ITERATION SCHEME FOR SOLVING FIXED POINT PROBLEMS AND EQUILIBRIUM PROBLEMS BY USING CONCEPT OF VISCOSITY METHODS

KATANYOO THAPTHAS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
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KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG
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Abstract

For the purpose of this research, we are using the concept of equilibrium problem and prove a strong convergence theorem by modifying the viscosity approximation methods for finding a common element of the set of fixed points of a finite family of κ_i-strictly pseudo-contractive mappings and of solutions of a finite family of equilibrium problems and variational inequality problems. Moreover, we apply our main theorem to prove strong convergence theorems involving optimization problems. Furthermore, we also give a numerical examples to support our main theorem.

Keywords : Viscosity approximation method, Strictly pseudo-contractive mapping, S-mapping, Variational inequality problem, The combination of equilibrium problem

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Chapter 1

Introduction

1.1 Research Motivation

Fixed point theory is an important area of mathematical analysis. The invention of the fixed point method is finding the solution in many area in Hilbert space such as nonlinear operator equations, variational inequality problems, equilibrium problems, optimization problems and minimal problems. The solution of some problems is an important and has many benefit in various disciplines such as physics, engineering, mathematics and economics. In the past few years, many researcher in mathematics has been developed and studied many topics in fixed point theory.

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. Recall that T is a κ*-strictly pseudo-contractive mapping* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C. \tag{1.1}$$

If $\kappa = 0$, then (1.1) reduces to nonexpansive mappings.

A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) = \{x \in C : Tx = x\}$.

Definition 1.1. Let $f : C \to C$ is said to be *contractive* if there exists a constant $\eta \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \eta \|x - y\|,$$

for all $x, y \in C$.

A bounded linear operator A on H is called *strongly positive* with coefficient $\overline{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2.$$

A mapping A of C into H is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \ \forall x, y \in C. \tag{1.2}$$

Let $A : C \to H$. *The variational inequality problem* is to find a point $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \ \forall v \in C. \tag{1.3}$$

The set of solutions of the variational inequality is denoted by $VI(C, A)$.

Variational inequalities were introduced and investigated by Stampacchia [8] in

1964. It is well known that variational inequalities cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance; see [9]-[11].

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium point, that is to find a point $x^* \in C$ such that $F(x^*, y) \geq 0$, for all $y \in C$.

The set of all solution of equilibrium problem is denoted by

$$EP(F) = \{x^* \in C : F(x^*, y) \geq 0, \forall y \in C\}. \tag{1.4}$$

The methods which are used to solve equilibrium problems have been applied in solving economic problem and some problems in pure and applied science; see [1, 2]. Many authors have studied an iterative scheme for the equilibrium problems; see, for example, [2]-[5].

In 2013, Suwannaut and Kangtunyakarn [15] introduced *the combination of equilibrium problem* which is to find $x \in C$ such that

$$\sum_{i=1}^{N} a_i F_i(x, y) \geq 0, \ \ \forall y \in C, \tag{1.5}$$

where $F_i : C \times C \to \mathbb{R}$ be bifunction and $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$, for every $i = 1, 2, ..., N$. The set of solution (1.5) is denoted by

$$EP\Big(\sum_{i=1}^{N} a_i F_i\Big) = \Big\{x \in C : \Big(\sum_{i=1}^{N} a_i F_i\Big)(x, y) \geq 0, \forall y \in C\Big\}.$$

If $F_i = F$, $\forall i = 1, 2, ..., N$, then (1.5) reduces to (1.4).

In 2007, Takahashi and Takahashi [5] proved the following theorem.

Theorem 1.1. Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying

$(A1)$ $F(x, x) = 0$ for all $x \in C$;

$(A2)$ F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

$(A3)$ For each $x, y, z \in C$,

$$\lim_{t \to 0^+} F\big(tz + (1 - t)x, y\big) \leq F(x, y);$$

$(A4)$ For each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous,

and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \ \ \forall y \in C,$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [0,1]$ satisfy some control conditions. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)} f(z)$.

The explicit viscosity method for nonexpansive mappings generates a sequence $\{x_n\}$ through the iteration process:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \ \ n \geq 0, \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. It is well known [6, 7] that under certain conditions, the sequence $\{x_n\}$ converges in norm to a fixed point q of T which solves the variational inequality

$$\langle (I-f)q, x-q\rangle \geq 0, \ \ x \in S, \tag{1.7}$$

where I is the identity of H and S is the set of fixed points of T, namely, $S = \{x \in H : Tx = x\}$.

Many authors proved a strong convergence theorem by using viscosity method; see, for example, [5, 6].

In 2010, Kangtunyakarn [12] proved a strong convergence theorem of the iterative scheme (1.9) to a common fixed point of $q \in \bigcap_{i=1}^{N} F(T_i)$ as follows:

Theorem 1.2. Let H be a Hilbert space, let f be an α-contraction on H and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\lambda}$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contraction of H into itself, for some $\kappa_i \in [0,1)$ and $\kappa = max\{\kappa_i : i = 1,2,...,N\}$, with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let S_n be the S-mappings generated by $T_1, T_2, ..., T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, ..., \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I, I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$, for all $j = 1,2,...,N-1, \kappa < c \leq \alpha_1^{n,N} \leq 1, \kappa \leq \alpha_3^{n,N} \leq d < 1, \kappa \leq \alpha_2^{n,j} \leq e < 1$, for all $j = 1,2,...,N$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)S_n x_n, \\ x_{n+1} = \alpha_n \gamma\big(a_n u + (1-a_n) f(x_n)\big) + (1-\alpha_n A)y_n, \ \ n \geq 1, \end{cases} \tag{1.8}$$

where $\{\beta_n\}$,$\{\alpha_n\}$ and $\{a_n\}$ are sequences in $[0,1]$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} a_n = 0$;

(ii) $\sum_{n=1}^{\infty} \left|\alpha_1^{n+1,j} - \alpha_1^{n,j}\right| < \infty, \ \sum_{n=1}^{\infty} \left|\alpha_3^{n+1,j} - \alpha_3^{n,j}\right| < \infty$, for all $j \in \{1,2,...,N\}$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \ \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;

(iii) $0 \leq \kappa \leq \beta_n < \theta < 1$, for all $n \geq 1$, for some $\theta \in (0,1)$.

Then both $\{x_n\}$ and $\{y_n\}$ strongly converges to $q \in \bigcap_{i=1}^{N} F(T_i)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \ \ \forall p \in \bigcap_{i=1}^{N} F(T_i). \tag{1.9}$$

From the work of Takahashi with Takahashi in [5] and Suwannaut with Kangtunyakarn in [15], we modify the viscosity methods which is different from the work of Kangtunyakarn in [12] as following:

For every $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ be bifunction which satisfy $(A1) - (A4)$ and $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$, $T_i : C \to C$ be κ_i-strictly pseudo-contractive mapping, for all $i = 1, 2, ..., N$ and $\kappa = max\{\kappa_i : i = 1, 2, ..., N\}$. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, for $x_1 \in C$ and sequence $\{x_n\}$ generated by

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n \big(\alpha_n f(x_n) + (1 - \alpha_n) S x_n\big) + (1 - \beta_n) P_C (I - \lambda A) u_n, \forall n \geq 1, \end{cases} \tag{1.10}$$

where $A : C \to H$ is α-inverse-strongly monotone mapping and $S : C \to C$ is S-mapping generated by a finite family of strictly pseudo-contractive mappings and a finite real numbers under suitable conditions of the parameters $\{\beta_n\}, \{\alpha_n\}, \{r_n\} \in [0, 1]$ and $\lambda \in (0, 2\alpha)$.

Motivated by the research going on in this direction, we prove a sequence $\{x_n\}$ generated by (1.10) converges strongly to a common element of the set of fixed points of a finite family of strictly pseudo-contractive mappings and of solutions of a finite family of equilibrium problems and variational inequality problems. Moreover, we apply our main theorem to prove strong convergence theorems involving optimization problems. Finally, we utilize our main theorem for the numerical examples.

1.2 Objectives of the study

1) To prove a strong convergence theorem by modifying the viscosity methods for finding a common element of the set of fixed points of a finite family of κ_i-strictly pseudo-contractive mappings and of solutions of a finite family of equilibrium problems and variational inequality problems.

2) For applying our main theorem to prove strong convergence theorems involving optimization problems.

3) To give numerical examples for our results to compare converge of them.

1.3 Scopes of the study

1) Study equilibrium problems in real Hilbert space.
2) Study variational inequality problems in real Hilbert space
3) Study the fixed point problems of strictly pseudo-contractive mappings and a convergence theorems involving optimization problems
4) All strong convergence theorems are considered and proved in a real Hilbert space.

1.4 Benefits of the study

1) To obtain new tools for fixed point problems on real Hilbert space.
2) To obtain a strong convergence theorem by the viscosity methods for finding a common element of the set of solutions of equilibrium problems and variational inequality problems.
3) To obtain a strong convergence theorem for finding a common element of the set of fixed point problems of strictly pseudo-contractive mappings and a convergence theorems involving optimization problems.

1.5 Research methodology

1) Study advanced topics in fixed point theory for a strictly pseudo-contractive mappings.
2) Study background in a real Hilbert space.
3) Collect and study research papers and textbooks concerning fixed point theorem.
4) Determine the objectives and scope of the research.
5) Produce tools for a strong convergence theorem of fixed point problems.
6) Prove a strong convergence theorem for fixed point problems in a real Hilbert space.
7) Provide examples and applications.
8) Conclude the results, make suggestions for further works and write the thesis.

Chapter 2

Preliminaries and Literature Reviews

The purpose of this chapter is to collect lemma, definition, theorem and terminology for used throughout of the thesis.

2.1 Fundamental properties in Hilbert spaces

Definition 2.1 (Normed space)**.** Let X be a vector space and the function $\|\cdot\| : X \to \mathbb{R}$ is norm if it satisfies the following conditions:

(i) $\|x\| \geq 0$;

(ii) $\|x\| = 0 \Leftrightarrow x = 0$;

(iii) $\|\alpha x\| = |\alpha|\|x\|, \forall \alpha \in \mathbb{F}$;

(iv) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$ (triangle inequality).

Then $(X, \|\cdot\|)$ or X is called normed space.

Definition 2.2 (Convergence in Normed Space [24])**.** Let $(E, \|\cdot\|)$ be a normed space. We say that a sequence $\{x_n\}$ of elements of E converges to some $x \in E$ if for every $\epsilon > 0$ there exists a number M such that for every $n \geq M$, we have $\|x_n - x\| < \epsilon$. In such a case we write $\lim_{n\to\infty} x_n = x$ or simply $x_n \to x$.

Definition 2.3 (Inner product space)**.** An inner product space is a vector space X with an inner product defined on X. Here, an inner product on X is a mapping of $X \times X$ into the scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; that is, with every pair of vector x and y there is associated a scalar which is written and is called the inner product of x and y, such that for all vectors x, y, z and scalar $\alpha \in \mathbb{F}$ we have:

(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \leftrightarrow x = 0$;

(ii) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$;

(iii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;

(iv) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, x\rangle$.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle}$.

Theorem 2.1 ([25])**.** If x and y are any two vectors in an inner product space X, then $|\langle x, y\rangle| \leq \|x\|\|y\|$.

Definition 2.4 (Hilbert space [24])**.** Let X be an inner product space and X is called Hilbert space if X is complete inner product space.

Lemma 2.2 ([29])**.** Let H be a real Hilbert space. Then the following results hold:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \text{for each } x, y \in H. \tag{2.1}$$

Definition 2.5 (Strong convergence [24])**.** A sequence $\{x_n\}$ of vectors in an inner product space K is called *strongly convergent* to a vector x in K if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.6 (Weak convergence [24])**.** A sequence $\{x_n\}$ of vectors in an inner product space K is called *weakly convergent* to a vector x in K if

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ as } n \to \infty \text{ for every } y \in K.$$

Theorem 2.3 ([24])**.** A strongly convergence sequence is weakly convergence (to the same limit), that is, $x_n \to x$ implies $x_n \rightharpoonup x$.

Remark 2.4 ([25])**.** If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Theorem 2.5 ([25])**.** Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Suppose that $\{x_n\} \subset C$ and $x_n \rightharpoonup x$. Then $x \in C$.

Definition 2.7 ([25])**.** Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \le a\} \text{ is closed.}$$

Moreover, f is called *convex* if for any $x_1, x_2 \in C$ and $t \in (0,1)$,

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2).$$

Similarly, f is said to be *concave* if for any $x_1, x_2 \in C$ and $t \in (0,1)$,

$$f(tx_1 + (1-t)x_2) \ge tf(x_1) + (1-t)f(x_2).$$

Theorem 2.6 ([25])**.** Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightharpoonup x_0$. Then

$$f(x_0) \le \liminf_{n\to\infty} f(x_n).$$

Lemma 2.7 ([16])**.** Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{u_n\} \subset H$ with $u_n \rightharpoonup u$, the inequality

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

holds for every $v \in H$ with $v \ne u$.

Theorem 2.8 ([25])**.** Let $\{a_n\}$ be a bounded of real numbers. Then, there exists subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Remark 2.9 ([25])**.** Let H be an inner product space. Then the following knowledge (i) and (ii) are equivalent:

(i) H is complete,

(ii) each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Definition 2.8 (Metric projection [25])**.** The (nearest point) projection P_C from H onto C assigns to each $x \in H$, the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

Lemma 2.10 ([26])**.** Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0 \quad \forall z \in C.$$

It is well-known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \forall x, y \in H.$$

Lemma 2.11 ([26])**.** Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.12 ([18])**.** Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.13 ([18])**.** Let $\{s_n\}$ be a sequence of nonnegative real number satisfying

$$s_{n+1} = (1-\alpha_n)s_n + \alpha_n\beta_n, \qquad \forall n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions

(i) $\{\alpha_n\} \subset [0,1], \ \sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$ or $\sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty.$

Then $\lim_{n\to\infty} s_n = 0.$

2.2 Fixed point theorems

Let X be a nonempty set and $T : X \to X$ a self-mapping. We say that $x \in X$ is a fixed point of T if and only if $Tx = x$ and $F(T)$ represents the set of all fixed points of T.

Example 2.14 ([27])**.** 1) If $X = \mathbb{R}$ and $T(x) = x^2 - 11x + 36$, then $F(T) = \{6\}$;

2) If $X = \mathbb{R}$ and $T(x) = 2x^2 - 3x$, then $F(T) = \{0, 2\}$;

3) If $X = \mathbb{R}$ and $T(x) = x + 5$, then $F(T) = \emptyset$;

4) If $X = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$;

Theorem 2.15 ([25])**.** Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Theorem 2.16 ([25])**.** Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.

Lemma 2.17 (Demiclosedness principle [28])**.** Assume that T is a nonexpansive self-mapping of closed convex C subset of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y it follows that $(I-T)x = y$, where I is the identity mapping of H.

In 2009, Kangtunyakarn and Suantai ([20]) introduced the S-mapping generated by a finite family of κ_i-strictly pseudo-contractions and a finite real numbers. The definition can be seen below:

Definition 2.9. Let C be a nonempty convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractions of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I, \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called an *S-mapping* generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 2.18. ([22]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractive mapping of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i = 1, 2, ..., N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0, 1], \alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j, \alpha_2^j \in (\kappa, 1)$, for all $i = 1, 2, ..., N-1$ and $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in (\kappa, 1], \alpha_2^j \in (\kappa, 1]$, for all $j = 1, 2, ..., N$, let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a nonexpansive mapping.

2.3 Equilibrium problems and generalized equilibrium problem in Hilbert spaces

The equilibrium problem provides us an natural to study problems arising in physics, optimization, economics, finance and certain fixed point problems.

For solving the equilibrium problems for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that $F : C \times C \to \mathbb{R}$ satisfy the following conditions:

$(A1)$ $F(x, x) = 0$ for all $x \in C$;

$(A2)$ F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

$(A3)$ For each $x, y, z \in C$,

$$\lim_{t \to 0^+} F\big(tz + (1-t)x, y\big) \leq F(x, y);$$

$(A4)$ For each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Example 2.19. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x, y) = -7x^2 + xy + 6y^2, \forall x, y \in \mathbb{R}.$$

Then a bifunction F satisfies the condition $(A1)$-$(A4)$ and $0 \in EP(F)$.
Solution. Let $x, y, z \in \mathbb{R}$. Since

$$F(x,x) = -7x^2 + x^2 + 6x^2 = 0,$$

thus we obtain $(A1)$ holds. Next, observe that

$$F(x,y) + F(y,x) = \left(-7x^2 + xy + 6y^2\right) + \left(-7y^2 + xy + 6x^2\right) = -x^2 + 2xy - y^2 = -(x-y)^2 \le 0.$$

This implies that F satisfies $(A2)$. Let $t \in [0,1]$. Consider

$$\lim_{t\to 0^+} F(tz+(1-t)x, y) = \lim_{t\to 0^+} (-7(tz+(1-t)x)^2+(tz+(1-t)x)y+6y^2) = -7x^2+xy+6y^2 = F(x,y).$$

Therefore, $(A3)$ is true. To show $(A4)$, first let $\alpha \in (0,1)$. Then we derive that

$$\begin{aligned}
&F(x, \alpha z + (1-\alpha)y) \\
&= -7x^2 + x(\alpha z + (1-\alpha)y) + 6(\alpha z + (1-\alpha)y)^2 \\
&= -7x^2 + \alpha xz + (1-\alpha)xy + 6(\alpha^2 z^2 + 2\alpha(1-\alpha)zy + (1-\alpha)^2 y^2) \\
&\le -7x^2 + \alpha xz + (1-\alpha)xy + 6(\alpha^2 z^2 + \alpha(1-\alpha)(z^2+y^2) + (1-\alpha)^2 y^2) \\
&= \alpha\left(-7x^2 + xz + 6(\alpha z^2 + (1-\alpha)z^2)\right) + (1-\alpha)\left(x^2 + xy + 6(\alpha y^2 + (1-\alpha)y^2)\right) \\
&= \alpha\left(-7x^2 + xz + 6z^2\right) + (1-\alpha)\left(-7x^2 + xy + 6y^2\right) \\
&= \alpha F(x,z) + (1-\alpha)F(x,y).
\end{aligned}$$

Hence F is a convex function. Let $\{y_n\} \subset \mathbb{R}$ with $y_n \to y$ as $n \to \infty$. Thus we get

$$\lim_{n\to\infty} F(x, y_n) = \lim_{n\to\infty} -7x^2 + xy_n + 6y_n^2 = -7x^2 + xy + 6y^2. \tag{2.2}$$

This yields that F is lower semicontinuous and $(A4)$ holds. Since $F(0,y) = 6y^2 \ge 0, \forall y \in \mathbb{R}$, thus we have $0 \in EP(F)$.

In 1994, Blum and Oettli [14] proved the following existence result:

Lemma 2.20 ([14])**.** Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C.$$

Lemma 2.21. ([17]) Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1)-(A4)$. For $r > 0$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \Big\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\Big\},$$

for all $x \in H$. Then, the following hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y \rangle ;$$

(iii) $F(T_r) = EP(F)$;

(iv) $EP(F)$ is closed and convex.

Lemma 2.22. ([15]) Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying $(A1) - (A4)$ with $\bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Then

$$EP\Big(\sum_{i=1}^{N} a_i F_i\Big) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0, 1)$, for every $i = 1, 2, ..., N$ and $\sum_{i=1}^{N} a_i = 1$.

Remark 2.23. ([15]) From Lemma 2.21 and Lemma 2.22, we have the following results;

(i) $\sum_{i=1}^{N} a_i F_i$ satisfying $(A1) - (A4)$;

(ii) $F(T_r) = \bigcap_{i=1}^{N} EP(F_i)$,

where $r > 0$ and $a_i \in (0, 1)$, for every $i = 1, 2, ..., N$ with $\sum_{i=1}^{N} a_i = 1$.

Chapter 3

A Theorem for Solving Equilibrium, Fixed Point and Variational Inequality Problems

In this chapter, we prove a strong convergence theorem for finding a common element of $\bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap VI(C,A)$ and apply theorem 3.1 for the numerical example.

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ be bifunction with satisfy $(A1) - (A4)$, $T_i : C \to C$ be κ_i-strictly pseudo-contractive mapping and let $A : C \to H$ be α-inverse strongly monotone mapping with $\mathcal{F} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap VI(C,A) \neq \emptyset$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, I = [0,1]$ with $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ and $\kappa < \alpha_1^j, \alpha_3^j < 1$, for all $i = 1, 2, ..., N-1, \kappa < \alpha_1^N \leq 1, \kappa \leq \alpha_3^N < 1, \kappa \leq \alpha_2^j < 1$, for all $j = 1, 2, ..., N$, where $\kappa = max\{\kappa_i : i = 1, 2, ..., N\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n\big(\alpha_n f(x_n) + (1-\alpha_n) S x_n\big) + (1-\beta_n) P_C(I - \lambda A) u_n, \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$ and $\lambda \in (0, 2\alpha)$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0,$

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, for all $n \geq 1$,

(iii) $f : C \to C$ be η-contraction,

(iv) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$, for all $i = 1, 2, ..., N$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.

Proof. First, we show that $(I - \lambda A)$ is a nonexpansive mapping. Let $x, y \in C$. Since A is α-inverse strongly monotone and $\lambda < 2\alpha$, we have

$$\begin{aligned} \|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2 \|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 - 2\alpha\lambda \|Ax-Ay\|^2 + \lambda^2 \|Ax-Ay\|^2 \\ &= \|x-y\|^2 + \lambda(\lambda - 2\alpha) \|Ax-Ay\|^2 \\ &\leq \|x-y\|^2. \end{aligned}$$

Then $(I-\lambda A)$ is a nonexpansive mapping. We will divide our proof into 5 steps.

Step 1. We show that the sequence $\{x_n\}$ is bounded. Since $\sum_{i=1}^{N} a_iF_i$ satisfies $(A1)-(A4)$ and

$$\sum_{i=1}^{N} a_iF_i(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, \forall y\in C,$$

by Lemma 2.21 and Remark 2.23, we have $u_n=T_{r_n}x_n$ and $\bigcap_{i=1}^{N} EP(F_i)=F(T_{r_n})$. Let $z\in\mathcal{F}$. By nonexpansiveness of $(I-\lambda A)$ and T_{r_n}, we obtain

$$\begin{aligned}
\|x_{n+1}-z\| =& \left\|\beta_n\big(\alpha_n f(x_n)+(1-\alpha_n)Sx_n\big)+(1-\beta_n)P_C(I-\lambda A)u_n-z\right\| \\
=& \left\|\beta_n\big(\alpha_n f(x_n)+(1-\alpha_n)Sx_n-z\big)+(1-\beta_n)\big(P_C(I-\lambda A)u_n-z\big)\right\| \\
\leq& \beta_n\left\|\alpha_n\big(f(x_n)-z\big)+(1-\alpha_n)(Sx_n-z)\right\| \\
&+(1-\beta_n)\|P_C(I-\lambda A)u_n-z\| \\
\leq& \beta_n\big(\alpha_n\|f(x_n)-f(z)\|+\alpha_n\|f(z)-z\|+(1-\alpha_n)\|Sx_n-z\|\big) \\
&+(1-\beta_n)\|P_C(I-\lambda A)u_n-z\| \\
\leq& \beta_n\big(\alpha_n\eta\|x_n-z\|+\alpha_n\|f(z)-z\|+(1-\alpha_n)\|x_n-z\|\big) \\
&+(1-\beta_n)\|u_n-z\| \\
=& \beta_n\Big(\big(1-\alpha_n(1-\eta)\big)\|x_n-z\|+\alpha_n\|f(z)-z\|\Big)+(1-\beta_n)\|x_n-z\| \\
\leq& \max\left\{\|x_1-z\|,\frac{\|f(z)-z\|}{1-\eta}\right\}.
\end{aligned}$$

By induction we can prove that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. By definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| =& \Big\|\Big(\beta_n\big(\alpha_n f(x_n)+(1-\alpha_n)Sx_n\big)+(1-\beta_n)P_C(I-\lambda A)u_n\Big) \\
&-\Big(\beta_{n-1}\big(\alpha_{n-1}f(x_{n-1})+(1-\alpha_{n-1})Sx_{n-1}\big) \\
&+(1-\beta_{n-1})P_C(I-\lambda A)u_{n-1}\Big)\Big\| \\
\leq& \beta_n\left\|\big(\alpha_n f(x_n)+(1-\alpha_n)Sx_n\big)-\big(\alpha_{n-1}f(x_{n-1})+(1-\alpha_{n-1})Sx_{n-1}\big)\right\| \\
&+|\beta_n-\beta_{n-1}|\,\|\alpha_{n-1}f(x_{n-1})+(1-\alpha_{n-1})Sx_{n-1}\| \\
&+(1-\beta_n)\|P_C(I-\lambda A)u_n-P_C(I-\lambda A)u_{n-1}\| \\
&+|\beta_{n-1}-\beta_n|\,\|P_C(I-\lambda A)u_{n-1}\| \\
\leq& \beta_n\big(\alpha_n\|f(x_n)-f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\,\|f(x_{n-1})\| \\
&+(1-\alpha_n)\|Sx_n-Sx_{n-1}\|+|\alpha_{n-1}-\alpha_n|\,\|Sx_{n-1}\|\big) \\
&+|\beta_n-\beta_{n-1}|\big(\alpha_{n-1}\|f(x_{n-1})\|+(1-\alpha_{n-1})\|Sx_{n-1}\|\big) \\
&+(1-\beta_n)\|P_C(I-\lambda A)u_n-P_C(I-\lambda A)u_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&+|\beta_{n-1}-\beta_n|\,\|P_C(I-\lambda A)u_{n-1}\| \\
\le&\beta_n\big(\alpha_n\eta\,\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\,\|f(x_{n-1})\| \\
&+(1-\alpha_n)\,\|x_n-x_{n-1}\|+|\alpha_{n-1}-\alpha_n|\,\|Sx_{n-1}\|\,\big) \\
&+|\beta_n-\beta_{n-1}|\big(\alpha_{n-1}\,\|f(x_{n-1})\|+(1-\alpha_{n-1})\,\|Sx_{n-1}\|\,\big) \\
&+(1-\beta_n)\,\|u_n-u_{n-1}\| \\
&+|\beta_{n-1}-\beta_n|\,\|P_C(I-\lambda A)u_{n-1}\| \\
\le&\beta_n\Big(|\alpha_n-\alpha_{n-1}|M+|\alpha_{n-1}-\alpha_n|M+\big(1-\alpha_n(1-\eta)\big)\,\|x_n-x_{n-1}\|\,\Big) \\
&+|\beta_n-\beta_{n-1}|\big(\alpha_{n-1}M+(1-\alpha_{n-1})M\big) \\
&+(1-\beta_n)\,\|u_n-u_{n-1}\| \\
&+|\beta_{n-1}-\beta_n|M \\
=&\beta_n\Big(2M|\alpha_n-\alpha_{n-1}|+\big(1-\alpha_n(1-\eta)\big)\,\|x_n-x_{n-1}\|\,\Big) \\
&+2M|\beta_n-\beta_{n-1}|+(1-\beta_n)\,\|u_n-u_{n-1}\|\,, \qquad (3.2)
\end{aligned}
$$

where $M=\max_{n\in\mathbb{N}}\{\,\|f(x_n)\|\,,\|Sx_n\|\,,\|P_C(I-\lambda A)u_n\|\,\}$.
Since $u_n=T_{r_n}x_n$ and definition of T_{r_n}, we obtain

$$\sum_{i=1}^{N}a_iF_i(T_{r_n}x_n,y)+\frac{1}{r_n}\langle y-T_{r_n}x_n,T_{r_n}x_n-x_n\rangle\ge 0,\forall y\in C \qquad (3.3)$$

and

$$\sum_{i=1}^{N}a_iF_i(T_{r_{n+1}}x_{n+1},y)+\frac{1}{r_{n+1}}\big\langle y-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\big\rangle\ge 0,\forall y\in C. \qquad (3.4)$$

From (3.3) and (3.4). It follow that

$$\sum_{i=1}^{N}a_iF_i(T_{r_n}x_n,T_{r_{n+1}}x_{n+1})+\frac{1}{r_n}\big\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n,T_{r_n}x_n-x_n\big\rangle\ge 0 \qquad (3.5)$$

and

$$\sum_{i=1}^{N}a_iF_i(T_{r_{n+1}}x_{n+1},T_{r_n}x_n)+\frac{1}{r_{n+1}}\big\langle T_{r_n}x_n-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\big\rangle\ge 0. \qquad (3.6)$$

From (3.5),(3.6) and the fact that $\sum_{i=1}^{N}a_iF_i$ satisfies $(A2)$, we have

$$
\begin{aligned}
&\frac{1}{r_n}\big\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n,T_{r_n}x_n-x_n\big\rangle \\
&+\frac{1}{r_{n+1}}\big\langle T_{r_n}x_n-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\big\rangle\ge 0.
\end{aligned}
$$

Which implies that

$$\Big\langle T_{r_n}x_n-T_{r_{n+1}}x_{n+1},\frac{T_{r_{n+1}}x_{n+1}-x_{n+1}}{r_{n+1}}-\frac{T_{r_n}x_n-x_n}{r_n}\Big\rangle\ge 0.$$

It follows that

$$\Big\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n,T_{r_n}x_n-T_{r_{n+1}}x_{n+1}+T_{r_{n+1}}x_{n+1}-x_n-\frac{r_n}{r_{n+1}}(T_{r_{n+1}}x_{n+1}-x_{n+1})\Big\rangle\ge 0. \qquad (3.7)$$

From (3.7), we obtain

$$\begin{aligned}\left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|^2 &\leq \left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n, T_{r_{n+1}}x_{n+1}-x_n-\frac{r_n}{r_{n+1}}(T_{r_{n+1}}x_{n+1}-x_{n+1})\right\rangle\\ &=\left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n, x_{n+1}-x_n+(1-\frac{r_n}{r_{n+1}})(T_{r_{n+1}}x_{n+1}-x_{n+1})\right\rangle\\ &\leq \left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|\Big[\,\|x_{n+1}-x_n\|\\ &\quad+|1-\frac{r_n}{r_{n+1}}|\left\|T_{r_{n+1}}x_{n+1}-x_{n+1}\right\|\Big]\\ &=\left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|\Big[\,\|x_{n+1}-x_n\|\\ &\quad+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\left\|T_{r_{n+1}}x_{n+1}-x_{n+1}\right\|\Big]\\ &\leq \left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|\Big[\,\|x_{n+1}-x_n\|\\ &\quad+\frac{1}{a}|r_{n+1}-r_n|\left\|T_{r_{n+1}}x_{n+1}-x_{n+1}\right\|\Big],\end{aligned}$$

which yields

$$\|u_{n+1}-u_n\| \leq \|x_{n+1}-x_n\|+\frac{1}{a}|r_{n+1}-r_n|\,\|u_{n+1}-x_{n+1}\|\,. \tag{3.8}$$

From (3.8), we have

$$\|u_n-u_{n-1}\| \leq \|x_n-x_{n-1}\|+\frac{1}{a}|r_n-r_{n-1}|\,\|u_n-x_n\|\,. \tag{3.9}$$

By substituting (3.9) into (3.2), we have

$$\begin{aligned}\|x_{n+1}-x_n\| \leq&\beta_n\Big(2M|\alpha_n-\alpha_{n-1}|+\big(1-\alpha_n(1-\eta)\big)\|x_n-x_{n-1}\|\Big)\\ &+2M|\beta_n-\beta_{n-1}|+(1-\beta_n)\|u_n-u_{n-1}\|\\ \leq&\beta_n\Big(2M|\alpha_n-\alpha_{n-1}|+\big(1-\alpha_n(1-\eta)\big)\|x_n-x_{n-1}\|\Big)\\ &+2M|\beta_n-\beta_{n-1}|+(1-\beta_n)\big(\,\|x_n-x_{n-1}\|+\frac{1}{a}|r_n-r_{n-1}|\,\|u_n-x_n\|\,\big)\\ \leq&\big(1-\beta_n\alpha_n(1-\eta)\big)\|x_n-x_{n-1}\|+2M|\alpha_n-\alpha_{n-1}|\\ &+2M|\beta_n-\beta_{n-1}|+(1-\beta_n)\frac{1}{a}|r_n-r_{n-1}|\,\|u_n-x_n\|\,.\end{aligned} \tag{3.10}$$

From (3.10), conditions (i),(v) and lemma 2.12, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.11}$$

Step 3. We will show that $\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|P_C(I-\lambda A)u_n-x_n\|=\lim_{n\to\infty}\|Sx_n-x_n\|=0$. Since T_{r_n} is a firmly nonexpansive mapping, then we obtain

$$\begin{aligned}\|z-T_{r_n}x_n\|^2 &=\|T_{r_n}z-T_{r_n}x_n\|^2\\ &\leq\langle T_{r_n}z-T_{r_n}x_n, z-x_n\rangle\\ &=\frac{1}{2}\big(\,\|T_{r_n}x_n-z\|^2+\|x_n-z\|^2-\|T_{r_n}x_n-x_n\|^2\,\big),\end{aligned}$$

which yields

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2 . \tag{3.12}$$

By nonexpansiveness of $P_C(I - \lambda A)$,(3.12) and definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 =& \left\|\beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n - z\big) + (1-\beta_n)\big(P_C(I-\lambda A)u_n - z\big)\right\|^2 \\
\leq& \beta_n \left\|\alpha_n\big(f(x_n) - z\big) + (1-\alpha_n)(Sx_n - z)\right\|^2 \\
&+ (1-\beta_n)\|P_C(I-\lambda A)u_n - z\|^2 \\
\leq& \beta_n\alpha_n \|f(x_n) - z\|^2 + \beta_n(1-\alpha_n)\|x_n - z\|^2 + (1-\beta_n)\|u_n - z\|^2 \\
\leq& \beta_n\alpha_n \|f(x_n) - z\|^2 + \beta_n(1-\alpha_n)\|x_n - z\|^2 \\
&+ (1-\beta_n)\big(\|x_n - z\|^2 - \|u_n - x_n\|^2\big) \\
\leq& \beta_n\alpha_n \|f(x_n) - z\|^2 + \|x_n - z\|^2 - (1-\beta_n)\|u_n - x_n\|^2 ,
\end{aligned}$$

which implies that

$$\begin{aligned}
(1-\beta_n)\|u_n - x_n\|^2 \leq& \beta_n\alpha_n \|f(x_n) - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
\leq& \beta_n\alpha_n \|f(x_n) - z\|^2 + \big(\|x_n - z\| + \|x_{n+1} - z\|\big)\|x_{n+1} - x_n\| .
\end{aligned} \tag{3.13}$$

By (3.11),(3.13), conditions (i) and (ii), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.14}$$

Put $w_n = \alpha_n f(x_n) + (1-\alpha_n)Sx_n$. By definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 =& \|\beta_n w_n + (1-\beta_n)P_C(I-\lambda A)u_n - z\|^2 \\
=& \left\|\beta_n(w_n - z) + (1-\beta_n)\big(P_C(I-\lambda A)u_n - z\big)\right\|^2 \\
\leq& \beta_n\|w_n - z\|^2 + (1-\beta_n)\|P_C(I-\lambda A)u_n - z\|^2 \\
&- \beta_n(1-\beta_n)\|w_n - P_C(I-\lambda A)u_n\|^2 \\
=& \beta_n\|\alpha_n f(x_n) + (1-\alpha_n)Sx_n - z\|^2 + (1-\beta_n)\|P_C(I-\lambda A)u_n - z\|^2 \\
&- \beta_n(1-\beta_n)\|w_n - P_C(I-\lambda A)u_n\|^2 \\
\leq& \beta_n\big(\alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|Sx_n - z\|^2\big) + (1-\beta_n)\|u_n - z\|^2 \\
&- \beta_n(1-\beta_n)\|w_n - P_C(I-\lambda A)u_n\|^2 \\
=& \beta_n\alpha_n\|f(x_n) - z\|^2 + \beta_n(1-\alpha_n)\|x_n - z\|^2 + (1-\beta_n)\|u_n - z\|^2 \\
&- \beta_n(1-\beta_n)\|w_n - P_C(I-\lambda A)u_n\|^2 \\
\leq& \beta_n\alpha_n\|f(x_n) - z\|^2 + \beta_n(1-\alpha_n)\|x_n - z\|^2 + (1-\beta_n)\|x_n - z\|^2 \\
&- \beta_n(1-\beta_n)\|w_n - P_C(I-\lambda A)u_n\|^2 \\
=& \beta_n\alpha_n\|f(x_n) - z\|^2 + (1-\beta_n\alpha_n)\|x_n - z\|^2 \\
&- \beta_n(1-\beta_n)\|w_n - P_C(I-\lambda A)u_n\|^2 \\
\leq& \beta_n\alpha_n\|f(x_n) - z\|^2 + \|x_n - z\|^2 \\
&- \beta_n(1-\beta_n)\|w_n - P_C(I-\lambda A)u_n\|^2 .
\end{aligned}$$

Which yields

$$\beta_n(1-\beta_n)\left\|w_n - P_C(I-\lambda A)u_n\right\|^2 \leq \beta_n\alpha_n\left\|f(x_n)-z\right\|^2 + \left\|x_n - z\right\|^2 - \left\|x_{n+1}-z\right\|^2$$
$$\leq \beta_n\alpha_n\left\|f(x_n)-z\right\|^2 + \big(\left\|x_n - z\right\| + \left\|x_{n+1}-z\right\|\big)\left\|x_{n+1}-x_n\right\|. \quad (3.15)$$

By (3.11),(3.15), conditions (i) and (ii), we have

$$\lim_{n\to\infty}\left\|w_n - P_C(I-\lambda A)u_n\right\| = 0. \quad (3.16)$$

By the definition of x_n, we obtain

$$x_{n+1} - P_C(I-\lambda A)u_n = \beta_n w_n - \beta_n P_C(I-\lambda A)u_n$$
$$= \beta_n\big(w_n - P_C(I-\lambda A)u_n\big). \quad (3.17)$$

By (3.17), we have

$$\left\|x_n - P_C(I-\lambda A)x_n\right\| = \left\|x_n - x_{n+1} + x_{n+1} - P_C(I-\lambda A)u_n + P_C(I-\lambda A)u_n - P_C(I-\lambda A)x_n\right\|$$
$$\leq \left\|x_n - x_{n+1}\right\| + \left\|x_{n+1} - P_C(I-\lambda A)u_n\right\| + \left\|P_C(I-\lambda A)u_n - P_C(I-\lambda A)x_n\right\|$$
$$\leq \left\|x_n - x_{n+1}\right\| + \beta_n\left\|w_n - P_C(I-\lambda A)u_n\right\| + \left\|u_n - x_n\right\|.$$

Form (3.11),(3.14) and (3.16), we have

$$\lim_{n\to\infty}\left\|x_n - P_C(I-\lambda A)x_n\right\| = 0. \quad (3.18)$$

Since

$$\left\|x_n - P_C(I-\lambda A)u_n\right\| = \left\|x_n - P_C(I-\lambda A)x_n + P_C(I-\lambda A)x_n - P_C(I-\lambda A)u_n\right\|$$
$$\leq \left\|x_n - P_C(I-\lambda A)x_n\right\| + \left\|P_C(I-\lambda A)x_n - P_C(I-\lambda A)u_n\right\|$$
$$\leq \left\|x_n - P_C(I-\lambda A)x_n\right\| + \left\|x_n - u_n\right\|.$$

From (3.14) and (3.18), we have

$$\lim_{n\to\infty}\left\|x_n - P_C(I-\lambda A)u_n\right\| = 0. \quad (3.19)$$

By the definition of x_n, we obtain

$$x_{n+1} - x_n = \beta_n\alpha_n\big(f(x_n) - x_n\big) + \beta_n(1-\alpha_n)(Sx_n - x_n)$$
$$+ (1-\beta_n)\big(P_C(I-\lambda A)u_n - x_n\big). \quad (3.20)$$

It follows that

$$\beta_n(1-\alpha_n)\left\|Sx_n - x_n\right\| \leq \beta_n\alpha_n\left\|f(x_n) - x_n\right\|$$
$$+ (1-\beta_n)\left\|P_C(I-\lambda A)u_n - x_n\right\| + \left\|x_{n+1} - x_n\right\|.$$

By (3.11),(3.19), conditions (i) and (ii), we have

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{3.21}$$

Step 4. We will show that $\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle \leq 0$, where $z = P_{\mathcal{F}} f(z)$.
To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle = \lim_{k\to\infty} \langle f(z) - z, x_{n_k} - z\rangle . \tag{3.22}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$, where $\omega \in C$.
From (3.14), we obtain $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$.
Assume that $\omega \notin VI(C, A)$. Since $VI(C, A) = F\big(P_C(I - \lambda A)\big)$, we have $\omega \neq P_C(I - \lambda A)\omega$.
By nonexpansiveness of $P_C(I - \lambda A)$,(3.18) and Opial's condition, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - P_C(I - \lambda A)\omega\| \\
&= \liminf_{k\to\infty} \|x_{n_k} - P_C(I - \lambda A)x_{n_k} + P_C(I - \lambda A)x_{n_k} - P_C(I - \lambda A)\omega\| \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - P_C(I - \lambda A)x_{n_k}\| \\
&\quad + \liminf_{k\to\infty} \|P_C(I - \lambda A)x_{n_k} - P_C(I - \lambda A)\omega\| \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - \omega\| .
\end{aligned}$$

This is a contradiction. Then we have

$$\omega \in VI(C, A). \tag{3.23}$$

Next, we will show that $\omega \in \bigcap_{i=1}^{N} F(T_i)$.
By Lemma 2.18, we have $F(S) = \bigcap_{i=1}^{N} F(T_i)$. Assume that $\omega \neq S\omega$. Using Opial's condition, (3.21), we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - S\omega\| \\
&= \liminf_{k\to\infty} \|x_{n_k} - Sx_{n_k} + Sx_{n_k} - S\omega\| \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - Sx_{n_k}\| + \liminf_{k\to\infty} \|Sx_{n_k} - S\omega\| \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - \omega\| .
\end{aligned}$$

This is a contradiction. Then we have

$$\omega \in F(S) = \bigcap_{i=1}^{N} F(T_i). \tag{3.24}$$

Next, we will show that $\omega \in \bigcap_{i=1}^{N} EP(F_i)$.
Since $\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C$ and $\sum_{i=1}^{N} a_i F_i$ satisfies condition

$(A1)-(A4)$, we obtain

$$\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq \sum_{i=1}^{N} a_iF_i(y,u_n), \forall y\in C.$$

In particular, it follows that

$$\left\langle y-u_{n_k},\frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle \geq \sum_{i=1}^{N} a_iF_i(y,u_{n_k}), \forall y\in C. \tag{3.25}$$

From (3.14),(3.25) and $(A4)$, we have

$$\sum_{i=1}^{N} a_iF_i(y,\omega)\leq 0, \forall y\in C. \tag{3.26}$$

Put $y_t:=ty+(1-t)\omega$, for all $t\in(0,1]$, we have $y_t\in C$. By using $(A1),(A4)$ and (3.26), we have

$$\begin{aligned}
0=&\sum_{i=1}^{N} a_iF_i(y_t,y_t)\\
=&\sum_{i=1}^{N} a_iF_i\big(y_t,ty+(1-t)\omega\big)\\
\leq& t\sum_{i=1}^{N} a_iF_i(y_t,y)+(1-t)\sum_{i=1}^{N} a_iF_i(y_t,\omega)\\
\leq& t\sum_{i=1}^{N} a_iF_i(y_t,y).
\end{aligned}$$

It implies that

$$0\leq\sum_{i=1}^{N} a_iF_i\big(ty+(1-t)\omega,y\big), \tag{3.27}$$

for all $t\in(0,1]$ and $y\in C$.

From (3.27), taking $t\to 0^+$ and using $(A3)$, we can conclude that

$$\begin{aligned}
0\leq&\lim_{t\to 0^+}\Big(\sum_{i=1}^{N} a_iF_i\big(ty+(1-t)\omega,y\big)\Big)\\
\leq&\sum_{i=1}^{N} a_iF_i(\omega,y), \forall y\in C.
\end{aligned}$$

Therefore, $\omega\in EP\Big(\sum_{i=1}^{N} a_iF_i\Big)$. By Lemma 2.22, we obtain $EP\Big(\sum_{i=1}^{N} a_iF_i\Big)=\bigcap_{i=1}^{N} EP(F_i)$. It follows that

$$\omega\in\bigcap_{i=1}^{N} EP(F_i). \tag{3.28}$$

From (3.23),(3.24) and (3.28), we can deduce that $\omega\in\mathcal{F}$.

Since $x_{n_k}\rightharpoonup\omega\in\mathcal{F}$ and Lemma 2.10, we can conclude that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(z)-z,x_n-z\rangle=&\lim_{k\to\infty}\langle f(z)-z,x_{n_k}-z\rangle\\
=&\langle f(z)-z,\omega-z\rangle\\
\leq& 0,
\end{aligned} \tag{3.29}$$

where $z = P_{\mathcal{F}} f(z)$.

Step 5. Finally, we will show that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$. By nonexpansive of S and $P_C(I - \lambda A)$, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \big\|\beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) + (1-\beta_n)P_C(I-\lambda A)u_n - z\big\|^2 \\
&= \big\|\beta_n\alpha_n\big(f(x_n) - z\big) + \beta_n(1-\alpha_n)(Sx_n - z) + (1-\beta_n)\big(P_C(I-\lambda A)u_n - z\big)\big\|^2 \\
&\le \big\|\beta_n(1-\alpha_n)(Sx_n - z) + (1-\beta_n)\big(P_C(I-\lambda A)u_n - z\big)\big\|^2 \\
&\quad + 2\beta_n\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle \\
&\le \big(\beta_n(1-\alpha_n)\|Sx_n - z\| + (1-\beta_n)\|P_C(I-\lambda A)u_n - z\|\big)^2 \\
&\quad + 2\beta_n\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle \\
&\le \big((1-\beta_n\alpha_n)\|x_n - z\| + (1-\beta_n)\|x_n - z\|\big)^2 \\
&\quad + 2\beta_n\alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle + 2\beta_n\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\le (1-\beta_n\alpha_n)^2\|x_n - z\|^2 + 2\beta_n\alpha_n\|f(x_n) - f(z)\|\|x_{n+1} - z\| \\
&\quad + 2\beta_n\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\le (1-\beta_n\alpha_n)^2\|x_n - z\|^2 + 2\beta_n\alpha_n\eta\|x_n - z\|\|x_{n+1} - z\| \\
&\quad + 2\beta_n\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\le (1-\beta_n\alpha_n)^2\|x_n - z\|^2 + \beta_n\alpha_n\eta\|x_n - z\|^2 + \beta_n\alpha_n\eta\|x_{n+1} - z\|^2 \\
&\quad + 2\beta_n\alpha_n\langle f(z) - z, x_{n+1} - z\rangle.
\end{aligned}$$

Which implies that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \frac{1 - \beta_n\alpha_n\eta - 2\beta_n\alpha_n + \beta_n^2\alpha_n^2 + 2\beta_n\alpha_n\eta}{1 - \beta_n\alpha_n\eta}\|x_n - z\|^2 \\
&\quad + \frac{2\beta_n\alpha_n}{1 - \beta_n\alpha_n\eta}\langle f(z) - z, x_{n+1} - z\rangle \\
&= \left(1 - \frac{2\beta_n\alpha_n(1-\eta)}{1 - \beta_n\alpha_n\eta}\right)\|x_n - z\|^2 + \frac{\beta_n^2\alpha_n^2}{1 - \beta_n\alpha_n\eta}\|x_n - z\|^2 \\
&\quad + \frac{2\beta_n\alpha_n}{1 - \beta_n\alpha_n\eta}\langle f(z) - z, x_{n+1} - z\rangle \\
&= \left(1 - \frac{2\beta_n\alpha_n(1-\eta)}{1 - \beta_n\alpha_n\eta}\right)\|x_n - z\|^2 + \frac{2\beta_n\alpha_n(1-\eta)}{1 - \beta_n\alpha_n\eta}\left(\frac{\beta_n\alpha_n}{2(1-\eta)}\|x_n - z\|^2\right. \\
&\quad \left. + \frac{1}{1-\eta}\langle f(z) - z, x_{n+1} - z\rangle\right).
\end{aligned}$$

Applying the conditions (i),(ii),(3.11),(3.29) and Lemma 2.12, we have the sequence $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$. From (3.14), we obtain $\{u_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$. This completes the proof. □

Next, we give the numerical example to support Theorem 3.1.

Example 3.2. Let $\mathbb{R}$ be the set of real numbers and let the mapping $A : \mathbb{R} \to \mathbb{R}$ defined by $Ax = \frac{2x}{3}, \forall x \in \mathbb{R}$. For all $i = 1, 2, ..., N$, let the mapping $T_i : \mathbb{R} \to \mathbb{R}$ defined by

$T_i x = \frac{6i}{6i+1}x, \ \forall x \in \mathbb{R}$

and let $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$F_i(x, y) = i(-7x^2 + xy + 6y^2), \forall x, y \in \mathbb{R}.$$

Furthermore, let $a_i = \frac{6}{7^i} + \frac{1}{N7^N}$, for every $i = 1, 2, ..., N$. Then we have

$$\sum_{i=1}^{N} a_i F_i(x, y) = \sum_{i=1}^{N} \Big(\frac{6}{7^i} + \frac{1}{N7^N}\Big) i(-7x^2 + xy + 6y^2) = E(-7x^2 + xy + 6y^2),$$

where $E = \sum_{i=1}^{N} \Big(\frac{6}{7^i} + \frac{1}{N7^N}\Big) i$, it is easy to check that $\sum_{i=1}^{N} a_i F_i$ satisfies all the conditions of Theorem 3.1. By the definition of F_i, we have

$$\begin{aligned}
0 \leq & \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \\
= & E(-7x^2 + xy + 6y^2) + \frac{1}{r_n}(y - u_n)(u_n - x_n) \\
= & E(-7x^2 + xy + 6y^2) + \frac{1}{r_n}(yu_n - yx_n - u_n^2 - u_n x_n) \\
\Leftrightarrow & \\
0 \leq & Er_n(-7x^2 + xy + 6y^2) + (yu_n - yx_n - u_n^2 - u_n x_n) \\
= & 6Er_n y^2 + Eu_n r_n y - 7Eu_n^2 r_n + yu_n - yx_n - u_n^2 - u_n x_n \\
= & 6Er_n y^2 + (u_n - x_n + Eu_n r_n)y + (-7Eu_n^2 r_n - u_n^2 - u_n x_n).
\end{aligned}$$

Let $G(y) = 6Er_n y^2 + \big(u_n(1 + Er_n) - x_n\big)y - 7Eu_n^2 r_n - u_n^2 - u_n x_n$. $G(y)$ is a quadratic function of y with coefficient $a = 6Er_n$, $b = u_n(1 + Er_n) - x_n$ and $c = -7Eu_n^2 r_n - u_n^2 - u_n x_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned}
\Delta = & b^2 - 4ac \\
= & \big(u_n(1 + Er_n) - x_n\big)^2 - 4(6Er_n)(-7Eu_n^2 r_n - u_n^2 - u_n x_n) \\
= & u_n^2(1 + Er_n)^2 - 2u_n x_n(1 + Er_n) + x_n^2 + 168E^2 u_n^2 r_n^2 + 24Eu_n^2 r_n - 24Eu_n x_n r_n \\
= & u_n^2 + 2Eu_n^2 r_n + E^2 u_n^2 r_n^2 - 2u_n x_n - 2Eu_n x_n r_n + x_n^2 + 168E^2 u_n^2 r_n^2 + 24Eu_n^2 r_n \\
& - 24Eu_n x_n r_n \\
= & u_n^2 + 26Eu_n^2 r_n + 169E^2 u_n^2 r_n^2 - 2u_n x_n - 26Eu_n x_n r_n + x_n^2 \\
= & (u_n + 13Eu_n r_n)^2 - 2x_n(u_n + 13Eu_n r_n) + x_n^2 \\
= & (u_n + 13Eu_n r_n - x_n)^2.
\end{aligned}$$

We know that $G(y) \geq 0, \forall y \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so we obtain

$$u_n = \frac{x_n}{1 + 13\sum_{i=1}^{N} \Big(\frac{6}{7^i} + \frac{1}{N7^N}\Big) i r_n}, \; for \; all \; n \in \mathbb{N}. \tag{3.30}$$

For every $j = 1, 2, ..., N$, let $\alpha_1^j = \frac{1}{2j}$, $\alpha_2^j = \frac{3j-1}{16j}$, $\alpha_3^j = \frac{13j-7}{16j}$. Then $\alpha_j = \Big(\frac{1}{2j}, \frac{3j-1}{16j}, \frac{13j-7}{16j}\Big)$, for all $j = 1, 2, ..., N$. Let S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. From the definition T_i, A and F_i, we have

$$\{0\} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A).$$

Put $\alpha_n = \frac{1}{3n}$, $\beta_n = \frac{4n+2}{17n}$, $r_n = \frac{n}{2n+1}$, $f(x) = \frac{3x}{5}$ and $\lambda = 1$, $\forall n \in \mathbb{N}$. From (3.30) we rewrite (3.1) as follows:

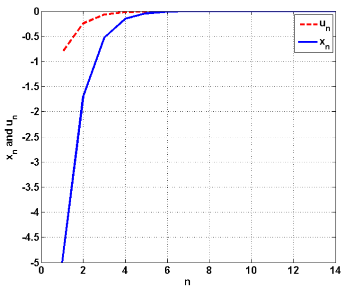
$$\begin{aligned} x_{n+1} =& \Big(\frac{4n+2}{17n}\Big)\Big(\frac{1}{3n}f(x_n) + \Big(1 - \frac{1}{3n}\Big)Sx_n\Big) \\ &+ \Big(1 - \frac{4n+2}{17n}\Big)(I - A)\frac{x_n}{1 + 13\sum_{i=1}^{N}\Big(\frac{6}{7^i} + \frac{1}{N7^N}\Big)ir_n}, \forall n \geq 1. \end{aligned} \tag{3.31}$$

It is clear that the sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy all the conditions of Theorem 3.1. From Theorem 3.1, we can conclude that the sequence $\{x_n\}$ and $\{u_n\}$ converges strongly to 0.

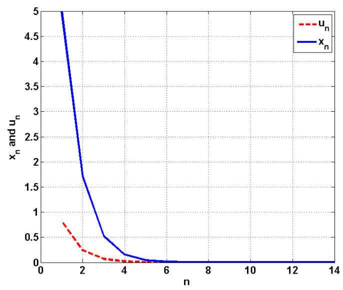
Table 1 shows that values of sequences $\{x_n\}$ and $\{u_n\}$, where $x_1 = -5$ and $x_1 = 5$ and $n = N = 14$.

Table 3.1: The values of $\{u_n\}$ and $\{x_n\}$ where $n = 14$.

	$x_1 = -5$		$x_1 = 5$	
n	u_n	x_n	u_n	x_n
1	-0.825688	-5.000000	0.825688	5.000000
2	-0.241546	-1.706927	0.241546	1.706927
3	-0.070026	-0.525198	0.070026	0.525198
4	-0.019977	-0.154636	0.019977	0.154636
5	-0.005630	-0.044447	0.005630	0.044447
⋮	⋮	⋮	⋮	⋮
8	-0.000120	-0.000980	0.000120	0.000980
⋮	⋮	⋮	⋮	⋮
11	-0.000002	-0.000020	0.000002	0.000020
12	-0.000001	-0.000006	0.000001	0.000006
13	-0.000000	-0.000002	0.000000	0.000002
14	-0.000000	-0.000000	0.000000	0.000000



(a) $x_1 = -5$



(b) $x_1 = 5$

Figure 3.1: The convergence comparison of the sequences $\{x_n\}$ and $\{u_n\}$ with different initial values x_1.

Next, we prove a strong convergence theorem for finding a common element of $EP(F) \cap \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A)$.

Corollary 3.3. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction with satisfy $(A1)-(A4)$, $T_i : C \to C$ be κ_i-strictly pseudo-contractive mapping, for all $i = 1, 2, ..., N$ and let $A : C \to H$ be α-inverse strongly monotone mapping with $\mathcal{F} = EP(F) \cap \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A) \neq \emptyset$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, I = [0, 1]$ with $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ and $\kappa < \alpha_1^j, \alpha_3^j < 1$, for all $i = 1, 2, ..., N-1, \kappa < \alpha_1^N \leq 1, \kappa \leq \alpha_3^N < 1, \kappa \leq \alpha_2^j < 1$, for all $j = 1, 2, ..., N$, where $\kappa = \max\{\kappa_i : i = 1, 2, ..., N\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n \big(\alpha_n f(x_n) + (1 - \alpha_n) S x_n\big) + (1 - \beta_n) P_C (I - \lambda A) u_n, \forall n \geq 1, \end{cases} \tag{3.32}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0, 1]$ and $\lambda \in (0, 2\alpha)$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0,$

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, for all $n \geq 1$,

(iii) $f : C \to C$ be η-contraction,

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.

Proof. First, we show that $(I - \lambda A)$ is a nonexpansive mapping. Let $x, y \in C$. Since A is α-inverse strongly monotone and $\lambda < 2\alpha$, we have

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\alpha\lambda \|Ax - Ay\|^2 + \lambda^2 \|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda(\lambda - 2\alpha) \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 . \end{aligned}$$

Then $(I - \lambda A)$ is a nonexpansive mapping. We will divide our proof into 5 steps.

Step 1. We show that the sequence $\{x_n\}$ is bounded. Since

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C.$$

Form Lemma 2.21, we have $u_n = T_{r_n} x_n$ and $EP(F) = F(T_{r_n})$.

Let $z \in F$. By nonexpansiveness of $(I - \lambda A)$ and T_{r_n}, we obtain

$$\begin{aligned}
\|x_{n+1} - z\| &= \big\|\beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) + (1-\beta_n)P_C(I-\lambda A)u_n - z\big\| \\
&= \big\|\beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n - z\big) + (1-\beta_n)\big(P_C(I-\lambda A)u_n - z\big)\big\| \\
&\leq \beta_n \big\|\alpha_n\big(f(x_n) - z\big) + (1-\alpha_n)(Sx_n - z)\big\| \\
&\quad + (1-\beta_n)\left\|P_C(I-\lambda A)u_n - z\right\| \\
&\leq \beta_n\big(\alpha_n \left\|f(x_n) - f(z)\right\| + \alpha_n \left\|f(z) - z\right\| + (1-\alpha_n)\left\|Sx_n - z\right\|\big) \\
&\quad + (1-\beta_n)\left\|P_C(I-\lambda A)u_n - z\right\| \\
&\leq \beta_n\big(\alpha_n\eta \left\|x_n - z\right\| + \alpha_n \left\|f(z) - z\right\| + (1-\alpha_n)\left\|x_n - z\right\|\big) \\
&\quad + (1-\beta_n)\left\|u_n - z\right\| \\
&= \beta_n\Big(\big(1 - \alpha_n(1-\eta)\big)\left\|x_n - z\right\| + \alpha_n \left\|f(z) - z\right\|\Big) + (1-\beta_n)\left\|x_n - z\right\| \\
&\leq \max\left\{\left\|x_1 - z\right\|, \frac{\left\|f(z) - z\right\|}{1-\eta}\right\}.
\end{aligned}$$

By induction we can prove that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. By definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \Big\|\Big(\beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) + (1-\beta_n)P_C(I-\lambda A)u_n\Big) \\
&\quad - \Big(\beta_{n-1}\big(\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})Sx_{n-1}\big) \\
&\quad + (1-\beta_{n-1})P_C(I-\lambda A)u_{n-1}\Big)\Big\| \\
&\leq \beta_n \big\|\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) - \big(\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})Sx_{n-1}\big)\big\| \\
&\quad + |\beta_n - \beta_{n-1}|\left\|\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})Sx_{n-1}\right\| \\
&\quad + (1-\beta_n)\left\|P_C(I-\lambda A)u_n - P_C(I-\lambda A)u_{n-1}\right\| \\
&\quad + |\beta_{n-1} - \beta_n|\left\|P_C(I-\lambda A)u_{n-1}\right\| \\
&\leq \beta_n\big(\alpha_n \left\|f(x_n) - f(x_{n-1})\right\| + |\alpha_n - \alpha_{n-1}|\left\|f(x_{n-1})\right\| \\
&\quad + (1-\alpha_n)\left\|Sx_n - Sx_{n-1}\right\| + |\alpha_{n-1} - \alpha_n|\left\|Sx_{n-1}\right\|\big) \\
&\quad + |\beta_n - \beta_{n-1}|\big(\alpha_{n-1}\left\|f(x_{n-1})\right\| + (1-\alpha_{n-1})\left\|Sx_{n-1}\right\|\big) \\
&\quad + (1-\beta_n)\left\|P_C(I-\lambda A)u_n - P_C(I-\lambda A)u_{n-1}\right\| \\
&\quad + |\beta_{n-1} - \beta_n|\left\|P_C(I-\lambda A)u_{n-1}\right\| \\
&\leq \beta_n\big(\alpha_n\eta\left\|x_n - x_{n-1}\right\| + |\alpha_n - \alpha_{n-1}|\left\|f(x_{n-1})\right\| \\
&\quad + (1-\alpha_n)\left\|x_n - x_{n-1}\right\| + |\alpha_{n-1} - \alpha_n|\left\|Sx_{n-1}\right\|\big) \\
&\quad + |\beta_n - \beta_{n-1}|\big(\alpha_{n-1}\left\|f(x_{n-1})\right\| + (1-\alpha_{n-1})\left\|Sx_{n-1}\right\|\big) \\
&\quad + (1-\beta_n)\left\|u_n - u_{n-1}\right\| + |\beta_{n-1} - \beta_n|\left\|P_C(I-\lambda A)u_{n-1}\right\|
\end{aligned}$$

$$
\begin{aligned}
&\le \beta_n\Big(|\alpha_n-\alpha_{n-1}|M+|\alpha_{n-1}-\alpha_n|M+\big(1-\alpha_n(1-\eta)\big)\left\|x_n-x_{n-1}\right\|\Big)\\
&\quad+|\beta_n-\beta_{n-1}|\big(\alpha_{n-1}M+(1-\alpha_{n-1})M\big)\\
&\quad+(1-\beta_n)\left\|u_n-u_{n-1}\right\|+|\beta_{n-1}-\beta_n|M\\
&=\beta_n\Big(2M|\alpha_n-\alpha_{n-1}|+\big(1-\alpha_n(1-\eta)\big)\left\|x_n-x_{n-1}\right\|\Big)\\
&\quad+2M|\beta_n-\beta_{n-1}|+(1-\beta_n)\left\|u_n-u_{n-1}\right\|,
\end{aligned}
\tag{3.33}
$$

where $M=\max_{n\in\mathbb{N}}\{\left\|f(x_n)\right\|,\left\|Sx_n\right\|,\left\|P_C(I-\lambda A)u_n\right\|\}$.
Since $u_n=T_{r_n}x_n$ and definition of T_{r_n}, we obtain

$$
F(T_{r_n}x_n,y)+\frac{1}{r_n}\left\langle y-T_{r_n}x_n,T_{r_n}x_n-x_n\right\rangle\ge 0,\forall y\in C \tag{3.34}
$$

and

$$
F(T_{r_{n+1}}x_{n+1},y)+\frac{1}{r_{n+1}}\left\langle y-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\right\rangle\ge 0,\forall y\in C. \tag{3.35}
$$

From (3.34) and (3.35). It follow that

$$
F(T_{r_n}x_n,T_{r_{n+1}}x_{n+1})+\frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n,T_{r_n}x_n-x_n\right\rangle\ge 0 \tag{3.36}
$$

and

$$
F(T_{r_{n+1}}x_{n+1},T_{r_n}x_n)+\frac{1}{r_{n+1}}\left\langle T_{r_n}x_n-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\right\rangle\ge 0. \tag{3.37}
$$

From (3.36),(3.37), we have

$$
\begin{aligned}
&\frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n,T_{r_n}x_n-x_n\right\rangle\\
&+\frac{1}{r_{n+1}}\left\langle T_{r_n}x_n-T_{r_{n+1}}x_{n+1},T_{r_{n+1}}x_{n+1}-x_{n+1}\right\rangle\ge 0.
\end{aligned}
$$

It follows that

$$
\left\langle T_{r_n}x_n-T_{r_{n+1}}x_{n+1},\frac{T_{r_{n+1}}x_{n+1}-x_{n+1}}{r_{n+1}}-\frac{T_{r_n}x_n-x_n}{r_n}\right\rangle\ge 0.
$$

Which implies that

$$
\left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n,T_{r_n}x_n-T_{r_{n+1}}x_{n+1}+T_{r_{n+1}}x_{n+1}-x_n-\frac{r_n}{r_{n+1}}(T_{r_{n+1}}x_{n+1}-x_{n+1})\right\rangle\ge 0. \tag{3.38}
$$

From (3.38), we obtain

$$
\begin{aligned}
\left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|^2&\le\left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n,T_{r_{n+1}}x_{n+1}-x_n-\frac{r_n}{r_{n+1}}(T_{r_{n+1}}x_{n+1}-x_{n+1})\right\rangle\\
&=\left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n,x_{n+1}-x_n+(1-\frac{r_n}{r_{n+1}})(T_{r_{n+1}}x_{n+1}-x_{n+1})\right\rangle\\
&\le\left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|\Big[\left\|x_{n+1}-x_n\right\|+|1-\frac{r_n}{r_{n+1}}|\\
&\quad\times\left\|T_{r_{n+1}}x_{n+1}-x_{n+1}\right\|\Big]\\
&=\left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|\Big[\left\|x_{n+1}-x_n\right\|+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\\
&\quad\times\left\|T_{r_{n+1}}x_{n+1}-x_{n+1}\right\|\Big]
\end{aligned}
$$

$$\leq \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \Big[\|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n| \times \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\| \Big],$$

which yields

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n| \, \|u_{n+1} - x_{n+1}\| . \tag{3.39}$$

From (3.39), we have

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{d}|r_n - r_{n-1}| \, \|u_n - x_n\| . \tag{3.40}$$

By substituting (3.40) into (3.33), we have

$$\begin{aligned}\|x_{n+1} - x_n\| \leq& \beta_n\Big(2M|\alpha_n - \alpha_{n-1}| + \big(1 - \alpha_n(1-\eta)\big) \|x_n - x_{n-1}\| \Big) \\ &+ 2M|\beta_n - \beta_{n-1}| + (1-\beta_n) \|u_n - u_{n-1}\| \\ \leq& \beta_n\Big(2M|\alpha_n - \alpha_{n-1}| + \big(1 - \alpha_n(1-\eta)\big) \|x_n - x_{n-1}\| \Big) \\ &+ 2M|\beta_n - \beta_{n-1}| + (1-\beta_n)\big(\|x_n - x_{n-1}\| + \frac{1}{d}|r_n - r_{n-1}| \, \|u_n - x_n\| \big) \\ =& \big(1 - \beta_n\alpha_n(1-\eta)\big) \|x_n - x_{n-1}\| + 2M|\alpha_n - \alpha_{n-1}| \\ &+ 2M|\beta_n - \beta_{n-1}| + (1-\beta_n)\frac{1}{d}|r_n - r_{n-1}| \, \|u_n - x_n\| .\end{aligned} \tag{3.41}$$

From (3.41), conditions (i),(iv) and lemma 2.12, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.42}$$

Step 3. We will show that $\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|P_C(I-\lambda A)u_n - x_n\| = \lim_{n\to\infty} \|Sx_n - x_n\| = 0$. Since T_{r_n} is a firmly nonexpansive mapping, then we obtain

$$\begin{aligned}\|z - T_{r_n}x_n\|^2 =& \|T_{r_n}z - T_{r_n}x_n\|^2 \\ \leq& \langle T_{r_n}z - T_{r_n}x_n, z - x_n\rangle \\ =& \frac{1}{2}\big(\|T_{r_n}x_n - z\|^2 + \|x_n - z\|^2 - \|T_{r_n}x_n - x_n\|^2 \big),\end{aligned}$$

which yields

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2 . \tag{3.43}$$

By nonexpansiveness of $P_C(I-\lambda A)$,(3.43) and definition of x_n, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 =& \left\|\beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n - z\big) + (1-\beta_n)\big(P_C(I-\lambda A)u_n - z\big)\right\|^2 \\ \leq& \beta_n \left\|\alpha_n\big(f(x_n) - z\big) + (1-\alpha_n)(Sx_n - z)\right\|^2 + (1-\beta_n) \\ &\times \|P_C(I-\lambda A)u_n - z\|^2 \\ \leq& \beta_n\alpha_n \|f(x_n) - z\|^2 + \beta_n(1-\alpha_n) \|x_n - z\|^2 + (1-\beta_n) \|u_n - z\|^2 \\ \leq& \beta_n\alpha_n \|f(x_n) - z\|^2 + \beta_n(1-\alpha_n) \|x_n - z\|^2 + (1-\beta_n) \\ &\times \big(\|x_n - z\|^2 - \|u_n - x_n\|^2 \big) \\ \leq& \beta_n\alpha_n \|f(x_n) - z\|^2 + \|x_n - z\|^2 - (1-\beta_n) \|u_n - x_n\|^2 ,\end{aligned}$$

which implies that

$$
\begin{aligned}
(1-\beta_n)\left\|u_n-x_n\right\|^2 \leq & \beta_n\alpha_n\left\|f(x_n)-z\right\|^2+\left\|x_n-z\right\|^2-\left\|x_{n+1}-z\right\|^2 \\
\leq & \beta_n\alpha_n\left\|f(x_n)-z\right\|^2+\big(\left\|x_n-z\right\|+\left\|x_{n+1}-z\right\|\big)\left\|x_{n+1}-x_n\right\|. \quad (3.44)
\end{aligned}
$$

By (3.42),(3.44), conditions (i) and (ii), we have

$$\lim_{n\to\infty}\left\|u_n-x_n\right\|=0. \quad (3.45)$$

Put $w_n=\alpha_n f(x_n)+(1-\alpha_n)Sx_n$. By definition of x_n, we have

$$
\begin{aligned}
\left\|x_{n+1}-z\right\|^2 = & \left\|\beta_n w_n+(1-\beta_n)P_C(I-\lambda A)u_n-z\right\|^2 \\
= & \big\|\beta_n(w_n-z)+(1-\beta_n)\big(P_C(I-\lambda A)u_n-z\big)\big\|^2 \\
\leq & \beta_n\left\|w_n-z\right\|^2+(1-\beta_n)\left\|P_C(I-\lambda A)u_n-z\right\|^2-\beta_n(1-\beta_n) \\
& \times\left\|w_n-P_C(I-\lambda A)u_n\right\|^2 \\
= & \beta_n\left\|\alpha_n f(x_n)+(1-\alpha_n)Sx_n-z\right\|^2+(1-\beta_n)\left\|P_C(I-\lambda A)u_n-z\right\|^2 \\
& -\beta_n(1-\beta_n)\left\|w_n-P_C(I-\lambda A)u_n\right\|^2 \\
\leq & \beta_n\big(\alpha_n\left\|f(x_n)-z\right\|^2+(1-\alpha_n)\left\|Sx_n-z\right\|^2\big)+(1-\beta_n)\left\|u_n-z\right\|^2 \\
& -\beta_n(1-\beta_n)\left\|w_n-P_C(I-\lambda A)u_n\right\|^2 \\
= & \beta_n\alpha_n\left\|f(x_n)-z\right\|^2+\beta_n(1-\alpha_n)\left\|x_n-z\right\|^2+(1-\beta_n)\left\|u_n-z\right\|^2 \\
& -\beta_n(1-\beta_n)\left\|w_n-P_C(I-\lambda A)u_n\right\|^2 \\
\leq & \beta_n\alpha_n\left\|f(x_n)-z\right\|^2+\beta_n(1-\alpha_n)\left\|x_n-z\right\|^2+(1-\beta_n)\left\|x_n-z\right\|^2 \\
& -\beta_n(1-\beta_n)\left\|w_n-P_C(I-\lambda A)u_n\right\|^2 \\
= & \beta_n\alpha_n\left\|f(x_n)-z\right\|^2+(1-\beta_n\alpha_n)\left\|x_n-z\right\|^2-\beta_n(1-\beta_n) \\
& \times\left\|w_n-P_C(I-\lambda A)u_n\right\|^2 \\
\leq & \beta_n\alpha_n\left\|f(x_n)-z\right\|^2+\left\|x_n-z\right\|^2-\beta_n(1-\beta_n) \\
& \times\left\|w_n-P_C(I-\lambda A)u_n\right\|^2.
\end{aligned}
$$

Which yields

$$
\begin{aligned}
\beta_n(1-\beta_n)\left\|w_n-P_C(I-\lambda A)u_n\right\|^2 \leq & \beta_n\alpha_n\left\|f(x_n)-z\right\|^2+\left\|x_n-z\right\|^2-\left\|x_{n+1}-z\right\|^2 \\
\leq & \beta_n\alpha_n\left\|f(x_n)-z\right\|^2+\big(\left\|x_n-z\right\|+\left\|x_{n+1}-z\right\|\big) \\
& \times\left\|x_{n+1}-x_n\right\|. \quad (3.46)
\end{aligned}
$$

By (3.42),(3.46), conditions (i) and (ii), we have

$$\lim_{n\to\infty}\left\|w_n-P_C(I-\lambda A)u_n\right\|=0. \quad (3.47)$$

By the definition of x_n, we obtain

$$
\begin{aligned}
x_{n+1}-P_C(I-\lambda A)u_n = & \beta_n w_n-\beta_n P_C(I-\lambda A)u_n \\
= & \beta_n\big(w_n-P_C(I-\lambda A)u_n\big). \quad (3.48)
\end{aligned}
$$

By (3.48), we have

$$\begin{aligned}\|x_n - P_C(I-\lambda A)x_n\| &= \|x_n - x_{n+1} + x_{n+1} - P_C(I-\lambda A)u_n + P_C(I-\lambda A)u_n \\ &\quad - P_C(I-\lambda A)x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - P_C(I-\lambda A)u_n\| \\ &\quad + \|P_C(I-\lambda A)u_n - P_C(I-\lambda A)x_n\| \\ &\le \|x_n - x_{n+1}\| + \beta_n \|w_n - P_C(I-\lambda A)u_n\| + \|u_n - x_n\|.\end{aligned}$$

Form (3.42),(3.45) and (3.47), we have

$$\lim_{n\to\infty} \|x_n - P_C(I-\lambda A)x_n\| = 0. \tag{3.49}$$

Since

$$\begin{aligned}\|x_n - P_C(I-\lambda A)u_n\| &= \|x_n - P_C(I-\lambda A)x_n + P_C(I-\lambda A)x_n - P_C(I-\lambda A)u_n\| \\ &\le \|x_n - P_C(I-\lambda A)x_n\| + \|P_C(I-\lambda A)x_n - P_C(I-\lambda A)u_n\| \\ &\le \|x_n - P_C(I-\lambda A)x_n\| + \|x_n - u_n\|.\end{aligned}$$

From (3.45) and (3.49), we have

$$\lim_{n\to\infty} \|x_n - P_C(I-\lambda A)u_n\| = 0. \tag{3.50}$$

By the definition of x_n, we obtain

$$\begin{aligned}x_{n+1} - x_n =& \beta_n\alpha_n\big(f(x_n) - x_n\big) + \beta_n(1-\alpha_n)(Sx_n - x_n) \\ &+ (1-\beta_n)\big(P_C(I-\lambda A)u_n - x_n\big).\end{aligned} \tag{3.51}$$

It follows that

$$\begin{aligned}\beta_n(1-\alpha_n)\|Sx_n - x_n\| \le& \beta_n\alpha_n \|f(x_n) - x_n\| \\ &+ (1-\beta_n)\|P_C(I-\lambda A)u_n - x_n\| + \|x_{n+1} - x_n\|.\end{aligned}$$

By (3.42),(3.50), conditions (i) and (ii), we have

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{3.52}$$

Step 4. We will show that $\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle \le 0$, where $z = P_{\mathcal{F}}f(z)$.
To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle = \lim_{k\to\infty} \langle f(z) - z, x_{n_k} - z\rangle. \tag{3.53}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k\to\infty$, where $\omega \in C$.
From (3.45), we obtain $u_{n_k} \rightharpoonup \omega$ as $k\to\infty$.

Assume that $\omega \notin VI(C,A)$. Since $VI(C,A) = F\big(P_C(I-\lambda A)\big)$, we have $\omega \neq P_C(I-\lambda A)\omega$. By nonexpansiveness of $P_C(I-\lambda A)$,(3.49) and Opial's condition, we have

$$\begin{aligned}\liminf_{k\to\infty}\|x_{n_k}-\omega\| &< \liminf_{k\to\infty}\|x_{n_k}-P_C(I-\lambda A)\omega\|\\ &= \liminf_{k\to\infty}\|x_{n_k}-P_C(I-\lambda A)x_{n_k}+P_C(I-\lambda A)x_{n_k}-P_C(I-\lambda A)\omega\|\\ &\leq \liminf_{k\to\infty}\|x_{n_k}-P_C(I-\lambda A)x_{n_k}\|\\ &\quad+\liminf_{k\to\infty}\|P_C(I-\lambda A)x_{n_k}-P_C(I-\lambda A)\omega\|\\ &\leq \liminf_{k\to\infty}\|x_{n_k}-\omega\|.\end{aligned}$$

This is a contradiction. Then we have

$$\omega \in VI(C,A). \tag{3.54}$$

Next, we will show that $\omega \in \bigcap_{i=1}^{N} F(T_i)$.

By Lemma 2.18, we have $F(S) = \bigcap_{i=1}^{N} F(T_i)$. Assume that $\omega \neq S\omega$. Using Opial's condition, (3.52), we obtain

$$\begin{aligned}\liminf_{k\to\infty}\|x_{n_k}-\omega\| &< \liminf_{k\to\infty}\|x_{n_k}-S\omega\|\\ &= \liminf_{k\to\infty}\|x_{n_k}-Sx_{n_k}+Sx_{n_k}-S\omega\|\\ &\leq \liminf_{k\to\infty}\|x_{n_k}-Sx_{n_k}\|+\liminf_{k\to\infty}\|Sx_{n_k}-S\omega\|\\ &\leq \liminf_{k\to\infty}\|x_{n_k}-\omega\|.\end{aligned}$$

This is a contradiction. Then we have

$$\omega \in F(S) = \bigcap_{i=1}^{N} F(T_i). \tag{3.55}$$

Next, we will show that $\omega \in EP(F)$.

Since

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, \forall y \in C,$$

By $(A2)$, we have

$$\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq F(y,u_n), \forall y \in C.$$

In particular, it follows that

$$\left\langle y-u_{n_k},\frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle \geq F(y,u_{n_k}), \forall y \in C.$$

Since $(1/r_{n_k})(u_{n_k}-x_{n_k}) \to 0$ and $u_{n_k} \rightharpoonup \omega$, from $(A4)$ we have

$$F(y,\omega) \leq \lim_{k\to\infty} F(y,u_{n_k}) \leq 0, \forall y \in C. \tag{3.56}$$

Replacing y with $y_t := ty+(1-t)\omega$, $t \in (0,1]$, we have $y_t \in C$. By using $(A1),(A4)$ and (3.56), we obtain

$$0 = F(y_t,y_t) \leq tF(y_t,y)+(1-t)F(y_t,\omega) \leq tF(y_t,y). \tag{3.57}$$

Hence, $F(ty+(1-t)\omega, y) \geq 0$, for all $t \in (0,1]$ and for all $y \in C$. Letting $t \to 0^+$ and using assumption $(A3)$, we can conclude that

$$F(\omega, y) \geq 0, y \in C. \tag{3.58}$$

Therefore, $\omega \in EP(F)$.
From (3.54),(3.55) and (3.58), we can deduce that $\omega \in \mathcal{F}$.
Since $x_{n_k} \rightharpoonup \omega \in \mathcal{F}$ and Lemma 2.10, we can conclude that

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(z)-z, x_n - z\rangle &= \lim_{k\to\infty} \langle f(z)-z, x_{n_k} - z\rangle \\
&= \langle f(z)-z, \omega - z\rangle \\
&\leq 0,
\end{aligned} \tag{3.59}$$

where $z = P_{\mathcal{F}} f(z)$.

Step 5. Finally, we will show that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.
By nonexpansive of S and $P_C(I-\lambda A)$, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \big\|\beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) + (1-\beta_n)P_C(I-\lambda A)u_n - z\big\|^2 \\
&= \big\|\beta_n\alpha_n\big(f(x_n)-z\big) + \beta_n(1-\alpha_n)(Sx_n - z) + (1-\beta_n)\big(P_C(I-\lambda A)u_n - z\big)\big\|^2 \\
&\leq \big\|\beta_n(1-\alpha_n)(Sx_n - z) + (1-\beta_n)\big(P_C(I-\lambda A)u_n - z\big)\big\|^2 \\
&\quad + 2\beta_n\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle \\
&\leq \big(\beta_n(1-\alpha_n)\|Sx_n - z\| + (1-\beta_n)\|P_C(I-\lambda A)u_n - z\|\big)^2 \\
&\quad + 2\beta_n\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle \\
&\leq \big((1-\beta_n\alpha_n)\|x_n - z\| + (1-\beta_n)\|x_n - z\|\big)^2 \\
&\quad + 2\beta_n\alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle + 2\beta_n\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\leq (1-\beta_n\alpha_n)^2\|x_n - z\|^2 + 2\beta_n\alpha_n\|f(x_n) - f(z)\|\,\|x_{n+1} - z\| \\
&\quad + 2\beta_n\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\leq (1-\beta_n\alpha_n)^2\|x_n - z\|^2 + 2\beta_n\alpha_n\eta\|x_n - z\|\,\|x_{n+1} - z\| \\
&\quad + 2\beta_n\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\leq (1-\beta_n\alpha_n)^2\|x_n - z\|^2 + \beta_n\alpha_n\eta\|x_n - z\|^2 + \beta_n\alpha_n\eta\|x_{n+1} - z\|^2 \\
&\quad + 2\beta_n\alpha_n\langle f(z) - z, x_{n+1} - z\rangle.
\end{aligned}$$

Which implies that

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq& \frac{1-\beta_n\alpha_n\eta-2\beta_n\alpha_n+\beta_n^2\alpha_n^2+2\beta_n\alpha_n\eta}{1-\beta_n\alpha_n\eta}\|x_n-z\|^2\\ &+\frac{2\beta_n\alpha_n}{1-\beta_n\alpha_n\eta}\langle f(z)-z, x_{n+1}-z\rangle\\ =&\left(1-\frac{2\beta_n\alpha_n(1-\eta)}{1-\beta_n\alpha_n\eta}\right)\|x_n-z\|^2+\frac{\beta_n^2\alpha_n^2}{1-\beta_n\alpha_n\eta}\|x_n-z\|^2\\ &+\frac{2\beta_n\alpha_n}{1-\beta_n\alpha_n\eta}\langle f(z)-z, x_{n+1}-z\rangle\\ =&\left(1-\frac{2\beta_n\alpha_n(1-\eta)}{1-\beta_n\alpha_n\eta}\right)\|x_n-z\|^2+\frac{2\beta_n\alpha_n(1-\eta)}{1-\beta_n\alpha_n\eta}\left(\frac{\beta_n\alpha_n}{2(1-\eta)}\|x_n-z\|^2\right.\\ &\left.+\frac{1}{1-\eta}\langle f(z)-z, x_{n+1}-z\rangle\right).\end{aligned}$$

Applying the conditions (i),(ii),(3.42),(3.59) and Lemma 2.12, we have the sequence $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}f(z)$. From (3.45), we obtain $\{u_n\}$ converges strongly to $z = P_{\mathcal{F}}f(z)$. This completes the proof. □

Next, we prove a strong convergence theorem for finding a common element of $\bigcap_{i=1}^{N} F(T_i) \cap VI(C, A)$.

Corollary 3.4. Let C be a nonempty closed convex subset of a real Hilbert space H. $T_i : C \to C$ be κ_i-strictly pseudo-contractive mapping, for all $i = 1, 2, ..., N$ and let $A : C \to H$ be α-inverse strongly monotone mapping with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A) \neq \emptyset$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, I = [0, 1]$ with $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ and $\kappa < \alpha_1^j, \alpha_3^j < 1$, for all $i = 1, 2, ..., N-1, \kappa < \alpha_1^N \leq 1, \kappa \leq \alpha_3^N < 1, \kappa \leq \alpha_2^j < 1$, for all $j = 1, 2, ..., N$, where $\kappa = max\{\kappa_i : i = 1, 2, ..., N\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$x_{n+1} = \beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) + (1-\beta_n)P_C(I-\lambda A)x_n, \forall n \geq 1, \tag{3.60}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0, 1]$ and $\lambda \in (0, 2\alpha)$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0,$

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, for all $n \geq 1$,

(iii) $f : C \to C$ be η-contraction,

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1}-\alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1}-\beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1}-r_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}f(z)$.

Proof. Put $F_i = 0, \forall i = 1, 2, ..., N$. Then we have $u_n = P_C x_n = x_n, \forall n \in \mathbb{N}$. Therefore the conclusion of Corollary 3.4 can be obtained by Theorem 3.1. □

Chapter 4

Applications

4.1 Constrained convex optimization problems

In this section, we apply our main theorem to prove strong convergence theorems involving optimization problems.

Let us recall the standard constrained convex optimization problem as follows:

$$\text{find } x^\star \in C \text{ such that } g(x^\star) = \min_{x\in C} g(x), \tag{4.1}$$

where $g : C \to \mathbb{R}$ is a convex, Fréchet differentiable function, C is closed-convex subset of H. The set of all solutions of (4.1) is denoted by Ω_g.
The following lemmas is important to prove Theorem 4.2.

Lemma 4.1. ([23]) (Optimality condition) A necessary condition of optimality for a point $x^\star \in C$ to be a solution of the minimization problem (4.1) is that $x^\star$ solves the variational inequality

$$\langle \nabla g(x^\star), x - x^\star\rangle \geq 0, \ \forall x \in C. \tag{4.2}$$

Equivalently, $x^\star \in C$ solves the fixed point equation

$$x^\star = P_C\big(x^\star - \lambda\nabla g(x^\star)\big),$$

for every constant $\lambda > 0$. if, in addition, g is convex, then the optimality condition (4.2) is also sufficient.

Next, we prove a strong convergence theorems involving optimization problems for finding a common element of $\bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap \Omega_g$ and apply theorem 4.2 for the numerical example.

Theorem 4.2. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $F_i : C\times C \to \mathbb{R}$ be bifunction with satisfy $(A1)-(A4)$, $g : C \to \mathbb{R}$ be a real value convex function with gradient ∇g is $\frac{1}{L}$-inverse strongly monotone and continuous function for all $L \geq 0$. Assume that $\mathcal{F} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap \Omega_g \neq \emptyset$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I\times I\times I, I = [0, 1]$ with $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ and $\kappa < \alpha_1^j, \alpha_3^j < 1$, for all $i = 1, 2, ..., N-1, \kappa < \alpha_1^N \leq 1, \kappa \leq \alpha_3^N < 1, \kappa \leq \alpha_2^j < 1$, for all $j = 1, 2, ..., N$, where $\kappa = \max\{\kappa_i : i = 1, 2, ..., N\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$\begin{cases} \displaystyle\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) + (1-\beta_n)P_C(I - \lambda\nabla g)u_n, \forall n \geq 1, \end{cases} \tag{4.3}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$ and $\lambda \in (0, \frac{2}{L})$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0,$

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, for all $n \geq 1$,

(iii) $f : C \to C$ be η-contraction,

(iv) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$, for all $i = 1, 2, ..., N$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$
$\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.

Proof. The conclusion of Theorem 4.2 can be obtained from Theorem 3.1 and Lemma 4.1. □

Next, we give the numerical example to support Theorem 4.2.

Example 4.3. In this example, we consider the same mappings and parameters as in Example 3.2 except the following mapping $g : \mathbb{R} \to \mathbb{R}$ be defined by $gx = 2x^2 + 1$. It is clear that

$$\{0\} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap \Omega_g.$$

Put $\lambda = \frac{1}{8}$. From (3.30), we rewrite (4.3) as follows:

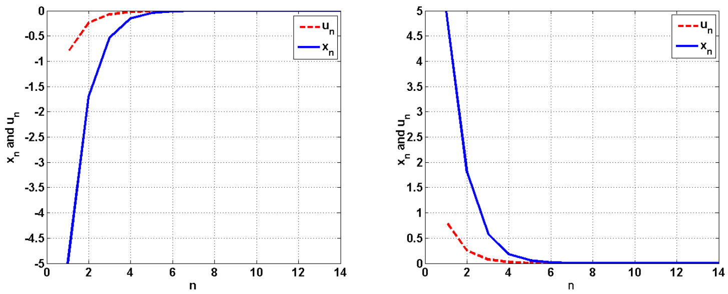
$$\begin{aligned} x_{n+1} =& \Big(\frac{4n+2}{17n}\Big)\Big(\frac{1}{3n} f(x_n) + \Big(1 - \frac{1}{3n}\Big) S x_n\Big) \\ &+ \Big(1 - \frac{4n+2}{17n}\Big)(I - \frac{1}{8}\nabla g) \frac{x_n}{1 + 13\sum_{i=1}^{N}\Big(\frac{6}{7^i} + \frac{1}{N7^N}\Big) i r_n}, \forall n \geq 1. \end{aligned} \tag{4.4}$$

It is clear that the sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy all the conditions of Theorem 4.2. From Theorem 4.2, we can conclude that the sequence $\{x_n\}$ and $\{u_n\}$ converges strongly to 0.

Table 2 shows that values of sequences $\{x_n\}$ and $\{u_n\}$, where $x_1 = -5$ and $x_1 = 5$ and $n = N = 14$.

Table 4.1: The values of $\{u_n\}$ and $\{x_n\}$ where $n = 14$.

	$x_1 = -5$		$x_1 = 5$	
n	u_n	x_n	u_n	x_n
1	-0.825688	-5.000000	0.825688	5.000000
2	-0.254147	-1.795972	0.254147	1.795972
3	-0.077666	-0.582496	0.077666	0.582496
4	-0.023370	-0.180898	0.023370	0.180898
5	-0.006949	-0.054859	0.006949	0.054859
⋮	⋮	⋮	⋮	⋮
8	-0.000175	-0.001422	0.000175	0.001422
⋮	⋮	⋮	⋮	⋮
11	-0.000004	-0.000035	0.000004	0.000035
12	-0.000001	-0.000010	0.000001	0.000010
13	-0.000000	-0.000003	0.000000	0.000003
14	-0.000000	-0.000001	0.000000	0.000001



(a) $x_1 = -5$ (b) $x_1 = 5$

Figure 4.1: The convergence comparison of the sequences $\{x_n\}$ and $\{u_n\}$ with different initial values x_1.

Remark 4.4.

(1) Table 3.1 and Figure 3.1 show that $\{x_n\}$ and $\{u_n\}$ converges to 0, where $\{0\} \in EP(F) \cap \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A)$. The convergence of $\{x_n\}$ and $\{u_n\}$ of Example 3.2 can be guaranteed by Theorem 3.1.

(2) Table 4.1 and Figure 4.1 show that $\{x_n\}$ and $\{u_n\}$ converges to 0, where $\{0\} \in EP(F) \cap \bigcap_{i=1}^{N} F(T_i) \cap \Omega_g$. The convergence of $\{x_n\}$ and $\{u_n\}$ of Example 4.3 can be guaranteed by Theorem 4.2.

(3) From these two Examples, we obtain that the sequence $\{x_n\}$ in Example 3.2 converges faster than the sequence $\{x_n\}$ in Example 4.3.

Chapter 5

Conclusions

In this chapter, we summarize all main theorems and applications obtained in this thesis.

According to the first objective of the study, we have proved strong convergence theorems by modifying the viscosity methods as follows.

(1) Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ be bifunction with satisfy $(A1) - (A4)$, $T_i : C \to C$ be κ_i-strictly pseudo-contractive mapping and let $A : C \to H$ be α-inverse strongly monotone mapping with $\mathcal{F} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A) \neq \emptyset$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, I = [0, 1]$ with $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ and $\kappa < \alpha_1^j, \alpha_3^j < 1$, for all $i = 1, 2, ..., N-1, \kappa < \alpha_1^N \leq 1, \kappa \leq \alpha_3^N < 1, \kappa \leq \alpha_2^j < 1$, for all $j = 1, 2, ..., N$, where $\kappa = max\{\kappa_i : i = 1, 2, ..., N\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n\big(\alpha_n f(x_n) + (1 - \alpha_n) S x_n\big) + (1 - \beta_n) P_C(I - \lambda A) u_n, \forall n \geq 1, \end{cases} \tag{5.1}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0, 1]$ and $\lambda \in (0, 2\alpha)$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, for all $n \geq 1$,

(iii) $f : C \to C$ be η-contraction,

(iv) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$, for all $i = 1, 2, ..., N$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction with satisfy $(A1) - (A4)$, $T_i : C \to C$ be κ_i-strictly pseudo-contractive mapping, for all $i = 1, 2, ..., N$ and let $A : C \to H$ be α-inverse strongly monotone mapping with $\mathcal{F} = EP(F) \cap \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A) \neq \emptyset$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, I = [0, 1]$ with $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ and $\kappa < \alpha_1^j, \alpha_3^j < 1$, for all $i = 1, 2, ..., N-1, \kappa < \alpha_1^N \leq 1, \kappa \leq \alpha_3^N < 1, \kappa \leq \alpha_2^j < 1$, for all $j = 1, 2, ..., N$, where $\kappa = \max\{\kappa_i : i = 1, 2, ..., N\}$.

Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) + (1-\beta_n)P_C(I - \lambda A)u_n, \forall n \geq 1, \end{cases} \tag{5.2}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$ and $\lambda \in (0, 2\alpha)$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, for all $n \geq 1$,

(iii) $f : C \to C$ be η-contraction,

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.

(3) Let C be a nonempty closed convex subset of a real Hilbert space H. $T_i : C \to C$ be κ_i-strictly pseudo-contractive mapping, for all $i = 1, 2, ..., N$ and let $A : C \to H$ be α-inverse strongly monotone mapping with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A) \neq \emptyset$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, I = [0,1]$ with $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ and $\kappa < \alpha_1^j, \alpha_3^j < 1$, for all $i = 1, 2, ..., N-1, \kappa < \alpha_1^N \leq 1, \kappa \leq \alpha_3^N < 1, \kappa \leq \alpha_2^j < 1$, for all $j = 1, 2, ..., N$, where $\kappa = max\{\kappa_i : i = 1, 2, ..., N\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$x_{n+1} = \beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) + (1-\beta_n)P_C(I - \lambda A)x_n, \forall n \geq 1, \tag{5.3}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$ and $\lambda \in (0, 2\alpha)$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, for all $n \geq 1$,

(iii) $f : C \to C$ be η-contraction,

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.

According to the second objective of the study, we have proved strong convergence theorems involving optimization problems as follows.

(4) Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ be bifunction with satisfy $(A1) - (A4)$, $g : C \to \mathbb{R}$ be a real value convex function with gradient ∇g is $\frac{1}{L}$-inverse strongly monotone and continuous function for all $L \geq 0$. Assume that $\mathcal{F} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap$

$\Omega_g \neq \emptyset$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, I = [0,1]$ with $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ and $\kappa < \alpha_1^j, \alpha_3^j < 1$, for all $i = 1, 2, ..., N-1, \kappa < \alpha_1^N \leq 1, \kappa \leq \alpha_3^N < 1, \kappa \leq \alpha_2^j < 1$, for all $j = 1, 2, ..., N$, where $\kappa = \max\{\kappa_i : i = 1, 2, ..., N\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$\begin{cases} \displaystyle\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n \big(\alpha_n f(x_n) + (1-\alpha_n) S x_n\big) + (1-\beta_n) P_C(I - \lambda \nabla g) u_n, \forall n \geq 1, \end{cases} \tag{5.4}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$ and $\lambda \in (0, \frac{2}{L})$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, for all $n \geq 1$,

(iii) $f : C \to C$ be η-contraction,

(iv) $\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$, for all $i = 1, 2, ..., N$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
$\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.

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