MATHEMATICAL ANALYSIS FOR CLASSICAL CHUA'S CIRCUIT WITH TWO NONLINEAR RESISTORS

NATCHAPHON LIMPHODAEN

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE

(APPLIED MATHEMATICS)

DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCE

KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

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Thesis Title	Mathematical Analysis for Classical Chua's Circuit with Two Nonlinear Resistors
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Abstract

This research provides a mathematical theory for classical Chua's circuit with two nonlinear resistors. We apply circuit analysis theory to formulate a mathematical model for the classical Chua's circuit in terms of a system of ordinary differential equations. Then, we find all equilibria of the above system and analyze the behavior of trajectories of system in a neighborhood of equilibria. Finally, numerical simulations for localization of a hidden attractor for the circuit are provided.

Keywords : circuit analysis, chaos theory, classical Chua's circuit, hidden attractor, nonlinear resistors.

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Natchaphon Limphodaen

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Chapter 1

Introduction

1.1 Research Motivation

Chaos system is a nonlinear dynamic system which has chaotic motion or random changing of waveform. It is sensitivity to initial condition and has the self-similarity property. Chaotic phenomenon has received much interest for a few decades. Such behavior has been successfully applied to signal transmission and cryptography [1-3]. Several types of oscillators have been studied and applied for generating chaos, e.g. Collpits, Wien bridge, Chua, Lorenz, etc.. Among those, Chua's circuit stands out, and always provides a reliable result. The original Chua's circuit has been modified to have only RCs and op-amps [4, 5], whose structures are compact and open for simple adjustments. More complicated dynamics of the Chua's circuit have been investigated. These include anti-monotonicity, and bubble formation [6, 7]. It is also possible to replace the piecewise linear characteristic of the Chua's diode with a smooth cubic function [8]. The chaotic circuit is also used in computer science, biology, communication, weather forecast, and other areas.

For example, turbulence in the tip vortex from an airplane wing in figure 1.1. Studies of the critical point beyond which a system creates turbulence were import for chaos theory.

Figure 1.1: turbulence in the tip vortex from an airplane wing for chaos theory

Chua's circuit [9, 10] is a simple electronic circuit that exhibits classic chaos theory behaviour [11-14]. This means, it is a "non-periodic oscillator".

Table 1.1: The detail of Chua's circuit

Map	Time Domain	Space Domain	Number of Space Dimensions	Number of parameters
Chua's circuit	continuous	real	3	3

It produces an oscillating waveform, which is not the same as a normal electronic oscillator. The main two famous circuit is Chua's and Jerk circuit [11], which use different methods of circuit connection to established chaos signal. Chua's circuit has a form in figure 1.2. The author expanded a classical Chua's circuit by connecting two nonlinear resistors to another form which is showed in figure 1.3 and full version in Figure 1.4.

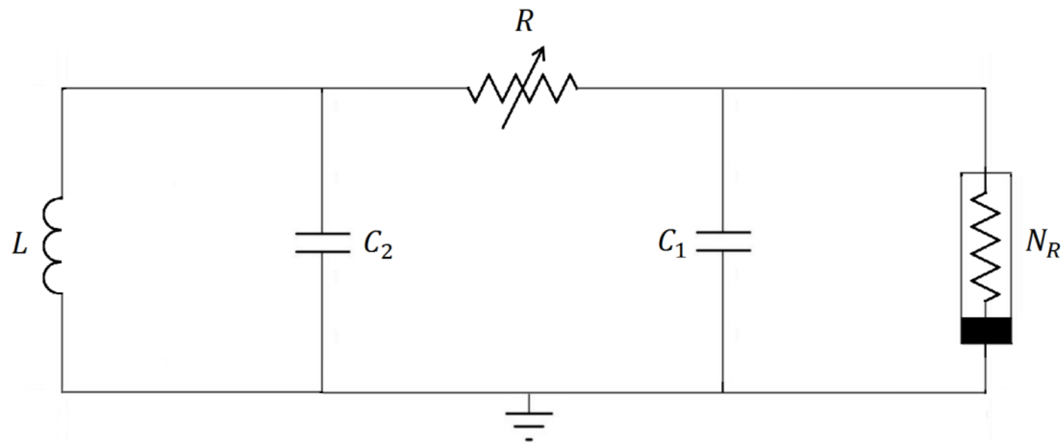

Figure 1.2: Classical Chua's Circuit

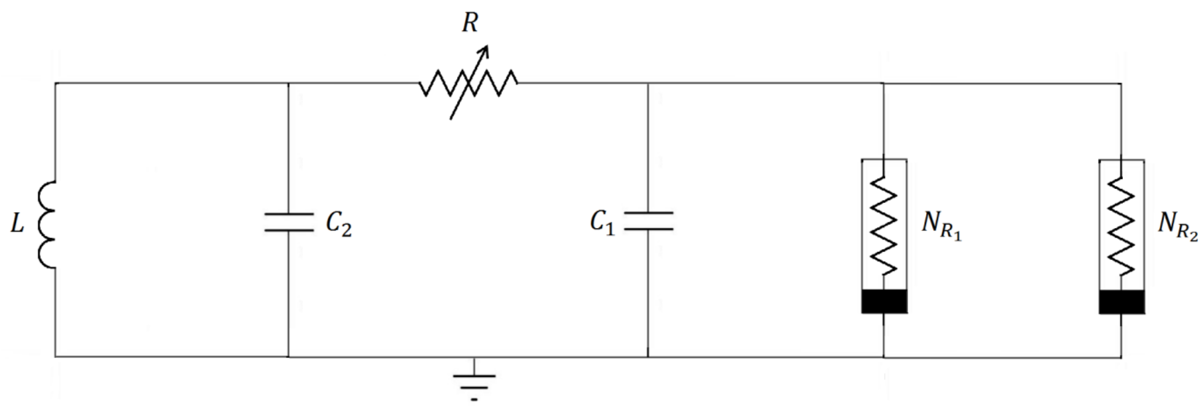

Figure 1.3: Classical Chua's Circuit with Two Nonlinear Resistor

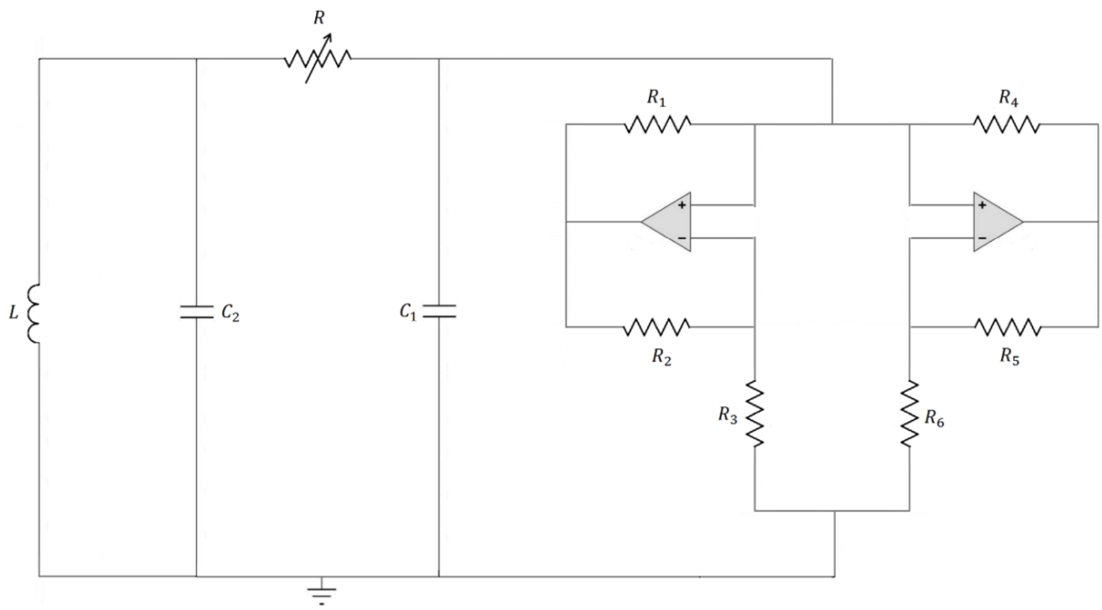

Figure 1.4: Fully Classical Chua's Circuit with Two Nonlinear Resistor

1.2 Objectives of the study

1) Investigate the chaos theory for classical Chua's circuit with nonlinear resistors.
2) Formulate of a classical Chua's circuit to a system of ordinary differential equations.
3) Analyze equilibria and eigenvalues from classical Chua's circuit with two nonlinear resistor.
4) Localize a hidden attractor in classical Chua's circuit with two nonlinear resistors.

1.3 Scopes of the study

1) Formulate a mathematical model for the classical Chua's circuit in terms of a system of ordinary differential equations, using circuit analysis theory.
2) Find all equilibria of the above system and analyze the behavior of trajectories of system in a neighborhood of equilibria.
3) Solve a polynomial of degree three equation (Cubic equation) to select types of cubic equation roots.
4) Make numerical simulations to localize a hidden attractor for the classical Chua's circuit.

1.4 Benefits of the study

To obtain a mathematical theory for analyzing the classical Chua's circuit.

1.5 Research methodology

1) Study advanced topics in Electrical Engineering
    - Electrical circuit analysis
    - Chua's circuits
    - Stability theory
    - Chaos theory
2) Study advanced topics in Mathematics

- Theory of ordinary differential equations, higher order differential equations, systems of ordinary differential equations
- Nonlinear dynamical systems
- Linear algebra
- Cubic equation

3) Formulate a mathematical model for the classical Chua's circuit in terms of a system of ordinary differential equations, using circuit analysis theory.
4) Find all equilibria of the above system and analyze the behavior of trajectories of system in a neighborhood of equilibria.
5) Solve a polynomial of degree three equation (cubic equation) to identify types of cubic equation roots.
6) Make numerical simulations to localize a hidden attractor for the classical Chua's circuit.

1.5.1 The research schedule

Table 1.2: The research schedule

Activity	Time frame							
	2017		2018				2019	
	Aug.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.	Jul.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.
Step 1	↔	↔						
Step 2		↔	↔					
Step 3				↔	↔			
Step 4						↔		
Step 5							↔	
Step 6								↔

Chapter 2

Preliminaries

2.1 Circuit Analysis

2.1.1 Ohm's law

The current through a conductor between two points is directly proportional to the voltage across the two points. Introducing the constant of proportionality, the resistance, one arrives at the usual mathematical equation that describes this relationship:

$$V = IR,$$

where I is the current through the conductor in units of amperes, V is the voltage measured across the conductor in units of volts, and R is the resistance of the conductor in units of ohms. See Figure 2.1.

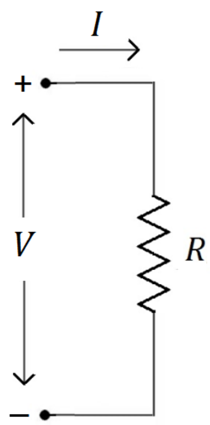

Figure 2.1: V, I and R parameters in ohm's law

2.1.2 Kirchhoff's current law

The principle of conservation of electric charge, combined with the very large repulsive Coulomb forces and at any node (junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node, this equation is

$$\sum_{k=1}^{n} I_k = 0,$$

where n is the total number of branches with currents flowing towards or away from the node.

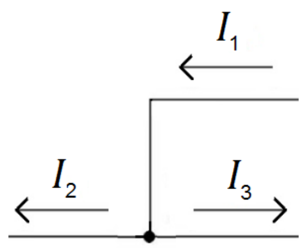

Figure 2.2: The sum of all currents at a node is equal to zero

$$(I_1 - I_2 - I_3 = 0)$$

2.1.3 Kirchhoff's voltage law

The directed sum of the electrical potential differences (voltage) around any closed network is zero. This equation is

$$\sum_{k=1}^{n} V_k = 0,$$

where n is the total number of voltages measured. See Figure 2.3.

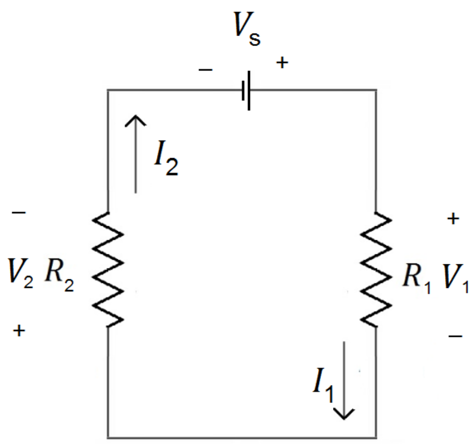

Figure 2.3: The sum of all voltages around a loop is equal to zero

$$(V_s - V_1 - V_2 = 0)$$

2.1.4 Current–voltage relation of the capacitor

The current through a capacitor between two points is directly proportional to the voltage across the two points. Introducing the constant of proportionality, the capacitance, one arrives at the usual mathematical equation that describes this relationship:

$$I = C\frac{dV}{dt},$$

where I is the current through the capacitor in units of amperes, V is the voltage measured across the capacitor in units of volts, and C is the capacitance of the capacitor in units of farads. See Figure 2.4.

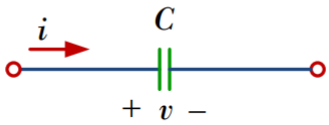

Figure 2.4: V, I and C parameters in Current–voltage relation of the capacitor

2.1.5 Current–voltage relation of the inductor

The current is allowed to pass through an inductor, the voltage across the inductor is directly proportional to the time rate of change of the current

$$V = L\frac{dI}{dt},$$

where I is the current through the inductor in units of amperes, V is the voltage measured across the inductor in units of volts, and L is the inductance of the inductor in units of henry. See Figure 2.5.

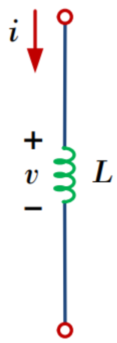

Figure 2.5: V, I and L parameters in Current–voltage relation of the inductor

2.1.6 Saturation in Op-amp

In the op-amp, the input using the voltage is supplied to the output of op-amp never goes above this voltage supply. Beside this the maximum output that we can obtain from an op-amp is about 85–90% of voltage supplied to op-amp. The maximum voltage reached is saturation voltage and this overall working is op-amp saturation. See Figure 2.6.

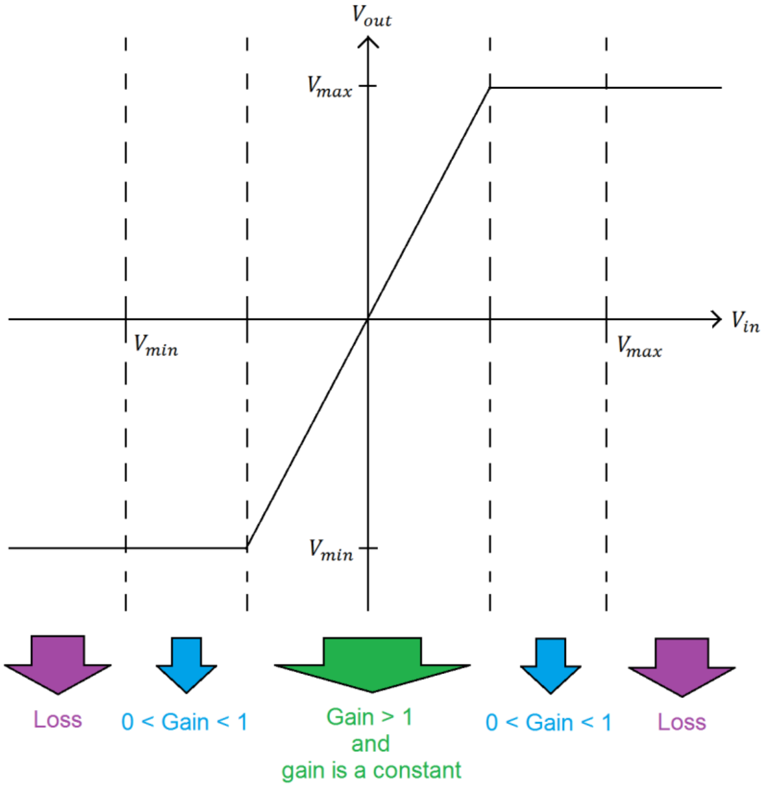

Figure 2.6: The saturation voltage in op-amp

2.1.7 Chaos circuit

Chaos circuit is a simple electronic circuit that exhibits classic chaotic behavior. This means roughly that it is a "non-periodic oscillator". It produces an oscillating waveform that, unlike an ordinary electronic oscillator, never "repeats". So, a simple nonlinear electrical circuit that can be used to study chaotic phenomena.

2.1.8 Hidden attractor

Hidden attractor has the small trajectory which is nearly the neighborhoods of any equilibrium points. In 2010, for the first time, a chaotic hidden attractor was discovered in Chua's circuit, which is described by a three-dimensional dynamical system.

For example of hidden attractor in Lorenz system

There is the Lorenz system

$$\dot{x} = 4(y - x),$$
$$\dot{y} = -xz + 29x - y,$$
$$\dot{z} = xy - 2z.$$

The initial point is $(5,5,5)$.

The equilibrium points of E_1 , E_2 and E_3 are

$$E_1 = (0,0,0),$$
$$E_2 = (7.4833, -7.4833, 28),$$
$$E_3 = (7.4833, 7.4833, 28).$$

The trajectory of hidden attractor is green line.

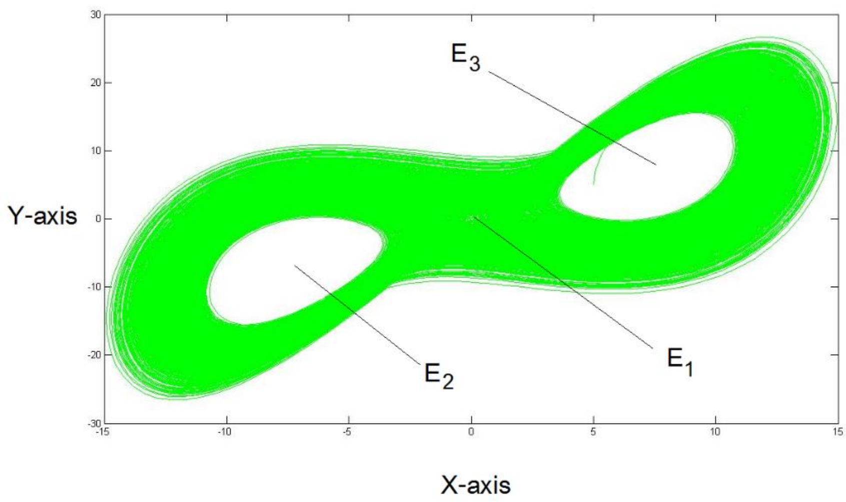

Figure 2.7: Hidden attractor in Lorenz system

2.2 Eigenvalue and Equilibrium Point

2.2.1 Eigenvalue

Let A be a matrix. If λ is a number such that the equation is

$$Ax = \lambda x.$$

It has a non-zero solution vector x, λ is called an eigenvalue of A**.** Eigenvalues are also called characteristic roots or characteristic value. The eigenvalue of a matrix are the roots of a certain polynomial associated with the matrix.

By definition, the equation is written

$$(A - \lambda I)x = 0,$$

where I is the identity matrix. Finding eigenvalues by using the determinate of above equation are

$$\det(A - \lambda I) = 0.$$

For example, find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Solution

$$\det(A - \lambda I) = 0,$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0,$$

$$(1-\lambda)^2 - 4 = 0,$$

$$\lambda^2 - 2\lambda - 3 = 0.$$

So, The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 1$.

2.2.2 Equilibrium point

It is a part of a dynamical system generated by an autonomous system of ordinary differential equations (ODEs) is a solution that does not change with time. Geometrically, equilibria are points in the system's phase space.

For example, $x + y - 5 = \dot{x}$ and $x - y = \dot{y}$

Solution given $\dot{x} = 0$ and $\dot{y} = 0$

Then $x + y = 5$ and $x - y = 0$

So, The equilibrium point is $(x, y) = (2.5, 2.5)$.

2.3 Cubic Equation

2.3.1 Third Polynomial Equation

In algebra, it is a function of the form

$$f(x) = ax^3 + bx^2 + cx + d,$$

where a is a non-zero. Setting $f(x) = 0$ produces its of the form

$$ax^3 + bx^2 + cx + d = 0,$$

where a, b, c and d are the coefficients of variable x. The solutions of this equation are called roots of the polynomial $f(x)$.

We shall find the solution of the cubic equation with $a = 1$

$$\lambda^3 + b\lambda^2 + c\lambda + d = 0.$$

The numbers of real and complex roots are determined by the discriminant of the cubic equation in reduced form:

$$\Delta = 18bcd - 4b^3d + b^2c^2 - 4c^3 - 27d^2,$$
$$\Delta_0 = b^2 - 3c,$$
$$\Delta_1 = 2b^3 - 9bc + 27d.$$

For Δ > 0, the equation has three distinct real roots. More precisely, substitutions $t - \frac{b}{3}$ into λ, we get $t^3 + pt + q = 0$, where $p = -\frac{\Delta_0}{3}$ and $q = \frac{\Delta_1}{27}$. The solution t will be in the form $t = u + v$, where

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} \quad \text{and} \quad v = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}.$$

Note that there three possible values of u, namely, if $u_1 = A$ is a third root then $u_2 = \omega A$ and $u_3 = \omega^2 A$ are also third roots, here $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Similarly, there are three positive values of v, namely, if $v_1 = B$ is a third root then $v_2 = \omega B$ and $v_3 = \omega^2 B$ are also third roots. However, the pair (u, v) must satisfies the condition $uv = -\frac{p}{3}$. Hence the solutions of the cubic equation with $a = 1$ are given by

$$\lambda_1 = A + B - \frac{b}{3},\ \lambda_2 = \omega A + \omega^2 B - \frac{b}{3},\ \lambda_3 = \omega^2 A + \omega B - \frac{b}{3}.$$

The solution of the cubic equation can be considered as follows. If Δ = 0, then the equation has a multiple root and all of its roots are real. There are two subcases:

- $\Delta_0 = 0$: it has a triple same root, $\lambda_1, \lambda_2, \lambda_3 = -\frac{b}{3}$.
- $\Delta_0 \neq 0$: it has a double same root, $\lambda_1, \lambda_2 = \frac{9d - bc}{\Delta^0}$ and a simple distinct root, $\lambda_3 = \frac{4bc - 9d - b^3}{\Delta^0}$.

For the case Δ < 0, the equation has one real root and two non-real complex conjugate roots.

The reduced equation of a C parameter that is easily eigenvalues analysis is

$$C = \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}.$$

So, in this case, we get

$$\lambda_1 = -\frac{1}{3}\left(b + C + \frac{\Delta_0}{C}\right), \lambda_2 = -\frac{1}{3}\left(b + \omega_2 C + \frac{\Delta_0}{\omega_2 C}\right), \lambda_3 = -\frac{1}{3}\left(b + \omega_3 C + \frac{\Delta_0}{\omega_3 C}\right).$$

Chapter 3

Mathematical Analysis for Classical Chua's Circuit with Two Nonlinear Resistors

3.1 Formulation of Classical Chua's Circuit to a System of ODEs

In this section, we formulate a mathematical model for the classical Chua's circuit (Figure 1.4) in terms of a system of nonlinear ordinary differential equations (ODEs).

We divide the circuit in Figure 1.4 into four parts as illustrated in Figures 3.1-3.4. Our analysis is based on fundamental theory of electrical circuit analysis such as Ohm's law, Kirchhoff's current law (KCL), Kirchhoff's voltage law (KVL).

There are three methods to formulate of classical Chua's circuit in Figure 1 to a system of ODEs.

Step I, divide classical Chua's circuit into each section

From Figure 1.4, it is calculated by using Kirchhoff's first law (KCL), Kirchhoff's second law (KVL) and Ohm's law can be expressed as

$$\sum_{k}^{n} I_k = 0,$$

$$\sum_{k}^{n} V_k = 0,$$

where n is the total number of branches with currents flowing towards or away from the node. And

$$V = IR,$$

consequently.

Step II, analyze in Figure 3.1.

For KVL

$$-V_L - V_{C_2} = 0, \tag{1}$$

Relationship between current and voltage of inductor is

$$V_L = L\frac{di_L}{dt}, \tag{2}$$

Substitute (2) into (1), we have

$$\frac{di_L}{dt} = -\frac{V_{C_2}}{L}, \tag{3}$$

There is 1^{st} ODE for Figure 3.1.

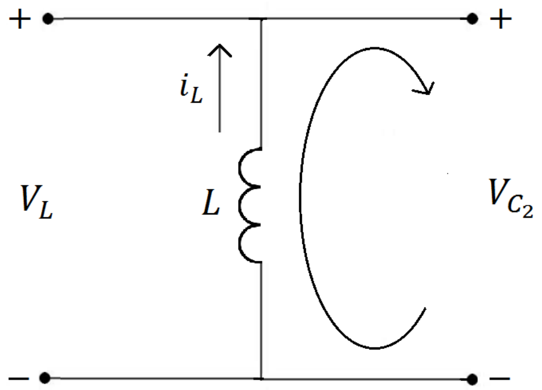

Figure 3.1: Chua's circuit part 1

For KCL at node a

$$i_L - i_{C_2} - i_R = 0, \tag{4}$$

For KVL

$$-V_{C_2} + V_R + V_{C_1} = 0, \tag{5}$$

Relationship between current and voltage of second capacitor is

$$i_{C_2} = C_2\frac{dV_{C_2}}{dt}, \tag{6}$$

There are the combine equations which are (4), (5) and (6)

$$\frac{dV_{C_2}}{dt} = \frac{V_{C_1} - V_{C_1}}{RC_2} + \frac{i_L}{C_2}, \tag{7}$$

There is 1^{st} ODE for Figure 3.2.

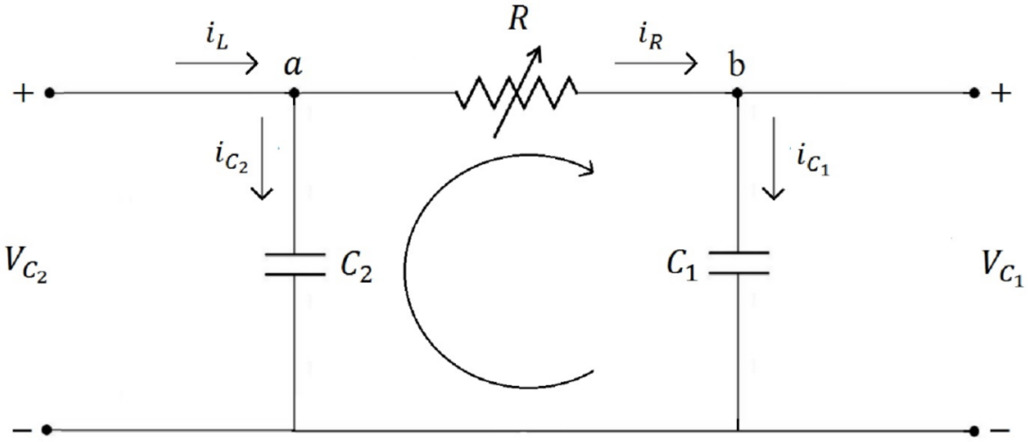

Figure 3.2: Chua's circuit part 2

For KCL at node b

$$i_R - i_{C_1} - i_{N_R} = 0, \tag{8}$$

For KVL

$$-V_{C_1} + V_{N_R} = 0, \tag{9}$$

Relationship between current and voltage of first capacitor is

$$i_{C_1} = C_1 \frac{dV_{C_1}}{dt}, \tag{10}$$

There are the combine equations which are (8), (9) and (10)

$$\frac{dV_{C_1}}{dt} = \frac{V_{C_2} - V_{C_1}}{RC_1} + \frac{i_{N_R}}{C_1}, \tag{11}$$

There is 1^{st} ODE for Figure 3.3.

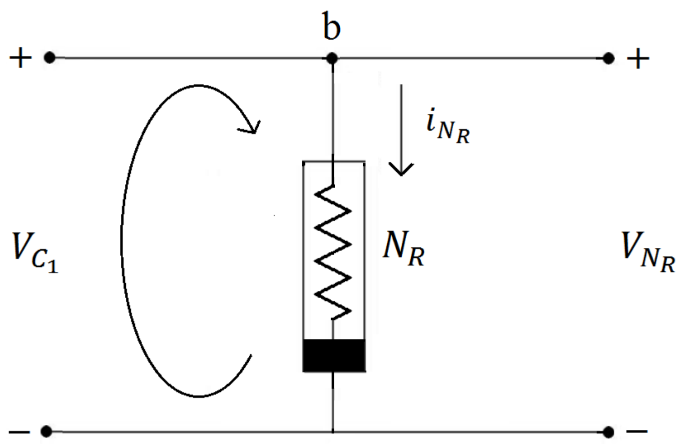

Figure 3.3: Chua's circuit part 3

It is straightforward to show that the circuit parts in Figures 3.1-3.3 can be described as (3), (7) and (11).

The circuit part in Figure 3.4 is a more complicated one since it consists of two nonlinear resistors.

Using KCL at node *c*, we have

$$i_{N_R} - i_x - i_y = 0, \tag{12}$$

For the nonlinear resistor on the left, using Ohm's law, we have

$$V_{N_R} = i_{R_3} R_3, \tag{13}$$

At the op amp output voltage (V_e) in left hand side is

$$V_e = (R_2 + R_3) i_{R_3}, \tag{14}$$

Then

$$V_{N_R} - V_e = i_x R_1, \tag{15}$$

Substitute (13) and (14) into (15)

$$i_x = -\frac{R_2}{R_1 R_3} V_{N_R}.$$

Combining these three equations to get

$$i_x = R_x V_{N_R},$$

where

$$R_x = -\frac{R_2}{R_1 R_3}.$$

For the nonlinear resistor on the right, using Ohm's law, we have

$$V_{N_R} = i_{R_6} R_6, \tag{16}$$

At the op amp output voltage (V_f) in right hand side is

$$V_f = (R_5 + R_6) i_{R_6}, \tag{17}$$

Then

$$V_{N_R} - V_f = i_y R_4, \tag{18}$$

Substitute (16) and (17) into (18)

$$i_y = -\frac{R_5}{R_4 R_6} V_{N_R},$$

Combining these three equations to get

$$i_y = R_y V_{N_R},$$

where

$$R_y = -\frac{R_5}{R_4 R_6},$$

Then the current i_{N_R} satisfies the relation

$$i_{N_R} = (R_x + R_y) V_{N_R},$$

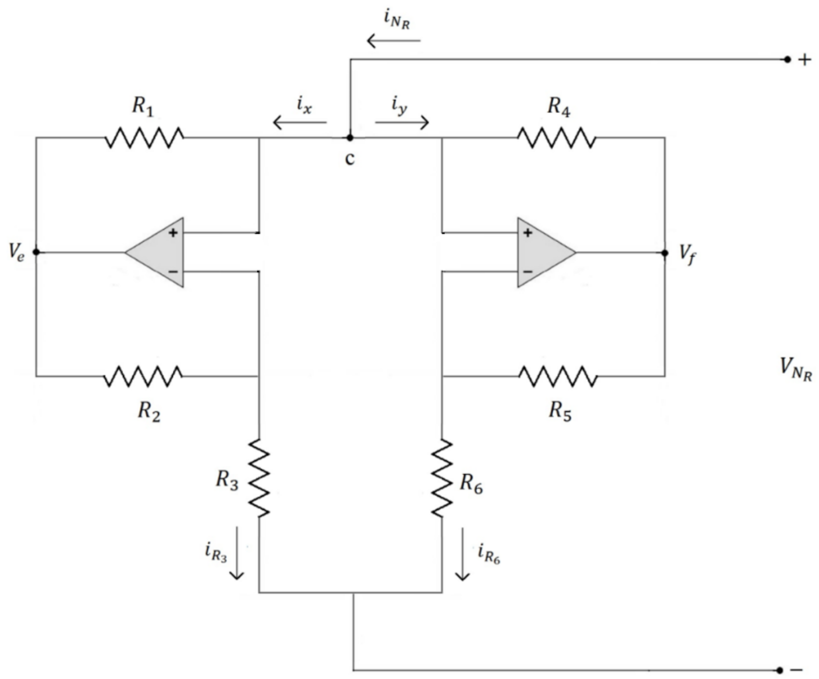

Figure 3.4: Chua's Circuit Part 4

However, as pointed out in [15], the behavior of i_{N_R} depends on the voltage V_{C_1}. Indeed, when $V_e < V_f$, the graph of i_{N_R} with respect to V_{C_1} is as follow

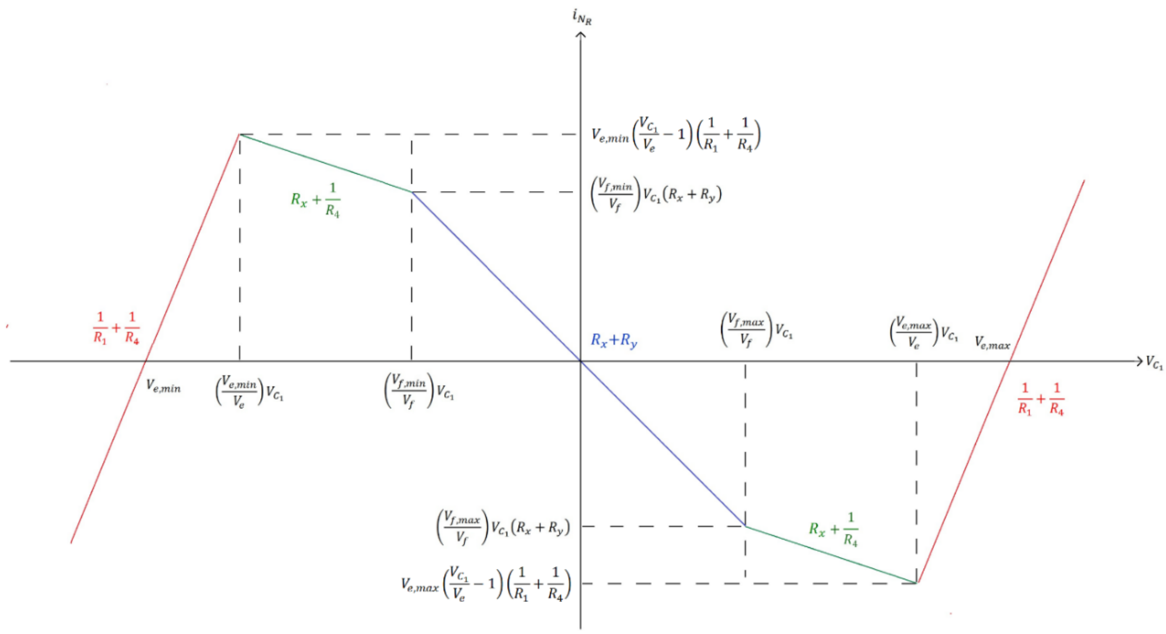

Figure 3.5: I-V Characteristic for nonlinear resistors ($V_e < V_f$)

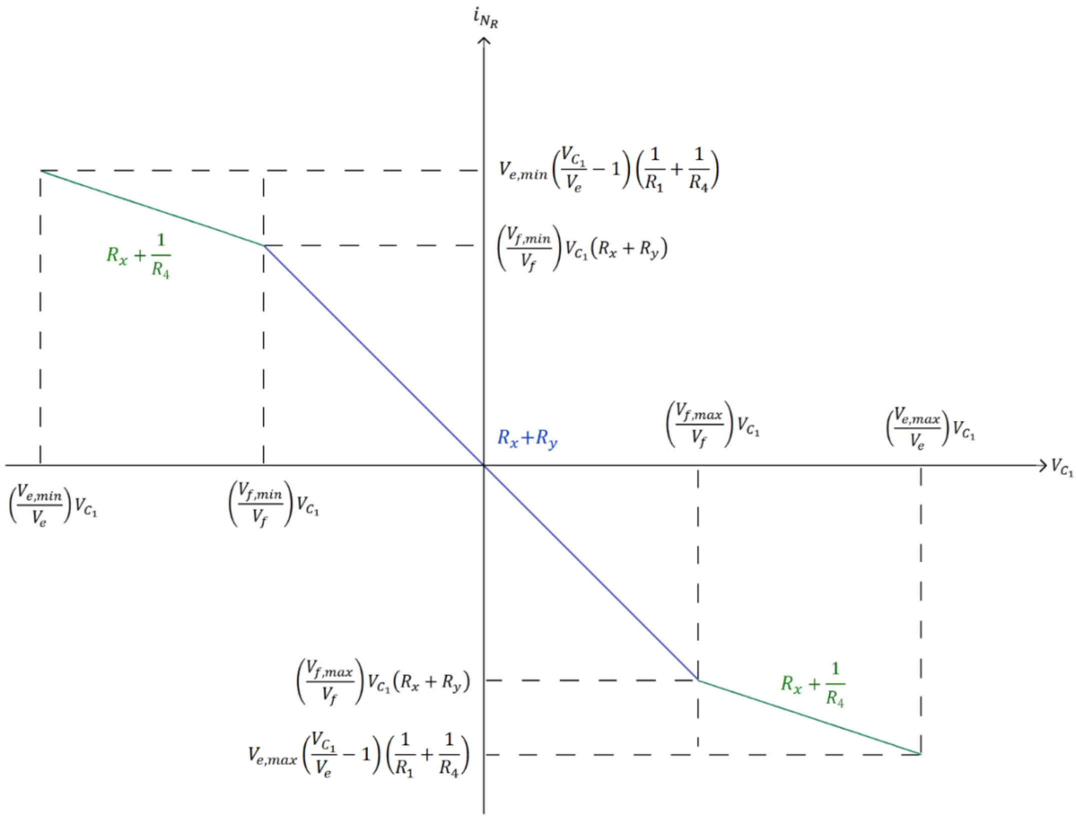

Figure 3.6: Reduced form of I-V Characteristic for nonlinear resistors ($V_e < V_f$)

From Figure 3.6,

$$i_{N_R} = \left(R_x + \frac{1}{R_4}\right)V_{C_1} + \frac{1}{2}\left(R_y - \frac{1}{R_4}\right)\left(\left|V_{C_1} + \frac{V_{f,max}}{V_f}V_{C_1}\right| - \left|V_{C_1} - \frac{V_{f,max}}{V_f}V_{C_1}\right|\right)$$

Introduce the following time-scale changing:

$$\tau = \frac{t}{RC_2}, x = \frac{V_f}{V_{f,max}}, y = \frac{V_f V_{C_2}}{V_{f,max}V_{C_1}}, z = \frac{V_f i_L R}{V_{f,max}V_{C_1}}.$$

Now, the equations $(3), (7)$ and (11) become the following systems of ODEs

$$\frac{dx}{d\tau} = \left(\frac{C_2}{C_1}\right)(-x + y - g(x)), \tag{19}$$

$$\frac{dy}{d\tau} = x - y + z, \tag{20}$$

$$\frac{dz}{d\tau} = -\left(\frac{R^2C_2}{L}\right)y, \tag{21}$$

where

$$g(x) = \left(R_x + \frac{1}{R_4}\right)x + \frac{1}{2}\left(R_y - \frac{1}{R_4}\right)(|x+1| - |x-1|) \tag{22}$$

We given $\zeta = \frac{C_2}{C_1}$, $\eta = \frac{R^2 C_2}{L}$, $m_0 = R_x + R_y$ and $m_1 = R_x + \frac{1}{R_4}$.

Finally, the following systems of ODEs from the classical Chua's circuit are

$$\dot{x} = \zeta\big(-x + y - g(x)\big), \tag{23}$$

$$\dot{y} = x - y + z, \tag{24}$$

$$\dot{z} = -\eta y, \tag{25}$$

where

$$g(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x+1| - |x-1|) \tag{26}$$

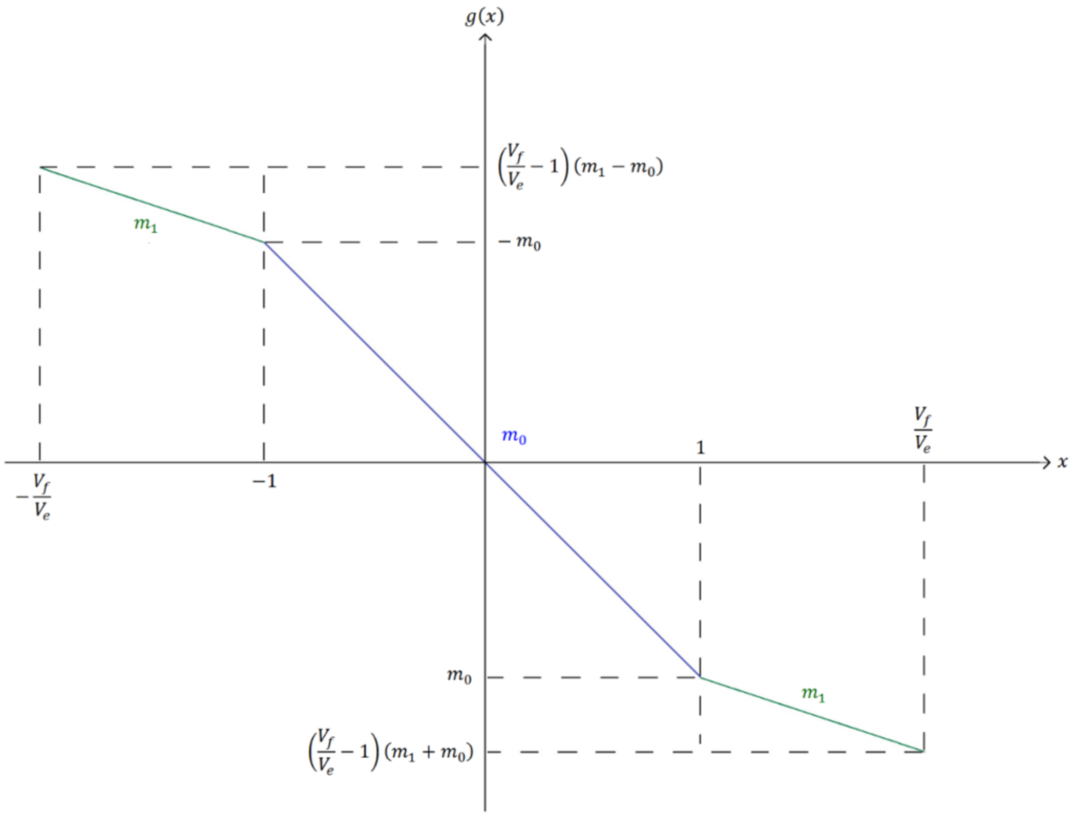

Figure 3.7: Changed Scale of I-V Characteristic for nonlinear resistors ($V_e < V_f$)

3.2 Equilibrium Points of the Classical Chua's System

From the characteristic of the classical Chua's system in Figure 3.6, we see that the behaviors of the current i_{N_R} depends on the voltage V_{C_1}, which is considered into three cases, namely, Case 1: $-1 \leq x \leq 1$, Case 2: $-\frac{V_f}{V_e} \leq x \leq -1$, Case 3: $1 \leq x \leq \frac{V_f}{V_e}$. Let E_i be the equilibrium point for Case i where $i = 1,2,3.$

For Case 1, we have

$$g(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(x + 1 - 1 - x),$$
$$= m_0 x.$$

It follows that the equations (23) to (25) become

$$\zeta(y - x) - \zeta m_0 x = 0, \tag{27}$$
$$x - y + z = 0, \tag{28}$$
$$-\eta y = 0, \tag{29}$$

Thus, the equilibrium for Case 1 is given by $(x_1, y_1, z_1) = (0,0,0)$.

For Case 2, we have

$$g(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(-x - 1 - 1 + x),$$
$$= m_1 x + m_0 - m_1,$$

Now, the equation (23) reduces to

$$\zeta(y - x) - \zeta(m_1 x - m_0 - m_1) = 0, \tag{30}$$

From the system of equations (28) to (30), we obtain the equilibrium point to be

$$(x_2, y_2, z_2) = (\tfrac{m_1 - m_0}{m_1 + 1}, 0, \tfrac{m_0 - m_1}{m_1 + 1}).$$

Finally for Case 3, we can see that

$$g(x) = m_1 x + m_1 - m_0,$$

and thus the equilibrium point is determined by

$$(x_3, y_3, z_3) = (\tfrac{m_1 - m_0}{m_1 + 1}, 0, \tfrac{m_0 - m_1}{m_1 + 1}).$$

3.3 Eigenvalues and Trajectories of the System

In this section, we find the eigenvalues for the classical Chua's system and analyze the behaviour of trajectories of the system in a neighborhood of each equilibrium point.

3.3.1 Finding eigenvalues

From the equations (23)-(26), we again consider three cases. For Case $1(-1 \leq x \leq 1)$, we obtain the linear system

$$\dot{X} = JX,$$

where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad J = \begin{bmatrix} -\zeta - \zeta m_0 & \zeta & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix}.$$

In order to get the solutions of the above system, we shall find the eigenvalues of the matrix J. Indeed, we have $\det(\lambda I_3 - J) = 0$ and thus the characteristic equation is given by

$$\lambda^3 + (\zeta + \zeta m_0 + 1)\lambda^2 + (\zeta m_0 + \eta)\lambda + (\zeta\eta + \zeta\eta m_0) = 0. \qquad (31)$$

For the second case $\left(-\frac{V_f}{V_e} \leq x \leq -1\right)$ and the third case $\left(1 \leq x \leq \frac{V_f}{V_e}\right)$, the coefficient matrix J is given by

$$J = \begin{bmatrix} -\zeta - \zeta m_1 & \zeta & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix}.$$

Similarly, the characteristic equation is

$$\lambda^3 + (\zeta + \zeta m_1 + 1)\lambda^2 + (\zeta m_1 + \eta)\lambda + (\zeta\eta + \zeta\eta m_1) = 0. \qquad (32)$$

Let us denode $b = \zeta + \zeta m + 1, c = \zeta m + \eta, d = \zeta\eta + \zeta\eta m,$ where $m = m_0$ or $m = m_1$.

3.3.2 Analysis for Trajectories of the System in a Neighborhood of Equilibrium

The classical Chua's system can be classified in terms of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ from the system $\dot{X} = JX$ as follows.

Case 1: $\lambda_1, \lambda_2, \lambda_3$ are negative real numbers. In this case, the equilibrium point is called a stable node. As the name suggested, the trajectories of $(x(t), y(t), z(t))$ will converge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 3.8.

Case 2: $\lambda_1, \lambda_2, \lambda_3$ are positive real numbers. In this case, the equilibrium point is called an unstable node. As the name suggested, the trajectories of $(x(t), y(t), z(t))$ will diverge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 3.9.

Case 3: λ_1 is a positive real number and λ_2, λ_3 are negative real numbers. In this case, the equilibrium point is called a saddle node. The trajectories lying on the x and y axes tend toward to equilibrium point, whereas the trajectories lying on the z axis tends away from equilibrium point. See Figure 3.10.

Case 4: λ_1 is a negative real number and λ_2, λ_3 are positive real numbers. In this case, the equilibrium point is called a saddle node. The trajectories of $\left(x(t), y(t)\right)$ will diverge but $z(t)$ will converge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 3.11.

Case 5: λ_1 is a negative real number, λ_2, λ_3 are complex numbers having negative real parts, and λ_2 is a conjugate of λ_3. In this case, the equilibrium point is called a stable focus node. As the name suggested, the trajectories of $\left(x(t), y(t)\right)$ will converge spiral form and $z(t)$ will converge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 3.12.

Case 6: λ_1 is a positive real numbers λ_2, λ_3 are complex numbers having positive real parts, and λ_2 is a conjugate of λ_3. In this case, the equilibrium point is called an unstable focus node. The trajectories of $\left(x(t), y(t)\right)$ will diverge spiral form and $z(t)$ will diverge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 3.13.

Case 7: λ_1 is a negative real numbers but λ_2, λ_3 have positive reals and (λ_2, λ_3) is a pair of complex-conjugate numbers. In this case, the equilibrium point is called a saddle focus node. As the name suggested, the trajectories of $\big(x(t), y(t)\big)$ will diverge spiral form but $z(t)$ will converge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 3.14.

Case 8: λ_1 is a positive real numbers but λ_2, λ_3 have negative reals and (λ_2, λ_3) is a pair of complex-conjugate numbers. In this case, the equilibrium point is called a saddle focus node. As the name suggested, the trajectories of $\big(x(t), y(t)\big)$ will converge spiral form but $z(t)$ will diverge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 3.15.

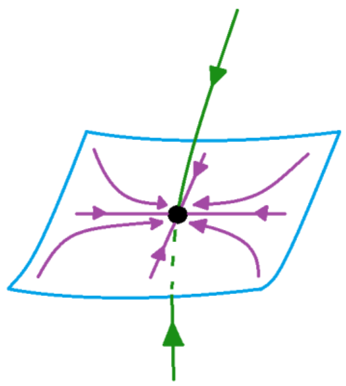

Figure 3.8: A trajectory of stable node

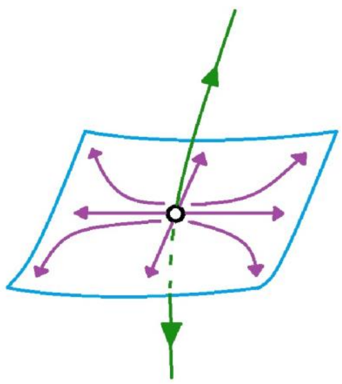

Figure 3.9: A trajectory of unstable node

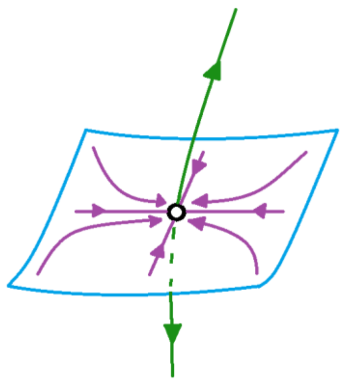

Figure 3.10: A trajectory of saddle node with $\lambda_1 > 0 > \lambda_{2,3}$

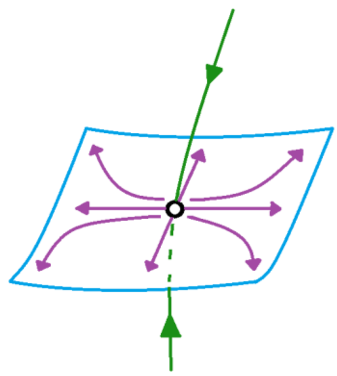

Figure 3.11: A trajectory of saddle node with $\lambda_{2,3} > 0 > \lambda_1$

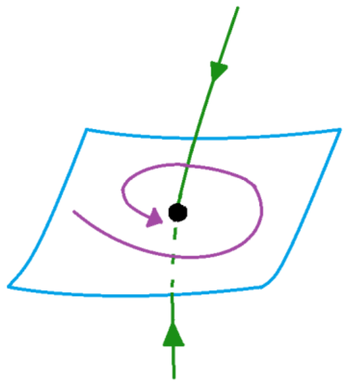

Figure 3.12: A trajectory of stable focus node

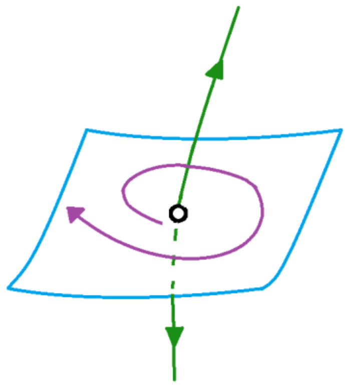

Figure 3.13: A trajectory of unstable focus node

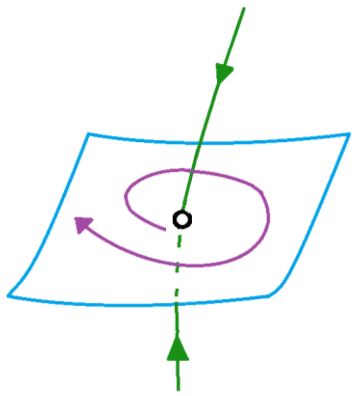

Figure 3.14: A trajectory of saddle focus node with $\mathrm{Re}(\lambda_2), \mathrm{Re}(\lambda_3) > 0 > \mathrm{Re}(\lambda_1)$

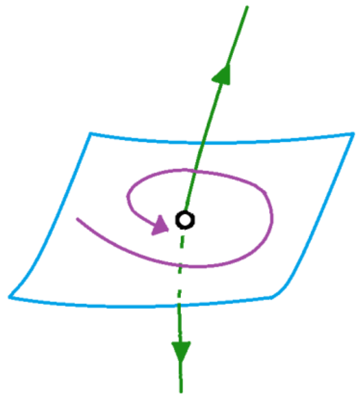

Figure 3.15: A trajectory of saddle focus node with $\mathrm{Re}(\lambda_1) > 0 > \mathrm{Re}(\lambda_2), \mathrm{Re}(\lambda_3)$

In summary, there are six types of equilibrium points, namely, stable node, unstable node, saddle node, stable focus node, unstable focus node, saddle focus node. The chaotic behaviour of the system occurs when the equilibrium point is a stable focus node (Figure 3.12) or a saddle focus node (Figures 3.14-3.15).

3.4 Reduction of the system

In this section, we show that our system can be reduced to a simpler system by introducing an invertible transformation.

From the equations $(23) - (26)$, we put them together in the following matrix form:

$$\frac{dM}{dt} = P_0 M + \mu Q(R^T M)\,, \tag{33}$$

where

$$P_0 = \begin{bmatrix} -\zeta(m_1+1+k) & \zeta & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix},\ \ Q = \begin{bmatrix} -\zeta \\ 0 \\ 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

From (33), let us introduce an invertible transformation matrix *S* such that $M = SY$. We shall find the explicit formula of *S* later. Multiplying S^{-1} to both sides of (33) yields

$$\frac{dY}{dt} = S^{-1}P_0 SY + \mu S^{-1}Q(R^T SY),$$

Let $A = S^{-1}P_0 S, B = S^{-1}Q, C^T = R^T S.$ Then we get

$$\frac{dY}{dt} = \mathrm{A}Y + \mu B(C^T Y), \tag{34}$$

where

$$A = \begin{bmatrix} 0 & -\alpha_0 & 0 \\ \alpha_0 & 0 & 0 \\ 0 & 0 & -\beta \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ c_3 \end{bmatrix},$$

Write *S* = [s_{ij}]. From the conditions $A = S^{-1}P_0 S, B = S^{-1}Q, C^T = R^T S,$ it is straightforward to deduce that

$$s_{11} = 1,$$

$$s_{12} = 0,$$

$$s_{13} = c_3,$$

$$s_{21} = m_1 + 1 + k,$$

$$s_{22} = -\frac{\alpha_0}{\zeta},$$

$$s_{23} = \frac{\beta}{\zeta} + c_3(m_1 + 1 + k),$$

$$s_{31} = m_1 + k - \frac{\alpha_0^2}{\zeta},$$

$$s_{32} = -\frac{\alpha_0}{\zeta} - \alpha_0(m_1 + 1 + k),$$

$$s_{33} = \frac{hd(1 + \zeta - d)}{\zeta} + c_3(m_1 + k)(1 - \beta).$$

Now, the transfer function of the system (33) is

$$W_{P_0}(p) = R^T(P_0 - pI)^{-1}Q, \tag{35}$$

and the transfer function of the system (34) is

$$W_A(p) = C^T(A - pI)^{-1}B. \tag{36}$$

We obtain that

$$(A - pI)^{-1} = \frac{1}{\det(A-pI)}\operatorname{adj}(A - pI)$$

$$= \frac{1}{(-p-\beta)(p^2+\alpha_0^2)}\begin{bmatrix} p^2 + \beta p & -\alpha_0 p - \alpha_0 \beta & 0 \\ \alpha_0 p + \alpha_0 \beta & p^2 + \beta p & 0 \\ 0 & 0 & p^2 + \alpha_0^2 \end{bmatrix}.$$

It follows that

$$W_A(p)$$

$$= [1 \quad 0 \quad c_3]\begin{bmatrix} \frac{p^2+\beta p}{(-p-\beta)(p^2+\alpha_0^2)} & \frac{-\alpha_0 p - \alpha_0\beta}{(-p-\beta)(p^2+\alpha_0^2)} & 0 \\ \frac{\alpha_0 p + \alpha_0\beta}{(-p-\beta)(p^2+\alpha_0^2)} & \frac{p^2+\beta p}{(-p-\beta)(p^2+\alpha_0^2)} & 0 \\ 0 & 0 & \frac{p^2+\alpha_0^2}{(-p-\beta)(p^2+\alpha_0^2)} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}$$

$$= \frac{b_1(p^2 + \beta p) + b_2(-\alpha_0 p - \alpha_0\beta) + c_3(p^2 + \alpha_0^2)}{(-p-\beta)(p^2+\alpha_0^2)}$$

$$= \frac{(b_1 + c_3)p^2 + (b_1\beta - b_2\alpha_0)p + (c_3\alpha_0^2 - b_2\alpha_0\beta)}{-p^3 - \beta p^2 - \alpha_0^2 p + \alpha_0^2\beta}, \tag{37}$$

Similarly,

$$W_{P_0}(p)$$

$$= \frac{(-\zeta)p^2 + (-\zeta)p}{-p^3 + (-\zeta m_1 - \zeta - \zeta k - 1)p^2 + (-\zeta m_1 - \zeta - \zeta k - \eta)p + (-\zeta m_1\eta - \zeta\eta - \zeta k\eta)}. \tag{38}$$

Since the system (33) and the system (34) are the same system, we have $W_A(p) = W_{P_0}(p)$. Now, we compare the numerator coefficients and the denominator coefficients of $W_A(p)$ and $W_{P_0}(p)$. We obtain that

$$c_3 = -\zeta - b_1,\ b_1\beta - b_2\alpha_0 = -\zeta, -c_3\alpha_0^2 + b_2\alpha_0\beta = \zeta\eta$$

$$\beta = \zeta m_1 + \zeta + \zeta k + 1, b_1\beta - b_2\alpha_0 = -\zeta, -c_3\alpha_0^2 + b_2\alpha_0\beta = \zeta\eta$$

These imply the following relations

$$\left.\begin{array}{c} k = \dfrac{\alpha_0^2 - \eta}{\zeta} - m_1, \qquad \beta = \alpha_0^2 + \zeta - \eta + 1, \qquad b_1 = \dfrac{\zeta(\beta - \alpha_0^2 - \eta)}{\alpha_0^2 + \beta^2}, \\ b_2 = \zeta\dfrac{\eta - \beta + \beta^2}{\alpha_0(\alpha_0^2 + \beta^2)}, \quad c_3 = \dfrac{-\zeta(\eta - \beta + \beta^2)}{\alpha_0^2 + \beta^2}. \end{array}\right\} \tag{39}$$

3.5 Localization of a Hidden Attractor for Classical Chua's Circuit

In this section, we will discuss the oscillation behaviour in the classical Chua's circuit with two nonlinear resistors, focused on hidden attractors.

In order to find a hidden attractor of the system, we will find a suitable initial point $(x(0), y(0), z(0))$ so that our system will have a chaos. The initial point depends on a parameter a_0, which is the solution of the equation $\Phi(a) = 0$ where Φ is the describing function [16] defined by

$$\Phi(a) = \int_0^{2\pi/\omega_0} \varphi(a\cos(\omega_0 t))\cos(\omega_0 t)\mathrm{dt}.$$

Here, $\varphi(x) = g(x) - kx$, where k is a coefficient of harmonic linearization, $x = \omega_0 t$ and

$$g(\omega_0 t) = \frac{1}{2}R(m_0 - m_1)(|\omega_0 t + 1| - |\omega_0 t - 1|).$$

Then, $g(\omega_0 t)$ is equal to $(m_0 - m_1)\omega_0 t$ for all $x \in [0, 1]$ and it equals the constant $m_0 - m_1$ when $x \in (1, 2\pi]$. It follows that

$$\Phi(a) = \int_0^{\tau} a(m_0 - m_1)\cos^2 t\mathrm{dt} + \int_{\tau}^{2\tau} (m_0 - m_1)\cos t\mathrm{dt} - \int_0^{2\pi} ak\cos^2 t\mathrm{dt},$$

where $\tau = \arccos\left(\frac{1}{a}\right)$. Thus, we have

$$\Phi(a_0) = (m_0 - m_1)\left[a_0 \int_0^{\arccos\left(\frac{1}{a_0}\right)} \cos^2 t \, \mathrm{d}t + \int_{\arccos\left(\frac{1}{a_0}\right)}^{2\arccos\left(\frac{1}{a_0}\right)} \cos t \mathrm{d}t\right] - a_0 k \int_0^{2\pi} \cos^2 t \mathrm{d}t,$$

$$= (m_0 - m_1)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + \frac{(1-2a_0)}{2a_0}\sin\left(\arccos\left(\frac{1}{a_0}\right)\right)\right] - a_0 k\pi,$$

$$= (m_0 - m_1)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + (1-2a_0)\frac{\sqrt{1+a_0^2}}{2a_0^2}\right] - a_0 k\pi. \quad (40)$$

From [16], the first step of multistage localization for our Chua's system is

$$M(0) = SY(0) = S\begin{bmatrix} a_0 + O(\mu) \\ 0 \\ \mathbf{O_{n-2}}(\mu) \end{bmatrix},$$

where $\mathbf{O}_{n-2}(\mu)$ is an *n*-2 dimensional vector such that all its coordinates are $O(\mu)$. We can approximate $O(\mu) \approx 0$ and $\mathbf{O}_{n-2}(\mu) \approx \mathbf{0}$. Thus,

$$\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = S\begin{bmatrix} a_0 \\ 0 \\ 0 \end{bmatrix},$$

which implies that

$$x(0) = a_0 s_{11} = a_0,$$
$$y(0) = a_0 s_{21} = a_0(m_1 + 1 + k),$$
$$z(0) = a_0 s_{31} = a_0\left(m_1 + k - \frac{\alpha_0^2}{\zeta}\right).$$

Chapter 4

Numerical Simulations

Example 4.1

Consider the classical Chua's circuit with two nonlinear resistors when the following parameters are given:

Table 4.1: parameters of classical Chua's circuit for Example 4.1

Parameters	Values	Units	Parameters	Values	Units
R	1000	Ω	R_6	400	Ω
R_1	250	Ω	C_1	10	μF
R_2	250	Ω	C_2	80	μF
R_3	500	Ω	L	70	mH
R_4	750	Ω	$V_{max,Op\ Amp}$	5	V
R_5	180	Ω			

We have the following parameters

$$\zeta = 8.4562, \qquad \eta = 12.0732, \qquad m_0 = -0.1768, \qquad m_1 = -1.1468.$$

From (39), we have $k = 0.2098.$ To find a suitable indicial point of the system, we have to solve the equation $\Phi(a_0) = 0,$ where Φ is given by the equation (39). Indeed, we have

$$0 = (m_0 - m_1)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + (1 - 2a_0)\frac{\sqrt{1 + a_0^2}}{2a_0^2}\right] - a_0 k\pi$$

$$= (0.97)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + (1 - 2a_0)\frac{\sqrt{1 + a_0^2}}{2a_0^2}\right] - a_0 0.2098\pi$$

An approximated solution a_0 via MATLAB is given by $a_0 = 9.4287.$ It follows from the previous discussion that an initial point is given by $(0) = 2.0392,\ y(0) = 0.5945,$ $z(0) = -13.4705$. A numerical simulation for the equations (23)-(25) with the initial point via MATLAB is illustrated in the following figures.

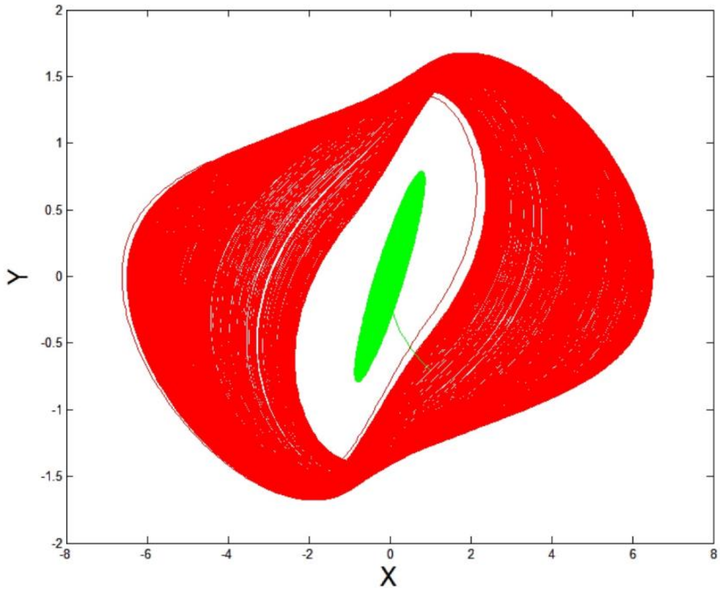

Figure 4.1: Attractors of the classical Chua's equations in (x-y axis) two dimensions for Example 4.1

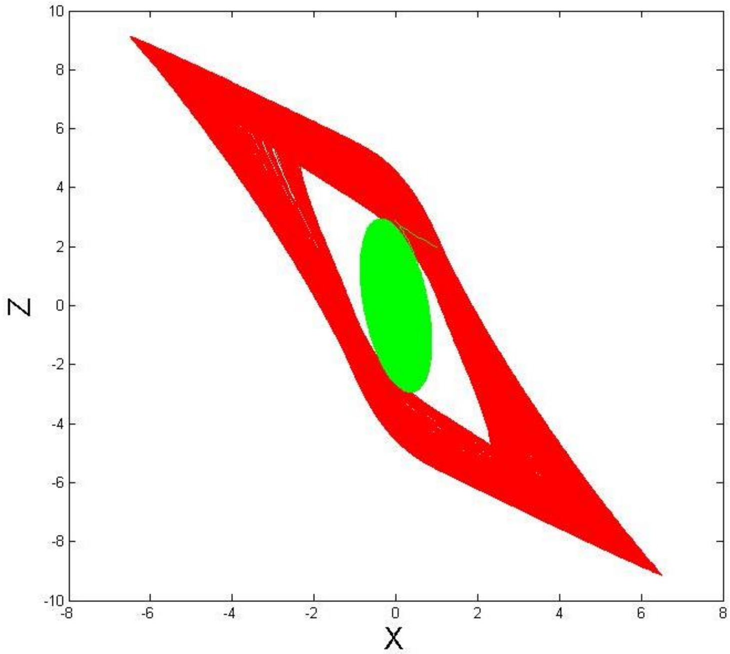

Figure 4.2: Attractors of the classical Chua's equations in (x-z axis) two dimensions for Example 4.1

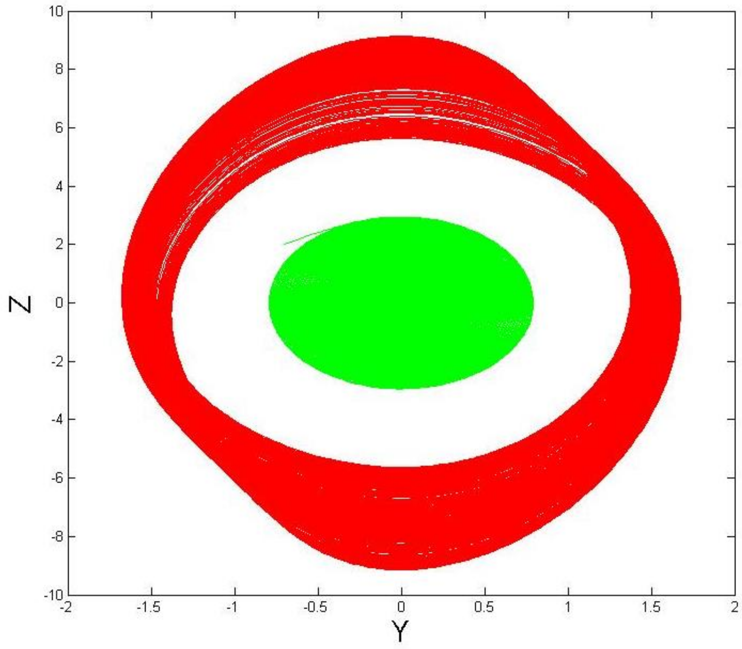

Figure 4.3: Attractors of the classical Chua's equations in (y-z axis) two dimensions for Example 4.1

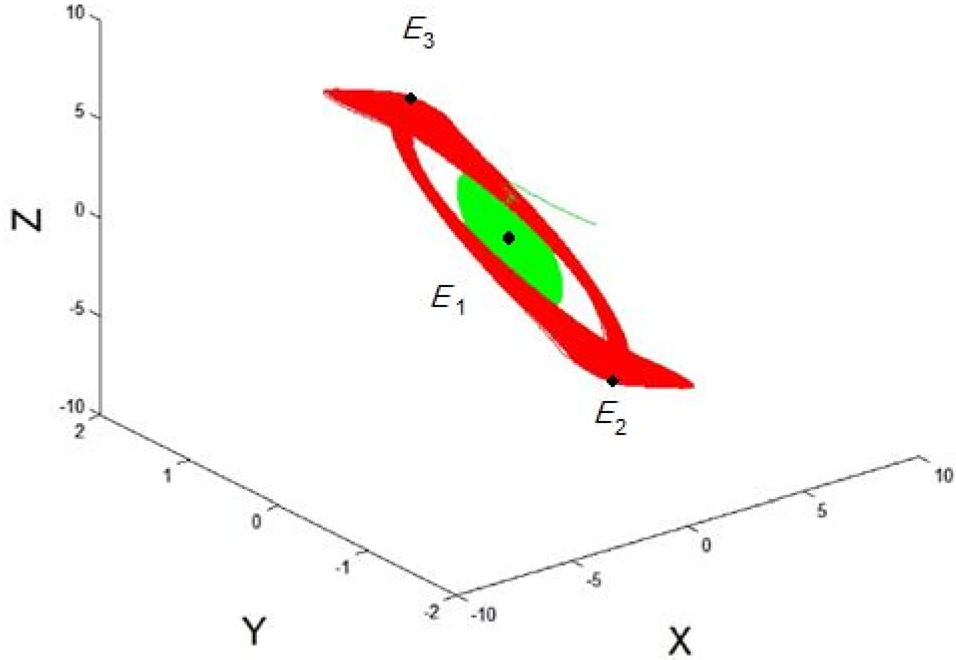

Figure 4.4: Attractors of the classical Chua's equations in three dimensions for Example 4.1

From the classical Chua's circuit, we formulate the following system

$$\frac{dx}{d\tau} = 8.4562\big(-x + y - g(x)\big),$$
$$\frac{dy}{d\tau} = x - y + z,$$
$$\frac{dz}{d\tau} = 12.0732y,$$

where $g(x) = -1.1468x + 0.485(|x+1| - |x-1|)$. This system has three equilibrium points, namely,

$$E_1 = (0,0,0),$$
$$E_2 = \left(\frac{m_1 - m_0}{m_1 + 1}, 0, \frac{m_0 - m_1}{m_1 + 1}\right) = (6.6076, 0, \ -6.6076),$$
$$E_3 = \left(\frac{m_0 - m_1}{m_1 + 1}, 0, \frac{m_1 - m_0}{m_1 + 1}\right) = (-6.6076, 0, 6.6076).$$

The eigenvalues of the system corresponding to each equilibrium point are given by

$$E_1: \lambda_1 = -7.9587,\ \lambda_2 = -0.0038 + 3.2494i,\ \lambda_3 = -0.0038 - 3.2494i,$$
$$E_2: \lambda_1 = 2.2193,\ \lambda_2 = -0.9916 + 2.4068i,\ \lambda_3 = -0.9916 - 2.4068i,$$
$$E_3: \lambda_1 = 2.2193,\ \lambda_2 = -0.9916 + 2.4068i,\ \lambda_3 = -0.9916 - 2.4068i.$$

Thus, the equilibrium point E_1 is a stable focus node, that is, the trajectory of $\big(x(t), y(t)\big)$ converges spiral form and $z(t)$ converges to the equilibrium point for any initial value $(x(0), y(0), z(0))$. On the other hands, the equilibrium points E_2 and E_3 are saddle focus nodes, that is, the trajectory of $\big(x(t), y(t)\big)$ will diverge spiral form but $z(t)$ converges to the equilibrium point for any initial value $(x(0), y(0), z(0))$.

Example 4.2

Consider the classical Chua's circuit with two nonlinear resistors when the following parameters are given:

Table 4.2: parameters of classical Chua's circuit for Example 4.2

Parameters	Values	Units	Parameters	Values	Units
R	1000	Ω	R_6	1000	Ω
R_1	2000	Ω	C_1	120	μF
R_2	5000	Ω	C_2	1	mF
R_3	500	Ω	L	80	H
R_4	4000	Ω	$V_{max,Op\ Amp}$	5	V
R_5	4000	Ω			

We have the following parameters

$$\zeta = 8.4, \qquad \eta = 12, \qquad m_0 = -1.2, \qquad m_1 = -0.05.$$

From (39), we have $k = -0.899.$ To find a suitable indicial point of the system, we have to solve the equation $\Phi(a_0) = 0,$ where Φ is given by the equation (39). Indeed, we have

$$0 = (m_0 - m_1)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + (1 - 2a_0)\frac{\sqrt{1 + a_0^2}}{2a_0^2}\right] - a_0 k\pi$$

$$= (-1.15)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + (1 - 2a_0)\frac{\sqrt{1 + a_0^2}}{2a_0^2}\right] - a_0(-0.899)\pi$$

An approximated solution a_0 via MATLAB is given by $a_0 = 1.5187$. It follows from the previous discussion that an initial point is given by $(0) = 1.5187,\ y(0) = 0.0926,$ $z(0) = -2.1682$. A numerical simulation for the equations (23)-(25) with the initial point via MATLAB is illustrated in the following figures.

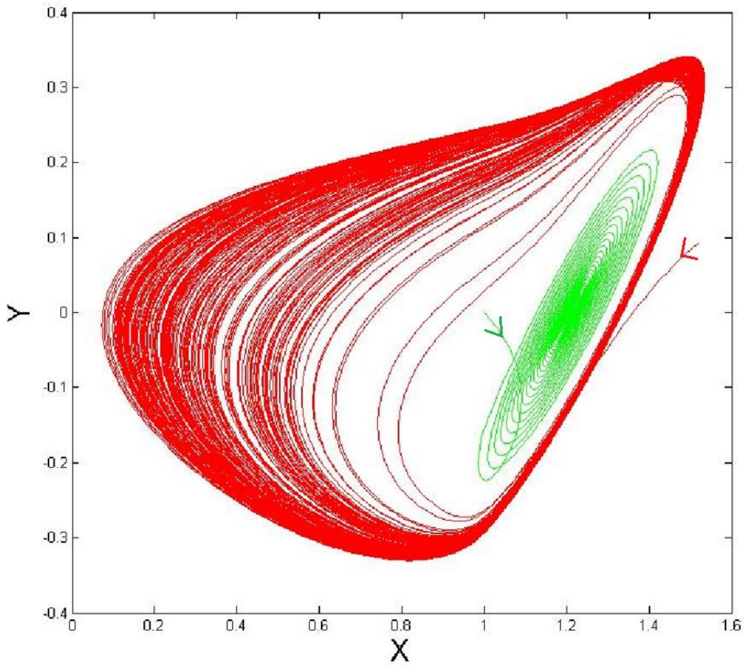

Figure 4.5: Attractors of the classical Chua's equations in (x-y axis) two dimensions for Example 4.2

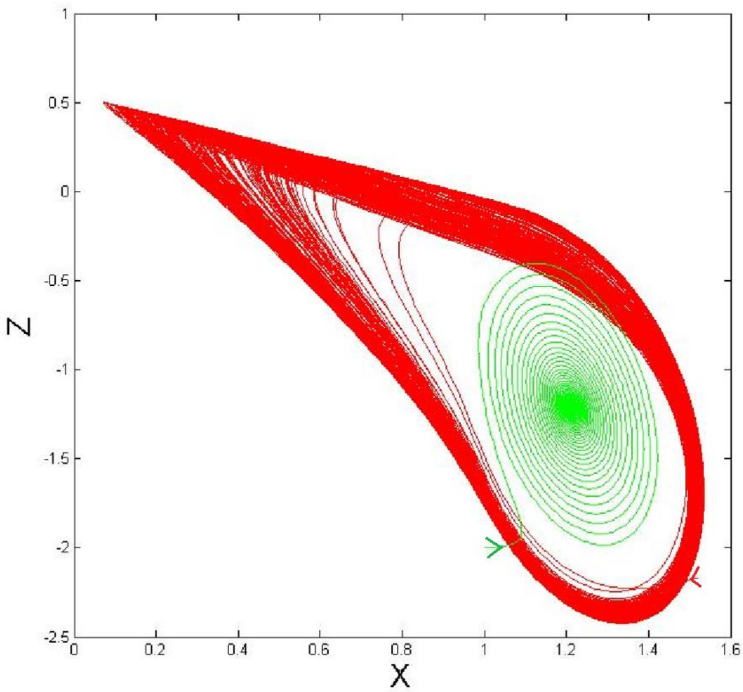

Figure 4.6: Attractors of the classical Chua's equations in (x-z axis) two dimensions for Example 4.2

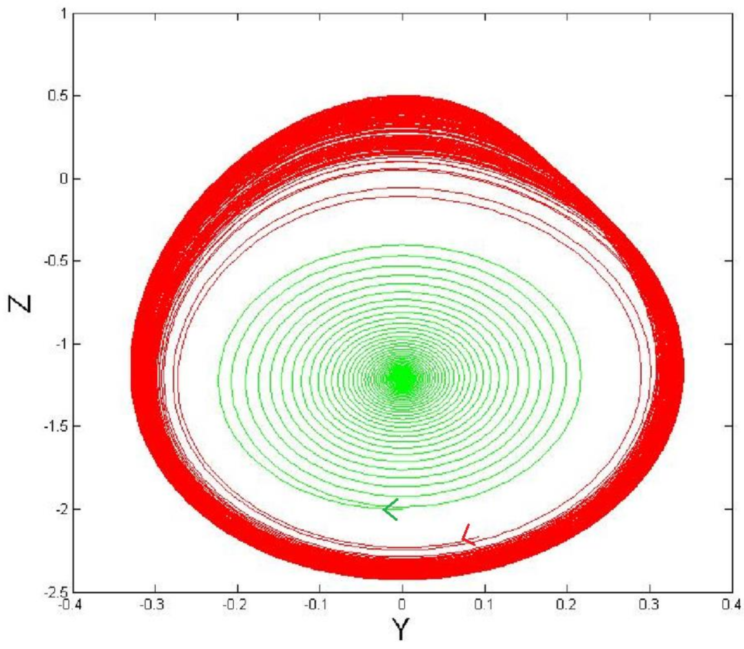

Figure 4.7: Attractors of the classical Chua's equations in (y-z axis) two dimensions for Example 4.2

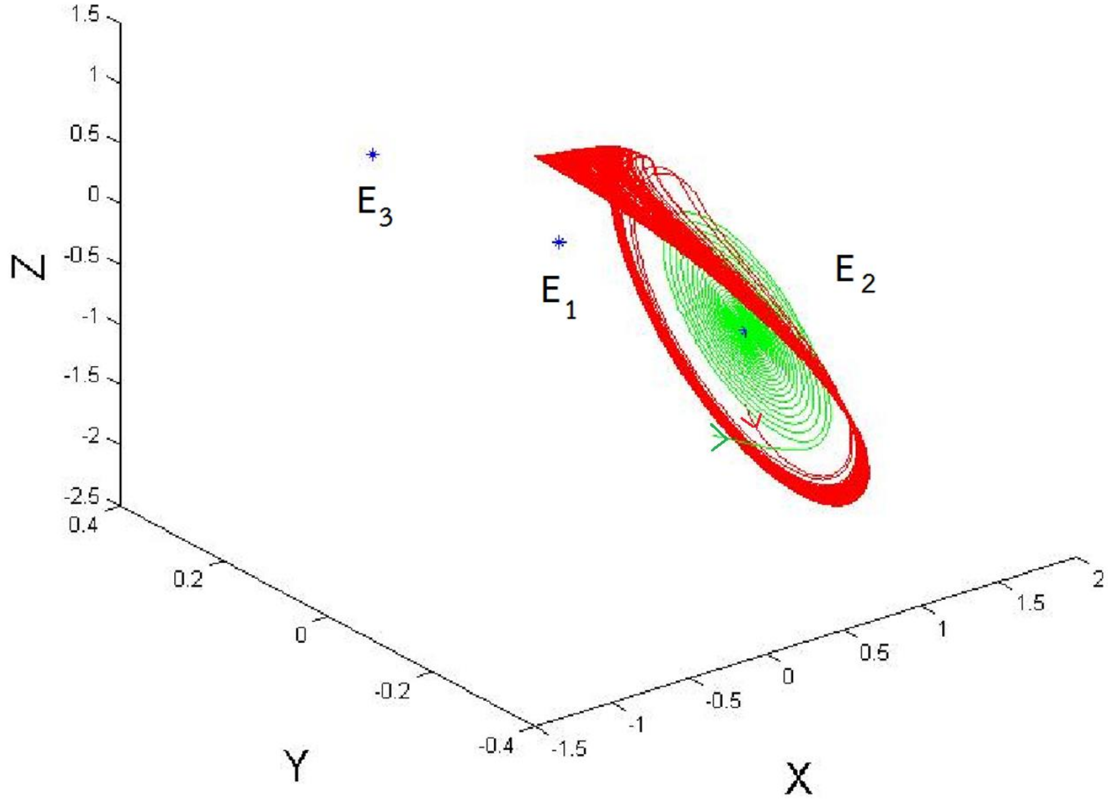

Figure 4.8: Attractors of the classical Chua's equations in three dimensions for Example 4.2

From the classical Chua's circuit, we formulate the following system

$$\frac{dx}{d\tau} = 8.4\big(-x + y - g(x)\big),$$
$$\frac{dy}{d\tau} = x - y + z,$$
$$\frac{dz}{d\tau} = -12y,$$

where $g(x) = -0.05x - 0.575(|x+1| - |x-1|)$. This system has three equilibrium points, namely,

$$E_1 = (0,0,0),$$
$$E_2 = \left(\frac{m_1-m_0}{m_1+1}, 0, \frac{m_0-m_1}{m_1+1}\right) = (1.2105, 0, \ -1.2105),$$
$$E_3 = \left(\frac{m_0-m_1}{m_1+1}, 0, \frac{m_1-m_0}{m_1+1}\right) = (-1.2105, 0, 1.2105).$$

The eigenvalues of the system corresponding to each equilibrium point are given by

$$E_1: \lambda_1 = 2.7123,\ \lambda_2 = -1.0162 + 2.5298i,\ \lambda_3 = -1.0162 - 2.5298i,$$
$$E_2: \lambda_1 = -8.8891,\ \lambda_2 = -0.0454 + 3.2818i,\ \lambda_3 = -0.0454 - 3.2818i,$$
$$E_3: \lambda_1 = -8.8891,\ \lambda_2 = -0.0454 + 3.2818i,\ \lambda_3 = -0.0454 - 3.2818i.$$

Thus, the equilibrium point E_1 is a saddle focus node, that is, the trajectory of $\big(x(t), y(t)\big)$ converges spiral form and $z(t)$ diverges to the equilibrium point for any initial value $(x(0), y(0), z(0))$. On the other hands, the equilibrium points E_2 and E_3 are stable focus node, that is, the trajectory of $\big(x(t), y(t)\big)$ will converge spiral form and $z(t)$ converges to the equilibrium point for any initial value $(x(0), y(0), z(0))$.

From Figures 4.1 to 4.8, we see that a chaotic behaviour occurs in our Chua's system has. The self-excited attractor of our system is appeared in the green lines, while the hidden attractor of our system is shown by the red lines in Figures 4.1 to 4.8.

Chapter 5

Conclusions and Suggestions

5.1 Conclusions

We apply fundamental laws in electrical engineering to formulate a mathematical model for the classical Chua's circuit with two nonlinear resistors (Figure 1.4) in terms of the system of ordinary nonlinear differential equations (23)-(26). Each nonlinear resistor in the circuit plays a role like an op-amp. The existence of two nonlinear resistors implies that the system has three equilibrium points. The behavior of the trajectory in a neighborhood of each equilibrium point depends on the eigenvalues of the system. To obtain the eigenvalues for each equilibrium point, we must solve a cubic polynomial equation. It turns out that all possible solutions of the cubic equation lead to six types of equilibrium points, namely, stable node in figure 3.8, unstable node in figure 3.9, saddle node in figure 3.10 and figure 3.11, stable focus node in figure 3.12, unstable focus node in figure 3.13, saddle focus node in figure 3.14 and figure 3.15. The chaotic behaviour of the circuit occurs when the equilibrium point is a stable focus node or a saddle focus node. The hidden attractor of our Chua's system is localized through a suitable initial point.

5.2 Suggestions

From this thesis, we present the classical Chua's circuit has been successfully applied to signal transmission and cryptography. The benefits of this thesis can be used in computer science, biology, communication, weather forecast, and other areas.

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Appendix

Appendix A.

THE RESEARCH PAPER

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26-Mar-2019

Dear Asst. Prof. Dr. Chansangiam:

It is a pleasure to accept your manuscript entitled "Mathematical Analysis for Classical Chua's Circuit with Two Nonlinear Resistors" in its current form for publication in the Songklanakarin Journal of Science and Technology. The comments of the reviewer(s) who reviewed your manuscript are included at the foot of this letter.

Thank you for your fine contribution. On behalf of the Editors of the Songklanakarin Journal of Science and Technology, we look forward to your continued contributions to the Journal.

Sincerely,
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proespichaya.k@psu.ac.th

Reviewer(s)' Comments to Author:

Reviewer: 1

Comments to the Author
Thanks for additional changes and adding a reference.

Associate Editor
Comments to the Author:
The revised paper is interesting.

SJST SONGKLANAKARIN JOURNAL OF SCIENCE AND TECHNOLOGY

Mathematical Analysis for Classical Chua's Circuit with Two Nonlinear Resistors

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Keyword:	chaos theory, circuit analysis, Chua's circuit, nonlinear resistors, hidden attractor

Author's Response

Thank for article processing. We have revised the manuscript as follows:

1. We correct them as suggested by a reviewer. Please see the yellow highlight in the revision file.

2. We correct the first sentence to "From the characteristic of nonlinear resistors in Figure 6 and the formula (7), ..."

3. We add the following sentence on Page 8: " We shall adopt a treatment on cubic equations (e.g. (Guilbeau L, 1930)) to this situation." and add the reference (Guilbeau L, 1930) for general treatment on cubic equations.

Please see the attach files.

Yours Sincerely,

Original Article

Mathematical Analysis for Classical Chua's Circuit with Two Nonlinear Resistors

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Abstract

We formulate a mathematical model for the classical Chua's circuit with two nonlinear resistors in terms of a system of nonlinear ordinary differential equations. The existence of two nonlinear resistors implies that the system has three equilibrium points. The behaviour of the trajectory in a neighbourhood of each equilibrium point depends on the eigenvalues of the system. The eigenvalues can be obtained from a cubic polynomial equation. It turns out that all possible solutions of the cubic equation lead to six types of equilibrium points, namely, stable node, unstable node, saddle node, stable focus node, unstable focus node, saddle focus node. The chaotic behaviour of the circuit occurs when the equilibrium point is a stable focus node or a saddle focus node. The hidden attractor of our Chua's system is localized through a suitable initial point.

Keywords: chaos theory, circuit analysis, Chua's circuit, nonlinear resistors, hidden attractor

1. Introduction

In nonlinear dynamical systems, it is well known that there are two types of oscillations, namely, periodic oscillation and chaotic oscillation. Chaos system is thus a nonlinear dynamic system which has chaotic motion or random changing of waveform. It is sensitive to initial conditions and has the self-similarity property. Chaotic phenomenon has been received much attention for a few decades. Such behaviour has been successfully applied to signal transmission and cryptography (Kolumban, Kennedy & Chua, 1998; Yang, Wu & Chua, 1997; Dmitriev, Panas & Starkov, 1995). Several types of oscillators have been studied and applied for generating chaos, e.g. Collpits, Wien bridge, Chua, Lorenz, etc. Among those, Chua's circuit is a famous one.

Chua's circuit (Chua, 1992; Sprott, 2000a) is a simple electronic circuit that exhibits classic chaos theory behaviour (Sprott, 2000b; Piper & Sprott, 2010; Chua & Lin, 1990; Sprott, 2011). It produces an oscillating waveform, which is different from usual electronic oscillators. The classical Chua's circuit, shown in Figure 1, consists of only resistors, capacitors, and a nonlinear resistor (Morgul, 1995; Aissi & Kazakos, 2008). The nonlinear resistor, also called Chua's diode, consists of many op-amps. In the literature, there are many ways to adjust the classical Chua's circuit to a more complicated one having chaotic behaviour. These include anti-monotonicity, and bubble formation (Kyprianidis, 2006; Stouboulos, Kyprianidis & Papadopoulou, 2008). It is also possible to replace the piecewise linear characteristic of the Chua's diode with a smooth cubic function (Kyprianidis & Fotiadou, 2006). Applications of Chua's chaotic systems go to computer science, mathematical biology, communication system, weather forecast and other branches of sciences.

In the literature, an oscillation in a dynamical system (e.g. Chua's system) can be localized numerically if initial conditions from its neighbourhood lead to asymptotic behaviour that approaches the oscillation. Such an oscillation is called an attractor, and its attracting set is called the basin of attraction. There are two types of attractors classified by the basin of attractions (Bragin, Vagaitsev, Kuznetsov & Leonov, 2011). A hidden attractor, discovered in (Kuznetsov, Leonov & Vagaitsev, 2010; Bragin, Vagaitsev, Kuznetsov & Leonov, 2011; Kuznetsov, Leonov & Seledzhi, 2011), is an attractor whose basin of attractions does not intersect with small neighbourhoods of equilibrium points; otherwise an attractor is called a self-excited attractor.

The present paper investigates the classical Chua's circuit by adding a nonlinear resistor, so that the circuit has two nonlinear resistors as shown in Figure 2. The circuit of each nonlinear resistor is shown in Figure 3. We apply fundamental laws in electrical engineering to make a mathematical model of the circuit; see Section 2. Such a model is described in terms of a system of nonlinear differential equations. Then we shall find all equilibrium points of the system; see Section 3. To investigate the trajectory behaviour about a neighbourhood of each equilibrium point, we shall classify the type of each equilibrium point through the associated eigenvalues; see Section 4. In Section 5, we show that our system can be reduced to a simpler one via an invertible transformation. Our system has one hidden attractor, and its localization is discussed in Section 6. Our theory is then illustrated with a numerical simulation in Section 7. We finish the paper with the conclusion in Section 8.

Figure 1: Classical Chua's circuit

Figure 2: Classical Chua's circuit with two nonlinear resistors

Figure 3: Fully classical Chua's circuit with two nonlinear resistors

2. Formulation of Classical Chua's Circuit to a System of ODEs

In this section, we formulate a mathematical model for the classical Chua's circuit (Figure 3) in terms of a system of nonlinear ordinary differential equations (ODEs).

We divide the circuit in Figure 3 into four parts as illustrated in Figures 4-5. Our analysis is based on fundamental theory of electrical circuit analysis such as Ohm's law, Kirchhoff's current law (KCL), Kirchhoff's voltage law (KVL).

Figure 4: Chua's circuit analysis

Figure 5: Two nonlinear resistors analysis in Chua's circuit

To analyse the circuit parts in Figure 4, we use the following notations. Let i_L and i_{N_R} be the currents through the inductor L and the nonlinear resistor N_R. Let V_{C_1} and V_{C_2} be the voltages measured across the capacitors C_1 and C_2. Let R be the resistance of the variable resistor. Now, the circuit parts in Figure 4 can be described as

$$\frac{di_L}{dt} = -\frac{V_{C_2}}{L}, \tag{1}$$

$$\frac{dV_{C_2}}{dt} = \frac{V_{C_1} - V_{C_2}}{RC_2} + \frac{i_L}{C_2}, \tag{2}$$

$$\frac{dV_{C_1}}{dt} = \frac{V_{C_2} - V_{C_1}}{RC_1} - \frac{i_{N_R}}{C_1}. \tag{3}$$

The circuit in Figure 5 is a more complicated one since it consists of two nonlinear resistors. For the nonlinear resistor on the left, using Ohm's law, we have $V_{N_R} = i_{R_3}R_3$, $V_e =$

$(R_2 + R_3)i_{R_3}$ and $V_{N_R} - V_e = i_x R_1$, where V_e is the voltage of the op-amp on the left hand side. Combining these three equations to get $i_x = R_x V_{N_R}$ where

$$R_x = -\frac{R_2}{R_1 R_3}.$$

Similarly, for the nonlinear resistor on the right, we obtain that $i_y = R_y V_{N_R}$ where

$$R_y = -\frac{R_5}{R_4 R_6}.$$

Using KCL at node c, we have $i_{N_R} - i_x - i_y = 0$. Then the current i_{N_R} satisfies the relation

$$i_{N_R} = (R_x + R_y)V_{N_R}.$$

However, as pointed out in (Chua & Ying, 1982), the behavior of i_{N_R} depends on the voltage V_{C_1}. Indeed, when $V_e < V_f$, the graph of i_{N_R} with respect to V_{C_1} is as follows.

Figure 6: Reduced form of I-V Characteristic for nonlinear resistors ($V_e < V_f$)

From Figure 6,

$$i_{N_R} = \left(R_x + \frac{1}{R_4}\right)V_{C_1} + \frac{1}{2}\left(R_y - \frac{1}{R_4}\right)\left(\left|V_{C_1} + \frac{V_{f,max}}{V_f}V_{C_1}\right| - \left|V_{C_1} - \frac{V_{f,max}}{V_f}V_{C_1}\right|\right)$$

where $V_{f,max}$ is the maximum voltage at a node f. Introduce the following time-scale changing:

$$\tau = \frac{t}{RC_2}, \qquad x = \frac{V_f}{V_{f,max}}, \qquad y = \frac{V_f V_{C_2}}{V_{f,max} V_{C_1}}, \qquad z = \frac{V_f i_L R}{V_{f,max} V_{C_1}}.$$

Now, the equations (1), (2) and (3) become the following system of ODEs

$$\frac{dx}{d\tau} = \left(\frac{C_2}{C_1}\right)(-x + y - g(x)), \tag{4}$$

$$\frac{dy}{d\tau} = x - y + z, \tag{5}$$

$$\frac{dz}{d\tau} = -\left(\frac{R^2C_2}{L}\right)y, \tag{6}$$

where

$$g(x) = R\left(R_x + \frac{1}{R_4}\right)x + \frac{1}{2}R\left(R_y - \frac{1}{R_4}\right)(|x+1| - |x-1|). \tag{7}$$

3. Equilibrium Points of the Classical Chua's System

From the characteristic of nonlinear resistors in Figure 6 and the formula (7), we see that the behavior of the current i_{N_R} depends on the voltage V_{C_1}, which is considered into three cases, namely, Case 1: $-1 \le x \le 1$, Case 2: $-\frac{V_f}{V_e} \le x \le 1$, Case 3: $-1 \le x \le \frac{V_f}{V_e}$. Let E_i be the equilibrium point for Case i where $i = 1,2,3$. Denote $\zeta = \frac{C_2}{C_1}$ and $\eta = \frac{R^2C_2}{L}$.

For Case 1, we have

$$g(x) = R\left(R_x + \frac{1}{R_4}\right)x + \frac{1}{2}R\left(R_y - \frac{1}{R_4}\right)(x+1-1-x)$$

$$= m_0 x,$$

where $m_0 = R(R_x + R_y)$. It follows that the equations (4) to (6) become

$$\zeta(y-x) - \zeta m_0 x = 0, \tag{8}$$

$$x - y + z = 0, \tag{9}$$

$$-\eta y = 0. \tag{10}$$

Thus, the equilibrium for Case 1 is given by $(x_1, y_1, z_1) = (0,0,0)$.

For Case 2, we have

$$g(x) = R\left(R_x + \frac{1}{R_4}\right)x + \frac{1}{2}R\left(R_y - \frac{1}{R_4}\right)(-x-1-1+x),$$

$$= m_1 x + m_0 - m_1,$$

where $m_1 = R\left(R_x + \frac{1}{R_4}\right)$. Now, the equation (4) reduces to

$$\zeta(y - x) - \zeta(m_1 x - m_0 - m_1) = 0. \quad (11)$$

From the system of equations (9) to (11), we obtain the equilibrium point to be

$$(x_2, y_2, z_2) = \left(\frac{m_1 - m_0}{m_1 + 1}, 0, \frac{m_0 - m_1}{m_1 + 1}\right).$$

Finally for Case 3, we can see that

$$g(x) = m_1 x + m_1 - m_0,$$

and thus the equilibrium point is determined by

$$(x_3, y_3, z_3) = \left(\frac{m_0 - m_1}{m_1 + 1}, 0, \frac{m_1 - m_0}{m_1 + 1}\right).$$

4. Eigenvalues and Trajectories of the System

In this section, we find the eigenvalues for the classical Chua's system and analyze the behaviour of trajectories of the system in a neighborhood of each equilibrium point.

4.1 Finding eigenvalues

We shall formulate our system into a vector differential equation. Let us denote

$$X(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \text{ and } \dot{X}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix},$$

where $\dot{x}(t), \dot{y}(t)$ and $\dot{z}(t)$ are the derivatives of $x(t)$, $y(t)$ and $z(t)$ with respect to the time t, respectively.

From the equations (4)-(7), we again consider three cases. For Case 1 ($-1 \leq x \leq 1$), we obtain the linear system

$$\dot{X}(t) = JX(t) \text{ where } J = \begin{bmatrix} -\zeta - \zeta m_0 & \zeta & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix}.$$

Recall the following the result:

Theorem 1 (see *e.g.* (Goode, 2000)) The initial value problem

$$X(t) = A(t)X(t) + B(t), \qquad X(t_0) = X_0,$$

where $A(t)$ and $B(t)$ are continuous vector-valued functions on an interval I, has a unique solution $X(t)$ on I.

This theorem guarantees the existence and the uniqueness of the trajectories $x(t)$, $y(t)$ and $z(t)$, provided that initial values $x(0), y(0)$ and $z(0)$ are given.

In order to get the solutions $x(t)$, $y(t)$ and $z(t)$ of the above system, we shall find the eigenvalues of the matrix J. Indeed, we have $\det(\lambda I_3 - J) = 0$ and thus the characteristic equation of J is given by

$$\lambda^3 + (\zeta + \zeta m_0 + 1)\lambda^2 + (\zeta m_0 + \eta)\lambda + (\zeta\eta + \zeta\eta m_0) = 0. \qquad (12)$$

For the second and the third cases $(-1 \le x \le 1)$, the coefficient matrix J is given by

$$J = \begin{bmatrix} -\zeta - \zeta m_1 & \zeta & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix}.$$

Similarly, its characteristic equation is

$$\lambda^3 + (\zeta + \zeta m_1 + 1)\lambda^2 + (\zeta m_1 + \eta)\lambda + (\zeta\eta + \zeta\eta m_1) = 0. \qquad (13)$$

We shall adopt a treatment on cubic equations (e.g. (Guilbeau, 1930)) to this situation. We shall find the solution of the cubic equation

$$\lambda^3 + b\lambda^2 + c\lambda + d = 0 \qquad (14)$$

in which $b = \zeta + \zeta m + 1, c = \zeta m + \eta$, and $d = \zeta\eta + \zeta\eta m$, where $m = m_0$ or $m = m_1$. The numbers of real and complex roots are determined by the discriminant of the cubic equation defined by

$$\Delta = 18bcd - 4b^3d + b^2c^2 - 4c^3 - 27d^2.$$

The general solution of the cubic equation involves calculating:

$$\Delta_0 = b^2 - 3c \text{ and } \Delta_1 = 2b^3 - 9bc + 27d.$$

For $\Delta > 0$, the equation has three distinct real roots. More precisely, substitutions $t - \frac{b}{3}$ into λ, we get $t^3 + pt + q = 0$, where $p = -\frac{\Delta_0}{3}$ and $q = \frac{\Delta_1}{27}$. The solution t will be in the form $t = u + v$, where

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} \text{ and } v = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}.$$

Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $A = |u|$ and $B = |v|$. Then, there are three possible values of u, namely $u_1 = A$, $u_2 = \omega A$ and $u_3 = \omega^2 A$. Similarly, there are three positive values of v, namely, $v_1 = B$, $v_2 = \omega B$ and $v_3 = \omega^2 B$. However, the pair (u, v) must satisfies the condition $uv = -\frac{p}{3}$. Hence, the solutions of (14) are given by

$$\lambda_1 = A + B - \frac{b}{3}, \qquad \lambda_2 = \omega A + \omega^2 B - \frac{b}{3}, \qquad \lambda_3 = \omega^2 A + \omega B - \frac{b}{3}.$$

For $\Delta = 0$, the equation has a multiple root and all of its roots are real. There are two subcases:

- $\Delta_0 = 0$: it has a triple same root $\lambda_1 = \lambda_2 = \lambda_3 = -\frac{b}{3}$.
- $\Delta_0 \neq 0$: it has a double same root $\lambda_1 = \lambda_2 = \frac{9d - bc}{\Delta_0}$ and a simple distinct root $\lambda_3 = \frac{4bc - 9d - b^3}{\Delta_0}$.

For $\Delta < 0$, the equation has one real root and two non-real complex conjugate roots.

4.2 Analysis for Trajectories of the System in a Neighborhood of Equilibrium

The classical Chua's system can be classified in terms of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ from the system $\dot{X}(t) = JX(t)$ as follows.

Case 1: λ_1, λ_2 and λ_3 are negative real numbers. In this case, the equilibrium point is called a stable node. Thus, the trajectories of $(x(t), y(t), z(t))$ will converge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 7.

Case 2: λ_1, λ_2 and λ_3 are positive real numbers. In this case, the equilibrium point is called an unstable node. Thus, the trajectories of $(x(t), y(t), z(t))$ will diverge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 8.

Case 3: λ_1 is a positive real number, λ_2 and λ_3 are negative real numbers. In this case, the equilibrium point is called a saddle node. The trajectories lying on the x and y axes tend toward to equilibrium point, whereas the trajectories lying on the z axis tends away from equilibrium point. See Figure 9 on the left hand side.

Case 4: λ_1 is a negative real number, λ_2 and λ_3 are positive real numbers. In this case, the equilibrium point is called a saddle node. The trajectories of $\big(x(t), y(t)\big)$ will diverge but $z(t)$ will converge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 9 on the right hand side.

Case 5: λ_1 is a negative real number, λ_2 and λ_3 are complex numbers having negative real parts, and λ_2 is a conjugate of λ_3. In this case, the equilibrium point is called a stable focus node. Thus, the trajectories of $\big(x(t), y(t)\big)$ will converge spiral form and $z(t)$ will converge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 10.

Case 6: λ_1 is a positive real numbers, λ_2 and λ_3 are complex numbers having positive real parts, and λ_2 is a conjugate of λ_3. In this case, the equilibrium point is called an unstable focus node. The trajectories of $(x(t), y(t))$ will diverge spiral form and $z(t)$ will diverge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 11.

Case 7: λ_1 is a negative real numbers but λ_2 and λ_3 are positive reals and pairs of complex-conjugate numbers. In this case, the equilibrium point is called a saddle focus node. Thus, the trajectories of $(x(t), y(t))$ will diverge spiral form but $z(t)$ will converge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 12 on the left hand side.

Case 8: λ_1 is a positive real numbers but λ_2 and λ_3 are negative reals and pairs of complex-conjugate numbers. In this case, the equilibrium point is called a saddle focus node. Thus, the trajectories of $(x(t), y(t))$ will converge spiral form but $z(t)$ will diverge to the equilibrium point for any initial value $(x(0), y(0), z(0))$. See Figure 12 on the right hand side.

Figure 7: A trajectory of stable node

Figure 8: A trajectory of unstable node

Figure 9: A trajectory of saddle node with $\lambda_1 > 0 > \lambda_2, \lambda_3$ from left hand side and $\lambda_2, \lambda_3 > 0 > \lambda_1$ from right hand side

Figure 10: A trajectory of stable focus node

Figure 11: A trajectory of unstable focus node

Figure 12: A trajectory of saddle focus node with $\mathrm{Re}(\lambda_2), \mathrm{Re}(\lambda_3) > 0 > \mathrm{Re}(\lambda_1)$ from left hand side and $\mathrm{Re}(\lambda_1) > 0 > \mathrm{Re}(\lambda_2), \mathrm{Re}(\lambda_3)$ from right hand side

In summary, there are six types of equilibrium points, namely, stable node, unstable node, saddle node, stable focus node, unstable focus node, and saddle focus node. The chaotic behaviour of the system occurs when the equilibrium point is a stable focus node (Figure 11) or a saddle focus node (Figures 12).

5. Reduction of the system

In this section, we show that our system can be reduced to a simpler system by introducing an invertible transformation.

From equations (4)-(7), we put them together in the following matrix form:

$$\frac{dM}{dt} = P_0 M + Q\mu(R^T M)\,, \tag{15}$$

where

$$P_0 = \begin{bmatrix} -\zeta(m_1+1+k) & \zeta & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix},\ Q = \begin{bmatrix} -\zeta \\ 0 \\ 0 \end{bmatrix}, \quad R^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

From (15), let us introduce an invertible transformation matrix S such that $M = SY$. We shall find the explicit formula of S later. Multiplying S^{-1} to both sides of (15) yields

$$\frac{dY}{dt} = S^{-1}P_0SY + S^{-1}Q\mu(R^TSY),$$

Let $A = S^{-1}P_0S, B = S^{-1}Q, C^T = R^TS$. Then we get

$$\frac{dY}{dt} = AY + B\mu(C^TY), \tag{16}$$

where

$$A = \begin{bmatrix} 0 & -\alpha_0 & 0 \\ \alpha_0 & 0 & 0 \\ 0 & 0 & -\beta \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}, C^T = \begin{bmatrix} 1 \\ 0 \\ c_3 \end{bmatrix},$$

Write $S = [s_{ij}]$. From the conditions $A = S^{-1}P_0S, B = S^{-1}Q, C^T = R^TS$, it is straightforward to deduce that

$$s_{11} = 1, s_{12} = 0, s_{13} = c_3,$$

$$s_{21} = m_1 + 1 + k, s_{22} = -\frac{\alpha_0}{\zeta}, s_{23} = \frac{\beta}{\zeta} + c_3(m_1 + 1 + k),$$

$$s_{31} = m_1 + k - \frac{\alpha_0^2}{\zeta}, s_{32} = -\frac{\alpha_0}{\zeta} - \alpha_0(m_1 + 1 + k), s_{33} = \frac{hd(1+\zeta-d)}{\zeta} + c_3(m_1 + k)(1 - \beta).$$

Now, the transfer functions of the system (15) and (16) are

$$W_{P_0}(p) = R^T(P_0 - pI)^{-1}Q, \tag{17}$$

$$W_A(p) = C^T(A - pI)^{-1}B. \tag{18}$$

We obtain that

$$(A - pI)^1 = \frac{1}{\det(A-pI)}\operatorname{adj}(A - pI)$$

$$= \frac{1}{(-p-\beta)(p^2+\alpha_0^2)}\begin{bmatrix} p^2 + \beta p & -\alpha_0 p - \alpha_0\beta & 0 \\ \alpha_0 p + \alpha_0\beta & p^2 + \beta p & 0 \\ 0 & 0 & p^2 + \alpha_0^2 \end{bmatrix}.$$

It follows that

$$W_A(p)$$

$$= [1 \quad 0 \quad c_3]\begin{bmatrix} \frac{p^2 + \beta p}{(-p-\beta)(p^2+\alpha_0^2)} & \frac{-\alpha_0 p - \alpha_0\beta}{(-p-\beta)(p^2+\alpha_0^2)} & 0 \\ \frac{\alpha_0 p + \alpha_0\beta}{(-p-\beta)(p^2+\alpha_0^2)} & \frac{p^2 + \beta p}{(-p-\beta)(p^2+\alpha_0^2)} & 0 \\ 0 & 0 & \frac{p^2 + \alpha_0^2}{(-p-\beta)(p^2+\alpha_0^2)} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}$$

$$= \frac{b_1(p^2 + \beta p) + b_2(-\alpha_0 p - \alpha_0\beta) + c_3(p^2 + \alpha_0^2)}{(-p-\beta)(p^2+\alpha_0^2)}$$

$$= \frac{(b_1 + c_3)p^2 + (b_1\beta - b_2\alpha_0)p + (c_3\alpha_0^2 - b_2\alpha_0\beta)}{-p^3 - \beta p^2 - \alpha_0^2 p + \alpha_0^2\beta}. \tag{19}$$

Similarly,

$$W_{P_0}(p) = \frac{(-\zeta)p^2 + (-\zeta)p}{-p^3 + (-\zeta m_1 - \zeta - \zeta k - 1)p^2 + (-\zeta m_1 - \zeta - \zeta k - \eta)p + (-\zeta m_1 \eta - \zeta\eta - \zeta k\eta)}. \quad (20)$$

Since the system (15) and the system (16) are the same system, we have $W_A(p) = W_{P_0}(p)$. Now, we compare the numerator and denominator coefficients of $W_A(p)$ and $W_{P_0}(p)$. We obtain that

$$c_3 = -\zeta - b_1,\ b_1\beta - b_2\alpha_0 = -\zeta,\ -c_3\alpha_0^2 + b_2\alpha_0\beta = \zeta\eta$$

$$\beta = \zeta m_1 + \zeta + \zeta k + 1, b_1\beta - b_2\alpha_0 = -\zeta, -c_3\alpha_0^2 + b_2\alpha_0\beta = \zeta\eta.$$

These imply the following relations

$$k = \frac{\alpha_0^2 - \eta}{\zeta} - m_1, \quad \beta = \alpha_0^2 + \zeta - \eta + 1, \quad b_1 = \frac{\zeta(\beta - \alpha_0^2 - \eta)}{\alpha_0^2 + \beta^2}, \quad (21)$$

$$b_2 = \zeta\frac{\eta - \beta + \beta^2}{\alpha_0(\alpha_0^2 + \beta^2)}, \quad c_3 = \frac{-\zeta(\eta - \beta + \beta^2)}{\alpha_0^2 + \beta^2}. \quad (22)$$

6. Localization of a Hidden Attractor for Classical Chua's Circuit

In this section, we will discuss the oscillation behaviour in the classical Chua's circuit with two nonlinear resistors, focused on hidden attractors.

In order to find a hidden attractor of the system, we will find a suitable initial point $(x(0), y(0), z(0))$ so that our system will have a chaos. The initial point depends on a parameter a_0, which is the solution of the equation $\Phi(a) = 0$ where Φ is the describing function (Bragin, Vagaitsev, Kuznetsov & Leonov, 2011) defined by

$$\Phi(a) = \int_0^{2\pi/\omega_0} \varphi(a\cos(\omega_0 t))\cos(\omega_0 t)\mathrm{d}t.$$

Here, $\varphi(a) = g(x) - kx$, where k is a coefficient of harmonic linearization, $x = \omega_0 t$, $g(x)$ is the function of x in (7) and

$$g(\omega_0 t) = \frac{1}{2} R(m_0 - m_1)(|\omega_0 t + 1| - |\omega_0 t - 1|).$$

Then, $g(\omega_0 t)$ is equal to $(m_0 - m_1)\omega_0 t$ for all $x \in [0, 1]$ and it equals the constant $m_0 - m_1$ when $x \in (1, 2\pi]$. It follows that

$$\Phi(a) = \int_0^{\tau} a(m_0 - m_1)\cos^2 t\mathrm{d}t + \int_{\tau}^{2\tau} (m_0 - m_1)\cos t\mathrm{d}t - \int_0^{2\pi} ak\cos^2 t\mathrm{d}t,$$

where $\tau = \arccos\left(\frac{1}{a}\right)$. Thus, we have

$$\Phi(a) = (m_0 - m_1)\left[a_0 \int_0^{\arccos\left(\frac{1}{a_0}\right)} \cos^2 t\,\mathrm{d}t + \int_{\arccos\left(\frac{1}{a_0}\right)}^{2\arccos\left(\frac{1}{a_0}\right)} \cos t\mathrm{d}t\right] - a_0 k \int_0^{2\pi} \cos^2 t\mathrm{d}t,$$

$$= (m_0 - m_1)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + \frac{(1 - 2a_0)}{2a_0}\sin\left(\arccos\left(\frac{1}{a_0}\right)\right)\right] - a_0 k\pi,$$

$$= (m_0 - m_1)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + (1 - 2a_0)\frac{\sqrt{1 - a_0^2}}{2a_0^2}\right] - a_0 k\pi. \qquad (25)$$

From (Bragin, Vagaitsev, Kuznetsov & G. A. Leonov, 2011), the first step of multistage localization for our Chua's system is

$$M(0) = SY(0) = S\begin{bmatrix} a_0 + \mathbf{O}(\mu) \\ 0 \\ \mathbf{O}_{n-2}(\mu) \end{bmatrix},$$

where $\mathbf{O}(\mu)$ is the big-O notation of order μ and $\mathbf{O}_{n-2}(\mu)$ is the (n-2)-dimensional big-O notation so that all its coordinates are big-O notations of order μ. We can approximate $\mathbf{O}(\mu) \approx 0$ and $\mathbf{O}_{n-2}(\mu) \approx 0$. Thus,

$$\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = S\begin{bmatrix} a_0 \\ 0 \\ 0 \end{bmatrix},$$

which implies that

$$x(0) = a_0 s_{11} = a_0,$$

$$y(0) = a_0 s_{21} = a_0(m_1 + 1 + k),$$

$$z(0) = a_0 s_{31} = a_0\left(m_1 + k - \frac{\alpha_0^2}{\zeta}\right).$$

7. Numerical Simulation

Consider the classical Chua's circuit with two nonlinear resistors when the following parameters are given: $R = 1000\ \Omega$, $R_1 = 250\ \Omega$, $R_2 = 250\ \Omega$, $R_3 = 500\ \Omega$, $R_4 = 750\ \Omega$, $R_5 = 180\ \Omega$, $R_6 = 400\ \Omega$, $C_1 = 10\ \mu F$, $C_2 = 80\ \mu F$, $L = 70\ mH$.

We have the following parameters

$$\zeta = 8.4562, \quad \eta = 12.0732, \quad m_0 = -0.1768, \quad m_1 = -1.1468.$$

From (21), we have $k = 0.2098$. To find a suitable indicial point of the system, we have to solve the equation $\Phi(a) = 0$, where Φ is given by the equation (21). Indeed, we have

$$0 = (m_0 - m_1)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + (1 - 2a_0)\frac{\sqrt{1 - a_0^2}}{2a_0^2}\right] - a_0 k\pi$$

$$= (-0.97)\left[\frac{a_0}{2}\arccos\left(\frac{1}{a_0}\right) + (1 - 2a_0)\frac{\sqrt{1 - a_0^2}}{2a_0^2}\right] - a_0 0.2098\pi$$

An approximated solution a_0 via MATLAB is given by $a_0 = 9.4287$. It follows from the previous discussion that an initial point is given by $x(0) = 2.0392$, $y(0) = 0.5945$, $z(0) = -13.4705$. A numerical simulation for the equations (4)-(7) with the initial point via MATLAB is illustrated in the following figures.

Figure 13: Attractors of the classical Chua's equations in two dimensions

Figure 14: Attractors of the classical Chua's equations in three dimensions

From the classical Chua's circuit, we formulate the following system

$$\frac{dx}{d\tau} = 8\big(-x + y - g(x)\big),$$

$$\frac{dy}{d\tau} = x - y + z,$$

$$\frac{dz}{d\tau} = -1142.8571y,$$

where $g(x) = -0.6667x - 0.9667(|x+1| - |x-1|)$. This system has three equilibrium points, namely,

$$E_1 = (0,0,0),$$

$$E_2 = \left(\frac{m_1 - m_0}{m_1 + 1}, 0, \frac{m_0 - m_1}{m_1 + 1}\right) = (6.6076, 0, \ -6.6076),$$

$$E_3 = \left(\frac{m_0 - m_1}{m_1 + 1}, 0, \frac{m_1 - m_0}{m_1 + 1}\right) = (-6.6076, 0, 6.6076).$$

The eigenvalues of the system corresponding to each equilibrium point are given by

$$E_1 \colon \lambda_1 = -7.9587,\ \lambda_2 = -0.0038 + 3.2494i,\ \lambda_3 = -0.0038 - 3.2494i,$$

$$E_2 \colon \lambda_1 = 2.2193,\ \lambda_2 = -0.9916 + 2.4068i,\ \lambda_3 = -0.9916 - 2.4068i,$$

$$E_3 \colon \lambda_1 = 2.2193,\ \lambda_2 = -0.9916 + 2.4068i,\ \lambda_3 = -0.9916 - 2.4068i.$$

Thus, the equilibrium point E_1 is a stable focus node, that is, the trajectory of $\big(x(t), y(t)\big)$ converges spiral form and $z(t)$ converges to the equilibrium point for any initial value $(x(0), y(0), z(0))$. On the other hands, the equilibrium points E_2 and E_3 are saddle focus nodes, that is, the trajectory of $\big(x(t), y(t)\big)$ will diverge spiral form but $z(t)$ converges to the equilibrium point for any initial value $(x(0), y(0), z(0))$.

From Figures 13 and 14, we see that a chaotic behaviour occurs in our Chua's system has. The self-excited attractor of our system is appeared in the green lines, while the hidden attractor of our system is shown by the red lines in Figures 13 and 14.

8. Conclusions

We apply fundamental laws in electrical engineering to formulate a mathematical model for the classical Chua's circuit with two nonlinear resistors (Figure 3) in terms of the system of ordinary nonlinear differential equations (4)-(6). Each nonlinear resistor in the circuit plays a role like an op-amp. The existence of two nonlinear resistors implies that the system has three equilibrium points. The behavior of the trajectory in a neighborhood of each equilibrium point depends on the eigenvalues of the system. To obtain the eigenvalues for each equilibrium point, we must solve a cubic polynomial equation. It turns out that all possible solutions of the cubic equation lead to six types of equilibrium points, namely, stable node, unstable node, saddle node, stable focus node, unstable focus node, saddle focus node. The chaotic behaviour of the circuit occurs when the equilibrium point is a stable focus node or a saddle focus node. The hidden attractor of our Chua's system is localized through a suitable initial point.

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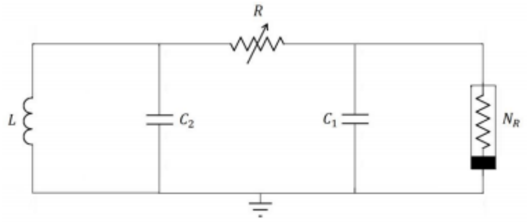

Figure 1: Classical Chua's circuit

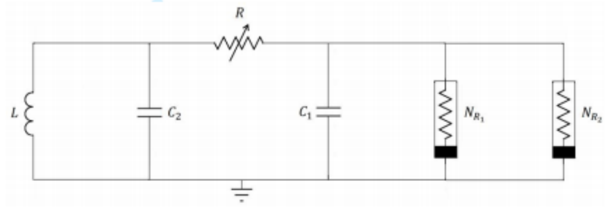

Figure 2: Classical Chua's circuit with two nonlinear resistors

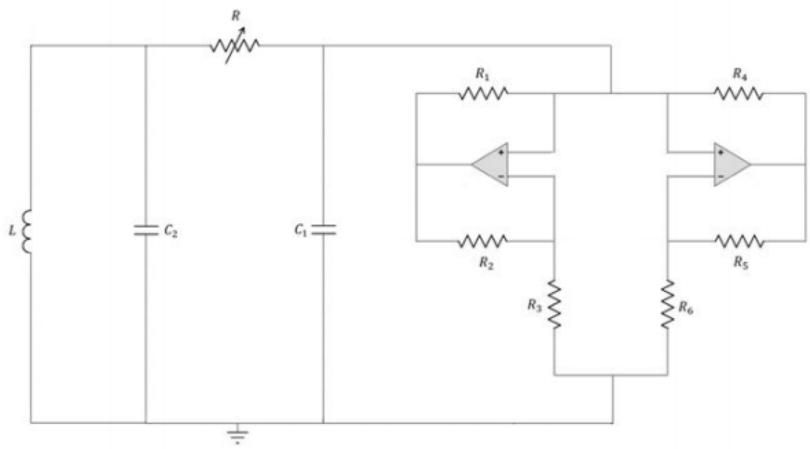

Figure 3: Fully classical Chua's circuit with two nonlinear resistors

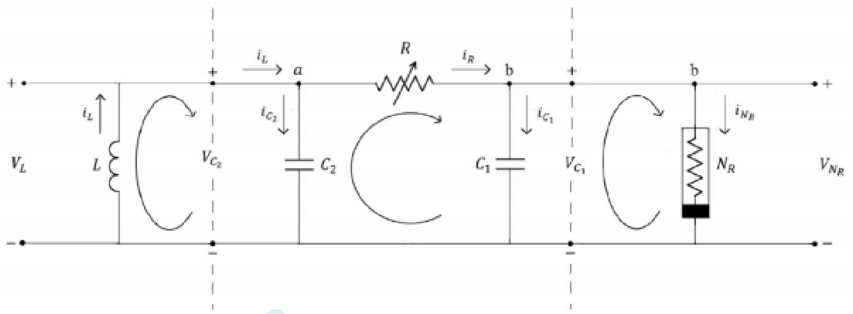

Figure 4: Chua's circuit analysis

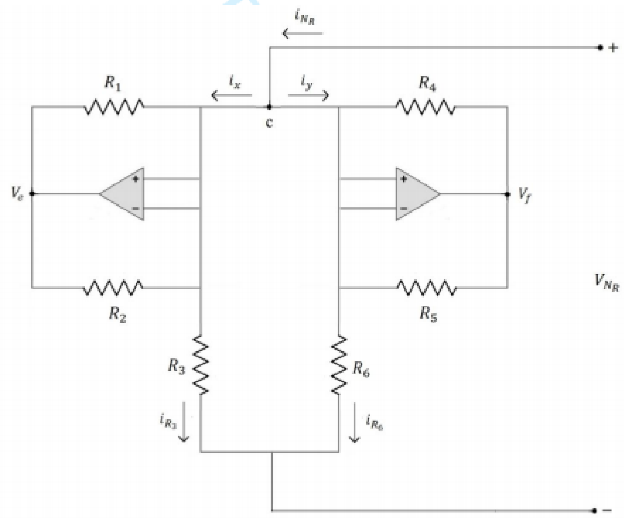

Figure 5: Two nonlinear resistors analysis in Chua's circuit

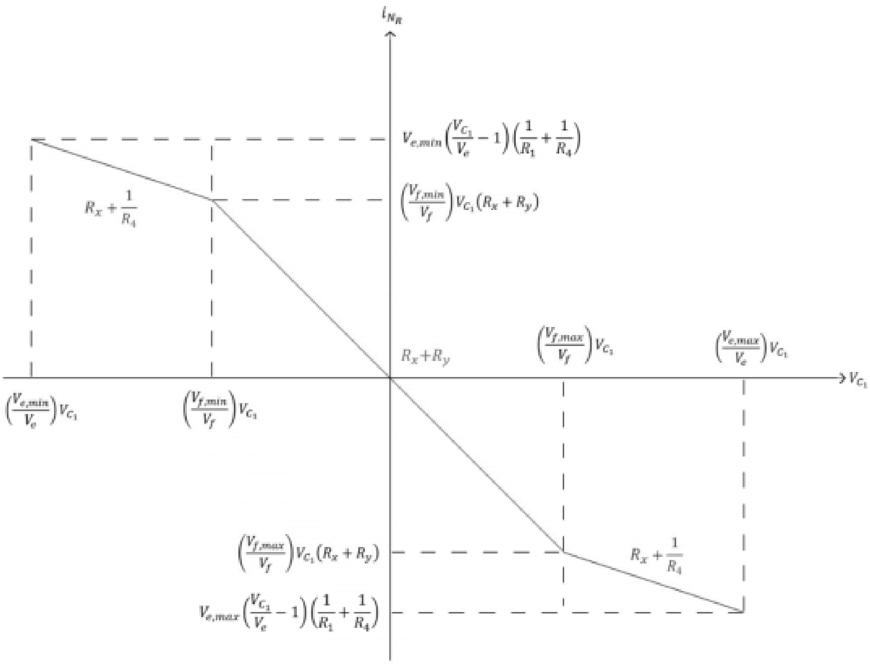

Figure 6: Reduced form of I-V Characteristic for nonlinear resistors ($V_e < V_f$)

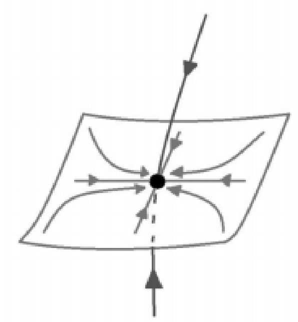

Figure 7: A trajectory of stable node

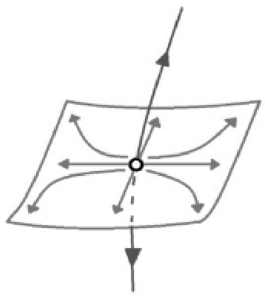

Figure 8: A trajectory of unstable node

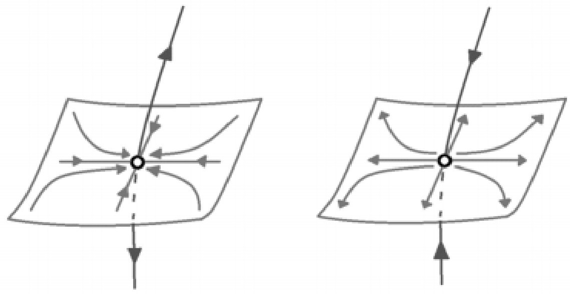

Figure 9: A trajectory of saddle node with $\lambda_1 > 0 > \lambda_{2,3}$ from left hand side and $\lambda_{2,3} > 0 > \lambda_1$ from right hand side

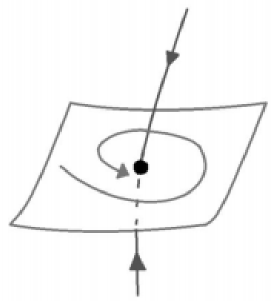

Figure 10: A trajectory of stable focus node

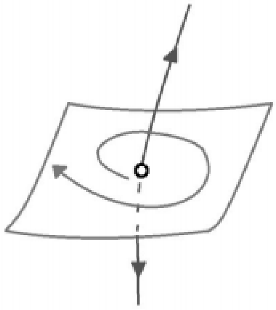

Figure 11: A trajectory of unstable focus node

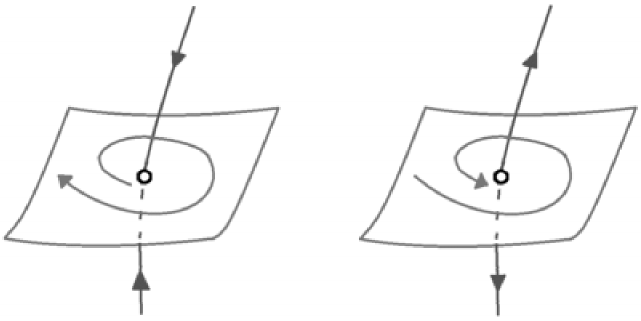

Figure 12: A trajectory of saddle focus node with $\mathrm{Re}(\lambda_2),\mathrm{Re}(\lambda_3) > 0 > \mathrm{Re}(\lambda_1)$ from left hand side and $\mathrm{Re}(\lambda_1) > 0 > \mathrm{Re}(\lambda_2),\mathrm{Re}(\lambda_3)$ from right hand side

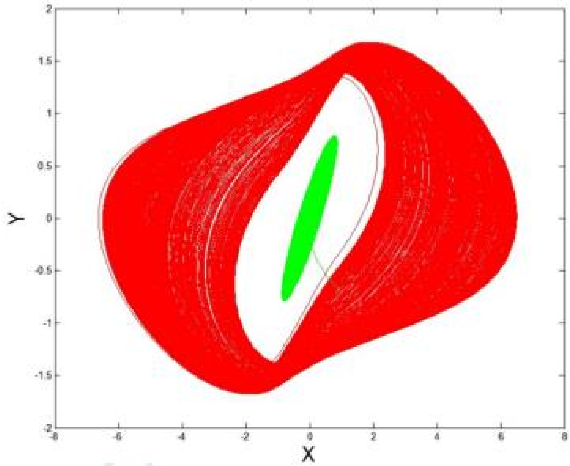

Figure 13: Attractors of the classical Chua's equations in two dimensions

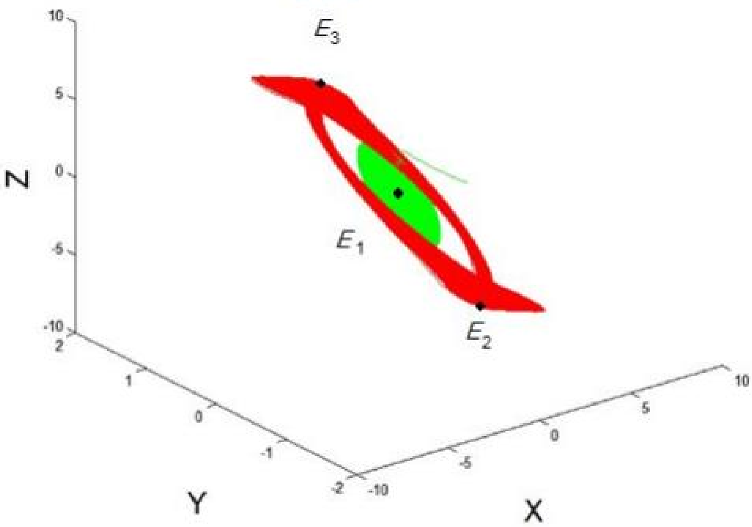

Figure 14: Attractors of the classical Chua's equations in three dimension

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