ทฤษฎีอแด็ปทีฟคอนโทรล: การออกแบบอแด็ปทีฟแบคสเต็ปปิงและทูนนิ่งฟังก์ชัน

ADAPTIVE CONTROL THEORY: THE ADAPTIVE BACKSTEPPING AND TUNING FUNCTION

โดย

นายณัฐพงศ์ ภัทรมานนท์

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ปริญญานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิศวกรรมศาสตร์บัณฑิต

สาขาวิชาวิศวกรรมระบบควบคุม

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Adaptive Control Theory: The Adaptive Backstepping And Tuning Function Design

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บทคัดย่อ

ในการออกแบบระบบควบคุมโดยทั่วไป (Non-adaptive) นั้น เราจะเลือกโครงสร้างของตัวควบคุม (เช่นการแทนที่โพล) ขึ้นมาก่อน แล้วจึงหา parameter ของตัวควบคุมจาก parameter ที่ทราบค่าของ plant ในการควบคุมแบบ Adaptive จะมีข้อแตกต่างที่หลักๆ คือ เราไม่ทราบค่า parameter ของ plant ดังนั้นเราจะได้ parameter ของตัวควบคุม จาก adaptation law ผลคือการออกแบบการควบคุมแบบ adaptive จะยุ่งยากขึ้น โดยสิ่งที่เพิ่มขึ้น คือ ความจำเป็นในการเลือก adaptation law และการพิสูจน์ว่า ระบบรวม (รวม adaptation law) มิเสถียรภาพ ปริญญานิพนธ์ฉบับนี้จะนำเสนอ เนื้อหาต่อไปนี้ : First Order Linear System with Unknown Constant Parameter : ซึ่งเป็นสาเหตุให้ใช้ทฤษฎี Adaptive Stability, ทฤษฎี Passivity, Lyapunov-Based Design, Feedback Linearization และ Zero Dynamics, Stabilization of Cascade Systems, Block Backstepping with Zero Dynamics, Adaptive Backstepping, Adaptive Block Backstepping และ Tuning Function Design ทฤษฎีส่วนใหญ่ข้างต้นสามารถหาได้จากหนังสือของ Krstic [48] ซึ่งจะมีรายละเอียดอย่างสมบูรณ์ของทฤษฎี Adaptive Block Backstepping และ Tuning Function Design

ADAPTIVE CONTROL THEORY: THE ADAPTIVE BACKSTEPPING AND TUNING FUNCTION

โดย
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ABSTRACT

In conventional (non-adaptive) control design, a controller structure (e.g., pole placement) is chosen first, and the parameters of the controller are then computed based on the known parameters of the plant. In adaptive control, the major difference is that the plant parameters are unknown, so that the controller parameters have to be provided by an adaptation law. As a result, the adaptive control design is more involved, with the additional needs of choosing an adaptation law and proving the stability of the system with adaptation. This thesis presents the following topics: First Order Linear System with Unknown Constant Parameter which gives the reason for using Adaptive Control, Stability theory, Passivity theory, Lyapunov-Based Design, Feedback Linearization and Zero Dynamics, Stabilization of Cascade Systems, Block Backstepping with Zero Dynamics, Adaptive Backstepping, Adaptive Block Backstepping and Tuning Function Design.

ADAPTIVE CONTROL THEORY : THE ADAPTIVE BACKSTEPPING AND TUNING FUNCTION DESIGN

P R O J E C T   R E P O R T

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February 2007

ADAPTIVE CONTROL THEORY : THE ADAPTIVE BACKSTEPPING AND TUNING FUNCTION DESIGN

PROJECT REPORT

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February 2007

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Dr.Poramate Pranayanuntana

MAR 2, 2007

Date

To our beloved

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ABSTRACT

In conventional (non-adaptive) control design, a controller structure (e.g., pole placement) is chosen first, and the parameters of the controller are then computed based on the known parameters of the plant. In adaptive control, the major difference is that the plant parameters are unknown, so that the controller parameters have to be provided by an adaptation law. As a result, the adaptive control design is more involved, with the additional needs of choosing an adaptation law and proving the stability of the system with adaptation. This thesis presents the following topics: First Order Linear System with Unknown Constant Parameter which gives the reason for using Adaptive Control, Stability theory, Passivity theory, Lyapunov-Based Design, Feedback Linearization and Zero Dynamics, Stabilization of Cascade Systems, Block Backstepping with Zero Dynamics, Adaptive Backstepping, Adaptive Block Backstepping and Tuning Function Design. Most of these theory can be found in Krstić et al. [48], which gives a full treatment of the theory of Adaptive backstepping and Tuning function design.

Contents

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Chapter 1

Introduction

1.1 First Order Linear System with Unknown Constant Parameter

Consider the first order linear system of the following form

$$\dot{x} = u - \theta x \tag{1.1.1}$$

where θ is a **positive unknown constant** as shown in Figure 1.1. This system is asymptotically stable (the equilibrium point $x = 0$ is an asymptotically stable equilibrium point) even when there is no external control input, that is, u is zero. The solution of this system is

$$x(t) = x(0)e^{-\theta t} + e^{-\theta t}\int_0^t e^{\theta s}u\, ds.$$

Therefore this system is uninteresting. What if the system we have is unstable to begin with? Consider the following system

$$\dot{x} = u + \theta x \tag{1.1.2}$$

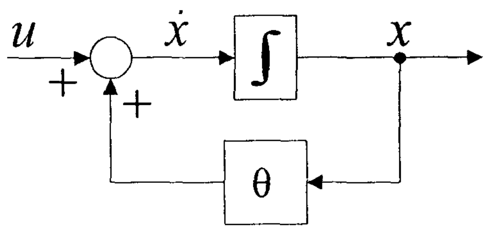

Figure 1.1: Linear Scalar Plant (1.1.2)

where θ is a **positive unknown constant**. The solution of (1.1.2) is of the form

$$\begin{aligned}
\dot{x} &= u + \theta x \\
\dot{x} - \theta x &= u \\
e^{-\theta t}\dot{x} - \theta e^{-\theta t}x &= ue^{-\theta t} \\
e^{-\theta t}\frac{d}{dt}x(t) + \frac{d}{dt}e^{-\theta t}x(t) &= e^{-\theta t}u(t) \\
\frac{d}{dt}(e^{-\theta t}x(t)) &= e^{-\theta t}u(t) \\
d(e^{-\theta s}x(t)) &= e^{-\theta s}u(s)ds \\
\int_{s=0}^{s=t} d(e^{-\theta s}x(s)) &= \int_{s=0}^{s=t} e^{-\theta s}u(s)ds \\
e^{-\theta s}x(s)|_{s=0}^{s=t} &= \int_0^t e{-}\theta s u(s)ds \\
e{-}\theta t x(t) - x(0) &= \int_0^t e^{-\theta s}u(s)ds \\
x(t) &= e^{\theta t}x(0) + \int_0^t e^{\theta(t-s)}u(s)ds
\end{aligned}$$

Unlike the system in (1.1.1), this system in (1.1.2) is unstable when $u = 0$.

What control law u will stabilize the system in (1.1.2)? If an a priori bound $\bar{\theta}$ on $|\theta|$ were known, $|\theta| \leq \bar{\theta}$, then $u = -2\bar{\theta}x$ would be a linear stabilizing controller. The system (1.1.2) with $u = -2\bar{\theta}x$ would becomes

$$\dot{x} = -(2\bar{\theta} - \theta)x.$$

Its only equilibrium point, $x = 0$, would be asymptotically stable since $2\bar{\theta} - \theta > 0$. (Its stability would be able to be investigated using phase line.) If such a bound is not known, no linear controller can be designed to guarantee stability of (1.1.2). For example, if $u = -kx$, where k is a positive constant, were applied; the system (1.1.2) would becomes

$$\dot{x} = -(k-\theta)x,$$

and the only equilibrium point, $x = 0$, would not be guaranteed to be asymptotically stable due to the fact that the constant θ would not be known. If the value of k picked were less than that of θ, then the equilibrium point $x = 0$ of the resulting system would be unstable.

To examine whether a static nonlinear controller can help, let us try the controller

$$u = -k_1 x - k_2 x^3. \tag{1.1.3}$$

where $k_1 > 0, k_2 > 0$. The resulting feedback system is

$$\dot{x} = (\theta - k_1)x - k_2 x^3. \tag{1.1.4}$$

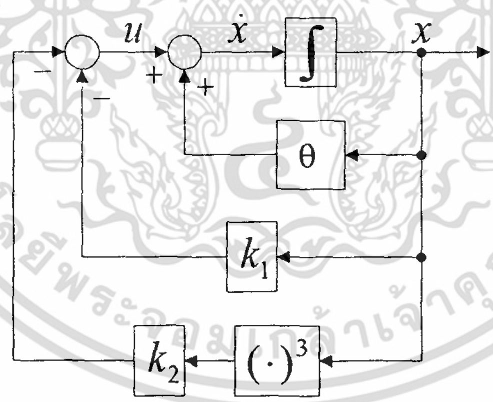

Figure 1.2: The resulting feedback system (1.1.4)

For $\theta > k_1$, the equilibrium $x = 0$ is unstable, but the nonlinear term $-k_2 x^3$ prevents $x(t)$ from growing unbounded. It is easy to see that $x(t)$ will converge to one of the two new equilibria $\pm\sqrt{\frac{\theta - k_1}{k_2}}$. Thus, the static nonlinear controller (1.1.5) has achieved boundedness of $x(t)$ without any knowledge of a bound on θ. Our goal is more ambitious than just boundedness of $x(t)$. We also want to achieve its regulation: $\lim_{t\to\infty} x(t) = 0$. Can this be accomplished by a dynamic nonlinear controller? The answer is affirmative: One such controller is

$$u = -(p+\xi)x, \qquad \dot{\xi} = x^2. \tag{1.1.5}$$

where $p > 0$ is a design parameter. The resulting feedback system is of second order:

$$\dot{x} = -(p+\xi)x + \theta x. \tag{1.1.6-a}$$

$$\dot{\xi} = x^2. \tag{1.1.6-b}$$

Its stability properties can be checked by examining the derivative of the Lyapunov function

$$V(x.\xi) = \frac{1}{2}x^2 + \frac{1}{2}(\xi - \theta)^2, \tag{1.1.7}$$

which turns out to be nonpositive:

$$\dot{V} = -px^2 - \xi x^2 + \theta x^2 + (\xi - \theta)x^2 = -px^2. \tag{1.1.8}$$

Thus, $V(x(t), \xi(t))$ evaluated along the solutions of (1.1.6-a), (1.1.6-b) is a nonincreasing function of time. This proves that $x(t)$ and $\xi(t)$ remain bounded for all $t \geq 0$. The proof that $\lim_{t\to\infty} x(t) = 0$ is also achieved can be given using Lasalle-Yoshizawa theorem (Theorem A.1.1).

How was the dynamics nonlinear controller (1.1.5) conceived? Not as a nonlinear controller, but rather as a parameter adaptation scheme! Its dynamics part $\dot{\xi} = x^2$ is, in fact, an update law for ξ as an estimate of θ. Consequently, the estimation error $\xi - \theta$ is penalized in Lyapunov function (1.1.7).

Chapter 2

Feedback Linearization and zero dynamics

2.1 Feedback Linearization and zero dynamics

One of the popular methods for nonlinear control design is feedback linearization, which employs a change of coordinates and feedback control to transform a nonlinear system into a system whose dynamics are linear (at least partially).

A great deal of research has been devoted to this subject over the last two decades, as evidenced by the comprehensive books of Isidori [9] and Nijmeies and Van der Schaft [33] and the references therein. Since feedback linearization is not a goal pursued in this book, we only briefly review some concepts needed for the remainder of the chapter.

For maximum accessibility we avoid the direct use of differential geometric notations, but we still refer to those notations for references.

Let us consider the nonlinear system.

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R} \\ y &= h(x), \quad y \in \mathbb{R} \end{aligned} \tag{2.1.1}$$

where f, g, h are smooth (that is, infinitely differentiable) vector functions.

The derivative of the output $y = h(x)$ is given by:

$$\dot{y} = \overbrace{\frac{\partial h}{\partial x}(x) f(x)}^{L_f h} + \overbrace{\frac{\partial h}{\partial x}(x) g(x) u}^{L_g h u} \tag{2.1.2}$$

If $\frac{\partial h}{\partial x}(x_0) g(x_0) \neq 0$, then the system (2.1.1) is said to have relative degree one at x_0. In our terminology, this implies that the output y separated from the input u by one integrator only.

If $\frac{\partial h}{\partial x}(x_0) g(x_0) = 0$, there are two cases:

(i) If there exist points x arbitrarily close to x_0 such that $\frac{\partial h}{\partial x}(x) g(x) \neq 0$, then (2.1.1) does not have a well-defined relative degree at x_0.

(ii) If there exists a neighborhood B_0 of x_0 such that $\frac{\partial h}{\partial x}(x) g(x) = 0$ for all $x \in B_0$, then the relative degree of (2.1.1) at x_0 may be well-defined.

In this case (ii), we define

$$\psi_1(x) = h(x), \quad \psi_2(x) = \frac{\partial h}{\partial x}(x) f(x) = L_f h \tag{2.1.3}$$

and compute the second derivative of y:

$$\begin{aligned} \ddot{y} &= \frac{\partial}{\partial x}\left(\frac{\partial h f}{\partial x}\right) f + \frac{\partial}{\partial x}\left(\frac{\partial h f}{\partial x}\right) g u \\ &= \frac{\partial \psi_2}{\partial x}(x) f(x) + \frac{\partial \psi_2}{\partial x}(x) g(x) u. \end{aligned} \tag{2.1.4}$$

If $\frac{\partial \psi_2}{\partial x}(x_0) g(x_0) \neq 0$, then (2.1.1) is said to have relative degree *two* at x_0.

If $\frac{\partial \psi_2}{\partial x}(x) g(x) = 0$ in a neighborhood of x_0, then we continue the differentiation procedure.

Definition 2.1.1 *The system* (2.1.1) *is said to have* relative degree ρ *at the point* x_0 *if there exists a neighborhood* B_0 *of* x_0 *on which*

$$\frac{\partial \psi_1}{\partial x}(x) g(x) = \frac{\partial \psi_2}{\partial x}(x) g(x) = \ldots = \frac{\partial \psi_{\rho-1}}{\partial x}(x) g(x) = 0 \tag{2.1.5}$$

$$\frac{\partial \psi_\rho}{\partial x}(x)g(x) \neq 0, \tag{2.1.6}$$

where

$$\psi_1(x) = h(x), \qquad \psi_i(x) = \frac{\partial \psi_{i-1}}{\partial x}(x)f(x), \qquad i = 2, ..., \rho \tag{2.1.7}$$

if (2.1.5) *and* (2.1.6) *are valid for all* $x \in \mathbb{R}^n$, *then the relative degree of* (2.1.1) *is said to be* globally defined.

Suppose now that (2.1.1) has relative degree ρ at x_0. Then we can use a change of coordinates and feedback control to locally transform of this system into the *cascade connection* a ρ-dimensional linear system and an $(n-\rho)$-dimensional nonlinear system.

In particular, after differentiating ρ times the output $y = h(x)$, the control u appears:

$$\begin{aligned} y^{(\rho)} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\cdots\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}f\right)f\cdots\right)f\right)f(x) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\cdots\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}f\right)f\cdots\right)f\right)g(x)u \\ &= \frac{\partial \psi_\rho}{\partial x}(x)f(x) + \frac{\partial \psi_\rho}{\partial x}(x)g(x)u = v \end{aligned} \tag{2.1.8}$$

Since $\frac{\partial \psi_\rho}{\partial x}g \neq 0$ in a neighborhood of x_0, we can linearize the input-output description of the system (2.1.1) using feedback to cancel the nonlinearities in (2.1.8):

$$u = \frac{1}{\frac{\partial \psi_\rho}{\partial x}(x)g(x)}\left[-\frac{\partial \psi_\rho}{\partial x}(x)f(x) + v\right]. \tag{2.1.9}$$

Then the dynamic of y and its derivatives are governed by a chain of ρ integrators: $y^{(\rho)} = v$.

Since our original system (2.1.1) has dimension n, we need to account for the remaining $n-\rho$ states.

Using differential geometric tools, it is easy to show that it is always possible to find $n-\rho$ functions $\psi_{\rho+1}(x), \ldots, \psi_n(x)$ with $\frac{\partial \psi_i}{\partial x}(x)g(x) = 0, i = \rho+1, \ldots, n$ such that the change of coordinates

$$
\begin{aligned}
\zeta_1 &= y = h(x) = \psi_1(x), \\
\zeta_2 &= \dot{y} = \psi_2(x), \\
&\vdots \\
\zeta_\rho &= y^{(\rho-1)} = \psi_\rho(x), \\
\zeta_{\rho+1} &= \psi_{\rho+1}(x), \\
&\vdots \\
\zeta_n &= \psi_n(x) \\
y &= \psi_1
\end{aligned}
\tag{2.1.10}
$$

is locally invertible and transforms, along with the feedback (2.1.9), the nonlinear system (2.1.1) into

$$
\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 \\
&\vdots \\
\dot{\zeta}_{\rho-1} &= \zeta_\rho \\
\dot{\zeta}_\rho &= v \\
\dot{\zeta}_{\rho+1} &= \frac{\partial \psi_{\rho+1}}{\partial x}(x) f(x) = \phi_{\rho+1}(\zeta) \\
&\vdots \\
\dot{\zeta}_n &= \frac{\partial \psi_n}{\partial x}(x) f(x) = \phi_n(\zeta) \\
y &= \zeta_1
\end{aligned}
\tag{2.1.11}
$$

As a cascade connection of a chain of ρ integrators with an $(n-\rho)$-dimensional nonlinear system, this system is a special case of the cascade systems to which we will apply backstepping in the following chapters.

The states $\zeta_{\rho+1}, ..., \zeta_n$ of the nonlinear subsystem in (2.1.11) have been rendered *unobservable* from the output y by the control (2.1.9). Hence, feedback linearization in this case is the nonlinear equation of placing ρ poles of a linear system at the origin and canceling the $(n-\rho)$ zeros with the remaining poles.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = Ax + Bu, sx = Ax + Bu$$

$$y = cx, y = cx$$

$$\begin{aligned} \frac{y(s)}{u(s)} &= \frac{\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{s^2+2s+1} \\ &= \frac{\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} s+2 \\ s+1 \end{bmatrix}}{s^2+2s+1} \\ &= \frac{c_1(s+2)+c_2(s+1)}{s^2+2s+1} \\ &= \frac{(c_1+c_2)s+(2c_1+c_2)}{s^2+2s+1} \\ &= \frac{s+5}{s^2+2s+1} \end{aligned}$$

from $\dot{y} = v$

$$\begin{aligned} \dot{y} &= C\dot{x} \\ &= C(Ax+Bu) \\ &= CAx + CBu \\ u &= \frac{1}{CB}(-CAx+v), CB = 1 \\ &= \frac{1}{1}(-CAx+v) \\ &= -\begin{bmatrix} 4 & -3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v \\ u &= -\begin{bmatrix} -4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v \\ u &= -kx + v \end{aligned}$$

$c_1 = 1$, $c_2 = 5$

$CB = \begin{bmatrix} 4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \neq 0$

relative degree = 1

$$y = Cx$$

$$x = Ax + Bu, u = -kx + v$$

$$\dot{x} = Ax + B(-k)x + Bv$$

$$\dot{x} = (A - Bk)x + Bv$$

$$A - Bk = \begin{bmatrix} 3 & -6 \\ 4 & -8 \end{bmatrix}$$

$$\frac{y(s)}{v(s)} = \begin{bmatrix} 4 & -3 \end{bmatrix} \begin{bmatrix} s-3 & 6 \\ -4 & s+8 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{v(s)}{v(s)}$$

$$= \frac{s+5}{s(s+5)}$$

Example 2.1.2

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x}_1 = -x_1 + x_{-2} + u$$

$$\dot{x}_2 = -x_2 + u$$

$$y = 4x_1 - 3x_2$$

$$u = \frac{1}{1}\{\begin{bmatrix} +4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v\}$$

$$= 4x_1 - 7x_2 + v$$

$$\dot{y} = 4\dot{x}_1 - 3\dot{x}_2$$

$$= 4(-x_1 + x_2 + u) - 3(-x_2 + u)$$

$$= -4x_1 + 7x_2 + u \quad , u = 4x_1 - 7x_2 + v$$

$$= v$$

$$\zeta = x_1 - x_2$$

$$\dot{\zeta} = \dot{x}_1 - \dot{x}_2$$

$$\dot{\zeta} = -x_1 + 2x_2$$

$$4\dot{\zeta} = -4x_1 + 8x_2$$

$$= -(4x_1 - 8x_2)$$

$$= -(4x_1 - 3x_2 - 5x_2)$$

To find x_1 , x_2 *in* y , ζ *form*

$$y = = 4x_1 - 3x_2$$

$$\zeta = x_1 - x_2$$

$$\begin{bmatrix} y \\ \zeta \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y \\ \zeta \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} y \\ \zeta \end{bmatrix}$$

$$x_1 = y - 3\zeta$$

$$x_2 = y - 4\zeta$$

Then we put

$$x_1 = y - 3\zeta \ ,$$

$$x_2 = y - 4\zeta \ \ \textit{in} \ -(4x_1 - 3x_2 - 5x_2)$$

We have now $\dot{\zeta} = y - 5\zeta$

Of course, to guarantee stability, the cancelled zeros must be stable.

In the nonlinear case, using the new control input v to stabilize the linear subsystem of (2.1.11) does not guarantee stability of the nonlinear whole system, unless the stability of the nonlinear part of (2.1.11) has been established separately.

When v is used to keep the output y equal to zero for all $t > 0$, that is, when $\zeta_1 \equiv \ldots \equiv \zeta_n \equiv 0$, the dynamics of $\zeta_{\rho+1}, \ldots, \zeta_n$ are described by

$$\begin{aligned} \dot{\zeta}_{\rho+1} &= \phi_{\rho+1}(0, \ldots, 0, \zeta_{\rho+1}, \ldots, \xi_n) \\ &\vdots \\ \dot{\zeta}_n &= \phi(0, \ldots . 0, \zeta_{\rho+1}, \ldots, \zeta_n) \end{aligned} \tag{2.1.12}$$

They are called *the zero dynamics* of (2.1.1), because they evolve on the subset of the state space on which the output of the system is identically zero. If the equilibrium at

$\zeta_{\rho+1} = \ldots = \zeta_n = 0$ of the zero dynamics (2.1.12) is asymptotically stable, the system (2.1.1) is said to be ***minimum phase***. With a slight abuse of notation, we will refer to the $(\zeta_{\rho+1}, ..., \zeta_n)$-subsystem as the zero dynamics subsystem of (2.1.1), even when $\xi_1, ..., \xi_\rho$ are not zero.

Example 2.1.3

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{y(s)}{u(s)} = G(s) = C[sI - A]^{-1}B$$
$$= \frac{2s+11}{(s+4)(s+5)}$$

then relative degree = 1

In (2.1.1) the output $y = h(x)$ is prespecified, possibly from a tracking objective, and the resulting cascade system is linear from the input v to the output y. This linearization process is usually called *input-output feedback linearization* [9]. If our goal is only to design a stabilizing controller, we may attempt to find an output with respect to which the relative degree is $\rho = n$. If such an output exists, the whole system is linearized without zero dynamics. This process is referred to as *full-state feedback linearization* [7, 8, 10, 44]. If such an output cannot be found, then we may look for an output which yields the highest relative degree, and thus results in a cascade system whose linear subsystem has the highest dimension [24]. It is desirable that with respect to the chosen output the system be minimum phase. The importance of this property will be clear in the following chapters which address problems of stabilization of cascade systems.

Chapter 3

Stabilization of cascade systems

3.1 Stabilization of cascade systems

We now consider cascade connections in which the nonlinear system is globally stable, but the input subsystem is more complex than just an integrator. We begin with the case where the input subsystem is linear:

$$\dot{x} = f(x) + g(x)y, \quad f(0) = 0, x \in \mathbb{R}^n, y \in \mathbb{R} \tag{3.1.1a}$$

$$\dot{\xi} = A\xi + Bu, \quad y = h\xi. \tag{3.1.1b}$$

We assume that when $y = 0$ the nonlinear system (3.1.1a) has a globally stable equilibrium at $x = 0$, and that an appropriate Lyapunov function $V(x)$ is known such that:

$$\frac{\partial V}{\partial x}(x)f(x) \leq -W(x) \leq 0. \tag{3.1.2}$$

The problem is to stabilize the linear subsystem (3.1.1b) without destabilizing the nonlinear subsystem (3.1.1a), and, if possible, to achieve GAS of the equilibrium of (3.1.1) at $(0,0)$. that is,

$$(x, \xi) = (0, 0)$$

This problem is not solvable in general. Here it will be solved by requiring the input subsystem (3.1.1b) to have the following passivity property:

Assumption 3.1.1 *The triple (A, b, h) is feedback positive real (FPR), that is, there exists a linear feedback transformation $u = K\xi + v$ such that $A + bK$ is Hurwitz and there are matrices $P > 0, Q \geq 0$ which satisfy*

$$(A + bK)^T P + P(A + bK) = -Q \quad (3.1.3a)$$

$$Pb = h^T. \quad (3.1.3b)$$

A sufficient condition for FPR is that there exists a feedback gain row vector K such that

(1) $A + bK$ is Hurwitz,

(2) the transfer function $Z(s) = h(sI - A - bK)^{-1}b$ is positive real (PR), and

(3) the pair $(A + bK, h)$ is observable.

It should be noted from (3.1.3b) *that the relative degree of PR transfer function is one because $b^T Pb = hb > 0$.*

Lemma 3.1.2 (Stabilization with FPR) *Let $V(x)$ be a Lyapunov function for* (3.1.1a) *satisfying* (3.1.2). *If the triple (A, b, h) is FPR, then a Lyapunov function for the cascade system* (3.1.1) *is*

$$V_a(x, \xi) = V(x) + \xi^T P\xi, \quad (3.1.4)$$

and the corresponding control law

$$u = \alpha_a(x, \xi) = K\xi - \frac{1}{2}\frac{\partial V}{\partial x}g(x) \quad (3.1.5)$$

guarantees that $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$ is globally bounded and converges to the largest invariant set M_a contained in the set $E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \middle| W(x) = 0, Q^{\frac{1}{2}}\xi = 0 \right\}$. If $W(x)$ is positive definite, that is, if the nonlinear subsystem (3.1.1a) *with $y = 0$ has a globally asymptotically stable equilibrium at $x = 0$, then the equilibrium $x = 0, \xi = 0$ is also GAS.*

Proof *Using* (3.1.2) *and* (3.1.3a) *and denoting* $u = K\xi + v$ *with* $v = -\frac{1}{2}\frac{\partial V}{\partial x}g(x)$ *from* (3.1.5)*, the derivative of* $V_a(x, \xi)$ *is*

$$\begin{aligned}\dot{V}_a &= \frac{\partial V}{\partial x}(x)[f(x) + g(x)y] \\ &\quad + \xi^T P[(A + bK)\xi + bv] + [(A + bK)\xi + bv]^T P\xi\end{aligned}$$

by (3.1.2) *and* (3.1.3a)

$$\leq -W(x) + \frac{\partial V}{\partial x}(x)g(x)y - \xi^T Q\xi + 2\xi^T Pbv$$

by (3.1.3b)

$$\begin{aligned}&= -W(x) + \frac{\partial V}{\partial x}(x)g(x)y - \xi^T Q\xi + 2y\left[-\frac{1}{2}\frac{\partial V}{\partial x}g(x)\right] \\ &= -W(x) - \xi^T Q\xi \leq 0.\end{aligned}$$

Since V_a is positive definite, radially unbounded and has a negative semidefinite derivative, $x(t)$ and $\xi(t)$ are globally bounded.

Furthermore LaSalle's theorem (Theorem A.1.2) guarantees convergence to the largest invariant set M_a in the set E_a.

If, in addition, $W(x)$ is positive definite, then the global asymptotic stability of $x = 0, \xi = 0$ is shown using Corollary A.1.3. From the positive definiteness of $W(x)$, the set E_a, on which $\dot{V}_a = 0$, is given by $E_a = \{(x, \xi) | x = 0, Q^{\frac{1}{2}}\xi = 0\}$.

Since $V(x)$ is positive definite, it has a minimum at $x = 0$, and thus $\frac{\partial V}{\partial x}(0) = 0$. This implies that on the set E_a the control term $v = -\frac{1}{2}\frac{\partial V}{\partial x}g(x)$ vanishes.
Hence, on the set E_a the state $\xi(t)$ satisfies

$$\dot{\xi} = (A + bK)\xi, \quad V_a(x, \xi) = \xi^T P\xi. \tag{3.1.6}$$

But V_a is constant on E_a, which means that $\xi^T P\xi$ must be constant on E_a. Since $A+bK$ is Hurwitz, $\xi = 0$ is the only solution of $\dot{\xi} = (A + bK)\xi$ that satisfies $\xi^T P\xi$ = constant. Thus, $\xi = 0$ on the largest invariant set contained in E_a. This implies that this invariant set M_a is just the equilibrium $x = 0, \xi = 0$, which by Corollary A.1.3, is GAS.

The stabilizing control law (3.1.5) consists of two terms, one linear and one nonlinear. The purpose of the latter is to preserve the stability of the nonlinear subsystem.

Example 3.1.3 *For a comparison with backstepping, let us first examine the second-order system stabilized in Example B.1.3:*

$$\dot{x} = x\xi \qquad (3.1.7a)$$

$$\dot{\xi} = u. \qquad (3.1.7b)$$

In this system we have $f(x) \equiv 0$, $g(x) = x$, $A = 0$, $B = 1$ *and* $y = \xi$. *Using* $V(x) = x^2$ *we see from Example 4.1.2 that* $W(x) \equiv 0$. *The FPR condition is trivially satisfied and the stabilizing control law is*

$$u = -k\xi - x^2, \qquad k > 0. \qquad (3.1.8)$$

With $k = 1$ *this is the same control law as* (B.1.38) *obtained by backstepping with* $\alpha(x) \equiv 0$. *We know from Example B.1.3 that this control law achieves GAS of the equilibrium* $(x, \xi) = (0, 0)$.

Next we consider a third-order system to which backstepping is not directly applicable:

$$\dot{x} = x(\xi_1 + \xi_2) \triangleq xy \qquad (3.1.9a)$$

$$\dot{\xi}_1 = \xi_2 \qquad (3.1.9b)$$

$$\dot{\xi}_2 = u. \qquad (3.1.9c)$$

In this case $b^T = [\,0 \quad 1\,]$ *and* $h = [\,1 \quad 1\,]$, *so that the condition* (3.1.3b) *yields*

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} p_{12} = 1 \\ p_{22} = 1. \end{matrix} \qquad (3.1.10)$$

With this case restriction on P *and with* $K = [\ -k_1 \quad -k_2\ ]$, (3.1.3a) *results in*

$$\begin{bmatrix} -2k_1 & p_{11} - k_1 - k_2 \\ p_{11} - k_1 - k_2 & 2 - 2k_2 \end{bmatrix} = - \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}. \tag{3.1.11}$$

For the simplest choice $q_{11} = q_{22} = 1$ *and* $q_{12} = 0$ *we get* $k_1 = 0.5$, $k_2 = 1.5$, $p_{11} = 2$. *Then the control law* (3.1.5) *is*

$$u = -\frac{1}{2}\xi_1 - \frac{3}{2}\xi_2 - x^2. \tag{3.1.12}$$

The equilibrium $(x, \xi_1, \xi_2) = (0, 0, 0)$ *is GAS because* $Q = I$ *is positive definite.* □

Example 3.1.4 *Let us now consider a system in which* $\dot{x} = f(x)$ *has a GAS equilibrium at* $x = 0$*:*

$$\dot{x} = -x^3 - x^3 \underbrace{(h_1\xi_1 + h_2\xi_2)}_{y} \tag{3.1.13a}$$

$$\dot{\xi}_1 = \xi_2 \tag{3.1.13b}$$

$$\dot{\xi}_2 = u \tag{3.1.13c}$$

When $h_2 = 0$ this system is stabilizable by two steps of integrator backstepping as in corollary (B.1.4). Thus, the case of interest is when $h_2 \neq 0$ and $h_1 h_2 \geq 0$ such that $h_1 \geq 0$, $h_2 > 0$. This includes the case $h_1 = 0$ when the transfer function $\frac{h_2 s + h_1}{s^2}$ of the linear part is only *weak minimum phase* [38] because it has a zero at $s = 0$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, h = [\ h_1 \quad h_2\ ]$$

$$Z(s) = h(sI - A)^{-1}b = [\ h_1 \quad h_2\ ] \frac{\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2}$$

$$= \frac{h_2 s + h_1}{s^2}$$

Choose a feedback $u = -k_1\xi_1 - k_2\xi_2 + v$ with $k_1, k_2 > 0$ which makes the polynomial $q(s) = s^2 + k_2 s + k_1$ Hurwitz and denote $p(s) = h_1 + h_2 s$.

$$\begin{aligned}
\dot{\xi} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, \qquad u = -k_1\xi_1 - k_2\xi_2 + v \\
\dot{\xi} &= \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix}v \\
0 &= s^2 + k_2 s + k_1 \\
s &= \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2}, \qquad k_2 > 0
\end{aligned}$$

$\therefore \Re(s) = \dfrac{-k_2}{2} < 0$, with $k_2 > 0$, then $q(s)$ is Hurwitz.

We can choose $k_1 = a^2$ and $k_2 = 2a$, with $a > \dfrac{h_1}{h_2}$, so that the transfer function $Z(s) = \dfrac{p(s)}{q(s)}$ is positive real

$$k_1 = a^2 \quad and \quad k_2 = 2a$$

$$Z(s) = \frac{p(s)}{q(s)} \quad is \quad P.R. \quad if \quad a > \frac{h_1}{h_2}$$

$$Note: \quad assume \quad h_1 > 0, \quad h_2 \geq 0 \qquad Z(s) = \frac{h_1 + h_2 s}{s^2 + 2as + a^2}$$

$$s = \sigma + j\omega \qquad \therefore Z(s) = \frac{(h_1 + h_2\sigma) + jh_2\omega}{(\sigma^2 + 2a\sigma - \omega^2 + a^2) + j2\omega(\sigma + a)}$$

$$Z(s) = \frac{[(h_1 + h_2\sigma) + jh_2\omega]\{(\sigma^2 + 2a\sigma - \omega^2 + a^2) - j2\omega(\sigma + a)\}}{(\sigma^2 + 2a\sigma - \omega^2 + a^2)^2 + [2\omega(\sigma + a)]^2}$$

$$\Re\{Z(s)\} = \frac{(h_1 + h_2\sigma)(\sigma^2 + 2a\sigma - \omega^2 + a^2) + 2h_2\omega^2(\sigma + a)}{(\sigma^2 + 2a\sigma - \omega^2 + a^2)^2 + [2\omega(\sigma + a)]^2}$$

$\therefore$ For $Z(s)$ to be P.R. we must have

$(h_1 + h_2\sigma)(\sigma^2 + 2a\sigma - \omega^2 + a^2) + 2h_2\omega^2(\sigma + a) \geq 0, \quad \forall\sigma \geq 0$

This can be achieved by nothing that when $\sigma = 0$

$$\begin{aligned}
h_1(a^2 - \omega^2) + 2h_2\omega^2 a &\geq 0 \\
h_1 a^2 + \omega^2(2h_2 a - h_1) &\geq 0, \qquad h_1 a^2 \geq 0 \\
2h_2 a - h_1 &\geq 0 \\
2h_2 a &\geq h_1 \\
a &\geq \frac{h_1}{2h_2}
\end{aligned}$$

If $a > \dfrac{h_1}{h_2} \geq \dfrac{h_1}{2h_2}$ and when $\sigma > 0$, the increased term

$$\begin{aligned}
(h_1 + h_2\sigma)(\sigma + a)^2 - h_1\omega^2 - h_2\sigma\omega^2 + 2h_2\omega^2\sigma + 2h_2\omega^2 a &= \\
(h_1 + h_2\sigma)(\sigma + a)^2 + h_2\sigma\omega^2 + \omega^2(2h_2 a - h_1) &\geq 0 \\
2h_2 a - h_1 &\geq 0 \\
2h_2 a &\geq h_1 \\
a &\geq \frac{h_1}{2h_2}
\end{aligned}$$

If $a > \dfrac{h_1}{h_2}$ then $a \geq \dfrac{h_1}{2h_2}$. So the value of $\Re\{Z(s)\}$ is always ≥ 0.

$A_* = A + bK$ is Hurwitz.

$Z(s) = h(sI - A_b K)^{-1} b$ is P.R.

$(A + bK, h)$ is observable

$$\mathcal{O} = \begin{bmatrix} h_1 & h_2 \\ -h_2 k_1 & h_1 - h_2 k_2 \end{bmatrix}$$

$$\begin{aligned}
det \quad \mathcal{O} &= h_1^2 - h_1 h_2 k_2 + h_2^2 k_1 \\
&= h_1^2 - s h_1 h_2 a + h_2^2 a^2 \\
&= (h_1 - h_2 a)^2 \geq 0 \\
det \quad \mathcal{O} \neq 0 \Leftrightarrow h_1 &\neq h_2 a \\
a &\neq \frac{h_1}{h_2}
\end{aligned}$$

$$a > \frac{h_1}{2h_2} \quad and \quad a \neq \frac{h_1}{h_2} \Rightarrow a > \frac{h_1}{h_2} \quad \text{is more strict.}$$

We can then write (3.1.13a) as $\dot{x} = f + gy, f = g = -x^3$. Clearly, $x = 0$ is a GAS equilibrium for $\dot{x} = f$, so the conditions of lemma (3.1.2) are satisfied. Using $V(x) = x^2$ in (3.1.5), we obtain the control law

$$u = -a^2\xi_1 - 2a\xi_2 + \overbrace{x^4}^{-\frac{1}{2}\cdot\frac{\partial(x^2)}{\partial x}\cdot(-x^3)} . \tag{3.1.14}$$

The situation is quite different when $h_1 h_2 < 0$, that is, when the transfer function $Z(s) = \dfrac{p(s)}{q(s)}$, is nonminimum phase. Then, Lemma 3.1.2 does not apply. In fact, a detailed calculation given in [19] shows that in this case the system cannot be globally stabilized.

The FPR property is a passivity property. Its nonlinear counterpart will be employed in the stabilization of the nonlinear cascade

$$\dot{x} = f(x,\xi) + g(x,\xi)y, \quad f(0,\xi) = 0, \quad \forall\xi \in \mathbb{R}^q, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R} \tag{3.1.15a}$$

$$\dot{\xi} = m(\xi) + \beta(\xi)u, \quad y = h(\xi), \quad h(0) = 0, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R}. \tag{3.1.15b}$$

Our key assumption is that (3.1.15b) can be rendered passive or strictly passive (cf. Appendix D) via a feedback transformation $u = k(\xi) + r(\xi)v$.

Definition 3.1.5 *The system*

$$\dot{\xi} = m(\xi) + \beta(\xi)u, \quad y = h(\xi), \quad h(0) = 0, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R} \tag{3.1.16}$$

is said to be feedback passive (FP) *if there exists a feedback transformation*

$$u = k(\xi) + r(\xi)v \tag{3.1.17}$$

such that the resulting system $\dot{\xi} = m(\xi) + \beta(\xi)k(\xi) + \beta(\xi)r(\xi)v, \quad y = h(\xi)$ *is passive with a storage function* $U(\xi)$ *which is positive definite and radially unbounded:*

$$\int_0^t y(\sigma)v(\sigma)d\sigma \geq U(\xi(t)) - U(\xi(0)). \tag{3.1.18}$$

The system (3.1.16) *is said to be* feedback strictly passive (FSP) *if the feedback* (3.1.17) *renders it strictly passive:*

$$\int_0^t y(\sigma)v(\sigma)d\sigma \geq U(\xi(t)) - U(\xi(0)) + \int_0^t \psi(\xi(\sigma))d\sigma, \tag{3.1.19}$$

where $\psi(\cdot)$ *is the positive definite dissipation rate.*

As in the linear case, FP systems of the form (3.1.16) must have relative degree one.

Lemma 3.1.6 (Stabilization with Passivity) *Let $V(x)$ be a radially unbounded Lyapunov function for $\dot{x} = f(x,\xi)$ satisfying*

$$\frac{\partial V}{\partial x}(x) f(x,\xi) \leq -W(x) \leq 0, \forall \xi \in \mathbb{R}^n, \qquad \forall \xi \in \mathbb{R} \tag{3.1.20}$$

and let (3.1.15b) *be FP as in Definition 3.1.5. Then, a Lyapunov function for the cascade system* (3.1.15a) *is*

$$V_a(x,\xi) \;=\; V(x) + U(\xi), \tag{3.1.21}$$

and the corresponding control law

$$u \;=\; \alpha_a(x,\xi) = k(\xi) - r(\xi)\frac{\partial V}{\partial x}(x) g(x,\xi) \tag{3.1.22}$$

guarantees that $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$ *is globally bounded and converges to the largest invariant set $\bar{M}_a$ contained in the set* $\bar{E}_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \middle| W(x) = 0 \right\}$. *If* (3.1.15b) *is FSP, then* (3.1.22) *guarantees convergence to the largest invariant set M_a contained in the set* $E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \middle| W(x) = 0, \xi = 0 \right\}$. *Finally, if* (3.1.15b) *is FSP and $W(x)$ is positive definite, that is, if $\dot{x} = f(x,\xi)$ has a GAS equilibrium at $x = 0$ uniformly in ξ, then the equilibrium $x = 0, \xi = 0$ of* (3.1.15a) *is also GAS.*

The closed-loop system (3.1.15) with the control (3.1.22) is

$$\begin{aligned} \dot{x} &= f(x,\xi) + g(x,\xi)y \\ \dot{\xi} &= m(\xi) + \beta(\xi)k(\xi) + \beta(\xi)r(\xi)v \\ y &= h(\xi), \quad v = -\frac{\partial V}{\partial x}(x) g(x,\xi). \end{aligned} \tag{3.1.23}$$

$V(x)$ P.D. and R.U. Lyapunov function for $\dot{x} = f(x,\xi)$ satisfying $\frac{\partial V}{\partial x}(x) f(x,\xi) \leq -W(x) \leq 0, \forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}$,

$$\dot{V} = \frac{\partial V}{\partial x}(f + gy) \leq -W(x) + \frac{\partial V}{\partial x} gy = -W(x) + \eta y \tag{3.1.24}$$

$$\int_0^t \eta(\sigma) y(\sigma) d\sigma \geq V(x(t)) - V(x(0)) + \int_0^t W(x(\sigma)) d\sigma. \tag{3.1.25}$$

$\therefore$ Σ_2 is passive since $W(x) \geq 0$. From Theorem C.1.4 we conclude that the negative feedback interconnection of Σ_1 and Σ_2 is passive with the positive definite and radially unbounded storage function $V_a(x, \xi) = V(x) + U(\xi)$. Lemma C.1.3 then states that $x = 0$, $\xi = 0$ is a globally stable equilibrium of

$$\begin{aligned}
\dot{x} &= f(x, \xi) + g(x, \xi)y \\
\dot{\xi} &= m(\xi) + \beta(\xi)k(\xi) + \beta(\xi)r(\xi)v \\
y &= h(\xi), \quad v = -\frac{\partial V}{\partial x}(x)g(x, \xi).
\end{aligned}$$

To see that $W(x) \to 0$ as $t \to \infty$, we differentiate $\int y(\sigma)v(\sigma)d(\sigma) \geq U(\xi(t) - U(\xi(0))$ and combine the result with $\frac{\partial V}{\partial x}(x)f(x, \xi) \leq -W(x) \leq 0 \; \forall x \in \mathbb{R}^n$, $\forall \xi \in \mathbb{R}$:

$$\begin{aligned}
\dot{V}_a &= \dot{V} + \dot{U} \leq \frac{\partial V}{\partial x}(f + gy) + yv \\
&\leq -W(x) + \frac{\partial V}{\partial x}gy + yv = -W(x) \leq 0.
\end{aligned} \tag{3.1.26}$$

Then, LaSalle's theorem (Theorem A.1.2) guarantees convergence to the set $\bar{M}_a \subseteq \{W(x) = 0\}$. If (3.1.15b) is FSP; that is,

$$\int_0^t y(\sigma)v(\sigma)d\sigma \geq U(\xi(t)) - U(\xi(0)) + \int_0^t \psi(\xi(\sigma)d\sigma \tag{3.1.27}$$

we replace (3.1.18) by (3.1.27). Then (3.1.26) becomes

$$\dot{v}_a \leq -W(x) - \varphi(\xi) \tag{3.1.28}$$

which, since $\psi(\xi)$ is positive definite, guarantees convergence to the set $M_a \subseteq \{W(x) = 0, \xi = 0\}$.

Finally, if $W(x)$ is also positive definite. we conclude from (3.1.28) and Theorem A.1.1 that $x = 0, \xi = 0$ is GAS.

$$\begin{aligned}
(\because V_a &= v(x) + U(\xi) \text{is p.d. and R.U. in})(x, \xi) \\
\dot{v}_a &= \dot{V} + \dot{U} \leq -W(x) - \psi(\xi)
\end{aligned}$$

Example 3.1.7 *consider the cascade system:*

$$\dot{x} = -x(1+e^{\xi}) + x^3\xi^2 \quad (3.1.29a)$$

$$\dot{\xi} = \xi u \quad (3.1.29b)$$

$$y = \xi^2$$

The choice of output $y = \xi^2$ *satisfies all the conditions of Lemma 3.1.6. First,* (3.1.29b) *is FSP: The feedback*

$$u = -\xi^2 + v \quad (3.1.30)$$

results in $\dot{\xi} = -\xi^3 + \xi v$, $y = \xi^2$, *which is strictly passive with storage function* $U(\xi) = \frac{1}{2}\xi^2$, *since*

$$\dot{U} = -\xi^4 - \xi^2 v = -\xi^4 + yv \quad (3.1.31)$$

implies that

$$\int_0^t y(\sigma)v(\sigma)d\sigma \geq U(\xi(t)) - U(\xi(0)) + \int_0^t \xi^4(\sigma)d\sigma. \quad (3.1.32)$$

Furthermore, (3.1.29a) *can be represented in the form* (3.1.15a) *with*

$$f(x,\xi) = -x(1+e^{\xi}), \quad g(x,\xi) = x^3 \quad (3.1.33)$$

and (3.1.20) *is satisfied with* $V(x) = \frac{1}{2}x^2$, *i.e.,*

$$\begin{aligned} \frac{\partial V}{\partial x}(x)f(x.\xi) &= x[-x(1+e^{\xi})] \\ &= -x^2(1+e^{\xi}) \leq -x^2 = -W(x) \leq 0, \quad \forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R} \\ W(x) &= x^2 \end{aligned}$$

Applying Lemma 3.1.6, we conclude that the control

$$u = -\xi^2 - x^4 \quad (3.1.34)$$

guarantees GAS of $x = 0$, $\xi = 0$. *Indeed, the derivative of the clf* $V_a(x,\xi) = \frac{1}{2}(x^2 + \xi^2)$ *is negative definite:*

$$\dot{V_a} = -x^2(1+e^{\xi}) + x^4\xi^2 - \xi^4 - x^4\xi^2 \leq -x^2 - \xi^4. \quad (3.1.35)$$

Chapter 4

Block backstepping with zero dynamics

4.1 Block backstepping with zero dynamics

Integrator backstepping (Lemma B.1.2) is a recursive design tool. Now we want to develop a similar tool for feedback stabilization of a system augmented by a dynamic block more complicated than just an integrator.

At first glance, it may appear that the cascade design in the preceding chapter provides us with such a tool. Not quite! The achievement of the cascade design is in being able to stabilize the input subsystem (3.1.1b) or (3.1.15b) *without destabilizing the original system*. What if the original system is not stable? Can we cascade it with a complicated input subsystem and still stabilize it in one step? We first show that this can be done with a linear input subsystem that is a minimum phase system with relative degree one. We then give a nonlinear extension of that result.

Example 4.1.1 *Let us start with an example in which we cascade the system* (3.1.29) *of Example 3.1.7 with a linear minimum phase system:*

$$\begin{array}{ll} x\text{-subsystem:} & \left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1(1+e^{x_2}) + x_1^3 x_2^2 \\ \dot{x}_2 & = & x_2 y \end{array} \right. \\ \xi\text{-subsystem:} & \left\{ \begin{array}{rcl} \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & u \\ y & = & \xi_1 + \xi_2. \end{array} \right. \end{array} \tag{4.1.1}$$

The transfer function of the input subsystem is $\frac{s+1}{s^2}$ *and its zero is at* $s = -1$. *One of its minimal realizations is*

$$\dot{y} = y - \xi_1 + u \tag{4.1.2a}$$

$$\dot{\xi}_1 = -\xi_1 + y. \tag{4.1.2b}$$

Its zero dynamics, that is, the dynamics constrained by $y(t) \equiv 0$, *are described by* $\dot{\xi}_1 = -\xi_1$.

The cascade design of the preceding subsection is not applicable to (4.1.1) *because the equilibrium* $x = 0$ *of the x-subsystem with* $y = 0$ *is unstable:* $\dot{x}_2 = x_2 y$ *and* $y = 0 \Rightarrow x_2$ *can be constant and* $x_2 \neq 0$. *Such as,* $x_2 = 2$:

$$\begin{array}{rcl} \dot{x}_1 & = & -x_1(1+e^2) + 4x_1^3 = ax_1^3 - bx_1 \\ & = & ax_1(x_1^2 - \frac{b}{a}); \qquad a, b > 0 \end{array}$$

linearizing around $(x_1, x_2) = (0, 0)$,

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1(1 + 1 + x_2 + \frac{x_2^2}{2!} + \ldots) + x_1^3 x_2^2 \\ \dot{x}_2 & = & 0 \end{array} \right.$$

we obtain

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -2x_1 \\ \dot{x}_2 & = & 0 \end{array} \right.$$

or

$$\dot{x} = \underbrace{\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{marginally stable}} x.$$

To circumvent this obstacle, we first convert (4.1.2a) *into an integrator via the feedback transformation*

$$u = \underbrace{-y + \xi_1}_{-\xi_2} + v, \tag{4.1.3}$$

where v is our new control variable. The system (4.1.1) *is then rewritten as*

$$\begin{aligned} \dot{x}_1 &= -x_1(1 + e^{x_2}) + x_1^3 x_2^2 \\ \dot{x}_2 &= x_2 y \\ \dot{y} &= v \\ \dot{\xi}_1 &= -\xi_1 + y. \end{aligned} \tag{4.1.4}$$

Now the subsystem consisting of the first three equations in (4.1.4) *is in a form convenient for integrator backstepping. From Example 3.1.7 we already know that the x-subsystem can be stabilized with y as its virtual control (cf.*(3.1.34)*):*

$$y_{des} = \alpha(x) = -x_1^4 - x_2^2. \tag{4.1.5}$$

The corresponding clf is $V(x, \xi) = \frac{1}{2}(x_1^2 + x_2^2)$*. Hence, we can achieve stabilization and regulation of* x_1, x_2, y *by a direct application of Lemma B.1.2. The resulting control law is*

$$u = -(y + x_1^4 + x_2^2) - \xi_2 - 2x_2^2 y + 4x_1^4(1 + e^{x_2} - x_1^2 x_2^2) - x_2^2. \tag{4.1.6}$$

This design ignored the presence of the zero dynamics subsystem $\dot{\xi}_1 = -\xi_1 + y$*. However, this subsystem is input-to-state stable (ISS) with respect to y, so that* ξ_1 *is bounded because* *y is bounded, and moreover* $\lim_{t\to\infty} \xi_1(t) = 0$ *since* $\lim_{t\to\infty} y(t) = 0$.

$$\begin{aligned}
\dot{\xi}_1 &= -\xi_1 + y \\
\dot{\xi}_1 + \xi_1 &= y \\
e^t\dot{\xi}_1 + e^t\xi_1 &= e^t y \\
\frac{d}{dt}(e^t\xi_1) &= e^t y \\
e^s\xi_1\Big|_0^t &= \int_0^t e^s y(s)ds \\
e^t\xi_1(t) - \xi_1(0) &= \int_0^t e^s y(s)ds \\
\xi_1(t) &= e^{-t}\xi_1(0) + e^{-t}\int_0^t e^s y(s)ds.
\end{aligned}$$

Since $\lim_{t\to\infty} y(t) = 0$ *then*

$$\begin{aligned}
\dot{\xi}_1 &= -\xi_1 + y = -\xi_1 \\
\xi_1(t) &= e^{-t}\xi_1(0) \to 0, \qquad as \quad t \to \infty.
\end{aligned}$$

We now want to generalize the above example and formulate design tools which allow the original system to be unstable when $y = 0$ and let us backstep more than a simple integrator at a time. Since we want to be able to apply these tools repeatedly, each lemma we formulate must guarantee for the cascade system all the properties assumed for the original system.

As we will see, the constructed $V_a(x, \xi)$ for the cascade system does not include the zero dynamics variables, but their boundedness is guaranteed by the boundedness of V_a.

Hence, we must reformulate Assumption B.1.1 to assume the same properties for the original system, by including the case when $V(x)$ is not positive definite:

Assumption 4.1.2 *Suppose Assumption B.1.1 is valid with* $V(x)$ *positive semidefinite, and the closed-loop system* (B.1.17) *with the control* (B.1.18) *has the property that* $x(t)$ *is bounded if* $V(x(t))$ *is bounded.*

Under this assumption, the control (B.1.18), applied to the system (B.1.17), guarantees not only global boundedness of $x(t)$, but also regulation of $W(x(t))$:

From (B.1.19) we conclude that $W(x(t))$ is integrable on $[0,\infty)$ and uniformly continuous, and hence converges to zero by Lemma A.3.5. Furthermore, since all solutions $x(t)$ are bounded, we can apply LaSalle's theorem (Theorem A.1.2) to conclude that $x(t)$ converges the largest invariant set M contained in the set $E = \{x \in \mathbb{R}^n | W(x) = 0\}$.

The following fact is easy to prove:

Corollary 4.1.3 *When Assumption B.1.1 is replaced by Assumption 4.1.2, then the boundedness and convergence properties in part (ii) of Lemma B.1.2 still hold.*

Lemma 4.1.4 (Linear Block Backstepping) *Consider the cascade system*

$$\dot{x} = f(x) + g(x)y, \quad f(0) = 0, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R} \tag{4.1.7a}$$

$$\dot{\xi} = A\xi + bu, \quad y = h\xi, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R} \tag{4.1.7b}$$

where (4.1.7b) *is a minimum phase system of relative degree one* ($hb \neq 0$).

If (4.1.7a) *satisfies Assumption 4.1.2 with* y *as its input, then there exists a feedback control which guarantees global boundedness and convergence of* $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$ *to the largest invariant set* M_a *contained in the set* $E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \middle| W(x) = 0, y = \alpha(x) \right\}$.

One choice for this control is

$$u = \frac{1}{hb}\Big\{ -c(y-\alpha(x)) - hA\xi + \frac{\partial \alpha}{\partial x}(x)[f(x)+g(x)y] - \frac{\partial V}{\partial x}(x)g(x)\Big\}, \quad c > 0. \tag{4.1.8}$$

Moreover, if $V(x)$ *and* $W(x)$ *are positive definite, then the equilibrium* $x = 0, \xi = 0$ *is GAS.*

Proof *We recall from [39] that the relativc-degree-one SISO linear system* (4.1.7b) *can be represented in the form*

$$\dot{y} = hA\xi + hbu \tag{4.1.9a}$$

$$\dot{\zeta} = A_0\zeta + b_0 y, \tag{4.1.9b}$$

where the eigenvalues of A_0 *are the (stable) zeros of the transfer function*

$$H(s) = h(sI - A)^{-1}b$$

of the minimum phase system (4.1.7b). *Using* (4.1.9) *and the feedback transformation*

$$u = \frac{1}{hb}(v - hA\xi), \tag{4.1.10}$$

we rewrite (4.1.7) *as follows:*

$$\dot{x} = f(x) + g(x)y \tag{4.1.11a}$$

$$\dot{y} = v \tag{4.1.11b}$$

$$\dot{\zeta} = A_0\zeta + b_0 y. \tag{4.1.11c}$$

We first ignore the zero dynamics (4.1.11c) *and, using Corollary 4.1.3, apply Lemma B.1.2 to* (4.1.11a)-(4.1.11b) *to achieve global boundedness of* x *and* y *and regulation of* $W(x(t))$ *and* $y(t) - \alpha(x(t))$. *In view of* (4.1.10) *and* (B.1.23), *one choice of control is given by* (4.1.8).

Retuning to (4.1.11c), *we note that* ζ *is bounded because* y *is bounded and* A_0 *is strictly Hurwitz. Thus,* ξ *is bounded. Since all solution of* (4.1.7) *are bounded, we can apply LaSalle's theorem (Theorem A.1.2) with* $\Omega = \mathbb{R}^{n+q}$ *to conclude convergence to the set* M_a.

From Lemma A.1.8 we also know that if $V(x)$ *and* $W(x)$ *are positive definite, then the equilibrium* $x = 0, y = 0$ *of* (4.1.11a)-(4.1.11b), *which is completely decoupled from* (4.1.11c), *is GAS. The fact that in this case the equilibrium* $x = 0, \xi = 0$ *of the cascade system* (4.1.7) *is also GAS follows immediately from the following lemma:*

Lemma 4.1.5 *Consider the cascade system with $\zeta \in \mathbb{R}^m, x \in \mathbb{R}^n$:*

$$\dot{\zeta} = A_0\zeta + b_0 y \tag{4.1.12a}$$

$$\dot{x} = f(x), f(0) = 0$$

$$y = h(x), h(0) = 0 \tag{4.1.12b}$$

If (4.1.12b) *is GAS and A_0 is strictly Hurwitz, then the equilibrium $\zeta = 0, x = 0$ of the cascade* (4.1.12) *is GAS.*

Proof *From the definition of GAS (Definition A.3.4) we know that the GAS property of* (4.1.12b) *implies the existence of class KL_∞ function β and β_1 such that*

$$|x(t)| \le \beta(|x(0)|, t), \qquad |y(t)| \le \beta_1(|x(0)|, t). \tag{4.1.13}$$

The solution of (4.1.12a)*, on the other hand, are given by*

$$\zeta(t) = e^{A_0 t}\zeta(0) + \int_0^t e^{A_0(t-\tau)} b_0 y(\tau) d\tau. \tag{4.1.14}$$

Since A_0 is strictly Hurwitz, we know that $|e^{A_0 t}| \le k_1 e^{-\alpha t}$:

$$\begin{aligned}
\forall \lambda_i \in \sigma(A_0), Re\lambda_i < 0 \;&\Leftrightarrow\; A_0 \text{ is strictly Hurwitz.} \\
&\Leftrightarrow\; \exists \alpha > 0 \text{ such that } \alpha I + A_0 \text{ is strictly Hurwitz.} \\
&\Leftrightarrow\; (e^{(\alpha I + A_0)t} = P e^{(\alpha I + D)t} P^{-1}) \\
&\Leftrightarrow\; |e^{(\alpha I + A_0)t}| \le k_1, \exists k_1 > 0 \\
&\Leftrightarrow\; |e^{\alpha t I} e^{A_0 t}| \le k_1, \exists k_1 > 0 \\
&\Leftrightarrow\; |e^{\alpha t} I e^{A_0 t}| \le k_1, \exists k_1 > 0 \\
&\Leftrightarrow\; |e^{\alpha t}| e^{A_0 t}| \le k_1, \exists k_1 > 0 \\
&\Leftrightarrow\; |e^{A_0 t}| \le k_1 e^{-\alpha t}, \exists k_1 > 0, \exists \alpha \in [0, |\lambda|_{min}).
\end{aligned}$$

Using this with (4.1.13) *in* (4.1.14)*, we obtain*

$$
\begin{aligned}
|\zeta(t)| &\leq |e^{A_0 t}||\zeta(0)| + \int_0^t |e^{A_0(t-\tau)}b_0||y(\tau)|d\tau \\
&\leq k_1 e^{-\alpha t}|\zeta(0)| + k_2 \int_0^t e^{-\alpha(t-\tau)}\beta_1(|x(0)|,\tau)d\tau \\
&\leq k_1 e^{-\alpha t}|\zeta(0)| + k_2 \sup_{0\leq\tau\leq t/2} \beta_1(|x(0)|,\tau) \int_0^{t/2} e^{-\alpha(t-\tau)}d\tau \\
&\quad + k_2 \sup_{t/2\leq\tau\leq t} \beta_1(|x(0)|,\tau) \int_{t/2}^t e^{-\alpha(t-\tau)}d\tau \\
&\leq k_1 e^{-\alpha t}|\zeta(0)| + k_2\beta_1(|x(0)|,0) \int_0^{t/2} e^{-\alpha(t-\tau)}d\tau \\
&\quad + k_2\beta_1(|x(0)|,t/2) \int_{t/2}^t e^{-\alpha(t-\tau)}d\tau \\
&= k_1 e^{-\alpha t}|\zeta(0)| + \frac{k_2}{\alpha}\beta_1(|x(0)|,0)e^{-\alpha t/2}(1-e^{-\alpha t/2}) \\
&\quad + \frac{k_2}{\alpha}\beta_1(|x(0)|,t/2)(1-e^{-\alpha t/2}) \\
&\leq k_1 e^{-\alpha t}|\zeta(0)| + \frac{k_2}{\alpha}\beta_1(|x(0)|,0)e^{-\alpha t/2} \\
&\quad + \frac{k_2}{\alpha}\beta_1(|x(0)|,t/2) \\
&\triangleq \beta_2\left(\left|\begin{array}{c}\zeta(0)\\ x(0)\end{array}\right|,t\right),
\end{aligned}
$$

where β_2 is a class KL_∞ function. Combining (4.1.13) *with* (4.1.15) *proves that* $\zeta = 0, x = 0$ *is GAS:*

$$
\left|\begin{array}{c}\zeta(0)\\ x(t)\end{array}\right| \leq \beta_3\left(\left|\begin{array}{c}\zeta(0)\\ x(0)\end{array}\right|,t\right), \quad \beta_3 \in KL_\infty. \tag{4.1.15}
$$

□

Comparing Lemma 3.1.2 and 4.1.4 we see that, instead of assuming global stability of $x = 0$ when $y = 0$, Lemma 4.1.4 assumes only global stabilizability of $x = 0$ through y. The corresponding assumptions on the input subsystem, however, reveal the price paid for this generalization: The minimum phase assumption of Lemma 4.1.4 is stronger than the FPR assumption of Lemma 3.1.2, which allows some zeros to be on the imaginary axis, that is, to be weak minimum phase.

Let as now examine the cascade system

$$\dot{x} = -x(1+e^{\xi_1}) + x^3\xi_1^2 \tag{4.1.16a}$$

$$\dot{\xi}_1 = \xi_1\xi_2^2 \tag{4.1.16b}$$

$$\dot{\xi}_2 = \xi_2 u. \tag{4.1.16c}$$

As we have already shown in Example 3.1.7, (4.1.16a)-(4.1.16b) is stabilizable through $y = \xi_2^2$, while (4.1.16c) with this output is FSP. However, if we try to stabilize the cascade (4.1.16), we run into difficulties because the relative degree of (4.1.16c) is not defined at $\xi_2 = 0$.

This example shows that we need to assume that the input subsystem

$$\dot{\xi} = m(\xi) + \beta(\xi)u, \quad y = h(\xi), \tag{4.1.17}$$

has a globally defined constant relative degree. For a nonlinear analog of Lemma 4.1.4, we also assume that the zero dynamics subsystem of (4.1.17) is ISS.

Lemma 4.1.6 (Nonlinear Block Backstepping) *Consider the cascade system:*

$$\dot{x} = f(x) + g(x)y, \quad f(0) = 0, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R} \quad . \tag{4.1.18a}$$

$$\dot{\xi}_1 = m(x,\xi) + \beta(x,\xi)u, \quad y = h(\xi), \quad h(0) = 0, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R}. \tag{4.1.18b}$$

Assume that (4.1.18b) *has globally defined and constant relative degree one uniformly in x, and that its zero dynamics subsystem is ISS with respect to x and y as its inputs. If* (4.1.18a) *satisfies Assumption 4.1.2 with y as its input, then there exists a feedback control which guarantees global boundedness and convergence of $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$ to the largest invariant set M_a contained in the set*

$$E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \middle| W(x) = 0, y = \alpha(x) \right\}.$$

One particular choice is

$$\begin{aligned} u &= \left(\frac{\partial h}{\partial \xi}(\xi)\beta(x,\xi)\right)^{-1}\Big\{ -c\big(y-\alpha(x)\big) - \frac{\partial h}{\partial \xi}(\xi)m(x,\xi) \\ &\quad + \frac{\partial \alpha}{\partial x}(x)\big[f(x)+g(x)y\big] - \frac{\partial V}{\partial x}(x)g(x)\Big\}, \quad c>0. \end{aligned} \tag{4.1.19}$$

Moreover, if $V(x)$ and $W(x)$ are positive definite, then the equilibrium $x = 0, \xi = 0$ is GAS.

Proof *Since the relative degree of the subsystem* (4.1.18b) *is globally defined and equal to one uniformly in x, there exists a global change of coordinates of the form* (2.1.10), *in particular $(y,\zeta) = (y,\phi(x,\xi))$ with $\frac{\partial \phi}{\partial \xi}\beta \equiv 0$, which transforms* (4.1.18b) *into*

$$\begin{aligned} \dot{y} &= \frac{\partial h}{\partial \xi}(\xi)m(x,\xi) + \frac{\partial h}{\partial \xi}(\xi)\beta(x,\xi)u \\ &\triangleq f_1(x,y,\zeta) + g_1(x,y,\zeta)u \end{aligned} \tag{4.1.20a}$$

$$\begin{aligned} \dot{\zeta} &= \frac{\partial \phi}{\partial x}(x,\xi)[f(x)+g(x)y] + \frac{\partial \phi}{\partial \xi}(x,\xi)m(x,\xi) \\ &\triangleq \Phi(\zeta,x,y). \end{aligned} \tag{4.1.20b}$$

We now consider the cascade system consisting of (4.1.18a) *and* (4.1.20a). *If we linearize* (4.1.20a) *with the feedback given by* (2.1.9),

$$u = \left(\frac{\partial h}{\partial \xi}\beta\right)^{-1}\left(v - \frac{\partial h}{\partial \xi}m\right), \tag{4.1.21}$$

we obtain $\dot{y} = v$. Then we can apply Lemma B.1.2, with v as the new control input, to guarantee global boundedness of x and y and regulation of $W(x(t))$ and $y(t) - \alpha(x(t))$. From (4.1.20b) *and the ISS assumption on the zero dynamics, ζ is also bounded, and thus ξ and u are bounded.*

Since all solution of (4.1.18) *are bounded, we can apply LaSalle's theorem (Theorem A.1.2) with* $\Omega = \mathbb{R}^{n+q}$ *to conclude convergence to the set* M_a. *combining* (4.1.21) *with* (B.1.23), *we see that a particular choice of control is given by* (4.1.19).

From Lemma(A.1.8) *we also know that if* $V(x)$ *and* $W(x)$ *are positive definite, then the equilibrium* $x = 0, y = 0$ *of* (4.1.18a) *and* (4.1.20a), *which is completely decoupled from* (4.1.20b), *is GAS. The fact that in this case the equilibrium* $x = 0, \xi = 0$ *of the cascade system* (4.1.18) *is also GAS follows from Lemma C.4 by noting that the state* (x, y) *of the GAS system* (4.1.18a) *and* (4.1.20a) *is the input of the ISS system* (4.1.20b). □

Lemma 4.1.6 relaxes the global stability assumption of Lemma 3.1.6 to global stabilizability of $x = 0$ through y. As in the case of Lemmas 3.1.2 and 4.1.4, however, the price paid for this generalization is the strengthening of the FP assumption of Lemma 3.1.6 to the ISS assumption of Lemma 4.1.6.

Chapter 5

Adaptive Backstepping

5.1 Adaptation as dynamic feedback

Consider the simplest nonlinear system

$$\dot{x} = u + \theta\varphi(x) \tag{5.1.1}$$

where θ is the unknown constant parameter.

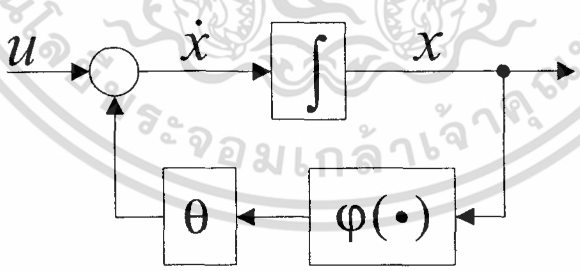

Figure 5.1: Block diagram of the system (5.1.1)

Even if we do not know a bound for θ, we can apply a nonlinear static feedback, as explained in preceding section, to guarantees global boundedness of $x(t)$. Now, we try to achieve a system regulation by applying a dynamic feedback controller.

If θ were known, the control

$$u = -\theta\varphi(x) - c_1 x, \quad c_1 > 0 \tag{5.1.2}$$

would render the derivative of $V_0(x) = \frac{1}{2}x^2$ negative definite: $\dot{V}_0 = -c_1 x^2$. Of course the control law (5.1.2) can not be implemented, since θ is unknown. Instead, one can employ its *certainty-equivalence* form in which θ is replaced by an estimate $\hat{\theta}$:

$$u = -\hat{\theta}\varphi x - c_1 x \tag{5.1.3}$$

Substituting (5.1.3) into (5.1.1), we obtain

$$\dot{x} = -c_1 x + \tilde{\theta}\varphi(x) \tag{5.1.4}$$

where $\tilde{\theta}$ is the *parameter error*:

$$\tilde{\theta} = \theta - \hat{\theta} \tag{5.1.5}$$

The derivative of $V_0(x) = \frac{1}{2}x^2$ becomes

$$\dot{V}_0 = -c_1 x^2 + \tilde{\theta} x \varphi(x) \tag{5.1.6}$$

Since the second term is indefinite and contains the unknown parameter error $\tilde{\theta}$, we can not conclude anything about the stability of (5.1.4). We make the controller dynamic with an update law for $\hat{\theta}$. To design this update law, we augment V_0 with a quadratic term in the parameter error $\tilde{\theta}$,

$$V_1(x, \tilde{\theta}) = \frac{1}{2}x^2 + \frac{1}{2\gamma}\tilde{\theta}^2 \tag{5.1.7}$$

where $\gamma > 0$ is the *adaptation gain*. The derivative of this function is

$$\begin{aligned} \dot{V}_1 &= x\dot{x} + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} \\ &= -c_1 x^2 + \tilde{\theta} x \varphi(x) + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} \\ &= -c_1 x^2 + \tilde{\theta}\left[x\varphi(x) + \frac{1}{\gamma}\dot{\tilde{\theta}}\right] \end{aligned} \tag{5.1.8}$$

The second term is still indefinite and contains $\tilde{\theta}$ as a factor. However, the situation is much better than in (5.1.6), because we now have the dynamics of $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$ at our disposal. With the appropriate choice of $\dot{\hat{\theta}}$ we can cancel the indefinite term. Thus, we choose the update law

$$\dot{\hat{\theta}} = -\dot{\tilde{\theta}} = \gamma x \varphi(x) \tag{5.1.9}$$

which yields

$$\dot{V}_1 = -c_1 x^2 \leq 0 \tag{5.1.10}$$

The resulting adaptive system consists of (5.1.1) with the control (5.1.3) and the update law (5.1.9), and is shown in Figure 5.2. In Figure 5.3, this system is redrawn in its closed-loop form consisting of (5.1.4) and (5.1.9), namely

$$\begin{aligned} \dot{x} &= -c_1 x + \tilde{\theta}\varphi(x) \\ \dot{\tilde{\theta}} &= -\gamma x \varphi(x) \end{aligned} \tag{5.1.11}$$

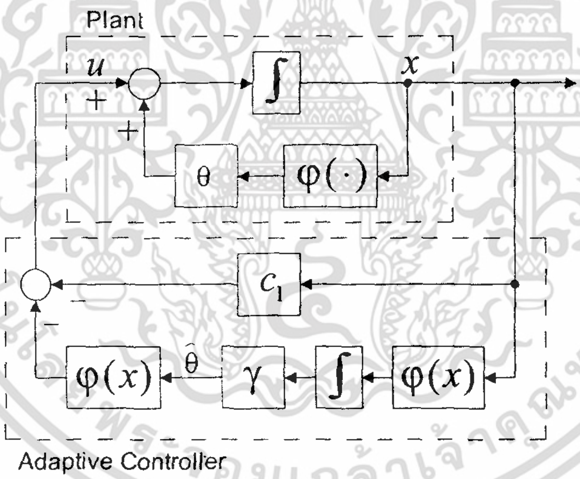

Figure 5.2: The closed-loop adaptive system (5.1.11)

Because $\dot{V}_1 \leq 0$, the equilibrium $x = 0$, $\tilde{\theta} = 0$ of (5.1.11) is globally stable. In addition, the desired regulation property $\lim_{t\to\infty} x(t) = 0$ follows from the LaSalle-Yoshizawa theorem. The adaptive nonlinear controller which guarantees these properties is given by (5.1.4) and (5.1.9):

$$\begin{aligned} u &= -c_1 x - \hat{\theta}\varphi(x) \\ \dot{\hat{\theta}} &= \gamma x \varphi(x) \end{aligned} \tag{5.1.12}$$

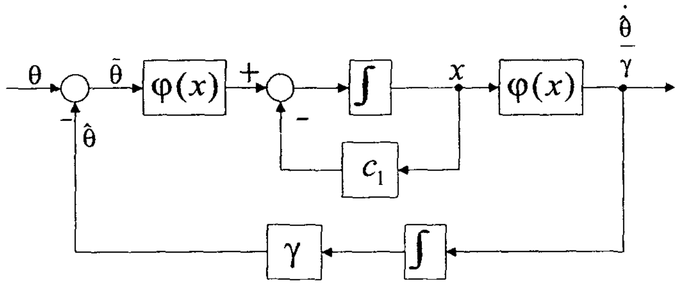

Figure 5.3: An equivalent representation of (5.1.11)

One may think that the above adaptive design is so straightforward because (5.1.1) is a first-order system. In fact, this is due to the *matching condition*: The terms containing unknown parameters in (5.1.1) are in the span of the control, that is, they can be directly cancelled by u when θ is known. To illustrate this point, let us consider the following second-order system, where again the uncertain term is "matched" by the control u:

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(x_1) \\ \dot{x}_2 &= \theta\varphi_2(x) + u \end{aligned} \tag{5.1.13}$$

If θ were known, we would be able to use a static design procedures as mentioned in preceding sections. First view x_2 as the virtual control, design the stabilizing function

$$\alpha_1(x_1) = -c_1x_1 - \varphi_1(x_1) \tag{5.1.14}$$

and then form the Lyapunov function

$$V_c(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - \alpha_1(x_1))^2 \tag{5.1.15}$$

whose derivative is rendered negative definite

$$\dot{V}_c(x) = -c_1x_1^2 - c_2(x_2 - \alpha_1)^2 \tag{5.1.16}$$

by the control

$$u = -c_2(x_2 - \alpha_1) - x_1 + \frac{\partial\alpha_1}{\partial x_1}(x_2 + \varphi_1) - \theta\varphi_2(x) \tag{5.1.17}$$

Since θ is unknown, we again replace it with its estimate $\hat{\theta}$ in (5.1.17) to obtain the adaptive control law:

$$u = -c_2(x_2 - \alpha_1) - x_1 + \frac{\partial \alpha_1}{\partial x_1}(x_2 + \varphi_1) - \hat{\theta}\varphi_2(x) \tag{5.1.18}$$

This results in the error system $(z_1 = x_1, z_2 = x_2 - \alpha_1)$:

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \varphi_2(x) \end{bmatrix}\tilde{\theta} \tag{5.1.19}$$

Then we augment (5.1.15) with a quadratic term in the parameter error $\tilde{\theta}$ to obtain the Lyapunov function:

$$V_1(z, \tilde{\theta}) = V_c + \frac{1}{2\gamma}\tilde{\theta}^2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}\tilde{\theta}^2 \tag{5.1.20}$$

Its derivative is

$$\dot{V}_1 = -c_1 z_1^2 - c_2 z_2^2 + \tilde{\theta}\left[z_2\varphi_2 - \frac{1}{\gamma}\dot{\hat{\theta}}\right] \tag{5.1.21}$$

The choice of update law

$$\dot{\hat{\theta}} = \gamma\varphi_2 z_2 \tag{5.1.22}$$

eliminates the $\tilde{\theta}$-term in (5.1.21) and renders the derivative of the Lyapunov function (5.1.20) nonpositive:

$$\dot{V}_1 = -c_1 z_1^2 - c_2 z_2^2 \leq 0 \tag{5.1.23}$$

This implies that the $z = 0, \tilde{\theta} = 0$ equilibrium point of the closed-loop adaptive system consisting of (5.1.19) and (5.1.22) (see block diagram in Figure 5.4)

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \varphi_2(x) \end{bmatrix}\tilde{\theta} \\ \dot{\tilde{\theta}} &= -\gamma\begin{bmatrix} 0 & \varphi_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned} \tag{5.1.24}$$

is globally stable and, in addition, $x(t) \to 0$ as $t \to \infty$.

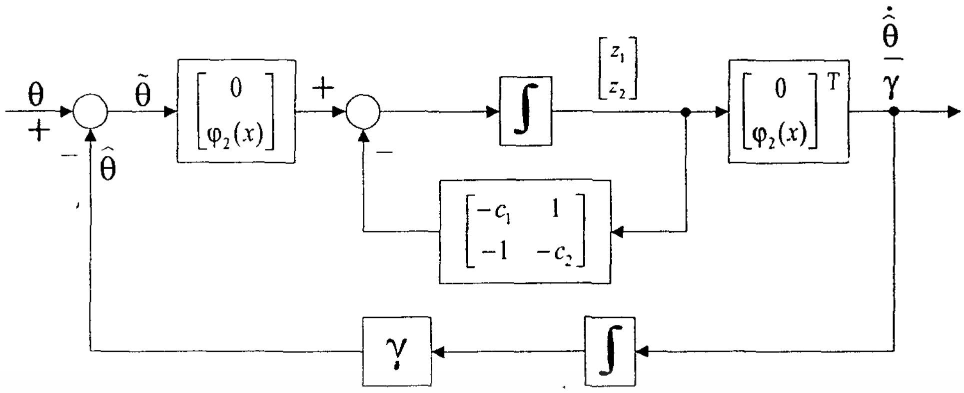

Figure 5.4: The closed-loop adaptive system (5.1.24)

5.2 Adaptive Integrator Backstepping

The adaptive design in example of Chapter 3 was simple because of the matching: The parametric uncertainty was in the span of the control. We now move to the more general case of ***extended matching***, where the parametric uncertainty enters the system one integrator before the control does:

$$\dot{x}_1 = x_2 + \theta\varphi(x_1) \tag{5.2.1a}$$

$$\dot{x}_2 = u. \tag{5.2.1b}$$

We use this example to introduce ***adaptive backstepping***.

If θ were known, we would apply Lemma B.1.2 to design the stabilizing function for x_2 as doing with the static part

$$\alpha_1(x_1, \theta) = -c_1 x_1 - \theta\varphi(x_1), \tag{5.2.2}$$

with the Lyapunov function

$$V_c(x, \theta) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - \alpha_1(x_1, \theta))^2 \tag{5.2.3}$$

whose derivative is rendered negative definite

$$\dot{V}_c(x, \theta) = -c_1 x_1^2 - c_2(x_2 - \alpha_1(x_1, \theta))^2 \tag{5.2.4}$$

by the control

$$u = -c_2(x_2 - \alpha_1(x_1, \theta)) - x_1 + \frac{\partial \alpha_1}{\partial x_1}(x_2 + \theta\varphi). \tag{5.2.5}$$

Since θ is unknown and appears one equation before the control does, we cannot apply Lemma B.1.2 because the dependence of $\alpha_1(x_1) = -c_1 x_1 - \theta\varphi(x_1)$ on the unknown parameter makes it impossible to continue the procedure. However, we can utilize the idea of integrator backstepping.

Step 1. If x_2 were the control, an adaptive controller for (5.2.1a) would be given by:

$$\alpha_1(x_1, \vartheta_1) = -c_1 z_1 - \vartheta_1\varphi(x_1) \tag{5.2.6}$$

$$\dot{\vartheta}_1 = \gamma z_1 \varphi(x_1) \tag{5.2.7}$$

where $z_1 = x_1$.

In the above equations we have replaced the parameter estimate $\hat{\theta}$ with the estimate ϑ_1, which denotes the estimate generated in this design step. As we will see, there will be another estimate generated in the next step. With (5.2.6) and the new error variable $z_2 = x_2 - \alpha_1$, the $\dot{z}_1$-equation becomes

$$\dot{z}_1 = -c_1 z_1 + z_2 + (\theta - \vartheta_1)\varphi. \tag{5.2.8}$$

The derivative of the Lyapunov function

$$V_1(x_1, \vartheta_1) = \frac{1}{2}z_1^2 + \frac{1}{2\gamma}(\theta - \vartheta_1)^2 \tag{5.2.9}$$

along the solutions of (5.2.8) is

$$\begin{aligned} \dot{V}_1 &= z_1\dot{z}_1 - \frac{1}{\gamma}(\theta - \vartheta_1)\dot{\vartheta}_1 \\ &= z_1 z_2 - c_1 z_1^2 + (\theta - \vartheta_1)\left[\varphi z_1 - \frac{1}{\gamma}\dot{\vartheta}_1\right] \\ &= z_1 z_2 - c_1 z_1^2. \end{aligned} \tag{5.2.10}$$

Step 2. The derivative of z_2 is now expressed as

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 \\ &= u - \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_1}{\partial \vartheta_1}\dot{\vartheta}_1. \end{aligned}$$

Substituting (5.2.1a) and the update law (5.2.7) results in

$$\begin{aligned} \dot{z}_2 &= u - \frac{\partial\alpha_1}{\partial x_1}(x_2 + \theta\varphi) - \frac{\partial\alpha_1}{\partial\vartheta}\gamma\varphi z_1 \\ &= u - \frac{\partial\alpha_1}{\partial x_1}x_2 - \frac{\partial\alpha_1}{\partial\vartheta_1}\gamma\varphi z_1 - \theta\frac{\partial\alpha_1}{\partial x_1}\varphi. \end{aligned} \tag{5.2.11}$$

At this point we need to select a Lyapunov function and design u to render its derivative nonpositive. Our first attempt is the augmented Lyapunov function

$$V_2(z_1, z_2, \vartheta_1) = V_1(z_1, \vartheta_1) + \frac{1}{2}z_2^2,$$

whose derivative, using (5.2.10) and (5.2.11), is

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 \\ &= -c_1z_1^2 + z_2\left[z_1 + u - \frac{\partial\alpha_1}{\partial x_1}x_2 - \frac{\partial\alpha_1}{\partial\vartheta_1}\gamma\varphi z_1 - \theta\frac{\partial\alpha_1}{\partial x_1}\varphi\right]. \end{aligned}$$

The control u should now be able to cancel the indefinite terms in $\dot{V}_2$. To deal with the terms containing the unknown parameter θ, we will try to employ the existing estimate ϑ_1:

$$u = -z_1 - c_2z_2 + \frac{\partial\alpha_1}{\partial x_1}x_2 + \frac{\partial\alpha_1}{\partial\vartheta_1}\gamma\varphi z_1 + \vartheta_1\frac{\partial\alpha_1}{\partial x_1}\varphi.$$

From the resulting derivative

$$\dot{V}_2 = -c_1z_1^2 - c_2z_2^2 - (\theta - \vartheta_1)\frac{\partial\alpha_1}{\partial x_1}\varphi z_2,$$

we see that we have no design freedom left to cancel the $(\theta - \vartheta_1)$-term. To overcome this difficulty, we replace ϑ_1 in the expression for u with a *new* estimate ϑ_2:

$$u = -z_1 - c_2z_2 + \frac{\partial\alpha_1}{\partial x_1}x_2 + \frac{\partial\alpha_1}{\partial\vartheta_1}\gamma\varphi z_1 + \vartheta_2\frac{\partial\alpha_1}{\partial x_1}\varphi. \tag{5.2.12}$$

With the choice (5.2.12), the $\dot{z}_2$-equation becomes

$$\dot{z}_2 = -c_2z_2 - z_1 - (\theta - \vartheta_2)\frac{\partial\alpha_1}{\partial x_1}\varphi. \tag{5.2.13}$$

The presence of the new parameter estimate ϑ_2 suggests the following augmentation of the Lyapunov function:

$$\begin{aligned} V_2(z_1, z_2, \vartheta_1, \vartheta_2) &= V_1 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}(\theta - \vartheta_2)^2 \\ &= \frac{1}{2}(z_1^2 + z_2^2) + \frac{1}{2\gamma}\left[(\theta - \vartheta_1)^2 + (\theta - \vartheta_2)^2\right]. \end{aligned} \tag{5.2.14}$$

The derivative of V_2 is

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 - \frac{1}{\gamma}(\theta - \vartheta_2)\dot{\vartheta}_2 \\
&= z_1 z_2 - c_1 z_1^2 + z_2\left[-c_2 z_2 - z_1 - (\theta - \vartheta_2)\frac{\partial\alpha_1}{\partial x_1}\varphi\right] - \frac{1}{\gamma}(\theta - \vartheta_2)\dot{\vartheta}_2 \\
&= -c_1 z_1^2 - c_2 z_2^2 - (\theta - \vartheta_2)\left(\frac{\partial\alpha_1}{\partial x_1}\varphi z_2 + \frac{1}{\gamma}\dot{\vartheta}_2\right).
\end{aligned} \tag{5.2.15}$$

Now the $(\theta - \vartheta_2)$-term can be eliminated with the update law

$$\dot{\vartheta}_2 = -\gamma\frac{\partial\alpha_1}{\partial x_1}\varphi z_2, \tag{5.2.16}$$

which yields

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2. \tag{5.2.17}$$

The equations (5.2.13) and (5.2.16) along with (5.2.8) and (5.2.7) form the error system representation of the resulting closed-loop adaptive system:

$$\begin{aligned}
\dot{z}_1 &= -c_1 z_1 + z_2 + (\theta - \vartheta_1)\varphi \\
\dot{z}_2 &= -c_2 z_2 - z_1 - (\theta - \vartheta_2)\frac{\partial\alpha_1}{\partial x_1}\varphi \\
\dot{\vartheta}_1 &= \gamma\varphi z_1 \\
\dot{\vartheta}_2 &= -\gamma\frac{\partial\alpha_1}{\partial x_1}\varphi z_2.
\end{aligned} \tag{5.2.18}$$

The matrix form of this system,

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \varphi & 0 \\ 0 & -\frac{\partial\alpha_1}{\partial x_1}\varphi \end{bmatrix}\begin{bmatrix} \theta - \vartheta_1 \\ \theta - \vartheta_2 \end{bmatrix} \\
\frac{d}{dt}\begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} &= \gamma\begin{bmatrix} \varphi & 0 \\ 0 & -\frac{\partial\alpha_1}{\partial x_1}\varphi \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix},
\end{aligned} \tag{5.2.19}$$

makes its properties more visible:

- The constant system matrix has negative terms along its diagonal, while its off-diagonal terms are skew-symmetric, and
- the matrix that multiplies the parameter errors in the $\dot{z}$-equation is used in the update laws for the parameter estimates.

The stability properties of (5.2.19) follow from (5.2.14) and (5.2.17): The LaSalle-Yoshizawa theorem (Theorem A.1.1) establishes that $z_1, z_2, \vartheta_1, \vartheta_2$ are bounded, and $z \to 0$ as $t \to \infty$. Since $z_1 = x_1$, x_1 is also bounded and converges to zero. The boundedness of x_2 then follows from the boundedness of α_1 (defined in (5.2.6)) and the fact that $x_2 = z_2 + \alpha_1$. Using (5.2.12) we conclude that the control u is also bounded. Finally, we note that the regulation of z and x_1 does not imply the regulation of x_2: From $z_2 = x_2 - \alpha_1$ and (5.2.6) we see that $x_2 + \vartheta_1\varphi(0)$ will converge to zero. Thus, x_2 is not guaranteed to converge to zero unless $\varphi(0) = 0$. However, x_2 will converge to a constant value:

$$\lim_{t\to\infty} x_2 = -\theta\varphi(0) \triangleq x_2^e. \tag{5.2.20}$$

This can be seen from (5.2.1a): Since x_1 and $\dot{x}_1$ converge to zero, so does $x_2 + \theta\varphi(0)$.

With the above example we have illustrated the idea of adaptive backstepping.

To formulate it as a design tool analogous to an integrator backstepping in Lemma B.1.2, we start with the assumption that an adaptive controller is known for an initial system.

Assumption 5.2.1 *Consider the system*

$$\dot{x} = f(x) + F(x)\theta + g(x)u, \tag{5.2.21}$$

where $x \in \mathbb{R}^n$ is the state, $\theta \in \mathbb{R}^q$ is a vector of unknown constant parameters, and $u \in \mathbb{R}$ is the control input. There exists an adaptive controller

$$\begin{aligned} u &= \alpha(x,\vartheta) \\ \dot{\vartheta} &= T(x,\vartheta), \end{aligned} \tag{5.2.22}$$

with parameter estimate $\vartheta \in \mathbb{R}^q$, and a smooth function $V(x,\vartheta) : \mathbb{R}^{n+q} \to \mathbb{R}$ which is positive definite and radially unbounded in the variables $(x, \vartheta - \theta)$ such that for all $(x,\vartheta) \in \mathbb{R}^{n+q}$:

$$\begin{aligned} \frac{\partial V}{\partial x}(x,\vartheta)[f(x) + F(x)\theta + g(x)\alpha(x,\vartheta)] + \frac{\partial V}{\partial \vartheta}(x,\vartheta)T(x,\vartheta) \\ \leq -W(x,\vartheta) \leq 0, \end{aligned} \tag{5.2.23}$$

where $W : \mathbb{R}^{n+q} \to \mathbb{R}$ is positive semidefinite. □

Under this assumption, the control (5.2.22), applied to the system (5.2.21), guarantees global boundedness of $x(t)$, $\vartheta(t)$ and, by the LaSalle-Yoshizawa theorem (Theorem A.1.1), regulation of $W(x(t),\vartheta(t))$. Adaptive backstepping allows us to achieve the same properties for the augmented system.

Lemma 5.2.2 (Adaptive Backstepping) *Let the system* (5.2.21) *be augmented by an integrator,*

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)\xi \\ \dot{\xi} &= u \end{aligned} \tag{5.2.24}$$

where $\xi \in \mathbb{R}$. Consider for this system the dynamic feedback controller

$$\begin{aligned} u &= -c(\xi - \alpha(x,\vartheta)) + \frac{\partial \alpha}{\partial x}(x,\vartheta)[f(x) + F(x)\bar{\vartheta} + g(x)\xi] + \frac{\partial \alpha}{\partial \vartheta}T(x,\vartheta) \\ &\quad - \frac{\partial V}{\partial x}(x,\vartheta)g(x), \quad c > 0 \end{aligned} \tag{5.2.25}$$

$$\dot{\vartheta} = T(x,\vartheta) \tag{5.2.26}$$

$$\dot{\bar{\vartheta}} = -\Gamma\left[\frac{\partial \alpha}{\partial x}(x,\vartheta)F(x)\right]^T(\xi - \alpha(x,\vartheta)), \tag{5.2.27}$$

where $\bar{\vartheta}$ is a new estimate of θ, $\Gamma = \Gamma^T > 0$ is the adaptation gain matrix. Under Assumption 5.2.1, *this adaptive controller guarantees global boundedness of $x(t)$, $\xi(t)$, $\vartheta(t)$, $\bar{\vartheta}(t)$ and regulation of $W(x(t),\vartheta(t))$ and $\xi(t) - \alpha(x(t),\vartheta(t))$. These properties can be established with the Lyapunov function*

$$V_a(x,\xi,\vartheta,\bar{\vartheta}) = V(x,\vartheta) + \frac{1}{2}[\xi - \alpha(x,\vartheta)]^2 + \frac{1}{2}(\theta - \bar{\vartheta})^T\Gamma^{-1}(\theta - \bar{\vartheta}). \tag{5.2.28}$$

Proof *With the error variable $z = \xi - \alpha(x,\vartheta)$,* (5.2.24) *is rewritten as*

$$\dot{x} = f(x) + F(x)\theta + g(x)[\alpha(x,\vartheta) + z] \tag{5.2.29a}$$

$$\dot{z} = u - \frac{\partial \alpha}{\partial x}[f(x) + F(x)\theta + g(x)(\alpha(x,\vartheta) + z)] - \frac{\partial \alpha}{\partial \vartheta}T(x,\vartheta). \tag{5.2.29b}$$

Note that in (5.2.29b) *the derivative of ϑ was replaced by the update law* (5.2.26). *Introducing a new parameter estimate $\bar{\vartheta}$, we augment the Lyapunov function:*

$$V_a(x,\xi,\vartheta,\bar{\vartheta}) = V(x,\vartheta) + \frac{1}{2}z^2 + \frac{1}{2}(\theta - \bar{\vartheta})^T\Gamma^{-1}(\theta - \bar{\vartheta}). \tag{5.2.30}$$

Using (5.2.23), *it is easy to show that the derivative of* (5.2.30) *satisfies*

$$\begin{aligned}
\dot{V}_a &= \frac{\partial V}{\partial x}(f + F\theta + g\alpha + gz) + \frac{\partial V}{\partial \vartheta}T \\
&\quad + z\left[u - \frac{\partial \alpha}{\partial x}(f + F\theta + g(\alpha + z)) - \frac{\partial \alpha}{\partial \vartheta}T\right] - \dot{\bar{\vartheta}}^T \Gamma^{-1}(\theta - \bar{\vartheta}) \\
&= \frac{\partial V}{\partial x}(f + F\theta + g\alpha) + \frac{\partial V}{\partial \vartheta}T \\
&\quad + z\left[u - \frac{\partial \alpha}{\partial x}(f + F\theta + g(\alpha + z)) - \frac{\partial \alpha}{\partial \vartheta}T + \frac{\partial V}{\partial x}g\right] - \dot{\bar{\vartheta}}^T \Gamma^{-1}(\theta - \bar{\vartheta}) \\
&\leq -W(x, \vartheta) + z\left[u - \frac{\partial \alpha}{\partial x}(f + F\bar{\vartheta} + g(\alpha + z)) - \frac{\partial \alpha}{\partial \vartheta}T + \frac{\partial V}{\partial x}g\right] \\
&\quad - \left[\frac{\partial \alpha}{\partial x}Fz + \dot{\bar{\vartheta}}^T \Gamma^{-1}\right](\theta - \bar{\vartheta}).
\end{aligned} \tag{5.2.31}$$

The $(\theta - \bar{\vartheta})$*-term is now eliminated with the update law*

$$\dot{\bar{\vartheta}} = -\Gamma\left(\frac{\partial \alpha}{\partial x}F\right)^T z. \tag{5.2.32}$$

and the control (5.2.25) *is chosen to make the bracketed term multiplying* z *in* (5.2.31) *equal to* $-cz$*:*

$$u = -cz + \frac{\partial \alpha}{\partial x}(f + F\bar{\vartheta} + g(\alpha + z)) + \frac{\partial \alpha}{\partial \vartheta}T - \frac{\partial V}{\partial x}g. \tag{5.2.33}$$

This results in the desired nonpositivity of $\dot{V}_a$*:*

$$\dot{V}_a \leq -W(x, \vartheta) - cz^2 \leq 0. \tag{5.2.34}$$

From (5.2.28) *and* (5.2.34) *we conclude that* $V(x, \vartheta)$, $\bar{\vartheta}$ *and* z *are bounded. By Assumption* 5.2.1, *this means that* $x(t)$ *and* $\vartheta(t)$ *are bounded. Hence,* $\xi = z + \alpha(x, \vartheta)$ *and* u *are bounded. By LaSalle-Yoshizawa theorem (Theorem A.1.1), the boundedness of all the signals combined with* (5.2.34) *proves the regulation of* $W(x(t), \vartheta(t))$ *and* $z(t)$. □

5.3 Adaptive Block Backstepping

We now extend the Adaptive Backstepping lemma (Lemma 5.2.2) by augmenting the initial system with a relative-degree-one nonlinear system whose zero dynamics subsystem is ISS just like we did in Lemmas B.1.2 and 4.1.6. The adaptive counterpart of

Assumption B.1.1 was Assumption 5.2.1. We now formulate the adaptive counterpart of Assumption 4.1.2, with analogous changes in the properties of $V(x, \vartheta)$ from Assumption 5.2.1.

Assumption 5.3.1 *Suppose Assumption 5.2.1 is valid, but $V(x, \vartheta)$ is only positive semidefinite, and the closed-loop system* (5.2.21) *with the adaptive controller* (5.2.22) *has the property that $x(t)$ and $\vartheta(t)$ are bounded if $V(x(t), \vartheta(t))$ is bounded.* □

Under this assumption, the control (5.2.22), applied to the system (5.2.21), guarantees global boundedness of $x(t), \vartheta(t)$ and, by Lemma A.3.5, regulation of $W(x(t), \vartheta(t))$.

Lemma 5.3.2 (Adaptive Block Backstepping) *Let the system* (5.2.21) *be augmented by a nonlinear system which is linear in the unknown parameter vector θ*

$$\dot{x} = f(x) + F(x)\theta + g(x)y \tag{5.3.1a}$$

$$\dot{\xi} = m(x,\xi) + M(x,\xi)\theta + \beta(x,\xi)u, \qquad y = h(\xi), \tag{5.3.1b}$$

where $\xi \in \mathbb{R}^q$, and suppose that (5.3.1b) *has relative degree one uniformly in x and that its zero dynamics subsystem is ISS with respect to y and x. Under Assumption 5.3.1, the feedback control*

$$\begin{aligned} u = {} & \left[\frac{\partial h}{\partial \xi}(\xi)\beta(x,\xi)\right]^{-1} \left\{ -c(y-\alpha(x,\vartheta)) - \frac{\partial h}{\partial \xi}(\xi)\left[m(x,\xi) + M(x,\xi)\bar{\vartheta}\right] \right. \\ & \left. + \frac{\partial \alpha}{\partial x}(x,\vartheta)\left[f(x) + F(x)\bar{\vartheta} + g(x)y\right] + \frac{\partial \alpha}{\partial \vartheta}T(x,\vartheta) - \frac{\partial V}{\partial x}(x,\vartheta)g(x)\right\}, \end{aligned} \tag{5.3.2}$$

with $c > 0$ and $\bar{\vartheta}$ a new estimate of θ, along with the update laws

$$\dot{\vartheta} = T(x,\vartheta) \tag{5.3.3}$$

$$\dot{\bar{\vartheta}} = \Gamma\left[\frac{\partial h}{\partial \xi}(\xi)M(x,\xi) - \frac{\partial \alpha}{\partial x}(x,\vartheta)F(x)\right]^T (y - \alpha(x,\vartheta)), \tag{5.3.4}$$

with the adaptation gain matrix $\Gamma = \Gamma^T > 0$, guarantees global boundedness of $x(t), \xi(t), \vartheta(t), \bar{\vartheta}(t)$ and regulation of $W(x(t), \vartheta(t))$ and $\xi(t) - \alpha(x(t), \vartheta(t))$.

Proof *As in Lemma 4.1.6, we employ the change of coordinates* $(y, \zeta) = (h(\xi), \phi(x, \xi))$, *with* $\frac{\partial \phi}{\partial \xi}\beta \equiv 0$, *to transform* (5.3.1b) *into the normal form*

$$\begin{aligned} \dot{y} &= \frac{\partial h}{\partial \xi}(\xi)[m(x,\xi) + M(x,\xi)\theta + \beta(x,\xi)u] \\ &= v + \frac{\partial h}{\partial \xi}(\xi)M(x,\xi)(\theta - \bar{\vartheta}) \qquad (5.3.5a) \\ \dot{\zeta} &= \frac{\partial \phi}{\partial x}(x,\xi)[f(x) + F(x)\theta + g(x)y] + \frac{\partial \phi}{\partial \xi}(x,\xi)[m(x,\xi) + M(x,\xi)\theta] \\ &\triangleq \Phi_0(x,y,\zeta) + \Phi(x,y.\zeta)\theta. \qquad (5.3.5b) \end{aligned}$$

Introducing a new parameter estimate $\bar{\vartheta}$, *we use the feedback transformation*

$$u = \left(\frac{\partial h}{\partial \xi}\beta\right)^{-1}\left\{v - \frac{\partial h}{\partial \xi}[m + M\bar{\vartheta}]\right\} \qquad (5.3.6)$$

to rewrite (5.3.1a) *and* (5.3.5a) *as*

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)y \qquad (5.3.7a) \\ \dot{y} &= v + \frac{\partial h}{\partial \xi}(\xi)M(x,\xi)(\theta - \bar{\vartheta}). \qquad (5.3.7b) \end{aligned}$$

We now apply Assumption 5.2.1 to (5.3.7). *The only difference between* (5.3.7) *and* (5.2.24) *is the presence of the additional parameter error term* $\frac{\partial h}{\partial \xi}M(\theta - \bar{\vartheta})$ *in* (5.3.7b). *This term can be eliminated in* $\dot{V}_a$ *by adding the term* $-\Gamma\left(\frac{\partial h}{\partial \xi}M\right)^T(y - \alpha)$ *to the update law* (5.2.27):

If θ *were known we would pick* $u = \frac{1}{b_0}[v - M(x,\xi)\theta]$ *so* (5.3.5a) *can be written in the form*

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)y \\ \dot{y} &= \dot{x_3} = v \end{aligned}$$

in which we could apply integrator backstepping. But since θ *is unknown, therefore we introduce a new parameter estimate* $\bar{\vartheta}$ *with the new clf*

$$
\begin{aligned}
V_2(x,y,\vartheta,\bar{\vartheta}) &= V_1(x,\vartheta) + \frac{1}{2}[y-\alpha(x,\vartheta)]^2 + \frac{1}{2}(\theta-\bar{\vartheta})^T\Gamma^{-1}(\theta-\bar{\vartheta}) \\
\dot{V}_2 &= \frac{\partial V}{\partial x}(f + F\theta + g\alpha + gz_3) + \frac{\partial V}{\partial \vartheta}T \\
&\quad + z_3\left[\underbrace{v + \frac{\partial h}{\partial \xi}(\xi)M(x,\xi)(\theta-\bar{\vartheta})}_{\dot{y}} \underbrace{- \frac{\partial \alpha}{\partial x}(f + F\theta + g(\alpha + z_3)) - \frac{\partial \alpha}{\partial \vartheta}T}_{\dot{\alpha}}\right] \\
&\quad - \dot{\bar{\vartheta}}^T\Gamma^{-1}(\theta-\bar{\vartheta}) \\
&= \frac{\partial V}{\partial x}(f + F\theta + g\alpha) + \frac{\partial V}{\partial \vartheta}T \\
&\quad + z_3\left[v + \frac{\partial h}{\partial \xi}(\xi)M(x,\xi)(\theta-\bar{\vartheta}) - \frac{\partial \alpha}{\partial x}(f + F\theta + g(\alpha+z)) - \frac{\partial \alpha}{\partial \vartheta}T + \frac{\partial V}{\partial x}g\right] \\
&\quad - \dot{\bar{\vartheta}}^T\Gamma^{-1}(\theta-\bar{\vartheta}) \\
&\le -W(x,\vartheta) + z_3\left[v - \frac{\partial \alpha}{\partial x}(f + F\theta + g(\alpha+z)) - \frac{\partial \alpha}{\partial \vartheta}T + \frac{\partial V}{\partial x}g\right] \\
&\quad - \Big[\frac{\partial \alpha}{\partial x}Fz_3 \underbrace{- \frac{\partial h}{\partial \xi}Mz_3}_{added} + \dot{\bar{\vartheta}}^T\Gamma^{-1}\Big](\theta-\bar{\vartheta})
\end{aligned}
$$

with

$$
v = -c_3 \underbrace{z_3}_{y-\alpha} + \frac{\partial \alpha}{\partial x}(f + F\bar{\vartheta} + g\underbrace{(\alpha + z_3)}_{y}) + \frac{\partial \alpha}{\partial \vartheta}T - \frac{\partial V}{\partial x}g
$$

or

$$
u = \left[\frac{\partial h}{\partial \xi}\beta\right]^{-1}\left[v - \frac{\partial h}{\partial \xi}[m + M\bar{\vartheta}]\right].
$$

Therefore, the control law

$$
\begin{aligned}
u &= \left[\frac{\partial h}{\partial \xi}\beta(x,\xi)\right]^{-1}\Big\{-c_3(y-\alpha(x,\vartheta)) - \frac{\partial h}{\partial \xi}(\xi)[m(x,\xi) + M(x,\xi)\bar{\vartheta}] \\
&\quad + \frac{\partial \alpha}{\partial x}(x,\vartheta)[f(x) + F(x)\bar{\vartheta} + g(x)y] + \frac{\partial \alpha}{\partial \vartheta}T(x,\vartheta) - \frac{\partial V}{\partial x}(x,\vartheta)g(x)\Big\},
\end{aligned}
$$

along with the update laws

$$
\begin{aligned}
\dot{\vartheta} &= T(x,\vartheta) \\
\dot{\bar{\vartheta}} &= \Gamma\left[\frac{\partial h}{\partial \xi}M(x,\xi) - \frac{\partial \alpha}{\partial x}(x,\theta)F(x)\right]^T (y - \alpha(x,\vartheta))
\end{aligned}
$$

will render the system stable. Combining this modification with (5.3.6), *we see that the resulting adaptive controller is given by* (5.3.2)-(5.3.4). *This guarantees the boundedness* *of* $x, \vartheta, \bar{\vartheta}, z$ *and the regulation of* $W(x, \vartheta)$ *and* z. *Hence,* $y = z + \alpha(x, \vartheta)$ *is bounded. Then, from* (5.3.5b) *and the ISS property of the zero dynamics,* ζ *is also bounded, and thus* ξ *and* u *are bounded* □

Chapter 6

Tuning Functions Design

6.1 Introduction

The adaptive backstepping designs for a plant with unknown parameters is a starting point for more elaborate adaptive designs which lead to new properties of the designed controller and the resulting feedback system. One of the improvement to be achieved with the tuning functions design in this chapter is the reduction of the dynamic order of the adaptive controller to its minimum: The number of parameter estimates is equal to the number of unknown parameters. This minimum-order design is advantageous not only for implementation, but also because it guarantees the strongest achievable stability and convergence properties.

In the tuning functions procedure the parameter update law is designed recursively. At each consecutive step, we design a tuning function as a potential update law. In contrast to adaptive backstepping. these intermediate update laws are not implemented. Instead, the controller uses them to compensate for the effect of parameter estimation transients. Only the final tuning function is used as the parameter update law.

6.2 Adaptive Control Lyapunov Functions

The basic idea of the Lyapunov approach to adaptive control is to design a control law and a parameter update law to guarantee that the derivative of a suitable Lyapunov function is nonpositive. We are therefore sent to search for a tripple: Lyapunov function, control law, and update law. For a class of nonlinear systems called parametric-strict-feedback systems we will be able to make this search systematic.

To begin with, let us investigate the possibility of adaptive design for the system

$$\dot{x} = f(x) + F(x)\theta + g(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R} \tag{6.2.1}$$

where $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters, and $f(x), F(x)$ and $g(x)$ are smooth. For simplicity let $f(0) = 0$, $F(0) = 0$, so that $x = 0$ is an equilibrium of the uncontrolled plant.

6.2.1 Departure from certainty equivalence

Much of the traditional adaptive control employs some form of "certainty equivalence" thinking. Following this path one first performs a design for the case when the exact value of θ is known. Suppose that this nontrivial task is completed and that its result is a feedback control $u = \alpha_c(x, \theta)$ which stabilizes the equilibrium $x = 0$ with respect to a known Lyapunov function $V_c(x, \theta)$. The subscript 'c' stands for "certainty equivalence". We know that $V_c(x, \theta)$ is positive definite and radially unbounded in x for all θ, and that there exists a function $W(x, \theta)$, which is also positive definite in x for all θ, such that

$$\frac{\partial V_c}{\partial x}[f(x) + F(x)\theta + g(x)\alpha_c(x, \theta)] \leq -W(x, \theta) \tag{6.2.2}$$

How can we exploit the knowledge of $\alpha_c(x, \theta)$ and $V_c(x, \theta)$ for adaptive design when θ is not known? The certainty equivalence idea is to replace θ by an estimate $\hat{\theta}(t)$ obtained from a parameter update law

$$\dot{\hat{\theta}} = \Gamma\tau(x, \hat{\theta}) \tag{6.2.3}$$

where the adaptation gain matrix Γ is positive definite. We want to select u and τ to guarantee that the derivative of a Lyapunov function is nonpositive. For the system (6.2.1),(6.2.3), a Lyapunov function candidate is

$$V(x,\hat{\theta}) = V_c(x,\hat{\theta}) + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{6.2.4}$$

where the "certainty equivalence" from of V_c is augmented by a term quadratic in the parameter estimation error

$$\tilde{\theta} = \theta - \hat{\theta} \tag{6.2.5}$$

Upon the substitution of $F(x)\theta = F(x)\hat{\theta} + F(x)\tilde{\theta}$, the derivative of $V(x,\hat{\theta})$ along the solutions of (6.2.1), (6.2.3) is

$$\dot{V} = \frac{\partial V_c}{\partial x}[f(x) + F(x)\hat{\theta} + g(x)u] + \frac{\partial V_c}{\partial x}\Gamma\tau + \tilde{\theta}^T\left(\frac{\partial V_c}{\partial x}F(x)\right)^T - \tilde{\theta}^T\tau \tag{6.2.6}$$

To eliminate the indefinite dependence of $\dot{V}$ on the unknown parameter error $\tilde{\theta}$, we select τ to cancel the last two terms in (6.2.6):

$$\tau(x,\hat{\theta}) = \left(\frac{\partial V_c}{\partial x}F(x)\right)^T \tag{6.2.7}$$

With this choice of $/ta$, the expression (6.2.6) is reduced to

$$\dot{V} = \frac{\partial V_c}{\partial x}[f(x) + F(x)\hat{\theta} + g(x)u] + \frac{\partial V_c}{\partial x}\Gamma\left(\frac{\partial V_c}{\partial x}F(x)\right)^T \tag{6.2.8}$$

Our next task is to select a control law $u = \alpha(x,\hat{\theta})$ to make $\dot{V}$ nonpositive. The "certainty equivalence" control $u = \alpha_c(x,\hat{\theta})$ fails to achieve this because then (6.2.2) and (6.2.8) yield

$$\dot{V} \leq -W(x,\hat{\theta}) + \frac{\partial V_c}{\partial \hat{\theta}}\Gamma\left(\frac{\partial V_c}{\partial x}F(x)\right)^T \tag{6.2.9}$$

Clearly, $\dot{V}$ is not nonpositive because a sign-indefinite term is added to $-W(x,\hat{x})$. In search of a better control law $\alpha(x,\hat{\theta})$. we augment $\alpha_c(x,\hat{\theta})$ by $\alpha_T(x,\hat{\theta})$,

$$\alpha(x,\hat{\theta}) = \alpha_c(x,\hat{\theta}) + \alpha_T(x,\hat{\theta}) \tag{6.2.10}$$

The substitution of (6.2.10) into (6.2.8) shows that the desired nonpositivity $\dot{V} \leq -W(x,\hat{\theta})$ will be achieved if α_T can be found to satisfy

$$\frac{\partial V_c}{\partial x}g(x)\alpha_T(x,\hat{\theta}) + \frac{\partial V_c}{\partial \hat{\theta}}\Gamma\left(\frac{\partial V_c}{\partial x}F(x)\right)^T = 0 \tag{6.2.11}$$

This condition for α_T demonstrates the difficulty of adaptive design for a general nonlinear system (6.2.1). It is easy to see that α_T satisfying (6.2.11) is unlikely to exist The scalar quantity $\frac{\partial V_c}{\partial x}g(x)$ may be zero at a set of points. Still, the condition (6.2.11) is of interest because of an important special case, which will be the starting point of our recursive design. The special case is the "extended matching" studied in the previous chapter. In this case, a smooth vector-valued function $\varphi : \mathbb{R}^{n+p} \to \mathbb{R}^p$ is known such that $\frac{\partial V_c}{\partial \hat{\theta}}$ can be factored as follows:

$$\frac{\partial V_c}{\partial \hat{\theta}} = \frac{\partial V_c}{\partial x} g(x) \varphi(x, \hat{\theta})^T \tag{6.2.12}$$

Then, irrespective of the zeros of $\frac{\partial V_c}{\partial x}g(x)$, an α_T which satisfies (6.2.11) is

$$\alpha_T(x, \hat{\theta}) = -\varphi(x, \hat{\theta})^T \Gamma \left(\frac{\partial V_c}{\partial x} F(x) \right)^T = -\varphi(x, \hat{\theta})^T \Gamma \tau(x, \hat{\theta}) \tag{6.2.13}$$

We observe that, in addition to its "certainty equivalence" part α_c, the adaptive control law α contains a part α_T which is proportional to τ, that is, to $\dot{\hat{\theta}}$ (see (6.2.3), (6.2.10), and (6.2.13)). In this way, the adaptive control law takes into account the parameter estimation transients. When the parameter estimate is constant, the control law reduces to the "certainty equivalence" control. Let eu examine an example of a system for which (6.2.12) is satisfied.

Example 6.2.1 *Consider the problem of designing an adaptive controller for the system*

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^T \theta \\ \dot{x}_2 &= u \end{aligned} \tag{6.2.14}$$

where $\theta = [\theta_1, \theta_2]^T$ is an unknown constant parameter vector, and the vector-valued function $\varphi(x_1) = [\varphi_1(x_1), \varphi_2(x_1)]^T$ is known and smooth. We dealt with this system in the extended-matching design. If the parameter θ were known, backstepping would result in the θ-dependent change of coordinates

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 + \varphi(x_1)^T \theta + c_1 x_1 \end{aligned} \tag{6.2.15}$$

and the control law

$$u = \alpha_c(x,\theta) = -z_1 - c_2 z_2 - \left(\frac{\partial\varphi^T}{\partial x_1} + c_1\right)\left(x_2 + \varphi(x_1)^T\theta\right) \tag{6.2.16}$$

which $c_1, c_2 > 0$, *which results in the closed-loop system*

$$\dot{z} = Az, \quad A = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \tag{6.2.17}$$

Due to the structure of A, *an appropriate Lyapunov function is*

$$v_c(x,\theta) = \frac{1}{2} z(x,\theta)^T z(x,\theta) \tag{6.2.18}$$

Observing from (6.2.1) *and* (6.2.14) *that*

$$f(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \quad F(x) = \begin{bmatrix} \varphi x_1{}^T \\ 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6.2.19}$$

and evaluating

$$\frac{\partial V_c}{\partial x} = z^T \begin{bmatrix} 1 & 0 \\ \frac{\partial\varphi^T}{\partial x_1}\theta & 1 \end{bmatrix} \tag{6.2.20}$$

with (6.2.19), (6.2.20), *and* (6.2.16), *it is easy to show that*

$$\frac{\partial V_c}{\partial x}[f(x) + F(x)\theta + g(x)\alpha_c(x,\theta)] = -c_1 z_1^2 - c_2 z_2^2 \tag{6.2.21}$$

Let us now evaluate the partial derivatives appearing in (6.2.11):

$$\frac{\partial V_c}{\partial \theta} = z^T e_2 \varphi^T = z_2 \varphi^T \tag{6.2.22}$$

$$\frac{\partial V_c}{\partial x} g = z^T e_2 = z_2 \tag{6.2.23}$$

where (6.2.23) *is immediate from* (6.2.20) *and* (6.2.19). *A comparison of* (6.2.22) *and* (6.2.23) *reveals that* $\frac{\partial V_c}{\partial \theta} = \frac{\partial V_c}{\partial x} g \varphi^T$, *so that* α_c *is given by* (6.2.13):

$$\alpha_\tau(x,\hat{\theta}) = -\varphi^T\Gamma\left(\frac{\partial V_c}{\partial x}F(x)\right)^T = -\varphi^T\Gamma\varphi\left[\ 1, \ \ \frac{\partial\varphi^T}{\partial x_1}\hat{\theta} + c_1\ \right] z \tag{6.2.24}$$

Taking for simplicity $\Gamma = I$, *the resulting adaptive control law is*

$$\begin{aligned} u &= \alpha(x,\hat{\theta}) = -z_1 - c_2 z_2 - \left(c_1 + \frac{\partial\varphi^T}{\partial x_1}\theta\right)\left(x_2 + \varphi(x_1)^T\theta\right) \\ &\quad -\varphi^T\varphi\left[1, \ \frac{\partial\varphi^T}{\partial x_1}\hat{\theta} + c_1\right] z \end{aligned} \tag{6.2.25}$$

and the corresponding parameter update law (6.2.7) *is*

$$\dot{\hat{\theta}} = \tau(x,\hat{\theta}) = \left(\frac{\partial V_c}{\partial x}F(x)\right)^T = \varphi\left[\begin{array}{ccc} 1 & , & \frac{\partial \varphi^T}{\partial x_1}\hat{\theta} + c_1 \end{array}\right] z \tag{6.2.26}$$

Note that in (6.2.25) *and* (6.2.26) *we use* $z(x,\hat{\theta})$ *instead of* $z(x,\theta)$. *With the choice of* α *and* τ *given by* (6.2.25) *and* (6.2.26), *the derivative* $\dot{V}$ *of the Lyapunov function* $V(x,\hat{\theta}) = \frac{1}{2}z(x,\hat{\theta})^T z(x,\hat{\theta}) + \frac{1}{2}\tilde{\theta}^T\tilde{\theta}$ *is guaranteed to be nonpositive:* $\dot{V} = -c_1 z_1^2 - c_2 z_2^2$. *This assures that both* x *and* $\hat{\theta}$ *are bounded. A standard argument using the LaSalle-Yoshizawa Theorem proves that also* $x(t) \to 0$. □

In the above example, the desired factorization (6.2.12) of $\frac{\partial V_c}{\partial \hat{\theta}}$ is a consequence of a particular feature of the system (6.2.14). The unknown parameter appears in the first, while the control appears only in the second equation. It is not hard to see that the same factorization (6.2.12) would be a possible for a higher-order plant, provided that *the unknown parameter is separated from the control input by at most one integrator.* So the factorization (6.2.12) is not a fortuitous event, but a structural property. For systems with this "extended matching" property, the above simple adaptive design is feasible. However, most systems fail to posses the "extended matching" property.

A benchmark example is the third-order system

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^T\theta \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \tag{6.2.27}$$

which has the form of (6.2.14) augmented by an integrator. In this system, θ and u are separated by two integrators and we unable to find α_T which satisfies (6.2.11). We will solve this problem with a recursive design which will circumvent the obstacle posed by the restrictive condition (6.2.11).

6.2.2 Certainty equivalence for a modified system

Condition (6.2.11) was dictated by our choice of the Lyapunov function $V_c(x, \hat{\theta})$ as the "certainty equivalence" form of $V_c(x, \theta)$. The only good thing we know about $V_c(x, \hat{\theta})$ is that it works when the factorization (6.2.12) is possible. Otherwise, we do not know how to remove the indefinite term preventing the nonpositivity of $\dot{V}$ in (6.2.9). Having recognized that a cause of our difficulties is $V_c(x, \theta)$, we now embark on a search for Lyapunov functions more suitable for adaptive control. The key idea is to counteract the effect of $\dot{\hat{\theta}}$ and thus prevent the parameter estimate transients from destroying the nonpositivity of the Lyapunov derivative.

We say that the system

$$\dot{x} = f(x) + F(x)\theta + g(x)u \tag{6.2.28}$$

is **globally adaptively stabilizable** if there exist a function $\alpha(x, \hat{\theta})$ smooth on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^p$ with $\alpha(0, \hat{\theta}) \equiv 0$, a smooth function $\tau(x, \hat{\theta})$, and a positive definite symmetric $p \times p$ matrix Γ, such that the dynamic controller

$$u = \alpha(x, \hat{\theta}) \tag{6.2.29}$$

$$\dot{\hat{\theta}} = \Gamma\tau(x, \hat{\theta}) \tag{6.2.30}$$

guarantees that the solution $(x(t), \hat{\theta}(t))$ is globally bounded, and $x(t) \to 0$ as $t \to \infty$, for all $\theta \in \mathbb{R}^p$.

Our approach is to replace the problem of adaptive stabilization of the original system (6.2.28) by a problem of nonadaptive stabilization of a modified system.

Definition 6.2.2 *A smooth function $V_a : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+$, positive definite and radially unbounded in x for each θ, is called an **adaptive control Lyapunov function (aclf)** for* (6.2.28)*if there exists a positive definite symmetric matrix $\Gamma \in \mathbb{R}^{p\times p}$ such that for each $\theta \in \mathbb{R}^p$, $V_a(x, \theta)$ is a **clf** for the modified system*

$$\dot{x} = f(x) + F(x)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^T\right) + g(x)u \tag{6.2.31}$$

that is, V_a satisfies

$$\inf_{u \in \mathbb{R}} \left\{ \frac{\partial V_a}{\partial x} \left[f(x) + F(x) \left(\theta + \Gamma \left(\frac{\partial V_a}{\partial \theta} \right)^T \right) + g(x) u \right] \right\} < 0 \tag{6.2.32}$$

□

We now show how to design an adaptive controller (6.2.29) - (6.2.30) when an aclf is known.

Theorem 6.2.3 *The following two statements are equivalent:*

1. *There exists a triple (α, V_a, Γ) such that $\alpha(x, \theta)$ globally asymptotically stabilizes* (6.2.32) *at $x = 0$ for each $\theta \in \mathbb{R}^p$ with respect to the Lyapunov function $V_a(x, \theta)$.*
2. *There exists an aclf $V_a(x, \theta)$ for* (6.2.28).

Moreover, if an aclf $V_a(x, \theta)$ exists, then (6.2.28) *is globally adaptively stabilizable.*

Proof (1 ⇒ 2) *Obvious because 1 implies that there exists a continuous function W : $\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+$, positive definite in x for each θ, such that*

$$\frac{\partial V_a}{\partial x} \left[f(x) + F(x) \left(\theta + \Gamma \left(\frac{\partial V_a}{\partial \theta} \right)^T \right) + g(x) \alpha(x.\theta) \right] \leq -W(x, \theta) \tag{6.2.33}$$

Thus $V_a(x, \theta)$ is a clf for (6.2.31) *for each $\theta \in \mathbb{R}^p$, and therefore it is an aclf for* (6.2.28).

(2 ⇒ 1) *The proof of this part is based on Sontag's constructive proof [171] of Artstein's theorem [4]. We assume that V_a is an aclf for* (6.2.28), *that is, a clf for* (6.2.31). *Sontag's formula* (??) *applied to* (6.2.31) *gives a control law smooth on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^p$:*

$$\alpha(x, \theta) = \begin{cases} -\dfrac{\frac{\partial V_a}{\partial x} \tilde{f} + \sqrt{\left(\frac{\partial V_a}{\partial x} \tilde{f} \right)^2 + \left(\frac{\partial V_a}{\partial x} g \right)^4}}{\frac{\partial V_a}{\partial x} g} & , \quad \frac{\partial V_a}{\partial x} g(x, \theta) \neq 0 \\ 0 & , \quad \frac{\partial V_a}{\partial x} g(x, \theta) = 0 \end{cases} \tag{6.2.34}$$

where

$$\tilde{f}(x, \vartheta) = f(x) + F(x) \left(\theta + \Gamma \left(\frac{\partial V_a}{\partial \theta} \right)^T \right) \tag{6.2.35}$$

With the choice (6.2.34),*inequality* (6.2.33) *is satisfied with the continuous function*

$$W(x,\theta) = \sqrt{\left(\frac{\partial V_a}{\partial x}\tilde{f}(x,\theta)\right)^2 + \left(\frac{\partial V_a}{\partial x}g(x,\theta)\right)^4} \tag{6.2.36}$$

which is positive definite in x *for each* θ*, because* (6.2.32) *implies that* $\frac{\partial V_a}{\partial x}\tilde{f}(x,\theta) < 0$ *whenever* $\frac{\partial V_a}{\partial x}g(x,\theta) = 0$ *and* $x \neq 0$. *We note that the control law* $\alpha(x,\theta)$ *will be continuous at* $x = 0$ *if and only if the aclf* V_a *satisfies the following property, called the* small control property *[171]: For each* $\theta \in \mathbb{R}^p$ *and for any* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that, if* $x \neq 0$ *satisfies* $|x| \leq \delta$*, then there is some* u *with* $|u| \leq \varepsilon$ *such that*

$$\frac{\partial V_a}{\partial x}\left[f(x) + F(x)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^T\right) + g(x)u\right] < 0 \tag{6.2.37}$$

Assuming the existence of an aclf we now show that (6.2.28) *is globally adaptively stabilizable. Since* ($2 \Rightarrow 1$)*, there exists a triple* (α, V_a, Γ) *and a function* W *such that* (6.2.33) *is satisfied, that is,*

$$\frac{\partial V_a}{\partial x}[f(x) + F(x)\theta + g(x)\alpha(x,\theta)] + \frac{\partial V_a}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial x}F(x)\right)^T \leq -W(x,\theta) \tag{6.2.38}$$

Consider the Lyapunov function candidate

$$V(x,\hat{\theta}) = V_a(x,\hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})^T\Gamma^{-1}(\theta - \hat{\theta}) \tag{6.2.39}$$

With the help of (6.2.38)*, the derivative of* V *along the solutions of* (6.2.28), (6.2.29), *and* (6.2.30) *is*

$$\begin{aligned}
\dot{V} &= \frac{\partial V_a}{\partial x}\left[f + F\theta + g\alpha(x,\hat{\theta})\right] + \frac{\partial V_a}{\partial \hat{\theta}}\Gamma\tau(x,\hat{\theta}) - \tilde{\theta}^T\tau(x,\hat{\theta}) \\
&= \frac{\partial V_a}{\partial x}\left[f + F\theta + g\alpha(x,\hat{\theta})\right] + \frac{\partial V_a}{\partial \hat{\theta}}\Gamma\tau(x,\hat{\theta}) + \frac{\partial V_a}{\partial x}F\tilde{\theta} - \tilde{\theta}^T\tau(x,\hat{\theta}) \\
&\leq -W(x,\hat{\theta}) - \frac{\partial V_a}{\partial \hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial x}F\right)^T + \frac{\partial V_a}{\partial \hat{\theta}}\Gamma\tau(x,\hat{\theta})
\end{aligned} \tag{6.2.40}$$

Choosing

$$\tau(x,\hat{\theta}) = \left(\frac{\partial V_a}{\partial x}(x,\hat{\theta})F(x)\right)^T \tag{6.2.41}$$

we get

$$\dot{V} \leq -W(x,\hat{\theta}), \quad \forall\theta \in \mathbb{R}^p \tag{6.2.42}$$

Thus, the equilibrium $x = 0$, $\hat{\theta} = \theta$ *of* (6.2.28), (6.2.29), *and* (6.2.30) *is globally stable, and by the LaSalle-Yoshizawa Theorem,* $x(t) \to 0$, *that is,* (6.2.28) *is globally adaptively stabilizable.* □

The adaptive controller constructed in the proof of Theorem 6.2.3 consists of a control law $u = \alpha(x, \theta)$ given by (6.2.34), and an update law $\dot{\hat{\theta}} = \Gamma\tau(x, \hat{\theta})$ with (6.2.41).

It is of interest to interpret this controller as a certainty equivalence controller. The control law $\alpha(x, \theta)$ given by (6.2.34) is stabilizing for the modified system (6.2.31) but may not be stabilizing for the original system (6.2.28). However, as the proof of Theorem 6.2.3 shows, its certainty equivalence form $\alpha(x, \hat{\theta})$ is an adaptive globally stabilizing control law for the original system (6.2.28). Hence, if a certainty equivalence approach is to be applied to a nonlinear system, the system is to be modified to require a control law which anticipates the parameter estimation transients. In the proof of Theorem 6.2.3, this is achieved by incorporating the *tuning function* τ in the control law α. Indeed, the formula (6.2.34) for α depends on τ via

$$\frac{\partial V_a}{\partial x}\tilde{f}(x, \theta) = \frac{\partial V_a}{\partial x} f + \tau(x, \theta)^T \left(\theta + \Gamma \left(\frac{\partial V_a}{\partial \theta} \right)^T \right) \tag{6.2.43}$$

which is obtained by combining (6.2.35) and (6.2.41). Using (6.2.41) to rewrite the inequality (6.2.38) as

$$\frac{\partial V_a}{\partial x}[f(x) + F(x)\theta + g(x)\alpha(x, \theta)] + \frac{\partial V_a}{\partial \theta}\Gamma\tau(x, \theta) \le -W(x, \theta) \tag{6.2.44}$$

it is not difficult to see that the control law (6.2.34) containing (6.2.43) prevents τ from destroying the nonpositivity of the Lyapunov derivative.

Remark 6.2.4 *A relevant question remains unanswered: If there exists an aclf for* (6.2.28), *is this system globally asymptotically stabilizable for each* θ *(and vice versa)? In other words, does the existence of a pair* α, V_a *satisfying* (6.2.33) *for some* $\Gamma > 0$ *imply the existence of a pair* (α^0, V_a^0) *satisfying* (6.2.33) *for* $\Gamma = 0$ *(and vice versa)? Adaptive Lyapunov designs available in the literature [59, 65, 69, 94, 156, 157, 186] are all for systems which are not only globally adaptively stabilizable, but aldo globally asymptotically stabilizable for each* θ.

As is always the case in adaptive control, in the proof of Theorem 6.2.3 we used a Lyapunov function $V(x,\hat{\theta})$ given by (6.2.39), which is quadratic in the parameter error $\theta - \hat{\theta}$. The quadratic form is suggested by the linear dependence of (6.2.28) on θ, and the fact that θ cannot be used for feedback. We will now show that the quadratic form of (6.2.39) is both necessary and sufficient for the existence of an aclf.

We say that system (6.2.28) is **globally adaptively quadratically stabilizable** if it is *globally adaptively stabilizable* and, in addition, these exist a smooth function $V_a(x,\theta)$ positive definite and radially unbounded in x for each θ, and a continuous function $W(x,\theta)$ positive definite in x for each θ, such that for all $(x(0),\hat{\theta}(0)) \in \mathbb{R}^{n+p}$ and all $\theta \in \mathbb{R}^p$, the derivative of (6.2.39) along the solutions of (6.2.28), (6.2.29), (6.2.30) is given by (6.2.42).

Corollary 6.2.5 *The system* (6.2.28) *is globally adaptively quadratically stabilizable if and only if there exists an aclf* $V_a(x,\theta)$.

Proof *The 'if' part is contained in the proof of Theorem 6.2.3 where the Lyapunov function* $V(x,\hat{\theta})$ *is in the form* (6.2.39). *To prove the 'only if' part, we start by assuming global adaptive quadratic stabilizability of* (6.2.28), *and first show that* $\tau(x,\hat{\theta})$ *must be given by* (6.2.41). *The derivative of* V *along the solutions of* (6.2.28), (6.2.29), (6.2.30) *given by* (6.2.40), *is rewritten as*

$$\begin{aligned}\dot{V} = \ & \frac{\partial V_a}{\partial x}\left[f + F\hat{\theta} + g\alpha(x,\hat{\theta})\right] + \frac{\partial V_a}{\partial \hat{\theta}}\Gamma\tau(x,\hat{\theta}) - \hat{\theta}^T\left(\left(\frac{\partial V_a}{\partial \theta}\right)^T - \tau\right) \\ + \ & \theta^T\left(\left(\frac{\partial V_a}{\partial \theta}\right)^T - \tau\right) \qquad (6.2.45)\end{aligned}$$

This expression has to be nonpositive to satisfy (6.2.42). *Since it is affine in* θ, *it can be nonpositive for all* $(x,\hat{\theta}) \in \mathbb{R}^{n+p}$ *and all* $\theta \in \mathbb{R}^p$ *only if the last term is zero, that is, only* *if* τ *is defined as in* (6.2.41). *Then, it is straightforward to verify that*

$$\begin{aligned} & \frac{\partial V_a}{\partial x}\left[f(x)+F(x)\left(\hat{\theta}+\Gamma\left(\frac{\partial V_a}{\partial \hat{\theta}}\right)^T\right)+g(x)\alpha(x,\hat{\theta})\right] \\ = & \ \dot{V}+\left(\hat{\theta}^T+\frac{\partial V_a}{\partial \hat{\theta}}\Gamma\right)\left(\tau-\left(\frac{\partial V_a}{\partial x}F\right)^T\right) \qquad (6.2.46) \\ \leq & \ -W(x,\hat{\theta}) \end{aligned}$$

for all $x, \hat{\theta} \in \mathbb{R}^{n+p}$. *By* $(1 \Rightarrow 2)$ *in Theorem 6.2.3,* $V_a(x,\theta)$ *is an aclf for* (6.2.28). □

The above analysis applies also to the case where the unknown parameter enter the control vector field:

$$\dot{x} = f(x) + F(x)\theta + [g(x) + G(x)\theta]u \qquad (6.2.47)$$

In this case, the existence of an aclf V_a is equivalent to the existence of a clf for the system

$$\begin{aligned} \dot{x} = & \ f(x)+F(x)\left(\theta+\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^T\right) \\ & +\left[g(x)+G(x)\left(\theta+\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^T\right)\right]u \qquad (6.2.48) \end{aligned}$$

The extension to the multi-input case is also straightforward.

It is of interest to examine the input-output properties of the system resulting from the application of the adaptive control law $\alpha(x,\hat{\theta})$ to the plant (6.2.1):

$$\dot{x} = f(x) + F(x)\hat{\theta} + g(x)\alpha(x,\hat{\theta}) + F(x)\tilde{\theta} \qquad (6.2.49)$$

6.2.3 Adaptive backstepping via aclf

With Theorem 6.2.3, the problem of adaptive stabilization is reduced to the problem of finding an aclf. We now address the problem of systematic construction of an aclf. Our aim is a recursive approach because we already know how to find aclf's for systems with the extended matching property, and expect to recursively enlarge this initial class of systems with repeated use of backstepping. So, we assume that an aclf is known for an initial system, and construct a new aclf for the initial system augmented by an integrator.

Lemma 6.2.6 *If the system*

$$\dot{x} = f(x) + F(x)\theta + g(x)u \tag{6.2.50}$$

is globally adaptively quadratically stabilizable with $\alpha \in \mathbb{C}^1$, then the augmented system

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)\xi \\ \dot{\xi} &= u \end{aligned} \tag{6.2.51}$$

is also globally adaptively quadratically stabilizable.

Proof *Since system* (6.2.50) *is globally adaptively stabilizable, then by Corollary 6.2.5 there exists an aclf $V_a(x,\theta)$, and by Theorem 6.2.3, it satisfies* (6.2.33) *with a control law $u = \alpha(x,\theta)$. We will now show that*

$$V_1(x,\xi,\theta) = V_a(x,\theta) + \frac{1}{2}(\xi - \alpha(x,\theta))^2 \tag{6.2.52}$$

is an aclf for the augmented system (6.2.51) *by showing that it satisfies*

$$\frac{\partial V_1}{\partial(x,\xi)} \begin{bmatrix} f + F\left(\theta + \Gamma\left(\frac{\partial V_1}{\partial \theta}\right)^T\right) + g\xi \\ \alpha_1(x,\xi,\theta) \end{bmatrix} \leq -W - (\xi - \alpha)^2 \tag{6.2.53}$$

with the control law

$$\begin{aligned} u = \alpha_1(x,\xi,\theta) = & -\frac{\partial V_a}{\partial x}g - (\xi - \alpha) + \frac{\partial \alpha}{\partial x}(f + F\theta + g\xi) \\ & + \frac{\partial \alpha}{\partial \theta}\Gamma\left(\frac{\partial V_1}{\partial x}F\right)^T + \frac{\partial V_a}{\partial \theta}\Gamma\left(\frac{\partial \alpha}{\partial x}F\right)^T \end{aligned} \tag{6.2.54}$$

Let us start by introducing for brevity $z = \xi - \alpha(x,\theta)$. With (6.2.52) *we compute*

$$\begin{aligned} \frac{\partial V_1}{\partial(x,\xi)} \begin{bmatrix} f + F\theta + g\xi \\ \alpha_1(x,\xi,\theta) \end{bmatrix} &= \frac{\partial V_1}{\partial x}(f + F\theta + g\xi) + \frac{\partial V_1}{\partial \xi}\alpha_1(x,\xi,\theta) \qquad (6\ 2.55) \\ &= \left(\frac{\partial V_a}{\partial x} - z\frac{\partial \alpha}{\partial x}\right)(f + F\theta + g\xi) + z\alpha_1 \\ &= \frac{\partial V_a}{\partial x}(f + F\theta + g\alpha) + \frac{\partial V_a}{\partial x}gz - z\frac{\partial \alpha}{\partial x}(f + F\theta + g\xi) + z\alpha_1 \\ &= \frac{\partial V_a}{\partial x}(f + F\theta + g\alpha) + z\left(\alpha_1 + \frac{\partial V_a}{\partial x}g - \frac{\partial \alpha}{\partial x}(f + F\theta + g\xi)\right) \end{aligned}$$

On the other hand, in view of (6.2.52), *we have*

$$\begin{aligned}\frac{\partial V_1}{\partial(x,\xi)}\begin{bmatrix} F\Gamma(\frac{\partial V_1}{\partial\theta})^T \\ 0 \end{bmatrix} &= \frac{\partial V_1}{\partial x}F\Gamma\left(\frac{\partial V_1}{\partial\theta}\right)^T \\ &= \left(\frac{\partial V_a}{\partial x} - z\frac{\partial\alpha}{\partial x}\right)F\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^T \\ &= \frac{\partial V_a}{\partial x}F\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^T - z\left(\frac{\partial\alpha}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial x}F\right)^T + \frac{\partial V_a}{\partial\theta}\Gamma\left(\frac{\partial\alpha}{\partial x}F\right)^T\right)\end{aligned} \tag{6.2.56}$$

Adding (6.2.55) *and* (6.2.56) *with* (6.2.33) *and* (6.2.54), *we get*

$$\begin{aligned}&\frac{\partial V_1}{\partial(x,\xi)}\begin{bmatrix} f + F\left(\theta + \Gamma(\frac{\partial V_1}{\partial\theta})^T\right) + g\xi \\ \alpha_1(x,\xi,\theta) \end{bmatrix} = \frac{\partial V_a}{\partial x}(f + F\theta + g\alpha) + \frac{\partial V_a}{\partial x}\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^T \\ &+z\left(\alpha_1 + \frac{\partial V_a}{\partial x}g - \frac{\partial\alpha}{\partial x}(f + F\theta + g\xi) - \frac{\partial\alpha}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial x}F\right)^T - \frac{\partial V_a}{\partial\theta}\Gamma\left(\frac{\partial\alpha}{\partial x}F\right)^T\right) \\ &\leq -W(x,\theta) - z^2\end{aligned} \tag{6.2.57}$$

This proves by Theorem 6.2.3 that $V_1(x,\xi,\theta)$ *is an aclf for system* (6.2.51), *and by Corollary 6.2.5, this system is globally adaptively quadratically stabilizable.* □

The new tuning function for system (6.2.51) is determined by the new aclf V_1 and given by

$$\begin{aligned}\tau_1(x,\xi,\theta) &= \left(\frac{\partial V_1}{\partial(x,\xi)}\begin{bmatrix} F \\ 0 \end{bmatrix}\right)^T = \left(\frac{\partial V_1}{\partial x}F\right)^T = \left[\left(\frac{\partial V_a}{\partial x} - (\xi-\alpha)\frac{\partial\alpha}{\partial x}\right)F\right]^T \\ &= \tau(x,\theta) - \left(\frac{\partial\alpha}{\partial x}F\right)^T(\xi - \alpha)\end{aligned} \tag{6.2.58}$$

We note that the new tuning function τ_1 is obtained by augmenting the initial tuning function τ with the term $-\left(\frac{\partial\alpha}{\partial x}F\right)^T(\xi-\alpha)$ which accounts for the fact that the aclf V_a is augmented by $\frac{1}{2}(\xi - \alpha(x,\theta))^2$

The form of the control law $\alpha_1(x,\xi,\theta)$ in (6.2.54) is of particular interest. It consists of two parts, $\alpha_1 = \alpha_{1,c} + \alpha_{1,\tau}$. The first part,

$$\alpha_{1,c}(x,\xi,\theta) = 3\frac{\partial V_a}{\partial x}g - (\xi - \alpha) + \frac{\partial\alpha}{\partial x}(f + F\theta + g\xi) \tag{6.2.59}$$

would become the "certainty equivalence" control law for the augmented system (6.2.51) if we were to set $\Gamma = 0$. The second part consists of two terms.

$$\alpha_{1,\tau}(x,\xi,\theta) = \frac{\partial \alpha}{\partial \theta}\Gamma\left(\frac{\partial V_1}{\partial x}F\right)^T + \frac{\partial V_a}{\partial \theta}\Gamma\left(\frac{\partial \alpha}{\partial x}F\right)^T \tag{6.2.60}$$

Their role is to produce $\frac{\partial V_a}{\partial x}F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^T$ in the aclf inequality (6.2.53). Observe that the first term in (6.2.60) incorporates $\tau_1 = \left(\frac{\partial V_1}{\partial x}F\right)^T$.

The control law $\alpha_1(x,\xi,\theta)$ in (6.2.54) is only one out of many possible control laws. Once we have shown that V_1 given by (6.2.52) is an aclf for (6.2.51), we can use, for example, the C^0 control law α_1 given by Sontag's formular (6.2.34) with $\frac{\partial V_1}{\partial(x,\xi)}g_1 = z$ and

$$\begin{aligned}\frac{\partial V_1}{\partial(x,\xi)}\tilde{f}_1(x,\xi,\theta) &= \frac{\partial V_1}{\partial(x,\xi)}\begin{bmatrix} f + F\left(\theta + \Gamma\left(\frac{\partial V_1}{\partial \theta}\right)^T\right) + g\xi \\ 0 \end{bmatrix} \\ &= \frac{\partial V_1}{\partial x}(f + g\xi) + \tau_1(x,\xi,\theta)^T\left(\theta + \Gamma\left(\frac{\partial V_1}{\partial \theta}\right)^T\right)\end{aligned} \tag{6.2.61}$$

It can be shown that the following function, used as a clf in [158], is a more general aclf than (6.2.52):

$$V_1(x,\xi,\theta) = V_a(x,\theta) + \int_0^{\xi-\alpha(x,\theta)} \eta(s)ds \tag{6.2.62}$$

where η is a C^0 function such that $s\eta(s) > 0$ whenever $s \neq 0$, $\eta'(0) > 0$, and $\eta \notin \mathcal{L}^1((-\infty,0]) \cup \mathcal{L}^1((0,+\infty])$.

The following example illustrates the use of Lemma 6.2.6.

Example 6.2.7 *Let us consider the system*

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi(x_1)^T\theta \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u\end{aligned} \tag{6.2.63}$$

we will treat the state x_3 as an integrator added to the (x_1, x_2)-subsystem from Example 6.2.1. In the example, we have already designed an adaptive control law for the system

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^T\theta \\ \dot{x}_2 &= x_3 \end{aligned} \tag{6.2.64}$$

considering x_3 as a control input. With (6.2.18), (6.2.19), (6.2.20), *and* (6.2.22), *it can be shown that*

$$\frac{\partial V_c}{\partial x}\left[f(x) + F(x)\left(\theta + \left(\frac{\partial V_c}{\partial \theta}\right)^T\right)\right] = -c_1 z_1^2 - c_2 z_2^2 \tag{6.2.65}$$

which means that $V_a(x_1, x_2, \theta) = V_c(x_1, x_2, te) = \frac{1}{2}(z_1^2 + z_2^2)$ is an aclf for the system (6.2.64) *considering x_3 as a control input. Therefore, Lemma 6.2.6, the function directly applicable. We define $z = x_3 - \alpha(x, \theta)$. By Lemma 6.2.6, the function*

$$V_1(x, \theta) = \frac{1}{2}\left(z_1^2 + z_2^2 + z_3^2\right) \tag{6.2.66}$$

is an aclf for the system (6.2.63). *With* (6.2.54) *and* (6.2.58), *we obtain*

$$\begin{aligned} \alpha_1(x, \theta) &= -z_1 - c_3 z_3 - \frac{\partial \alpha}{\partial(x_1, x_2)}\begin{bmatrix} x_2 + \varphi^T\theta \\ x_3 \end{bmatrix} + \frac{\partial \alpha}{\partial \theta}\tau_1 \\ &\quad + z_2 \varphi^T \frac{\partial \alpha}{\partial x_1}\varphi \end{aligned} \tag{6.2.67}$$

$$\tau_1(x, \theta) = \tau - \frac{\partial \alpha}{\partial x_1}\varphi z_3 \tag{6.2.68}$$

With the following adaptive control law and the parameter update law:

$$u = \alpha_1(x, \hat{\theta}) \tag{6.2.69}$$

$$\dot{\hat{\theta}} = \tau_1(x, \hat{\theta}) \tag{6.2.70}$$

it is straightforward to verify that the closed-loop adaptive system is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 - \frac{\partial \alpha}{\partial x_1}|\varphi|^2 \\ 0 & -1 + \frac{\partial \alpha}{\partial x_1}|\varphi|^2 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{\partial \varphi^T}{\partial x_1}\hat{\theta} + c_1 \\ -\frac{\partial \alpha}{\partial x_1} \end{bmatrix} \varphi^T \tilde{\theta} \tag{6.2.71}$$

$$\dot{\hat{\theta}} = \varphi \begin{bmatrix} 1, & \frac{\partial \varphi^T}{\partial x_1}\hat{\theta} + c_1, & -\frac{\partial \alpha}{\partial x_1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{6.2.72}$$

where z_1, z_2, z_3 are used with $\hat{\theta}$ as an argument. The global stability of this system is established using the Lyapunov function $V(x, \hat{\theta}) = V_1(x, \hat{\theta}) + \frac{1}{2}\tilde{\theta}^T\tilde{\theta}$. □

While in Lemma 6.2.6 the initial system is augmented only by an integrator, a minor modification is sufficient to obtain an analogous result for the more general system

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)\xi \\ \dot{\xi} &= u + F_1(x, \xi)\theta \end{aligned} \tag{6.2.73}$$

Corollary 6.2.8 *The function $V_1(x, \xi, \theta)$ defined in* (6.2.52) *is an aclf for the system* (6.2.73) *with the control law and the tuning function given as*

$$\alpha_1(x, \xi, \theta) = \alpha_1(x, \xi, \theta) - F_1(x, \xi)\left(\theta + \Gamma\left(\frac{\partial V_c}{\partial \theta}\right)^T\right) \tag{6.2.74}$$

$$\tau_1(x, \xi, \theta) = \tau_1(x, \xi, \theta) + (\xi - \alpha)F_1(x, \xi)^T \tag{6.2.75}$$

□

A repeated application of Corollary 6.2.8 will further extend the class of nonlinear systems for this type of adaptive design. With the knowledge of V_a, τ, and α for the system (6.2.73), it is not hard to see that by applying Corollary 6.2.8 twice we can find V_2, τ_2, and α_2 for the system

$$\begin{aligned}\dot{x} &= f(x) + F(x)\theta + g(x)\xi_1 \\ \dot{\xi}_1 &= \xi_2 + F_1(x, \theta_1)\theta \\ \dot{\xi}_2 &= u + F_2(x, \xi_1, \xi_2)\theta\end{aligned} \tag{6.2.76}$$

In fact, it is clear that an n-fold application of Corollary 6.2.8 will provide us with V_n, τ_n, and α_n for the system

$$\begin{aligned}\dot{x} &= f(x) + F(x)\theta + g(x)\xi_1 \\ \dot{\xi}_1 &= \xi_2 + F_1(x, \xi_1)\theta \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_n + F_{n-1}(x, \xi_1, \ldots, \xi_{n-1})\theta \\ \dot{\xi}_n &= u + F_n(x, \xi_1, \ldots, \xi_n)\theta\end{aligned} \tag{6.2.77}$$

We will now develop a detailed design procedure for such systems.

6.3 Set-Point Regulation

With repeated use of Corollary 6.2.8, we can design an adaptive controller to globally stabilize a desired equilibrium x^e of the *parametric strict-feedback system* (??):

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi_1(x_1)^T\theta \\ \dot{x}_2 &= x_3 + \varphi_2(x_1, x_2)^T\theta \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \varphi_{n-1}(x_1, \ldots, x_{n-1})^T\theta \\ \dot{x}_n &= \beta(x)u + \varphi_n(x)^T\theta\end{aligned} \tag{6.3.1}$$

where $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters, β and

$$F = [\varphi_1, \ldots, \varphi_n] \tag{6.3.2}$$

are smooth nonlinear functions taking arguments in $\mathbb{R}^n$, and $\beta(x) \neq 0$, $\forall x \in \mathbb{R}^n$.

In this section, we develop a procedure for adaptive regulation of the output $y = x_1$ to a given set-point y_s. With a constant control u^e, the first n-1 equilibrium equations of $\dot{x}^e = 0$ in (6.3.1) can be successively solved for $x_2^e, \ldots, x_n^e$ as functions of x_1^e and θ:

$$\begin{aligned} x_2^e &= -\varphi_1(x_1^e)^T\theta \\ x_3^e &= -\varphi_2(x_1^e, x_2^e)^T\theta \\ &\vdots \\ x_n^e &= -\varphi_{n-1}(x_1^e, \ldots, x_{n-1}^e)^T\theta \end{aligned} \tag{6.3.3}$$

Then the n^{th} equation $\dot{x}_n^e = 0$ yields a relationship between x_1^e, u^e, and θ. When θ is known, then $\dot{x}_n^e = 0$ can be solved for u^e needed to keep x_1^e at a desired set-point $x_1^e = y_s$. The corresponding values $x_2^e, \ldots, x_n^e$ will be dictated by (6.3.3). Therefore, for each value of θ and a prescribed y_s, the equilibrium x^e and the corresponding control value u^e are uniquely defined. In the special case where $\varphi_1(0) = \cdots = \varphi_{n-1}(0) = 0$, the choice $y_s = 0$ results in the equilibrium being $x^e = 0$ for all values of θ.

Our problem now is to globally stabilize this equilibrium when θ is unknown and also to achieve set-point regulation: $x(t) \to x^e$ as $t \to \infty$.

Comparing the systems (6.3.1) and (6.2.73), we observe that if x_3 were the control variable, then Corollary 6.2.8 would provide the desired adaptive control for the subsystem made of the first two equations of (6.3.1). Therefore, we can initiate our recursive design procedure by augmenting his subsystem by the third equation, as in (6.2.76). For convenience, we will do this in a self-contained fashion, independent of Section 6.2. An additional feature of the procedure in this section in a set of error coordinates in which the stability properties of the resulting closed-loop adaptive system are clearly displayed without an explicit use of the aclf concept.

6.3.1 Design procedure

We will start by adaptively stabilizing the first equation (6.3.1) considering x_2 to be its control. At each subsequence step, we will augment the designed subsystem by one equation. At the i^{th} step, an i^{th}-order subsystem is stabilized with respect to a Lyapunov function V_i by the design of a *stabilizing function* α_i and a *tuning function* τ_i. The update law for the parameter estimate $\hat{\theta}(t)$ and the adaptive feedback control u are designed at the final step. The third step is crucial for understanding the general design procedure.

Step 1. Introducing the first two error variables

$$z_1 = x_1 - y_s \tag{6.3.4}$$

$$z_2 = x_2 - \alpha_1 \tag{6.3.5}$$

we rewrite $\dot{x}_1 = x_2 + \varphi_1(x_1)^T\theta$, the first equation of (6.3.1), as

$$\dot{z}_1 = z_2 + \alpha_1 + w_1(x_1)^T\theta \tag{6.3.6}$$

where, for uniformly with subsequence steps, we have defined the first regressor vector as

$$w_1(x_1) = \varphi_1(x_1) \tag{6.3.7}$$

Our task in this step is to stabilize (6.3.6) with respect to the Lyapunov function

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{6.3.8}$$

whose derivative along the solutions of (6.3.6) is

$$\dot{V}_1 = z_1(z_2 + \alpha_1 + w_1^T\theta) - \tilde{\theta}^T\Gamma^{-1}(\dot{\hat{\theta}} - \Gamma w_1 z_1) \tag{6.3.9}$$

We can eliminate $\tilde{\theta}$ from $\dot{V}_1$ with the update law $\dot{\hat{\theta}} = \Gamma\tau_1$, where

$$\tau_1(x_1) = w_1(x_1)z_1 \tag{6.3.10}$$

If x_2 were actual control, we would let $z_2 \equiv 0$, that is, $x_2 \equiv \alpha_1$. Then, to make $\dot{V}_1 = -c_1z_1^2$, we would choose

$$\alpha_1(x_1, \hat{\theta}) = -c_1z_1 - w_1(x_1)^T\hat{\theta} \tag{6.3.11}$$

Since x_2 is not our control, we have $z_2 \not\equiv 0$, and we do not use $\dot{\hat{\theta}} = \Gamma\tau_1$ as an update law. Instead, we retain τ_1 as our first *tuning function* and tolerate the presence of $\tilde{\theta}$ in $\dot{V}_1$:

$$\dot{V}_1 = -c_1z_1^2 + z_1z_2 + \tilde{\theta}^T(\Gamma^{-1}\dot{\hat{\theta}} - \tau_1) \tag{6.3.12}$$

The second term $z_1 z_2$ in $\dot{V}_1$ will be cancelled at the next step. With $\alpha_1(x_1, \hat{\theta})$ as in (6.3.11), the z_1-system becomes

$$\dot{z}_1 = -c_1 z_1 + z_2 + w_1(x_1)^T \tilde{\theta} \tag{6.3.13}$$

Step 2. We now consider that x_3 is the control variable in the second equation of (6.3.1). Introducing

$$z_3 = x_3 - \alpha_2 \tag{6.3.14}$$

we rewrite $\dot{x}_2 = x_3 + \varphi_2(x_1, x_2)^T \theta$ as

$$\dot{z}_2 = z_3 + \alpha_2 - \frac{\partial \alpha_1}{\partial x_1} x_2 + w_2(x_1, x_2, \hat{\theta})^T \theta - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} \tag{6.3.15}$$

where the second regressor vector w_2 is defined as

$$w_2(x_1, x_2, \hat{\theta}) = \varphi_2 - \frac{\partial \alpha_1}{\partial x_1} \varphi_1 \tag{6.3.16}$$

Our task in this step is to stabilize the (z_1, z_2)-system (6.3.13), (6.3.15) with respect to

$$V_2 = V_1 + \frac{1}{2} z_2^2 \tag{6.3.17}$$

whose derivative along the solutions of (6.3.13) and (6.3.15) is

$$\begin{aligned} \dot{V}_2 = \quad & -c_1 z_1^2 + z_2 \left[z_1 + z_3 + \alpha_2 - \frac{\partial \alpha_1}{\partial x_1} x_2 + w_2^T \hat{\theta} - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} \right] \\ & + \tilde{\theta}^T \left(\tau_1 + w_2 z_2 - \Gamma^{-1} \dot{\hat{\theta}} \right) \end{aligned} \tag{6.3.18}$$

We can eliminate $\tilde{\theta}$ from $\dot{V}_2$ with the update law $\dot{\hat{\theta}} = \Gamma \tau_2$, where

$$\tau_2(x_1, x_2, \hat{\theta}) = \tau_1 + w_2 z_2 = \begin{bmatrix} w_1 & , & w_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{6.3.19}$$

If x_3 were our actual control and, hence, $z_3 \equiv 0$, we would achieve $\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2$ by designing α_2 to make the bracketed term multiplying z_2 in (6.3.18) equal to $-c_2 z_2$, namely

$$\alpha_2(x_1, x_2, \hat{\theta}) = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 - w_2^T \hat{\theta} + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2 \tag{6.3.20}$$

We retain τ_2 as our second tuning function in the term $\Gamma\tau_2$ which replaces $\dot{\hat{\theta}}$ in (6.3.20). However, we do not use $\dot{\hat{\theta}} = \Gamma\tau_2$ as an update law, so that the resulting $\dot{V}_2$ is

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}(\Gamma\tau_2 - \dot{\hat{\theta}}) + \tilde{\theta}^T(\tau_2 - \Gamma^{-1}\dot{\hat{\theta}}) \tag{6.3.21}$$

The first two terms in $\dot{V}_2$ are negative definite, the third term will be cancelled at the next step, while the discrepancy between $\Gamma\tau_2$ and $\dot{\hat{\theta}}$ in the last two terms remains. By substituting (6.3.20) into (6.3.15), the (z_1, z_2)-subsystem becomes

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ z_3 + \frac{\partial \alpha_1}{\alpha \hat{\theta}}(\Gamma\tau_2 - \dot{\hat{\theta}}) \end{bmatrix} \tag{6.3.22}$$

Step 3. Proceeding to the third equation in (6.3.1), we introduce

$$z_4 = x_4 - \alpha_3 \tag{6.3.23}$$

and rewrite $\dot{x}_3 = x_4 + \varphi_3(x_1, x_2, x_3)^T\theta$ as

$$\dot{z}_3 = z_4 + \alpha_3 - \frac{\partial \alpha_2}{\partial x_1}x_2 - \frac{\partial \alpha_2}{\partial x_2}x_3 + w_3(x_1, x_2, x_3, \hat{\theta})^T - \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}} \tag{6.3.24}$$

where the third regressor vector w_3 is defined as

$$w_3(x_1, x_2, x_3, \hat{\theta}) = \varphi_3 - \frac{\partial \alpha_2}{\partial x_1}\varphi_1 - \frac{\partial \alpha_2}{\partial x_1}\varphi_2 \tag{6.3.25}$$

Our task is to stabilize the (z_1, z_2, z_3)-system with respect to

$$V_3 = V_2 + \frac{1}{2}z_3^2 \tag{6.3.26}$$

whose derivative along (6.3.22) and (6.3.24) is

$$\begin{aligned} \dot{V}_3 \;=\; & -c_1 z_1^2 - c_2 z_2^2 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}(\Gamma\tau_2 - \dot{\hat{\theta}}) \\ & + z_3 \left[z_2 + z_4 + \alpha_3 - \frac{\partial \alpha_2}{\partial x_1}x_2 - \frac{\partial \alpha_2}{\partial x_2}x_3 + w_3^T\hat{\theta} - \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}} \right] \\ & + \tilde{\theta}^T\left(\tau_2 + w_3 z_3 - \Gamma^{-1}\dot{\hat{\theta}}\right) \end{aligned} \tag{6.3.27}$$

We can eliminate $\tilde{\theta}$ from $\dot{V}_3$ with the update law $\dot{\hat{\theta}} = \Gamma\tau_3$, where τ_3 is our tuning function

$$\tau_3(x_1, x_2, x_3, \hat{\theta}) = \tau_2 + w_3 z_3 = \begin{bmatrix} w_1, & w_2, & w_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{6.3.28}$$

If x_3 were our actual control, we could have $z_4 \equiv 0$ and achieve $\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2$ by designing α_3 to make the bracketed term multiplying z_3 equal to $-c_3 z_3$, namely

$$\alpha_3(x_1, x_2, x_3, \hat{\theta}) = -z_2 - c_3 z_3 + \frac{\partial \alpha_2}{\partial x_1} x_2 + \frac{\partial \alpha_2}{\partial x_2} x_3 - w_3^T \hat{\theta} + \frac{\partial \alpha_2}{\partial \hat{\theta}} \Gamma \tau_3 + \nu_3 \qquad (6.3.29)$$

where ν_3 is a correction term yet to be chosen. Substituting (6.3.29) into (6.3.27), and noting that

$$\begin{aligned} \dot{\hat{\theta}} - \Gamma \tau_2 &= \dot{\hat{\theta}} - \Gamma \tau_3 + \Gamma \theta_3 - \Gamma \tau_2 \\ &= \dot{\hat{\theta}} - \Gamma \tau_3 + \Gamma w_3 z_3 \end{aligned} \qquad (6.3.30)$$

(6.3.27) is written as

$$\begin{aligned} \dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 + z_3 \left(\nu_3 - \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma w_3 z_2 \right) + z_3 z_4 \\ &\quad + \left(z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} + z_3 \frac{\partial \alpha_2}{\partial \hat{\theta}} \right) (\Gamma \tau_3 - \dot{\hat{\theta}}) + \tilde{\theta}^T (\tau_3 - \Gamma^{-1} \dot{\hat{\theta}}) \end{aligned} \qquad (6.3.31)$$

and the (z_1, z_2, z_3)-subsystem becomes

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} &= \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 \\ 0 & -1 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ -\frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma w_3 z_3 \\ \nu_3 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_3 - \dot{\hat{\theta}}) \\ z_4 + \frac{\partial \alpha_2}{\partial \hat{\theta}} (\Gamma \tau_3 - \dot{\hat{\theta}}) \end{bmatrix} \end{aligned} \qquad (6.3.32)$$

If x_4 were our control, we would have $z_4 = 0$, and with the update law $\dot{\hat{\theta}} = \Gamma \tau_3$, the last vector in (6.3.32) would be zero. However, the potentially destabilizing term $-\frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma w_3 z_3$ would still remain. This unmatched term must be accommodated by a choice of the correction term ν_3. From (6.3.31), the choice of ν_3 is immediate.

$$\nu_3(x_1, x_2, x_3, \hat{\theta}) = \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma w_3 z_2 \qquad (6.3.33)$$

We again postpone the decision about $\dot{\hat{\theta}}$ and do not use $\dot{\hat{\theta}} = \Gamma \tau_3$ as an update law. The resulting $\dot{V}_3$ is

$$\begin{aligned} \dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^3 + z_3 z_4 + \left(z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} + z_3 \frac{\partial \alpha_2}{\partial \hat{\theta}} \right) (\Gamma \tau_3 - \dot{\hat{\theta}}) \\ &\quad + \tilde{\theta}^T (\tau_3 - \Gamma^{-1} \dot{\hat{\theta}}) \end{aligned} \qquad (6.3.34)$$

and the (z_1, z_2, z_3)-subsystem becomes

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma w_3 \\ 0 & -1 + \frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma w_3 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} \tilde{\theta}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial\alpha_1}{\partial\hat{\theta}} \\ \frac{\partial\alpha_2}{\partial\hat{\theta}} \end{bmatrix} (\Gamma\tau_3 - \dot{\hat{\theta}}) \tag{6.3.35}$$

The 'system matrix' in (6.3.35) has a significant property: the skew symmetry of the nonlinear term $\frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma w_3$ achieved by the choice of ν_3 in (6.3.33). This term is analogous to the second term in (6.2.60) and the skew symmetry is crucial for stabilization.

Step i. Introducing

$$z_{i+1} = x_{i+1} - \alpha_i \tag{6.3.36}$$

we rewrite $\dot{x}_i = x_{i+1} + \varphi(x_1, \ldots, x_i)^T\theta$ as

$$\dot{z}_i = z_{i+1} + \alpha_i - \sum_{k=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_k} x_{k+1} + w_i(x_1, \ldots, x_i, \hat{\theta})^T\theta - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} \tag{6.3.37}$$

where the i^{th} regressor vector is defined as

$$w_i(x_1, \ldots, x_i, \hat{\theta}) = \varphi_i - \sum_{k=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_k}\varphi_k \tag{6.3.38}$$

Our objective is to stabilize the $(z_1, \ldots, z_i)$-system with respect to

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 \tag{6.3.39}$$

whose derivative is

$$\begin{aligned} \dot{V}_i = & -\sum_{k=1}^{i-1} c_k z_k^2 + \left(\sum_{k=1}^{i-2} z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\theta}}\right)(\Gamma\tau_{i-1} - \dot{\hat{\theta}}) \\ & + z_i\left[z_{i-1} + z_{i+1} + \alpha_i - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}x_{k+1} + w_i^T\hat{\theta} - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\right] \\ & + \tilde{\theta}^T\left(\tau_{i-1} + w_i z_i - \Gamma^{-1}\dot{\hat{\theta}}\right) \end{aligned} \tag{6.3.40}$$

We can eliminate $\tilde{\theta}$ from $\dot{V}_i$ with the update law $\dot{\hat{\theta}} = \Gamma\tau_i$, where

$$\begin{aligned} \tau_i(x_1, \ldots, x_i, \hat{\theta}) &= \tau_{i-1} + z_i w_i \\ &= \begin{bmatrix} w_1 & \ldots & w_i \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_i \end{bmatrix} \end{aligned} \tag{6.3.41}$$

Then, in the absence of z_{i+1}, we would achieve $\dot{V}_i = -\sum_{k=1}^{i} c_k z_k^2$, by designing α_i to make the bracketed term multiplying z_i equal to $-c_i z_i$, namely

$$\begin{aligned} \alpha_i(x_1, \ldots, x_i, \hat{\theta}) &= -z_{i-1} - c_i z_i + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} - w_i^T \hat{\theta} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i \\ &\quad + \nu_i \end{aligned} \tag{6.3.42}$$

where ν_i is a correction term yet to be chosen. Nothing that

$$\begin{aligned} \dot{\hat{\theta}} - \Gamma\tau_{i-1} &= \dot{\hat{\theta}} - \Gamma\tau_i + \Gamma\tau_i - \Gamma\tau_{i-1} \\ &= \dot{\hat{\theta}} - \Gamma\tau_i + \Gamma w_i z_i \end{aligned} \tag{6.3.43}$$

we rewrite $\dot{V}_i$ as

$$\begin{aligned} \dot{V}_i &= -\sum_{k=1}^{i-1} c_k z_k^2 + z_i \left[z_{i+1} + \nu_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} (\Gamma\tau_i - \dot{\hat{\theta}}) \right] \\ &\quad + \left(\sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) (\Gamma\tau_{i-1} - \dot{\hat{\theta}}) + \tilde{\theta}^T (\tau_i - \Gamma^{-1} \dot{\hat{\theta}}) \\ &= \sum_{k=1}^{i-1} c_k z_k^2 + z_i \left[z_{i+1} + \nu_i - \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \Gamma w_i \right] \\ &\quad + \left(\sum_{k=1}^{i-1} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) (\Gamma\tau_i - \dot{\hat{\theta}}) - \tilde{\theta}^T (\tau_i - \Gamma^{-1} \dot{\hat{\theta}}) \end{aligned} \tag{6.3.44}$$

and represent the $(z_1, \ldots, z_i)$-subsystem as

$$
\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & -c_2 & 1+\sigma_{23} & \cdots & \sigma_{2,i-1} & 0 \\ 0 & -1-\sigma_{23} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1+\sigma_{i-2,i-1} & 0 \\ 0 & -\sigma_{2,i-1} & \cdots & -1-\sigma_{i-2,i-1} & -c_{i-1} & 1 \\ 0 & 0 & \cdots & 0 & -1 & -c_i \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_i \end{bmatrix}
$$

$$
+ \begin{bmatrix} w_1^T \\ \vdots \\ w_i^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ \sigma_{2,i} z_i \\ \vdots \\ \sigma_{i-1,i} z_i \\ \nu_i \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ z_{i+1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial \alpha_1}{\partial \hat{\theta}} \\ \vdots \\ \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \end{bmatrix} (\Gamma \tau_i - \dot{\hat{\theta}}) \tag{6.3.45}
$$

where

$$
\sigma_{jk}(x, \hat{\theta}) = -\frac{\partial \alpha_{j-1}}{\alpha \hat{\theta}} \Gamma w_k \tag{6.3.46}
$$

Now the correction term is chosen as

$$
\nu_i(x_1, \ldots, x_i, \hat{\theta}) = \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \Gamma w_i = -\sum_{k=2}^{i-1} \sigma_{k,i} z_k \tag{6.3.47}
$$

Because we do not use $\dot{\hat{\theta}} = \Gamma \tau_i$ as an update law, the resulting $\dot{V}_i$ is

$$
\dot{V}_i = \sum_{k=1}^{i} c_k z_k^2 + z_i z_{i+1} + \left(\sum_{k=1}^{i-1} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) (\Gamma \tau_i - \dot{\hat{\theta}}) + \tilde{\theta}^T (\tau_i - \Gamma^{-1} \dot{\hat{\theta}}) \tag{6.3.48}
$$

and the $(z_1, \ldots, z_i)$-subsystem becomes

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 & 1+\sigma_{23} & \cdots & \sigma_{2i} \\ 0 & -1-\sigma_{23} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1+\sigma_{i-1,i} \\ 0 & -\sigma_{2i} & \cdots & -1-\sigma_{i-1,i} & -c_i \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_i \end{bmatrix}$$

$$+ \begin{bmatrix} w_1^T \\ \vdots \\ w_i^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ z_{i+1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial \alpha_1}{\partial \hat{\theta}} \\ \vdots \\ \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \end{bmatrix} (\Gamma \tau_i - \dot{\hat{\theta}}) \tag{6.3.49}$$

Step n. At the final step, we introduce

$$z_n = x_n - \alpha_{n-1} \tag{6.3.50}$$

and rewrite the last equation $\dot{x}_n = \beta(x)u + \varphi_n(x)^T \theta$ as

$$\dot{z}_n = \beta u + \varphi_n^T \theta - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}(x_{k+1} + \varphi_k^T \theta) - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \tag{6.3.51}$$

where the last regressor vector is defined as

$$w_n(x, \hat{\theta}) = \varphi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} \varphi_k \tag{6.3.52}$$

In this equation, the actual control input is at our disposal. We are finally in the position to design our actual update law $\dot{\hat{\theta}} = \Gamma \tau_n$ and feedback control u to stabilize the full z-system with respect to

$$\begin{aligned} V_n &= V_{n-1} + \frac{1}{2} z_n^2 \\ &= \frac{1}{2} z^T z + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \end{aligned} \tag{6.3.53}$$

Our goal is to make $\dot{V}_n$ nonpositive:

$$\begin{aligned} \dot{V}_n = & - \sum_{k=1}^{n-1} c_k z_k^2 + \left(\sum_{k=1}^{n-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) (\Gamma \tau_{n-1} - \dot{\hat{\theta}}) \\ & + z_n \left[z_{n-1} + \beta u - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} + w_n^T \hat{\theta} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right] \\ & + \tilde{\theta}^T \left(\tau_{n-1} + w_n z_n - \Gamma^{-1} \dot{\hat{\theta}} \right) \end{aligned} \tag{6.3.54}$$

To eliminate $\tilde{\theta}$ from $\dot{V}_n$, we choose the update law

$$\begin{aligned}\dot{\hat{\theta}} &= \Gamma\tau_n(z,\hat{\theta}) = \Gamma\tau_{n-1} + \Gamma w_n z_n \\ &= \Gamma W(z,\hat{\theta})z\end{aligned} \quad (6.3.55)$$

where the regressor matrix W is composed of the regressor vector $w_1, \ldots, w_n$:

$$W(z,\hat{\theta}) = [w_1, \ldots, w_n] \quad (6.3.56)$$

We choose the control u to make the bracketed term multiplying z_n equal to $-c_n z_n$:

$$u = \frac{1}{\beta}\left(-z_{n-1} - c_n z_n + \sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}x_{k+1} - w_n^T z_n + \frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\Gamma\tau_n + \nu_n\right) \quad (6.3.57)$$

where ν_n is a correction term yet to be chosen. With (6.3.57), $\dot{V}_n$ becomes

$$\dot{V}_n = -\sum_{k=1}^{n-1} c_k z_k^2 + \left(\sum_{k=1}^{n-2} z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\theta}}\right)(\Gamma\tau_{n-1} - \dot{\hat{\theta}}) + z_n\nu_n \quad (6.3.58)$$

Then, noting that

$$\dot{\hat{\theta}} - \Gamma\tau_{n-1} = \Gamma\tau_n - \Gamma\tau_{n-1} = \Gamma w_n z_n \quad (6.3.59)$$

we rewrite $\dot{V}_n$ as

$$\dot{V}_n = -\sum_{k=1}^{n-1} c_k z_k^2 + z_n\left(\nu_n - \sum_{k=1}^{n-2} z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\theta}}\Gamma w_n\right) \quad (6.3.60)$$

Now the correction term ν_n is chosen as

$$\nu_n(x,\hat{\theta}) = \sum_{k=1}^{n-2} z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\theta}}\Gamma w_n = -\sum_{k=2}^{n-1}\gamma_{k,n} z_k \quad (6.3.61)$$

We have thus reached our goal:

$$\dot{V}_n = -\sum_{k=1}^{n} c_k z_k^2 \quad (6.3.62)$$

The overall closed-loop system is

$$\dot{z} = A_z(z,\hat{\theta})z + W(z,\theta)^T\tilde{\theta} \quad (6.3.63)$$

$$\dot{\hat{\theta}} = \Gamma W(z,\hat{\theta})z \quad (6.3.64)$$

where

$$A_z(z,\hat{\theta}) = \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 & 1+\sigma_{23} & \cdots & \sigma_{2n} \\ 0 & -1-\sigma_{23} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1+\sigma_{n-1,n} \\ 0 & -\sigma_{2n} & \cdots & -1-\sigma_{n-1,n} & -c_n \end{bmatrix} \quad (6.3.65)$$

The system (6.3.63) will be referred to as the *error system.* It is important to note that a major portion of the designed effort was invested into achieving

$$A_z(z,\hat{\theta}) + A_z(z,\hat{\theta})^T = -2\begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}, \quad \forall (z,\hat{\theta}) \in \mathbb{R}^{n+p} \quad (6.3.66)$$

which yields (6.3.62) with the simple quadratic Lyapunov function (6.3.53). We observe that, as desired, the system (6.3.63)-(6.3.64) has an equilibrium at $(z,\tilde{\theta}) = (0,0)$. The stability properties of this equilibrium will be established in Section 6.3.2.

Example 6.3.1 *In application of the tuning functions procedure, we do not need to repeat the Lyapunov argument. All we need for a specific design are the final analytical expressions provided by the procedure. Let us now illustrate this by designing an adaptive controller for the benchmark system from Example 6.2.7:*

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^T\theta \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \quad (6.3.67)$$

The design objective is the regulation of the output $y = x_1$ to the set-point y_s. The first three expressions provided by the procedure are the definitions (6.3.4), (6.3.5), *and* (6.3.14) *of the error variables*

$$\begin{aligned} z_1 &= x_1 - y_s \\ z_2 &= x_2 - \alpha_1(x_1,\hat{\theta}) \\ z_3 &= x_3 - \alpha_2(x_1,x_2,\hat{\theta}) \end{aligned} \quad (6.3.68)$$

where α_1 *and* α_2 *are the stabilizing functions given by* (6.3.11) *and* (6.3.20):

$$\begin{aligned}\alpha_1 &= -c_1 z_1 - \varphi^T\hat{\theta} \\ \alpha_2 &= -c_2 z_2 - z_1 + \frac{\partial\alpha_1}{\partial x_1}(x_2 + \varphi^T\hat{\theta}) + \frac{\partial\alpha_1}{\partial\hat{\theta}}\tau_2\end{aligned} \qquad (6.3.69)$$

The tuning functions, determined from (6.3.10), (6.3.19), *and* (6.3.28), *are*

$$\begin{aligned}\tau_1 &= z_1\varphi \\ \tau_2 &= \tau_1 - z_2\frac{\partial\alpha_1}{\partial x_1}\varphi \\ \tau_3 &= \tau_2 - z_3\frac{\partial\alpha_2}{\partial x_1}\varphi\end{aligned} \qquad (6.3.70)$$

With the above expressions and the choice $\Gamma = I$, *the parameter update law and the feedback control are obtained from* (6.3.55) *and* (6.3.29), *respectively. They are*

$$\dot{\hat{\theta}} = \tau_3 = z_1\varphi - z_2\frac{\partial\alpha_1}{\partial x_1}\varphi - z_3\frac{\partial\alpha_2}{\partial x_1}\varphi \qquad (6.3.71)$$

$$\begin{aligned}u &= -c_3 z_3 - z_2 + \frac{\partial\alpha_2}{\partial x_1}(x_2 + \varphi^T\hat{\theta}) + \frac{\partial\alpha_2}{\partial x_2}x_3 + \frac{\partial\alpha_2}{\partial\hat{\theta}}\tau_3 \\ &\quad -z_2\frac{\partial\alpha_2}{\partial\hat{\theta}}\frac{\partial\alpha_2}{\partial x_1}\varphi\end{aligned} \qquad (6.3.72)$$

This completes the design of the adaptive controller for (6.3.67). *In the* $(z, \tilde{\theta})$*-coordinates the designed system is*

$$\dot{z} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 - \frac{\partial\alpha_2}{\partial x_1}|\varphi|^2 \\ 0 & -1 + \frac{\partial\alpha_2}{\partial x_1}|\varphi|^2 & -c_3 \end{bmatrix} z + \begin{bmatrix} 1 \\ -\frac{\partial\alpha_1}{\partial x_1} \\ -\frac{\partial\alpha_2}{\partial x_1} \end{bmatrix}\varphi^T\tilde{\theta} \qquad (6.3.73)$$

$$\dot{\tilde{\theta}} = -\varphi\left[\, 1, \; -\frac{\partial\alpha_1}{\partial x_1}, \; -\frac{\partial\alpha_2}{\partial x_1} \,\right] z \qquad (6.3.74)$$

It is of interest to relate the stabilizing functions α_1 *and* α_2 *and the control law* u *to the material from Section 6.2. The stabilizing function* α_1 *has a "certainty equivalence" form. The stabilizing function* α_2 *has the term* $-\frac{\partial\alpha_1}{\partial\hat{\theta}}\tau_2$ *which accounts for parameter estimation transients, while the rest of it is in the "certainty equivalence" form. The control law* u *departs from the "certainty equivalence" form in the last two terms whose role in the same as that of* (6.2.60). *The last term in* u *is particularly important. Since* $-\frac{\partial\alpha_1}{\partial\hat{\theta}} = -\varphi^T$, *this term contributes with* $+ - \frac{\partial\alpha_2}{\partial x_1}|\varphi|^2$ *in the 1system matrix' in* (6.3.73) *and achieves the skew symmetry, which is crucial for stability.* □

6.3.2 Stability and convergence

To investigate stability properties of the closed-loop adaptive system (6.3.63)-(6.3.64), we express φ_i, α_i, τ_i, and w_i in the z-coordinates. Then, by the uniform stability theorem, the global stability of the equilibrium $(z, \tilde{\theta}) = 0$ follows from the fact that the derivative $\dot{V}_n$ of V_n along the solutions of (6.3.63)-(6.3.64) is given by (6.3.62).

From LaSalle's Invariance Theorem, it further follows that the ($(n+p)$-dimensional) state $(z(t), \tilde{\theta}(t))$ converges to the largest invariant set where $\dot{V}_n = 0$. This means, in particular, that $z(t) \to 0$ as $t \to 0$.

We now set out to determine M. On this invariant set, we have $z \equiv 0$ and $\dot{z} \equiv 0$. Setting $z = 0$, $\dot{z} = 0$ in (6.3.63) we obtain $\dot{\hat{\theta}} = 0$ and

$$W(z, \hat{\theta})^T (\theta - \hat{\theta}) = 0, \quad \forall (z, \hat{\theta}) \in M \tag{6.3.75}$$

From (6.3.38) and (6.3.56), it is easily seen that

$$W(z, \hat{\theta})^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{\partial \alpha_1}{\partial x_1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\frac{\partial \alpha_{n-1}}{\partial x_1} & \cdots & -\frac{\partial \alpha_{n-1}}{\partial x_{n-1}} & 1 \end{bmatrix} F(x)^T \triangleq N(z, \hat{\theta}) F(x)^T \tag{6.3.76}$$

Since $N(z, \hat{\theta})$ is obviously nonsingular for all $(z, \hat{\theta}) \in M$, then (6.3.75) and (6.3.76) imply

$$F(x)^T (\theta - \hat{\theta}) = 0 \quad on\ M \tag{6.3.77}$$

Now we show that $x = x^e$ on M. Since $z_1 = x_1 - y_s$ then $x_1 = y_s = x_1^e$ on M. In view of (6.3.77), we get

$$(\theta - \hat{\theta})^T \varphi_1(x_1^e) = 0 \quad on\ M \tag{6.3.78}$$

Recall from (6.3.11) that $\alpha_1 = -c_1 z_1 - \hat{\theta}^T \varphi_1$. Therefore, on M, we have $\alpha_1 = -\hat{\theta}^T \varphi_1(x_1^e)$. Combining this with $z_2 = 0 = x_2 - \alpha_1$ and (6.3.3), we get $x_2 = x_2^e$ on M. Using (6.3.77), we obtain

$$(\theta - \hat{\theta})^T \varphi_2(x_1^e, x_2^e) = 0 \quad on\ M \tag{6.3.79}$$

Continuing in the same fashion, we prove that $x_i = x_i^e$ and $(\theta - \hat{\theta})^T \varphi_i(x_1^e, \ldots, x_i^e) = 0$ on M, $i = 1, \ldots, n$. Thus, the largest invariant set M in E is

$$\begin{aligned} M &= \left\{(z, \tilde{\theta}) \in \mathbb{R}^{n+p} \mid z = 0,\ F_e^T \tilde{\theta} = 0\right\} \\ &= \left\{(x, \hat{\theta}) \in \mathbb{R}^{n+p} \mid x = 0,\ F_e^T \hat{\theta} = F_e^T \theta\right\} \end{aligned} \tag{6.3.80}$$

where $F_e = F(x^e)$. The two equivalent expressions for M and the convergence of $(z(t), \tilde{\theta}(t))$ to M prove that $x(t) \to x^e$ as $t \to \infty$.

An important property of M is its dimension, $p - rank\{F_e\}$. When $rank\{F_e\} = p$, then $\dim M = 0$, that is, M becomes the equilibrium point $x = x^e$, $\hat{\theta} = \theta$. This means that the parameter estimates converge to their true values, so that the equilibrium $x = x^e$, $\hat{\theta} = \theta$ is globally *asymptotically* stable.

The above facts prove the following result:

Theorem 6.3.2 *The closed-loop adaptive system consisting of the plant* (6.3.1), *the controller* (6.3.57), *and the update law* (6.3.55) *has a globally stable equilibrium* $(x, \hat{\theta}) = (x^e, \theta)$. *Furthermore, its state* $(x(t), \hat{\theta}(t))$ *converges to the* $(p - rank\{F_e\})$*-dimensional equilibrium manifold* M *given by* (6.3.80), *which means, in particular, that*

$$\lim_{t \to \infty} x(t) = x^e \tag{6.3.81}$$

If $y_s = 0$ *and* $F(0) = 0$, *then* $\lim_{t \to \infty} x(t) = 0$. *The equilibrium* $x = x^e$, $\hat{\theta} = \theta$ *is globally asymptotically stable if and only if* $rank\{F_e\} = p$. □

As the dimension of M reduces, the stability properties of the adaptive system improve. The most desirable case is when M is an equilibrium point, in which case, this equilibrium is globally asymptotically stable, and the parameter estimates converge to the actual parameter values. Global asymptotic stability can be achieved with as many as $p = n$ unknown parameters. This is among the main advantages of eliminating overparametrization.

We now discuss the basic stability properties established in Theorem 6.2.12 on a simple example.

Example 6.3.3 *We consider the second order system with an unknown parameter vector* $\theta \in \mathbb{R}^p$*:*

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(x_1)^T\theta \\ \dot{x}_2 &= u + \varphi_2(x)^T\theta \end{aligned} \tag{6.3.82}$$

The control objective is to regulate x *to zero* ($x_1^e = 0$)*. We define the error variable*

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 - \alpha_1(x_1, \hat{\theta}) \end{aligned} \tag{6.3.83}$$

The controller is designed applying (6.3.11) *and* (6.3.20) *as*

$$\begin{aligned} \alpha_1 &= -c_1 z_1 - \varphi_1(x_1)^T\hat{\theta} \\ \frac{\partial\alpha_1}{\partial x_1} &= -c_1 - \frac{\partial\varphi_1 x_1{}^T}{\partial x_1}\hat{\theta}, \qquad \frac{\partial\alpha_1}{\partial\hat{\theta}} = -\varphi_1(x_1)^T \\ u &= -z_1 - c_2 z_2 + \frac{\partial\alpha_1}{\partial x_1}x_2 - \left(\varphi_2(x)^T - \frac{\partial\alpha_1}{\partial x_1}\varphi_1(x_1)^T\right)\hat{\theta} + \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} \end{aligned} \tag{6.3.84}$$

while the parameter update law is

$$\dot{\hat{\theta}} = \Gamma\left[\begin{array}{cc} \varphi_1, & \varphi_2 - \frac{\partial\alpha_1}{\partial x_1}\varphi_1 \end{array}\right] z \tag{6.3.85}$$

The resulting error system is

$$\dot{z} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} z + \begin{bmatrix} \varphi_1^T \\ \varphi_2^T - \frac{\partial\alpha_1}{\partial x_1}\varphi_1^T \end{bmatrix} \tilde{\theta} \tag{6.3.86}$$

Now we illustrate and discuss the stability properties established by Theorem 6.3.2. From (6.3.82) we see that $x_1^e = 0$, $x_2^e = -\varphi_1(0)^T\theta$. By Theorem 6.3.2, the point

$$\begin{bmatrix} x_1 \\ x_2 \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ -\varphi_1(0)^T\theta \\ \theta \end{bmatrix} \tag{6.3.87}$$

is a globally stable equilibrium, and the stable of the closed-loop system converges to the equilibrium manifold

$$\begin{aligned} M &= \left\{ (x, \hat{\theta}) \in \mathbb{R}^{2+p} \mid \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\varphi_1(0)^T\theta \end{bmatrix} \right. \\ & \qquad \left. , \begin{bmatrix} \varphi_1(0)^T \\ \varphi_2(0, -\varphi_1(0)^T\theta)^T \end{bmatrix} (\theta - \hat{\theta}) = 0 \right\} \end{aligned} \tag{6.3.88}$$

A basic question that one would ask is: What type of figure in $\mathbb{R}^{2+p}$ is M? Our further discussion will, without loss of generality, be limited to $p \leq 2$.

In the simplest case where $\dim\theta = p = 1$, the following two possibilities exist:

- If both $\varphi_1(0) = 0$ and $\varphi_2(0,0) = 0$, then the manifold M is the subspace $x = 0 \in \mathbb{R}^3$, that is, M is the $\hat{\theta}$-axis

- If either $\varphi_1 \neq 0$ or $\varphi_2(0, -\varphi_1(0)\theta) \neq 0$, then the manifold M is the single point $x_1 = 0$, $x_2 = -\varphi_1(0)\theta$, $\hat{\theta} = \theta$. This point is an equilibrium which is not only globally stable, but also globally *asymptotically* stable.

Next, we analyze the case $p = 2$.

- Suppose $\begin{bmatrix} \varphi_1(x_1)^T \\ \varphi_2(x_1)^T \end{bmatrix} = \begin{bmatrix} x_1^2 & e^{x_1} \\ \cos x_1 & 0 \end{bmatrix}$. Since $\begin{bmatrix} \varphi_1(0)^T \\ \varphi_2(0)^T \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has full rank, the manifold M is the single point $x_1 = 0$, $x_2 = -\theta_2$, $\hat{\theta}_1 = \theta_1$, $\hat{\theta}_2 = \theta_2$, which is a globally *asymptotically* stable equilibrium.

- Suppose $\begin{bmatrix} \varphi_1(x_1)^T \\ \varphi_2(x_1)^T \end{bmatrix} = \begin{bmatrix} -\cos x_1 & e^{x_1} \\ \sin x_1 & 0 \end{bmatrix}$. Since $\begin{bmatrix} \varphi_1(0)^T \\ \varphi_2(0)^T \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, the manifold M is the linear variety $x_1 = 0$, $x_2 = \theta_1 - \theta_2$, $\hat{\theta}_2 - \hat{\theta}_1 = \theta_2 - \theta_1$. Neither of the parameter estimates in guaranteed to converge to the actual parameter value, but they are jointly converging to the line $\hat{\theta}_2 = \hat{\theta}_1 + \theta_2 - \theta_1$ in the plane $x_1 = 0$, $x_2 = \theta_1 - \theta_2$.

- Suppose $\begin{bmatrix} \varphi_1(x_1)^T \\ \varphi_2(x_1)^T \end{bmatrix} = \begin{bmatrix} x_1^2 & e^{x_1} - 1 \\ \sin x_1 & 0 \end{bmatrix}$. Since $\begin{bmatrix} \varphi_1(0)^T \\ \varphi_2(0)^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the manifold M is the plane (linear variety) $x = 0$. This is the case of the weakest convergence properties because one cannot guarantee that the parameter estimates converge to any submanifold in the plane M.

6.4 Tracking

The set-point regulation design is readily extended to the task of tracking.

The control objective is to free the output $y = x_1$ of the system (6.3.1)

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(x_1)^T\theta \\ \dot{x}_2 &= x_3 + \varphi_2(x_1, x_2)^T\theta \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \varphi_{n-1}(x_1, \ldots, x_{n-1})^T\theta \\ \dot{x}_n &= \beta(x)u + \varphi_n(x)^T\theta \end{aligned} \tag{6.4.1}$$

to asymptotically track the reference output $y_r(t)$ whose first n derivative are assumed to be known, bounded, and piecewise continuous.

An alternative control objective would be asymptotically track the output of a known asymptotically stable linear reference model

$$y_r = G_m(s)r(s) = \frac{k_m}{s^n + m_{n-1}z^{n-1} + \ldots + m_0} r(s) \tag{6.4.2}$$

where the denominator is Hurwitz, $k_m > 0$, and $r(t)$ is bounded and piecewise continuous. A realization which is of particular interest is

$$\begin{aligned} \dot{x}_m &= \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -m_0 & \cdots & \cdots & -m_{n-1} \end{bmatrix} x_m + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_m \end{bmatrix} \\ y_r &= x_{m,1} \end{aligned} \tag{6.4.3}$$

because, in this case, the derivatives of y_r are available as the states of the reference model: $y_r^{(i)} = x_{m,i+1}$, $i = 0, \ldots, n-1$.

These functions are used in the design for tracking.

$$z_i = x_i - y_r^{i-1} - \alpha_{i-1} \tag{6.4.4}$$

$$\alpha_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-1)}) = -z_{i-1} - c_i z_i - w_i^T\hat{\theta} + \sum_{k=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_k}x_{k+1} + \frac{\partial\alpha_{i-1}}{\partial y_r^{(k-1)}}y_r^{(k)}\right) - k_i|w_i|^2 z_i + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma\tau_i + \sum_{k=2}^{i-1}\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\Gamma w_i z_k \tag{6.4.5}$$

$$\tau_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-1)}) = \tau_{i-1} + w_i z_i \tag{6.4.6}$$

$$w_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-2)}) = \varphi_i - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}\varphi_k \tag{6.4.7}$$

for $i = 1, \ldots, n$, $\bar{x}_i = (x_1, \ldots, x_i)$ and $\bar{y}_r^{(i)} = (y_r, \dot{y}_r, \ldots, y_r^{(i)})$

Adaptive control law:

$$u = \frac{1}{\beta(x)}[\alpha_n(x, \hat{\theta}, \bar{y}_r^{(n-1)}) + y_r^{(n)}] \tag{6.4.8}$$

Parameter update law:

$$\dot{\hat{\theta}} = \Gamma\tau_n(x, \hat{\theta}, \bar{y}_r^{(n-1)}) = \Gamma W z \tag{6.4.9}$$

The closed-loop adaptive system consisting of the plant (6.4.1), the controller (6.4.8), and the update law (6.4.9) has a globally uniformly stable equilibrium at $(z, \tilde{\theta}) = 0$, and $\lim_{t\to\infty} = 0$, which means, in particular. that global asymptotic tracking is achieved:

$$\lim_{t\to\infty}[y(t) - y_r(t)] = 0 \tag{6.4.10}$$

Moreover, if $\lim_{t\to\infty} y_r^{(i)} = 0$, $i = 0, \ldots, n-1$, and $F(0) = -$, then $\lim_{t\to\infty} x(t) = 0$

6.5 Unknown virtual control coefficients

For the sake of clarity, the adaptive design in this chapter was presented for the class of parametric strict-feedback system. We now give an extension of the tuning function design for systems with unknown virtual control coefficient. Consider systems of the form

$$\begin{aligned}
\dot{x}_1 &= b_1 x_2 + \varphi_1(x_1)^T\theta \\
\dot{x}_2 &= b_2 x_3 + \varphi_2(x_1, x_2)^T\theta \\
&\vdots \\
\dot{x}_i &= b_i x_{i+1} + \varphi_i(x_1, \ldots, x_i)^T\theta, \qquad i = 1, \ldots, n-1 \\
&\vdots \\
\dot{x}_n &= b_n\beta(x)u + \varphi_n(x_1, \ldots x_n)^T\theta
\end{aligned} \tag{6.5.1}$$

where, in addition to the unknown vector θ, the constant coefficients b_i are also unknown. We refer to the coefficients b_i as the 'virtual control coefficients' The occurrence of the unknown b_i-coefficients is frequent in applications ranging from electric motors and robotic manipulators to flight dynamics.

When the signs of b_i, $i = 1, \ldots, n$, are known.

We consider two special cases of (6.5.1). The extension to the general case is straight-forward but tedious.

The first special case is when the only unknown virtual control coefficient is the 'high-frequency gain' b_n:

$$\begin{aligned}
\dot{x}_i &= x_{i+1} + \varphi_i(x_1, \ldots, x_i)^T\theta, \qquad i = 1, \ldots, n-1 \\
\dot{x}_n &= b_n\beta(x)u + \varphi_n(x_1, \ldots, x_n)^T\theta
\end{aligned} \tag{6.5.2}$$

For this case the modification of the tuning functions design is simple. In the design of tuning function for tracking, we only need to change the control law (6.4.8):

$$u = \frac{\hat{\varrho}}{\beta(x)}[\alpha_n(x, \hat{\theta}, \bar{y}_r^{(n-1)}) + y_r^{(n)}] \tag{6.5.3}$$

where $\hat{\varrho}$ is the estimate of $\varrho = 1/b_n$ computed as

$$\dot{\hat{\varrho}} = -\gamma sgn(b_n)\left(\alpha_n + y_r^{(n)}\right) z_n, \qquad \gamma > 0 \tag{6.5.4}$$

We are only using the knowledge of the *sign* of the unknown parameter b_n. In this simple case it is not necessary to estimate b_n itself. It can be checked that the resulting error system has the form (??) with an additional term due to $\tilde{\varrho} = \varrho - \hat{\varrho}$:

$$\dot{z} = A_z(z,\hat{\theta},t)z + W(z,\hat{\theta},t)^T\tilde{\theta} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -b_n\left(\alpha_n + y_r^{(n)}\right) \end{bmatrix} \tilde{\varrho} \tag{6.5.5}$$

Consider the Lyapunov function

$$V = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \frac{|b_n|}{2\gamma}\tilde{\varrho}^2 \tag{6.5.6}$$

Its derivative along the solutions of (6.5.4) and (6.5.5) is

$$\dot{V} \leq -c_0|z|^2 \tag{6.5.7}$$

which satisfies tracking conditions, which means, all the states are bounded and asymptotic tracking is achieved.

Now we move on to a more difficult case:

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \varphi_i(x_1,\ldots,x_i)^T\theta, \qquad i = 1,\ldots,m-1,m+1,\ldots,n-1 \\ \dot{x}_m &= b_m x_{m+1} + \varphi_m(x_1,\ldots,x_m)^T\theta \\ \dot{x}_n &= \beta(x)u + \varphi_n(x_1,\ldots,x_n)^T\theta \end{aligned} \tag{6.5.8}$$

where b_m, $m < n$, is the only unknown coefficient. From step m on, the design procedure for this case differs considerably from the tracking design procedure. We now need $\hat{b}_m$ and $\hat{\varrho}$ the estimates of b_m and $\varrho = 1/b_m$. The estimate $\hat{\varrho}$ is introduced to avoid the division by $\hat{b}_m(t)$ which can occasionally take value zero. The complete design procedure is given by the following expressions (with $z_0 = 0$, $\alpha_0 = 0$, $\tau_0 = 0$):

Coordinate transformation:

$$z_i = x_i - y_r^{(i-1)} - \alpha_{i-1}, \qquad i = 1,\ldots,m \tag{6.5.9}$$

$$z_j = x_j - \hat{\varrho}y_r^{(j-1)} - \alpha_{j-1} \qquad j = m+1,\ldots,n \tag{6.5.10}$$

Regressor:

$$w_i = \varphi_i - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}\varphi_k, \qquad i = 1,\ldots,n \tag{6.5.11}$$

Tuning function for $\hat{\theta}$:

$$\tau_i = \tau_{i+1} + w_i z_i, \qquad i = 1, \ldots, n \tag{6.5.12}$$

Tuning function for $\hat{b}_m$:

$$\pi_m = z_{m+1} z_m \tag{6.5.13}$$

$$\pi_j = \pi_{j-1} - \frac{\partial \alpha_m}{\partial x_m} x_{m+1} z_j, \qquad j = m+1, \ldots, n \tag{6.5.14}$$

Stabilizing functions:

$$\alpha_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-1)}) = \bar{\alpha}_i, \qquad i = 1, \ldots, m-1 \tag{6.5.15}$$

$$\alpha_m(\bar{x}_m, \hat{\theta}, \bar{y}_r^{(m-1)}, \hat{\varrho}) = \hat{\varrho}\bar{\alpha}_m \tag{6.5.16}$$

$$\alpha_j(\bar{x}_j, \hat{\theta}, \bar{y}_r^{(j-1)}, \hat{b}_m, \hat{\varrho}) = \bar{\alpha}_j, \qquad j = m+1, \ldots, n \tag{6.5.17}$$

$$\begin{aligned}
\bar{\alpha}_i &= -z_{i-1} - c_i z_i - w_i^T \hat{\theta} + \sum_{k=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) \\
&\quad + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i + \sum_{k=2}^{i-1} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma w_i z_k, \qquad i = 1, \ldots, m
\end{aligned} \tag{6.5.18}$$

$$\begin{aligned}
\bar{\alpha}_{m+1} &= -\hat{b}_m z_m - c_{m+1} z_{m+1} - w_{m+1}^T \hat{\theta} + \sum_{k=1}^{m-1} \frac{\partial \alpha_m}{\partial x_k} x_{k+1} + \hat{b}_m \frac{\partial \alpha_m}{\partial x_m} x_{m+1} \\
&\quad + \sum_{k=1}^{m} \frac{\partial \alpha_m}{\partial y_r^{(k-1)}} y_r^{(k)} + \frac{\partial \alpha_m}{\partial \hat{\theta}} \Gamma \tau_{m+1} + \left(y_r^{(m)} + \frac{\partial \alpha_m}{\partial \hat{\varrho}} \right) \dot{\hat{\varrho}} \\
&\quad + \sum_{k=2}^{m} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma w_{m+1} z_k
\end{aligned} \tag{6.5.19}$$

$$\begin{aligned}
\bar{\alpha}_j &= -z_{j-1} - c_j z_j - w_j^T \hat{\theta} + \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_k} x_{k+1} + \hat{b}_m \frac{\partial \alpha_{j-1}}{\partial x_m} x_{m+1} \\
&\quad + \sum_{k=1 \neq m}^{j-1} \frac{\partial \alpha_{j-1}}{\partial y_r^{(k-1)}} y_r^{(k)} + \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma \tau_j + \frac{\partial \alpha_{j-1}}{\partial \hat{b}_m} \gamma \pi_j + \left(y_r^{(j-1)} + \frac{\partial \alpha_{j-1}}{\partial \hat{\varrho}} \right) \dot{\hat{\varrho}} \\
&\quad + \sum_{k=2}^{j-1} \frac{\partial \alpha_{k-1}}{\partial \hat{b}_m} \gamma \frac{\partial \alpha_{j-1}}{\partial x_m} x_{m+1} z_k, \qquad j = m+2, \ldots, n
\end{aligned} \tag{6.5.20}$$

Adaptive control law:

$$u = \frac{1}{\beta(x)} [\alpha_n + \hat{\varrho} y_r^{(n)}] \tag{6.5.21}$$

Parameter update laws:

$$\dot{\hat{\theta}} = \Gamma\tau_n = \Gamma W_z \quad (6.5.22)$$

$$\dot{\hat{b}}_m = \gamma\pi_n = \gamma\left[z_{m+1}z_m - \sum_{j=m+1}^{n}\frac{\partial\alpha_{j-1}}{\partial x_m}x_{m+1}z_j\right] \quad (6.5.23)$$

$$\dot{\hat{\varrho}} = -\gamma sgn(b_m)(y_r^{(m)} + \bar{\alpha}_m)z_m \quad (6.5.24)$$

Lengthy but straightforward calculations show that the design procedure (6.5.9) - (6.5.24) results in the closed-loop system

$$\begin{aligned}
\dot{z}_i &= -c_i z_i - \sum_{k=2}^{i-1}\sigma_{ki}z_k - z_{i-1} + \sum_{k=i+1}^{n}\sigma_{ik}z_k + w_i^T\tilde{\theta} \\
&\quad , i = 1, \ldots, m-1 && (6.5.25) \\
\dot{z}_m &= -c_m z_m - \sum_{k=2}^{m-1}\sigma_{km}z_k - z_{m-1} + \hat{b}_m z_{m+1} + \sum_{k=m+1}^{n}\sigma_{mk}z_k \\
&\quad + w_m^T\tilde{\theta} - b_m(y_r^{(m)} - \bar{\alpha}_m)\tilde{\varrho} + z_{m+1}\tilde{b}_m && (6.5.26) \\
\dot{z}_{m+1} &= -c_{m+1}z_{m+1} - \sum_{k=2}^{m}\sigma_{k,m+1}z_k - b_m z_m + z_{m+2} + \sum_{k=m+2}^{n}\sigma_{m+1,k}z_k \\
&\quad + w_{m+1}^T\tilde{\theta} - \frac{\partial\alpha_m}{\partial x_m}x_{m+1}\tilde{b}_m && (6.5.27) \\
\dot{z}_j &= -c_j z_j - \sum_{k=2}^{j-1}\sigma_{kj}z_k - z_{j-1} + z_{j+1} + \sum_{k=j+1}^{n}\sigma_{jk}z_k \\
&\quad + w_j^T\tilde{\theta} - \frac{\partial\alpha_{j-1}}{\partial x_m}x_{m+1}\tilde{b}_m, \quad i = m+2, \ldots, n && (6.5.28)
\end{aligned}$$

where σ_{ik} is defined for $k = i+1, \ldots, n$ as

$$\sigma_{ik} = \begin{cases} 0, & i = 1 \\ -\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma w_k, & i = 2, \ldots, m+1 \\ -\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma w_k + \frac{\partial\alpha_{i-1}}{\partial\hat{b}_m}\gamma\frac{\partial\alpha_{k-1}}{\partial x_m}x_{m+1}, & i = m+2, \ldots, n-1 \end{cases} \quad (6.5.29)$$

A Lyapunov function for this system is

$$V = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \frac{1}{2\gamma}\tilde{b}_m^2 + \frac{|b_m|}{2\gamma}\tilde{\varrho}^2 \quad (6.5.30)$$

Its derivative along the solutions of (6.5.22)-(6.5.24) and (6.5.25)-(??),

$$\dot{V} = -\sum_{k=1}^{n}c_k z_k^2 \quad (6.5.31)$$

lead us to the same conclusion as in tracking design procedures which all the states are bounded and asymptotic tracking is achieved.

Bibliography

[1] G. Bastin and G. Campion, "Indirect adaptive control of linearly parametrized nonlinear system," *Proceedings of the 3rd IFAC Symposium on Adaptive Systems in Control and Signal Processing,* Glasgow, UK, 1989.

[2] C. I. Byrnes and A. Isidori, "New results and examples in nonlinear feedback stabilization," *Systems and Control Letters*, vol. 12, pp. 437-442, 1989.

[3] G. Campion and G. Bastin, "Indirect adaptive state-feedback control of linearly parametrized nonlinear systems" *International Journal of Adaptive Control and Signal Processing.* vol. 4, pp. 345-358, 1990.

[4] J.J. Craig, *Adaptive Control of Mechnical Manipulators,* Reading, MA: Addison-Wesley, 1988.

[5] W. Dayawansa, W. M. Boothby and D. L. Elliott, "Global state and feedback equivalence of nonlinear systems," *Systems and Control Letters*, vol. 6, pp. 229-234, 1985.

[6] A. Feuer, A. S. Morse, "Local stability of parameter adaptive control systems," *Proceeding of the 1978 Conference on Information Sciences and Systems,* Johns Hopkins, Baltimore, MD, pp. 107-111, 1978.

[7] L. R. Hunt, R. Su, and G. Meyer, "Global transformations of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 28, pp. 24-31, 1983.

[8] L. R. Hunt, R. Su, and G. Meyer, "Design for multi-input nonlinear system," in *Differential Geometric Control Theory*, R. W. Brockett, R. S. Millman, and H. S. Sussmann, Eds., Boston: Birkhäuser, 1983.

[9] A. Isidori, *Nonlinear Control Systems*, Berlin: Springer-Verlag, 1989.

[10] B. Jakubczyk, W. Respondek, "On linearization of systems." *Bulletin of the Polish Academy of Science, Series on Mathematical Science*, vol. 28, no. 9-10, pp. 517-522, 1980.

[11] Z. P. Jiang and L. Praly, "Iterative designs of adaptive controllers for systems with nonlinear integrators," *Proceeding of the 30th IEEE Conference on Decision and Control*, Brington, UK, December 1991, pp. 2482- 2487.

[12] I. Kanellakopoulos, *Adaptive Control of Nonlinear Systems*, Ph. D. Dissertation, University of Illinois, Urbana, 1991.

[13] I. Kanellakopoulos, P. V. Kokotovic, and R. Marino, "An extended direct scheme for robust adaptive nonlinear control," *Automatica*, vol. 27, pp. 247-255, 1991.

[14] I. Kanellakopoulos, P. V. Kokotovic, and R. H. Middleton, "Observer based adaptive control of nonlinear systems under matching conditions," *Proceeding of the 1990 American Control Conference*, San Diego, CA, pp. 549-552.

[15] I. Kanellakopoulos, P. V. Kokotovic, and R. H. Middleton, "Indirect adaptive output-feedback control of a class of nonlinear systems," *Proceeding of the 29th IEEE conference on Decision and Control*, Honolulu, HI, December 1990, pp. 2714-2719.

[16] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Transactions on Automatic Control*, vol. 36, pp. 1241-1253, 1991.

[17] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "Adaptive output feedback control of systems with output nonlinearities," pp. 495-525 in [20].

[18] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "Adaptive output feedback control of systems with output nonlinearities," *IEEE Transactions on Automatic Control*, vol. 37, pp. 1266-1282, 1992.

[19] P. V. Kokotović and H. J. Sussmann, "A positive of real condition for global stabilization of nonlinear system," *Systems and Control Letters*, vol. 13, pp. 125-133, 1989.

[20] P. V. Kokotovic, Ed., *Foundations of Adaptive Control*, Berlin : Springer-Verlag, 1991.

[21] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "Adaptive feedback linearization of nonlinear systems," pp. 311-346 in [20].

[22] M. Krstić, *Adaptive Nonlinear Control*, Ph. D. Dissertation, University of California, Santa Barbara, 1994.

[23] M. Krstić, I. Kanellakopoulos, and P. V. Kokotovic, "Adaptive nonlinear control without overparametrization," *Systems and Control Letters*, vol. 19, pp. 177-185, 1992.

[24] R. Marino, "On the largest feedback linearizable subsystem," *Systems and Control Letters*, vol. 6, pp. 345-351, 1986.

[25] R. Marino, W. M. Boothby, and D. L. Elliot, "Geometric properties of linearizable control systems," *Mathematical Systems Theory*, vol. 18, pp. 97-123, 1985.

[26] R. Marino and P. Tomei, "Dynamic output-feedback linearization and global stabilization," *Systems and Control Letters*, vol. 17, pp. 115-121, 1991.

[27] R. Marino and P. Tomei, "Global adaptive observers for nonlinear systems via filtered transformations," *IEEE Transaction on Automatic Control*, vol. 37, pp. 1239-1245, 1992.

[28] R. Marino and P. Tomei, "Global adaptive observers and output feedback stabilization for a class of nonlinear systems," pp. 455-493 in [20]

[29] R. Marino and P. Tomei, "Global Adaptive output-feedback control of nonlinear systems, Part I : linear parametrization," *IEEE Transactions on Automatic Control*, vol. 38, pp. 17-32, 1993.

[30] R. Marino and P. Tomei, "Global adaptive output-feedback control of nonlinear systems, PartII : nonlinear parametrization," *IEEE Transactions on Automatic Control*, vol. 38, pp. 33-49, 1993.

[31] R. H. Middleton and G. C. Goodwin, "Adaptive computed torque control for rigid link manipulators," *System and Control Letters*, vol. 10, pp. 9-16, 1988.

[32] K. Nam and A. Arapostathis, "A model-reference adaptive control scheme for pure-feedback nonlinear systems," *IEEE Transactions on Automatic Control*; vol. 33, pp. 803-811, 1988.

[33] H. Nijmeijer and A. van der Schaft, *Nonlinear Dynamical Control Systems*, New York: Springer-Verlag, 1990.

[34] R.Ortega and M. W. Spong, "Adaptive motion control of rigid robots : a tutorial," *Automatica*, vol. 25, pp. 877-888, 1989.

[35] J. B. Pomet and L. Praly, "Indirect adaptive nonlinear control," *Proceeding of the 27th IEEE Conference on Decision and Control*, Austin, TX, December 1988, pp. 2414-2415.

[36] J. B. Pomet and L. Praly, "Adaptive nonlinear control : an estimation based algorithm ," in *New Trends in Nonlinear Control Theory*, J. Descusse, M. Fliess, A. Isidori, and D. Leborgne, Eds., Spinger-Verlag, New York, 1989.

[37] J. B. Pomet and L. Praly, "Adaptive nonlinear regulation : estimation from the Lyapunov equation," *IEEE Transactions on Automatic Control*, vol. 37, pp. 729-740, 1992.

[38] A. Saberi, P. V. Kokotović, and H. J. Sussmann, "Global stabilization of partially linear composite systems," *SIAM J. Control Opt.*, 1990, vol. 28, pp. 1491-1503.

[39] P. Sannuti and A. Saberi, "Special coordinate basis for multivariable linear systems–finite and infinite zero structure, squaring down and decoupling," *International Journal of Control*, vol. 45, pp. 1655-1704, 1987.

[40] S. S. Sastry and M. Bodson, *Adaptive Control : Stability, Convergence and Robustness*, Englewood Cliffs, NJ: Prentice-Hall, 1989.

[41] J.-J. E. Slotine and W. Li, "On the adaptive control of robot manipulators," *International Journal of Robotics Research*, vol. 6, pp. 49-59, 1987.

[42] J.-J. E. Slotine and W. Li, "Adaptive manipulator control : a case study," *IEEE Transactions on Automatic Control*, vol. 33, pp. 995-1003, 1988.

[43] E. D. Sontag and H. J. Sussmann, "Further comments on the stability of the angular velocity of a rigid body," *System and Control Letters*, vol. 12, pp. 437-442, 1988.

[44] R. Su, "On the linear equivalents of nonlinear systems," *Systems and Control Letters*, vol. 2, pp. 48-52, 1982.

[45] D. Taylor, P. V. Kokotović, R. Marino and I. Kanellakopoulos, "Adaptive regulation of nonlinear systems with unmodeled dynamics," *IEEE Transactions on Automatic Control*, vol. 34, pp. 405-412, 1989.

[46] A. R. Teel, R. R. Kadiyala, P. V. Kokotović and S. S. Sastry, "Indirect techniques for adaptive input-output linearization of nonlinear systems," *International Journal of Control*, vol. 53, pp. 193-222, 1991.

[47] J. Tsinias, "Sufficient Lyapunov-like conditions for Stabilization," *Mathematics of Control, Signal and Systems*, vol. 2, pp. 343-357, 1989.

[48] M. Krstić, I. Kanellakopoulos and P. Kokotović, *Nonlinear and Adaptive Control Design*, New York: Wiley, 1995.

Appendix A

Stability

A.1 Main Stability Theorems

Lyapunov Stability. To begin with, we remind reader that Lyapunov stability, asymptotic stability, uniform stability, uniform asymptotic stability, etc., are properties not of a dynamic system as a whole, but rather of its individual solutions. Consider the time-varying system

$$\dot{x} = f(x,t), \tag{A.1.1}$$

where $x \in \mathbb{R}^n$, and $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x. The solution of (A.1.1) which starts from the point x_0 at time $t_0 \geq 0$ is denoted as $x(t; x_0, t_0)$ with $x(t_0; x_0, t_0) = x_0$. Lyapunov stability concepts describe continuity properties of $x(t; x_0, t_0)$ with respect to x_0. If the initial condition x_0 is perturbed to $\tilde{x}_0$, then, for stability, the resulting perturbed solution $x(t; \tilde{x}_0, t_0)$ is required to stay close to $x(t; x_0, t_0)$ for all $t \geq t_0$. In addition, for asymptotic stability, the error $x(t; \tilde{x}_0, t_0) - x(t; x_0, t_0)$ is required to vanish as $t \to \infty$. So, the solution $x(t; x_0, t_0)$ of (A.1.1) is

- ***bounded***, if there exist a constant $B(x_0, t_0) > 0$ such that

$$|x(t; x_0, t_0)| < B(x_0, t_0), \quad \forall t \geq t_0; \tag{A.1.2}$$

- ***stable***, if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon, t_0) > 0$ such that

$$|\tilde{x}_0 - x_0| < \delta \Rightarrow |x(t; \tilde{x}_0, t_0) - x(t; x_0, t_0)| < \varepsilon, \qquad \forall t \geq t_0; \tag{A.1.3}$$

- ***attractive***, if there exists an $r(t_0) > 0$ and, for each $\varepsilon > 0$, a $T(\varepsilon, t_0) > 0$ such that

$$|\tilde{x}_0 - x_0| < r \Rightarrow |x(t; \tilde{x}_0, t_0) - x(t; x_0, t_0)| < \varepsilon, \qquad \forall t \geq t_0 + T; \tag{A.1.4}$$

- ***asymptotically stable***, if it is stable and attractive; and

- ***unstable***, if it is not stable.

The stability properties of $x(t; x_0, t_0)$ in general depend on the initial time t_0. For different t_0, different values of $B(x_0, t_0)$, $\delta(\varepsilon, t_0)$, $r(t_0)$, and $T(\varepsilon, t_0)$ may be needed to satisfy (A.1.2), (A.1.3) and (A.1.4). When these constants are independent of t_0, the corresponding properties are *uniform* (*Clearly, all properties are uniform if the system is time-invariant:* $\dot{x} = f(x)$). For adaptive systems, *uniform stability* is more desirable than just stability. Even more desirable is ***uniform asymptotic stability***, often shortened to UAS. The solution $x(t; x_0, t_0)$ is UAS if it is *uniformly stable and uniformly attractive*, that is, if $\delta(\varepsilon, t_0) = \delta(\varepsilon)$, $r(t_0) = r$, and $T(\varepsilon, t_0) = T(\varepsilon)$ do not depend on t_0.

Some solutions of a given system may be stable and others unstable. In particular, (A.1.1) may have stable and unstable ***equilibria***, that is constant solutions $x(t; x_0, t_0) \equiv x_e$ satisfying $f(x_e, t) \equiv 0$. If an equilibrium x_e is asymptotically stable, then it has a ***region of attraction*** — a set Ω of initial states x_0 such that $x(t; x_0, t_0) \to x_e$ as $t \to \infty$ for all $x_0 \in \Omega$ (*When x_e is only stable, then the solutions starting in Ω remain close to x_e in the sense of* (A.1.3)). In this report, the stability properties for which an estimate of the region of attraction is given are referred to as ***regional***. Otherwise they are called *local*. When the region of attraction is the whole space $\mathbb{R}^n$, then the stability properties are ***global***.

Any equilibrium under investigation can be translated to the origin by redefining the state x as $z = x - x_e$. Such a translation $z = x - x(t; x_0, t_0)$ can be defined for any solution $x(t; x_0, t_0)$ so that the solution under investigation can always be considered to be an equilibrium at the origin with a corresponding redefinition of $f(x,t)$ into $\bar{f}(z,t)$ such that $\bar{f}(0,t) \equiv 0$, namely;

$$\dot{z} = f(z + x(t; x_0, t_0), t) - f(x(t; x_0, t_0), t) \triangleq \bar{f}(z, t). \tag{A.1.5}$$

Therefore, there is no loss of generality in standardizing the stability results for the zero solution $z(t; 0, t_0) \equiv 0$. In adaptive tracking problems, this zero solution is particulary meaningful when the state z represents the tracking error and its derivatives.

To be of practical interest, stability conditions must not required that explicitly solve (A.1.1). The direct method of Lyapunov aims at determining the stability properties of $x(t; x_0, t_0)$ from the properties of $f(x,t)$ and its relationship with a positive definite function $V(x,t)$. For global result, this function must be radially unbounded, that is $V(x,t) \to \infty$ as $|x| \to \infty$ uniformly in t. For simplicity, we will assume that the translation to the origin has been performed, that is, $f(0,t) \equiv 0$, and thus the solution under investigation is $x \equiv 0$.

Uniform asymptotic stability is a desirable property, because systems that possess it can deal better with perturbation and disturbances. We shall see that, in general, adaptive designs achieve less than uniform asymptotic stability. However, they achieve more than uniform stability because they force the tracking error to converge to zero. This key property is referred to as ***regulation*** when the reference signal is constant, and ***tracking*** when it is a time-varying signal. For convergence analysis, a powerful tool is the following theorem due to LaSalle and Yoshizawa:

Theorem A.1.1 (LaSalle-Yoshizawa) *Let $x = 0$ be an equilibrium point of* (A.1.1) *and suppose f is locally Lipschitz in x uniformly in t. Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable, positive definite and radially unbounded function $V(x)$ such that*

$$\dot{V} = \frac{\partial V}{\partial x}(x) f(x,t) \leq -W(x) \leq 0, \qquad \forall t \geq 0, \forall x \in \mathbb{R}^n, \tag{A.1.6}$$

where W is a continuous function. Then, all solutions of (A.1.1) *are* ***globally uni-***

formly bounded *and satisfy*

$$\lim_{t\to\infty} W(x(t)) = 0. \tag{A.1.7}$$

In addition, if $W(x)$ is positive definite, then the equilibrium $x = 0$ is ***globally uniformly asymptotically stable*** *(GUAS).*

For regulation task, the designed system is usually time-invariant,

$$\dot{x} = f(x), \tag{A.1.8}$$

in which case we are interested in its ***invariant sets***. A set M is called an invariant set of (A.1.8) if any solution $x(t)$ that belong to M at some time constant t_1 must belong to M for all future and past time:

$$x(t_1) \in M \Rightarrow x(t) \in M, \qquad \forall t \in \mathbb{R}. \tag{A.1.9}$$

A set Ω is *positively invariant* if this is true for all future time only:

$$x(t_1) \in \Omega \Rightarrow x(t) \in \Omega, \qquad \forall t \geq t_1. \tag{A.1.10}$$

Can we guarantee convergence to a desired invariant set? A rewarding answer to this question is provided by LaSalle's Invariance Theorem and its asymptotic stability corollary:

Theorem A.1.2 (LaSalle) *Let Ω be a positively invariant set of (A.1.8). Let $V : \Omega \to \mathbb{R}_+$ be a continuously differentiable function $V(x)$ such that $\dot{V}(x) \leq 0, \forall x \in \Omega$. Let $E = \left\{x \in \Omega | \dot{V}(x) = 0\right\}$, and let M be the largest invariant set contained in E. Then, every bounded solution $x(t)$ starting in Ω converges to M as $t \to \infty$.*

Corollary A.1.3 (Asymptotic Stability) *Let $x = 0$ be the only equilibrium of (A.1.8). Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable, positive definite, radially unbounded* *function $V(x)$ such that $\dot{V}(x) \le 0, \forall x \in \mathbb{R}^n$. Let $E = \left\{x \in \mathbb{R}^n | \dot{V}(x) = 0\right\}$, and suppose that no solution other than $x(t) \equiv 0$ can stay forever in E. Then the origin is globally asymptotically stable (GAS).*

These invariance results will motivate us to closely examine the invariant subsets of E. As we shall see, the convergence properties of the designed system are stronger if the dimension of M is lower. In the most favorable case of asymptotic stability, the largest invariant subset M of E is just the origin $x = 0$. Our aim will thus be render the dimension of M as low as possible.

Input-to-State Stability. Another stability concept which is used throughout the report is that of input-to-state stability (ISS), the system

$$\dot{x} = f(x, u) \tag{A.1.11}$$

is said to be *input-to-state stable (ISS)* if for any $x(0)$ and for any input $u(.)$ continuous and bounded on $[0, \infty)$ the solution exists for all $t \ge 0$ and satisfies

$$|x(t)| \le \beta(|x(0)|, t) + \gamma\Big(\sup_{0 \le \tau \le t} |u(\tau)|\Big), \qquad \forall t \ge 0, \tag{A.1.12}$$

where $\beta(s, t)$ and $\gamma(s)$ are strictly increasing functions of $s \in \mathbb{R}_+$ with $\beta(0, t) = 0, \gamma(0) = 0$, while β is a increasing function of t with $\lim_{t \to \infty} \beta(s, t) = 0, \forall s \in \mathbb{R}_+$.

A.2 Lyapunov's Direct Method

The basic philosophy of Lyapunov's direct method is the mathematical extension of a fundamental physical observation: if the total *energy* of a mechanical (or electrical) system is continuously dissipated, then the system, *whether linear or nonlinear*, must eventually settle down to an equilibrium point. Thus, we may conclude the stability of a system by examining the variation of a single *scalar* function.

The total mechanical energy of the nonlinear mass-spring-damper system

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_0 x + k_1 x^3 = 0 \tag{A.2.1}$$

is the sum of its kinetic energy and its potential energy.

$$V(x) = \frac{1}{2}m\dot{x}^2 + \int_0^x (k_0 s + k_1 s^3)ds = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4 \qquad \text{(A.2.2)}$$

Comparing the definitions of stability and mechanical energy, one can easily see some relations between the mechanical energy and the stability concepts described earlier:

- zero energy corresponds to the equilibrium point $(x = 0, \dot{x} = 0)$
- asymptotic stability implies the convergence of mechanical energy to zero
- instability is related to the growth of mechanical energy

These relations indicate that the value of a scalar quantity, the mechanical energy, indirectly reflects the magnitude of the state vector: and furthermore, that the stability properties of the system can be characterized by the variation of the mechanical energy of the system.

The rate of the energy variation during the system's motion is obtained easily by differentiating the first equality in (A.2.2)

$$\dot{V}(x) = m\dot{x}\ddot{x} + k_0 x + k_1 x^3 \dot{x} = \dot{x}(-b\dot{x}\,|\dot{x}|) = -b|\dot{x}|^3 \qquad \text{(A.2.3)}$$

The direct method of Lyapunov is based on a generalization of the concept in the mass-spring-damper system to more complex system. Faced with a set of nonlinear differential equations, the basic procedure of Lyapunov's direct method is to generate a scalar "energy-like" function for the dynamic system, and examine the time variation of that scalar function. In this way, conclusions may be drawn on the stability of the set of differential equations without using the difficult stability definitions or requiring explicit knowledge of solutions.

Theorem A.2.1 (Local Stability) *if, in a ball* $\mathbf{B}_{Ro}$, *there exists a scalar function* $V(x)$ *with continuous first partial derivatives such that*

- *$V(x)$ is positive definite (locally in $\mathbf{B}_{Ro}$)*
- *$\dot{V}(x)$ is negative semi-definite (locally in $\mathbf{B}_{Ro}$)*

then the equilibrium point x=0 is stable. If, actually, the derivative $\dot{V}(x)$ is locally negative definite in $\mathbf{B}_{Ro}$, then the stability is asymptotic.

Theorem A.2.2 (Global Stability) *assume that there exists a scalar function V of the state $\boldsymbol{x}$, with continuous first order derivatives such that*

- *$V(x)$ is positive definite*
- *$\dot{V}(x)$ is negative definite*
- *$V(x) \to \infty$ as $||x|| \to \infty$*

then the equilibrium at the origin is globally asymptotically stable.

A.3 Lyapunov Stability

Consider the non-autonomous system

$$\dot{x} = f(x, t) \tag{A.3.1}$$

Definition A.3.1 *The origin $x = 0$ is equilibrium point for* (A.3.1) *if*

$$f(0, t) = 0, \forall t \geq 0. \tag{A.3.2}$$

Definition A.3.2 *A continuous function $\gamma : [0, a) \to \mathbb{R}_+$ is said to belong to class K if it is strictly increasing and $\gamma(0) = 0$. It is said to belong to class K_∞ if $a = \infty$ and $\gamma(r) \to \infty$ as $r \to \infty$.*

Definition A.3.3 *A continuous function* $\beta : [0, a) \times \mathbb{R}_+ \to \mathbb{R}_+$ *is said to belong to class* KL *if for each fixed* s *the mapping* $\beta(r, s)$ *belong to class* K *with respect to* r*, and for each fixed* r *the mapping* $\beta(r, s)$ *is decreasing with respect to* s *and* $\beta(r, s) \to 0$ *as* $s \to \infty$*. It is said to belong to class* KL_∞ *if, in addition, for is fixed* s *the mapping* $\beta(r, s)$ *belong to class* K_∞ *with respect to* r*.*

Definition A.3.4 *The equilibrium point* $x = 0$ *of* (A.3.1) *is*

- *uniformly stable, if there exists a class* K *function* $\gamma(\cdot)$ *and a positive constant* c*, independent of* t_0*, such that*

$$|x(t)| \leq \gamma(|x(t_0)|), \quad \forall t \geq t_0 \geq 0, \quad \forall x(t_0)||x(t_0)| < c\,; \tag{A.3.3}$$

- *uniformly asymptotically stable, if there exist a class* KL *function* $\beta(\cdot, \cdot)$ *and a positive constant* c*, independent of* t_0*, such that*

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0), \quad \forall t \geq t_0 \geq 0, \quad \forall x(t_0)||x(t_0)| < c; \tag{A.3.4}$$

- *exponentially stable, if* (A.3.4) *is satisfied with* $\beta(r, s) = kre^{\alpha s}, k > 0, \alpha > 0$*;*
- *globally uniformly stable, if* (A.3.3) *is satisfied with* $\gamma \in K_\infty$ *for any initial state* $x(t_0)$*;*
- *globally uniformly asymptotically stable, if* (A.3.4) *is satisfied with* $\beta \in KL_\infty$ *for any initial state* $x(t_0)$*; and*
- *globally exponentially stable, if* (A.3.4) *is satisfied for any initial state* $x(t_0)$ *and with* $\beta(r, s) = kre^{-\alpha s}, k > 0, \alpha > 0$*.*

Lemma A.3.5 **(Barbalat)** *Consider the function* $\phi : \mathbb{R}_+ \to \mathbb{R}$*. If* ϕ *is uniformly continuous and* $\lim_{t \to \infty} \int_0^\infty \phi(\tau) d\tau$ *exists and is finite, then*

$$\lim_{t \to \infty} \phi(t) = 0 \tag{A.3.5}$$

Appendix B

Backstepping

B.1 Integrator Backstepping

The simplicity of scalar designs motivates us to use them as starting points of recursive designs for higher-order systems. Consider a scalar system as in (B.1.1a) augmented with an integrator:

$$\dot{x} = \cos x - x^3 + \xi \quad \text{(B.1.1a)}$$

$$\dot{\xi} = u. \quad \text{(B.1.1b)}$$

Let the design objective be the regulation of $x(t)$, that is, $x(t) \to 0$ as $t \to \infty$, for all $x(0)$, $\xi(0)$. Of course, $\xi(t)$ must remain bounded. Form (B.1.1a), the only equilibrium with $x = 0$ is at $(x, \xi) = (0, -1)$. We will meet our design objective by rendering this equilibrium GAS. In the block diagram in Figure B.1 the scalar system (B.1.1a) appears in the dashed box. To construct a clf [1] we will exploit the fact that a clf is known for its subsystem in the dashed box. Indeed, if ξ were the control input, then the corresponding clf and control law would be $V(x) = \frac{1}{2}x^2$ and $\xi = -c_1 x - \cos x$ clf. Of course ξ is just a state variable and not the control. Nevertheless, as its "desired value" we prescribe

$$\xi_{des} = -c_1 x - \cos x \triangleq \alpha(x). \quad \text{(B.1.2)}$$

[1] A system for which a good choice of $V(x)$ and $W(x)$ is said to posses a clf.

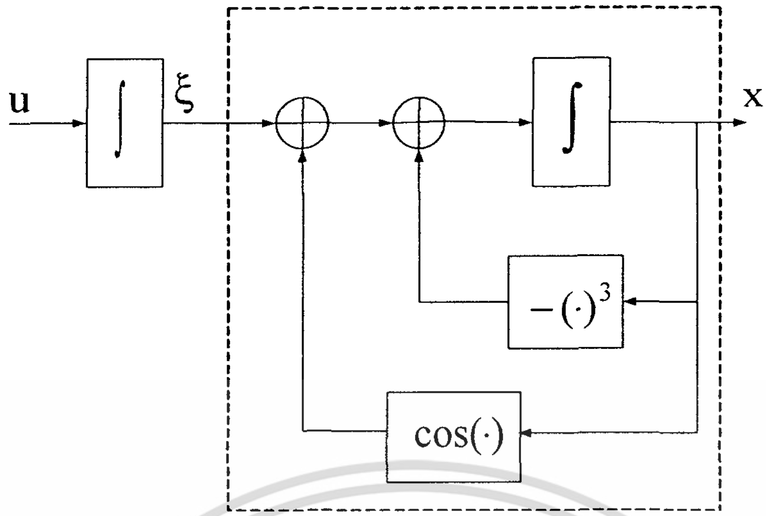

Figure B.1: The block diagram of the system (B.1.1a), (B.1.1b)

Let z be the deviation of ξ from its desired value:

$$z = \xi - \xi_{des} = \xi - \alpha(x) = \xi + c_1 x + \cos x. \quad \text{(B.1.3)}$$

We call ξ a *virtual control*, and its desired value $\alpha(x)$ a *stabilizing function*. The variable z is the corresponding *error variable*. Now we rewrite the system (B.1.1) in the (x, z)-coordinates in which it takes on a more convenient form, as illustrated in Figure B.2 and Figure B.3. Starting from (B.1.1) and Figure B.1, we add and subtract the stabilizing function $\alpha(x)$ to the $\dot{x}_1$-equation as shown in Figure B.2. Then we use $\alpha(x)$ as the feedback control inside the dashed box and "backstep" $-\alpha(x)$ through the integrator, as in Figure B.3. In the new coordinates (x, z) the system is expressed as

$$\dot{x} = \cos x - x^3 + [\xi + c_1 x + \cos x] - c_1 x - \cos x = -c_1 x - x^3 + z \quad \text{(B.1.4a)}$$

$$\dot{z} = \dot{\xi} - \dot{\alpha} = \dot{\xi} + (c_1 - \sin x)\dot{x} = u + (c_1 - \sin x)(-c_1 x - x^3 + z). \quad \text{(B.1.4b)}$$

The first key feature of backstepping is that we don't use a differentiator to implement the time derivative $\dot{\alpha}$ in (B.1.4b); since $\alpha(x)$ is a known function, it is easy to compute its time derivative analytically as

$$\dot{\alpha} = \frac{\partial \alpha}{\partial x}\dot{x} = -(c_1 - \sin x)(-c_1 x - x^3 + z). \quad \text{(B.1.5)}$$

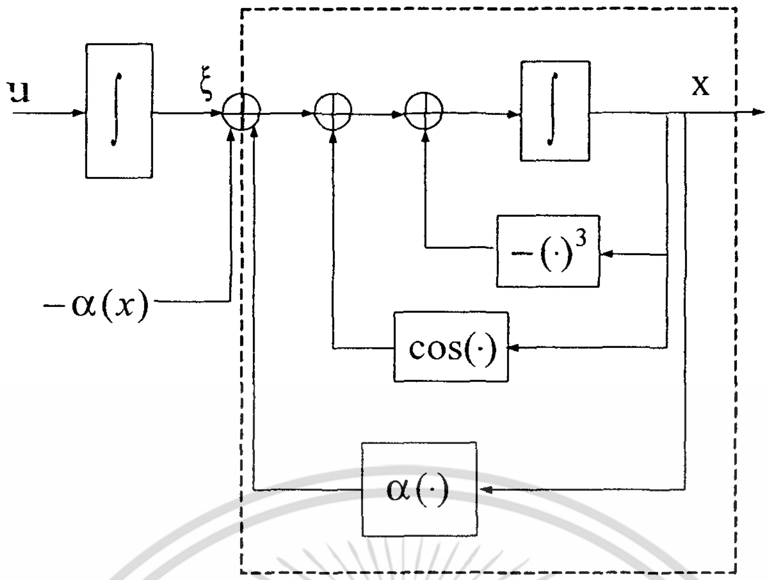

Figure B.2: Introducing $\alpha(x)$ as the desired value for ξ.

We now need to select a clf V_a for the system (B.1.1). Let us try to construct it by augmenting $V(x)$ with a quadratic term in the error variable z:

$$V_a(x,\xi) = V(x) + \frac{1}{2}z^2 = \frac{1}{2}x^2 + \frac{1}{2}(\xi + c_1 x + \cos x)^2. \tag{B.1.6}$$

The derivative of V_a along the solution of (B.1.4) is computed as

$$\begin{aligned} \dot{V}_a(x,z,u) &= x[-c_1x - x^3 + z] + z[u + (c_1 - \sin x)(-c_1x - x^3 + z)] \\ &= -c_1x^2 - x^4 + z[x + u + (c_1 - \sin x)(-c_1x - x^3 + z)]. \end{aligned} \tag{B.1.7}$$

As always, we let $\dot{V}_a$ be an explicit function of u and design u to satisfy the clf inequality clf. For this reason, the cross-term xz, which is due to the presence of z in (B.1.4a), is grouped together with u. This is possible because u is also multiplied by z due to the chosen form of V_a. This is the second key feature of backstepping. Now we choose the control u to make $\dot{V}_a$ negative definite in x and z. The simplest way to achieve this is to make the bracketed term in (B.1.7) equal to $-c_2z^2$, where $c_2 > 0$:

$$\begin{aligned} u &= -c_2z - x - (c_1 - \sin x)(-c_1x - x^3 + z) \\ &= -c_2(\xi + c_1x + \cos x) - x - (c_1 - \sin x)(\xi + \cos x - x^3). \end{aligned} \tag{B.1.8}$$

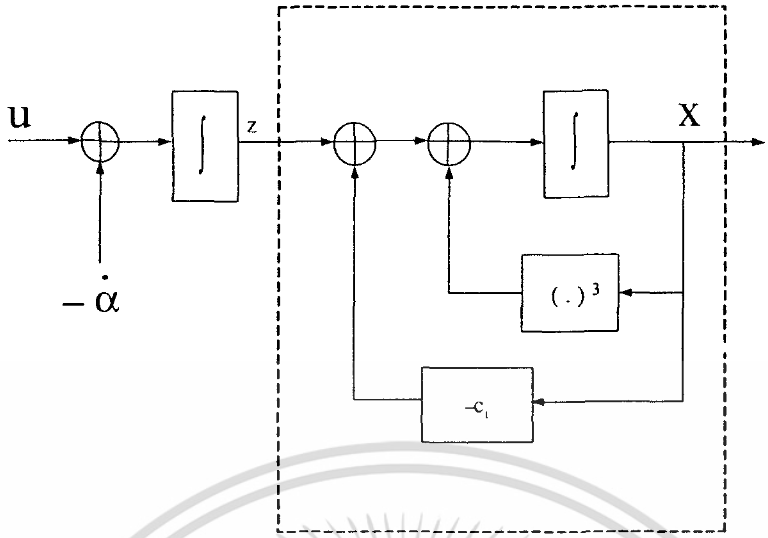

Figure B.3: Closing the feedback loop in the dashed box with $+\alpha$ and "backstepping" $-\alpha$ through the integrator.

With this control, the clf derivative is

$$\dot{V}_a = -c_1 x^2 - c_2 z^2, \tag{B.1.9}$$

which proves that in the $(x. z)$-coordinates the equilibrium $(0, 0)$ is GAS. In view of (B.1.3), the equilibrium $(0. -1)$ in the (x, ξ)-coordinates has the same property.

The resulting closed-loop system in the (x, z)-coordinates is

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -c_1 - x^2 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \tag{B.1.10}$$

Although written in a linear-like form, this system is nonlinear. An important structural property of this system is that its nonlinear "system matrix" is the sum of a *negative diagonal* and a *skew-symmetric* matrix function of x. This is the third key feature of backstepping, which will be extremely useful in other designs.

Avoiding cancelations. The above control law is not the best way to achieve negativity of $\dot{V}_a$, because it involves at least one unnecessary cancelation. A closer examination of (B.1.7) reveals that the term $-z^2 \sin x$ need not be canceled because it can be dominated by $-c_2 z^2$. A control law which avoids this cancelation is

$$u = -c_2 z - x - (c_1 - \sin x)(-c_1 x - x^3), \quad c_2 > c_1 + 1. \tag{B.1.11}$$

With this control, the clf derivative is

$$\dot{V}_a = -c_1 x^2 - x^4 - (c_2 - c_1 + \sin x) z^2. \tag{B.1.12}$$

Although more complicated than (B.1.9), this function is easily rendered negative definite by the choice $c_2 > c_1 + 1$. The resulting system in the (x, z)-coordinates preserves its skew-symmetric from

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -c_1 - x^2 & 1 \\ -1 & -c_2 + c_1 - \sin x \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \tag{B.1.13}$$

The simplified control law (B.1.11) is an illustration of design flexibilities in satisfying the clf inequality $\dot{V}_a \leq 0$ and at the same time avoiding unnecessary cancelations. In fact, more detailed calculations show that the control law can be further simplified to

$$u = -k_1 z - k_2 x^2 z, \tag{B.1.14}$$

with

$$k_1 > c_2 + c_1 + 1 + \frac{(c_1^2 + c_1 + 1)^2}{2c_1}, \quad k_2 \geq \frac{(c_1 + 1)^2}{4}. \tag{B.1.15}$$

Using this control we obtain

$$\dot{V}_a \leq -\frac{1}{2} c_1 x^2 - c_2 z^2. \tag{B.1.16}$$

Integrator backstepping as a general design tool is based on the following assumption:

Assumption B.1.1 *Consider the system*

$$\dot{x} = f(x) + g(x) u, \quad f(0) = 0, \tag{B.1.17}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}$ is the control input. There exist a continuously differentiable feedback control law

$$u = \alpha(x), \quad \alpha(0) = 0, \tag{B.1.18}$$

and a smooth, positive definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(x)[f(x) + g(x)\alpha(x)] \leq -W(x) \leq 0, \quad \forall x \in \mathbb{R}^n, \tag{B.1.19}$$

where $W : \mathbb{R}^n \to \mathbb{R}$ is positive semidefinite.

Under this assumption, the control (B.1.18), applied to the system (B.1.17), guarantees global boundedness of $x(t)$, and via the LaSalle-Yoshizawa theorem (Theorem A.1.1), the regulation of $W(x(t))$:

$$\lim_{t\to\infty} W(x(t)) = 0. \tag{B.1.20}$$

A strong convergence result is obtained using LaSalle's theorem (Theorem A.1.2) with $\Omega = \mathbb{R}^n$: $x(t)$ converges to the largest invariant set M contained in the set $E = \{x \in \mathbb{R}^n | W(x) = 0\}$. Clearly, if $W(x)$ is positive definite, the control (B.1.18) renders $x = 0$ the GAS equilibrium of(B.1.17)

Lemma B.1.2 (Integrator Backstepping) *Let the system* (B.1.17) *be augmented by an integrator:*

$$\dot{x} = f(x) + g(x)\xi \tag{B.1.21a}$$

$$\dot{\xi} = u, \tag{B.1.21b}$$

and suppose that (B.1.21a) *satisfies Assumption* (B.1.1) *with* $\xi \in \mathbb{R}$ *as its control.*

(i) If $W(x)$ *is positive definite, then*

$$V_a(x,\xi) = V(x) + \frac{1}{2}\left[\xi - \alpha(x)\right]^2 \tag{B.1.22}$$

is a clf for the full system (B.1.21), *that is, there exists a feedback control* $u = \alpha_a(x,\xi)$ *which renders* $x = 0, \xi = 0$ *the GAS equilibrium of* (B.1.21). *One such control is*

$$u = -c(\xi - \alpha(x)) + \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\xi] - \frac{\partial V}{\partial x}(x)g(x), \quad c > 0. \tag{B.1.23}$$

(ii) If $W(x)$ is only positive semidefinite, then there exists a feedback control which renders $\dot{V}_a \leq -W_a(x,\xi) \leq 0$ such that $W_a(x,\xi) > 0$ whenever $W(x) > 0$ or $\xi \neq \alpha(x)$. This guarantees global boundedness and convergence of $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$ to the largest invariant set M_a contained in the set $E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+1} \middle| W(x) = 0, \xi = \alpha(x) \right\}$.

□

Proof Introducing the error variable

$$z = \xi - \alpha(x), \tag{B.1.24}$$

and differentiating[2] with respect to time, (B.1.21) is rewritten as

$$\begin{aligned} \dot{x} &= f(x) + g(x)[\alpha(x) + z] && \text{(B.1.25a)} \\ \dot{z} &= u - \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)(\alpha(x) + z)] && \text{(B.1.25b)} \end{aligned}$$

using (B.1.19), the derivative of (B.1.22) along the solution of (B.1.25) is

$$\begin{aligned} \dot{V}_a &= \frac{\partial V}{\partial x}(f + g\alpha + gz) + z\left[u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z))\right] \\ &= \frac{\partial V}{\partial x}(f + g\alpha) + z\left[u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) + \frac{\partial V}{\partial x}g\right] \\ &\leq -W(x) + z\left[u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) + \frac{\partial V}{\partial x}g\right], \end{aligned} \tag{B.1.26}$$

where the term containing z as a factor have been grouped together. By the LaSalle-Yoshizawa theorem (Theorem A.1.1). any choice of the control u which renders $\dot{V}_a \leq -W_a(x,\xi) \leq -W(x)$, with W_a positive definite in $z = \xi - \alpha(a)$, guarantee global boundedness of x, z and $\xi = z + \alpha(x)$, and regulation of $W(x)$ and $z(t)$. Furthermore, LaSalle's theorem (Theorem A.1.2) guarantees convergence of $\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$ to the largest invariant set

[2] Once again, note that the time derivative $\dot{\alpha}$ in (B.1.25b) is implemented analytically without the need for a differentiator

contained in the set $\left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{n+1} \middle| W(x) = 0, z = 0 \right\}$. Again, the simplest way to make $\dot{V}_a$ negative definite in z is to choose the control (B.1.23), which renders the bracketed term in (B.1.26) equal to $-cz$ and yields

$$\dot{V}_a \leq -W(x) - cz^2 \triangleq -W_a(x, \xi) \leq 0 \tag{B.1.27}$$

Clearly, if $W(x)$ is positive definite. Theorem A.1.1 guarantees the global asymptotic stability of $x = 0, z = 0$, which in turn implies that $V_a(x, \xi)$ is a clf and $x = 0, \xi = 0$ is the GAS equilibrium of (B.1.21).

While the choice of control (B.1.23) is simple, this control may not be desirable because it involves cancelation of nonlinearities, some of which may be useful. As illustrated by (B.1.8) and (B.1.9), the requirement that $\dot{V}_a$ in (B.1.26) be made negative by u allows considerable freedom in the choice of control law $u = \alpha_a(a, \xi)$ such that

$$\dot{V}_a \leq -W(x) + z[\alpha_a(x, \xi) - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) + \frac{\partial V}{\partial x} g] = -W_a(x, \xi) \leq 0. \tag{B.1.28}$$

We stress that the main result of backstepping is not the specific form of the control law (B.1.23), but rather the construction of a Lyapunov function whose derivative can be made negative by a wide variety of control laws. In this way, the design of a stabilizing state-feedback controller is effectively reduced to satisfying the scalar inequality (B.1.28).

Example B.1.3 *As a design tool, backstepping is less restrictive than feedback linearization. In some situations it can overcome singularities such as lack of controllability. This is illustrated by the system*

$$\dot{x} = x\xi \tag{B.1.29}$$

$$\dot{\xi} = u, \tag{B.1.30}$$

which is uncontrollable at $x = 0$. *Comparing with* (B.1.21), *we see that* $f(x) = 0$, $g(x) = x$. *Applying* (B.1.2) *with* $V(x) = \frac{1}{2}x^2$ *we can choose*

$$\alpha(x) = -x^2, \quad z = \xi - \alpha(x) = \xi + x^2, \tag{B.1.31}$$

so that $W(x)$ *in* (B.1.19) *is positive definite:* $W(x) = x^4$. *The substitution of* (B.1.31) *into* (B.1.29) *yields*

$$\dot{x} = -x^3 + xz \qquad \text{(B.1.32a)}$$

$$\dot{z} = u + 2x^2(z - x^2), \qquad \text{(B.1.32b)}$$

The control (B.1.23) *which renders* $\dot{V}_a = -x^4 - z^2$ *is*

$$u = -z - x^2 - 2x^2z + 2x^4 = -\xi - 2x^2 - 2x^2\xi. \qquad \text{(B.1.33)}$$

The resulting system in the (x, ξ)*-coordinates is*

$$\dot{x} = x\xi \qquad \text{(B.1.34a)}$$

$$\dot{\xi} = -\xi - 2x^2 - 2x^2\xi, \qquad \text{(B.1.34b)}$$

and its equilibrium $(0, 0)$ *is GAS.*

A significant design flexibility of backstepping is in the choice of $\alpha(x)$. *For the system* (B.1.29), *instead of* (B.1.31) *we can choose*

$$\alpha(x) \equiv 0, \quad z \equiv \xi, \qquad \text{(B.1.35)}$$

so that $W(x) \equiv 0$ *is semidefinite and*

$$V_a = \frac{1}{2}x^2 + \frac{1}{2}\xi^2. \qquad \text{(B.1.36)}$$

The derivative of V_a *along the solutions of* (B.1.29) *is*

$$\dot{V}_a = x^2\xi + \xi u = \xi(u + x^2). \qquad \text{(B.1.37)}$$

In this case the best we can do is to render $\dot{V}_a$ *negative semidefinite: The control*

$$u = -\xi - x^2 \qquad \text{(B.1.38)}$$

yields the closed-loop system

$$\dot{x} = x\xi \quad \text{(B.1.39a)}$$

$$\dot{\xi} = -\xi - x^2 \quad \text{(B.1.39b)}$$

and the Lyapunov derivative $\dot{V}_a = -\xi^2$. *Then,* (B.1.2)*(ii) guarantees that* $(x(t), \xi(t))$ *is bounded and converges to the largest invariant set* M_a *of* (B.1.39) *contained in the set* E_a *where* $\xi = 0$. *But* $\xi(t) \equiv 0$ *implies* $x(t) \equiv 0$. *Applying Corollary A.1.3, we conclude that the equilibrium* $(0, 0)$ *is GAS.*
Comparing the two control laws (B.1.33) *and* (B.1.38) *we see that the choice* $\alpha(x) \equiv 0$ *simplified the control by eliminating the* x^4*-term. Lemma B.1.2 shows how to add a single integrator. This lemma can be repeatedly applied to add a whole chain of integrators.*

Corollary B.1.4 (Chain of Integrators) *Let the system* (B.1.17) *satisfying Assumption B.1.1 with* $\alpha(x) = \alpha_0(x)$ *be augmented by a chain of* k *integrators so that* u *is replaced by* ξ_1, *the state of the last integrator in the chain:*

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{k-1} &= \xi_k \\ \dot{\xi}_k &= u. \end{aligned} \quad \text{(B.1.40)}$$

For this system, repeated application of Lemma B.1.2 with $\xi_1, \ldots, \xi_k$ *as virtual controls, results in the Lyapunov function*

$$V_a(x, \xi_1, \ldots, \xi_k) = V(x) + \frac{1}{2}\sum_{i=1}^{k}[\xi_i - \alpha_{i-1}(x, \xi_1, \ldots, \xi_{i-1})]^2. \quad \text{(B.1.41)}$$

Any choice of feedback control which renders $\dot{V}_a \leq -W_a(x, \xi_1, \ldots, \xi_k) \leq 0$, *with* $W_a(x, \xi_1, \ldots, \xi_k) = 0$ *only if* $W(x) = 0$ *and* $\xi_i = \alpha_{i-1}(x, \xi_1, \ldots, \xi_{i-1})$, $i = 1, \ldots, k$, *guarantees that* $[x^T(t), \xi_1(t), \ldots, \xi_k(t)]^T$ *is globally bounded and converges to the largest invariant set* M_a *contained in the set*

$$E_a = \left\{[x^T, \xi_1, \ldots, \xi_k]^T \in \mathbb{R}^{n+k} | W(x) = 0, \xi_i = \alpha_{i-1}(x, \xi_1, \ldots, \xi_{i-1}), i = 1, \ldots, k\right\}.$$

Furthermore, if $W(x)$ *is positive definite, that is, if* $x = 0$ *can be rendered GAS through* ξ_1*, then* (B.1.41) *is a clf for* (B.1.40) *and the equilibrium* $x = 0, \xi_1 = \cdots = \xi_k = 0$ *can be rendered GAS through* u*.*

Appendix C

Passivity

C.1 Passivity

we consider systems of the form

$$\begin{aligned} \dot{x} &= f(x,t) + g(x,t)u \\ y &= h(x,t), \end{aligned} \tag{C.1.1}$$

with $x \in \mathbb{R}^n, y \in \mathbb{R}^m, u \in \mathbb{R}^m$, and f, g, h continuous in t and smooth in x. Suppose $f(0,t) = 0$ and $h(0,t) = 0$ for all $t \geq 0$.

Definition C.1.1 *The system* (C.1.1) *is said to be passive if there exists a continuous nonnegative ("storage") function* $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$, *which satisfies* $V(0,t) = 0, \;\; \forall t \geq 0$, *such that for all* $u \in C^0, x(0) \in \mathbb{R}^n, \; t \geq t_0 \geq 0$

$$\int_{t_0}^{t} y^T(\sigma)u(\sigma)\, d\sigma \;\; \geq \;\; V(x(t),t) - V(x(t_0),t_0). \tag{C.1.2}$$

Definition C.1.2 *The system* (C.1.1) *is said to be strictly passive if there exists a continuous nonnegative (storage) function* $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$, *which satisfies* $V(0,t) = 0, \;\; \forall t \geq 0$, *and a positive definite function (dissipation rate)* $\psi : \mathbb{R}^n \to \mathbb{R}_+$, *such that for* *all* $u \in C^0$, $x(0) \in \mathbb{R}^n$, $t \geq t_0 \geq 0$

$$\int_{t_0}^{t} y^T(\sigma)u(\sigma)\,d\sigma \;\geq\; V(x(t),t) - V(x(t_0),t_0) + \int_{t_0}^{t} \psi(x(\sigma))\,d\sigma. \quad \text{(C.1.3)}$$

Passivity and Lyapunov stability are closely related concepts.

Lemma C.1.3 *Suppose the system* (C.1.1) *is (strictly) passive. If* V *is positive definite, radially unbounded, and decrescent, that is, if there exist class* $\mathcal{K}_\infty$ *functions* γ_1 *and* γ_2 *such that* $\gamma_1(|x|) \leq V(x,t) \leq \gamma_2(|x|)$, $\forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, *then, for* $u \equiv 0$, *the equilibrium* $x = 0$ *of* (C.1.1) *is globally uniformly (asymptotically) stable.*

Proof *When* $u \equiv 0$, *in the case of strict passivity, differentiating* (C.1.3), *we have*

$$\dot{V} \leq -\psi(x). \quad \text{(C.1.4)}$$

Thus, the equilibrium $x = 0$ *is globally uniformly asymptotically stable. The case of passivity is analogous.* □

Many problems in parameter identification and adaptive control can be studied as feedback interconnections of passive systems (see Figure C.1):

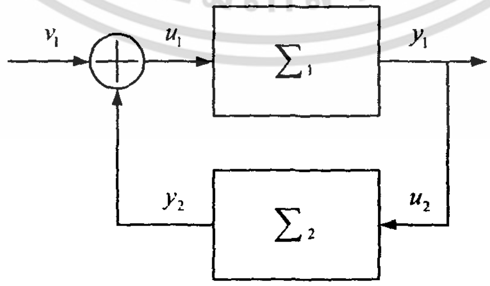

Figure C.1: Feedback interconnection of two passive systems.

$$\Sigma_1 \quad : \quad \begin{cases} \dot{x}_1 &= f_1(x,t) + g_1(x,t)u_1 \\ y_1 &= h_1(x,t) \end{cases} \tag{C.1.5}$$

$$\Sigma_2 \quad : \quad \begin{cases} \dot{x}_2 &= f_2(x,t) + g_2(x,t)u_2 \\ y_2 &= h_2(x,t) \end{cases} \tag{C.1.6}$$

connected by the relations

$$u_1 = -y_2 + v_1 \tag{C.1.7}$$

$$u_2 = y_1, \tag{C.1.8}$$

where v_1 is a new input to the system.

Theorem C.1.4 *Suppose the system Σ_1 is (strictly) passive with storage function V_1 (and dissipation rate ψ_1) independent of x_2. Likewise, suppose the system Σ_2 is (strictly) passive with storage function V_2 (and dissipation rate ψ_2) independent of x_1. Then the interconnected system* (C.1.5)-(C.1.8) *with input v_1 and output y_1 is*

1. strictly passive *if both Σ_1 and Σ_2 are strictly passive,*
2. passive *if at least one of the systems Σ_1 and Σ_2 is passive but not strictly passive.*

Moreover, when $v_1 \equiv 0$, if Σ_1 is strictly passive and Σ_2 is passive, then the equilibrium $x = 0$ is globally uniformly stable and $\lim_{t\to\infty} x_1(t) = 0$.

Proof *Let us first assume that Σ_1 and Σ_2 are both strictly passive. Then, in view of* (C.1.7)-(C.1.8) *we have*

$$\int_{t_0}^{t} y_1^T[v_1 - y_2]\, d\sigma \ \geq \ V_1(x_1(t),t) - V_1(x_1(t_0),t_0) + \int_{t_0}^{t} \psi_1(x_1)\, d\sigma \tag{C.1.9}$$

$$\int_{t_0}^{t} y_2^T y_1\, d\sigma \ \geq \ V_2(x_2(t),t) - V_2(x_2(t_0),t_0) + \int_{t_0}^{t} \psi_2(x_2)\, d\sigma \tag{C.1.10}$$

Adding inequalities (C.1.9) *and* (C.1.10), *we obtain*

$$\int_{t_0}^{t} y_1^T(\sigma) v_1(\sigma)\, d\sigma \;\geq\; V(x(t),t) - V(x(t_0),t_0) + \int_{t_0}^{t} \psi(x)\, d\sigma, \tag{C.1.11}$$

where the storage function V and the dissipation rate ψ for the complete x-system are defined as

$$V(x,t) = V_1(x_1,t) + V_2(x_2,t) \tag{C.1.12}$$

$$\psi(x) = \psi_1(x_1) + \psi_2(x_2). \tag{C.1.13}$$

Since V is positive definite, radially unbounded and decrescent, and ψ is positive definite, this proves the strict passivity. If at least one of the systems Σ_1 and Σ_2 is passive but not strictly passive, then its dissipation rate ψ_i is at best positive semidefinite but not positive definite, and the overall system is only passive. Finally, when $v_1 \equiv 0$, if Σ_1 is strictly passive and Σ_2 is passive, then ψ_2 is positive semidefinite, and by differentiating (C.1.11) *we get*

$$\dot{V} \leq -\psi_1(x_1). \tag{C.1.14}$$

Thus, by Theorem A.1.1, $x = 0$ is globally uniformly stable and $\lim_{t\to\infty} x_1(t) = 0$. □

The quadratic nonnegative terms in the foregoing equation represent the dissipation rate. The dissipation rate takes different forms, which we illustrate by various special cases of the network.

Case 1: Take $R_1 = R_3 = \infty$ and $R_2 = 0$. Then,

$$uy = \dot{V}.$$

In this case, there is no energy dissipation in the network; that is, the system is lossless.

Case 2: Take $R_2 = 0$ and $R_3 = \infty$. Then,

$$uy = \dot{V} + \frac{1}{R_1} u^2.$$

This dissipation rate is proportional to u^2. There is no energy dissipation if and only if $u(t)$ is identically zero.

Case 3: Take $R_1 = R_3 = \infty$. Then,

$$uy = \dot{V} + R_2 y^2.$$

where we have used the fact that in this case $y = x_1$. The dissipation rate is proportional to y^2. There is no energy dissipation if and only if $y(t)$ is identically zero.

Case 4: Take $R_1 = \infty$. Then,

$$uy = \dot{V} + R_2 x_1^2 + \frac{1}{R_3} x_2^2.$$

The dissipation rate is a positive definite function of the state x. There is no energy dissipation if and only if $x(t)$ is identically zero.

Case 5: Take $R_1 = \infty$, $R_2 = 0$. Then,

$$uy = \dot{V} + \frac{1}{R_3} x_2^2.$$

This dissipation rate is a positive semidefinite function of the state. Notice, however,

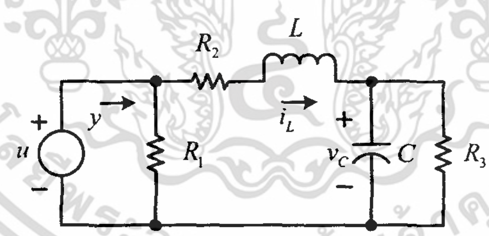

Figure C.2: RLC Circuit Illustration of Passivity Concept.

that from the second state equation, we have

$$x_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

irrespective of the input u. Therefore, like the previous case, there is no energy dissipation if and only if $x(t)$ is identically zero.

These five cases illustrate four basic forms of the dissipation rate: no dissipation, strict dissipation when the input is not identically zero, strict dissipation when the output is not identically zero, and strict dissipation when the state is not identically zero. These four basic forms will be captured in Definition C.1.1. It is clear that combinations of these forms are also possible. For example, for the complete circuit when all resistors are present, we have

$$uy = \dot{V} + \frac{1}{R_1}u^2 + R_2 x_1^2 + \frac{1}{R_3}x_2^2$$

whose dissipation rate is the sum of a quadratic term in the input and a positive definite function of the state.