LOWER BOUNDS OF MULTICOLOR BIPARTITE RAMSEY NUMBERS FOR EVEN CYCLES

WARAPORN SUMMART

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE IN APPLIED MATHEMATICS

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

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(Assoc.Prof.Dr. DusaneeThanaboripat)

Dean

Date 28/07/15

หัวข้อวิทยานิพนธ์	ขอบเขตล่างของจำนวนแรมเซย์สองส่วนหลายสีสำหรับวงคู่
ชื่อนักศึกษา	นางสาววราภรณ์ สุ่มมาตย์
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บทคัดย่อ

จำนวนแรมเซย์สองส่วนหลายสี เขียนแทนด้วย $br(G_1, G_2, \ldots, G_k)$ คือ จำนวนเต็มบวก n ที่น้อยที่สุด ที่ทำให้การระบายสีเส้นเชื่อมของกราฟ $K_{n,n}$ ด้วยสี k สี แล้วจะเกิดกราฟที่สมสัณฐานกับ G_i ในสีที่ i สำหรับบางสี i โดยที่ $G_1, G_2, \ldots, G_k$ เป็นกราฟสองส่วน ในวิทยานิพนธ์ฉบับนี้เราได้ศึกษา ขอบเขตล่างของ $br_k(C_{2m})$ และขอบเขตล่างบางจำนวนของจำนวนแรมเซย์สองส่วน 3 สีของ C_{2m} เมื่อ m เป็นจำนวนเต็มบวกซึ่งมากกว่า 2

คำสำคัญ : กราฟ จำนวนแรมเซย์ จำนวนแรมเซย์สองส่วน จำนวนแรมเซย์สองส่วนหลายสี

Thesis Title	Lower Bounds of Multicolor Bipartite Ramsey Numbers for Even Cycles
Student Name	Ms. Waraporn Summart
Student ID	56605060
Degree	Master of Science
Department	Applied Mathematics
Year	2015
Thesis Advisor	Dr. Decha Samana

Abstract

The multicolor bipartite Ramsey number denoted by $br(G_1, G_2, \ldots, G_k)$, is the smallest positive integer n such that any edge coloring of $K_{n,n}$ with k colors will result in a monochromatic isomorphic to G_i in the i^{th} color for some i, where $G_1, G_2, \ldots, G_k$ are bipartite graphs. In this thesis, we show the lower bound for $br_k(C_{2m})$ and some lower bounds of 3-color bipartite Ramsey numbers of C_{2m} where m is a positive integer grater than 2.

Keywords : Graph, Ramsey number, bipartite Ramsey number, multicolor bipartite Ramsey number.

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Waraporn Summart

Table of Contents

List of Figures

List of Tables

Table **Page**

Chapter 1
Introduction

Frank Ramsey began to study Ramsey theory. He was interested in Ramsey number which is about the definition and properties of Ramsey Number. This number can be used to solve some problems by applying into a graph. It can be a diagram consisting points and lines.

The famous problem is the party problem which is the problem to find the minimum number of people at a birthday party to guarantee that three of them mutually know each other or three of them mutually do not know each other.

Considering one person for one vertex as well as red edge for the mutual know to each other and blue edge for mutually do not know each other.

Figure 1.1 Graph represent relation of 5 person in the party problem.

Figure 1.1, there is no forming for blue or red edge. Therefore, we could make a conclusion that a triangle is not formed with red or blue edges if number of vertices are less than 6.

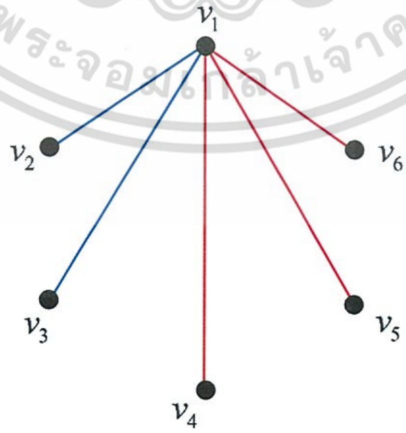


Figure 1.2 Edges coloring incident v_1 with other 5 vertices.

Figure 1.2, consider the vertex v_1. It must join with other 5 vertices and at least 3 edges will be red or blue. Assume v_1v_2, v_1v_3 and v_1v_4 are red edge. Therefore, if either v_2v_3, v_3v_4 or v_2v_4 edges are red without loss of generality, we let v_1v_2 are red , then there will be a red triangle as shown in Figure 1.3.

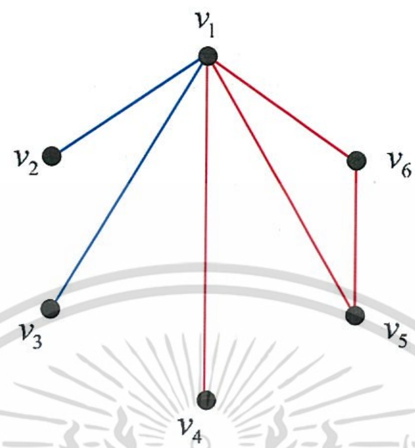


Figure 1.3 Edge coloring v_5v_6 with red.

But if none of them are red, there will be a blue triangle as shown in Figure 1.4. Hence, six is the least number for this problem.

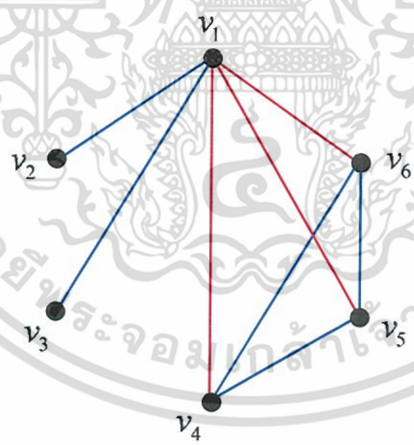


Figure 1.4 Edges coloring v_4v_6, v_4v_5 and v_5v_6 with blue.

There will be more formation of a triangle with red or blue edges if more vertices are added. This problem is the beginning of the study about definition and properties of Ramsey numbers of complete graph (K_n) and the development of other forms of graph for example, cycles (C_n), wheels (W_n), and books (B_n), as well as increasing of two-relation to the muti-relation (two-color to muti-color).

The progress on evaluating the basic number themselves has been very unsatisfactory for a long time. However, considerable progress has been made in this area, mostly by using computer algorithms.

In 1991, J.A. Bondy and P. Erdös [3] proved that

$$R(C_n, C_n) \geq 2n-1 \qquad \text{where } n \text{ is odd.}$$

In 2006, S. Yongqi, Y. Yuansheng, X. Feng, and L. Bingxi [16] proved that

$$R_k(C_{2m}) \geq 2(k-1)(m-1)+2, \text{ where } m,k \geq 2.$$

In 1975, L. W. Beneke and A.J. Schwenk [1] proved that

$$br_2(C_4) = 5.$$

In 1991, G. Exoo [9] proved that

$$br_3(C_4) = 11.$$

In 2013, J. Dybizbanski, T. Dzido, and S. Radziszowski [7] proved that

$$br_4(C_4) = 19,$$

$$26 \leq br_5(C_4) \leq 28,$$

$$br_k(C_4) \geq k^2+1, \quad \text{where } k \text{ is a prime power.}$$

In 2002, V. Longani [12] studied the methods for finding some bipartite Ramsey numbers. He found that

$$br\left(K_{1,n}, K_{1,n}\right) = 2n-1 \quad (n = 1,2,3,\ldots),$$

$$br\left(K_{2,2}, K_{2,5}\right) = 5,$$

$$br\left(K_{2,3}, K_{2,3}\right) = 9.$$

In 2013, R. Zhang, Y. Sun, and Y. Wu [17] proved that

$$br(C_{2m}, C_{2m}) \geq \begin{cases} m+n-1, & ; m \neq n \\ 2m, & ; m = n \end{cases} \quad \text{where } m,n \geq 2.$$

In [8] T. Dzido, A. Nowik and P. Szuca showed that

$$R_k(C_{2m}) \geq (k-1)m$$

for all integer $m \geq 2$ and an odd integer $k \geq 1$.

This thesis is divided into four chapters. In the first chapter, we provide the basic ideas of Ramsey number. The second chapter discusses graph theory and Ramsey number. In the third chapter, we prove our results on the lower bounds of multicolor bipartite Ramsey number for even cycles and lower bound of 3-color bipartite Ramsey numbers $br(C_6, C_{2m}, C_{2m+4})$ where $m \geq 4$. The final chapter summarizes the results on the lower bounds of multicolor bipartite Ramsey number for even cycles and lower bound of $br(C_6, C_{2m}, C_{2m+4})$.

Chapter 2
Preliminaries and Basic Concepts

The purpose of this chapter is to explain definitions of graph, preliminaries, and Ramsey number that are used throughout this thesis. Definition in this chapter we can learn more from [2], [4], [5], [9], [13], and [15].

2.1 Graph

Definition 2.1.1 A *graph* G is a finite nonempty set of objects called *vertices* together with a (possibly empty) set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$, while the edge set is denoted by $E(G)$.

Definition 2.1.2 The edge $e=\{u,v\}$ is said to *join* the vertices u and v. If $e=\{u,v\}$ is an edge of a graph G, then u and v are *adjacent vertices*, while u and e are *incident*, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges*. It is convenient to henceforth denoted an edge by uv or vu rather than by $\{u,v\}$.

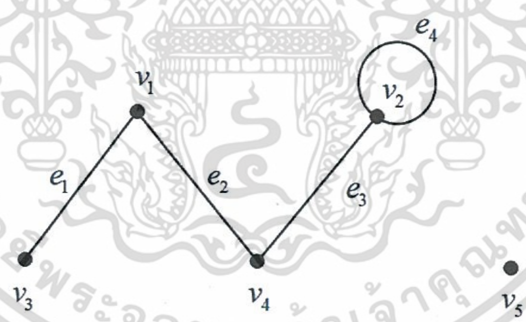
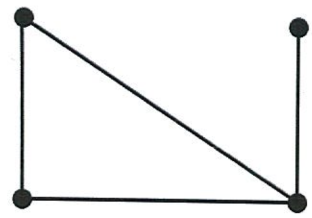


Figure 2.1 Graph $G=(V(G),E(G))$.

In Figure 2.1 v_1 adjacent v_3 but not adjacent to v_4. v_3 and v_1 is incident to e_1 but v_4 is not incident to e_1.

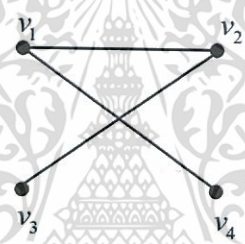
Definition 2.1.3 The cardinality of the vertex set of a graph $G(V,E)$ is called the *order* of G and is commonly denoted by $|V(G)|$ while the cardinality of its edge set is the *size* of G and is often denoted by $|E(G)|$.



G

Figure 2.2 Graph G with $|V(G)|=4$ and $|E(G)|=4$.

Definition 2.1.4 Let v be a vertex of G. The *degree* of v is the number of edges meeting at v, and is denoted by $\deg v$. In Figure 2.6, $\deg v_1 = 2$ and $\deg v_3 = 1$.



G

Figure 2.3 Graph with $\deg v_1 = 2$ and $\deg v_3 = 1$.

Definition 2.1.5 A *loop* is an edge that joins a single endpoint to itself and a vertex of degree 1 is an *endpoint.*

Definition 2.1.6 A *multiple edges* is a collection of two or more edges having identical endpoint.

Definition 2.1.7 A *simple graph* is a graph having no loop and multiple edges.

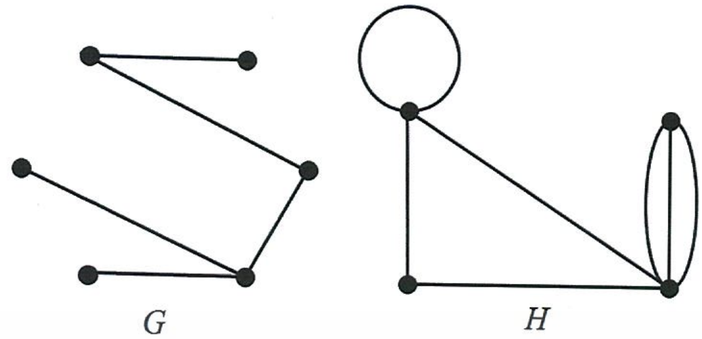


Figure 2.4 A simple graph G and a non simple graph H.

From Figure 2.4, graph G is a simple graph because it hasn't a loop and multiple edges and graph H is not a simple graph because it has a loop and multiple edges.

Definition 2.1.8 Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *subgraph* of G is a graph all of whose vertices belong to $V(G)$ and all of whose edges belong to $E(G)$.

For example, if G is the connected graph in Figure 2.4, where $V(G)=\{v_1,v_2,v_3,v_4\}$ and $E(G) = \{e_1,e_2,e_3,e_4,e_5\}$.

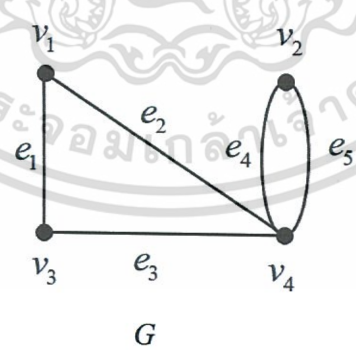


Figure 2.5 Graph G with $V(G)=\{v_1,v_2,v_3,v_4\}$ and $E(G)=\{e_1,e_2,e_3,e_4,e_5\}$.

then the graphs H_1 and H_2 in Figure 2.6 are subgraphs of G :

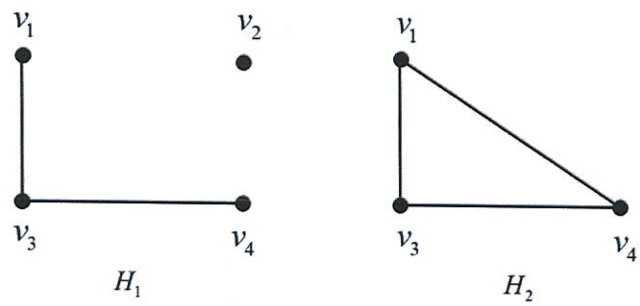


Figure 2.6 H_1 and H_2 are subgraphs of G in Figure 2.5 .

Note that a subgraph of G must actually be a graph, and G is regarded as a subgraph of itself.

Definition 2.1.9 A *walk* in a graph G is an alternating sequence of vertices and edges,

$$W = v_0, e_1, v_1, e_2, \ldots, e_n, v_n$$

such that for $j = 1, \ldots, n$, the vertices v_{j-1} and v_j are the endpoints of the edge e_j.

- In a simple graph, a walk may be represented simply by listing a sequence of vertices: $W = v_0, v_1, \ldots, v_n$ such that for $j = 1, \ldots, n$, the vertices v_{j-1} and v_j are adjacent.
- The *initial vertex* is v_0.
- The *final vertex* (or *terminal vertex*) is v_n.
- An *internal vertex* is a vertex that is neither initial nor final.

Definition 2.1.10 The number of edges in a walk is called its *length*.

Definition 2.1.11 A walk is *closed* if the initial vertex is also the final vertex; otherwise, it is *open*.

Definition 2.1.12 A *trail* in a graph is a walk such that no edge occurs more than once.

Definition 2.1.13 A *path* in a graph is a trail such that no internal vertex is repeated.

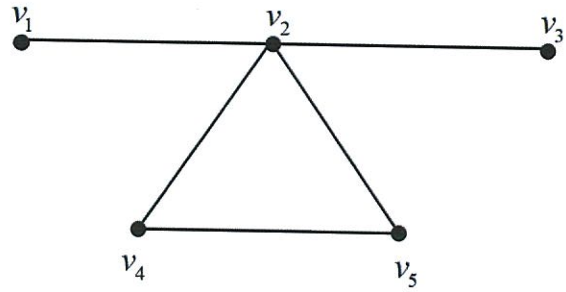


Figure 2.7 A graph used to explain walk, trail and path.

In the graph shown in Figure 2.7, $v_1, v_1v_2, v_2, v_2v_4, v_4, v_2v_4, v_2, v_2v_3, v_3$ represents a walk that is not a trail, and $v_1, v_1v_2, v_2, v_2v_4, v_4, v_4v_5, v_5, v_2v_5, v_2, v_2v_3, v_3$ represents a trail is not a path.

Definition 2.1.14 A nontrivial closed trail of a graph G is referred to as a *circuit* of G, and a circuit $v_1, v_2, \ldots, v_n, v_1$ $(n \geq 3)$ whose all vertices are distinct is called a *cycle*. The cycle graph with n vertices is denoted by C_n.

Figure 2.8 A cycle graph C_6.

Definition 2.1.15 A simple graph in which each pair of distinct vertices are adjacent is a *complete graph*. We denote the complete graph on n vertices by K_n.

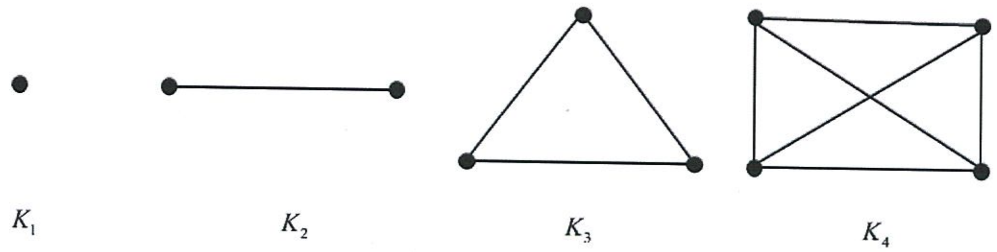


Figure 2.9 Complete graphs K_1 ,K_2, K_3, and K_4.

Definition 2.1.16 The *complement* $\overline{G}$ of a graph G is that graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G.

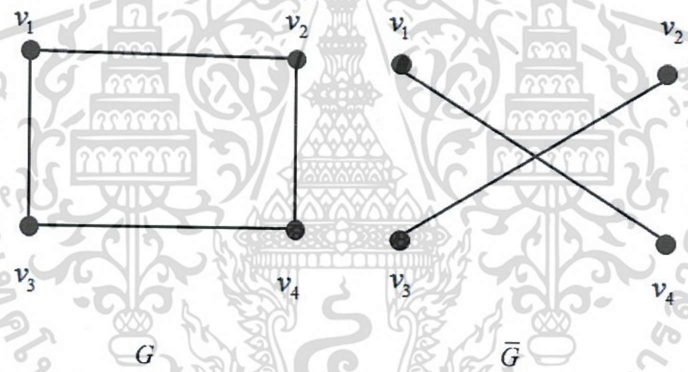


Figure 2.10 $\overline{G}$ is a complement graph of G.

Definition 2.1.17 If the vertex set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B, than G is a *bipartite graph*.

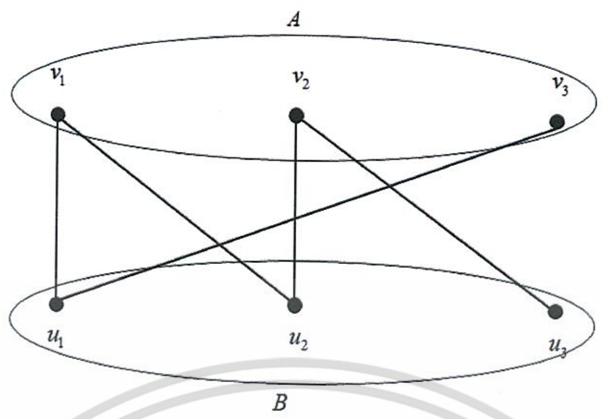


Figure 2.11 A bipartite graph.

Definition 2.1.18 A *complete bipartite graph* is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge. We denote the bipartite graph with m vertices and n vertices by $K_{m,n}$.

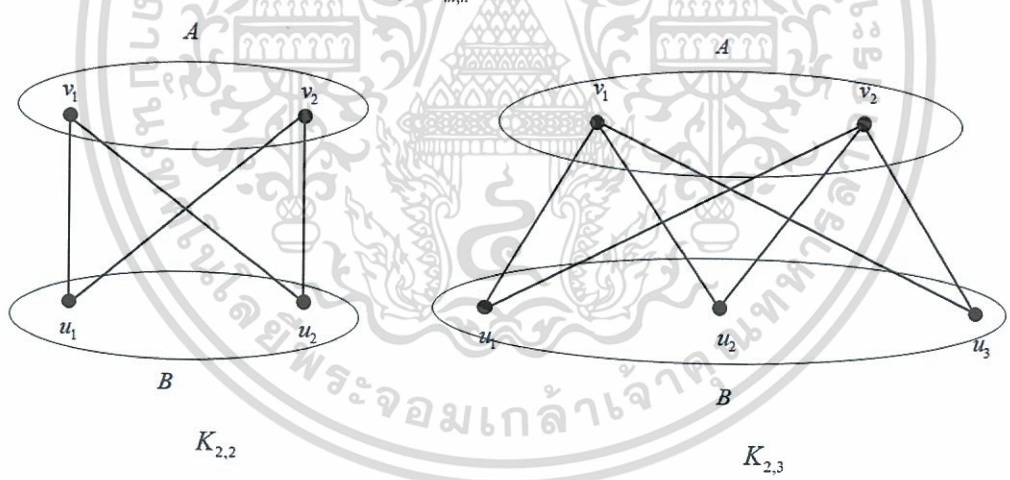


Figure 2.12 Complete bipartite graphs $K_{2,2}$ and $K_{2,3}$.

Definition 2.1.19 An *edge coloring* of graph G is an assignment of color (which are actually considers as elements of some set) to the edges of G one color to each edge, so that adjacent edges are assigned different colors.

Definition 2.1.20 An edge coloring of G use k colors is called k*-edge coloring.*

Definition 2.1.21 A graph G is *k-edge-colorable* if there exists an l-edge coloring of G for some $l \le k$. The minimum k for which a graph G is k-edge colorable is its *edge chromatic number* or *chromatic index* and is denoted by $\chi_l(G)$.

Example k edge coloring.

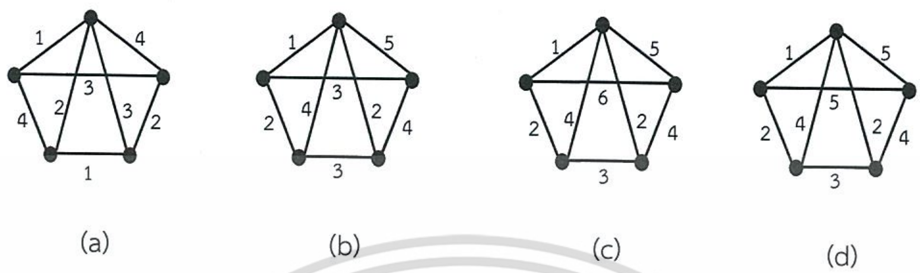


Figure 2.13 k-edge-colorings.

We usually display a k-edge coloring by writing the number $1,2,3,\ldots,k$ next to the appropriate edges. For example, Figure 2.13 (a), (b) and (c) above illustrate a 4-edge coloring, 5-edge coloring, and 6–edge-coloring of a graph G with eight edges; Figure 2.13 (d) is not a permissible coloring, since two of edges colored 2 meet at common vertex. It follows that $\chi'(G) \le 4$, since G has a 4-edge coloring [Figure 2.13 (a)]. On the other hand, $\chi'(G) \ge 4$, since contains four edges meeting at a common vertex (that is, a vertex of degree 4), which must be assigned different colors. So $\chi'(G) = 4$.

Definition 2.1.22 A graph G is said to be *factorable* into the factors $G_1, G_2, \ldots, G_t$ if these factors are pairwise edge-disjoint and $\bigcup_{i=1}^{t} E(G_i) = E(G)$. If G is factored into $G_1, G_2, \ldots, G_t$, then we represent this by $G = G_1 \oplus G_2 \oplus \cdots \oplus G_t$ which is called a *factorization* of G.

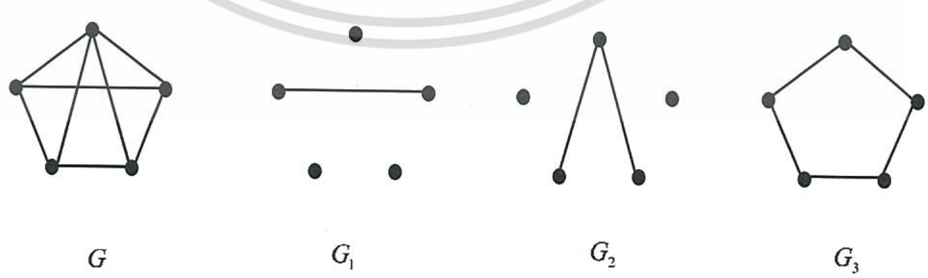


Figure 2.14 Factorization of $G = G_1 \oplus G_2 \oplus G_3$.

2.2 Ramsey numbers and bipartite Ramsey numbers.

Notation For any graph G_1 and G_2, we let a red G_1 (blue G_2) be a subgraph $G_1(G_2)$ which all of whose edges are colored red (blue).

Definition 2.2.1 For integers s and t, the *classical Ramsey number* $R(s,t)$ is the least positive integer n such that if the edges of the complete graph K_n are arbitrarily colored using the colors red and bule, then either a red K_s or a blue K_t will be forced as a subgraph of K_n.

Table 2.1 show nontrivial values and bounds for classical Ramsey numbers $R(s,t)$ [12].

$R(s,t)$		s													
		3	4	5	6	7	8	9	10	11	12	13	14	15	
t	3	6	9	14	18	23	28	36	40 42	47 50	52 59	59 68	66 77	73 87	lower exact value upper
	4		18	25	36 41	49 61	58 84	73 115	92 149	98 191	128 238	133 291	141 349	153 417	
	5			43 49	58 87	80 143	101 216	126 316	144 442	171 633	191 848	213 1138	239 1461	265 1878	
	6				102 165	113 298	132 495	169 780	179 1171	253 1804	263 2566	317 3703	5033	401 6911	
	7					205 540	217 1031	241 1713	289 2826	405 4553	417 6954	511 10578	15263	22112	
	8						282 1870	317 3583	6090	10630	16944	817 27485	41525	861 63609	
	9							565 6588	581 12677	22325	38832	64864			
	10								798 23556	45881	81123			1265	

Table 2.1 Some exact values and bounds of Ramsey numbers $R(s,t)$.

Theorem 2.2.1 [4] (**Ramsey Theorem**) For every positive integers s and t the Ramsey number $R(s,t)$ always exists; moreover,

$$R(s,t) \le \binom{s+t-2}{s-1}.$$

Example 2.2.1 Find $R(3,4)$ by using Theorem 2.2.1, we have

$$R(3,4) \le \binom{3+4-2}{3-1} = \binom{5}{2} = \frac{5!}{(5-2)!2!} = 10.$$

Definition 2.2.2 For given graph G_1 and G_2, *Ramsey number* $R(G_1,G_2)$ is the least positive integer n such that when the edges of K_n are colored arbitrarily red or blue, there necessarily exists either a red G_1 or a blue G_2 as a subgraph of K_n.

Example 2.2.2 [12] Let n be a positive integer.

$$R(C_3,C_3)=6,$$
$$R(C_4,C_4)=6,$$
$$R(C_3,C_n)=2n-1 \text{ for } n\ge 4,$$
$$R(C_4,C_n)=n+1 \text{ for } n\ge 6,$$
$$R(C_5,C_n)=2n-1 \text{ for } n\ge 5,$$
$$R(C_6,C_6)=8.$$

Definition 2.2.3 For given graphs $G_1,G_2,\ldots,G_k$, $k\ge 2$, the *multicolor Ramsey number* $R(G_1,G_2,\ldots,G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n with k colors, then it always contains a monochromatic copy of G_i colored with i, for some $1\le i\le k$. We denote such a number by $R_k(G)$ if $G=G_1=G_2=\cdots=G_k$.

The exact values and bounds for $R_k(C_m)$ [12] as shown in Table 2.2.

k \ m	3	4	5	6	7	8	
3	17	11	17	12	25	16	Upper exact value lower
4	51 62	18	33 158	18 20	49	20	
5	162 307	27 29	65	26	97	28	
6	538 1838	32 43	129		193		

Table 2.2 Exact values and bounds for $R_k(C_m)$ for $3 \le k \le 6$ and $3 \le m \le 8$.

Next, we will describe the definitions of bipartite Ramsey number as follows:

Definition 2.2.4 For given bipartite graphs $K_{s,s}$ and $K_{t,t}$, the *bipartite Ramsey number* $br(s,t)$ is the least positive integer n such that if the edges of $K_{n,n}$ are colored with two colors (red or blue), then there always exists either a red $K_{s,s}$ or a blue $K_{t,t}$.

Some exact values and upper bound of bipartite Ramsey numbers as shown in Table 2.3.

$br(s,t)$		$K_{s,s}$				
		2	3	4	5	6
$K_{t,t}$	2	5	9	14	≤ 19	≤ 25
	3		17	≤ 29	≤ 41	≤ 56
	4			≤ 48	≤ 72	≤ 101
	5				≤ 115	≤ 168

Table 2.3 Some exact values and upper bounds of bipartite Ramsey numbers.

Theorem 2.2.2 [4] For every positive integers s and t the bipartite Ramsey number $br(K_{s,s},K_{t,t})$ exists; moreover,

$$br(K_{s,s},K_{t,t}) \le \binom{s+t}{s} - 1.$$

For example,

$$\begin{aligned} br(K_{2,2},K_{2,2}) &\le \binom{2+2}{2} - 1 \\ &= \binom{4}{2} - 1 = \frac{4!}{2!2!} - 1 = \frac{12}{2} - 1 = 5. \end{aligned}$$

Definition 2.2.5 For given bipartite graphs G_1 and G_2, the *bipartite Ramsey number* $br(G_1,G_2)$ is the least positive integer n such that if the edges of $K_{n,n}$ are colored with two colors (red or blue), then there always exists either a red G_1 or a blue G_2.

According to the definition of bipartite Ramsey number, V. Longani [18], has found that

$$br\left(K_{1,n},K_{1,n}\right) = 2n-1 \quad (n = 1,2,3,\ldots)$$

$$br\left(K_{2,2},K_{2,5}\right) = 5$$

$$br\left(K_{2,3},K_{2,3}\right) = 9.$$

L.W. Beineke and A.J. Schwenk [1] have also found that

$$br\left(K_{2,2},K_{2,5}\right) = 5,$$

$$br\left(K_{3,3},K_{3,3}\right) = 17.$$

Theorem 2.2.3 [16], $br\left(C_6,C_{2m}\right) \ge m+2$ for $m \ge 4$.

Definition 2.2.6 The *multicolor bipartite Ramsey number* $br(G_1,G_2,\ldots,G_k)$ is the smallest integer n such that any coloring of edges of $K_{n,n}$ with k colors will result in an isomorphic monochromatic copy of G_i in the i^{th} color for some $1 \le i \le k$, where $G_1,G_2,\ldots,G_k$ are bipartite graphs. We will denote $br\left(K_{p,p},K_{q,q}\right)$ by $br(p,q)$ and $br(G_1,G_2,\ldots,G_k)$ where $G = G_1 = G_2 = \cdots = G_k$ by $br_k(G)$.

Some exact values and of multicolor bipartite Ramsey numbers are as following:

G. Exoo [8] proved that

$$br_3(C_4) = 11.$$

J. Dybizbanski, T. Dzido, and S. Radziszowski [6] proved that

$$br_4(C_4) = 19,$$

$$26 \le br_5(C_4) \le 28,$$

$$br_k(C_4) \ge k^2 + 1, \quad \text{where } k \text{ is a prime power.}$$

Next chapter, we will show that lower bounds of multicolor bipartite Ramsey number for even cycle and lower bounds of 3-color Ramsey number for even cycle.

Chapter 3

Main Results

In this chapter, we will find lower bounds of multicolor bipartite Ramsey number for even cycles with the algorithm to construct the graph without even cycles. Moreover, we will show lower bounds of 3-color bipartite Ramsey number $br(C_6, C_{2m}, C_{2m+4})$ where $m \geq 4$.

3.1 LOWER BOUND OF $br_k(C_{2m})$

According to the work of T. Dzido, A. Nowik and P. Szuca [7], they have shown the algorithm to find lower bounds of multicolor Ramsey number for even cycles, that is $R_k(C_{2m}) \geq (k-1)m$ for all integer $m \geq 2$ and an odd integer $k \geq 1$. We modify the algorithm to find lower bound of multicolor bipartite Ramsey number for even cycles.

Theorem 3.1.1 For all integers $m, k \geq 2$,

$$br_k(C_{2m}) \geq k(m-1) + \left\lfloor \frac{k}{2} \right\rfloor + 1.$$

Proof Let $K_{n,n}$ be a complete bipartite graph with $n = k(m-1) + \left\lfloor \frac{k}{2} \right\rfloor$, where $K_{n,n}$ are separated into subgraphs G and S, that is $K_{n,n} = G \oplus S$ where $G = G(V_1(G), V_2(G))$ and $S = S(V_1(G), V_2(G))$ are complete bipartite graphs with $|G| = 2k(m-1)$ and $|S| = 2\left\lfloor \frac{k}{2} \right\rfloor$, respectively.

Partition each partition set of $G(V_1(G), V_2(G))$ and $S(V_1(G), V_2(G))$ into k set. That is, $V_1(G) = F_1 \cup F_2 \cup \cdots \cup F_k$ where $|F_i| = m-1$ and $V_2(G) = H_1 \cup H_2 \cup \cdots \cup H_k$ where $|H_j| = m-1$.

Let $E_r(F_i H_j)$ be an edge set of an edge coloring between F_i and H_j with color r as condition:

$$E_r(F_i H_j) = \{ f_{ip} h_{jq} \mid j-1 \equiv (i+r-2) \bmod k \},\ \forall f_{ip} \in F_i,\ \forall h_{jq} \in H_j,$$
$$p, q \in \{1, 2, \ldots, m-1\}, \text{ and } r = 1, 2, \ldots, k.$$

For example the partition when $k = 4$ and $m = 4$ as shown in Figure 3.1.

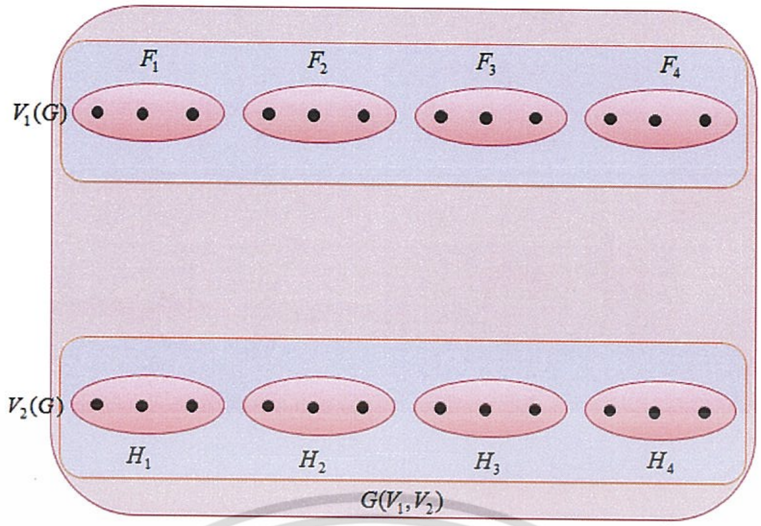


Figure 3.1 Partition the vertices of $V_1(G)$ and $V_2(G)$ when $k=4$ and 4 color.

Figure 3.2 presents an edge coloring between F_i and H_j with 4 colors.

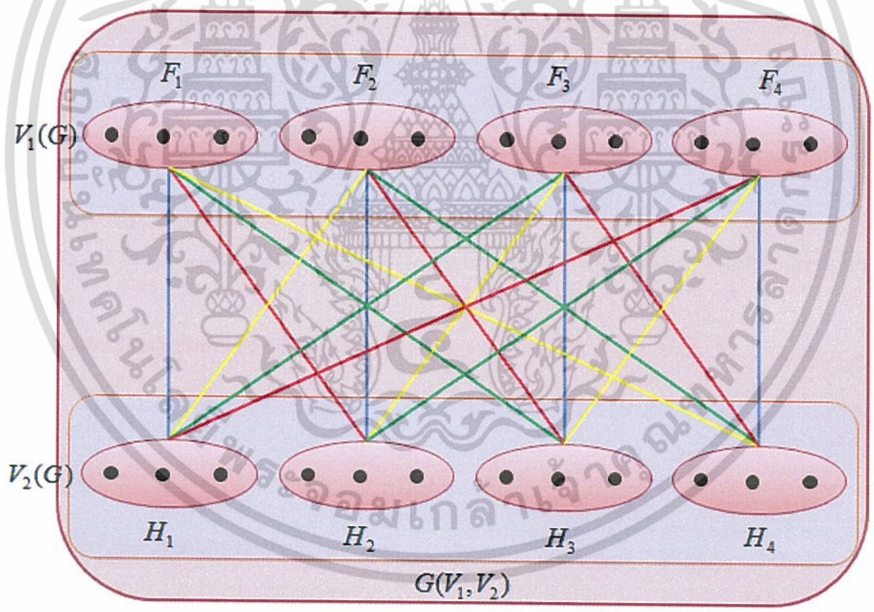


Figure 3.2 Edges coloring between F_i and H_j with 4 colors.

It is easy to see that, $G(V_1,V_2)$ does not contain any monochromatic cycle of length more than $2m-2$ vertices.

Next, let $S(V_1,V_2)$ be a complete bipartite graph with $V_1(S)=\{v_1,v_3,\dots,v_{2s-1}\}$ and $V_2(S)=\{v_2,v_4,\dots,v_{2s}\}$, where $s=1,2,\dots,\left\lfloor \frac{k}{2} \right\rfloor$. We color all edge in $S(V_1,V_2)$ with an edge coloring c, where $c(v_a v_b)$ is an edge coloring of $v_a v_b$ defined as follows:

$$c(v_a v_b)=\begin{cases} a & if\ a<b \\ b & if\ a>b \end{cases}, v_a \in V_1(S),\ v_b \in V_2(S).$$

Thus, there are no monochromatic cycles in a subgraph induced by S.

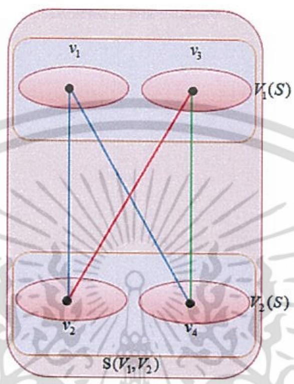


Figure 3.3 Edge coloring in $S(V_1,V_2)$.

Next, we extend a graph G by combining a graph G with a graph S. We color all edges between $G(V_1,V_2)$ and $S(V_1,V_2)$ with color i, where i is a color of edges that incident to v_i and all vertices in $V_2(G)$ $(V_1(G))$ if $v_i \in V_1(S)$ $(v_i \in V_2(S))$.

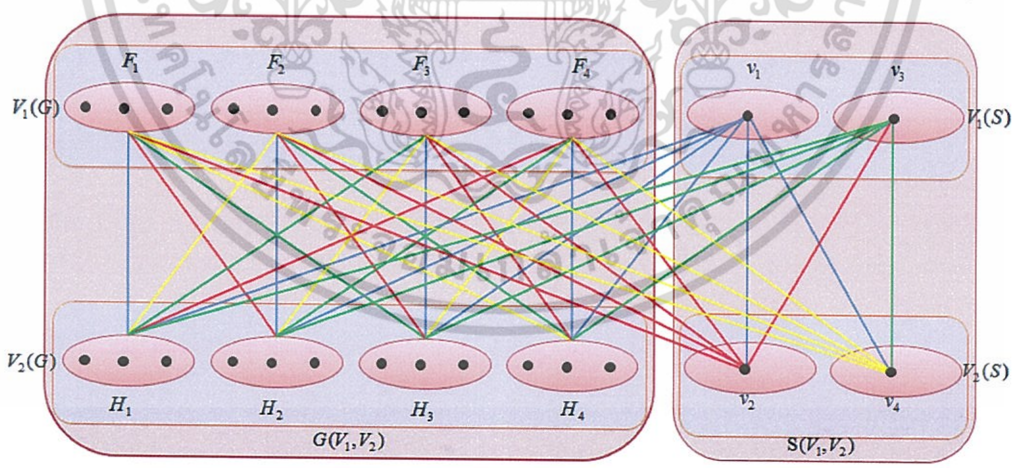


Figure 3.4 An illustration of an edge coloring in the proof of Theorem 3.1.1 for $k=4$ and $m=4$.

Finally, we will show that in $K_{n,n} = G \oplus S$ there are no monochromatic cycle with length at least $2m$. By using contradiction, suppose to the contrary that a graph $G \oplus S$ contains a cycle C of color d of length more than $2m-1$. Since G has no monochromatic cycle of length more than $2m-2$, we have $V(C) \cap V(S) \neq \varnothing$. That is, there exists $v_d \in V(S)$, $d \in \left\{1,2,\ldots,2\left\lfloor \frac{k}{2} \right\rfloor\right\}$, which is adjacent by an edge of color d with G. Assume that every $v_t \in V(S)$ and v_t adjacent to v_d. If $d < t$, then $c(v_d v_t) = d$. However, all edges which join v_t with G are colored with color t. This implies that there is no monochromatic path of length more than $2m-1$. If $d > t$, then $c(v_d v_t) = t$. It is easily to see that there is no monochromatic path of length more than $2m-2$. Therefore, the length of C is less than $2m$, we get a contradiction.

□

Consider $k=2$, we have $br(C_{2m}, C_{2m}) \geq 2m$. This lower bound is the same result as in [18]. The following corollary is trivial from Theorem 3.1.1 when $k=3$.

Corollary 3.1.2. For all integer $m \geq 2$,

$$br_3(C_{2m}) \geq 3m-1.$$

3.2 LOWER BOUND OF $br(C_6, C_{2m}, C_{2m+4})$

R. Zhang, Y. Sun, and Y. Wu [8], showed $br(C_6, C_{2m}) = m+2$. We modify such an algorithm to find a lower bound of 3-color bipartite Ramsey numbers for $br(C_6, C_{2m}, C_{2m+4})$.

Theorem 3.2.1 For all integer $m \geq 4$,

$$br(C_6, C_{2m}, C_{2m+4}) \geq 2m+3.$$

Proof Let $K_{n,n}$ be a complete bipartite graph with $n=2m+2$. Partition $K_{n,n}$ be factor into subgraphs G and S, that is, $K_{n,n} = G \oplus S$. This implies that, $G(V_1(G), V_2(G))$ is a complete bipartite graph with $|G| = 2(m+1)$ and $S(V_1(G), V_2(G))$ is a complete bipartite graph with $|S| = 2(m+1)$, respectively (see Figure 3.5).

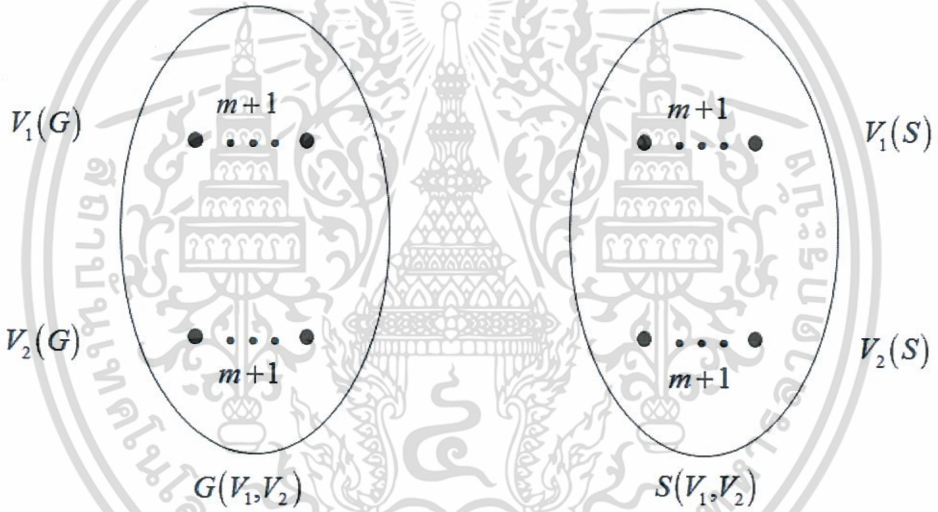


Figure 3.5 Partition of the vertex set of $G(V_1,V_2)$ and $S(V_1,V_2)$.

By using Theorem 2.2.3, we color all edges of $G(V_1,V_2)$ and $S(V_1,V_2)$ with 2 colors. Then $G(V_1,V_2)$ does not contain subgraphs C_6, C_{2m} and neither does $S(V_1,V_2)$.

Next, we color the edges between $G(V_1,V_2)$ and $S(V_1,V_2)$ with color 3 as shown in Figure 3.6.

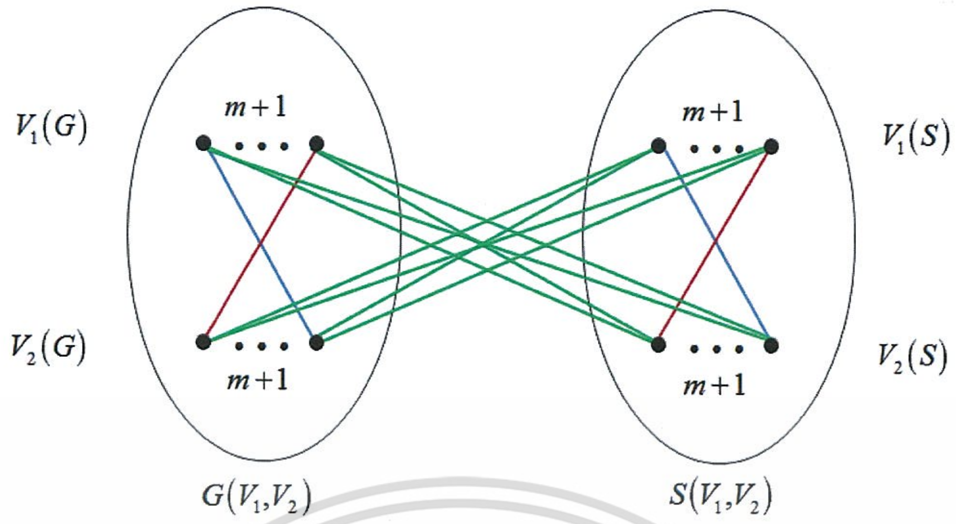


Figure 3.6 The edges coloring between $G(V_1,V_2)$ and $S(V_1,V_2)$.

We obtain that the maximum length of cycle is $2m+2$.

Therefore, $br(C_6,C_{2m},C_{2m+4}) \geq 2m+3$.

□

Chapter 4

Conclusions

From our study in Chapter 3, we obtained lower bound of multicolor bipartite Ramsey number for even cycle and lower bound of 3-color bipartite Ramsey numbers. We conclude the following results.

4.1. For all integers $m, k \geq 2$,

$$br_k(C_{2m}) \geq k(m-1) + \left\lfloor \frac{k}{2} \right\rfloor + 1.$$

4.2. For all integer $m \geq 2$,

$$br_3(C_{2m}) \geq 3m-1.$$

4.3. For all integer $m \geq 4$,

$$br\left(C_6, C_{2m}, C_{2m+4}\right) \geq 2m+3.$$

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Appendix

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Lower Bound of Multicolor Bipartite Ramsey Number for Even Cycles

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Abstract. The multicolor bipartite Ramsey number $br(G_1,G_2,\ldots,G_k)$ is the smallest positive n such that any coloring of edges of $K_{n,n}$ with k colors will result in an isomorphic monochromatic copy of G_i in the i^{th} color for some i, where $G_1,G_2,\ldots,G_k$ are bipartite graphs. In this paper, we show that lower bound for $k-color$, $br(C_{2m},C_{2m},\ldots,C_{2m})$ where $m\geq 2$, and C_{2m} is the cycle on $2m$ vertices.

Keywords: Multicolor bipartite Ramsey number, Ramsey number, Graph

INTRODUCTION

All graphs will as usual be undirected, finite, and have no loops or multiple edges. K_n will denote the complete graph on n vertices and $K_{m,n}$ the complete bipartite graph with parts containing m and n vertices, respectively. If G is a graph, $V(G)$ will denote its vertex set and $E(G)$ its edge set. If G is factored into $G_1,G_2,\ldots,G_k$, than we represent this by $G=G_1\oplus G_2\oplus\ldots\oplus G_k$. Let $E_r(V_1V_2)$ be a set of coloring all edges between V_1 and V_2 by color r. And let $c(uv)$ be a color of edge $\{uv\}$.

For given graphs $G_1,G_2,\ldots G_k$, $k\geq 2$, the multicolor Ramsey number $R(G_1,G_2,\ldots,G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n with k color, then it always contains a monochromatic copy of G_i colored with i, for some $1\leq i\leq k$. We denote such a number by $R_k(G)$ if $G=G_1=G_2=\cdots=G_k$. In the case of 2 colors $(k=2)$ we deal with classical Ramsey numbers, which have been studied extensively for 50 years.

More generally, the multicolor bipartite Ramsey number $br(G_1,G_2,\ldots,G_k)$ is the smallest integer n such that any coloring of edges of $K_{n,n}$ with k color will result in an isomorphic monochromatic copy of G_i in the i^{th} color for some i, where $G_1,G_2,\ldots,G_k$ are bipartite graphs. We will denote $br\left(K_{p,p},K_{q,q}\right)$ by $br(p,q)$ and $br(G_1,G_2,\ldots,G_k)$ where $G=G_1=G_2=\cdots=G_k$ by $br_k(G)$.

In 1991, J.A. Bondy and P. Erdos [3] proved that

$$R\left(C_n,C_n\right)=2n-1 \qquad \text{if } n \text{ is odd.}$$

In 2006, S. Yongqi, Y. Yuansheng, X. Feng, and L. Bingxi [8] proved that

$$R_k(C_{2m}) = 2(k-1)(m-1)+2.$$

In 1975, L. W.Beneke and A .J. Schwenk [5] proved that

$$br_2(C_4) = 5.$$

In 1991, G.Exoo [2] proved that

$$br_3(C_4) = 11.$$

In 2013, J. Dybizbanski,T. Dzido, and S. Radziszowski [4] proved that

$$br_4(C_4) = 19,$$
$$26 \le br_5(C_4) \le 28,$$
$$br_k(C_4) \ge k^2+1, \quad k \text{ is prime power.}$$

In 2013, R. Zhang, Y. Sun, and Y. Wu [6] proved that

$$br(C_{2m}, C_{2m}) \ge \begin{cases} m+n-1, & m \ne n, \\ 2m, & m = n. \end{cases}$$

In [7], D. Samana and V. Longani proved that

$$br_k(C_{2m}) = k(m-1)+2.$$

In [9], They have shown $R_k(C_{2m}) \ge (k-1)m$ for all integer $m \ge 2$ and an odd integer $k \ge 1$. We use a slight modification of a similar construction for lower bound of multicolor bipartite Ramsey number for even cycles.

LOWER BOUND OF $br_k(C_{2m})$

Theorem 1. For all integer $m \ge 2$ and integer $k \ge 2$,

$$br_k(C_{2m}) \ge k(m-1) + \left\lfloor \frac{k}{2} \right\rfloor + 1.$$

Proof Let $K_{n,n}$ be a complete bipartite graph with $n = k(m-1)+\left\lfloor \frac{k}{2} \right\rfloor$, and its factor into G and S, $K_{n,n} = G \oplus S$, $G=(V_1,V_2)$ and $S=(V_1,V_2)$ are bipartite complete graphs of order $2k(m-1)$ and $k-1$, respectively. Partition the vertices of $V_1(G)$ and $V_2(G)$ into k sets, that is, $V_1(G)=F_1 \cup F_2 \cup \cdots \cup F_k$ and $V_2(G)=H_1 \cup H_2 \cup \cdots \cup H_k$ where $|F_i| = |H_j| = m-1$. We will give a k - coloring of all edges of $G=(V_1,V_2)$ as follows:

$$E_r(F_iH_j)=\left\{ f_{ip}h_{jq} \mid j-1 \equiv (i+r-2) \bmod k \right\}, \forall f_{ip} \in F_i, \ \forall h_{jq} \in H_j \text{ and } p,q \in \{1,2,\ldots,m-1\}.$$

Obviously, such a graph $G=(V_1,V_2)$ contains no monochromatic cycle of more than $2m-2$ vertices.

Proceedings of the 10th IMT-GT ICMSA2014
14-16 October, Kuala Terengganu, Malaysia

Next, let $S=(V_1,V_2)$ be a complete bipartite graph with $V_1(S)=\{v_1,v_3,\dots,v_{2s-1}\}$ and $V_2(S)=\{v_2,v_4,\dots,v_{2s}\}$ where $s=1,2,\dots,\left\lfloor \frac{k}{2}\right\rfloor$. The edges in $S=(V_1,V_2)$ are colored as follows:

$$c(v_a v_b)=\begin{cases} a & \text{if } a<b \\ b & \text{if } a>b \end{cases},\ v_a\in V_1(S),\ v_b\in V_2(S).$$

Thus, there are no monochromatic cycle in the subgraph induced by S.

More formally, extend graph G to graph S by adding $2\left\lfloor \frac{k}{2}\right\rfloor$ new vertices $S=\left\{v_1,\dots,v_{2\left\lfloor \frac{k}{2}\right\rfloor}\right\}$. Now, color all edges between v_i and G with the color i.

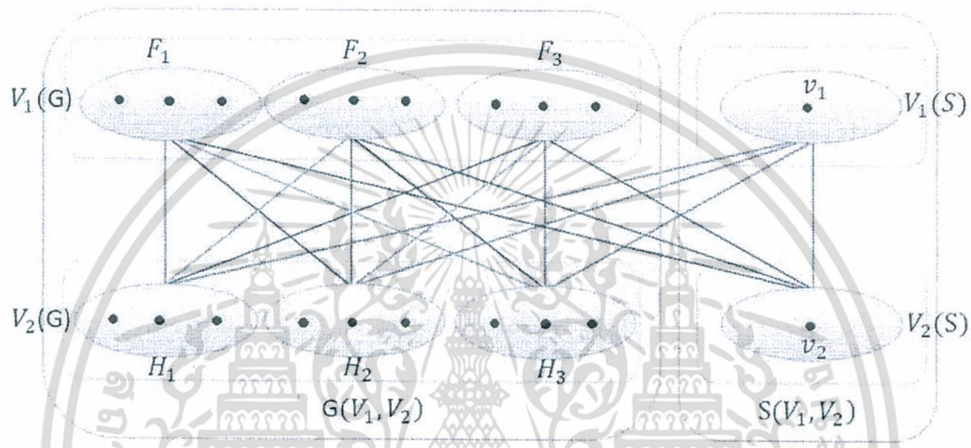

FIGURE 1. An illustration of coloring from the proof Theorem1 for the case $k=3$ and $m=4$.

Finally, we will show that in $K_{n,n}=G\oplus S$ there are no monochromatic cycle of length at least $2m$. In contrary to our claim, suppose that graph $G\oplus S$ contains a cycle C of color d longer than $2m-1$. Since in G there is no monochromatic cycle of length greater than $2m-2$, we have $V(C)\cap V(S)\neq\varnothing$. That is, there exists $v_d\in V(S)$, $d\in\left\{1,2,\dots,2\left\lfloor \frac{k}{2}\right\rfloor\right\}$ which is adjacent by an edge of color d with G. Assume that any $v_t\in V(S)$ and v_t adjacent to v_d. If $d<t$, then $c(v_d v_t)=d$. But all edges which join v_t with G are colored with color t, this implies that there is no monochromatic path of length more than $2m-1$. If $d>t$, then $c(v_d v_t)=t$. It is easily seen that there is no monochromatic path length more than $2m-2$. Therefore, the length of C is less than $2m$, We get a contradiction. □

Corollary 1. For all integer $m\geq 2$,

$$br_3(C_{2m})\geq 3m-1.$$

Consider $k=2$, we have $br(C_{2m},C_{2m})=2m$ which lower bound is the same result as [4]. This is trivial form Theorem 1 when we consider in the case of $k=3$.

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