NONHOMOGENEOUS SYSTEM OF COUPLED LINEAR MATRIX FRACTIONAL DYNAMICAL DIFFERENTIAL EQUATIONS IN CAPUTO'S SENSE

SIREETON WINTACHAI

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER IN SCIENCE (APPLIED MATHEMATICS)

DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

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Abstract

In this research, we investigate a nonhomogeneous system of coupled linear matrix fractional dynamical differential equations. The fractional derivative considered here is taken in Caputo's sense. We obtain an explicit form of its general solution in terms of the Kronecker product, the vector operator and matrix series concerning Mittag-Leffler functions.

Keywords : Linear matrix fractional differential equation, Caputo's derivative, vector operator, Kronecker product, Mittag-Leffler functions.

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Table of Contents

	Page
Abstract in English	i
Acknowledgements	ii
Table of Contents	iii
List of Tables	iv
Chapter 1. Introduction	1
1.1 Research Motivation	1
1.2 Objectives of the study	2
1.3 Scopes of the study	3
1.4 Benefits of the Study	3
1.5 Research methodology	3
Chapter 2. Preliminaries	5
2.1 Kronecker product	5
2.2 The vector operator	6
2.3 Functions of complex variables defined by power series	6
2.4 Special functions	7
2.5 Functions of complex matrices defined by power series	7
2.6 Mittage-Leffler functions of complex matrices	9
2.7 Caputo fractional derivative	11
Chapter 3. System of linear coupled matrix fractional differential equations	13
3.1 The main result	15
3.2 Special cases	18
Chapter 4. Examples	25
4.1 An auxiliary lemma for computing Mittage-Leffler functions of matrices	25
4.2 Examples	26
Chapter 5. Conclusions and Suggestions	32
5.1 System of linear coupled matrix fractional differential equations and A general solution of the system	32
5.2 Suggestions	36
References	37
Appendix/Appendices	39
Appendix A	40
Author Biography	53

List of Tables

Table **Page**

1.1 The research schedule 4

Chapter 1

Introduction

1.1 Research Motivation

Fractional calculus was introduced more than 300 years ago. It is a branch of mathematical analysis that studies the several different possibilities of defining real of complex number powers of the differentiation and integration operators and developing a calculus for such operators generalizing the classical one. It is applied into many branches of mathematical science and engineering. It is dominating by modern examples of applications in differential and integral equations. The fractional derivative of the exponential obtained by Liouville in 1832, and the fractional of power function got by Riemann [1] in 1847. The Riemann-Liouville fractional derivative was failed in the description and modeling of some complex phenomena. Thus, Caputo's fractional derivative was introduced in 1967 by Caputo [2].

Linear matrix (fractional) differential equations are important in various fields which including applied science, engineering, economics. For a detail survey with collections of applications in various fields, see Miller and Ross [3], Podlubny [4], Kilbas and Saigo [5], and Kilbas et al. [6]. The simplest form of linear matrix differential equations is shown below:

$$X'(t) = AX(t). \tag{1.1}$$

Here, $A \in M_n$ and $X(t)$ is an unknown matrix-valued function to be solved. The solution of (1.1) is given by

$$X(t) = e^{A(t-t_0)}X(t_0), \tag{1.2}$$

see more detail in Ben and Rachidi [7],[8], Cheng and Yau [9] and Leonard [10]. A general system of nonhomogeneous linear matrix differential equations takes the form

$$X'(t) = AX(t) + U(t). \tag{1.3}$$

here, $U(t)$ is a given matrix-valued function. The solution of (1.3) is given by

$$X(t) = e^{(t-t_0)A}X(t_0) + e^{tA} * U(t), \tag{1.4}$$

where $*$ denotes the matrix convolution product. A general system of nonhomogeneous coupled linear matrix ordinary differential equations takes in the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t). \end{aligned} \tag{1.5}$$

Here, A,B,C,D,E,F,G,H are given matrix-valued functions, and $X(t), Y(t)$ are unknown matrix-valued functions. The solution is given in terms of Kronecker products, the vector operator and matrix series concerning exponential and hyperbolic functions. A nonhomogeneous case of (1.5) was discussed in Kilman and Al-Zhour [11] when $E = C,\ F = D,\ G = A,\ H = B$ and $U(t) = V(t) = 0$. The system (1.5) was investigate in Kongyaksee and Chansangiam [12] under the assumption that $AC = CA$ and $BD = DB$.

A simple system of homogeneous linear matrix fractional dynamical differential equations takes the form

$$X^{(\alpha)}(t) \ = \ AX(t). \tag{1.6}$$

The solution of (1.6) is given by (see Balanchan and Kokila [13],[14])

$$X(t) \ = \ E_\alpha(A(t-t_0)^\alpha)C, \tag{1.7}$$

where E_α is Mittag-Leffler functions with parameter $\alpha > 0$. The simplest form of non-homogeneous linear matrix fractional dynamical differential equations with delays in control is shown below:

$$X^{(\alpha)}(t) \ = \ AX(t) + U(t). \tag{1.8}$$

The solution of (1.8) is as follows (see Balanchan et al. [14])

$$X(t) \ = \ E_\alpha(At^\alpha + \int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)u(s)ds. \tag{1.9}$$

A general system of nonhomogeneous coupled linear matrix differential equations takes the form

$$\begin{aligned} X^{(\alpha)}(t) \ &= \ AX(t)B + CY(t)D + U(t), \\ Y^{(\alpha)}(t) \ &= \ EX(t)F + GY(t)H + V(t), \end{aligned} \tag{1.10}$$

The system was investigated in Leonard [11] under the assumption that $AC = CA$ and $BD = DB$.

In the present work, we consider a generalization of the system (1.10), namely,

$$\begin{aligned} X^{(\alpha)}(t) \ &= \ \sum_{i=0}^{p} A_iX(t)B_i + \sum_{i=0}^{q} C_iY(t)D_i + U(t), \\ Y^{(\alpha)}(t) \ &= \ \sum_{i=0}^{r} E_iX(t)F_i + \sum_{i=0}^{s} G_iY(t)H_i + V(t). \end{aligned}$$

where $0 < \alpha \leq 1$ and all derivatives are in Caputo's sense. To obtain an explicit formula of the solution, we impose an assumption on the coefficient matrices. Our result includes the results in Al-Zhour [11], Kongyaksee and Chansangiam [12] and Killiman and Al-Zhour [15]. We also provide and an illustrative example of the main result.

1.2 Objectives of the study

To investigate a system of nonhomogeneous linear matrix fractional dynamical differential equations.

1.3 Scopes of the study

We solve the following couple nonhomogeneous system of linear matrix fractional dynamical differential equations:

$$\begin{aligned} X^{(\alpha)}(t) &= \sum_{i=0}^{p} A_i X(t) B_i + \sum_{i=0}^{q} C_i Y(t) D_i + U(t), \\ Y^{(\alpha)}(t) &= \sum_{i=0}^{r} E_i X(t) F_i + \sum_{i=0}^{s} G_i Y(t) H_i + V(t). \end{aligned}$$

Here, $0 < \alpha \leq 1$ and $A,\ B,\ C,\ D,\ E,\ F,\ G,\ H$ are given square matrices satisfying assumptions,$U(t),\ V(t)$ are given matrix-valued functions, and $X(t),\ Y(t)$ are unknown matrix-valued functions.

1.4 Benefits of the Study

To provide further theory for nonhomogeneous linear matrix fractional dynamical differential equations.

1.5 Research methodology

1) Study advanced topics in matrix analysis.
2) Study research and papers concerning Kronecker products and the vector operator.
3) Study background in Mittag-Leffler matrix functions.
4) Study research papers about linear matrix differential equations.
5) Study research and papers about linear matrix fractional differential equations.
6) Solve a couple of non-homogeneous linear matrix fractional dynamical differential equations by using Kronecker products and vector operator.
7) Investigate initial value problems for systems of nonhomogeneous linear matrix fractional dynamical differential equations in Caputo's sense.
8) Provide numerical examples of the main result.

Table 1.1*:* The research schedule

Activity	Time frame							
	2017		2018				2019	
	Aug.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.	Aug.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.
Step 1	⟷							
Step 2		⟷	⟷					
Step 3			⟷	⟷				
Step 4					⟷			
Step 5						⟷		
Step 6							⟷	
Step 7							⟷	
Step 8								⟷

Chapter 2

Preliminaries

In this section, we provide adequate tools for solving system of linear matrix differential equations. We shall denote the set of all m-by-n complex matrices by $M_{m,n}$ and we set $M_n = M_{n,n}$. We shall denote the set of all m-by-n real matrices by $M_{m,n}(\mathbb{R})$ and $M_n(\mathbb{R}) = M_{n,n}(\mathbb{R})$.

2.1 Kronecker product

Definition 2.1. Let $A = (a_{ij}) \in M_m$ and $B \in M_n$. The Kronecker product of A and B is defined by

$$A \otimes B = (a_{ij}B) \in M_{mn}.$$

Example 2.2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$. Find $A \otimes B$.

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 1B & 2B \\ 3B & 4B \end{bmatrix} \\ &= \begin{bmatrix} 1\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} & 2\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \\ 3\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} & 4\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 2 \\ 2 & 2 & 4 & 4 \\ 0 & 3 & 0 & 4 \\ 6 & 6 & 8 & 8 \end{bmatrix}. \end{aligned}$$

Lemma 2.3 (see e.g. Horn and Johnson [16]). The following properties hold for matrices of appropriate sizes:

1. $I_m \otimes I_n = I_{mn}$,
2. $(kA) \otimes B = k(A \otimes B) = A \otimes (kB)$, for all $k \in \mathbb{C}$,
3. $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$,
4. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$,
5. $(A \otimes B)^T = A^T \otimes B^T$,
6. $(A \otimes B)(C \otimes D) = AC \otimes BD$.

2.2 The vector operator

For each $A = (a_{ij}) \in M_m$, we define

$$\text{Vec}\, A = [a_{11} \dots a_{m1} \dots a_{12} \dots a_{m2} \dots a_{1m} \dots a_{mm}]^T.$$

Example 2.4. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 8 & 3 \\ 1 & 0 & 5 \end{bmatrix}$. Then

$$\text{Vec}(A) = \begin{bmatrix} 1 & 2 & 1 & 0 & 8 & 0 & 2 & 3 & 5 \end{bmatrix}^T.$$

Lemma 2.5 (see e.g. Horn and Johnson [16])**.** The following properties hold for matrices of appropriate sizes:

1. $\text{Vec}(kA) = k\, \text{Vec}\, A$,
2. $\text{Vec}(A + B) = \text{Vec}\, A + \text{Vec}\, B$.

Theorem 2.6 (see e.g. Horn and Johnson [16])**.** Consider matrices $A \in M_{m,n}, B \in M_{p,q}$ and $X \in M_{n,p}$. The Kronecker product and the vector operator are related by

$$\text{Vec}(AXB) = (B^T \otimes A)\, \text{Vec}\, X.$$

2.3 Functions of complex variables defined by power series

Let $f(z)$ be a function of the complex variable z. For a power series f defined as

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

where $a_k \in \mathbb{C}$, The radius of convergence R of this series is a non negative real number such that the series converges if $|z - z_0| < R$ and diverges if $|z - z_0| > R$. In the case when this series converges for all $z \in \mathbb{C}$, we write $R = \infty$.

In particular, we have the following complex-valued functions represented by power series:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2k!},$$

$$\sin(z) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(2k-1)!},$$

$$\cosh(z) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(1 - (-1)^k) z^k}{k!},$$

$$\sinh(z) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(1 + (-1)^k) z^k}{k!}.$$

2.4 Special functions

Recall that the Gamma function Γ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \qquad \text{Re}(z) > 0.$$

Then, we have the following properties:

1. $\Gamma(n) = (n-1)!$ for every $n \in N$.
2. $\Gamma(n+1) = n\Gamma(n)$ for every $n > 0$,
3. $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin(\pi n)}$,
4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

The Mittag-Leffler function $E_{\alpha,\beta}$ is a special function depending on two parameters $\alpha, \beta > 0$. It may be defined by the following series when the real part of α is strictly positive:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

In particular, the following functions:

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z - 1}{z},$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} = \frac{e^z - 1 - z}{z^2},$$

$$E_{2,1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh(z),$$

$$E_{2,2}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \frac{\sinh(z)}{z},$$

$$E_{2,1}(-z^2) = \sum_{k=0}^{\infty} \frac{(-z^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos(z),$$

$$E_{2,2}(-z^2) = \sum_{k=0}^{\infty} \frac{(-z^2)^k}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{z(2k+1)!} = \frac{\sin(z)}{z}.$$

2.5 Functions of complex matrices defined by power series

Definition 2.7 (see e.g. Gradshteyn and Ryzhik [18]). Let A be an nxn matrix with complex element with eigenvalues $\lambda_1, ...\lambda_n$. Then the spectral radius $\rho(A)$ of A is

$$\rho(A) = max(|\lambda_1|, ..., |\lambda_n|).$$

Consider $A \in M_n$ and an analytic function f defined on region in a complex plane containing the origin and the spectrum of A. Then there is positive constant R such that f admits the Taylor series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z)^k \quad \text{for} \quad |z| < R,$$

If spectral radius of A is less than R, than the matrix power series $\sum_{k=0}^{\infty} a_k A^k$ converges, denoted by $f(A)$.

In particular, the following matrix series converge for any $A \in M_n$:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

$$\cos(A) = \sum_{k=0}^{\infty} \frac{(-1)^k A^{2k}}{2k!},$$

$$\sin(A) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^{k-1} A^{2k-1}}{(2k-1)!},$$

$$\cosh(A) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(1-(-1)^k) A^k}{k!},$$

$$\sinh(A) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(1+(-1)^k) A^k}{k!}.$$

Definition 2.8. Let $A \in M_n$ be a diagonalizable matrix and let f be a complex-valued function defined on the spectrum of A, that is $\sigma(A) \subseteq Dom(f)$. Let $S \in M_n$ be an invertible matrix such that,

$$A = SDS^{-1} \quad \text{where} \quad D = diag(\lambda_1, ..., \lambda_n).$$

Primary matrix function f of A is defined by

$$f(A) = S \begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n \end{bmatrix} S^{-1}.$$

Example 2.9. Let A = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find $f(A)$ and e^A.

Write $A = SDS^{-1}$.

The eigenvalues of A are $\lambda = 3, -1$ with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We have

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1},$$

so,

$$f(A) = \frac{1}{2}\begin{bmatrix} f(3)+f(-1) & f(3)-f(-1) \\ f(3)-f(-1) & f(3)+f(-1) \end{bmatrix}.$$

and

$$e^A = \frac{1}{2}\begin{bmatrix} e^3+e^{-1} & e^3-e^{-1} \\ e^3-e^{-1} & e^3+e^{-1} \end{bmatrix}.$$

2.6 Mittage-Leffler functions of complex matrices

Definition 2.10. The Mittage-Leffler function with parameters $\alpha > 0$ and $\beta > 0$ of $A \in M_n$ is defined by

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} A^k = I_n + \frac{1}{\Gamma(\alpha+\beta)} A + \frac{1}{\Gamma(2\alpha+\beta)} A^2 + \cdots$$

where $\beta = 1$, we set $E_\alpha := E_{\alpha,1}$.

In particular, we have the following functions:

$$E_{1,1}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{A^k}{k!} = e^A,$$

$$E_{2,1}(A^2) = \sum_{k=0}^{\infty} \frac{A^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{A^{2k}}{(2k)!} = \cosh(A),$$

$$E_{2,2}(A^2) = \sum_{k=0}^{\infty} \frac{A^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{A^{2k+1}}{(2k+1)!} = A^{-1}\sinh(A),$$

$$E_{2,1}(-A^2) = \sum_{k=0}^{\infty} \frac{(-A^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k A^{2k}}{(2k)!} = \cos(A),$$

$$E_{2,2}(-A^2) = \sum_{k=0}^{\infty} \frac{(-A^2)^k}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k A^{2k+1}}{A(2k+1)!} = A^{-1}\sin(A).$$

Lemma 2.11 (see e.g. Steeb and Hardy [17])**.** Let f be an analytic function defined on a region including the origin and the spectrum of A. Then

$$f(I \otimes A) = I \otimes f(A),$$
$$f(A \otimes I) = f(A) \otimes I.$$

In particular, the following relations hold for any complex square matrix A :

$$E_\alpha(A \otimes I) = E_\alpha(A) \otimes I,$$
$$E_\alpha(I \otimes A) = I \otimes E_\alpha(A).$$

Lemma 2.12 (see e.g Killicman and Al-Zhour [15])**.** The following properties hold for matrices of appropriate sizes:

1. If $AB = BA$, then $E_\alpha(A+B) = E_\alpha(A)E_\alpha(B)$,
2. $(E_\alpha(A))^T = E_\alpha(A^T)$,
3. $E_\alpha(A)$ is always invertible and $(E_\alpha(A))^{-1} = E_\alpha(-A)$,
4. $E_\alpha(A \otimes I_n) = E_\alpha(A) \otimes I_n$ and $E_\alpha(I_n \otimes A) = I_n \otimes E_\alpha(A)$.

Proof. 1. If $AB = BA$, then we have

$$\begin{aligned}
E_\alpha(A)E_\alpha(B) &= \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k+1)} \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(\alpha k+1)} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{A^m B^n}{\Gamma(\alpha m+1)\Gamma(\alpha n+1)} \\
&= \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{\Gamma(\alpha l+1))}{\Gamma(\alpha l+1)} \frac{A^m B^{l-m}}{\Gamma(\alpha m+1)\Gamma(\alpha(l-m)+1)} \\
&= \sum_{l=0}^{\infty} \frac{1}{\Gamma(\alpha l+1)} \sum_{m=0}^{l} \frac{\Gamma(\alpha l+1)A^m B^{l-m}}{\Gamma(\alpha m+1)\Gamma(\alpha(l-m)+1)} \\
&= \sum_{k=0}^{\infty} \frac{(A+B)^l}{\Gamma(\alpha l+1)} \\
&= E_\alpha(A+B).
\end{aligned}$$

2. We have

$$\begin{aligned}
E_\alpha(A^T) &= \sum_{k=0}^{\infty} \frac{(A^T)^k}{\Gamma(\alpha k+1)} \\
&= \sum_{k=0}^{\infty} \frac{(A^k)^T}{\Gamma(\alpha k+1)} \\
&= \sum_{k=0}^{\infty} (\frac{A^k}{\Gamma(\alpha k+1)})^T \\
&= (\sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k+1)})^T \\
&= (E_\alpha(A))^T.
\end{aligned}$$

3. Putting $B = -A$ in 1, we have

$$E_\alpha(A - A) = E_\alpha(A)E_\alpha(-A),$$

and thus

$$I = E_\alpha(0) = E_\alpha(A)E_\alpha(-A).$$

So, $E_\alpha(-A)$ is the inverse of $E_\alpha(A)$.

4. Using Lemma 2.11, we have $E_\alpha(A \otimes I_n) = E_\alpha(A) \otimes I_n$ and $E_\alpha(I_n \otimes A) = I_n \otimes E_\alpha(A)$.

□

2.7 Caputo fractional derivative

Definition 2.13. Let f be a piecewise continuous function on (t_0, ∞) which is integrable on any finite subinterval of $[t_0, \infty)$. Let $\alpha > 0$ and let $n \in \mathbb{N}$ be such that $n - 1 < \alpha \leq n$. The Caputo's derivative of f of order α is defined by

$$D^\alpha f(t) = \begin{cases} D^n f(t), & \alpha = n, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{t_0}^{t} \frac{D^n f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & n-1 < \alpha < n. \end{cases}$$

Here, D is the usual differential operator.

Example 2.14. Let $\alpha = \frac{1}{2}, n = 1, f(t) = t.$ we get

$$D^{1/2} f(t) = \frac{1}{\Gamma(1/2)} \int_{t_0}^{t} \frac{D^1 f(\tau)}{(t-\tau)^{1/2}} d\tau.$$

Taking into account the properties of Gamma function and using the substitution $u := (t-\tau)^{1/2}$ the final result for the Caputo fractional derivative of the function $f(t) = t$ is obtained as

$$\begin{aligned} D^{1/2} f(t) &= -\frac{1}{\sqrt{\pi}} \int_{t_0}^{t} \frac{1}{(t-\tau)^{1/2}} d(t-\tau) \\ &= -\frac{1}{\sqrt{\pi}} \int_{\sqrt{t}}^{t_0} \frac{1}{u} du) \\ &= \frac{1}{\sqrt{\pi}} \int_{t_0}^{\sqrt{t}} \frac{2u}{u} du) \\ &= \frac{2}{\sqrt{\pi}} (\sqrt{t} - t_0). \end{aligned}$$

Thus, it holds

$$D^{1/2} f(t) = \frac{2(\sqrt{t} - t_0)}{\sqrt{\pi}}.$$

The class of Caputo's fractional derivative include the following function:

Theorem 2.15. Let $n-1<\alpha<n$, $n\in\mathbb{N}$, $a,\beta\in\mathbb{R}$. Then

1. $D^{\alpha}x^{\beta}=\dfrac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)\Gamma(n-\alpha)}x^{\beta-\alpha}B_{\tau}(n-\alpha,\beta-n+1),\tau=\dfrac{x-t_0}{x},t_0>0,$
2. $D^{\alpha}e^{ax}=a^{n}e^{at_0}(x-t_0)^{n-\alpha}E_{1,n-\alpha+1}(a(x-t_0)),$
3. $D^{\alpha}\sin(ax)=-\frac{1}{2}i(ia)^{n}(x-t_0)^{n-\alpha}[e^{iat_0}E_{1,n-\alpha+1}(ia(x-t_0))-(-1)^{n}e^{-iat_0}E_{1,n-\alpha+1}(-ia(x-t_0))],$
4. $D^{\alpha}\cos(ax)=-\frac{1}{2}(ia)^{n}(x-t_0)^{n-\alpha}[e^{iat_0}E_{1,n-\alpha+1}(ia(x-t_0))+(-1)^{n}e^{-iat_0}E_{1,n-\alpha+1}(-ia(x-t_0))],$
5. $D^{\alpha}\sinh(ax)=\frac{1}{2}(a)^{n}(x-t_0)^{n-\alpha}[e^{at_0}E_{1,n-\alpha+1}(a(x-t_0))-(-1)^{n}e^{-at_0}E_{1,n-\alpha+1}(-a(x-t_0))],$
6. $D^{\alpha}\cosh(ax)=\frac{1}{2}(a)^{n}(x-t_0)^{n-\alpha}[e^{at_0}E_{1,n-\alpha+1}(a(x-t_0))+(-1)^{n}e^{-at_0}E_{1,n-\alpha+1}(-a(x-t_0))].$

Chapter 3

System of linear coupled matrix fractional differential equations

In this chapter, we investigate a system of coupled nonhomogeneous linear matrix fractional differential equations. From now on, let $A, B, C, D, E, F, G, H \in M_n(\mathbb{R})$ be given constant matrices and let $U(t), V(t) \in M_n(\mathbb{R})$ be given matrix-valued function. We start with two auxiliary lemmas.

Lemma 3.1. For any $A, B \in M_n(\mathbb{C})$, we have

$$E_\alpha\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \begin{bmatrix} E_\alpha(A) & 0 \\ 0 & E_\alpha(B) \end{bmatrix}.$$

Proof. Using standard techniques in matrix analysis, we obtain

$$\begin{aligned}
E_\alpha\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^k \\
&= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} A^k & 0 \\ 0 & B^k \end{bmatrix} \\
&= \sum_{k=0}^{\infty} \begin{bmatrix} \dfrac{A^k}{\Gamma(\alpha k+1)} & 0 \\ 0 & \dfrac{B^k}{\Gamma(\alpha k+1)} \end{bmatrix} \\
&= \begin{bmatrix} \displaystyle\sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k+1)} & 0 \\ 0 & \displaystyle\sum_{k=0}^{\infty} \frac{B^k}{\Gamma(\alpha k+1)} \end{bmatrix} \\
&= \begin{bmatrix} E_\alpha(A) & 0 \\ 0 & E_\alpha(B) \end{bmatrix}.
\end{aligned}$$

□

Lemma 3.2. For any $A, B \in M_n(\mathbb{C})$, we have

$$E_\alpha\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \begin{bmatrix} E_{2\alpha,1}(AB) & \big(E_{2\alpha,\alpha+1}(AB)\big)A \\ \big(E_{2\alpha,\alpha+1}(BA)\big)B & E_{2\alpha,1}(BA) \end{bmatrix},$$

$$E_\alpha\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} E_\alpha(A)+E_\alpha(-A) & E_\alpha(A)-E_\alpha(-A) \\ E_\alpha(A)-E_\alpha(-A) & E_\alpha(A)+E_\alpha(-A) \end{bmatrix}.$$

Proof. A direct computation reveals that

$$\begin{aligned}
E_\alpha\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+1)}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^k \\
&= \sum_{even}\frac{1}{\Gamma(\alpha k+1)}\begin{bmatrix} (AB)^k & 0 \\ 0 & (BA)^k \end{bmatrix} + \sum_{odd}\frac{1}{\Gamma(\alpha k+1)}\begin{bmatrix} 0 & (AB)^k A \\ (BA)^k B & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha(2k)+1)}(AB)^k & \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha(2k+1)+1)}(AB)^k A \\ \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha(2k+1)+1)}(BA)^k B & \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha(2k)+1)}(BA)^k \end{bmatrix} \\
&= \begin{bmatrix} E_{2\alpha,1}(AB) & \big(E_{2\alpha,\alpha+1}(AB)\big)A \\ \big(E_{2\alpha,\alpha+1}(BA)\big)B & E_{2\alpha,1}(BA) \end{bmatrix}.
\end{aligned}$$

We also have

$$\begin{aligned}
E_\alpha\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) &= \sum_{even}\frac{1}{\Gamma(\alpha k+1)}\begin{bmatrix} A^k & 0 \\ 0 & A^k \end{bmatrix} + \sum_{odd}\frac{1}{\Gamma(\alpha k+1)}\begin{bmatrix} 0 & A^k \\ A^k & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{even}\frac{1}{(\alpha k+1)}A^k & \sum_{odd}\frac{1}{(\alpha k+1)}A^k \\ \sum_{even}\frac{1}{(\alpha k+1)}A^k & \sum_{odd}\frac{1}{(\alpha k+1)}A^k \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty}\left(\frac{1+(-1)^k}{2\Gamma(2k+1)}\right)A^k & \sum_{k=0}^{\infty}\left(\frac{1-(-1)^k}{2\Gamma(2k+1)}\right)A^k \\ \sum_{k=0}^{\infty}\left(\frac{1-(-1)^k}{2\Gamma(2k+1)}\right)A^k & \sum_{k=0}^{\infty}\left(\frac{1+(-1)^k}{2\Gamma(2k+1)}\right)A^k \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} E_\alpha(A)+E_\alpha(-A) & E_\alpha(A)-E_\alpha(-A) \\ E_\alpha(A)-E_\alpha(-A) & E_\alpha(A)+E_\alpha(-A) \end{bmatrix}.
\end{aligned}$$

□

Now, we are in position to prove the main result of this thesis.

3.1 The main result

Theorem 3.3. Let $0 < \alpha \leq 1$. Assume that

$$\sum_{i=1}^{p}\sum_{j=1}^{q}(D_jB_i)^T \otimes (A_iC_j) \;=\; \sum_{i=1}^{q}\sum_{j=1}^{s}(H_jD_i)^T \otimes (C_iG_j)\,, \tag{3.1}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{r}(F_jH_i)^T \otimes (G_iE_j) \;=\; \sum_{i=1}^{r}\sum_{j=1}^{p}(B_jF_i)^T \otimes (E_iA_j)\,. \tag{3.2}$$

Then the general solution of the system of nonhomogeneous coupled linear matrix fractional differential equations of order α :

$$\begin{aligned} X^{(\alpha)}(t) &= \sum_{i=1}^{p} A_iX(t)B_i + \sum_{i=1}^{q} C_iY(t)D_i + U(t), \\ Y^{(\alpha)}(t) &= \sum_{i=1}^{r} E_iX(t)F_i + \sum_{i=1}^{s} G_iY(t)H_i + V(t), \end{aligned} \tag{3.3}$$

subject to $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$\begin{aligned} \mathrm{Vec}\, X(t) \;&= E_\alpha((t-t_0)^\alpha K)E_{2\alpha,1}((t-t_0)^{2\alpha}LM)\,\mathrm{Vec}\, W_1 \\ &+ E_\alpha((t-t_0)^\alpha K)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha}LM))M\,\mathrm{Vec}\, W_2 \\ &+ \int_{t_0}^{t}(t-s)^{\alpha-1}E_\alpha((t-s)^\alpha K)E_{2\alpha,1}((t-s)^{2\alpha}LM)\,\mathrm{Vec}\, U(s)ds \\ &+ \int_{t_0}^{t}(t-s)^{\alpha-1}E_\alpha((t-s)^\alpha K)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha}LM))M\,\mathrm{Vec}\, V(s)ds, \\ \mathrm{Vec}\, Y(t) \;&= E_\alpha((t-t_0)^\alpha N)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha}ML))L\,\mathrm{Vec}\, W_1 \\ &+ E_\alpha((t-t_0)^\alpha N)E_{2\alpha,1}((t-t_0)^{2\alpha}ML)\,\mathrm{Vec}\, W_2 \\ &+ \int_{t_0}^{t}(t-s)^{\alpha-1}E_\alpha((t-s)^\alpha N)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha}ML))L\,\mathrm{Vec}\, U(s)ds \\ &+ \int_{t_0}^{t}(t-s)^{\alpha-1}E_\alpha((t-s)^\alpha N)E_{2\alpha,1}((t-s)^{2\alpha}ML)\,\mathrm{Vec}\, V(s)ds, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} K &= \sum_{i=1}^{p} B_i^T \otimes A_i,\;\; N \;=\; \sum_{i=1}^{s} H_i^T \otimes G_i, \\ M &= \sum_{i=1}^{r} F_i^T \otimes E_i,\;\; L \;=\; \sum_{i=1}^{q} D_i^T \otimes C_i. \end{aligned} \tag{3.5}$$

Proof. Using Lemmas 2.3, 2.5 and 2.11, we have

$$\begin{aligned}
\text{Vec}\, X^{(\alpha)}(t) &= \text{Vec}\left(\sum_{i=1}^{p} A_i X(t) B_i + \sum_{i=1}^{q} C_i Y(t) D_i + U(t)\right) \\
&= \text{Vec}\left(\sum_{i=1}^{p} A_i X(t) B_i\right) + \text{Vec}\left(\sum_{i=1}^{q} C_i Y(t) D_i\right) + \text{Vec}\, U(t) \\
&= \sum_{i=1}^{p} \text{Vec}\,(A_i X(t) B_i) + \sum_{i=1}^{q} \text{Vec}\,(C_i Y(t) D_i) + \text{Vec}\, U(t) \\
&= \sum_{i=1}^{p} \left(B_i^T \otimes A_i\right) \text{Vec}\, X(t) + \sum_{i=1}^{q} \left(D_i^T \otimes C_i\right) \text{Vec}\, Y(t) + \text{Vec}\, U(t) \\
&= K\, \text{Vec}\, X(t) + L\, \text{Vec}\, Y(t) + \text{Vec}\, U(t),
\end{aligned}$$

and

$$\begin{aligned}
\text{Vec}\, Y^{(\alpha)}(t) &= \text{Vec}\left(\sum_{i=1}^{r} E_i X(t) F_i + \sum_{i=1}^{s} G_i Y(t) H_i + V(t)\right) \\
&= \text{Vec}\left(\sum_{i=1}^{r} E_i X(t) F_i\right) + \text{Vec}\left(\sum_{i=1}^{s} G_i Y(t) H_i\right) + \text{Vec}\, V(t) \\
&= \sum_{i=1}^{r} \text{Vec}\,(E_i X(t) F_i) + \sum_{i=1}^{s} \text{Vec}\,(G_i Y(t) H_i) + \text{Vec}\, V(t) \\
&= \sum_{i=1}^{r} \left(F_i^T \otimes E_i\right) \text{Vec}\, X(t) + \sum_{i=1}^{s} \left(H_i^T \otimes G_i\right) \text{Vec}\, Y(t) + \text{Vec}\, V(t) \\
&= M\, \text{Vec}\, X(t) + N\, \text{Vec}\, Y(t) + \text{Vec}\, V(t).
\end{aligned}$$

Thus, the system (3.3) is transformed to the following equivalent system:

$$\begin{bmatrix} \text{Vec}\, X^{(\alpha)}(t) \\ \text{Vec}\, Y^{(\alpha)}(t) \end{bmatrix} = \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{bmatrix} \text{Vec}\, X(t) \\ \text{Vec}\, Y(t) \end{bmatrix} + \begin{bmatrix} \text{Vec}\, U(t) \\ \text{Vec}\, V(t) \end{bmatrix}.$$

Let us denote $S = P + Q$ where

$$P = \begin{bmatrix} K & 0 \\ 0 & N \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & L \\ M & 0 \end{bmatrix}.$$

This system has the following solution:

$$\begin{bmatrix} \text{Vec}\, X(t) \\ \text{Vec}\, Y(t) \end{bmatrix} = E_\alpha((t-t_0)^\alpha S) \begin{bmatrix} \text{Vec}\, X(t_0) \\ \text{Vec}\, Y(t_0) \end{bmatrix} + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha S) \begin{bmatrix} \text{Vec}\, U(s) \\ \text{Vec}\, V(s) \end{bmatrix} ds.$$

Now, we compute $E_\alpha(S)$. We have

$$\begin{aligned}
PQ &= \begin{bmatrix} K & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} 0 & L \\ M & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & KL \\ NM & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \sum_{i=1}^{p} \left(B_i^T \otimes A_i\right) \sum_{i=1}^{q} \left(D_i^T \otimes C_i\right) \\ \sum_{i=1}^{s} \left(H_i^T \otimes G_i\right) \sum_{i=1}^{r} \left(F_i^T \otimes E_i\right) & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \sum_{i=1}^{p} \sum_{j=1}^{q} (D_j B_i)^T \otimes (A_i C_j) \\ \sum_{i=1}^{s} \sum_{j=1}^{r} (F_j H_i)^T \otimes (G_i E_j) & 0 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
PQ &= \begin{bmatrix} 0 & L \\ M & 0 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & N \end{bmatrix} \\
&= \begin{bmatrix} 0 & LN \\ MK & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \sum_{i=1}^{q} \left(D_i^T \otimes C_i\right) \sum_{i=1}^{s} \left(H_i^T \otimes G_i\right) \\ \sum_{i=1}^{r} \left(F_i^T \otimes E_i\right) \sum_{i=1}^{p} \left(B_i^T \otimes A_i\right) & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \sum_{i=1}^{q} \sum_{j=1}^{s} (H_j D_i)^T \otimes (C_i G_j) \\ \sum_{i=1}^{r} \sum_{j=1}^{p} (B_j F_i)^T \otimes (E_i A_j) & 0 \end{bmatrix}
\end{aligned}$$

Using the hypothesis (3.1) and (3.2) together with Lemmas 2.3, 2.5 and 2.11 we can deduce that $KL = LN$ and $NM = MK$. Thus, $PQ = QP$. From which it follows from Lemma 6 that

$$E_\alpha(S) = E_\alpha(P+Q) = E_\alpha(P)E_\alpha(Q).$$

By Lemma 3.1, we have

$$E_\alpha(P) = \begin{bmatrix} E_\alpha(K) & 0 \\ 0 & E_\alpha(N) \end{bmatrix}.$$

By Lemma 3.2, we have

$$E_\alpha(Q) = \begin{bmatrix} E_{2\alpha,1}(LM) & (E_{2\alpha,\alpha+1}(LM))M \\ (E_{2\alpha,\alpha+1}(ML))L & E_{2\alpha,1}(ML) \end{bmatrix}.$$

Hence,

$$\begin{aligned} E_\alpha(S) &= \begin{bmatrix} E_\alpha(K) & 0 \\ 0 & E_\alpha(N) \end{bmatrix} \begin{bmatrix} E_{2\alpha,1}(LM) & (E_{2\alpha,\alpha+1}(LM))M \\ (E_{2\alpha,\alpha+1}(ML))L & E_{2\alpha,1}(ML) \end{bmatrix} \\ &= \begin{bmatrix} E_\alpha(K)E_{2\alpha,1}(LM) & E_\alpha(K)(E_{2\alpha,\alpha+1}(LM))M \\ E_\alpha(N)(E_{2\alpha,\alpha+1}(ML))L & E_\alpha(N)E_{2\alpha,1}(ML) \end{bmatrix}. \end{aligned}$$

Therefore, the general solution of (3.3) is given by (3.4). □

In Theorem 3.3, the hypotheses (3.1) and (3.2) is not too restrictive since it includes many interesting special cases.

3.2 Special cases

Corollary 3.4. Let $0 < \alpha \leq 1$. Assume that $(DB)^T \otimes (AC) = (HD)^T \otimes (CG)$ and $(FH)^T \otimes (GE) = (BF)^T \otimes (EA)$. Then the general solution of the system:

$$\begin{aligned} X^{(\alpha)}(t) &= AX(t)B + CY(t)D + U(t), \\ Y^{(\alpha)}(t) &= EX(t)F + GY(t)H + V(t), \end{aligned} \tag{3.6}$$

subject to $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$\begin{aligned} \operatorname{Vec} X(t) &= E_\alpha((t-t_0)^\alpha R_1)E_{2\alpha,1}((t-t_0)^{2\alpha}R_2R_3)\operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-t_0)^\alpha R_1)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha}R_2R_3))R_3 \operatorname{Vec} W_2 \\ &\quad + \int_{t_0}^t (t-s)^{\alpha-1}E_\alpha((t-s)^\alpha R_1)E_{2\alpha,1}((t-s)^{2\alpha}R_2R_3)\operatorname{Vec} U(s)ds \\ &\quad + \int_{t_0}^t (t-s)^{\alpha-1}E_\alpha((t-t_0)^\alpha R_1)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha}R_2R_3))R_3 \operatorname{Vec} V(s)ds, \\ \operatorname{Vec} Y(t) &= E_\alpha((t-t_0)^\alpha R_4)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha}R_3R_2))R_2 \operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-t_0)^\alpha R_4)E_{2\alpha,1}((t-t_0)^{2\alpha}R_3R_2)\operatorname{Vec} W_2 \\ &\quad + \int_{t_0}^t (t-s)^{\alpha-1}E_\alpha((t-s)^\alpha R_4)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha}R_3R_2))R_2 \operatorname{Vec} U(s)ds \\ &\quad + \int_{t_0}^t (t-s)^{\alpha-1}E_\alpha((t-s)^\alpha N)E_{2\alpha,1}((t-s)^{2\alpha}R_3R_2)\operatorname{Vec} V(s)ds, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} R_1 &= B^T \otimes A, \; R_2 = D^T \otimes C, \\ R_3 &= F^T \otimes E, \; R_4 = H^T \otimes G. \end{aligned} \tag{3.8}$$

Proof. Put $p = q = r = s = 1$ in Theorem 3.3. We obtain

$$
\begin{aligned}
\operatorname{Vec} X(t) &= E_\alpha((t-t_0)^\alpha(B^T \otimes A))E_{2\alpha,1}((t-t_0)^{2\alpha}((FD)^T \otimes CE)\operatorname{Vec} W_1 \\
&\quad + E_\alpha((t-t_0)^\alpha(B^T \otimes A))(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha}((FD)^T \otimes CE)))(F^T \otimes E)\operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t}(t-t_0)^{\alpha-1}E_\alpha((t-t_0)^\alpha(B^T \otimes A))E_{2\alpha,1}((t-s)^{2\alpha}((FD)^T \otimes CE))\operatorname{Vec} U(s)ds \\
&\quad + \int_{t_0}^{t}(t-s)^{\alpha-1}E_\alpha((t-s)^\alpha(B^T \otimes A))(E_{2\alpha,\alpha+1}((t-s)^{2\alpha}((FD)^T \otimes CE)))(F^T \otimes E)\operatorname{Vec} V(s)ds \\
&= E_\alpha((t-t_0)^\alpha R_1)E_{2\alpha,1}((t-t_0)^{2\alpha}R_2R_3)\operatorname{Vec} W_1 \\
&\quad + E_\alpha((t-s)^\alpha R_1)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha}R_2R_3))R_3\operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t}(t-s)^{\alpha-1}E_\alpha((t-s)^\alpha R_1)E_{2\alpha,1}((t-s)^{2\alpha}R_2R_3)\operatorname{Vec} U(s)ds \\
&\quad + \int_{t_0}^{t}(t-s)^{\alpha-1}E_\alpha((t-t_0)^\alpha R_1)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha}R_2R_3))R_3\operatorname{Vec} V(s)ds, \\
\operatorname{Vec} Y(t) &= E_\alpha((t-t_0)^\alpha(H^T \otimes G))(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha}((DF)^T \otimes EC))(D^T \otimes C)\operatorname{Vec} W_1 \\
&\quad + E_\alpha((t-t_0)^\alpha(H^T \otimes G))E_{2\alpha,1}((t-t_0)^{2\alpha}((DF)^T \otimes EC))\operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t}(t-s)^{\alpha-1}E_\alpha((t-s)^\alpha(H^T \otimes G))(E_{2\alpha,\alpha+1}((t-s)^{2\alpha}((DF)^T \otimes EC)))(D^T \otimes C)\operatorname{Vec} U(s)ds \\
&\quad + \int_{t_0}^{t}(t-s)^{\alpha-1}E_\alpha((t-s)^\alpha(H^T \otimes G))E_{2\alpha,1}((t-s)^{2\alpha}((DF)^T \otimes EC))\operatorname{Vec} V(s)ds \\
&= E_\alpha((t-t_0)^\alpha R_4)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha}R_3R_2))R_2\operatorname{Vec} W_1 \\
&\quad + E_\alpha((t-t_0)^\alpha R_4)E_{2\alpha,1}((t-t_0)^{2\alpha}R_3R_2)\operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t}(t-s)^{\alpha-1}E_\alpha((t-s)^\alpha R_4)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha}R_3R_2))R_2\operatorname{Vec} U(s)ds \\
&\quad + \int_{t_0}^{t}(t-s)^{\alpha-1}E_\alpha((t-s)^\alpha N)E_{2\alpha,1}((t-s)^{2\alpha}R_3R_2)\operatorname{Vec} V(s)ds.
\end{aligned}
$$

□

Corollary 3.5. Let $0 < \alpha \leq 1$. Denote K, M, N, L as in (3.5). Assume that (3.1) and (3.2) hold. Then the general solution of the system:

$$
\begin{aligned}
X^{(\alpha)}(t) &= \sum_{i=1}^{p} A_iX(t)B_i + \sum_{i=1}^{q} C_iY(t)D_i, \\
Y^{(\alpha)}(t) &= \sum_{i=1}^{r} E_iX(t)F_i + \sum_{i=1}^{s} G_iY(t)H_i,
\end{aligned}
\tag{3.9}
$$

subject to $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$
\begin{aligned}
\operatorname{Vec} X(t) &= E_\alpha((t-t_0)^\alpha K)E_{2\alpha,1}((t-t_0)^{2\alpha}LM)\operatorname{Vec} W_1 \\
&\quad + E_\alpha((t-t_0)^\alpha K)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha}LM))M\operatorname{Vec} W_2, \\
\operatorname{Vec} Y(t) &= E_\alpha((t-t_0)^\alpha N)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha}ML))L\operatorname{Vec} W_1 \\
&\quad + E_\alpha((t-t_0)^\alpha N)E_{2\alpha,1}((t-t_0)^{2\alpha}ML)\operatorname{Vec} W_2.
\end{aligned}
\tag{3.10}
$$

Proof. Put $U(t) = V(t) = 0$ in Theorem 3.3. We obtain

$$\begin{aligned}
\operatorname{Vec} X(t) &= E_\alpha((t-t_0)^\alpha K) E_{2\alpha,1}((t-t_0)^{2\alpha} LM) \operatorname{Vec} W_1 \\
&\quad + E_\alpha((t-t_0)^\alpha K)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} LM)) M \operatorname{Vec} W_2 \\
&\quad + 0 + 0 \\
&= E_\alpha((t-t_0)^\alpha K) E_{2\alpha,1}((t-t_0)^{2\alpha} LM) \operatorname{Vec} W_1 \\
&\quad + E_\alpha((t-t_0)^\alpha K)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} LM)) M \operatorname{Vec} W_2, \\
\operatorname{Vec} Y(t) &= E_\alpha((t-t_0)^\alpha N)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} ML)) L \operatorname{Vec} W_1 \\
&\quad + E_\alpha((t-t_0)^\alpha N) E_{2\alpha,1}((t-t_0)^{2\alpha} ML) \operatorname{Vec} W_2 \\
&\quad + 0 + 0 \\
&= E_\alpha((t-t_0)^\alpha N)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} ML)) L \operatorname{Vec} W_1 \\
&\quad + E_\alpha((t-t_0)^\alpha N) E_{2\alpha,1}((t-t_0)^{2\alpha} ML) \operatorname{Vec} W_2.
\end{aligned}$$

□

Corollary 3.6. Denote K, M, N, L as in (3.5). Assume that (3.1) and (3.2) hold. Then the general solution of the system:

$$\begin{aligned}
X'(t) &= \sum_{i=1}^{p} A_i X(t) B_i + \sum_{i=1}^{q} C_i Y(t) D_i + U(t), \\
Y'(t) &= \sum_{i=1}^{r} E_i X(t) F_i + \sum_{i=1}^{s} G_i Y(t) H_i + V(t),
\end{aligned} \tag{3.11}$$

subject to $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)K} E_{2,1}((t-t_0)^2 LM) \operatorname{Vec} W_1 \\
&\quad + e^{(t-t_0)K}(E_{2,2}((t-t_0)^2 LM)) M \operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t} e^{(t-s)K} E_{2,1}((t-s)^2 LM) \operatorname{Vec} U(s) ds \\
&\quad + \int_{t_0}^{t} e^{(t-s)K}(E_{2,2}((t-s)^2 LM)) M \operatorname{Vec} V(s) ds, \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)N}(E_{2,2}((t-t_0)^2 ML)) L \operatorname{Vec} W_1 \\
&\quad + e^{(t-t_0)N} E_{2,1}((t-t_0)^2 ML) \operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t} e^{(t-s)N} e^{(t-s)N}(E_{2,2}((t-s)^2 ML)) L \operatorname{Vec} U(s) ds \\
&\quad + \int_{t_0}^{t} e^{(t-s)N} E_{2,1}((t-s)^2 ML) \operatorname{Vec} V(s) ds,
\end{aligned} \tag{3.12}$$

Proof. Put $\alpha = 1$ in Theorem 3.3. We obtain

$$\begin{aligned}
\operatorname{Vec} X(t) &= E_1((t-t_0)K)E_{2,1}((t-t_0)^2LM)\operatorname{Vec} W_1 \\
&\quad + E_1((t-t_0)K)(E_{2,2}((t-t_0)^2LM))M\operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t} E_1((t-s)K)E_{2,1}((t-s)^2LM)\operatorname{Vec} U(s)ds \\
&\quad + \int_{t_0}^{t} E_1((t-s)K)(E_{2,2}((t-s)^2LM))M\operatorname{Vec} V(s)ds \\
&= e^{(t-t_0)K}E_{2,1}((t-t_0)^2LM)\operatorname{Vec} W_1 \\
&\quad + e^{(t-t_0)K}(E_{2,2}((t-t_0)^2LM))M\operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t} e^{(t-s)K}E_{2,1}((t-s)^2LM)\operatorname{Vec} U(s)ds \\
&\quad + \int_{t_0}^{t} e^{(t-t_0)K}(E_{2,2}((t-s)^2LM))M\operatorname{Vec} V(s)ds, \\
\operatorname{Vec} Y(t) &= E_1(t-t_0)N(E_{2,2}((t-t_0)^2ML))L\operatorname{Vec} W_1 \\
&\quad + E_1(t-t_0)NE_{2,1}((t-t_0)^2ML)\operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t} E_1(t-s)Ne^{(t-s)N}(E_{2,2}((t-s)^2ML))L\operatorname{Vec} U(s)ds \\
&\quad + \int_{t_0}^{t} e_1(t-s)NE_{2,1}((t-s)^2ML)\operatorname{Vec} V(s)ds \\
&= e^{(t-t_0)N}(E_{2,2}((t-t_0)^2ML))L\operatorname{Vec} W_1 \\
&\quad + e^{(t-t_0)N}E_{2,1}((t-t_0)^2ML)\operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t} e^{(t-s)N}e^{(t-s)N}(E_{2,2}((t-s)^2ML))L\operatorname{Vec} U(s)ds \\
&\quad + \int_{t_0}^{t} e^{(t-s)N}E_{2,1}((t-s)^2ML)\operatorname{Vec} V(s)ds.
\end{aligned}$$

□

Corollary 3.7. Denote K, M, N, L as in (3.5). Assume that (3.1) and (3.2) hold. Then the general solution of the system:

$$\begin{aligned}
X'(t) &= \sum_{i=1}^{p} A_iX(t)B_i + \sum_{i=1}^{q} C_iY(t)D_i, \\
Y'(t) &= \sum_{i=1}^{r} E_iX(t)F_i + \sum_{i=1}^{s} G_iY(t)H_i,
\end{aligned} \tag{3.13}$$

subject to $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)K}E_{2,1}((t-t_0)^2LM)\operatorname{Vec} W_1 + e^{(t-t_0)K}(E_{2,2}((t-t_0)^2LM))M\operatorname{Vec} W_2, \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)N}(E_{2,2}((t-t_0)^2ML))L\operatorname{Vec} W_1 + e^{(t-t_0)N}E_{2,1}((t-t_0)^2ML)\operatorname{Vec} W_2.
\end{aligned} \tag{3.14}$$

Proof. Put $U(t) = V(t) = 0$ in Corollary 3.6. We obtain

$$\begin{aligned}
\operatorname{Vec} X(t) &= E_1((t-t_0)K)E_{2,1}((t-t_0)^2LM)\operatorname{Vec} W_1 \\
&\quad + E_1((t-t_0)K)(E_{2,2}((t-t_0)^2LM))M\operatorname{Vec} W_2 \\
&\quad + 0 + 0 \\
&= e^{(t-t_0)K}E_{2,1}((t-t_0)^2LM)\operatorname{Vec} W_1 + e^{(t-t_0)K}(E_{2,2}((t-t_0)^2LM))M\operatorname{Vec} W_2, \\
\operatorname{Vec} Y(t) &= E_1(t-t_0)N(E_{2,2}((t-t_0)^2ML))L\operatorname{Vec} W_1 \\
&\quad + E_1(t-t_0)NE_{2,1}((t-t_0)^2ML)\operatorname{Vec} W_2 \\
&\quad + 0 + 0 \\
&= e^{(t-t_0)N}(E_{2,2}((t-t_0)^2ML))L\operatorname{Vec} W_1 + e^{(t-t_0)N}E_{2,1}((t-t_0)^2ML)\operatorname{Vec} W_2.
\end{aligned}$$

□

The next result was firstly obtained in Kongyaksee and Chansangiam [12].

Corollary 3.8. Let $0 < \alpha \leq 1$. Denote R_1, R_2, R_3, R_4 as in (3.8). Assume that $DB = HD, AC = CG$ and $FH = BF, GE = EA$. The general solution of the system:

$$\begin{aligned}
X'(t) &= AX(t)B + CY(t)D + U(t), \\
Y'(t) &= EX(t)F + GY(t)H + V(t),
\end{aligned} \tag{3.15}$$

subject to $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)R_1}E_{2,1}((t-t_0)^2R_2R_3)\operatorname{Vec} W_1 \\
&\quad + e^{(t-t_0)R_1}(E_{2,2}((t-t_0)^2R_2R_3))R_2\operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t} e^{(t-s)R_1}E_{2,1}((t-s)^2R_2R_3)\operatorname{Vec} U(s)ds \\
&\quad + \int_{t_0}^{t} e^{(t-s)R_1}(E_{2,2}((t-s)^2R_2R_3))R_2\operatorname{Vec} V(s)ds, \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)R_4}(E_{2,2}((t-t_0)^2R_3R_2))R_3\operatorname{Vec} W_1 \\
&\quad + e^{(t-t_0)R_4}E_{2,1}((t-t_0)^2R_3R_2)\operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t} e^{(t-s)R_4}(E_{2,2}((t-s)^2R_3R_2))R_3\operatorname{Vec} U(s)ds \\
&\quad + \int_{t_0}^{t} e^{(t-s)R_4}E_{2,1}((t-s)^2R_3R_2)\operatorname{Vec} V(s)ds.
\end{aligned} \tag{3.16}$$

Proof. Put $\alpha = 1$ in Corollary 3.4. We obtain

$$\begin{aligned}
\operatorname{Vec} X(t) &= E_1((t-t_0)R_1)E_{2,1}((t-t_0)^2R_2R_3)\operatorname{Vec} W_1\\
&\quad + E_1((t-t_0)R_1)(E_{2,2}((t-t_0)^2R_2R_3))R_3\operatorname{Vec} W_2\\
&\quad + \int_{t_0}^{t} E_1((t-s)R_1)E_{2,1}((t-s)^2R_2R_3)\operatorname{Vec} U(s)ds\\
&\quad + \int_{t_0}^{t} E_1((t-t_0)R_1)(E_{2,2}((t-t_0)^2R_2R_3))R_3\operatorname{Vec} V(s)ds\\
&= e^{(t-t_0)R_1}E_{2,1}((t-t_0)^2R_2R_3)\operatorname{Vec} W_1\\
&\quad + e^{(t-t_0)R_1}(E_{2,2}((t-t_0)^2R_2R_3))R_2\operatorname{Vec} W_2\\
&\quad + \int_{t_0}^{t} e^{(t-s)R_1}E_{2,1}((t-s)^2R_2R_3)\operatorname{Vec} U(s)ds\\
&\quad + \int_{t_0}^{t} e^{(t-s)R_1}(E_{2,2}((t-s)^2R_2R_3))R_2\operatorname{Vec} V(s)ds,\\
\operatorname{Vec} Y(t) &= E_1((t-t_0)R_4)(E_{2,2}((t-t_0)^2R_3R_2))R_2\operatorname{Vec} W_1\\
&\quad + E_1((t-t_0)R_4)E_{2,1}((t-t_0)^2R_3R_2)\operatorname{Vec} W_2\\
&\quad + \int_{t_0}^{t} E_1((t-s)R_4)(E_{2,2}((t-s)^2R_3R_2))R_2\operatorname{Vec} U(s)ds\\
&\quad + \int_{t_0}^{t} E_((t-s)N)E_{2,1}((t-s)^2R_3R_2)\operatorname{Vec} V(s)ds\\
&= e^{(t-t_0)R_4}(E_{2,2}((t-t_0)^2R_3R_2))R_3\operatorname{Vec} W_1\\
&\quad + e^{(t-t_0)R_4}E_{2,1}((t-t_0)^2R_3R_2)\operatorname{Vec} W_2\\
&\quad + \int_{t_0}^{t} e^{(t-s)R_4}(E_{2,2}((t-s)^2R_3R_2))R_3\operatorname{Vec} U(s)ds\\
&\quad + \int_{t_0}^{t} e^{(t-s)R_4}E_{2,1}((t-s)^2R_3R_2)\operatorname{Vec} V(s)ds.
\end{aligned}$$

□

The next result was firstly obtained in Al-Zhour [11].

Corollary 3.9. Let $0 < \alpha \leq 1$. Denote R_1, R_2 as in (18). Assume that $AC = CA, BD = DB$. The general solution of the system:

$$\begin{aligned}
X^{(\alpha)}(t) &= AX(t)B + CY(t)D,\\
Y^{(\alpha)}(t) &= CX(t)D + AY(t)B,
\end{aligned} \tag{3.17}$$

subject to $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$\begin{aligned}
\operatorname{Vec} X(t) &= \frac{1}{2}E_\alpha((t-t_0)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) + E_\alpha((t-t_0)^\alpha R_2))\operatorname{Vec} W_1\\
&\quad + \frac{1}{2}E_\alpha((t-t_0)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) - E_\alpha(-(t-t_0)^\alpha R_2))\operatorname{Vec} W_2,\\
\operatorname{Vec} Y(t) &= \frac{1}{2}E_\alpha((t-t_0)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) - E_\alpha((t-t_0)^\alpha R_2))\operatorname{Vec} W_1\\
&\quad + \frac{1}{2}E_\alpha((t-t_0)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) + E_\alpha(-(t-t_0)^\alpha R_2))\operatorname{Vec} W_2.
\end{aligned} \tag{3.18}$$

Proof. From Corollary 3.4, put $U(t) = V(t) = 0$ and $E = C, F = D, G = A, H = B$. $\square$

Chapter 4

Examples

4.1 An auxiliary lemma for computing Mittage-Leffler functions of matrices

In this section, we provide an example in order to illustrate our main result in Section 3. The next lemma is used to compute certain Mittage-Leffler function.

Lemma 4.1. For any $a, c, d \in \mathbb{R}$ with $a \neq d$, we have

$$E_\alpha\left(\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}\right) = \begin{bmatrix} E_\alpha(a) & 0 \\ \frac{c}{a-d}(E_\alpha(a) - E_\alpha(d)) & E_\alpha(d) \end{bmatrix}. \tag{4.1}$$

Proof. By expanding, we have

$$\begin{aligned} E_\alpha\left(\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}\right) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}^k \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} a^k & 0 \\ c\left(\frac{a^k - d^k}{a-d}\right) & d^k \end{bmatrix} \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} \frac{a^k}{\Gamma(\alpha k+1)} & 0 \\ \frac{c}{a-d}\left(\frac{a^k - d^k}{\Gamma(\alpha k+1)}\right) & \frac{d^k}{\Gamma(\alpha k+1)} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{a^k}{\Gamma(\alpha k+1)} & 0 \\ \frac{c}{a-d}\left(\sum_{k=0}^{\infty} \frac{a^k - d^k}{\Gamma(\alpha k+1)}\right) & \sum_{k=0}^{\infty} \frac{d^k}{\Gamma(\alpha k+1)} \end{bmatrix} \\ &= \begin{bmatrix} E_\alpha(a) & 0 \\ \frac{c}{a-d}(E_\alpha(a) - E_\alpha(d)) & E_\alpha(d) \end{bmatrix}. \end{aligned}$$

□

4.2 Examples

Example 4.2. Let $0 < \alpha \leq 1$. Consider the following coupled matrix fractional differential equations:

$$\begin{aligned} X^{(\alpha)}(t) &= AX(t) + CY(t), \\ Y^{(\alpha)}(t) &= CX(t) + AY(t), \end{aligned} \tag{4.2}$$

under the initial contidions $X(0) = W_1$ and $Y(0) = W_2$, where

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}, W_1 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \text{ and } W_2 = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}.$$

First, note that $AC = CA$. From Corollary 3.9, we have

$$\begin{aligned} \text{Vec}\, X(t) &= \frac{1}{2}E_\alpha(t^\alpha(I \otimes A))(E_\alpha(t^\alpha(I \otimes C)) + E_\alpha(-t^\alpha(I \otimes C)))\, \text{Vec}\, W_1 \\ &+ \frac{1}{2}E_\alpha(t^\alpha(I \otimes A))(E_\alpha(t^\alpha(I \otimes C)) - E_\alpha(-t^\alpha(I \otimes C)))\, \text{Vec}\, W_2, \\ \text{Vec}\, Y(t) &= \frac{1}{2}E_\alpha(t^\alpha(I \otimes A))(E_\alpha(t^\alpha(I \otimes C)) - E_\alpha(-t^\alpha(I \otimes C)))\, \text{Vec}\, W_1 \\ &+ \frac{1}{2}E_\alpha(t^\alpha(I \otimes A))(E_\alpha(t^\alpha(I \otimes C)) + E_\alpha(-t^\alpha(I \otimes C)))\, \text{Vec}\, W_2. \end{aligned}$$

To obtain the solution of the system, we compute

$$\begin{aligned} E_\alpha(t^\alpha(I \otimes A)) &= I \otimes E_\alpha(t^\alpha A) \\ &= I \otimes \begin{bmatrix} E_\alpha(2t^\alpha) & 0 \\ -2E_\alpha(t^\alpha + 2E_\alpha(2t^\alpha) & E_\alpha(2t^\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \eta_1(t) & 0 & 0 & 0 \\ \eta_2(t) & \eta_3(t) & 0 & 0 \\ 0 & 0 & \eta_1(t) & 0 \\ 0 & 0 & \eta_2(t) & \eta_3(t) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} E_\alpha(t^\alpha(I \otimes C)) &= I \otimes E_\alpha(t^\alpha C) \\ &= I \otimes \begin{bmatrix} E_\alpha(2t^\alpha) & 0 \\ 2E_\alpha(t^\alpha) - 2E_\alpha(2t^\alpha) & E_\alpha(t^\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \eta_3(t) & 0 & 0 & 0 \\ -\eta_2(t) & \eta_1(t) & 0 & 0 \\ 0 & 0 & \eta_3(t) & 0 \\ 0 & 0 & -\eta_2(t) & \eta_1(t) \end{bmatrix}, \end{aligned}$$

where $\eta_1(t) = E_\alpha(t^\alpha), \eta_2(t) = -2E_\alpha(t^\alpha) + 2E_\alpha(2t^\alpha), \eta_3(t) = E_\alpha(2t^\alpha)$, and

$$
\begin{aligned}
&E_\alpha(-t^\alpha(I \otimes C)) \\
&\quad = I \otimes E_\alpha(-t^\alpha C) \\
&\quad = I \otimes (E_\alpha(t^\alpha C))^{-1} \\
&\quad = I \otimes \left(\frac{1}{-2E_\alpha(t^\alpha) + E_\alpha(t^\alpha)E_\alpha(2t^\alpha) + 2E_\alpha(2t^\alpha)} \right) \begin{bmatrix} E_\alpha(t^\alpha) & 0 \\ -2E_\alpha(t^\alpha) + 2E_\alpha(2t^\alpha) & E_\alpha(2t^\alpha) \end{bmatrix} \\
&\quad = \frac{1}{\eta_1(t)\eta_3(t) + \eta_2(t)} \begin{bmatrix} \eta_1(t) & 0 & 0 & 0 \\ \eta_2(t) & \eta_3(t) & 0 & 0 \\ 0 & 0 & \eta_1(t) & 0 \\ 0 & 0 & \eta_2(t) & \eta_3(t) \end{bmatrix}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&E_\alpha(t^\alpha(I \otimes A))(E_\alpha(t^\alpha(I \otimes C)) + (E_\alpha(t^\alpha(I \otimes C)))^{-1}) \\
&\quad = \frac{1}{\kappa_2(t)} \begin{bmatrix} \kappa_1(t) & 0 & 0 & 0 \\ \kappa_3(t) & \kappa_4(t) & 0 & 0 \\ 0 & 0 & \kappa_1(t) & 0 \\ 0 & 0 & \kappa_3(t) & \kappa_4(t) \end{bmatrix}.
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_1(t) &= \eta_1^2(t) + \eta_1^2(t)\eta_3^2(t) + \eta_1(t)\eta_2(t)\eta_3(t), \\
\kappa_2(t) &= \eta_1(t)\eta_3(t) + \eta_2(t), \\
\kappa_3(t) &= \eta_1(t)\eta_2(t) + \eta_2(t)\eta_3(t), \\
\kappa_4(t) &= \eta_1^2(t)\eta_3^2(t) + \eta_1(t)\eta_2(t)\eta_3(t) + \eta_3^2(t).
\end{aligned}
$$

Similary, we have

$$
\begin{aligned}
&E_\alpha(t^\alpha(I \otimes A))(E_\alpha(t^\alpha(I \otimes C)) - (E_\alpha(t^\alpha(I \otimes C)))^{-1}) \\
&\quad = \frac{1}{\kappa_2(t)} \begin{bmatrix} \kappa_5(t) & 0 & 0 & 0 \\ -\kappa_3(t) & \kappa_6(t) & 0 & 0 \\ 0 & 0 & \kappa_5(t) & 0 \\ 0 & 0 & -\kappa_3(t) & \kappa_6(t) \end{bmatrix}.
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_5(t) &= -\eta_1^2(t) + \eta_1^2(t)\eta_3^2(t) + \eta_1(t)\eta_2(t)\eta_3(t), \\
\kappa_6(t) &= \eta_1^2(t)\eta_3^2(t) + \eta_1(t)\eta_2(t)\eta_3(t) - \eta_3(t).
\end{aligned}
$$

Hence, by Corollary 3.9, we have

$$\begin{aligned}
\operatorname{Vec} X(t) &= \frac{1}{2\kappa_2(t)}\left(\begin{bmatrix} \kappa_1(t) & 0 & 0 & 0 \\ \kappa_3(t) & \kappa_4(t) & 0 & 0 \\ 0 & 0 & \kappa_1(t) & 0 \\ 0 & 0 & \kappa_3(t) & \kappa_4(t) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} \kappa_5(t) & 0 & 0 & 0 \\ -\kappa_3(t) & \kappa_6(t) & 0 & 0 \\ 0 & 0 & \kappa_5(t) & 0 \\ 0 & 0 & -\kappa_3(t) & \kappa_6(t) \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}\right) \\
&= \frac{1}{2\kappa_2(t)}\begin{bmatrix} \kappa_1(t)+\kappa_5(t) \\ \kappa_4(t) - 2\kappa_6(t) \\ 0 \\ 2\kappa_4(t) + 2\kappa_6(t) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\operatorname{Vec} Y(t) &= \frac{1}{2\kappa_2(t)}\left(\begin{bmatrix} \kappa_5(t) & 0 & 0 & 0 \\ -\kappa_3(t) & \kappa_6(t) & 0 & 0 \\ 0 & 0 & \kappa_5(t) & 0 \\ 0 & 0 & -\kappa_3(t) & \kappa_6(t) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} \kappa_1(t) & 0 & 0 & 0 \\ \kappa_3(t) & \kappa_4(t) & 0 & 0 \\ 0 & 0 & \kappa_1(t) & 0 \\ 0 & 0 & \kappa_3(t) & \kappa_4(t) \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}\right) \\
&= \frac{1}{2\kappa_2(t)}\begin{bmatrix} \kappa_1(t)+\kappa_5(t) \\ -2\kappa_4(t) + \kappa_6(t) \\ 0 \\ 2\kappa_4(t) + 2\kappa_6(t) \end{bmatrix}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
X(t) &= \frac{1}{2\kappa_2(t)}\begin{bmatrix} \kappa_1(t)+\kappa_5(t) & \kappa_4(t) - 2\kappa_6(t) \\ 0 & 2\kappa_4(t) + 2\kappa_6(t) \end{bmatrix}, \\
Y(t) &= \frac{1}{2\kappa_2(t)}\begin{bmatrix} \kappa_1(t)+\kappa_5(t) & -2\kappa_4(t) + \kappa_6(t) \\ 0 & 2\kappa_4(t) + 2\kappa_6(t) \end{bmatrix}.
\end{aligned}$$

Example 4.3. Let $0 < \alpha \leq 1$. Consider the following coupled matrix fractional differential equations:

$$\begin{aligned}
X^{(\alpha)}(t) &= AX(t)B + CY(t)D, \\
Y^{(\alpha)}(t) &= CX(t)D + AY(t)B,
\end{aligned} \tag{4.3}$$

under the initial contidions $X(0) = W_1$ and $Y(0) = W_2$, where

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

and

$$W_1 = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}, W_2 = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}.$$

First, note that $AC = CA$ and $BD = DB$. From Corollary 3.9, we have

$$\begin{aligned}
\operatorname{Vec} X(t) &= \frac{1}{2}E_\alpha(t^\alpha(B^T \otimes A))(E_\alpha(t^\alpha(D^T \otimes C)) + E_\alpha(-t^\alpha(D^T \otimes C)))\operatorname{Vec} W_1 \\
&+ \frac{1}{2}E_\alpha(t^\alpha(B^T \otimes A))(E_\alpha(t^\alpha(D^T \otimes C)) - E_\alpha(-t^\alpha(D^T \otimes C)))\operatorname{Vec} W_2, \\
\operatorname{Vec} Y(t) &= \frac{1}{2}E_\alpha(t^\alpha(B^T \otimes A))(E_\alpha(t^\alpha(D^T \otimes C)) - E_\alpha(-t^\alpha(D^T \otimes C)))\operatorname{Vec} W_1 \\
&+ \frac{1}{2}E_\alpha(t^\alpha(B^T \otimes A))(E_\alpha(t^\alpha(D^T \otimes C)) + E_\alpha(-t^\alpha(D^T \otimes C)))\operatorname{Vec} W_2.
\end{aligned}$$

To obtain the solution of the system, we compute $E_\alpha(B^T \otimes A)$. We have

$$B^T \otimes A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of $B^T \otimes A$ are $\lambda = 1, 2, -1, -2$.

Write $E_\alpha(X) = \sum_{k=0}^{3} r_k(t)X^k$. We have

$$\begin{bmatrix} E_\alpha(t^\alpha) \\ E_\alpha(2t^\alpha) \\ E_\alpha(-t^\alpha) \\ E_\alpha(-2t^\alpha) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & 4 & -8 \end{bmatrix} \begin{bmatrix} r_0(t) \\ r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix},$$

so,

$$\begin{aligned}
r_0(t) &= \frac{1}{12}(8E_\alpha(t^\alpha) - 2E_\alpha(2t^\alpha) + 8E_\alpha(-t^\alpha) - 2E_\alpha(-2t^\alpha)), \\
r_1(t) &= \frac{1}{12}(8E_\alpha(t^\alpha) - E_\alpha(2t^\alpha) - 8E_\alpha(-t^\alpha) + E_\alpha(-2t^\alpha)), \\
r_2(t) &= \frac{1}{12}(-2E_\alpha(t^\alpha) + 2E_\alpha(2t^\alpha) - 2E_\alpha(-t^\alpha) + 2E_\alpha(-2t^\alpha)), \\
r_3(t) &= \frac{1}{12}(-2E_\alpha(t^\alpha) + E_\alpha(2t^\alpha) + 2E_\alpha(-t^\alpha) - E_\alpha(-2t^\alpha)).
\end{aligned}$$

Thus,

$$\begin{aligned}
E_\alpha t^\alpha(B^T \otimes A) &= r_0(t)I + r_1(t)(B^T \otimes A) + r_2(t)(B^T \otimes A)^2 + r_3(t)(B^T \otimes A)^3 \\
&= r_0(t)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + r_1(t)\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix} + r_2(t)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & 4 \end{bmatrix} + r_3(t)\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 14 & 8 \\ 1 & 0 & 0 & 0 \\ 14 & 8 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} r_0(t) + r_2(t) & 0 & r_1(t) + r_3(t) & 0 \\ 6r_2(t) & r_0(t) + 4r_2(t) & 2r_1(t) + 14r_3(t) & 2r_1(t) + 8r_3(t) \\ r_1(t) + r_3(t) & 0 & r_0(t) + r_2(t) & 0 \\ 2r_1(t) + 14r_3(t) & 2r_1(t) + 6r_3(t) & 6r_2(t) & r_0(t) + 4r_2(t) \end{bmatrix}.
\end{aligned}$$

To compute $E_\alpha(D^T \otimes C)$, we have

$$D^T \otimes C = \begin{bmatrix} 2 & 0 & 2 & 0 \\ -2 & 1 & -2 & 1 \\ 2 & 0 & 2 & 0 \\ -2 & 1 & -2 & 1 \end{bmatrix}.$$

Consider

$$\begin{aligned}
E_\alpha(D^T \otimes C) &= E_\alpha\left(\begin{bmatrix} C & C \\ C & C \end{bmatrix}\right) \\
&= E_\alpha\left(\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}\right) E_\alpha\left(\begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}\right) \\
&= E_\alpha\left(\begin{bmatrix} E_\alpha(C) & 0 \\ 0 & E_\alpha(C) \end{bmatrix}\right) \frac{1}{2}E_\alpha\left(\begin{bmatrix} E_\alpha(C)+E_\alpha(C) & E_\alpha(C)-E_\alpha(C) \\ E_\alpha(C)-E_\alpha(C) & E_\alpha(C)+E_\alpha(C) \end{bmatrix}\right) \\
&= \frac{1}{2}\begin{bmatrix} (E_\alpha(C))^2 + E_\alpha(C)E_\alpha(-C) & (E_\alpha(C))^2 - E_\alpha(C)E_\alpha(-C) \\ (E_\alpha(C))^2 - E_\alpha(C)E_\alpha(-C) & (E_\alpha(C))^2 + E_\alpha(C)E_\alpha(-C) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} (E_\alpha(C))^2 + I & (E_\alpha(C))^2 - I \\ (E_\alpha(C))^2 - I & (E_\alpha(C))^2 + I \end{bmatrix}.
\end{aligned}$$

It follows that

$$\begin{aligned}
E_\alpha(C) &= E_\alpha\left(\begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} E_\alpha(2) & 0 \\ -2E_\alpha(2) + E_\alpha(1) & E_\alpha(1) \end{bmatrix},
\end{aligned}$$

and

$$(E_\alpha(C))^2 = \begin{bmatrix} (E_\alpha(2))^2 & 0 \\ -2(E_\alpha(2))^2 - E_\alpha(1)E_\alpha(2) + (E_\alpha(1))^2 & (E_\alpha(1))^2 \end{bmatrix}.$$

Thus,

$$E_\alpha((t)^\alpha(D^T \otimes C)) = \frac{1}{2}\begin{bmatrix} \omega_1(t) & 0 & \omega_2(t) & 0 \\ \omega_3(t) & \omega_1(t) & \omega_4(t) & \omega_2(t) \\ \omega_2(t) & 0 & \omega_1(t) & 0 \\ \omega_4(t) & \omega_2(t) & \omega_3(t) & \omega_1(t) \end{bmatrix}$$

where

$$\begin{aligned}
\omega_1(t) &= (E_\alpha(2t^\alpha))^2 + 1, \\
\omega_2(t) &= (E_\alpha(t^\alpha))^2 - 1, \\
\omega_3(t) &= -2(E_\alpha(2t^\alpha))^2 - E_\alpha(t^\alpha)E_\alpha(2) + (E_\alpha(t^\alpha))^2 + 1, \\
\omega_4(t) &= -2(E_\alpha(2t^\alpha))^2 - E_\alpha(t^\alpha)E_\alpha(2t^\alpha) + (E_\alpha(t^\alpha))^2 - 1.
\end{aligned}$$

Hence, by Corollary 3.9, we have

$$\begin{aligned}
\text{Vec}\, X(t) &= \frac{1}{2}\begin{bmatrix} r_0(t)+r_2(t) & 0 & r_1(t)+r_3(t) & 0 \\ 6r_2(t) & r_0(t)+4r_2(t) & 2r_1(t)+14r_3(t) & 2r_1(t)+8r_3(t) \\ r_1(t)+r_3(t) & 0 & r_0(t)+r_2(t) & 0 \\ 2r_1(t)+14r_3(t) & 2r_1(t)+6r_3(t) & 6r_2(t) & r_0(t)+4r_2(t) \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix} \\
&+ \frac{1}{4}\begin{bmatrix} \omega_1(t) & 0 & \omega_2(t) & 0 \\ \omega_3(t) & \omega_1(t) & \omega_4(t) & \omega_2(t) \\ \omega_2(t) & 0 & \omega_1(t) & 0 \\ \omega_4(t) & \omega_2(t) & \omega_3(t) & \omega_1(t) \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \\
&= \frac{1}{4}\begin{bmatrix} 2r_0(t)+2r_2(t)-\omega_2(t) \\ 6r_2(t)-4r_1(t)+36r_2(t)-16r_3(t)+\omega_1(t)+2\omega_2(t)-\omega_4(t) \\ 2r_1(t)+2r_3(t)-\omega_1(t) \\ -2r_0(t)+16r_1(t)-8r_2(t)+64r_3(t)+2\omega_1 v+\omega_2(t)-\omega_3(t) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\text{Vec}\, Y(t) &= \frac{1}{4}\begin{bmatrix} \omega_1(t) & 0 & \omega_2(t) & 0 \\ \omega_3(t) & \omega_1(t) & \omega_4(t) & \omega_2(t) \\ \omega_2(t) & 0 & \omega_1(t) & 0 \\ \omega_4(t) & \omega_2(t) & \omega_3(t) & \omega_1(t) \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix} \\
&+ \frac{1}{2}\begin{bmatrix} r_0(t)+r_2(t) & 0 & r_1(t)+r_3(t) & 0 \\ 6r_2(t) & r_0(t)+4r_2(t) & 2r_1(t)+14r_3(t) & 2r_1(t)+8r_3(t) \\ r_1(t)+r_3(t) & 0 & r_0+r_2(t) & 0 \\ 2r_1(t)+14r_3(t) & 2r_1(t)+6r_3(t) & 6r_2(t) & r_0(t)+4r_2(t) \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \\
&= \frac{1}{4}\begin{bmatrix} -2r_1(t)-2r_3(t)+\omega_1(t) \\ 2r_0(t)+4r_1(t)+40r_2(t)-28r_3(t)+3\omega_1(t)-\omega_2(t)+\omega_3(t) \\ -2r_0(t)-2r_2(t)+\omega_2(t) \\ 4r_2(t)+4r_1(t)+28r_2(t)+12r_3(t)-\omega_1(t)+3\omega_2(t)+\omega_4(t) \end{bmatrix}.
\end{aligned}$$

Therefore,

$$X(t) = \frac{1}{4}\begin{bmatrix} 2r_0(t)+2r_2(t)-\omega_2(t) & 6r_2(t)-4r_1(t)+36r_2(t)-16r_3(t)+\omega_1(t)+2\omega_2(t)-\omega_4(t) \\ 2r_1(t)+2r_3(t)-\omega_1(t) & -2r_0(t)+16r_1(t)-8r_2(t)+64r_3(t)+2\omega_1 v+\omega_2(t)-\omega_3(t) \end{bmatrix},$$

$$Y(t) = \frac{1}{4}\begin{bmatrix} -2r_1(t)-2r_3(t)+\omega_1(t) & 2r_0(t)+4r_1(t)+40r_2(t)-28r_3(t)+3\omega_1(t)-\omega_2(t)+\omega_3(t) \\ -2r_0(t)-2r_2(t)+\omega_2(t) & 4r_2(t)+4r_1(t)+28r_2(t)+12r_3(t)-\omega_1(t)+3\omega_2(t)+\omega_4(t) \end{bmatrix}.$$

Chapter 5

Conclusions and Suggestions

We solve a nonhomogeneous system of coupled linear matrix fractional dynamical differential equations. We consider the fractional derivative taken in Caputo's sense. We have an explicit form of the general solution to this obtained in terms of Kronecker product, the vector operator and Mittage-Leffler functions.

5.1 System of linear coupled matrix fractional differential equations and A general solution of the system

1. Nonhomogeneous system of coupled linear matrix fractional dynamical differential equations

$$\begin{aligned} X^{(\alpha)}(t) &= \sum_{i=1}^{p} A_i X(t) B_i + \sum_{i=1}^{q} C_i Y(t) D_i + U(t), \\ Y^{(\alpha)}(t) &= \sum_{i=1}^{r} E_i X(t) F_i + \sum_{i=1}^{s} G_i Y(t) H_i + V(t), \end{aligned}$$

Assumption

$$\begin{aligned} \sum_{i=1}^{p}\sum_{j=1}^{q} (D_j B_i)^T \otimes (A_i C_j) &= \sum_{i=1}^{q}\sum_{j=1}^{s} (H_j D_i)^T \otimes (C_i G_j), \\ \sum_{i=1}^{s}\sum_{j=1}^{r} (F_j H_i)^T \otimes (G_i E_j) &= \sum_{i=1}^{r}\sum_{j=1}^{p} (B_j F_i)^T \otimes (E_i A_j). \end{aligned}$$

A general solution of system

$$\begin{aligned} \operatorname{Vec} X(t) &= E_\alpha((t-t_0)^\alpha K) E_{2\alpha,1}((t-t_0)^{2\alpha} LM) \operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-t_0)^\alpha K)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} LM)) M \operatorname{Vec} W_2 \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha K) E_{2\alpha,1}((t-s)^{2\alpha} LM) \operatorname{Vec} U(s) ds \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha K)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha} LM)) M \operatorname{Vec} V(s) ds, \\ \operatorname{Vec} Y(t) &= E_\alpha((t-t_0)^\alpha N)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} ML)) L \operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-t_0)^\alpha N) E_{2\alpha,1}((t-t_0)^{2\alpha} ML) \operatorname{Vec} W_2 \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha N)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha} ML)) L \operatorname{Vec} U(s) ds \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha N) E_{2\alpha,1}((t-s)^{2\alpha} ML) \operatorname{Vec} V(s) ds, \end{aligned}$$

where

$$\begin{aligned} K &= \sum_{i=1}^{p} B_i^T \otimes A_i, \; N = \sum_{i=1}^{s} H_i^T \otimes G_i, \\ M &= \sum_{i=1}^{r} F_i^T \otimes E_i, \; L = \sum_{i=1}^{q} D_i^T \otimes C_i. \end{aligned}$$

2. Nonhomogeneous system of coupled linear matrix fractional dynamical differential equations

$$\begin{aligned} X^{(\alpha)}(t) &= AX(t)B + CY(t)D + U(t), \\ Y^{(\alpha)}(t) &= EX(t)F + GY(t)H + V(t), \end{aligned}$$

Assumption $(DB)^T \otimes (AC) = (HD)^T \otimes (CG)$ and $(FH)^T \otimes (GE) = (BF)^T \otimes (EA)$.
A general solution of system

$$\begin{aligned} \operatorname{Vec} X(t) &= E_\alpha((t-t_0)^\alpha R_1)E_{2\alpha,1}((t-t_0)^{2\alpha} R_2R_3) \operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-s)^\alpha R_1)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha} R_2R_3))R_3 \operatorname{Vec} W_2 \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha R_1)E_{2\alpha,1}((t-s)^{2\alpha} R_2R_3) \operatorname{Vec} U(s)ds \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-t_0)^\alpha R_1)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} R_2R_3))R_3 \operatorname{Vec} V(s)ds, \\ \operatorname{Vec} Y(t) &= E_\alpha((t-t_0)^\alpha R_4)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} R_3R_2))R_2 \operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-t_0)^\alpha R_4)E_{2\alpha,1}((t-t_0)^{2\alpha} R_3R_2) \operatorname{Vec} W_2 \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha R_4)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha} R_3R_2))R_2 \operatorname{Vec} U(s)ds \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha N)E_{2\alpha,1}((t-s)^{2\alpha} R_3R_2) \operatorname{Vec} V(s)ds. \end{aligned}$$

where

$$\begin{aligned} R_1 &= B^T \otimes A, \; R_2 = D^T \otimes C, \\ R_3 &= F^T \otimes E, \; R_4 = H^T \otimes G. \end{aligned}$$

3. Nonhomogeneous system of coupled linear matrix fractional dynamical differential equations

$$\begin{aligned} X^{(\alpha)}(t) &= \sum_{i=1}^{p} A_iX(t)B_i + \sum_{i=1}^{q} C_iY(t)D_i, \\ Y^{(\alpha)}(t) &= \sum_{i=1}^{r} E_iX(t)F_i + \sum_{i=1}^{s} G_iY(t)H_i, \end{aligned}$$

Assumption

$$\begin{aligned} \sum_{i=1}^{p}\sum_{j=1}^{q} (D_jB_i)^T \otimes (A_iC_j) &= \sum_{i=1}^{q}\sum_{j=1}^{s} (H_jD_i)^T \otimes (C_iG_j), \\ \sum_{i=1}^{s}\sum_{j=1}^{r} (F_jH_i)^T \otimes (G_iE_j) &= \sum_{i=1}^{r}\sum_{j=1}^{p} (B_jF_i)^T \otimes (E_iA_j). \end{aligned}$$

A general solution of system

$$\begin{aligned} \operatorname{Vec} X(t) &= E_\alpha((t-t_0)^\alpha K)E_{2\alpha,1}((t-t_0)^{2\alpha} LM) \operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-t_0)^\alpha K)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} LM))M \operatorname{Vec} W_2, \\ \operatorname{Vec} Y(t) &= E_\alpha((t-t_0)^\alpha N)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} ML))L \operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-t_0)^\alpha N)E_{2\alpha,1}((t-t_0)^{2\alpha} ML) \operatorname{Vec} W_2, \end{aligned}$$

where

$$K = \sum_{i=1}^{p} B_i^T \otimes A_i, \; N = \sum_{i=1}^{s} H_i^T \otimes G_i,$$
$$M = \sum_{i=1}^{r} F_i^T \otimes E_i, \; L = \sum_{i=1}^{q} D_i^T \otimes C_i.$$

4. Nonhomogeneous system of coupled linear matrix fractional dynamical differential equations

$$X'(t) = \sum_{i=1}^{p} A_i X(t) B_i + \sum_{i=1}^{q} C_i Y(t) D_i + U(t),$$
$$Y'(t) = \sum_{i=1}^{r} E_i X(t) F_i + \sum_{i=1}^{s} G_i Y(t) H_i + V(t),$$

Assumption

$$\sum_{i=1}^{p}\sum_{j=1}^{q} (D_j B_i)^T \otimes (A_i C_j) = \sum_{i=1}^{q}\sum_{j=1}^{s} (H_j D_i)^T \otimes (C_i G_j),$$
$$\sum_{i=1}^{s}\sum_{j=1}^{r} (F_j H_i)^T \otimes (G_i E_j) = \sum_{i=1}^{r}\sum_{j=1}^{p} (B_j F_i)^T \otimes (E_i A_j).$$

A general solution of system

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)K} E_{2,1}((t-t_0)^2 LM) \operatorname{Vec} W_1 \\
&+ e^{(t-t_0)K} (E_{2,2}((t-t_0)^2 LM)) M \operatorname{Vec} W_2 \\
&+ \int_{t_0}^{t} e^{(t-s)K} E_{2,1}((t-s)^2 LM) \operatorname{Vec} U(s) ds \\
&+ \int_{t_0}^{t} e^{(t-s)K} (E_{2,2}((t-s)^2 LM)) M \operatorname{Vec} V(s) ds, \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)N} (E_{2,2}((t-t_0)^2 ML)) L \operatorname{Vec} W_1 \\
&+ e^{(t-t_0)N} E_{2,1}((t-t_0)^2 ML) \operatorname{Vec} W_2 \\
&+ \int_{t_0}^{t} e^{(t-s)N} e^{(t-s)N} (E_{2,2}((t-s)^2 ML)) L \operatorname{Vec} U(s) ds \\
&+ \int_{t_0}^{t} e^{(t-s)N} E_{2,1}((t-s)^2 ML) \operatorname{Vec} V(s) ds,
\end{aligned} \tag{5.1}$$

where

$$K = \sum_{i=1}^{p} B_i^T \otimes A_i, \; N = \sum_{i=1}^{s} H_i^T \otimes G_i,$$
$$M = \sum_{i=1}^{r} F_i^T \otimes E_i, \; L = \sum_{i=1}^{q} D_i^T \otimes C_i.$$

5. Nonhomogeneous system of coupled linear matrix fractional dynamical differential equations

$$X'(t) = \sum_{i=1}^{p} A_i X(t) B_i + \sum_{i=1}^{q} C_i Y(t) D_i,$$
$$Y'(t) = \sum_{i=1}^{r} E_i X(t) F_i + \sum_{i=1}^{s} G_i Y(t) H_i,$$

Assumption

$$\sum_{i=1}^{p}\sum_{j=1}^{q}(D_jB_i)^T\otimes(A_iC_j) = \sum_{i=1}^{q}\sum_{j=1}^{s}(H_jD_i)^T\otimes(C_iG_j),$$
$$\sum_{i=1}^{s}\sum_{j=1}^{r}(F_jH_i)^T\otimes(G_iE_j) = \sum_{i=1}^{r}\sum_{j=1}^{p}(B_jF_i)^T\otimes(E_iA_j).$$

A general solution of system

$$\text{Vec}\,X(t) = e^{(t-t_0)K}E_{2,1}((t-t_0)^2LM)\,\text{Vec}\,W_1 + e^{(t-t_0)K}(E_{2,2}((t-t_0)^2LM))M\,\text{Vec}\,W_2,$$
$$\text{Vec}\,Y(t) = e^{(t-t_0)N}(E_{2,2}((t-t_0)^2ML))L\,\text{Vec}\,W_1 + e^{(t-t_0)N}E_{2,1}((t-t_0)^2ML)\,\text{Vec}\,W_2,$$

where

$$K = \sum_{i=1}^{p}B_i^T\otimes A_i,\ N = \sum_{i=1}^{s}H_i^T\otimes G_i,$$
$$M = \sum_{i=1}^{r}F_i^T\otimes E_i,\ L = \sum_{i=1}^{q}D_i^T\otimes C_i.$$

6. Nonhomogeneous system of coupled linear matrix fractional dynamical differential equations

$$X'(t) = AX(t)B + CY(t)D + U(t),$$
$$Y'(t) = EX(t)F + GY(t)H + V(t),$$

Assumption $DB = HD$,$AC = CG$ $FH = BF$ and $GE = EA$.
A general solution of system

$$\begin{aligned}\text{Vec}\,X(t) &= e^{(t-t_0)R_1}E_{2,1}((t-t_0)^2R_2R_3)\,\text{Vec}\,W_1\\ &+ e^{(t-t_0)R_1}(E_{2,2}((t-t_0)^2R_2R_3))R_2\,\text{Vec}\,W_2\\ &+ \int_{t_0}^{t}e^{(t-s)R_1}E_{2,1}((t-s)^2R_2R_3)\,\text{Vec}\,U(s)ds\\ &+ \int_{t_0}^{t}e^{(t-s)R_1}(E_{2,2}((t-s)^2R_2R_3))R_2\,\text{Vec}\,V(s)ds,\\ \text{Vec}\,Y(t) &= e^{(t-t_0)R_4}(E_{2,2}((t-t_0)^2R_3R_2))R_3\,\text{Vec}\,W_1\\ &+ e^{(t-t_0)R_4}E_{2,1}((t-t_0)^2R_3R_2)\,\text{Vec}\,W_2\\ &+ \int_{t_0}^{t}e^{(t-s)R_4}(E_{2,2}((t-s)^2R_3R_2))R_3\,\text{Vec}\,U(s)ds\\ &+ \int_{t_0}^{t}e^{(t-s)R_4}E_{2,1}((t-s)^2R_3R_2)\,\text{Vec}\,V(s)ds,\end{aligned}$$

where

$$R_1 = B^T\otimes A,\ R_2 = D^T\otimes C,$$
$$R_3 = F^T\otimes E,\ R_4 = H^T\otimes G.$$

7. Nonhomogeneous system of coupled linear matrix fractional dynamical differential equations

$$X^{\alpha}(t) = AX(t)B + CY(t)D,$$
$$Y^{\alpha}(t) = CX(t)D + AY(t)B,$$

Assumption $AC = CA$ and $BD = DB$.
A general solution of system

$$\begin{aligned}
\text{Vec}\, X(t) \; &= \frac{1}{2} E_\alpha((t-t_0)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) + E_\alpha((t-t_0)^\alpha R_2))\, \text{Vec}\, W_1 \\
&+ \frac{1}{2} E_\alpha((t-t_0)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) - E_\alpha(-(t-t_0)^\alpha R_2))\, \text{Vec}\, W_2, \\
\text{Vec}\, Y(t) \; &= \frac{1}{2} E_\alpha((t-t_0)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) - E_\alpha((t-t_0)^\alpha R_2))\, \text{Vec}\, W_1 \\
&+ \frac{1}{2} E_\alpha((t-t_0)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) + E_\alpha(-(t-t_0)^\alpha R_2))\, \text{Vec}\, W_2,
\end{aligned}$$

where

$$R_1 \;=\; B^T \otimes A, \;\; R_2 \;=\; D^T \otimes C.$$

5.2 Suggestions

In this thesis, we investigate initial value problems for the Systems of Nonhomogeneous linear matrix fractional dynamical differential equations in Caputo’s sense and provide examples numerical method of the main result. Whereas in fractional calculus have other methods to calculated such as Riemann–Liouville fractional derivative, Weyl’s derivative derivative and etc. Systems of Nonhomogeneous linear matrix fractional dynamical differential equations in Chapter 3 can be calculated by using all method had mentioned.

References

[1] Herrmann. Fractional Calculus: An Introduction for Physicists. Singapore: World Scientific: 2011.

[2] Caputo MR. Linear modles of dissipation whose Q is almost frequency independent–II. Geophysical Journal International. 1967:13(5):529–539.

[3] Miller KS, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. New York: Wiley-Interscience: 1993

[4] Podlubny I. Fractional differential equations. New York: Academic Press: 1999.

[5] Kilbas AA, Saigo M. Fractional integrals and derivatives of Mittag-Leffler type function. Doklady Akademii Nauk Belarusi. 1995;39(4):22–26.

[6] Kilbas AA, Srivastava HM, Tryjillo JJ. Theory and Applications of Fractional differential Equations. Amsterdam: 2006.

[7] Ben Taher R, Rachidi M. Linear recurrence relations in the algebra of matrices and applications. Linear Algebra Appl. 2001;330(1-3):15–24.

[8] Ben Taher R, Rachidi M. Some explicit formulas for the polynomial decomposition of the matrix exponential and applications. Linear Algebra Appl. 2002;350(1-3):171–184

[9] Chang HW, Yau SST. On more explicit formulas for matrix exponential. Linear Algebra Appl. 1997;262:102–110

[10] Leonard IE. The Matrix Exponential. SIAM Review. 1996;38(3):507–512.

[11] Al-Zhour ZAA. The general (vectors) solutions of such linear (coupled) matrix fractional differential equations by using Kronecker structures. Applied Mathematics and Computer. 2014;232:498–510.

[12] Kongyaksee R, Chansangiam P. Solving System of Nonhomogeneous Coupled Linear Matrix Differential Equations in Terms of Mittag-Leffler Matrix Functions. Journal of Computation Analysis and Applications. 2019;26(1):1150–1160.

[13] Balanchan K, Kokila J. On the controllability dynamic systems. Int J Appl Math Comput Sci. 2012;22(3):523–531.

[14] Balanchan K, Kokila J, Trujillo JJ. Relative controllability of fractional dynamical system with moltiple delays in control. Comput Math Appl. 2012;64:3037–3045.

[15] Killicman A, Al-Zhour ZAA. The general common exact solutions of coupled linear matrix and matrix differential equations. The 2nd International Conference on Research and Education in Mathematics. 2005;p.75–78.

[16] Horn RA, Johnson CR. Topics in Matrix Anaiysis. New York: Cambridge University Press; 1991.

[17] Steeb HW, Hardy Y. Matrix Calculus and Kronecker Product. A Practical Approach to Linear and Multilinear Algebra. Singapore: World Scientific: 2011.

[18] Gradshteyn IS, Ryzhik IM. Tables of Integrals, Series, and Products, 6th ed. San Diego, CA: Academic Press, pp. 1115-1116, 2000.

Appendix

Appendix A

The research paper

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Nonhomogeneous system of coupled linear matrix fractional dynamical differential equations in Caputo's sense with control delays

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Abstract: In this paper, we investigate a nonhomogeneous system of coupled linear matrix fractional dynamical differential equations with delays in control. The fractional derivative considered here is taken in Caputo's sense. We obtain an explicit form of its general solution in terms of the Kronecker product, the vector operator, and matrix series concerning Mittag-Leffler functions.

Keywords: Linear matrix fractional differential equation; Caputo's derivative; Kronecker product; vector operator; Mittage-Leffler functions

AMS Math Classification (2010) : 15A16, 15A69, 26A33, 33E12, 34A50

1 Introduction

Fractional calculus was introduced more than 300 years ago. It is a branch of mathematical analysis that studies the several different possibilities of defining real of complex number powers of the differentiation and integration operators and developing a calculus for such operators generalizing the classical one. It is applied into many branches of mathematical science and engineering. It is dominating by modern examples of applications in differential and integral equations. The fractional derivative of the exponential obtained by Liouville in 1832, and the fractional of power function got by Riemann [1] in 1847. The Riemann-Liouville fractional derivative was failed in the description and modeling of some complex phenomena. Thus, Caputo's fractional derivative was introduced in 1967 by Caputo [2].

Linear matrix (fractional) differential equations are important in various fields which including applied science, engineering, economics. For a detail survey with collections of applications in various fields, see Miller and Ross [3], Podlubny [4], Kilbas and Saigo [5], and Kilbas et al. [6]. The simplest form of linear matrix differential equations is shown below:

$$X(t) \;=\; AX(t). \tag{1}$$

Here, $A \in M_n$ and $X(t)$ is an unknown matrix-valued function to be solved. The solution of (1) is given by

$$X(t) \;=\; e^{A(t-t_0)}X(t_0), \tag{2}$$

see more detail in Ben and Rachidi [7],[8], Cheng and Yau [9] and Leonard [10]. A general system of nonhomogeneous linear matrix differential equations takes the form

$$X'(t) \;=\; AX(t) + U(t). \tag{3}$$

here, $U(t)$ is a given matrix-valued function. The solution of (3) is given by

$$X(t) \;=\; e^{(t-t_0)A}X(t_0) + e^{tA} * U(t), \tag{4}$$

where $*$ denotes the matrix convolution product. A general system of nonhomogeneous coupled linear matrix ordinary differential equations takes in the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t). \end{aligned} \tag{5}$$

Here, A,B,C,D,E,F,G,H are given matrix-valued functions, and $X(t), Y(t)$ are unknown matrix-valued functions. The solution is given in terms of Kronecker products, the vector operator and matrix series concerning exponential and hyperbolic functions. A nonhomogeneous case of (5) was discussed in Kilman and Al-Zhour [11] when $E = C$, $F = D$, $G = A$, $H = B$ and $U(t) = V(t) = 0$. The system (5) was investigate in Kongyaksee and Chansangiam [12] under the assumption that $AC = CA$ and $BD = DB$.

A simple system of homogeneous linear matrix fractional dynamical differential equations takes the form

$$X^{(\alpha)}(t) = AX(t). \tag{6}$$

The solution of (6) is given by (see Balanchan and Kokila [13],[14])

$$X(t) = E_\alpha(A(t-t_0)^\alpha)C, \tag{7}$$

where E_α is Mittag-Leffler functions with parameter $\alpha > 0$. The simplest form of nonhomogeneous linear matrix fractional dynamical differential equations with delays in control is shown below:

$$X^{(\alpha)}(t) = AX(t) + U(t). \tag{8}$$

The solution of (8) is as follows (see Balanchan et al. [14])

$$X(t) = E_\alpha(At^\alpha + \int_0^t (t-s)^{\alpha-1} E_\alpha(A(t-s)^\alpha)u(s)ds. \tag{9}$$

A general system of nonhomogeneous coupled linear matrix differential equations takes the form

$$\begin{aligned} X^{(\alpha)}(t) &= AX(t)B + CY(t)D + U(t), \\ Y^{(\alpha)}(t) &= EX(t)F + GY(t)H + V(t), \end{aligned} \tag{10}$$

The system was investigated in [11] under the assumption that $AC = CA$ and $BD = DB$.

In the present work, we consider a generalization of the system (10), namely,

$$\begin{aligned} X^{(\alpha)}(t) &= \sum_{i=0}^{p} A_i X(t) B_i + \sum_{i=0}^{q} C_i Y(t) D_i + U(t), \\ Y^{(\alpha)}(t) &= \sum_{i=0}^{r} E_i X(t) F_i + \sum_{i=0}^{p} G_i Y(t) H_i + V(t). \end{aligned}$$

where $0 < \alpha \leq 1$ and all derivatives are in Caputo's sense. To obtain an explicit formula of the solution, we impose an assumption on the coefficient matrices. Our result includes the results in Al-Zhour [11], Kongyaksee and Chansangiam [12] and Killiman and Al-Zhour [15]. We also provide and an illustrative example of the main result.

2 Preliminaries

In this section, we provide adequate tools for solving system of linear matrix differential equations. We shall denote the set of all m-by-n complex matrices by $M_{m,n}$ and we set $M_n = M_{n,n}$. We shall denote the set of all m-by-n real matrices by $M_{m,n}(\mathbb{R})$ and $M_n(\mathbb{R}) = M_{n,n}(\mathbb{R})$.

2.1 Kronecker Product

Definition 1. *Let $A = (a_{ij}) \in M_m$ and $B \in M_n$. The Kronecker product of A and B is defined by*

$$A \otimes B = (a_{ij}B) \in M_{mn}.$$

Lemma 2 (see e.g. Horn and Johnson [16]). *The following properties hold for matrices of appropriate sizes:*

1. $I_m \otimes I_n = I_{mn}$,
2. $(kA) \otimes B = k(A \otimes B) = A \otimes (kB)$, *for all* $k \in \mathbb{C}$,
3. $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$,
4. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$,
5. $(A \otimes B)^T = A^T \otimes B^T$,
6. $(A \otimes B)(C \otimes D) = AC \otimes BD$.

2.2 Vector operator

For each $A = (a_{ij}) \in M_m$, we define

$$\operatorname{Vec} A = [a_{11} \dots a_{m1} \dots a_{12} \dots a_{m2} \dots a_{1m} \dots a_{mm}]^T.$$

Lemma 3 (see e.g. Horn and Johnson [16]). *The following properties hold for matrices of appropriate sizes:*

1. $\operatorname{Vec}(kA) = k \operatorname{Vec} A$,
2. $\operatorname{Vec}(A + B) = \operatorname{Vec} A + \operatorname{Vec} B$.

Theorem 4 (see e.g. Horn and Johnson [16]). *Consider matrices $A \in M_{m,n}$, $B \in M_{p,q}$ and $X \in M_{n,p}$. The Kronecker product and the vector operator are related by*

$$\operatorname{Vec}(AXB) = (B^T \otimes A) \operatorname{Vec} X.$$

2.3 Mittage-Leffler function

The Mittag-Leffler function $E_{\alpha,\beta}$ is a special function depending on two parameters $\alpha, \beta > 0$. It may be defined by the following series when the real part of α is strictly positive:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

When the Gamma function Γ is defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \operatorname{Re}(z) > 0.$$

Definition 5. *The Mittage-Leffler function with parameters $\alpha > 0$ and $\beta > 0$ of $A \in M_n$ is defined by*

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} A^k = I_n + \frac{1}{\Gamma(\alpha + \beta)} A + \frac{1}{\Gamma(2\alpha + \beta)} A^2 + \cdots$$

where $\beta = 1$, we set $E_\alpha := E_{\alpha,1}$.

Lemma 6 (see e.g Killicman and Al-Zhour [15]). *The following properties hold for matrices of appropriate sizes:*

1. $E_\alpha(A)$ *is always invertible,*
2. $(E_\alpha(a))^{-1} = E_\alpha(-A)$ *and* $(E_\alpha(a))^T = E_\alpha(A^T)$,
3. *If* $AB = BA$, *then* $E_\alpha(A + B) = E_\alpha(A) E_\alpha(B)$,
4. $E_\alpha(A \otimes I_n) = E_\alpha(A) \otimes I_n$ *and* $E_\alpha(I_n \otimes A) = I_n \otimes E_\alpha(A)$.

2.4 Caputo's fractional derivative

Definition 7. *Let f be a piecewise continuous function on $(0,\infty)$ which is integrable on any finite subinterval of $[0,\infty)$. Let $\alpha > 0$ and let $n \in \mathbb{N}$ be such that $n-1 < \alpha \leq n$. The Caputo's derivative of f of order α is defined by*

$$D^{\alpha} f(t) = \begin{cases} D^n f(t), & \alpha = n, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \frac{D^n f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & n-1 < \alpha \leq n. \end{cases}$$

Here, D is the usual differential operator.

For a matrix-valued function $X(t) = [x_{ij}(t)]$ and $\alpha > 0$, we defined $X^{(\alpha)}(t) = [D^{\alpha} x_{ij}(t)]$, provided that $D^{\alpha} x_{ij}(t)$ exists for each i, j. In particular, $X'(t) = [x'_{ij}(t)]$.

3 System of linear coupled matrix fractional differential equations

In this section, we investigate a system of coupled nonhomogeneous linear matrix fractional differential equations. From now on, let $A, B, C, D, E, F, G, H \in M_n(\mathbb{R})$ be given constant matrices and let $U(t), V(t) \in M_n(\mathbb{R})$ be given matrix-valued function. We start with two auxiliary lemmas.

Lemma 8. *For any $A, B \in M_n(\mathbb{C})$, we have*

$$E_{\alpha}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \begin{bmatrix} E_{\alpha}(A) & 0 \\ 0 & E_{\alpha}(B) \end{bmatrix}.$$

Proof. Using standard techniques in matrix analysis, we obtain

$$\begin{aligned} E_{\alpha}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^k \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} A^k & 0 \\ 0 & B^k \end{bmatrix} \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} \dfrac{A^k}{\Gamma(\alpha k+1)} & 0 \\ 0 & \dfrac{B^k}{\Gamma(\alpha k+1)} \end{bmatrix} \\ &= \begin{bmatrix} \displaystyle\sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k+1)} & 0 \\ 0 & \displaystyle\sum_{k=0}^{\infty} \frac{B^k}{\Gamma(\alpha k+1)} \end{bmatrix} \\ &= \begin{bmatrix} E_{\alpha}(A) & 0 \\ 0 & E_{\alpha}(B) \end{bmatrix}. \end{aligned}$$

□

Lemma 9. *For any $A, B \in M_n(\mathbb{C})$*

$$E_{\alpha}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \begin{bmatrix} E_{2\alpha,1}(AB) & \big(E_{2\alpha,\alpha+1}(AB)\big)A \\ \big(E_{2\alpha,\alpha+1}(BA)\big)B & E_{2\alpha,1}(BA) \end{bmatrix},$$

$$E_{\alpha}\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} E_{\alpha}(A)+E_{\alpha}(-A) & E_{\alpha}(A)-E_{\alpha}(-A) \\ E_{\alpha}(A)-E_{\alpha}(-A) & E_{\alpha}(A)+E_{\alpha}(-A) \end{bmatrix}.$$

Proof. A direct computation reveals that

$$
\begin{aligned}
E_\alpha\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^k \\
&= \sum_{even} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} (AB)^k & 0 \\ 0 & (BA)^k \end{bmatrix} + \sum_{odd} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} 0 & (AB)^k A \\ (BA)^k B & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k)+1)} (AB)^k & \sum\limits_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+1)+1)} (AB)^k A \\ \sum\limits_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+1)+1)} (BA)^k B & \sum\limits_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k)+1)} (BA)^k \end{bmatrix} \\
&= \begin{bmatrix} E_{2\alpha,1}(AB) & \big(E_{2\alpha,\alpha+1}(AB)\big)A \\ \big(E_{2\alpha,\alpha+1}(BA)\big)B & E_{2\alpha,1}(BA) \end{bmatrix}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
E_\alpha\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) &= \sum_{even} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} A^k & 0 \\ 0 & A^k \end{bmatrix} + \sum_{odd} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} 0 & A^k \\ A^k & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{even} \frac{1}{(\alpha k+1)} A^k & \sum\limits_{odd} \frac{1}{(\alpha k+1)} A^k \\ \sum\limits_{even} \frac{1}{(\alpha k+1)} A^k & \sum\limits_{odd} \frac{1}{(\alpha k+1)} A^k \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{k=0}^{\infty} \left(\frac{1+(-1)^k}{2\Gamma(2k+1)}\right) A^k & \sum\limits_{k=0}^{\infty} \left(\frac{1-(-1)^k}{2\Gamma(2k+1)}\right) A^k \\ \sum\limits_{k=0}^{\infty} \left(\frac{1-(-1)^k}{2\Gamma(2k+1)}\right) A^k & \sum\limits_{k=0}^{\infty} \left(\frac{1+(-1)^k}{2\Gamma(2k+1)}\right) A^k \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} E_\alpha(A)+E_\alpha(-A) & E_\alpha(A)-E_\alpha(-A) \\ E_\alpha(A)-E_\alpha(-A) & E_\alpha(A)+E_\alpha(-A) \end{bmatrix}.
\end{aligned}
$$

□

Now, we are in position to prove the main result of this paper.

Theorem 10. *Let* $0 < \alpha \leq 1$. *Assume that*

$$\sum_{i=1}^{p} \sum_{j=1}^{q} (D_j B_i) \otimes (A_i C_j)^T \;=\; \sum_{i=1}^{q} \sum_{j=1}^{s} (H_j D_i)^T \otimes (C_i G_j)\,, \tag{11}$$

$$\sum_{i=1}^{s} \sum_{j=1}^{r} (F_j H_i)^T \otimes (G_i E_j) \;=\; \sum_{i=1}^{r} \sum_{j=1}^{p} (B_j F_i)^T \otimes (E_i A_j)\,. \tag{12}$$

Then the general solution of the system of nonhomogeneous coupled linear matrix fractional differential equations with delays in control of order α *:*

$$
\begin{aligned}
X^{(\alpha)}(t) &= \sum_{i=1}^{p} A_i X(t) B_i + \sum_{i=1}^{q} C_i Y(t) D_i + U(t), \\
Y^{(\alpha)}(t) &= \sum_{i=1}^{r} E_i X(t) F_i + \sum_{i=1}^{s} G_i Y(t) H_i + V(t),
\end{aligned} \tag{13}
$$

subject to the control delays $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$\begin{aligned}
\operatorname{Vec} X(t) \;&= E_\alpha((t-t_0)^\alpha K) E_{2\alpha,1}((t-t_0)^{2\alpha} LM) \operatorname{Vec} W_1 \\
&+ E_\alpha((t-t_0)^\alpha K)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} LM)) M \operatorname{Vec} W_2 \\
&+ \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha K) E_{2\alpha,1}((t-s)^{2\alpha} LM) \operatorname{Vec} U(s) ds \\
&+ \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha K)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha} LM)) M \operatorname{Vec} V(s) ds, \\
\operatorname{Vec} Y(t) \;&= E_\alpha((t-t_0)^\alpha N)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} ML)) L \operatorname{Vec} W_1 \\
&+ E_\alpha((t-t_0)^\alpha N) E_{2\alpha,1}((t-t_0)^{2\alpha} ML) \operatorname{Vec} W_2 \\
&+ \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha N)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha} ML)) L \operatorname{Vec} U(s) ds \\
&+ \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha N) E_{2\alpha,1}((t-s)^{2\alpha} ML) \operatorname{Vec} V(s) ds,
\end{aligned} \tag{14}$$

where,

$$\begin{aligned}
K \;&=\; \sum_{i=1}^{p} B_i^T \otimes A_i, \; N \;=\; \sum_{i=1}^{s} H_i^T \otimes G_i, \\
M \;&=\; \sum_{i=1}^{r} F_i^T \otimes E_i, \; L \;=\; \sum_{i=1}^{q} D_i^T \otimes C_i.
\end{aligned} \tag{15}$$

Proof. Using Lemmas 2, 3 and 6, we have

$$\begin{aligned}
\operatorname{Vec} X^{(\alpha)}(t) \;&=\; K \operatorname{Vec} X(t) + L \operatorname{Vec} Y(t) + \operatorname{Vec} U(t), \\
\operatorname{Vec} X^{(\alpha)}(t) \;&=\; M \operatorname{Vec} X(t) + N \operatorname{Vec} Y(t) + \operatorname{Vec} V(t).
\end{aligned}$$

Thus, the system (13) is transformed to the following equivalent system:

$$\begin{bmatrix} \operatorname{Vec} X^{(\alpha)}(t) \\ \operatorname{Vec} Y^{(\alpha)}(t) \end{bmatrix} \;=\; \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix} + \begin{bmatrix} \operatorname{Vec} U(t) \\ \operatorname{Vec} V(t) \end{bmatrix}.$$

Let us denote $S = P + Q$ where

$$P \;=\; \begin{bmatrix} K & 0 \\ 0 & N \end{bmatrix} \text{ and } Q \;=\; \begin{bmatrix} 0 & L \\ M & 0 \end{bmatrix}.$$

This system has the following solution:

$$\begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix} \;=\; E_\alpha((t-t_0)^\alpha S) \begin{bmatrix} \operatorname{Vec} X(t_0) \\ \operatorname{Vec} Y(t_0) \end{bmatrix} + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha S) \begin{bmatrix} \operatorname{Vec} U(s) \\ \operatorname{Vec} V(s) \end{bmatrix} ds.$$

Now, we compute $E_\alpha(S)$. We have

$$PQ \;=\; \begin{bmatrix} 0 & KL \\ NM & 0 \end{bmatrix} \text{ and } QP \;=\; \begin{bmatrix} 0 & LN \\ MK & 0 \end{bmatrix}.$$

Using the hypothesis (11) and (12) together with Lemmas 2, 3 and 6 we can deduce that $KL = LN$ and $NM = MK$. Thus, $PQ = QP$. From which it follows from Lemma 6 that

$$E_\alpha(S) \;=\; E_\alpha(P+Q) \;=\; E_\alpha(P) E_\alpha(Q).$$

By Lemma 8, we have

$$E_\alpha(P) \;=\; \begin{bmatrix} E_\alpha(K) & 0 \\ 0 & E_\alpha(N) \end{bmatrix}.$$

By Lemma 9, we have

$$E_\alpha(Q) \;=\; \begin{bmatrix} E_{2\alpha,1}(LM) & (E_{2\alpha,\alpha+1}(LM))M \\ (E_{2\alpha,\alpha+1}(ML))L & E_{2\alpha,1}(ML) \end{bmatrix}.$$

Hence,

$$\begin{aligned} E_\alpha(S) \;&=\; \begin{bmatrix} E_\alpha(K) & 0 \\ 0 & E_\alpha(N) \end{bmatrix} \begin{bmatrix} E_{2\alpha,1}(LM) & (E_{2\alpha,\alpha+1}(LM))M \\ (E_{2\alpha,\alpha+1}(ML))L & E_{2\alpha,1}(ML) \end{bmatrix} \\ &=\; \begin{bmatrix} E_\alpha(K)E_{2\alpha,1}(LM) & E_\alpha(K)(E_{2\alpha,\alpha+1}(LM))M \\ E_\alpha(N)(E_{2\alpha,\alpha+1}(ML))L & E_\alpha(N)E_{2\alpha,1}(ML) \end{bmatrix}. \end{aligned}$$

Therefore, the general solution of (13) is given by (14). □

In Theorem 10, the hypotheses (11) and (12) is not too restrictive since it includes many interesting special cases.

Corollary 11. *Let $0 < \alpha \leq 1$. Assume that $(DB)^T \otimes (AC) = (HD)^T \otimes (CG)$ and $(FH)^T \otimes (GE) = (BF)^T \otimes (EA)$. Then the general solution of the system:*

$$\begin{aligned} X^{(\alpha)}(t) \;&=\; AX(t)B + CY(t)D + U(t), \\ Y^{(\alpha)}(t) \;&=\; EX(t)F + GY(t)H + V(t), \end{aligned} \tag{16}$$

subject to $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$\begin{aligned} \operatorname{Vec} X(t) \;&= E_\alpha((t-t_0)^\alpha R_1)E_{2\alpha,1}((t-t_0)^{2\alpha} R_2R_3) \operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-s)^\alpha R_1)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha} R_2R_3))R_3 \operatorname{Vec} W_2 \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha R_1)E_{2\alpha,1}((t-s)^{2\alpha} R_2R_3) \operatorname{Vec} U(s)ds \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-t_0)^\alpha R_1)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} R_2R_3))R_3 \operatorname{Vec} V(s)ds, \\ \operatorname{Vec} Y(t) \;&= E_\alpha((t-t_0)^\alpha R_4)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} R_3R_2))R_2 \operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-t_0)^\alpha R_4)E_{2\alpha,1}((t-t_0)^{2\alpha} R_3R_2) \operatorname{Vec} W_2 \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha R_4)(E_{2\alpha,\alpha+1}((t-s)^{2\alpha} R_3R_2))R_2 \operatorname{Vec} U(s)ds \\ &\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} E_\alpha((t-s)^\alpha N)E_{2\alpha,1}((t-s)^{2\alpha} R_3R_2) \operatorname{Vec} V(s)ds. \end{aligned} \tag{17}$$

where,

$$\begin{aligned} R_1 \;&=\; B^T \otimes A,\; R_2 \;=\; D^T \otimes C, \\ R_3 \;&=\; F^T \otimes E,\; R_4 \;=\; H^T \otimes G. \end{aligned} \tag{18}$$

Proof. Put $p = q = r = s = 1$ in Theorem 10. □

Corollary 12. *Let $0 < \alpha \leq 1$. Denote K, M, N, L as in (15). Assume that (11) and (12) hold. Then the general solution of the system:*

$$\begin{aligned} X^{(\alpha)}(t) \;&=\; \sum_{i=1}^{p} A_iX(t)B_i + \sum_{i=1}^{q} C_iY(t)D_i, \\ Y^{(\alpha)}(t) \;&=\; \sum_{i=1}^{r} E_iX(t)F_i + \sum_{i=1}^{s} G_iY(t)H_i, \end{aligned} \tag{19}$$

subject to $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$\begin{aligned} \operatorname{Vec} X(t) \;&= E_\alpha((t-t_0)^\alpha K)E_{2\alpha,1}((t-t_0)^{2\alpha} LM) \operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-t_0)^\alpha K)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} LM))M \operatorname{Vec} W_2, \\ \operatorname{Vec} Y(t) \;&= E_\alpha((t-t_0)^\alpha N)(E_{2\alpha,\alpha+1}((t-t_0)^{2\alpha} ML))L \operatorname{Vec} W_1 \\ &\quad + E_\alpha((t-t_0)^\alpha N)E_{2\alpha,1}((t-t_0)^{2\alpha} ML) \operatorname{Vec} W_2. \end{aligned} \tag{20}$$

Proof. Put $U(t) = V(t) = 0$ in Theorem 10. □

Corollary 13. *Denote K, M, N, L as in (15). Assume that (11) and (12) hold. Then the general solution of the system:*

$$\begin{aligned} X'(t) &= \sum_{i=1}^{p} A_i X(t) B_i + \sum_{i=1}^{q} C_i Y(t) D_i + U(t), \\ Y'(t) &= \sum_{i=1}^{r} E_i X(t) F_i + \sum_{i=1}^{s} G_i Y(t) H_i + v(t), \end{aligned} \tag{21}$$

subject to $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)K} E_{2,1}((t-t_0)^2 LM) \operatorname{Vec} W_1 \\ &+ e^{(t-t_0)K} (E_{2,2}((t-t_0)^2 LM)) M \operatorname{Vec} W_2 \\ &+ \int_{t_0}^{t} e^{(t-s)K} E_{2,1}((t-s)^2 LM) \operatorname{Vec} U(s) ds \\ &+ \int_{t_0}^{t} e^{(t-s)K} (E_{2,2}((t-s)^2 LM)) M \operatorname{Vec} V(s) ds, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)N} (E_{2,2}((t-t_0)^2 ML)) L \operatorname{Vec} W_1 \\ &+ e^{(t-s)N} E_{2,1}((t-s)^2 ML) \operatorname{Vec} W_2 \\ &+ \int_{t_0}^{t} e^{(t-s)N} e^{(t-s)N} (E_{2,2}((t-s)^2 ML)) L \operatorname{Vec} U(s) ds \\ &+ \int_{t_0}^{t} e^{(t-s)N} E_{2,1}((t-s)^2 ML) \operatorname{Vec} V(s) ds, \end{aligned} \tag{22}$$

Proof. Put $\alpha = 1$ in Theorem 10. □

Corollary 14. *Denote K, M, N, L as in (15). Assume that (11) and (12) hold. Then the general solution of the system:*

$$\begin{aligned} X'(t) &= \sum_{i=1}^{p} A_i X(t) B_i + \sum_{i=1}^{q} C_i Y(t) D_i, \\ Y'(t) &= \sum_{i=1}^{r} E_i X(t) F_i + \sum_{i=1}^{s} G_i Y(t) H_i, \end{aligned} \tag{23}$$

subject to $X(t_0) = W_1$ and $Y(t_0) = W_2$ is given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)K} E_{2,1}((t-t_0)^2 LM) \operatorname{Vec} W_1 + e^{(t-t_0)K} (E_{2,2}((t-t_0)^2 LM)) M \operatorname{Vec} W_2, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)N} (E_{2,2}((t-t_0)^2 ML)) L \operatorname{Vec} W_1 + e^{(t-t_0)N} E_{2,1}((t-t_0)^2 ML) \operatorname{Vec} W_2, \end{aligned} \tag{24}$$

Proof. Put $U(t) = V(t) = 0$ in Corollary 13. □

The next result was firstly obtained in Kongyaksee and Chansangiam [12]

Corollary 15. *Let $0 < \alpha \leq 1$. Denote R_1, R_2, R_3, R_4 as in (18). Assume that $DB = HD, AC = CG$ and $FH = BF, GE = EA$. The general solution of the system:*

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + v(t), \end{aligned} \tag{25}$$

subject to $X(t_0) = W_1$ *and* $Y(t_0) = W_2$ *is given by*

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)R_1} E_{2,1}((t-t_0)^2 R_2 R_3) \operatorname{Vec} W_1 \\
&\quad + e^{(t-t_0)R_1}(E_{2,2}((t-t_0)^2 R_2 R_3)) R_2 \operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t} e^{(t-s)R_1} E_{2,1}((t-s)^2 R_2 R_3) \operatorname{Vec} U(s) ds \\
&\quad + \int_{t_0}^{t} e^{(t-s)R_1}(E_{2,2}((t-s)^2 R_2 R_3)) R_2 \operatorname{Vec} V(s) ds, \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)R_4}(E_{2,2}((t-t_0)^2 R_3 R_2)) R_3 \operatorname{Vec} W_1 \\
&\quad + e^{(t-t_0)R_4} E_{2,1}((t-t_0)^2 R_3 R_2) \operatorname{Vec} W_2 \\
&\quad + \int_{t_0}^{t} e^{(t-s)R_4}(E_{2,2}((t-s)^2 R_3 R_2)) R_3 \operatorname{Vec} U(s) ds \\
&\quad + \int_{t_0}^{t} e^{(t-s)R_4} E_{2,1}((t-s)^2 R_3 R_2) \operatorname{Vec} V(s) ds,
\end{aligned} \tag{26}$$

Proof. Put $\alpha = 1$ in Corollary 11. □

The next result was firstly obtained in Al-Zhour [11]

Corollary 16. *Let* $0 < \alpha \leq 1$. *Denote* R_1, R_2 *as in (18). Assume that* $AC = CA, BD = DB$. *The general solution of the system:*

$$\begin{aligned}
X^{(\alpha)}(t) &= AX(t)B + CY(t)D, \\
Y^{(\alpha)}(t) &= CX(t)D + AY(t)B,
\end{aligned} \tag{27}$$

subject to $X(t_0) = W_1$ *and* $Y(t_0) = W_2$ *is given by*

$$\begin{aligned}
\operatorname{Vec} X(t) &= \frac{1}{2} E_\alpha((t-t_0)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) + E_\alpha((t-t_0)^\alpha R_2)) \operatorname{Vec} W_1 \\
&\quad + \frac{1}{2} E_\alpha((t-s)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) - E_\alpha(-(t-t_0)^\alpha R_2)) \operatorname{Vec} W_2, \\
\operatorname{Vec} Y(t) &= \frac{1}{2} E_\alpha((t-t_0)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) - E_\alpha((t-t_0)^\alpha R_2)) \operatorname{Vec} W_1 \\
&\quad + \frac{1}{2} E_\alpha((t-t_0)^\alpha R_1)(E_\alpha((t-t_0)^\alpha R_2) + E_\alpha(-(t-t_0)^\alpha R_2)) \operatorname{Vec} W_2.
\end{aligned} \tag{28}$$

Proof. From Corollary 11, put $U(t) = V(t) = 0$ and $E = C, F = D, G = A, H = B$. □

4 Example

In this section, we provide an example in order to illustrate our main result in Section 3. The next lemma is used to compute certain Mittage-Leffler function.

Lemma 17. *For any* $a, c, d \in \mathbb{R}$ *with* $a \neq d$, *we have*

$$E_\alpha\left(\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}\right) = \begin{bmatrix} E_\alpha(a) & 0 \\ \frac{c}{a-d}(E_\alpha(a) - E_\alpha(d)) & E_\alpha(d) \end{bmatrix}. \tag{29}$$

Proof. By expanding, we have

$$\begin{aligned}
E_\alpha\left(\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}\right) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}^k \\
&= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \begin{bmatrix} a^k & 0 \\ c\left(\dfrac{a^k - d^k}{a-d}\right) & d^k \end{bmatrix} \\
&= \sum_{k=0}^{\infty} \begin{bmatrix} \dfrac{a^k}{\Gamma(\alpha k+1)} & 0 \\ \dfrac{c}{a-d}\left(\dfrac{a^k - d^k}{\Gamma(\alpha k+1)}\right) & \dfrac{d^k}{\Gamma(\alpha k+1)} \end{bmatrix} \\
&= \begin{bmatrix} \displaystyle\sum_{k=0}^{\infty} \frac{a^k}{\Gamma(\alpha k+1)} & 0 \\ \dfrac{c}{a-d}\left(\displaystyle\sum_{k=0}^{\infty} \frac{a^k - d^k}{\Gamma(\alpha k+1)}\right) & \displaystyle\sum_{k=0}^{\infty} \frac{B^k}{\Gamma(\alpha k+1)} \end{bmatrix} \\
&= \begin{bmatrix} E_\alpha(a) & 0 \\ \dfrac{c}{a-d}(E_\alpha(a) - E_\alpha(d)) & E_\alpha(d) \end{bmatrix}.
\end{aligned}$$

□

Example 18. *Let $0 < \alpha \leq 1$. Consider the following coupled matrix fractional differential equations:*

$$\begin{aligned}
X^{(\alpha)}(t) &= AX(t) + CY(t), \\
Y^{(\alpha)}(t) &= CX(t) + AY(t),
\end{aligned} \tag{30}$$

under the initial contidions $X(t_0) = W_1$ and $Y(t_0) = W_2$, where

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}, W_1 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \text{ and } W_2 = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}.$$

First, note that $AC = CA$. From Corollary 16, we have

$$\begin{aligned}
\operatorname{Vec} X(t) &= \frac{1}{2} E_\alpha((t-t_0)^\alpha (I \otimes A))(E_\alpha((t-t_0)^\alpha (I \otimes C)) + E_\alpha(-(t-t_0)^\alpha (I \otimes C))) \operatorname{Vec} W_1 \\
&\quad + \frac{1}{2} E_\alpha((t-t_0)^\alpha (I \otimes A))(E_\alpha((t-t_0)^\alpha (I \otimes C)) - E_\alpha(-(t-t_0)^\alpha (I \otimes C))) \operatorname{Vec} W_2, \\
\operatorname{Vec} Y(t) &= \frac{1}{2} E_\alpha((t-t_0)^\alpha (I \otimes A))(E_\alpha((t-t_0)^\alpha (I \otimes C)) - E_\alpha(-(t-t_0)^\alpha (I \otimes C))) \operatorname{Vec} W_1 \\
&\quad + \frac{1}{2} E_\alpha((t-t_0)^\alpha (I \otimes A))(E_\alpha((t-t_0)^\alpha (I \otimes C)) + E_\alpha(-(t-t_0)^\alpha (I \otimes C))) \operatorname{Vec} W_2.
\end{aligned}$$

To obtain the solution of the system, we compute

$$\begin{aligned}
E_\alpha((t-t_0)^\alpha (I \otimes A)) &= I \otimes (E_\alpha(t-t_0)^\alpha A) \\
&= I \otimes \begin{bmatrix} E_\alpha(2t^\alpha) & 0 \\ -2E_\alpha(t^\alpha + 2E_\alpha(2t^\alpha) & E_\alpha(2t^\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \eta_1 & 0 & 0 & 0 \\ \eta_2 & \eta_3 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 \\ 0 & 0 & \eta_2 & \eta_3 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned} E_\alpha((t-t_0)^\alpha(I\otimes C)) &= I\otimes(E_\alpha(t-t_0)^\alpha C)\\ &= I\otimes\begin{bmatrix} E_\alpha(2t^\alpha) & 0\\ 2E_\alpha(t^\alpha)-2E_\alpha(2t^\alpha) & E_\alpha(t^\alpha)\end{bmatrix}\\ &= \begin{bmatrix} \eta_3 & 0 & 0 & 0\\ -\eta_2 & \eta_1 & 0 & 0\\ 0 & 0 & \eta_3 & 0\\ 0 & 0 & -\eta_2 & \eta_1\end{bmatrix}, \end{aligned}$$

where $\eta_1 = E_\alpha(t^\alpha), \eta_2 = -2E_\alpha(t^\alpha)+2E_\alpha(2t^\alpha), \eta_3 = E_\alpha(2t^\alpha)$*, and*

$$\begin{aligned} E_\alpha(-(t-t_0)^\alpha(I\otimes C)) &= I\otimes E_\alpha(-(t-t_0)^\alpha C)\\ &= I\otimes(E_\alpha((t-t_0)^\alpha C))^{-1}\\ &= I\otimes\left(\frac{1}{-2E_\alpha(t^\alpha)+E_\alpha(t^\alpha)E_\alpha(2t^\alpha)+2E_\alpha(2t^\alpha)}\right)\begin{bmatrix} E_\alpha(t^\alpha) & 0\\ -2E_\alpha(t^\alpha)+2E_\alpha(2t^\alpha) & E_\alpha(2t^\alpha)\end{bmatrix}\\ &= \frac{1}{\eta_1\eta_3+\eta_2}\begin{bmatrix} \eta_1 & 0 & 0 & 0\\ \eta_2 & \eta_3 & 0 & 0\\ 0 & 0 & \eta_1 & 0\\ 0 & 0 & \eta_2 & \eta_3\end{bmatrix}, \end{aligned}$$

it following that

$$E_\alpha((t-t_0)^\alpha(I\otimes A))(E_\alpha((t-t_0)^\alpha(I\otimes C))+(E_\alpha((t-t_0)^\alpha(I\otimes C)))^{-1}) = \frac{1}{\kappa_2}\begin{bmatrix} \kappa_1 & 0 & 0 & 0\\ \kappa_3 & \kappa_4 & 0 & 0\\ 0 & 0 & \kappa_1 & 0\\ 0 & 0 & \kappa_3 & \kappa_4\end{bmatrix}.$$

where $\kappa_1 = \eta_1^2+\eta_1^2\eta_3^2+\eta_1\eta_2\eta_3,\ \kappa_2 = \eta_1\eta_3+\eta_2,\ \kappa_3 = \eta_1\eta_2+\eta_2\eta_3,\ \kappa_4 = \eta_1^2\eta_3^2+\eta_1\eta_2\eta_3+\eta_3^2.$

Similary, we have

$$E_\alpha((t-t_0)^\alpha(I\otimes A))(E_\alpha((t-t_0)^\alpha(I\otimes C))-(E_\alpha((t-t_0)^\alpha(I\otimes C)))^{-1}) = \frac{1}{\kappa_2}\begin{bmatrix} \kappa_5 & 0 & 0 & 0\\ -\kappa_3 & \kappa_6 & 0 & 0\\ 0 & 0 & \kappa_5 & 0\\ 0 & 0 & -\kappa_3 & \kappa_6\end{bmatrix}.$$

where $\kappa_5 = -\eta_1^2+\eta_1^2\eta_3^2+\eta_1\eta_2\eta_3,\ \kappa_6 = \eta_1^2\eta_3^2+\eta_1\eta_2\eta_3-\eta_3^2$*. Hence, by Corollary 16, we have*

$$\begin{aligned} \operatorname{Vec} X(t) &= \frac{1}{2\kappa_2}\left(\begin{bmatrix} \kappa_1 & 0 & 0 & 0\\ \kappa_3 & \kappa_4 & 0 & 0\\ 0 & 0 & \kappa_1 & 0\\ 0 & 0 & \kappa_3 & \kappa_4\end{bmatrix}\begin{bmatrix}1\\1\\0\\2\end{bmatrix}+\begin{bmatrix} \kappa_5 & 0 & 0 & 0\\ -\kappa_3 & \kappa_6 & 0 & 0\\ 0 & 0 & \kappa_5 & 0\\ 0 & 0 & -\kappa_3 & \kappa_6\end{bmatrix}\begin{bmatrix}1\\-2\\0\\2\end{bmatrix}\right)\\ &= \frac{1}{2\kappa_2}\begin{bmatrix}\kappa_1+\kappa_5\\ \kappa_4-2\kappa 6\\ 0\\ 2\kappa_4+2\kappa_6\end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \operatorname{Vec} Y(t) &= \frac{1}{2\kappa_2}\left(\begin{bmatrix} \kappa_5 & 0 & 0 & 0\\ -\kappa_3 & \kappa_6 & 0 & 0\\ 0 & 0 & \kappa_5 & 0\\ 0 & 0 & -\kappa_3 & \kappa_6\end{bmatrix}\begin{bmatrix}1\\1\\0\\2\end{bmatrix}+\begin{bmatrix} \kappa_1 & 0 & 0 & 0\\ \kappa_3 & \kappa_4 & 0 & 0\\ 0 & 0 & \kappa_1 & 0\\ 0 & 0 & \kappa_3 & \kappa_4\end{bmatrix}\begin{bmatrix}1\\-2\\0\\2\end{bmatrix}\right)\\ &= \frac{1}{2\kappa_2}\begin{bmatrix}\kappa_1+\kappa_5\\ -2\kappa_4+\kappa 6\\ 0\\ 2\kappa_4+2\kappa_6\end{bmatrix}. \end{aligned}$$

5 Conclusions

We solve a nonhomogeneous system of coupled linear matrix fractional dynamical differential equations. We consider the fractional derivative taken in Caputo's sense. We have an explicit form of the general solution to this obtained in terms of Kronecker product, the vector operator and Mittage-Leffler functions.

References

[1] Herrmann R. Fractional Calculus: An Introduction for Physicists. Singapore: World Scientific; 2011.

[2] Caputo MR. Linear models of dissipation whose Q is almost frequency indepemdent–II. Geophysical Journal International. 1967;13(5):529–539.

[3] Miller KS, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. New York: Wiley-Interscience; 1993.

[4] Podlubny I. Fractional differential equations. New York: Academic Press; 1999.

[5] Kilbas AA, Saigo M. Fractional integrals and derivatives of Mittag-Leffler type function. Doklady Akademii Nauk Belarusi. 1995;39(4):22–26.

[6] Kilbas AA, Srivastava HM, Tryjillo JJ. Theory and Applications of Fractional differential Equations. Amsterdam; 2006.

[7] Ben Taher R, Rachidi M. Linear recurrence relations in the algebra of matrices and applications. Linear Algebra Appl. 2001;330(1-3):15–24.

[8] Ben Taher R, Rachidi M. Some explicit formulas for the polynomial decomposition of the matrix exponential and applications. Linear Algebra Appl. 2002;350(1-3):171–184.

[9] Cheng HW, Yau SST. On more explicit formulas for matrix exponential . Linear Algebra Appl. 1997;262:102–110.

[10] Leonard IE. The Matrix Exponential. SIAM Review. 1996;38(3):507–512.

[11] Al-Zhour ZAA. The general (vectors) solutions of such linear (coupled) matrix fractional differential equations by using Kronecker structures. Applied Mathematics and Computer. 2014;232:498–510.

[12] Kongyaksee R, Changiam P. Solving Systems of Nonhomogeneous Coupled Linear Matrix Differential Equations in Terms of Mittag-Leffler Matrix Functions. Journal of Computational Analysis and Applications. 2019;26(1):1150–1160.

[13] Balanchan K, Kokila J. On the controllability dynamic systems. Int J Appl Math Comput Sci. 2012;22(3):523–531.

[14] Balanchan K, Kokila J, Trujillo JJ. Relative controllability of fractional dynamical system with moltiple delays in control. Comput Math Appl. 2012;64:3037–3045.

[15] Killicman A, Al-Zhour ZAA. The general common exact solutions of coupled linear matrix and matrix differential equations. The 2nd International Conference on Research and Education in Mathematics. 2005;p. 75–87.

[16] Horn RA, Johnson CR. Topics in Matrix Analysis. New York: Cambridge University Press; 1991.

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