

**สำนักหอสมุดกลาง พระจอมเกล้าลาดกระบัง**

การประยุกต์ของการออกแบบทูนนิ่งฟังก์ชันในการควบคุมปรากฏการณ์การสั่น  
อันดับ 2

APPLICATION OF TUNNING FUCTION DESIGNS ON SECOND ORDER  
OSCILLATION CONTROL



โดย

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ปริญญานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิศวกรรมศาสตรบัณฑิต  
สาขาวิศวกรรมระบบควบคุม  
สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง

ปีการศึกษา 2548

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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ปริญญานิพนธ์ปีการศึกษา 2548

ภาควิชาวิศวกรรมระบบควบคุม

คณะวิศวกรรมศาสตร์ สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง

เรื่อง การประยุกต์ของการออกแบบทูนนิ่งฟังก์ชันในการควบคุมปรากฏการณ์การสั่นอันดับ 2  
Application of Tuning Function Design on Second Order Oscillation Control

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## Application of Tuning Function Designs on Second Order Oscillation Control

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### บทคัดย่อ

ในการออกแบบระบบควบคุมโดยทั่วไป (Non-adaptive) นั้น เราจะเลือกโครงสร้างของตัวควบคุม (เช่น การแทนที่โพล) ขึ้นมาก่อน แล้วจึงหา parameter ของตัวควบคุมจาก parameter ที่ทราบค่าของ plant ในการควบคุมแบบ Adaptive จะมีข้อแตกต่างที่หลักๆ คือ เราไม่ทราบค่า parameter ของ plant ดังนั้นเราจะได้ parameter ของตัวควบคุมจาก adaptation law ผลคือการออกแบบการควบคุมแบบ adaptive จะยุ่งยากขึ้น โดยสิ่งที่เพิ่มขึ้นคือ ความจำเป็นในการเลือก adaptation law และการพิสูจน์ว่าระบบรวม (รวม adaptation law) มีเสถียรภาพ ปริญญาบัตรฉบับนี้จะนำเสนอเนื้อหาต่อไปนี้: First Order Linear System with Unknown Constant Parameter: ซึ่งเป็นสาเหตุให้ใช้ทฤษฎี Adaptive Stability, ทฤษฎี Passivity, Lyapunov-Based Design, Feedback Linearization และ Zero Dynamics, Stabilization of Cascade Systems, Block Backstepping with Zero Dynamics, Adaptive Backstepping, Adaptive Block Backstepping และ Tuning Function Design ทฤษฎีส่วนใหญ่ข้างต้นสามารถหาได้จากหนังสือของ Krstić [48] ซึ่งจะมีรายละเอียดอย่างสมบูรณ์ของทฤษฎี Adaptive Block Backstepping และ Tuning Function Design และเพื่อที่จะให้อ่านเห็นประโยชน์จากทฤษฎีเหล่านั้น เราได้ประยุกต์ใช้ทฤษฎีเหล่านั้นไปควบคุมตำแหน่งของแขนกลแบบข้อต่อเดียวขับโดยมอเตอร์แปรปรวน ชนิดแม่เหล็กถาวร

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้าไม่ว่ากรณีใดๆ ทั้งสิ้น อีกทั้งห้ามมิให้ดัดแปลงเนื้อหา และต้องอ้างอิงถึงเจ้าของเอกสารทุกครั้งที่มีการนำไปใช้

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## Application of Tuning Function Design on Second Order Oscillation Control

โดย

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### ABSTRACT

In conventional (non-adaptive) control design, a controller structure (e.g., pole placement) is chosen first, and the parameters of the controller are then computed based on the known parameters of the plant. In adaptive control, the major difference is that the plant parameters are unknown, so that the controller parameters have to be provided by an adaptation law. As a result, the adaptive control design is more involved, with the additional needs of choosing an adaptation law and proving the stability of the system with adaptation. This thesis presents the following topics: First Order Linear System with Unknown Constant Parameter which gives the reason for using Adaptive Control, Stability theory, Passivity theory, Lyapunov-Based Design, Feedback Linearization and Zero Dynamics, Stabilization of Cascade Systems, Block Backstepping with Zero Dynamics, Adaptive Backstepping, Adaptive Block Backstepping and Tuning Function Design. Most of these theory can be found in Krstić et al. [48], which gives a full treatment of the theory of Adaptive Backstepping and Tuning Function Design. In order to see the usefulness of these theory we apply them to the position control of a single-link direct-drive manipulator actuated by a permanent magnet brush dc motor.

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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**APPLICATION OF TUNING FUNCTION  
DESIGN ON SECOND ORDER OSCILLATION  
CONTROL**

**P R O J E C T R E P O R T**

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**APPLICATION OF TUNING FUNCTION  
DESIGN ON SECOND ORDER OSCILLATION  
CONTROL**

**PROJECT REPORT**

Submitted in Partial Fulfillment  
of the REQUIREMENTS for the

Degree of  
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*P. Pranayanuntana*

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*March 28, 2006*

Date



เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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Last but not least we would like to thank to each other for the term work that lead this project to success.

## ABSTRACT

In conventional (non-adaptive) control design, a controller structure (e.g., pole placement) is chosen first, and the parameters of the controller are then computed based on the known parameters of the plant. In adaptive control, the major difference is that the plant parameters are unknown, so that the controller parameters have to be provided by an adaptation law. As a result, the adaptive control design is more involved, with the additional needs of choosing an adaptation law and proving the stability of the system with adaptation. This thesis presents the following topics: First Order Linear System with Unknown Constant Parameter which gives the reason for using Adaptive Control, Stability theory, Passivity theory, Lyapunov-Based Design, Feedback Linearization and Zero Dynamics, Stabilization of Cascade Systems, Block Backstepping with Zero Dynamics, Adaptive Backstepping, Adaptive Block Backstepping and Tuning Function Design. Most of these theory can be found in Krstić et al. [48], which gives a full treatment of the theory of Adaptive backstepping and Tuning function design. In order to see the usefulness of these theory we apply them to the position control of a single-link direct-drive manipulator actuated by a permanent magnet brush dc motor.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Early results . . . . .	1
1.2	First Order Linear System with Unknown Constant Parameter . . . . .	3
<b>2</b>	<b>Lyapunov-Based Design</b>	<b>7</b>
2.1	Lyapunov-Based Design . . . . .	7
<b>3</b>	<b>Feedback Linearization and zero dynamics</b>	<b>9</b>
3.1	Feedback Linearization and zero dynamics . . . . .	9
<b>4</b>	<b>Stabilization of cascade systems</b>	<b>14</b>
4.1	Stabilization of cascade systems . . . . .	14
<b>5</b>	<b>Block backstepping with zero dynamics</b>	<b>25</b>
5.1	Block backstepping with zero dynamics . . . . .	25
<b>6</b>	<b>Adaptive Backstepping</b>	<b>36</b>
6.1	Adaptive Integrator Backstepping . . . . .	36

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6.2	Adaptive Block Backstepping . . . . .	43
<b>7</b>	<b>Tuning Functions Design</b>	<b>47</b>
7.1	Introduction . . . . .	47
7.2	Adaptive Control Lyapunov Functions . . . . .	48
7.2.1	Departure from certainty equivalence . . . . .	48
7.2.2	Certainty equivalence for a modified system . . . . .	53
7.2.3	Adaptive backstepping via aclf . . . . .	59
7.3	Set-Point Regulation . . . . .	64
7.3.1	Design procedure . . . . .	66
<b>8</b>	<b>System Model</b>	<b>77</b>
<b>9</b>	<b>Derivation of u</b>	<b>81</b>
<b>10</b>	<b>DC Motor Interfacing</b>	<b>90</b>
10.1	DC Motor Interfacing . . . . .	90
10.2	C/C++ interface Serial Port using polling technique . . . . .	91
10.3	DC Motors . . . . .	97
10.4	A/D and D/A . . . . .	99
10.5	CP-PIC877 V2 . . . . .	100
10.6	Circuit Driver . . . . .	102
<b>A</b>	<b>Stability</b>	<b>106</b>

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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A.1 Main Stability Theorems . . . . .	106
A.2 Lyapunov's Direct Method . . . . .	110
A.3 Lyapunov Stability . . . . .	112
<b>B Backstepping</b>	<b>114</b>
B.1 Integrator Backstepping . . . . .	114
<b>C Passivity</b>	<b>125</b>
C.1 Passivity . . . . .	125



เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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# List of Figures

1.1	Linear Scalar Plant (1.2.2)	3
1.2	The resulting feedback system (1.2.4)	5
2.1	Lyapunov-based adaptive scheme for the scalar system (1.2.2)	8
8.1	schematic of the mechanical system	77
8.2	Moments of Inertia of rigid body	79
8.3	Schematic of electromechanical	80
9.1	MATLAB Simulink simulation of the system.	88
9.2	Result from MATLAB Simulink.	89
10.1	Block diagram for motor control system.	91
10.2	Computer to PIC16F877.	96
10.3	Motor with Gearbox	97
10.4	Encoder Wheels	98
10.5	Feedback by Encoder	98
10.6	Application Diagram	99

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10.7 CP-PIC877 v2 . . . . .	100
10.8 ET-PSPI . . . . .	101
10.9 Motor Driver Amp Diagram . . . . .	102
10.10 ET-SMCC Board . . . . .	103
10.11 PWM 1 . . . . .	103
10.12 PWM 2 . . . . .	104
10.13 PWM 3 . . . . .	104
10.14 Computer to PIC16F877. . . . .	105
B.1 The block diagram of the system (B.1.1a), (B.1.1b) . . . . .	115
B.2 Introducing $\alpha(x)$ as the desired value for $\xi$ . . . . .	116
B.3 Closing the feedback loop in the dashed box with $+\alpha$ and “backstepping” $-\alpha$ through the integrator. . . . .	117
C.1 Feedback interconnection of two passive systems. . . . .	126
C.2 RLC Circuit Illustration of Passivity Concept. . . . .	129

# Chapter 1

## Introduction

### 1.1 Early results

Interest in adaptive control of nonlinear systems was stimulated by major advances in the differential-geometric theory of nonlinear feedback control in the mid-1980's. A thorough treatment of this theory was given by Isidori in his seminal book [9] which unified a decade of results by many researchers. Particularly popular were the results on “feedback linearization,” that is, the state and feedback transformation of nonlinear system into linear ones [5, 7, 10, 24, 25]. This methodology helped convert many previously intractable nonlinear problem into much simpler problems solvable by familiar linear methods.

It soon became clear, however, that along with their many advantages, the nonlinear geometric methods have some shortcomings. One of is their inability to handle the presence of unknown parameters. This motivated the first series of adaptive nonlinear control schemes. They were all restricted to systems satisfying the *matching condition*. Examples of such systems are rigid models of robotic manipulators. While the first robotic adaptive scheme by Craig [4] required measurement of joint accelerations, this impractical assumption was soon removed by Slotine and Li [41, 42], Middleton and Goodwin [31], and Ortega and Spong [34], among others. A more general treatment of adaptive non-linear regulation under the matching condition was given by Taylor, Kokotović, Marino

and Kanellakopoulos [45], including unmodeled dynamics which violated the matching condition.

The matching condition was relaxed to the *extended matching* condition by Kanellakopoulos, Kokotović and Marino [13] and Campion and Bastin [1, 3]. For a period, the extended matching condition was the frontier which could not be crossed by Lyapunov-base designs. Nam and Arapostathis [32] and Sastry and Isidori [40] combined feedback linearization with adaptation techniques from adaptive linear control. However, to achieve global stability, these schemes required that the nonlinearities be restricted by linear growth conditions. Similar restriction on system nonlinearities were imposed by Kanellakopoulos, Kokotović and Middleton [14, 15] and Teel, Kadiyala, Kokotović and Sastry [46].

The only nonlinear estimation-base results which went beyond the linear growth constraints were obtained by Pomet and Praly [35, 36, 37], who used Lyapunov functions to characterize relationships between nonlinear growth constraints and controller stabilizing properties. In the absence of matching conditions, their schemes still involved some growth restrictions but were able to handle the benchmark third-order example.

The state-of-the-art of adaptive control, including adaptive nonlinear control, was reviewed in the 1990 Grainger lecture [20]. One of these lectures presented the result of Kanellakopoulos, Kokotović and Morse [16, 21], which finally broke the extended matching barrier. This was achieved with a new recursive design procedure called *adaptive backstepping*. Adaptive backstepping, developed by Ioannis Kanellakopoulos [12] in collaboration with Peter Kokotović and Steve Morse, emerged as a confluence of the adaptive estimation idea, on one side, and, on the other side, nonlinear control idea expressed in works of Tsiniias [47], Byrnes and Isidori [2], Sontag and Sussmann [43], Kokotović and Sussmann [19], and Saberi, Kokotović and Sussmann [38]. Adaptive backstepping was also strongly influenced by the properties of an early adaptive scheme by Feuer and Morse [6], which, although designed for linear systems, preserved global stability under output feedback for a class of output nonlinearities, as shown by Kanellakopoulos, Kokotović

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and Morse [17, 18]. Adaptive backstepping influenced further developments in adaptive nonlinear control. Mario and Tomei [28, 29, 30] combined it with their filtered transformations [26, 27] to solve the adaptive output-feedback problem for a class of nonlinear systems that has not since been enlarged. Adaptive backstepping also stimulated efforts to reduce its overparametrization. A partial reduction was achieved by Jiang and Praly [11]. With the invention of *tuning functions*, Miroslav Krstić [22, 23], introduced a new design which completely removed the overparameterization.

## 1.2 First Order Linear System with Unknown Constant Parameter

Consider the first order linear system of the following form

$$\dot{x} = u - \theta x \quad (1.2.1)$$

where  $\theta$  is a **positive unknown constant**. This system is asymptotically stable (the equilibrium point  $x = 0$  is an asymptotically stable equilibrium point) even when there is no external control input, that is,  $u$  is zero. The solution of this system is

$$x(t) = x(0)e^{-\theta t} + e^{-\theta t} \int_0^t e^{\theta s} u \, ds.$$

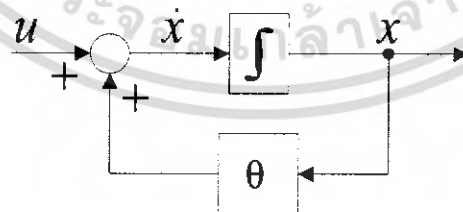


Figure 1.1: Linear Scalar Plant (1.2.2)

Therefore this system is uninteresting. What if the system we have is unstable to begin with? Consider the following system

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where  $\theta$  is a **positive unknown constant**. The solution of (1.2.2) is of the form

$$x(t) = x(0)e^{\theta t} + e^{\theta t} \int_0^t e^{-\theta s} u \, ds.$$

Unlike the system in (1.2.1), this system in (1.2.2) is unstable when  $u = 0$ .

What control law  $u$  will stabilize the system in (1.2.2)? If an a priori bound  $\bar{\theta}$  on  $|\theta|$  were known,  $|\theta| \leq \bar{\theta}$ , then  $u = -2\bar{\theta}x$  would be a linear stabilizing controller. The system (1.2.2) with  $u = -2\bar{\theta}x$  would become

$$\dot{x} = -(2\bar{\theta} - \theta)x.$$

Its only equilibrium point,  $x = 0$ , would be asymptotically stable since  $2\bar{\theta} - \theta > 0$ . (Its stability would be able to be investigated using phase line.) If such a bound is not known, no linear controller can be designed to guarantee stability of (1.2.2). For example, if  $u = -kx$ , where  $k$  is a positive constant, were applied; the system (1.2.2) would become

$$\dot{x} = -(k - \theta)x,$$

and the only equilibrium point,  $x = 0$ , would not be guaranteed to be asymptotically stable due to the fact that the constant  $\theta$  would not be known. If the value of  $k$  picked were less than that of  $\theta$ , then the equilibrium point  $x = 0$  of the resulting system would be unstable.

To examine whether a static nonlinear controller can help, let us try the controller

$$u = -k_1x - k_2x^3. \quad (1.2.3)$$

where  $k_1 > 0, k_2 > 0$ . The resulting feedback system is

$$\dot{x} = (\theta - k_1)x - k_2x^3. \quad (1.2.4)$$

For  $\theta > k_1$ , the equilibrium  $x = 0$  is unstable, but the nonlinear term  $-k_2x^3$  prevents  $x(t)$  from growing unbounded. It is easy to see that  $x(t)$  will converge to one of the two new equilibria  $\pm\sqrt{\frac{\theta-k_1}{k_2}}$ . Thus, the static nonlinear controller (1.2.5) has achieved

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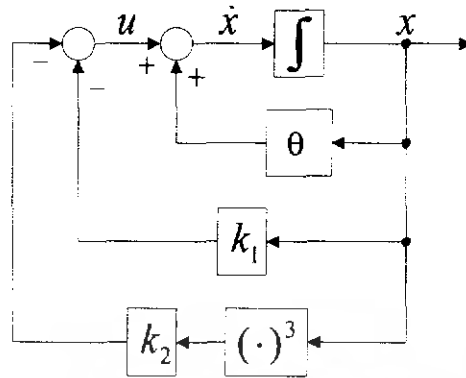


Figure 1.2: The resulting feedback system (1.2.4)

boundedness of  $x(t)$  without any knowledge of a bound on  $\theta$ . Our goal is more ambitious than just boundedness of  $x(t)$ . We also want to achieve its regulation:  $\lim_{t \rightarrow \infty} x(t) = 0$ . Can this be accomplished by a dynamic nonlinear controller? The answer is affirmative: One such controller is

$$u = -(p + \xi)x, \quad \dot{\xi} = x^2. \tag{1.2.5}$$

where  $p > 0$  is a design parameter. The resulting feedback system is of second order:

$$\dot{x} = -(p + \xi)x + \theta x. \tag{1.2.6-a}$$

$$\dot{\xi} = x^2. \tag{1.2.6-b}$$

Its stability properties can be checked by examining the derivative of the Lyapunov function

$$V(x, \xi) = \frac{1}{2}x^2 + \frac{1}{2}(\xi - \theta)^2, \tag{1.2.7}$$

which turns out to be nonpositive:

$$\dot{V} = -px^2 - \xi x^2 + \theta x^2 + (\xi - \theta)x^2 = -px^2. \tag{1.2.8}$$

Thus,  $V(x(t), \xi(t))$  evaluated along the solutions of (1.2.6-a), (1.2.6-b) is a nonincreasing function of time. This proves that  $x(t)$  and  $\xi(t)$  remain bounded for all  $t \geq 0$ . The

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proof that  $\lim_{t \rightarrow \infty} x(t) = 0$  is also achieved can be given using Lasalle-Yoshizawa theorem (Theorem A.1.1).

How was the dynamics nonlinear controller (1.2.5) conceived? Not as a nonlinear controller, but rather as a parameter adaptation scheme! Its dynamics part  $\dot{\xi} = x^2$  is, in fact, an update law for  $\xi$  as an estimate of  $\theta$ . Consequently, the estimation error  $\xi - \theta$  is penalized in Lyapunov function (1.2.7).



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# Chapter 10

## DC Motor Interfacing

### 10.1 DC Motor Interfacing

Since the control objective is to develop a link position regulation controller for the electromechanical dynamics of despite parametric uncertainty. We designed  $u$  in the MATLAB SIMULINK and interface with the motor through the serial port. SIMULINK must receive required reading of angular position, angular velocity, current through A/D and then microcontroller to feed to the computer via the the serial port. Also SIMULINK must transmit the signal level back to the microcontroller and this signal will be converted to driving voltage or pulse for the motor

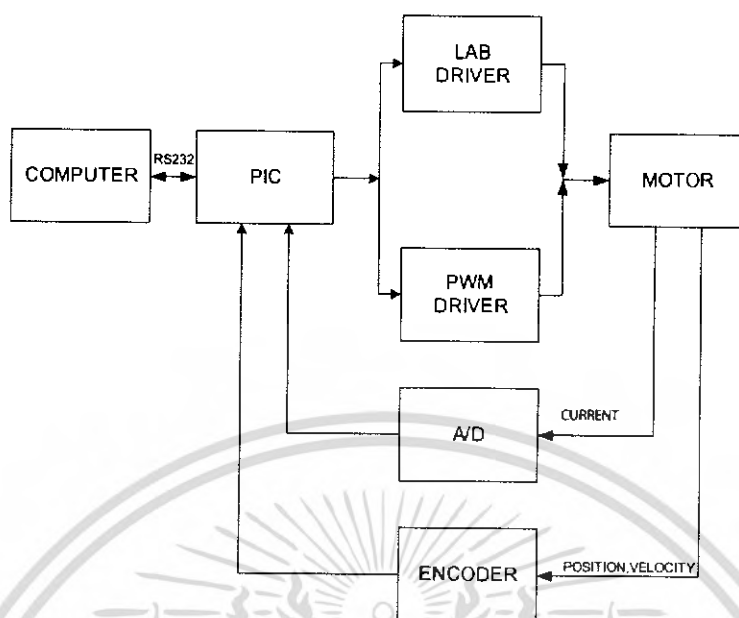


Figure 10.1: Block diagram for motor control system.

## 10.2 C/C++ interface Serial Port using polling technique

We present here C programming since it is easier to understand than Visual C programming that runs on windows. Also we can test the program easily and the size of the compiler is small with the size of less than 1 MByte.

In serial interface using polling technique the flow of data is not regular study, we cannot predict the arrival time of next data chunk. Therefore we have to keep checking (polling) for the arrival of new data and at the same time do the transmitting and receiving operation. The polling rate is higher than the rate of data arrival.

For serial interface, the comport setting has details follows:

C programming code(comport setting) for serial port interface has the details as follows:

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1.Code for assigning address for each comport.

```
#define COMP1 0x3F8 //Assign address for each comport in HEXA
//COM1=0x3F8
//COM2=0x2F8
//COM3=0x3E8
//COM4=0x2E8
```

2.The following is the code for setting comport for RS-232 interface.

```
outp(COMP1+1,0); //turn off interrupt comport1
outp(COMP1+3,0x80); //turn on for communication
outp(COMP1+0,0x03); //low byte baud rate setting
//0x03=38,400 BPS
//0x01=115,200 BPS
//0x02=57,600 BPS
//0x06=19,200 BPS
//0x0C=9,600 BPS
//0x18=4,800 BPS
//0x30=2,400 BPS
outp(1+1,0x00); //High Byte baud rate setting
outp(COMP1+3,0x03); //8 bit translation,non-parity,1 stop bit
outp(COMP1+2,0x07); //control FIFO register
outp(COMP1+4,0x0B); //start transmission DTR, RTS and OUT2
```

The above code can be implemented in the main program for polling as follows:

```
#include<dos.h>
```

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```

#include<conio.h>

#define COMP1 0x3F8 //assign address for each comport using
//HEXA DECIMAL
//COM1=0x3F8
//COM2=0x2F8
//COM3=0x3E8
//COM4=0x2E8

void main(void)//main routine
{
int valA; //variable assigning in polling routine
int strg; //Declare variable type string for storing transmit/recieve
//character at comport.
outp(COMP1+1,0);//turn off interrupt for com part1
//Set com port for transmission
outp(COMP1+3,0x80);//turn on com port for translation
outp(COMP1+0,0X03);//Low byte baud rate setting
//0x03=38,400 BPS
//0x01=115,200 BPS
//0x02=57,600 BPS
//0x06=19,200 BPS
//0x0C=9,600 BPS
//0x18=4,800 BPS
//0x30=2,400 BPS
outp(1+1, 0x00); //High Byte baud rate setting
outp(COMP1+3, 0x03)//8-bit transmission, nonparity, 1 slop bit
outp(COMP1+2, 0xC7)//control FIFO register.
outp(COMP1+4, 0x0B)//start transmission OTR, RTS and OUT2

```

```
printf("\ nCommunicate's Polling. Press ESC to Exit\ n")
```

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```
do{valA=inp(COMP1+5); //check
//Already recieved data
if(valA & 1){strg=inp(COMP1); //if so, store data
printf("%valA", strg);} //display on monitor
if(kbhit()){strg=getch(); //if
outp(COMP1,strg);} //then send the charactor to serial port
}while(strg!=27); //exit program if the esc key was hit
}
```



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```

/*****

```

## Code Implemented

```

/*****

```

```

void main(void) //main routine
{
int valA;
int strg;
outp(COMP1 + 1 , 0);
//set comport for transmission outp(COMP1 + 3 , 0x80);
outp(COMP1 + 0 ,0x03);
outp(PORT1 + 1 , 0x00);
outp(PORT1 + 3 , 0x03);
outp(PORT1 + 2 , 0xC7);
outp(PORT1 + 4 , 0x0B);
printf(" nCommunicate's Polling. Press ESC to Exit ");
do { valA = inp(COMP1 + 5);
if (valA & 1) {strg = inp(COMP1);
printf("if (kbhit()){strg = getch();
outp(COMP1,strg);}
} while (ch !=27);
}
}
/*****

```

Due to computer have characteristic in signal difference from microcontroller. Therefore, it must have IC to translate signal which can use together. This IC is MAX232

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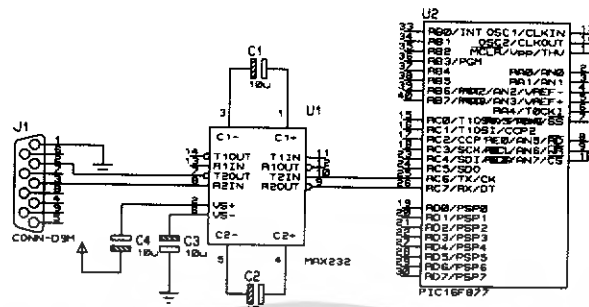


Figure 10.2: Computer to PIC16F877.

In the part of PIC, beside hardware and C-compiler have function for using, for example.

```
#include <16F877.h>
#fuses HS,NOWDT,NOPROTECT,NOLVP
#use delay(clock=20000000)
#use rs232(baud=9600, xmit=PIN_C6, rcv=PIN_C7)
void main() {
char ch;
printf("Hello World!!\n\n");
printf("If you OK please press I\n\n");
ch=getch();
while(ch!='1') {
printf("Please try agian.\n\n");
ch=getc();
}
printf("Thank you.\n\n");
while(TRUE);
}
```

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## 10.3 DC Motors

Motors of every shape and size can be found in the surplus market, in toys and in old electronic gadgets.

### Gear Reduction

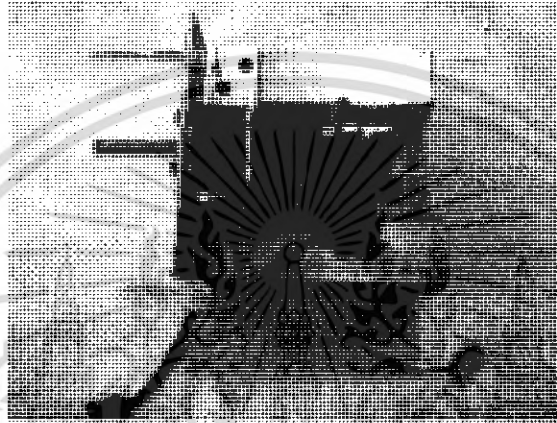


Figure 10.3: Motor with Gearbox

Dc motors tend to spin too quickly and do not have enough torque to turn the robotic load directly so some kind of gear reduction will have to be used. The gears both slow down the speed and increase the torque of the output shaft. This gearbox can be created out of individual gears or it can be part of the motor as in the Escap motor pictured above.

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## Encoder Wheels

Another necessity when using DC motors is an encoder disk on the robot's drive shaft.

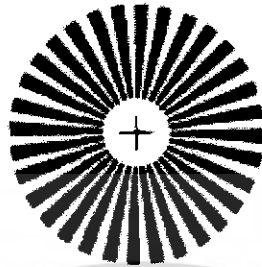


Figure 10.4: Encoder Wheels

This shaft encoder provides a pulse train that can be counted and allows the microcontroller to determine how much the robot arm has turned and how fast the turning is spinning.

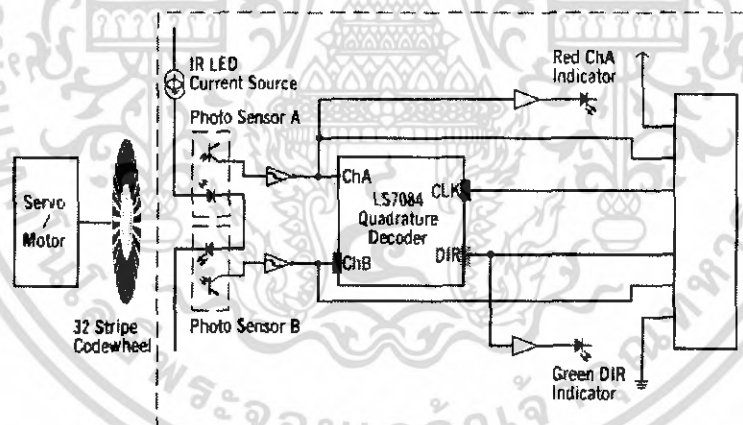


Figure 10.5: Feedback by Encoder

The encoder wheel can be a solid disk with alternating black and white stripes like the one shown above. An IR light source is aimed at the disk. As the wheel spins, the black stripes absorb the light while the white stripes reflect the light back to a phototransistor. The wheel can also be a disk with slots cut into it like those found in computer mice. In this case, the IR light source is on one side of the disk and the phototransistor is on the other side.

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other side of the disk. In both cases the phototransistor converts the pulses of light into a signal that is sent back to the microcontroller to be counted.

## 10.4 A/D and D/A

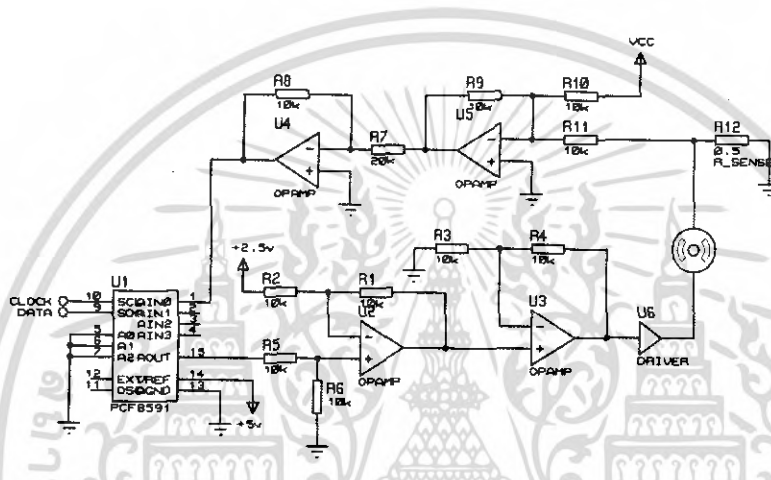


Figure 10.6: Application Diagram

For receive/transmit analog signal, use pcf8591 is the IC analog to digital 4 channel and digital to analog 1 channel by data transmission in i2c. From the system schematic, is a transmit signal digital to analog 0-5v to amplifier circuit. For receive voltage level which appropriate for motor then transmit to drive circuit, the current is increased. Then this value is transmitted to motor. Other side of motor there is  $R_{sense}$  connect to ground, measure voltage for calculate current which is input of motor. The voltage is sent to amplifier then output 0-5v will send to pdf8591, convert analog to digital.

## 10.5 CP-PIC877 V2

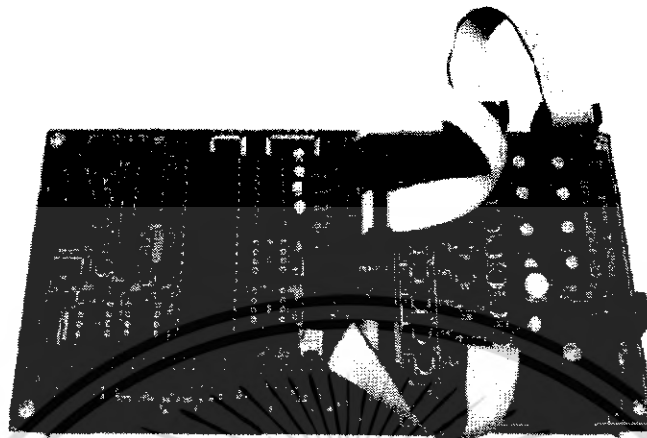


Figure 10.7: CP-PIC877 v2

### Hardware's Specifications of CP-PIC877 V1.0 & CP-PIC877 V2.0 Board

- CPU PIC 16F877-20 On BOARD RUN 4 MHz
- ON CHIP FLASH PROGRAM MEMORY 8K x 14 WORDS
- ON CHIP 368 BYTES RAM / 256 BYTES EEPROM
- CPU's 31 BIT I/O PORT ON CP-PIC877 V2.0 Board(34 PIN I/O ET-BUS)
- SPI PORT, 12C PORT, RS232 PORT, RS422/485 PORT (OPTIONS)
- HIGH SINK / SOURCE CURRENT 25 MA
- TWO CAPTURE, COMPARE, PWM MODULES
- RTC IC No.DS1307 (OPTIONS)
- Serial EEPROM 24xx (OPTIONS)
- POWER ON RESET/WATCHDOG TIMER
- A-TO-D 10 BIT 8CH
- LCD PORT 14 PIN ET-BUS FOR CHARACTER TYPE LCD
- VR 10K x 8 FOR TEST ANALOG PORT
- LED x 8 FOR TEST OUTPUT
- SWITCH x 8 FOR TEST INPUT
- 7805 POWER SUPPLY ON BOARD

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- PCB SIZE CP-PIC877 V2 16.5 x 8.5 cm

### Using CP-PIC877 V2 Board

1. Connect download cable between CP-PIC877 Board and Parallel Port of PC. On ETCAB10PIN

V2 Board, you put a tip of download cable into female connector named

ET-PSPI and other side put into female connector on board named

ET-PSPL as in the picture.

2. 9-12VDC-800mA Power Supply (OUTPUT Connector Female 2.5mm center nega-

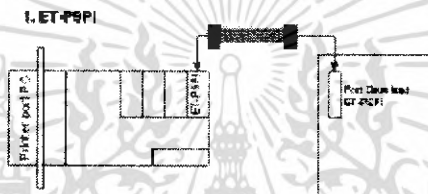


Figure 10.8: ET-PSPI

tive) or DC ADAPTER ET-ADAPTER 10V on CP-PIC877 V2

## 10.6 Circuit Driver

In this part we study 2 possibilities of the driving circuit.

1. class AB analog drive which is the circuit that give analog continuous output voltage. The circuit passes the signal from A/D to opamp which is the zero span circuit and diff. amp then class AB power amp.

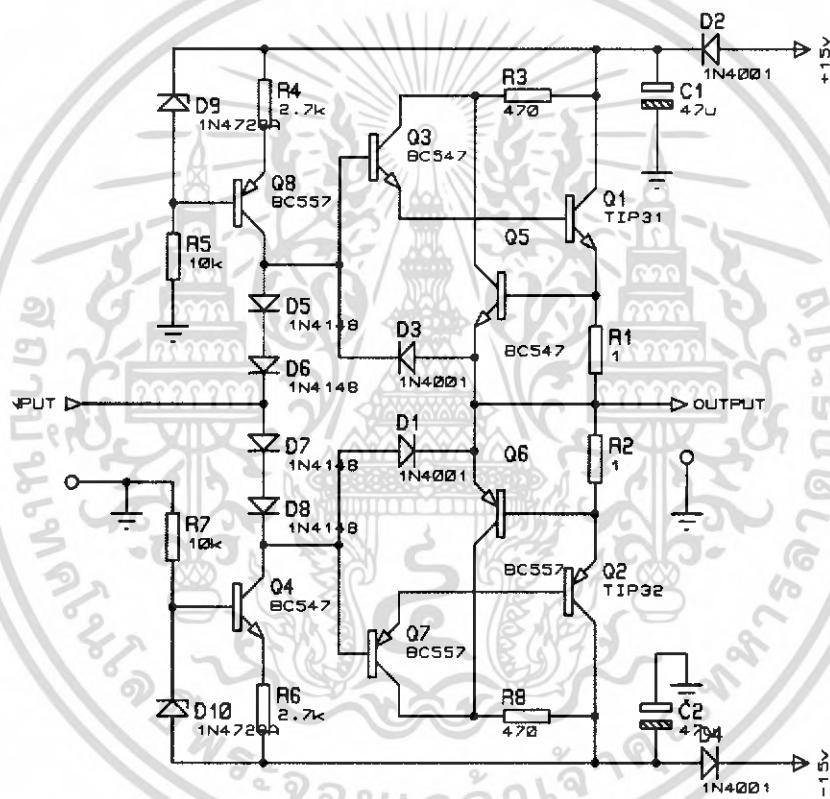


Figure 10.9: Motor Driver Amp Diagram

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2. PWM drive which is driving digitally.

We can connect use CP-PIC877 V2 board with ET-SMCC V2.0 board which has L298 as DRIVER IC

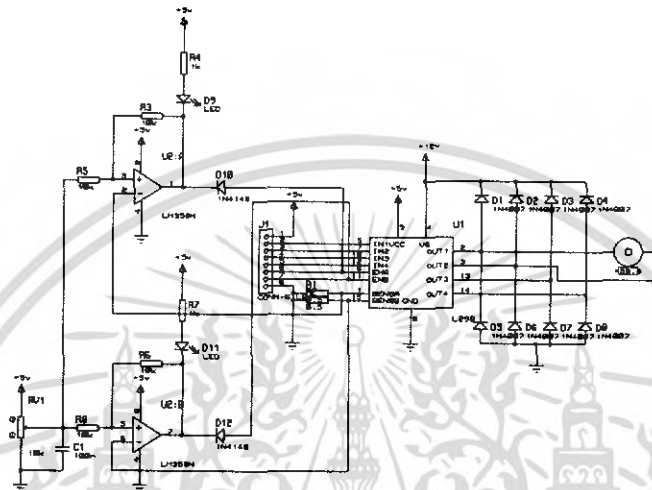


Figure 10.10: ET-SMCC Board

### Speed Control

Applying full power to the motor causes it to spin at top speed and removing power causes the motor to stop spinning. The H-bridge discussed above is capable of these two states. In order to get the motor to spin at lower speeds, a technique known as Pulse Width Modulation (PWM) can be used.

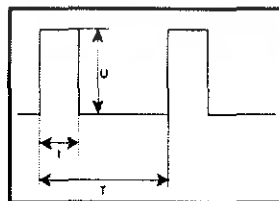


Figure 10.11: PWM 1

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The period of the pulse train is T. In each pulse the time for which the signal level is of U (logic 1) is t seconds. Therefore the average voltage is  $V_{avg} = \frac{Ut}{T}$

PWM is an efficient way for digital circuits to simulate a range of analog values. By

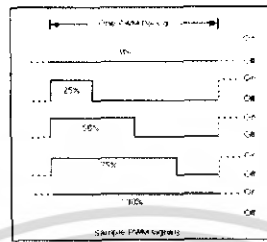


Figure 10.12: PWM 2

rapidly switching between full power and no power, an average value somewhere in between can be simulated. Three signals are shown in the Figure 2.11, 2.12, 2.13. All three signals share the same frequency, but the width of the pulses are different. By varying the duration of the “on” time, the motor can be made to “see” an average value anywhere between 0V and V+

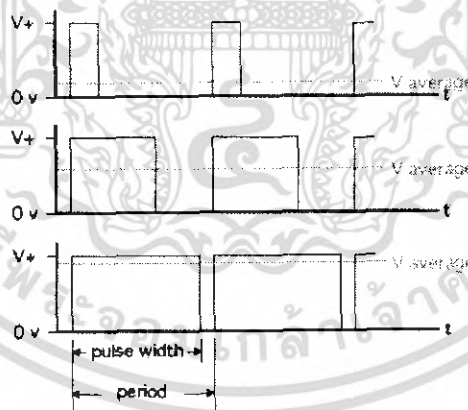


Figure 10.13: PWM 3

As for the frequency of this signal, anything above 10 kHz should be good. If you hear a high pitched whining coming from your motors then you might want to use a frequency above human hearing, say 20 kHz or 25 kHz.

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# Chapter 2

## Lyapunov-Based Design

### 2.1 Lyapunov-Based Design

The controller (1.2.5) is an outcome of a systematic Lyapunov design procedure. In this procedure we seek a parameter update law for the estimate  $\hat{\theta}(t)$ ,

$$\dot{\hat{\theta}} = \tau(x, \hat{\theta}), \quad (2.1.1)$$

which, along with a control law  $u = \alpha(x, \hat{\theta})$  will make the Lyapunov function

$$V(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}(\hat{\theta} - \theta)^2 \quad (2.1.2)$$

a nonincreasing function of time:

$$V(x(t), \hat{\theta}(t)) \leq V(x(t_0), \hat{\theta}(t_0)), \quad \forall t \geq t_0, \quad \forall t_0 \geq 0. \quad (2.1.3)$$

To this end, we express  $\dot{V}$  as a function of  $u$  and  $\hat{\theta}$  and seek  $\alpha(x, \hat{\theta})$  and  $\tau(x, \hat{\theta})$  to guarantee that  $\dot{V} \leq -px^2$  with  $p > 0$ , namely

$$\dot{V} = x(u + \theta x) + (\hat{\theta} - \theta)\dot{\hat{\theta}} \leq -px^2. \quad (2.1.4)$$

Rearranging term we get

$$xu + \hat{\theta}\dot{\hat{\theta}} + \theta(x^2 - \dot{\hat{\theta}}) \leq -px^2. \quad (2.1.5)$$

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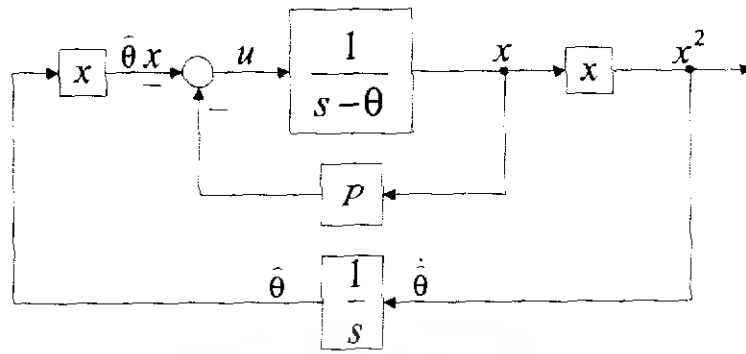


Figure 2.1: Lyapunov-based adaptive scheme for the scalar system (1.2.2)

Since neither  $\alpha(x, \hat{\theta})$  nor  $\tau(x, \hat{\theta})$  is allowed to depend on the unknown  $\theta$ , we must take  $\tau(x, \hat{\theta}) = x^2$ , that is

$$\dot{\hat{\theta}} = x^2. \quad (2.1.6)$$

The remaining condition

$$xu + \hat{\theta}x^2 \leq -px^2 \quad (2.1.7)$$

allows us to select  $\alpha(x, \hat{\theta})$  in various ways. The choice which result in the dynamic nonlinear controller (1.2.5) is

$$u = -(p + \hat{\theta})x. \quad (2.1.8)$$

# Chapter 3

## Feedback Linearization and zero dynamics

### 3.1 Feedback Linearization and zero dynamics

One of the popular methods for nonlinear control design is feedback linearization, which employs a change of coordinates and feedback control to transform a nonlinear system into a system whose dynamics are linear (at least partially).

A great deal of research has been devoted to this subject over the last two decades, as evidenced by the comprehensive books of Isidori [9] and Nijmeies and Van der Schaft [33] and the references therein. Since feedback linearization is not a goal pursued in this book, we only briefly review some concepts needed for the remainder of the chapter.

For maximum accessibility we avoid the direct use of differential geometric notations, but we still refer to those notations for references.

Let us consider the nonlinear system.

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R} \\ y &= h(x), \quad y \in \mathbb{R} \end{aligned} \tag{3.1.1}$$

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where  $f, g, h$  are smooth (that is, infinitely differentiable) vector functions.

The derivative of the output  $y = h(x)$  is given by:

$$\dot{y} = \overbrace{\frac{\partial h}{\partial x}(x)f(x)}^{L_f h} + \overbrace{\frac{\partial h}{\partial x}(x)g(x)u}^{L_g h} \quad (3.1.2)$$

If  $\frac{\partial h}{\partial x}(x_0)g(x_0) \neq 0$ , then the system (3.1.1) is said to have relative degree one at  $x_0$ .

In our terminology, this implies that the output  $y$  separated from the input  $u$  by one integrator only.

If  $\frac{\partial h}{\partial x}(x_0)g(x_0) = 0$ , there are two cases:

(i) If there exist points  $x$  arbitrarily close to  $x_0$  such that  $\frac{\partial h}{\partial x}(x)g(x) \neq 0$ , then (3.1.1) does not have a well-defined relative degree at  $x_0$ .

(ii) If there exists a neighborhood  $B_0$  of  $x_0$  such that  $\frac{\partial h}{\partial x}(x)g(x) = 0$  for all  $x \in B_0$ , then the relative degree of (3.1.1) at  $x_0$  may be well-defined.

In this case(ii), we define

$$\psi_1(x) = h(x), \quad \psi_2(x) = \frac{\partial h}{\partial x}(x)f(x) = L_f h \quad (3.1.3)$$

and compute the second derivative of  $y$ :

$$\begin{aligned} \ddot{y} &= \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \right) f + \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \right) g u \\ &= \frac{\partial \psi_2}{\partial x}(x)f(x) + \frac{\partial \psi_2}{\partial x}(x)g(x)u. \end{aligned} \quad (3.1.4)$$

If  $\frac{\partial \psi_2}{\partial x}(x_0)g(x_0) \neq 0$ , then (3.1.1) is said to have relative degree *two* at  $x_0$ .

If  $\frac{\partial \psi_2}{\partial x}(x)g(x) = 0$  in a neighborhood of  $x_0$ , then we continue the differentiation procedure.

**Definition 3.1.1** *The system (3.1.1) is said to have relative degree  $\rho$  at the point  $x_0$  if there exists a neighborhood  $B_0$  of  $x_0$  on which*

$$\frac{\partial \psi_1}{\partial x}(x)g(x) = \frac{\partial \psi_2}{\partial x}(x)g(x) = \dots = \frac{\partial \psi_{\rho-1}}{\partial x}(x)g(x) = 0 \quad (3.1.5)$$

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$$\frac{\partial \psi_\rho}{\partial x}(x)g(x) \neq 0, \quad (3.1.6)$$

where

$$\psi_1(x) = h(x), \quad \psi_i(x) = \frac{\partial \psi_{i-1}}{\partial x}(x)f(x), \quad i = 2, \dots, \rho \quad (3.1.7)$$

if (3.1.5) and (3.1.6) are valid for all  $x \in \mathbb{R}^n$ , then the relative degree of (3.1.1) is said to be globally defined.

Suppose now that (3.1.1) has relative degree  $\rho$  at  $x_0$ . Then we can use a change of coordinates and feedback control to locally transform of this system into the *cascade connection* a  $\rho$ -dimensional linear system and an  $(n - \rho)$ -dimensional nonlinear system.

In particular, after differentiating  $\rho$  times the output  $y = h(x)$ , the control  $u$  appears:

$$\begin{aligned} y^{(\rho)} &= \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left( \dots \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} f \right) f \dots \right) f \right) f(x) + \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left( \dots \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} f \right) f \dots \right) f \right) g(x)u \\ &= \frac{\partial \psi_\rho}{\partial x}(x)f(x) + \frac{\partial \psi_\rho}{\partial x}(x)g(x)u = v \end{aligned} \quad (3.1.8)$$

Since  $\frac{\partial \psi_\rho}{\partial x}g \neq 0$  in a neighborhood of  $x_0$ , we can linearize the input-output description of the system (3.1.1) using feedback to cancel the nonlinearities in (3.1.8):

$$u = \frac{1}{\frac{\partial \psi_\rho}{\partial x}(x)g(x)} \left[ -\frac{\partial \psi_\rho}{\partial x}(x)f(x) + v \right]. \quad (3.1.9)$$

Then the dynamic of  $y$  and its derivatives are governed by a chain of  $\rho$  integrators:

$$y^{(\rho)} = v.$$

Since our original system (3.1.1) has dimension  $n$ , we need to account for the remaining  $n - \rho$  states.

Using differential geometric tools, it is easy to show that it is always possible to find  $n - \rho$  functions  $\psi_{\rho+1}(x), \dots, \psi_n(x)$  with  $\frac{\partial \psi_i}{\partial x}(x)g(x) = 0, i = \rho + 1, \dots, n$  such that the change

of coordinates

$$\begin{aligned}
 \zeta_1 &= y = h(x) = \psi_1(x), \\
 \zeta_2 &= \dot{y} = \psi_2(x), \\
 &\vdots \\
 \zeta_\rho &= y^{(\rho-1)} = \psi_\rho(x), \\
 \zeta_{\rho+1} &= \psi_{\rho+1}(x), \\
 &\vdots \\
 \zeta_n &= \psi_n(x) \\
 y &= \psi_1
 \end{aligned} \tag{3.1.10}$$

is locally invertible and transforms, along with the feedback (3.1.9), the nonlinear system (3.1.1) into

$$\begin{aligned}
 \dot{\zeta}_1 &= \zeta_2 \\
 &\vdots \\
 \dot{\zeta}_{\rho-1} &= \zeta_\rho \\
 \dot{\zeta}_\rho &= v \\
 \dot{\zeta}_{\rho+1} &= \frac{\partial \psi_{\rho+1}}{\partial x}(x) f(x) = \phi_{\rho+1}(\zeta) \\
 &\vdots \\
 \dot{\zeta}_n &= \frac{\partial \psi_n}{\partial x}(x) f(x) = \phi_n(\zeta) \\
 y &= \zeta_1
 \end{aligned} \tag{3.1.11}$$

As a cascade connection of a chain of  $\rho$  integrators with an  $(n - \rho)$ -dimensional nonlinear system, this system is a special case of the cascade systems to which we will apply backstepping in the following chapters.

The states  $\zeta_{\rho+1}, \dots, \zeta_n$  of the nonlinear subsystem in (3.1.11) have been rendered *unobservable* from the output  $y$  by the control (3.1.9). Hence, feedback linearization in this case is the nonlinear equation of placing  $\rho$  poles of a linear system at the origin and canceling the  $(n - \rho)$  zeros with the remaining poles.

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Of course, to guarantee stability, the canceled zeros must be stable.

In the nonlinear case, using the new control input  $v$  to stabilize the linear subsystem of (3.1.11) does not guarantee stability of the nonlinear whole system, unless the stability of the part of (3.1.11) has been established separately.

When  $v$  is used to keep the output  $y$  equal to zero for all  $t > 0$ , that is, when  $\zeta_1 \equiv \dots \equiv \zeta_n \equiv 0$ , the dynamics of  $\zeta_{\rho+1}, \dots, \zeta_n$  are described by

$$\begin{aligned}\dot{\zeta}_{\rho+1} &= \phi_{\rho+1}(0, \dots, 0, \zeta_{\rho+1}, \dots, \zeta_n) \\ &\vdots \\ \dot{\zeta}_n &= \phi_n(0, \dots, 0, \zeta_{\rho+1}, \dots, \zeta_n)\end{aligned}\tag{3.1.12}$$

They are called *the zero dynamics* of (3.1.1), because they evolve on the subset of the state space on which the output of the system is identically zero. If the equilibrium at  $\zeta_{\rho+1} = \dots = \zeta_n = 0$  of the zero dynamics (3.1.12) is asymptotically stable, the system (3.1.1) is said to be *minimum phase*. With a slight abuse of notation, we will refer to the  $(\zeta_{\rho+1}, \dots, \zeta_n)$ -subsystem as the zero dynamics subsystem of (3.1.1), even when  $\xi_1, \dots, \xi_\rho$  are not zero.

In (3.1.1) the output  $y = h(x)$  is prespecified, possibly from a tracking objective, and the resulting cascade system is linear from the input  $v$  to the output  $y$ . This linearization process is usually called *input-output feedback linearization* [9]. If our goal is only to design a stabilizing controller, we may attempt to find an output with respect to which the relative degree is  $\rho = n$ . If such an output exists, the whole system is linearized without zero dynamics. This process is referred to as *full-state feedback linearization* [7, 8, 10, 44]. If such an output cannot be found, then we may look for an output which yields the highest relative degree, and thus results in a cascade system whose linear subsystem has the highest dimension [24]. It is desirable that with respect to the chosen output the system be minimum phase. The importance of this property will be clear in the following chapters which address problems of stabilization of cascade systems.

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# Chapter 4

## Stabilization of cascade systems

### 4.1 Stabilization of cascade systems

We now consider cascade connections in which the nonlinear system is globally stable, but the input subsystem is more complex than just an integrator. We begin with the case where the input subsystem is linear:

$$\dot{x} = f(x) + g(x)y, \quad f(0) = 0, x \in \mathbb{R}^n, y \in \mathbb{R} \quad (4.1.1a)$$

$$\dot{\xi} = A\xi + Bu, \quad y = h\xi. \quad (4.1.1b)$$

We assume that when  $y = 0$  the nonlinear system (4.1.1a) has a globally stable equilibrium at  $x = 0$ , and that an appropriate Lyapunov function  $V(x)$  is known such that:

$$\frac{\partial V}{\partial x}(x)f(x) \leq -W(x) \leq 0. \quad (4.1.2)$$

The problem is to stabilize the linear subsystem (4.1.1b) without destabilizing the nonlinear subsystem (4.1.1a), and, if possible, to achieve GAS of the equilibrium of (4.1.1) at  $(0, 0)$ . that is,

$$(x, \xi) = (0, 0)$$

This problem is not solvable in general. Here it will be solved by requiring the input subsystem (4.1.1b) to have the following passivity property:

**Assumption 4.1.1** *The triple  $(A, b, h)$  is feedback positive real (FPR), that is, there exists a linear feedback transformation  $u = K\xi + v$  such that  $A + bK$  is Hurwitz and there are matrices  $P > 0, Q \geq 0$  which satisfy*

$$(A + bK)^T P + P(A + bK) = -Q \quad (4.1.3a)$$

$$Pb = h^T. \quad (4.1.3b)$$

A sufficient condition for FPR is that there exists a feedback gain row vector  $K$  such that

- (1)  $A + bK$  is Hurwitz,
- (2) the transfer function  $Z(s) = h(sI - A - bK)^{-1}b$  is positive real (PR), and
- (3) the pair  $(A + bK, h)$  is observable.

It should be noted from (4.1.3b) that the relative degree of PR transfer function is one because  $b^T P b = h b > 0$ .

**Lemma 4.1.2 (Stabilization with FPR)** *Let  $V(x)$  be a Lyapunov function for (4.1.1a) satisfying (4.1.2). If the triple  $(A, b, h)$  is FPR, then a Lyapunov function for the cascade system (4.1.1) is*

$$V_a(x, \xi) = V(x) + \xi^T P \xi, \quad (4.1.4)$$

and the corresponding control law

$$u = \alpha_a(x, \xi) = K\xi - \frac{1}{2} \frac{\partial V}{\partial x} g(x) \quad (4.1.5)$$

guarantees that  $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$  is globally bounded and converges to the largest invariant set  $M_a$

contained in the set  $E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \mid W(x) = 0, Q^{\frac{1}{2}} \xi = 0 \right\}$ . If  $W(x)$  is positive definite, that is, if the nonlinear subsystem (4.1.1a) with  $y = 0$  has a globally asymptotically stable equilibrium at  $x = 0$ , then the equilibrium  $x = 0, \xi = 0$  is also GAS.

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 ไม่สามารถแก้ไข หรือทำซ้ำโดยไม่ได้รับอนุญาต  
 ไม่สามารถนำข้อมูลไปใช้ทางอื่นได้

**Proof** Using (4.1.2) and (4.1.3a) and denoting  $u = K\xi + v$  with  $v = -\frac{1}{2}\frac{\partial V}{\partial x}g(x)$  from (4.1.5), the derivative of  $V_a(x, \xi)$  is

$$\begin{aligned}\dot{V}_a &= \frac{\partial V}{\partial x}(x)[f(x) + g(x)y] \\ &\quad + \xi^T P[(A + bK)\xi + bv] + [(A + bK)\xi + bv]^T P\xi\end{aligned}$$

by (4.1.2) and (4.1.3a)

$$\leq -W(x) + \frac{\partial V}{\partial x}(x)g(x)y - \xi^T Q\xi + 2\xi^T Pbv$$

by (4.1.3b)

$$\begin{aligned}&= -W(x) + \frac{\partial V}{\partial x}(x)g(x)y - \xi^T Q\xi + 2y \left[ -\frac{1}{2}\frac{\partial V}{\partial x}g(x) \right] \\ &= -W(x) - \xi^T Q\xi \leq 0.\end{aligned}$$

Since  $V_a$  is positive definite, radially unbounded and has a negative semidefinite derivative,  $x(t)$  and  $\xi(t)$  are globally bounded.

Furthermore LaSalle's theorem (Theorem A.1.2) guarantees convergence to the largest invariant set  $M_a$  in the set  $E_a$ .

If, in addition,  $W(x)$  is positive definite, then the global asymptotic stability of  $x = 0, \xi = 0$  is shown using Corollary 2.3. From the positive definiteness of  $W(x)$ , the set  $E_a$ , on which  $\dot{V}_a = 0$ , is given by  $E_a = \{(x, \xi) | x = 0, Q^{\frac{1}{2}}\xi = 0\}$ .

Since  $V(x)$  is positive definite, it has a minimum at  $x = 0$ , and thus  $\frac{\partial V}{\partial x}(0) = 0$ . This implies that on the set  $E_a$  the control term  $v = -\frac{1}{2}\frac{\partial V}{\partial x}g(x)$  vanishes.

Hence, on the set  $E_a$  the state  $\xi(t)$  satisfies

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With this case restriction on  $P$  and with  $K = [ -k_1 \quad -k_2 ]$ , (4.1.3a) results in

$$\begin{bmatrix} -2k_1 & p_{11} - k_1 - k_2 \\ p_{11} - k_1 - k_2 & 2 - 2k_2 \end{bmatrix} = - \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}. \quad (4.1.11)$$

For the simplest choice  $q_{11} = q_{22} = 1$  and  $q_{12} = 0$  we get  $k_1 = 0.5$ ,  $k_2 = 1.5$ ,  $p_{11} = 2$ .

Then the control law (4.1.5) is

$$u = -\frac{1}{2}\xi_1 - \frac{3}{2}\xi_2 - x^2. \quad (4.1.12)$$

The equilibrium  $(x, \xi_1, \xi_2) = (0, 0, 0)$  is GAS because  $Q = I$  is positive definite.  $\square$

**Example 4.1.4** Let us now consider a system in which  $\dot{x} = f(x)$  has a GAS equilibrium at  $x = 0$ :

$$\dot{x} = -x^3 - x^3 \underbrace{(h_1\xi_1 + h_2\xi_2)}_u \quad (4.1.13a)$$

$$\dot{\xi}_1 = \xi_2 \quad (4.1.13b)$$

$$\dot{\xi}_2 = u \quad (4.1.13c)$$

When  $h_2 = 0$  this system is stabilizable by two steps of integrator backstepping as in corollary (B.1.4). Thus, the case of interest is when  $h_2 \neq 0$  and  $h_1 h_2 \geq 0$  such that  $h_1 \geq 0$ ,  $h_2 > 0$ . This includes the case  $h_1 = 0$  when the transfer function  $\frac{h_2 s + h_1}{s^2}$  of the linear part is only *weak minimum phase* [38] because it has a zero at  $s = 0$ .

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, h = [ h_1 \quad h_2 ]$$

$$Z(s) = h(sI - A)^{-1}b = [ h_1 \quad h_2 ] \frac{\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2}$$

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Choose a feedback  $u = -k_1\xi_1 - k_2\xi_2 + v$  with  $k_1, k_2 > 0$  which makes the polynomial  $q(s) = s^2 + k_2s + k_1$  Hurwitz and denote  $p(s) = h_1 + h_2s$ .

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad u = -k_1\xi_1 - k_2\xi_2 + v$$

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$0 = s^2 + k_2s + k_1$$

$$s = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2}, \quad k_2 > 0$$

$\therefore \Re(s) = \frac{-k_2}{2} < 0$ , with  $k_2 > 0$ , then  $q(s)$  is Hurwitz.

We can choose  $k_1 = a^2$  and  $k_2 = 2a$ , with  $a > \frac{h_1}{h_2}$ , so that the transfer function  $Z(s) = \frac{p(s)}{q(s)}$  is positive real

$$k_1 = a^2 \quad \text{and} \quad k_2 = 2a$$

$$Z(s) = \frac{p(s)}{q(s)} \quad \text{is P.R. if } a > \frac{h_1}{h_2}$$

Note : assume  $h_1 > 0, h_2 \geq 0$   $Z(s) = \frac{h_1 + h_2s}{s^2 + 2as + a^2}$

$$s = \sigma + j\omega \quad \therefore Z(s) = \frac{(h_1 + h_2\sigma) + jh_2\omega}{(\sigma^2 + 2a\sigma - \omega^2 + a^2) + j2\omega(\sigma + a)}$$

$$Z(s) = \frac{[(h_1 + h_2\sigma) + jh_2\omega]\{(\sigma^2 + 2a\sigma - \omega^2 + a^2) - j2\omega(\sigma + a)\}}{(\sigma^2 + 2a\sigma - \omega^2 + a^2)^2 + [2\omega(\sigma + a)]^2}$$

$$\Re\{Z(s)\} = \frac{(h_1 + h_2\sigma)(\sigma^2 + 2a\sigma - \omega^2 + a^2) + 2h_2\omega^2(\sigma + a)}{(\sigma^2 + 2a\sigma - \omega^2 + a^2)^2 + [2\omega(\sigma + a)]^2}$$

$\therefore$  For  $Z(s)$  to be P.R. we must have

$$(h_1 + h_2\sigma)(\sigma^2 + 2a\sigma - \omega^2 + a^2) + 2h_2\omega^2(\sigma + a) \geq 0, \quad \forall \sigma \geq 0$$

This can be achieved by nothing that when  $\sigma = 0$

$$h_1(a^2 - \omega^2) + 2h_2\omega^2a \geq 0$$

$$h_1a^2 + \omega^2(2h_2a - h_1) \geq 0, \quad h_1a^2 \geq 0$$

$$2h_2a - h_1 \geq 0$$

$$2h_2a \geq h_1$$

$$a \geq \frac{h_1}{2h_2}$$

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษา  $a \geq \frac{h_1}{2h_2}$  ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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If  $a > \frac{h_1}{h_2} \geq \frac{h_1}{2h_2}$  and when  $\sigma > 0$ , the increased term

$$\begin{aligned} (h_1 + h_2\sigma)(\sigma + a)^2 - h_1\omega^2 - h_2\sigma\omega^2 + 2h_2\omega^2\sigma + 2h_2\omega^2a &= \\ (h_1 + h_2\sigma)(\sigma + a)^2 + h_2\sigma\omega^2 + \omega^2(2h_2a - h_1) &\geq 0 \\ 2h_2a - h_1 &\geq 0 \\ 2h_2a &\geq h_1 \\ a &\geq \frac{h_1}{2h_2} \end{aligned}$$

If  $a > \frac{h_1}{h_2}$  then  $a \geq \frac{h_1}{2h_2}$ . So the value of  $\Re\{Z(s)\}$  is always  $\geq 0$ .

$A_* = A + bK$  is Hurwitz.

$Z(s) = h(sI - A_*K)^{-1}b$  is P.R.

$(A + bK, h)$  is observable

$$\mathcal{O} = \begin{bmatrix} h_1 & h_2 \\ -h_2k_1 & h_1 - h_2k_2 \end{bmatrix}$$

$$\begin{aligned} \det \mathcal{O} &= h_1^2 - h_1h_2k_2 + h_2^2k_1 \\ &= h_1^2 - sh_1h_2a + h_2^2a^2 \\ &= (h_1 - h_2a)^2 \geq 0 \\ \det \mathcal{O} \neq 0 &\Leftrightarrow h_1 \neq h_2a \\ &a \neq \frac{h_1}{h_2} \end{aligned}$$

$$a > \frac{h_1}{2h_2} \quad \text{and} \quad a \neq \frac{h_1}{h_2} \Rightarrow a > \frac{h_1}{h_2} \quad \text{is more strict.}$$

We can then write (4.1.13a) as  $\dot{x} = f + gy$ ,  $f = g = -x^3$ . Clearly,  $x = 0$  is a GAS equilibrium for  $\dot{x} = f$ , so the conditions of lemma (4.1.2) are satisfied. Using  $V(x) = x^2$  in (4.1.5), we obtain the control law

$$u = -a^2\xi_1 - 2a\xi_2 + \underbrace{-\frac{1}{2} \frac{d(x^2)}{dt}}_{x^4} \cdot (-x^3). \quad (4.1.14)$$

The situation is quite different when  $h_1h_2 < 0$ , that is, when the transfer function  $Z(s) = \frac{p(s)}{q(s)}$ , is nonminimum phase. Then, Lemma 4.1.2 does not apply. In fact, a เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปเผยแพร่บนสื่อออนไลน์ การค้าไม่ว่ากรณีใดๆ ทั้งสิ้น อีกทั้งห้ามมิให้ดัดแปลงเนื้อหา และต้องอ้างอิงถึงเจ้าของเอกสารทุกครั้งที่มีการนำไปใช้

detailed calculation given in [19] shows that in this case the system cannot be globally stabilized.

The FPR property is a passivity property. Its nonlinear counterpart will be employed in the stabilization of the nonlinear cascade

$$\dot{x} = f(x, \xi) + g(x, \xi)y, \quad f(0, \xi) = 0, \quad \forall \xi \in \mathbb{R}^q, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R} \quad (4.1.15a)$$

$$\dot{\xi} = m(\xi) + \beta(\xi)u, \quad y = h(\xi), \quad h(0) = 0, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R}. \quad (4.1.15b)$$

Our key assumption is that (4.1.15b) can be rendered passive or strictly passive (cf. Appendix D) via a feedback transformation  $u = k(\xi) + r(\xi)v$ .

**Definition 4.1.5** *The system*

$$\dot{\xi} = m(\xi) + \beta(\xi)u, \quad y = h(\xi), \quad h(0) = 0, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R} \quad (4.1.16)$$

*is said to be feedback passive (FP) if there exists a feedback transformation*

$$u = k(\xi) + r(\xi)v \quad (4.1.17)$$

*such that the resulting system  $\dot{\xi} = m(\xi) + \beta(\xi)k(\xi) + \beta(\xi)r(\xi)v$ ,  $y = h(\xi)$  is passive with a storage function  $U(\xi)$  which is positive definite and radially unbounded:*

$$\int_0^t y(\sigma)v(\sigma)d\sigma \geq U(\xi(t)) - U(\xi(0)). \quad (4.1.18)$$

*The system (4.1.16) is said to be feedback strictly passive (FSP) if the feedback (4.1.17) renders it strictly passive:*

$$\int_0^t y(\sigma)v(\sigma)d\sigma \geq U(\xi(t)) - U(\xi(0)) + \int_0^t \psi(\xi(\sigma))d\sigma, \quad (4.1.19)$$

*where  $\psi(\cdot)$  is the positive definite dissipation rate.*

As in the linear case, FP systems of the form (4.1.16) must have relative degree one.

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**Lemma 4.1.6 (Stabilization with Passivity)** *Let  $V(x)$  be a radially unbounded Lyapunov function for  $\dot{x} = f(x, \xi)$  satisfying*

$$\frac{\partial V}{\partial x}(x)f(x, \xi) \leq -W(x) \leq 0, \forall \xi \in \mathbb{R}^n, \quad \forall \xi \in \mathbb{R} \quad (4.1.20)$$

and let (4.1.15b) be FP as in Definition 4.1.5. Then, a Lyapunov function for the cascade system (4.1.15a) is

$$V_a(x, \xi) = V(x) + U(\xi), \quad (4.1.21)$$

and the corresponding control law

$$u = \alpha_a(x, \xi) = k(\xi) - r(\xi) \frac{\partial V}{\partial x}(x)g(x, \xi) \quad (4.1.22)$$

guarantees that  $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$  is globally bounded and converges to the largest invariant set

$\bar{M}_a$  contained in the set  $\bar{E}_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \mid W(x) = 0 \right\}$ . If (4.1.15b) is FSP, then (4.1.22) guarantees convergence to the largest invariant set  $M_a$  contained in the set  $E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \mid W(x) = 0, \xi = 0 \right\}$ . Finally, if (4.1.15b) is FSP and  $W(x)$  is positive definite, that is, if  $\dot{x} = f(x, \xi)$  has a GAS equilibrium at  $x = 0$  uniformly in  $\xi$ , then the equilibrium  $x = 0, \xi = 0$  of (4.1.15a) is also GAS.

The closed-loop system (4.1.15) with the control (4.1.22) is

$$\begin{aligned} \dot{x} &= f(x, \xi) + g(x, \xi)y \\ \dot{\xi} &= m(\xi) + \beta(\xi)k(\xi) + \beta(\xi)r(\xi)v \\ y &= h(\xi), \quad v = -\frac{\partial V}{\partial x}(x)g(x, \xi). \end{aligned} \quad (4.1.23)$$

$V(x)$  P.D. and R.U. Lyapunov function for  $\dot{x} = f(x, \xi)$  satisfying  $\frac{\partial V}{\partial x}(x)f(x, \xi) \leq -W(x) \leq 0, \forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}$ ,

$$\dot{V} = \frac{\partial V}{\partial x}(f + gy) \leq -W(x) + \frac{\partial V}{\partial x}gy = -W(x) + \eta y \quad (4.1.24)$$

$$\int_0^t \eta(\sigma)y(\sigma)d\sigma \geq V(x(t)) - V(x(0)) + \int_0^t W(x(\sigma))d\sigma. \quad (4.1.25)$$

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$\therefore \Sigma_2$  is passive since  $W(x) \geq 0$ . From Theorem C.1.4 we conclude that the negative feedback interconnection of  $\Sigma_1$  and  $\Sigma_2$  is passive with the positive definite and radially unbounded storage function  $V_a(x, \xi) = V(x) + U(\xi)$ . Lemma C.1.3 then states that  $x = 0, \xi = 0$  is a globally stable equilibrium of

$$\begin{aligned}\dot{x} &= f(x, \xi) + g(x, \xi)y \\ \dot{\xi} &= m(\xi) + \beta(\xi)k(\xi) + \beta(\xi)r(\xi)v \\ y &= h(\xi), \quad v = -\frac{\partial V}{\partial x}(x)g(x, \xi).\end{aligned}$$

To see that  $W(x) \rightarrow 0$  as  $t \rightarrow \infty$ , we differentiate  $\int y(\sigma)v(\sigma)d(\sigma) \geq U(\xi(t)) - U(\xi(0))$  and combine the result with  $\frac{\partial V}{\partial x}(x)f(x, \xi) \leq -W(x) \leq 0 \forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}$ :

$$\begin{aligned}\dot{V}_a &= \dot{V} + \dot{U} \leq \frac{\partial V}{\partial x}(f + gy) + yv \\ &\leq -W(x) + \frac{\partial V}{\partial x}gy + yv = -W(x) \leq 0.\end{aligned}\tag{4.1.26}$$

Then, LaSalle's theorem (Theorem A.1.2) guarantees convergence to the set  $\bar{M}_a \subseteq \{W(x) = 0\}$ . If (4.1.15b) is FSP; that is,

$$\int_0^t y(\sigma)v(\sigma)d\sigma \geq U(\xi(t)) - U(\xi(0)) + \int_0^t \psi(\xi(\sigma))d\sigma\tag{4.1.27}$$

we replace (4.1.18) by (4.1.27). Then (4.1.26) becomes

$$\dot{v}_a \leq -W(x) - \psi(\xi)\tag{4.1.28}$$

which, since  $\psi(\xi)$  is positive definite, guarantees convergence to the set  $M_a \subseteq \{W(x) = 0, \xi = 0\}$ .

Finally, if  $W(x)$  is also positive definite, we conclude from (4.1.28) and Theorem A.1.1 that  $x = 0, \xi = 0$  is GAS.

$$(\because V_a = v(x) + U(\xi) \text{ is p.d. and R.U. in } (x, \xi))$$

$$\dot{v}_a = \dot{V} + \dot{U} \leq -W(x) - \psi(\xi)$$

**Example 4.1.7** consider the cascade system:

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$$\dot{x} = -x(1 + e^\xi) + x^3\xi^2 \quad (4.1.29a)$$

$$\dot{\xi} = \xi v \quad (4.1.29b)$$

$$y = \xi^2$$

The choice of output  $y = \xi^2$  satisfies all the conditions of Lemma 4.1.6. First, (4.1.29b) is FSP: The feedback

$$u = -\xi^2 + v \quad (4.1.30)$$

results in  $\dot{\xi} = -\xi^3 + \xi v$ ,  $y = \xi^2$ , which is strictly passive with storage function  $U(\xi) = \frac{1}{2}\xi^2$ , since

$$\dot{U} = -\xi^4 - \xi^2 v = -\xi^4 + yv \quad (4.1.31)$$

implies that

$$\int_0^t y(\sigma)v(\sigma)d\sigma \geq U(\xi(t)) - U(\xi(0)) + \int_0^t \xi^4(\sigma)d\sigma. \quad (4.1.32)$$

Furthermore, (4.1.29a) can be represented in the form (4.1.15a) with

$$f(x, \xi) = -x(1 + e^\xi), \quad g(x, \xi) = x^3 \quad (4.1.33)$$

and (4.1.20) is satisfied with  $V(x) = \frac{1}{2}x^2$ , i.e.,

$$\begin{aligned} \frac{\partial V}{\partial x}(x)f(x, \xi) &= x[-x(1 + e^\xi)] \\ &= -x^2(1 + e^\xi) \leq -x^2 = -W(x) \leq 0, \quad \forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R} \end{aligned}$$

$$W(x) = x^2$$

Applying Lemma 4.1.6, we conclude that the control

$$u = -\xi^2 - x^4 \quad (4.1.34)$$

guarantees GAS of  $x = 0$ ,  $\xi = 0$ . Indeed, the derivative of the clf  $V_a(x, \xi) = \frac{1}{2}(x^2 + \xi^2)$  is negative definite:

$$\dot{V}_a = -x^2(1 + e^\xi) + x^4\xi^2 - \xi^4 - x^4\xi^2 \leq -x^2 - \xi^4. \quad (4.1.35)$$

□

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# Chapter 5

## Block backstepping with zero dynamics

### 5.1 Block backstepping with zero dynamics

Integrator backstepping (Lemma B.1.2) is a recursive design tool. Now we want to develop a similar tool for feedback stabilization of a system augmented by a dynamic block more complicated than just an integrator.

At first glance, it may appear that the cascade design in the preceding chapter provides us with such a tool. Not quite! The achievement of the cascade design is in being able to stabilize the input subsystem (4.1.1b) or (4.1.15b) *without destabilizing the original system*. What if the original system is not stable? Can we cascade it with a complicated input subsystem and still stabilize it in one step? We first show that this can be done with a linear input subsystem that is a minimum phase system with relative degree one. We then give a nonlinear extension of that result.

**Example 5.1.1** *Let us start with an example in which we cascade the system (4.1.29) of Example 4.1.7 with a linear minimum phase system:*

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$$\begin{aligned}
 x\text{-subsystem: } & \begin{cases} \dot{x}_1 = -x_1(1 + e^{x_2}) + x_1^3 x_2^2 \\ \dot{x}_2 = x_2 y \end{cases} \\
 \xi\text{-subsystem: } & \begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = u \\ y = \xi_1 + \xi_2. \end{cases}
 \end{aligned} \tag{5.1.1}$$

The transfer function of the input subsystem is  $\frac{s+1}{s^2}$  and its zero is at  $s = -1$ .

One of its minimal realizations is

$$\dot{y} = y - \xi_1 + u \tag{5.1.2a}$$

$$\dot{\xi}_1 = -\xi_1 + y. \tag{5.1.2b}$$

Its zero dynamics, that is, the dynamics constrained by  $y(t) \equiv 0$ , are described by  $\dot{\xi}_1 = -\xi_1$ .

The cascade design of the preceding subsection is not applicable to (5.1.1) because the equilibrium  $x = 0$  of the  $x$ -subsystem with  $y = 0$  is unstable:  $\dot{x}_2 = x_2 y$  and  $y = 0 \Rightarrow x_2$  can be constant and  $x_2 \neq 0$ . Such as,  $x_2 = 2$ :

$$\begin{aligned}
 \dot{x}_1 &= -x_1(1 + e^2) + 4x_1^3 = ax_1^3 - bx_1 \\
 &= ax_1\left(x_1^2 - \frac{b}{a}\right); \quad a, b > 0
 \end{aligned}$$

linearizing around  $(x_1, x_2) = (0, 0)$ ,

$$\begin{cases} \dot{x}_1 = -x_1(1 + 1 + x_2 + \frac{x_2^2}{2!} + \dots) + x_1^3 x_2^2 \\ \dot{x}_2 = 0 \end{cases}$$

we obtain

$$\begin{cases} \dot{x}_1 = -2x_1 \\ \dot{x}_2 = 0 \end{cases}$$

or

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} x.$$

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้ marginally stable ศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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To circumvent this obstacle, we first convert (5.1.2a) into an integrator via the feedback transformation

$$u = \underbrace{-y + \xi_1}_{-\xi_2} + v, \quad (5.1.3)$$

where  $v$  is our new control variable. The system (5.1.1) is then rewritten as

$$\begin{aligned} \dot{x}_1 &= -x_1(1 + e^{x_2}) + x_1^3 x_2^2 \\ \dot{x}_2 &= x_2 y \\ \dot{y} &= v \\ \dot{\xi}_1 &= -\xi_1 + y. \end{aligned} \quad (5.1.4)$$

Now the subsystem consisting of the first three equations in (5.1.4) is in a form convenient for integrator backstepping. From Example 4.1.7 we already know that the  $x$ -subsystem can be stabilized with  $y$  as its virtual control (cf.(4.1.34)):

$$y_{des} = \alpha(x) = -x_1^4 - x_2^2. \quad (5.1.5)$$

The corresponding clf is  $V(x, \xi) = \frac{1}{2}(x_1^2 + x_2^2)$ . Hence, we can achieve stabilization and regulation of  $x_1, x_2, y$  by a direct application of Lemma B.1.2. The resulting control law is

$$u = -(y + x_1^4 + x_2^2) - \xi_2 - 2x_2^2 y + 4x_1^4(1 + e^{x_2} - x_1^2 x_2^2) - x_2^2. \quad (5.1.6)$$

This design ignored the presence of the zero dynamics subsystem  $\dot{\xi}_1 = -\xi_1 + y$ . However, this subsystem is input-to-state stable (ISS) with respect to  $y$ , so that  $\xi_1$  is bounded because

$y$  is bounded, and moreover  $\lim_{t \rightarrow \infty} \xi_1(t) = 0$  since  $\lim_{t \rightarrow \infty} y(t) = 0$ .

$$\begin{aligned}\dot{\xi}_1 &= -\xi_1 + y \\ \dot{\xi}_1 + \xi_1 &= y \\ e^t \dot{\xi}_1 + e^t \xi_1 &= e^t y \\ \frac{d}{dt}(e^t \xi_1) &= e^t y \\ e^s \xi_1 \Big|_0^t &= \int_0^t e^s y(s) ds \\ e^t \xi_1(t) - \xi_1(0) &= \int_0^t e^s y(s) ds \\ \xi_1(t) &= e^{-t} \xi_1(0) + e^{-t} \int_0^t e^s y(s) ds.\end{aligned}$$

Since  $\lim_{t \rightarrow \infty} y(t) = 0$  then

$$\begin{aligned}\dot{\xi}_1 &= -\xi_1 + y = -\xi_1 \\ \xi_1(t) &= e^{-t} \xi_1(0) \rightarrow 0, \quad \text{as } t \rightarrow \infty.\end{aligned}$$

We now want to generalize the above example and formulate design tools which allow the original system to be unstable when  $y = 0$  and let us backstep more than a simple integrator at a time. Since we want to be able to apply these tools repeatedly, each lemma we formulate must guarantee for the cascade system all the properties assumed for the original system.

As we will see, the constructed  $V_\alpha(x, \xi)$  for the cascade system does not include the zero dynamics variables, but their boundedness is guaranteed by the boundedness of  $V_\alpha$ .

Hence, we must reformulate Assumption B.1.1 to assume the same properties for the original system, by including the case when  $V(x)$  is not positive definite:

**Assumption 5.1.2** *Suppose Assumption B.1.1 is valid with  $V(x)$  positive semidefinite, and the closed-loop system (B.1.17) with the control (B.1.18) has the property that  $x(t)$  is bounded if  $V(x(t))$  is bounded.*

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Under this assumption, the control (B.1.18), applied to the system (B.1.17), guarantees not only global boundedness of  $x(t)$ , but also regulation of  $W(x(t))$ :

From (B.1.19) we conclude that  $W(x(t))$  is integrable on  $[0, \infty)$  and uniformly continuous, and hence converges to zero by Lemma A.3.5. Furthermore, since all solutions  $x(t)$  are bounded, we can apply LaSalle's theorem (Theorem A.1.2) to conclude that  $x(t)$  converges to the largest invariant set  $M$  contained in the set  $E = \{x \in \mathbb{R}^n | W(x) = 0\}$ .

The following fact is easy to prove:

**Corollary 5.1.3** *When Assumption B.1.1 is replaced by Assumption 5.1.2, then the boundedness and convergence properties in part (ii) of Lemma B.1.2 still hold.*

**Lemma 5.1.4 (Linear Block Backstepping)** *Consider the cascade system*

$$\dot{x} = f(x) + g(x)y, \quad f(0) = 0, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R} \quad (5.1.7a)$$

$$\dot{\xi} = A\xi + bu, \quad y = h\xi, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R} \quad (5.1.7b)$$

where (5.1.7b) is a minimum phase system of relative degree one ( $hb \neq 0$ ).

If (5.1.7a) satisfies Assumption 5.1.2 with  $y$  as its input, then there exists a feedback control which guarantees global boundedness and convergence of  $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$  to the largest

invariant set  $M_a$  contained in the set  $E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \mid W(x) = 0, y = \alpha(x) \right\}$ .

One choice for this control is

$$u = \frac{1}{hb} \left\{ -c(y - \alpha(x)) - hA\xi + \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)y] - \frac{\partial V}{\partial x}(x)g(x) \right\}, \quad c > 0. \quad (5.1.8)$$

Moreover, if  $V(x)$  and  $W(x)$  are positive definite, then the equilibrium  $x = 0, \xi = 0$  is

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**Proof** We recall from [39] that the relative-degree-one SISO linear system (5.1.7b) can be represented in the form

$$\dot{y} = hA\xi + hbu \quad (5.1.9a)$$

$$\dot{\zeta} = A_0\zeta + b_0y, \quad (5.1.9b)$$

where the eigenvalues of  $A_0$  are the (stable) zeros of the transfer function

$$H(s) = h(sI - A)^{-1}b$$

of the minimum phase system (5.1.7b). Using (5.1.9) and the feedback transformation

$$u = \frac{1}{hb}(v - hA\xi), \quad (5.1.10)$$

we rewrite (5.1.7) as follows:

$$\dot{x} = f(x) + g(x)y \quad (5.1.11a)$$

$$\dot{y} = v \quad (5.1.11b)$$

$$\dot{\zeta} = A_0\zeta + b_0y. \quad (5.1.11c)$$

We first ignore the zero dynamics (5.1.11c) and, using Corollary 5.1.3, apply Lemma B.1.2 to (5.1.11a)-(5.1.11b) to achieve global boundedness of  $x$  and  $y$  and regulation of  $W(x(t))$  and  $y(t) - \alpha(x(t))$ . In view of (5.1.10) and (B.1.23), one choice of control is given by (5.1.8).

Returning to (5.1.11c), we note that  $\zeta$  is bounded because  $y$  is bounded and  $A_0$  is strictly Hurwitz. Thus,  $\xi$  is bounded. Since all solution of (5.1.7) are bounded, we can apply LaSalle's theorem (Theorem A.1.2) with  $\Omega = \mathbb{R}^{n+q}$  to conclude convergence to the set  $M_\alpha$ .

From Lemma A.1.8 we also know that if  $V(x)$  and  $W(x)$  are positive definite, then the equilibrium  $x = 0, y = 0$  of (5.1.11a)-(5.1.11b), which is completely decoupled from (5.1.11c), is GAS. The fact that in this case the equilibrium  $x = 0, \xi = 0$  of the cascade system (5.1.7) is also GAS follows immediately from the following lemma:

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**Lemma 5.1.5** Consider the cascade system with  $\zeta \in \mathbb{R}^m, x \in \mathbb{R}^n$  :

$$\dot{\zeta} = A_0\zeta + b_0y \quad (5.1.12a)$$

$$\dot{x} = f(x), f(0) = 0$$

$$y = h(x), h(0) = 0 \quad (5.1.12b)$$

If (5.1.12b) is GAS and  $A_0$  is strictly Hurwitz, then the equilibrium  $\zeta = 0, x = 0$  of the cascade (5.1.12) is GAS.

**Proof** From the definition of GAS (Definition A.3.4) we know that the GAS property of (5.1.12b) implies the existence of class  $KL_\infty$  function  $\beta$  and  $\beta_1$  such that

$$|x(t)| \leq \beta(|x(0)|, t), \quad |y(t)| \leq \beta_1(|x(0)|, t). \quad (5.1.13)$$

The solution of (5.1.12a), on the other hand, are given by

$$\zeta(t) = e^{A_0 t} \zeta(0) + \int_0^t e^{A_0(t-\tau)} b_0 y(\tau) d\tau. \quad (5.1.14)$$

Since  $A_0$  is strictly Hurwitz, we know that  $|e^{A_0 t}| \leq k_1 e^{-\alpha t}$ .

$$\begin{aligned} \forall \lambda_i \in \sigma(A_0), \operatorname{Re} \lambda_i < 0 &\Leftrightarrow A_0 \text{ is strictly Hurwitz.} \\ &\Leftrightarrow \exists \alpha > 0 \text{ such that } \alpha I + A_0 \text{ is strictly Hurwitz.} \\ &\Leftrightarrow (e^{(\alpha I + A_0)t} = P e^{(\alpha I + D)t} P^{-1}) \\ &\Leftrightarrow |e^{(\alpha I + A_0)t}| \leq k_1, \exists k_1 > 0 \\ &\Leftrightarrow |e^{\alpha t I} e^{A_0 t}| \leq k_1, \exists k_1 > 0 \\ &\Leftrightarrow |e^{\alpha t} I e^{A_0 t}| \leq k_1, \exists k_1 > 0 \\ &\Leftrightarrow |e^{\alpha t}| |e^{A_0 t}| \leq k_1, \exists k_1 > 0 \\ &\Leftrightarrow |e^{A_0 t}| \leq k_1 e^{-\alpha t}, \exists k_1 > 0, \exists \alpha \in [0, |\lambda|_{\min}). \end{aligned}$$

Using this with (5.1.13) in (5.1.14), we obtain

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$$\begin{aligned}
|\zeta(t)| &\leq |e^{A_0 t} \zeta(0)| + \int_0^t |e^{A_0(t-\tau)} b_0| |y(\tau)| d\tau \\
&\leq k_1 e^{-\alpha t} |\zeta(0)| + k_2 \int_0^t e^{-\alpha(t-\tau)} \beta_1(|x(0)|, \tau) d\tau \\
&\leq k_1 e^{-\alpha t} |\zeta(0)| + k_2 \sup_{0 \leq \tau \leq t/2} \beta_1(|x(0)|, \tau) \int_0^{t/2} e^{-\alpha(t-\tau)} d\tau \\
&\quad + k_2 \sup_{t/2 \leq \tau \leq t} \beta_1(|x(0)|, \tau) \int_{t/2}^t e^{-\alpha(t-\tau)} d\tau \\
&\leq k_1 e^{-\alpha t} |\zeta(0)| + k_2 \beta_1(|x(0)|, 0) \int_0^{t/2} e^{-\alpha(t-\tau)} d\tau \\
&\quad + k_2 \beta_1(|x(0)|, t/2) \int_{t/2}^t e^{-\alpha(t-\tau)} d\tau \\
&= k_1 e^{-\alpha t} |\zeta(0)| + \frac{k_2}{\alpha} \beta_1(|x(0)|, 0) e^{-\alpha t/2} (1 - e^{-\alpha t/2}) \\
&\quad + \frac{k_2}{\alpha} \beta_1(|x(0)|, t/2) (1 - e^{-\alpha t/2}) \\
&\leq k_1 e^{-\alpha t} |\zeta(0)| + \frac{k_2}{\alpha} \beta_1(|x(0)|, 0) e^{-\alpha t/2} \\
&\quad + \frac{k_2}{\alpha} \beta_1(|x(0)|, t/2) \\
&\triangleq \beta_2 \left( \begin{array}{c} |\zeta(0)| \\ |x(0)| \end{array}, t \right),
\end{aligned}$$

where  $\beta_2$  is a class  $KL_\infty$  function. Combining (5.1.13) with (5.1.15) proves that  $\zeta = 0, x = 0$  is GAS:

$$\left| \begin{array}{c} \zeta(0) \\ x(t) \end{array} \right| \leq \beta_3 \left( \left| \begin{array}{c} \zeta(0) \\ x(0) \end{array} \right|, t \right), \quad \beta_3 \in KL_\infty. \quad (5.1.15)$$

□

Comparing Lemma 4.1.2 and 5.1.4 we see that, instead of assuming global stability of  $x = 0$  when  $y = 0$ , Lemma 5.1.4 assumes only global stabilizability of  $x = 0$  through  $y$ .

The corresponding assumptions on the input subsystem, however, reveal the price paid

for this generalization: The minimum phase assumption of Lemma 5.1.4 is stronger than เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่นอนุญาตให้นำไปใช้ประโยชน์ด้านการค้า ไม่ว่ากรณีใดๆ ทั้งสิ้น อีกทั้งห้ามมิให้ดัดแปลงเนื้อหา และต้องอ้างอิงถึงเจ้าของเอกสารทุกครั้งที่มีการนำไปใช้

the FPR assumption of Lemma 4.1.2, which allows some zeros to be on the imaginary axis, that is, to be weak minimum phase.

Let us now examine the cascade system

$$\dot{x} = -x(1 + e^{\xi_1}) + x^3 \xi_1^2 \quad (5.1.16a)$$

$$\dot{\xi}_1 = \xi_1 \xi_2^2 \quad (5.1.16b)$$

$$\dot{\xi}_2 = \xi_2 u. \quad (5.1.16c)$$

As we have already shown in Example 4.1.7, (5.1.16a)-(5.1.16b) is stabilizable through  $y = \xi_2^2$ , while (5.1.16c) with this output is FSP. However, if we try to stabilize the cascade (5.1.16), we run into difficulties because the relative degree of (5.1.16c) is not defined at  $\xi_2 = 0$ .

This example shows that we need to assume that the input subsystem

$$\dot{\xi} = m(\xi) + \beta(\xi)u, \quad y = h(\xi), \quad (5.1.17)$$

has a globally defined constant relative degree. For a nonlinear analog of Lemma 5.1.4, we also assume that the zero dynamics subsystem of (5.1.17) is ISS.

**Lemma 5.1.6 (Nonlinear Block Backstepping)** *Consider the cascade system:*

$$\dot{x} = f(x) + g(x)y, \quad f(0) = 0, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R} \quad (5.1.18a)$$

$$\dot{\xi}_1 = m(x, \xi) + \beta(x, \xi)u, \quad y = h(\xi), \quad h(0) = 0, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R}. \quad (5.1.18b)$$

*Assume that (5.1.18b) has globally defined and constant relative degree one uniformly in  $x$ , and that its zero dynamics subsystem is ISS with respect to  $x$  and  $y$  as its inputs.*

*If (5.1.18a) satisfies Assumption 5.1.2 with  $y$  as its input, then there exists a feedback control which guarantees global boundedness and convergence of  $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$  to the largest*

*invariant set  $M_a$  contained in the set*

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$$E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \mid W(x) = 0, y = \alpha(x) \right\}.$$

One particular choice is

$$u = \left( \frac{\partial h}{\partial \xi}(\xi) \beta(x, \xi) \right)^{-1} \left\{ -c(y - \alpha(x)) - \frac{\partial h}{\partial \xi}(\xi) m(x, \xi) + \frac{\partial \alpha}{\partial x}(x) [f(x) + g(x)y] - \frac{\partial V}{\partial x}(x) g(x) \right\}, \quad c > 0. \quad (5.1.19)$$

Moreover, if  $V(x)$  and  $W(x)$  are positive definite, then the equilibrium  $x = 0, \xi = 0$  is GAS.

**Proof** Since the relative degree of the subsystem (5.1.18b) is globally defined and equal to one uniformly in  $x$ , there exists a global change of coordinates of the form (3.1.10), in particular  $(y, \zeta) = (y, \phi(x, \xi))$  with  $\frac{\partial \phi}{\partial \xi} \beta \equiv 0$ , which transforms (5.1.18b) into

$$\begin{aligned} \dot{y} &= \frac{\partial h}{\partial \xi}(\xi) m(x, \xi) + \frac{\partial h}{\partial \xi}(\xi) \beta(x, \xi) u \\ &\triangleq f_1(x, y, \zeta) + g_1(x, y, \zeta) u \end{aligned} \quad (5.1.20a)$$

$$\begin{aligned} \dot{\zeta} &= \frac{\partial \phi}{\partial x}(x, \xi) [f(x) + g(x)y] + \frac{\partial \phi}{\partial \xi}(x, \xi) m(x, \xi) \\ &\triangleq \Phi(\zeta, x, y). \end{aligned} \quad (5.1.20b)$$

We now consider the cascade system consisting of (5.1.18a) and (5.1.20a). If we linearize (5.1.20a) with the feedback given by (3.1.9),

$$u = \left( \frac{\partial h}{\partial \xi} \beta \right)^{-1} \left( v - \frac{\partial h}{\partial \xi} m \right), \quad (5.1.21)$$

we obtain  $\dot{y} = v$ . Then we can apply Lemma B.1.2, with  $v$  as the new control input, to guarantee global boundedness of  $x$  and  $y$  and regulation of  $W(x(t))$  and  $y(t) - \alpha(x(t))$ .

From (5.1.20b) and the ISS assumption on the zero dynamics,  $\zeta$  is also bounded, and thus  $\xi$  and  $u$  are bounded.

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Since all solution of (5.1.18) are bounded, we can apply LaSalle's theorem (Theorem A.1.2) with  $\Omega = \mathbb{R}^{n+q}$  to conclude convergence to the set  $M_a$ . combining (5.1.21) with (B.1.23), we see that a particular choice of control is given by (5.1.19).

From Lemma(A.1.8) we also know that if  $V(x)$  and  $W(x)$  are positive definite, then the equilibrium  $x = 0, y = 0$  of (5.1.18a) and (5.1.20a), which is completely decoupled from (5.1.20b), is GAS. The fact that in this case the equilibrium  $x = 0, \xi = 0$  of the cascade system (5.1.18) is also GAS follows from Lemma C.4 by noting that the state  $(x, y)$  of the GAS system (5.1.18a) and (5.1.20a) is the input of the ISS system (5.1.20b).  $\square$

Lemma 5.1.6 relaxes the global stability assumption of Lemma 4.1.6 to global stabilizability of  $x = 0$  through  $y$ . As in the case of Lemmas 4.1.2 and 5.1.4, however, the price paid for this generalization is the strengthening of the FP assumption of Lemma 4.1.6 to the ISS assumption of Lemma 5.1.6.

# Chapter 6

## Adaptive Backstepping

### 6.1 Adaptive Integrator Backstepping

The adaptive design in example of Chapter 3 was simple because of the matching: The parametric uncertainty was in the span of the control. We now move to the more general case of *extended matching*, where the parametric uncertainty enters the system one integrator before the control does:

$$\dot{x}_1 = x_2 + \theta\varphi(x_1) \quad (6.1.1a)$$

$$\dot{x}_2 = u. \quad (6.1.1b)$$

We use this example to introduce *adaptive backstepping*.

If  $\theta$  were known, we would apply Lemma B.1.2 to design the stabilizing function for  $x_2$  as doing with the static part

$$\alpha_1(x_1, \theta) = -c_1 x_1 - \theta\varphi(x_1), \quad (6.1.2)$$

with the Lyapunov function

$$V_c(x, \theta) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - \alpha_1(x_1, \theta))^2 \quad (6.1.3)$$

whose derivative is rendered negative definite

$$\dot{V}_c(x, \theta) = -c_1 x_1^2 - c_2 (x_2 - \alpha_1(x_1, \theta))^2 \quad (6.1.4)$$

by the control

$$u = -c_2 (x_2 - \alpha_1(x_1, \theta)) - x_1 + \frac{\partial \alpha_1}{\partial x_1} (x_2 + \theta \varphi). \quad (6.1.5)$$

Since  $\theta$  is unknown and appears one equation before the control does, we cannot apply Lemma B.1.2 because the dependence of  $\alpha_1(x_1) = -c_1 x_1 - \theta \varphi(x_1)$  on the unknown parameter makes it impossible to continue the procedure. However, we can utilize the idea of integrator backstepping.

**Step 1.** If  $x_2$  were the control, an adaptive controller for (6.1.1a) would be given by:

$$\alpha_1(x_1, \vartheta_1) = -c_1 z_1 - \vartheta_1 \varphi(x_1) \quad (6.1.6)$$

$$\dot{\vartheta}_1 = \gamma z_1 \varphi(x_1) \quad (6.1.7)$$

where  $z_1 = x_1$ .

In the above equations we have replaced the parameter estimate  $\hat{\theta}$  with the estimate  $\vartheta_1$ , which denotes the estimate generated in this design step. As we will see, there will be another estimate generated in the next step. With (6.1.6) and the new error variable  $z_2 = x_2 - \alpha_1$ , the  $\dot{z}_1$ -equation becomes

$$\dot{z}_1 = -c_1 z_1 + z_2 + (\theta - \vartheta_1) \varphi. \quad (6.1.8)$$

The derivative of the Lyapunov function

$$V_1(x_1, \vartheta_1) = \frac{1}{2} z_1^2 + \frac{1}{2\gamma} (\theta - \vartheta_1)^2 \quad (6.1.9)$$

along the solutions of (6.1.8) is

$$\begin{aligned} \dot{V}_1 &= z_1 \dot{z}_1 - \frac{1}{\gamma} (\theta - \vartheta_1) \dot{\vartheta}_1 \\ &= z_1 z_2 - c_1 z_1^2 + (\theta - \vartheta_1) \left[ \varphi z_1 - \frac{1}{\gamma} \dot{\vartheta}_1 \right] \\ &= z_1 z_2 - c_1 z_1^2 \end{aligned} \quad (6.1.10)$$

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**Step 2.** The derivative of  $z_2$  is now expressed as

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 \\ &= u - \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1 - \frac{\partial \alpha_1}{\partial \vartheta_1} \dot{\vartheta}_1. \end{aligned}$$

Substituting (6.1.1a) and the update law (6.1.7) results in

$$\begin{aligned} \dot{z}_2 &= u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \theta \varphi) - \frac{\partial \alpha_1}{\partial \vartheta} \gamma \varphi z_1 \\ &= u - \frac{\partial \alpha_1}{\partial x_1} x_2 - \frac{\partial \alpha_1}{\partial \vartheta_1} \gamma \varphi z_1 - \theta \frac{\partial \alpha_1}{\partial x_1} \varphi. \end{aligned} \quad (6.1.11)$$

At this point we need to select a Lyapunov function and design  $u$  to render its derivative nonpositive. Our first attempt is the augmented Lyapunov function

$$V_2(z_1, z_2, \vartheta_1) = V_1(z_1, \vartheta_1) + \frac{1}{2} z_2^2,$$

whose derivative, using (6.1.10) and (6.1.11), is

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + z_2 \dot{z}_2 \\ &= -c_1 z_1^2 + z_2 \left[ z_1 + u - \frac{\partial \alpha_1}{\partial x_1} x_2 - \frac{\partial \alpha_1}{\partial \vartheta_1} \gamma \varphi z_1 - \theta \frac{\partial \alpha_1}{\partial x_1} \varphi \right]. \end{aligned}$$

The control  $u$  should now be able to cancel the indefinite terms in  $\dot{V}_2$ . To deal with the terms containing the unknown parameter  $\theta$ , we will try to employ the existing estimate  $\vartheta_1$ :

$$u = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \vartheta_1} \gamma \varphi z_1 + \vartheta_1 \frac{\partial \alpha_1}{\partial x_1} \varphi.$$

From the resulting derivative

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 - (\theta - \vartheta_1) \frac{\partial \alpha_1}{\partial x_1} \varphi z_2,$$

we see that we have no design freedom left to cancel the  $(\theta - \vartheta_1)$ -term. To overcome this difficulty, we replace  $\vartheta_1$  in the expression for  $u$  with a *new* estimate  $\vartheta_2$ :

$$u = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \vartheta_1} \gamma \varphi z_1 + \vartheta_2 \frac{\partial \alpha_1}{\partial x_1} \varphi. \quad (6.1.12)$$

With the choice (6.1.12), the  $z_2$ -equation becomes

$$\dot{z}_2 = -c_2 z_2 - z_1 - (\theta - \vartheta_2) \frac{\partial \alpha_1}{\partial x_1} \varphi. \quad (6.1.13)$$

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The presence of the new parameter estimate  $\vartheta_2$  suggests the following augmentation of the Lyapunov function:

$$\begin{aligned} V_2(z_1, z_2, \vartheta_1, \vartheta_2) &= V_1 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}(\theta - \vartheta_2)^2 \\ &= \frac{1}{2}(z_1^2 + z_2^2) + \frac{1}{2\gamma}[(\theta - \vartheta_1)^2 + (\theta - \vartheta_2)^2]. \end{aligned} \quad (6.1.14)$$

The derivative of  $V_2$  is

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 - \frac{1}{\gamma}(\theta - \vartheta_2)\dot{\vartheta}_2 \\ &= z_1\dot{z}_2 - c_1z_1^2 + z_2 \left[ -c_2z_2 - z_1 - (\theta - \vartheta_2)\frac{\partial\alpha_1}{\partial x_1}\varphi \right] - \frac{1}{\gamma}(\theta - \vartheta_2)\dot{\vartheta}_2 \\ &= -c_1z_1^2 - c_2z_2^2 - (\theta - \vartheta_2) \left( \frac{\partial\alpha_1}{\partial x_1}\varphi z_2 + \frac{1}{\gamma}\dot{\vartheta}_2 \right). \end{aligned} \quad (6.1.15)$$

Now the  $(\theta - \vartheta_2)$ -term can be eliminated with the update law

$$\dot{\vartheta}_2 = -\gamma \frac{\partial\alpha_1}{\partial x_1}\varphi z_2, \quad (6.1.16)$$

which yields

$$\dot{V}_2 = -c_1z_1^2 - c_2z_2^2. \quad (6.1.17)$$

The equations (6.1.13) and (6.1.16) along with (6.1.8) and (6.1.7) form the error system representation of the resulting closed-loop adaptive system:

$$\begin{aligned} \dot{z}_1 &= -c_1z_1 + z_2 + (\theta - \vartheta_1)\varphi \\ \dot{z}_2 &= -c_2z_2 - z_1 - (\theta - \vartheta_2)\frac{\partial\alpha_1}{\partial x_1}\varphi \\ \dot{\vartheta}_1 &= \gamma\varphi z_1 \\ \dot{\vartheta}_2 &= -\gamma \frac{\partial\alpha_1}{\partial x_1}\varphi z_2. \end{aligned} \quad (6.1.18)$$

The matrix form of this system,

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \varphi & 0 \\ 0 & -\frac{\partial\alpha_1}{\partial x_1}\varphi \end{bmatrix} \begin{bmatrix} \theta - \vartheta_1 \\ \theta - \vartheta_2 \end{bmatrix} \\ \frac{d}{dt} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} &= \gamma \begin{bmatrix} \varphi & 0 \\ 0 & -\frac{\partial\alpha_1}{\partial x_1}\varphi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \end{aligned} \quad (6.1.19)$$

makes its properties more visible:

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- The constant system matrix has negative terms along its diagonal, while its off-diagonal terms are skew-symmetric, and
- the matrix that multiplies the parameter errors in the  $\dot{z}$ -equation is used in the update laws for the parameter estimates.

The stability properties of (6.1.19) follow from (6.1.14) and (6.1.17): The LaSalle-Yoshizawa theorem (Theorem A.1.1) establishes that  $z_1, z_2, \vartheta_1, \vartheta_2$  are bounded, and  $z \rightarrow 0$  as  $t \rightarrow \infty$ . Since  $z_1 = x_1$ ,  $x_1$  is also bounded and converges to zero. The boundedness of  $x_2$  then follows from the boundedness of  $\alpha_1$  (defined in (6.1.6)) and the fact that  $x_2 = z_2 + \alpha_1$ . Using (6.1.12) we conclude that the control  $u$  is also bounded. Finally, we note that the regulation of  $z$  and  $x_1$  does not imply the regulation of  $x_2$ : From  $z_2 = x_2 - \alpha_1$  and (6.1.6) we see that  $x_2 + \vartheta_1 \varphi(0)$  will converge to zero. Thus,  $x_2$  is not guaranteed to converge to zero unless  $\varphi(0) = 0$ . However,  $x_2$  will converge to a constant value:

$$\lim_{t \rightarrow \infty} x_2 = -\theta \varphi(0) \triangleq x_2^e. \quad (6.1.20)$$

This can be seen from (6.1.1a): Since  $x_1$  and  $\dot{x}_1$  converge to zero, so does  $x_2 + \theta \varphi(0)$ .

With the above example we have illustrated the idea of adaptive backstepping.

To formulate it as a design tool analogous to an integrator backstepping in Lemma B.1.2, we start with the assumption that an adaptive controller is known for an initial system.

**Assumption 6.1.1** Consider the system

$$\dot{x} = f(x) + F(x)\theta + g(x)u, \quad (6.1.21)$$

where  $x \in \mathbb{R}^n$  is the state,  $\theta \in \mathbb{R}^q$  is a vector of unknown constant parameters, and  $u \in \mathbb{R}$  is the control input. There exists an adaptive controller

$$\begin{aligned} u &= \alpha(x, \vartheta) \\ \dot{\vartheta} &= T(x, \vartheta), \end{aligned} \quad (6.1.22)$$

with parameter estimate  $\vartheta \in \mathbb{R}^q$ , and a smooth function  $V(x, \vartheta) : \mathbb{R}^{n+q} \rightarrow \mathbb{R}$  which is positive definite and radially unbounded in the variables  $(x, \vartheta - \theta)$  such that for all

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$(x, \vartheta) \in \mathbb{R}^{n+q}$ :

$$\begin{aligned} \frac{\partial V}{\partial x}(x, \vartheta)[f(x) + F(x)\theta + g(x)\alpha(x, \vartheta)] + \frac{\partial V}{\partial \vartheta}(x, \vartheta)T(x, \vartheta) \\ \leq -W(x, \vartheta) \leq 0, \end{aligned} \quad (6.1.23)$$

where  $W : \mathbb{R}^{n+q} \rightarrow \mathbb{R}$  is positive semidefinite.  $\square$

Under this assumption, the control (6.1.22), applied to the system (6.1.21), guarantees global boundedness of  $x(t)$ ,  $\vartheta(t)$  and, by the LaSalle-Yoshizawa theorem (Theorem A.1.1), regulation of  $W(x(t), \vartheta(t))$ . Adaptive backstepping allows us to achieve the same properties for the augmented system.

**Lemma 6.1.2 (Adaptive Backstepping)** *Let the system (6.1.21) be augmented by an integrator,*

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)\xi \\ \dot{\xi} &= u \end{aligned} \quad (6.1.24)$$

where  $\xi \in \mathbb{R}$ . Consider for this system the dynamic feedback controller

$$\begin{aligned} u &= -c(\xi - \alpha(x, \vartheta)) + \frac{\partial \alpha}{\partial x}(x, \vartheta)[f(x) + F(x)\vartheta + g(x)\xi] + \frac{\partial \alpha}{\partial \vartheta}T(x, \vartheta) \\ &\quad - \frac{\partial V}{\partial x}(x, \vartheta)g(x), \quad c > 0 \end{aligned} \quad (6.1.25)$$

$$\dot{\vartheta} = T(x, \vartheta) \quad (6.1.26)$$

$$\dot{\vartheta} = -\Gamma \left[ \frac{\partial \alpha}{\partial x}(x, \vartheta)F(x) \right]^T (\xi - \alpha(x, \vartheta)), \quad (6.1.27)$$

where  $\bar{\vartheta}$  is a new estimate of  $\theta$ ,  $\Gamma = \Gamma^T > 0$  is the adaptation gain matrix. Under Assumption 6.1.1, this adaptive controller guarantees global boundedness of  $x(t)$ ,  $\xi(t)$ ,  $\vartheta(t)$ ,  $\bar{\vartheta}(t)$  and regulation of  $W(x(t), \vartheta(t))$  and  $\xi(t) - \alpha(x(t), \vartheta(t))$ . These properties can be established with the Lyapunov function

$$V_a(x, \xi, \vartheta, \bar{\vartheta}) = V(x, \vartheta) + \frac{1}{2}[\xi - \alpha(x, \vartheta)]^2 + \frac{1}{2}(\theta - \bar{\vartheta})^T \Gamma^{-1}(\theta - \bar{\vartheta}). \quad (6.1.28)$$

**Proof** With the error variable  $z = \xi - \alpha(x, \vartheta)$ , (6.1.24) is rewritten as

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$$\dot{x} = f(x) + F(x)\theta + g(x)[\alpha(x, \vartheta) + z] \quad (6.1.29a)$$

$$\dot{z} = u - \frac{\partial \alpha}{\partial x}[f(x) + F(x)\theta + g(x)(\alpha(x, \vartheta) + z)] - \frac{\partial \alpha}{\partial \vartheta}T(x, \vartheta). \quad (6.1.29b)$$

Note that in (6.1.29b) the derivative of  $\vartheta$  was replaced by the update law (6.1.26). Introducing a new parameter estimate  $\bar{\vartheta}$ , we augment the Lyapunov function:

$$V_a(x, \xi, \vartheta, \bar{\vartheta}) = V(x, \vartheta) + \frac{1}{2}z^2 + \frac{1}{2}(\theta - \bar{\vartheta})^T \Gamma^{-1}(\theta - \bar{\vartheta}). \quad (6.1.30)$$

Using (6.1.23), it is easy to show that the derivative of (6.1.30) satisfies

$$\begin{aligned} \dot{V}_a &= \frac{\partial V}{\partial x}(f + F\theta + g\alpha + gz) + \frac{\partial V}{\partial \vartheta}T \\ &\quad + z \left[ u - \frac{\partial \alpha}{\partial x}(f + F\theta + g(\alpha + z)) - \frac{\partial \alpha}{\partial \vartheta}T \right] - \dot{\vartheta}^T \Gamma^{-1}(\theta - \bar{\vartheta}) \\ &= \frac{\partial V}{\partial x}(f + F\theta + g\alpha) + \frac{\partial V}{\partial \vartheta}T \\ &\quad + z \left[ u - \frac{\partial \alpha}{\partial x}(f + F\theta + g(\alpha + z)) - \frac{\partial \alpha}{\partial \vartheta}T + \frac{\partial V}{\partial x}g \right] - \dot{\vartheta}^T \Gamma^{-1}(\theta - \bar{\vartheta}) \\ &\leq -W(x, \vartheta) + z \left[ u - \frac{\partial \alpha}{\partial x}(f + F\bar{\vartheta} + g(\alpha + z)) - \frac{\partial \alpha}{\partial \vartheta}T + \frac{\partial V}{\partial x}g \right] \\ &\quad - \left[ \frac{\partial \alpha}{\partial x}Fz + \dot{\vartheta}^T \Gamma^{-1} \right] (\theta - \bar{\vartheta}). \end{aligned} \quad (6.1.31)$$

The  $(\theta - \bar{\vartheta})$ -term is now eliminated with the update law

$$\dot{\vartheta} = -\Gamma \left( \frac{\partial \alpha}{\partial x}F \right)^T z, \quad (6.1.32)$$

and the control (6.1.25) is chosen to make the bracketed term multiplying  $z$  in (6.1.31) equal to  $-cz$ :

$$u = -cz + \frac{\partial \alpha}{\partial x}(f + F\bar{\vartheta} + g(\alpha + z)) + \frac{\partial \alpha}{\partial \vartheta}T - \frac{\partial V}{\partial x}g. \quad (6.1.33)$$

This results in the desired nonpositivity of  $\dot{V}_a$ :

$$\dot{V}_a \leq -W(x, \vartheta) - cz^2 \leq 0. \quad (6.1.34)$$

From (6.1.28) and (6.1.34) we conclude that  $V(x, \vartheta)$ ,  $\bar{\vartheta}$  and  $z$  are bounded. By Assumption 6.1.1, this means that  $x(t)$  and  $\vartheta(t)$  are bounded. Hence,  $\xi = z + \alpha(x, \vartheta)$  and  $u$  are bounded. By LaSalle-Yoshizawa theorem (Theorem A.1.1), the boundedness of all the signals combined with (6.1.34) proves the regulation of  $W(x(t), \vartheta(t))$  and  $z(t)$ .  $\square$

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## 6.2 Adaptive Block Backstepping

We now extend the Adaptive Backstepping lemma (Lemma 6.1.2) by augmenting the initial system with a relative-degree-one nonlinear system whose zero dynamics subsystem is ISS just like we did in Lemmas B.1.2 and 5.1.6. The adaptive counterpart of Assumption B.1.1 was Assumption 6.1.1. We now formulate the adaptive counterpart of Assumption 5.1.2, with analogous changes in the properties of  $V(x, \vartheta)$  from Assumption 6.1.1.

**Assumption 6.2.1** *Suppose Assumption 6.1.1 is valid, but  $V(x, \vartheta)$  is only positive semidefinite, and the closed-loop system (6.1.21) with the adaptive controller (6.1.22) has the property that  $x(t)$  and  $\vartheta(t)$  are bounded if  $V(x(t), \vartheta(t))$  is bounded.  $\square$*

Under this assumption, the control (6.1.22), applied to the system (6.1.21), guarantees global boundedness of  $x(t), \vartheta(t)$  and, by Lemma A.3.5, regulation of  $W(x(t), \vartheta(t))$ .

**Lemma 6.2.2 (Adaptive Block Backstepping)** *Let the system (6.1.21) be augmented by a nonlinear system which is linear in the unknown parameter vector  $\theta$*

$$\dot{x} = f(x) + F(x)\theta + g(x)y \quad (6.2.1a)$$

$$\dot{\xi} = m(x, \xi) + M(x, \xi)\theta + \beta(x, \xi)u, \quad y = h(\xi), \quad (6.2.1b)$$

where  $\xi \in \mathbb{R}^q$ , and suppose that (6.2.1b) has relative degree one uniformly in  $x$  and that its zero dynamics subsystem is ISS with respect to  $y$  and  $x$ . Under Assumption 6.2.1, the feedback control

$$u = \left[ \frac{\partial h}{\partial \xi}(\xi)\beta(x, \xi) \right]^{-1} \left\{ -c(y - \alpha(x, \vartheta)) - \frac{\partial h}{\partial \xi}(\xi) [m(x, \xi) + M(x, \xi)\bar{\vartheta}] + \frac{\partial \alpha}{\partial x}(x, \vartheta) [f(x) + F(x)\bar{\vartheta} + g(x)y] + \frac{\partial \alpha}{\partial \vartheta} T(x, \vartheta) - \frac{\partial V}{\partial x}(x, \vartheta)g(x) \right\}, \quad (6.2.2)$$

with  $c > 0$  and  $\bar{\vartheta}$  a new estimate of  $\theta$ , along with the update laws

$$\dot{\vartheta} = T(x, \vartheta) \quad (6.2.3)$$

$$\dot{\bar{\vartheta}} = \Gamma \left[ \frac{\partial h}{\partial \xi}(\xi)M(x, \xi) - \frac{\partial \alpha}{\partial x}(x, \vartheta)F(x) \right]^T (y - \alpha(x, \vartheta)), \quad (6.2.4)$$

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with the adaptation gain matrix  $\Gamma = \Gamma^T > 0$ , guarantees global boundedness of  $x(t), \xi(t), \vartheta(t), \bar{\vartheta}(t)$  and regulation of  $W(x(t), \vartheta(t))$  and  $\xi(t) - \alpha(x(t), \vartheta(t))$ .

**Proof** As in Lemma 5.1.6, we employ the change of coordinates  $(y, \zeta) = (h(\xi), \phi(x, \xi))$ , with  $\frac{\partial \phi}{\partial \xi} \beta \equiv 0$ , to transform (6.2.1b) into the normal form

$$\begin{aligned} \dot{y} &= \frac{\partial h}{\partial \xi}(\xi)[m(x, \xi) + M(x, \xi)\theta + \beta(x, \xi)u] \\ &= v + \frac{\partial h}{\partial \xi}(\xi)M(x, \xi)(\theta - \bar{\vartheta}) \end{aligned} \quad (6.2.5a)$$

$$\begin{aligned} \dot{\zeta} &= \frac{\partial \phi}{\partial x}(x, \xi)[f(x) + F(x)\theta + g(x)y] + \frac{\partial \phi}{\partial \xi}(x, \xi)[m(x, \xi) + M(x, \xi)\theta] \\ &\triangleq \Phi_0(x, y, \zeta) + \Phi(x, y, \zeta)\theta. \end{aligned} \quad (6.2.5b)$$

Introducing a new parameter estimate  $\bar{\vartheta}$ , we use the feedback transformation

$$u = \left( \frac{\partial h}{\partial \xi} \beta \right)^{-1} \left\{ v - \frac{\partial h}{\partial \xi} [m + M\bar{\vartheta}] \right\} \quad (6.2.6)$$

to rewrite (6.2.1a) and (6.2.5a) as

$$\dot{x} = f(x) + F(x)\theta + g(x)y \quad (6.2.7a)$$

$$\dot{y} = v + \frac{\partial h}{\partial \xi}(\xi)M(x, \xi)(\theta - \bar{\vartheta}). \quad (6.2.7b)$$

We now apply Assumption 6.1.1 to (6.2.7). The only difference between (6.2.7) and (6.1.24) is the presence of the additional parameter error term  $\frac{\partial h}{\partial \xi} M(\theta - \bar{\vartheta})$  in (6.2.7b).

This term can be eliminated in  $\dot{V}_a$  by adding the term  $-\Gamma \left( \frac{\partial h}{\partial \xi} M \right)^T (y - \alpha)$  to the update law (6.1.27):

If  $\theta$  were known we would pick  $u = \frac{1}{b_0} [v - M(x, \xi)\theta]$  so (6.2.5a) can be written in the form

$$\dot{x} = f(x) + F(x)\theta + g(x)y$$

$$\dot{y} = \dot{x}_3 = v$$

in which we could apply integrator backstepping. But since  $\theta$  is unknown, therefore we introduce a new parameter estimate  $\bar{\vartheta}$  with the new clf

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of  $x, \vartheta, \bar{\vartheta}, z$  and the regulation of  $W(x, \vartheta)$  and  $z$ . Hence,  $y = z + \alpha(x, \vartheta)$  is bounded. Then, from (6.2.5b) and the ISS property of the zero dynamics,  $\zeta$  is also bounded, and thus  $\xi$  and  $u$  are bounded  $\square$



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# Chapter 7

## Tuning Functions Design

### 7.1 Introduction

The adaptive backstepping solution to the problem of nonlinear stabilization and tracking in the presence of unknown parameters is a starting point for more elaborate adaptive designs which lead to new properties of the designed controller and the resulting feedback system. One of the improvement to be achieved with the tuning functions design in this chapter is the reduction of the dynamic order of the adaptive controller to its minimum: The number of parameter estimates is equal to the number of unknown parameters. This minimum-order design is advantageous not only for implementation, but also because it guarantees the strongest achievable stability and convergence properties.

In the tuning functions procedure the parameter update law is designed recursively. At each consecutive step we design a tuning function as a potential update law. In contrast to adaptive backstepping in the previous chapter. These intermediate update laws are not implemented. Instead, the controller uses them to compensate for the effect of parameter estimation transients. Only the final tuning function is used as the parameter update law.

## 7.2 Adaptive Control Lyapunov Functions

The basic idea of the Lyapunov approach to adaptive control is to design a control law and a parameter update law to guarantee that the derivative of a suitable Lyapunov function is nonpositive. We are therefore sent to search for a tripple: Lyapunov function, control law, and update law. For a class of nonlinear systems called parametric-strict-feedback systems we will be able to make this search systematic.

To begin with, let us investigate the possibility of adaptive design for the system

$$\dot{x} = f(x) + F(x)\theta + g(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R} \quad (7.2.1)$$

where  $\theta \in \mathbb{R}^p$  is a vector of unknown constant parameters, and  $f(x)$ ,  $F(x)$  and  $g(x)$  are smooth. For simplicity let  $f(0) = 0$ ,  $F(0) = 0$ , so that  $x = 0$  is an equilibrium of the uncontrolled plant.

### 7.2.1 Departure from certainty equivalence

Much of the traditional adaptive control employs some form of “certainty equivalence” thinking. Following this path one first performs a design for the case when the exact value of  $\theta$  is known. Suppose that this nontrivial task is completed and that its result is a feedback control  $u = \alpha_c(x, \theta)$  which stabilizes the equilibrium  $x = 0$  with respect to a known Lyapunov function  $V_c(x, \theta)$ . The subscript ‘c’ stands for “certainty equivalence”. We know that  $V_c(x, \theta)$  is positive definite and radially unbounded in  $x$  for all  $\theta$ , and that there exists a function  $W(x, \theta)$ , which is also positive definite in  $x$  for all  $\theta$ , such that

$$\frac{\partial V_c}{\partial x} [f(x) + F(x)\theta + g(x)\alpha_c(x, \theta)] \leq -W(x, \theta) \quad (7.2.2)$$

How can we exploit the knowledge of  $\alpha_c(x, \theta)$  and  $V_c(x, \theta)$  for adaptive design when  $\theta$  is not known? The certainty equivalence idea is to replace  $\theta$  by an estimate  $\hat{\theta}(t)$  obtained from a parameter update law

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where the adaptation gain matrix  $\Gamma$  is positive definite. We want to select  $u$  and  $\tau$  to guarantee that the derivative of a Lyapunov function is nonpositive. For the system (7.2.1),(7.2.3), a Lyapunov function candidate is

$$V(x, \hat{\theta}) = V_c(x, \hat{\theta}) + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}, \quad (7.2.4)$$

where the “certainty equivalence” from of  $V_c$  is augmented by a term quadratic in the parameter estimation error

$$\tilde{\theta} = \theta - \hat{\theta}. \quad (7.2.5)$$

Upon the substitution of  $F(x)\theta = F(x)\hat{\theta} + F(x)\tilde{\theta}$ , the derivative of  $V(x, \hat{\theta})$  along the solutions of (7.2.1), (7.2.3) is

$$\dot{V} = \frac{\partial V_c}{\partial x} (f(x) + F(x)\hat{\theta} + g(x)u) + \frac{\partial V_c}{\partial x} \Gamma \tau + \tilde{\theta}^T \left( \frac{\partial V_c}{\partial x} F(x) \right)^T - \tilde{\theta}^T \tau \quad (7.2.6)$$

To eliminate the indefinite dependence of  $\dot{V}$  on the unknown parameter error  $\tilde{\theta}$ , we select  $\tau$  to cancel the last two terms in (7.2.6):

$$\tau(x, \hat{\theta}) = \left( \frac{\partial V_c}{\partial x} F(x) \right)^T. \quad (7.2.7)$$

With this choice of  $\tau$ , the expression (7.2.6) is reduced to

$$\dot{V} = \frac{\partial V_c}{\partial x} (f(x) + F(x)\hat{\theta} + g(x)u) + \frac{\partial V_c}{\partial \hat{\theta}} \Gamma \left( \frac{\partial V_c}{\partial x} F(x) \right)^T. \quad (7.2.8)$$

Our next task is to select a control law  $u = \alpha(x, \hat{\theta})$  to make  $\dot{V}$  nonpositive. The “certainty equivalence” control  $u = \alpha_c(x, \hat{\theta})$  fails to achieve this because then (7.2.2) and (7.2.8) yield

$$\dot{V} \leq -W(x, \hat{\theta}) + \frac{\partial V_c}{\partial \hat{\theta}} \Gamma \left( \frac{\partial V_c}{\partial x} F(x) \right)^T. \quad (7.2.9)$$

Clearly,  $\dot{V}$  is not nonpositive because a sign-indefinite term is added to  $-W(x, \hat{\theta})$ . In search of a better control law  $\alpha(x, \hat{\theta})$ , we augment  $\alpha_c(x, \hat{\theta})$  by  $\alpha_T(x, \hat{\theta})$ ,

$$\alpha(x, \hat{\theta}) = \alpha_c(x, \hat{\theta}) + \alpha_T(x, \hat{\theta}). \quad (7.2.10)$$

The substitution of (7.2.10) into (7.2.8) shows that the desired nonpositivity  $\dot{V} \leq -W(x, \hat{\theta})$  will be achieved if  $\alpha_T$  can be found to satisfy

$$\frac{\partial V_c}{\partial x} g(x) \alpha_T(x, \hat{\theta}) + \frac{\partial V_c}{\partial \hat{\theta}} \Gamma \left( \frac{\partial V_c}{\partial x} F(x) \right)^T = 0. \quad (7.2.11)$$

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This condition for  $\alpha_T$  demonstrates the difficulty of adaptive design for a general nonlinear system (7.2.1). It is easy to see that  $\alpha_T$  satisfying (7.2.11) is unlikely to exist. The scalar quantity  $\frac{\partial V_c}{\partial x}g(x)$  may be zero at a set of points. Still, the condition (7.2.11) is of interest because of an important special case, which will be the starting point of our recursive design. The special case is the “extended matching” studied in the previous chapter. In this case, a smooth vector-valued function  $\varphi : \mathbb{R}^{n+p} \rightarrow \mathbb{R}^p$  is known such that  $\frac{\partial V_c}{\partial \hat{\theta}}$  can be factored as follows:

$$\frac{\partial V_c}{\partial \hat{\theta}} = \frac{\partial V_c}{\partial x}g(x)\varphi(x, \hat{\theta})^T. \quad (7.2.12)$$

Then, irrespective of the zeros of  $\frac{\partial V_c}{\partial x}g(x)$ , an  $\alpha_T$  which satisfies (7.2.11) is

$$\alpha_T(x, \hat{\theta}) = -\varphi(x, \hat{\theta})^T \Gamma \left( \frac{\partial V_c}{\partial x}F(x) \right)^T = -\varphi(x, \hat{\theta})^T \Gamma \tau(x, \hat{\theta}). \quad (7.2.13)$$

We observe that, in addition to its “certainty equivalence” part  $\alpha_c$ , the adaptive control law  $\alpha$  contains a part  $\alpha_T$  which is proportional to  $\tau$ , that is, to  $\dot{\hat{\theta}}$  (see (7.2.3), (7.2.10), and (7.2.13)). In this way, the adaptive control law takes into account the parameter estimation transients. When the parameter estimate is constant, the control law reduces to the “certainty equivalence” control. Let us examine an example of a system for which (7.2.12) is satisfied.

**Example 7.2.1** Consider the problem of designing an adaptive controller for the system

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^T \theta \\ \dot{x}_2 &= u, \end{aligned} \quad (7.2.14)$$

where  $\theta = [\theta_1, \theta_2]^T$  is an unknown constant parameter vector, and the vector-valued function  $\varphi(x_1) = [\varphi_1(x_1), \varphi_2(x_1)]^T$  is known and smooth. We dealt with this system in the extended-matching design. If the parameter  $\theta$  were known, backstepping would result in the  $\theta$ -dependent change of coordinates

$$z_1 = x_1$$

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and the control law

$$u = \alpha_c(x, \theta) = -z_1 - c_2 z_2 - \left( \frac{\partial \varphi^T}{\partial x_1} + c_1 \right) \left( x_2 + \varphi(x_1)^T \theta \right) \quad (7.2.16)$$

which  $c_1, c_2 > 0$ , which results in the closed-loop system

$$\dot{z} = Az, \quad A = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix}. \quad (7.2.17)$$

Due to the structure of  $A$ , an appropriate Lyapunov function is

$$v_c(x, \theta) = \frac{1}{2} z(x, \theta)^T z(x, \theta), \quad (7.2.18)$$

Observing from (7.2.1) and (7.2.14) that

$$f(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \quad F(x) = \begin{bmatrix} \varphi x_1^T \\ 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (7.2.19)$$

and evaluating

$$\frac{\partial V_c}{\partial x} = z^T \begin{bmatrix} 1 & 0 \\ \frac{\partial \varphi^T}{\partial x_1} \theta & 1 \end{bmatrix}, \quad (7.2.20)$$

with (7.2.19), (7.2.20), and (7.2.16), it is easy to show that

$$\frac{\partial V_c}{\partial x} [f(x) + F(x)\theta + g(x)\alpha_c(x, \theta)] = -c_1 z_1^2 - c_2 z_2^2. \quad (7.2.21)$$

Let us now evaluate the partial derivatives appearing in (7.2.11):

$$\frac{\partial V_c}{\partial \theta} = z^T e_2 \varphi^T = z_2 \varphi^T \quad (7.2.22)$$

$$\frac{\partial V_c}{\partial x} g = z^T e_2 = z_2 \quad (7.2.23)$$

where (7.2.23) is immediate from (7.2.20) and (7.2.19). A comparison of (7.2.22) and (7.2.23) reveals that  $\frac{\partial V_c}{\partial \theta} = \frac{\partial V_c}{\partial x} g \varphi^T$ , so that  $\alpha_c$  is given by (7.2.13):

$$\alpha_r(x, \hat{\theta}) = -\varphi^T \Gamma \left( \frac{\partial V_c}{\partial x} F(x) \right)^T = -\varphi^T \Gamma \varphi \left[ 1, \frac{\partial \varphi^T}{\partial x_1} \hat{\theta} + c_1 \right] z. \quad (7.2.24)$$

Taking for simplicity  $\Gamma = I$ , the resulting adaptive control law is

$$u = \alpha(x, \hat{\theta}) = -z_1 - c_2 z_2 - \left( c_1 + \frac{\partial \varphi^T}{\partial x_1} \theta \right) \left( x_2 + \varphi(x_1)^T \theta \right)$$

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(7.2.25)

and the corresponding parameter update law (7.2.7) is

$$\dot{\hat{\theta}} = \tau(x, \hat{\theta}) = \left( \frac{\partial V_c}{\partial x} F(x) \right)^T = \varphi \left[ 1, \frac{\partial \varphi^T}{\partial x_1} \hat{\theta} + c_1 \right] z. \quad (7.2.26)$$

Note that in (7.2.25) and (7.2.26) we use  $z(x, \hat{\theta})$  instead of  $z(x, \theta)$ . With the choice of  $\alpha$  and  $\tau$  given by (7.2.25) and (7.2.26), the derivative  $\dot{V}$  of the Lyapunov function  $V(x, \hat{\theta}) = \frac{1}{2} z(x, \hat{\theta})^T z(x, \hat{\theta}) + \frac{1}{2} \tilde{\theta}^T \tilde{\theta}$  is guaranteed to be nonpositive:  $\dot{V} = -c_1 z_1^2 - c_2 z_2^2$ . This assures that both  $x$  and  $\hat{\theta}$  are bounded. A standard argument using the LaSalle-Yoshizawa Theorem proves that also  $x(t) \rightarrow 0$ .

In the above example, the desired factorization (7.2.12) of  $\frac{\partial V_c}{\partial \hat{\theta}}$  is a consequence of a particular feature of the system (7.2.14). The unknown parameter appears in the first, while the control appears only in the second equation. It is not hard to see that the same factorization (7.2.12) would be a possible for a higher-order plant, provided that the unknown parameter is separated from the control input by at most one integrator. So the factorization (7.2.12) is not a fortuitous event, but a structural property. For systems with this “extended matching” property, the above simple adaptive design is feasible. However, most systems fail to possess the “extended matching” property.

A benchmark example is the third-order system

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^T \theta \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \quad (7.2.27)$$

which has the form of (7.2.14) augmented by an integrator. In this system,  $\theta$  and  $u$  are separated by two integrators and we are unable to find  $\alpha_T$  which satisfies (7.2.11). We will solve this problem with a recursive design which will circumvent the obstacle posed by the restrictive condition (7.2.11).

## 7.2.2 Certainty equivalence for a modified system

Condition (7.2.11) was dictated by our choice of the Lyapunov function  $V_c(x, \hat{\theta})$  as the “certainty equivalence” form of  $V_c(x, \theta)$ . The only good thing we know about  $V_c(x, \hat{\theta})$  is that it works when the factorization (7.2.12) is possible. Otherwise, we do not know how to remove the indefinite term preventing the nonpositivity of  $\dot{V}$  in (7.2.9). Having recognized that a cause of our difficulties is  $V_c(x, \theta)$ , we now embark on a search for Lyapunov functions more suitable for adaptive control. The key idea is to counteract the effect of  $\dot{\hat{\theta}}$  and thus prevent the parameter estimate transients from destroying the nonpositivity of the Lyapunov derivative

We say that the system

$$\dot{x} = f(x) + F(x)\theta + g(x)u \quad (7.2.28)$$

is **globally adaptively stabilizable** if there exist a function  $\alpha(x, \hat{\theta})$  smooth on  $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^p$  with  $\alpha(0, \hat{\theta}) \equiv 0$ , a smooth function  $\tau(x, \hat{\theta})$ , and a positive definite symmetric  $p \times p$  matrix  $\Gamma$ , such that the dynamic controller

$$u = \alpha(x, \hat{\theta}) \quad (7.2.29)$$

$$\dot{\hat{\theta}} = \Gamma\tau(x, \hat{\theta}) \quad (7.2.30)$$

guarantees that the solution  $(x(t), \hat{\theta}(t))$  is globally bounded, and  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ , for all  $\theta \in \mathbb{R}^p$ .

Our approach is to replace the problem of adaptive stabilization of the original system (7.2.28) by a problem of nonadaptive stabilization of a modified system.

**Definition 7.2.2** A smooth function  $V_a : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}_+$ , positive definite and radially unbounded in  $x$  for each  $\theta$ , is called an **adaptive control Lyapunov function (aclf)** for (7.2.28) if there exists a positive definite symmetric matrix  $\Gamma \in \mathbb{R}^{p \times p}$  such that for each  $\theta \in \mathbb{R}^p$ ,  $V_a(x, \theta)$  is a **clf** for the modified system

$$\dot{x} = f(x) + F(x) \left( \theta + \Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T \right) + g(x)u, \quad (7.2.31)$$

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that is,  $V_a$  satisfies

$$\inf_{u \in \mathbb{R}} \left\{ \frac{\partial V_a}{\partial x} \left[ f(x) + F(x) \left( \theta + \Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T \right) + g(x)u \right] \right\} < 0. \quad (7.2.32)$$

We now show how to design an adaptive controller (7.2.29) - (7.2.30) when an aclf is known.

**Theorem 7.2.3** *The following two statements are equivalent:*

1. *There exists a triple  $(\alpha, V_a, \Gamma)$  such that  $\alpha(x, \theta)$  globally asymptotically stabilizes (7.2.32) at  $x = 0$  for each  $\theta \in \mathbb{R}^p$  with respect to the Lyapunov function  $V_a(x, \theta)$ .*
2. *There exists an aclf  $V_a(x, \theta)$  for (7.2.28).*

Moreover, if an aclf  $V_a(x, \theta)$  exists, then (7.2.28) is globally adaptively stabilizable.

**Proof** (1  $\Rightarrow$  2) Obvious because 1 implies that there exists a continuous function  $W : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}_+$ , positive definite in  $x$  for each  $\theta$ , such that

$$\frac{\partial V_a}{\partial x} \left[ f(x) + F(x) \left( \theta + \Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T \right) + g(x)\alpha(x, \theta) \right] \leq -W(x, \theta). \quad (7.2.33)$$

Thus  $V_a(x, \theta)$  is a clf for (7.2.31) for each  $\theta \in \mathbb{R}^p$ , and therefore it is an aclf for (7.2.28).

(2  $\Rightarrow$  1) The proof of this part is based on Sontag's constructive proof [171] of Artstein's theorem [4]. We assume that  $V_a$  is an aclf for (7.2.28), that is, a clf for (7.2.31). Sontag's formula applied to (7.2.31) gives a control law smooth on  $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^p$ :

$$\alpha(x, \theta) = \begin{cases} \frac{\frac{\partial V_a}{\partial x} \tilde{f} + \sqrt{\left( \frac{\partial V_a}{\partial x} \tilde{f} \right)^2 + \left( \frac{\partial V_a}{\partial x} g \right)^4}}{\frac{\partial V_a}{\partial x} g}, & \frac{\partial V_a}{\partial x} g(x, \theta) \neq 0 \\ 0, & \frac{\partial V_a}{\partial x} g(x, \theta) = 0 \end{cases} \quad (7.2.34)$$

where

$$\tilde{f}(x, \theta) = f(x) + F(x) \left( \theta + \Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T \right). \quad (7.2.35)$$

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With the choice (7.2.34), inequality (7.2.33) is satisfied with the continuous function

$$W(x, \theta) = \sqrt{\left(\frac{\partial V_a}{\partial x} \bar{f}(x, \theta)\right)^2 + \left(\frac{\partial V_a}{\partial x} g(x, \theta)\right)^4}, \quad (7.2.36)$$

which is positive definite in  $x$  for each  $\theta$ , because (7.2.32) implies that  $\frac{\partial V_a}{\partial x} \bar{f}(x, \theta) < 0$  whenever  $\frac{\partial V_a}{\partial x} g(x, \theta) = 0$  and  $x \neq 0$ . We note that the control law  $\alpha(x, \theta)$  will be continuous at  $x = 0$  if and only if the aclf  $V_a$  satisfies the following property, called the *small control property* [171]: For each  $\theta \in \mathbb{R}^p$  and for any  $\varepsilon > 0$  there is a  $\delta > 0$  such that, if  $x \neq 0$  satisfies  $|x| \leq \delta$ , then there is some  $u$  with  $|u| \leq \varepsilon$  such that

$$\frac{\partial V_a}{\partial x} \left[ f(x) + F(x) \left( \theta + \Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T \right) + g(x)u \right] < 0 \quad (7.2.37)$$

Assuming the existence of an aclf we now show that (7.2.28) is globally adaptively stabilizable. Since (2  $\Rightarrow$  1), there exists a triple  $(\alpha, V_a, \Gamma)$  and a function  $W$  such that (7.2.33) is satisfied, that is,

$$\frac{\partial V_a}{\partial x} [f(x) + F(x)\theta + g(x)\alpha(x, \theta)] + \frac{\partial V_a}{\partial \theta} \Gamma \left( \frac{\partial V_a}{\partial x} F(x) \right)^T \leq -W(x, \theta) \quad (7.2.38)$$

Consider the Lyapunov function candidate

$$V(x, \hat{\theta}) = V_a(x, \hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})^T \Gamma^{-1}(\theta - \hat{\theta}) \quad (7.2.39)$$

With the help of (7.2.38), the derivative of  $V$  along the solutions of (7.2.28), (7.2.29), and (7.2.30) is

$$\begin{aligned} \dot{V} &= \frac{\partial V_a}{\partial x} [f + F\theta + g\alpha(x, \hat{\theta})] + \frac{\partial V_a}{\partial \theta} \Gamma \tau(x, \hat{\theta}) - \tilde{\theta}^T \tau(x, \hat{\theta}) \\ &= \frac{\partial V_a}{\partial x} [f + F\theta + g\alpha(x, \hat{\theta})] + \frac{\partial V_a}{\partial \theta} \Gamma \tau(x, \hat{\theta}) + \frac{\partial V_a}{\partial x} F \tilde{\theta} - \tilde{\theta}^T \tau(x, \hat{\theta}) \\ &\leq -W(x, \hat{\theta}) - \frac{\partial V_a}{\partial \theta} \Gamma \left( \frac{\partial V_a}{\partial x} F \right)^T + \frac{\partial V_a}{\partial \theta} \Gamma \tau(x, \hat{\theta}) \end{aligned} \quad (7.2.40)$$

$$+ \tilde{\theta}^T \left( \frac{V_a}{x} F \right)^T - \tilde{\theta}^T \tau(x, \hat{\theta}). \quad (7.2.41)$$

Choosing

$$\tau(x, \hat{\theta}) = \left( \frac{\partial V_a}{\partial x}(x, \hat{\theta}) F(x) \right)^T \quad (7.2.42)$$

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we get

$$\dot{V} \leq -W(x, \hat{\theta}), \quad \forall \theta \in \mathbb{R}^p. \quad (7.2.43)$$

Thus, the equilibrium  $x = 0, \hat{\theta} = \theta$  of (7.2.28), (7.2.29), and (7.2.30) is globally stable, and by the LaSalle-Yoshizawa Theorem,  $x(t) \rightarrow 0$ , that is, (7.2.28) is globally adaptively stabilizable.

The adaptive controller constructed in the proof of Theorem 7.2.3 consists of a control law  $u = \alpha(x, \theta)$  given by (7.2.34), and an update law  $\dot{\hat{\theta}} = \Gamma\tau(x, \hat{\theta})$  with (7.2.42).

It is of interest to interpret this controller as a certainty equivalence controller. The control law  $\alpha(x, \theta)$  given by (7.2.34) is stabilizing for the modified system (7.2.31) but may not be stabilizing for the original system (7.2.28). However, as the proof of Theorem 7.2.3 shows, its certainty equivalence form  $\alpha(x, \hat{\theta})$  is an adaptive globally stabilizing control law for the original system (7.2.28). Hence, if a certainty equivalence approach is to be applied to a nonlinear system, the system is to be modified to require a control law which anticipates the parameter estimation transients. In the proof of Theorem 7.2.3, this is achieved by incorporating the *tuning function*  $\tau$  in the control law  $\alpha$ . Indeed, the formula (7.2.34) for  $\alpha$  depends on  $\tau$  via

$$\frac{\partial V_a}{\partial x} \tilde{f}(x, \theta) = \frac{\partial V_a}{\partial x} f + \tau(x, \theta)^T \left( \theta + \Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T \right), \quad (7.2.44)$$

which is obtained by combining (7.2.35) and (7.2.42). Using (7.2.42) to rewrite the inequality (7.2.38) as

$$\frac{\partial V_a}{\partial x} [f(x) + F(x)\theta + g(x)\alpha(x, \theta)] + \frac{\partial V_a}{\partial \theta} \Gamma\tau(x, \theta) \leq -W(x, \theta) \quad (7.2.45)$$

it is not difficult to see that the control law (7.2.34) containing (7.2.44) prevents  $\tau$  from destroying the nonpositivity of the Lyapunov derivative.

**Remark 7.2.4** *A relevant question remains unanswered: If there exists an aclf for (7.2.28), is this system globally asymptotically stabilizable for each  $\theta$  (and vice versa)? In other words, does the existence of a pair  $\alpha, V_a$  satisfying (7.2.33) for some  $\Gamma > 0$  imply*

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if  $\tau$  is defined as in (7.2.42). Then, it is straightforward to verify that

$$\begin{aligned} \frac{\partial V_a}{\partial x} \left[ f(x) + F(x) \left( \hat{\theta} + \Gamma \left( \frac{\partial V_a}{\partial \hat{\theta}} \right)^T \right) + g(x) \alpha(x, \hat{\theta}) \right] &= \dot{V} + \left( \hat{\theta}^T + \frac{\partial V_a}{\partial \hat{\theta}} \Gamma \right) \left( \tau - \left( \frac{\partial V_a}{\partial x} F \right)^T \right) \\ &\leq -W(x, \hat{\theta}) \end{aligned} \quad (7.2.47)$$

for all  $x, \hat{\theta} \in \mathbb{R}^{n+p}$ . By (1  $\Rightarrow$  2) in Theorem 7.2.3,  $V_a(x, \theta)$  is an aclf for (7.2.28).

The above analysis applies also to the case where the unknown parameter enter the control vector field:

$$\dot{x} = f(x) + F(x)\theta + [g(x) + G(x)\theta]u \quad (7.2.48)$$

In this case, the existence of an aclf  $V_a$  is equivalent to the existence of a clf for the system

$$\dot{x} = f(x) + F(x) \left( \theta + \Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T \right) + \left[ g(x) + G(x) \left( \theta + \Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T \right) \right] u \quad (7.2.49)$$

The extension to the multi-input case is also straightforward.

It is of interest to examine the input-output properties of the system resulting from the application of the adaptive control law  $\alpha(x, \hat{\theta})$  to the plant (7.2.1):

$$\dot{x} = f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta}) + F(x)\tilde{\theta}. \quad (7.2.50)$$

In early Lyapunov designs for linear systems of relative degree one, an important property was the strict positive realness of the transfer function between the parameter error and the output error [142]. For an analogous passivity property of the nonlinear system (7.2.50), let us consider that its input is  $\tilde{\theta}$ .

**Corollary 7.2.6** (*Passivity*) *Suppose a function  $V_a(x, \theta)$  is known to be an aclf with an associate control law  $\alpha(x, \theta)$ . The the system*

$$\begin{aligned} \dot{x} &= f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta}) + F(x)\tilde{\theta} \\ \tau &= \left( \frac{\partial V_a}{\partial x}(x, \hat{\theta}) F(x) \right)^T \end{aligned} \quad (7.2.51)$$

with  $\tilde{\theta}$  as the the input and  $\tau$  as the output is strictly passive. ญาติให้เข้าไปใช้ประโยชน์ด้านการค้า  
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### 7.2.3 Adaptive backstepping via aclf

With Theorem 7.2.3, the problem of adaptive stabilization is reduced to the problem of finding an aclf. We now address the problem of systematic construction of an aclf. Our aim is a recursive approach because we already know how to find aclf's for systems with the extended matching property, and expect to recursively enlarge this initial class of systems with repeated use of backstepping. So, we assume that an aclf is known for an initial system, and construct a new aclf for the initial system augmented by an integrator.

**Lemma 7.2.7** *If the system*

$$\dot{x} = f(x) + F(x)\theta + g(x)u, \quad (7.2.52)$$

*is globally adaptively quadratically stabilizable with  $\alpha \in \mathbb{C}^1$ , then the augmented system*

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)\xi \\ \dot{\xi} &= u, \end{aligned} \quad (7.2.53)$$

*is also globally adaptively quadratically stabilizable.*

**Proof** Since system (7.2.52) is globally adaptively stabilizable, then by Corollary 7.2.5 there exists an aclf  $V_a(x, \theta)$ , and by Theorem 7.2.3, it satisfies (7.2.33) with a control law  $u = \alpha(x, \theta)$ . We will now show that

$$V_1(x, \xi, \theta) = V_a(x, \theta) + \frac{1}{2}(\xi - \alpha(x, \theta))^2 \quad (7.2.54)$$

is an aclf for the augmented system (7.2.53) by showing that it satisfies

$$\frac{\partial V_1}{\partial(x, \xi)} \begin{bmatrix} f + F \left( \theta + \Gamma \left( \frac{\partial V_1}{\partial \theta} \right)^T \right) + g\xi \\ \alpha_1(x, \xi, \theta) \end{bmatrix} \leq -W - (\xi - \alpha)^2 \quad (7.2.55)$$

with the control law

$$\begin{aligned} u = \alpha_1(x, \xi, \theta) = & - \frac{\partial V_a}{\partial x} g - (\xi - \alpha) + \frac{\partial \alpha}{\partial x} (f + F\theta + g\xi) \\ & + \frac{\partial \alpha}{\partial \theta} \Gamma \left( \frac{\partial V_1}{\partial x} F \right)^T + \frac{\partial V_a}{\partial \theta} \Gamma \left( \frac{\partial \alpha}{\partial x} F \right)^T \end{aligned} \quad (7.2.56)$$

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Let us start by introducing for brevity  $z = \xi - \alpha(x, \theta)$ . With (7.2.54) we compute

$$\begin{aligned}
\frac{\partial V_1}{\partial(x, \xi)} \begin{bmatrix} f + F\theta + g\xi \\ \alpha_1(x, \xi, \theta) \end{bmatrix} &= \frac{\partial V_1}{\partial x}(f + F\theta + g\xi) + \frac{\partial V_1}{\partial \xi} \alpha_1(x, \xi, \theta) \\
&= \left( \frac{\partial V_a}{\partial x} - z \frac{\partial \alpha}{\partial x} \right) (f + F\theta + g\xi) + z \alpha_1 \\
&= \frac{\partial V_a}{\partial x}(f + F\theta + g\alpha) + \frac{\partial V_a}{\partial x} g z - z \frac{\partial \alpha}{\partial x}(f + F\theta + g\xi) + z \alpha_1 \\
&= \frac{\partial V_a}{\partial x}(f + F\theta + g\alpha) + z \left( \alpha_1 + \frac{\partial V_a}{\partial x} g - \frac{\partial \alpha}{\partial x}(f + F\theta + g\xi) \right)
\end{aligned}$$

On the other hand, in view of (7.2.54), we have

$$\begin{aligned}
\frac{\partial V_1}{\partial(x, \xi)} \begin{bmatrix} F\Gamma \left( \frac{\partial V_1}{\partial \theta} \right)^T \\ 0 \end{bmatrix} &= \frac{\partial V_1}{\partial x} F\Gamma \left( \frac{\partial V_1}{\partial \theta} \right)^T \\
&= \left( \frac{\partial V_a}{\partial x} - z \frac{\partial \alpha}{\partial x} \right) F\Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T \\
&= \frac{\partial V_a}{\partial x} F\Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T \\
&\quad - z \left( \frac{\partial \alpha}{\partial \theta} \Gamma \left( \frac{\partial V_1}{\partial x} F \right)^T + \frac{\partial V_a}{\partial \theta} \Gamma \left( \frac{\partial \alpha}{\partial x} F \right)^T \right) \quad (7.2.57)
\end{aligned}$$

Adding (7.2.57) and (7.2.57) with (7.2.33) and (7.2.56), we get

$$\begin{aligned}
\frac{\partial V_1}{\partial(x, \xi)} \begin{bmatrix} f + F \left( \theta + \Gamma \left( \frac{\partial V_1}{\partial \theta} \right)^T \right) + g\xi \\ \alpha_1(x, \xi, \theta) \end{bmatrix} &= \frac{\partial V_a}{\partial x}(f + F\theta + g\alpha) + \frac{\partial V_a}{\partial x} \Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T \\
&\quad + z \left( \alpha_1 + \frac{\partial V_a}{\partial x} g - \frac{\partial \alpha}{\partial x}(f + F\theta + g\xi) \right. \\
&\quad \left. - \frac{\partial \alpha}{\partial \theta} \Gamma \left( \frac{\partial V_1}{\partial x} F \right)^T - \frac{\partial V_a}{\partial \theta} \Gamma \left( \frac{\partial \alpha}{\partial x} F \right)^T \right) \\
&\leq -W(x, \theta) - z^2 \quad (7.2.58)
\end{aligned}$$

This proves by Theorem 7.2.3 that  $V_1(x, \xi, \theta)$  is an acfl for system (7.2.53), and by Corollary 7.2.5, this system is globally adaptively quadratically stabilizable.

The new tuning function for system (7.2.53) is determined by the new acf  $V_1$  and given by

$$\begin{aligned}\tau_1(x, \xi, \theta) &= \left( \frac{\partial V_1}{\partial(x, \xi)} \begin{bmatrix} F \\ 0 \end{bmatrix} \right)^T = \left( \frac{\partial V_1}{\partial x} F \right)^T = \left[ \left( \frac{\partial V_a}{\partial x} - (\xi - \alpha) \frac{\partial \alpha}{\partial x} \right) F \right]^T \\ &= \tau(x, \theta) - \left( \frac{\partial \alpha}{\partial x} F \right)^T (\xi - \alpha)\end{aligned}\quad (7.2.59)$$

We note that the new tuning function  $\tau_1$  is obtained by augmenting the initial tuning function  $\tau$  with the term  $-\left(\frac{\partial \alpha}{\partial x} F\right)^T (\xi - \alpha)$  which accounts for the fact that the acf  $V_a$  is augmented by  $\frac{1}{2}(\xi - \alpha(x, \theta))^2$ .

The form of the control law  $\alpha_1(x, \xi, \theta)$  in (7.2.56) is of particular interest. It consists of two parts,  $\alpha_1 = \alpha_{1,c} + \alpha_{1,\tau}$ . The first part,

$$\alpha_{1,c}(x, \xi, \theta) = -\frac{\partial V_a}{\partial x} g - (\xi - \alpha) + \frac{\partial \alpha}{\partial x} (f + F\theta + g\xi), \quad (7.2.60)$$

would become the ‘‘certainty equivalence’’ control law for the augmented system (7.2.53) if we were to set  $\Gamma = 0$ . The second part consists of two terms.

$$\alpha_{1,\tau}(x, \xi, \theta) = \frac{\partial \alpha}{\partial \theta} \Gamma \left( \frac{\partial V_1}{\partial x} F \right)^T + \frac{\partial V_a}{\partial \theta} \Gamma \left( \frac{\partial \alpha}{\partial x} F \right)^T. \quad (7.2.61)$$

Their role is to produce  $\frac{\partial V_a}{\partial x} F \Gamma \left( \frac{\partial V_a}{\partial \theta} \right)^T$  in the acf inequality (7.2.55). Observe that the first term in (7.2.61) incorporates  $\tau_1 = \left( \frac{\partial V_1}{\partial x} F \right)^T$ .

The control law  $\alpha_1(x, \xi, \theta)$  in (7.2.56) is only one out of many possible control laws. Once we have shown that  $V_1$  given by (7.2.54) is an acf for (7.2.53), we can use, for example, the  $C^0$  control law  $\alpha_1$  given by Sontag’s formula (7.2.34) with  $\frac{\partial V_1}{\partial(x, \xi)} g_1 = z$  and

$$\frac{\partial V_1}{\partial(x, \xi)} \tilde{f}_1(x, \xi, \theta) = \frac{\partial V_1}{\partial(x, \xi)} \begin{bmatrix} f + F \left( \theta + \Gamma \left( \frac{\partial V_1}{\partial \theta} \right)^T \right) + g\xi \\ 0 \end{bmatrix} \quad (7.2.62)$$

$$= \frac{\partial V_1}{\partial x} (f + g\xi) + \tau_1(x, \xi, \theta)^T \left( \theta + \Gamma \left( \frac{\partial V_1}{\partial \theta} \right)^T \right) \quad (7.2.63)$$

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It can be shown that the following function, used as a clf in [158], is a more general aclf than (7.2.54):

$$V_1(x, \xi, \theta) = V_a(x, \theta) + \int_0^{\xi - \alpha(x, \theta)} \eta(s) ds, \quad (7.2.64)$$

where  $\eta$  is a  $C^0$  function such that  $s\eta(s) > 0$  whenever  $s \neq 0$ ,  $\eta'(0) > 0$ , and  $\eta \notin \mathcal{L}^1((-\infty, 0]) \cup \mathcal{L}^1([0, +\infty))$ .

The following example illustrates the use of Lemma 7.2.7.

**Example 7.2.8** *Let us consider the system*

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^T \theta \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u. \end{aligned} \quad (7.2.65)$$

*we will treat the state  $x_3$  as an integrator added to the  $(x_1, x_2)$ -subsystem from Example ???. In the example, we have already designed an adaptive control law for the system*

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^T \theta \\ \dot{x}_2 &= x_3, \end{aligned} \quad (7.2.66)$$

*considering  $x_3$  as a control input. With (7.2.18), (7.2.19), (7.2.20), and (7.2.22), it can be shown that*

$$\frac{\partial V_c}{\partial x} \left[ f(x) + F(x) \left( \theta + \left( \frac{\partial V_c}{\partial \theta} \right)^T \right) \right] = -c_1 z_1^2 - c_2 z_2^2, \quad (7.2.67)$$

*which means that  $V_a(x_1, x_2, \theta) = V_c(x_1, x_2, \theta) = \frac{1}{2}(z_1^2 + z_2^2)$  is an aclf for the system (7.2.66) considering  $x_3$  as a control input. Therefore, Lemma 7.2.7, the function directly applicable. We define  $z_3 = x_3 - \alpha(x, \theta)$ . By Lemma 7.2.7, the function*

$$V_1(x, \theta) = \frac{1}{2} (z_1^2 + z_2^2 + z_3^2) \quad (7.2.68)$$

*is an aclf for the system (7.2.65). With (7.2.56) and (7.2.59) we obtain*

$$\alpha_1(x, \theta) = -z_1 - c_3 z_3 - \frac{\partial \alpha}{\partial (x_1, x_2)} \begin{bmatrix} x_2 + \varphi^T \theta \\ x_3 \end{bmatrix} + \frac{\partial \alpha}{\partial \theta} \tau_1 + z_2 \varphi^T \frac{\partial \alpha}{\partial x_1} \varphi \quad (7.2.69)$$

เอกสารนี้เป็น  $\tau_1(x, \theta) = \tau_1 - \frac{\partial \alpha}{\partial x_1} \varphi z_3$ ; งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์อื่นใด (7.2.70) คำ  
ไม่ว่ากรณีใดๆ ทั้งสิ้น อีกทั้งห้ามมิให้ดัดแปลงเนื้อหา และต้องอ้างอิงถึงเจ้าของเอกสารทุกครั้งที่มีการนำไปใช้

With the following adaptive control law and the parameter update law:

$$u = \alpha_1(x, \hat{\theta}) \quad (7.2.71)$$

$$\dot{\hat{\theta}} = \tau_1(x, \hat{\theta}), \quad (7.2.72)$$

it is straightforward to verify that the closed-loop adaptive system is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 - \frac{\partial \alpha}{\partial x_1} |\varphi|^2 \\ 0 & -1 + \frac{\partial \alpha}{\partial x_1} |\varphi|^2 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{\partial \varphi^T}{\partial x_1} \hat{\theta} + c_1 \\ -\frac{\partial \alpha}{\partial x_1} \end{bmatrix} \varphi^T \tilde{\theta} \quad (7.2.73)$$

$$\dot{\hat{\theta}} = \varphi \begin{bmatrix} 1, & \frac{\partial \varphi^T}{\partial x_1} \hat{\theta} + c_1, & -\frac{\partial \alpha}{\partial x_1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad (7.2.74)$$

where  $z_1, z_2, z_3$  are used with  $\hat{\theta}$  as an argument. The global stability of this system is established using the Lyapunov function  $V(x, \hat{\theta}) = V_1(x, \hat{\theta}) + \frac{1}{2} \tilde{\theta}^T \tilde{\theta}$ .

While in Lemma 7.2.7 the initial system is augmented only by an integrator, a minor modification is sufficient to obtain an analogous result for the more general system

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)\xi \\ \dot{\xi} &= u + F_1(x, \xi)\theta. \end{aligned} \quad (7.2.75)$$

**Corollary 7.2.9** The function  $V_1(x, \xi, \theta)$  defined in (7.2.54) is an acf for the system (7.2.75) with the control law and the tuning function given as

$$\alpha_1(x, \xi, \theta) = \alpha_1(x, \xi, \theta) - F_1(x, \xi) \left( \theta + \Gamma \left( \frac{\partial V_c}{\partial \theta} \right)^T \right) \quad (7.2.76)$$

$$\tau_1(x, \xi, \theta) = \tau_1(x, \xi, \theta) + (\xi - \alpha) F_1(x, \xi)^T. \quad (7.2.77)$$

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
ไม่ว่ากรณีใดๆ ทั้งสิ้น อีกทั้งห้ามมิให้ดัดแปลงเนื้อหา และต้องอ้างอิงถึงเจ้าของเอกสารทุกครั้งที่มีการนำไปใช้

A repeated application of Corollary 7.2.9 will further extend the class of nonlinear systems for this type of adaptive design. With the knowledge of  $V_a$ ,  $\tau$ , and  $\alpha$  for the system (7.2.75), it is not hard to see that by applying Corollary 7.2.9 twice we can find  $V_2$ ,  $\tau_2$ , and  $\alpha_2$  for the system

$$\begin{aligned}\dot{x} &= f(x) + F(x)\theta + g(x)\xi_1 \\ \dot{\xi}_1 &= \xi_2 + F_1(x, \xi_1)\theta \\ \dot{\xi}_2 &= u + F_2(x, \xi_1, \xi_2)\theta.\end{aligned}\tag{7.2.78}$$

In fact, it is clear that an  $n$ -fold application of Corollary 7.2.9 will provide us with  $V_n$ ,  $\tau_n$ , and  $\alpha_n$  for the system

$$\begin{aligned}\dot{x} &= f(x) + F(x)\theta + g(x)\xi_1 \\ \dot{\xi}_1 &= \xi_2 + F_1(x, \xi_1)\theta \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_n + F_{n-1}(x, \xi_1, \dots, \xi_{n-1})\theta \\ \dot{\xi}_n &= u + F_n(x, \xi_1, \dots, \xi_n)\theta.\end{aligned}\tag{7.2.79}$$

We will now develop a detailed design procedure for such systems.

### 7.3 Set-Point Regulation

With repeated use of Corollary 7.2.9, we can design an adaptive controller to globally stabilize a desired equilibrium  $x^e$  of the *parametric strict-feedback system*:

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi_1(x_1)^T \theta \\ \dot{x}_2 &= x_3 + \varphi_2(x_1, x_2)^T \theta \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \varphi_{n-1}(x_1, \dots, x_{n-1})^T \theta \\ \dot{x}_n &= \beta(x)u + \varphi_n(x)^T \theta\end{aligned}\tag{7.3.1}$$

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า ไม่ว่าจะกรณีใดๆ ทั้งสิ้น อีกทั้งห้ามมิให้ดัดแปลงเนื้อหา และต้องอ้างอิงถึงเจ้าของเอกสารทุกครั้งที่มีการนำไปใช้

where  $\theta \in \mathbb{R}^p$  is a vector of unknown constant parameters,  $\beta$  and

$$F = [\varphi_1, \dots, \varphi_n] \quad (7.3.2)$$

are smooth nonlinear functions taking arguments in  $\mathbb{R}^n$ , and  $\beta(x) \neq 0, \forall x \in \mathbb{R}^n$ .

In this section we develop a procedure for adaptive regulation of the output  $y = x_1$  to a given set-point  $y_s$ . With a constant control  $u^e$ , the first  $n-1$  equilibrium equations of  $\dot{x}^e = 0$  in (7.3.1) can be successively solved for  $x_2^e, \dots, x_n^e$  as functions of  $x_1^e$  and  $\theta$ :

$$\begin{aligned} x_2^e &= -\varphi_1(x_1^e)^T \theta \\ x_3^e &= -\varphi_2(x_1^e, x_2^e)^T \theta \\ &\vdots \\ x_n^e &= -\varphi_{n-1}(x_1^e, \dots, x_{n-1}^e)^T \theta \end{aligned} \quad (7.3.3)$$

Then the  $n$ th equation  $\dot{x}_n^e = 0$  yields a relationship between  $x_1^e$ ,  $u^e$ , and  $\theta$ . When  $\theta$  is known, then  $\dot{x}_n^e = 0$  can be solved for  $u^e$  needed to keep  $x_1^e$  at a desired set-point  $x_1^e = y_s$ . The corresponding values  $x_2^e, \dots, x_n^e$  will be dictated by (7.3.3). Therefore, for each value of  $\theta$  and a prescribed  $y_s$ , the equilibrium  $x^e$  and the corresponding control value  $u^e$  are uniquely defined. In the special case where  $\varphi_1(0) = \dots = \varphi_{n-1}(0) = 0$ , the choice  $y_s = 0$  results in the equilibrium being  $x^e = 0$  for all values of  $\theta$ .

Our problem now is to globally stabilize this equilibrium when  $\theta$  is unknown and also to achieve set-point regulation:  $x(t) \rightarrow x^e$  as  $t \rightarrow \infty$ .

Comparing the systems (7.3.1) and (7.2.75), we observe that if  $x_3$  were the control variable, then Corollary 7.2.9 would provide the desired adaptive control for the subsystem made of the first two equations of (7.3.1). Therefore, we can initiate our recursive design procedure by augmenting this subsystem by the third equation, as in (7.2.78). For convenience, we will do this in a self-contained fashion, independent of Section 7.2. An additional feature of the procedure in this section is a set of error coordinates in which the stability properties of the resulting closed-loop adaptive system are clearly displayed without an explicit use of the aclf concept.

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ไม่ว่ากรณีใดๆ ทั้งสิ้น อีกทั้งห้ามมิให้ดัดแปลงเนื้อหา และต้องอ้างอิงถึงเจ้าของเอกสารทุกครั้งที่มีการนำไปใช้

### 7.3.1 Design procedure

We will start by adaptively stabilizing the first equation (7.3.1) considering  $x_2$  to be its control. At each subsequent step we will augment the designed subsystem by one equation. At the  $i$ th step, an  $i$ th-order subsystem is stabilized with respect to a Lyapunov function  $V_i$  by the design of a *stabilizing function*  $\alpha_i$  and a *tuning function*  $\tau_i$ . The update law for the parameter estimate  $\hat{\theta}(t)$  and the adaptive feedback control  $u$  are designed at the final step. The third step is crucial for understanding the general design procedure.

**Step 1.** Introducing the first two error variables

$$z_1 = x_1 - y_s \quad (7.3.4)$$

$$z_2 = x_2 - \alpha_1 \quad (7.3.5)$$

we rewrite  $\dot{x}_1 = x_2 + \varphi_1(x_1)^T \theta$ , the first equation of (7.3.1), as

$$\dot{z}_1 = z_2 + \alpha_1 + w_1(x_1)^T \theta, \quad (7.3.6)$$

where, for uniformly with subsequent steps, we have defined the first regressor vector as

$$w_1(x_1) = \varphi_1(x_1). \quad (7.3.7)$$

Our task in this step is to stabilize (7.3.6) with respect to the Lyapunov function

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}, \quad (7.3.8)$$

whose derivative along the solutions of (7.3.6) is

$$\dot{V}_1 = z_1(z_2 + \alpha_1 + w_1^T \tilde{\theta}) - \tilde{\theta}^T \Gamma^{-1} (\dot{\tilde{\theta}} - \Gamma w_1 z_1). \quad (7.3.9)$$

We can eliminate  $\dot{\tilde{\theta}}$  from  $\dot{V}_1$  with the update law  $\dot{\hat{\theta}} = \Gamma \tau_1$ , where

$$\tau_1(x_1) = w_1(x_1) z_1. \quad (7.3.10)$$

If  $x_2$  were actual control, we would let  $z_2 \equiv 0$ , that is,  $x_2 \equiv \alpha_1$ . Then, to make  $\dot{V}_1 = -c_1 z_1^2$ , we would choose

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับ  $\alpha_1(x_1, \hat{\theta}) = -c_1 z_1 - w_1(x_1)^T \hat{\theta}$ . ไม่อนุญาตให้นำไปใช้ประโยชน์ (7.3.11) ใดๆ  
ไม่ว่ากรณีใดๆ ทั้งสิ้น อีกทั้งห้ามมิให้ดัดแปลงเนื้อหา และต้องอ้างอิงถึงเจ้าของเอกสารทุกครั้งที่มีการนำไปใช้

Since  $x_2$  is not our control, we have  $z_2 \neq 0$ , and we do not use  $\dot{\hat{\theta}} = \Gamma \tau_1$  as an update law. Instead, we retain  $\tau_1$  as our first *tuning function* and tolerate the presence of  $\tilde{\theta}$  in  $\dot{V}_1$ :

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 - \tilde{\theta}^T (\Gamma^{-1} \dot{\hat{\theta}} - \tau_1). \quad (7.3.12)$$

The second term  $z_1 z_2$  in  $\dot{V}_1$  will be cancelled at the next step. With  $\alpha_1(x_1, \hat{\theta})$  as in (7.3.11), the  $z_1$ -system becomes

$$\dot{z}_1 = -c_1 z_1 + z_2 + w_1(x_1)^T \tilde{\theta} \quad (7.3.13)$$

**Step 2.** We now consider that  $x_3$  is the control variable in the second equation of (7.3.1).

Introducing

$$z_3 = x_3 - \alpha_2, \quad (7.3.14)$$

we rewrite  $\dot{x}_2 = x_3 + \varphi_2(x_1, x_2)^T \theta$  as

$$\dot{z}_2 = z_3 + \alpha_2 - \frac{\partial \alpha_1}{\partial x_1} x_2 + w_2(x_1, x_2, \hat{\theta})^T \theta - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} \quad (7.3.15)$$

where the second regressor vector  $w_2$  is defined as

$$w_2(x_1, x_2, \hat{\theta}) = \varphi_2 - \frac{\partial \alpha_1}{\partial x_1} \varphi_1. \quad (7.3.16)$$

Our task in this step is to stabilize the  $(z_1, z_2)$ -system (7.3.13), (7.3.15) with respect to

$$V_2 = V_1 + \frac{1}{2} z_2^2 \quad (7.3.17)$$

whose derivative along the solutions of (7.3.13) and (7.3.15) is

$$\begin{aligned} \dot{V}_2 = -c_1 z_1^2 + z_2 \left[ z_1 + z_3 + \alpha_2 - \frac{\partial \alpha_1}{\partial x_1} x_2 + w_2^T \hat{\theta} - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} \right] \\ + \tilde{\theta}^T \left( \tau_1 + w_2 z_2 - \Gamma^{-1} \dot{\hat{\theta}} \right). \end{aligned} \quad (7.3.18)$$

We can eliminate  $\tilde{\theta}$  from  $\dot{V}_2$  with the update law  $\dot{\hat{\theta}} = \Gamma \tau_2$ , where

$$\tau_2(x_1, x_2, \hat{\theta}) = \tau_1 + w_2 z_2 = \begin{bmatrix} w_1 & , & w_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \quad (7.3.19)$$

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If  $x_3$  were our actual control and, hence,  $z_3 \equiv 0$ , we would achieve  $\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2$  by designing  $\alpha_2$  to make the bracketed term multiplying  $z_2$  in (7.3.18) equal to  $-c_2 z_2$ , namely

$$\alpha_2(x_1, x_2, \hat{\theta}) = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 - w_2^T \hat{\theta} + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2. \quad (7.3.20)$$

We retain  $\tau_2$  as our second tuning function in the term  $\Gamma \tau_2$  which replaces  $\dot{\hat{\theta}}$  in (7.3.20). However, we do not use  $\dot{\hat{\theta}} = \Gamma \tau_2$  as an update law, so that the resulting  $\dot{V}_2$  is

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) + \bar{\theta}^T (\tau_2 - \Gamma^{-1} \dot{\hat{\theta}}). \quad (7.3.21)$$

The first two terms in  $\dot{V}_2$  are negative definite, the third term will be cancelled at the next step, while the discrepancy between  $\Gamma \tau_2$  and  $\dot{\hat{\theta}}$  in the last two terms remains. By substituting (7.3.20) into (7.3.15), the  $(z_1, z_2)$ -subsystem becomes

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \bar{\theta} + \begin{bmatrix} 0 \\ z_3 + \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) \end{bmatrix} \quad (7.3.22)$$

**Step 3.** Proceeding to the third equation in (7.3.1) we introduce

$$z_4 = x_4 - \alpha_3 \quad (7.3.23)$$

and rewrite  $\dot{x}_3 = x_4 + \varphi_3(x_1, x_2, x_3)^T \theta$  as

$$\dot{z}_3 = z_4 + \alpha_3 - \frac{\partial \alpha_2}{\partial x_1} x_2 - \frac{\partial \alpha_2}{\partial x_2} x_3 + w_3(x_1, x_2, x_3, \hat{\theta})^T \theta - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}}, \quad (7.3.24)$$

where the third regressor vector  $w_3$  is defined as

$$w_3(x_1, x_2, x_3, \hat{\theta}) = \varphi_3 - \frac{\partial \alpha_2}{\partial x_1} \varphi_1 - \frac{\partial \alpha_2}{\partial x_1} \varphi_2. \quad (7.3.25)$$

Our task is to stabilize the  $(z_1, z_2, z_3)$ -system with respect to

$$V_3 = V_2 + \frac{1}{2} z_3^2 \quad (7.3.26)$$

whose derivative along (7.3.22) and (7.3.24) is

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) + z_3 \left[ z_2 + z_4 + \alpha_3 - \frac{\partial \alpha_2}{\partial x_1} x_2 - \frac{\partial \alpha_2}{\partial x_2} x_3 + w_3^T \hat{\theta} - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} \right] + \bar{\theta}^T (\tau_2 + w_3 z_3 - \Gamma^{-1} \dot{\hat{\theta}}) \quad (7.3.27)$$

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ไม่ว่ากรณีใดๆ ทั้งสิ้น อีกทั้งห้ามมิให้ดัดแปลงเนื้อหา และต้องอ้างอิงถึงเจ้าของเอกสารทุกครั้งที่มีการนำไปใช้

We can eliminate  $\dot{\tilde{\theta}}$  from  $\dot{V}_3$  with the update law  $\dot{\hat{\theta}} = \Gamma\tau_3$ , where  $\tau_3$  is our tuning function

$$\tau_3(x_1, x_2, x_3, \hat{\theta}) = \tau_2 + w_3 z_3 = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad (7.3.28)$$

If  $x_3$  were our actual control, we could have  $z_4 \equiv 0$  and achieve  $\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2$  by designing  $\alpha_3$  to make the bracketed term multiplying  $z_3$  equal to  $-c_3 z_3$ , namely

$$\alpha_3(x_1, x_2, x_3, \hat{\theta}) = -z_2 - c_3 z_3 + \frac{\partial \alpha_2}{\partial x_1} x_2 + \frac{\partial \alpha_2}{\partial x_2} x_3 - w_3^T \hat{\theta} + \frac{\partial \alpha_2}{\partial \hat{\theta}} \Gamma \tau_3 + \nu_3, \quad (7.3.29)$$

where  $\nu_3$  is a correction term yet to be chosen. Substituting (7.3.29) into (7.3.27), and noting that

$$\begin{aligned} \dot{\hat{\theta}} - \Gamma \tau_2 &= \dot{\hat{\theta}} - \Gamma \tau_3 + \Gamma \theta_3 - \Gamma \tau_2 \\ &= \dot{\hat{\theta}} - \Gamma \tau_3 + \Gamma w_3 z_3. \end{aligned} \quad (7.3.30)$$

(7.3.27) is written as

$$\begin{aligned} \dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 + z_3 \left( \nu_3 - \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma w_3 z_2 \right) + z_3 z_4 + \left( z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} + z_3 \frac{\partial \alpha_2}{\partial \hat{\theta}} \right) (\Gamma \tau_3 - \dot{\hat{\theta}}) \\ &\quad + \tilde{\theta}^T (\tau_3 - \Gamma^{-1} \dot{\hat{\theta}}) \end{aligned} \quad (7.3.31)$$

and the  $(z_1, z_2, z_3)$ -subsystem becomes

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} &= \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 \\ 0 & -1 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ -\frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma w_3 z_2 \\ \nu_3 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_3 - \dot{\hat{\theta}}) \\ z_4 + \frac{\partial \alpha_2}{\partial \hat{\theta}} (\Gamma \tau_3 - \dot{\hat{\theta}}) \end{bmatrix} \end{aligned} \quad (7.3.32)$$

If  $x_4$  were our control, we would have  $z_4 = 0$ , and with the update law  $\dot{\hat{\theta}} = \Gamma\tau_3$ , the last vector in (7.3.32) would be zero. However, the potentially destabilizing term  $-\frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma w_3 z_2$  would still remain. This unmatched term must be accommodated by a choice of the correction term  $\nu_3$ . From (7.3.31), the choice of  $\nu_3$  is immediate:

$$\nu_3(x_1, x_2, x_3, \hat{\theta}) = \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma w_3 z_2. \quad (7.3.33)$$

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We again postpone the decision about  $\hat{\theta}$  and do not use  $\dot{\hat{\theta}} = \Gamma\tau_3$  as an update law. The resulting  $\dot{V}_3$  is

$$\begin{aligned} \dot{V}_3 = & -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3 z_4 + \left( z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} + z_3 \frac{\partial \alpha_2}{\partial \hat{\theta}} \right) (\Gamma\tau_3 - \dot{\hat{\theta}}) \\ & + \tilde{\theta}^T (\tau_3 - \Gamma^{-1} \dot{\hat{\theta}}) \end{aligned} \quad (7.334)$$

and the  $(z_1, z_2, z_3)$ -subsystem becomes

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = & \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 - \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma w_3 \\ 0 & -1 + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma w_3 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} \tilde{\theta} \\ & + \begin{bmatrix} 0 \\ 0 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial \alpha_1}{\partial \hat{\theta}} \\ \frac{\partial \alpha_2}{\partial \hat{\theta}} \end{bmatrix} (\Gamma\tau_3 - \dot{\hat{\theta}}), \end{aligned} \quad (7.335)$$

The 'system matrix' in (7.335) has a significant property: the skew symmetry of the nonlinear term  $\frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma w_3$  achieved by the choice of  $\nu_3$  in (7.333). This term is analogous to the second term in (7.2.61) and the skew symmetry is crucial for stabilization.

**Step  $i$ .** Introducing

$$z_{i+1} = x_{i+1} - \alpha_i, \quad (7.336)$$

we rewrite  $\dot{x}_i = x_{i+1} + \varphi_i(x_1, \dots, x_i)^T \theta$  as

$$\dot{z}_i = z_{i+1} + \alpha_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + w_i(x_1, \dots, x_i, \hat{\theta})^T \theta - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}, \quad (7.337)$$

where the  $i$ th regressor vector is defined as

$$w_i(x_1, \dots, x_i, \hat{\theta}) = \varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \varphi_k. \quad (7.338)$$

Our objective is to stabilize the  $(z_1, \dots, z_i)$ -system with respect to

$$V_i = V_{i-1} + \frac{1}{2} z_i^2, \quad (7.339)$$

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whose derivative is

$$\begin{aligned}\dot{V}_i = & - \sum_{k=1}^{i-1} c_k z_k^2 + \left( \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) (\Gamma \tau_{i-1} - \dot{\hat{\theta}}) \\ & + z_i \left[ z_{i-1} + z_{i+1} + \alpha_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + w_i^T \hat{\theta} - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right] \\ & + \tilde{\theta}^T (\tau_{i-1} + w_i z_i - \Gamma^{-1} \dot{\hat{\theta}}).\end{aligned}\quad (7.3.40)$$

We can eliminate  $\dot{\hat{\theta}}$  from  $\dot{V}_i$  with the update law  $\dot{\hat{\theta}} = \Gamma \tau_i$ , where

$$\begin{aligned}\tau_i(x_1, \dots, x_i, \hat{\theta}) &= \tau_{i-1} + z_i w_i \\ &= \begin{bmatrix} w_1 & \dots & w_i \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_i \end{bmatrix}.\end{aligned}\quad (7.3.41)$$

Then, in the absence of  $z_{i+1}$ , we would achieve  $\dot{V}_i = -\sum_{k=1}^i c_k z_k^2$ , by designing  $\alpha_i$  to make the bracketed term multiplying  $z_i$  equal to  $-c_i z_i$ , namely

$$\alpha_i(x_1, \dots, x_i, \hat{\theta}) = -z_{i-1} - c_i z_i + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} - w_i^T \hat{\theta} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i + \nu_i, \quad (7.3.42)$$

where  $\nu_i$  is a correction term yet to be chosen. Nothing that

$$\begin{aligned}\dot{\hat{\theta}} - \Gamma \tau_{i-1} &= \dot{\hat{\theta}} - \Gamma \tau_i + \Gamma \tau_i - \Gamma \tau_{i-1} \\ &= \dot{\hat{\theta}} - \Gamma \tau_i + \Gamma w_i z_i,\end{aligned}\quad (7.3.43)$$

we rewrite  $\dot{V}_i$  as

$$\begin{aligned}\dot{V}_i &= - \sum_{k=1}^{i-1} c_k z_k^2 + z_i \left[ z_{i+1} + \nu_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} (\Gamma \tau_i - \dot{\hat{\theta}}) \right] \\ &\quad + \left( \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) (\Gamma \tau_{i-1} - \dot{\hat{\theta}}) + \tilde{\theta}^T (\tau_i - \Gamma^{-1} \dot{\hat{\theta}}) \\ &= - \sum_{k=1}^{i-1} c_k z_k^2 + z_i \left[ z_{i+1} + \nu_i - \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \Gamma w_i \right] \\ &\quad + \left( \sum_{k=1}^{i-1} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) (\Gamma \tau_i - \dot{\hat{\theta}}) + \tilde{\theta}^T (\tau_i - \Gamma^{-1} \dot{\hat{\theta}}),\end{aligned}\quad (7.3.44)$$

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and represent the  $(z_1, \dots, z_i)$ -subsystem as

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & -c_2 & 1 + \sigma_{23} & \cdots & \sigma_{2,i-1} & 0 \\ 0 & -1 - \sigma_{23} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 + \sigma_{i-2,i-1} & 0 \\ 0 & -\sigma_{2,i-1} & \cdots & -1 - \sigma_{i-2,i-1} & -c_{i-1} & 1 \\ 0 & 0 & \cdots & 0 & -1 & -c_i \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_i \end{bmatrix} + \begin{bmatrix} w_1^T \\ \vdots \\ w_i^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ \sigma_{2,i} z_i \\ \vdots \\ \sigma_{i-1,i} z_i \\ \nu_i \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ z_{i+1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial \alpha_1}{\partial \hat{\theta}} \\ \vdots \\ \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \end{bmatrix} (\Gamma \tau_i - \dot{\hat{\theta}}), \quad (7.3.45)$$

where

$$\sigma_{jk}(x, \hat{\theta}) = -\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_k. \quad (7.3.46)$$

Now the correction term is chosen as

$$\nu_i(x_1, \dots, x_i, \hat{\theta}) = \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \Gamma w_i = -\sum_{k=2}^{i-1} \sigma_{k,i} z_k. \quad (7.3.47)$$

Because we do not use  $\dot{\hat{\theta}} = \Gamma \tau_i$  as an update law, the resulting  $\dot{V}_i$  is

$$\dot{V}_i = \sum_{k=1}^i c_k z_k^2 + z_i z_{i+1} + \left( \sum_{k=1}^{i-1} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) (\Gamma \tau_i - \dot{\hat{\theta}}) + \tilde{\theta}^T (\tau_i - \Gamma^{-1} \dot{\hat{\theta}}), \quad (7.3.48)$$

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and the  $(z_1, \dots, z_i)$ -subsystem becomes

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_i \end{bmatrix} &= \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 & 1 + \sigma_{23} & \cdots & \sigma_{2i} \\ 0 & -1 - \sigma_{23} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 + \sigma_{i-1,i} \\ 0 & -\sigma_{2i} & \cdots & -1 - \sigma_{i-1,i} & -c_i \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_i \end{bmatrix} \\ &+ \begin{bmatrix} w_1^T \\ \vdots \\ w_i^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ z_{i+1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial \alpha_1}{\partial \hat{\theta}} \\ \vdots \\ \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \end{bmatrix} (\Gamma \tau_i - \dot{\hat{\theta}}). \end{aligned} \quad (7.3.49)$$

**Step  $n$ .** At the final step, we introduce

$$z_n = x_n - \alpha_{n-1} \quad (7.3.50)$$

and rewrite the last equation  $\dot{x}_n = \beta(x)u + \varphi_n(x)^T \theta$  as

$$\dot{z}_n = \beta u + \varphi_n^T \theta - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} (x_{k+1} + \varphi_k^T \theta) - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}, \quad (7.3.51)$$

where the last regressor vector is defined as

$$w_n(x, \hat{\theta}) = \varphi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} \varphi_k. \quad (7.3.52)$$

In this equation, the actual control input is at our disposal. We are finally in the position to design our actual update law  $\dot{\hat{\theta}} = \Gamma \tau_n$  and feedback control  $u$  to stabilize the full  $z$ -system with respect to

$$\begin{aligned} V_n &= V_{n-1} + \frac{1}{2} z_n^2 \\ &= \frac{1}{2} z^T z + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}. \end{aligned} \quad (7.3.53)$$

Our goal is to make  $\dot{V}_n$  nonpositive:

$$\begin{aligned} \dot{V}_n &= - \sum_{k=1}^{n-1} c_k z_k^2 + \left( \sum_{k=1}^{n-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) (\Gamma \tau_{n-1} - \dot{\hat{\theta}}) \\ &+ z_n \left[ z_{n-1} + \beta u - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} + w_n^T \hat{\theta} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right] \\ &+ \tilde{\theta}^T (\tau_{n-1} + w_n z_n - \Gamma^{-1} \dot{\hat{\theta}}). \end{aligned} \quad (7.3.54)$$

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับใช้ภายในเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ในประการ  
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To eliminate  $\dot{\tilde{\theta}}$  from  $\dot{V}_n$  we choose the update law

$$\begin{aligned}\dot{\tilde{\theta}} &= \Gamma\tau_n(z, \hat{\theta}) = \Gamma\tau_{n-1} + \Gamma w_n z_n \\ &= \Gamma W(z, \hat{\theta})z,\end{aligned}\quad (7.3.55)$$

where the regressor matrix  $W$  is composed of the regressor vectors  $w_1, \dots, w_n$ :

$$W(z, \hat{\theta}) = [w_1, \dots, w_n]. \quad (7.3.56)$$

We choose the control  $u$  to make the bracketed term multiplying  $z_n$  equal to  $-c_n z_n$ :

$$u = \frac{1}{\beta} \left( -z_{n-1} - c_n z_n + \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} - w_n^T z_n + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \Gamma \tau_n + \nu_n \right), \quad (7.3.57)$$

where  $\nu_n$  is a correction term yet to be chosen. With (7.3.57),  $\dot{V}_n$  becomes

$$\dot{V}_n = -\sum_{k=1}^{n-1} c_k z_k^2 + \left( \sum_{k=1}^{n-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) (\Gamma \tau_{n-1} - \dot{\tilde{\theta}}) + z_n \nu_n. \quad (7.3.58)$$

Then, noting that

$$\dot{\tilde{\theta}} - \Gamma \tau_{n-1} = \Gamma \tau_n - \Gamma \tau_{n-1} = \Gamma w_n z_n, \quad (7.3.59)$$

we rewrite  $\dot{V}_n$  as

$$\dot{V}_n = -\sum_{k=1}^{n-1} c_k z_k^2 + z_n \left( \nu_n - \sum_{k=1}^{n-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \Gamma w_n \right). \quad (7.3.60)$$

Now the correction term  $\nu_n$  is chosen as

$$\nu_n(x, \hat{\theta}) = \sum_{k=1}^{n-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \Gamma w_n = -\sum_{k=2}^{n-1} \gamma_{k,n} z_k. \quad (7.3.61)$$

We have thus reached our goal:

$$\dot{V}_n = -\sum_{k=1}^n c_k z_k^2. \quad (7.3.62)$$

The overall closed-loop system is

$$\dot{z} = A_z(z, \hat{\theta})z + W(z, \hat{\theta})^T \tilde{\theta} \quad (7.3.63)$$

$$\dot{\tilde{\theta}} = \Gamma W(z, \hat{\theta})z, \quad (7.3.64)$$

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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where

$$A_z(z, \hat{\theta}) = \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 & 1 + \sigma_{23} & \cdots & \sigma_{2n} \\ 0 & -1 - \sigma_{23} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 + \sigma_{n-1,n} \\ 0 & -\sigma_{2n} & \cdots & -1 - \sigma_{n-1,n} & -c_n \end{bmatrix} \quad (7.3.65)$$

The system (7.3.63) will be referred to as the *error system*. It is important to note that a major portion of the designed effort was invested into achieving

$$A_z(z, \hat{\theta}) + A_z(z, \hat{\theta})^T = -2 \begin{bmatrix} c_1 & & & \\ & \ddots & & \\ & & & c_n \end{bmatrix}, \quad \forall (z, \hat{\theta}) \in \mathbb{R}^{n+p} \quad (7.3.66)$$

which yields (7.3.62) with the simple quadratic Lyapunov function (7.3.53). We observe that, as desired, the system (7.3.63)-(7.3.64) has an equilibrium at  $(z, \tilde{\theta}) = (0, 0)$ . The stability properties of this equilibrium will be established in Section ??.

**Example 7.3.1** *In applications of the tuning functions procedure we do not need to repeat the Lyapunov argument. All we need for a specific design are the final analytical expressions provided by the procedure. Let us now illustrate this by designing an adaptive controller for the benchmark system from Example 7.2.8:*

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^T \theta \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u. \end{aligned} \quad (7.3.67)$$

*The design objective is the regulation of the output  $y = x_1$  to the set-point  $y_s$ . The first three expressions provided by the procedure are the definitions (7.3.4), (7.3.5), and (7.3.14) of the error variables*

$$\begin{aligned} z_1 &= x_1 - y_s \\ z_2 &= x_2 - \alpha_1(x_1, \hat{\theta}) \end{aligned} \quad (7.3.68)$$

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับ  $z_3 = x_3 - \alpha_2(x_1, x_2, \hat{\theta})$ , ไม่นอนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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where  $\alpha_1$  and  $\alpha_2$  are the stabilizing functions given by (7.3.11) and (7.3.20):

$$\begin{aligned}\alpha_1 &= -c_1 z_1 - \varphi^T \hat{\theta} \\ \alpha_2 &= -c_2 z_2 - z_1 + \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \hat{\theta}) + \frac{\partial \alpha_1}{\partial \hat{\theta}} \tau_2.\end{aligned}\quad (7.3.69)$$

The tuning functions, determined from (7.3.10), (7.3.19), and (7.3.28), are

$$\begin{aligned}\tau_1 &= z_1 \varphi \\ \tau_2 &= \tau_1 - z_2 \frac{\partial \alpha_1}{\partial x_1} \varphi \\ \tau_3 &= \tau_2 - z_3 \frac{\partial \alpha_2}{\partial x_1} \varphi.\end{aligned}\quad (7.3.70)$$

With the above expressions and the choice  $\Gamma = I$ , the parameter update law and the feedback control are obtained from (7.3.55) and (7.3.29), respectively. They are

$$\dot{\hat{\theta}} = \tau_3 = z_1 \varphi - z_2 \frac{\partial \alpha_1}{\partial x_1} \varphi - z_3 \frac{\partial \alpha_2}{\partial x_1} \varphi \quad (7.3.71)$$

$$u = -c_3 z_3 - z_2 + \frac{\partial \alpha_2}{\partial x_1} (x_2 + \varphi^T \hat{\theta}) + \frac{\partial \alpha_2}{\partial x_2} x_3 + \frac{\partial \alpha_2}{\partial \hat{\theta}} \tau_3 - z_2 \frac{\partial \alpha_2}{\partial \hat{\theta}} \frac{\partial \alpha_2}{\partial x_1} \varphi. \quad (7.3.72)$$

This completes the design of the adaptive controller for (7.3.67). In the  $(z, \hat{\theta})$ -coordinates the designed system is

$$\dot{z} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 - \frac{\partial \alpha_2}{\partial x_1} |\varphi|^2 \\ 0 & -1 + \frac{\partial \alpha_2}{\partial x_1} |\varphi|^2 & -c_3 \end{bmatrix} z + \begin{bmatrix} 1 \\ -\frac{\partial \alpha_1}{\partial x_1} \\ -\frac{\partial \alpha_2}{\partial x_1} \end{bmatrix} \varphi^T \hat{\theta} \quad (7.3.73)$$

$$\dot{\hat{\theta}} = -\varphi \left[ 1, -\frac{\partial \alpha_1}{\partial x_1}, -\frac{\partial \alpha_2}{\partial x_1} \right] z \quad (7.3.74)$$

It is of interest to relate the stabilizing functions  $\alpha_1$  and  $\alpha_2$  and the control law  $u$  to the material from Section 7.2. The stabilizing function  $\alpha_1$  has a “certainty equivalence” form. The stabilizing function  $\alpha_2$  has the term  $-\frac{\partial \alpha_1}{\partial \hat{\theta}} \tau_2$  which accounts for parameter estimation transients, while the rest of it is in the “certainty equivalence” form. The control law  $u$  departs from the “certainty equivalence” form in the last two terms whose role is the same as that of (7.2.61). The last term in  $u$  is particularly important. Since  $-\frac{\partial \alpha_1}{\partial \hat{\theta}} = -\varphi^T$ , this term contributes with  $+\frac{\partial \alpha_2}{\partial x_1} |\varphi|^2$  in the 1system matrix’ in (7.3.73) and achieves the skew symmetry, which is crucial for stability.

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# Chapter 8

## System Model

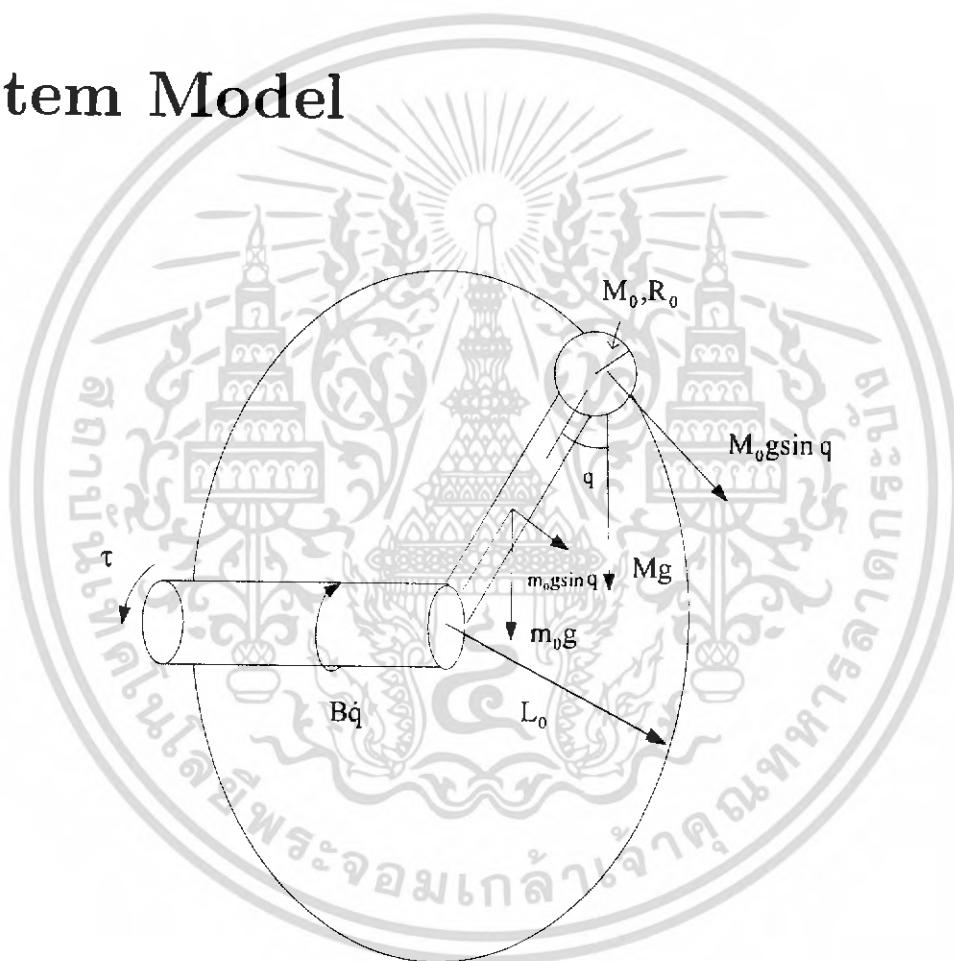


Figure 8.1: schematic of the mechanical system

An extended body free to rotate about an axis will do so under the application of a net torque. The resulting angular acceleration is given by Newton's Second Law for rotation:

$$\sum \tau = J\ddot{q} \tag{8.0.1}$$

where  $\tau$  is the applied torque (including any torque caused by friction),  $\ddot{q}$  is the an angular

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acceleration, and  $\hat{J}$  is the moment of inertia. The moment of inertia of a body depends on its mass distribution about an axis of rotation. Consequently, the same body can have different moments of inertia depending on where the axis of rotation is chosen.

For our setting (8.0.1) can be written as

$$\tau - \hat{B}\dot{q} - L_0 M_0 g \sin q - \frac{L_0}{2} m_0 g \sin q = \hat{J}\ddot{q} \quad (8.0.2)$$

where

- $\tau$  is the torque applied by the motor
- $J$  is the rotor inertia;
- $m_0$  is the link mass;
- $M_0$  is the load mass;
- $L_0$  is the link length;
- $R_0$  is the radius of load;
- $g$  is the gravitational coefficient;
- $\hat{B}$  is the coefficient of viscous friction at the joint;
- $q(t)$  is angular motor position (and hence the position of the load)
- $I(t)$  is motor armature current
- $K_\tau$  is the coefficient which characterizes the electromechanical conversion of armature current to torque.
- $\hat{J}$  is the total moment of inertia of our system.

$$\begin{aligned} \tau &= \hat{J}\ddot{q} + L_0 M_0 g \sin q + \frac{L_0}{2} m_0 g \sin q \\ &= \hat{J}\ddot{q} + \underbrace{\hat{B}\dot{q} (L_0 M_0 g + \frac{L_0}{2} m_0 g)}_{\hat{N}} \sin q \end{aligned} \tag{8.0.3}$$

$$\begin{aligned} K_\tau I &= \tau = \hat{J}\ddot{q} + \hat{B}\dot{q} + \hat{N} \sin q \\ I &= \frac{\hat{J}}{K_\tau} \ddot{q} + \frac{\hat{B}}{K_\tau} \dot{q} + \frac{\hat{N}}{K_\tau} \sin q \end{aligned}$$

letting  $M = \frac{\hat{J}}{K_\tau}$ ,  $B = \frac{\hat{B}}{K_\tau}$ ,  $N = \frac{\hat{N}}{K_\tau}$ , then (8.0.3) can be written as

$$\begin{aligned} I &= M\ddot{q} + B\dot{q} + N \sin q \\ \ddot{q} &= \frac{I}{M} - \frac{B}{M} \dot{q} - \frac{N}{M} \sin q \end{aligned} \tag{8.0.4}$$

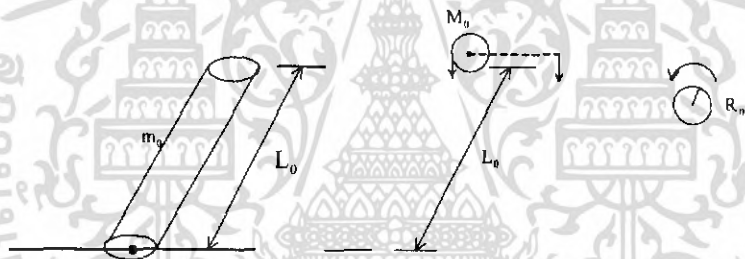


Figure 8.2: Moments of Inertia of rigid body

Moments of Inertia of long thin rod with rotation axis through end:  $I = \frac{1}{3} m_0 L_0^2$

Moments of Inertia of solid sphere:  $I = \frac{2}{5} M_0 R_0^2$

Moments of Inertia of point mass:  $I = M_0 L_0^2$

$$\hat{J} = J + \frac{m_0 L_0^2}{3} + M_0 L_0^2 + \frac{2}{5} M_0 R_0^2$$

From figure 8.3, applying KVL, we can obtain

$$V_e = IR + L\dot{i} + K_B \dot{q}$$

$$L\dot{i} = V_e - IR - K_B \dot{q}$$

$$\dot{i} = \frac{V_e}{L} - \frac{IR}{L} - \frac{K_B}{L} \dot{q}$$

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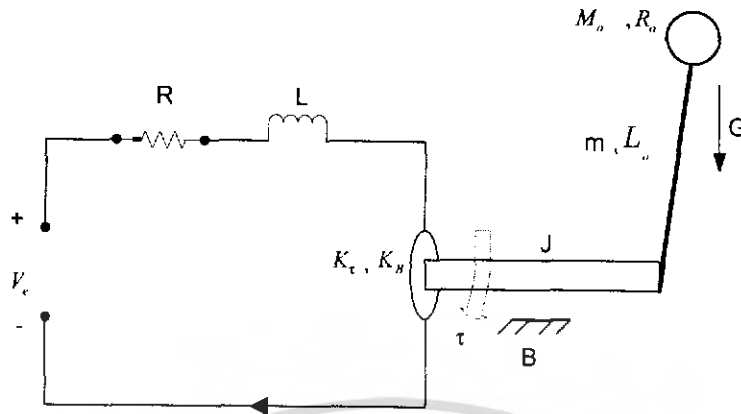


Figure 8.3: Schematic of electromechanical

where

- $L$  is the armature inductance.
- $R$  is the armature resistance.
- $K_B$  is the back-emf coefficient.
- $V_e$  is the input control voltage.

The electromechanical dynamics of the can be written as

$$\begin{aligned} \dot{q} &= \dot{q} \\ \ddot{q} &= \frac{I}{M} - \frac{N}{M} \sin q - \frac{B}{M} \dot{q} \\ \dot{i} &= \frac{V_e}{L} - \frac{K_B}{L} \dot{q} - \frac{R}{L} i. \end{aligned} \tag{8.0.5}$$

We will assume that all of the parametric quantities defined in (8.0.5) lie between known constant bounds. For example, the inductance  $L$  and the inertial quantity  $M$  are assume to be bounded in the following manner

$$m_1 \leq M \leq m_2 \quad l_1 \leq L \leq l_2$$

where  $m_1, m_2, l_1$  and  $l_2$  are known positive, scalar bounding constants. □

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# Chapter 9

## Derivation of $u$

The dynamic given by (8.0.5) can also be written in the following form

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= b_1 x_3 + \theta_1 \sin(x_1) + \theta_2 x_2 \\ \dot{x}_3 &= b_0 u + \theta_3 x_2 + \theta_4 x_3\end{aligned}\tag{9.0.1}$$

where  $x_1 = q$ ,  $x_2 = \dot{q}$ ,  $x_3 = I$ ,  $u = V_e$ ,  $b_0 = \frac{1}{L}$ ,  $b_1 = \frac{1}{M}$ ,  $\theta_1 = \frac{-N}{M}$ ,  $\theta_2 = \frac{-B}{M}$ ,  $\theta_3 = \frac{-K_B}{L}$ , and  $\theta_4 = \frac{-R}{L}$ .

The control objective is to develop a link position regulate controller for the electro-mechanical dynamics of (8.0.5) despite parametric uncertainty.

Beginning with we are trying to find a control law for the following second-order system, where the uncertain terms are “matched” by the virtual control  $x_3$ :

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= b_1 x_3 + \theta_1 \sin(x_1) + \theta_2 x_2.\end{aligned}\tag{9.0.2}$$

Since our aim is the position control, that is, we are trying to move  $x_1$  (angular position) to some set point  $x_{1d}$ .

We then first consider the scalar system

$$\dot{x}_1 = x_2\tag{9.0.3}$$

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with  $x_2$  as its virtual control. Next we form a control Lyapunov function  $V_0(x_1) = \frac{1}{2}(x_1 - x_{1d})^2$ , whose derivative is rendered negative definite via a stabilizing function

$$x_2 = \alpha_0(x_1) = -c_1(x_1 - x_{1d}) = x_{2d}, \quad c_1 > 0. \quad (9.0.4)$$

That is,

$$\begin{aligned} \dot{V}_0(x_1) &= (x_1 - x_{1d})\dot{x}_1 \\ &= -c_1(x_1 - x_{1d})^2. \end{aligned} \quad (9.0.5)$$

Since  $x_2$  is not our real control input, we let  $z_2 = x_2 - x_{2d}$  be an error variable. The system (9.0.2) in terms of error variables  $z_1 = x_1 - x_{1d}$ ,  $z_2 = x_2 - x_{2d} = x_2 - \alpha_0(x_1)$  is

$$\begin{aligned} \dot{z}_1 &= z_2 + \alpha_0(x_1) \\ \dot{z}_2 &= b_1 x_3 + \theta_1 \sin x_1 + \theta_2 x_2 + c_1 x_2. \end{aligned} \quad (9.0.6)$$

Now with the clf

$$\begin{aligned} V_1(x_1, x_2) &= \frac{1}{2}(x_1 - x_{1d})^2 + \frac{1}{2}(x_2 - x_{2d})^2 \\ &= \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 \end{aligned} \quad (9.0.7)$$

whose derivative is

$$\begin{aligned} \dot{V}_1 &= z_1 \dot{z}_1 + z_2 \dot{z}_2 \\ &= z_1(z_2 - c_1 z_1) + z_2(b_1 x_3 + \theta_1 \sin x_1 + \theta_2 x_2 + c_1 x_2). \end{aligned} \quad (9.0.8)$$

If  $\theta_1, \theta_2$  were known,  $\dot{V}_1$  in (9.0.8) can be rendered negative definite via the stabilizing control

$$x_3 = \frac{1}{b_1}[-z_1 - c_2 z_2 - \theta_1 \sin x_1 - \theta_2 x_2 - c_1 x_2], \quad (9.0.9)$$

that is,

$$\begin{aligned} \dot{V}_1 &= z_1 z_2 - c_1 z_1^2 + z_2(b_1 x_3 + \theta_1 \sin x_1 + \theta_2 x_2 + c_1 x_2) \\ &= z_1 z_2 - c_1 z_1^2 + z_2(-z_1 - c_2 z_2) \end{aligned}$$

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But since  $\theta_1, \theta_2$  are unknown. Then we replace  $\theta_1, \theta_2$  in (9.0.9) with parameter estimates  $\vartheta_1, \vartheta_2$ , respectively, and we augment  $V_1$  with quadratic terms in the parameter error  $(\theta_1 - \vartheta_1)$  and  $(\theta_2 - \vartheta_2)$  to obtain the new Lyapunov function:

$$\begin{aligned} V_1(x, \vartheta) &= V_1(x_1, x_2, \vartheta_1, \vartheta_2) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma_1}(\theta_1 - \vartheta_1)^2 + \frac{1}{2\gamma_2}(\theta_2 - \vartheta_2)^2 \\ &= \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2} \underbrace{\begin{bmatrix} \theta_1 - \vartheta_1 & \theta_2 - \vartheta_2 \end{bmatrix}}_{\Gamma_1^{-1}} \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}^{-1} \begin{bmatrix} \theta_1 - \vartheta_1 \\ \theta_2 - \vartheta_2 \end{bmatrix} \end{aligned} \quad (9.0.11)$$

Its derivative is

$$\begin{aligned} \dot{V}_1 &= -c_1z_1^2 - c_2z_2^2 + z_2 \underbrace{\begin{bmatrix} \sin x_1 & x_2 \end{bmatrix}}_{\hat{F}(x)} \begin{bmatrix} \theta_1 - \vartheta_1 \\ \theta_2 - \vartheta_2 \end{bmatrix} - \begin{bmatrix} \dot{\vartheta}_1 \\ \dot{\vartheta}_2 \end{bmatrix}^T \Gamma_1^{-1} \begin{bmatrix} \theta_1 - \vartheta_1 \\ \theta_2 - \vartheta_2 \end{bmatrix} \\ &= -c_1z_1^2 - c_2z_2^2 + (z_2\hat{F}(x) - \dot{\vartheta}^T\Gamma_1^{-1})(\theta - \vartheta). \end{aligned} \quad (9.0.12)$$

If  $x_3$  were our control input, then the adaptive control law

$$x_{3d} = \frac{1}{b_1}[-z_1 - c_2z_2 - \vartheta_1 \sin x_1 - \vartheta_2 x_2 - c_1x_2] = \alpha(x, \vartheta) \quad (9.0.13)$$

with the update law

$$\begin{aligned} \dot{\vartheta} &= z_2\Gamma_1\hat{F}(x)^T = z_2 \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \begin{bmatrix} \sin x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} z_2\gamma_1 \sin x_1 \\ z_2\gamma_2 x_2 \end{bmatrix} = T(x, \vartheta) \end{aligned} \quad (9.0.14)$$

would stabilize the system (9.0.2), and regulate the states  $x_1$  to  $x_{1d}$  and  $x_2$  to 0 by LaSalle-Yoshizawa theorem, since  $\dot{V}_1$  in (9.0.12) would become

$$\dot{V}_1 = -c_1z_1^2 - c_2z_2^2 = -W(x, \vartheta) \leq 0 \quad (9.0.15)$$

where  $W(x, \vartheta) = c_1z_1^2 + c_2z_2^2$  is positive semidefinite and (by LaSalle-Yoshizawa theorem (Theorem (A.1.1))) we have  $\lim_{t \rightarrow \infty} W(x, \vartheta) = 0$ .

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Now let us go back to the third-order system (9.0.1) which can be written as

$$\dot{x} = f(x) + F(x)\theta + g(x)y, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (9.0.16a)$$

$$\dot{x}_3 = m(x, x_3) + M(x, x_3)\theta + b_0u, \quad y = x_3 \quad (9.0.16b)$$

where  $f(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ ;  $F(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sin x_1 & x_2 & 0 & 0 \end{bmatrix}$ ;  $g(x) = \begin{bmatrix} 0 \\ b_1 \end{bmatrix}$ ;  $m(x, x_3) = 0$ ;

$$M(x, x_3) = \begin{bmatrix} 0 & 0 & x_2 & x_3 \end{bmatrix}; \theta = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix}^T.$$

That is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sin x_1 & x_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} + \begin{bmatrix} 0 \\ b_1 \end{bmatrix} \underbrace{x_3}_{y=x_3} \quad (9.0.17a)$$

$$\dot{x}_3 = \begin{bmatrix} 0 & 0 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} + b_0u, \quad y = x_3. \quad (9.0.17b)$$

We have already seen that (9.0.16a) has  $y$  as its virtual input, with the clf

$$V_1(x_1, x_2, \vartheta) = \frac{1}{2} \underbrace{(x_1 - x_{1d})^2}_{z_1} + \frac{1}{2} \underbrace{(x_2 - x_{2d})^2}_{z_2} + \frac{1}{2} [\theta - \vartheta]_{2 \times 1}^T \Gamma_1^{-1} [\theta - \vartheta]_{2 \times 1} \quad (9.0.18)$$

which can be rendered N.S.D., that is,

$$\begin{aligned} \dot{V}_1 &= \frac{\partial V_1}{\partial x} \dot{x} = \frac{\partial V_1}{\partial x} (f + F\theta + g x_3) \\ &= -c_1 z_1^2 - c_2 z_2^2 + \left[ z_2 \hat{F}(x) - \dot{\vartheta}^T \Gamma_1^{-1} \right] (\theta - \vartheta)_{2 \times 1} \\ &= -c_1 z_1^2 - c_2 z_2^2 \\ &= -W(x, \vartheta), \quad W(x, \vartheta) = c_1 z_1^2 + c_2 z_2^2, \end{aligned} \quad (9.0.19)$$

via the control law

$$x_{3d} = \alpha(x, \vartheta) = \frac{1}{2} \begin{bmatrix} -z_1 - c_2 z_2 - \underbrace{\vartheta_1 \sin x_1 - \vartheta_2 x_2}_{\hat{F}(x)\vartheta} - c_1 x_2 \end{bmatrix} \quad (9.0.20)$$

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and the adaptation law

$$\dot{\vartheta}_{2 \times 1} = \begin{bmatrix} \dot{\vartheta}_1 \\ \dot{\vartheta}_2 \end{bmatrix} = z_2 \Gamma_1^T \hat{F}(x)^T = z_2 \underbrace{\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}}_{\Gamma_1^T = \Gamma_1} \underbrace{\begin{bmatrix} \sin x_1 \\ x_2 \end{bmatrix}}_{\hat{F}(x)^T} = \begin{bmatrix} z_2 \gamma_1 \sin x_1 \\ z_2 \gamma_2 x_2 \end{bmatrix} = T(x, \vartheta). \quad (9.0.21)$$

In (9.0.16), if  $\theta$  were known we would pick  $u = \frac{1}{b_0} [v - M(x, \xi)\theta]$  so it can be written as

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)y \\ \dot{y} &= \dot{x}_3 = v \end{aligned} \quad (9.0.22)$$

where we could apply integrator backstepping. But since  $\theta$  is unknown, therefore we introduce a new parameter estimate  $\bar{\vartheta}$  with a new clf

$$V_2(x, y, \vartheta, \bar{\vartheta}) = V_1(x, \vartheta) + \frac{1}{2} [y - \alpha(x, \vartheta)]^2 + \frac{1}{2} (\theta - \bar{\vartheta})^T \Gamma_2^{-1} (\theta - \bar{\vartheta}). \quad (9.0.23)$$

Therefore, with  $z_3 = y - \alpha(x, \vartheta) = x_3 - \alpha(x, \vartheta)$ , we obtain the derivative of the clf  $V_2$  as

$$\begin{aligned} \dot{V}_2 &= \frac{\partial V_1}{\partial x} (f + F\theta + g\alpha + gz_3) + \frac{\partial V_1}{\partial \vartheta} T \\ &+ z_3 \left[ \underbrace{v + \frac{\partial h}{\partial x_3} (x_3) M(x, x_3) (\theta - \bar{\vartheta})}_{y=x_3} - \underbrace{\frac{\partial \alpha}{\partial x} (f + F\theta + g(\alpha + z_3)) - \frac{\partial \alpha}{\partial \vartheta} T}_{-\dot{\alpha}} \right] \\ &- \dot{\bar{\vartheta}}^T \Gamma_2^{-1} (\theta - \bar{\vartheta}) \\ &= \frac{\partial V_1}{\partial x} (f + F\theta + g\alpha) + \frac{\partial V_1}{\partial \vartheta} T \\ &+ z_3 \left[ v + \frac{\partial h}{\partial x_3} (x_3) M(x, x_3) (\theta - \bar{\vartheta}) - \frac{\partial \alpha}{\partial x} (f + F\theta + g(\alpha + z_3)) - \frac{\partial \alpha}{\partial \vartheta} T + \frac{\partial V_1}{\partial x} g \right] \\ &- \dot{\bar{\vartheta}}^T \Gamma_2^{-1} (\theta - \bar{\vartheta}) \\ &\leq -W(x, \vartheta) + z_3 \left[ v - \frac{\partial \alpha}{\partial x} (f + F\bar{\vartheta} + g(\alpha + z_3)) - \frac{\partial \alpha}{\partial \vartheta} T + \frac{\partial V_1}{\partial x} g \right] \\ &- \left[ \frac{\partial \alpha}{\partial x} F z_3 - \underbrace{\frac{\partial h}{\partial x_3} (x_3) M z_3 + \dot{\bar{\vartheta}}^T \Gamma_2^{-1}}_{\text{added}} (\theta - \bar{\vartheta}) \right] \\ &= -W(x, \vartheta) - c_3 z_3^2 \end{aligned} \quad (9.0.24)$$

through

$$v = -c_3 \underbrace{z_3}_{y-\alpha} + \frac{\partial \alpha}{\partial x} (f + F\bar{\vartheta} + g(\alpha + z_3)) + \frac{\partial \alpha}{\partial \vartheta} T - \frac{\partial V_1}{\partial x} g$$

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ไม่ว่ากรณีใดๆ ทั้งสิ้น อีกทั้งห้ามมิให้ดัดแปลงเนื้อหา และต้องอ้างอิงถึงเจ้าของเอกสารทุกครั้งที่มีการนำไปใช้

or

$$u = \left[ \frac{\partial h}{\partial x_3}(x_3)b_0 \right]^{-1} \left[ v - \frac{\partial h}{\partial x_3}(x_3)[m + M\bar{\vartheta}] \right]$$

and

$$\dot{\vartheta} = \Gamma \left[ \frac{\partial h}{\partial x_3}(x_3)M(x, x_3) - \frac{\partial \alpha}{\partial x}(x, \vartheta)F(x) \right]^T (y - \alpha(x, \vartheta)).$$

Therefore, the control law

$$u = \left[ \frac{\partial h}{\partial x_3}(x_3)b_0 \right]^{-1} \left\{ -c_3(y - \alpha(x, \vartheta)) - \frac{\partial h}{\partial x_3}(x_3) [m(x, x_3) + M(x, x_3)\bar{\vartheta}] + \frac{\partial \alpha}{\partial x}(x, \vartheta) [f(x) + F(x)\bar{\vartheta} + g(x)y] + \frac{\partial \alpha}{\partial \vartheta}T(x, \vartheta) - \frac{\partial V}{\partial x}(x, \vartheta)g(x) \right\}, \quad (9.0.25)$$

with the update law

$$\dot{\vartheta} = T(x, \vartheta) = \begin{bmatrix} z_2\gamma_1 \sin x_1 \\ z_2\gamma_2 x_2 \\ 0 \\ 0 \end{bmatrix} \quad (9.0.26)$$

$$\dot{\vartheta} = \Gamma_2 \left[ \frac{\partial h}{\partial x_3}(x_3)M(x, x_3) - \frac{\partial \alpha}{\partial x}(x, \vartheta)F(x) \right]^T (y - \alpha(x, \vartheta)) \quad (9.0.27)$$

will rendered global boundedness (GB) of  $\begin{bmatrix} z_1 & z_2 & z_3 & \theta - \bar{\vartheta} & \theta - \bar{\vartheta} \end{bmatrix}^T$  and regulation of  $W(x, \vartheta) = c_1 z_1^2 + c_2 z_2^2 = c_1(x_1 - x_{1d})^2 + c_2(x_2 - x_{2d})^2$  and  $x_3(t) - \alpha(x(t), \vartheta(t))$  by LaSalle-Yoshizawa theorem (Theorem (A.1.1)). That is,

$$x_1 \rightarrow x_{1d},$$

$$x_2 \rightarrow x_{2d} = \alpha_0(x_1) \rightarrow \alpha_0(x_{1d}) = 0,$$

$$x_3 \rightarrow \alpha(x, \vartheta),$$

$$\text{as } t \rightarrow \infty.$$

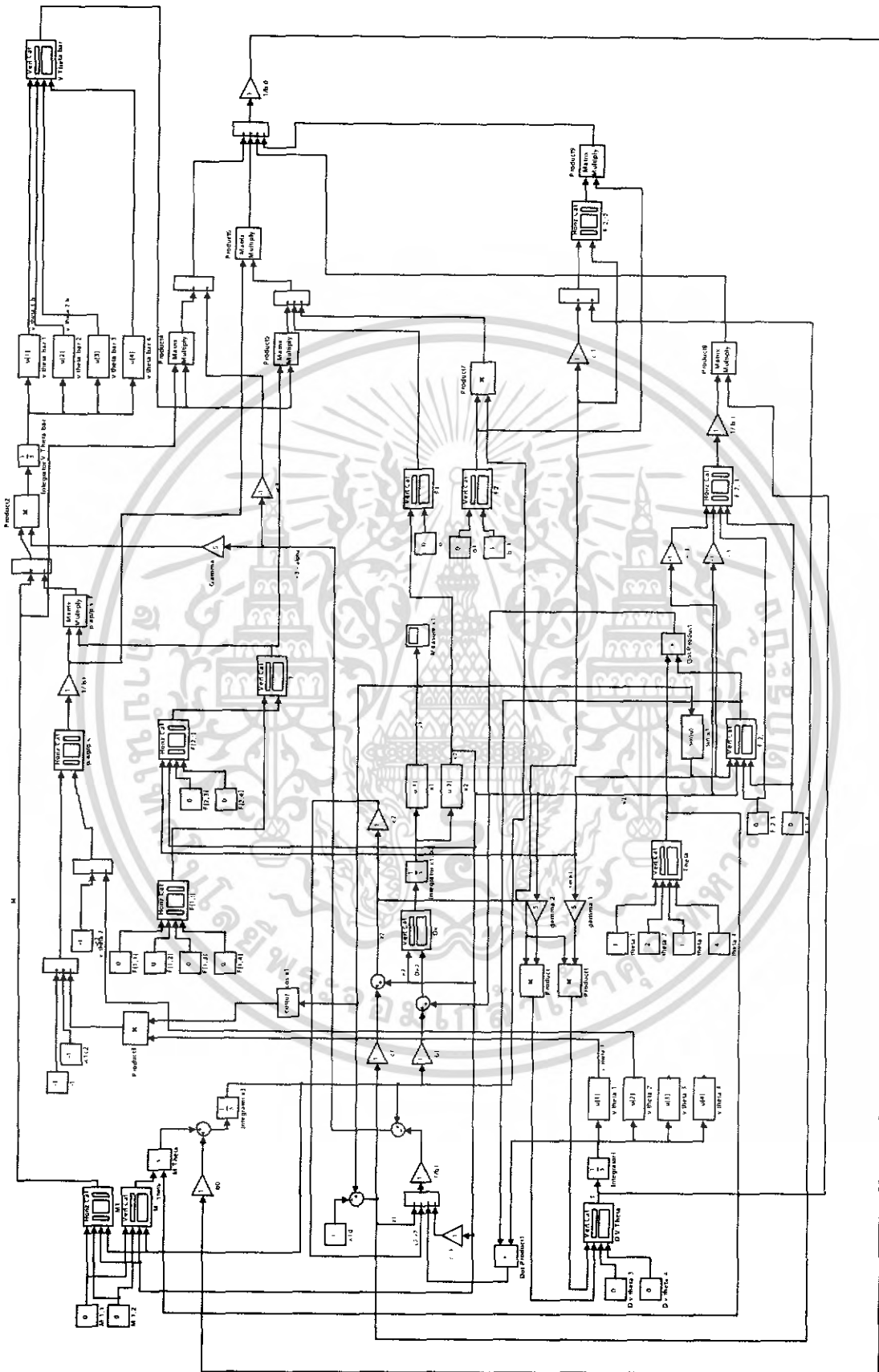
เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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Our  $u$  looks like

$$\begin{aligned}
 u = & \frac{1}{b_0} \left\{ -c_3(x_3 - \alpha(x, \vartheta)) - \begin{bmatrix} 0 & 0 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \bar{\vartheta}_1 \\ \bar{\vartheta}_2 \\ \bar{\vartheta}_3 \\ \bar{\vartheta}_4 \end{bmatrix} \right. \\
 & + \frac{1}{b_1} \left[ -1 - c_1 c_2 - \vartheta_1 \cos x_1 \quad -c_2 - \vartheta_2 \right] \times \\
 & \left( \underbrace{\begin{bmatrix} x_2 \\ 0 \end{bmatrix}}_f + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sin x_1 & x_2 & 0 & 0 \end{bmatrix}}_F \begin{bmatrix} \bar{\vartheta}_1 \\ \bar{\vartheta}_2 \\ \bar{\vartheta}_3 \\ \bar{\vartheta}_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ b_1 \end{bmatrix}}_g x_3 \right) \\
 & + \frac{1}{b_1} \underbrace{\begin{bmatrix} -\sin x_1 & -x_2 & 0 & 0 \end{bmatrix}}_{\frac{\partial \alpha}{\partial \vartheta}} \underbrace{\begin{bmatrix} z_2 \gamma_1 \sin x_1 \\ z_2 \gamma_2 x_2 \\ 0 \\ 0 \end{bmatrix}}_T \\
 & - \underbrace{\begin{bmatrix} z_1 + c_1 z_2 \\ z_2 \end{bmatrix}}_{\frac{\partial V}{\partial z}} \underbrace{\begin{bmatrix} 0 \\ b_1 \end{bmatrix}}_g \left. \right\}. \tag{9.0.28}
 \end{aligned}$$

This is similar to Lemma 6.2.2 (Adaptive Block Backstepping) with no zero dynamics system. □

เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับการใช้งานเพื่อการศึกษาเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
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เอกสารนี้เป็นเอกสารที่สงวนไว้สำหรับครูใช้งานเพื่อการศึกษเท่านั้น ไม่อนุญาตให้นำไปใช้ประโยชน์ด้านการค้า  
 Figure 9.1: MATLAB Simulink simulation of the system.  
 ไม่ว่ากรรมใดๆ ทั้งสิ้น อีกทั้งห้ามมิให้ตัดแปลงเนื้อหา และต้องอ้างอิงถึงเจ้าของเอกสารทุกครั้งที่มีการนำไปใช้

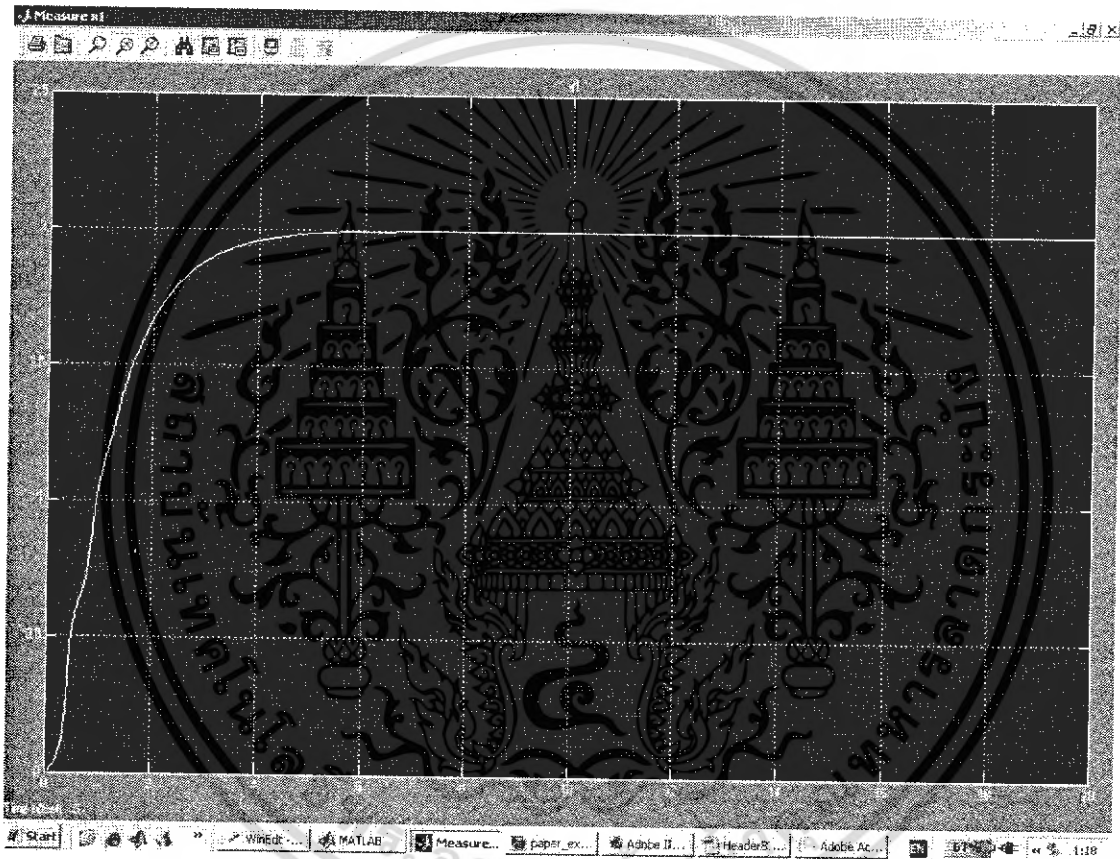


Figure 9.2: Result from MATLAB Simulink.

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# Appendix A

## Stability

### A.1 Main Stability Theorems

**Lyapunov Stability.** To begin with, we remind reader that Lyapunov stability, asymptotic stability, uniform stability, uniform asymptotic stability, etc., are properties not of a dynamic system as a whole, but rather of its individual solutions. Consider the time-varying system

$$\dot{x} = f(x, t), \quad (\text{A.1.1})$$

where  $x \in \mathbb{R}^n$ , and  $f : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is piecewise continuous in  $t$  and locally Lipschitz in  $x$ . The solution of (A.1.1) which starts from the point  $x_0$  at time  $t_0 \geq 0$  is denoted as  $x(t; x_0, t_0)$  with  $x(t_0; x_0, t_0) = x_0$ . Lyapunov stability concepts describe continuity properties of  $x(t; x_0, t_0)$  with respect to  $x_0$ . If the initial condition  $x_0$  is perturbed to  $\tilde{x}_0$ , then, for stability, the resulting perturbed solution  $x(t; \tilde{x}_0, t_0)$  is required to stay close to  $x(t; x_0, t_0)$  for all  $t \geq t_0$ . In addition, for asymptotic stability, the error  $x(t; \tilde{x}_0, t_0) - x(t; x_0, t_0)$  is required to vanish as  $t \rightarrow \infty$ . So, the solution  $x(t; x_0, t_0)$  of (A.1.1) is

- **bounded**, if there exist a constant  $B(x_0, t_0) > 0$  such that

$$|x(t; x_0, t_0)| < B(x_0, t_0), \quad \forall t \geq t_0; \quad (\text{A.1.2})$$

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- **stable**, if for each  $\varepsilon > 0$  there exists a  $\delta(\varepsilon, t_0) > 0$  such that

$$|\tilde{x}_0 - x_0| < \delta \Rightarrow |x(t; \tilde{x}_0, t_0) - x(t; x_0, t_0)| < \varepsilon, \quad \forall t \geq t_0; \quad (\text{A.1.3})$$

- **attractive**, if there exists an  $r(t_0) > 0$  and, for each  $\varepsilon > 0$ , a  $T(\varepsilon, t_0) > 0$  such that

$$|\tilde{x}_0 - x_0| < r \Rightarrow |x(t; \tilde{x}_0, t_0) - x(t; x_0, t_0)| < \varepsilon, \quad \forall t \geq t_0 + T; \quad (\text{A.1.4})$$

- **asymptotically stable**, if it is stable and attractive; and
- **unstable**, if it is not stable.

The stability properties of  $x(t; x_0, t_0)$  in general depend on the initial time  $t_0$ . For different  $t_0$ , different values of  $B(x_0, t_0)$ ,  $\delta(\varepsilon, t_0)$ ,  $r(t_0)$ , and  $T(\varepsilon, t_0)$  may be needed to satisfy (A.1.2), (A.1.3) and (A.1.4). When these constants are independent of  $t_0$ , the corresponding properties are *uniform* (Clearly, all properties are uniform if the system is time-invariant:  $\dot{x} = f(x)$ ). For adaptive systems, *uniform stability* is more desirable than just stability. Even more desirable is **uniform asymptotic stability**, often shortened to UAS. The solution  $x(t; x_0, t_0)$  is UAS if it is *uniformly stable and uniformly attractive*, that is, if  $\delta(\varepsilon, t_0) = \delta(\varepsilon)$ ,  $r(t_0) = r$ , and  $T(\varepsilon, t_0) = T(\varepsilon)$  do not depend on  $t_0$ .

Some solutions of a given system may be stable and others unstable. In particular, (A.1.1) may have stable and unstable **equilibria**, that is constant solutions  $x(t; x_0, t_0) \equiv x_e$  satisfying  $f(x_e, t) \equiv 0$ . If an equilibrium  $x_e$  is asymptotically stable, then it has a **region of attraction** — a set  $\Omega$  of initial states  $x_0$  such that  $x(t; x_0, t_0) \rightarrow x_e$  as  $t \rightarrow \infty$  for all  $x_0 \in \Omega$  (When  $x_e$  is only stable, then the solutions starting in  $\Omega$  remain close to  $x_e$  in the sense of (A.1.3)). In this report, the stability properties for which an estimate of the region of attraction is given are referred to as **regional**. Otherwise they are called *local*. When the region of attraction is the whole space  $\mathbb{R}^n$ , then the stability properties are **global**.

Any equilibrium under investigation can be translated to the origin by redefining the state  $x$  as  $z = x - x_e$ . Such a translation  $z = x - x(t; x_0, t_0)$  can be defined for any

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*formly bounded and satisfy*

$$\lim_{t \rightarrow \infty} W(x(t)) = 0. \quad (\text{A.1.7})$$

*In addition, if  $W(x)$  is positive definite, then the equilibrium  $x = 0$  is **globally uniformly asymptotically stable (GUAS)**.*

For regulation task, the designed system is usually time-invariant,

$$\dot{x} = f(x), \quad (\text{A.1.8})$$

in which case we are interested in its **invariant sets**. A set  $M$  is called an invariant set of (A.1.8) if any solution  $x(t)$  that belong to  $M$  at some time constant  $t_1$  must belong to  $M$  for all future and past time:

$$x(t_1) \in M \Rightarrow x(t) \in M, \quad \forall t \in \mathbb{R}. \quad (\text{A.1.9})$$

A set  $\Omega$  is **positively invariant** if this is true for all future time only:

$$x(t_1) \in \Omega \Rightarrow x(t) \in \Omega, \quad \forall t \geq t_1. \quad (\text{A.1.10})$$

Can we guarantee convergence to a desired invariant set? A rewarding answer to this equation is provided by LaSalle's Invariance Theorem and its asymptotic stability corollary:

**Theorem A.1.2 (LaSalle)** *Let  $\Omega$  be a positively invariant set of (A.1.8). Let  $V : \Omega \rightarrow \mathbb{R}_+$  be a continuously differentiable function  $V(x)$  such that  $\dot{V}(x) \leq 0, \forall x \in \Omega$ . Let  $E = \{x \in \Omega | \dot{V}(x) = 0\}$ , and let  $M$  be the largest invariant set contained in  $E$ . Then, every bounded solution  $x(t)$  starting in  $\Omega$  converges to  $M$  as  $t \rightarrow \infty$ .*

**Corollary A.1.3 (Asymptotic Stability)** *Let  $x = 0$  be the only equilibrium of (A.1.8).*

*Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be a continuously differentiable, positive definite, radially unbounded*

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function  $V(x)$  such that  $\dot{V}(x) \leq 0, \forall x \in \mathbb{R}^n$ . Let  $E = \{x \in \mathbb{R}^n | \dot{V}(x) = 0\}$ , and suppose that no solution other than  $x(t) \equiv 0$  can stay forever in  $E$ . Then the origin is globally asymptotically stable (GAS).

These invariance results will motivate us to closely examine the invariant subsets of  $E$ . As we shall see, the convergence properties of the designed system are stronger if the dimension of  $M$  is lower. In the most favorable case of asymptotic stability, the largest invariant subset  $M$  of  $E$  is just the origin  $x = 0$ . Our aim will thus be render the dimension of  $M$  as low as possible.

**Input-to-State Stability.** Another stability concept which is used throughout the report is that of input-to-state stability (ISS), the system

$$\dot{x} = f(x, u) \quad (\text{A.1.11})$$

is said to be *input-to-state stable (ISS)* if for any  $x(0)$  and for any input  $u(\cdot)$  continuous and bounded on  $[0, \infty)$  the solution exists for all  $t \geq 0$  and satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma\left(\sup_{0 \leq \tau \leq t} |u(\tau)|\right), \quad \forall t \geq 0, \quad (\text{A.1.12})$$

where  $\beta(s, t)$  and  $\gamma(s)$  are strictly increasing functions of  $s \in \mathbb{R}_+$  with  $\beta(0, t) = 0, \gamma(0) = 0$ , while  $\beta$  is a increasing function of  $t$  with  $\lim_{t \rightarrow \infty} \beta(s, t) = 0, \forall s \in \mathbb{R}_+$ .

## A.2 Lyapunov's Direct Method

The basic philosophy of Lyapunov's direct method is the mathematical extension of a fundamental physical observation: if the total *energy* of a mechanical (or electrical) system is continuously dissipated, then the system, *whether linear or nonlinear*, must eventually settle down to an equilibrium point. Thus, we may conclude the stability of a system by examining the variation of a single *scalar* function.

The total mechanical energy of the nonlinear mass-spring-damper system

$$m\ddot{x} + b\dot{x} + k_0x + k_1x^3 = 0 \quad (\text{A.2.1})$$

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is the sum of its kinetic energy and its potential energy.

$$V(x) = \frac{1}{2}m\dot{x}^2 + \int_0^x (k_0s + k_1s^3)ds = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0x^2 + \frac{1}{4}k_1x^4 \quad (\text{A.2.2})$$

Comparing the definitions of stability and mechanical energy, one can easily see some relations between the mechanical energy and the stability concepts described earlier:

- zero energy corresponds to the equilibrium point ( $x = 0, \dot{x} = 0$ )
- asymptotic stability implies the convergence of mechanical energy to zero
- instability is related to the growth of mechanical energy

These relations indicate that the value of a scalar quantity, the mechanical energy, indirectly reflects the magnitude of the state vector: and furthermore, that the stability properties of the system can be characterized by the variation of the mechanical energy of the system.

The rate of the energy variation during the system's motion is obtained easily by differentiating the first equality in (A.2.2)

$$\dot{V}(x) = m\dot{x}\ddot{x} + k_0x + k_1x^3\dot{x} = \dot{x}(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3 \quad (\text{A.2.3})$$

The direct method of Lyapunov is based on a generalization of the concept in the mass-spring-damper system to more complex system. Faced with a set of nonlinear differential equations, the basic procedure of Lyapunov's direct method is to generate a scalar "energy-like" function for the dynamic system, and examine the time variation of that scalar function. In this way, conclusions may be drawn on the stability of the set of differential equations without using the difficult stability definitions or requiring explicit knowledge of solutions.

**Theorem A.2.1 (Local Stability)** *if, in a ball  $B_{R_0}$ , there exists a scalar function  $V(x)$  with continuous first partial derivatives such that*

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- $V(x)$  is positive definite (locally in  $\mathbf{B}_{R_0}$ )
- $\dot{V}(x)$  is negative semi-definite (locally in  $\mathbf{B}_{R_0}$ )

then the equilibrium point  $\mathbf{x}=\mathbf{0}$  is stable. If, actually, the derivative  $\dot{V}(x)$  is locally negative definite in  $\mathbf{B}_{R_0}$ , then the stability is asymptotic.

**Theorem A.2.2 (Global Stability)** assume that there exists a scalar function  $V$  of the state  $\mathbf{x}$ , with continuous first order derivatives such that

- $V(x)$  is positive definite
- $\dot{V}(x)$  is negative definite
- $V(x) \rightarrow \infty$  as  $\|x\| \rightarrow \infty$

then the equilibrium at the origin is globally asymptotically stable.

### A.3 Lyapunov Stability

Consider the non-autonomous system

$$\dot{x} = f(x, t) \tag{A.3.1}$$

**Definition A.3.1** The origin  $x = 0$  is equilibrium point for (A.3.1) if

$$f(0, t) = 0, \forall t \geq 0. \tag{A.3.2}$$

**Definition A.3.2** A continuous function  $\gamma : [0, a) \rightarrow \mathbb{R}_+$  is said to belong to class  $K$  if it is strictly increasing and  $\gamma(0) = 0$ . It is said to belong to class  $K_\infty$  if  $a = \infty$  and  $\gamma(r) \rightarrow \infty$  as  $r \rightarrow \infty$ .

**Definition A.3.3** A continuous function  $\beta : [0, a) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to class  $KL$  if for each fixed  $s$  the mapping  $\beta(r, s)$  belong to class  $K$  with respect to  $r$ , and for each fixed  $r$  the mapping  $\beta(r, s)$  is decreasing with respect to  $s$  and  $\beta(r, s) \rightarrow 0$  as  $s \rightarrow \infty$ . It is said to belong to class  $KL_\infty$  if, in addition, for is fixed  $s$  the mapping  $\beta(r, s)$  belong to class  $K_\infty$  with respect to  $r$ .

**Definition A.3.4** The equilibrium point  $x = 0$  of (A.3.1) is

- uniformly stable, if there exists a class  $K$  function  $\gamma(\cdot)$  and a positive constant  $c$ , independent of  $t_0$ , such that

$$|x(t)| \leq \gamma(|x(t_0)|), \quad \forall t \geq t_0 \geq 0, \quad \forall x(t_0) \mid |x(t_0)| < c; \quad (\text{A.3.3})$$

- uniformly asymptotically stable, if there exist a class  $KL$  function  $\beta(\cdot, \cdot)$  and a positive constant  $c$ , independent of  $t_0$ , such that

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0), \quad \forall t \geq t_0 \geq 0, \quad \forall x(t_0) \mid |x(t_0)| < c; \quad (\text{A.3.4})$$

- exponentially stable, if (A.3.4) is satisfied with  $\beta(r, s) = kre^{\alpha s}$ ,  $k > 0, \alpha > 0$ ;
- globally uniformly stable, if (A.3.3) is satisfied with  $\gamma \in K_\infty$  for any initial state  $x(t_0)$ ;
- globally uniformly asymptotically stable, if (A.3.4) is satisfied with  $\beta \in KL_\infty$  for any initial state  $x(t_0)$ ; and
- globally exponentially stable, if (A.3.4) is satisfied for any initial state  $x(t_0)$  and with  $\beta(r, s) = kre^{-\alpha s}$ ,  $k > 0, \alpha > 0$ .

**Lemma A.3.5 (Barbalat)** Consider the function  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ . If  $\phi$  is uniformly continuous and  $\lim_{t \rightarrow \infty} \int_0^t \phi(\tau) d\tau$  exists and is finite, then

$$\lim_{t \rightarrow \infty} \phi(t) = 0 \quad (\text{A.3.5})$$

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# Appendix B

## Backstepping

### B.1 Integrator Backstepping

The simplicity of scalar designs motivates us to use them as starting points of recursive designs for higher-order systems. Consider a scalar system as in (B.1.1a) augmented with an integrator:

$$\dot{x}_1 = \cos x - x^3 + \xi \quad (\text{B.1.1a})$$

$$\dot{\xi} = u. \quad (\text{B.1.1b})$$

Let the design objective be the regulation of  $x(t)$ , that is,  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ , for all  $x(0), \xi(0)$ . Of course,  $\xi(t)$  must remain bounded. From (B.1.1a), the only equilibrium with  $x = 0$  is at  $(x, \xi) = (0, -1)$ . We will meet our design objective by rendering this equilibrium GAS. In the block diagram in Figure B.1 the scalar system (B.1.1a) appears in the dashed box. To construct a clf<sup>1</sup> we will exploit the fact that a clf is known for its subsystem in the dashed box. Indeed, if  $\xi$  were the control input, then the corresponding clf and control law would be  $V(x) = \frac{1}{2}x^2$  and  $\xi = -c_1x - \cos x$  clf. Of course  $\xi$  is just a state variable and not the control. Nevertheless, as its “desired value” we prescribe

$$\xi_{dcs} = -c_1x - \cos x \triangleq \alpha(x). \quad (\text{B.1.2})$$

<sup>1</sup>A system for which a good choice of  $V(x)$  and  $W(x)$  (as in Assumption ??) is said to possess a clf. คำว่ากรณีนี้อาจใช้แทนที่คำอื่น ๆ ได้ เช่น คำว่า “Lyapunov function” หรือ “energy function” ก็ได้

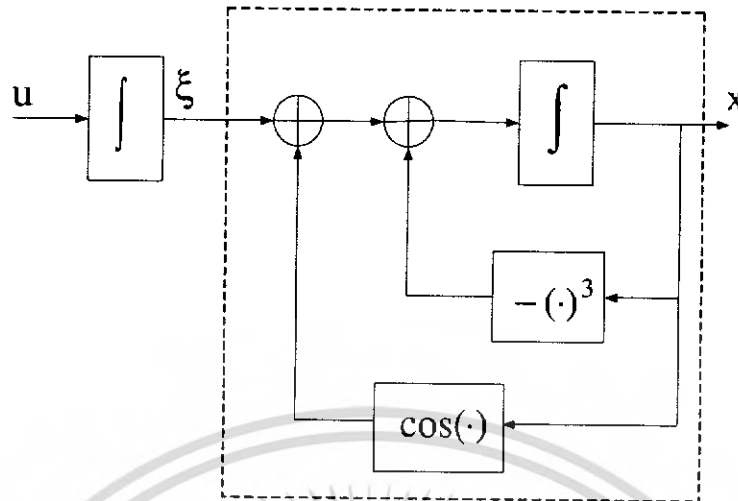


Figure B.1: The block diagram of the system (B.1.1a), (B.1.1b)

Let  $z$  be the deviation of  $\xi$  from its desired value:

$$z = \xi - \xi_{des} = \xi - \alpha(x) = \xi + c_1 x + \cos x. \quad (\text{B.1.3})$$

We call  $\xi$  a *virtual control*, and its desired value  $\alpha(x)$  a *stabilizing function*. The variable  $z$  is the corresponding *error variable*. Now we rewrite the system (B.1.1) in the  $(x, z)$ -coordinates in which it takes on a more convenient form, as illustrated in Figure B.2 and Figure B.3. Starting from (B.1.1) and Figure B.1, we add and subtract the stabilizing function  $\alpha(x)$  to the  $\dot{x}_1$ -equation as shown in Figure B.2. Then we use  $\alpha(x)$  as the feedback control inside the dashed box and “backstep”  $-\alpha(x)$  through the integrator, as in Figure B.3. In the new coordinates  $(x, z)$  the system is expressed as

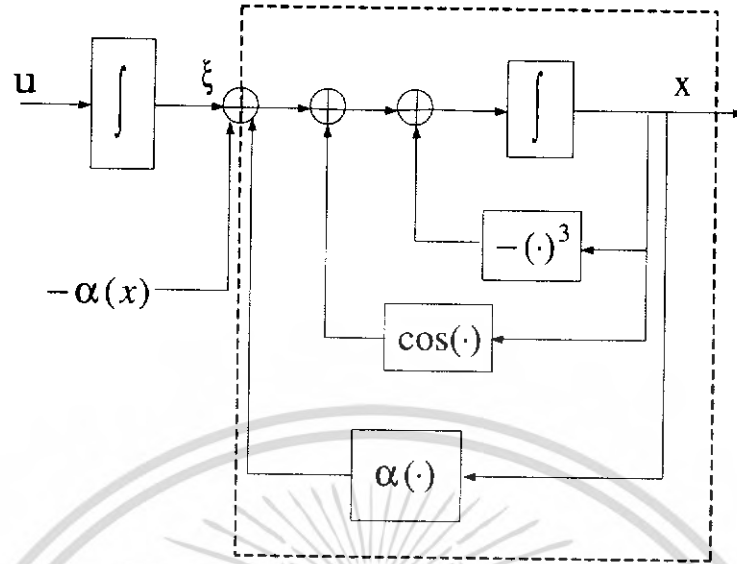
$$\dot{x} = \cos x - x^3 + [\xi + c_1 x + \cos x] - c_1 x - \cos x = -c_1 x - x^3 + z \quad (\text{B.1.4a})$$

$$\dot{z} = \dot{\xi} - \dot{\alpha} = \dot{\xi} + (c_1 - \sin x)\dot{x} = u + (c_1 - \sin x)(-c_1 x - x^3 + z). \quad (\text{B.1.4b})$$

The first key feature of backstepping is that we don’t use a differentiator to implement the time derivative  $\dot{\alpha}$  in (B.1.4b); since  $\alpha(x)$  is a known function, it is easy to compute its time derivative analytically as

$$\dot{\alpha} = \frac{\partial \alpha}{\partial x} \dot{x} = (c_1 - \sin x)(-c_1 x - x^3 + z). \quad (\text{B.1.5})$$

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Figure B.2: Introducing  $\alpha(x)$  as the desired value for  $\xi$ .

We now need to select a clf  $V_a$  for the system (B.1.1). Let us try to construct it by augmenting  $V(x)$  with a quadratic term in the error variable  $z$ :

$$V_a(x, \xi) = V(x) + \frac{1}{2}z^2 = \frac{1}{2}x^2 + \frac{1}{2}(\xi + c_1x + \cos x)^2. \quad (\text{B.1.6})$$

The derivative of  $V_a$  along the solution of (B.1.4) is computed as

$$\begin{aligned} \dot{V}_a(x, z, u) &= x[-c_1x - x^3 + z] + z[u + (c_1 - \sin x)(-c_1x - x^3 + z)] \\ &= -c_1x^2 - x^4 + z[x + u + (c_1 - \sin x)(-c_1x - x^3 + z)]. \end{aligned} \quad (\text{B.1.7})$$

As always, we let  $\dot{V}_a$  be an explicit function of  $u$  and design  $u$  to satisfy the clf inequality. For this reason, the cross-term  $xz$ , which is due to the presence of  $z$  in (B.1.4a), is grouped together with  $u$ . This is possible because  $u$  is also multiplied by  $z$  due to the chosen form of  $V_a$ . This is the second key feature of backstepping. Now we choose the control  $u$  to make  $\dot{V}_a$  negative definite in  $x$  and  $z$ . The simplest way to achieve this is to make the bracketed term in (B.1.7) equal to  $-c_2z^2$ , where  $c_2 > 0$ :

$$\begin{aligned} u &= -c_2z - x - (c_1 - \sin x)(-c_1x - x^3 + z) \\ &= -c_2(\xi + c_1x + \cos x) - x - (c_1 - \sin x)(\xi + \cos x - x^3). \end{aligned} \quad (\text{B.1.8})$$

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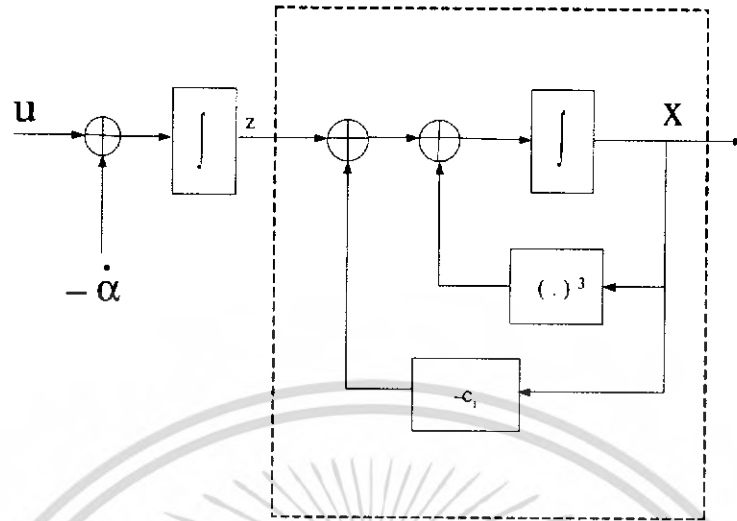


Figure B.3: Closing the feedback loop in the dashed box with  $+\alpha$  and “backstepping”  $-\dot{\alpha}$  through the integrator.

With this control, the clf derivative is

$$\dot{V}_a = -c_1 x^2 - c_2 z^2, \tag{B.1.9}$$

which proves that in the  $(x, z)$ -coordinates the equilibrium  $(0, 0)$  is GAS. In view of (B.1.3), the equilibrium  $(0, -1)$  in the  $(x, \xi)$ -coordinates has the same property.

The resulting closed-loop system in the  $(x, z)$ -coordinates is

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -c_1 - x^2 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{B.1.10}$$

Although written in a linear-like form, this system is nonlinear. An important structural property of this system is that its nonlinear “system matrix” is the sum of a *negative diagonal* and a *skew-symmetric* matrix function of  $x$ . This is the third key feature of backstepping, which will be extremely useful in other designs.

**Avoiding cancelations.** The above control law is not the best way to achieve negativity of  $\dot{V}_a$ , because it involves at least one unnecessary cancellation. A closer examination of (B.1.7) reveals that the term  $-z^2 \sin x$  need not be canceled because it can be dominated by  $-c_2 z^2$ . A control law which avoids this cancellation is

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With this control, the clf derivative is

$$\dot{V}_a = -c_1x^2 - x^4 - (c_2 - c_1 + \sin x)z^2. \quad (\text{B.1.12})$$

Although more complicated than (B.1.9), this function is easily rendered negative definite by the choice  $c_2 > c_1 + 1$ . The resulting system in the  $(x, z)$ -coordinates preserves its skew-symmetric form

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -c_1 - x^2 & 1 \\ -1 & -c_2 + c_1 - \sin x \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \quad (\text{B.1.13})$$

The simplified control law (B.1.11) is an illustration of design flexibilities in satisfying the clf inequality  $\dot{V}_a \leq 0$  and at the same time avoiding unnecessary cancelations. In fact, more detailed calculations show that the control law can be further simplified to

$$u = -k_1z - k_2x^2z, \quad (\text{B.1.14})$$

with

$$k_1 > c_2 + c_1 + 1 + \frac{(c_1^2 + c_1 + 1)^2}{2c_1}, \quad k_2 \geq \frac{(c_1 + 1)^2}{4}. \quad (\text{B.1.15})$$

Using this control we obtain

$$\dot{V}_a \leq -\frac{1}{2}c_1x^2 - c_2z^2. \quad (\text{B.1.16})$$

Integrator backstepping as a general design tool is based on the following assumption:

**Assumption B.1.1** Consider the system

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0, \quad (\text{B.1.17})$$

where  $x \in \mathbb{R}^n$  is the state and  $u \in \mathbb{R}$  is the control input. There exist a continuously differentiable feedback control law

$$u = \alpha(x), \quad \alpha(0) = 0, \quad (\text{B.1.18})$$

and a smooth, positive definite, radially unbounded function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\frac{\partial V}{\partial x}(x)[f(x) + g(x)\alpha(x)] \leq -W(x) \leq 0, \quad \forall x \in \mathbb{R}^n, \quad (\text{B.1.19})$$

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where  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  is positive semidefinite.

Under this assumption, the control (B.1.18), applied to the system (B.1.17), guarantees global boundedness of  $x(t)$ , and via the LaSalle-Yoshizawa theorem (Theorem A.1.1), the regulation of  $W(x(t))$ :

$$\lim_{t \rightarrow \infty} W(x(t)) = 0. \quad (\text{B.1.20})$$

A strong convergence result is obtained using LaSalle's theorem (Theorem A.1.2) with  $\Omega = \mathbb{R}^n : x(t)$  converges to the largest invariant set  $M$  contained in the set  $E = \{x \in \mathbb{R}^n | W(x) = 0\}$ . Clearly, if  $W(x)$  is positive definite, the control (B.1.18) renders  $x = 0$  the GAS equilibrium of (B.1.17)

**Lemma B.1.2 (Integrator Backstepping)** *Let the system (B.1.17) be augmented by an integrator:*

$$\dot{x} = f(x) + g(x)\xi \quad (\text{B.1.21a})$$

$$\dot{\xi} = u, \quad (\text{B.1.21b})$$

and suppose that (B.1.21a) satisfies Assumption (B.1.1) with  $\xi \in \mathbb{R}$  as its control.

(i) *If  $W(x)$  is positive definite, then*

$$V_a(x, \xi) = V(x) + \frac{1}{2} [\xi - \alpha(x)]^2 \quad (\text{B.1.22})$$

is a clf for the full system (B.1.21), that is, there exists a feedback control  $u = \alpha_a(x, \xi)$  which renders  $x = 0, \xi = 0$  the GAS equilibrium of (B.1.21). One such control is

$$u = -c(\xi - \alpha(x)) + \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\xi] - \frac{\partial V}{\partial x}(x)g(x), \quad c > 0. \quad (\text{B.1.23})$$

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(ii) If  $W(x)$  is only positive semidefinite, then there exists a feedback control which renders  $\dot{V}_a \leq -W_a(x, \xi) \leq 0$  such that  $W_a(x, \xi) > 0$  whenever  $W(x) > 0$  or  $\xi \neq \alpha(x)$ . This guarantees global boundedness and convergence of  $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$  to the largest invariant set

$$M_a \text{ contained in the set } E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+1} \mid W(x) = 0, \xi = \alpha(x) \right\}.$$

□

**Proof** Introducing the error variable

$$z = \xi - \alpha(x), \quad (\text{B.1.24})$$

and differentiating<sup>2</sup> with respect to time, (B.1.21) is rewritten as

$$\dot{x} = f(x) + g(x)[\alpha(x) + z] \quad (\text{B.1.25a})$$

$$\dot{z} = u - \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)(\alpha(x) + z)] \quad (\text{B.1.25b})$$

using (B.1.19), the derivative of (B.1.22) along the solution of (B.1.25) is

$$\begin{aligned} \dot{V}_a &= \frac{\partial V}{\partial x}(f + g\alpha + gz) + z \left[ u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) \right] \\ &= \frac{\partial V}{\partial x}(f + g\alpha) + z \left[ u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) + \frac{\partial V}{\partial x}g \right] \\ &\leq -W(x) + z \left[ u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) + \frac{\partial V}{\partial x}g \right], \end{aligned} \quad (\text{B.1.26})$$

where the term containing  $z$  as a factor have been grouped together. By the LaSalle-Yoshizawa theorem (Theorem A.1.1), any choice of the control  $u$  which renders  $\dot{V}_a \leq -W_a(x, \xi) \leq -W(x)$ , with  $W_a$  positive definite in  $z = \xi - \alpha(x)$ , guarantee global boundedness of  $x, z$  and  $\xi = z + \alpha(x)$ , and regulation of  $W(x)$  and  $z(t)$ . Furthermore, LaSalle's theorem (Theorem A.1.2) guarantees convergence of  $\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$  to the largest invariant set

<sup>2</sup>Once again, note that the time derivative  $\dot{\alpha}$  in (B.1.25b) is implemented analytically without the need for a differentiator.   
 ๒. อีกครั้งหนึ่ง โปรดทราบว่า การหาอนุพันธ์ของ  $\alpha$  ใน (B.1.25b) นี้ได้ดำเนินการขึ้นโดยไม่จำเป็นต้องใช้ตัวแยกค่าอนุพันธ์   
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contained in the set  $\left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{n+1} \mid W(x) = 0, z = 0 \right\}$ . Again, the simplest way to make  $\dot{V}_a$  negative definite in  $z$  is to choose the control (B.1.23), which renders the bracketed term in (B.1.26) equal to  $-cz$  and yields

$$\dot{V}_a \leq -W(x) - cz^2 \triangleq -W_a(x, \xi) \leq 0 \quad (\text{B.1.27})$$

Clearly, if  $W(x)$  is positive definite. Theorem A.1.1 guarantees the global asymptotic stability of  $x = 0, z = 0$ , which in turn implies that  $V_a(x, \xi)$  is a clf and  $x = 0, \xi = 0$  is the GAS equilibrium of (B.1.21).

While the choice of control (B.1.23) is simple, this control may not be desirable because it involves cancelation of nonlinearities, some of which may be useful. As illustrated by (B.1.8) and (B.1.9), the requirement that  $\dot{V}_a$  in (B.1.26) be made negative by  $u$  allows considerable freedom in the choice of control law  $u = \alpha_a(a, \xi)$  such that

$$\dot{V}_a \leq -W(x) + z[\alpha_a(x, \xi) - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) + \frac{\partial V}{\partial x}g] = -W_a(x, \xi) \leq 0. \quad (\text{B.1.28})$$

We stress that the main result of backstepping is not the specific form of the control law (B.1.23), but rather the construction of a Lyapunov function whose derivative can be made negative by a wide variety of control laws. In this way, the design of a stabilizing state-feedback controller is effectively reduced to satisfying the scalar inequality (B.1.28).

**Example B.1.3** *As a design tool, backstepping is less restrictive than feedback linearization. In some situations it can overcome singularities such as lack of controllability. This is illustrated by the system*

$$\dot{x} = x\xi \quad (\text{B.1.29})$$

$$\dot{\xi} = u, \quad (\text{B.1.30})$$

which is uncontrollable at  $x = 0$ . Comparing with (B.1.21), we see that  $f(x) = 0, g(x) = x$ . Applying (B.1.2) with  $V(x) = \frac{1}{2}x^2$  we can choose

เอกสารนี้เป็นเอกสารที่สงวนไว้  $\alpha(x) = -x^2$ , และ  $z = \xi - \alpha(x) = \xi + x^2$ , อนุญาตให้นำไปใช้ประโยชน์ (B.1.31) รัค้ำ  
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so that  $W(x)$  in (B.1.19) is positive definite:  $W(x) = x^4$ . The substitution of (B.1.31) into (B.1.29) yields

$$\dot{x} = -x^3 + xz \quad (\text{B.1.32a})$$

$$\dot{z} = u + 2x^2(z - x^2), \quad (\text{B.1.32b})$$

The control (B.1.23) which renders  $\dot{V}_a = -x^4 - z^2$  is

$$u = -z - x^2 - 2x^2z + 2x^4 = -\xi - 2x^2 - 2x^2\xi. \quad (\text{B.1.33})$$

The resulting system in the  $(x, \xi)$ -coordinates is

$$\dot{x} = x\xi \quad (\text{B.1.34a})$$

$$\dot{\xi} = -\xi - 2x^2 - 2x^2\xi, \quad (\text{B.1.34b})$$

and its equilibrium  $(0, 0)$  is GAS.

A significant design flexibility of backstepping is in the choice of  $\alpha(x)$ . For the system (B.1.29), instead of (B.1.31) we can choose

$$\alpha(x) \equiv 0, \quad z \equiv \xi, \quad (\text{B.1.35})$$

so that  $W(x) \equiv 0$  is semidefinite and

$$V_a = \frac{1}{2}x^2 + \frac{1}{2}\xi^2. \quad (\text{B.1.36})$$

The derivative of  $V_a$  along the solutions of (B.1.29) is

$$\dot{V}_a = x^2\xi + \xi u = \xi(u + x^2). \quad (\text{B.1.37})$$

In this case the best we can do is to render  $\dot{V}_a$  negative semidefinite: The control

$$u = -\xi - x^2 \quad (\text{B.1.38})$$

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yields the closed-loop system

$$\dot{x} = x\xi \quad (\text{B.1.39a})$$

$$\dot{\xi} = -\xi - x^2 \quad (\text{B.1.39b})$$

and the Lyapunov derivative  $\dot{V}_a = -\xi^2$ . Then, (B.1.2)(ii) guarantees that  $(x(t), \xi(t))$  is bounded and converges to the largest invariant set  $M_a$  of (B.1.39) contained in the set  $E_a$  where  $\xi = 0$ . But  $\xi(t) \equiv 0$  implies  $x(t) \equiv 0$ . Applying Corollary A.1.3, we conclude that the equilibrium  $(0, 0)$  is GAS.

Comparing the two control laws (B.1.33) and (B.1.38) we see that the choice  $\alpha(x) \equiv 0$  simplified the control by eliminating the  $x^4$ -term. Lemma B.1.2 shows how to add a single integrator. This lemma can be repeatedly applied to add a whole chain of integrators.

**Corollary B.1.4 (Chain of Integrators)** Let the system (B.1.17) satisfying Assumption B.1.1 with  $\alpha(x) = \alpha_0(x)$  be augmented by a chain of  $k$  integrators so that  $u$  is replaced by  $\xi_1$ , the state of the last integrator in the chain:

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{k-1} &= \xi_k \\ \dot{\xi}_k &= u. \end{aligned} \quad (\text{B.1.40})$$

For this system, repeated application of Lemma B.1.2 with  $\xi_1, \dots, \xi_k$  as virtual controls, results in the Lyapunov function

$$V_a(x, \xi_1, \dots, \xi_k) = V(x) + \frac{1}{2} \sum_{i=1}^k [\xi_i - \alpha_{i-1}(x, \xi_1, \dots, \xi_{i-1})]^2. \quad (\text{B.1.41})$$

Any choice of feedback control which renders  $\dot{V}_a \leq -W_a(x, \xi_1, \dots, \xi_k) \leq 0$ , with

$W_a(x, \xi_1, \dots, \xi_k) = 0$  only if  $W(x) = 0$  and  $\xi_i = \alpha_{i-1}(x, \xi_1, \dots, \xi_{i-1})$ ,  $i = 1, \dots, k$ ,  
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guarantees that  $[x^T(t), \xi_1(t), \dots, \xi_k(t)]^T$  is globally bounded and converges to the largest invariant set  $M_a$  contained in the set

$$E_a = \{[x^T, \xi_1, \dots, \xi_k]^T \in \mathbb{R}^{n+k} | W(x) = 0, \xi_i = \alpha_{i-1}(x, \xi_1, \dots, \xi_{i-1}), i = 1, \dots, k\}.$$

Furthermore, if  $W(x)$  is positive definite, that is, if  $x = 0$  can be rendered GAS through  $\xi_1$ , then (B.1.41) is a clf for (B.1.40) and the equilibrium  $x = 0, \xi_1 = \dots = \xi_k = 0$  can be rendered GAS through  $u$ .



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# Appendix C

## Passivity

### C.1 Passivity

we consider systems of the form

$$\begin{aligned} \dot{x} &= f(x, t) + g(x, t)u \\ y &= h(x, t), \end{aligned} \tag{C.1.1}$$

with  $x \in \mathbb{R}^n, y \in \mathbb{R}^m, u \in \mathbb{R}^m$ , and  $f, g, h$  continuous in  $t$  and smooth in  $x$ . Suppose  $f(0, t) = 0$  and  $h(0, t) = 0$  for all  $t \geq 0$ .

**Definition C.1.1** *The system (C.1.1) is said to be passive if there exists a continuous nonnegative (“storage”) function  $V : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , which satisfies  $V(0, t) = 0, \forall t \geq 0$ , such that for all  $u \in C^0, x(0) \in \mathbb{R}^n, t \geq t_0 \geq 0$*

$$\int_{t_0}^t y^T(\sigma)u(\sigma) d\sigma \geq V(x(t), t) - V(x(t_0), t_0). \tag{C.1.2}$$

**Definition C.1.2** *The system (C.1.1) is said to be strictly passive if there exists a continuous nonnegative (storage) function  $V : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , which satisfies  $V(0, t) = 0, \forall t \geq 0$ , and a positive definite function (dissipation rate)  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , such that for*

all  $u \in C^0$ ,  $x(0) \in \mathbb{R}^n$ ,  $t \geq t_0 \geq 0$

$$\int_{t_0}^t y^T(\sigma)u(\sigma) d\sigma \geq V(x(t), t) - V(x(t_0), t_0) + \int_{t_0}^t \psi(x(\sigma)) d\sigma. \quad (\text{C.1.3})$$

Passivity and Lyapunov stability are closely related concepts.

**Lemma C.1.3** *Suppose the system (C.1.1) is (strictly) passive. If  $V$  is positive definite, radially unbounded, and decrescent, that is, if there exist class  $\mathcal{K}_\infty$  functions  $\gamma_1$  and  $\gamma_2$  such that  $\gamma_1(|x|) \leq V(x, t) \leq \gamma_2(|x|)$ ,  $\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}_+$ , then, for  $u \equiv 0$ , the equilibrium  $x = 0$  of (C.1.1) is globally uniformly (asymptotically) stable.*

**Proof** When  $u \equiv 0$ , in the case of strict passivity, differentiating (C.1.3), we have

$$\dot{V} \leq -\psi(x). \quad (\text{C.1.4})$$

Thus, the equilibrium  $x = 0$  is globally uniformly asymptotically stable. The case of passivity is analogous.  $\square$

Many problems in parameter identification and adaptive control can be studied as feedback interconnections of passive systems (see Figure C.1):

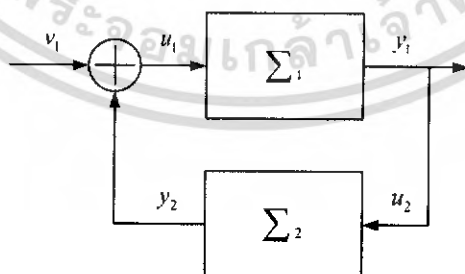


Figure C.1: Feedback interconnection of two passive systems.

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$$\Sigma_1 : \begin{cases} \dot{x}_1 = f_1(x, t) + g_1(x, t)u_1 \\ y_1 = h_1(x, t) \end{cases} \quad (\text{C.1.5})$$

$$\Sigma_2 : \begin{cases} \dot{x}_2 = f_2(x, t) + g_2(x, t)u_2 \\ y_2 = h_2(x, t) \end{cases} \quad (\text{C.1.6})$$

connected by the relations

$$u_1 = -y_2 + v_1 \quad (\text{C.1.7})$$

$$u_2 = y_1, \quad (\text{C.1.8})$$

where  $v_1$  is a new input to the system.

**Theorem C.1.4** *Suppose the system  $\Sigma_1$  is (strictly) passive with storage function  $V_1$  (and dissipation rate  $\psi_1$ ) independent of  $x_2$ . Likewise, suppose the system  $\Sigma_2$  is (strictly) passive with storage function  $V_2$  (and dissipation rate  $\psi_2$ ) independent of  $x_1$ . Then the interconnected system (C.1.5)-(C.1.8) with input  $v_1$  and output  $y_1$  is*

1. strictly passive if both  $\Sigma_1$  and  $\Sigma_2$  are strictly passive,
2. passive if at least one of the systems  $\Sigma_1$  and  $\Sigma_2$  is passive but not strictly passive.

Moreover, when  $v_1 \equiv 0$ , if  $\Sigma_1$  is strictly passive and  $\Sigma_2$  is passive, then the equilibrium  $x = 0$  is globally uniformly stable and  $\lim_{t \rightarrow \infty} x_1(t) = 0$ .

**Proof** Let us first assume that  $\Sigma_1$  and  $\Sigma_2$  are both strictly passive. Then, in view of (C.1.7)-(C.1.8) we have

$$\int_{t_0}^t y_1^T [v_1 - y_2] d\sigma \geq V_1(x_1(t), t) - V_1(x_1(t_0), t_0) + \int_{t_0}^t \psi_1(x_1) d\sigma \quad (\text{C.1.9})$$

$$\int_{t_0}^t y_2^T y_1 d\sigma \geq V_2(x_2(t), t) - V_2(x_2(t_0), t_0) + \int_{t_0}^t \psi_2(x_2) d\sigma. \quad (\text{C.1.10})$$

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Adding inequalities (C.1.9) and (C.1.10), we obtain

$$\int_{t_0}^t y_1^T(\sigma)v_1(\sigma) d\sigma \geq V(x(t), t) - V(x(t_0), t_0) + \int_{t_0}^t \psi(x) d\sigma, \quad (\text{C.1.11})$$

where the storage function  $V$  and the dissipation rate  $\psi$  for the complete  $x$ -system are defined as

$$V(x, t) = V_1(x_1, t) + V_2(x_2, t) \quad (\text{C.1.12})$$

$$\psi(x) = \psi_1(x_1) + \psi_2(x_2). \quad (\text{C.1.13})$$

Since  $V$  is positive definite, radially unbounded and decrescent, and  $\psi$  is positive definite, this proves the strict passivity. If at least one of the systems  $\Sigma_1$  and  $\Sigma_2$  is passive but not strictly passive, then its dissipation rate  $\psi_i$  is at best positive semidefinite but not positive definite, and the overall system is only passive. Finally, when  $v_1 \equiv 0$ , if  $\Sigma_1$  is strictly passive and  $\Sigma_2$  is passive, then  $\psi_2$  is positive semidefinite, and by differentiating (C.1.11) we get

$$\dot{V} \leq -\psi_1(x_1). \quad (\text{C.1.14})$$

Thus, by Theorem A.1.1,  $x = 0$  is globally uniformly stable and  $\lim_{t \rightarrow \infty} x_1(t) = 0$ .  $\square$

The quadratic nonnegative terms in the foregoing equation represent the dissipation rate. The dissipation rate takes different forms, which we illustrate by various special cases of the network.

Case 1: Take  $R_1 = R_3 = \infty$  and  $R_2 = 0$ . Then,

$$uy = \dot{V}.$$

In this case, there is no energy dissipation in the network; that is, the system is lossless.

Case 2: Take  $R_2 = 0$  and  $R_3 = \infty$ . Then,

$$uy = \dot{V} + \frac{1}{R_1}u^2.$$

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This dissipation rate is proportional to  $u^2$ . There is no energy dissipation if and only if  $u(t)$  is identically zero.

Case 3: Take  $R_1 = R_3 = \infty$ . Then,

$$uy = \dot{V} + R_2 y^2.$$

where we have used the fact that in this case  $y = x_1$ . The dissipation rate is proportional to  $y^2$ . There is no energy dissipation if and only if  $y(t)$  is identically zero.

Case 4: Take  $R_1 = \infty$ . Then,

$$uy = \dot{V} + R_2 x_1^2 + \frac{1}{R_3} x_2^2.$$

The dissipation rate is a positive definite function of the state  $x$ . There is no energy dissipation if and only if  $x(t)$  is identically zero.

Case 5: Take  $R_1 = \infty, R_2 = 0$ . Then,

$$uy = \dot{V} + \frac{1}{R_3} x_2^2.$$

This dissipation rate is a positive semidefinite function of the state. Notice, however,

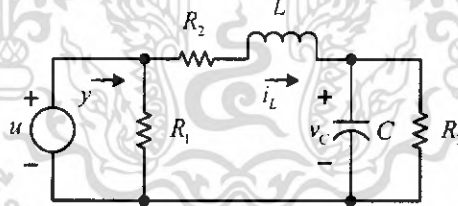


Figure C.2: RLC Circuit Illustration of Passivity Concept.

that from the second state equation, we have

$$x_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

irrespective of the input  $u$ . Therefore, like the previous case, there is no energy dissipation if and only if  $x(t)$  is identically zero.

These five cases illustrate four basic forms of the

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dissipation rate: no dissipation, strict dissipation when the input is not identically zero, strict dissipation when the output is not identically zero, and strict dissipation when the state is not identically zero. These four basic forms will be captured in Definition C.1.1. It is clear that combinations of these forms are also possible. For example, for the complete circuit when all resistors are present, we have

$$uy = \dot{V} + \frac{1}{R_1}u^2 + R_2x_1^2 + \frac{1}{R_3}x_2^2$$

whose dissipation rate is the sum of a quadratic term in the input and a positive definite function of the state.

