoretical analysis, data acquisition, the interpretation of the solutions. Both authors read and approved the final manuscript.

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## Research Article

# Numerical Simulations of Water Quality Measurement Model in an Opened-Closed Reservoir with Contaminant Removal Mechanism

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The mathematical simulation of water contaminant measurement is often used to assess the water quality. The monitoring point placement for water quality measurement in an opened-closed reservoir can give accurate or inaccurate assessment. In this research, the mathematical model of the approximated water quality in an opened-closed reservoir with removal mechanism system is proposed. The water quality model consists of the hydrodynamic model and the dispersion model. The hydrodynamic model is used to describe the water current in the opened-closed reservoir. The transient advection-diffusion equation with removal mechanism provides the water pollutant concentration. The water velocity from the hydrodynamic model is plugged into the dispersion model. The finite difference techniques are used to approximate the solution of the water quality model. The proposed numerical simulations give a suitable area of zonal removal mechanism placement. The proposed simulations also give the overall and specified approximated water quality for each point and time when the exit gate is opened on the different periods of time. In addition, the proposed techniques can give a suitable period of time to open the exit gate to achieve a good agreement water quality by using contaminant removal mechanism.

## 1. Introduction

Field measurement and mathematical simulation are methods to detect the amount of the level of pollutants in water area. In water quality modeling for reservoir, the general governing equations used are the hydrodynamic model and the dispersion model. The two-dimensional shallow water equation and the advection-diffusion-reaction equation govern the first and the second models, respectively.

The several numerical techniques for solving such models were available. In [1–3], they used the hydrodynamic model and the dispersion model with the finite element method to approximate the velocity of the water current in bay, estuaries, and open reservoir, respectively. In [4], the finite element method was used for solving the water pollution levels to the optimal control of the water treatment plants to achieve minimum cost. In [5], the method of the characteristic technique

combined with Galerkin finite element method is used to solve the shallow water mass transport problems. In [6, 7], the numerical techniques are used to solve the nonuniform flow of stream water quality model with the advection-dispersion-reaction equations. In [8], the Crank-Nicolson method is used to solve the hydrodynamic model and the backward time central space (BTCS) for the dispersion model. In [9], the approximated solutions of the hydrodynamic model and advection-diffusion-reaction equation in a uniform reservoir are proposed. In [10], a nondimensional form of the hydrodynamic model with variable coefficients using Lax-Wendroff method is presented. In [11], the Lax-Wendroff finite difference method is also proposed to approximate the water elevation and water flow velocity with a rectangular domain. In [12], the mathematical models and numerical methods for approximating water current and pollutant concentration level in Rama-nine reservoirs are presented. In [13],

the Lax-Wendroff method for solving the dimensional form of shallow water equation in spherical model with Matlab program is proposed. In [14], an analytical solution to a hydrodynamic model in an open uniform reservoir with the specified tidal wave functions is proposed.

In this research, the mathematical models for water quality measurement which consist of the hydrodynamic model and the dispersion model, used to simulate water quality in a water flow systems, were considered. The first is a hydrodynamic model that provides the water current and the elevation of water in an opened-closed reservoir. The second is a dispersion model that gives the concentration of pollutant in an opened-closed reservoir with the contaminant removal mechanism. For numerical techniques, we used the Lax-Wendroff method to the system of the hydrodynamic model and the forward in time central in space (FTCS) to the dispersion model. The results from the shallow water equation of the hydrodynamic model are the water flow velocity which are input data for advection-diffusion-reaction equation which provides the level of pollutant concentration field. Averaging the equation over the depth with anisotropic bottom topography and discarding the term due to the Coriolis force, surface wind effect, and external forces, it follows that the two-dimensional shallow water and advection-diffusion-reaction equations are applicable.

## 2. Water Quality Measurement in Opened-Closed Reservoir with Removal Mechanism Model

The mathematical models for water quality measurement in opened-closed reservoir with removal mechanism are described. They are used to simulate time-varying pollutant levels caused by waste water discharges from external source into an opened-closed reservoir with removal mechanism and drain water at the exit gate. The first model is a hydrodynamic model that determined the velocity and elevation of the water at any locations in the reservoir with anisotropic bottom topography, while the second model is a pollutant dispersion with removal mechanism model that determined the pollutant level at any points in the reservoir.

**2.1. Hydrodynamic Model.** The two-dimensional unsteady water current into and out of the reservoir can be determined by using the system of shallow water equations as the conservation of mass and conservation of momentum. It is taken into account that the equations of the system of shallow water can be derived from depth-averaging Navier-Stokes equations in the vertical direction, neglecting the diffusion of momentum due to turbulence and discarding the terms expressing the effects of friction, surface wind, Coriolis factor, and shearing stresses. The continuity and momentum equation governs the hydrodynamic behavior of the reservoir [15]. The well known two-dimensional shallow water equations are [1, 2, 13, 16]

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0,$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial (u^2h + (1/2)gh^2)}{\partial x} + \frac{\partial (uvh)}{\partial y} = 0,$$

$$\frac{\partial (vh)}{\partial t} + \frac{\partial (uvh)}{\partial x} + \frac{\partial (v^2h + (1/2)gh^2)}{\partial y} = 0,$$

(1)

where  $x$  is the longitudinal distance along the reservoir (m),  $y$  is the transverse distance along the reservoir (m),  $t$  is the time (s),  $\eta(x, y)$  is the depth measured from the mean water level to the reservoir bed (m),  $\zeta(x, y, t)$  is the elevation of water surface from the mean water level in reservoir (m),  $h(x, y, t) = \eta(x, y) + \zeta(x, y, t)$  is the elevation of water surface measured from the mean water level to the bed of the reservoir,  $u(x, y, t)$  is velocity in  $x$ -direction (m/s),  $v(x, y, t)$  is velocity in  $y$ -direction (m/s), and  $g$  is gravitational constant ( $9.81 \text{ m/s}^2$ ), for all  $(x, y, t) \in \Omega = [0, l] \times [0, m] \times [0, T]$ .

Such independent variables  $x, y$ , and  $t$  make up the special dimensions and time. The dependent variables are the depth  $h$  with respect to the surface and the two-dimensional velocities  $u$  and  $v$ . The partial derivatives taken with respect to the same term ( $\partial t, \partial x, \partial y$ ) are grouped into vectors and rewritten as a single hyperbolic partial differential [13, 15],

$$\frac{\partial H}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \quad (2)$$

where

$$H = \begin{pmatrix} h \\ uh \\ vh \end{pmatrix},$$

$$U = \begin{pmatrix} uh \\ u^2h + \frac{1}{2}gh^2 \\ uvh \end{pmatrix}, \quad (3)$$

$$V = \begin{pmatrix} vh \\ uvh \\ v^2h + \frac{1}{2}gh^2 \end{pmatrix}.$$

## 2.2. Dispersion Model

**2.2.1. Water Pollutant Dispersion Model.** Applying the distributed pollutant process, including the transportation and diffusion, we have the mass transfer equation. There is a representation simplified by averaging the equation over the depth, generating the advection-diffusion equation as follows [4]:

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right), \quad (4)$$

where  $C(x, y, t)$  is the depth averaged water pollutant concentration at the point  $(x, y)$  and at the time  $t$  ( $\text{kg/m}^3$ ) and  $D$  is the pollutant dispersion coefficient ( $\text{m}^2/\text{s}$ ).

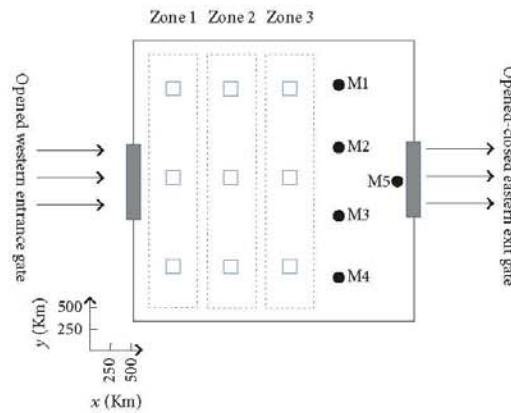


FIGURE 1: An opened-closed reservoir with contaminant removal mechanism zones and monitoring points M1, M2, M3, M4, and M5.

2.2.2. *Water Pollutant Dispersion with Removal Mechanism Model.* The mechanisms of pollutant removal are introduced by decaying chemical reaction and absorptive reduction. A representation is modified by generating the advection-diffusion-reaction with sink term [2],

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) - RC - Q(x, y, t), \tag{5}$$

where  $C(x, y, t)$  is the depth averaged water pollutant concentration at the point  $(x, y)$  and at the time  $t$  ( $\text{kg}/\text{m}^3$ ),  $D$  is the pollutant dispersion coefficient ( $\text{m}^2/\text{s}$ ),  $R \geq 0$  is the water pollutant decaying rate ( $\text{s}^{-1}$ ), and  $Q(x, y, t)$  is the decreasing rate of water pollutant concentration due to a water pollutant sink ( $\text{kg}/\text{m}^3\text{s}$ ).

2.3. *Initial and Boundary Conditions of Hydrodynamic Model in Opened-Closed Reservoir.* The initial conditions of (2) are assumed to be motionless,

$$\begin{aligned} u(x, y, 0) &= 0, \\ v(x, y, 0) &= 0, \\ h(x, y, 0) &= 0. \end{aligned} \tag{6}$$

An opened-closed reservoir supplied by the outer water wave is going to flow into the reservoir such that the water is drained as shown in Figure 1. The elevation of water on the opened gate is assumed to be a wave maker function, say  $f$ . The elevation of water on the exit gates is assumed to be  $\partial h/\partial x = \kappa$ , where  $\kappa < 0$  is the rate of change between inner and outer water level through the eastern exit gate.

The boundary conditions of the model in an opened-closed reservoir are assumed as follows:

- (i)  $u = 0, \partial v/\partial y = 0$ , and  $\partial h/\partial y = 0$ , for all  $(x, y)$  lie on the horizontal edges of the reservoir.

- (ii)  $v = 0, \partial u/\partial x = 0$ , and  $\partial h/\partial x = 0$  for all  $(x, y)$  lie on the vertical edges of the reservoir.

- (iii)  $v = 0, \partial u/\partial x = 0$ , and  $h = f$  for all  $(x, y)$  lie on the flowing water into the opened entrance gate.

- (iv)  $v = 0, \partial u/\partial x = u_e$ , and  $\partial h/\partial x = \kappa$  for all  $(x, y)$  lie on the flowing water at the opened-closed exit gate. The initial and boundary conditions are also shown in Figure 2.

2.4. *Initial and Boundary Conditions of Water Pollutant Dispersion Model in Opened-Closed Reservoir.* The initial pollutant concentration in reservoir is  $c_0$  ( $\text{kg}/\text{m}^3$ ). The water pollutant is discharged from the open gate into the opened-closed reservoir which are assumed to be  $C(x, y, t) = c_1$ , where  $c_1$  ( $\text{kg}/\text{m}^3$ ) is the averaged discharged pollutant concentration along the entrance gate. The opened-closed reservoir drain water at the exit gate by assuming rate of water drain as  $\partial C/\partial x = -c_2$ , where  $c_2$  is a nonnegative rate of change of pollutant concentration across the exit gate. The embankment of reservoir is a nonabsorbing boundary bank. Consequently, there is no rate of change of pollutant concentration at the boundary of opened-closed reservoir;  $\partial C/\partial n = 0$ , where  $n$  is a normal vector, as shown in Figure 3.

2.5. *Removal Pollutant Concentration Mechanism in Water Pollutant Dispersion Model.* The removal terms can model a variety of different phenomena, that is, the removal of pollutant concentration in (5) with absorptive rate function  $Q = Q(x, y, t)$  ( $\text{kg}/\text{m}^3\text{s}$ ), where  $Q$  is a nonnegative real value function.

The term  $R$  in (5) arises when there is pollutant concentration loss from a water surface; the reaction term is used to describe the decaying rate of pollutant concentration, where  $R$  is a nonnegative constant.

### 3. Numerical Techniques

The hydrodynamic model provides the velocity field and elevation of the water. Then the calculated results will be input into the dispersion model that provides the pollutant concentration results.

3.1. *Lax-Wendroff Method for the Hydrodynamic Model.* We apply the numerical method for solving the single hyperbolic partial differential equations known as the Lax-Wendroff method. The Lax-Wendroff method involves starting to calculate a first half step and then using the result from the half step to calculate the full step [7, 8]. To find the water velocity and water elevation, we now discretize (2) on a rectangle reservoir with length  $l$  and width  $m$ . The domain  $[0, l] \times [0, m]$  is divided into  $L$  and  $M$  subintervals such that  $L\Delta x = l$  and  $M\Delta y = m$ , respectively. The time intervals  $[0, k]$  are also divided into  $T$  subintervals such that  $T\Delta t = k$ . We can then approximate  $H(x_i, y_j, t_n)$  by  $H_{i,j}^n$  and the value of the difference approximation of  $H(x, y, t)$  at points  $x = i\Delta x$  and  $y = j\Delta y$  and time  $t = n\Delta t$  where  $0 \leq i \leq L, 0 \leq j \leq M$ , and  $0 \leq n \leq T$ , similarly defined for  $U_{i,j}^n$  and  $V_{i,j}^n$ . The grid

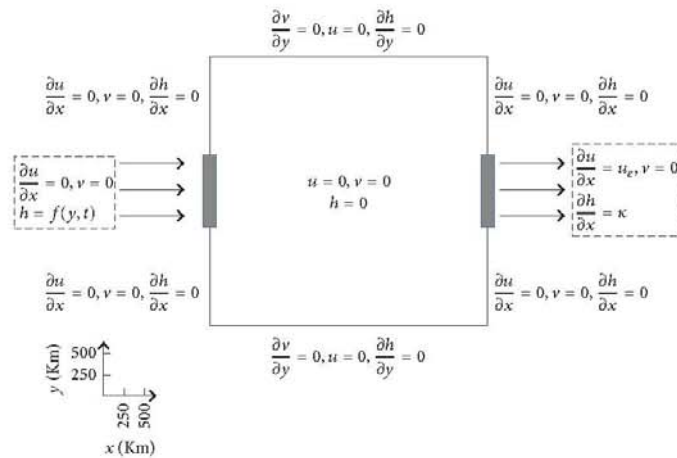


FIGURE 2: Initial and boundary conditions of hydrodynamic model in opened-closed reservoir.

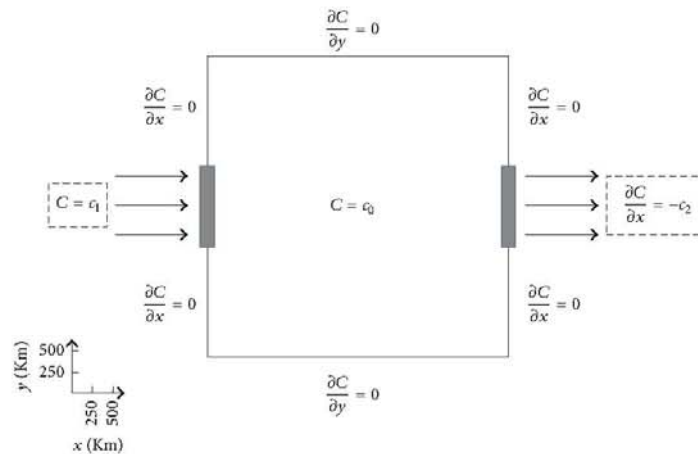


FIGURE 3: The initial and boundary conditions of water pollutant dispersion model in opened-closed reservoir.

points are defined by  $x_i = i\Delta x$  for all  $i = 0, 1, 2, \dots, L$ ,  $y_j = j\Delta y$  for all  $j = 0, 1, 2, \dots, M$ , and  $t_n = n\Delta t$  for all  $n = 0, 1, 2, \dots, T$  in which  $L, M$ , and  $T$  are positive integers. Using the Lax-Wendroff method [7, 8] on (2), we can obtain the following finite difference equation of hyperbolic partial differential equations.

The first half step defines values of  $H$  at time step  $n + 1/2$  and the midpoint of the grid cell; the first half steps are stored in separate matrices  $H_x$  and  $H_y$ , to be reintroduced during the second half step as shown in Figure 4. The first half steps for  $U$  and  $V$  are approximated in a similar process and are stored in  $U_x, U_y$  and  $V_x, V_y$  as follows [13, 15]:

$$H_{x_{i+1/2,j}}^{n+1/2} = \frac{1}{2} (H_{i+1,j}^n + H_{i,j}^n) - \frac{\Delta t}{2\Delta x} (U_{i+1,j}^n - U_{i,j}^n), \quad (7)$$

$$H_{y_{i,j+1/2}}^{n+1/2} = \frac{1}{2} (H_{i,j+1}^n + H_{i,j}^n) - \frac{\Delta t}{2\Delta y} (V_{i,j+1}^n - V_{i,j}^n), \quad (8)$$

$$U_{x_{i+1/2,j}}^{n+1/2} = \frac{1}{2} (U_{i+1,j}^n + U_{i,j}^n) - \frac{\Delta t}{2\Delta x} \left[ \left( (U_{i+1,j}^n)^2 H_{i+1,j}^n - (U_{i,j}^n)^2 H_{i,j}^n \right) + \left( \frac{g}{2} (H_{i+1,j}^n)^2 - \frac{g}{2} (H_{i,j}^n)^2 \right) \right], \quad (9)$$

$$U_{y_{i,j+1/2}}^{n+1/2} = \frac{1}{2} (U_{i,j+1}^n + U_{i,j}^n) - \frac{\Delta t}{2\Delta y} \left[ (U_{i,j+1}^n V_{i,j+1}^n H_{i,j+1}^n) - (U_{i,j}^n V_{i,j}^n H_{i,j}^n) \right], \quad (10)$$

$$V_{x_{i+1/2,j}}^{n+1/2} = \frac{1}{2} (V_{i+1,j}^n + V_{i,j}^n) - \frac{\Delta t}{2\Delta x} \left[ (U_{i+1,j}^n V_{i+1,j}^n H_{i+1,j}^n) - (U_{i,j}^n V_{i,j}^n H_{i,j}^n) \right], \quad (11)$$

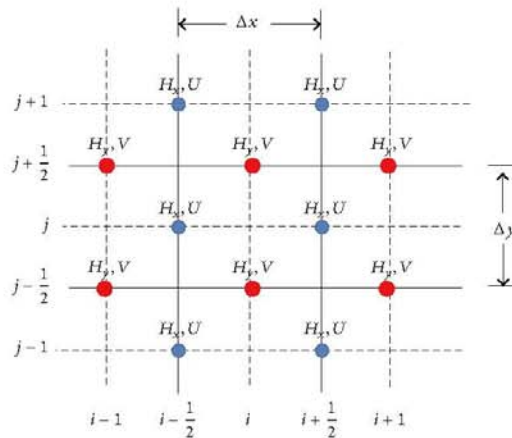


FIGURE 4: The values of vector  $H_x$ ,  $H_y$ ,  $U$ , and  $V$  represented the solution at the midpoints of the grid cell.

$$\begin{aligned}
 V_{y_{j+1/2}}^{n+1/2} &= \frac{1}{2} (V_{i,j+1}^n + V_{i,j}^n) \\
 &- \frac{\Delta t}{2\Delta y} \left[ \left( (U_{i,j+1}^n)^2 H_{i,j+1}^n - (U_{i,j}^n)^2 H_{i,j}^n \right) \right. \\
 &\left. + \left( \frac{g}{2} (H_{i,j+1}^n)^2 - \frac{g}{2} (H_{i,j}^n)^2 \right) \right]. \quad (12)
 \end{aligned}$$

The second half step completes the time step  $n+1$  by using the values calculated in the first half step to calculate new values at the centre of the grid cell [13, 15]:

$$\begin{aligned}
 H_{i,j}^{n+1} &= H_{i,j}^n - \frac{\Delta t}{\Delta x} (U_{x_{i+1/2,j}}^{n+1/2} - U_{x_{i-1/2,j}}^{n+1/2}) - \frac{\Delta t}{\Delta y} (V_{y_{j+1/2}}^{n+1/2} \\
 &- V_{y_{j-1/2}}^{n+1/2}), \\
 U_{i,j}^{n+1} &= U_{i,j}^n \\
 &- \frac{\Delta t}{\Delta x} \left[ \left( (U_{x_{i+1/2,j}}^{n+1/2})^2 (H_{x_{i+1/2,j}}^{n+1/2}) + \frac{g}{2} (H_{x_{i+1/2,j}}^{n+1/2})^2 \right) \right. \\
 &- \left. \left( (U_{x_{i-1/2,j}}^{n+1/2})^2 (H_{x_{i-1/2,j}}^{n+1/2}) + \frac{g}{2} (H_{x_{i-1/2,j}}^{n+1/2})^2 \right) \right] \\
 &- \frac{\Delta t}{\Delta y} \left[ (U_{y_{j+1/2}}^{n+1/2}) (V_{y_{j+1/2}}^{n+1/2}) (H_{y_{j+1/2}}^{n+1/2}) \right. \\
 &- \left. (U_{y_{j-1/2}}^{n+1/2}) (V_{y_{j-1/2}}^{n+1/2}) (H_{y_{j-1/2}}^{n+1/2}) \right], \\
 V_{i,j}^{n+1} &= V_{i,j}^n - \frac{\Delta t}{\Delta x} \left[ (U_{x_{i+1/2,j}}^{n+1/2}) (V_{x_{i+1/2,j}}^{n+1/2}) (H_{x_{i+1/2,j}}^{n+1/2}) \right. \\
 &- \left. (U_{x_{i-1/2,j}}^{n+1/2}) (V_{x_{i-1/2,j}}^{n+1/2}) (H_{x_{i-1/2,j}}^{n+1/2}) \right] \\
 &- \frac{\Delta t}{\Delta y} \left[ \left( (V_{y_{j+1/2}}^{n+1/2})^2 (H_{y_{j+1/2}}^{n+1/2}) + \frac{g}{2} (H_{y_{j+1/2}}^{n+1/2})^2 \right) \right. \\
 &- \left. \left( (V_{y_{j-1/2}}^{n+1/2})^2 (H_{y_{j-1/2}}^{n+1/2}) + \frac{g}{2} (H_{y_{j-1/2}}^{n+1/2})^2 \right) \right]. \quad (13)
 \end{aligned}$$

The finite difference method computes a numerical approximation to the boundary conditions of the reservoir. For the left boundary condition, we have  $i = 0$  and  $1 \leq j \leq M$ , letting  $H_{0,j}^n = H_{1,j}^n$ ,  $U_{0,j}^n = U_{1,j}^n$ , and  $V_{0,j}^n = V_{1,j}^n$ . Substituting the approximate unknown vector nodes  $H_{0,j}^n$ ,  $U_{0,j}^n$ , and  $V_{0,j}^n$  of the left boundary into (7), (9), and (11), respectively,

$$\begin{aligned}
 H_{x_{1/2,j}}^{n+1/2} &= \frac{1}{2} (H_{1,j}^n + H_{0,j}^n) - \frac{\Delta t}{2\Delta x} (U_{1,j}^n - U_{0,j}^n) = H_{1,j}^n, \\
 U_{x_{1/2,j}}^{n+1/2} &= \frac{1}{2} (U_{1,j}^n + U_{0,j}^n) \\
 &- \frac{\Delta t}{2\Delta x} \left[ \left( (U_{1,j}^n)^2 H_{1,j}^n - (U_{0,j}^n)^2 H_{0,j}^n \right) \right. \\
 &\left. + \left( \frac{g}{2} (H_{1,j}^n)^2 - \frac{g}{2} (H_{0,j}^n)^2 \right) \right] = U_{1,j}^n, \\
 V_{x_{1/2,j}}^{n+1/2} &= \frac{1}{2} (V_{1,j}^n + V_{0,j}^n) - \frac{\Delta t}{2\Delta x} \left[ (U_{1,j}^n V_{1,j}^n H_{1,j}^n) \right. \\
 &- \left. (U_{0,j}^n V_{0,j}^n H_{0,j}^n) \right] = V_{1,j}^n.
 \end{aligned} \quad (14)$$

For the right boundary condition, we have  $i = L$  and  $1 \leq j \leq M$ , letting  $H_{L+1,j}^n = H_{L,j}^n$ ,  $U_{L+1,j}^n = U_{L,j}^n$ , and  $V_{L+1,j}^n = V_{L,j}^n$ . Substituting the approximate unknown vector nodes  $H_{L+1,j}^n$ ,  $U_{L+1,j}^n$ , and  $V_{L+1,j}^n$  of the right boundary into (7), (9), and (11), respectively,

$$\begin{aligned}
 H_{x_{L+1/2,j}}^{n+1/2} &= \frac{1}{2} (H_{L+1,j}^n + H_{L,j}^n) - \frac{\Delta t}{2\Delta x} (U_{L+1,j}^n - U_{L,j}^n) \\
 &= H_{L,j}^n, \\
 U_{x_{L+1/2,j}}^{n+1/2} &= \frac{1}{2} (U_{L+1,j}^n + U_{L,j}^n) \\
 &- \frac{\Delta t}{2\Delta x} \left[ \left( (U_{L+1,j}^n)^2 H_{L+1,j}^n - (U_{L,j}^n)^2 H_{L,j}^n \right) \right. \\
 &\left. + \left( \frac{g}{2} (H_{L+1,j}^n)^2 - \frac{g}{2} (H_{L,j}^n)^2 \right) \right] = U_{L,j}^n, \\
 V_{x_{L+1/2,j}}^{n+1/2} &= \frac{1}{2} (V_{L+1,j}^n + V_{L,j}^n) \\
 &- \frac{\Delta t}{2\Delta x} \left[ (U_{L+1,j}^n V_{L+1,j}^n H_{L+1,j}^n) - (U_{L,j}^n V_{L,j}^n H_{L,j}^n) \right] \\
 &= V_{L,j}^n.
 \end{aligned} \quad (15)$$

For the lower boundary condition, we have  $1 \leq i \leq L$  and  $j = 0$ , letting  $H_{i,0}^n = H_{i,1}^n$ ,  $U_{i,0}^n = U_{i,1}^n$ , and  $V_{i,0}^n = V_{i,1}^n$ . Substituting the approximate unknown vector nodes  $H_{i,0}^n$ ,  $U_{i,0}^n$ , and  $V_{i,0}^n$  of the lower boundary into (8), (10), and (12), respectively,

$$\begin{aligned}
 H_{y_{1/2}}^{n+1/2} &= \frac{1}{2} (H_{i,1}^n + H_{i,0}^n) - \frac{\Delta t}{2\Delta y} (V_{i,1}^n - V_{i,0}^n) = H_{i,1}^n, \\
 U_{y_{1/2}}^{n+1/2} &= \frac{1}{2} (U_{i,1}^n + U_{i,0}^n) - \frac{\Delta t}{2\Delta y} \left[ (U_{i,1}^n V_{i,1}^n H_{i,1}^n) \right. \\
 &- \left. (U_{i,0}^n V_{i,0}^n H_{i,0}^n) \right] = U_{i,1}^n,
 \end{aligned}$$

$$\begin{aligned}
V_{y,i/2}^{n+1/2} &= \frac{1}{2} (V_{i,1}^n + V_{i,0}^n) \\
&\quad - \frac{\Delta t}{2\Delta y} \left[ \left( (U_{i,1}^n)^2 H_{i,1}^n - (U_{i,0}^n)^2 H_{i,0}^n \right) \right. \\
&\quad \left. + \left( \frac{g}{2} (H_{i,1}^n)^2 - \frac{g}{2} (H_{i,0}^n)^2 \right) \right] = V_{i,1}^n.
\end{aligned} \tag{16}$$

For the upper boundary condition, we have  $1 \leq i \leq L$  and  $j = M$ , letting  $H_{i,M+1}^n = H_{i,M}^n$ ,  $U_{i,M+1}^n = U_{i,M}^n$  and  $V_{i,M+1}^n = V_{i,M}^n$ . Substituting the approximate unknown vector nodes  $H_{i,M+1}^n$ ,  $U_{i,M+1}^n$ , and  $V_{i,M+1}^n$  of the upper boundary into (8), (10), and (12), respectively,

$$\begin{aligned}
H_{y,M+1/2}^{n+1/2} &= \frac{1}{2} (H_{i,M+1}^n + H_{i,M}^n) - \frac{\Delta t}{2\Delta y} (V_{i,M+1}^n - V_{i,M}^n) \\
&= H_{i,M}^n, \\
U_{y,M+1/2}^{n+1/2} &= \frac{1}{2} (U_{i,M+1}^n + U_{i,M}^n) \\
&\quad - \frac{\Delta t}{2\Delta y} \left[ (U_{i,M+1}^n V_{i,M+1}^n H_{i,M+1}^n) - (U_{i,M}^n V_{i,M}^n H_{i,M}^n) \right] \\
&= U_{i,M}^n,
\end{aligned} \tag{17}$$

$$\begin{aligned}
V_{y,M+1/2}^{n+1/2} &= \frac{1}{2} (V_{i,M+1}^n + V_{i,M}^n) \\
&\quad - \frac{\Delta t}{2\Delta y} \left[ \left( (U_{i,M+1}^n)^2 H_{i,M+1}^n - (U_{i,M}^n)^2 H_{i,M}^n \right) \right. \\
&\quad \left. + \left( \frac{g}{2} (H_{i,M+1}^n)^2 - \frac{g}{2} (H_{i,M}^n)^2 \right) \right] = V_{i,M}^n.
\end{aligned}$$

**3.2. Forward Time Central Space Method for Water Pollutant Dispersion with Removal Mechanism Model.** We can then approximate  $C(x_i, y_j, t_n)$  by  $C_{i,j}^n$ ; the value of the difference approximation of  $C(x, y, t)$  at point the grid point  $(x_i, y_j, t_n)$  is defined by  $x_i = i\Delta x$  for all  $i = 0, 1, 2, \dots, L$ ,  $y_j = j\Delta y$  for all  $j = 0, 1, 2, \dots, M$ , and  $t_n = n\Delta t$  for all  $n = 0, 1, 2, \dots, T$ , in which  $L, M$ , and  $T$  are positive integers. We use the forward differences in time and central difference in space in advection-diffusion equation; we get the finite difference equations as follows [15]:

$$\begin{aligned}
C &\approx C_{i,j}^n, \\
\frac{\partial C}{\partial t} &\approx \frac{C_{i,j}^{n+1} - C_{i,j}^n}{\Delta t}, \\
\frac{\partial C}{\partial x} &\approx \frac{C_{i+1,j}^n - C_{i-1,j}^n}{2\Delta x}, \\
\frac{\partial C}{\partial y} &\approx \frac{C_{i,j+1}^n - C_{i,j-1}^n}{2\Delta y}, \\
\frac{\partial^2 C}{\partial x^2} &\approx \frac{C_{i+1,j}^n - 2C_{i,j}^n + C_{i-1,j}^n}{(\Delta x)^2},
\end{aligned}$$

$$\frac{\partial^2 C}{\partial y^2} \approx \frac{C_{i,j+1}^n - 2C_{i,j}^n + C_{i,j-1}^n}{(\Delta y)^2},$$

$$u \approx U_{i,j}^n$$

$$v \approx V_{i,j}^n$$

$$Q = Q_{i,j}^n.$$

(18)

Taking the forward time central space technique [15] into (5), we get the following finite difference equation:

$$\begin{aligned}
&\frac{C_{i,j}^{n+1} - C_{i,j}^n}{\Delta t} + u_{i,j}^n \left( \frac{C_{i+1,j}^n + C_{i-1,j}^n}{2\Delta x} \right) \\
&\quad + v_{i,j}^n \left( \frac{C_{i,j+1}^n + C_{i,j-1}^n}{2\Delta y} \right) \\
&= D \left( \frac{C_{i+1,j}^n - 2C_{i,j}^n + C_{i-1,j}^n}{(\Delta x)^2} \right. \\
&\quad \left. + \frac{C_{i,j+1}^n - 2C_{i,j}^n + C_{i,j-1}^n}{(\Delta y)^2} \right) - RC_{i,j}^n - Q_{i,j}^n.
\end{aligned} \tag{19}$$

Rearrangement of (19) gives

$$\begin{aligned}
C_{i,j}^n &= (\gamma_x - \lambda_x u_{i,j}^n) C_{i+1,j}^n + (\gamma_y - \lambda_y v_{i,j}^n) C_{i,j+1}^n \\
&\quad + (\gamma_x + \lambda_x u_{i,j}^n) C_{i-1,j}^n + (\gamma_y + \lambda_y v_{i,j}^n) C_{i,j-1}^n \\
&\quad + (1 - 2\gamma_x - 2\gamma_y - R\Delta t) C_{i,j}^n - Q_{i,j}^n \Delta t,
\end{aligned} \tag{20}$$

where  $\lambda_x = \Delta t/2\Delta x$ ,  $\lambda_y = \Delta t/2\Delta y$ ,  $\gamma_x = D\Delta t/(\Delta x)^2$ ,  $\gamma_y = D\Delta t/(\Delta y)^2$ ,  $D$  is the diffusion coefficient ( $\text{m}^2/\text{s}$ ),  $R$  is the substance decaying rate ( $\text{s}^{-1}$ ), and  $Q(x, y, t)$  is the decreasing rate of water pollutant concentration due to a water pollutant sink ( $\text{kg}/\text{m}^3\text{s}$ ).

If  $C_{i,j}^n$  is on the boundary of the opened-closed reservoir, it is calculated by applying the backward difference scheme to the right boundary condition and the upper boundary condition. We also take the forward difference scheme to the left boundary condition and the lower boundary condition.

For the left boundary condition, we have  $i = 1$  and  $1 \leq j \leq M$ , letting  $C_{0,j}^n = C_{1,j}^n$ . Substitute the approximate unknown vector nodes  $C_{0,j}^n$  of the left boundary into (20) as follows:

$$\begin{aligned}
&\frac{C_{1,j}^{n+1} - C_{1,j}^n}{\Delta t} + u_{1,j}^n \left( \frac{C_{2,j}^n + C_{0,j}^n}{2\Delta x} \right) \\
&\quad + v_{1,j}^n \left( \frac{C_{1,j+1}^n + C_{1,j-1}^n}{2\Delta y} \right) \\
&= D \left( \frac{C_{2,j}^n - C_{1,j}^n}{(\Delta x)^2} + \frac{C_{1,j+1}^n - 2C_{1,j}^n + C_{1,j-1}^n}{(\Delta y)^2} \right) \\
&\quad - RC_{1,j}^n - Q_{1,j}^n.
\end{aligned} \tag{21}$$

For the right boundary condition, we have  $i = L$  and  $1 \leq j \leq M$ , letting  $C_{L+1,j}^n = C_{L,j}^n$ . Substitute the approximate unknown vector nodes  $C_{L+1,j}^n$  of the right boundary into (20) as follows:

$$\begin{aligned} & \frac{C_{L,j}^{n+1} - C_{L,j}^n}{\Delta t} + u_{L,j}^n \left( \frac{C_{L+1,j}^n + C_{L-1,j}^n}{2\Delta x} \right) \\ & + v_{L,j}^n \left( \frac{C_{L,j+1}^n + C_{L,j-1}^n}{2\Delta y} \right) \\ & = D \left( \frac{-C_{L,j}^n - C_{L-1,j}^n}{(\Delta x)^2} + \frac{C_{L,j+1}^n - 2C_{L,j}^n + C_{L,j-1}^n}{(\Delta y)^2} \right) \\ & - RC_{L,j}^n - Q_{L,j}^n. \end{aligned} \quad (22)$$

For the lower boundary condition, we have  $1 \leq i \leq L$  and  $j = 1$ , letting  $C_{i,0}^n = C_{i,1}^n$ . Substitute the approximate unknown vector nodes  $C_{i,0}^n$  of the lower boundary into (20) as follows:

$$\begin{aligned} & \frac{C_{i,1}^{n+1} - C_{i,1}^n}{\Delta t} + u_{i,1}^n \left( \frac{C_{i+1,1}^n + C_{i-1,1}^n}{2\Delta x} \right) \\ & + v_{i,j}^n \left( \frac{C_{i,2}^n + C_{i,0}^n}{2\Delta y} \right) \\ & = D \left( \frac{C_{i+1,1}^n - 2C_{i,1}^n + C_{i-1,1}^n}{(\Delta x)^2} + \frac{C_{i,2}^n - C_{i,1}^n}{(\Delta y)^2} \right) \\ & - RC_{i,1}^n - Q_{i,1}^n. \end{aligned} \quad (23)$$

For the upper boundary condition, we have  $1 \leq i \leq L$  and  $j = M$ , letting  $C_{i,M+1}^n = C_{i,M}^n$ . Substitute the approximate unknown vector nodes  $C_{i,M+1}^n$  of the upper boundary into (20) as follows:

$$\begin{aligned} & \frac{C_{i,M}^{n+1} - C_{i,M}^n}{\Delta t} + u_{i,M}^n \left( \frac{C_{i+1,M}^n + C_{i-1,M}^n}{2\Delta x} \right) \\ & + v_{i,j}^n \left( \frac{C_{i,M+1}^n + C_{i,M-1}^n}{2\Delta y} \right) \\ & = D \left( \frac{C_{i+1,M}^n - 2C_{i,M}^n + C_{i-1,M}^n}{(\Delta x)^2} + \frac{-C_{i,M}^n + C_{i,M-1}^n}{(\Delta y)^2} \right) \\ & - RC_{i,M}^n - Q_{i,M}^n. \end{aligned} \quad (24)$$

#### 4. Numerical Simulations

Assume that the western gate is opened and the water elevation along the gate is described as a function  $f = 1 + (0.25 \sin(0.1t))$ . The elevation of water on the eastern exit gates is assumed to be  $\partial h / \partial x = \kappa = 0.05$ , where  $\kappa$  is the rate of change between inner and outer water level through the eastern exit gate. The initial pollutant concentration in reservoir is  $c_0 = 1.50$  (kg/m<sup>3</sup>). The water pollutant is released from the open western gate into the reservoir, which is the averaged pollutant concentration along the entrance gate

TABLE 1: Removal mechanism locations  $(x, y)$  (meter, meter).

Removal mechanism	Northern	Center	Southern
Zone 1	(200, 1600)	(200, 1080)	(200, 400)
Zone 2	(800, 1600)	(800, 1080)	(800, 400)
Zone 3	(1200, 1600)	(1200, 1080)	(1200, 400)

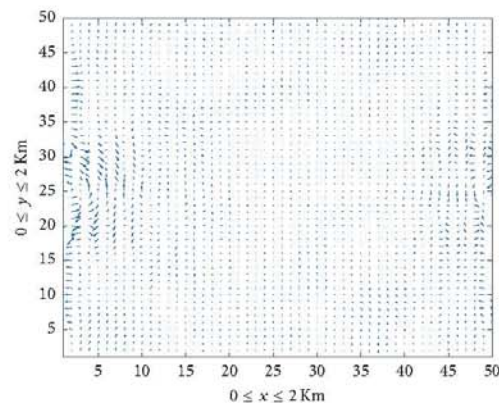


FIGURE 5: Approximated water velocity in an opened-closed reservoir.

defined by  $c_1 = 10.00$  (kg/m<sup>3</sup>). The reservoir has drained water off through the exit eastern gate with the rate of change of pollutant concentration across the gate defined by  $c_2 = -0.001$ . The embankment of reservoir is a nonabsorbing boundary bank. Consequently, there is no rate of change of pollutant concentration at the boundary of opened-closed reservoir,  $\partial C / \partial n = 0$ , where  $n$  is a normal vector. There are three contaminant removal mechanisms in the reservoir as shown in Figure 1. There are 3 mechanisms such as northern, center, and southern mechanisms which are contained in each zone. The removal mechanisms for each zone are located as shown in Table 1. If the contaminant removal mechanism is added, assume that the removal mechanisms have the same absorptive rate that can be described as a function  $Q(x, y, t) = 0.10$  (kg/m<sup>3</sup>s). There is a pollutant concentration which is less from the water surface, the decaying rate  $R = 0.1 \times 10^{-6}$  (s<sup>-1</sup>).

The water flow velocity in  $x$ - and  $y$ -directions and the water elevations are approximated by using (13) as shown in Tables 3 and 4 and Figure 5. The water pollutant concentrations are approximated by using (20). The approximated pollutant concentrations in an opened-closed reservoir are shown in Table 5.

*4.1. Simulation 1: The Different Activated Zonal Removal Mechanism Systems Effect.* The approximated pollutant concentration levels in removal mechanism system and non-removal mechanism system are compared in Figure 6. The approximated pollutant concentrations, when the first zonal removal mechanism systems are activated, are shown as a surface plot in Figure 7. The approximated pollutant

TABLE 2: Monitoring point locations  $(x, y)$  (meter, meter).

Monitoring points	M1	M2	M3	M4	M5
Locations	(1600, 1600)	(1600, 1200)	(1600, 800)	(1600, 400)	(2000, 1000)

TABLE 3: Approximated water flow velocity in  $x$ -direction  $u(x, y, t)$  (m/s).

$y/x$ (m)	0	400	800	1200	1600	2000
800	0.0000	0.0870	-0.0356	0.0411	0.0219	-0.1939
1000	0.0000	-0.0344	-0.0021	0.0103	-0.0487	0.0305
1200	0.0000	-0.1260	0.0491	-0.0584	0.0283	0.1866

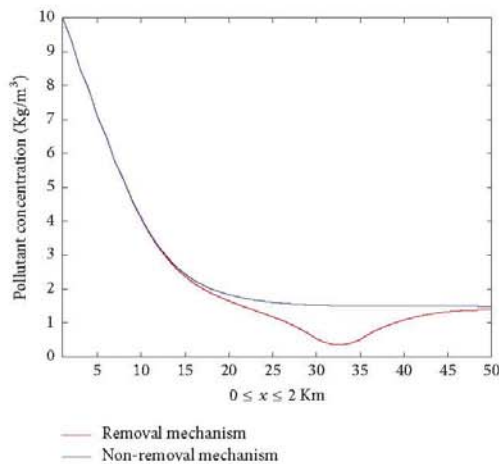


FIGURE 6: Comparison of approximated pollutant concentrations in removal mechanism system and nonremoval mechanism system.

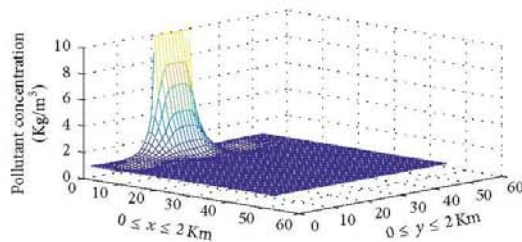


FIGURE 7: Approximated pollutant concentrations when the first zonal removal mechanism system is activated.

concentrations, when the second zonal removal mechanism systems are activated, are shown as a surface plot in Figure 8. The approximated pollutant concentrations, when the third zonal removal mechanism systems are activated, are shown as a surface plot in Figure 9.

The approximated pollutant concentrations along the northern part,  $C(x, y, t)$ , for all  $0 \leq x \leq 2000$  m,  $y = 1200$  m, and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated,

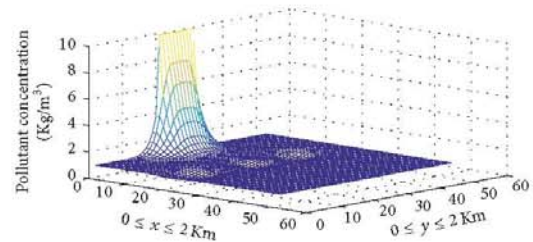


FIGURE 8: Approximated pollutant concentrations when the second zonal removal mechanism system is activated.

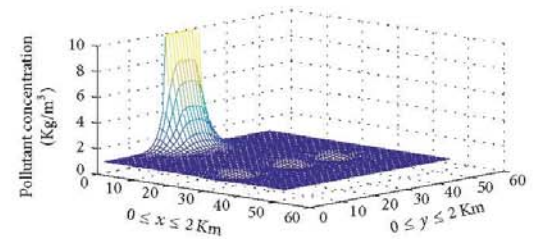


FIGURE 9: Approximated pollutant concentrations when the third zonal removal mechanism system is activated.

are compared in Figure 10. The approximated pollutant concentrations along the middle part,  $C(x, y, t)$ , for all  $0 \leq x \leq 2000$  m,  $y = 1000$  m, and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated, are compared in Figure 11. The approximated pollutant concentrations along the southern part,  $C(x, y, t)$ , for all  $0 \leq x \leq 2000$  m,  $y = 800$  m, and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated, are compared in Figure 12.

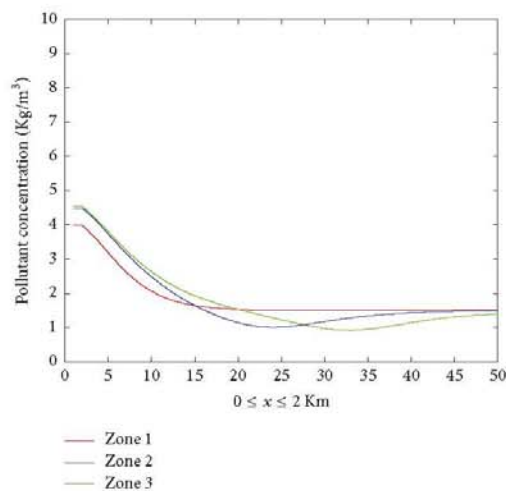
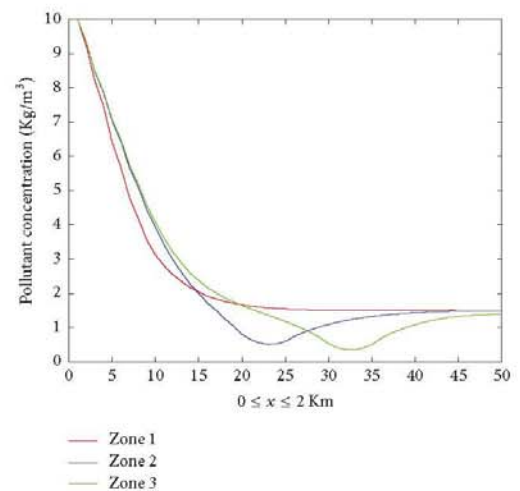
**4.2. Simulation 2: The Exit Gate Opening on Different Periods of Time Effect.** Assuming that the third zonal removal mechanism systems are activated. The approximated pollutant concentrations over 5 monitoring nodes, when the third zonal removal mechanism systems are activated and the eastern exit gate is opened, are shown in Figure 13. Five monitoring nodes are located as shown in Table 2. If the eastern exit gate is opened or shut on different periods of

TABLE 4: Approximated water flow velocity in  $y$ -direction  $v(x, y, t)$  (m/s).

$y/x$ (m)	0	400	800	1200	1600	2000
800	0.6213	0.0256	-0.1082	0.0412	0.0323	0.1251
1000	-0.2959	0.4853	-0.1423	-0.0040	-0.1365	-0.5382
1200	0.5779	-0.0048	-0.1376	0.1127	0.0168	0.1442

TABLE 5: Approximated pollutant concentration  $C(x, y, t)$  ( $\text{kg}/\text{m}^3$ ).

$y/x$ (m)	0	400	800	1200	1600	2000
800	5.7854	2.9295	1.5580	0.7936	1.0637	1.3736
1000	10.0000	4.0758	1.6448	0.5297	1.0873	1.3920
1200	5.8178	2.9421	1.5798	0.9594	1.1355	1.3860

FIGURE 10: Comparison of approximated pollutant concentrations along the northern part,  $C(x, y, t)$ , for all  $0 \leq x \leq 2000$  m and  $y = 1200$  m and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated.FIGURE 11: Comparison of approximated pollutant concentrations along the middle part,  $C(x, y, t)$ , for all  $0 \leq x \leq 2000$  m,  $y = 1000$  m, and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated.

time such as 0–100 min, 33–100 min, and 66–100 min, the approximated pollutant concentrations over 5 monitoring nodes are also compared in Figure 13. The comparison of approximated pollutant concentrations along the eastern exit gate,  $C(x, y, t)$ , for all  $800 \leq y \leq 1200$  m,  $x = 2000$  m, and  $0 \leq t \leq 100$  min, when the eastern exit gate is opened or shut on the difference periods of time such as 0–100 min, 33–100 min, and 66–100 min, is shown in Figure 14. The overall approximated pollutant concentrations, when the eastern exit gate is opened on the period 0–100 min, are shown in Figure 15. The overall approximated pollutant concentrations, when the eastern exit gate is opened on the period 33–100 min, are shown in Figure 16. The overall approximated pollutant concentrations, when the eastern exit gate is opened on the period 66–100 min, are also shown in Figure 17.

## 5. Discussion

In simulation 1, the different activated zonal removal mechanism systems are considered. The approximated pollutant concentration levels with the contaminant removal mechanism system are shown in Figure 6. We can see that the removal mechanism can be used to reduce the pollutant concentration in the opened-closed reservoir. The approximated pollutant concentrations, when the first/second/third zonal removal mechanism systems are activated, are shown as a surface plot in Figures 7, 8, and 9. The approximated pollutant concentrations along the northern/middle/southern parts, when the first, second, and third zonal removal mechanism systems are activated, are compared in Figures 10, 11, and 12. We can see that the third zonal removal mechanism systems give better water quality than another activated zones.

In simulation 2, the eastern exit gate opening on different periods of time is considered. The suitable third zonal

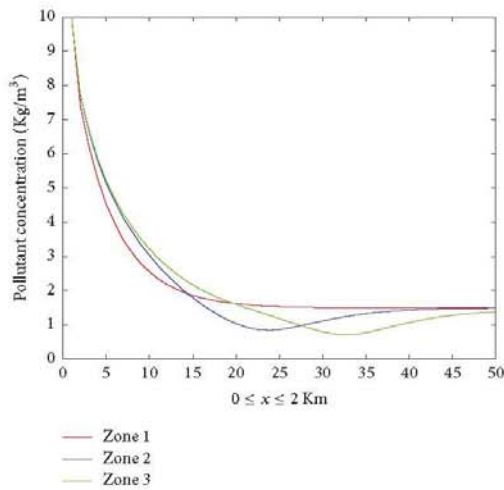


FIGURE 12: Comparison of approximated pollutant concentrations along the southern part,  $C(x, y, t)$ , for all  $0 \leq x \leq 2000$  m,  $y = 800$  m, and  $0 \leq t \leq 100$  min, when the first, second, and third zonal removal mechanism systems are activated.

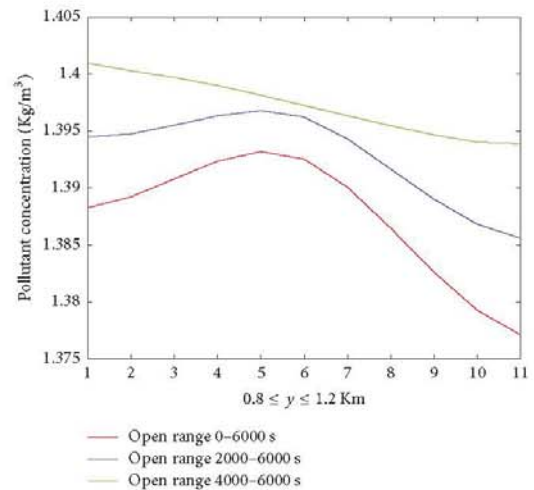


FIGURE 14: The approximated pollutant concentrations over the eastern exit gate is opened on 0–100 min, 33–100 min, and 66–100 min, when  $C(x, y, t)$  for all  $800 \leq y \leq 1200$  m and  $x = 2000$  m.

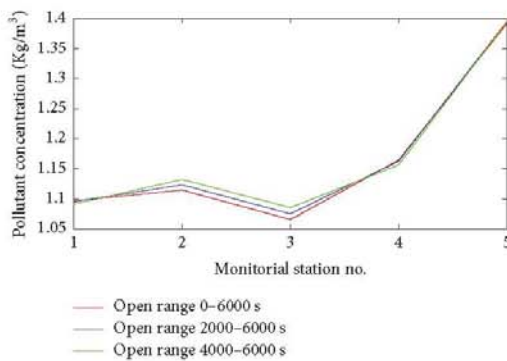


FIGURE 13: The approximated pollutant concentrations over 5 monitoring nodes, when the eastern exit gate is opened on 0–100 min, 33–100 min, and 66–100 min.

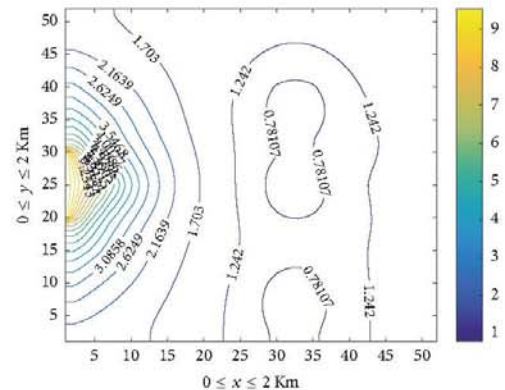


FIGURE 15: The overall approximated pollutant concentrations when the eastern exit gate is opened on the period 0–100 min.

removal mechanism system has been chosen to be activated. If the eastern exit gate is opened or shut on different periods of time such as 0–100 min, 33–100 min, and 66–100 min, the approximated pollutant concentrations over 5 monitoring nodes are shown in Figure 13. We can see that the full periods (0–100 min) of opened exit gate give better water quality for all 5 monitoring nodes.

If we consider the approximated pollutant concentrations along the eastern exit gate, when the the eastern exit gate is opened or shut on the difference periods of time as shown in Figure 14, we can also see that the long opened exit gate period (0–100 min) gives the better monitored water quality than another periods as well.

However, the overall approximated pollutant concentrations, when the eastern exit gate is opened on the different period of times, are shown in Figures 15, 16, and 17. We can see that the period (66–100 min) of the opened exit gate, which is shortest, gives better overall water quality than other periods. These mean that the monitoring point placement for water quality measurement in an opened-closed reservoir affect to water quality assessment. These mean that the approximated water quality measurement on monitoring points in an opened-closed reservoir affect the accuracy assessment. The overall simulated pollutant concentration method gives more precise water quality assessment than the monitoring assessment method.

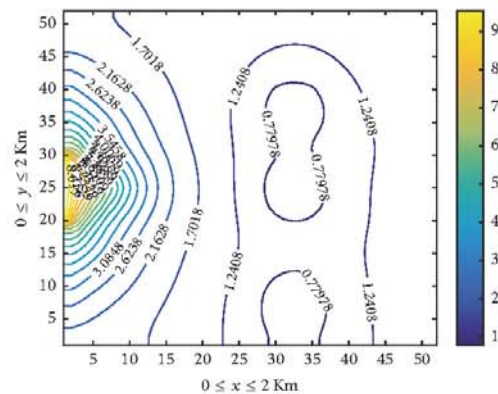


FIGURE 16: The overall approximated pollutant concentrations when the eastern exit gate is opened on the period 33–100 min.

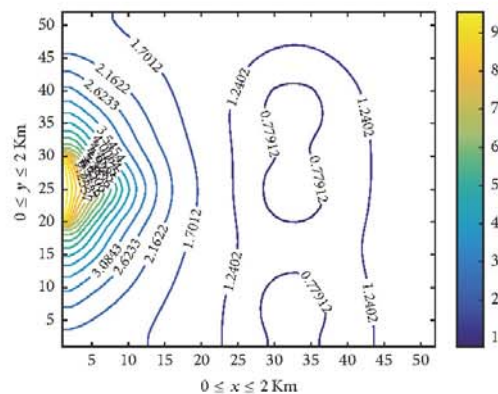


FIGURE 17: The overall approximated pollutant concentrations when the eastern exit gate is opened on the period 66–100 min.

## 6. Conclusion

The monitoring point placement for water quality measurement in an opened-closed reservoir can give accurate or inaccurate assessment. However, the proposed numerical simulations have given the overall approximated water quality when the removal mechanism systems are activated in different zones. The proposed techniques give the zonal removal mechanism placement to specialists who want to control the water quality. Moreover, we obtain the overall and specified approximated water quality for each points and time when the exit gate is opened on the different periods of time. The techniques can give a suitable period of time to open the exit gate to specialists who want to get a good agreement water quality by using contaminant removal mechanism.

## Conflicts of Interest

The authors declare no conflicts of interest.

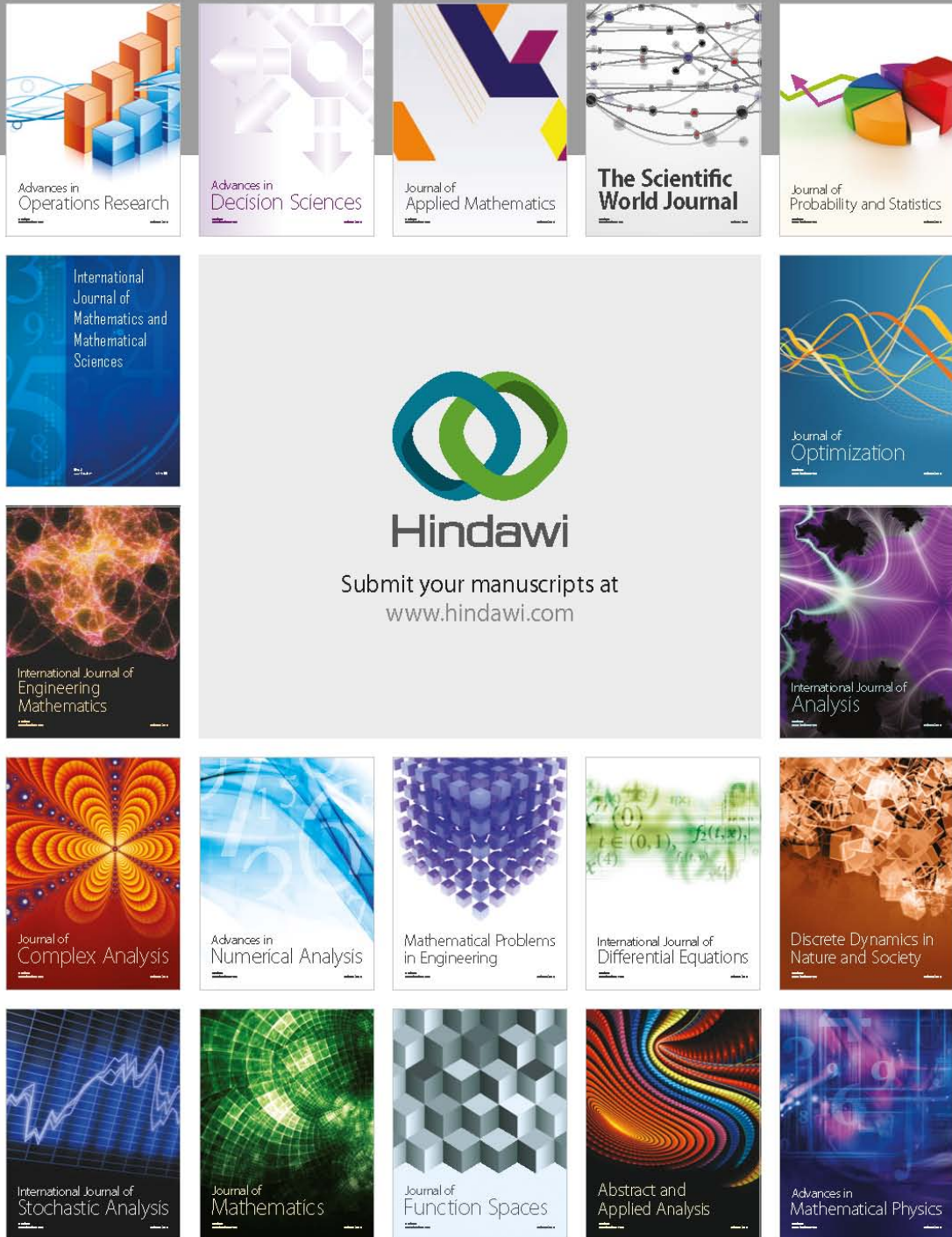
## Acknowledgments

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Appendix B  
Program

#### A4. Source Code of ADRE Removal Mechanism

```

clf;
clear all;
% define the grid size
n = 50;
dt = 0.01;
dx = 1;
dy = 1;
g = 9.8;
H = ones(n+2,n+2); % displacement matrix (this is what gets drawn)
U = zeros(n+2,n+2); % x velocity
bt= 6000;
V = zeros(n+2,n+2); % y velocity
%draw the mesh
% grid = surf(H);
% axis([1 n 1 n -5 5]);
% hold all;
% create initial displacement
[x,y] = meshgrid( linspace(-3,3,10) );
%R = sqrt(x.^2 + y.^2) + eps;
%Z = (sin(R)./R);
%Z = max(Z,0);
R = 0;
Z = (sin(R)./R);
Z = max(Z,0);
%Z=zeros(n+2,n+2);
% add displacement to the height matrix
w = size(Z,1);
i = 10:w+9;
j = 20:w+19;
H(i,j) = H(i,j) + Z;
% empty matrix for half-step calculations
Hx = zeros(n+1,n+1);
Hy = zeros(n+1,n+1);
Ux = zeros(n+1,n+1);
Uy = zeros(n+1,n+1);
Vx = zeros(n+1,n+1);
Vy = zeros(n+1,n+1);
u = zeros(n+2,n+2,bt);
v = zeros(n+2,n+2,bt);
nn=1;
% Pumping rate
pmp = 0.001;

```

```

% difference of depth level between inside and outside lake
dffh = 0.05;
while nn<=bt
% redraw the mesh
% set(grid, 'zdata', H);
% draw the vector
quiver(U,V);
drawnow
% blending the edges keeps the function stable
H(:,1) = H(:,2);
H(:,n+2) = H(:,n+1);
H(1,:) = H(2,:);
H(n+2,:) = H(n+1,:);
    %opened west gate
    for k=20:30
        H(k,1) = 1+(0.25*sin(0.01*nn)); %good
    end
%OC=bt*(1/3); %opened/closed 1 time
OC1=bt*(1/4); % opened/closed 2 time
OC2=bt*(1/2); % opened/closed 2 time
OC3=bt*(3/4); % opened/closed 2 time
OC4=bt*(4/4); % opened/closed 2 time
    %opened east gate
    if (nn>OC1) && (nn<OC2)
        %if nn>=OC
        for k=20:30
            H(k,n+2) = H(k,n+1)-dffh;
        end
    end
    if (nn>OC3) && (nn<OC4)
        %if nn>=OC
        for k=20:30
            H(k,n+2) = H(k,n+1)-dffh;
        end
    end
end

% reverse direction at the x edges
U(1,:) = -U(2,:);
U(n+2,:) = -U(n+1,:);
% reverse direction at the y edges
V(:,1) = -V(:,2);
V(:,n+2) = -V(:,n+1);
% ----- Lax-Wendroff-----%
% First half step

```

```

i = 1:n+1;
j = 1:n+1;
% height
Hx(i,j) = (H(i+1,j+1)+H(i,j+1))/2 - dt/(2*dx)*(U(i+1,j+1)-U(i,j+1));
Hy(i,j) = (H(i+1,j+1)+H(i+1,j))/2 - dt/(2*dy)*(V(i+1,j+1)-V(i+1,j));
% x momentum
Ux(i,j) = (U(i+1,j+1)+U(i,j+1))/2 - ...
dt/(2*dx)*( U(i+1,j+1).^2./H(i+1,j+1) - U(i,j+1).^2./H(i,j+1) + g/2*H(i+1,j+1).^2 - g/2*H(i,j+1).^2);
Uy(i,j) = (U(i+1,j+1)+U(i+1,j))/2 - dt/(2*dy)*( (V(i+1,j+1).*U(i+1,j+1))./H(i+1,j+1)) - (V(i+1,j).*U(i+1,j))./H(i+1,j) );
% y momentum
Vx(i,j) = (V(i+1,j+1)+V(i,j+1))/2 - dt/(2*dx)*((U(i+1,j+1).*V(i+1,j+1))./H(i+1,j+1)) - (U(i,j+1).*V(i,j+1))./H(i,j+1));
Vy(i,j) = (V(i+1,j+1)+V(i+1,j))/2 - dt/(2*dy)*((V(i+1,j+1).^2./H(i+1,j+1) + g/2*H(i+1,j+1).^2)-(V(i+1,j).^2./H(i+1,j)
+ g/2*H(i+1,j).^2));

% Second half step
i = 2:n+1;
j = 2:n+1;
% height
H(i,j) = H(i,j) - (dt/dx)*(Ux(i,j-1)-Ux(i-1,j-1)) - (dt/dy)*(Vy(i-1,j)-Vy(i-1,j-1));
% x momentum
U(i,j) = U(i,j) - (dt/dx)*((Ux(i,j-1).^2./Hx(i,j-1) + g/2*Hx(i,j-1).^2) - (Ux(i-1,j-1).^2./Hx(i-1,j-1) + g/2*Hx(i-1,j-1).^2)) - (dt/dy)*((Vy(i-1,j).*Uy(i-1,j))./Hy(i-1,j)) - (Vy(i-1,j-1).*Uy(i-1,j-1))./Hy(i-1,j-1)));
% y momentum
V(i,j) = V(i,j) - (dt/dx)*((Ux(i,j-1).*Vx(i,j-1))./Hx(i,j-1)) - (Ux(i-1,j-1).*Vx(i-1,j-1))./Hx(i-1,j-1)) - (dt/dy)*((Vy(i-1,j).^2./Hy(i-1,j) + g/2*Hy(i-1,j).^2) - (Vy(i-1,j-1).^2./Hy(i-1,j-1) + g/2*Hy(i-1,j-1).^2));
%----- Lax-Wendroff-----%
%opened east gate
if (nn>OC1) && (nn<OC2)
%if nn>=OC
for k=20:30
H(k,n+2) = H(k,n+1)-dffh;
U(k,n+2) = U(k,n+1)+(pmp*dx);
end
end
if (nn>OC3) && (nn<OC4)
for k=20:30
H(k,n+2) = H(k,n+1)-dffh;
U(k,n+2) = U(k,n+1)+(pmp*dx);
end
end
u(i,j,nn) = U(i,j);
v(i,j,nn) = V(i,j);
nn=nn+1;
end

```

```

% -----Dispersion model with removal mechanism -----%
C = zeros(n+2,n+2,bt);
C1 = zeros(n+2,n+2);
q = zeros(n+2,n+2);
cm1 = zeros(n+2,1);
cm2 = zeros(n+2,1);
cm3 = zeros(n+2,1);
cm4 = zeros(n+2,1);
cm5 = zeros(n+2,1);
% Parameters
% Reaction rate
rc=0.0010;

%Diffusion coefficient
D=1.0000;

% Initial pollutant concentration
for i=1:n+2
    for j=1:n+2
        C(i,j,1) = 1.00;
        C1(i,j) = C(i,j,1);
    end
end

%Initial values at monitoring points
cm1(1,1) = C1(10,40);
cm2(1,1) = C1(20,40);
cm3(1,1) = C1(30,40);
cm4(1,1) = C1(40,40);
cm5(1,1) = C1(25,50);
mincm1=100;
mincm2=100;
mincm3=100;
mincm4=100;
mincm5=100;

%Discharged pollutant concentration
cp=10.0;
for nn=1:bt
    for i=20:30
        C(i,1,nn) = cp;
    end
end
end

```

```
%Removal factor rate of the removal mechanism
rmv=0.10; %removal
%rmv=0; %non removal
%Zonal selected removal 1,2,3
chk=1;
% removal zone No.1
if chk==1
    for i=5:10
        for j=5:10
            q(i,j) = rmv;
        end
    end
end
for i=22:27
    for j=5:10
        q(i,j) = rmv;
    end
end
for i=35:40
    for j=5:10
        q(i,j) = rmv;
    end
end
end
% removal zone No.2
if chk==2
    for i=5:10
        for j=20:25
            q(i,j) = rmv;
        end
    end
end
for i=22:27
    for j=20:25
        q(i,j) = rmv;
    end
end
for i=35:40
    for j=20:25
        q(i,j) = rmv;
    end
end
end
% removal zone No.3
if chk==3
    for i=5:10
```

```

    for j=30:35
        q(i,j) = rmv;
    end
end
for i=22:27
    for j=30:35
        q(i,j) = rmv;
    end
end
for i=35:40
    for j=30:35
        q(i,j) = rmv;
    end
end
end

%===== ADRE FD approximation FTCS
for nn=1:bt
    for i=2:n+1
        for j=2:n+1
            %case1 Lower and left BCs
            if (i==2)
                if (j==2)
                    C(i,j,nn+1) = -(dt/(2*dx))*u(i,j,nn)+(dt*D)/(dx^2)*C(i+1,j,nn)+(-
(dt/(2*dy))*v(i,j,nn)+(dt*D)/(dy^2))*C(i,j+1,nn)+((dt/(2*dx))*u(i,j,nn)+((dt*D)/(dx^2))*C(i-
1,j,nn)+((dt/(2*dy))*v(i,j,nn)+((dt*D)/(dy^2))*C(i,j-1,nn)+ (1-(dt*rc)-((2*dt*D)/(dx^2))-
((2*dt*D)/(dy^2)))*C(i,j,nn)-(dt*q(i,j));
                    C(i,j-1,nn+1) = C(i,j,nn+1);
                    C(i-1,j,nn+1) = C(i,j,nn+1);
                end
            %case2 Lower BC
            if (j>2) && (j<n+1)
                C(i,j,nn+1) = -(dt/(2*dx))*u(i,j,nn)+(dt*D)/(dx^2)*C(i+1,j,nn)+(-
(dt/(2*dy))*v(i,j,nn)+(dt*D)/(dy^2))*C(i,j+1,nn)+((dt/(2*dx))*u(i,j,nn)+((dt*D)/(dx^2))*C(i-
1,j,nn)+((dt/(2*dy))*v(i,j,nn)+((dt*D)/(dy^2))*C(i,j-1,nn)+ (1-(dt*rc)-((2*dt*D)/(dx^2))-
((2*dt*D)/(dy^2)))*C(i,j,nn)-(dt*q(i,j));
                C(i-1,j,nn+1) = C(i,j,nn+1);
            end
            %case3 Right and Lower BCs
            if (j==n+1)
                C(i,j,nn+1) = -(dt/(2*dx))*u(i,j,nn)+(dt*D)/(dx^2)*C(i+1,j,nn)+(-
(dt/(2*dy))*v(i,j,nn)+(dt*D)/(dy^2))*C(i,j+1,nn)+((dt/(2*dx))*u(i,j,nn)+((dt*D)/(dx^2))*C(i-
1,j,nn)+((dt/(2*dy))*v(i,j,nn)+((dt*D)/(dy^2))*C(i,j-1,nn)+ (1-(dt*rc)-((2*dt*D)/(dx^2))-
((2*dt*D)/(dy^2)))*C(i,j,nn)-(dt*q(i,j));
            end
        end
    end
end

```

```

C(i,j+1,nn+1) = C(i,j,nn+1);
C(i-1,j,nn+1) = C(i,j,nn+1);
end
end
if (i>2) && (i<n+1)
if (j==2)
if (i<20) %case4 Lower left BC
C(i,j,nn+1) = (-(dt/(2*dx))*u(i,j,nn)+(dt*D)/(dx^2))*C(i+1,j,nn)+(-
(dt/(2*dy))*v(i,j,nn)+(dt*D)/(dy^2))*C(i,j+1,nn)+((dt/(2*dx))*u(i,j,nn)+((dt*D)/(dx^2))*C(i-
1,j,nn)+((dt/(2*dy))*v(i,j,nn)+((dt*D)/(dy^2))*C(i,j-1,nn)+ (1-(dt*rc)-((2*dt*D)/(dx^2))-
((2*dt*D)/(dy^2)))*C(i,j,nn)-(dt*q(i,j));
C(i,j-1,nn+1) = C(i,j,nn+1);
end
%case5 Entrance gate
if (i>=20) && (i<=30)
C(i,1,nn) = cp;
C(i,1,nn+1) = cp;
%C(i,j,nn+1) = 2;
C(i,j,nn+1) = (-(dt/(2*dx))*u(i,j,nn)+(dt*D)/(dx^2))*C(i+1,j,nn)+(-
(dt/(2*dy))*v(i,j,nn)+(dt*D)/(dy^2))*C(i,j+1,nn)+((dt/(2*dx))*u(i,j,nn)+((dt*D)/(dx^2))*C(i-
1,j,nn)+((dt/(2*dy))*v(i,j,nn)+((dt*D)/(dy^2))*C(i,j-1,nn)+ (1-(dt*rc)-((2*dt*D)/(dx^2))-
((2*dt*D)/(dy^2)))*C(i,j,nn)-(dt*q(i,j));
end
%case6 Upper left BC
if (i>30)
C(i,j,nn+1) = (-(dt/(2*dx))*u(i,j,nn)+(dt*D)/(dx^2))*C(i+1,j,nn)+(-
(dt/(2*dy))*v(i,j,nn)+(dt*D)/(dy^2))*C(i,j+1,nn)+((dt/(2*dx))*u(i,j,nn)+((dt*D)/(dx^2))*C(i-
1,j,nn)+((dt/(2*dy))*v(i,j,nn)+((dt*D)/(dy^2))*C(i,j-1,nn)+ (1-(dt*rc)-((2*dt*D)/(dx^2))-
((2*dt*D)/(dy^2)))*C(i,j,nn)-(dt*q(i,j));
C(i,j-1,nn+1) = C(i,j,nn+1);
end
end
%case7 inner reservoir
if (j>2) && (j<n+1)
C(i,j,nn+1) = (-(dt/(2*dx))*u(i,j,nn)+(dt*D)/(dx^2))*C(i+1,j,nn)+(-
(dt/(2*dy))*v(i,j,nn)+(dt*D)/(dy^2))*C(i,j+1,nn)+((dt/(2*dx))*u(i,j,nn)+((dt*D)/(dx^2))*C(i-
1,j,nn)+((dt/(2*dy))*v(i,j,nn)+((dt*D)/(dy^2))*C(i,j-1,nn)+ (1-(dt*rc)-((2*dt*D)/(dx^2))-
((2*dt*D)/(dy^2)))*C(i,j,nn)-(dt*q(i,j));
end
%case8 Right BC
if (j==n+1)
C(i,j,nn+1) = (-(dt/(2*dx))*u(i,j,nn)+(dt*D)/(dx^2))*C(i+1,j,nn)+(-
(dt/(2*dy))*v(i,j,nn)+(dt*D)/(dy^2))*C(i,j+1,nn)+((dt/(2*dx))*u(i,j,nn)+((dt*D)/(dx^2))*C(i-

```

```

1,j,nn)+((dt/(2*dy))*v(i,j,nn)+((dt*D)/(dy^2))*C(i,j-1,nn)+ (1-(dt*rc)-((2*dt*D)/(dx^2))-
((2*dt*D)/(dy^2))*C(i,j,nn)-(dt*q(i,j));
    C(i,j+1,nn+1) = C(i,j,nn+1);
    %exit gate polluted water is released
    if (i>=20) && (i<=30)
        C(i,j+1,nn+1) = C(i,j,nn+1)-(0.001*dx);
    end
end
end
if (i==n+1)
    if (j==2) %case9 Left and upper BCs
        C(i,j,nn+1) = -(dt/(2*dx))*u(i,j,nn)+(dt*D)/(dx^2)*C(i+1,j,nn)+(-
(dt/(2*dy))*v(i,j,nn)+(dt*D)/(dy^2))*C(i,j+1,nn)+((dt/(2*dx))*u(i,j,nn)+((dt*D)/(dx^2))*C(i-
1,j,nn)+((dt/(2*dy))*v(i,j,nn)+((dt*D)/(dy^2))*C(i,j-1,nn)+ (1-(dt*rc)-((2*dt*D)/(dx^2))-
((2*dt*D)/(dy^2))*C(i,j,nn)-(dt*q(i,j));
        C(i,j-1,nn+1) = C(i,j,nn+1);
        C(i+1,j,nn+1) = C(i,j,nn+1);
    end
    %case10 Upper BCs
    if (j>2) && (j<n+1)
        C(i,j,nn+1) = -(dt/(2*dx))*u(i,j,nn)+(dt*D)/(dx^2)*C(i+1,j,nn)+(-
(dt/(2*dy))*v(i,j,nn)+(dt*D)/(dy^2))*C(i,j+1,nn)+((dt/(2*dx))*u(i,j,nn)+((dt*D)/(dx^2))*C(i-
1,j,nn)+((dt/(2*dy))*v(i,j,nn)+((dt*D)/(dy^2))*C(i,j-1,nn)+ (1-(dt*rc)-((2*dt*D)/(dx^2))-
((2*dt*D)/(dy^2))*C(i,j,nn)-(dt*q(i,j));
        C(i+1,j,nn+1) = C(i,j,nn+1);
    end
    %case11 Right and upper BCs
    if (j==n+1)
        C(i,j,nn+1) = -(dt/(2*dx))*u(i,j,nn)+(dt*D)/(dx^2)*C(i+1,j,nn)+(-
(dt/(2*dy))*v(i,j,nn)+(dt*D)/(dy^2))*C(i,j+1,nn)+((dt/(2*dx))*u(i,j,nn)+((dt*D)/(dx^2))*C(i-
1,j,nn)+((dt/(2*dy))*v(i,j,nn)+((dt*D)/(dy^2))*C(i,j-1,nn)+ (1-(dt*rc)-((2*dt*D)/(dx^2))-
((2*dt*D)/(dy^2))*C(i,j,nn)-(dt*q(i,j));
        C(i,j+1,nn+1) = C(i,j,nn+1);
        C(i+1,j,nn+1) = C(i,j,nn+1);
    end
end
end
% Non-negative solution checker
if C(i,j,nn+1)<0
    chkx=i;
    chky=j;
    chkt=nn+1;
    %return
end
end

```

```

end
% Four corner points approximations
C(1,1,nn+1) = 0.5*(C(2,1,nn+1) + C(1,2,nn+1));
C(n+2,1,nn+1) = 0.5*(C(n+1,1,nn+1) + C(n+2,2,nn+1));
C(1,n+2,nn+1) = 0.5*(C(2,n+2,nn+1) + C(1,n+1,nn+1));
C(n+2,n+2,nn+1) = 0.5*(C(n+1,n+2,nn+1) + C(n+2,n+1,nn+1));
    %Stored values
    for i=1:n+2
        for j=1:n+2
            C1(i,j) = C(i,j,nn+1);
        end
    end
end
%stored values at monitoring points
cm1(nn+1,1) = C1(10,40);
cm2(nn+1,1) = C1(20,40);
cm3(nn+1,1) = C1(30,40);
cm4(nn+1,1) = C1(40,40);
cm5(nn+1,1) = C1(25,50);
%Minimum finding at monitoring points
if mincm1 >= cm1(nn+1,1)
    mincm1 = cm1(nn+1,1);
end
if mincm2 >= cm2(nn+1,1)
    mincm2 = cm2(nn+1,1);
end
if mincm3 >= cm3(nn+1,1)
    mincm3 = cm3(nn+1,1);
end
if mincm4 >= cm4(nn+1,1)
    mincm4 = cm4(nn+1,1);
end
if mincm5 >= cm5(nn+1,1)
    mincm5 = cm5(nn+1,1);
end
end

% Stored minimum concentration over 4 monitoring points
mincma(1,1) = mincm1;
mincma(2,1) = mincm2;
mincma(3,1) = mincm3;
mincma(4,1) = mincm4;
mincma(5,1) = mincm5;
%=====END=====

```

## Author Biography

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7. บทความเรื่อง "การอ้างอิงตามทฤษฎีบทที่ว่าแน่นอนไม่แท้เพราะผู้ใช้งาน", วารสารคณิตศาสตร์เพชรยอดมงกุฎ, ประจำปี 2556 ครั้งที่ 16, โครงการคณิตศาสตร์เพชรยอดมงกุฎ โดยมูลนิธิร่มฉัตรและสำนักงานคณะกรรมการการศึกษาขั้นพื้นฐาน
8. Presentation in the Asian Mathematical Conference 2013 (AMC 2013), " An approach to incremental support vector machine learning algorithm for classification", June 30th-July 4th, 2013, Bexco, Busan, Republic of Korea.
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