ตัวแบบเชิงคณิตศาสตร์ของการไหลของเลือดที่มีแอมปลิจูดขนาดเล็กในหลอดเลือดที่มีภาวะขัด

A MATHEMATICAL MODEL OF A SMALL AMPLITUDE OF BLOOD FLOW IN A BLOOD VESSEL WITH BLOCKAGE

ชีวิน ปิ่นมุข

CHEWIN PINMOOK

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ประยุกต์

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สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง

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หัวข้อวิทยานิพนธ์	ตัวแบบเชิงคณิตศาสตร์ของการไหลของเลือดที่มีแอมปลิจูดขนาดเล็กในหลอดเลือดที่มีภาวะขัด
นักศึกษา	นายชีวิน ปิ่นมุข
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พ.ศ.	2555
อาจารย์ที่ปรึกษา	ผศ.ดร.นพรัตน์ โพธิ์ชัย

บทคัดย่อ

การศึกษาการไหลของเลือดที่มีภาวะขัดสามารถนำไปประยุกต์ใช้ในการวินิจฉัยกับผู้ป่วยที่มีภาวะไขมันอุดตันในเส้นเลือดหรือการพัฒนาเครื่องมือเพื่อใช้รักษาผู้ป่วยที่มีภาวะดังกล่าว ซึ่งงานวิจัยนี้จำลองแบบของการไหลของเลือดภายในหลอดเลือดทั้งในกรณีที่มีและไม่มีภาวะขัด หรือการกีดขวางของสิ่งอุดตันโดยเงื่อนไขค่าขอบเขตและเงื่อนไขค่าเริ่มต้นที่เหมาะสม ซึ่งตัวแบบกำหนดโดยระบบสมการคลื่นที่มีแอมปลิจูดขนาดเล็กในหนึ่งมิติโดยหาผลเฉลยเชิงตัวเลขด้วยวิธีผลต่างสืบเนื่องแบบแครงก์–นิโคลสัน (Crank - Nicolson) และแบบแล็กซ์ – เวนดรอฟฟ์ (Lax - Wendroff) ที่มีความแม่นยำอันดับสองทั้งตำแหน่งและเวลา ผลที่ได้จากงานวิจัยนี้พบว่าสามารถจำลองพฤติกรรมของการไหลของเลือดภายในหลอดเลือดทั้งในสภาวะปกติ และสภาวะขัดซึ่งส่งผลให้ค่าสูงสุดของแอมปลิจูดของความดันและความเร็วของการไหลของเลือดภายในหลอดเลือดมีค่าต่ำลงเมื่อระยะตำแหน่งมากขึ้น หรือ สัมประสิทธิ์การขัด (dragged blockage coefficient) มีค่ามากขึ้น รวมทั้งสามารถวิเคราะห์หาวิธีการผลต่างสืบเนื่องที่เหมาะสมกับแต่ละปัญหาได้

คำสำคัญ : ระบบสมการคลื่นที่มีแอมปลิจูดขนาดเล็ก, หลอดเลือด, ภาวะขัด, วิธีผลต่างสืบเนื่องแบบแครงก์ – นิโคลสัน, วิธีผลต่างสืบเนื่องแบบแล็กซ์ – เวนดรอฟฟ์, สัมประสิทธิ์การขัด

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ABSTRACT

The study of blood flow in a blood vessel with blockage can be applied to helping the patient with atherosclerosis decease or development of medical devices which is related. This research simulates both solutions of the blood flow in a blood vessel with and without blockage which is determined with small amplitude wave equations system in one dimension by using the proper initial condition and boundary conditions and it is solved with finite difference by Crank - Nicolson and Lax – Wendroff methods in both of second-order accuracy of time and space. This result will simulate the behavior of blood flow in a blood vessel with blockage which the maximum of pressure and velocity of blood flow decrease for long distance or more dragged blockage coefficient. Their methods can be analyzed to the proper cases.

Keywords : Small Amplitude Wave system, Blood Vessel, Blockage, Finite difference by Crank - Nicolson method, dragged blockage coefficient

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CHAPTER 1

Introduction

The comprehension of physical phenomena obviously describes to the mathematical model creation. However, to simplify with the correctness of modeling, scientists might proof these equations by using numerical methods. Include, the problem of velocity and pressure characteristics from blood flows in a blood vessel with blockage can be expressed to mathematic modeling for solving the certainly results. We will describe in this chapter with regard to the background of blood flow and atherosclerotic blockage in human body, the numerical methods, objectives,and field of study.

1. Background of Human Physiology : Circulation system

1.1 Blood flow

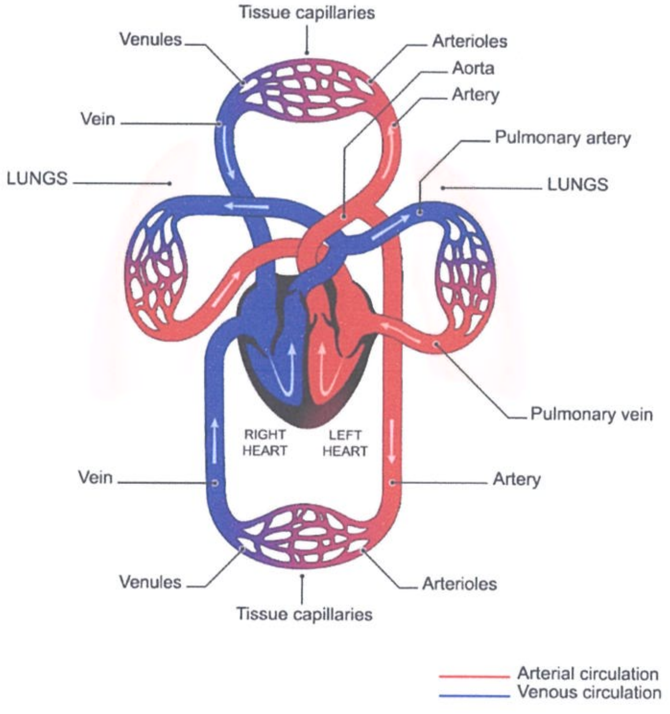

Figure 1.1 the circulatory of blood flow [1]

The circulation of blood flow is the one of most important to all humans and animals because there are related to transportation about many chemical processes in creatures, e.g. hemoglobin from respiratory system, nutrition of foods from alimentary progression. The procedure of blood flow direction can be described to the circulatory of blood in figure 1.1 begins with the pumping of red blood from left ventricle (the chamber from left below of heart) to aorta (the largest of red blood vessel) separates to arteries, arteriole, and exchanges nutrients and oxygen to capillaries. This system of red blood flow direction is called "systemic arteries". Conversely, for oxygenated blood flow in systemic veins, the direction is from small vessels to largers i.e. venules, small veins, and venae cavae respectively [9].

1.2 Atherosclerotic blockage

Atherosclerotic blockage is the reductor of blood flow in blood vessels in Arteries. It possibly develops atherosclerotic plaque which consists of an accumulation of bad cholesterol or low – density lipoprotein (LDL) under endothelium layer in initial stage [7], [8], [10] with figure1.2.

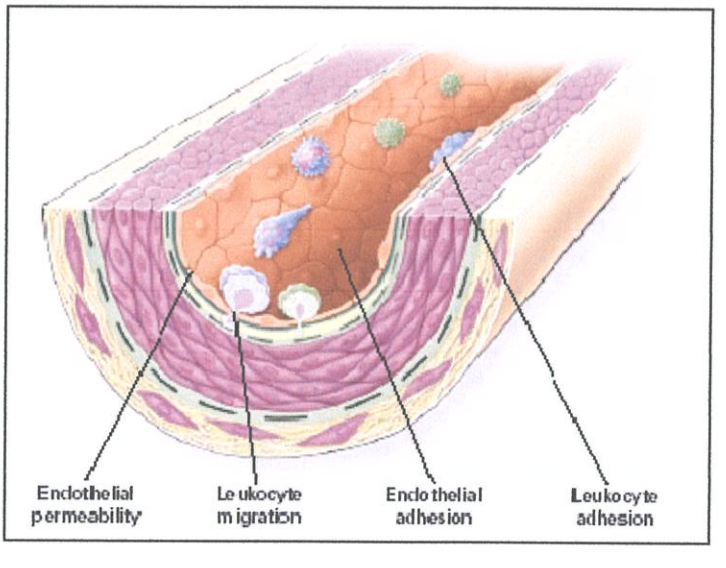

Figure 1.2 The initial stage of atherosclerotic plaque [7]

The accumulated cholesterol, in figure 1.3, products will be transformed to free radicals, the oxidative wastes, by blood vessels cells. Chemical reaction of these free radicals attract monocytes, one types of white blood cells, and permanently adhesive in these sites. The greatly of monocytes are called macrophages and will be foam cell when packs with fatty droplets from LDL oxidation. For larger of foam cells, These accumulative are called fatty streaks which is initial form of atherosclerotic plaque [7], [8], [10].

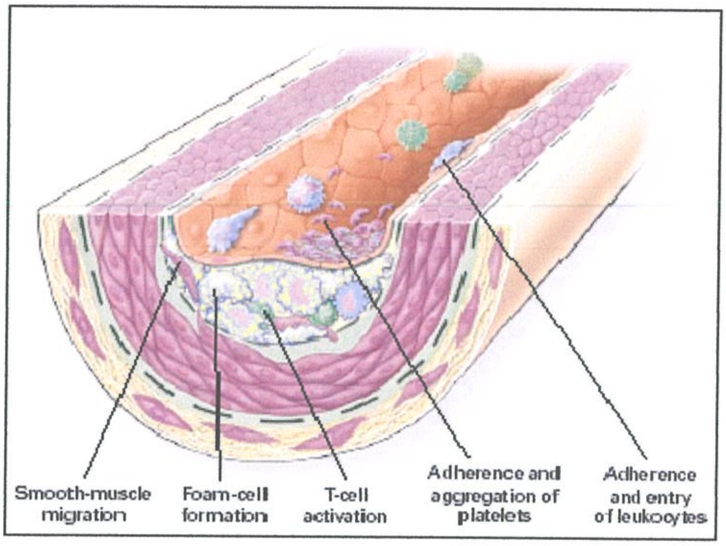

Figure 1.3 The earlist form of atherosclerotic plaque (fatty streak) [7]

The plaque thicking makes degeneration of wall in the vicinity of the plaque which is called fibroblasts (figure 1.4) which form a connective tissue cap over the plaque. "Atherosclerosis" term is this progressive because of this mean excessive growth of fibrous connective tissue within arteriole blood vessel.

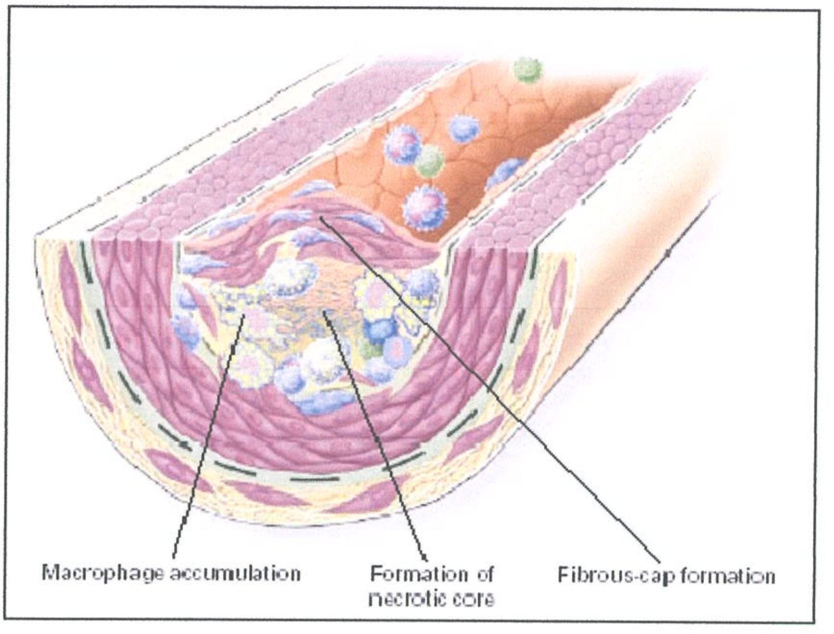

Figure 1.4 The fibroblast stage of atherosclerotic plaque [7]

In the last stage, the enlarging atherosclerotic plaque will be completely blocks the vessels and this blockage is called "thrombus" which is at site of the vessels.

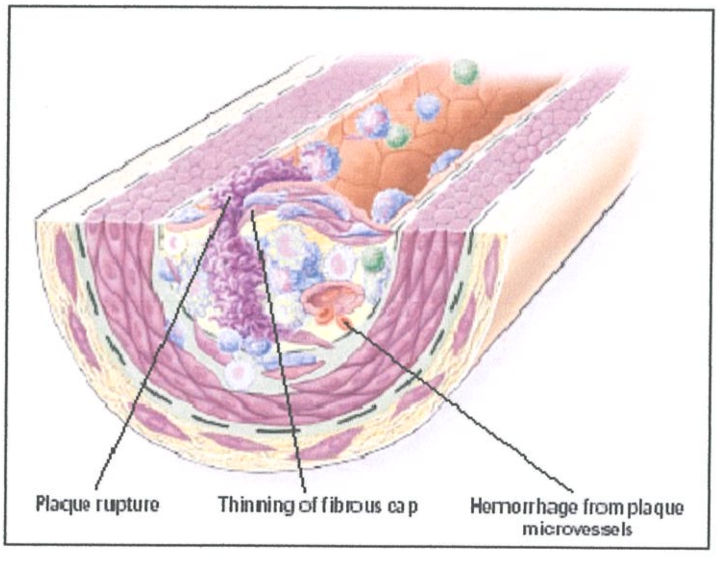

Figure 1.5 The last stage of atherosclerotic plaque [7]

1.3 Arteries blood vessel

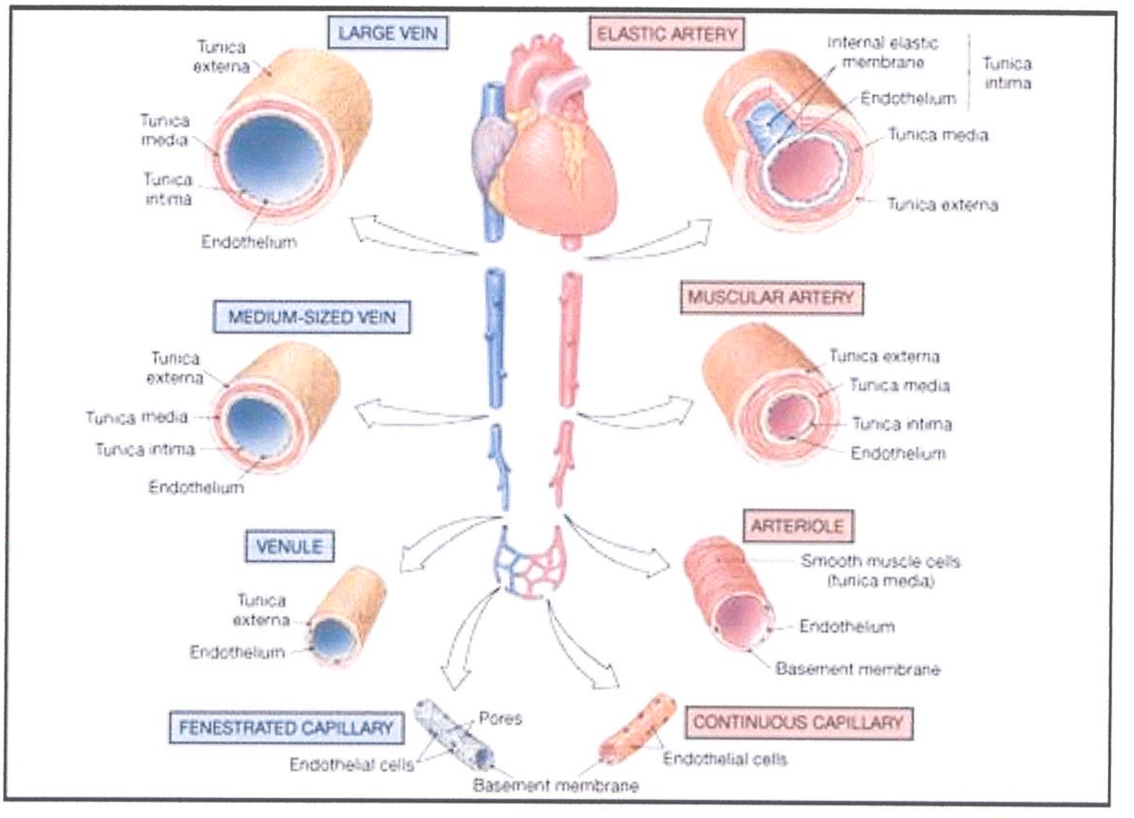

Figure 1.6 Histological structure of blood vessels [16]

The Arteries blood vessel is small arterial branches from arteries. It has important properties are elasticity and contractility which is processes of vasoconstriction and vasodilation. The vasoconstriction is process of decreasing vessel diameter for income blood flow by heart pressure control and vasodilation is relaxation of arteriole contraction after [16].

Vessel	D (cm)	A (cm^2)	P(mm Hg)	v (cm/s)
Aorta	2.5	2.5	100	33
Small arteries	0.5	20	100	30
Arterioles	3×10^{-3}	40	85	15
Capillaries	6×10^{-4}	2500	30	0.03
Venules	2×10^{-3}	250	10	0.5
Small veins	0.5	80		2
Venae cavae	3.0	8	2	20

Table 1.1 Diameter, total cross sectional area, mean blood pressure at entrance, and mean fluid velocity of blood vessels [8]

2. Small Amplitude Wave Theory

In the mention of small amplitude case, the assumption of dynamical fluid flow is steady flow which is satisfied by Bernoulli's principle in this condition [18]

$$\frac{1}{2}u^2 + \int \frac{dP}{\rho} + gz = C \qquad (1.1)$$

where u is fluid velocity,

P is fluid pressure,

z is height of fluid,

ρ is fluid density,

g is gravitational acceleration,

C is a constant.

Consider the principle of small amplitude, the amplitude of wave is tiny with respect to wavelength. This assumption is similar to the definition of the shallow water equation. However, the shallow water is defined by small variation of wave with respect to height reference of water [17], [18].

Equation (1.1) can be rearranged by dividing the gravitational acceleration g and we assume that the fluid density is not respect to pressure function, thus

$$\frac{u^2}{2g} + \frac{P}{\rho g} + z = \mathrm{C}$$

For the small amplitude condition, its amplitude of wave is tiny with respect to wavelength. It can be derived in mathematical equation as

$$z = -\frac{u^2}{2g} - \frac{P}{\rho g}$$

So, the height of fluid wave is referred to fluid velocity and pressure. For small amplitude, we need to set z tends to zero. It makes fluid velocity and fluid pressure also tend to zero [9].

3. Finite Difference Method : Crank – Nicolson Method

Crank - Nicolson method is widely method in numerical solution for finite difference in parabolic equation. This technique can be adapted to the problem solving of first order hyperbolic systems with initial-boundary problems numerically [12]. Its accuracy is maintained to second order [11], [12] and stability of this technique is unconditionally stable which is described in Appendix B.1.

The Crank - Nicolson technique by finite difference method for one dimension space is

$$U_j^{n+1} - \frac{lA^{n+1}}{4h}\left[U_{j+1}^{n+1} - U_{j-1}^{n+1}\right] = U_j^n + \frac{lA^n}{4h}\left[U_{j+1}^n - U_{j-1}^n\right],$$

where U_j^n is approximate solution for first order hyperbolic systems in discrete space.

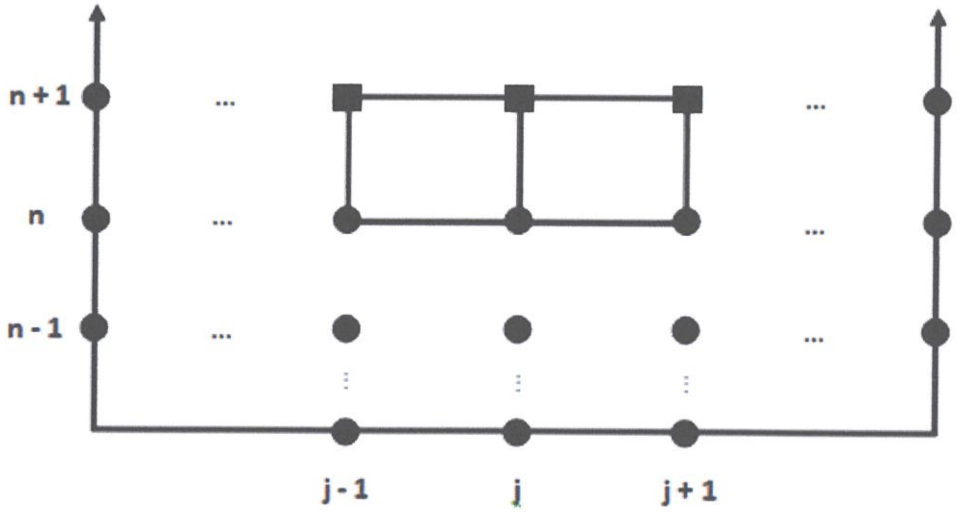

Figure 1.7 Scheme of finite difference with Crank - Nicolson method

4. Finite Difference Method : Lax – Wendroff Method

Lax – Wendroff method is the finite difference method explicitly. It is originated to solve the hyperbolic systems. Its accuracy is maintained to second order in both space and time [11],[12].

The Lax – Wendroff technique by finite difference method for one dimension space is

$$U_j^{n+1} = U_j^n + \frac{l}{4h}\left(a^n + a^{n+1}\right)\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{l^2\left(a^n\right)^2}{2h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right),$$

where U_j^n is approximate solution for the first order hyperbolic systems in discrete space.

However, the stability for this method converges to analytic solution when $\left|\frac{a_n l}{h}\right| < 1$ which a_n is eigenvalues of A^n by von Neumann stability analysis method [11] (see appendix B2).

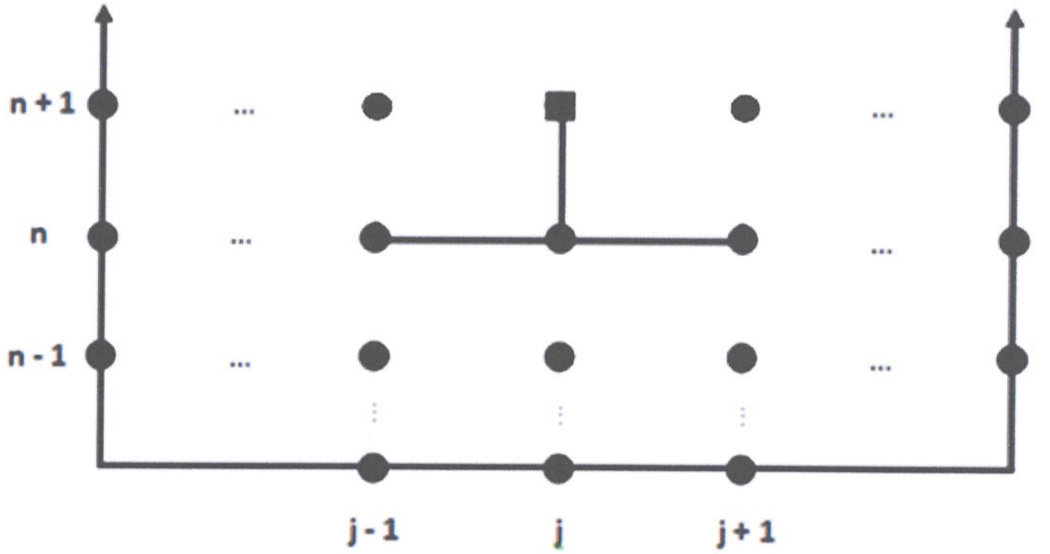

Figure 1.8 Scheme of finite difference with Lax - Wendroff method

5. Objectives for this study

1. To adapted the governing equation of blood flow, small amplitude wave pressure equation, to the blood flow with blockage equation
2. To find initial condition and boundary conditions of blood flow which satisfy to the governing equations.
3. To use numerical methods to compute the solutions of blood flow with blockage case and without blockage, blood pressure and blood velocity solutions.
4. To find analytical solution of blood flow in without blockage and blockage case.
5. To find the precision of solutions by comparing the numerical solution from distinct methods to analytic.

6. Scope of this study

1. To study mathematical model of blood flow in small amplitude wave equation in average and incompressible flow.
2. To study blood flow in slow velocity and small pressure with linear velocity dragged blockage.
3. To study blood flow in the constant compliance blood vessel.

CHAPTER 2

Literature Review

2.1 Small Amplitude Wave Pressure Equations

The small amplitude wave equation describes to phenomena of blood flow in small of velocity and pressure [9]. This theory can be applied to the study of blood flow in blood vessel, especially in arteriole vessels, such as atherosclerosis condition [8], [10], peristaltic motion, the characteristic of blood flow by contraction and dilation of blood vessel [3], [4] ,etc.

The small amplitude wave pressure equations were created by J. Keener and J. Sneyd in 1998 [9]. They used the model of blood flow by assumption that small amplitude wave behavior is reducing of nonlinear terms, velocity and pressure. By conservative of mass or continuity equation and conservative of momentum with non - viscous blood flow (incompressible), they derived the small amplitude wave equation of blood flow (with non - stenose) such that

$$\rho\frac{\partial u}{\partial t}+\frac{\partial P}{\partial x}=0\,, \tag{2.1.1}$$

$$a\frac{\partial P}{\partial t}+A_0\frac{\partial u}{\partial x}=0\,, \tag{2.1.2}$$

where u is velocity of blood flow (cm/s^3),

P is pressure of blood flow (mmHg),

ρ denotes density of blood flow (g/cm^{3}s),

c denotes compliance factor of blood vessel (cm^2/mmHg),

A_0 denotes cross – section of blood vessel (cm^2).

2.2 Finite Difference Method : Crank - Nicolson Method

Crank - Nicolson method is well-known in numerical solution for the finite difference method. In first order hyperbolic systems, it solves with initial – boundary conditions. Its accuracy is maintained to second order and unconditionally stable [11], [12].

Proving of Crank - Nicolson method begins with Taylor series expansion on advection equation such that

$$\frac{\partial S}{\partial t} = A(t)\frac{\partial S}{\partial x}. \tag{2.2.1}$$

where $A(t)$ is coefficient matrix with respect to time t and S is the vector of solutions.

By Taylor series expansion, the forward time difference, we find that

$$S(x,t+l) = S(x,t) + l\frac{\partial S}{\partial t} + \ldots \tag{2.2.2}$$

where $l = \Delta t$.

From equation (2.2.2), we rearrange $\frac{\partial S}{\partial t}$ as

$$\frac{\partial}{\partial t}S(x,t) = \frac{S(x,t+l) - S(x,t)}{l}. \tag{2.2.3}$$

For centered – difference at time t, we find that

$$S(x+h,t) = S(x,t) + h\frac{\partial}{\partial x}S(x,t) + O(h^2,k^2), \tag{2.2.4}$$

$$S(x-h,t) = S(x,t) - h\frac{\partial}{\partial x}S(x,t) + O(h^2,k^2). \tag{2.2.5}$$

where $h = \Delta x$.

Taking equation (2.2.4) – (2.2.5), therefore

$$\frac{\partial}{\partial x}S(x,t) = \frac{S(x+h,t) - S(x-h,t)}{2h}. \tag{2.2.6}$$

Substituting (2.2.3) and (2.2.6) into (2.2.1), therefore

$$\frac{S(x,t+l)-S(x,t)}{l}=A(t)\frac{S(x+h,t)-S(x-h,t)}{2h}. \qquad (2.2.7)$$

For advection equation at time $t+l$, we find that

$$\frac{\partial S}{\partial t}=A(t+l)\frac{\partial S}{\partial x}, \qquad (2.2.8)$$

which $A(t+l)$ is coefficient with respect to time $t+l$.

And Taylor series expansion of forward time difference for time $t+l$ is

$$S(x+h,t+l)=S(x,t)+h\frac{\partial}{\partial x}S(x,t)+l\frac{\partial}{\partial t}S(x,t)+O(h^2,l^2), \qquad (2.2.9)$$

$$S(x-h,t+l)=S(x,t)+h\frac{\partial}{\partial x}S(x,t)-l\frac{\partial}{\partial t}S(x,t)+O(h^2,l^2). \qquad (2.2.10)$$

Taking equation (2.2.9) – (2.2.10), therefore

$$\frac{\partial}{\partial x}S(x,t)=\frac{S(x+h,t+l)-S(x-h,t+l)}{2h}. \qquad (2.2.11)$$

Substituting (2.2.3) and (2.2.11) into (2.2.8), therefore

$$\frac{S(x,t+l)-S(x,t)}{l}=A(t+l)\frac{S(x+h,t+l)-S(x-h,t+l)}{2h}. \qquad (2.2.12)$$

Averaging the eqs (2.2.7) and (2.2.12), we can see that

$$\frac{S(x,t+l)-S(x,t)}{l}=\frac{1}{2}\left[\left(A(t)\frac{S(x+h,t)-S(x-h,t)}{2h}\right)+\left(A(t+l)\frac{S(x+h,t+l)-S(x-h,t+l)}{2h}\right)\right],$$

$$\frac{S(x,t+l)-S(x,t)}{l}=\left(A(t)\frac{S(x+h,t)-S(x-h,t)}{4h}\right)+\left(A(t+l)\frac{S(x+h,t+l)-S(x-h,t+l)}{4h}\right),$$

$$S(x,t+l)-S(x,t)=\left(\frac{lA(t)}{4h}\big(S(x+h,t)-S(x-h,t)\big)\right)+\left(\frac{lA(t+l)}{4h}\big(S(x+h,t+l)-S(x-h,t+l)\big)\right),$$

$$S(x,t+l)-\frac{lA(t+l)}{4h}\big(S(x+h,t+l)-S(x-h,t+l)\big)=S(x,t)+\frac{lA(t)}{4h}\big(S(x+h,t)-S(x-h,t)\big). \qquad (2.2.13)$$

From differential equation of (2.2.13) we can approximate to difference equation is [14]

$$\tilde{S}(x,t+l)-\frac{l\tilde{A}(t+l)}{4h}\left(\tilde{S}(x+h,t+l)-\tilde{S}(x-h,t+l)\right)=\tilde{S}(x,t)+\frac{l\tilde{A}(t)}{4h}\left(\tilde{S}(x+h,t)-\tilde{S}(x-h,t)\right), \quad (2.2.14)$$

where $\tilde{S}(x,t)$ and $\tilde{A}(t)$ are difference functions that satisfy to $S(x,t)$ and $A(t)$ respectively.

We can approximate $S_j^n \approx S(x_j,t_n)$ and $A^n \approx A(t_n)$ and substitutes in (2.2.14) is

$$S_j^{n+1}-\frac{lA^{n+1}}{4h}\left(S_{j+1}^{n+1}-S_{j-1}^{n+1}\right)=S_j^n+\frac{lA^n}{4h}\left(S_{j+1}^n-S_{j-1}^n\right). \quad (2.2.15)$$

The equation (2.2.15) is called Crank - Nicolson finite difference equation. For stability analysis, this method is unconditionally stable by von Neumann stability analysis method [11].

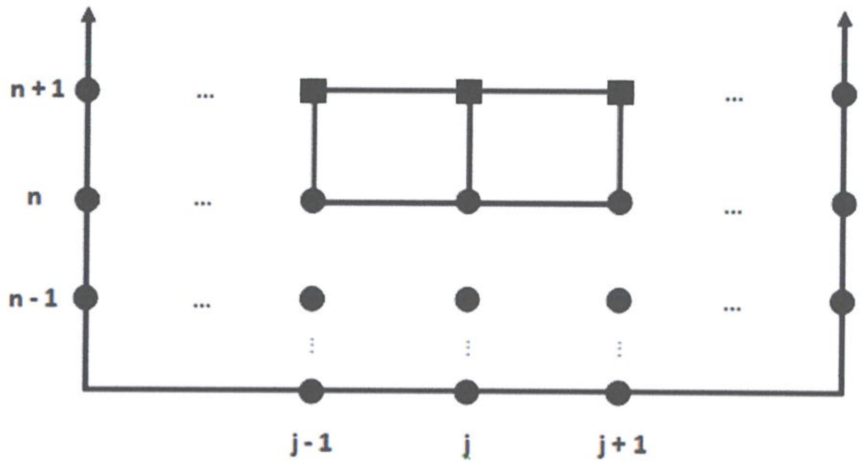

Figure 2.1 Scheme of finite difference Crank - Nicolson method

2.3 Finite Difference Method : Lax - Wendroff Method

The proof of Lax - Wendroff method begins with Taylor series expansion on advection equation such that

$$\frac{\partial S}{\partial t}=A(t)\frac{\partial S}{\partial x}, \quad (2.3.1)$$

where $A(t)$ is coefficient matrix with respect to time t and S is the vector of solutions.

By Taylor series expansion, forward time difference, we can see that

$$S(x,t+l)=S(x,t)+l\frac{\partial}{\partial t}S(x,t)+\frac{l^2}{2}\frac{\partial^2}{\partial t^2}S(x,t)+\dots. \tag{2.3.2}$$

Substituting (2.3.1) into (2.3.2), we obtain that

$$S(x,t+l)=S(x,t)+lA(t)\frac{\partial}{\partial x}S(x,t)+\frac{l^2}{2}\frac{\partial^2}{\partial t^2}S(x,t)+\dots. \tag{2.3.3}$$

Furthermore, we can replace $\frac{\partial^2}{\partial t^2}S(x,t)$ by using

$$\frac{\partial^2}{\partial t^2}S(x,t)=\frac{\partial A}{\partial t}\frac{\partial S}{\partial x}+A(t)\frac{\partial^2 S}{\partial t\partial x}, \tag{2.3.4}$$

$$\frac{\partial^2 S}{\partial x\partial t}=A(t)\frac{\partial^2 S}{\partial x\partial x}. \tag{2.3.5}$$

Since $\frac{\partial^2 S}{\partial t\partial x}=\frac{\partial^2 S}{\partial x\partial t}$, and substituting (2.3.5) into (2.3.4), so

$$\frac{\partial^2}{\partial t^2}S(x,t)=\frac{\partial A}{\partial t}\frac{\partial S}{\partial x}+A^2\frac{\partial^2 S}{\partial x\partial x}. \tag{2.3.6}$$

Substituting (2.3.6) into (2.3.3), we obtain that

$$S(x,t+l)=S(x,t)+lA(t)\frac{\partial}{\partial x}S(x,t)+\frac{l^2}{2}\left(\frac{\partial A}{\partial t}\frac{\partial S}{\partial x}+A^2\frac{\partial^2 S}{\partial x^2}\right)+\dots,$$

$$S(x,t+l)=S(x,t)+\left(lA(t)+\frac{l^2}{2}\frac{\partial A}{\partial t}\right)\frac{\partial S}{\partial x}+\frac{l^2}{2}A^2(t)\frac{\partial^2 S}{\partial x^2}+\dots \tag{2.3.7}$$

From differential equation of (2.3.7), we can approximate to difference equation is

$$\tilde{S}(x,t+l)=\tilde{S}(x,t)+\left(lA(t)+\frac{l^2}{2}\frac{\partial A}{\partial t}\right)\frac{\partial \tilde{S}}{\partial x}+\frac{l^2}{2}A^2(t)\frac{\partial^2 \tilde{S}}{\partial x^2}+\dots. \tag{2.3.8}$$

where $\tilde{S}(x,t)$ is difference function that satisfies to $S(x,t)$.

We can approximate derivative difference of $\frac{\partial}{\partial x}\tilde{S}(x,t)$ and $\frac{\partial^2}{\partial x^2}\tilde{S}(x,t)$. By using centered space – finite difference, they are

$$\frac{\partial}{\partial x}\tilde{S}(x,t) \approx \frac{S_{j+1}^{n} - S_{j-1}^{n}}{2h}, \tag{2.3.9}$$

$$\frac{\partial^2}{\partial x^2}\tilde{S}(x,t) \approx \frac{S_{j+1}^{n} - 2S_j^n + S_{j-1}^{n}}{h^2}. \tag{2.3.10}$$

where $S_j^n \approx S(x_j, t_n)$.

Substituting (2.3.9), (2.3.10) ,and rearrange to finite difference equation as

$$S_j^{n+1} = S_j^n + \left(lA(t) + \frac{l^2}{2}\frac{\partial A}{\partial t}\right)\left(\frac{S_{j+1}^{n} - S_{j-1}^{n}}{2h}\right) + \frac{l^2}{2}A^2(t)\left(\frac{S_{j+1}^{n} - 2S_j^n + S_{j-1}^{n}}{h^2}\right),$$

$$S_j^{n+1} = S_j^n + \left(\frac{lA(t)}{2h} + \frac{l^2}{4h}\frac{\partial A}{\partial t}\right)\left(S_{j+1}^{n} - S_{j-1}^{n}\right) + \frac{l^2 A^2(t)}{2h^2}\left(S_{j+1}^{n} - 2S_j^n + S_{j-1}^{n}\right). \tag{2.3.11}$$

From (2.3.11), $\frac{\partial A}{\partial t} \approx \frac{A(t+l) - A(t)}{l} = \frac{A^{n+1} - A^n}{l}$ for $A^n = A(t)$. So,

$$S_j^{n+1} = S_j^n + \left(\frac{lA^n}{2h} + \frac{l}{4h}\left(A^{n+1} - A^n\right)\right)\left(S_{j+1}^{n} - S_{j-1}^{n}\right) + \frac{l^2\left(A^n\right)^2}{2h^2}\left(S_{j+1}^{n} - 2S_j^n + S_{j-1}^{n}\right),$$

$$S_j^{n+1} = S_j^n + \left(\frac{l}{4h}\left(A^{n+1} + A^n\right)\right)\left(S_{j+1}^{n} - S_{j-1}^{n}\right) + \frac{l^2\left(A^n\right)^2}{2h^2}\left(S_{j+1}^{n} - 2S_j^n + S_{j-1}^{n}\right). \tag{2.3.12}$$

The equation (2.3.12) is called Lax – Wendroff finite difference equation.

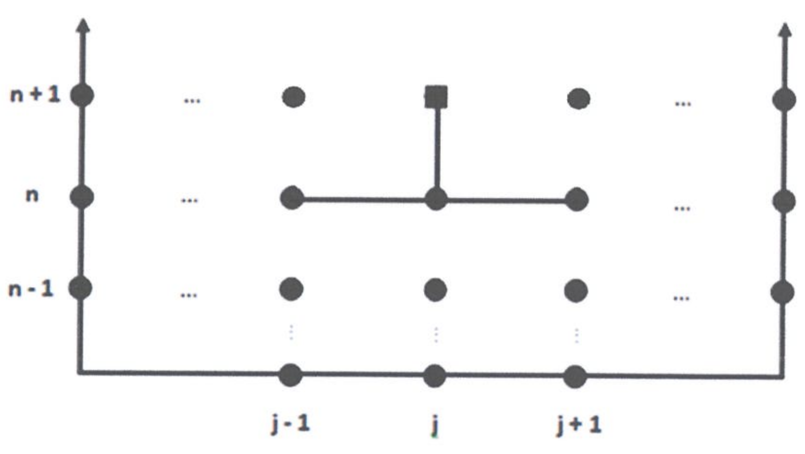

Figure 2.2 Scheme of finite difference with Lax - Wendroff method

2.4 The compliance of blood vessel model

The compliance of blood vessel model is the simple form of blood flow with elastic blood vessel [9]. The definition in first paper is defined by Spensor and Denisson in 1963. The variation of arterial blood volume due to a given change in arterial blood pressure is described to mathematical equation as

$$C = \frac{\Delta V}{\Delta P}, \tag{2.3.1}$$

where C as arterial blood vessel compliance, V is arterial blood volume, and P is arterial blood pressure.

Despite that this definition rearranges to cross – sectional compliance model which is utilized in clinical practice. In 1986, Reneman et al. have determined this principle was the ratio between variation of cross – sectional area and blood pressure which were described to

$$c = \frac{\Delta A}{\Delta P}, \tag{2.3.2}$$

where c as cross section of arterial blood vessel compliance, A is cross section of arterial blood volume, and P is arterial blood pressure.

In this research, we choose the definition of arterial compliance model by equation (2.3.2) to derive the equation of blood flow with stenostic blockage in section 3.4.

2.5 Drag and Damped force

The drag force is the reaction force from the Newton's third law with respect to relative motion of moving bodies in fluid. It is defined as [20],

$$f_{drag} = -cv \tag{2.4.1}$$

where f_{drag} is linear drag force, v is moving bodies velocity within fluid, and c is drag coefficient. By using Newton's third law : active force from the moving bodies in fluid is equal to passive force from fluid for resistant of bodies dynamics within fluid. We can proximate the fluid flow in small velocity. However, it cannot be referred in large velocity, turbulent flow of fluid. While damped force is represented to the friction force of "simple harmonics motion" (figure 2.3) which is written in Newton's law of motion as [21], [22]

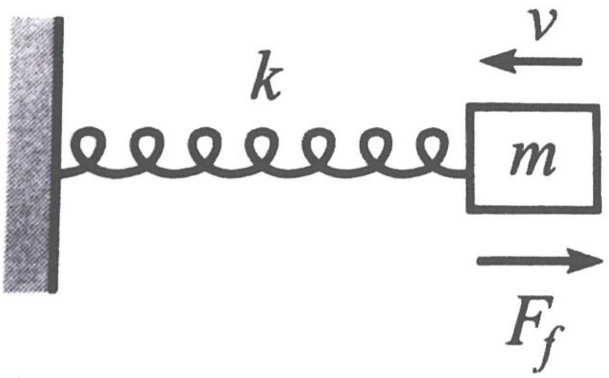

Figure 2.2 Damped harmonics oscillator [21]

$$m\frac{dv}{dt} = -bv - kx\,, \tag{2.4.2}$$

where m is mass at the end of spring, b is frictional coefficient, and k is spring constant.

We can write the derivative from of the equation as

$$m\ddot{x} + b\dot{x} + kx = 0\,, \tag{2.4.3}$$

Where $\dot{x} = \dfrac{dx}{dt}$.

CHAPTER 3

Research Methodology

3.1 The Crank – Nicolson method for the blood flow in small amplitude wave equation without blockage

We assume the equation systems of blood flow in small amplitude without blockage,

$$\rho\frac{\partial u}{\partial t}+\frac{\partial P}{\partial x}=0\,, \tag{2.1.1}$$

$$c\frac{\partial P}{\partial t}+A_0\frac{\partial u}{\partial x}=0\,. \tag{2.1.2}$$

Rearrange (2.1.1) by divided with ρ and (2.1.2) with c. Then,

$$\frac{\partial u}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial x}=0\,, \tag{3.1.1}$$

$$\frac{\partial P}{\partial t}+\frac{A_0}{c}\frac{\partial u}{\partial x}=0\,. \tag{3.1.2}$$

From (3.1.1) and (3.1.2), we can write these equations into matrix form as

$$\frac{\partial}{\partial t}\begin{pmatrix}u\\P\end{pmatrix}+\begin{bmatrix}0 & \frac{1}{\rho}\\ \frac{A_0}{c} & 0\end{bmatrix}\frac{\partial}{\partial x}\begin{pmatrix}u\\P\end{pmatrix}=\underline{0}\,. \tag{3.1.3}$$

Equation (3.1.3), the finite difference method for hyperbolic system, is solved by using Crank - Nicolson method. Let $U(x,t)=\begin{pmatrix}u(x,t)\\P(x,t)\end{pmatrix}$ which $u(x,t)$ and $P(x,t)$ are the solution of velocity and pressure of blood flow, respectively. So, eq (3.1.3) can be written as

$$\frac{\partial}{\partial t}U + \alpha \frac{\partial}{\partial x}U = \underline{0}, \tag{3.1.4}$$

where $\alpha = \begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{A_0}{c} & 0 \end{bmatrix}$.

However, in discrete space, we can let. Let $u(x_m, t_n)$ and $P(x_m, t_n)$ are velocity and pressure of blood flow respectively, which $x_m = m\Delta x$ and $t_n = n\Delta t$. We can let

$$U_m^n = \begin{pmatrix} u(x_m, t_n) \\ P(x_m, t_n) \end{pmatrix} = \begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix}. \tag{3.1.5}$$

By using Crank – Nicolson equation in (2.2.15), we can derive the finite difference equation of Crank - Nicolson method for equation (3.1.4) which can be written as

$$\begin{Bmatrix} u_m^{n+1} \\ P_m^{n+1} \end{Bmatrix} - \frac{\nu}{4}\left(\begin{Bmatrix} P_{m+1}^{n+1} \\ u_{m+1}^{n+1} \end{Bmatrix} - \begin{Bmatrix} P_{m-1}^{n+1} \\ u_{m-1}^{n+1} \end{Bmatrix} \right) = \begin{Bmatrix} u_m^n \\ P_m^n \end{Bmatrix} + \frac{\nu}{4}\left(\begin{Bmatrix} P_{m+1}^n \\ u_{m+1}^n \end{Bmatrix} - \begin{Bmatrix} P_{m-1}^n \\ u_{m-1}^n \end{Bmatrix} \right), \tag{3.1.6}$$

where $\nu = \frac{\Delta t}{\Delta x}\begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{A_0}{c} & 0 \end{bmatrix}$ as

$$u_m^{n+1} - \frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\left(\frac{1}{\rho}\right)\left[P_{m+1}^{n+1} - P_{m-1}^{n+1}\right] = u_j^n + \frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\left(\frac{1}{\rho}\right)\left[P_{m+1}^n - P_{m-1}^n\right], \tag{3.1.7}$$

$$P_m^{n+1} - \frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\left(\frac{A_0}{c}\right)\left[u_{m+1}^{n+1} - u_{m-1}^{n+1}\right] = P_j^n + \frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\left(\frac{A_0}{c}\right)\left[u_{m+1}^n - u_{m-1}^n\right]. \tag{3.1.8}$$

However, eq (3.1.6) can be rewritten in matrix form as

$$C^{n+1}\mathbf{W}^{n+1} = D^n\mathbf{W}^n + \mathbf{F}^n. \tag{3.1.7}$$

where

$$C^{n+1} = \begin{bmatrix} 1 & 0 & 0 & -\frac{\lambda}{4}\frac{1}{\rho} & 0 & 0 & \cdots & 0 \\ 0 & 1 & -\frac{\lambda}{4}\frac{A_0}{c} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\lambda}{4}\frac{1}{\rho} & 1 & 0 & 0 & -\frac{\lambda}{4}\frac{1}{\rho} & \cdots & 0 \\ \frac{\lambda}{4}\frac{A_0}{c} & 0 & 0 & 1 & -\frac{\lambda}{4}\frac{A_0}{c} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \frac{\lambda}{4}\frac{1}{\rho} & 1 & 0 \\ 0 & \cdots & 0 & 0 & \frac{\lambda}{4}\frac{A_0}{c} & 0 & 0 & 1 \end{bmatrix},$$

$$D^{n} = \begin{bmatrix} 1 & 0 & 0 & \frac{\lambda}{4}\frac{1}{\rho} & 0 & 0 & \cdots & 0 \\ 0 & 1 & \frac{\lambda}{4}\frac{A_0}{c} & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\frac{\lambda}{4}\frac{1}{\rho} & 1 & 0 & 0 & \frac{\lambda}{4}\frac{1}{\rho} & \cdots & 0 \\ -\frac{\lambda}{4}\frac{A_0}{c} & 0 & 0 & 1 & \frac{\lambda}{4}\frac{A_0}{c} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & -\frac{\lambda}{4}\frac{1}{\rho} & 1 & 0 \\ 0 & \cdots & 0 & 0 & -\frac{\lambda}{4}\frac{A_0}{c} & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{F}^{n} = \begin{Bmatrix} -\frac{\lambda}{4}\frac{1}{\rho}P_0^{n+1} - \frac{\lambda}{4}\frac{1}{\rho}P_0^{n} \\ -\frac{\lambda}{4}\frac{A_0}{c}u_0^{n+1} - \frac{\lambda}{4}\frac{A_0}{c}u_0^{n} \\ 0 \\ 0 \\ \vdots \\ \frac{\lambda}{4}\frac{1}{\rho}P_M^{n+1} + \frac{\lambda}{4}\frac{1}{\rho}P_M^{n} \\ \frac{\lambda}{4}\frac{A_0}{c}u_M^{n+1} + \frac{\lambda}{4}\frac{A_0}{c}u_M^{n} \end{Bmatrix},$$

$$\mathbf{W}^n = \begin{Bmatrix} u_1^n \\ P_1^n \\ \vdots \\ u_{M-1}^n \\ P_{M-1}^n \end{Bmatrix},$$

where $t_n = n\Delta t$, $m = 1,2,3,\cdots,M-1$, $n = 0,1,2,\cdots,N$, and $\lambda = \dfrac{\Delta t}{\Delta x}$. The results of this method shows at results section. The stability condition is unconditional stable (see appendix B.1).

3.2 The Lax – Wendroff method for the blood flow in small amplitude wave equation without blockage

We assume the equation systems of blood flow in small amplitude without blockage which is

$$\frac{\partial}{\partial t}U + \alpha\frac{\partial}{\partial x}U = \underline{0}, \qquad (3.1.4)$$

where $\alpha = \begin{bmatrix} 0 & \dfrac{1}{\rho} \\ \dfrac{A_0}{c} & 0 \end{bmatrix}$ and $U = \begin{pmatrix} u(x,t) \\ P(x,t) \end{pmatrix}$.

If we let the solution in discrete space be

$$U_m^n = \begin{pmatrix} u(x_m,t_n) \\ P(x_m,t_n) \end{pmatrix} = \begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix}. \qquad (3.1.5)$$

By using Lax - Wendroff equation in (2.3.12), we can derive the finite difference equation of Lax – Wendroff method for equation (3.1.4) as

$$\begin{pmatrix} u_m^{n+1} \\ P_m^{n+1} \end{pmatrix} = \begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix} + \frac{l}{4h}\left(a^n + a^{n+1}\right)\left(\begin{pmatrix} u_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - \begin{pmatrix} u_{m-1}^n \\ P_{m-1}^n \end{pmatrix}\right) + \frac{l^2\left(a^n\right)^2}{2h^2}\left(\begin{pmatrix} u_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - 2\begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix} + \begin{pmatrix} u_{m-1}^n \\ P_{m-1}^n \end{pmatrix}\right), \qquad (3.2.1)$$

Where $l = \Delta t, h = \Delta x$ and $a^n = a(t_n)$.

However, we find that coefficient matrix $a(t)$ in equation (3.1.4) is $\alpha = \begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{A_0}{c} & 0 \end{bmatrix}$ which does not depend on time. So that,

$$a^n = a^{n+1}. \tag{3.2.2}$$

Thus,

$$\begin{pmatrix} u_m^{n+1} \\ P_m^{n+1} \end{pmatrix} = \begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix} + \frac{l}{4h}(a^n + a^n)\left(\begin{pmatrix} u_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - \begin{pmatrix} u_{m-1}^n \\ P_{m-1}^n \end{pmatrix}\right) + \frac{l^2(a^n)^2}{2h^2}\left(\begin{pmatrix} u_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - 2\begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix} + \begin{pmatrix} u_{m-1}^n \\ P_{m-1}^n \end{pmatrix}\right),$$

$$\begin{pmatrix} u_m^{n+1} \\ P_m^{n+1} \end{pmatrix} = \begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix} + \frac{la^n}{2h}\left(\begin{pmatrix} u_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - \begin{pmatrix} u_{m-1}^n \\ P_{m-1}^n \end{pmatrix}\right) + \frac{l^2(a^n)^2}{2h^2}\left(\begin{pmatrix} u_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - 2\begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix} + \begin{pmatrix} u_{m-1}^n \\ P_{m-1}^n \end{pmatrix}\right),$$

$$\begin{pmatrix} u_m^{n+1} \\ P_m^{n+1} \end{pmatrix} = \begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix} + \frac{l}{2h}\begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{A_0}{c} & 0 \end{bmatrix}\left(\begin{pmatrix} u_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - \begin{pmatrix} u_{m-1}^n \\ P_{m-1}^n \end{pmatrix}\right) + \frac{l^2}{2h^2}\begin{bmatrix} \frac{A_0}{\rho c} & 0 \\ 0 & \frac{A_0}{\rho c} \end{bmatrix}\left(\begin{pmatrix} u_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - 2\begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix} + \begin{pmatrix} u_{m-1}^n \\ P_{m-1}^n \end{pmatrix}\right). \tag{3.2.3}$$

The equation (3.2.3) is used to find numerical data of blood flow with non – blockage. The stability condition is $0 < \nu \leq 1$ which $\nu = \frac{l}{h}\begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{A_0}{c} & 0 \end{bmatrix}$ (see appendix B.2).

3.3 The Crank – Nicolson method of the blood flow in small amplitude wave equation with linear velocity dragged blockage

The equation systems of blood flow in small amplitude is added the dragged blockage term in equation (2.2.1), we can obtain that

$$\rho\frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} = -ku, \tag{2.2.1}$$

$$c\frac{\partial P}{\partial t} + A_0\frac{\partial u}{\partial x} = 0. \tag{2.2.2}$$

Dividing (2.2.1) with ρ and (2.2.2) with c. Then,

$$\frac{\partial u}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial x}=-\frac{k}{\rho}u, \tag{3.3.1}$$

$$\frac{\partial P}{\partial t}+\frac{A_0}{c}\frac{\partial u}{\partial x}=0. \tag{3.3.2}$$

We are going to reform the equations (3.3.1) and (3.3.2) into homogeneous equations, which each one of them is equal to zero. Therefore, we assume $u(x,t)=v(x,t)e^{-\frac{k}{\rho}t}$. It follows by [17] and substitutes into equation (3.3.1) such that

$$\frac{\partial}{\partial t}\left(v(x,t)e^{-\frac{k}{\rho}t}\right)+\frac{1}{\rho}\frac{\partial}{\partial x}P=-\frac{k}{\rho}\left(v(x,t)e^{-\frac{k}{\rho}t}\right), \tag{3.3.3}$$

$$e^{-\frac{k}{\rho}t}\left(\frac{\partial}{\partial t}v(x,t)\right)+v(x,t)\left(-\frac{k}{\rho}e^{-\frac{k}{\rho}t}\right)+\frac{1}{\rho}\frac{\partial}{\partial x}P=-\frac{k}{\rho}\left(v(x,t)e^{-\frac{k}{\rho}t}\right), \tag{3.3.4}$$

$$e^{-\frac{k}{\rho}t}\left(\frac{\partial}{\partial t}v(x,t)\right)+\frac{1}{\rho}\frac{\partial}{\partial x}P=-\frac{k}{\rho}\left(v(x,t)e^{-\frac{k}{\rho}t}\right)-v(x,t)\left(-\frac{k}{\rho}e^{-\frac{k}{\rho}t}\right), \tag{3.3.5}$$

$$e^{-\frac{k}{\rho}t}\left(\frac{\partial}{\partial t}v(x,t)\right)+\frac{1}{\rho}\frac{\partial}{\partial x}P=-\frac{k}{\rho}v(x,t)e^{-\frac{k}{\rho}t}+v(x,t)\frac{k}{\rho}e^{-\frac{k}{\rho}t}, \tag{3.3.6}$$

$$e^{-\frac{k}{\rho}t}\left(\frac{\partial}{\partial t}v(x,t)\right)+\frac{1}{\rho}\frac{\partial}{\partial x}P=0. \tag{3.3.7}$$

Multiplying $e^{\frac{k}{\rho}t}$ into equation (3.3.7),

$$\left(\frac{\partial}{\partial t}v(x,t)\right)+\frac{1}{\rho}e^{\frac{k}{\rho}t}\frac{\partial}{\partial x}P=0,$$

$$v_t+\frac{1}{\rho}e^{\frac{k}{\rho}t}P_x=0. \tag{3.3.8}$$

Similarly, substituting $u(x,t)=v(x,t)e^{-\frac{k}{\rho}t}$ into equation (3.3.2), then

$$\frac{\partial}{\partial t}P+\frac{A_0}{c}\frac{\partial}{\partial x}\left(v(x,t)e^{-\frac{k}{\rho}t}\right)=0,$$

$$\frac{\partial}{\partial t}P+\frac{A_0}{c}e^{-\frac{k}{\rho}t}\frac{\partial}{\partial x}\left(v(x,t)\right)=0\,,$$

$$P_t+\frac{A_0}{c}e^{-\frac{k}{\rho}t}v_x=0\,. \tag{3.3.9}$$

Equations (3.3.8) and (3.3.9), can be written in a matrix form,

$$\frac{\partial}{\partial t}\begin{pmatrix}v\\P\end{pmatrix}+\begin{bmatrix}0 & \frac{1}{\rho}e^{\frac{k}{\rho}t}\\ \frac{A_0}{c}e^{-\frac{k}{\rho}t} & 0\end{bmatrix}\frac{\partial}{\partial x}\begin{pmatrix}v\\P\end{pmatrix}=\underline{0} \tag{3.3.10}$$

Let $U(x,t)=\begin{pmatrix}u(x,t)\\P(x,t)\end{pmatrix}$ which $u(x,t)$ and $P(x,t)$ are the solution of velocity and pressure of blood flow, respectively. So, eq (3.3.10) can be written as

$$\frac{\partial}{\partial t}U+\alpha\frac{\partial}{\partial x}U=\underline{0}\,, \tag{3.3.11}$$

where $\alpha=\begin{bmatrix}0 & \frac{1}{\rho}e^{\frac{k}{\rho}t}\\ \frac{A_0}{c}e^{-\frac{k}{\rho}t} & 0\end{bmatrix}$.

We can see that the equation (3.3.11) consists of two homogeneous equations. Therefore, we can find the approximate solution in equation (3.3.11) by using Crank - Nicolson method which can be rewritten in Crank - Nicolson equation (2.2.15) as

$$v_m^{n+1}-\frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\left(\frac{1}{\rho}e^{\frac{k}{\rho}(t+dt)}\right)\left[P_{m+1}^{n+1}-P_{m-1}^{n+1}\right]=v_j^n+\frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\left(\frac{1}{\rho}e^{\frac{k}{\rho}t}\right)\left[P_{m+1}^{n}-P_{m-1}^{n}\right], \tag{3.3.12}$$

$$P_m^{n+1}-\frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\left(\frac{A_0}{c}e^{-\frac{k}{\rho}(t+dt)}\right)\left[v_{m+1}^{n+1}-v_{m-1}^{n+1}\right]=P_j^n+\frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\left(\frac{A_0}{c}e^{-\frac{k}{\rho}t}\right)\left[v_{m+1}^{n}-v_{m-1}^{n}\right]. \tag{3.3.13}$$

However, eq (3.3.10) can be rewritten in matrix form as

$$G^{n+1}\mathbf{W}^{n+1}=E^n\mathbf{W}^n+\mathbf{F}^n\,. \tag{3.3.14}$$

where

$$G^{n+1}=\begin{bmatrix} 1 & 0 & 0 & -\frac{\lambda}{4}a_1^{n+1} & 0 & 0 & \cdots & 0 \\ \frac{\lambda}{4}a_2^{n+1} & 1 & -\frac{\lambda}{4}a_2^{n+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\lambda}{4}a_1^{n+1} & 1 & 0 & 0 & -\frac{\lambda}{4}a_1^{n+1} & \cdots & 0 \\ \frac{\lambda}{4}a_2^{n+1} & 0 & 0 & 1 & -\frac{\lambda}{4}a_2^{n+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \frac{\lambda}{4}a_1^{n+1} & 1 & -\frac{\lambda}{4}a_1^{n+1} \\ 0 & \cdots & 0 & 0 & \frac{\lambda}{4}a_2^{n+1} & 0 & -\frac{\lambda}{4}a_2^{n} & 1 \end{bmatrix},$$

$$G^{n}=\begin{bmatrix} 1 & 0 & 0 & \frac{\lambda}{4}a_1^{n} & 0 & 0 & \cdots & 0 \\ -\frac{\lambda}{4}a_2^{n} & 1 & \frac{\lambda}{4}a_2^{n} & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\frac{\lambda}{4}a_1^{n} & 1 & 0 & 0 & \frac{\lambda}{4}a_1^{n} & \cdots & 0 \\ -\frac{\lambda}{4}a_2^{n} & 0 & 0 & 1 & \frac{\lambda}{4}a_2^{n} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & -\frac{\lambda}{4}a_1^{n} & 1 & \frac{\lambda}{4}a_1^{n} \\ 0 & \cdots & 0 & 0 & -\frac{\lambda}{4}a_2^{n} & 0 & \frac{\lambda}{4}a_2^{n} & 1 \end{bmatrix},$$

$$\mathbf{F}^{n}=\begin{Bmatrix} -\frac{\lambda}{4}a_1^{n+1}P_0^{n+1}-\frac{\lambda}{4}\frac{1}{\rho}a_1^{n}P_0^{n} \\ -\frac{\lambda}{4}a_2^{n+1}\Delta x e^{-\frac{k}{\rho}t_{n+1}}v_0^{n+1}-\frac{\lambda}{4}a_2^{n+1}\Delta x e^{-\frac{k}{\rho}t_{n}}v_0^{n} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix},$$

$$\mathbf{W}^n = \begin{Bmatrix} v_1^n \\ P_1^n \\ \vdots \\ v_{M-1}^n \\ v_{M-1}^n \end{Bmatrix},$$

Which $a_1^n = \frac{1}{\rho} e^{\frac{k}{\rho} t_n}$, $a_2^n = \frac{A_0}{c} e^{-\frac{k}{\rho} t_n}$, $t_n = n\Delta t$, $m = 1,2,3,\cdots, M-1$, $n = 0,1,2,\cdots, N$, and

$\lambda = \frac{\Delta t}{\Delta x}$. The results of this method show in the results section. The stability condition is

unconditional stable (see appendix B.1).

3.4 The Lax – Wendroff method for the blood flow in small amplitude wave equation with linear velocity dragged blockage

Similarly, in equation (3.3.11)

$$\frac{\partial}{\partial t} U + \alpha \frac{\partial}{\partial x} U = \underline{0}, \tag{3.3.11}$$

where $\alpha = \begin{bmatrix} 0 & \frac{1}{\rho} e^{\frac{k}{\rho} t} \\ \frac{A_0}{c} e^{-\frac{k}{\rho} t} & 0 \end{bmatrix}$.

We can find the approximate solution in equation (3.3.11) by using Lax - Wendroff method which can be rewritten in Lax - Wendroff equation (3.1.4) as

$$\begin{aligned} \begin{pmatrix} v_m^{n+1} \\ P_m^{n+1} \end{pmatrix} = \begin{pmatrix} v_m^n \\ P_m^n \end{pmatrix} + \frac{\Delta t}{4\Delta x} \begin{bmatrix} 0 & \frac{1}{\rho}\left(e^{\frac{k}{\rho}(n+1)\Delta t} + e^{\frac{k}{\rho} n\Delta t} \right) \\ \frac{A_0}{c}\left(e^{-\frac{k}{\rho}(n+1)\Delta t} + e^{-\frac{k}{\rho} n\Delta t} \right) & 0 \end{bmatrix} \left(\begin{pmatrix} v_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - \begin{pmatrix} v_{m-1}^n \\ P_{m-1}^n \end{pmatrix} \right) \\ + \frac{(\Delta t)^2}{2(\Delta x)^2} \begin{bmatrix} \frac{A_0}{\rho c} & 0 \\ 0 & \frac{A_0}{\rho c} \end{bmatrix} \left(\begin{pmatrix} v_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - 2\begin{pmatrix} v_m^n \\ P_m^n \end{pmatrix} + \begin{pmatrix} v_{m-1}^n \\ P_{m-1}^n \end{pmatrix} \right) \end{aligned} . \tag{3.4.1}$$

Equation (3.4.1) solves the solution of blood pressure and blood vessel with blockage which is described in section 4.2.

3.5 The initial and boundary conditions in small amplitude wave equation without blockage

The initial and boundary conditions are determined by the normal healthy human : blood pressure varies between 80 mmHg to 120 mmHg and blood velocity average is 30 cm/s [8], [9]. We assume that blood wave characteristic at the boundary points and the initial points are

(i) At the initial rod (x = 0)

$$P(0,t)=100+20\cos\left(\sqrt{\frac{A_0}{\rho c}}t\right), \tag{3.5.1}$$

$$u(0,t)=30-20\cos\left(\sqrt{\frac{A_0}{\rho c}}t\right). \tag{3.5.2}$$

(ii) At the end of rod (x = L)

$$P(L,t)=100+20\cos\left(L-\sqrt{\frac{A_0}{\rho c}}t\right), \tag{3.5.3}$$

$$u(L,t)=30-20\cos\left(L-\sqrt{\frac{A_0}{\rho c}}t\right). \tag{3.5.4}$$

(iii) At the initial time (t = 0)

$$P(x,0)=120\text{ mmHg}, \tag{3.5.5}$$

$$u(x,0)=10\text{ cm/s}. \tag{3.5.6}$$

3.6 The initial and boundary conditions in small amplitude wave equation with blockage

The initial and boundary conditions are determined by the normal healthy human which is blood pressure varies between 80 mmHg to 120 mmHg and blood velocity average is 30 cm/s [8], [9]. Furthermore, we assume that both of maximum of blood pressure and blood velocity at the end of blood vessel are decreased by linear dragged blockage. We assume that blood wave characteristic at the boundary points and the initial points are

(i) At the initial rod (x = 0)

$$P(0,t) = 100 + 20\sin\left(\frac{k}{\rho}t\right), \tag{3.6.1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = -20\frac{ck}{\rho A_0}\cos\left(\frac{k}{\rho}t\right). \tag{3.6.2}$$

(ii) At the end of rod (x = L)

$$\left.\frac{\partial P}{\partial x}\right|_{x=L} = 0\,, \tag{3.6.3}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0\,. \tag{3.6.4}$$

(iii) At the initial time (t = 0)

$$P(x,0) = 100 \text{ mmHg}, \tag{3.6.5}$$

$$u(x,0) = 30 \text{ cm/s}. \tag{3.6.6}$$

For the boundary conditions in section 3.5 and 3.6, the relation of sine and cosine function is $\sin(x) = \cos(x+\phi)$ when $\phi = \frac{\pi}{2}$. However, we can replace inital and boundary conditions to other wave functions for the other problems.

3.7 An analytical solutions of blood flows in small amplitude wave equation without blockage

In this research, we transform the first order partial differential equation system to two second order wave equations. The separation of variable method is used to solve both solutions of blood flow in terms of velocity and pressure without blockage.

From using equation (2.1.1) and (2.1.2), such that

$$\rho\frac{\partial}{\partial t}u+\frac{\partial}{\partial x}P=0, \tag{2.1.1}$$

$$c\frac{\partial}{\partial t}P+A_0\frac{\partial}{\partial x}u=0. \tag{2.1.2}$$

Using differentiates (2.1.1) with $\frac{\partial}{\partial t}$, we obtain that

$$\rho\frac{\partial^2}{\partial t^2}u+\frac{\partial^2}{\partial t\partial x}P=0,$$

$$\frac{\partial^2}{\partial t\partial x}P=-\rho\frac{\partial^2}{\partial t^2}u. \tag{3.7.1}$$

Similarly, differentiates (2.1.2) with $\frac{\partial}{\partial x}$, so

$$c\frac{\partial^2}{\partial x\partial t}P+A_0\frac{\partial^2}{\partial x^2}u=0,$$

$$\frac{\partial^2}{\partial x\partial t}P=-\frac{A_0}{c}\frac{\partial^2}{\partial x^2}u. \tag{3.7.2}$$

By Schwarz – Clairaut theorem, symmetric of second partial derivatives, we can assume that

$$\frac{\partial^2}{\partial x\partial t}P=\frac{\partial^2}{\partial t\partial x}P. \tag{3.7.3}$$

Takes eqs (3.7.1) and (3.7.2) into (3.7.3), consequence

$$-\frac{A_0}{c}\frac{\partial^2}{\partial x^2}u=-\rho\frac{\partial^2}{\partial t^2}u,$$

$$\frac{\partial^2}{\partial x^2}u = \frac{\rho c}{A_0}\frac{\partial^2}{\partial t^2}u\,. \tag{3.7.4}$$

Equation (3.7.4) is wave equation of blood flow velocity without blockage.We similarly use this method for derive wave equation of blood pressure without blockage.

To differentiates (2.1.1) with $\frac{\partial}{\partial x}$, we obtain that

$$\rho\frac{\partial^2}{\partial x \partial t}u + \frac{\partial^2}{\partial x^2}P = 0\,,$$

$$\frac{\partial^2}{\partial x \partial t}u = -\frac{1}{\rho}\frac{\partial^2}{\partial x^2}P\,. \tag{3.7.5}$$

Similarly, differentiates (2.1.2) with $\frac{\partial}{\partial t}$, so

$$c\frac{\partial^2}{\partial t^2}P + A_0\frac{\partial^2}{\partial t \partial x}u = 0\,,$$

$$\frac{\partial^2}{\partial t \partial x}u = -\frac{c}{A_0}\frac{\partial^2}{\partial t^2}P\,. \tag{3.7.6}$$

By Schwarz – Clairaut theorem, symmetric of second partial derivatives, we can assume that

$$\frac{\partial^2}{\partial x \partial t}u = \frac{\partial^2}{\partial t \partial x}u\,. \tag{3.7.7}$$

Takes eqs (3.7.5) and (3.7.6) into (3.7.7), consequence

$$-\frac{1}{\rho}\frac{\partial^2}{\partial x^2}P = -\frac{c}{A_0}\frac{\partial^2}{\partial t^2}P\,,$$

$$\frac{\partial^2}{\partial x^2}P = \frac{\rho c}{A_0}\frac{\partial^2}{\partial t^2}P\,. \tag{3.7.8}$$

Equation (3.7.8) is wave equation of blood pressure without blockage.

By separation of variables, we can split the function $u(x,t)$ and $P(x,t)$ into 2 independent variables i.e.

$$u(x,t) = \left(\Phi_1(x)\right)\left(\Theta_1(t)\right)\,. \tag{3.7.9}$$

Which $\Phi_1(x)$ and $\Theta_1(t)$ are the blood velocity function in space and time respectively.

Similarly,

$$P(x,t)=(\Phi_2(x))(\Theta_2(t)), \tag{3.7.10}$$

where $\Phi_2(x)$ and $\Theta_2(t)$ are the blood pressure function in space and time respectively.

Takes (3.7.9) into (3.7.4), we find the blood flows velocity solution in wave equation are

$$\frac{\partial^2}{\partial x^2}\{(\Phi_1(x))(\Theta_1(t))\}=\frac{\rho c}{A_0}\frac{\partial^2}{\partial t^2}\{(\Phi_1(x))(\Theta_1(t))\},$$

$$\frac{1}{(\Phi_1(x))}\frac{\partial^2}{\partial x^2}\Phi_1(x)=\frac{\rho c}{A_0}\frac{1}{(\Theta_1(t))}\frac{\partial^2}{\partial t^2}\Theta_1(t). \tag{3.7.11}$$

Since (3.7.11) is completely separation of variables, we assume that this equation is constant. Consequently,

$$\frac{1}{(\Phi_1(x))}\frac{\partial^2}{\partial x^2}\Phi_1(x)=\frac{\rho c}{A_0}\frac{1}{(\Theta_1(t))}\frac{\partial^2}{\partial t^2}\Theta_1(t)=-\alpha, \tag{3.7.12}$$

where $\alpha\in\mathbb{R}^+$

By, (3.7.12), we can solve $\Phi_1(x)$ and $\Theta_1(t)$ in second order ordinary differential equation form as

$$\frac{1}{(\Phi_1(x))}\frac{\partial^2}{\partial x^2}\Phi_1(x)=-\alpha \text{ and } \frac{\rho c}{A_0}\frac{1}{(\Theta_1(t))}\frac{\partial^2}{\partial t^2}\Theta_1(t)=-\alpha. \tag{3.7.13}$$

Let $\Phi_1(x)=e^{\beta_1 x}$ and $\Theta_1(t)=e^{\beta_2 t}$, we can write the solution of $\Phi_1(x)$ and $\Theta_1(t)$ as

$$\Phi_1(x)=e^{(i\sqrt{\alpha})x}+e^{(-i\sqrt{\alpha})x}, \tag{3.7.14}$$

$$\Theta_1(t)=e^{(i\sqrt{\alpha\frac{A_0}{\rho c}})t}+e^{(-i\sqrt{\alpha\frac{A_0}{\rho c}})t}. \tag{3.7.15}$$

However, the solution of blood velocity is $u(x,t)=(\Phi_1(x))(\Theta_1(t))$, hence

$$u(x,t)=\left(e^{(i\sqrt{\alpha})x}+e^{(-i\sqrt{\alpha})x}\right)\left(e^{(i\sqrt{\alpha\frac{A_0}{\rho c}})t}+e^{(-i\sqrt{\alpha\frac{A_0}{\rho c}})t}\right). \tag{3.7.16}$$

By Euler's identity, eq (3.7.16) can be written as

$$u(x,t)=A_1\cos\left(\sqrt{\alpha}\left(x-\sqrt{\frac{A_0}{\rho c}}t\right)\right)+A_2\sin\left(\sqrt{\alpha}\left(x-\sqrt{\frac{A_0}{\rho c}}t\right)\right). \tag{3.7.17}$$

where A_1, A_2 are constants.

Similarly, we can solve the blood pressure in wave solution as

$$P(x,t)=B_1\cos\left(\sqrt{\alpha}\left(x-\sqrt{\frac{A_0}{\rho c}}t\right)\right)+B_2\sin\left(\sqrt{\alpha}\left(x-\sqrt{\frac{A_0}{\rho c}}t\right)\right). \tag{3.7.18}$$

where B_1 and B_2 are constants.

3.8 An analytical solution of blood flows in small amplitude wave equation with blockage

Eventhough the analytical damped blood velocity solution is complexible, the solution of damping blood pressure can be solved to compare with the numerical solution. From the governing equations,

$$u_t+\frac{1}{\rho}P_x=-\frac{k}{\rho}u, \tag{3.2.1}$$

$$P_t+\frac{A_0}{c}u_x=0. \tag{3.2.2}$$

Assuming that $P=e^{i\frac{k}{\rho}t}q(x)$ and $u=e^{i\frac{k}{\rho}t}w(x)$, equations (3.2.1) and (3.2.2) become

$$\frac{\partial}{\partial t}\left(e^{i\frac{k}{\rho}t}w(x)\right)+\frac{1}{\rho}\frac{\partial}{\partial x}\left(e^{i\frac{k}{\rho}t}q(x)\right)=-\frac{k}{\rho}\left(e^{i\frac{k}{\rho}t}w(x)\right), \tag{3.8.1}$$

$$\frac{\partial}{\partial t}\left(e^{i\frac{k}{\rho}t}q(x)\right)+\frac{A_0}{c}\frac{\partial}{\partial x}\left(e^{i\frac{k}{\rho}t}w(x)\right)=0. \tag{3.8.2}$$

Consequence

$$i\frac{k}{\rho}e^{i\frac{k}{\rho}t}w(x)+\frac{1}{\rho}e^{i\frac{k}{\rho}t}\frac{\partial q}{\partial x}=-\frac{k}{\rho}e^{i\frac{k}{\rho}t}w(x), \tag{3.8.3}$$

$$i\frac{k}{\rho}e^{i\frac{k}{\rho}t}q(x)+\frac{A_0}{c}e^{i\frac{k}{\rho}t}\frac{\partial w}{\partial x}=0. \tag{3.8.4}$$

Thus

$$(i+1)\frac{k}{\rho}w+\frac{1}{\rho}\frac{\partial q}{\partial x}=0, \tag{3.8.5}$$

$$i\frac{k}{\rho}q+\frac{A_0}{c}\frac{\partial w}{\partial x}=0. \tag{3.8.6}$$

Differentiate Eq (3.8.5) with respect to x,

$$(i+1)\frac{k}{\rho}\frac{\partial w}{\partial x}+\frac{1}{\rho}\frac{\partial^2 q}{\partial x^2}=0, \tag{3.8.7}$$

Takes the condition from (3.8.6), so that

$$\frac{\partial w}{\partial x}=-i\frac{ck}{A_0\rho}q \tag{3.8.8}$$

Takes (3.8.8) into (3.8.7), we get that

$$(i+1)\frac{k}{\rho}\left(-i\frac{ck}{A_0\rho}q\right)+\frac{1}{\rho}\frac{\partial^2 q}{\partial x^2}=0,$$

$$\frac{\partial^2 q}{\partial x^2}+(1-i)\left(\frac{ck^2}{A_0\rho}q\right)=0,$$

$$\frac{\partial^2 q}{\partial x^2}=(i-1)\left(\frac{ck^2}{A_0\rho}\right)q. \tag{3.8.9}$$

From (3.8.9), we can solve $q(x)$ as

$$q(x) = Ae^{mx} + Be^{-mx}, \tag{3.8.10}$$

where $m^2 = (i-1)\dfrac{ck^2}{\rho A_0}$.

Equation (3.8.10) can be rewritten as

$$q(x) = A\frac{e^{mx}e^{-mL}}{e^{-mL}} + B\frac{e^{-mx}e^{mL}}{e^{mL}},$$

$$q(x) = Ce^{-mL+mx} + De^{mL-mx},$$

$$q(x) = Ce^{-m(L-x)} + De^{m(L-x)}. \tag{3.8.11}$$

where C, D are arbitrary constants.

And we can rewrite (3.8.11) as

$$q(x) = a\cosh(m(L-x)). \tag{3.8.12}$$

where a is arbitrary constant.

By boundary condition (3.6.1), $P(0,t) = 100 + 20\cos(\frac{k}{\rho}t)$ when $t > 0$, it implies that

$$P(0,t) = 100 + 20\cos\left(\frac{k}{\rho}t\right),$$

$$P(0,t) = 20\left(50 + \cos\left(\frac{k}{\rho}t\right)\right). \tag{3.8.13}$$

However, $P(x,t) = e^{i\frac{k}{\rho}t}q(x)$, it implies (3.8.3) that

$$e^{i\frac{k}{\rho}t}q(0) = 20\left(50 + \cos\left(\frac{k}{\rho}t\right)\right). \tag{3.8.14}$$

$$q(0) = 20. \tag{3.8.15}$$

From (3.8.12), we give that

$$q(0) = a\cosh(mL). \tag{3.8.16}$$

It implies to (3.8.15) that

$$20 = a\cosh(mL). \tag{3.8.17}$$

Consequently,

$$a = \frac{20}{\cosh(mL)}. \tag{3.8.18}$$

Substituting (3.8.18) into (3.8.12), so

$$q(x) = 20\frac{\cosh(m(L-x))}{\cosh(mL)}. \tag{3.8.19}$$

Consider m from (3.8.10),

$$m^2 = (i-1)\frac{ck^2}{\rho A_0}. \tag{3.8.20}$$

So, we give

$$m = \left(k\sqrt{\frac{c}{\rho A_0}}\right)\sqrt{i-1}. \tag{3.8.21}$$

Since $\sqrt{i-1} = 2^{1/4}e^{\frac{3\pi}{8}i}$, we have

$$m = k\sqrt{\frac{c}{\rho A_0}}2^{1/4}e^{\frac{3}{8}\pi i}. \tag{3.8.22}$$

Equation (3.8.22) can be rewritten in trigonometric form as

$$m = k\sqrt{\frac{c}{\rho A_0}}2^{1/4}\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right). \tag{3.8.23}$$

Let $k\sqrt{\frac{c}{\rho A_0}}2^{1/4}\cos\frac{3\pi}{8}=\alpha, k\sqrt{\frac{c}{\rho A_0}}2^{1/4}\sin\frac{3\pi}{8}=\beta$, we therefore give that

$$m=\alpha+i\beta. \tag{3.8.24}$$

From (3.8.19), we transform $q(x)$ for easy to solve it is

$$q(x)=20\left(\frac{e^{m(L-x)}+e^{-m(L-x)}}{2}\right)\left(\frac{2}{e^{mL}+e^{-mL}}\right),$$

$$q(x)=20\left(\frac{e^{m(L-x)}+e^{-m(L-x)}}{e^{mL}+e^{-mL}}\right),$$

$$q(x)=20\left(\frac{e^{(\alpha+i\beta)(L-x)}+e^{-(\alpha+i\beta)(L-x)}}{e^{(\alpha+i\beta)L}+e^{-(\alpha+i\beta)L}}\right),$$

$$q(x)=20\left(\frac{e^{\alpha(L-x)}\left(\cos(\beta(L-x))+i\sin(\beta(L-x))\right)+e^{-\alpha(L-x)}\left(\cos(\beta(L-x))-i\sin(\beta(L-x))\right)}{e^{\alpha L}\left(\cos(\beta L)+i\sin(\beta L)\right)+e^{-\alpha L}\left(\cos(\beta L)-i\sin(\beta L)\right)}\right),$$

$$q(x)=20\left(\frac{\cos(\beta(L-x))\left(e^{\alpha(L-x)}+e^{-\alpha(L-x)}\right)+i\sin(\beta(L-x))\left(e^{\alpha(L-x)}-e^{-\alpha(L-x)}\right)}{\cos(\beta L)\left(e^{\alpha L}+e^{-\alpha L}\right)+i\sin(\beta L)\left(e^{\alpha L}-e^{-\alpha L}\right)}\right),$$

$$q(x)=20\left(\frac{\cos(\beta(L-x))\left(\frac{e^{\alpha(L-x)}+e^{-\alpha(L-x)}}{2}\right)+i\sin(\beta(L-x))\left(\frac{e^{\alpha(L-x)}-e^{-\alpha(L-x)}}{2}\right)}{\cos(\beta L)\left(\frac{e^{\alpha L}+e^{-\alpha L}}{2}\right)+i\sin(\beta L)\left(\frac{e^{\alpha L}-e^{-\alpha L}}{2}\right)}\right). \tag{3.8.25}$$

Since $\frac{e^{\theta}+e^{-\theta}}{2}=\cosh(\theta), \frac{e^{\theta}-e^{-\theta}}{2}=\sinh(\theta)$, equation (3.8.25) therefore rearranges as

$$q(x)=20\left(\frac{\cos(\beta(L-x))\cosh(\alpha(L-x))+i\sin(\beta(L-x))\sinh(\alpha(L-x))}{\cos(\beta L)\cosh(\alpha L)+i\sin(\beta L)\sinh(\alpha L)}\right),$$

$$\begin{aligned}&=20\left(\frac{\cos(\beta(L-x))\cosh(\alpha(L-x))\cos(\beta L)\cosh(\alpha L)+\sin(\beta(L-x))\sinh(\alpha(L-x))\sin(\beta L)\sinh(\alpha L)}{\cos^2(\beta L)\cosh^2(\alpha L)+\sin^2(\beta L)\sinh^2(\alpha L)}\right)\\&+20i\left(\frac{\sin(\beta(L-x))\sinh(\alpha(L-x))\cos(\beta L)\cosh(\alpha L)-\sin(\beta L)\sinh(\alpha L)\cos(\beta(L-x))\cosh(\alpha(L-x))}{\cos^2(\beta L)\cosh^2(\alpha L)+\sin^2(\beta L)\sinh^2(\alpha L)}\right).\end{aligned} \tag{3.8.26}$$

However, $P(x,t)=e^{i\frac{k}{\rho}t}q(x)$, multiplying (3.8.26) with $e^{i\frac{k}{\rho}t}$ and considering in imaginary part only become

$$P(x,t)=100+20\left(\sin\frac{k}{\rho}t\right)\left(\frac{\cos(\beta(L-x))\cosh(\alpha(L-x))\cos(\beta L)\cosh(\alpha L)+\sin(\beta(L-x))\sinh(\alpha(L-x))\sin(\beta L)\sinh(\alpha L)}{\cos^2(\beta L)\cosh^2(\alpha L)+\sin^2(\beta L)\sinh^2(\alpha L)}\right).$$
$$+20\left(\cos\frac{k}{\rho}t\right)\left(\frac{\sin(\beta(L-x))\sinh(\alpha(L-x))\cos(\beta L)\cosh(\alpha L)-\sin(\beta L)\sinh(\alpha L)\cos(\beta(L-x))\cosh(\alpha(L-x))}{\cos^2(\beta L)\cosh^2(\alpha L)+\sin^2(\beta L)\sinh^2(\alpha L)}\right)$$

(3.8.27)

This is the solution of blood pressure for linear dragged blockage. At the end point, $x=L$, equation (3.8.27) can be rewritten as

$$P(L,t)=100+20\left(\frac{\sin\left(\frac{k}{\rho}t\right)\cos(\beta L)\cosh(\alpha L)-\cos\left(\frac{k}{\rho}t\right)\sin(\beta L)\sinh(\alpha L)}{\cos^2(\beta L)\cosh^2(\alpha L)+\sin^2(\beta L)\sinh^2(\alpha L)}\right). \quad (3.8.28)$$

CHAPTER 4

Numerical Results

In this section, both of the governing equations in (2.1.1) and (2.1.2), blood flow of small amplitude equation for non - blockage, and equations (2.2.1) and (2.2.2), blood flow of small amplitude equation with linear velocity damped blockage, we use the m - files of MATLAB (R2010a) for calculation of the Crank - Nicolson and Lax – Wendroff methods. It turns out that the blood pressure and velocity datas.

4.1 The numerical solution of small amplitude wave pressure equations

4.1.1 The numerical solution of blood pressure equations without blockage by Crank – Nicolson method

We simulate the solutions by determine length of blood vessel is L = 1 cm within time interval T = 40 s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with $\Delta t = 0.00125$ cm (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Crank – Nicolson equation in equations (3.1.7) and (3.1.8), we obtain the numerical solution as show on table 4.1 and figure 4.1.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000
	10	83.2186	82.9796	84.0457	83.3175	83.7799	87.5033	86.7875	82.9605	81.7330	82.3728	81.7774
	20	108.1616	113.9600	116.2831	118.7394	117.8238	116.9050	117.9557	117.5946	119.8761	119.5157	119.7741
	30	103.0850	96.6196	94.8126	94.8140	94.9661	91.1616	89.7444	90.0488	89.4454	84.9459	85.0388
	40	86.6612	86.1954	86.7585	85.3831	87.9168	88.8801	93.6853	100.2804	100.4981	101.3745	105.3329

Table 4.1 Table of blood pressure of small amplitude equation without blockage by Crank – Nicolson finite difference method

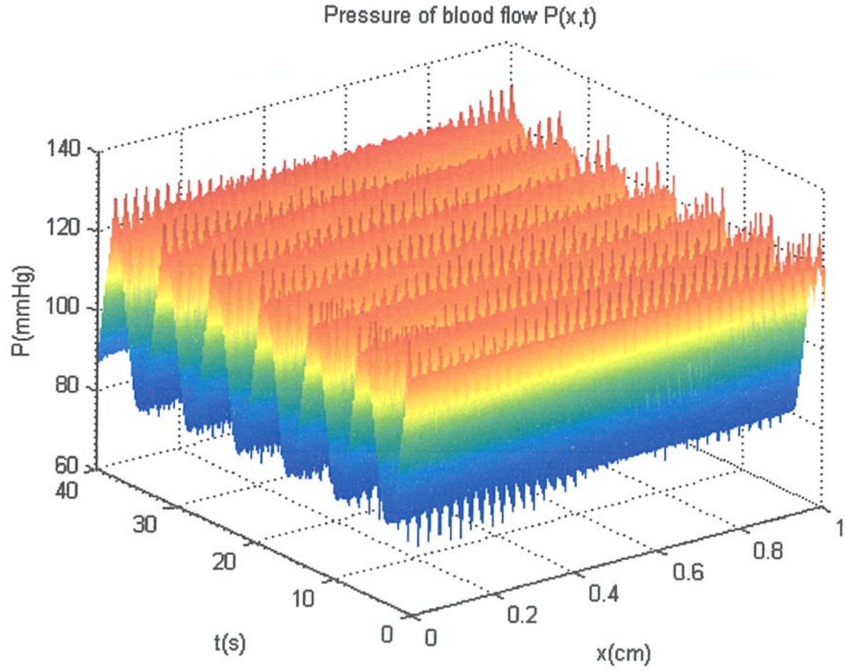

Figure 4.1 Numerical solution of blood pressure $P(x,t)$ with small amplitude condition

by Crank – Nicolson finite difference method

4.1.2 Numerical solution of blood velocity without blockage by Crank – Nicolson method

Similarly, from those conditions which is described in 4.1.1, we take the Crank - Nicolson finite difference equations (3.1.7) - (3.1.8) to obtain the numerical solution of blood flow velocity as shown on table 4.2 and figure 4.2

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000
	10	46.7814	47.0204	45.9543	46.6825	46.2201	42.4967	43.2125	47.0395	48.2670	46.9305	48.2226
	20	21.8384	16.0400	13.7169	11.2606	12.1762	13.0950	12.0443	12.4054	10.1239	10.4843	10.2259
	30	26.9150	33.3804	35.1874	35.1860	35.0339	38.8384	40.2556	39.9512	40.5546	45.0541	44.9612
	40	43.3388	43.8046	43.2415	44.6169	42.0832	41.1199	36.3147	29.7196	29.5019	28.6255	24.6671

Table 4.2 Table of blood velocity of small amplitude equation without blockage

by Crank – Nicolson finite difference method

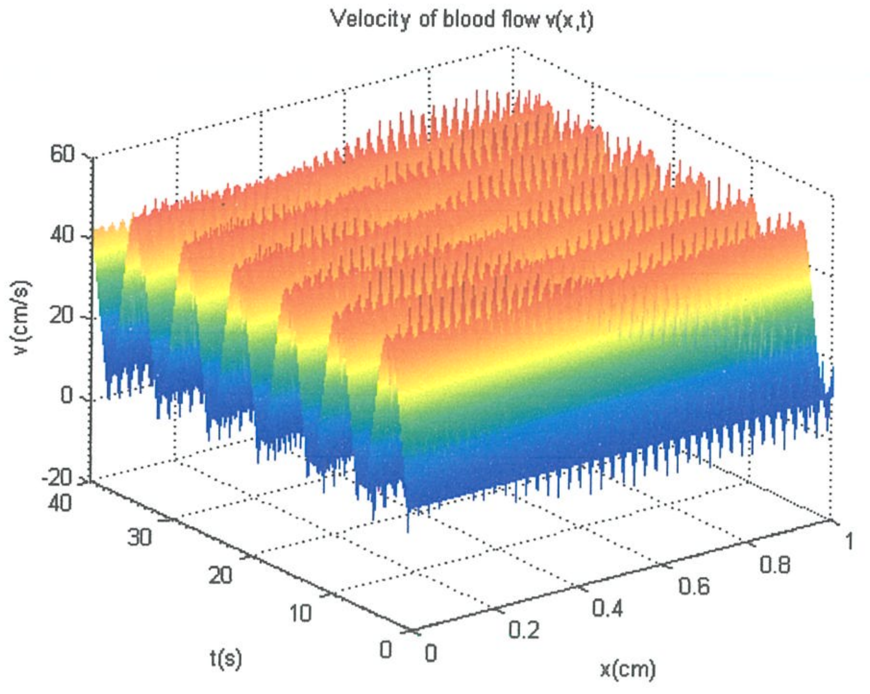

Figure 4.2 Numerical solution of blood velocity $u(x,t)$ with small amplitude condition by Crank – Nicolson finite difference method

For the time processing of this method and meshing, we find that elapsed time is 18.410203 s.

4.1.3 Numerical solution of blood pressure without blockage by Lax – Wendroff method

We simulate the solutions by determine length of blood vessel is L = 1 cm within time interval T = 40 s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with $\Delta t = 0.00125$ cm (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Lax - Wendroff equation in equation (3.2.3), we obtain the numerical solution as shown on table 4.3 and figure 4.3.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000	120.0000
	10	83.2186	82.0448	81.1797	80.5452	80.1244	79.9158	79.9199	80.1346	80.5503	81.1349	81.7774
	20	108.1616	109.9353	111.6631	113.2732	114.7425	116.0554	117.1983	118.1593	118.9283	119.4900	119.7741
	30	103.0850	101.2824	99.2479	97.1805	95.1355	93.1409	91.2189	89.3914	87.6852	86.1581	85.0388
	40	86.6612	87.9126	89.5991	91.4584	93.4207	95.4552	97.5377	99.6433	101.7377	103.7386	105.3329

Table 4.3 Table of blood pressure of small amplitude equation without blockage by Lax - Wendroff finite difference method

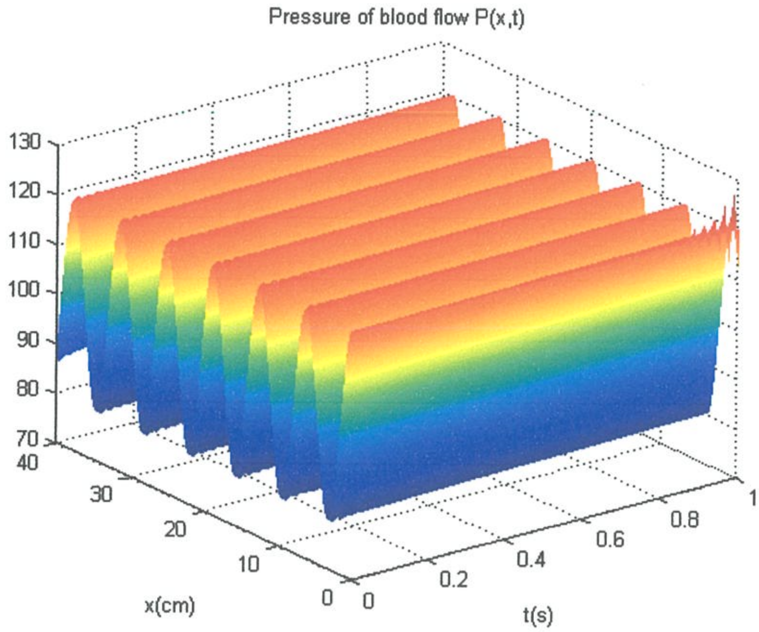

Figure 4.3 Numerical solution of blood pressure $P(x,t)$ with small amplitude condition by Lax - Wendroff finite difference method

4.1.4 Numerical solution of blood velocity without blockage by Lax - Wendroff method

Similarly, from those conditions which is described in 4.1.1, we take the Lax - Wendroff finite difference equation (3.2.3) to obtain the numerical solution of blood flow velocity as shown on table 4.4 and figure 4.4

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000
	10	46.7814	47.6706	48.4877	49.1393	49.5958	49.8467	49.8883	49.7216	49.3550	48.8196	48.2226
	20	21.8384	19.9469	18.2198	16.6229	15.1675	13.8689	12.7408	11.7950	11.0418	10.4960	10.2259
	30	26.9150	29.1999	31.2812	33.3093	35.2952	37.2237	39.0752	40.8291	42.4596	43.9109	44.9612
	40	43.3388	41.3957	39.6302	37.8236	35.9464	34.0088	32.0298	30.0323	28.0492	26.1595	24.6671

Table 4.4 Table of blood velocity of small amplitude equation without blockage

by Lax - Wendroff finite difference method

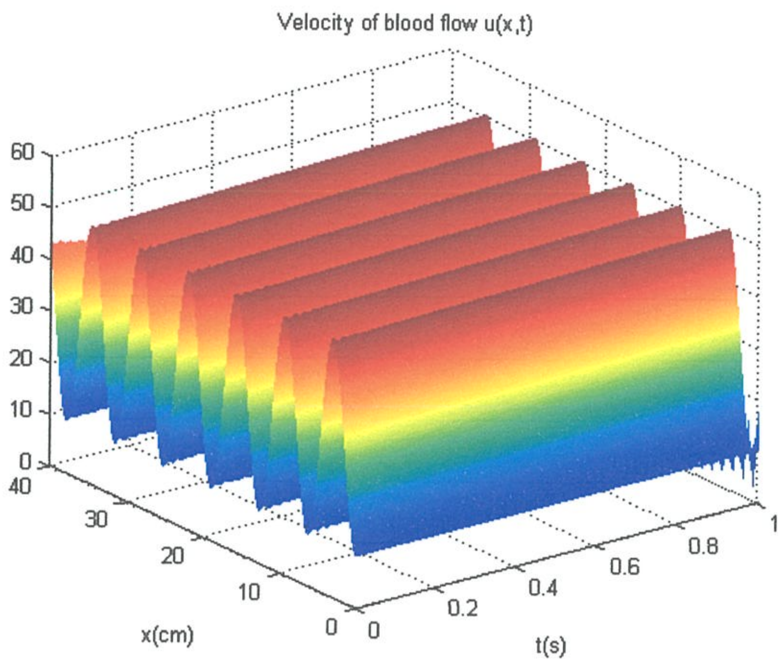

Figure 4.4 Numerical solution of blood velocity $u(x,t)$ with small amplitude condition

by Lax - Wendroff finite difference method

For the time processing of this method and meshing, we find that elapsed time is 0.259717s.

4.2 Numerical solution of blood flow with dragged blockage by using Crank – Nicolson Method

4.2.1 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 0.0 g/cm^{3}s

We simulate the solution sby determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 0.0 g/cm^{3}s. This meshing uses the space domain with Δx = 0.0125 cm (m = 80) and time domain with Δt = 0.00125 cm (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Crank - Nicolson equation in equations (3.3.12) and (3.3.13), we obtain the numerical solution as shown on table 4.5 and figure 4.5.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	20	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	30	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	40	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000

Table 4.5 Table of blood pressure by using Crank – Nicolson method

with dragged coefficient = 0.0 g/cm^{3}s

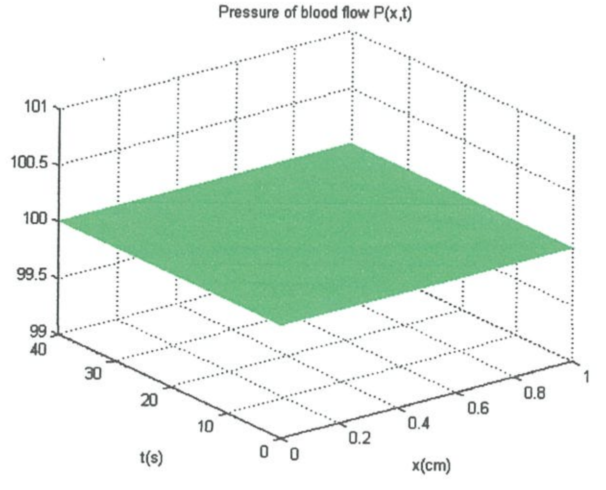

Figure 4.5 Numerical solution of blood pressure $P(x,t)$

with dragged coefficient = 0.0 g/cm^{3}s

4.2.2 Numerical solution of blood velocity with dragged blockage with dragged coefficient $k = 0.0$ g/cm^{3}s

In the same way, from those conditions which are described in 4.2.1, we take the Crank - Nicolson finite difference equations (3.3.12) and (3.3.13) to obtain the numerical solution of blood flows velocity as shown on table 4.6 and figure 4.6

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	20	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	30	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	40	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000

Table 4.6 Table of blood velocity by using Crank – Nicolson method

with dragged coefficient = 0.0 g/cm^{3}s

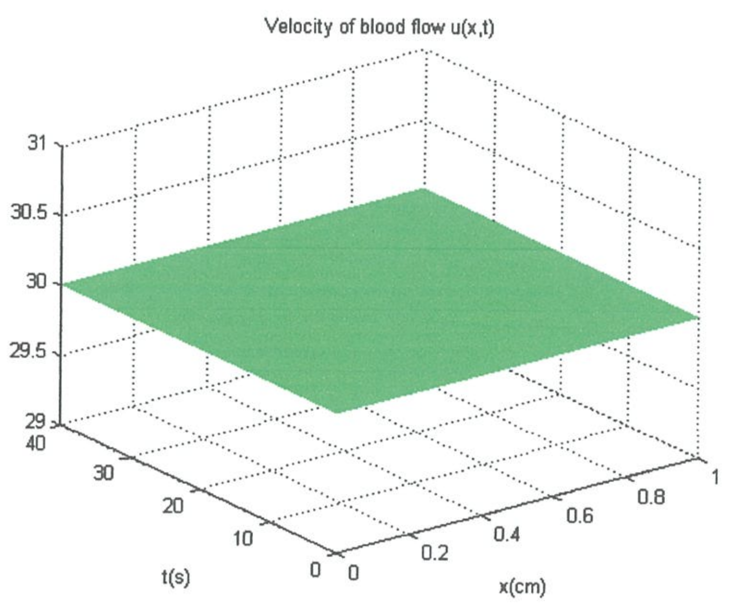

Figure 4.6 Numerical solution of blood velocity $u(x,t)$

with dragged coefficient = 0.0 g/cm^{3}s

For the time processing of this method and meshing, we find that elapsed time is 18.530330 s.

4.2.3 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 1.06 g/cm^{3}s

We simulate the solutions by determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 1.06 g/cm^{3}s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with $\Delta t = 0.00125$ s (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Crank – Nicolson equation in equations (3.3.12) and (3.3.13), we obtain the numerical solution as shown on table 4.7 and figure 4.7.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	89.1196	91.7622	94.2868	96.6432	98.7878	100.6812	102.2829	103.5558	104.4844	105.0486	105.2367
	20	118.2589	117.4635	116.6082	115.7377	114.8934	114.1124	113.4272	112.8652	112.4479	112.1912	112.1045
	30	80.2394	78.9071	77.7938	76.8766	76.1331	75.5428	75.0872	74.7501	74.5189	74.3840	74.3396
	40	114.9023	117.9334	120.6568	123.0665	125.1583	126.9299	128.3797	129.5074	130.3127	130.7958	130.9568

Table 4.7 Table of blood pressure by using Crank – Nicolson method

with dragged coefficient = 1.06 g/cm^{3}s

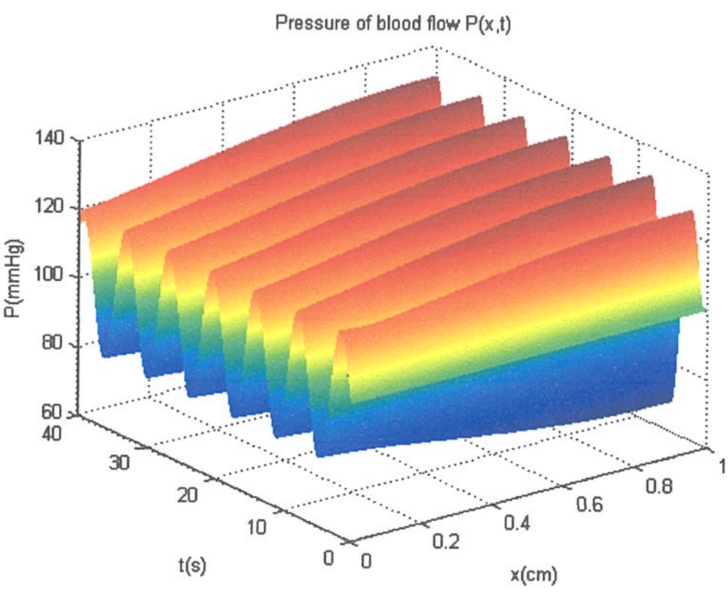

Figure 4.7 Numerical solution of blood pressure $P(x,t)$

with dragged coefficient = 1.06 g/cm^{3}s

4.2.4 Numerical solution of blood velocity with dragged blockage with dragged coefficient $k = 1.06$ g/cm^{3}s

Similarly, from those conditions which are described in 4.2.3, we take the Crank - Nicolson finite difference equations (3.3.12) and (3.3.13) to obtain the numerical solution of blood flows velocity as shown on table 4.8 and figure 4.8

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	56.0399	53.9812	51.8813	49.5612	47.0547	44.3929	41.5958	38.6895	35.7082	32.6740	30.3823
	20	8.4423	9.5927	10.9893	12.7017	14.6946	16.9309	19.3716	21.9762	24.7034	27.5110	29.6438
	30	40.2064	40.3317	40.0810	39.5182	38.6733	37.5804	36.2767	34.8021	33.1990	31.5111	30.2166
	40	34.4288	33.0682	32.0923	31.3245	30.7494	30.3474	30.0948	29.9648	29.9280	29.9531	29.9928

Table 4.8 Table of blood velocity by using Crank – Nicolson method

with dragged coefficient = 1.06 g/cm^{3}s

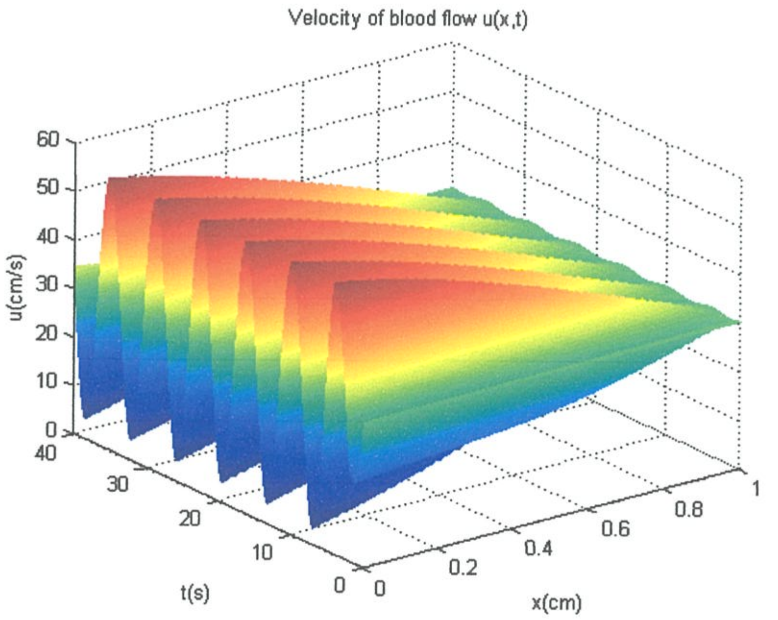

Figure 4.8 Numerical solution of blood velocity $u(x,t)$

with linear dragged coefficient = 1.06 g/cm^{3}s

For the time processing of this method and meshing, we find that elapsed time is 62.433004 s.

4.2.5 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 2.12 g/cm^{3}s

We simulate by determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 2.12 g/cm^{3}s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with Δt = 0.00125 s (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Crank – Nicolson equation in equations (3.3.12) and (3.3.13), we obtain the numerical solution as shown on table 4.9 and figure 4.9.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	89.1196	93.2203	97.0215	100.4733	103.5351	106.1741	108.3647	110.0871	111.3271	112.0749	112.3247
	20	118.2589	115.2398	112.2926	109.5001	106.9351	104.6604	102.7285	101.1821	100.0542	99.3681	99.1378
	30	80.2394	81.2051	82.3499	83.5843	84.8268	86.0051	87.0566	87.9293	88.5822	88.9858	89.1223
	40	114.9023	116.3007	117.3269	118.0479	118.5276	118.8249	118.9923	119.0742	119.1065	119.1153	119.1165

Table 4.9 Table of blood pressure by using Crank – Nicolson method

with dragged coefficient = 2.12 g/cm^{3}s

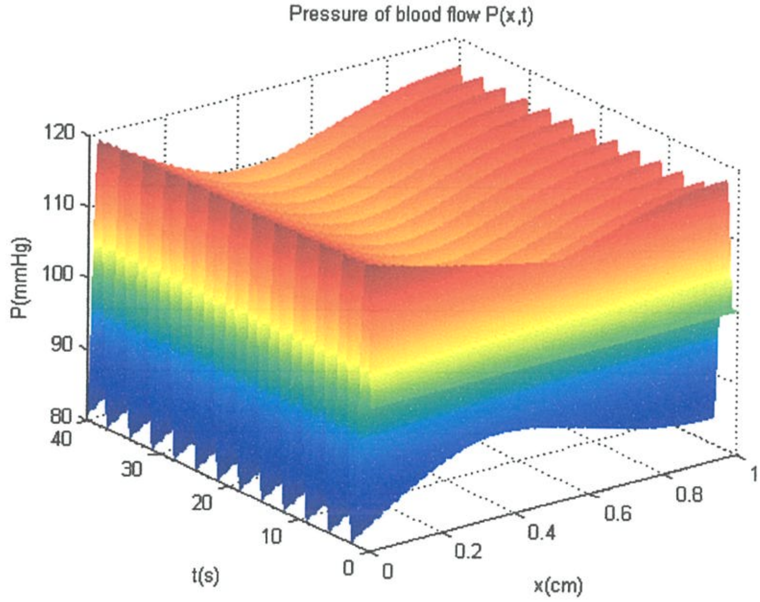

Figure 4.9 Numerical solution of blood pressure $P(x,t)$

with dragged coefficient = 2.12 g/cm^{3}s

4.2.6 Numerical solution of blood velocity with dragged blockage with dragged coefficient $k = 2.12\ g/cm^3 s$

In the same way, from those conditions which are described in 4.2.5, we take the Crank - Nicolson finite difference equations (3.3.12) and (3.3.13) to obtain the numerical solution of blood flows velocity as shown on table 4.10 and figure 4.10

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	13.6434	15.7157	17.8077	19.9481	22.0006	23.8722	25.5136	26.9171	28.1112	29.1531	29.8804
	20	18.2649	15.9065	14.8710	14.7546	15.4613	16.8797	18.8900	21.3692	24.1940	27.2429	29.6043
	30	36.7781	32.7809	29.8437	27.6082	26.1324	25.4186	25.4181	26.0381	27.1497	28.5964	29.7966
	40	47.2671	46.3631	45.0014	43.2933	41.3822	39.3812	37.3704	35.3973	33.4797	31.6116	30.2297

Table 4.10 Table of blood velocity by using Crank – Nicolson method

with dragged coefficient = 2.12 $g/cm^3 s$

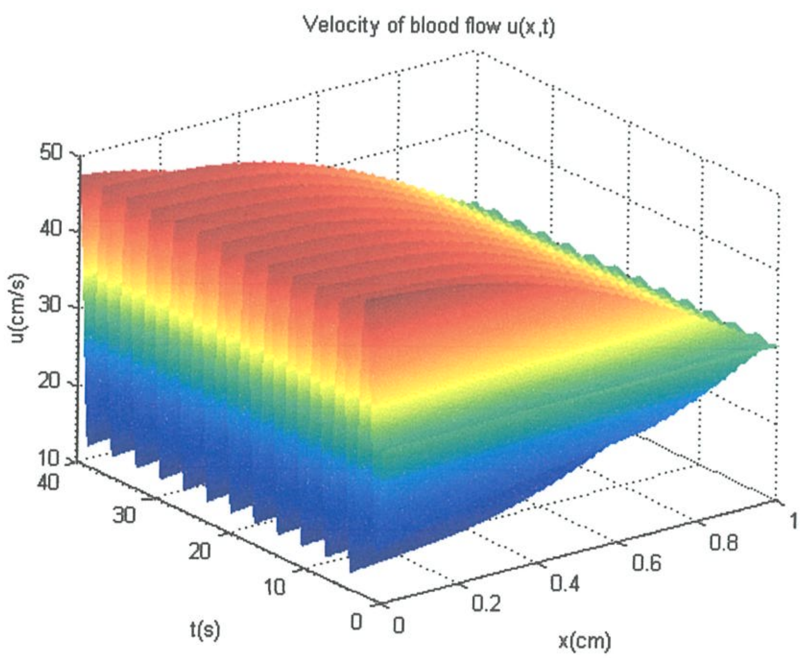

Figure 4.10 Numerical solution of blood velocity $u(x,t)$

with dragged coefficient = 2.12 $g/cm^3 s$

For the time processing of this method and meshing, we find that elapsed time is 85.794434 s.

4.2.7 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 3.18 g/cm^{3}s

We simulate by determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 3.18 g/cm^{3}s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with $\Delta t = 0.00125$ s (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Crank – Nicolson equation in equations (3.3.12) and (3.3.13), we obtain the numerical solution as shown on table 4.11 and figure 4.11.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	89.1196	93.5234	97.4724	100.9454	103.9343	106.4400	108.4696	110.0331	111.1407	111.8013	112.0208
	20	118.2589	114.4123	110.7397	107.3308	104.2578	101.5775	99.3334	97.5581	96.2747	95.4988	95.2392
	30	80.2394	82.2907	84.5049	86.7525	88.9205	90.9127	92.6490	94.0648	95.1109	95.7523	95.9685
	40	114.9023	115.3064	115.2633	114.9005	114.3352	113.6723	113.0026	112.4020	111.9299	111.6293	111.5263

Table 4.11 Table of blood pressure by using Crank – Nicolson method

with dragged coefficient = 3.18 g/cm^{3}s

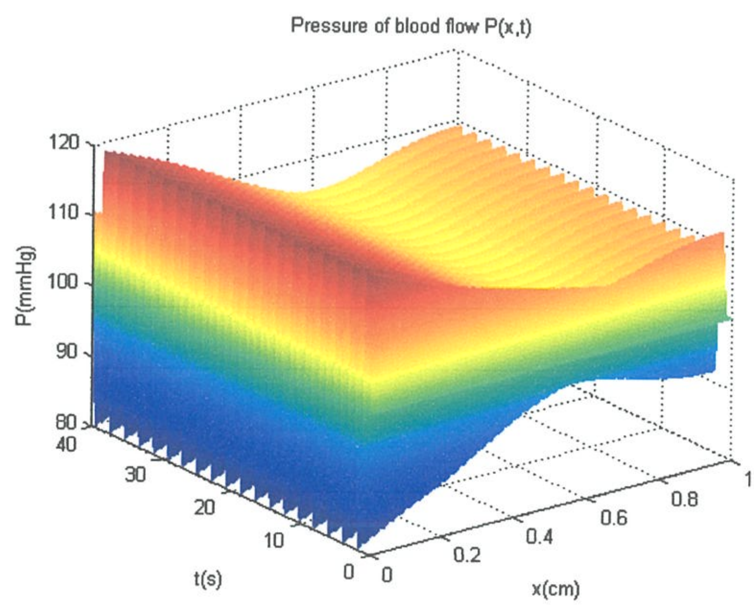

Figure 4.11 Numerical solution of blood pressure $P(x,t)$

with dragged coefficient = 3.18 g/cm^{3}s

4.2.8 Numerical solution of blood velocity with dragged blockage with dragged coefficient $k = 3.18$ g/cm^{3}s

Similarly, from those conditions which are described in 4.2.7, we take the Crank - Nicolson finite difference equations (3.3.12) and (3.3.13) to obtain the numerical solution of blood flows velocity as shown on table 4.12 and figure 4.12

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	42.0780	42.0160	40.5009	38.2253	35.7263	33.4359	31.6437	30.4826	29.9357	29.8602	29.9756
	20	40.1367	33.8965	29.0828	25.3689	22.9472	21.8465	21.9700	23.1337	25.0991	27.5978	29.6523
	30	21.0492	19.1861	19.2162	20.3460	22.0979	24.0487	25.8790	27.3991	28.5523	29.3988	29.9171
	40	17.1020	22.7674	27.5904	31.6528	34.6150	36.3175	36.7587	36.0639	34.4543	32.2167	30.3221

Table 4.12 Table of blood velocity by using Crank – Nicolson method

with dragged coefficient = 3.18 g/cm^{3}s

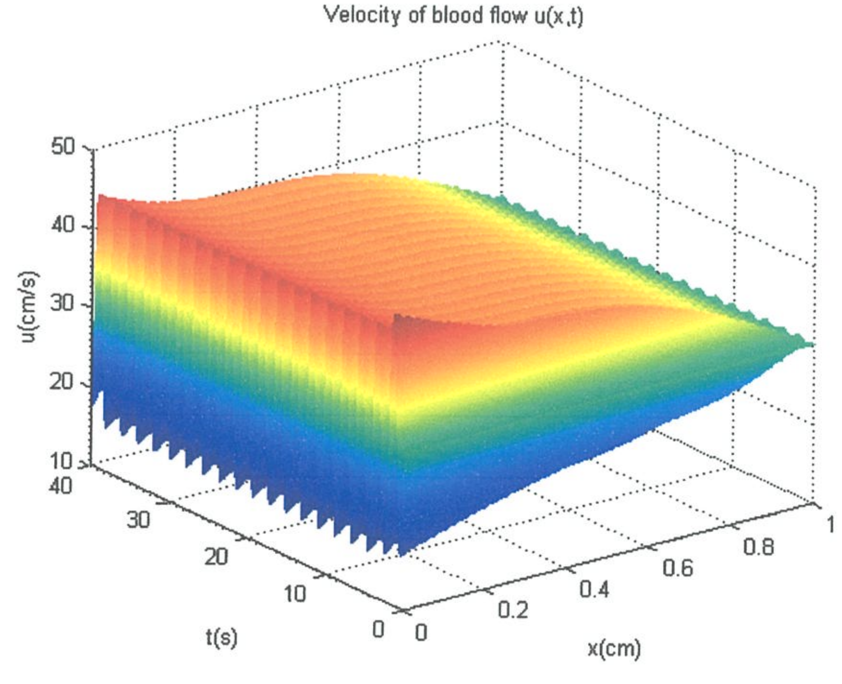

Figure 4.12 Numerical solution of blood velocity $u(x,t)$

with dragged coefficient = 3.18 g/cm^{3}s

For the time processing of this method and meshing, we find that elapsed time is 114.946201 s.

4.2.9 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 4.24 g/cm^{3}s

We simulate by determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 4.24 g/cm^{3}s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with $\Delta t = 0.00125$ s (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Crank – Nicolson equation in equations (3.3.12) and (3.3.13), we obtain the numerical solution as shown on table 4.13 and figure 4.13.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	89.1196	93.6527	97.5785	100.9084	103.6712	105.9061	107.6568	108.9661	109.8716	110.4025	110.5774
	20	118.2589	114.0080	110.0273	106.4049	103.2021	100.4591	98.1999	96.4375	95.1775	94.4215	94.1695
	30	80.2394	82.8419	85.5983	88.3493	90.9630	93.3328	95.3746	97.0240	98.2338	98.9720	99.2202
	40	114.9023	114.7857	114.1408	113.1467	111.9633	110.7295	109.5622	108.5567	107.7865	107.3036	107.1392

Table 4.13 Table of blood pressure by using Crank – Nicolson method

with dragged coefficient = 4.24 g/cm^{3}s

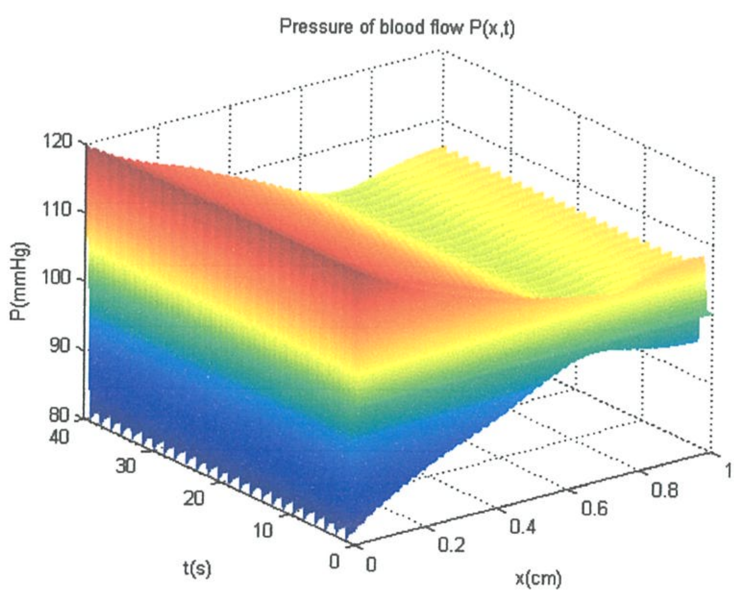

Figure 4.13 Numerical solution of blood pressure $P(x,t)$

with dragged coefficient = 4.24 g/cm^{3}s

4.2.10 Numerical solution of blood velocity with dragged blockage with dragged coefficient $k = 4.24$ g/cm^{3}s

In the same way, from those conditions which are described in 4.2.9, we take the Crank - Nicolson finite difference equations (3.3.12) and (3.3.13) to obtain the numerical solution of blood flows velocity as shown on table 4.14 and figure 4.14

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	22.8977	19.2243	19.4120	21.7684	25.1186	28.4121	30.8910	32.1629	32.1949	31.2532	31.2532
	20	46.2816	43.3860	38.7452	33.6757	29.2643	26.1685	24.6410	24.6172	25.8226	27.8677	29.6882
	30	15.3869	22.9226	28.9250	33.3306	35.8645	36.7002	36.2585	35.0180	33.3779	31.5914	30.2277
	40	33.2105	26.0544	22.6887	21.8817	22.9132	24.8943	27.0110	28.6894	29.6717	30.0097	30.0081

Table 4.14 Table of blood velocity by using Crank – Nicolson method

with dragged coefficient = 4.24 g/cm^{3}s

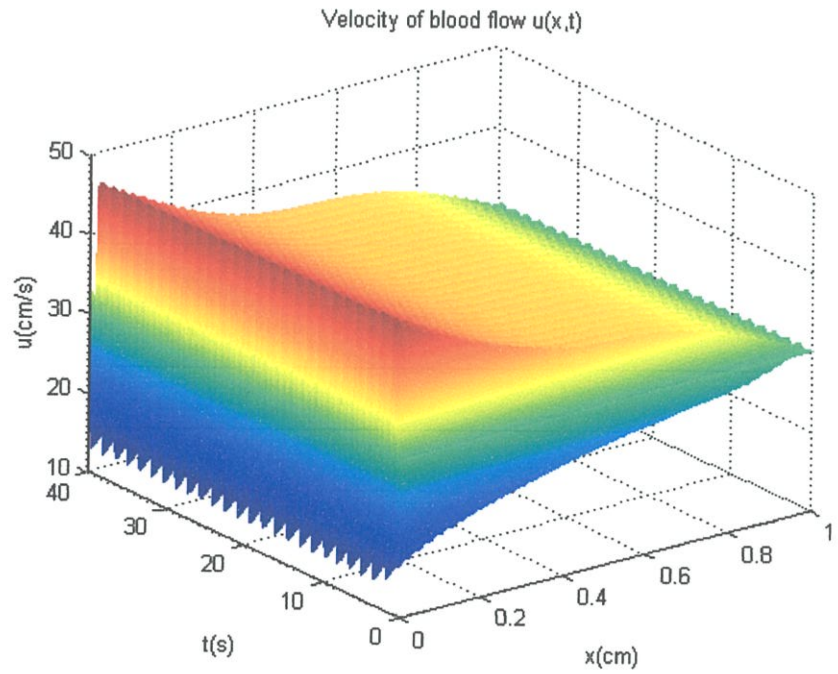

Figure 4.14 Numerical solution of blood velocity $u(x,t)$

with dragged coefficient = 4.24 g/cm^{3}s

For the time processing of this method and meshing, we find that elapsed time is 96.916271 s.

4.2.11 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 5.30 g/cm^{3}s

We simulate by determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 5.30 g/cm^{3}s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with $\Delta t = 0.00125$ s (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Crank – Nicolson equation in equations (3.3.12) and (3.3.13), we obtain the numerical solution as shown on table 4.15 and figure 4.15.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	89.1196	93.7887	97.6970	100.8884	103.4293	105.3975	106.8734	107.9324	108.6390	109.0424	109.1734
	20	118.2589	113.7324	109.5730	105.8654	102.6565	99.9655	97.7925	96.1265	94.9520	94.2543	94.0230
	30	80.2394	83.1767	86.2586	89.2987	92.1516	94.7072	96.8847	98.6269	99.8951	100.6649	100.9229
	40	114.9023	114.4996	113.4872	112.0931	110.5145	108.9169	107.4358	106.1781	105.2243	104.6302	104.4285

Table 4.15 Table of blood pressure by using Crank – Nicolson method with dragged coefficient = 5.30 g/cm^{3}s

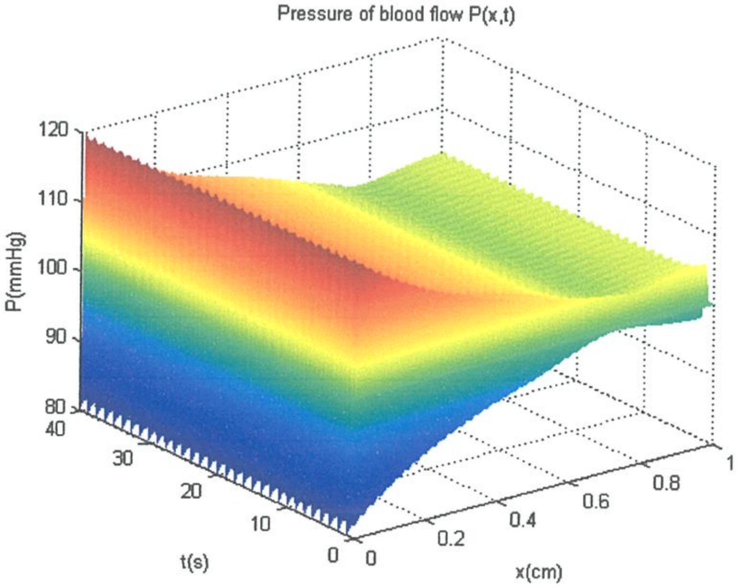

Figure 4.15 Numerical solution of blood pressure $P(x,t)$ with dragged coefficient = 5.30 g/cm^{3}s

4.2.12 Numerical solution of blood velocity with dragged blockage with dragged coefficient $k = 5.30\ g/cm^3s$

In the same way, from those conditions which are described in 4.2.11, we take the Crank - Nicolson finite difference equations (3.3.12) and (3.3.13) to obtain the numerical solution of blood flows velocity as shown on table 4.16 and figure 4.16

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	28.2040	36.3729	38.8764	37.8012	34.7381	31.2805	28.5930	27.2599	27.3381	28.5059	28.5059
	20	32.4889	39.0167	39.8287	37.4430	33.6801	30.0875	27.6264	26.6467	27.0332	28.3988	29.7627
	30	36.6049	41.0341	40.0975	36.5681	32.3687	28.8922	26.8292	26.2707	26.9377	28.4045	29.7657
	40	40.2582	42.2784	39.6589	35.2330	30.8913	27.7745	26.2542	26.1561	27.0569	28.5221	29.7851

Table 4.16 Table of blood velocity by using Crank – Nicolson method

with dragged coefficient = $5.30\ g/cm^3s$

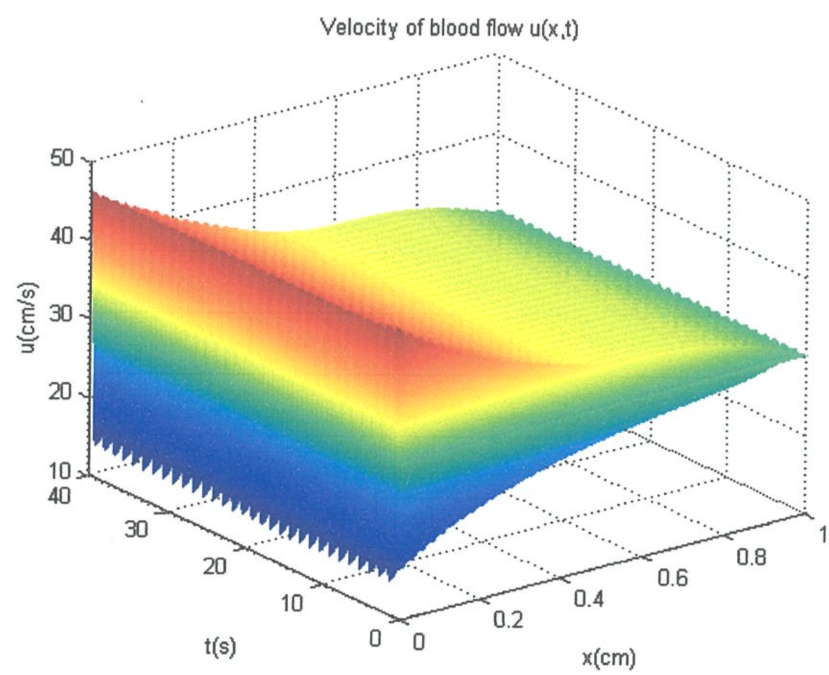

Figure 4.16 Numerical solution of blood velocity $u(x,t)$

with dragged coefficient = $5.30\ g/cm^3s$

For the time processing of this method and meshing, we find that elapsed time is 99.929708 s.

4.3 Numerical solution of blood flow with dragged blockage by using Lax - Wendroff Method

4.3.1 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 0.0 g/cm^{3}s

We simulate by determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 0.0 g/cm^{3}s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with $\Delta t = 0.00125$ s (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Lax - Wendroff equation in equation (3.4.1), we obtain the numerical solution as shown on table 4.17 and figure 4.17.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	20	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	30	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	40	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000

Table 4.17 Table of blood pressure by using Lax – Wendroff method

with dragged coefficient = 0.0 g/cm^{3}s

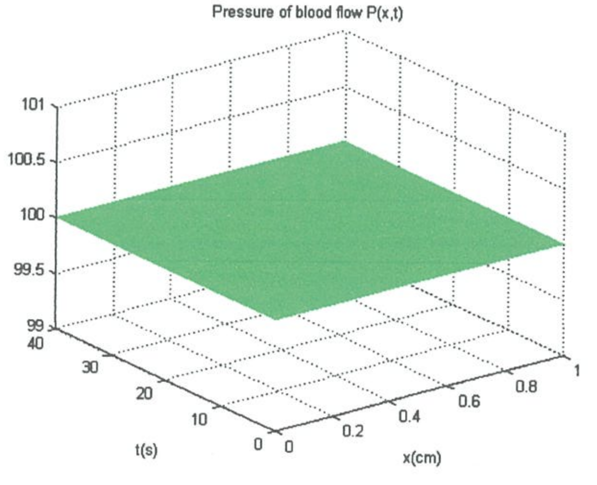

Figure 4.17 Numerical solution of blood pressure $P(x,t)$

with dragged coefficient = 0.0 g/cm^{3}s

4.3.2 Numerical solution of blood velocity with dragged blockage with dragged coefficient $k = 0.0$ g/cm^{3}s

Similarly, from those conditions which are described in 4.2.1, we take the Lax - Wendroff finite difference equation (3.4.1) to obtain the numerical solution of blood flows velocity as shown on table 4.18 and figure 4.18

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	20	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	30	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	40	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000

Table 4.18 Table of blood velocity by using Lax – Wendroff method

with dragged coefficient = 0.0 g/cm^{3}s

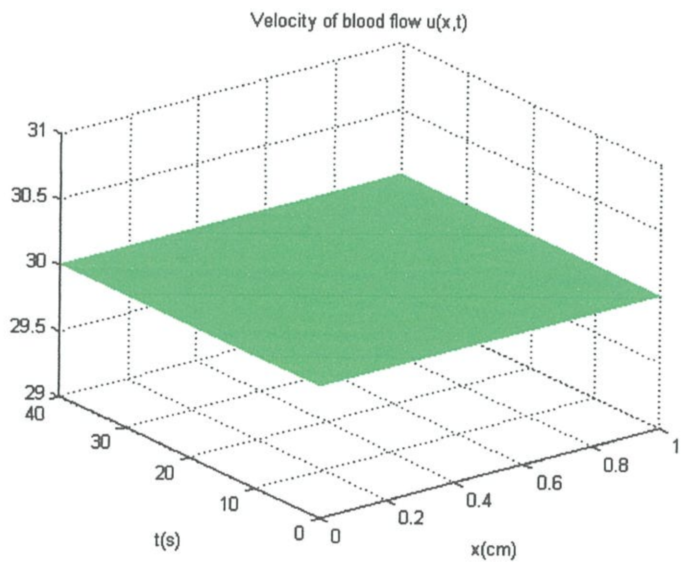

Figure 4.18 Numerical solution of blood velocity $u(x,t)$

with dragged coefficient = 0.0 g/cm^{3}s

For the time processing of this method and meshing, we find that elapsed time is 9.708310 s.

4.3.3 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 1.06 g/cm^{3}s

We simulate by determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 1.06 g/cm^{3}s. This meshing uses the space domain with Δx = 0.0125 cm (m = 80) and time domain with Δt = 0.00125 s (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Lax - Wendroff equation in equation (3.4.1), we obtain the numerical solution as shown on table 4.19 and figure 4.19.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	89.1196	92.0236	94.7549	97.3454	99.7837	102.0519	104.1334	106.0149	107.6847	109.1279	110.2972
	20	118.2589	115.9717	113.6559	111.3250	109.0093	106.7414	104.5522	102.4700	100.5215	98.7391	97.1962
	30	80.2394	81.1737	82.3288	83.6500	85.0980	86.6358	88.2281	89.8411	91.4412	92.9894	94.4095
	40	114.9023	115.6215	115.9989	116.1126	115.9984	115.6856	115.2027	114.5781	113.8413	113.0258	112.1855

Table 4.19 Table of blood pressure by using Lax – Wendroff method

with dragged coefficient = 1.06 g/cm^{3}s

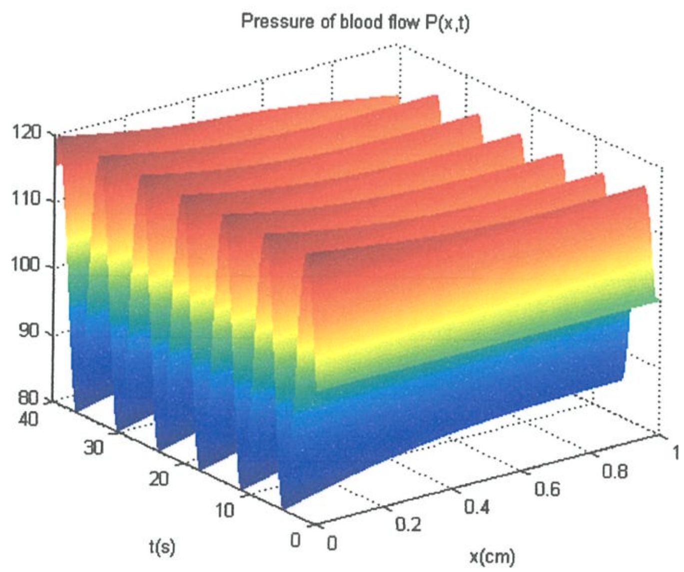

Figure 4.19 Numerical solution of blood pressure $P(x,t)$

with dragged coefficient = 1.06 g/cm^{3}s

4.3.4 Numerical solution of blood velocity with dragged blockage with dragged coefficient $k = 1.06$ g/cm^{3}s

In the same way, from those conditions which are described in 4.2.3, we take the Lax - Wendroff finite difference equation (3.4.1) to obtain the numerical solution of blood flows velocity as shown on table 4.20 and figure 4.20

		Distance (cm)										
		0.00	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.00**
Time (s)	**0**	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	44.9901	43.3804	41.6914	40.0046	38.3543	36.7620	35.2442	33.8140	32.4825	31.2641	30.2037
	20	10.9851	11.8600	12.9240	14.1259	15.4333	16.8201	18.2629	19.7398	21.2296	22.7046	24.0968
	30	46.9293	47.0709	46.9742	46.6441	46.1004	45.3653	44.4619	43.4135	42.2447	40.9879	39.7118
	40	20.6050	19.4926	18.5908	17.9429	17.5480	17.3947	17.4679	17.7504	18.2219	18.8561	19.6055

Table 4.20 Table of blood velocity by using Lax – Wendroff method

with dragged coefficient = 1.06 g/cm^{3}s

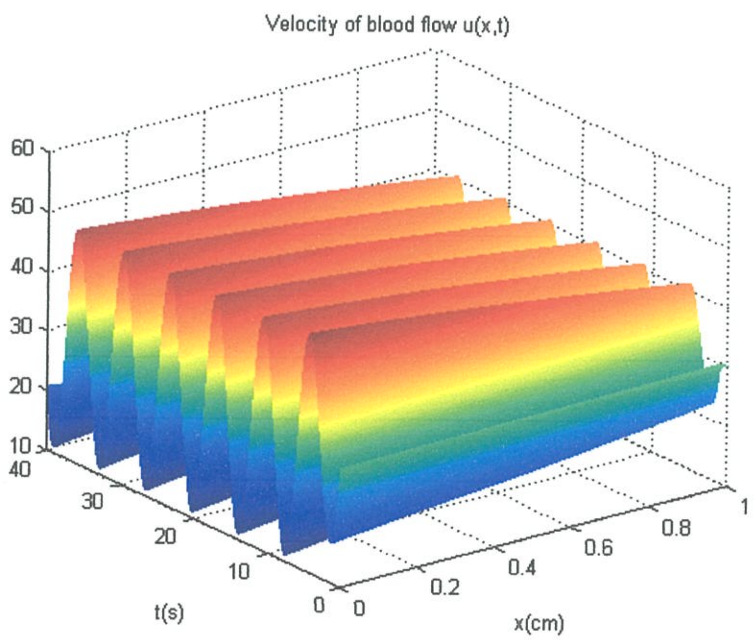

Figure 4.20 Numerical solution of blood velocity $u(x,t)$

with dragged coefficient = 1.06 g/cm^{3}s

For the time processing of this method and meshing, we find that elapsed time is 9.765926 s.

4.3.5 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 2.12 g/cm^{3}s

We simulate by determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 2.12 g/cm^{3}s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with $\Delta t = 0.00125$ s (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Lax - Wendroff equation in equation (3.4.1), we obtain the numerical solution as shown on table 4.21 and figure 4.21.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	118.2589	114.4965	110.6481	106.8868	103.3749	100.2406	97.5711	95.4139	93.7821	92.6632	92.0478
	20	114.9023	115.8269	115.5678	114.4297	112.6314	110.3796	107.8726	105.2930	102.8034	100.5504	98.7065
	30	93.9038	98.4226	102.0612	104.8954	106.9412	108.2394	108.8643	108.9175	108.5187	107.7999	106.9114
	40	80.1222	82.8857	86.1145	89.5658	93.0338	96.3450	99.3621	101.9852	104.1493	105.8157	106.9344

Table 4.21 Table of blood pressure by using Lax – Wendroff method

with dragged coefficient = 2.12 g/cm^{3}s

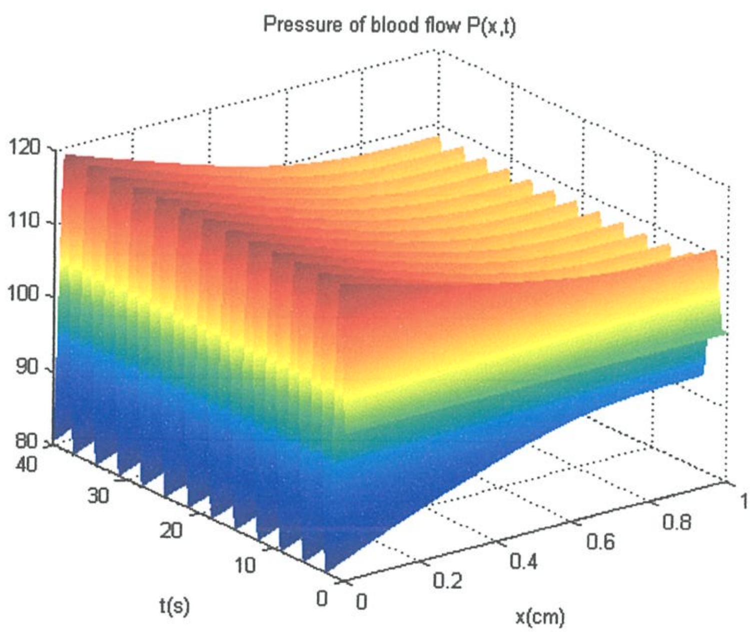

Figure 4.21 Numerical solution of blood pressure $P(x,t)$

with dragged coefficient = 2.12 g/cm^{3}s

4.3.6 Numerical solution of blood velocity with dragged blockage with dragged coefficient k = 2.12 g/cm^{3}s

In the same way, from those conditions which are described in 4.2.5, we take the Lax - Wendroff finite difference equation (3.4.1) to obtain the numerical solution of blood flows velocity as shown on table 4.22 and figure 4.22

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	14.6747	16.5470	18.8668	21.3858	23.9094	26.2841	28.3935	30.1578	31.5321	32.5019	33.0586
	20	22.4756	20.4417	19.1232	18.5558	18.6537	19.3003	20.3711	21.7430	23.3002	24.9311	26.4903
	30	39.1963	35.6639	32.2678	29.2855	26.8414	24.9940	23.7579	23.1126	23.0085	23.3690	24.0840
	40	45.0301	44.1810	42.7277	40.8611	38.7684	36.6139	34.5343	32.6358	30.9936	29.6569	28.6813

Table 4.22 Table of blood velocity by using Lax – Wendroff method

with dragged coefficient = 2.12 g/cm^{3}s

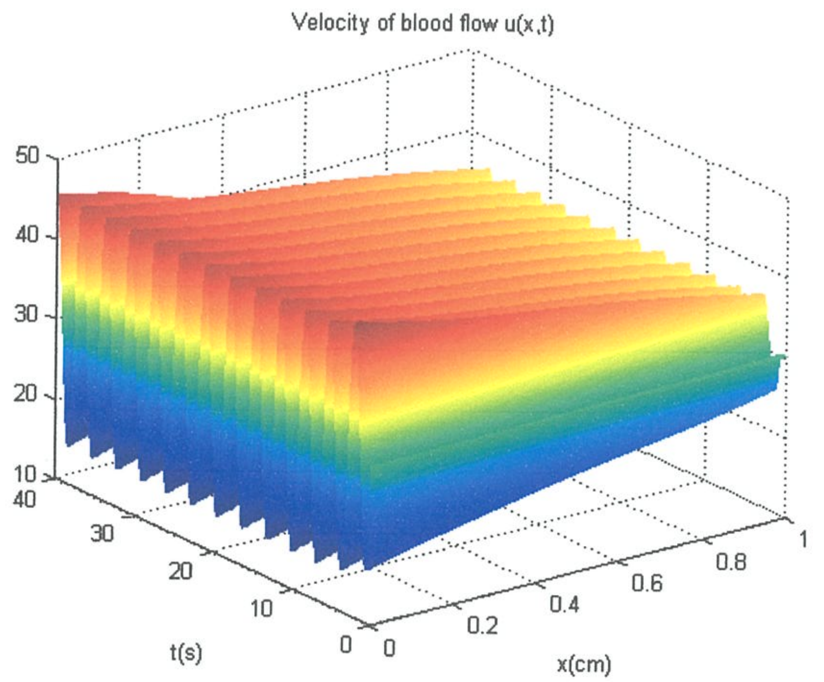

Figure 4.22 Numerical solution of blood velocity $u(x,t)$

with dragged coefficient = 2.12 g/cm^{3}s

For the time processing of this method and meshing, we find that elapsed time is 9.928366 s.

4.3.7 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 3.18 g/cm^{3}s

We simulate by determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 3.18 g/cm^{3}s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with Δt = 0.00125 s (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Lax - Wendroff equation in equation (3.4.1), we obtain the numerical solution as shown on table 4.23 and figure 4.23.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	80.2394	82.8427	86.8150	91.2916	95.6282	99.3651	102.2342	104.1440	105.1474	105.3998	105.1094
	20	93.9038	100.8833	105.8337	108.8290	110.0127	109.6753	108.2105	106.0525	103.6222	101.2947	99.4204
	30	117.8799	117.4341	114.9932	111.4447	107.4773	103.6400	100.3225	97.7501	95.9998	95.0317	94.7459
	40	111.6122	104.4952	98.7918	94.7019	92.2942	91.4478	91.8892	93.2537	95.1440	97.1729	98.9589

Table 4.23 Table of blood pressure by using Lax – Wendroff method

with dragged coefficient = 3.18 g/cm^{3}s

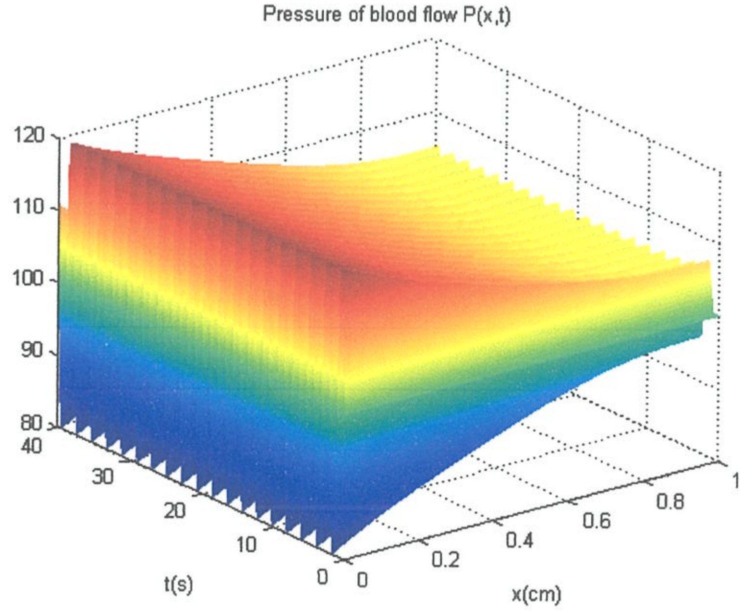

Figure 4.23 Numerical solution of blood pressure $P(x,t)$

with dragged coefficient = 3.18 g/cm^{3}s

4.3.8 Numerical solution of blood velocity with dragged blockage with dragged coefficient $k = 3.18$ g/cm^{3}s

Similarly, from those conditions which are described in 4.2.7, we take the Lax - Wendroff finite difference equation (3.4.1) to obtain the numerical solution of blood flows velocity as shown on table 4.24 and figure 4.24

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	44.1309	43.8621	42.0650	39.2672	36.0457	32.8944	30.1843	28.1494	26.8905	26.3907	26.5441
	20	40.2494	34.9982	30.2474	26.5635	24.1286	22.9123	22.7499	23.4015	24.6003	26.0835	27.5773
	30	19.0472	17.6959	18.0270	19.6878	22.1575	24.9325	27.5915	29.8263	31.4536	32.4095	32.7154
	40	16.3717	21.2060	26.0590	30.2553	33.4520	35.5245	36.5072	36.5450	35.8483	34.6599	33.2604

Table 4.24 Table of blood velocity by using Lax – Wendroff method

with dragged coefficient = 3.18 g/cm^{3}s

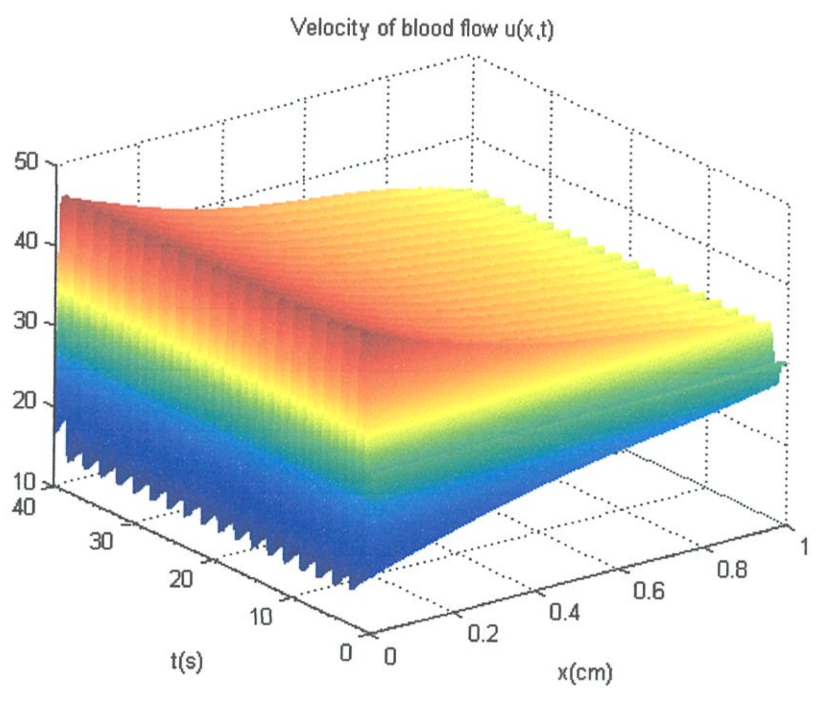

Figure 4.24 Numerical solution of blood velocity $u(x,t)$

with linear dragged coefficient = 3.18 g/cm^{3}s

For the time processing of this method and meshing, we find that elapsed time is 9.517268 s.

4.3.9 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 4.24 g/cm^3s

We simulate by determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 4.24 g/cm^3s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with $\Delta t = 0.00125$ s (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 $cm^2/mmHg$. By using Lax - Wendroff equation in equation (3.4.1), we obtain the numerical solution as shown on table 4.25 and figure 4.25.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	114.9023	115.9826	113.3634	108.9360	104.1781	100.1496	97.4289	96.1445	96.0818	96.8224	97.8669
	20	80.1222	86.3135	93.2794	99.3837	103.6972	105.9150	106.2290	105.1442	103.2920	101.2855	99.6610
	30	111.6122	102.2803	95.6145	91.9061	90.9166	91.9927	94.2997	97.0359	99.5733	101.5127	102.6375
	40	104.3885	110.6450	112.5705	111.4129	108.4195	104.7665	101.3752	98.8104	97.2781	96.6978	96.8219

Table 4.25 Table of blood pressure by using Lax – Wendroff method

with dragged coefficient = 4.24 g/cm^3s

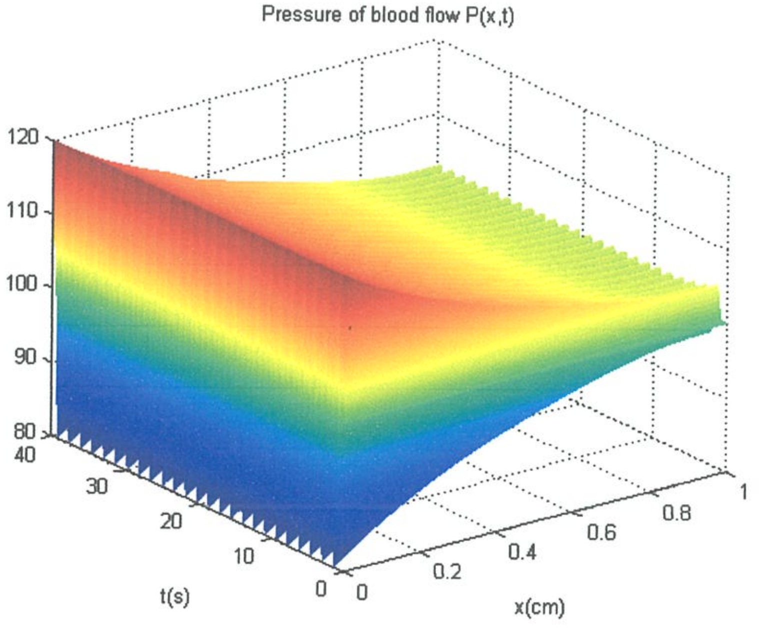

Figure 4.25 Numerical solution of blood pressure $P(x,t)$

with dragged coefficient = 4.24 g/cm^3s

4.3.10 Numerical solution of blood velocity with dragged blockage with dragged coefficient $k = 4.24$ g/cm^{3}s

Similarly, from those conditions which are described in 4.2.9, we take the Lax - Wendroff finite difference equation (3.4.1) to obtain the numerical solution of blood flows velocity as shown on table 4.26 and figure 4.26

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	21.9576	18.9501	18.8602	20.9869	24.2537	27.6753	30.5353	32.4396	33.2902	33.2177	32.5068
	20	45.9118	43.3716	39.0253	34.2570	30.1296	27.2360	25.7377	25.4763	26.1112	27.2429	28.4728
	30	16.8360	23.2317	29.1186	33.3515	35.5893	36.0263	35.1634	33.6063	31.9070	30.4680	29.5366
	40	31.6478	25.6569	22.1507	21.2728	22.4153	24.7260	27.3753	29.7137	31.3453	32.1329	32.1455

Table 4.26 Table of blood velocity by using Lax – Wendroff method

with dragged coefficient = 4.24 g/cm^{3}s

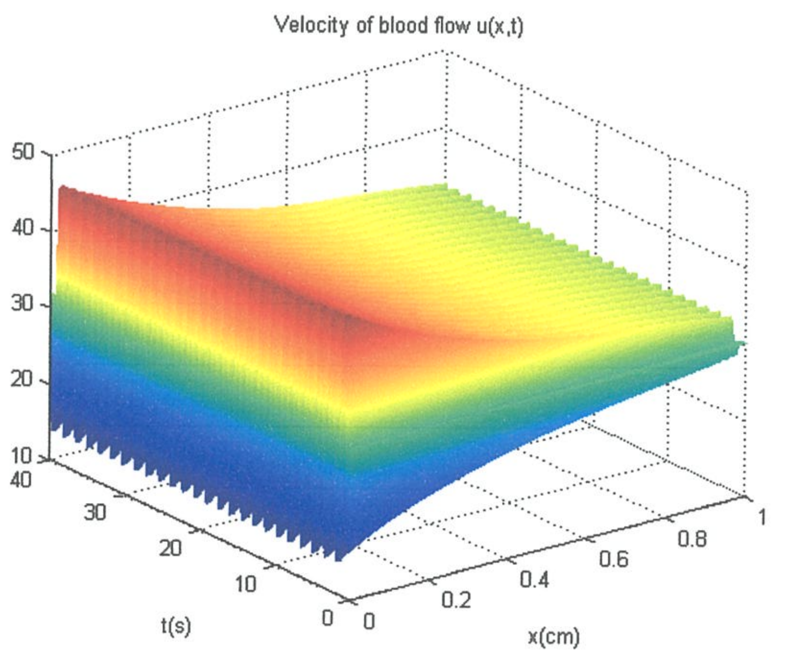

Figure 4.26 Numerical solution of blood velocity $u(x,t)$ with dragged coefficient = 4.24 g/cm^{3}s

For the time processing of this method and meshing, we find that elapsed time is 9.638287 s.

4.3.11 Numerical solution of blood pressure with dragged blockage with dragged coefficient k = 5.30 g/cm^{3}s

We simulate by determine length of blood vessel is L = 1 cm within time interval T = 40 s. The dragged coefficient is k = 5.30 g/cm^{3}s. This meshing uses the space domain with $\Delta x = 0.0125$ cm (m = 80) and time domain with $\Delta t = 0.00125$ s (n = 32000). The physical parameters of blood density ρ is 1.06 g/cm^3, cross section of blood arteries A_0 is 23 cm^2, and compliance parameter c is 23 cm^2/mmHg. By using Lax - Wendroff equation in equation (3.4.1), we obtain the numerical solution as shown on table 4.27 and figure 4.27.

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
	10	94.7525	88.0530	87.7159	91.1475	95.8939	100.0679	102.6487	103.4557	102.8978	101.6562	101.6562
	20	89.8727	85.7653	87.6781	92.5097	97.7299	101.6653	103.6323	103.7653	102.6820	101.1449	99.8440
	30	85.7025	84.4828	88.5192	94.4198	99.7550	103.1827	104.4039	103.8590	102.3306	100.6095	99.3234
	40	82.5341	84.2879	90.1653	96.7218	101.7983	104.4783	104.8686	103.6840	101.8178	100.0334	98.8524

Table 4.27 Table of blood pressure by using Lax – Wendroff method

with dragged coefficient = 5.30 g/cm^{3}s

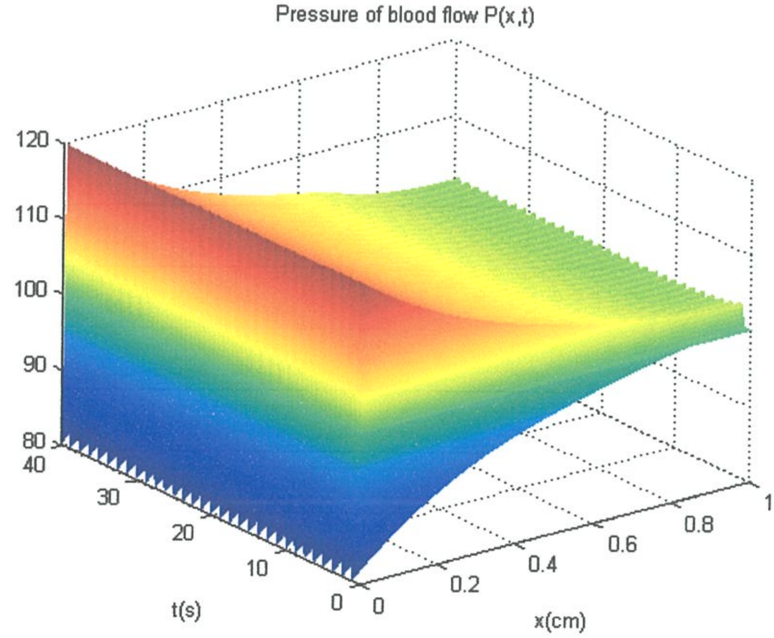

Figure 4.27 Numerical solution of blood pressure $P(x,t)$ with dragged coefficient = 5.30 g/cm^{3}s

4.3.12 Numerical solution of blood velocity with dragged blockage with dragged coefficient k = 5.30 g/cm^{3}s

In the same way, from those conditions which are described in 4.2.11, we take the Lax - Wendroff finite difference equation (3.4.1) to obtain the numerical solution of blood flows velocity as shown on table 4.28 and figure 4.28

		Distance (cm)										
		0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Time (s)	0	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000	30.0000
	10	29.2419	36.0691	38.9072	38.1565	35.3335	32.0144	29.3273	27.7956	27.4215	27.8855	28.7358
	20	33.4677	38.8251	39.9191	37.7735	34.1893	30.7307	28.3334	27.2721	27.3371	28.0830	29.0215
	30	37.4666	40.9786	40.2515	36.8605	32.7656	29.4086	27.4677	26.9497	27.4476	28.4214	29.3805
	40	40.9428	42.3635	39.8661	35.4674	31.1485	28.1284	26.7799	26.8414	27.7373	28.8706	29.7830

Table 4.28 Table of blood velocity by using Lax – Wendroff method with dragged coefficient = 5.30 g/cm^{3}s

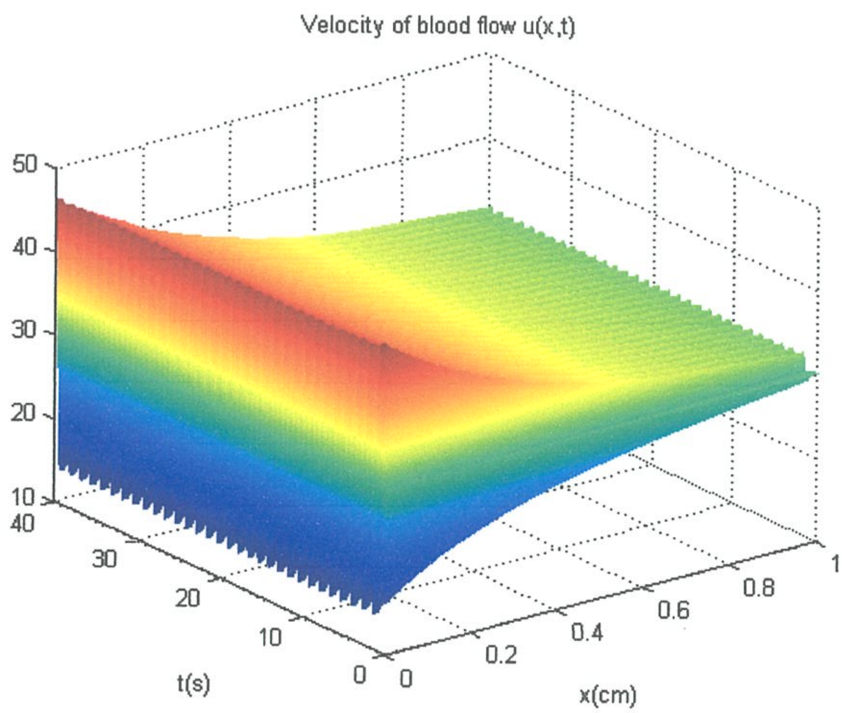

Figure 4.28 Numerical solution of blood velocity $u(x,t)$ with dragged coefficient = 5.30 g/cm^{3}s

For the time processing of this method and meshing, we find that elapsed time is 9.631317 s.

4.4 The tendencies of numerical solutions of blood pressure and blood velocity with blockage in terms of dragged coefficient

4.4.1 Numerical solution of blood velocity and blood pressure by Crank – Nicolson method for x = 0.4 cm

We compare the solutions of blood flow, blood pressure and blood velocity, from the numerical methods (Finite difference equation). By using the solutions on x = 0.4 cm, we varies the dragged coefficient k from 1.06 g/cm^3s to 5.30 g/cm^3s for time interval = 6.25 s.

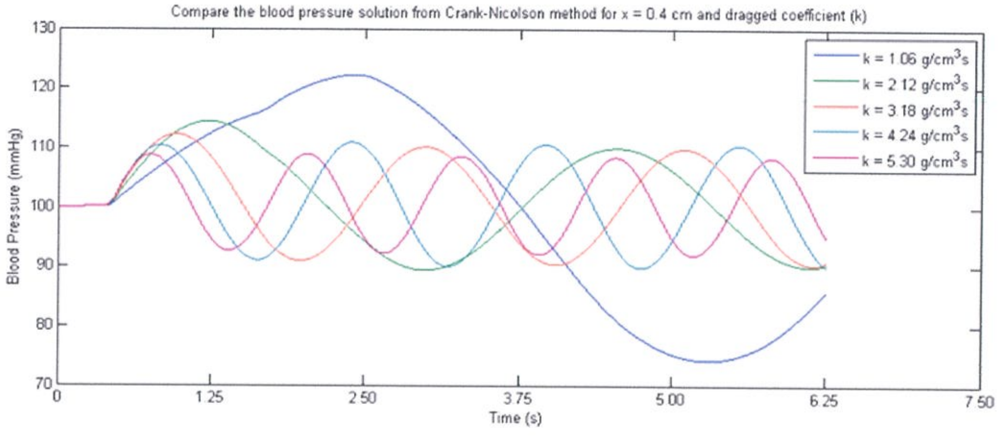

Figure 4.29 Comparing graph of blood pressure solutions from Crank – Nicolson method with the varying of dragged coefficient (k) which x = 0.4 cm

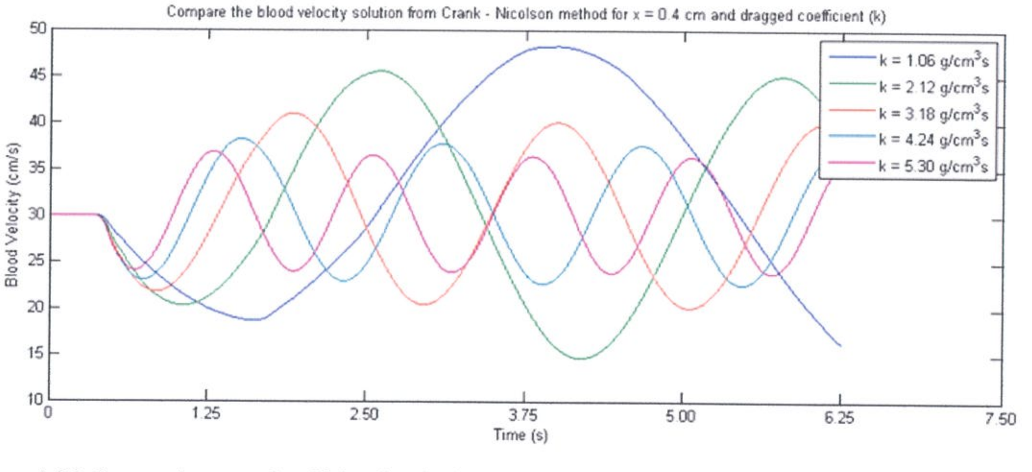

Figure 4.30 Comparing graph of blood velocity solutions from Crank – Nicolson method with the varying of dragged coefficient (k) which x = 0.4 cm

From figure 4.29 and figure 4.30, we find that the maximum of overall solutions decrease when the dragged coefficient k is large. The blue lines represent to the solutions of blood flow with dragged coefficient k = 1.06 g/cm^{3}s but their maximum of solutions are largest. While the purple lines are the solutions of largest dragged coefficient, k = 5.30 g/cm^{3}s. They are smallest of amplitude waves. Furthermore, the both graphs show the frequency of wave solutions which relate to the number of dragged coefficient also. By the large dragged coefficient, the solutions have the frequently wave sequences more than the wave solutions which is tiny coefficient.

4.4.2 Numerical solution of blood velocity and blood pressure by Lax - Wendroff method for x = 0.4 cm

Similarly to Crank – Nicolson method, we find that the maximum of the solutions of blood pressure and blood velocity are decreased by the large of dragged coefficient. The frequency of blood flow solutions with blockage relate to it. Their relations follows by the figure 4.31 and figure 4.32.

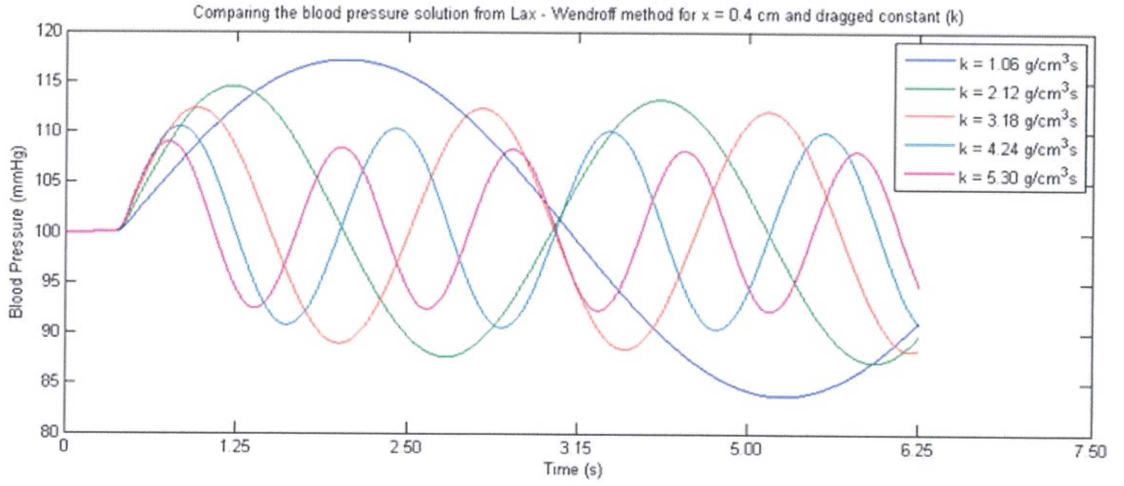

Figure 4.31 Comparing graph of blood pressure solutions from Lax - Wendroff method with the varying of dragged coefficient (k) which x = 0.4 cm

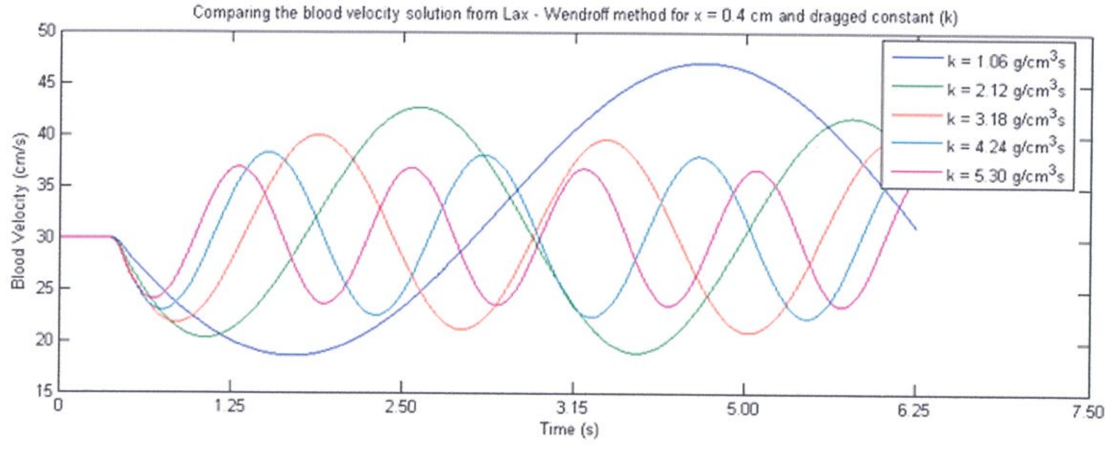

Figure 4.32 Comparing graph of blood velocity solutions from Lax - Wendroff method with the varying of dragged coefficient (k) which x = 0.4 cm

CHAPTER 5

Precision comparing and Error estimation

5.1 Error estimation of solutions from finite difference method

We have 2 sections for analyzing the consistents of solution from finite difference.

5.1.1 Error estimation of method for small amplitude wave pressure equation without blockage

By the analytical solution, we find that the solutions of blood pressure and velocity are

$$P(x,t) = 100 + 20\cos\left(x - \sqrt{\frac{A_0}{\rho c}}t\right), \tag{5.1}$$

$$u(x,t) = 30 - 20\cos\left(x - \sqrt{\frac{A_0}{\rho c}}t\right). \tag{5.2}$$

respectively.

We examine the solutions of blood pressure and blood velocity with x = 0.5 cm, at the middle of blood vessel. Furthermore, we compare the solutions of blood pressure and velocity with analytical solution which follow the table 5.1, 5.2 ,and figure 5.1and 5.2.

Time	Blood Pressure (mmHg)		
(s)	Lax - Wendroff	Crank - Nicolson	Analytics
0	120.0000	120.0000	117.4304
5	95.6579	91.2586	95.4913
10	79.9122	87.5947	80.0363
15	92.9458	88.3309	93.1828
20	116.0858	116.8458	116.0962
25	116.1800	117.2710	115.9490
30	93.0936	91.2151	92.9521
35	79.9018	81.7984	80.0526
40	95.5042	88.8907	95.7313

Table 5.1 Blood pressure solutions without blockage from finite difference methods and analytical solution

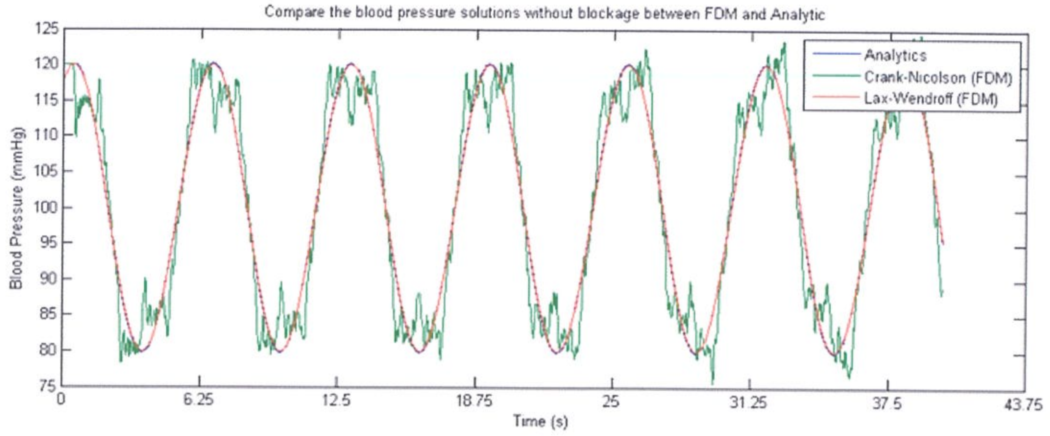

Figure 5.1 Comparing graph of blood pressure solutions without blockage from finite difference method with analytical solution

Time (s)	Blood Velocity (cm/s)		
	Lax - Wendroff	Crank - Nicolson	Analytics
0	10.0000	10.0000	12.5696
5	34.7585	38.7414	34.5087
10	49.8490	42.4053	49.9637
15	36.5023	41.6691	36.8172
20	13.8399	13.1542	13.9038
25	14.3296	12.7290	14.0510
30	37.2699	38.7849	37.0479
35	49.7948	48.2016	49.9474
40	33.9601	41.1093	34.2687

Table 5.2 Blood velocity solutions without blockage from finite difference methods and analytical solution

From the data of section 5.1.1, we find that the percentage of error for blood pressure solution without blockage in Lax – Wendroff method is 0.16% while the error of blood pressure solution without blockage in Crank – Nicolson is 3.28%. Likewise, in the solution of blood velocity for non-blockage, The percentage error of Lax – Wendroff and Crank – Nicolson are 0.92% and 14.82% respectively.

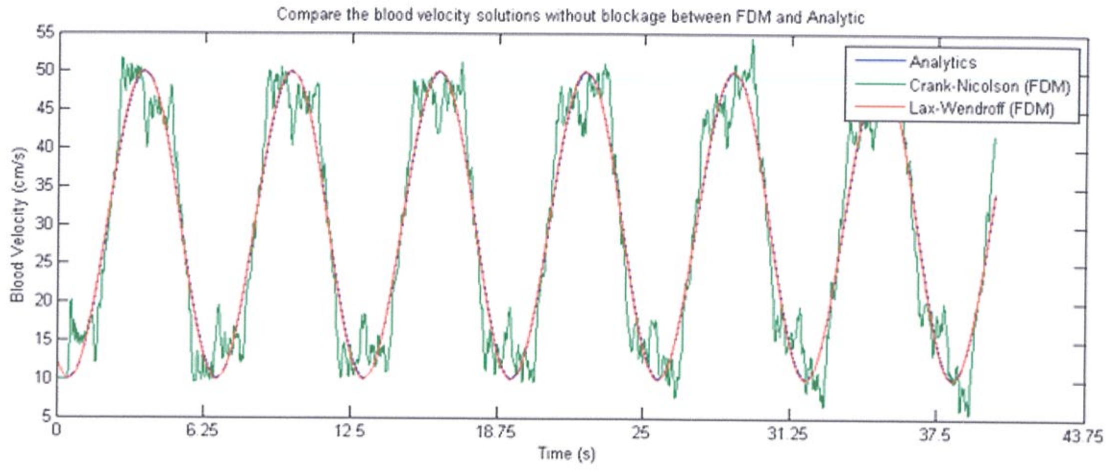

Figure 5.2 Comparing graph of blood velocity solutions without blockage from finite difference method with analytical solution

5.1.2 Error estimation of method for small amplitude wave pressure equation with linear dragged blockage

We can see that the solution of blood velocity with blockage is more complicate. Consider the end of blood vessel (x = L), the solution of blood pressure on the right ended of Eq (3.6.27) is

$$P(L,t) = 100 + 20\left(\frac{\sin\left(\frac{k}{\rho}t\right)\cos(\beta L)\cosh(\alpha L) - \cos\left(\frac{k}{\rho}t\right)\sin(\beta L)\sinh(\alpha L)}{\cos^2(\beta L)\cosh^2(\alpha L) + \sin^2(\beta L)\sinh^2(\alpha L)}\right). \tag{3.8.28}$$

In this thesis, we assume that the length of blood vessel is $L = 1$ cm, and we compare the suitable of numerical method and analytic i.e.

1. The linear dragged blockage is $k = 1.06$ g/cm^{3}s

We can see that the solution of blood pressure in Crank – Nicolson method satisfies the analytical solution. While solution from Lax – Wendroff shifts which follow by table 5.3 and figure 5.3.

Time	Blood Pressure (mmHg)		
(s)	Lax - Wendroff	Crank - Nicolson	Analytics
0	78.9235	78.9235	78.9235
5	72.2702	69.2784	92.4768
10	105.3447	105.2501	110.2941
15	130.7620	130.7595	113.2057
20	112.1073	112.1058	97.1962
25	76.1068	76.1066	85.2036
30	74.3375	74.3396	94.4095
35	109.3342	109.3358	111.6247
40	130.9581	130.9568	112.1855

Table 5.3 Blood pressure solutions from finite difference methods

and analytical solution for k = 1.06 g/cm^{3}s

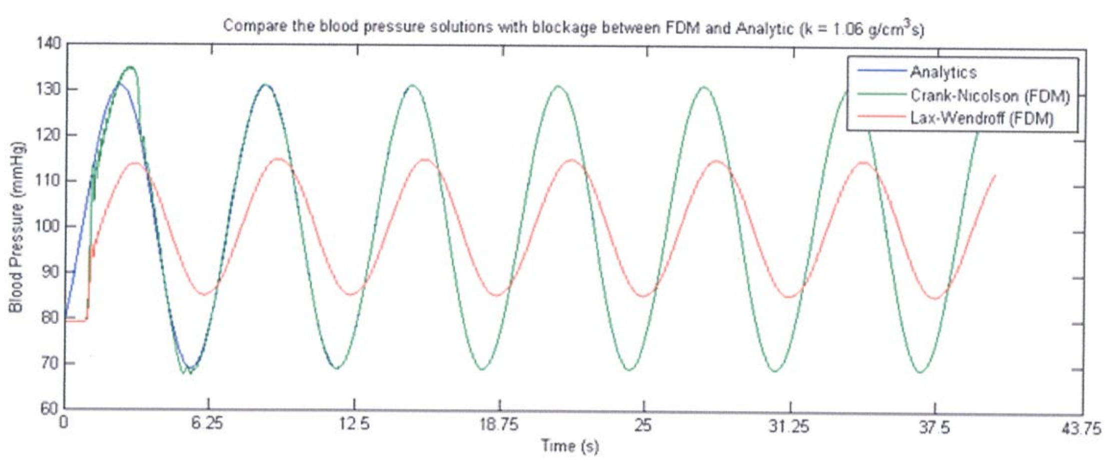

Figure 5.3 Comparing graph of blood pressure solutions from finite difference method with

analytical solution for k = 1.06 g/cm^{3}s

From this data, the percentage error of blood pressure solution at the end point by Lax – Wendroff and Crank – Nicolson are 13.66% and 0.74% respectively.

2. The linear dragged blockage is $k = 2.12$ g/cm^{3}s

We can see that the solution of blood pressure in Crank – Nicolson method satisfies the analytic solution while solution from Lax – Wendroff has more error which follow by table 5.4 and figure 5.4.

Time (s)	Blood Pressure (mmHg)		
	Lax - Wendroff	Crank - Nicolson	Analytics
0	89.4253	89.4253	89.4253
5	115.3630	115.2811	107.5634
10	84.7935	84.7992	92.0201
15	110.1558	110.1521	105.5179
20	98.1637	98.1639	98.7065
25	92.9258	92.9292	96.6523
30	113.7078	113.7020	106.9114
35	84.0705	84.0769	91.7493
40	113.0241	113.0192	106.9344

Table 5.4 Blood pressure solutions from finite difference methods and analytical solution for k = 2.12 g/cm^{3}s

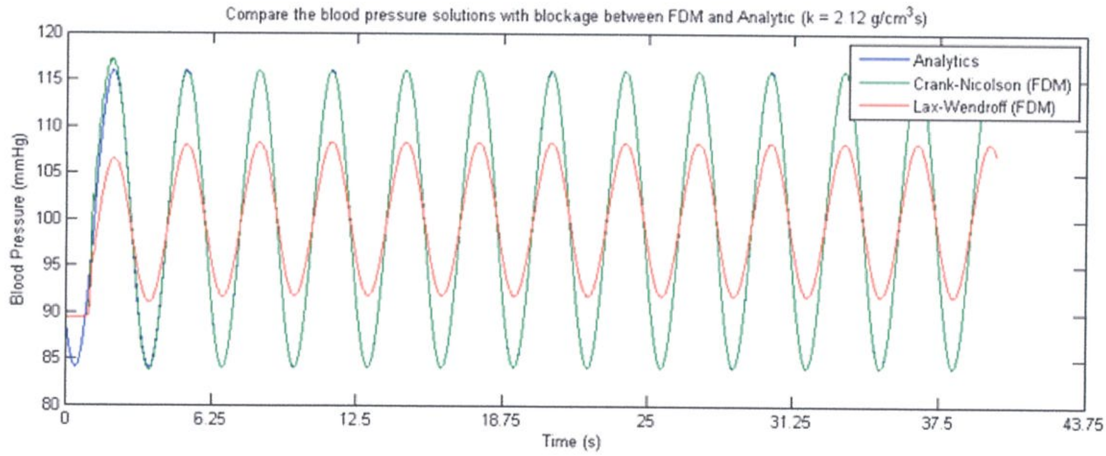

Figure 5.4 Comparing graph of blood pressure solutions from finite difference method with analytical solution for k = 2.12 g/cm^{3}s

From this data, the percentage error of blood pressure solution at the end point by Lax – Wendroff and Crank – Nicolson are 5.22% and 0.52% respectively.

3. The linear dragged blockage is $k = 3.18$ g/cm^{3}s

We can see that the solution of blood pressure in Crank – Nicolson method satisfies the analytic solution while solution from Lax – Wendroff has more error which follow by table 5.5 and figure 5.5.

Time	Blood Pressure (mmHg)		
(s)	Lax - Wendroff	Crank - Nicolson	Analytics
0	102.0720	102.0720	102.0720
5	92.5207	92.5730	95.8715
10	109.2918	109.2817	105.1319
15	93.3615	93.3718	97.0464
20	100.7945	100.7890	99.4206
25	105.4313	105.4294	103.8398
30	90.9533	90.9617	94.7459
35	108.3141	108.3032	104.1433
40	96.4145	96.4226	98.9589

Table 5.5 Blood pressure solutions from finite difference methods

and analytical solution for k = 3.18 g/cm^{3}s

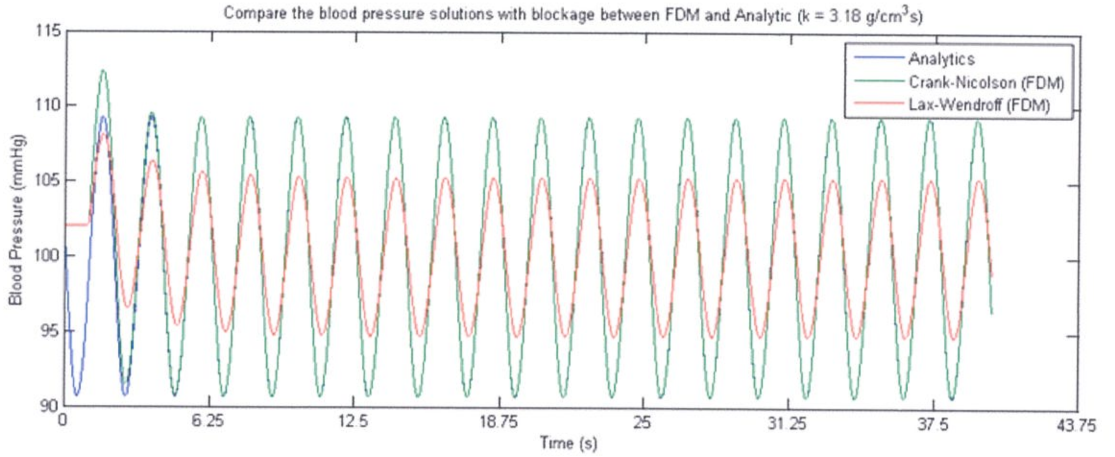

Figure 5.5 Comparing graph of blood pressure solutions from finite difference method with analytical solution for k = 3.18 g/cm^{3}s

From this data, the percentage error of blood pressure solution at the end point by Lax – Wendroff and Crank – Nicolson are 2.82% and 0.26% respectively.

4. The linear dragged blockage is $k = 4.24$ g/cm^{3}s

We can see that the solution of blood pressure in Crank – Nicolson method satisfies the analytic solution while solution from Lax – Wendroff has more error which follow by table 5.6 and figure 5.6.

Time	Blood Pressure (mmHg)		
(s)	Lax - Wendroff	Crank - Nicolson	Analytics
0	106.1557	106.1557	106.1557
5	101.3929	101.8514	102.8002
10	94.9812	95.0140	98.0264
15	94.5109	94.5235	96.9372
20	100.5388	100.5293	99.6642
25	105.9289	105.9088	102.8156
30	104.3001	104.2933	102.6376
35	97.5807	97.5952	99.3376
40	93.7253	93.7440	96.8219

Table 5.6 Blood pressure solutions from finite difference methods

and analytical solution for k = 4.24 g/cm^{3}s

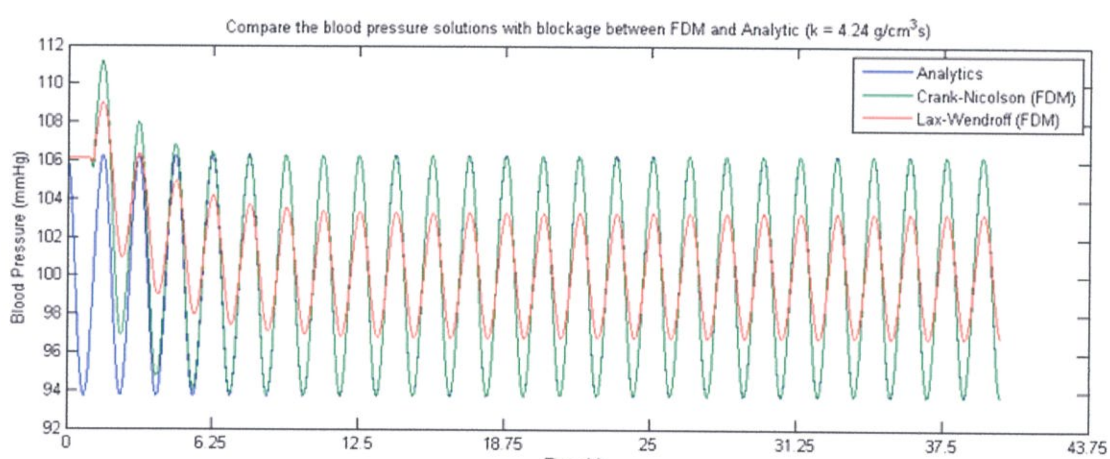

Figure 5.6 Comparing graph of blood pressure solutions from finite difference method with analytical solution for k = 4.24 g/cm^{3}s

From this data, the percentage error of blood pressure solution at the end point by Lax – Wendroff and Crank – Nicolson are 2.06% and 0.21% respectively.

5. The linear dragged blockage is $k = 5.30$ g/cm^{3}s

We can see that the solution of blood pressure in Crank – Nicolson method satisfies the analytic solution while solution from Lax – Wendroff has more error which follow by table 5.7 and figure 5.7.

Time	Blood Pressure (mmHg)		
(s)	Lax - Wendroff	Crank - Nicolson	Analytics
0	102.2219	102.2219	102.2219
5	101.7892	102.2329	101.5054
10	101.3251	101.3445	100.5383
15	100.8376	100.8275	100.1392
20	100.3354	100.3255	99.8478
25	99.8273	99.8200	99.5800
30	99.3222	99.3178	99.3236
35	98.8291	98.8277	99.0798
40	98.3565	98.3582	98.8524

Table 5.7 Blood pressure solutions from finite difference methods

and analytical solution for k = 5.30 g/cm^{3}s

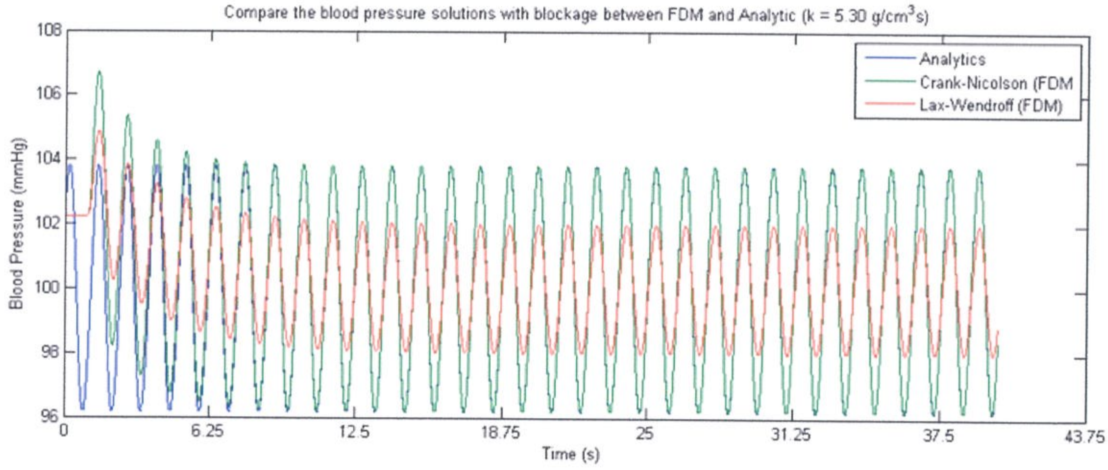

Figure 5.7 Comparing graph of blood pressure solutions from finite difference method with analytical solution for k = 5.30 g/cm^{3}s

From this data, the percentage error of blood pressure solution at the end point by Lax – Wendroff and Crank – Nicolson are 1.24% and 0.16% respectively.

CHAPTER 6

Conclusion

1. The numerical solutions of blood pressure and blood velocity

We can find the solutions of blood flow in a blood vessel within the small amplitude condition. The solutions of blood flow without blockage are uniform waves (in the section 4.1). While the tendency of maximum blood flow with blockage solutions are decreasing, for long distance, which is influenced by dragged blockage (in the section 4.2, 4.3). Furthermore, the numerical solutions with blockage show that there are more frequency of wave solutions of blood pressure and blood velocity for larger dragged coefficients (in section 4.4). Their blockage solutions are related to satisfy to the common sense of rapidly breathing of atherosclerosis people.

2. The initial condition and boundary conditions of blood flow

The initial and boundary conditions without blockage are referred to normal conditions of human body (in the section 3.5). The conditions of governing equation with blockage is defined by the common sense of the restraining of blood velocity and the slow down of blood pressure at the end of rod which are represented to derivative condition (in the section 3.6).

3. The analytical solution of blood pressure and blood velocity

For section 3.7, the solutions of blood flow without blockage can be derived in both of blood pressure and blood velocity solutions. They are uniform wave solution. However, they are only derived in blood pressure at the end of the rod for the blood flow with blockage, as the condition of velocity solution is complicate (in the section 3.8).

4. The precision of numerical approximation

The numerical solution of Lax – Wendroff method, explicitly finite difference, is more accutate than Crank – Nicolson for the solutions of blood flow without blockage analytically (in section 5.1.1). Conversely, the analytic solution of blood flow with blockage satisfies to the numerical solutions from Crank – Nicolson method which is more accurate than Lax - Wendroff (in section 5.1.2). Furthermore, both of finite difference methods make blood pressure solutions with small percentage error for larger dragged coefficient k (in section 5.1.2).

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Appendix

APPENDIX A

Matlab Codes for Blood Flows in Small Amplitude System

A.1 Matlab code for small amplitude system with non – blockage by using Crank – Nicolson finite difference method

```
%-----------------------------------------------------------------------
% Small amplitude system without linear drag blockage (CN - Method)
%-----------------------------------------------------------------------

clear all;

%% Meshing Domain Section
x0 = 0;      xl = 1;      dx = 0.0125;       M = (xl-x0)/dx;
t0 = 0;      tl = 40;      dt = 0.00125;       N = (tl-t0)/dt;

%% Physical Parameter Declaration
    rho = 1.060;      % density of blood
    A0 = 20;          % cross sectional area of aorta
    c = 20;           % compliance parameter

% ---- Matrices section ----%
v = zeros(N+1,M+1);  % Velocity of blood flow
P = zeros(N+1,M+1);  % Pressure of blood flow

A_matrix_new = zeros(2*(M-1),2*(M-1));  % Coefficient matrix
A_matrix_old = zeros(2*(M-1),2*(M-1));

X = zeros(2*M-2,1); % Variables matrix
F = zeros(2*M-2,1); % Remainder matrix

% ---- Boundary conditions ----%
% --- Left Boundaries
    n = 0:N;
    Bul = 20*cos(sqrt(A0/(rho*c))*n*dt);
    BPl = 100-20*cos(sqrt(A0/(rho*c))*n*dt);

% --- Right Boundaries
    Bur = 20*cos(xl- sqrt(A0/(rho*c))*n*dt);
    BPr = 100-20*cos(xl- sqrt(A0/(rho*c))*n*dt);

% ---- Initial conditions ----%
    IP = 80;
    Iu = 20;

%% Crank - Nicolson programming

for i = 1:N      % Time loop intervals
    if i == 1    % (Initial time t = t0 // Initial conditions)
        v(i,:) = Iu;
        P(i,:) = IP;
```

```
        X(1:2:end) = v(1,2:end-1)';
        X(2:2:end) = P(1,2:end-1)';
        r = 3;
        q = 4;
        for j = 1:M+1 % Space loop intervals

% ---- stability coefficient
        alpha_v = 0.25*(dt/dx)*(1/rho);
        alpha_P = 0.25*(dt/dx)*(A0/c);

            if j == 1 % (Left boundary conditions)
                P(i+1,j) = BPl(i+1)';   %Left Boundary
                v(i+1,j) = Bul(i+1)';
                P(i+1,M+1) = BPr(i+1)'; %Right Boundary
                v(i+1,M+1) = Bur(i+1)';

            elseif j ~= 1 && j ~= M+1 % j == 2:M

                if j == 2
                    F(1) = -alpha_v*P(i+1,j-1)-alpha_v*P(i,j-1);
                    F(2) = -alpha_P*v(i+1,j-1)-alpha_P*v(i,j-1);

                elseif j ~= 2 && j ~= M
                    F(r) = 0;
                    F(q) = 0;
                    r = r+2;
                    q = q+2;

                else
                    F(M+j-3) = alpha_v*P(i+1,j+1)+alpha_v*P(i,j+1);
                    F(M+j-2) = alpha_P*v(i+1,j+1)+alpha_P*v(i,j+1);
                end

            else % (Right Right boundary conditions j == M+1)

            end

        end
        % ---- Create coefficient matrix A
            for l = 1:M-1
                A_matrix_new(2*l-1,(2*l-1)+3) = -alpha_v;
                A_matrix_new(2*l,(2*l)+1) = -alpha_P;
                A_matrix_new(2*l+1,(2*l+1)-1) = alpha_v;
                A_matrix_new(2*l+2,(2*l+2)-3) = alpha_P;
                A_matrix_new(2*l-1,2*l-1) = 1;
                A_matrix_new(2*l,2*l) = 1;
                A_matrix_old(2*l-1,(2*l-1)+3) = alpha_v;
                A_matrix_old(2*l,(2*l)+1) = alpha_P;
                A_matrix_old(2*l+1,(2*l+1)-1) = -alpha_v;
                A_matrix_old(2*l+2,(2*l+2)-3) = -alpha_P;
                A_matrix_old(2*l-1,2*l-1) = 1;
                A_matrix_old(2*l,2*l) = 1;
            end
        % ----
        % ---- Edit Coefficient matrix to suitable
                A_matrix_new = A_matrix_new(1:2*(M-1),1:2*(M-1));
                A_matrix_new(2*(M)-2,2*(M)-2) = 1;
                A_matrix_new(2*(M)-2,2*(M)-3) = 0;
                A_matrix_new(2*(M)-3,2*(M)-3) = 1;
```

```
            A_matrix_new(2*(M)-3,2*(M)-2) = 0;
            A_matrix_new(2,1) = 0;

            A_matrix_old = A_matrix_old(1:2*(M-1),1:2*(M-1));
            A_matrix_old(2*(M)-2,2*(M)-2) = 1;
            A_matrix_old(2*(M)-2,2*(M)-3) = 0;
            A_matrix_old(2*(M)-3,2*(M)-3) = 1;
            A_matrix_old(2*(M)-3,2*(M)-2) = 0;
            A_matrix_old(2,1) = 0;

    XX = (A_matrix_old)*X+F;

    % ---- Solving variable vector
    X =(A_matrix_new)\((A_matrix_old)*X+F);

    % ---- Input v,P from F(v,P)
    v(i+1,2:end-1) = X(1:2:end)';
    P(i+1,2:end-1) = X(2:2:end)';

else

    r = 3;
    q = 4;

    for j = 1:M+1 % Space loop intervals

        if j == 1 % (Left boundary conditions)
            P(i+1,j) = BPl(i+1)';   %Left Boundary
            v(i+1,j) = Bul(i+1)';
            P(i+1,M+1) = BPr(i+1)'; %Right Boundary
            v(i+1,M+1) = Bur(i+1)';

        elseif j ~= 1 && j ~= M+1 % j == 2:M

            if j == 2
                F(1) = -alpha_v*P(i+1,j-1)-alpha_v*P(i,j-1);
                F(2) = -alpha_P*v(i+1,j-1)-alpha_P*v(i,j-1);

            elseif j ~= 2 && j ~= M
                F(r) = 0;
                F(q) = 0;
                r = r+2;
                q = q+2;

            else
                F(M+j-3) = alpha_v*P(i+1,j+1)+alpha_v*P(i,j+1);
                F(M+j-2) = alpha_P*v(i+1,j+1)+alpha_P*v(i,j+1);
            end

        else

        end

    end

    % ---- Create coefficient matrix A
        for l = 1:M-1
            A_matrix_new(2*l-1,(2*l-1)+3) = -alpha_v;
```

```
                A_matrix_new(2*l,(2*l)+1) = -alpha_P;
                A_matrix_new(2*l+1,(2*l+1)-1) = alpha_v;
                A_matrix_new(2*l+2,(2*l+2)-3) = alpha_P;
                A_matrix_new(2*l-1,2*l-1) = 1;
                A_matrix_new(2*l,2*l) = 1;

                A_matrix_old(2*l-1,(2*l-1)+3) = alpha_v;
                A_matrix_old(2*l,(2*l)+1) = alpha_P;
                A_matrix_old(2*l+1,(2*l+1)-1) = -alpha_v;
                A_matrix_old(2*l+2,(2*l+2)-3) = -alpha_P;
                A_matrix_old(2*l-1,2*l-1) = 1;
                A_matrix_old(2*l,2*l) = 1;
            end

        % ---- Edit Coefficient matrix to suitable
                A_matrix_new = A_matrix_new(1:2*(M-1),1:2*(M-1));
                A_matrix_new(2*(M)-2,2*(M)-2) = 1;
                A_matrix_new(2*(M)-2,2*(M)-3) = 0;
                A_matrix_new(2*(M)-3,2*(M)-3) = 1;
                A_matrix_new(2*(M)-3,2*(M)-2) = 0;
                A_matrix_new(2,1) = 0;

                A_matrix_old = A_matrix_old(1:2*(M-1),1:2*(M-1));
                A_matrix_old(2*(M)-2,2*(M)-2) = 1;
                A_matrix_old(2*(M)-2,2*(M)-3) = 0;
                A_matrix_old(2*(M)-3,2*(M)-3) = 1;
                A_matrix_old(2*(M)-3,2*(M)-2) = 0;
                A_matrix_old(2,1) = 0;

        XX = (A_matrix_old)*X+F;

        % ---- Solving variable vector
        X =(A_matrix_new)\((A_matrix_old)*X+F);

        % ---- Input v,P from F(v,P)

        v(i+1,2:end-1) = X(1:2:end)';
        P(i+1,2:end-1) = X(2:2:end)';

    end

end
```

A.2 Matlab code for small amplitude system with linear velocity dragged blockage by using Crank – Nicolson finite difference method

```
%-------------------------------------------------------------------------
% Small amplitude system with linear drag blockage (CN - Method)
%-------------------------------------------------------------------------

%% Meshing Domain Section

x0 = 0;      xl = 1;      dx = 0.125;       M = (xl-x0)/dx;
t0 = 0;      tl = 40;      dt = 0.0125;       N = (tl-t0)/dt;

%% Physical Parameter Declaration

    rho = 1.060;      % density of blood
    A0 = 23;          % cross sectional area of aorta
    c = 23;           % compliance parameter
    k = 1.06;         % dragging constant

%% Declaration of matrices & IBVP

% ---- Matrices section ----%

u = zeros(N+1,M+1);  % Velocity of blood flow
v = zeros(N+1,M+1);  % V - Matrix (Auxilary of velocity matrix)
P = zeros(N+1,M+1);  % Pressure of blood flow

A_matrix_new = zeros(2*(M-1),2*(M-1));  % Coefficient matrix
A_matrix_old = zeros(2*(M-1),2*(M-1));

X = zeros(2*M-2,1); % Variables matrix
F = zeros(2*M-2,1); % Remainder matrix

% ----

% ---- Boundary conditions ----%

    % --- Left Boundaries
            n = 0:N;
            u(n+1,M+1) = 0;
            BPl = 100+20*sin((k/rho)*n*dt);
    % ----

% ---- Initial conditions ----%

    m = 0:M;
    IP = 100;
    Iu = 0;


%% Crank - Nicolson programming

for i = 1:N      % Time loop intervals

    if i == 1    % (Initial time t = t0 // Initial conditions)
```

```
        u(i,:) = Iu;
        v(i,:) = u(i,:);
        P(i,:) = IP;

        X(1:2:end) = v(1,2:end-1)';
        X(2:2:end) = P(1,2:end-1)';

        r = 3;
        q = 4;

        for j = 1:M+1 % Space loop intervals

            % ---- stability coefficient
                alpha_v_old = 0.25*(dt/dx)*(1/rho)*exp((k/rho)*(i-
1)*dt);
                alpha_P_old = 0.25*(dt/dx)*(A0/c)*exp(-(k/rho)*(i-
1)*dt);

                alpha_v_new =
0.25*(dt/dx)*(1/rho)*exp((k/rho)*(i)*dt);
                alpha_P_new = 0.25*(dt/dx)*(A0/c)*exp(-
(k/rho)*(i)*dt);

            % ----

            if j == 1 % (initial space x = x0 // Left boundary
conditions)

                P(i+1,j) = BPl(i+1)';    %Left Boundary

            elseif j ~= 1 && j ~= M+1 % j == 2:M

                if j == 2

                    F(1) = -alpha_v_new*P(i+1,j-1)-alpha_v_old*P(i,j-
1);
                    F(2) = -alpha_P_new*dx*exp(-
(k/rho)*i*dt)*v(i+1,j-1)-alpha_P_old*dx*exp(-(k/rho)*(i-1)*dt)*v(i,j-
1);

                elseif j ~= 2 && j ~= M

                    F(r) = 0;
                    F(q) = 0;

                    r = r+2;
                    q = q+2;

                else

                    F(M+j-3) = 0;
                    F(M+j-2) = 0;

                end
```

```
            else % (Right boundary x = xl // Right boundary
conditions j == M+1)

            end

        end

        % ---- Create coefficient matrix A
            for l = 1:M-1
                A_matrix_new(2*l-1,(2*l-1)+3) = -alpha_v_new;
                A_matrix_new(2*l,(2*l)+1) = -alpha_P_new;
                A_matrix_new(2*l+1,(2*l+1)-1) = alpha_v_new;
                A_matrix_new(2*l+2,(2*l+2)-3) = alpha_P_new;
                A_matrix_new(2*l-1,2*l-1) = 1;
                A_matrix_new(2*l,2*l) = 1;

                A_matrix_old(2*l-1,(2*l-1)+3) = alpha_v_old;
                A_matrix_old(2*l,(2*l)+1) = alpha_P_old;
                A_matrix_old(2*l+1,(2*l+1)-1) = -alpha_v_old;
                A_matrix_old(2*l+2,(2*l+2)-3) = -alpha_P_old;
                A_matrix_old(2*l-1,2*l-1) = 1;
                A_matrix_old(2*l,2*l) = 1;
            end
        % ----
        % ---- Edit Coefficient matrix to suitable
                A_matrix_new = A_matrix_new(1:2*(M-1),1:2*(M-1));
                A_matrix_new(2*(M)-2,2*(M)-2) = 1;
                A_matrix_new(2*(M)-2,2*(M)-3) = alpha_P_new;
                A_matrix_new(2*(M)-3,2*(M)-3) = 1;
                A_matrix_new(2*(M)-3,2*(M)-2) = -alpha_v_new;
                A_matrix_new(2,1) = alpha_P_new;

                A_matrix_old = A_matrix_old(1:2*(M-1),1:2*(M-1));
                A_matrix_old(2*(M)-2,2*(M)-2) = 1;
                A_matrix_old(2*(M)-2,2*(M)-3) = -alpha_P_old;
                A_matrix_old(2*(M)-3,2*(M)-3) = 1;
                A_matrix_old(2*(M)-3,2*(M)-2) = alpha_v_old;
                A_matrix_old(2,1) = -alpha_P_old;

        XX = (A_matrix_old)*X+F;

        % ---- Solving variable vector
        X =(A_matrix_new)\((A_matrix_old)*X+F);

        % ---- Input v,P from F(v,P)

        v(i+1,2:end-1) = X(1:2:end)';
        v(i+1,1) = v(i+1,2)-(dx*exp(-(k/rho)*(i-
1)*dt)*(c/A0)*(k/rho)*(20*cos((k/rho)*(i-1)*dt)));

        u(i+1,1:end-1) = v(i+1,1:end-1).*exp(-(k/rho)*(i)*dt);
        P(i+1,2:end-1) = X(2:2:end)';

        u(i+1,M+1) = u(i+1,M);
        v(i+1,M+1) = v(i+1,M);
        P(i+1,M+1) = P(i+1,M);

    else
```

```
        r = 3;
        q = 4;

        for j = 1:M+1 % Space loop intervals

            % ---- stability coefficient
                alpha_v_old = 0.25*(dt/dx)*(1/rho)*exp((k/rho)*(i-
1)*dt);
                alpha_P_old = 0.25*(dt/dx)*(A0/c)*exp(-(k/rho)*(i-
1)*dt);

                alpha_v_new =
0.25*(dt/dx)*(1/rho)*exp((k/rho)*(i)*dt);
                alpha_P_new = 0.25*(dt/dx)*(A0/c)*exp(-
(k/rho)*(i)*dt);

            % ----

            if j == 1 % (initial space x = x0 // Left boundary
conditions)

                P(i+1,j) = BPl(i+1)';   %Left Boundary

            elseif j ~= 1 && j ~= M+1 % j == 2:M

                if j == 2

                    F(1) = -alpha_v_new*P(i+1,j-1)-alpha_v_old*P(i,j-
1);
                    F(2) = -alpha_P_new*dx*exp(-
(k/rho)*i*dt)*v(i+1,j-1)-alpha_P_old*dx*exp(-(k/rho)*(i-1)*dt)*v(i,j-
1);

                elseif j ~= 2 && j ~= M

                    F(r) = 0;
                    F(q) = 0;

                    r = r+2;
                    q = q+2;

                else

                    F(M+j-3) = 0;
                    F(M+j-2) = 0;

                end

            else % (Right boundary x = xl // Right boundary
conditions j == M+1)

            end

        end

        % ---- Create coefficient matrix A
            for l = 1:M-1
```

```
                A_matrix_new(2*l-1,(2*l-1)+3) = -alpha_v_new;
                A_matrix_new(2*l,(2*l)+1) = -alpha_P_new;
                A_matrix_new(2*l+1,(2*l+1)-1) = alpha_v_new;
                A_matrix_new(2*l+2,(2*l+2)-3) = alpha_P_new;
                A_matrix_new(2*l-1,2*l-1) = 1;
                A_matrix_new(2*l,2*l) = 1;

                A_matrix_old(2*l-1,(2*l-1)+3) = alpha_v_old;
                A_matrix_old(2*l,(2*l)+1) = alpha_P_old;
                A_matrix_old(2*l+1,(2*l+1)-1) = -alpha_v_old;
                A_matrix_old(2*l+2,(2*l+2)-3) = -alpha_P_old;
                A_matrix_old(2*l-1,2*l-1) = 1;
                A_matrix_old(2*l,2*l) = 1;
            end
        % ----
        % ---- Edit Coefficient matrix to suitable
                A_matrix_new = A_matrix_new(1:2*(M-1),1:2*(M-1));
                A_matrix_new(2*(M)-2,2*(M)-2) = 1;
                A_matrix_new(2*(M)-2,2*(M)-3) = alpha_P_new;
                A_matrix_new(2*(M)-3,2*(M)-3) = 1;
                A_matrix_new(2*(M)-3,2*(M)-2) = -alpha_v_new;
                A_matrix_new(2,1) = alpha_P_new;

                A_matrix_old = A_matrix_old(1:2*(M-1),1:2*(M-1));
                A_matrix_old(2*(M)-2,2*(M)-2) = 1;
                A_matrix_old(2*(M)-2,2*(M)-3) = -alpha_P_old;
                A_matrix_old(2*(M)-3,2*(M)-3) = 1;
                A_matrix_old(2*(M)-3,2*(M)-2) = alpha_v_old;
                A_matrix_old(2,1) = -alpha_P_old;

        XX = (A_matrix_old)*X+F;

        % ---- Solving variable vector
        X =(A_matrix_new)\((A_matrix_old)*X+F);

        % ---- Input v,P from F(v,P)

        v(i+1,2:end-1) = X(1:2:end)';
        v(i+1,1) = v(i+1,2)-(dx*exp((k/rho)*(i-
1)*dt)*(c/A0)*(k/rho)*(20*cos((k/rho)*(i-1)*dt)));
        u(i+1,1:end-1) = v(i+1,1:end-1).*exp(-(k/rho)*(i)*dt);
        P(i+1,2:end-1) = X(2:2:end)';

        u(i+1,M+1) = u(i+1,M);
        v(i+1,M+1) = v(i+1,M);
        P(i+1,M+1) = P(i+1,M);
    end

end
```

A.3 Matlab code for small amplitude system without linear velocity dragged blockage by using Lax – Wendroff finite difference method

```
%----------------------------------------------------------------------
% Small amplitude system without linear drag blockage (LW - Method)
%----------------------------------------------------------------------
clc;
clear all;
warning off;

% Set constant of small amplitude equation
rho =1.060;          % density of blood
A0 = 23;             % regular cross - section of blood vessel (artery)
c = 23 ;             % compliance of blood vessel in artery

%----------------------------------------------------------------------
% Part II. Numerical parameters
%----------------------------------------------------------------------

xl=0; xr=1;                    % x domain [xl,xr]
dx = 0.025;                         % J: number of division for x
J = (xr-xl) / dx;               % dx: mesh size

tl=0; tf = 100;                  % final simulation time
dt = 0.025;
Nt = (tf-tl)/dt;                % Nt: number of time steps

% Set eigenmatrix

a=(dt/dx)*(1/rho);

if a>1
    disp('a Stability aborted');
end

b=(dt/dx)*(A0/c);

if b>1
    disp('b Stability aborted');
end

%----------------------------------------------------------------------
% Part III. Lax - Wenroff method
%----------------------------------------------------------------------

% Meshing section
x = xl : dx : xr;                 % generate the grid point on x - space
tt = 0 : dt : tf;                 % generate the grid point on y - space
t = 0 : dt : tf;                  % time variable

IU = 20;
IP = 80;

% Define initial condition
I1 = IU;                 % u(x,0): dim f1 = (1:J+1)
I2 = IP;                   % P(x,0): dim f2 = (1:J+1)
```

```
% Define boundary condition
B1l = (20*cos(xl- sqrt(A0/(rho*c))*t));
B1r = (20*cos(xr- sqrt(A0/(rho*c))*t));
B2l = (100-20*cos(xl- sqrt(A0/(rho*c))*t));% P(0,t): dim B1l =
(1:J+1)
B2r = (100-20*cos(xr- sqrt(A0/(rho*c))*t));

% store the solution at all grid points for all time steps

P = zeros(Nt+1,(J+1));
v = zeros(Nt+1,(J+1));

% Find the approximate solution at each time step
for n = 1:Nt+1

    if n==1                                  % first time step

         v(1:end,1) = B1l(1:end);
         P(1:end,1) = B2l(1:end);

         v(1:end,J+1) = B1r(1:end);
         P(1:end,J+1) = B2r(1:end);

       for j=1:J+1                            % interior nodes

       v(n,j) = I1;
       P(n,j) = I2;

       end

    else                                     % for n > 1

       for j=2:J                             % interior nodes

                v(n,j) = v(n-1,j)+0.5*(dt/dx)*(1/rho)*((P(n-1,j+1))-
(P(n-1,j-1)))+...
                           0.5*((dt/dx)^2)*(A0/(rho*c))*((v(n-
1,j+1))-2*(v(n-1,j))+(v(n-1,j-1)));

                P(n,j) = P(n-1,j)+0.5*(dt/dx)*(A0/c)*((v(n-1,j+1))-
(v(n-1,j-1)))+...
                           0.5*((dt/dx)^2)*(A0/(rho*c))*((P(n-
1,j+1))-2*(P(n-1,j))+(P(n-1,j-1)));

       end

    end

end
```

A.4 Matlab code for small amplitude system with linear velocity dragged blockage by using Lax – Wendroff finite difference method

```
%-------------------------------------------------------------------
% Small amplitude system with linear drag blockage (LW - Method)
%-------------------------------------------------------------------
clc;
clear all;
warning off;
tic;

% Set constant of small amplitude equation
rho =1.060;         % density of blood
A0 = 23;            % regular cross - section of blood vessel (artery)
c = 23 ;            % compliance of blood vessel in artery
k1 = 1.06;

%-------------------------------------------------------------------
% Part II. Numerical parameters
%-------------------------------------------------------------------

xl=0; xr=1;                  % x domain [xl,xr]
dx = 0.01;                       % J: number of division for x
J = (xr-xl) / dx;             % dx: mesh size

tl=0; tf = 40;                % final simulation time
dt = 0.001;
Nt = (tf-tl)/dt;              % Nt: number of time steps

% Set eigenmatrix

a=(dt/dx)*(1/rho);

if a>1
    disp('a Stability aborted');
end

b=(dt/dx)*(A0/c);

if b>1
    disp('b Stability aborted');
end

%-------------------------------------------------------------------
% Part III. Lax - Wenroff method
%-------------------------------------------------------------------

% Meshing section
x = xl : dx : xr;              % generate the grid point on x - space
t = 0 : dt : tf;               % time variable

IU = 0;
IP = 100;

% Define initial condition
I1 = IU;            % u(x,0): dim f1 = (1:J+1)
```

```
I2 = IP;                    % P(x,0): dim f2 = (1:J+1)

% Define boundary condition
B21 = (100+20*sin((k1/rho)*t));% P(0,t): dim B11 = (1:J+1)

% store the solution at all grid points for all time steps

u = zeros(Nt+1,(J+1));
P = zeros(Nt+1,(J+1));
v = zeros(Nt+1,(J+1));

% Find the approximate solution at each time step
for n = 1:Nt+1

    if n==1                                  % first time step

        u(1:end,1) = 0; %B11(1:end);
        v(1:end,1) = 0; %B11(1:end);
        P(1:end,1) = B21(1:end);

       for j=1:J+1                             % interior nodes

       v(n,:) = I1;
       P(n,:) = I2;

       u(n,:) = v(n,:);

       end

    else                                     % for n > 1

       for j=1:J                             % interior nodes

           if j == 1

               %v(n,j) = u(n,j).*exp((k1/rho)*(n-1)*dt);

           else

               v(n,j) = v(n-
1,j)+0.25*((dt/dx)*(1/rho)*((exp((k1/rho)*(n)*dt))+exp((k1/rho)*(n-
1)*dt))).*((P(n-1,j+1))-(P(n-1,j-1)))+...
                              0.5*((dt/dx)^2)*(A0/(rho*c))*((v(n-
1,j+1))-2*(v(n-1,j))+(v(n-1,j-1)));

               u(n,2:end) = v(n,2:end).*exp(-(k1/rho)*(n-1)*dt);

               P(n,j) = P(n-1,j)+0.25*((dt/dx)*(A0/c)*(exp(-
(k1/rho)*(n)*dt)+exp(-(k1/rho)*(n-1)*dt))).*((v(n-1,j+1))-(v(n-1,j-
1)))+...
                              0.5*((dt/dx)^2)*(A0/(rho*c))*((P(n-
1,j+1))-2*(P(n-1,j))+(P(n-1,j-1)));

               v(n,j+1) = v(n,j);
               u(n,j+1) = u(n,j);
               P(n,j+1) = P(n,j);
           end
```

```
        end
        v(n,1) = v(n,2)-(dx*exp(-(k1/rho)*(n-
2)*dt)*(c/A0)*(k1/rho)*(20*cos((k1/rho)*(n-2)*dt)));
        u(n,1) = v(n,1).*exp(-(k1/rho)*(n-1)*dt);
    end

end
```

APPENDIX B

Von Neumann Stability Analysis

B.1 Von Neumann stabity analysis method for Crank – Nicolson finite difference method

B.1.1 Stability of Crank – Nicolson method for small amplitude wave system without linear drag blockage

Consider discrete solution U_j^n, by von Neumann stability analysis method, it can be assumed that

$$U_m^n = Ae^{\alpha nl} e^{\beta mhi}. \tag{B.1.1}$$

Consider the small amplitude wave system without linear drag blockage in Crank - Nicolson form

$$\begin{Bmatrix} u_m^{n+1} \\ P_m^{n+1} \end{Bmatrix} - \frac{\nu}{4}\left(\begin{Bmatrix} P_{m+1}^{n+1} \\ u_{m+1}^{n+1} \end{Bmatrix} - \begin{Bmatrix} P_{m-1}^{n+1} \\ u_{m-1}^{n+1} \end{Bmatrix} \right) = \begin{Bmatrix} u_m^{n} \\ P_m^{n} \end{Bmatrix} + \frac{\nu}{4}\left(\begin{Bmatrix} P_{m+1}^{n} \\ u_{m+1}^{n} \end{Bmatrix} - \begin{Bmatrix} P_{m-1}^{n} \\ u_{m-1}^{n} \end{Bmatrix} \right), \tag{3.1.6}$$

Equation (3.1.6) can be rearranged as

$$U_m^{n+1} - \frac{\nu}{4}\left(U_{m+1}^{n+1} - U_{m-1}^{n+1}\right) = U_m^n + \frac{\nu}{4}\left(U_{m+1}^n - U_{m-1}^n\right), \tag{B.1.2}$$

where $U_m^n = \begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix}$ and $\nu = \frac{l}{h}\begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{A_0}{c} & 0 \end{bmatrix}$.

Substituting equation (B.1.1) into (B.1.2), so that

$$\begin{aligned} &Ae^{\alpha(n+1)l} e^{m\beta hi} - \frac{\nu}{4}\left(Ae^{\alpha(n+1)l} e^{(m+1)\beta hi} - Ae^{\alpha(n+1)l} e^{(m-1)\beta hi}\right) \\ &= Ae^{\alpha nl} e^{m\beta hi} + \frac{\nu}{4}\left(Ae^{\alpha nl} e^{(m+1)\beta hi} - Ae^{\alpha nl} e^{(m-1)\beta hi}\right) \end{aligned}. \tag{B.1.3}$$

Dividing (B.1.3) with $Ae^{\alpha nl} e^{m\beta hi}$, we have

$$e^{\alpha l} - \frac{\nu}{4}\left(e^{\alpha l} e^{\beta hi} - e^{\alpha l} e^{-\beta hi}\right) = 1 + \frac{\nu}{4}\left(e^{\beta hi} - e^{-\beta hi}\right),$$

$$e^{\alpha l}\left(1-\frac{\nu}{4}\left(e^{\beta hi}-e^{-\beta hi}\right)\right)=1+\frac{\nu}{4}\left(e^{\beta hi}-e^{-\beta hi}\right),$$

$$e^{\alpha l}=\frac{1+\frac{\nu}{4}\left(e^{i\beta h}-e^{-i\beta h}\right)}{1-\frac{\nu}{4}\left(e^{i\beta h}-e^{-i\beta h}\right)}. \tag{B.1.4}$$

Since $\frac{e^{i\beta h}-e^{-i\beta h}}{2i}=\sin(\beta h)$, we have

$$e^{\alpha l}=\frac{1+i\frac{\nu}{2}\sin(\beta h)}{1-i\frac{\nu}{2}\sin(\beta h)}. \tag{B.1.5}$$

The amplification factor, $\left|e^{\alpha l}\right|$, is considered by norm of equation (B.1.5). Nevertheless, this equation is complex number. It therefore finds norm of this equation as

$$\left|e^{\alpha l}\right|^2=\left(\frac{\left(1+i\frac{\nu}{2}\sin(\beta h)\right)\left(1-i\frac{\nu}{2}\sin(\beta h)\right)}{\left(1-i\frac{\nu}{2}\sin(\beta h)\right)\left(1+i\frac{\nu}{2}\sin(\beta h)\right)}\right)=1. \tag{B.1.6}$$

Therefore,

$$\left|e^{\alpha l}\right|=1. \tag{B.1.7}$$

This is a condition of the stability of Crank – Nicolson finite difference method without linear drag blockage in (3.1.6) which is called that unconditional stable.

B.1.2 Stability of Crank – Nicolson method for small amplitude wave system with linear drag blockage

Consider the small amplitude wave system with linear drag blockage in Crank - Nicolson form

$$U_m^{n+1} - \frac{lA^{n+1}}{4h}\left[U_{m+1}^{n+1} - U_{m-1}^{n+1}\right] = U_m^n + \frac{lA^n}{4h}\left[U_{m+1}^n - U_{m-1}^n\right], \tag{2.2.15}$$

where $U_m^n = \begin{Bmatrix} v_m^n \\ P_m^n \end{Bmatrix}$ and $A^n = \begin{bmatrix} 0 & \frac{1}{\rho} e^{\frac{k}{\rho} n\Delta t} \\ \frac{A_0}{c} e^{-\frac{k}{\rho} n\Delta t} & 0 \end{bmatrix}$.

Similarly to (B.1.1), we assume that $U_m^n = Ae^{\alpha nl} e^{m\beta hi}$ and substitutes in (2.2.15). So

$$\begin{aligned} Ae^{\alpha(n+1)l} e^{m\beta hi} - \frac{lA^{n+1}}{4h}\left[Ae^{\alpha(n+1)l} e^{\beta(m+1)hi} - Ae^{\alpha(n+1)l} e^{\beta(m-1)hi}\right] \\ = Ae^{\alpha nl} e^{m\beta hi} + \frac{lA^n}{4h}\left[Ae^{\alpha nl} e^{\beta(m+1)hi} - Ae^{\alpha nl} e^{\beta(m-1)hi}\right] \end{aligned},$$

$$\begin{aligned} Ae^{\alpha(n+1)l} e^{m\beta hi} - \frac{lA^{n+1}}{4h}\left[Ae^{\alpha(n+1)l} e^{\beta(m+1)hi} - Ae^{\alpha(n+1)l} e^{\beta(m-1)hi}\right] \\ = Ae^{\alpha nl} e^{m\beta hi} + \frac{lA^n}{4h}\left[Ae^{\alpha nl} e^{\beta(m+1)hi} - Ae^{\alpha nl} e^{\beta(m-1)hi}\right] \end{aligned}, \tag{B.1.8}$$

Dividing (B.1.8) with $Ae^{\alpha nl} e^{m\beta hi}$, we have

$$e^{\alpha l}\left(1 - \frac{lA^{n+1}}{4h}\left[e^{\beta hi} - e^{-\beta hi}\right]\right) = 1 + \frac{lA^n}{4h}\left[e^{\beta hi} - e^{-\beta hi}\right],$$

$$e^{\alpha l} = \frac{1 + \frac{lA^n}{4h}\left[e^{\beta hi} - e^{-\beta hi}\right]}{1 - \frac{lA^{n+1}}{4h}\left[e^{\beta hi} - e^{-\beta hi}\right]}. \tag{B.1.9}$$

Since $\frac{e^{i\beta h} - e^{-i\beta h}}{2i} = \sin(\beta h)$, we have

$$e^{\alpha l} = \frac{1 + i\frac{lA^n}{4h}\sin(\beta h)}{1 - i\frac{lA^{n+1}}{4h}\sin(\beta h)}. \tag{B.1.10}$$

The amplification factor, $\left|e^{\alpha l}\right|$, is considered by norm of equation (B.1.5). Nevertheless, this equation is complex number. It therefore finds norm of this equation as

$$\left|e^{\alpha l}\right|^2 = \frac{\left(1+i\frac{lA^n}{4h}\sin(\beta h)\right)\left(1-i\frac{lA^n}{4h}\sin(\beta h)\right)}{\left(1-i\frac{lA^{n+1}}{4h}\sin(\beta h)\right)\left(1+i\frac{lA^{n+1}}{4h}\sin(\beta h)\right)},$$

$$\left|e^{\alpha l}\right|^2 = \frac{\left(1+\left(\frac{l}{4h}\right)^2\left(A^n\right)^2\sin^2(\beta h)\right)}{\left(1+\left(\frac{l}{4h}\right)^2\left(A^{n+1}\right)^2\sin^2(\beta h)\right)}. \tag{B.1.11}$$

Because of $A^n = \begin{bmatrix} 0 & \frac{1}{\rho}e^{\frac{k}{\rho}n\Delta t} \\ \frac{A_0}{c}e^{-\frac{k}{\rho}n\Delta t} & 0 \end{bmatrix}$, so

$$\left(A^n\right)^2 = \begin{bmatrix} 0 & \frac{1}{\rho}e^{\frac{k}{\rho}n\Delta t} \\ \frac{A_0}{c}e^{-\frac{k}{\rho}n\Delta t} & 0 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{\rho}e^{\frac{k}{\rho}n\Delta t} \\ \frac{A_0}{c}e^{-\frac{k}{\rho}n\Delta t} & 0 \end{bmatrix},$$

$$\left(A^n\right)^2 = \begin{bmatrix} \frac{A_0}{\rho c} & 0 \\ 0 & \frac{A_0}{\rho c} \end{bmatrix}. \tag{B.1.12}$$

It is similar with $\left(A^{n+1}\right)^2$ which is

$$\left(A^{n+1}\right)^2 = \begin{bmatrix} 0 & \frac{1}{\rho}e^{\frac{k}{\rho}(n+1)\Delta t} \\ \frac{A_0}{c}e^{-\frac{k}{\rho}(n+1)\Delta t} & 0 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{\rho}e^{\frac{k}{\rho}(n+1)\Delta t} \\ \frac{A_0}{c}e^{-\frac{k}{\rho}(n+1)\Delta t} & 0 \end{bmatrix},$$

$$\left(A^{n+1}\right)^2 = \begin{bmatrix} \dfrac{A_0}{\rho c} & 0 \\ 0 & \dfrac{A_0}{\rho c} \end{bmatrix}. \tag{B.1.13}$$

From (B.1.12) and (B.1.13), we can obtain that

$$\left(A^n\right)^2 = \left(A^{n+1}\right)^2. \tag{B.1.14}$$

Substituting (B.1.14) into (B.1.11), we have

$$\left|e^{\alpha l}\right|^2 = 1. \tag{B.1.15}$$

Which shows that the small amplitude wave system with linear drag blockage in Crank - Nicolson form has unconditional stable.

B.2 Von Neumann stabity analysis method for Lax - Wendroff finite difference method

B.2.1 Stability of Lax - Wendroff method for small amplitude wave system without linear drag blockage

Consider discrete solution U_j^n, by von Neumann stability analysis method, it can be assumed that

$$U_m^n = Ae^{\alpha nl}e^{m\beta hi}. \tag{B.1.1}$$

Consider the small amplitude wave system without linear drag blockage in Lax – Wendroff form

$$\begin{pmatrix} u_m^{n+1} \\ P_m^{n+1} \end{pmatrix} = \begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix} + \frac{l}{2h}\begin{bmatrix} 0 & \dfrac{1}{\rho} \\ \dfrac{A_0}{c} & 0 \end{bmatrix}\left(\begin{pmatrix} u_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - \begin{pmatrix} u_{m-1}^n \\ P_{m-1}^n \end{pmatrix}\right) + \frac{l^2}{2h^2}\begin{bmatrix} \dfrac{A_0}{\rho c} & 0 \\ 0 & \dfrac{A_0}{\rho c} \end{bmatrix}\left(\begin{pmatrix} u_{m+1}^n \\ P_{m+1}^n \end{pmatrix} - 2\begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix} + \begin{pmatrix} u_{m-1}^n \\ P_{m-1}^n \end{pmatrix}\right). \tag{3.2.3}$$

Equation (3.2.3) can be rearranged as

$$U_m^{n+1} = U_m^n + \frac{\nu}{2}\left(U_{m+1}^n - U_{m-1}^n\right) + \frac{\nu^2}{2}\left(U_{m+1}^n - 2U_m^n + U_{m-1}^n\right), \tag{B.2.1}$$

where $U_m^n = \begin{pmatrix} u_m^n \\ P_m^n \end{pmatrix}$ and $\nu = \frac{l}{h}\begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{A_0}{c} & 0 \end{bmatrix}$.

Substituting equation (B.1.1) into (B.2.2), so that

$$\begin{aligned} Ae^{\alpha(n+1)l}e^{m\beta hi} &= Ae^{\alpha nl}e^{m\beta hi} + \frac{\nu}{2}\left(Ae^{\alpha nl}e^{(m+1)\beta hi} - Ae^{\alpha nl}e^{(m-1)\beta hi}\right) \\ &+ \frac{\nu^2}{2}\left(Ae^{\alpha nl}e^{(m+1)\beta hi} - 2Ae^{\alpha nl}e^{m\beta hi} + Ae^{\alpha nl}e^{(m-1)\beta hi}\right) \end{aligned} \quad \text{(B.2.2)}$$

Dividing (B.2.2) with $Ae^{\alpha nl}e^{m\beta hi}$, we have

$$e^{\alpha l} = 1 + \frac{\nu}{2}\left(e^{i\beta h} - e^{-i\beta h}\right) + \frac{\nu^2}{2}\left(e^{i\beta h} - 2 + e^{-i\beta h}\right). \quad \text{(B.2.3)}$$

Since $\frac{e^{i\beta h} - e^{-i\beta h}}{2i} = \sin(\beta h)$ and $\frac{e^{i\beta h} + e^{-i\beta h}}{2} = \cos(\beta h)$, we have

$$e^{\alpha l} = 1 + i\nu\sin(\beta h) - \nu^2\left(1 - \cos(\beta h)\right). \quad \text{(B.2.4)}$$

Since $\cos(2A) = 1 - 2\sin^2(A)$, we have

$$e^{\alpha l} = 1 + i\nu\sin(\beta h) - \nu^2\left(1 - \left(1 - 2\sin^2\left(\frac{\beta h}{2}\right)\right)\right),$$

$$e^{\alpha l} = 1 + i\nu\sin(\beta h) - 2\nu^2\sin^2\left(\frac{\beta h}{2}\right). \quad \text{(B.2.5)}$$

The amplification factor, $\left|e^{\alpha l}\right|$, is considered by norm of equation (B.2.5). Nevertheless, this equation is complex number. It therefore finds norm of this equation as

$$\left|e^{\alpha l}\right|^2 = \left(1 - 2\nu^2\sin^2\left(\frac{\beta h}{2}\right)\right)^2 + \left(\nu\sin(\beta h)\right)^2,$$

$$\left|e^{\alpha l}\right|^2 = 1 - 4\nu^2\sin^2\left(\frac{\beta h}{2}\right) + 4\nu^4\sin^4\left(\frac{\beta h}{2}\right) + \nu^2\sin^2(\beta h). \quad \text{(B.2.6)}$$

Because of $\sin(2A) = 2\sin(A)\cos(A)$ consequently it implies that

$$\left|e^{\alpha l}\right|^2 = 1 - 4\nu^2\sin^2\left(\frac{\beta h}{2}\right) + 4\nu^4\sin^4\left(\frac{\beta h}{2}\right) + 4\nu^2\sin^2\left(\frac{\beta h}{2}\right)\cos^2\left(\frac{\beta h}{2}\right),$$

$$\left|e^{\alpha l}\right|^2 = 1 - 4\nu^2 \sin^2\left(\frac{\beta h}{2}\right)\left(1 - \cos^2\left(\frac{\beta h}{2}\right)\right) + 4\nu^4 \sin^4\left(\frac{\beta h}{2}\right). \tag{B.2.7}$$

From the identity of triginometry, we have $1 - \cos^2\left(\frac{\beta h}{2}\right) = \sin^2\left(\frac{\beta h}{2}\right)$, it follows that

$$\left|e^{\alpha l}\right|^2 = 1 - 4\nu^2 \sin^2\left(\frac{\beta h}{2}\right)\left(\sin^2\left(\frac{\beta h}{2}\right)\right) + 4\nu^4 \sin^4\left(\frac{\beta h}{2}\right),$$

$$= 1 - 4\nu^2 \sin^4\left(\frac{\beta h}{2}\right) + 4\nu^4 \sin^4\left(\frac{\beta h}{2}\right),$$

$$= 1 - 4\nu^2 \sin^4\left(\frac{\beta h}{2}\right)\left(1 - \nu^2\right). \tag{B.2.8}$$

Consider equation (B.2.8), maximum of amplification factor is equal to 1. Therefore,

$$\left|e^{\alpha l}\right|^2 \le 1 \text{ or } 0 \le 1 - 4\nu^2 \sin^4\left(\frac{\beta h}{2}\right)\left(1 - \nu^2\right) \le 1. \tag{B.2.9}$$

We can obtain that $\max\left(\sin^4\left(\frac{\beta h}{2}\right)\right) = 1$,

$$1 - 4\nu^2\left(1 - \nu^2\right) \le 1,$$

$$-4\nu^2\left(1 - \nu^2\right) \le 0,$$

$$4\nu^2\left(\nu^2 - 1\right) \le 0. \tag{B.2.10}$$

Since $\nu^2 > 0$, we can obtain that

$$\nu^2 \le 1, \tag{B.2.11}$$

which (B.2.11) implies that

$$\left|e^{\alpha l}\right| = \nu \le 1. \tag{B.2.12}$$

This is a condition of the stability of Lax – Wendroff finite difference method without linear drag blockage in (3.2.3).

B.2.2 Stability of Lax - Wendroff method for small amplitude wave system with linear drag blockage

Consider the small amplitude wave system with linear drag blockage in Lax – Wendroff form

$$U_m^{n+1} = U_m^n + \frac{l}{4h}\left(A^{n+1} - A^n\right)\left(U_{m+1}^n - U_{m-1}^n\right) + \frac{l^2}{2h^2}\left(A^n\right)^2\left(U_{m+1}^n - 2U_m^n + U_{m-1}^n\right) \qquad .(3.1.15)$$

where $U_m^n = \begin{Bmatrix} v_m^n \\ P_m^n \end{Bmatrix}$ and $A^n = \begin{bmatrix} 0 & \frac{1}{\rho}e^{\frac{k}{\rho}n\Delta t} \\ \frac{A_0}{c}e^{-\frac{k}{\rho}n\Delta t} & 0 \end{bmatrix}$.

We can let $U_m^n = Ae^{\alpha nl}e^{m\beta hi}$, so

$$e^{\alpha k} = 1 + \frac{\Delta t}{4\Delta x}\left(A^{n+1} - A^n\right)\left(e^{\beta hi} - e^{-\beta hi}\right) + \frac{(\Delta t)^2}{2(\Delta x)^2}\left(A^n\right)^2\left(e^{\beta hi} + e^{-\beta hi} - 2\right). \qquad \text{(B.2.14)}$$

Since $\dfrac{e^{i\beta h} - e^{-i\beta h}}{2i} = \sin(\beta h)$ and $\dfrac{e^{i\beta h} + e^{-i\beta h}}{2} = \cos(\beta h)$, we have

$$e^{\alpha l} = 1 + i\frac{l}{2h}\left(A^{n+1} - A^n\right)\sin(\beta h) - \frac{l^2}{h^2}\left(A^n\right)^2\left(1 - \cos(\beta h)\right). \qquad \text{(B.2.15)}$$

Since $\cos(2A) = 1 - 2\sin^2(A)$, we have

$$e^{\alpha l} = 1 + i\frac{l}{2h}\left(A^{n+1} - A^n\right)\sin(\beta h) - \frac{l^2}{h^2}\left(A^n\right)^2\left(1 - \left(1 - 2\sin^2\left(\frac{\beta h}{2}\right)\right)\right),$$

$$e^{\alpha l} = 1 - 2\frac{l^2}{h^2}\left(A^n\right)^2\sin^2\left(\frac{\beta h}{2}\right) + i\frac{l}{2h}\left(A^{n+1} - A^n\right)\sin(\beta h), \qquad \text{(B.2.16)}$$

The amplification factor, $\left|e^{\alpha l}\right|$, is considered by norm of equation (B.2.16). Nevertheless, this equation is complex number. It therefore finds norm of this equation as

$$\left|e^{\alpha l}\right|^2 = \left(1 - 2\frac{l^2}{h^2}\left(A^n\right)^2\sin^2\left(\frac{\beta h}{2}\right)\right)^2 + \left(\frac{l}{2h}\left(A^{n+1} - A^n\right)\sin(\beta h)\right)^2,$$

$$\begin{aligned}\left|e^{\alpha l}\right|^2 = {} & 1 - 4\frac{l^2}{h^2}\left(A^n\right)^2\sin^2\left(\frac{\beta h}{2}\right) + 4\frac{l^4}{h^4}\left(A^n\right)^4\sin^4\left(\frac{\beta h}{2}\right) \\ & + \left(\frac{l}{2h}\right)^2\left(A^{n+1} - A^n\right)^2\sin^2(\beta h)\end{aligned}. \qquad \text{(B.2.17)}$$

Since $\sin(2A) = 2\sin(A)\cos(A)$, consequently it implies that

$$\left|e^{\alpha l}\right|^2 = 1 - 4\frac{l^2}{h^2}\left(A^n\right)^2 \sin^2\left(\frac{\beta h}{2}\right) + 4\frac{l^4}{h^4}\left(A^n\right)^4 \sin^4\left(\frac{\beta h}{2}\right) + \left(\frac{l}{h}\right)^2 \left(A^{n+1} - A^n\right)^2 \sin^2\left(\frac{\beta h}{2}\right)\cos^2\left(\frac{\beta h}{2}\right),$$

$$\left|e^{\alpha l}\right|^2 = 1 - 4\frac{l^2}{h^2}\left(A^n\right)^2 \sin^2\left(\frac{\beta h}{2}\right) + 4\frac{l^4}{h^4}\left(A^n\right)^4 \sin^4\left(\frac{\beta h}{2}\right) + \left(\frac{l}{h}\right)^2 \left(A^{n+1} - A^n\right)^2 \sin^2\left(\frac{\beta h}{2}\right)\cos^2\left(\frac{\beta h}{2}\right). \tag{B.2.18}$$

Consider $\left(A^n\right)^2, \left(A^n\right)^4$ and $\left(A^{n+1} - A^n\right)^2$ such that

$$\left(A^n\right)^2 = \begin{bmatrix} 0 & \frac{1}{\rho}e^{\frac{k}{\rho}n\Delta t} \\ \frac{A_0}{c}e^{-\frac{k}{\rho}n\Delta t} & 0 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{\rho}e^{\frac{k}{\rho}n\Delta t} \\ \frac{A_0}{c}e^{-\frac{k}{\rho}n\Delta t} & 0 \end{bmatrix} = \begin{bmatrix} \frac{A_0}{\rho c} & 0 \\ 0 & \frac{A_0}{\rho c} \end{bmatrix}, \tag{B.2.19}$$

$$\left(A^n\right)^4 = \left(A^n\right)^2\left(A^n\right)^2 = \begin{bmatrix} \frac{A_0}{\rho c} & 0 \\ 0 & \frac{A_0}{\rho c} \end{bmatrix}\begin{bmatrix} \frac{A_0}{\rho c} & 0 \\ 0 & \frac{A_0}{\rho c} \end{bmatrix} = \begin{bmatrix} \left(\frac{A_0}{\rho c}\right)^2 & 0 \\ 0 & \left(\frac{A_0}{\rho c}\right)^2 \end{bmatrix}, \tag{B.2.20}$$

$$\left(A^{n+1} - A^n\right)^2 = \left(2 - e^{\frac{k}{\rho}\Delta t} - e^{-\frac{k}{\rho}\Delta t}\right)\begin{bmatrix} \frac{A_0}{\rho c} & 0 \\ 0 & \frac{A_0}{\rho c} \end{bmatrix}, \tag{B.2.21}$$

substituting (B.2.19), (B.2.20), (b.2.21) into Eq(B.2.18), and let $\begin{bmatrix} \frac{A_0}{\rho c} & 0 \\ 0 & \frac{A_0}{\rho c} \end{bmatrix} = B$,so we can obtain that

$$\left|e^{\alpha l}\right|^2 = 1 - 4\frac{l^2}{h^2}B\sin^2\left(\frac{\beta h}{2}\right) + 4\frac{l^4}{h^4}\left(B\right)^2 \sin^4\left(\frac{\beta h}{2}\right) + \left(\frac{l}{h}\right)^2\left(2 - e^{\frac{k}{\rho}\Delta t} - e^{-\frac{k}{\rho}\Delta t}\right)B\sin^2\left(\frac{\beta h}{2}\right)\cos^2\left(\frac{\beta h}{2}\right),$$

$$\left|e^{\alpha l}\right|^2 = 1 - 4\frac{l^2}{h^2}B\sin^2\left(\frac{\beta h}{2}\right) + 4\frac{l^4}{h^4}(B)^2\sin^4\left(\frac{\beta h}{2}\right) + \left(\frac{l}{h}\right)^2\left(2 - \left(e^{\frac{k}{\rho}\Delta t} + e^{-\frac{k}{\rho}\Delta t}\right)\right)B\sin^2\left(\frac{\beta h}{2}\right)\cos^2\left(\frac{\beta h}{2}\right). \tag{B.2.22}$$

Since $\dfrac{e^{\frac{k}{\rho}\Delta t} + e^{-\frac{k}{\rho}\Delta t}}{2} = \cosh\left(\dfrac{k}{\rho}\Delta t\right)$, we have

$$\left|e^{\alpha l}\right|^2 = 1 - 4\frac{l^2}{h^2}B\sin^2\left(\frac{\beta h}{2}\right) + 4\frac{(\Delta t)^4}{(\Delta x)^4}(B)^2\sin^4\left(\frac{\beta h}{2}\right) + 2\left(\frac{l}{h}\right)^2\left(1 - \cosh\left(\frac{k}{\rho}\Delta t\right)\right)B\sin^2\left(\frac{\beta h}{2}\right)\cos^2\left(\frac{\beta h}{2}\right). \tag{B.2.23}$$

Since $\cos^2\left(\dfrac{\beta h}{2}\right) = 1 - \sin^2\left(\dfrac{\beta h}{2}\right)$, we have

$$\left|e^{\alpha l}\right|^2 = 1 - 4\frac{l^2}{h^2}B\sin^2\left(\frac{\beta h}{2}\right) + 4\frac{l^4}{h^4}(B)^2\sin^4\left(\frac{\beta h}{2}\right) + 2\left(\frac{l}{h}\right)^2\left(1 - \cosh\left(\frac{k}{\rho}\Delta t\right)\right)B\sin^2\left(\frac{\beta h}{2}\right)\left(1 - \sin^2\left(\frac{\beta h}{2}\right)\right),$$

$$\left|e^{\alpha l}\right|^2 = 1 - 4\frac{l^2}{h^2}B\sin^2\left(\frac{\beta h}{2}\right) + 4\frac{l^4}{h^4}(B)^2\sin^4\left(\frac{\beta h}{2}\right) + 2\left(\frac{l}{h}\right)^2\left(1 - \cosh\left(\frac{k}{\rho}\Delta t\right)\right)B\sin^2\left(\frac{\beta h}{2}\right) - 2\left(\frac{l}{h}\right)^2\left(1 - \cosh\left(\frac{k}{\rho}\Delta t\right)\right)B\sin^4\left(\frac{\beta h}{2}\right). \tag{B.2.24}$$

Consider equation (B.2.24), maximum of amplification factor is equal to 1. Therefore,

$$4\frac{l^4}{h^4}(B)^2\sin^4\left(\frac{\beta h}{2}\right) + 2\left(\frac{l}{h}\right)^2\left(1 - \cosh\left(\frac{k}{\rho}\Delta t\right)\right)B\sin^2\left(\frac{\beta h}{2}\right) \le 2\left(\frac{l}{h}\right)^2\left(1 - \cosh\left(\frac{k}{\rho}\Delta t\right)\right)B\sin^4\left(\frac{\beta h}{2}\right) + 4\frac{l^2}{h^2}B\sin^2\left(\frac{\beta h}{2}\right). \tag{B.2.25}$$

We can obtain that $\max\left(\sin^4\left(\dfrac{\beta h}{2}\right)\right) = \max\left(\sin^2\left(\dfrac{\beta h}{2}\right)\right) = 1$, it follows that

$$4\frac{l^4}{h^4}(B)^2 + 2\left(\frac{l}{h}\right)^2\left(1 - \cosh\left(\frac{k}{\rho}\Delta t\right)\right)B \le 2\left(\frac{l}{h}\right)^2\left(1 - \cosh\left(\frac{k}{\rho}\Delta t\right)\right)B + 4\frac{l^2}{h^2}B. \tag{B.2.26}$$

Rearranging (B.2.26), we can obtain that

$$\frac{l^4}{h^4}(B)^2 \le \frac{l^2}{h^2}B. \tag{B.2.27}$$

Since $\frac{l^2}{h^2}B=\frac{(\Delta t)^2}{(\Delta x)^2}B>0$, we have

$$\left(\frac{\Delta t}{\Delta x}\right)^2 B\le 1, \tag{B.2.28}$$

where $B=\begin{bmatrix}\frac{A_0}{\rho c} & 0\\ 0 & \frac{A_0}{\rho c}\end{bmatrix}$.

From (B.2.19), $\left(A^n\right)^2=\begin{bmatrix}0 & \frac{1}{\rho}e^{\frac{k}{\rho}n\Delta t}\\ \frac{A_0}{c}e^{-\frac{k}{\rho}n\Delta t} & 0\end{bmatrix}\begin{bmatrix}0 & \frac{1}{\rho}e^{\frac{k}{\rho}n\Delta t}\\ \frac{A_0}{c}e^{-\frac{k}{\rho}n\Delta t} & 0\end{bmatrix}=\begin{bmatrix}\frac{A_0}{\rho c} & 0\\ 0 & \frac{A_0}{\rho c}\end{bmatrix}=B$, it implies that

$$\left(\frac{\Delta t}{\Delta x}\right)^2\left(A^n\right)^2\le 1, \tag{B.2.29}$$

where $A^n=\begin{bmatrix}0 & \frac{1}{\rho}e^{\frac{k}{\rho}n\Delta t}\\ \frac{A_0}{c}e^{-\frac{k}{\rho}n\Delta t} & 0\end{bmatrix}$.

So the amplication factor $\left|e^{\alpha l}\right|$ is implies that

$$\frac{\Delta t}{\Delta x}(D)\le 1, \tag{B.2.30}$$

where $D=\begin{bmatrix}\sqrt{\frac{A_0}{\rho c}} & 0\\ 0 & \sqrt{\frac{A_0}{\rho c}}\end{bmatrix}$.

The inequality (B.2.30) is the stability of the small amplitude wave system with linear drag blockage in Lax – Wendroff form.

APPENDIX C

Percentage of Errors

The percentage of errors is the values for indicating of compatibality of method in statistics. Its definitions gives that the percentage error is

$$\delta = \left| \frac{v - v_{approx}}{v} \right| \times 100\% \,, \tag{C.1}$$

where v is true solution and v_{approx} is approximation solution.

However, in mathematically, we find that the numerical solutions is equivalent to the approximation solutions which tends to the analytical solutions (true solutions). So,

$$\eta = \left| \frac{S - S_{num}}{S} \right| \times 100\% \,, \tag{C.2}$$

where S is analytical solution and S_{num} is numerical solution.

Furthermore, we find the mean percentage error (MPE) is

$$MPE = \frac{100}{N} \sum_{i=1}^{N} \left| \frac{S - S_{num}}{S} \right| \%$$

where S is analytical solution, S_{num} is numerical solution ,and N is a number of solutions.

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A Mathematical Model of Blood Flow in a Vessel with Blockage

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Abstract: A Mathematical model of blood flow in a vessel with blockage can simulate flowing behavior. It is important to gain an understanding of the physical mechanism underlying the shape of the pulse in abnormal physiology. In this research, a blood flow in a vessel with blockage is governed by small-amplitude pressure wave equations. The approximate solution is determined by using a Lax-Wendroff method.

Keywords: Blood flow; Blockage; Blood vessel; Small-amplitude pressure wave; Lax-Wendroff method.

2000 Mathematics Subject Classification: 65N06; 92C30

1 Introduction

Atherosclerosis disease, the disease in circulatory system begins with accumulative of fat tissue within a vessel [1], is the fat blockage in the vessel which leads to critical blood vessel diseases such as stokes, paralysis, hemiparesis etc. Furthermore, many researches supported that the blockage was main cause of the heart failure or cardiovascular disease [6]. One of the systems of equations called "The small Amplitude wave equation" was derived by J. Keener and J. Sneyd. It described the phenomena of blood flow on a small-scale of velocity and pressure conditions [2]. In [1] and [3], they studied blood flow in a vessel, especially in artery blood vessels. The modified form of small amplitude wave equation could be computed to resolve about this problem.

2 Governing Equations

In [2], mathematical model of blood flow in vessel by small amplitude condition was created. They derived it under the assumption of conservative of mass or continuity equation and conservative of momentum with small amplitude condition such that

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$$\rho u_t + P_x = 0, \tag{2.1}$$

$$cP_t + A_0 u_x = 0, \tag{2.2}$$

where u is velocity of blood flow (cm/s), P is pressure of blood flow (mmHg), ρ is density of blood flow (g / cm^3), c is compliance factor of blood vessel (cm^2 / mmHg) and A_0 is cross – section of blood vessel (cm^2).

Of course, since the affect of the blockage, we modified the equation by assuming that there is drag force in terms of blood velocity as shown in figure 1. Hence, by physical situation, the drag force of fat blockage is linear velocity drag force the equation systems of blood flow in small amplitude with linear drag force becomes,

$$u_t + \frac{1}{\rho} P_x = -\frac{\eta}{\rho} u, \tag{2.3}$$

$$P_t + \frac{A_0}{c} u_x = 0. \tag{2.4}$$

With initial condition:

$$u(x,0) = 0.3 + 0.1\sin\left(\frac{2\pi}{\lambda}x\right)$$

$$P(x,0) = 100 + 20\sin\left(\frac{2\pi}{\lambda}x\right)$$

for all $0 \le x \le 1$ at $t = 0$.
The left boundary condition:

$$u(0,t) = 0.3 - 0.1e^{-\frac{\eta}{\rho}t}\sin\left(\frac{2\pi}{\tau}t\right)$$

$$P(0,t) = 100 + 20\sin\left(\frac{2\pi}{\tau}t\right)$$

for all $0 \le t \le T$ at $x = 0$. Another right boundary condition is

$$\frac{\partial u}{\partial x}(x,t) = \frac{\partial P}{\partial x}(x,t) = 0$$

where λ is wavelength of blood flow in artery, τ is wave period of blood flow in artery, and η is damping constant of blood flow.

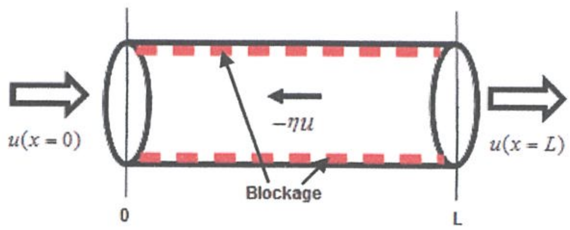

Figure 1 Illustration of blood flow in fat blockage (red dash line) which assumes this affect by dragged force inside.

Let $u(x,t)=v(x,t)e^{-\eta t/\rho}$. Substituting them into equations (2.3)-(2.4), yields

$$v_t+\frac{1}{\rho}e^{\frac{\eta}{\rho}t}P_x=0 \tag{2.5}$$

$$P_t+\frac{A_0}{c}e^{-\frac{\eta}{\rho}t}v_x=0 \tag{2.6}$$

It can written in the matrix form as

$$U_t+AU_x=0, \tag{2.7}$$

where

$$U=\begin{Bmatrix} v \\ P \end{Bmatrix} \text{ and } A=\begin{bmatrix} 0 & \dfrac{e^{\eta t/\rho}}{\rho} \\ \dfrac{A_0e^{-\eta t/\rho}}{c} & 0 \end{bmatrix}.$$

3 Numerical Technique

Lax-Wendroff method

We now discretize Eq.(2.7) by dividing the interval [0,1] into M and N subintervals and such that $M\Delta x=L$, and the intervals $[0,T]$ into N subintervals such that $N\Delta t=T$. The grid point (x_m,t_n) are defined by $x_m=m\Delta x$ for all $m=0,1,2,\ldots,M$ and $t_n=n\Delta t$ for all $n=0,1,2,\ldots,N$ in which M and N are positive integers. We can then approximate $v(x_m,t_n)$ by v_m^n, the value of the difference approximation of $v(x,t)$ at point $x=m\Delta x$ and $t=n\Delta t$, where $0\le m\le M$ and $0\le n\le N$, and similarly defined pressure $P(x_m,t_n)$ for P_m^n. Using the Lax-Wendroff method [4] on Eq.(2.7), we can obtain the following finite difference equation:

$$U_m^{n+1} = U_m^n - \frac{ak}{2h}\left(U_{m+1}^n - U_{m-1}^n\right) + \frac{a^2k^2}{2h^2}\left(U_{m-1}^n - 2U_m^n + U_{m+1}^n\right) \quad (2.8)$$

The Lax-Wendroff method is second order accurate and it is stable if amplification factor $\left|\frac{ak}{h}\right| < 1$, [5].

4 Numerical Result

4.1 Approximate solution for non-blockage blood flow (undamped case)

In this section, approximation solutions are separated into two sections. First, there are solutions of blood velocity and pressure in undamped ($\eta = 0$) or non fat blockage condition. We choose $\Delta x = 0.02$ and $\Delta t = 1$ with amplification factor is 0.7494

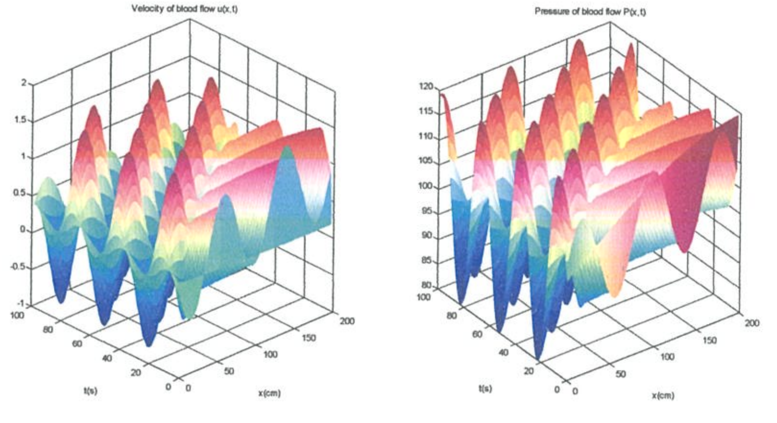

Figure 2 The tendency of solution of velocity and pressure of blood flow with undamped ($\eta = 0$).

4.2 Approximate solution for blockage blood flow (damped case)

Another, solutions of blood velocity and pressure in blockage case ($\eta > 0$) are determined by $\Delta x = 0.02$ and $\Delta t = 1$ with amplification factor is 0.7494. We compare about vary damping characteristics by using many damping constants which follows that

Case 1 : $\eta = 10$

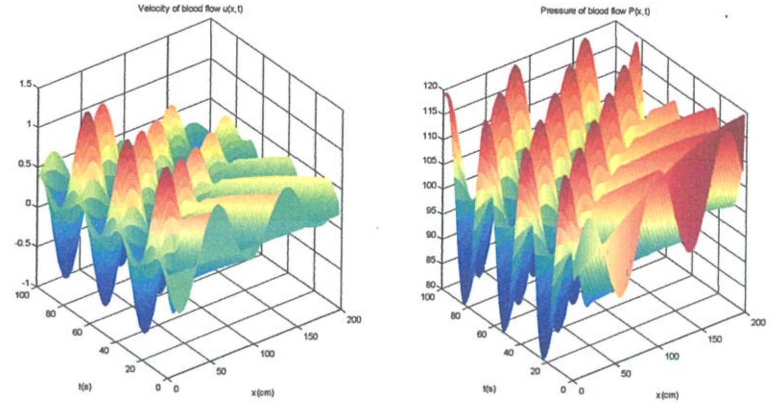

Figure 3 The tendency of solution of velocity and pressure of blood flow with $\eta = 10$.

Case 2 : $\eta = 30$

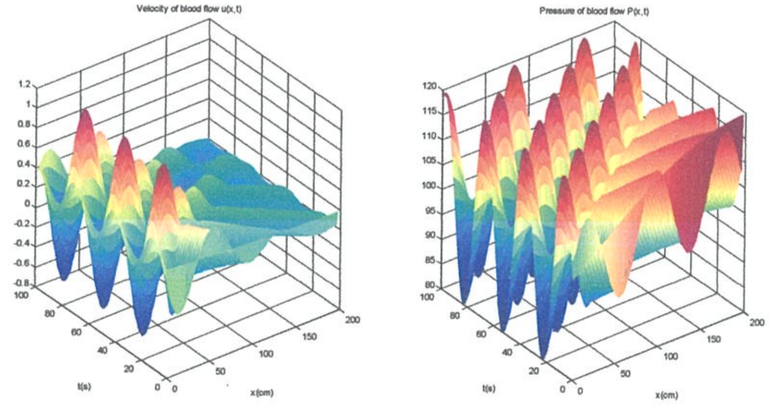

Figure 4 The tendency of solutions of velocity and pressure of blood flow with $\eta = 30$.

Case 3 : $\eta = 50$

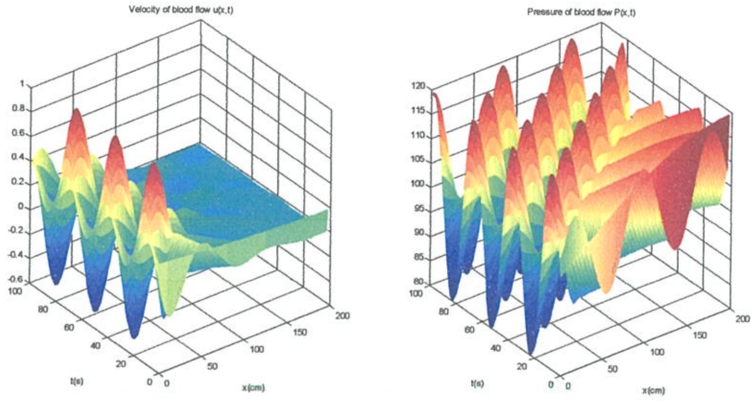

Figure 5 The tendency of solutions of velocity and pressure of blood flow with $\eta = 50$.

5 Conclusions

From this research, we only find that the velocity of a blood flow decreases along with x-direction (space dimension) for increasing the damping constant. While, blood pressure do not decrease although there is increasing about damping which is factor of plaque size into a blood vessels.

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