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THURDKWAN CHANGPUEK

A THESIS SUMMITTED IN FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN APPLIED MATHEMATICS

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บทคัดย่อ

ในงานวิจัยนี้เป็นการสร้างแบบจำลองทางคณิตศาสตร์เพื่อศึกษาการแพร่ระบาดของโรคไข้หวัดหมูโดยแบ่งตามช่วงอายุของประชากร อ้างอิงข้อมูลจากรายงานการเฝ้าระวังโรคกรมควบคุมโรค กระทรวงสาธารณสุข ซึ่งแบบจำลองนี้ได้พัฒนามาจากแบบจำลองทางคณิตศาสตร์ของ วรรณิกา จำเป็น และคณะ โดยแบ่งกลุ่มประชากรออกเป็น 3 กลุ่ม คือกลุ่มอายุ 1-10 ปี อายุ 11-20 ปี และอายุตั้งแต่ 20 ปีขึ้นไป แต่ละกลุ่มแบ่งเป็น 5 กลุ่มย่อย คือ กลุ่มคนที่เสี่ยงต่อการติดเชื้อ กลุ่มคนที่ติดเชื้อแฝง กลุ่มคนที่ติดเชื้อ กลุ่มคนที่ถูกกักกันโรค และกลุ่มคนที่ฟื้นไข้ เมื่อสร้างแบบจำลองที่เป็นระบบสมการเชิงอนุพันธ์ไม่เชิงเส้นได้แล้ว จะทำการหาจุดสมดุลพร้อมกับพิจารณาเงื่อนไขที่จำเป็นสำหรับพารามิเตอร์ที่ทำให้เกิดความเสถียรภายใต้สภาวะไร้โรคและสภาวะระบาดอย่างเรื้อรัง โดยพิจารณาทั้งความเสถียรแบบเฉพาะที่และความเสถียรแบบวงกว้าง นอกจากนี้ยังแสดงผลเฉลยเชิงตัวเลขเพื่อเป็นการสนับสนุนทฤษฎีบท จากนั้นได้จำลองสถานการณ์แบบเครือข่ายที่ประกอบด้วยประชากรและสถานที่สาธารณะ โดยให้ประชากรมีการเคลื่อนที่แบบสุ่มไปยังสถานที่สาธารณะ และแสดงผลเฉลยเชิงตัวเลขตามการแจกแจงของเวลาภายใต้สถานการณ์ที่แตกต่างกัน

คำสำคัญ : ความเสถียรแบบเฉพาะที่, ความเสถียรแบบวงกว้าง, ฟังก์ชัน Lyapunov, แบบจำลอง SEIQR

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Abstract

In this research, we formulate the mathematical models by incorporating the age structure of human population to study the transmission of swine flu. The data of swine flu from the report of Department of Disease Control, Ministry of Public Health, Thailand are analyzed. We modify the mathematical model proposed by Jumpen et al. The human population is separated into three groups such as 1-10 years, 11-20 years and more than 20 years, respectively. Each group is subdivided into five classes: susceptible, exposed, infected, quarantined and recovered. The mathematical model is described by the system of nonlinear differential equations. We find the equilibrium states and consider the necessary conditions of the parameters for the disease free and endemic equilibrium states to be local and global asymptotically stables. Numerical solutions are showed to support the theoretical predictions. Furthermore, we propose the network model which contains people and public places. We suppose that the people randomly move to any public places. Numerical results are shown by the time distribution dealing with different situations.

Keywords: global asymptotically stable, local asymptotically stable, Lyapunov function, SEIQR model

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Chapter 1
Introduction

1.1 Statement and Significant of the Problems

Swine flu is an Emerging Infectious Disease (EID) such as SARS and avian influenza. The swine flu caused by a new strain of type A influenza virus designated as influenza H1N1 which has not previously circulated in human. Swine flu is a highly contagious, rapidly and widely spreading disease. Beginning in March 2009, the swine flu was first detected in Mexico and then spread to the United States and then to Europe. In April 2009, nine countries officially reported with confirmed cases of swine flu. There are Mexico, the United State, Austria, Canada, Germany, Israel, New Zealand, Spain and the United Kingdom [1]. The World Health Organization (WHO) closely monitored the situation daily and revised prevention and control measure. As of June 2009, The WHO announced the pandemic alert level at phase 6 which is the highest phase of a pandemic alert. It indicates that human to human spread of the virus in at least one other country in another WHO region [2]. The total report of swine flu cases worldwide more than 207 countries were 622,482 persons by November 2009 [3]. There were at least 16,713 confirmed swine flu worldwide deaths as of March 2010 and WHO expected that the number of deaths from swine flu is higher than the reported [4]. The number of deaths classified by world zones as follows: Africa 167 people, Americas 7,567 people, Eastern Mediterranean 1,019 people, Europe 4,571 people, South-East Asia 1,664 people and Western Pacific 1,716 people.

In Thailand, two cases of swine flu were first confirmed in May 2009, both of them had recently travelled back from Mexico [5]. Thailand was the 31st country in the world with reported cases of the swine flu. From May 2009 to March 2010, there were 35,446 confirmed swine flu patients and the number of deaths were 208 persons, 105 males and 113 females [6]. Recently, the Bureau of Epidemiology at the Ministry of Public Health reported total H1N1 cases of 28,607 persons (morbidity rate was 45.03 per 100,000 population) and total H1N1 deaths of 31 persons (mortality rate was 0.05 per 100,000 population) between January and March 2014. It found that the most H1N1 cases were patients aged 7-9 years, 10-14 years and 25-34 years, respectively. The highest number of patient was found in Lampang, Rayong, Chiang Mai, Bangkok and Uttaradit, respectively [7].

As above mentioned, we analyze the reported data from the Bureau of Epidemiology, Ministry of Public Health, Thailand in year 2009. The purposes of this research are formulating and analyzing the mathematical model for swine flu with the age structure of human population. The work provides the results for predicting the spread of the disease.

1.2 Objectives of the study

The objectives of this thesis are to study and develop the mathematical model for swine flu transmission incorporated with the age structure of human population. More specially, the research aims to:

1) To study the swine flu transmission and develop a mathematical model for the disease based on the Susceptible-Exposed-Infected-Quarantined-Recovered (SEIQR) model by introducing age structure into the SEIQR model.
2) To analyze the local stability and global stability of the model and also establish the conditions of parameters for stability.
3) To investigate the relationships between susceptible, exposed, infected, quarantined and recovered classes of each age group during the swine flu outbreaks.
4) To provide the results of mathematical model for analyzing, predicting, preventing, decreasing morbidity rate and controlling the endemic of swine flu.

1.3 Scope of the study

The scope of the study are as follow:

1) Analyze and study the reported data from the Bureau of Epidemiology, Ministry of Public Health, Thailand in year 2009.
2) Formulate the mathematical model for swine flu transmission incorporated with the age structure of human population. We suppose that the total population, the total number for each age group remain constant. The transmission rates of the disease in all age groups are assumed to be different.
3) Analyze the local stability of equilibrium states by applying the Routh–Hurwitz stability criterion and Hopf bifurcation.

4) Construct the Lypunov functions with the conditions to examine the global stability of equilibrium states.
5) Propose a SEIQR network model to study an epidemic network distribution.

1.4 Process of the study

The process of the study are as follow:

1) Study the reported data from the Bureau of Epidemiology, Ministry of Public Health, Thailand in year 2009.
2) Study definition, theoretical background and review the related modeling work.
3) Formulate the mathematical models.
4) Analyze the mathematical models.
5) Modify and develop the mathematical models.
6) Discuss the results in research.

1.5 Benefits

The results of this research would be a guideline to reduce the epidemic of swine flu.

Chapter 2
Literature Review

In this chapter, we introduce preliminary knowledge of swine flu and some theoretical background that related to this research.

2.1 Background of Swine Flu

2.1.1 Swine Flu Virus

The three types of influenza viruses that cause human flu are influenza A, influenza B and influenza C. Influenza A virus infects many animals such as pigs, horses, cats, dogs and birds. Influenza B virus infects only human and cause seasonal flu which is not severe. Influenza C infects pigs but it does not infect birds [8]. Each type of influenza virus is subdivided into different subtypes based on the antibody response to these viruses. There are at least 18 different H (hemagglutinin) subtypes which are named as H1 to H18 and 9 different N(neuraminidase) subtypes which are named as N1 to N9. The first three hemagglutinins(H1, H2 and H3) and two neuraminidase(N1 and N2) are found in human influenza virus [9]. Swine flu is caused by influenza A subtype H1N1, derived originally from a strain of virus that is endemic in pigs [10]. The swine flu virus resulted from reassortment, a process through two or more influenza viruses can swap genes, produce new and dangerous strains [11]. The virus has some gene segments from each of the infected parent viruses and may have different characteristics than either of the parental viruses [12]. Pigs can be both infected by avian flu viruses and by human flu viruses, therefore pig represent a host which produced new variant viruses and can pass to human [13]. The Swine flu virus has been determined as a reassortment of five different flu viruses, strains originated from birds, humans, North American pigs and Eurasian pigs [14]. At this time, there are four main influenza A subtypes that have been isolated in pigs: H1N1, H1N2, H3N2 and H3N1 [15]. The new strain is named as influenza A H3N2v (v means the virus is a variant that normally infects only pigs but has begun to infect human) caused an outbreak of swine flu in 2011 [16].

2.1.2 Swine Flu Transmission and Symptoms

The Swine flu virus spreads from human to human similarity to other influenza viruses. It spreads when infected people cough or sneeze, then droplets are diffused through the air and deposited on the mouth or nose of other people nearby. The virus can also survive on hard surfaces for up to 24 hours and soft surfaces for around 20 minutes in the surrounding environment. Sometimes people may become infected by touching droplets on a surface with virus on it and then they touch their faces. The transmission of virus from swine to human is irregularly possible. It is believed to occur mainly in swine farms where persons are in closed contact with infected pigs. The symptoms of swine flu are similar to those of regular human flu. Symptoms are fever, cough, sore throat, muscle pain, headache, runny nose, chills and fatigue [11]. Some people also are reported vomiting and diarrhea associated with swine flu [17]. The majority of swine flu cases have occurred in people under the age of 25 years, but most cases of severe and fatal infections have been in adults between 30 and 50 years old [18]. Some people are at higher risk of more complicated or severe illness when they get swine flu such as pregnant women, children younger than 5 years, people aged 65 years and older, people with medical conditions like asthma, diabetes, obesity, liver disease, heart disease, kidney disease and chronic respiratory [19,20].

2.1.3 Treatment and Medicine

For people who experience flu-like symptoms, stay at home for at least 7 days after the start of symptoms and limit contact with others to avoid spreading infection to others, until fully recovered. In addition: take plenty of rest, drink plenty of fluids, avoid smoking and eat nutritious food. Most flu people are able to recover at home without antiviral medicine or medical attention. People who are from higher risk group and who have persistent or rapidly worsening symptoms should be treated with antivirals. These symptoms include difficulty breathing or a high fever for more than three days [21]. For treatment, antiviral drugs work best if the medicines taken within 48 hours of the start of symptoms. There are four different antiviral drugs that are licensed for using to the treatment of influenza: amantadine, rimantadine, oseltamivir and zanamivir. The swine flu virus, isolated from humans, are resistant to amantadine and rimantadine. CDC (Centers for Disease Control and Prevention) recommended the use of oseltamivir or zanamivir for the treatment of infection with swine flu virus [11]. However, WHO reported 314 samples of the swine flu cases tested worldwide have shown resistance to oseltamivir [22], but nobody has any resistance to zanamivir [23]. Antiviral drugs must be taken for five days to treat flu symptoms. Oseltamivir is a tablet, the recommended dose is one 75 mg capsule twice a day for people aged 13 years and older. Dose varies by child's weight for children younger than 13 years: weight 15 kg or less is 30 mg twice a day, weight 16-23 kg is 45 mg twice a day, weight 24-40 kg is 60 mg twice a day and weight 41 kg or more is 75 mg twice a day. Zanamivir is taken as a power by inhalation and do not recommend for children younger than 5 years, the recommended dose is two inhalations (2x5mg) twice daily [21]. Both medications may have side effects such as lightheadedness, chills, nausea, vomiting, loss of appetite and trouble breathing [24].

2.1.4 Prevention and Vaccination

The swine flu virus was a new virus when it emerged, therefore most people had no or little immunity to it and vaccines from human seasonal flu provided little protection from swine flu virus. However, a study at CDC found that children had no preexisting immunity but adults 18-64 years old had 6-9%, and older adults had some degree of immunity 33% [25, 26]. It would be possible that the higher immunity in older adults may be due to previous exposure to similar seasonal influenza virus. Nowadays, there are two types of H1N1 vaccine available to protect against swine flu. One is an injectable vaccine that is usually given in the arm with a needle and the other is a nasal spray vaccine usually sprayed to inhale through the nose. Persons 10 years old and older will need one dose of vaccine. For children aged 6 months to 9 years will need two doses of vaccine, separated by 4 weeks. Infants younger than 6 months are too young to get any influenza vaccine. The H1N1 influenza vaccine is recommended for people who live with infants younger than 6 months, pregnant women, health care worker, anyone 6 months to 24 years of age and people age of 25 to 64 years at higher risk group. There are other ways to prevent the spread of illness, take everyday actions as follows: get a nose and mouth cover with a tissue when cough or sneeze, wash hands often with soap and water, especially after cough or sneeze, avoid touching eyes, nose or mouth, germs spread this way, try to avoid closed contact with sick people [27].

2.2 Prior Mathematical Modeling Studies

In 2009, Jumpen et al. [28] proposed a mathematical model based on SEIQR model for describing the pandemic influenza. The SEIQR model consists of 5 classes: susceptible (S(t)), individuals who can be infected with the disease; exposed (E(t)), individuals who are latently infected; (I(t)), individuals who have been infected and can transmit the disease to the susceptible individuals; quarantined (Q(t)), individuals who are isolated to stop the interactions with others in order to reduce the transmission of disease to susceptible individuals; and recovered (R(t)), individuals who have been infected and then recovered from the disease. Both classes are not able to be infected again since they have immunity to the disease. The dynamical transfer of the population is depicted in the following diagram.

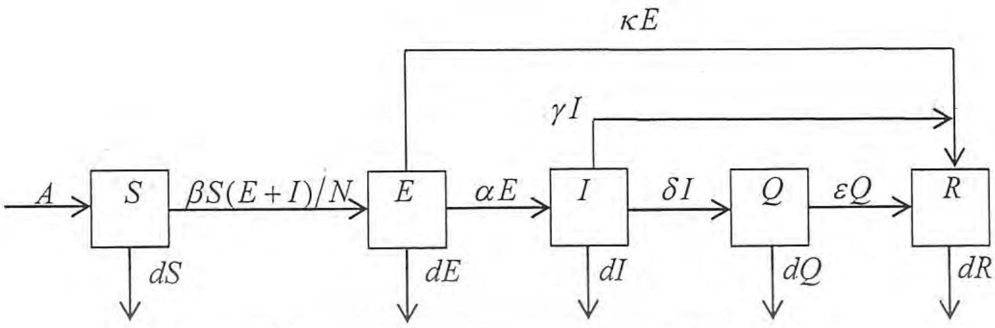

Figure 2.1 The transfer diagram for SEIQR model.

In above diagram, parameters $b, d, \beta, \alpha, \kappa, \gamma, \delta$ and ε are non-negative constants. Let $A = bS_0$ is a constant recruitment in which the constant b is the natural birth rate and S_0 is the initial value of susceptible individual. The other parameters are defined as follows: d is the natural mortality rate; $\beta = \beta_1 c$ in which β_1 is the probability of catching the disease per contact to the infected/ exposed person and c is the average number of people contacted by each person per day; α is the rate at which the exposed individuals E become the infected individuals I; δ is the rate at which the individuals leave the infective individuals I for the quarantined individuals Q; and $\kappa, \gamma, \varepsilon$ are the rates at which individuals in the E, I and Q classes recover from the disease.

The system of differential equations for SEIQR model is:

$$\frac{dS}{dt} = A - \frac{\beta S(E+I)}{N} - dS, \tag{2.1}$$

$$\frac{dE}{dt} = \frac{\beta S(E+I)}{N} - (d+\alpha+\kappa)E, \tag{2.2}$$

$$\frac{dI}{dt} = \alpha E - (d+\gamma+\delta)I, \tag{2.3}$$

$$\frac{dQ}{dt} = \delta I - (d+\varepsilon)Q, \tag{2.4}$$

$$\frac{dR}{dt} = \kappa E + \gamma I + \varepsilon Q - dR, \tag{2.5}$$

with the condition $N(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$.

They used standard dynamical analysis to determine the conditions on the values of the parameters which is form of the threshold number R_0 for local and global asymptotically stables of disease-free equilibrium, condition for local asymptotically stable of endemic equilibrium state is obtained. The basic reproductive number $R_0' = \sqrt{R_0}$ is defined as the average number of secondary infectious cases produced from primary infectious cases. The Routh-Hurwitz criteria is applied for locally asymptotically stable. If $R_0 \leq 1$, the disease-free equilibrium state is locally asymptotically stable and the endemic equilibrium state is locally asymptotically stable when $R_0 > 1$. The disease-free equilibrium state is shown to be globally asymptotically stable by constructing Lyapunov function.

In 2003, Pongsumpun and Tang [29] formulated the transmission model based on SIR (susceptible-infected-recovered) dynamical behavior for dengue hemorrhagic fever which included an age structure in the human population. The human population is divided into juvenile and adult classes with the different contact rates between two classes. The transmission rate of virus to a susceptible adult being much lower than that to a susceptible juvenile. They also simplified their model in the absence of infected adults because data records showed that the adults have only a small or no chance of becoming sick with DHF.

In 2009, Pongsumpun and Tang [30] formulated the transmission network model of Plasmodium Vivax malaria. They considered the dynamical transmission of P.vivax malaria between houses and villages. The human population is divided into four classes: susceptible, infected, dormant and recovered classes. There are q villages, L persons and u houses in each village. The persons move to any houses in this village by random process which random the 1^{st} person to the L^{th} person go to each house every day, one person can go only one time in one house in each day. The probability for each person to visit each house are equivalent and no person come from the outside of this village. At the first day, there is only one infected human in only one house, the consequences of changing the number of villages, number of houses in each village, contact rates and relapse rates are simulated.

In 2011, Jumpen et al. [31] proposed an SEIQR – SIS epidemic network model to study pandemic influenza. The network model contains two populations representing people and public places which called people nodes and hub nodes. The susceptible people nodes can be infected by infectious hub nodes and the susceptible hub nodes can be infected by infectious people nodes. They derived the approximate threshold condition to examine the stability of the model and presented a numerical simulation of the disease transmission in the network model. The results showed that the network parameters including hub radius, contact radius and visiting probability affected on the disease transmission. The larger size of hub radius and the higher value of visiting probability speed up the outbreak of the disease and heighten the number of infected people.

In 2013, Pongsumpun et al. [32] analyzed the infectious wave solutions of an SEIR model via a contact infection process. They showed the propagating infection wave and examined the change in shape of the disease wave under the different ranges of the contact and recovery rates. These parameters effect to the epidemic peak, the time of the epidemic peak and the length of outburst. The shapes of the waves for the various values of parameter with fixed velocities give the same time for the epidemic peaks but different slopes of the curves. They compared the infection waves of SEIR and SIR models, the peak of infection wave and the length of the outburst in the SEIR model is less than in the SIR model.

Chapter 3
Mathematical Model for Swine Flu

3.1 Analysis of Data and Formulating the Mathematical Model

In Thailand, the data of swine flu cases during May 2009 and July 2009, there were 2,925 cases with 12 deaths reported. The incidences are increased to 10,043 cases with 81 deaths in August 2009. A total of cases were reported from 67 provinces. The highest proportion of cases by age group were 11–20 years old 58.7%, 6–10 years old 18.7%, 21–30 years old 8.8%, 1–5 years old 4%, 31–40 years old 3.7%, 41–50 years old 3.2%, 51–60 years old 1.3%, more than 60 years old 0.5%, respectively. Figure 3.1 shows the reported data from the Bureau of Epidemiology, Ministry of Public Health, Thailand in year 2009 [32].

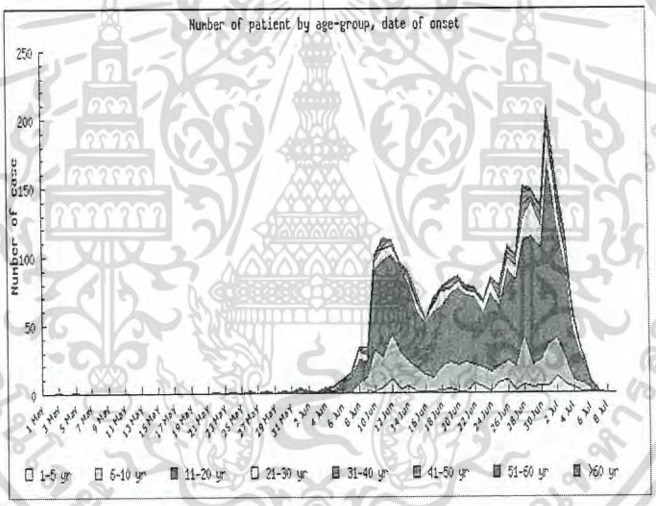

Figure 3.1 The number of cases due to swine flu by age group from date of onset.

There is the different distribution of this disease in each age group. We propose a mathematical model to study the transmission of swine flu by introducing age structure into the SEIQR model. The human population is classified into three age groups such as groups of people 1–10 years, 11 – 20 years, and more than 20 years, respectively. Each group has constant in size and is divided into five classes, i.e. susceptible means individuals who can be infected with the disease, exposed means individuals who are latently infected, infectious means individuals who have been infected and can transmit the disease to the susceptible individuals, quarantined means individuals who are isolated to stop the interactions with others in order to reduce the transmission of disease to susceptible individuals and recovered means individuals who have been infected and then recovered from the disease. The transmission diagram of swine flu is represented by the following figure.

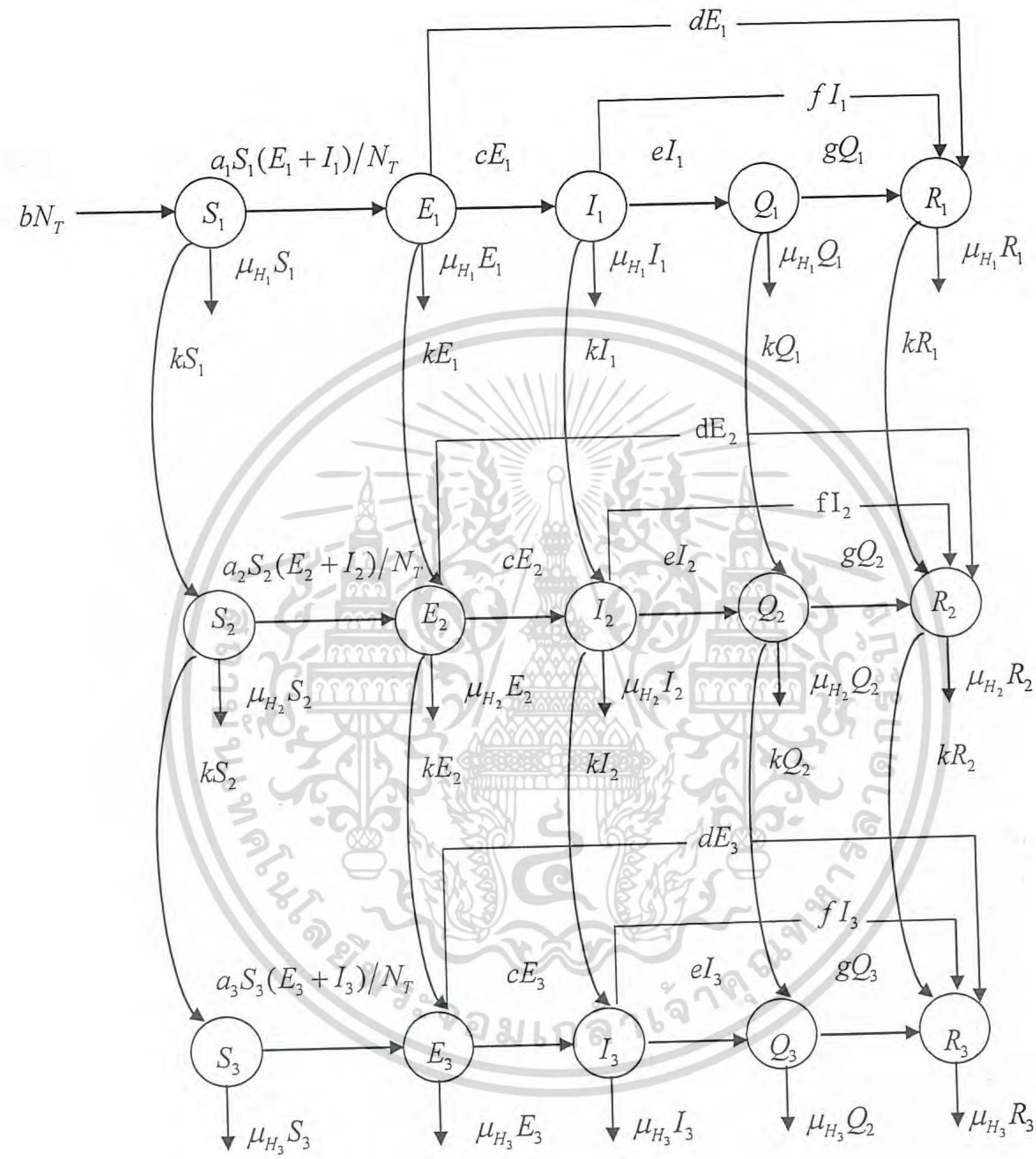

Figure 3.2 The diagram of the transmission model of swine flu.

Therefore, the SEIQR model with the age structure of human is described by the following details.

For the susceptible population:

The rate of change of the susceptible population of the first age group (per unit time) = the number of newborn (per unit time) – the number of the susceptible individuals of the first age group become the exposed individuals of the first age group (per unit time) – the number of deaths of susceptible individuals of the first age group (per unit time) – the number of susceptible individuals of the first age group pass into susceptible individuals of the second age group (per unit time).

The rate of change of the susceptible population of the second age group (per unit time)
= the number of susceptible individuals of the first age group pass to susceptible individuals of the second age group (per unit time) – the number of the susceptible individuals of the second age group become the exposed individuals of the second age group (per unit time) – the number of deaths of susceptible individuals of the second age group (per unit time) – the number of susceptible individuals of the second age group pass into susceptible individuals of the third age group (per unit time).

The rate of change of the susceptible population of the third age group (per unit time)
= the number of susceptible individuals of the second age group pass to susceptible individuals of the third age group (per unit time) – the number of the susceptible individuals of the third age group become the exposed individuals of the third age group (per unit time) – the number of deaths of susceptible individuals of the third age group (per unit time).

For the exposed population:

The rate of change of the exposed population of the first age group (per unit time)
= the number of the susceptible individuals of the first age group become the exposed individuals of the first age group (per unit time) – the number of deaths of exposed individuals of the first age group (per unit time) – the number of exposed individuals of the first age group pass into exposed individuals of the second age group (per unit time) – the number of the exposed individuals of the first age group become the infected individuals of the first age group (per unit time) – the number of the exposed individuals of the first age group recover from the disease (per unit time).

The rate of change of the exposed population of the second age group (per unit time)
= the number of the susceptible individuals of the second age group become the exposed individuals of the second age group (per unit time) + the number of exposed individuals of the first age group pass to be exposed individuals of the second age group (per unit time) – the number of deaths of exposed individuals of the second age group (per unit time) – the number of exposed individuals of the second age group pass into exposed individuals of the third age group (per unit time) – the number of the exposed individuals of the second age group become the infected individuals of the second age group (per unit time) – the number of the exposed individuals of the second age group recover from the disease (per unit time).

The rate of change of the exposed population of the third age group (per unit time)
= the number of the susceptible individuals of the third age group become the exposed individuals of the third age group (per unit time) + the number of exposed individuals of the second age group pass to be exposed individuals of the third age group (per unit time) – the number of deaths of exposed individuals of the third age group (per unit time) – the number of the exposed individuals of the third age group become the infected individuals of the third age group (per unit time) – the number of the exposed individuals of the third age group recover from the disease (per unit time).

For the infected population:

The rate of change of the infected population of the first age group (per unit time)
= the number of the exposed individuals of the first age group become the infected individuals of the first age group (per unit time) – the number of deaths of infected individuals of the first age group (per unit time) – the number of infected individuals of the first age group pass to be infected individuals of the second age group (per unit time) – the number of the infected individuals of the first age group become the quarantined individuals of the first age group (per unit time) – the number of the infected individuals of the first age group recover from the disease (per unit time).

The rate of change of the infected population of the second age group (per unit time)
= the number of the exposed individuals of the second age group become the infected individuals of the second age group (per unit time) + the number of infected individuals of the first age group pass to infected individuals of the second age group

(per unit time) – the number of deaths of infected individuals of the second age group (per unit time) – the number of infected individuals of the second age group pass to be infected individuals of the third age group (per unit time) – the number of the infected individuals of the second age group become the quarantined individuals of the second age group (per unit time) – the number of the infected individuals of the second age group recover from the disease (per unit time).

The rate of change of the infected population of the third age group (per unit time)
= the number of the exposed individuals of the third age group become the infected individuals of the third age group (per unit time) + the number of infected individuals of the second age group pass to be infected individuals of the third age group (per unit time) – the number of deaths of infected individuals of the third age group (per unit time) – the number of the infected individuals of the third age group become the quarantined individuals of the third age group (per unit time) – the number of the infected individuals of the third age group recover from the disease (per unit time).

For the quarantined population:

The rate of change of the quarantined population of the first age group (per unit time)
= the number of the infected individuals of the first age group become the quarantined individuals of the first age group (per unit time) – the number of deaths of quarantined individuals of the first age group (per unit time) – the number of quarantined individuals of the first age group pass to be quarantined individuals of the second age group (per unit time) – the number of the quarantined individuals of the first age group recover from the disease (per unit time).

The rate of change of the quarantined population of the second age group (per unit time)
= the number of the infected individuals of the second age group become the quarantined individuals of the second age group (per unit time) + the number of quarantined individuals of the first age group pass to be quarantined individuals of the second age group (per unit time) – the number of deaths of quarantined individuals of the second age group (per unit time) – the number of quarantined individuals of the second age group pass into quarantined individuals of the third age group (per unit time) – the number of the quarantined individuals of the second age group recover from the disease (per unit time).

The rate of change of the quarantined population of the third age group (per unit time)
= the number of the infected individuals of the third age group become the quarantined individuals of the third age group (per unit time) + the number of quarantined individuals of the second age group pass to be quarantined individuals of the third age group (per unit time) – the number of deaths of quarantined individuals of the third age group (per unit time) – the number of the quarantined individuals of the third age group recover from the disease (per unit time).

For the recovered population:

The rate of change of the recovered population of the first age group (per unit time)
= the number of the exposed individuals of the first age group recover from the disease (per unit time) + the number of the infected individuals of the first age group recover from the disease (per unit time) + the number of the quarantined individuals of the first age group recover from the disease (per unit time) – the number of deaths of recovered individuals of the first age group (per unit time) – the number of recovered individuals of the first age group pass to be recovered individuals of the second age group (per unit time).

The rate of change of the recovered population of the second age group (per unit time)
= the number of the exposed individuals of the second age group recover from the disease (per unit time) + the number of the infected individuals of the second age group recover from the disease (per unit time) + the number of the quarantined individuals of the second age group recover from the disease (per unit time) + the number of recovered individuals of the first age group pass to be recovered individuals of the second age group (per unit time) – the number of deaths of recovered individuals of the second age group (per unit time) – the number of recovered individuals of the second age group pass into recovered individuals of the third age group (per unit time).

The rate of change of the recovered population of the third age group (per unit time)
= the number of the exposed individuals of the third age group recover from the disease (per unit time) + the number of the infected individuals of the third age group recover from the disease (per unit time) + the number of the quarantined individuals of the third age group recover from the disease (per unit time) + the number of recovered individuals of the second age group pass to be recovered individuals of the third age group (per unit time) – the number of deaths of recovered individuals of the third age group (per unit time).

The rate of change of the susceptible population of the first age group, the second age group and the third age group (per unit time), respectively.

$$\frac{dS_1}{dt} = bN_T - \frac{a_1 S_1 (E_1 + I_1)}{N_T} - (\mu_{H_1} + k)S_1, \tag{3.1}$$

$$\frac{dS_2}{dt} = kS_1 - \frac{a_2 S_2 (E_2 + I_2)}{N_T} - (\mu_{H_2} + k)S_2, \tag{3.2}$$

$$\frac{dS_3}{dt} = kS_2 - \frac{a_3 S_3 (E_3 + I_3)}{N_T} - \mu_{H_3} S_3, \tag{3.3}$$

The rate of change of the exposed population of the first age group, the second age group and the third age group (per unit time), respectively.

$$\frac{dE_1}{dt} = \frac{a_1 S_1 (E_1 + I_1)}{N_T} - (\mu_{H_1} + c + d + k)E_1, \tag{3.4}$$

$$\frac{dE_2}{dt} = kE_1 + \frac{a_2 S_2 (E_2 + I_2)}{N_T} - (\mu_{H_2} + c + d + k)E_2, \tag{3.5}$$

$$\frac{dE_3}{dt} = kE_2 + \frac{a_3 S_3 (E_3 + I_3)}{N_T} - (\mu_{H_3} + c + d)E_3, \tag{3.6}$$

The rate of change of the infected population of the first age group, the second age group and the third age group (per unit time), respectively.

$$\frac{dI_1}{dt} = cE_1 - (\mu_{H_1} + e + f + k)I_1, \tag{3.7}$$

$$\frac{dI_2}{dt} = kI_1 + cE_2 - (\mu_{H_2} + e + f + k)I_2, \tag{3.8}$$

$$\frac{dI_3}{dt} = kI_2 + cE_3 - (\mu_{H_3} + e + f)I_3, \tag{3.9}$$

The rate of change of the quarantined population of the first age group, the second age group and the third age group (per unit time), respectively.

$$\frac{dQ_1}{dt} = eI_1 - (\mu_{H_1} + g + k)Q_1, \tag{3.10}$$

$$\frac{dQ_2}{dt} = kQ_1 + eI_2 - (\mu_{H_2} + g + k)Q_2, \tag{3.11}$$

$$\frac{dQ_3}{dt} = kQ_2 + eI_3 - (\mu_{H_3} + g)Q_3, \tag{3.12}$$

The rate of change of the recovered population of the first age group, the second age group and the third age group (per unit time), respectively

$$\frac{dR_1}{dt} = dE_1 + fI_1 + gQ_1 - (\mu_{H_1} + k)R_1, \tag{3.13}$$

$$\frac{dR_2}{dt} = kR_1 + dE_2 + fI_2 + gQ_2 - (\mu_{H_2} + k)R_2, \tag{3.14}$$

$$\frac{dR_3}{dt} = kR_2 + dE_3 + fI_3 + gQ_3 - \mu_{H_3}R_3, \tag{3.15}$$

where the parameters are defined as follows:

N_T is the total population,

N_{T_1} is the total number of the first age group,

N_{T_2} is the total number of the second age group,

N_{T_3} is the total number of the third age group,

b is the natural birth rate,

μ_{H_1} is the natural mortality rate of the first age group,

μ_{H_2} is the natural mortality rate the second age group,

μ_{H_3} is the natural mortality rate third age group,

k is the rate at which the first age group pass into the second age group and also the second age group pass into the third age group,

a_1 is equal to $\varepsilon_1\delta_n$ in which ε_1 is the probability of catching the disease per contact to the infected/exposed person and δ_n is the average number of people contacted by each person per day,

a_2 is equal to $\varepsilon_2\beta_n$ in which ε_2 is the probability of catching the disease per contact to the infected/exposed person and β_n is the average number of people contacted by each person per day,

a_3 is equal to $\varepsilon_3\alpha_n$ in which ε_3 is the probability of catching the disease per contact to the infected/exposed person and α_n is the average number of people contacted by each person per day,

c is the rate at which the exposed individuals E become the infected individuals I,

e is the rate at which the individuals leave the infective individuals I for the quarantined individuals Q,

d, f, g are the rate at which individuals in the E, I, Q classes recover from the disease, respectively.

Chapter 4
Local Stability Analysis

4.1 Mathematical Model

The SEIQR model in an age structural population is described by the following system of differential equations:

$$\frac{dS_1}{dt} = bN_T - \frac{a_1S_1(E_1+I_1)}{N_T} - (\mu_{H_1}+k)S_1, \tag{4.1}$$

$$\frac{dE_1}{dt} = \frac{a_1S_1(E_1+I_1)}{N_T} - (\mu_{H_1}+c+d+k)E_1, \tag{4.2}$$

$$\frac{dI_1}{dt} = cE_1 - (\mu_{H_1}+e+f+k)I_1, \tag{4.3}$$

$$\frac{dQ_1}{dt} = eI_1 - (\mu_{H_1}+g+k)Q_1, \tag{4.4}$$

$$\frac{dR_1}{dt} = dE_1 + fI_1 + gQ_1 - (\mu_{H_1}+k)R_1, \tag{4.5}$$

$$\frac{dS_2}{dt} = kS_1 - \frac{a_2S_2(E_2+I_2)}{N_T} - (\mu_{H_2}+k)S_2, \tag{4.6}$$

$$\frac{dE_2}{dt} = kE_1 + \frac{a_2S_2(E_2+I_2)}{N_T} - (\mu_{H_2}+c+d+k)E_2, \tag{4.7}$$

$$\frac{dI_2}{dt} = kI_1 + cE_2 - (\mu_{H_2}+e+f+k)I_2, \tag{4.8}$$

$$\frac{dQ_2}{dt} = kQ_1 + eI_2 - (\mu_{H_2}+g+k)Q_2, \tag{4.9}$$

$$\frac{dR_2}{dt} = kR_1 + dE_2 + fI_2 + gQ_2 - (\mu_{H_2}+k)R_2, \tag{4.10}$$

$$\frac{dS_3}{dt} = kS_2 - \frac{a_3S_3(E_3+I_3)}{N_T} - \mu_{H_3}S_3, \tag{4.11}$$

$$\frac{dE_3}{dt} = kE_2 + \frac{a_3S_3(E_3+I_3)}{N_T} - (\mu_{H_3}+c+d)E_3, \tag{4.12}$$

$$\frac{dI_3}{dt} = kI_2 + cE_3 - (\mu_{H_3}+e+f)I_3, \tag{4.13}$$

$$\frac{dQ_3}{dt} = kQ_2 + eI_3 - (\mu_{H_3} + g)Q_3, \tag{4.14}$$

$$\frac{dR_3}{dt} = kR_2 + dE_3 + fI_3 + gQ_3 - \mu_{H_3}R_3, \tag{4.15}$$

with the conditions

$N_{T_1} = S_1 + E_1 + I_1 + Q_1 + R_1$,

$N_{T_2} = S_2 + E_2 + I_2 + Q_2 + R_2$,

and $N_{T_3} = S_3 + E_3 + I_3 + Q_3 + R_3$.

We assume that the total population, total number of the first age group, total number of the second age group, and total number of the third age group are all constant. If we add equations (4.1) – (4.15) together, we get

$$\frac{d}{dt}\left(S_1 + E_1 + I_1 + Q_1 + R_1 + S_2 + E_2 + I_2 + Q_2 + R_2 + S_3 + E_3 + I_3 + Q_3 + R_3\right)$$
$$= bN_T - \mu_{H_1}\left(S_1 + E_1 + I_1 + Q_1 + R_1\right) - \mu_{H_2}\left(S_2 + E_2 + I_2 + Q_2 + R_2\right)$$
$$-\mu_{H_3}\left(S_3 + E_3 + I_3 + Q_3 + R_3\right),$$

or
$$\frac{dN_T}{dt} = bN_T - \mu_{H_1}N_{T_1} - \mu_{H_2}N_{T_2} - \mu_{H_3}N_{T_3}$$
$$= b\left(N_{T_1} + N_{T_2} + N_{T_3}\right) - \mu_{H_1}N_{T_1} - \mu_{H_2}N_{T_2} - \mu_{H_3}N_{T_3}$$
$$= \left(b - \mu_{H_1}\right)N_{T_1} + \left(b - \mu_{H_2}\right)N_{T_2} + \left(b - \mu_{H_3}\right)N_{T_3}. \tag{4.16}$$

Since the total population is constant, i.e. $\frac{dN_T}{dt} = 0$, thus the birth rate would have to be equal to the mortality rate of each group, $b = \mu_{H_1} = \mu_{H_2} = \mu_{H_3}$. If we add only equations (4.1) – (4.5) together, we get

$$\frac{d}{dt}\left(S_1 + E_1 + I_1 + Q_1 + R_1\right) = bN_T - (\mu_{H_1} + k)\left(S_1 + E_1 + I_1 + Q_1 + R_1\right),$$

or
$$\frac{dN_{T_1}}{dt} = bN_T - (b + k)N_{T_1}. \tag{4.17}$$

Since total number of the first age group is constant, i.e. $\frac{dN_{T_1}}{dt}=0$, equation (4.17) would give us the ratio between total number of the first age group and total population,

$$\frac{N_{T_1}}{N_T}=\frac{b}{b+k}. \tag{4.18}$$

If we add only equations (4.6) – (4.10) together, we get

$$\frac{d}{dt}\left(S_2+E_2+I_2+Q_2+R_2\right)$$

$$=k\left(S_1+E_1+I_1+Q_1+R_1\right)-(\mu_{H_2}+k)\left(S_2+E_2+I_2+Q_2+R_2\right),$$

or

$$\frac{dN_{T_2}}{dt}=kN_{T_1}-(b+k)N_{T_2}. \tag{4.19}$$

Since total number of the second age group is constant, i.e. $\frac{dN_{T_2}}{dt}=0$, equation (4.19) would give us the ratio between total number of the first age group and total number of the second age group,

$$\frac{N_{T_1}}{N_{T_2}}=\frac{b+k}{k}. \tag{4.20}$$

Similarly, by adding equations (4.11) – (4.15), we get

$$\frac{d}{dt}\left(S_3+E_3+I_3+Q_3+R_3\right)=k\left(S_2+E_2+I_2+Q_2+R_3\right)-\mu_{H_3}\left(S_3+E_3+I_3+Q_3+R_3\right),$$

or

$$\frac{dN_{T_3}}{dt}=kN_{T_2}-bN_{T_3}, \tag{4.21}$$

and substituting $\frac{dN_{T_3}}{dt}=0$ into (4.21) then gives the ratio between total number of the second age group and total number of the third age group,

$$\frac{N_{T_2}}{N_{T_3}}=\frac{b}{k}. \tag{4.22}$$

From equation (4.20), we obtain $N_{T_1} = N_{T_2}\left(\frac{b+k}{k}\right)$. Substituting N_{T_1} into equation (4.18) then gives

$$\frac{N_{T_2}}{N_T}\left(\frac{b+k}{k}\right) = \frac{b}{b+k},$$

or

$$\frac{N_{T_2}}{N_T} = \frac{bk}{(b+k)^2}. \tag{4.23}$$

Similarly, by substituting $N_{T_2} = N_{T_3}\left(\frac{b}{k}\right)$ from equation (4.22) into equation (4.23), we obtain

$$\frac{N_{T_3}}{N_T}\left(\frac{b}{k}\right) = \frac{bk}{(b+k)^2},$$

or

$$\frac{N_{T_3}}{N_T} = \frac{k^2}{(b+k)^2}. \tag{4.24}$$

To normalize equations (4.1) – (4.15), we let

$$\overline{S}_1 = \frac{S_1}{N_{T_1}},\ \overline{E}_1 = \frac{E_1}{N_{T_1}},\ \overline{I}_1 = \frac{I_1}{N_{T_1}},\ \overline{Q}_1 = \frac{Q_1}{N_{T_1}},\ \overline{R}_1 = \frac{R_1}{N_{T_1}},$$

$$\overline{S}_2 = \frac{S_2}{N_{T_2}},\ \overline{E}_2 = \frac{E_2}{N_{T_2}},\ \overline{I}_2 = \frac{I_2}{N_{T_2}},\ \overline{Q}_2 = \frac{Q_2}{N_{T_2}},\ \overline{R}_2 = \frac{R_2}{N_{T_2}},$$

$$\overline{S}_3 = \frac{S_3}{N_{T_3}},\ \overline{E}_3 = \frac{E_3}{N_{T_3}},\ \overline{I}_3 = \frac{I_3}{N_{T_3}},\ \overline{Q}_3 = \frac{Q_3}{N_{T_3}} \text{ and } \overline{R}_3 = \frac{R_3}{N_{T_3}}.$$

We obtain the new three conditions:

$\overline{S}_1 + \overline{E}_1 + \overline{I}_1 + \overline{Q}_1 + \overline{R}_1 = 1$, $\overline{S}_2 + \overline{E}_2 + \overline{I}_2 + \overline{Q}_2 + \overline{R}_2 = 1$, and $\overline{S}_3 + \overline{E}_3 + \overline{I}_3 + \overline{Q}_3 + \overline{R}_3 = 1$.

Hence the equations (4.1) – (4.15) can be written as

$$\frac{d\overline{S}_1}{dt} = m_1(1-\overline{S}_1) - m_2\overline{S}_1(\overline{E}_1+\overline{I}_1), \tag{4.25}$$

$$\frac{d\overline{E}_1}{dt} = m_2\overline{S}_1(\overline{E}_1+\overline{I}_1) - m_3\overline{E}_1, \tag{4.26}$$

$$\frac{d\overline{I}_1}{dt} = c\overline{E}_1 - m_4\overline{I}_1, \tag{4.27}$$

$$\frac{d\overline{Q}_1}{dt} = e\overline{I}_1 - m_5\overline{Q}_1, \tag{4.28}$$

$$\frac{d\overline{S}_2}{dt} = m_1\left(\overline{S}_1-\overline{S}_2\right) - m_6\overline{S}_2(\overline{E}_2+\overline{I}_2), \tag{4.29}$$

$$\frac{d\overline{E}_2}{dt} = m_1\overline{E}_1 + m_6\overline{S}_2(\overline{E}_2+\overline{I}_2) - m_3\overline{E}_2, \tag{4.30}$$

$$\frac{d\overline{I}_2}{dt} = m_1\overline{I}_1 + c\overline{E}_2 - m_4\overline{I}_2, \tag{4.31}$$

$$\frac{d\overline{Q}_2}{dt} = m_1\overline{Q}_1 + e\overline{I}_2 - m_5\overline{Q}_2, \tag{4.32}$$

$$\frac{d\overline{S}_3}{dt} = b\left(\overline{S}_2-\overline{S}_3\right) - m_7\overline{S}_3(\overline{E}_3+\overline{I}_3), \tag{4.33}$$

$$\frac{d\overline{E}_3}{dt} = b\overline{E}_2 + m_7\overline{S}_3(\overline{E}_3+\overline{I}_3) - m_8\overline{E}_3, \tag{4.34}$$

$$\frac{d\overline{I}_3}{dt} = b\overline{I}_2 + c\overline{E}_3 - m_9\overline{I}_3, \tag{4.35}$$

$$\frac{d\overline{Q}_3}{dt} = b\overline{Q}_2 + e\overline{I}_3 - m_{10}\overline{Q}_3, \tag{4.36}$$

where $m_1 = b+k$, $m_2 = \dfrac{a_1 b}{b+k}$, $m_3 = b+c+d+k$, $m_4 = b+e+f+k$, $m_5 = b+g+k$,

$m_6 = \dfrac{a_2 bk}{(b+k)^2}$, $m_7 = \dfrac{a_3 k^2}{(b+k)^2}$, $m_8 = b+c+d$, $m_9 = b+e+f$ and $m_{10} = b+g$.

4.2 Analysis of the Mathematical Model

In this section, we use standard dynamical analysis to determine the conditions of parameters for local asymptotically stability.

4.2.1 The Steady State Solutions

From (4.25) – (4.28) with the right hand side equal to zero, it can be seen that the equilibrium points must satisfy the following relations:

$$\overline{S}_1 = \frac{m_1}{m_2(\overline{E}_1 + \overline{I}_1) + m_1}, \tag{4.37}$$

$$\overline{I}_1 = \frac{c}{m_4}\overline{E}_1, \tag{4.38}$$

$$\overline{Q}_1 = \frac{e}{m_5}\overline{I}_1. \tag{4.39}$$

Substituting equations (4.37) and (4.38) into equation (4.26), we obtain $\overline{E}_1$ which is a solution of the following quadratic equation:

$$m_2 m_3 (c + m_4)\overline{E}_1^2 + (m_1 m_3 m_4 - m_1 m_2 m_4 - m_1 m_2 c)\overline{E}_1 = 0. \tag{4.40}$$

Setting the right hand side of (4.29) – (4.32) equal to zero, we obtain the following relations:

$$\overline{S}_2 = \frac{m_1 \overline{S}_1}{m_6(\overline{E}_2 + \overline{I}_2) + m_1}, \tag{4.41}$$

$$\overline{I}_2 = \frac{m_1 \overline{I}_1 + c\overline{E}_2}{m_4}, \tag{4.42}$$

$$\overline{Q}_2 = \frac{m_1 \overline{Q}_1 + e\overline{I}_2}{m_5}. \tag{4.43}$$

Substituting equations (4.41) and (4.42) into equation (4.30), we obtain $\overline{E}_2$ which is a solution of the following quadratic equation:

$$m_3 m_6 (c + m_4)\overline{E}_2^2 + \left[m_1 m_3 m_4 + m_1 m_6 \left(m_3 \overline{I}_1 - (c + m_4)\overline{S}_1 - (c + m_4)\overline{E}_1 \right) \right] \overline{E}_2$$
$$- \left[m_1^2 m_6 \overline{I}_1 (\overline{S}_1 + \overline{E}_1) + m_1^2 m_4 \overline{E}_1 \right] = 0. \tag{4.44}$$

Then, let right hand side of (4.33) – (4.36) equal to zero, we get the following relations:

$$\overline{S}_3 = \frac{b\overline{S}_2}{m_7(\overline{E}_3+\overline{I}_3)+b}, \quad (4.45)$$

$$\overline{I}_3 = \frac{b\overline{I}_2 + c\overline{E}_3}{m_9}, \quad (4.46)$$

$$\overline{Q}_3 = \frac{b\overline{Q}_2 + e\overline{I}_3}{m_{10}}. \quad (4.47)$$

Substituting equations (4.45) and (4.46) into equation (4.44), we obtain $\overline{E}_3$ which is a solution of the following quadratic equation:

$$m_7m_8(c+m_9)\overline{E}_3^2 + \left[bm_8m_9 + bm_7\left(m_8\overline{I}_2 - (c+m_9)\overline{S}_2 - (c+m_9)\overline{E}_2\right)\right]\overline{E}_3$$
$$-\left[b^2m_7\overline{I}_2(\overline{S}_2+\overline{E}_2) + b^2m_9\overline{E}_2\right] = 0. \quad (4.48)$$

Next, we consider all equilibrium points that satisfy equations (4.37) – (4.48).

First, The solutions of equation (4.40) are $\overline{E}_1 = 0$ or $\overline{E}_1 = \dfrac{m_1m_2(c+m_4) - m_1m_3m_4}{m_2m_3(c+m_4)}$.

If we substitute $\overline{E}_1 = 0$ in equations (4.37) – (4.39), we get $\overline{S}_1 = 1,\ \overline{I}_1 = 0,$ and $\overline{Q}_1 = 0$, and equation (4.44) becomes

$$m_3m_6(c+m_4)\overline{E}_2^2 + \left[m_1m_3m_4 - m_1m_6(c+m_4)\right]\overline{E}_2 = 0. \quad (4.49)$$

The solutions of equation (4.49) are $\overline{E}_2 = 0$ and $\overline{E}_2 = \dfrac{m_1m_6(c+m_4) - m_1m_3m_4}{m_3m_6(c+m_4)}$.

If we substitute $\overline{S}_1 = 1,\ \overline{E}_1 = 0,\ \overline{I}_1 = 0,\ \overline{Q}_1 = 0$ and $\overline{E}_2 = 0$ in equations (4.41) – (4.43), we get $\overline{S}_2 = 1,\ \overline{I}_2 = 0,$ and $\overline{Q}_2 = 0$, and equation (4.48) becomes

$$m_7m_8(c+m_9)\overline{E}_3^2 + \left[bm_8m_9 - bm_7(c+m_9)\overline{S}_2\right]\overline{E}_3 = 0. \quad (4.50)$$

Substituting the solutions $\overline{E}_3 = 0$ and $\overline{E}_3 = \dfrac{bm_7(c+m_9) - bm_8m_9}{m_7m_8(c+m_9)}$ of equation (4.50) with values: $\overline{S}_1 = 1,\ \overline{E}_1 = 0,\ \overline{I}_1 = 0,\ \overline{Q}_1 = 0,\ \overline{S}_2 = 1,\ \overline{E}_2 = 0,\ \overline{I}_2 = 0,$ and $\overline{Q}_2 = 0$ in equations (4.45) – (4.47), we obtain two equilibrium points:

$$P_0 = (1,0,0,0,1,0,0,0,1,0,0,0) \text{ and } \hat{P} = (1,0,0,0,1,0,0,0,\hat{S}_3,\hat{E}_3,\hat{I}_3,\hat{Q}_3),$$

where $\hat{S}_3 = \frac{1}{\hat{R}_0}$, $\hat{E}_3 = \frac{b}{m_8\hat{R}_0}\left[\hat{R}_0 - 1\right]$, $\hat{I}_3 = \frac{bc}{m_8 m_9 \hat{R}_0}\left[\hat{R}_0 - 1\right]$,

$$\hat{Q}_3 = \frac{bce}{m_8 m_9 m_{10}\hat{R}_0}\left[\hat{R}_0 - 1\right] \text{ and } \hat{R}_0 = \frac{m_7(c+m_9)}{m_8 m_9}.$$

The third equilibrium point is found from substituting: $\overline{S}_1 = 1$, $\overline{E}_1 = 0$, $\overline{I}_1 = 0$, $\overline{Q}_1 = 0$ and $\overline{E}_2 = \frac{m_1 m_6(c+m_4) - m_1 m_3 m_4}{m_3 m_6 (c+m_4)}$ (or rewritten as $\overline{E}_2 = \tilde{E}_2$) in equations (4.41) – (4.43), we get $\overline{S}_2 = \tilde{S}_2$, $\overline{I}_2 = \tilde{I}_2$, and $\overline{Q}_2 = \tilde{Q}_2$, and equation (4.48) becomes

$$m_7 m_8 (c+m_9)\overline{E}_3^2 + \left[bm_8 m_9 + bm_7\left(m_8\tilde{I}_2 - (c+m_9)\tilde{S}_2 - (c+m_9)\tilde{E}_2\right)\right]\overline{E}_3$$
$$-\left[b^2 m_7 \tilde{I}_2(\tilde{S}_2 + \tilde{E}_2) + b^2 m_9 \tilde{E}_2\right] = 0,$$

or

$$A_1\overline{E}_3^2 + A_2\overline{E}_3 + A_3 = 0, \qquad (4.51)$$

where

$$A_1 = m_7 m_8 (c+m_9),$$

$$A_2 = bm_8 m_9 + bm_7\left(m_8\tilde{I}_2 - (c+m_9)\tilde{S}_2 - (c+m_9)\tilde{E}_2\right),$$

and

$$A_3 = -\left[b^2 m_7 \tilde{I}_2(\tilde{S}_2 + \tilde{E}_2) + b^2 m_9 \tilde{E}_2\right].$$

The solutions of (4.51) are given by

$$\overline{E}_3 = \frac{-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}}{2A_1}.$$

Since A_1 is positive and A_3 is negative, it is easy to see that the term in square root of $\overline{E}_3$, $A_2^2 - 4A_1A_3$ is positive and $A_2^2 - 4A_1A_3$ is greater than A_2. It implies that $-A_2 - \sqrt{A_2^2 - 4A_1A_3}$ is less than zero. Thus, there is the only one solution $\overline{E}_3 = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}$ of equation (4.51).

Substituting the solution $\overline{E}_3 = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}$ (or rewritten as $\overline{E}_3 = \tilde{E}_3$) of equation (4.51) with values: $\overline{S}_2 = \tilde{S}_2$, $\overline{I}_2 = \tilde{I}_2$, $\overline{E}_2 = \tilde{E}_2$, and $\overline{Q}_2 = \tilde{Q}_2$ in equations (4.45) – (4.47), we obtain the third equilibrium point

$$\tilde{P} = (1,0,0,0,\tilde{S}_2,\tilde{E}_2,\tilde{I}_2,\tilde{Q}_2,\tilde{S}_3,\tilde{E}_3,\tilde{I}_3,\tilde{Q}_3),$$

where $\tilde{S}_2 = \frac{1}{\tilde{R}_0}$, $\tilde{E}_2 = \frac{m_1}{m_3\tilde{R}_0}\left[\tilde{R}_0 - 1\right]$, $\tilde{I}_2 = \frac{m_1 c}{m_3 m_4 \hat{R}_0}\left[\tilde{R}_0 - 1\right]$,

$$\tilde{Q}_2 = \frac{m_1 ce}{m_3 m_4 m_5 \tilde{R}_0}\left[\tilde{R}_0 - 1\right],\ \tilde{S}_3 = \frac{b\tilde{S}_2}{m_7(\tilde{E}_3 + \tilde{I}_3) + b},\ \tilde{E}_3 = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1},$$

$$\tilde{I}_3 = \frac{b\tilde{I}_2 + c\tilde{E}_3}{m_9},\ \tilde{Q}_3 = \frac{b\tilde{Q}_2 + e\tilde{I}_3}{m_{10}},\text{ and } \tilde{R}_0 = \frac{m_6(c+m_4)}{m_3 m_4}.$$

The forth equilibrium point is obtained from substituting $\overline{E}_1 = \frac{m_1 m_2(c+m_4) - m_1 m_3 m_4}{m_2 m_3 (c+m_4)}$ (or rewritten as $\overline{E}_1 = E_1^*$), we get $\overline{S}_1 = S_1^*$, $\overline{I}_1 = I_1^*$ and $\overline{Q}_1 = Q_1^*$, and equation (4.44) becomes

$$m_3 m_6 (c+m_4)\overline{E}_2^2 + \left[m_1 m_3 m_4 + m_1 m_6\left(m_3 I_1^* - (c+m_4)S_1^* - (c+m_4)E_1^*\right)\right]\overline{E}_2$$
$$-\left[m_1^2 m_6 I_1^*(S_1^* + E_1^*) - m_1^2 m_4 E_1^*\right] = 0,$$

or

$$B_1\overline{E}_2^2 + B_2\overline{E}_2 + B_3 = 0, \tag{4.52}$$

where $B_1 = m_3 m_6 (c+m_4)$,

$B_2 = m_1 m_3 m_4 + m_1 m_6\left(m_3 I_1^* - (c+m_4)S_1^* - (c+m_4)E_1^*\right)$,

and $B_3 = -\left[m_1^2 m_6 I_1^*(S_1^* + E_1^*) + m_1^2 m_4 E_1^*\right]$.

We use the same method as above, there is the only one solution $\overline{E}_2 = \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1}$ of equation (4.52). Substituting the solution $\overline{E}_2 = \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1}$ (or rewritten as $\overline{E}_2 = E_2^*$) of equation (4.52) with values: $\overline{S}_1 = S_1^*$, $\overline{I}_1 = I_1^*$, $\overline{E}_1 = E_1^*$, $\overline{Q}_1 = Q_1^*$, and $\overline{E}_2 = E_2^*$ in equations (4.41) – (4.43), we get $\overline{S}_2 = S_2^*$, $\overline{I}_2 = I_2^*$, and $\overline{Q}_2 = Q_2^*$, and equation (4.48) becomes

$$m_7 m_8 (c+m_9)\overline{E}_3^2 + \left[b m_8 m_9 + b m_7 \left(m_8 I_2^* - (c+m_9) S_2^* - (c+m_9) E_2^* \right) \right] \overline{E}_3$$

$$- \left[b^2 m_7 I_2^* (S_2^* + E_2^*) + b^2 m_9 E_2^* \right] = 0 ,$$

or
$$C_1 \overline{E}_3^2 + C_2 \overline{E}_3 + C_3 = 0 , \tag{4.53}$$

where $C_1 = m_7 m_8 (c+m_9)$,

$$C_2 = b m_8 m_9 + b m_7 \left(m_8 I_2^* - (c+m_9) S_2^* - (c+m_9) E_2^* \right),$$

and $C_3 = -\left[b^2 m_7 I_2^* (S_2^* + E_2^*) + b^2 m_9 E_2^* \right]$.

With the same method as above, there is the only one solution $\overline{E}_3 = \dfrac{-C_2 + \sqrt{C_2^2 - 4C_1 C_3}}{2C_1}$ of equation (4.53). Substituting the solution $\overline{E}_3 = \dfrac{-C_2 + \sqrt{C_2^2 - 4C_1 C_3}}{2C_1}$ (or rewritten as $\overline{E}_3 = E_3^*$) of equation (4.53) with values: $\overline{S}_2 = S_2^*$, $\overline{E}_2 = E_2^*$, $\overline{I}_2 = I_2^*$, $\overline{Q}_2 = Q_2^*$, and $\overline{E}_3 = E_3^*$ in equations (4.45) – (4.47), we get the forth equilibrium point

$$P^* = (S_1^*, E_1^*, I_1^*, Q_1^*, S_2^*, E_2^*, I_2^*, Q_2^*, S_3^*, E_3^*, I_3^*, Q_3^*) ,$$

where $S_1^* = \dfrac{1}{R_0^*}$, $E_1^* = \dfrac{m_1}{m_3 R_0^*}\left[R_0^* - 1 \right]$, $I_1^* = \dfrac{m_1 c}{m_3 m_4 R_0^*}\left[R_0^* - 1 \right]$,

$$Q_1^* = \frac{m_1 ce}{m_3 m_4 m_5 R_0^*}\left[R_0^* - 1 \right],\ S_2^* = \frac{m_1 S_1^*}{m_6 (E_2^* + I_2^*) + m_1},\ E_2^* = \frac{-B_2 + \sqrt{B_2^2 - 4B_1 B_3}}{2B_1}$$

$$I_2^* = \frac{m_1 I_1^* + cE_2^*}{m_4},\ Q_2^* = \frac{m_1 Q_1^* + eI_2^*}{m_5},\ S_3^* = \frac{bS_2^*}{m_7 (E_3^* + I_3^*) + b},$$

$$E_3^* = \frac{-C_2 + \sqrt{C_2^2 - 4C_1 C_3}}{2C_1},\ I_3^* = \frac{bI_2^* + cE_3^*}{m_9},\ Q_3^* = \frac{bQ_2^* + eI_3^*}{m_{10}},$$

and $R_0^* = \dfrac{m_2 (c+m_4)}{m_3 m_4}$.

4.2.2 The local stability Analysis

The local stability of the equilibrium solutions can be determined from the Jacobian matrix of the linearized system (4.25) – (4.36) evaluated at the equilibrium solutions. The eigenvalues are obtained by solving the characteristic equations; $\det(J(p)-\lambda I_{12})=0$, where $J(p)$ is the Jacobian matrix at equilibrium point p, I_{12} is the identity matrix dimension 12×12. If all eigenvalues are negative real parts, then the equilibrium solution is locally stable.

First, we consider the Jacobian matrix corresponding to equations (4.25) – (4.36) is the 12×12 matrix given by

$$J=\begin{bmatrix} D_1 & 0 & 0 \\ D_2 & D_3 & 0 \\ 0 & D_4 & D_5 \end{bmatrix},$$

where

$$D_1=\begin{bmatrix} -m_1-m_2(\overline{E}_1+\overline{I}_1) & -m_2\overline{S}_1 & -m_2\overline{S}_1 & 0 \\ m_2(\overline{E}_1+\overline{I}_1) & m_2\overline{S}_1-m_3 & m_2\overline{S}_1 & 0 \\ 0 & c & -m_4 & 0 \\ 0 & 0 & e & -m_5 \end{bmatrix},$$

$$D_2=\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_1 \end{bmatrix},$$

$$D_3=\begin{bmatrix} -m_1-m_6(\overline{E}_2+\overline{I}_2) & -m_6\overline{S}_2 & -m_6\overline{S}_2 & 0 \\ m_6(\overline{E}_2+\overline{I}_2) & m_6\overline{S}_2-m_3 & m_6\overline{S}_2 & 0 \\ 0 & c & -m_4 & 0 \\ 0 & 0 & e & -m_5 \end{bmatrix},$$

$$D_4=\begin{bmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix},$$

and $$D_5 = \begin{bmatrix} -b - m_7(\overline{E}_3 + \overline{I}_3) & -m_7\overline{S}_3 & -m_7\overline{S}_3 & 0 \\ m_7(\overline{E}_3 + \overline{I}_3) & m_7\overline{S}_3 - m_8 & m_7\overline{S}_3 & 0 \\ 0 & c & -m_9 & 0 \\ 0 & 0 & e & -m_{10} \end{bmatrix}.$$

J evaluated at $P_0 = (1,0,0,0,1,0,0,0,1,0,0,0)$ is

$$J(P_0) = \begin{bmatrix} D_{0_1} & 0 & 0 \\ D_{0_2} & D_{0_3} & 0 \\ 0 & D_{0_4} & D_{0_5} \end{bmatrix}$$

where

$$D_{0_1} = \begin{bmatrix} -m_1 & -m_2 & -m_2 & 0 \\ m_2 & m_2 - m_3 & m_2 & 0 \\ 0 & c & -m_4 & 0 \\ 0 & 0 & e & -m_5 \end{bmatrix}, \quad D_{0_2} = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_1 \end{bmatrix},$$

$$D_{0_3} = \begin{bmatrix} -m_1 & -m_6 & -m_6 & 0 \\ m_6 & m_6 - m_3 & m_6 & 0 \\ 0 & c & -m_4 & 0 \\ 0 & 0 & e & -m_5 \end{bmatrix}, \quad D_{0_4} = \begin{bmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix},$$

and $$D_{0_5} = \begin{bmatrix} -b & -m_7 & -m_7 & 0 \\ 0 & m_7 - m_8 & m_7 & 0 \\ 0 & c & -m_9 & 0 \\ 0 & 0 & e & -m_{10} \end{bmatrix}.$$

The characteristic equation of above matrix is

$$(\lambda + b)(\lambda + m_1)^2(\lambda + m_5)^2(\lambda + m_{10})\left[\lambda^2 + r_1\lambda + r_2\right]$$

$$\left[\lambda^2 + s_1\lambda + s_2\right]\left[\lambda^2 + t_1\lambda + t_2\right] = 0, \qquad (4.54)$$

where $r_1 = (m_3 + m_4 - m_2)$, $r_2 = m_3m_4 - m_2(c + m_4)$, $s_1 = (m_3 + m_4 - m_6)$,

$s_2 = m_8m_9 - m_7(c + m_9)$, $t_1 = (m_8 + m_9 - m_7)$, and $t_2 = m_8m_9 - m_7(c + m_9)$.

From the characteristic equation (4.54), we see that the first six eigenvalues are

$$\lambda_1 = -b, \lambda_{2,3} = -m_1, \lambda_{4,5} = -m_5 \text{ and } \lambda_6 = -m_{10},$$

all of these eigenvalues are negative. The remaining eigenvalues are found by solving

$$\lambda^2 + r_1\lambda + r_2 = 0, \lambda^2 + s_1\lambda + s_2 = 0 \text{ and } \lambda^2 + t_1\lambda + t_2 = 0.$$

For determine the stability of solutions, we only need the signs of all eigenvalues are negative. Thus, we apply the Routh–Hurwitz criteria to the equation in form $\lambda^2 + A\lambda + B = 0$, the stability holds if and only if $A > 0$ and $B > 0$.

For $\lambda^2 + r_1\lambda + r_2 = 0$, if $R_0^* = \dfrac{m_2(c+m_4)}{m_3m_4} < 1$, we have

$$r_1 = m_3 + m_4 - m_2 = \frac{(1-R_0^*)m_3m_4 + cm_3}{c+m_4} + m_4 > 0,$$

and

$$r_2 = m_3m_4 - m_2(c+m_4) = m_3m_4\left(1-R_0^*\right) > 0.$$

For $\lambda^2 + s_1\lambda + s_2 = 0$, if $\tilde{R}_0 = \dfrac{m_6(c+m_4)}{m_3m_4} < 1$, we have

$$s_1 = m_3 + m_4 - m_6 = \frac{(1-\tilde{R}_0)m_3m_4 + cm_3}{c+m_4} + m_4 > 0,$$

and

$$s_2 = m_3m_4 - m_6(c+m_4) = m_3m_4\left(1-\tilde{R}_0\right) > 0.$$

For $\lambda^2 + t_1\lambda + t_2 = 0$, if $\hat{R}_0 = \dfrac{m_7(c+m_9)}{m_8m_9} < 1$, we have

$$t_1 = m_8 + m_9 - m_7 = \frac{(1-\hat{R}_0)m_8m_9 + cm_8}{c+m_9} + m_9 > 0,$$

and

$$t_2 = m_8m_9 - m_7(c+m_9) = m_8m_9\left(1-\hat{R}_0\right) > 0.$$

Thus all roots of the three characteristic equations have negative real parts if $\hat{R}_0 < 1, \tilde{R}_0 < 1$ and $R_0^* < 1$. Therefore, the disease-free equilibrium P_0 is locally asymptotically stable when $R_0 = \max\left\{\hat{R}_0, \tilde{R}_0, R_0^*\right\} < 1$.

J evaluated at $\hat{P}=(1,0,0,0,1,0,0,0,\hat{S}_3,\hat{E}_3,\hat{I}_3,\hat{Q}_3)$ is

$$J(\hat{P})=\begin{bmatrix}\hat{D}_1 & 0 & 0\\ \hat{D}_2 & \hat{D}_3 & 0\\ 0 & \hat{D}_4 & \hat{D}_5\end{bmatrix},$$

where

$$\hat{D}_1=\begin{bmatrix}-m_1 & -m_2 & -m_2 & 0\\ m_2 & m_2-m_3 & m_2 & 0\\ 0 & c & -m_4 & 0\\ 0 & 0 & e & -m_5\end{bmatrix},\ \hat{D}_2=\begin{bmatrix}m_1 & 0 & 0 & 0\\ 0 & m_1 & 0 & 0\\ 0 & 0 & m_1 & 0\\ 0 & 0 & 0 & m_1\end{bmatrix},$$

$$\hat{D}_3=\begin{bmatrix}-m_1 & -m_6 & -m_6 & 0\\ m_6 & m_6-m_3 & m_6 & 0\\ 0 & c & -m_4 & 0\\ 0 & 0 & e & -m_5\end{bmatrix},\ \hat{D}_4=\begin{bmatrix}b & 0 & 0 & 0\\ 0 & b & 0 & 0\\ 0 & 0 & b & 0\\ 0 & 0 & 0 & b\end{bmatrix},$$

and $$\hat{D}_5=\begin{bmatrix}-b-m_7(\hat{E}_3+\hat{I}_3) & -m_7\hat{S}_3 & -m_7\hat{S}_3 & 0\\ m_7(\hat{E}_3+\hat{I}_3) & m_7\hat{S}_3-m_8 & m_7\hat{S}_3 & 0\\ 0 & c & -m_9 & 0\\ 0 & 0 & e & -m_{10}\end{bmatrix}.$$

The characteristic equation of above matrix is

$$(\lambda+m_1)^2(\lambda+m_5)^2(\lambda+m_{10})\left[\lambda^2+r_1\lambda+r_2\right]$$
$$\left[\lambda^2+s_1\lambda+s_2\right]\left[\lambda^3+t_1\lambda^2+t_2\lambda+t_3\right]=0, \qquad (4.55)$$

where $r_1=(m_3+m_4-m_2)$, $r_2=m_3m_4-m_2(c+m_4)$, $s_1=(m_3+m_4-m_6)$,

$s_2=m_8m_9-m_7(c+m_9)$, $t_1=b+m_8+m_9+m_7(\hat{E}_3+\hat{I}_3-\hat{S}_3)$,

$t_2=bm_8+bm_9+m_8m_9+m_7(m_8+m_9)(\hat{E}_3+\hat{I}_3)-m_7\hat{S}_3(b+c+m_9)$,

and $t_3=bm_8m_9+m_7m_8m_9(\hat{E}_3+\hat{I}_3)-bm_7(c+m_9)\hat{S}_3$.

From the characteristic equation (4.55), we see that the first five eigenvalues are

$$\lambda_{1,2} = -m_1, \lambda_{3,4} = -m_5 \text{ and } \lambda_5 = -m_{10},$$

all of these eigenvalues are negative. The remaining eigenvalues are found by solving

$$\lambda^2 + r_1\lambda + r_2 = 0, \lambda^2 + s_1\lambda + s_2 = 0 \text{ and } \lambda^3 + t_1\lambda^2 + t_2\lambda + t_3 = 0.$$

For $\lambda^2 + r_1\lambda + r_2 = 0$ and $\lambda^2 + s_1\lambda + s_2 = 0$, using the same method as above, all roots of the two characteristic equations have negative real parts if $R_0^* < 1$ and $\tilde{R}_0 < 1$, respectively.

Thus, we apply the Routh–Hurwitz criteria to the equation in form $\lambda^3 + t_1\lambda^2 + t_2\lambda + t_3 = 0$, the stability holds if and only if $t_1 > 0$, $t_3 > 0$ and $t_1t_2 - t_3 > 0$.

If $\hat{R}_0 = \dfrac{m_7(c+m_9)}{m_8m_9} > 1$, we have

$$\begin{aligned}
t_1 &= b + m_8 + m_9 + m_7(\hat{E}_3 + \hat{I}_3 - \hat{S}_3) \\
&= b + m_9 + \frac{cm_7}{m_9\hat{R}_0} + m_7(\hat{E}_3 + \hat{I}_3) \\
&= m_9 + \frac{cm_7}{m_9\hat{R}_0} + b\hat{R}_0 > 0, \\
t_2 &= bm_8 + bm_9 + m_8m_9 + m_7(m_8 + m_9)(\hat{E}_3 + \hat{I}_3) - m_7\hat{S}_3(b + c + m_9) \\
&= bm_9 + \frac{bcm_7}{m_9\hat{R}_0} + m_7\left(m_8 + m_9\right)(\hat{E}_3 + \hat{I}_3) \\
&= bm_9\hat{R}_0 + \frac{bcm_7}{m_9\hat{R}_0} + bm_8(\hat{R}_0 - 1) > 0, \\
t_3 &= bm_8m_9 + m_7m_8m_9(\hat{E}_3 + \hat{I}_3) - bm_7(c + m_9)\hat{S}_3 \\
&= m_7m_8m_9(\hat{E}_3 + \hat{I}_3) \\
&= bm_8m_9(\hat{R}_0 - 1) > 0.
\end{aligned}$$

It can be easily seen that

$$t_1t_2 - t_3 = \left[bm_9^2\hat{R}_0 + \frac{bcm_7}{\hat{R}_0} + bm_8m_9(\hat{R}_0 - 1) + bcm_7 + b\left(\frac{cm_7}{m_9\hat{R}_0}\right)^2 + \frac{bcm_7m_8}{m_9\hat{R}_0}(\hat{R}_0 - 1) \right.$$

$$\left. + m_9\left(b\hat{R}_0\right)^2 + \frac{b^2cm_7}{m_9} + b^2m_8\hat{R}_0(\hat{R}_0 - 1) \right] - bm_8m_9(\hat{R}_0 - 1)$$

$$= \left[bm_9^2\hat{R}_0 + \frac{bcm_7}{\hat{R}_0} + bcm_7 + b\left(\frac{cm_7}{m_9\hat{R}_0}\right)^2 + \frac{bcm_7m_8}{m_9\hat{R}_0}(\hat{R}_0 - 1) \right.$$

$$\left. + m_9\left(b\hat{R}_0\right)^2 + \frac{b^2cm_7}{m_9} + b^2m_8\hat{R}_0(\hat{R}_0 - 1) \right] > 0.$$

Hence the Routh-Hurwitz conditions are satisfied. Thus the third age group endemic equilibrium state $\hat{P}$ is locally asymptotically stable when $R_0 = \max\left\{\tilde{R}_0, R_0^*\right\} < 1$ and $\hat{R}_0 > 1$.

J evaluated at $\tilde{P} = (1,0,0,0,\tilde{S}_2,\tilde{E}_2,\tilde{I}_2,\tilde{Q}_2,\tilde{S}_3,\tilde{E}_3,\tilde{I}_3,\tilde{Q}_3)$ is

$$J(\tilde{P}) = \begin{bmatrix} \tilde{D}_1 & 0 & 0 \\ \tilde{D}_2 & \tilde{D}_3 & 0 \\ 0 & \tilde{D}_4 & \tilde{D}_5 \end{bmatrix},$$

where

$$\tilde{D}_1 = \begin{bmatrix} -m_1 & -m_2 & -m_2 & 0 \\ m_2 & m_2 - m_3 & m_2 & 0 \\ 0 & c & -m_4 & 0 \\ 0 & 0 & e & -m_5 \end{bmatrix}, \tilde{D}_2 = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_1 \end{bmatrix},$$

$$\tilde{D}_3 = \begin{bmatrix} -m_1 - m_6(\tilde{E}_2 + \tilde{I}_2) & -m_6\tilde{S}_2 & -m_6\tilde{S}_2 & 0 \\ m_6(\tilde{E}_2 + \tilde{I}_2) & m_6\tilde{S}_2 - m_3 & m_6\tilde{S}_2 & 0 \\ 0 & c & -m_4 & 0 \\ 0 & 0 & e & -m_5 \end{bmatrix}, \tilde{D}_4 = \begin{bmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix},$$

and $$\tilde{D}_5 = \begin{bmatrix} -b - m_7(\tilde{E}_3 + \tilde{I}_3) & -m_7\tilde{S}_3 & -m_7\tilde{S}_3 & 0 \\ m_7(\tilde{E}_3 + \tilde{I}_3) & m_7\tilde{S}_3 - m_8 & m_7\tilde{S}_3 & 0 \\ 0 & c & -m_9 & 0 \\ 0 & 0 & e & -m_{10} \end{bmatrix}.$$

The characteristic equation of above matrix is

$$(\lambda + m_1)(\lambda + m_5)^2(\lambda + m_{10})\left[\lambda^2 + r_1\lambda + r_2\right]$$

$$\left[\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3\right]\left[\lambda^3 + t_1\lambda^2 + t_2\lambda + t_3\right] = 0, \tag{4.56}$$

where $r_1 = (m_3 + m_4 - m_2)$, $r_2 = m_3m_4 - m_2(c + m_4)$, $s_1 = m_1 + m_3 + m_4 + m_6(\tilde{E}_2 + \tilde{I}_2 - \tilde{S}_2)$,

$s_2 = m_1m_3 + m_1m_4 + m_3m_4 + m_6(m_3 + m_4)(\tilde{E}_2 + \tilde{I}_2) - m_6\tilde{S}_2(m_1 + c + m_4)$,

$s_3 = m_1m_3m_4 + m_6m_3m_4(\tilde{E}_2 + \tilde{I}_2) - m_1m_6(c + m_4)\tilde{S}_2$, $t_1 = b + m_8 + m_9 + m_7(\tilde{E}_3 + \tilde{I}_3 - \tilde{S}_3)$,

$t_2 = bm_8 + bm_9 + m_8m_9 + m_7(m_8 + m_9)(\tilde{E}_3 + \tilde{I}_3) - m_7\tilde{S}_3(b + c + m_9)$,

and $t_3 = bm_8m_9 + m_7m_8m_9(\tilde{E}_3 + \tilde{I}_3) - bm_7(c + m_9)\tilde{S}_3$.

From the characteristic equation (4.56), we see that the first four eigenvalues are

$$\lambda_1 = -m_1, \lambda_{2,3} = -m_5 \text{ and } \lambda_4 = -m_{10},$$

all of these eigenvalues are negative. The remaining eigenvalues are found by solving

$$\lambda^2 + r_1\lambda + r_2 = 0,\ \lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 = 0 \text{ and } \lambda^3 + t_1\lambda^2 + t_2\lambda + t_3 = 0.$$

For $\lambda^2 + r_1\lambda + r_2 = 0$, it same as above, all the roots of the characteristic equations have negative real parts if $R_0^* < 1$.

For $\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 = 0$, if $\tilde{R}_0 = \dfrac{m_6(c + m_4)}{m_3m_4} > 1$, we have

$$s_1 = m_1 + m_3 + m_4 + m_6(\tilde{E}_2 + \tilde{I}_2 - \tilde{S}_2)$$

$$= m_1 + m_4 + \frac{cm_6}{m_4\tilde{R}_0} + m_6(\tilde{E}_2 + \tilde{I}_2)$$

$$= m_4 + \frac{cm_6}{m_4\tilde{R}_0} + m_1\tilde{R}_0 > 0,$$

$$s_2 = m_1m_3 + m_1m_4 + m_3m_4 + m_6(m_3+m_4)(\tilde{E}_2+\tilde{I}_2) - m_6\tilde{S}_2(m_1+c+m_4)$$

$$= m_1m_4 + \frac{m_1cm_6}{m_4\tilde{R}_0} + \left(m_3+m_4\right)m_6(\tilde{E}_2+\tilde{I}_2)$$

$$= m_1m_4\tilde{R}_0 + \frac{m_1cm_6}{m_4\tilde{R}_0} + m_1m_3(\tilde{R}_0-1) > 0,$$

$$s_3 = m_1m_3m_4 + m_6m_3m_4(\tilde{E}_2+\tilde{I}_2) - m_1m_6(c+m_4)\tilde{S}_2$$

$$= m_6m_3m_4(\tilde{E}_2+\tilde{I}_2)$$

$$= m_1m_3m_4(\tilde{R}_0-1) > 0.$$

It can be easily seen that

$$s_1s_2 - s_3 = \left[m_1m_4^2\tilde{R}_0 + \frac{cm_1m_6}{\tilde{R}_0} + m_1m_3m_4(\tilde{R}_0-1) + cm_1m_6 + c\left(\frac{m_1m_6}{m_4\tilde{R}_0}\right)^2 + \frac{cm_1m_3m_6}{m_4\tilde{R}_0}(\tilde{R}_0-1)\right.$$

$$\left. + m_4\left(m_1\tilde{R}_0\right)^2 + \frac{m_1^2cm_6}{m_4} + m_1^2m_3\tilde{R}_0(\tilde{R}_0-1)\right] - m_1m_3m_4(\tilde{R}_0-1)$$

$$= \left[m_1m_4^2\tilde{R}_0 + \frac{cm_1m_6}{\tilde{R}_0} + cm_1m_6 + c\left(\frac{m_1m_6}{m_4\tilde{R}_0}\right)^2 + \frac{cm_1m_3m_6}{m_4\tilde{R}_0}(\tilde{R}_0-1)\right.$$

$$\left. + m_4\left(m_1\tilde{R}_0\right)^2 + \frac{m_1^2cm_6}{m_4} + m_1^2m_3\tilde{R}_0(\tilde{R}_0-1)\right] > 0.$$

Hence the Routh–Hurwitz conditions are satisfied.

For $\lambda^3 + t_1\lambda^2 + t_2\lambda + t_3 = 0$, with $\tilde{R}_0 = \dfrac{m_6(c+m_4)}{m_3m_4} > 1$, we have

$$t_1 = b + m_8 + m_9 + m_7(\tilde{E}_3 + \tilde{I}_3 - \tilde{S}_3),$$

$$t_2 = bm_8 + bm_9 + m_8m_9 + m_7(m_8+m_9)(\tilde{E}_3+\tilde{I}_3) - m_7\tilde{S}_3(b+c+m_9),$$

$$t_3 = bm_8m_9 + m_7m_8m_9(\tilde{E}_3+\tilde{I}_3) - bm_7(c+m_9)\tilde{S}_3,$$

where $\tilde{E}_3, \tilde{I}_3$ and $\tilde{S}_3$ are defined in terms of $\tilde{E}_2, \tilde{I}_2$ and $\tilde{S}_2$ as before.

It might be complicated to show by hand that $\lambda^3+t_1\lambda^2+t_2\lambda+t_3=0$ satisfy the Routh–Hurwitz conditions, use *MATLAB* to graph the conditions, shown in the following figure by assigning various values of a_2 in which $\tilde{R}_0=\dfrac{m_6(c+m_4)}{m_3m_4}>1$ and the other parameters are fixed.

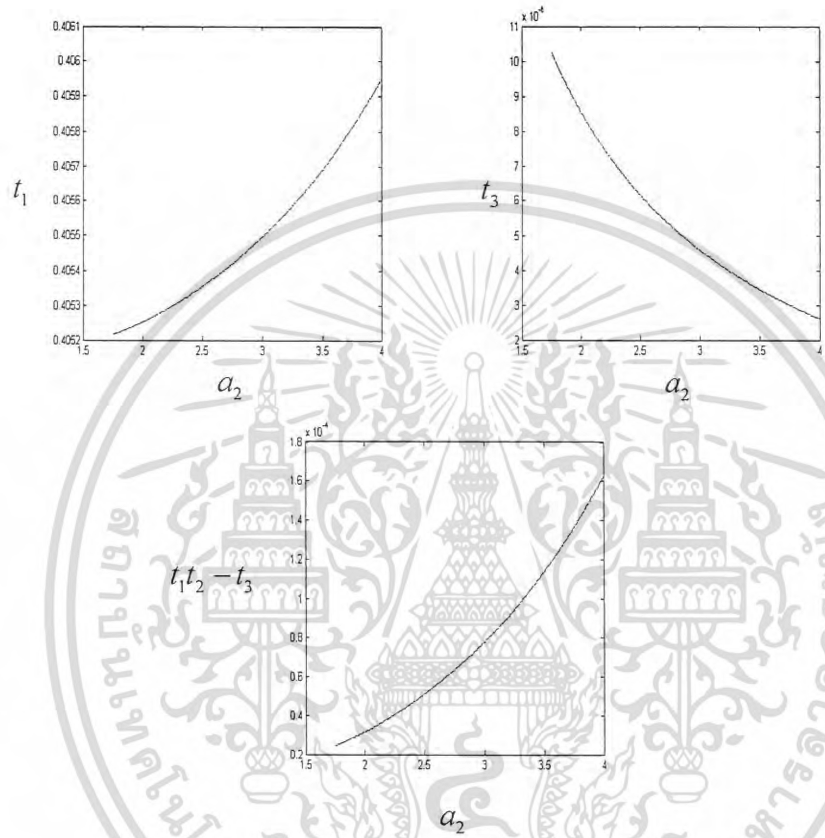

Figure 4.1 The parameter spaces for equation $\lambda^3+t_1\lambda^2+t_2\lambda+t_3=0$ which satisfies the Routh– Hurwitz conditions. The values of other parameters are k = 0.000273973, b = 0.000039139, c = 0.111111, d = 0.142857, e = 0.2, f = 0.142857, g = 0.7 and $a_3=1$.

From the above figure, the Routh–Hurwitz conditions are satisfied for $\widetilde{\mathrm{R}}_0>1$.
Thus, the second and third age group endemic equilibrium state $\widetilde{\mathrm{P}}$ is locally asymptotically stable when $\mathrm{R}_0^*<1$ and $\widetilde{\mathrm{R}}_0>1$.

J evaluated at $P^* = (S_1^*, E_1^*, I_1^*, Q_1^*, S_2^*, E_2^*, I_2^*, Q_2^*, S_3^*, E_3^*, I_3^*, Q_3^*)$ is

$$J = \begin{bmatrix} D_1^* & 0 & 0 \\ D_2^* & D_3^* & 0 \\ 0 & D_4^* & D_5^* \end{bmatrix},$$

where

$$D_1^* = \begin{bmatrix} -m_1 - m_2(E_1^* + I_1^*) & -m_2 S_1^* & -m_2 S_1^* & 0 \\ m_2(E_1^* + I_1^*) & m_2 S_1^* - m_3 & m_2 S_1^* & 0 \\ 0 & c & -m_4 & 0 \\ 0 & 0 & e & -m_5 \end{bmatrix},$$

$$D_2^* = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_1 \end{bmatrix},$$

$$D_3^* = \begin{bmatrix} -m_1 - m_6(E_2^* + I_2^*) & -m_6 S_2^* & -m_6 S_2^* & 0 \\ m_6(E_2^* + I_2^*) & m_6 S_2^* - m_3 & m_6 S_2^* & 0 \\ 0 & c & -m_4 & 0 \\ 0 & 0 & e & -m_5 \end{bmatrix},$$

$$D_4^* = \begin{bmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix},$$

and $$D_5^* = \begin{bmatrix} -b - m_7(E_3^* + I_3^*) & -m_7 S_3^* & -m_7 S_3^* & 0 \\ m_7(E_3^* + I_3^*) & m_7 S_3^* - m_8 & m_7 S_3^* & 0 \\ 0 & c & -m_9 & 0 \\ 0 & 0 & e & -m_{10} \end{bmatrix}.$$

The characteristic equation of above matrix is

$$(\lambda+m_5)^2(\lambda+m_{10})\left[\lambda^3+r_1\lambda^2+r_2\lambda+r_3\right]\times$$
$$\left[\lambda^3+s_1\lambda^2+s_2\lambda+s_3\right]\left[\lambda^3+t_1\lambda^2+t_2\lambda+t_3\right]=0, \qquad (4.57)$$

where $r_1=m_1+m_3+m_4+m_2(E_1^*+I_1^*-S_1^*)$,

$r_2=m_1m_3+m_1m_4+m_3m_4+m_2(m_3+m_4)(E_1^*+I_1^*)-m_2S_1^*(m_1+c+m_4)$,

$r_3=m_1m_3m_4+m_2m_3m_4(E_1^*+I_1^*)-m_1m_2(c+m_4)S_1^*$,

$s_1=m_1+m_3+m_4+m_6(E_2^*+I_2^*-S_2^*)$,

$s_2=m_1m_3+m_1m_4+m_3m_4+m_6(m_3+m_4)(E_2^*+I_2^*)-m_6S_2^*(m_1+c+m_4)$,

$s_3=m_1m_3m_4+m_6m_3m_4(E_2^*+I_2^*)-m_1m_6(c+m_4)S_2^*$,

$t_1=b+m_8+m_9+m_7(E_3^*+I_3^*-S_3^*)$,

$t_2=bm_8+bm_9+m_8m_9+m_7(m_8+m_9)(E_3^*+I_3^*)-m_7S_3^*(b+c+m_9)$,

and $t_3=bm_8m_9+m_7m_8m_9(E_3^*+I_3^*)-bm_7(c+m_9)S_3^*$.

From the characteristic equation (4.57), we see that the first four eigenvalues are

$$\lambda_{1,2}=-m_5 \text{ and } \lambda_3=-m_{10}.$$

All of these eigenvalues are negative. The remaining eigenvalues are found by solving

$$\lambda^2+r_1\lambda+r_2=0,\ \lambda^3+s_1\lambda^2+s_2\lambda+s_3=0 \text{ and } \lambda^3+t_1\lambda^2+t_2\lambda+t_3=0.$$

For $\lambda^2+r_1\lambda+r_2=0$, if $R_0^*=\dfrac{m_2(c+m_4)}{m_3m_4}>1$, we have

$$r_1=m_1+m_3+m_4+m_2(E_1^*+I_1^*-S_1^*)$$
$$=m_1+m_4+\frac{cm_2}{m_4R_0^*}+m_2(E_1^*+I_1^*)$$
$$=m_4+\frac{cm_2}{m_4R_0^*}+m_1R_0^*>0,$$

$$r_2 = m_1m_3 + m_1m_4 + m_3m_4 + m_2(m_3+m_4)(E_1^* + I_1^*) - m_2S_1^*(m_1 + c + m_4)$$

$$= m_1m_4 + \frac{cm_1m_2}{m_4R_0^*} + m_2\left(m_3+m_4\right)\left(E_1^* + I_1^*\right)$$

$$= m_1m_4R_0^* + \frac{cm_1m_2}{m_4R_0^*} + m_1m_3(R_0^* - 1) > 0\,,$$

$$r_3 = m_1m_3m_4 + m_2m_3m_4(E_1^* + I_1^*) - m_1m_2(c+m_4)S_1^*$$

$$= m_2m_3m_4(E_1^* + I_1^*)$$

$$= m_1m_3m_4(R_0^* - 1) > 0\,.$$

It can be easily seen that

$$r_1r_2 - r_3 = \left[m_1m_4^2\tilde{R}_0 + \frac{cm_1m_2}{R_0^*} + m_1m_3m_4(R_0^*-1) + cm_1m_2 + m_1\left(\frac{cm_2}{m_4R_0^*}\right)^2 + \frac{cm_1m_2m_3}{m_4R_0^*}(R_0^*-1) \right.$$

$$\left. + m_4\left(m_1R_0^*\right)^2 + \frac{cm_1^2m_2}{m_4} + m_1^2m_3R_0^*(R_0^*-1) \right] - m_1m_3m_4(R_0^*-1)$$

$$= \left[m_1m_4^2\tilde{R}_0 + \frac{cm_1m_2}{R_0^*} + cm_1m_2 + m_1\left(\frac{cm_2}{m_4R_0^*}\right)^2 + \frac{cm_1m_2m_3}{m_4R_0^*}(R_0^*-1) \right.$$

$$\left. + m_4\left(m_1R_0^*\right)^2 + \frac{cm_1^2m_2}{m_4} + m_1^2m_3R_0^*(R_0^*-1) \right] > 0.$$

Hence the Routh–Hurwitz conditions are satisfied.

For $\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 = 0$ and $\lambda^3 + t_1\lambda^2 + t_2\lambda + t_3 = 0$, with $R_0^* = \dfrac{m_2(c+m_4)}{m_3m_4} > 1$,

we have

$$s_1 = m_1 + m_3 + m_4 + m_6(E_2^* + I_2^* - S_2^*)\,,$$

$$s_2 = m_1m_3 + m_1m_4 + m_3m_4 + m_6(m_3+m_4)(E_2^* + I_2^*) - m_6S_2^*(m_1 + c + m_4)\,,$$

$$s_3 = m_1m_3m_4 + m_6m_3m_4(E_2^* + I_2^*) - m_1m_6(c+m_4)S_2^*\,,$$

$$t_1 = b + m_8 + m_9 + m_7(E_3^* + I_3^* - S_3^*)\,,$$

$$t_2 = bm_8 + bm_9 + m_8m_9 + m_7(m_8+m_9)(E_3^* + I_3^*) - m_7S_3^*(b + c + m_9)\,,$$

$$t_3 = bm_8m_9 + m_7m_8m_9(E_3^* + I_3^*) - bm_7(c+m_9)S_3^*\,,$$

where $E_2^*, I_2^*, S_2^*, E_3^*, I_3^*$ and S_3^* are defined in terms of E_1^*, I_1^* and S_1^* as before.

Proceeding with the same manner as above, use *MATLAB* to graph the conditions of the Routh–Hurwitz, shown in the following figure by assigning various values of a_1 in which $R_0^* = \dfrac{m_2(c+m_4)}{m_3 m_4} > 1$ and the other parameters are fixed.

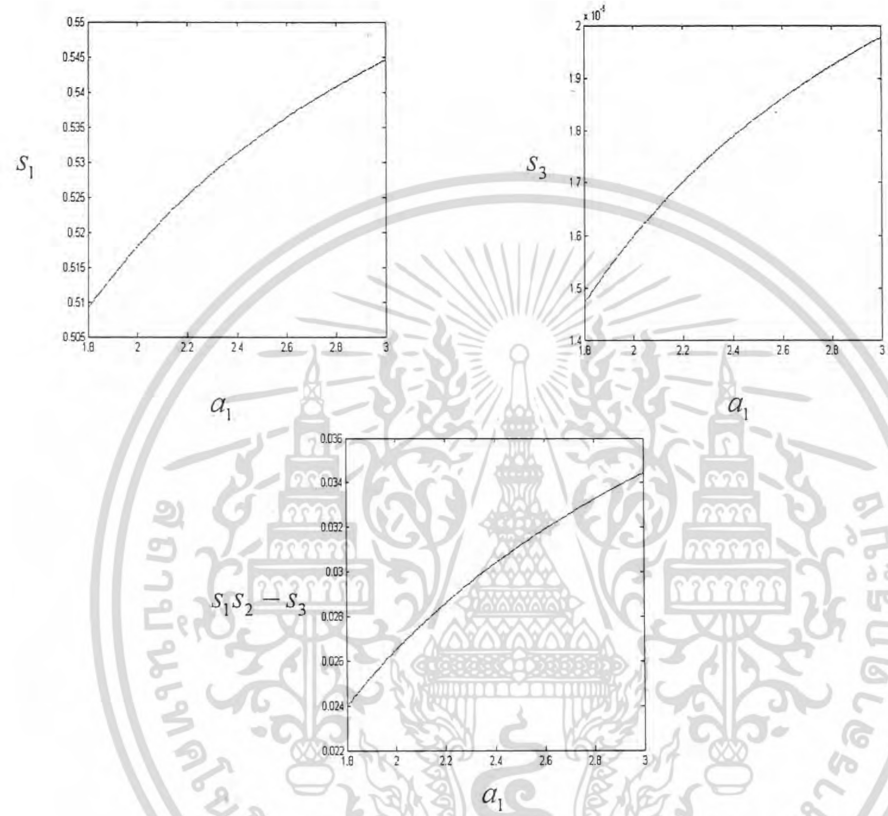

Figure 4.2 The parameter spaces for equation $\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 = 0$ which satisfies the Routh– Hurwitz conditions. The values of other parameters are k = 0.000273973, b = 0.000039139, c = 0.111111, d = 0.142857, e = 0.2, f = 0.142857, g = 0.7 and $a_2 = 0.95$.

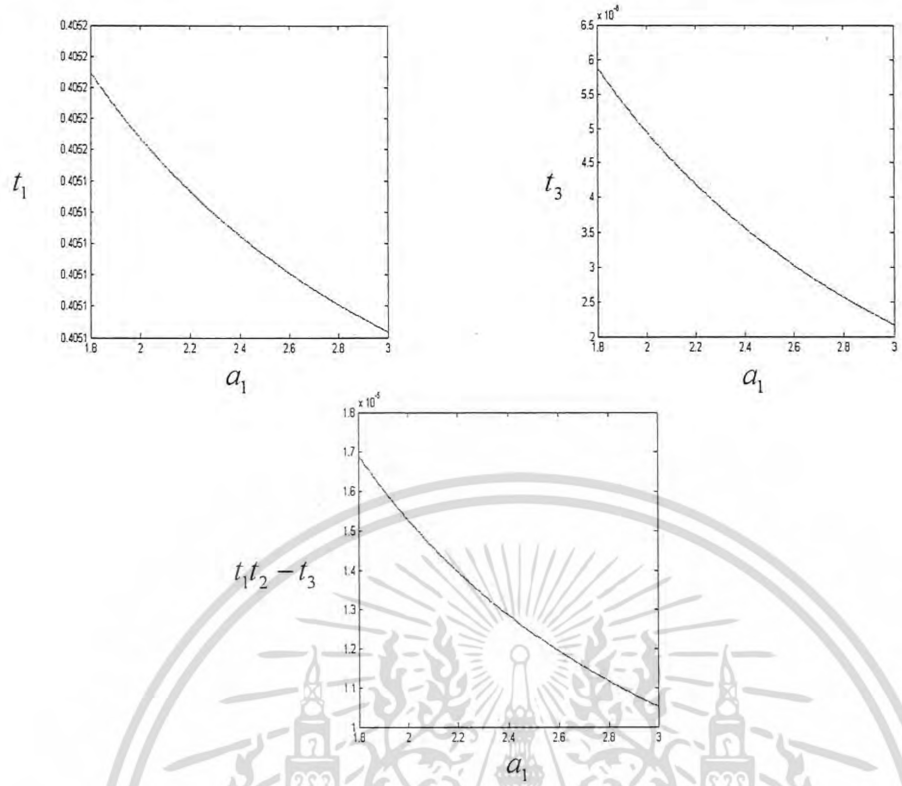

Figure 4.3 The parameter spaces for equation $\lambda^3 + t_1\lambda^2 + t_2\lambda + t_3 = 0$ which satisfies the Routh– Hurwitz conditions. The values of other parameters are k = 0.000273973, b = 0.000039139, c = 0.111111, d = 0.142857, e = 0.2, f = 0.142857, g = 0.7, a_2 = 0.95 and a_3 = 0.8.

From the above figure, the Routh–Hurwitz conditions are satisfied for $R_0^* > 1$.
Thus, the endemic equilibrium P^* is locally asymptotically stable when $R_0^* > 1$.

4.3 Numerical Results

We consider the numerical solutions for the transmission of swine flu in each age group. The trajectories of the solutions when the parameter values will lead to each equilibrium state are shown in the figures. The values of parameters used in this study are determined by the real life observations. The values of the parameters are as follows: $b = 1/(365 \times 70)$ day^{-1}, corresponding to a life expectancy of 70 years for human; each age group develops into another age group is 10 years, so $k = 1/(365 \times 10)$ day^{-1}. The other parameters are arbitrarily chosen. Numerical solutions of system of differential equations (4.25) – (4.36) are shown in the following figures.

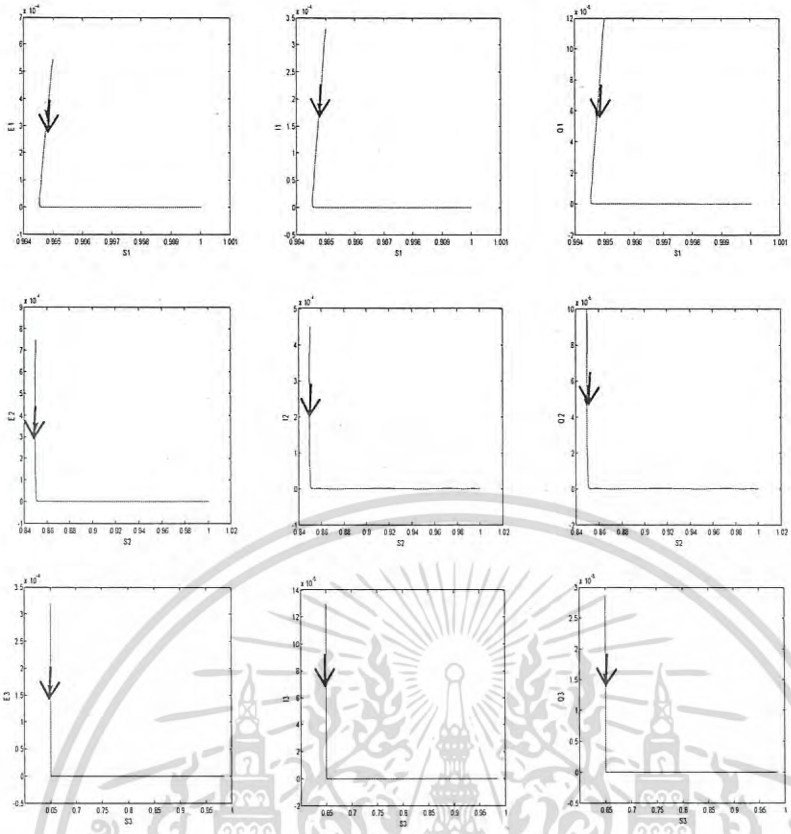

Figure 4.4 Numerical solutions demonstrate the solution trajectories, projected onto $(S_1,E_1),(S_1,I_1),(S_1,Q_1),(S_2,E_2),(S_2,I_2),(S_2,Q_2),(S_3,E_3),(S_3,I_3),(S_3,Q_3)$,respectively. For $R_0^*<1$, $\tilde{R}_0<1$ and $\hat{R}_0<1$ with values of parameters are $b=1/(365\times70)$ day^{-1}, $k=1/(365\times10)$ day^{-1}, $c=1/9$ day^{-1}, d = 1/7 day^{-1}, e = 1/5 day^{-1}, f = 1/7 day^{-1}, g = 0.7 day^{-1}, $a_1=0.6, a_2=0.9, a_3=0.15$, $R_0^*=0.39044693$, $\tilde{R}_0=0.51246160$ and $\hat{R}_0=0.59863369$. The proportions of population $(S_1,E_1,I_1,Q_1,S_2,E_2,I_2,Q_2,S_3,E_3,I_3,Q_3)$ approach to the disease free state $(1,0,0,0,1,0,0,0,1,0,0,0)$.

Note. By spanning scale of the above figure, we see that these graphs are smooth curve at the angle point. The enlarge graphs are shown in the following figure.

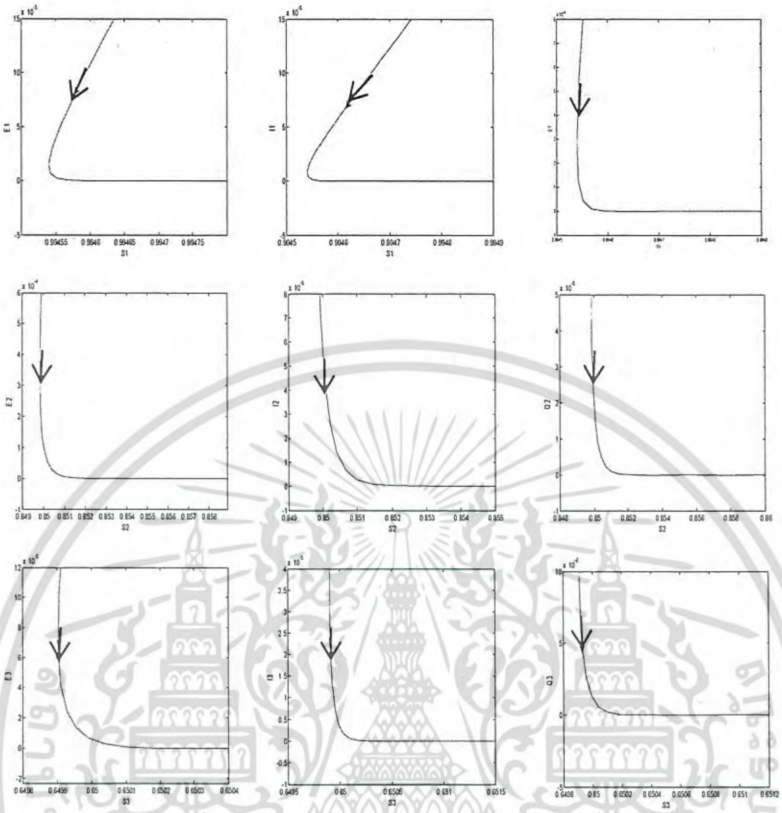

Figure 4.5 The enlarge graphs on the angle point of figure 4.4.

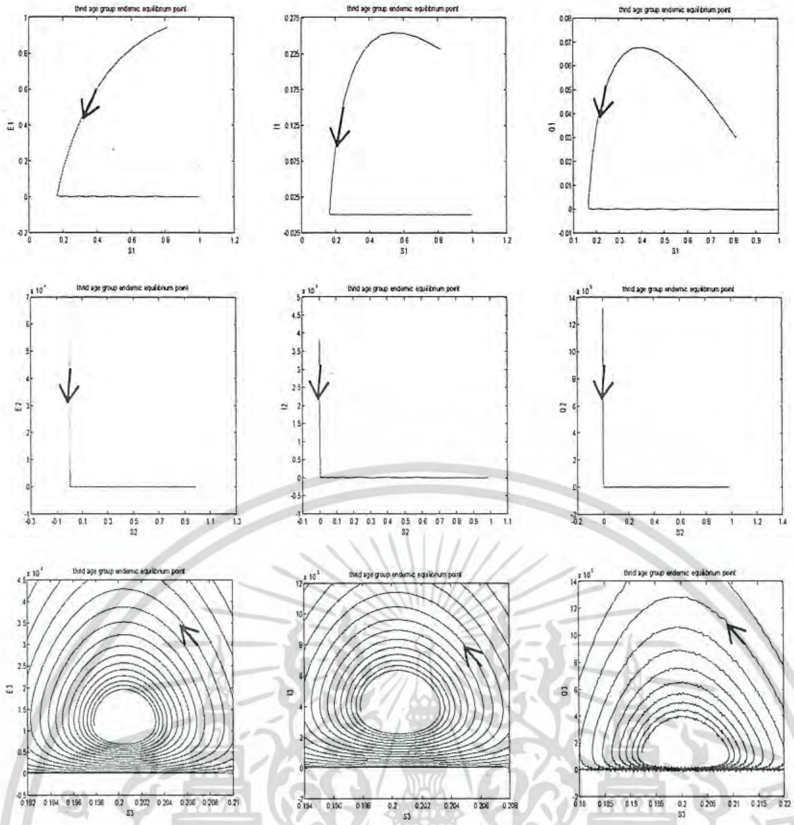

Figure 4.6 Numerical solutions demonstrate the solution trajectories, projected onto (S_1,E_1), (S_1,E_1), (S_1,I_1), (S_1,Q_1), (S_2,E_2), (S_2,I_2), (S_2,Q_2), (S_3,E_3), (S_3,I_3), (S_3,Q_3) ,respectively. For $R_0^* < 1$, $\tilde{R}_0 < 1$ and $\hat{R}_0 > 1$ with values of parameters are $a_1 = 1.45$, $a_2 = 1.65$, $a_3 = 1.25$, $R_0^* = 0.94358008$, $\tilde{R}_0 =$ 0.93951293, and $\hat{R}_0 = 4.98861411$ but the other parameters are same as in figure 4.4. The proportions of population $(S_1,E_1,I_1,Q_1,S_2,E_2,I_2,Q_2,S_3,E_3,I_3,Q_3)$ spiral into the third age group endemic equilibrium state (1, 0, 0, 0, 1, 0, 0, 0, 0.20045647, 0.00012320, 0.00003992, 0.00001141).

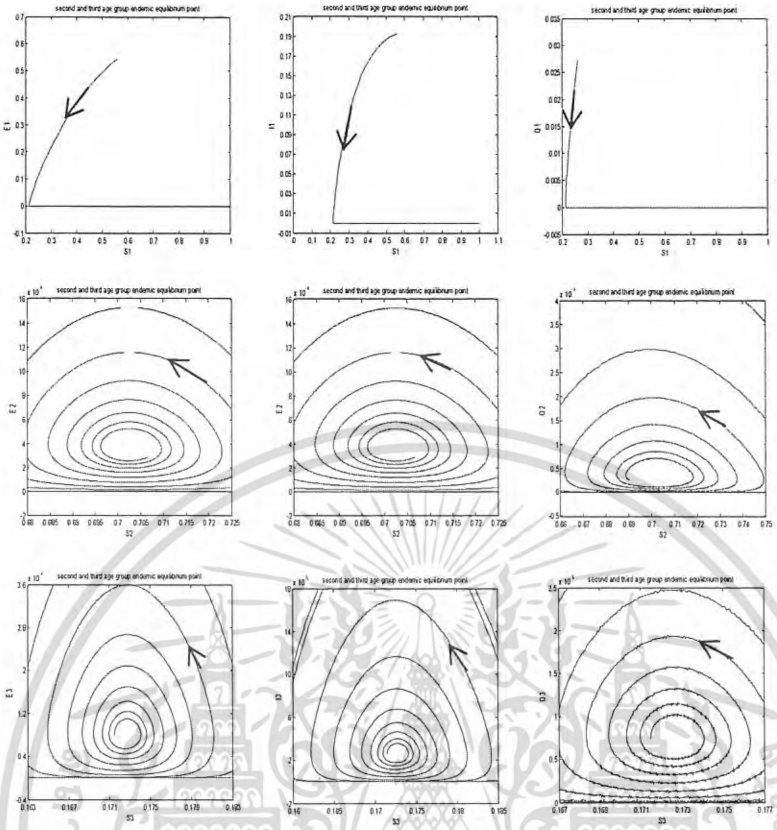

Figure 4.7 Numerical solutions demonstrate the solution trajectories, projected onto (S_1, E_1), (S_1, I_1), (S_1, Q_1), (S_2, E_2), (S_2, I_2), (S_2, Q_2), (S_3, E_3), (S_3, I_3), (S_3, Q_3), respectively. For $R_0^* < 1$, $\tilde{R}_0 > 1$ with values of parameters are $a_1 = 1.52$, $a_2 = 2.5, a_3 = 1.45$, $R_0^* = 0.98913223$, $\tilde{R}_0 = 1.42350444$ but the other parameters are same as in figure 4.4. The proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ spiral into the second and the third age group endemic equilibrium state (1, 0, 0, 0, 0.70249166, 0.00036634, 0.00011861, 0.00003387, 0.17266629, 0.00008170, 0.00002649, 0.00000757).

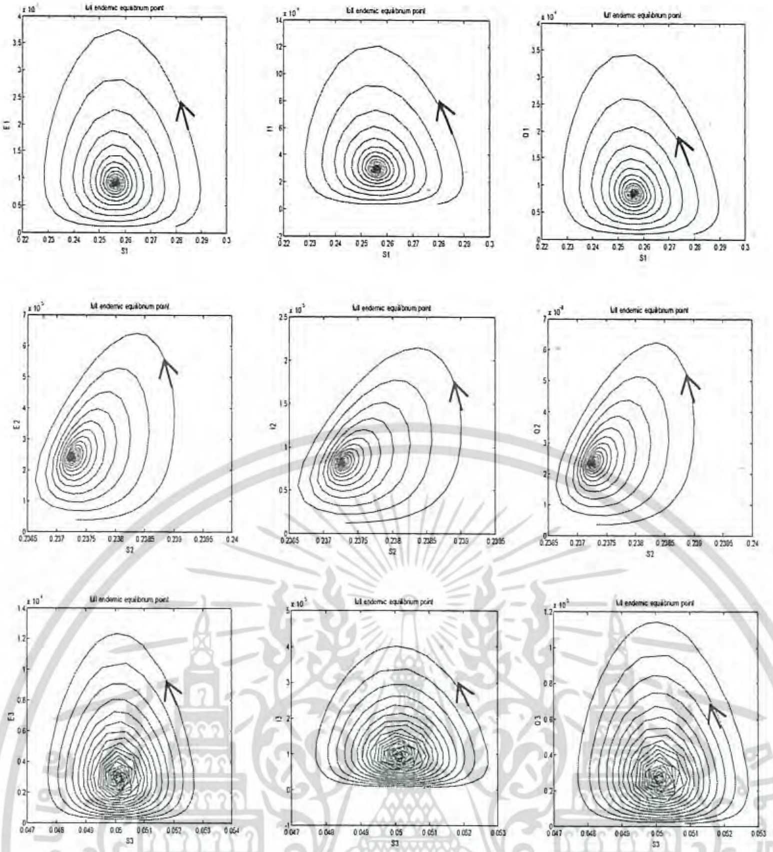

Figure 4.8 Numerical solutions demonstrate the solution trajectories, projected onto (S_1,E_1), (S_1,I_1), (S_1,Q_1), (S_2,E_2), (S_2,I_2), (S_2,Q_2), (S_3,E_3), (S_3,I_3), (S_3,Q_3), respectively. For $R_0^*>1$ with values of parameters are $a_1=6$, $a_2=7$, $a_3=5$ and $R_0^*=3.90446932$ but the other parameters are same as in figure 4.4 The proportions of population $(S_1,E_1,I_1,Q_1,S_2,E_2,I_2,Q_2,S_3,E_3,I_3,Q_3)$ spirals or oscillates into the full endemic equilibrium state (0.25611675, 0.00091599, 0.00029658, 0.00008470, 0.23726898, 0.00002434, 0.00000815, 0.00000237, 0.05010638, 0.00002884, 0.00000935, 0.00000267).

The bifurcation diagrams of (4.25) – (4.36) are shown in the following figures.

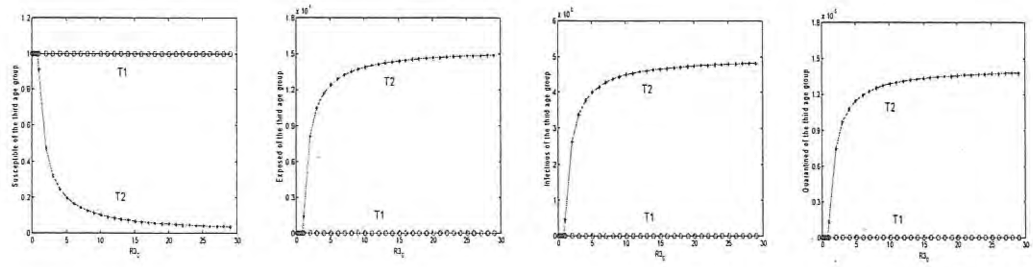

Figure 4.9 Bifurcation diagrams of the solutions of (4.25) – (4.36), plotted onto $(\hat{R}_0, S_3), (\hat{R}_0, E_3), (\hat{R}_0, I_3), (\hat{R}_0, Q_3)$, respectively. For the different values of $\hat{R}_0$, while *-*-* denotes the stable solutions and 0-0-0 denotes the unstable solutions.

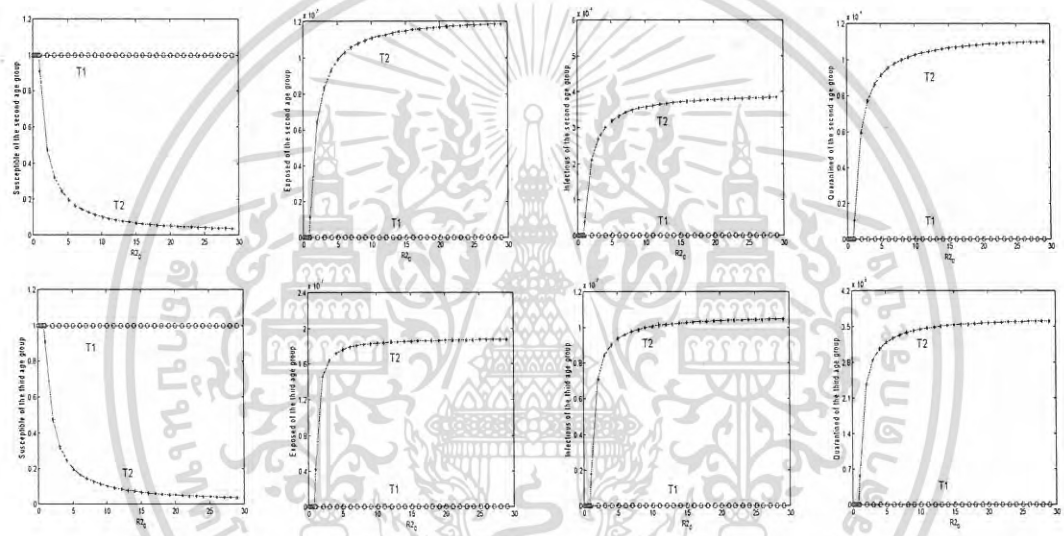

Figure 4.10 Bifurcation diagrams of the solutions of (4.25) – (4.36) plotted onto $(\tilde{R}_0, S_2), (\tilde{R}_0, E_2), (\tilde{R}_0, I_2), (\tilde{R}_0, Q_2)$, $(\tilde{R}_0, S_3), (\tilde{R}_0, E_3)$, $(\tilde{R}_0, I_3), (\tilde{R}_0, Q_3)$, respectively. For the different values of $\tilde{R}_0$, while *-*-* denotes the stable solutions and 0-0-0 denotes the unstable solutions.

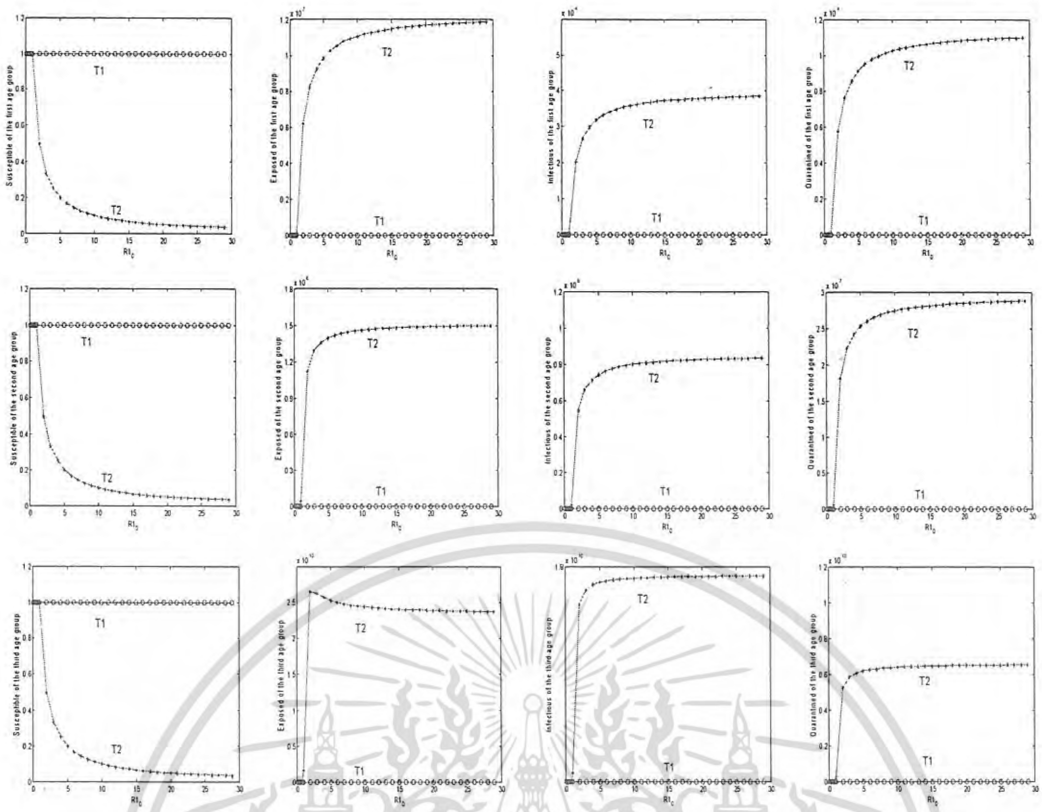

Figure 4.11 Bifurcation diagrams of the solutions of (4.25) – (4.36), plotted onto $(R_0^*, S_1), (R_0^*, E_1), (R_0^*, I_1), (R_0^*, Q_1), (R_0^*, S_2), (R_0^*, E_2), (R_0^*, I_2), (R_0^*, Q_2), (R_0^*, S_3),$ $(R_0^*, E_3), (R_0^*, I_3), (R_0^*, Q_3)$ respectively. For the different values of R_0^*, while *-*-* denotes the stable solutions and 0-0-0 denotes the unstable solutions.

Chapter 5
Global Stability Analysis

5.1 Mathematical Model

The SEIQR model is applied to the human population by age group. The differential equations are given by

$$\frac{dS_1}{dt} = bN_T - \frac{a_1 S_1 (E_1 + I_1)}{N_T} - (b+k)S_1, \tag{5.1}$$

$$\frac{dE_1}{dt} = \frac{a_1 S_1 (E_1 + I_1)}{N_T} - (b+c+d+k)E_1, \tag{5.2}$$

$$\frac{dI_1}{dt} = cE_1 - (b+e+f+k)I_1, \tag{5.3}$$

$$\frac{dQ_1}{dt} = eI_1 - (b+g+k)Q_1, \tag{5.4}$$

$$\frac{dR_1}{dt} = dE_1 + fI_1 + gQ_1 - (b+k)R_1, \tag{5.5}$$

$$\frac{dS_2}{dt} = kS_1 - \frac{a_2 S_2 (E_2 + I_2)}{N_T} - (b+k)S_2, \tag{5.6}$$

$$\frac{dE_2}{dt} = kE_1 + \frac{a_2 S_2 (E_2 + I_2)}{N_T} - (b+c+d+k)E_2, \tag{5.7}$$

$$\frac{dI_2}{dt} = kI_1 + cE_2 - (b+e+f+k)I_2, \tag{5.8}$$

$$\frac{dQ_2}{dt} = kQ_1 + eI_2 - (b+g+k)Q_2, \tag{5.9}$$

$$\frac{dR_2}{dt} = kR_1 + dE_2 + fI_2 + gQ_2 - (b+k)R_2, \tag{5.10}$$

$$\frac{dS_3}{dt} = kS_2 - \frac{a_3 S_3 (E_3 + I_3)}{N_T} - bS_3, \tag{5.11}$$

$$\frac{dE_3}{dt} = kE_2 + \frac{a_3 S_3 (E_3 + I_3)}{N_T} - (b+c+d)E_3, \tag{5.12}$$

$$\frac{dI_3}{dt} = kI_2 + cE_3 - (b+e+f)I_3, \tag{5.13}$$

$$\frac{dQ_3}{dt} = kQ_2 + eI_3 - (b+g)Q_3, \tag{5.14}$$

$$\frac{dR_3}{dt} = kR_2 + dE_3 + fI_3 + gQ_3 - bR_3, \tag{5.15}$$

with the conditions $N_{T_1} = S_1 + E_1 + I_1 + Q_1 + R_1$, $N_{T_2} = S_2 + E_2 + I_2 + Q_2 + R_2$, and $N_{T_3} = S_3 + E_3 + I_3 + Q_3 + R_3$.

If we add (5.1) – (5.5), (5.6) – (5.10), and (5.11) – (5.15) then

$$N_{T_1}'(t) = bN_T - (b+k)N_{T_1},\ N_{T_2}'(t) = kN_{T_1} - (b+k)N_{T_2},\ N_{T_3}'(t) = kN_{T_2} - bN_{T_3}. \tag{5.16}$$

We assume that total number of the first age group, the second age group and the third age group remain constant. Therefore $N_{T_1}'(t) = N_{T_2}'(t) = N_{T_3}'(t) = 0$. Setting the right hand side of (5.16) to be zero, we obtain the following three relations: $\frac{N_1}{N_T} = \frac{b}{b+k}$, $\frac{N_2}{N_T} = \frac{bk}{(b+k)^2}$ and $\frac{N_3}{N_T} = \frac{k^2}{(b+k)^2}$.

We can see that all equations described by (5.1) – (5.15), the non-negative octant R_+^{15} is positively invariant (where R_+^{15} denotes the non-negative region). With respect to (5.1) – (5.15), we have the following results:

Proposition 1 Let $(S_1(t), E_1(t), I_1(t), Q_1(t), R_1(t), S_2(t), E_2(t), I_2(t), Q_2(t), R_2(t), S_3(t), E_3(t), I_3(t), Q_3(t), R_3(t))$ be the solution of (5.1) – (5.15) with the initial condition $(S_1(0), E_1(0), I_1(0), Q_1(0), R_1(0), S_2(0), E_2(0), I_2(0), Q_2(0), R_2(0), S_3(0), E_3(0), I_3(0), Q_3(0), R_3(0))$ and the compact set

$$\Omega_1 = \Big\{(S_1, E_1, I_1, Q_1, R_1, S_2, E_2, I_2, Q_2, R_2, S_3, E_3, I_3, Q_3, R_3) \in R_+^{15},$$
$$J_1 \le N_{T_1} = \left(\frac{b}{b+k}\right)N_T,\ J_2 \le N_{T_2} = \left(\frac{bk}{(b+k)^2}\right)N_T,\ J_3 \le N_{T_3} = \left(\frac{k^2}{(b+k)^2}\right)N_T\Big\}.$$

Then, under the flow described by (5.1) – (5.15), Ω_1 is positively invariant set that attracts all solutions in R_+^{15}.

Proof We choose the Lyapunov function

$$J(t) = (J_1(t), J_2(t), J_3(t))$$
$$= (S_1 + E_1 + I_1 + Q_1 + R_1, S_2 + E_2 + I_2 + Q_2 + R_2, S_3 + E_3 + I_3 + Q_3 + R_3)$$

positive definite on R_+^{15} and we have

$$\frac{dJ}{dt} = \left(\frac{dJ_1}{dt}, \frac{dJ_2}{dt}, \frac{dJ_3}{dt}\right)$$

$$= \left(\frac{d}{dt}S_1 + \frac{d}{dt}E_1 + \frac{d}{dt}I_1 + \frac{d}{dt}Q_1 + \frac{d}{dt}R_1,\right.$$

$$\frac{d}{dt}S_2 + \frac{d}{dt}E_2 + \frac{d}{dt}I_2 + \frac{d}{dt}Q_2 + \frac{d}{dt}R_2,$$

$$\left.\frac{d}{dt}S_3 + \frac{d}{dt}E_3 + \frac{d}{dt}I_3 + \frac{d}{dt}Q_3 + \frac{d}{dt}R_3\right)$$

$$= \big(bN_T - (b+k)(S_1 + E_1 + I_1 + Q_1 + R_1),$$

$$k(S_1 + E_1 + I_1 + Q_1 + R_1) - (b+k)(S_2 + E_2 + I_2 + Q_2 + R_2),$$

$$k(S_2 + E_2 + I_2 + Q_2 + R_2) - b(S_3 + E_3 + I_3 + Q_3 + R_3)\big)$$

$$= \big(bN_T - (b+k)N_1, kN_1 - (b+k)N_2, kN_2 - bN_3\big).$$

We use the fact that $N_1 = \frac{b}{b+k}N_T$, $N_2 = \frac{bk}{(b+k)^2}N_T$ and $N_3 = \frac{k^2}{(b+k)^2}N_T$, then we can prove that

$$\frac{dJ_1}{dt} = bN_T - (b+k)J_1 \le 0, \quad \text{for } J_1 \ge \frac{bN_T}{b+k} \tag{5.17}$$

$$\frac{dJ_2}{dt} = \frac{bk}{b+k}N_T - (b+k)J_2 \le 0, \quad \text{for } J_2 \ge \frac{bkN_T}{(b+k)^2} \tag{5.18}$$

$$\frac{dJ_3}{dt} = \frac{bk^2}{(b+k)^2}N_T - bJ_3 \le 0, \quad \text{for } J_3 \ge \frac{k^2N_T}{(b+k)^2}. \tag{5.19}$$

From (5.17) – (5.19), one has $\frac{dJ}{dt} \le 0$ which implies that Ω_1 is a positively invariant set. In other words, by solving (5.17) – (5.19), we obtain

$$0 \le \left(J_1(t), J_2(t), J_3(t)\right)$$

$$\le \left(\left(bN_T/(b+k)\right) + J_1(0)e^{-(b+k)t},\ \left(bkN_T/(b+k)^2\right) + J_2(0)e^{-(b+k)t},\right.$$

$$\left.\left(k^2N_T/(b+k)^2\right) + J_3(0)e^{-bt}\right),$$

where $J_1(0), J_2(0)$ and $J_3(0)$ are the initial conditions of $J_1(t), J_2(t)$ and $J_3(t)$. Thus, as $t \to \infty$,

$$0 \le \left(J_1(t), J_2(t), J_3(t)\right) \le \left(bN_T/(b+k), bkN_T/(b+k)^2, k^2N_T/\ (b+k)^2\right) = (N_1, N_2, N_3)$$

and one can conclude that Ω_1 is an attractive set.

5.2 Equilibrium Points

From (5.1) – (5.15), we let $\gamma_1 = bN_T$, $\gamma_2 = b+k$, $\gamma_3 = \dfrac{a_1}{N_T}$, $\gamma_4 = b+e+f+k$, $\gamma_5 = b+g+k$, $\gamma_6 = b+c+d+k$, $\gamma_7 = \dfrac{a_2}{N_T}$, $\gamma_8 = \dfrac{a_3}{N_T}$, $\gamma_9 = b+e+f$, $\gamma_{10} = b+g$ and $\gamma_{11} = b+c+d$.

The equilibrium points are obtained by setting the right hand side of (5.1) – (5.15) to zero.

Then we obtain

$$\overline{S}_1 = \frac{\gamma_1}{\gamma_2 + \gamma_3(\overline{E}_1 + \overline{I}_1)}, \tag{5.20}$$

$$\overline{E}_1 = \frac{\gamma_3 \overline{S}_1 \overline{I}_1}{\gamma_6 - \gamma_3 \overline{S}_1} \tag{5.21}$$

$$\overline{I}_1 = \frac{c}{\gamma_4} \overline{E}_1, \tag{5.22}$$

$$\overline{Q}_1 = \frac{e}{\gamma_5} \overline{I}_1, \tag{5.23}$$

$$\overline{R}_1 = \frac{d\overline{E}_1 + f\overline{I}_1 + g\overline{Q}_1}{\gamma_2}. \tag{5.24}$$

Substituting Equations (5.20) and (5.22) in equation (5.21), we obtain that $\overline{E}_1$ must be a solution of the following quadratic equation:

$$\gamma_3\gamma_6(c+\gamma_4)\overline{E}_1^2+(\gamma_2\gamma_4\gamma_6-\gamma_1\gamma_3\gamma_4-\gamma_1\gamma_3 c)\overline{E}_1=0. \tag{5.25}$$

From (5.6) – (5.10), we obtain

$$\overline{S}_2=\frac{k\overline{S}_1}{\gamma_2+\gamma_7(\overline{E}_2+\overline{I}_2)}, \tag{5.26}$$

$$\overline{E}_2=\frac{k\overline{E}_1+\gamma_7\overline{S}_2\overline{I}_2}{\gamma_6-\gamma_7\overline{S}_2}, \tag{5.27}$$

$$\overline{I}_2=\frac{k\overline{I}_1+c\overline{E}_2}{\gamma_4}, \tag{5.28}$$

$$\overline{Q}_2=\frac{k\overline{Q}_1+e\overline{I}_2}{\gamma_5}, \tag{5.29}$$

$$\overline{R}_2=\frac{k\overline{R}_1+d\overline{E}_2+f\overline{I}_2+g\overline{Q}_2}{\gamma_2}. \tag{5.30}$$

Substituting Equations (5.26) and (5.28) in equation (5.27), we obtain that $\overline{E}_2$ must be a solution of the following quadratic equation:

$$\gamma_6\gamma_7(c+\gamma_4)\overline{E}_2^2+\left[\gamma_2\gamma_4\gamma_6+\gamma_7 k\left(\gamma_6\overline{I}_1-(c+\gamma_4)\overline{S}_1-(c+\gamma_4)\overline{E}_1\right)\right]\overline{E}_2$$
$$-\left[k^2\gamma_7\overline{I}_1(\overline{S}_1+\overline{E}_1)+\gamma_2\gamma_4 k\overline{E}_1\right]=0. \tag{5.31}$$

From (5.11) – (5.15), we obtain

$$\overline{S}_3=\frac{k\overline{S}_2}{b+\gamma_8(\overline{E}_3+\overline{I}_3)}, \tag{5.32}$$

$$\overline{E}_3=\frac{k\overline{E}_2+\gamma_8\overline{S}_3\overline{I}_3}{\gamma_{11}-\gamma_8\overline{S}_3}, \tag{5.33}$$

$$\overline{I}_3=\frac{k\overline{I}_2+c\overline{E}_3}{\gamma_9}, \tag{5.34}$$

$$\overline{Q}_3=\frac{k\overline{Q}_2+e\overline{I}_3}{\gamma_{10}}, \tag{5.35}$$

$$\overline{R}_3=\frac{k\overline{R}_2+d\overline{E}_3+f\overline{I}_3+g\overline{Q}_3}{b}. \tag{5.36}$$

Substituting Equations (5.32) and (5.34) in equation (5.33), we obtain that $\overline{E}_3$ must be a solution of the following quadratic equation:

$$\gamma_8\gamma_{11}(c+\gamma_9)\overline{E}_3^2+\left[b\gamma_9\gamma_{11}+k\gamma_8\left(\gamma_{11}\overline{I}_2-(c+\gamma_9)\overline{S}_2-(c+\gamma_9)\overline{E}_2\right)\right]\overline{E}_3$$
$$-\left[k^2\gamma_8\overline{I}_2(\overline{S}_2+\overline{E}_2)+b\gamma_9k\overline{E}_2\right]=0. \qquad (5.37)$$

From (5.25) we obtain $\overline{E}_1=0$ and $\overline{E}_1=\dfrac{\gamma_1\gamma_3(c+\gamma_4)-\gamma_2\gamma_4\gamma_6}{\gamma_3\gamma_6(c+\gamma_4)}$.

If we substitute $\overline{E}_1=0$ in equations (5.20), (5.22) - (5.24), we get $\overline{S}_1=\dfrac{\gamma_1}{\gamma_2}$, $\overline{I}_1=0$, $\overline{Q}_1=0$, $\overline{R}_1=0$ and equation (5.31) becomes

$$\gamma_6\gamma_7(c+\gamma_4)\overline{E}_2^2+\left[\gamma_2\gamma_4\gamma_6-\gamma_7k(c+\gamma_4)\frac{\gamma_1}{\gamma_2}\right]\overline{E}_2=0. \qquad (5.38)$$

The solutions of equation (5.38) are $\overline{E}_2=0$ and $\overline{E}_2=\dfrac{\gamma_1\gamma_7k(c+\gamma_4)-\gamma_2^2\gamma_4\gamma_6}{\gamma_2\gamma_6\gamma_7(c+\gamma_4)}$.

If we substitute $\overline{S}_1=\dfrac{\gamma_1}{\gamma_2}$, $\overline{E}_1=0$, $\overline{I}_1=0$, $\overline{Q}_1=0$, $\overline{R}_1=0$ and $\overline{E}_2=0$ in equations (5.26), (5.28) – (5.30), we get $\overline{S}_2=\dfrac{k\gamma_1}{\gamma_2^2}$, $\overline{I}_2=0$, $\overline{Q}_2=0$, $\overline{R}_2=0$ and equation (5.37) becomes

$$\gamma_8\gamma_{11}(c+\gamma_9)\overline{E}_3^2+\left[b\gamma_9\gamma_{11}-k\gamma_8(c+\gamma_9)\frac{k\gamma_1}{\gamma_2^2}\right]\overline{E}_3=0. \qquad (5.39)$$

Substituting the solutions $\overline{E}_3=0$ and $\overline{E}_3=\dfrac{\gamma_1\gamma_8k^2(c+\gamma_9)-\gamma_2^2\gamma_9\gamma_{11}b}{\gamma_2^2\gamma_8\gamma_{11}(c+\gamma_9)}$ of equation (5.39) with values: $\overline{S}_1=\dfrac{\gamma_1}{\gamma_2}$, $\overline{E}_1=0$, $\overline{I}_1=0$, $\overline{Q}_1=0$, $\overline{R}_1=0$, $\overline{S}_2=\dfrac{k\gamma_1}{\gamma_2^2}$, $\overline{E}_2=0$, $\overline{I}_2=0$, $\overline{Q}_2=0$ and $\overline{R}_2=0$ in equations (5.32), (5.34) – (5.36), we obtain two equilibrium points:

$$P_0=(\frac{\gamma_1}{\gamma_2},0,0,0,0,\frac{k\gamma_1}{\gamma_2^2},0,0,0,0,\frac{k^2\gamma_1}{b\gamma_2^2},0,0,0,0)$$

and $\hat{P}=(\dfrac{\gamma_1}{\gamma_2},0,0,0,0,\dfrac{k\gamma_1}{\gamma_2^2},0,0,0,0,\hat{S}_3,\hat{E}_3,\hat{I}_3,\hat{Q}_3,\hat{R}_3)$, where $\hat{S}_3 = \frac{k^2\gamma_1}{b\gamma_2^2}\left(\frac{1}{\hat{R}_0}\right)$, $\hat{E}_3 = \frac{\gamma_1 k^2}{\gamma_2^2\gamma_{11}}\left(1-\frac{1}{\hat{R}_0}\right)$, $\hat{I}_3 = \frac{c\gamma_1 k^2}{\gamma_2^2\gamma_9\gamma_{11}}\left(1-\frac{1}{\hat{R}_0}\right)$,

$$\hat{Q}_3 = \frac{ce\gamma_1 k^2}{\gamma_2^2\gamma_9\gamma_{10}\gamma_{11}}\left(1-\frac{1}{\hat{R}_0}\right),\ \hat{R}_3 = \frac{\gamma_9\gamma_{10}d + cf\gamma_{10} + ceg}{b\gamma_9\gamma_{10}}\left(1-\frac{1}{\hat{R}_0}\right)$$

and $\hat{R}_0 = \frac{\gamma_1\gamma_8 k^2(c+\gamma_9)}{\gamma_2^2\gamma_9\gamma_{11}b}$.

The third equilibrium point is found from substituting: $\overline{S}_1 = \frac{\gamma_1}{\gamma_2}$, $\overline{E}_1 = 0$, $\overline{I}_1 = 0$, $\overline{Q}_1 = 0$, $\overline{R}_1 = 0$ and $\overline{E}_2 = \frac{\gamma_1\gamma_7 k(c+\gamma_4) - \gamma_2^2\gamma_4\gamma_6}{\gamma_2\gamma_6\gamma_7(c+\gamma_4)}$ (or rewritten as $\overline{E}_2 = \tilde{E}_2$) in equations (5.26), (5.28) – (5.30), we get $\overline{S}_2 = \tilde{S}_2$, $\overline{I}_2 = \tilde{I}_2$, $\overline{Q}_2 = \tilde{Q}_2$ and $\overline{R}_2 = \tilde{R}_2$, and equation (5.37) becomes

$$\gamma_8\gamma_{11}(c+\gamma_9)\overline{E}_3^2 + \left[b\gamma_9\gamma_{11} + \gamma_8 k\left(\gamma_{11}\tilde{I}_2 - (c+\gamma_9)\tilde{S}_2 - (c+\gamma_9)\tilde{E}_2\right)\right]\overline{E}_3 - \left[k^2\gamma_8\tilde{I}_2(\tilde{S}_2+\tilde{E}_2) + bk\gamma_9\tilde{E}_2\right] = 0,$$

or

$$L_1\overline{E}_3^2 + L_2\overline{E}_3 + L_3 = 0, \tag{5.40}$$

where $L_1 = \gamma_8\gamma_{11}(c+\gamma_9)$, $L_2 = b\gamma_9\gamma_{11} + \gamma_8 k\left(\gamma_{11}\tilde{I}_2 - (c+\gamma_9)\tilde{S}_2 - (c+\gamma_9)\tilde{E}_2\right)$,

and $L_3 = -\left[k^2\gamma_8\tilde{I}_2(\tilde{S}_2+\tilde{E}_2) + bk\gamma_9\tilde{E}_2\right]$.

The solutions of (5.40) are given by

$$\overline{E}_3 = \frac{-L_2 \pm \sqrt{L_2^2 - 4L_1L_3}}{2L_1}.$$

Since L_1 is positive and L_3 is negative, it is easy to see that the term in square root of $\overline{E}_3$, $L_2^2 - 4L_1L_3$ is positive and $L_2^2 - 4L_1L_3$ is greater than L_2. It implies that $-L_2 - \sqrt{L_2^2 - 4L_1L_3}$ is less than zero. Thus, there is the only one solution $\overline{E}_3 = \frac{-L_2 + \sqrt{L_2^2 - 4L_1L_3}}{2L_1}$ of equation (5.40).

Substituting the solution $\overline{E}_3 = \frac{-L_2 + \sqrt{L_2^2 - 4L_1L_3}}{2L_1}$ (or rewritten as $\overline{E}_3 = \tilde{E}_3$) of equation (5.40) with values: $\overline{S}_2 = \tilde{S}_2$, $\overline{I}_2 = \tilde{I}_2$, $\overline{E}_2 = \tilde{E}_2$, $\overline{Q}_2 = \tilde{Q}_2$ and $\overline{R}_2 = \tilde{R}_2$ in equations (5.32), (5.34) – (5.36), we obtain the third equilibrium point

$$\tilde{P}=(\frac{\gamma_1}{\gamma_2},0,0,0,0,\tilde{S}_2,\tilde{E}_2,\tilde{I}_2,\tilde{Q}_2,\tilde{R}_2,\tilde{S}_3,\tilde{E}_3,\tilde{I}_3,\tilde{Q}_3,\tilde{R}_3),$$

where $\tilde{S}_2=\frac{\gamma_1 k}{\gamma_2^2}\left(\frac{1}{\tilde{R}_0}\right)$, $\tilde{E}_2=\frac{\gamma_1 k}{\gamma_2\gamma_6}\left(1-\frac{1}{\tilde{R}_0}\right)$, $\tilde{I}_2=\frac{c\gamma_1 k}{\gamma_2\gamma_4\gamma_6}\left(1-\frac{1}{\tilde{R}_0}\right)$,

$\tilde{Q}_2=\frac{ce\gamma_1 k}{\gamma_2\gamma_4\gamma_5\gamma_6}\left(1-\frac{1}{\tilde{R}_0}\right)$, $\tilde{R}_2=\frac{\gamma_4\gamma_5 d+cf\gamma_5+ceg}{\gamma_2\gamma_4\gamma_5}\left(1-\frac{1}{\tilde{R}_0}\right)$,

$\tilde{S}_3=\frac{k\tilde{S}_2}{\gamma_8(\tilde{E}_3+\tilde{I}_3)+b}$, $\tilde{E}_3=\frac{-L_2+\sqrt{L_2^2-4L_1L_3}}{2L_1}$, $\tilde{I}_3=\frac{k\tilde{I}_2+c\tilde{E}_3}{\gamma_9}$, $\tilde{Q}_3=\frac{k\tilde{Q}_2+e\tilde{I}_3}{\gamma_{10}}$,

$\tilde{R}_3=\frac{k\tilde{R}_2+d\tilde{E}_3+f\tilde{I}_3+g\tilde{Q}_3}{b}$, and $\tilde{R}_0=\frac{\gamma_1\gamma_7 k(c+\gamma_4)}{\gamma_2^2\gamma_4\gamma_6}$.

The forth equilibrium point is obtained from substituting $\overline{E}_1=\frac{\gamma_1\gamma_3(c+\gamma_4)-\gamma_2\gamma_4\gamma_6}{\gamma_3\gamma_6(c+\gamma_4)}$ (or rewritten as $\overline{E}_1=E_1^*$), we get $\overline{S}_1=S_1^*$, $\overline{I}_1=I_1^*$, $\overline{Q}_1=Q_1^*$ and $\overline{R}_1=R_1^*$, and equation (5.31) becomes

$$\gamma_6\gamma_7(c+\gamma_4)\overline{E}_2^2+\left[\gamma_2\gamma_4\gamma_6+\gamma_7 k\left(\gamma_6 I_1^*-(c+\gamma_4)S_1^*-(c+\gamma_4)E_1^*\right)\right]\overline{E}_2$$
$$-\left[k^2\gamma_7 I_1^*(S_1^*+E_1^*)+\gamma_2\gamma_4 kE_1^*\right]=0.$$

or
$$M_1\overline{E}_2^2+M_2\overline{E}_2+M_3=0, \tag{5.41}$$

where $M_1=\gamma_6\gamma_7(c+\gamma_4)$, $M_2=\gamma_2\gamma_4\gamma_6+\gamma_7 k\left(\gamma_6 I_1^*-(c+\gamma_4)S_1^*-(c+\gamma_4)E_1^*\right)$,

and $M_3=-\left[k^2\gamma_7 I_1^*(S_1^*+E_1^*)+\gamma_2\gamma_4 kE_1^*\right]$.

We use the same method as above, there is the only one solution $\overline{E}_2=\frac{-M_2+\sqrt{M_2^2-4M_1M_3}}{2M_1}$ of equation (5.41). Substituting the solution $\overline{E}_2=\frac{-M_2+\sqrt{M_2^2-4M_1M_3}}{2M_1}$ (or rewritten as $\overline{E}_2=E_2^*$) of equation (5.41) with values: $\overline{S}_1=S_1^*$, $\overline{I}_1=I_1^*$, $\overline{E}_1=E_1^*$, $\overline{Q}_1=Q_1^*$, $\overline{R}_1=R_1^*$ and $\overline{E}_2=E_2^*$ in equations (5.32), (5.34) – (5.36), we get $\overline{S}_2=S_2^*$, $\overline{I}_2=I_2^*$, $\overline{Q}_2=Q_2^*$, $\overline{R}_2=R_2^*$ and equation (5.40) becomes

$$\gamma_8\gamma_{11}(c+\gamma_9)\overline{E}_3^2+\left[b\gamma_9\gamma_{11}+k\gamma_8\left(\gamma_{11}I_2^*-(c+\gamma_9)S_2^*-(c+\gamma_9)E_2^*\right)\right]\overline{E}_3$$
$$-\left[k^2\gamma_8 I_2^*(S_2^*+E_2^*)+b\gamma_9 kE_2^*\right]=0.$$

or
$$K_1\overline{E}_3^2 + K_2\overline{E}_3 + K_3 = 0\,, \tag{5.42}$$

where $K_1 = \gamma_8\gamma_{11}(c+\gamma_9)$, $K_2 = b\gamma_9\gamma_{11} + k\gamma_8\left(\gamma_{11}I_2^* - (c+\gamma_9)S_2^* - (c+\gamma_9)E_2^*\right)$,

and $K_3 = -\left[k^2\gamma_8 I_2^*(S_2^* + E_2^*) + b\gamma_9 kE_2^*\right]$.

We use the same method as above, there is the only one solution $\overline{E}_3 = \dfrac{-K_2 + \sqrt{K_2^2 - 4K_1K_3}}{2K_1}$ of equation (5.42). Substituting the solution $\overline{E}_3 = \dfrac{-K_2 + \sqrt{K_2^2 - 4K_1K_3}}{2K_1}$ (or rewritten as $\overline{E}_3 = E_3^*$) of equation (5.42) with values: $\overline{S}_2 = S_2^*$, $\overline{E}_2 = E_2^*$, $\overline{I}_2 = I_2^*$, $\overline{Q}_2 = Q_2^*$, $\overline{R}_2 = R_2^*$ and $\overline{E}_3 = E_3^*$ in equations (5.32) , (5.34) – (5.36), we get the forth equilibrium point

$$P^* = (S_1^*, E_1^*, I_1^*, Q_1^*, R_1^*, S_2^*, E_2^*, I_2^*, Q_2^*, R_2^*, S_3^*, E_3^*, I_3^*, Q_3^*, R_3^*),$$

where $S_1^* = \dfrac{\gamma_1}{\gamma_2}\left(\dfrac{1}{R_0^*}\right)$, $E_1^* = \dfrac{\gamma_1}{\gamma_6}\left(1 - \dfrac{1}{R_0^*}\right)$, $I_1^* = \dfrac{\gamma_1 c}{\gamma_4\gamma_6}\left[1 - \dfrac{1}{R_0^*}\right]$,

$$Q_1^* = \frac{\gamma_1 ce}{\gamma_4\gamma_5\gamma_6}\left[1 - \frac{1}{R_0^*}\right],\ R_1^* = \frac{\gamma_4\gamma_5 d + cf\gamma_5 + ceg}{\gamma_2\gamma_4\gamma_5}\left(1 - \frac{1}{R_0^*}\right),$$

$$S_2^* = \frac{kS_1^*}{\gamma_2 + \gamma_7(E_2^* + I_2^*)},\ E_2^* = \frac{-M_2 + \sqrt{M_2^2 - 4M_1M_3}}{2M_1},\ I_2^* = \frac{kI_1^* + cE_2^*}{\gamma_4},$$

$$Q_2^* = \frac{kQ_1^* + eI_2^*}{\gamma_5},\ R_2^* = \frac{kR_1^* + dE_2^* + fI_2^* + gQ_2^*}{\gamma_2},\ S_3^* = \frac{kS_2^*}{\gamma_8(E_3^* + I_3^*) + b},$$

$$E_3^* = \frac{-K_2 + \sqrt{K_2^2 - 4K_1K_3}}{2K_1},\ I_3^* = \frac{kI_2^* + cE_3^*}{\gamma_9},\ Q_3^* = \frac{kQ_2^* + eI_3^*}{\gamma_{10}},$$

$$R_3^* = \frac{kR_2^* + dE_3^* + fI_3^* + gQ_3^*}{b} \text{ and } R_0^* = \frac{\gamma_1\gamma_3(c+\gamma_4)}{\gamma_2\gamma_4\gamma_6}.$$

We have four equilibrium points:

i) The disease free equilibrium point

$$P_1\left(\overline{S_1},\overline{E}_1,\overline{I_1},\overline{Q}_1,\overline{R_1},\overline{S_2},\overline{E_2},\overline{I_2},\overline{Q_2},\overline{R_2},\overline{S_3},\overline{E_3},\overline{I_3},\overline{Q_3},\overline{R_3}\right)$$

$$=P_1\left(\frac{bN_T}{b+k},0,0,0,0,\frac{bkN_T}{(b+k)^2},0,0,0,0,\frac{k^2N_T}{(b+k)^2},0,0,0,0\right)$$

ii) The third age group endemic equilibrium point

$$P_2\left(\hat{S}_1,\hat{E}_1,\hat{I}_1,\hat{Q}_1,\hat{R}_1,\hat{S}_2,\hat{E}_2,\hat{I}_2,\hat{Q}_2,\hat{R}_2,\hat{S}_3,\hat{E}_3,\hat{I}_3,\hat{Q}_3,\hat{R}_3\right)$$

$$=P_2\left(\frac{bN_T}{b+k},0,0,0,0,\frac{bkN_T}{(b+k)^2},0,0,0,0,\hat{S}_3,\hat{E}_3,\hat{I}_3,\hat{Q}_3,\hat{R}_3\right)$$

where $\hat{S}_3=\frac{k^2N_T}{(b+k)^2}\left(\frac{1}{\hat{R}_0}\right)$, $\hat{E}_3=\frac{bk^2N_T}{(b+k)^2(b+c+d)}\left(1-\frac{1}{\hat{R}_0}\right)$,

$$\hat{I}_3=\frac{bck^2N_T}{(b+k)^2(b+c+d)(b+e+f)}\left(1-\frac{1}{\hat{R}_0}\right),$$

$$\hat{Q}_3=\frac{bcek^2N_T}{(b+k)^2(b+c+d)(b+e+f)(b+g)}\left(1-\frac{1}{\hat{R}_0}\right),$$

$$\hat{R}_3=\frac{(b+e+f)(b+g)d+cf(b+g)+ceg}{b(b+e+f)(b+g)}\left(1-\frac{1}{\hat{R}_0}\right),$$

with $\hat{R}_0=\frac{a_3k^2(b+c+e+f)}{(b+k)^2(b+c+d)(b+e+f)}$.

iii) The second and third age group endemic equilibrium point

$$P_3\left(\widetilde{S}_1,\widetilde{E}_1,\widetilde{I}_1,\widetilde{Q}_1,\widetilde{R}_1,\widetilde{S}_2,\widetilde{E}_2,\widetilde{I}_2,\widetilde{Q}_2,\widetilde{R}_2,\widetilde{S}_3,\widetilde{E}_3,\widetilde{I}_3,\widetilde{Q}_3,\widetilde{R}_3\right)$$

$$=P_3\left(\frac{bN_T}{b+k},0,0,0,0,\widetilde{S}_2,\widetilde{E}_2,\widetilde{I}_2,\widetilde{Q}_2,\widetilde{R}_2,\widetilde{S}_3,\widetilde{E}_3,\widetilde{I}_3,\widetilde{Q}_3,\widetilde{R}_3\right)$$

where $\widetilde{S}_2 = \frac{bkN_T}{(b+k)^2}\left(\frac{1}{\widetilde{R}_0}\right)$, $\widetilde{E}_2 = \frac{bkN_T}{(b+k)(b+c+d+k)}\left(1-\frac{1}{\widetilde{R}_0}\right)$,

$$\widetilde{I}_2 = \frac{bckN_T}{(b+k)(b+c+d+k)(b+e+f+k)}\left(1-\frac{1}{\widetilde{R}_0}\right),$$

$$\widetilde{Q}_2 = \frac{bcekN_T}{(b+k)(b+c+d+k)(b+e+f+k)(b+g+k)}\left(1-\frac{1}{\widetilde{R}_0}\right),$$

$$\widetilde{R}_2 = \frac{(b+e+f+k)(b+g+k)d+cf(b+g+k)+ceg}{(b+k)(b+e+f+k)(b+g+k)}\left(1-\frac{1}{\widetilde{R}_0}\right),$$

$$\widetilde{S}_3 = \frac{k\widetilde{S}_2}{\frac{a_3}{N_T}\left(\widetilde{E}_3+\widetilde{I}_3\right)+b},\ \widetilde{E}_3 = \frac{-L_2+\sqrt{L_2^2-4L_1L_3}}{2L_1},\ \widetilde{I}_3 = \frac{k\widetilde{I}_2+c\widetilde{E}_3}{b+e+f}$$

$$\widetilde{Q}_3 = \frac{k\widetilde{Q}_2+e\widetilde{I}_3}{b+g},\ \widetilde{R}_3 = \frac{k\widetilde{R}_2+d\widetilde{E}_3+f\widetilde{I}_3+g\widetilde{Q}_3}{b}$$

with $\widetilde{R}_0 = \frac{a_2bk(b+c+e+f+k)}{(b+k)^2(b+c+d+k)(b+e+f+k)}$, $L_1 = \frac{a_3}{N_T}(b+c+d)(b+c+e+f)$

$$L_2 = b(b+c+d)(b+e+f)+\frac{a_3k}{N_T}\left((b+c+d)\widetilde{I}_2-(b+c+e+f)\left(\widetilde{S}_2+\widetilde{E}_2\right)\right),$$

and $L_3 = -\left[\frac{k^2a_3}{N_T}\widetilde{I}_2\left(\widetilde{S}_2+\widetilde{E}_2\right)+bk(b+e+f)\widetilde{E}_2\right]$.

iv) The full endemic equilibrium point

$$P_4(S_1^*,E_1^*,I_1^*,Q_1^*,R_1^*,S_2^*,E_2^*,I_2^*,Q_2^*,R_2^*,S_3^*,E_3^*,I_3^*,Q_3^*,R_3^*)$$

where $S_1^* = \frac{bN_T}{b+k}\left(\frac{1}{R_0^*}\right)$, $E_1^* = \frac{bN_T}{(b+c+d+k)}\left(1-\frac{1}{R_0^*}\right)$,

$$I_1^* = \frac{bcN_T}{(b+c+d+k)(b+e+f+k)}\left(1-\frac{1}{R_0^*}\right),$$

$$Q_1^* = \frac{bceN_T}{(b+c+d+k)(b+e+f+k)(b+g+k)}\left(1-\frac{1}{R_0^*}\right),$$

$$R_1^* = \frac{(b+e+f+k)(b+g+k)d+cf(b+g+k)+ceg}{(b+k)(b+e+f+k)(b+g+k)}\left(1-\frac{1}{R_0^*}\right),$$

$$S_2^* = \frac{kS_1^*}{\frac{a_2}{N_T}\left(E_2^* + I_2^*\right) + \left(b+k\right)}, \quad E_2^* = \frac{-M_2 + \sqrt{M_2^2 - 4M_1M_3}}{2M_1}, \quad I_2^* = \frac{kI_1^* + cE_2^*}{b+e+f+k},$$

$$Q_2^* = \frac{kQ_1^* + eI_2^*}{b+g+k}, \quad R_2^* = \frac{kR_1^* + dE_2^* + fI_2^* + gQ_2^*}{b+k}, \quad S_3^* = \frac{kS_2^*}{\frac{a_3}{N_T}\left(E_3^* + I_3^*\right) + b},$$

$$E_3^* = \frac{-K_2 + \sqrt{K_2^2 - 4K_1K_3}}{2K_1}, \quad I_3^* = \frac{kI_2^* + cE_3^*}{b+e+f}, \quad Q_3^* = \frac{kQ_2^* + eI_3^*}{b+g},$$

$$R_3^* = \frac{kR_2^* + dE_3^* + fI_3^* + gQ_3^*}{b}$$

with $$R_0^* = \frac{a_1 b\left(b+c+e+f+k\right)}{\left(b+k\right)\left(b+c+d+k\right)\left(b+e+f+k\right)},$$

$$M_1 = \frac{a_2}{N_T}\left(b+c+d+k\right)\left(b+c+e+f+k\right),$$

$$M_2 = \left(b+k\right)\left(b+c+d+k\right)\left(b+e+f+k\right) + \frac{a_2 k}{N_T}\left(\left(b+c+d+k\right)I_1^* - \left(b+c+e+f+k\right)\left(S_1^* + E_1^*\right)\right),$$

$$M_3 = -\left[\frac{a_2 k^2}{N_T} I_1^*\left(S_1^* + E_1^*\right) + k\left(b+k\right)\left(b+e+f+k\right)E_1^*\right]$$

$$K_1 = \frac{a_3}{N_T}\left(b+c+d\right)\left(b+c+e+f\right),$$

$$K_2 = b\left(b+c+d\right)\left(b+e+f\right) + \frac{a_3 k}{N_T}\left(\left(b+c+d\right)I_2^* - \left(b+c+e+f\right)\left(S_2^* + E_2^*\right)\right),$$

and $$K_3 = -\left[\frac{k^2 a_3}{N_T} I_2^*\left(S_2^* + E_2^*\right) + bk\left(b+e+f\right)E_2^*\right].$$

5.3 Global Stability of the Equilibrium States

The global behavior of the equilibrium states for equations (5.1) – (5.15) is determined from Lyapunov techniques.

Theorem 1 Assume that

$$\left.\begin{aligned} a_1 &= \frac{bN_T}{\overline{S}_1} \\ a_2 &= \frac{bN_T}{\overline{S}_2} \\ a_3 &= \frac{bN_T}{\overline{S}_3} \end{aligned}\right\} \tag{5.43}$$

for $R_0 = \max\left\{\hat{R}_0, \tilde{R}_0, R_0^*\right\} < 1$, the disease free equilibrium point P_1 is globally asymptotically stable on Ω_1.

Proof Let us consider on Ω_1 , the Lyapunov function

$$\rho(t) = \overline{S}_1\left(\frac{S_1}{\overline{S}_1} - \ln\frac{S_1}{\overline{S}_1}\right) + E_1 + I_1 + Q_1 + R_1 + \overline{S}_2\left(\frac{S_2}{\overline{S}_2} - \ln\frac{S_2}{\overline{S}_2}\right) + E_2 + I_2 + Q_2 + R_2$$

$$+\overline{S}_3\left(\frac{S_3}{\overline{S}_3} - \ln\frac{S_3}{\overline{S}_3}\right) + E_3 + I_3 + Q_3 + R_3.$$

The derivative with respect to time yields

$$\frac{d}{dt}\rho(t) = \frac{d}{dt}S_1\left(1 - \frac{\overline{S}_1}{S_1}\right) + \frac{d}{dt}E_1 + \frac{d}{dt}I_1 + \frac{d}{dt}Q_1 + \frac{d}{dt}R_1$$

$$+\frac{d}{dt}S_2\left(1 - \frac{\overline{S}_2}{S_2}\right) + \frac{d}{dt}E_2 + \frac{d}{dt}I_2 + \frac{d}{dt}Q_2 + \frac{d}{dt}R_2$$

$$+\frac{d}{dt}S_3\left(1 - \frac{\overline{S}_3}{S_3}\right) + \frac{d}{dt}E_3 + \frac{d}{dt}I_3 + \frac{d}{dt}Q_3 + \frac{d}{dt}R_3$$

$$=\left(bN_T-\left(\frac{a_1\left(E_1+I_1\right)}{N_T}+b+k\right)S_1\right)\left(1-\frac{\bar{S}_1}{S_1}\right)+\left(\frac{a_1S_1\left(E_1+I_1\right)}{N_T}-\left(b+c+d+k\right)E_1\right)$$

$$+\left(cE_1-\left(b+e+f+k\right)I_1\right)+\left(eI_1-\left(b+g+k\right)Q_1\right)+\left(dE_1+fI_1+gQ_1-\left(b+k\right)R_1\right)$$

$$+\left(kS_1-\left(\frac{a_2\left(E_2+I_2\right)}{N_T}+b+k\right)S_2\right)\left(1-\frac{\bar{S}_2}{S_2}\right)$$

$$+\left(kE_1+\frac{a_2S_2\left(E_2+I_2\right)}{N_T}-\left(b+c+d+k\right)E_2\right)$$

$$+\left(kI_1+cE_2-\left(b+e+f+k\right)I_2\right)+\left(kQ_1+eI_2-\left(b+g+k\right)Q_2\right)$$

$$+\left(kR_1+dE_2+fI_2+gQ_2-\left(b+k\right)R_2\right)+\left(kS_2-\left(\frac{a_3\left(E_3+I_3\right)}{N_T}+b\right)S_3\right)\left(1-\frac{\bar{S}_3}{S_3}\right)$$

$$+\left(kE_2+\frac{a_3S_3\left(E_3+I_3\right)}{N_T}-\left(b+c+d\right)E_3\right)+\left(kI_2+cE_3-\left(b+e+f\right)I_3\right)$$

$$+\left(kQ_2+eI_3-\left(b+g\right)Q_3\right)+\left(kR_2+dE_3+fI_3+gQ_3-bR_3\right)$$

$$=bN_T\left(1-\frac{\bar{S}_1}{S_1}\right)+\left(b+k\right)\bar{S}_1-bS_1-k\frac{S_1}{S_2}\bar{S}_2+\left(b+k\right)\bar{S}_2-bS_2-k\frac{S_2}{S_3}\bar{S}_3+b\bar{S}_3-bS_3$$

$$+\left(E_1+I_1\right)\left(\frac{a_1\bar{S}_1}{N_T}-b\right)+\left(E_2+I_2\right)\left(\frac{a_2\bar{S}_2}{N_T}-b\right)+\left(E_3+I_3\right)\left(\frac{a_3\bar{S}_3}{N_T}-b\right)$$

$$-bQ_1-bR_1-bQ_2-bR_2-bQ_3-bR_3. \tag{5.44}$$

By using conditions in (5.43), we obtain

$$\frac{d}{dt}\rho(t)=bN_T\left(1-\frac{\bar{S}_1}{S_1}\right)+\left(b+k\right)\bar{S}_1\left(1-\frac{S_1}{\bar{S}_1}\right)+kS_1\left(1-\frac{\bar{S}_2}{S_2}\right)+\left(b+k\right)\bar{S}_2\left(1-\frac{S_2}{\bar{S}_2}\right)$$

$$+kS_2\left(1-\frac{\bar{S}_3}{S_3}\right)+b\bar{S}_3\left(1-\frac{S_3}{\bar{S}_3}\right)-bQ_1-bR_1-bQ_2-bR_2-bQ_3-bR_3$$

$$=bN_T\left(1-\frac{\bar{S}_1}{S_1}\right)+\left(b+k\right)\bar{S}_1\left(1-\frac{S_1}{\bar{S}_1}\right)+k\bar{S}_1\left(\frac{S_1}{\bar{S}_1}-\frac{S_1}{\bar{S}_1}\frac{\bar{S}_2}{S_2}\right)+\left(b+k\right)\bar{S}_2\left(1-\frac{S_2}{\bar{S}_2}\right)$$

$$+k\bar{S}_2\left(\frac{S_2}{\bar{S}_2}-\frac{S_2}{\bar{S}_2}\frac{\bar{S}_3}{S_3}\right)+b\bar{S}_3\left(1-\frac{S_3}{\bar{S}_3}\right)-bQ_1-bR_1-bQ_2-bR_2-bQ_3-bR_3. \tag{5.45}$$

Note that on Ω_1, we have $\overline{S_1}=\dfrac{bN_T}{b+k}, \overline{S_2}=\dfrac{k}{b+k}\overline{S_1}$ and $\overline{S_3}=\dfrac{k}{b}\overline{S_2}$. The above equation becomes

$$\frac{d}{dt}\rho(t)=bN_T\left(1-\frac{\overline{S}_1}{S_1}\right)+bN_T\left(1-\frac{S_1}{\overline{S}_1}\right)+k\overline{S}_1\left(\frac{S_1}{\overline{S}_1}-\frac{S_1}{\overline{S}_1}\frac{\overline{S}_2}{S_2}\right)+k\overline{S}_1\left(1-\frac{S_2}{\overline{S}_2}\right)$$

$$+k\overline{S}_2\left(\frac{S_2}{\overline{S}_2}-\frac{S_2}{\overline{S}_2}\frac{\overline{S}_3}{S_3}\right)+k\overline{S}_2\left(1-\frac{S_3}{\overline{S}_3}\right)-bQ_1-bR_1-bQ_2-bR_2-bQ_3-bR_3$$

$$=bN_T\left(2-\frac{\overline{S}_1}{S_1}-\frac{S_1}{\overline{S}_1}\right)+k\overline{S}_1\left(\frac{S_1}{\overline{S}_1}-\frac{S_1}{\overline{S}_1}\frac{\overline{S}_2}{S_2}+1-\frac{S_2}{\overline{S}_2}\right)+k\overline{S}_2\left(\frac{S_2}{\overline{S}_2}-\frac{S_2}{\overline{S}_2}\frac{\overline{S}_3}{S_3}+1-\frac{S_3}{\overline{S}_3}\right)$$

$$-bQ_1-bR_1-bQ_2-bR_2-bQ_3-bR_3$$

$$=-bN_T\frac{\left(\overline{S}_1-S_1\right)^2}{S_1\overline{S}_1}-k\overline{S}_1\left(\left(1-\frac{\overline{S}_2}{S_2}\right)\left(\frac{S_2}{\overline{S}_2}-\frac{S_1}{\overline{S}_1}\right)\right)-k\overline{S}_2\left(\left(1-\frac{\overline{S}_3}{S_3}\right)\left(\frac{S_3}{\overline{S}_3}-\frac{S_2}{\overline{S}_2}\right)\right)$$

$$-bQ_1-bR_1-bQ_2-bR_2-bQ_3-bR_3. \tag{5.46}$$

If $\dfrac{S_2}{\overline{S}_2}\ge\dfrac{S_1}{\overline{S}_1}$ for all $S_1\ge\overline{S_1}$ and $\dfrac{S_2}{\overline{S}_2}\le\dfrac{S_1}{\overline{S}_1}$ for all $0<S_1\le\overline{S_1}$, then

$-k\overline{S}_1\left(\left(1-\dfrac{\overline{S}_2}{S_2}\right)\left(\dfrac{S_2}{\overline{S}_2}-\dfrac{S_1}{\overline{S}_1}\right)\right)\le 0.$

If $\dfrac{S_3}{\overline{S}_3}\ge\dfrac{S_2}{\overline{S}_2}$ for all $S_2\ge\overline{S_2}$ and $\dfrac{S_3}{\overline{S}_3}\le\dfrac{S_2}{\overline{S}_2}$ for all $0<S_2\le\overline{S_2}$,

then $-k\overline{S}_2\left(\left(1-\dfrac{\overline{S}_3}{S_3}\right)\left(\dfrac{S_3}{\overline{S}_3}-\dfrac{S_2}{\overline{S}_2}\right)\right)\le 0.$

We can see that all of terms in (5.46) are always non – positive. From equation (5.46), $\dfrac{d}{dt}\rho(t)\le 0$, then the function $\dfrac{d}{dt}\rho(t)$ is negative definite. The limit set of each solution is contained in the largest invariant set for which $S_1=\overline{S_1}, S_2=\overline{S_2}, S_3=\overline{S_3}, Q_1=0, R_1=0, Q_2=0, R_2=0,\ Q_3=0$ and $R_3=0$ which is singleton $\{P_1\}$. LaSalle's invariant principle implies that the disease free equilibrium P_1 is globally asymptotically stable on Ω_1.

To prove the global stability of the third age group endemic equilibrium point P_2, we consider the following theorem.

Theorem 2 If $\hat{R}_0 > 1$ and $R_0 = \max\left\{\tilde{R}_0, R_0^*\right\} < 1$ then the third age group endemic equilibrium state

$$P_2\left(\hat{S}_1, \hat{E}_1, \hat{I}_1, \hat{Q}_1, \hat{R}_1, \hat{S}_2, \hat{E}_2, \hat{I}_2, \hat{Q}_2, \hat{R}_2, \hat{S}_3, \hat{E}_3, \hat{I}_3, \hat{Q}_3, \hat{R}_3\right) \in \Omega_1$$

exists and is globally asymptotically stable on Ω_1 if

$$\left.\begin{aligned} a_1 &= \frac{bN_T}{\hat{S}_1} \\ a_2 &= \frac{bN_T}{\hat{S}_2} \\ a_3 &= \frac{(b+d)N_T}{\hat{S}_3} \\ d &= e+f \end{aligned}\right\} \tag{5.47}$$

Proof The Lyapunov function of the form

$$\psi(t) = \hat{S}_1\left(\frac{S_1}{\hat{S}_1} - \ln\frac{S_1}{\hat{S}_1}\right) + E_1 + I_1 + Q_1 + R_1 + \hat{S}_2\left(\frac{S_2}{\hat{S}_2} - \ln\frac{S_2}{\hat{S}_2}\right) + E_2 + I_2 + Q_2 + R_2$$
$$+\hat{S}_3\left(\frac{S_3}{\hat{S}_3} - \ln\frac{S_3}{\hat{S}_3}\right) + E_3 + I_3 + \hat{Q}_3\left(\frac{Q_3}{\hat{Q}_3} - \ln\frac{Q_3}{\hat{Q}_3}\right)$$

Its derivative along the trajectories of (5.1) – (5.15)

$$\frac{d}{dt}\psi(t) = \frac{d}{dt}S_1\left(1 - \frac{\hat{S}_1}{S_1}\right) + \frac{d}{dt}E_1 + \frac{d}{dt}I_1 + \frac{d}{dt}Q_1 + \frac{d}{dt}R_1$$
$$+\frac{d}{dt}S_2\left(1 - \frac{\hat{S}_2}{S_2}\right) + \frac{d}{dt}E_2 + \frac{d}{dt}I_2 + \frac{d}{dt}Q_2 + \frac{d}{dt}R_2$$
$$+\frac{d}{dt}S_3\left(1 - \frac{\hat{S}_3}{S_3}\right) + \frac{d}{dt}E_3 + \frac{d}{dt}I_3 + \frac{d}{dt}Q_3\left(1 - \frac{\hat{Q}_3}{Q_3}\right)$$

$$=\left(bN_T-\left(\frac{a_1(E_1+I_1)}{N_T}+b+k\right)S_1\right)\left(1-\frac{\hat{S}_1}{S_1}\right)+\left(\frac{a_1S_1(E_1+I_1)}{N_T}-(b+c+d+k)E_1\right)$$

$$+\left(cE_1-(b+e+f+k)I_1\right)+\left(eI_1-(b+g+k)Q_1\right)+\left(dE_1+fI_1+gQ_1-(b+k)R_1\right)$$

$$+\left(kS_1-\left(\frac{a_2(E_2+I_2)}{N_T}+b+k\right)S_2\right)\left(1-\frac{\hat{S}_2}{S_2}\right)+\left(kE_1+\frac{a_2S_2(E_2+I_2)}{N_T}-(b+c+d+k)E_2\right)$$

$$+\left(kI_1+cE_2-(b+e+f+k)I_2\right)+\left(kQ_1+eI_2-(b+g+k)Q_2\right)$$

$$+\left(kR_1+dE_2+fI_2+gQ_2-(b+k)R_2\right)+\left(kS_2-\left(\frac{a_3(E_3+I_3)}{N_T}+b\right)S_3\right)\left(1-\frac{\hat{S}_3}{S_3}\right)$$

$$+\left(kE_2+\frac{a_3S_3(E_3+I_3)}{N_T}-(b+c+d)E_3\right)+\left(kI_2+cE_3-(b+e+f)I_3\right)$$

$$+\left(kQ_2+eI_3-(b+g)Q_3\right)\left(1-\frac{\hat{Q}_3}{Q_3}\right)$$

$$=bN_T\left(1-\frac{\hat{S}_1}{S_1}\right)+(b+k)\hat{S}_1-bS_1-k\frac{S_1}{S_2}\hat{S}_2+(b+k)\hat{S}_2-bS_2-k\frac{S_2}{S_3}\hat{S}_3+b\hat{S}_3-bS_3$$

$$+(E_1+I_1)\left(\frac{a_1\hat{S}_1}{N_T}-b\right)+(E_2+I_2)\left(\frac{a_2\hat{S}_2}{N_T}-b\right)+\left(\frac{a_3\hat{S}_3}{N_T}-b-d\right)E_3+\left(\frac{a_3\hat{S}_3}{N_T}-b-e-f\right)I_3$$

$$-bQ_1-bR_1-\left(b+k\frac{\hat{Q}_3}{Q_3}\right)Q_2-(b+k)R_2+eI_3\left(1-\frac{\hat{Q}_3}{Q_3}\right)+(b+g)\hat{Q}_3\left(1-\frac{Q_3}{\hat{Q}_3}\right). \qquad (5.48)$$

Substituting four conditions of (5.24) into (5.25), we have

$$\frac{d}{dt}\psi(t)=bN_T\left(1-\frac{\hat{S}_1}{S_1}\right)+(b+k)\hat{S}_1-bS_1-k\frac{S_1}{S_2}\hat{S}_2+(b+k)\hat{S}_2-bS_2-k\frac{S_2}{S_3}\hat{S}_3+b\hat{S}_3-bS_3$$

$$-bQ_1-bR_1-\left(b+k\frac{\hat{Q}_3}{Q_3}\right)Q_2-(b+k)R_2+eI_3\left(1-\frac{\hat{Q}_3}{Q_3}\right)+(b+g)\hat{Q}_3\left(1-\frac{Q_3}{\hat{Q}_3}\right)$$

$$= bN_T\left(1-\frac{\hat{S}_1}{S_1}\right)+(b+k)\hat{S}_1\left(1-\frac{S_1}{\hat{S}_1}\right)+k\hat{S}_1\left(\frac{S_1}{\hat{S}_1}-\frac{S_1}{\hat{S}_1}\frac{\hat{S}_2}{S_2}\right)+(b+k)\hat{S}_2\left(1-\frac{S_2}{\hat{S}_2}\right)$$

$$+k\frac{\hat{S}_2}{\hat{R}_0}\left(\frac{S_2}{\hat{S}_2/\hat{R}_0}-\frac{S_2}{\hat{S}_2/\hat{R}_0}\frac{\hat{S}_3}{S_3}\right)+b\hat{S}_3\left(1-\frac{S_3}{\hat{S}_3}\right)-bQ_1-bR_1-\left(b+k\frac{\hat{Q}_3}{Q_3}\right)Q_2$$

$$-(b+k)R_2+e\hat{I}_3\left(\frac{I_3}{\hat{I}_3}-\frac{I_3}{\hat{I}_3}\frac{\hat{Q}_3}{Q_3}\right)+(b+g)\hat{Q}_3\left(1-\frac{Q_3}{\hat{Q}_3}\right). \tag{5.49}$$

Next, using the endemic relations in the third age group endemic equilibrium state, we have $\hat{S}_1=\frac{bN_T}{b+k}, \hat{S}_2=\frac{k}{b+k}\hat{S}_1, b\hat{S}_3=\frac{k\hat{S}_2}{\hat{R}_0}$ and $e\hat{I}_3=(b+g)\hat{Q}_3$, equation (5.49) becomes

$$\frac{d}{dt}\psi(t)=bN_T\left(1-\frac{\hat{S}_1}{S_1}\right)+bN_T\left(1-\frac{S_1}{\hat{S}_1}\right)+k\hat{S}_1\left(\frac{S_1}{\hat{S}_1}-\frac{S_1}{\hat{S}_1}\frac{\hat{S}_2}{S_2}\right)+k\hat{S}_1\left(1-\frac{S_2}{\hat{S}_2}\right)$$

$$+b\hat{S}_3\left(\frac{S_2}{\hat{S}_2/\hat{R}_0}-\frac{S_2}{\hat{S}_2/\hat{R}_0}\frac{\hat{S}_3}{S_3}\right)+b\hat{S}_3\left(1-\frac{S_3}{\hat{S}_3}\right)-bQ_1-bR_1-\left(b+k\frac{\hat{Q}_3}{Q_3}\right)Q_2-(b+k)R_2$$

$$+e\hat{I}_3\left(\frac{I_3}{\hat{I}_3}-\frac{I_3}{\hat{I}_3}\frac{\hat{Q}_3}{Q_3}\right)+e\hat{I}_3\left(1-\frac{Q_3}{\hat{Q}_3}\right)$$

$$=bN_T\left(2-\frac{\hat{S}_1}{S_1}-\frac{S_1}{\hat{S}_1}\right)+k\hat{S}_1\left(\frac{S_1}{\hat{S}_1}-\frac{S_1}{\hat{S}_1}\frac{\hat{S}_2}{S_2}+1-\frac{S_2}{\hat{S}_2}\right)$$

$$+b\hat{S}_3\left(\frac{S_2}{\hat{S}_2/\hat{R}_0}-\frac{S_2}{\hat{S}_2/\hat{R}_0}\frac{\hat{S}_3}{S_3}+1-\frac{S_3}{\hat{S}_3}\right)-bQ_1-bR_1-\left(b+k\frac{\hat{Q}_3}{Q_3}\right)Q_2-(b+k)R_2$$

$$+e\hat{I}_3\left(\frac{I_3}{\hat{I}_3}-\frac{I_3}{\hat{I}_3}\frac{\hat{Q}_3}{Q_3}+1-\frac{Q_3}{\hat{Q}_3}\right),$$

$$=-bN_T\frac{\left(\hat{S}_1-S_1\right)^2}{S_1\hat{S}_1}-k\hat{S}_1\left(\left(1-\frac{\hat{S}_2}{S_2}\right)\left(\frac{S_2}{\hat{S}_2}-\frac{S_1}{\hat{S}_1}\right)\right)-b\hat{S}_3\left(\left(1-\frac{\hat{S}_3}{S_3}\right)\left(\frac{S_3}{\hat{S}_3}-\frac{S_2}{\hat{S}_2/\hat{R}_0}\right)\right)$$

$$-bQ_1-bR_1-\left(b+k\frac{\hat{Q}_3}{Q_3}\right)Q_2-(b+k)R_2-e\hat{I}_3\left(\left(1-\frac{\hat{Q}_3}{Q_3}\right)\left(\frac{Q_3}{\hat{Q}_3}-\frac{I_3}{\hat{I}_3}\right)\right). \tag{5.50}$$

The second term of (5.50) is same as before in Theorem 1, for the third term, we see that, if $\frac{S_3}{\hat{S}_3} \geq \frac{S_2}{\hat{S}_2/\hat{R}_0}$ for all $S_2 \geq \hat{S}_2/\hat{R}_0$ and $\frac{S_3}{\hat{S}_3} \leq \frac{S_2}{\hat{S}_2/\hat{R}_0}$ for all $0 < S_2 \leq \hat{S}_2/\hat{R}_0$, therefore $-b\hat{S}_3\left(\left(1-\frac{\hat{S}_3}{S_3}\right)\left(\frac{S_3}{\hat{S}_3}-\frac{S_2}{\hat{S}_2/\hat{R}_0}\right)\right) \leq 0$.

For the last term, if $\frac{Q_3}{\hat{Q}_3} \geq \frac{I_3}{\hat{I}_3}$ for all $I_3 \geq \hat{I}_3$ and $\frac{Q_3}{\hat{Q}_3} \leq \frac{I_3}{\hat{I}_3}$ for all $0 < I_3 \leq \hat{I}_3$ thus $-e\hat{I}_3\left(\left(1-\frac{\hat{Q}_3}{Q_3}\right)\left(\frac{Q_3}{\hat{Q}_3}-\frac{I_3}{\hat{I}_3}\right)\right) \leq 0$.

Therefore, all terms in (5.50) are always non – positive and $\frac{d}{dt}\psi(t) \leq 0$. The limit set of each solution is contained in the largest invariant set for which $S_1 = \hat{S}_1, S_2 = \hat{S}_2, S_3 = \hat{S}_3, Q_1 = 0, R_1 = 0, Q_2 = 0, R_2 = 0$ and $Q_3 = \hat{Q}_3$ which is singleton $\{P_2\}$. Hence, by LaSalle's invariant principle, the third age group endemic equilibrium P_2 is globally asymptotically stable on Ω_1.

Next, we consider the global stability of the second and third age group endemic equilibrium point P_3 .

Theorem 3 If $\tilde{R}_0 > 1$ and $R_0^* < 1$ then the second and third age group endemic equilibrium state $P_3(\tilde{S}_1, \tilde{E}_1, \tilde{I}_1, \tilde{Q}_1, \tilde{R}_1, \tilde{S}_2, \tilde{E}_2, \tilde{I}_2, \tilde{Q}_2, \tilde{R}_2, \tilde{S}_3, \tilde{E}_3, \tilde{I}_3, \tilde{Q}_3, \tilde{R}_3) \in \Omega_1$ exists and is globally asymptotically stable on Ω_1 when

$$\left.\begin{aligned} a_1 &= \frac{bN_T}{\tilde{S}_1} \\ a_2 &= \frac{(b+d)N_T}{\tilde{S}_2} \\ a_3 &= \frac{(b+d)N_T}{\tilde{S}_3} \\ d &= e+f. \end{aligned}\right\} \tag{5.51}$$

Proof. Lyapunov function can be written in the form of

$$\eta(t) = \tilde{S}_1\left(\frac{S_1}{\tilde{S}_1}-\ln\frac{S_1}{\tilde{S}_1}\right)+E_1+I_1+Q_1+R_1+\tilde{S}_2\left(\frac{S_2}{\tilde{S}_2}-\ln\frac{S_2}{\tilde{S}_2}\right)+E_2+I_2+\tilde{Q}_2\left(\frac{Q_2}{\tilde{Q}_2}-\ln\frac{Q_2}{\tilde{Q}_2}\right)$$
$$+\tilde{S}_3\left(\frac{S_3}{\tilde{S}_3}-\ln\frac{S_3}{\tilde{S}_3}\right)+E_3+I_3+\tilde{Q}_3\left(\frac{Q_3}{\tilde{Q}_3}-\ln\frac{Q_3}{\tilde{Q}_3}\right)$$

$$\frac{d}{dt}\eta(t)=\frac{d}{dt}S_1\left(1-\frac{\widetilde{S}_1}{S_1}\right)+\frac{d}{dt}E_1+\frac{d}{dt}I_1+\frac{d}{dt}Q_1+\frac{d}{dt}R_1$$

$$+\frac{d}{dt}S_2\left(1-\frac{\widetilde{S}_2}{S_2}\right)+\frac{d}{dt}E_2+\frac{d}{dt}I_2+\frac{d}{dt}Q_2\left(1-\frac{\widetilde{Q}_2}{Q_2}\right)$$

$$+\frac{d}{dt}S_3\left(1-\frac{\widetilde{S}_3}{S_3}\right)+\frac{d}{dt}E_3+\frac{d}{dt}I_3+\frac{d}{dt}Q_3\left(1-\frac{\widetilde{Q}_3}{Q_3}\right)$$

$$=\left(bN_T-\left(\frac{a_1(E_1+I_1)}{N_T}+b+k\right)S_1\right)\left(1-\frac{\widetilde{S}_1}{S_1}\right)+\left(\frac{a_1S_1(E_1+I_1)}{N_T}-(b+c+d+k)E_1\right)$$

$$+\left(cE_1-(b+e+f+k)I_1\right)+\left(eI_1-(b+g+k)Q_1\right)+\left(dE_1+fI_1+gQ_1-(b+k)R_1\right)$$

$$+\left(kS_1-\left(\frac{a_2(E_2+I_2)}{N_T}+b+k\right)S_2\right)\left(1-\frac{\widetilde{S}_2}{S_2}\right)+\left(kE_1+\frac{a_2S_2(E_2+I_2)}{N_T}-(b+c+d+k)E_2\right)$$

$$+\left(kI_1+cE_2-(b+e+f+k)I_2\right)+\left(kQ_1+eI_2-(b+g+k)Q_2\right)\left(1-\frac{\widetilde{Q}_2}{Q_2}\right)$$

$$+\left(kS_2-\left(\frac{a_3(E_3+I_3)}{N_T}+b\right)S_3\right)\left(1-\frac{\widetilde{S}_3}{S_3}\right)+\left(kE_2+\frac{a_3S_3(E_3+I_3)}{N_T}-(b+c+d)E_3\right)$$

$$+\left(kI_2+cE_3-(b+e+f)I_3\right)+\left(kQ_2+eI_3-(b+g)Q_3\right)\left(1-\frac{\widetilde{Q}_3}{Q_3}\right),$$

$$=bN_T\left(1-\frac{\widetilde{S}_1}{S_1}\right)+(b+k)\widetilde{S}_1-bS_1-k\frac{S_1}{S_2}\widetilde{S}_2+(b+k)\widetilde{S}_2-bS_2-k\frac{S_2}{S_3}\widetilde{S}_3+b\widetilde{S}_3-bS_3$$

$$+(E_1+I_1)\left(\frac{a_1\widetilde{S}_1}{N_T}-b\right)-\left(b+k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1-(b+k)R_1+E_2\left(\frac{a_2\widetilde{S}_2}{N_T}-b-d\right)+I_2\left(\frac{a_2\widetilde{S}_2}{N_T}-b-e-f\right)$$

$$+(b+g+k)\widetilde{Q}_2\left(1-\frac{Q_2}{\widetilde{Q}_2}\right)+e\widetilde{I}_2\left(1-\frac{\widetilde{Q}_2}{Q_2}\right)+\left(\frac{a_3\widetilde{S}_3}{N_T}-b-d\right)E_3+\left(\frac{a_3\widetilde{S}_3}{N_T}-b-e-f\right)I_3$$

$$+(kQ_2+eI_3)\left(1-\frac{\widetilde{Q}_3}{Q_3}\right)+(b+g)\widetilde{Q}_3\left(1-\frac{Q_3}{\widetilde{Q}_3}\right). \qquad (5.52)$$

Substituting a_1, a_2, a_3 and d of (5.51) into (5.52), we have

$$\frac{d}{dt}\eta(t) = bN_T\left(1-\frac{\widetilde{S}_1}{S_1}\right) + (b+k)\widetilde{S}_1 - bS_1 - k\frac{S_1}{S_2}\widetilde{S}_2 + (b+k)\widetilde{S}_2 - bS_2 - k\frac{S_2}{S_3}\widetilde{S}_3 + b\widetilde{S}_3 - bS_3$$

$$-\left(b + k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1 - (b+k)R_1 + (b+g+k)\widetilde{Q}_2\left(1-\frac{Q_2}{\widetilde{Q}_2}\right) + eI_2\left(1-\frac{\widetilde{Q}_2}{Q_2}\right)$$

$$+(kQ_2 + eI_3)\left(1-\frac{\widetilde{Q}_3}{Q_3}\right) + (b+g)\widetilde{Q}_3\left(1-\frac{Q_3}{\widetilde{Q}_3}\right)$$

$$= bN_T\left(1-\frac{\widetilde{S}_1}{S_1}\right) + (b+k)\widetilde{S}_1\left(1-\frac{S_1}{\widetilde{S}_1}\right) + k\frac{\widetilde{S}_1}{\widetilde{R}_0}\left(\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0} - \frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}\frac{\widetilde{S}_2}{S_2}\right) + (b+k)\widetilde{S}_2\left(1-\frac{S_2}{\widetilde{S}_2}\right)$$

$$+ k\widetilde{S}_2\left(\frac{S_2}{\widetilde{S}_2} - \frac{S_2}{\widetilde{S}_2}\frac{\widetilde{S}_3}{S_3}\right) + \left(b\widetilde{S}_3 + \frac{a_3}{N_T}\left(\widetilde{E}_3 + \widetilde{I}_3\right)\widetilde{S}_3\right)\left(1-\frac{S_3}{\widetilde{S}_3}\right) - \frac{a_3}{N_T}\left(\widetilde{E}_3 + \widetilde{I}_3\right)\widetilde{S}_3\left(1-\frac{S_3}{\widetilde{S}_3}\right)$$

$$-\left(b + k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1 - (b+k)R_1 + (b+g+k)\widetilde{Q}_2\left(1-\frac{Q_2}{\widetilde{Q}_2}\right) + e\widetilde{I}_2\left(\frac{I_2}{\widetilde{I}_2} - \frac{I_2}{\widetilde{I}_2}\frac{\widetilde{Q}_2}{Q_2}\right)$$

$$+\left(k\widetilde{Q}_2 + e\widetilde{I}_3\right)\left(\frac{kQ_2 + eI_3}{k\widetilde{Q}_2 + e\widetilde{I}_3} - \frac{kQ_2 + eI_3}{k\widetilde{Q}_2 + e\widetilde{I}_3}\frac{\widetilde{Q}_3}{Q_3}\right) + (b+g)\widetilde{Q}_3\left(1-\frac{Q_3}{\widetilde{Q}_3}\right). \tag{5.53}$$

Next, using the endemic relations in the second and third age group endemic equilibrium state, we have $\widetilde{S}_1 = \frac{bN_T}{b+k}$, $(b+k)\widetilde{S}_2 = \frac{k\widetilde{S}_1}{\widetilde{R}_0}$, $b\widetilde{S}_3 + \frac{a_3}{N_T}\left(\widetilde{E}_3 + \widetilde{I}_3\right)\widetilde{S}_3 = k\widetilde{S}_2$, $e\widetilde{I}_2 = (b+g+k)\widetilde{Q}_2$ and $k\widetilde{Q}_2 + e\widetilde{I}_3 = (b+g)\widetilde{Q}_3$, equation (5.53) becomes

$$\frac{d}{dt}\eta(t) = bN_T\left(1-\frac{\widetilde{S}_1}{S_1}\right) + bN_T\left(1-\frac{S_1}{\widetilde{S}_1}\right) + (b+k)\widetilde{S}_2\left(\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0} - \frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}\frac{\widetilde{S}_2}{S_2}\right) + (b+k)\widetilde{S}_2\left(1-\frac{S_2}{\widetilde{S}_2}\right)$$

$$+ k\widetilde{S}_2\left(\frac{S_2}{\widetilde{S}_2} - \frac{S_2}{\widetilde{S}_2}\frac{\widetilde{S}_3}{S_3}\right) + k\widetilde{S}_2\left(1-\frac{S_3}{\widetilde{S}_3}\right) - \left(k\widetilde{S}_2 - b\widetilde{S}_3\right)\left(1-\frac{S_3}{\widetilde{S}_3}\right)$$

$$-\left(b + k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1 - (b+k)R_1 + e\widetilde{I}_2\left(1-\frac{Q_2}{\widetilde{Q}_2}\right) + e\widetilde{I}_2\left(\frac{I_2}{\widetilde{I}_2} - \frac{I_2}{\widetilde{I}_2}\frac{\widetilde{Q}_2}{Q_2}\right)$$

$$+(b+g)\widetilde{Q}_3\left(\frac{kQ_2 + eI_3}{k\widetilde{Q}_2 + e\widetilde{I}_3} - \frac{kQ_2 + eI_3}{k\widetilde{Q}_2 + e\widetilde{I}_3}\frac{\widetilde{Q}_3}{Q_3}\right) + (b+g)\widetilde{Q}_3\left(1-\frac{Q_3}{\widetilde{Q}_3}\right),$$

$$\frac{d}{dt}\eta(t)=bN_T\left(2-\frac{\widetilde{S}_1}{S_1}-\frac{S_1}{\widetilde{S}_1}\right)+(b+k)\widetilde{S}_2\left(\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}-\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}\frac{\widetilde{S}_2}{S_2}+1-\frac{S_2}{\widetilde{S}_2}\right)+k\widetilde{S}_2\left(\frac{S_2}{\widetilde{S}_2}-\frac{S_2}{\widetilde{S}_2}\frac{\widetilde{S}_3}{S_3}+1-\frac{S_3}{\widetilde{S}_3}\right)$$

$$-b\left(\frac{k\widetilde{S}_2}{b}-\widetilde{S}_3\right)\left(1-\frac{S_3}{\widetilde{S}_3}\right)-\left(b+k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1-(b+k)R_1+e\widetilde{I}_2\left(\frac{I_2}{\widetilde{I}_2}-\frac{I_2}{\widetilde{I}_2}\frac{\widetilde{Q}_2}{Q_2}+1-\frac{Q_2}{\widetilde{Q}_2}\right)$$

$$+(b+g)\widetilde{Q}_3\left(\frac{kQ_2+eI_3}{k\widetilde{Q}_2+e\widetilde{I}_3}-\frac{kQ_2+eI_3}{k\widetilde{Q}_2+e\widetilde{I}_3}\frac{\widetilde{Q}_3}{Q_3}+1-\frac{Q_3}{\widetilde{Q}_3}\right),$$

$$\frac{d}{dt}\eta(t)=-bN_T\frac{\left(\widetilde{S}_1-S_1\right)^2}{S_1\widetilde{S}_1}-(b+k)\widetilde{S}_2\left(\left(1-\frac{\widetilde{S}_2}{S_2}\right)\left(\frac{S_2}{\widetilde{S}_2}-\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}\right)\right)$$

$$-k\widetilde{S}_2\left(\left(1-\frac{\widetilde{S}_3}{S_3}\right)\left(\frac{S_3}{\widetilde{S}_3}-\frac{S_2}{\widetilde{S}_2}\right)\right)-b\left(\frac{k\widetilde{S}_2}{b}-\widetilde{S}_3\right)\left(1-\frac{S_3}{\widetilde{S}_3}\right)-\left(b+k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1-(b+k)R_1$$

$$-e\widetilde{I}_2\left(\left(1-\frac{\widetilde{Q}_2}{Q_2}\right)\left(\frac{Q_2}{\widetilde{Q}_2}-\frac{I_2}{\widetilde{I}_2}\right)\right)-(b+g)\widetilde{Q}_3\left(\left(1-\frac{\widetilde{Q}_3}{Q_3}\right)\left(\frac{Q_3}{\widetilde{Q}_3}-\frac{kQ_2+eI_3}{k\widetilde{Q}_2+e\widetilde{I}_3}\right)\right). \quad (5.54)$$

The third term of (5.54) is same as before in Theorem 1, for the second term,

if $\frac{S_2}{\widetilde{S}_2}\geq\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}$ for all $S_1\geq\widetilde{S}_1/\widetilde{R}_0$ and $\frac{S_2}{\widetilde{S}_2}\leq\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}$ for all $0<S_1\leq\widetilde{S}_1/\widetilde{R}_0$,

therefore $-(b+k)\widetilde{S}_2\left(\left(1-\frac{\widetilde{S}_2}{S_2}\right)\left(\frac{S_2}{\widetilde{S}_2}-\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}\right)\right)\leq 0$.

For the forth term, if $\frac{k\widetilde{S}_2}{b}\leq\widetilde{S}_3$ for all $S_3\geq\widetilde{S}_3$ and $\frac{k\widetilde{S}_2}{b}\geq\widetilde{S}_3$ for all $0<S_3\leq\widetilde{S}_3$,

then $-b\left(\frac{k\widetilde{S}_2}{b}-\widetilde{S}_3\right)\left(1-\frac{S_3}{\widetilde{S}_3}\right)\leq 0$.

For the seventh term, if $\frac{Q_2}{\widetilde{Q}_2}\geq\frac{I_2}{\widetilde{I}_2}$ for all $I_2\geq\widetilde{I}_2$ and $\frac{Q_2}{\widetilde{Q}_2}\leq\frac{I_2}{\widetilde{I}_2}$ for all $0<I_2\leq\widetilde{I}_2$,

then $-e\widetilde{I}_2\left(\left(1-\frac{\widetilde{Q}_2}{Q_2}\right)\left(\frac{Q_2}{\widetilde{Q}_2}-\frac{I_2}{\widetilde{I}_2}\right)\right)\leq 0$.

For the last term, if $\frac{Q_3}{\widetilde{Q}_3}\geq\frac{kQ_2+eI_3}{k\widetilde{Q}_2+e\widetilde{I}_3}$ for all $Q_2\geq\widetilde{Q}_2$ and $I_3\geq\widetilde{I}_3$, $\frac{Q_3}{\widetilde{Q}_3}\leq\frac{kQ_2+eI_3}{k\widetilde{Q}_2+e\widetilde{I}_3}$ for all $0<Q_2\leq\widetilde{Q}_2$ and $0<I_3\leq\widetilde{I}_3$, then $-(b+g)\widetilde{Q}_3\left(\left(1-\frac{\widetilde{Q}_3}{Q_3}\right)\left(\frac{Q_3}{\widetilde{Q}_3}-\frac{kQ_2+eI_3}{k\widetilde{Q}_2+e\widetilde{I}_3}\right)\right)\leq 0$.

Therefore, all of terms in (31) are always non – positive and $\frac{d}{dt}\eta(t)\le 0$. The limit set of each solution is contained in the largest invariant set for which $S_1=\widetilde{S}_1$, $S_2=\widetilde{S}_2$, $S_3=\widetilde{S}_3$, $Q_1=0$, $R_1=0$, $Q_2=\widetilde{Q}_2$, and $Q_3=\widetilde{Q}_3$ which is singleton $\{P_3\}$. Hence, by LaSalle's invariant principle, the second and third age group endemic equilibrium P_3 is globally asymptotically stable on Ω_1.

Last, we consider the global stability of the full endemic equilibrium point P_4 .

Theorem 4 If $R_0^*>1$, then the full endemic equilibrium state $P_4\left(S_1^*,E_1^*,I_1^*,Q_1^*,R_1^*,S_2^*,E_2^*,I_2^*,Q_2^*,R_2^*,S_3^*,E_3^*,I_3^*,Q_3^*,R_3^*\right)\in\Omega_1$ exists and is globally asymptotically stable on Ω_1 if

$$\left.\begin{aligned} a_1 &= \frac{(b+d)N_T}{S_1^*} \\ a_2 &= \frac{(b+d)N_T}{S_2^*} \\ a_3 &= \frac{(b+d)N_T}{S_3^*} \\ d &= e+f \end{aligned}\right\} \tag{5.55}$$

Proof. The Lyapunov function can be written in the form

$$\begin{aligned}\kappa(t) &= S_1^*\left(\frac{S_1}{S_1^*}-\ln\frac{S_1}{S_1^*}\right)+E_1+I_1+Q_1^*\left(\frac{Q_1}{Q_1^*}-\ln\frac{Q_1}{Q_1^*}\right) \\ &+S_2^*\left(\frac{S_2}{S_2^*}-\ln\frac{S_2}{S_2^*}\right)+E_2+I_2+Q_2^*\left(\frac{Q_2}{Q_2^*}-\ln\frac{Q_2}{Q_2^*}\right) \\ &+S_3^*\left(\frac{S_3}{S_3^*}-\ln\frac{S_3}{S_3^*}\right)+E_3+I_3+Q_3^*\left(\frac{Q_3}{Q_3^*}-\ln\frac{Q_3}{Q_3^*}\right)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\kappa(t) &= \frac{d}{dt}S_1\left(1-\frac{S_1^*}{S_1}\right)+\frac{d}{dt}E_1+\frac{d}{dt}I_1+\frac{d}{dt}Q_1\left(1-\frac{Q_1^*}{Q_1}\right) \\ &+\frac{d}{dt}S_2\left(1-\frac{S_2^*}{S_2}\right)+\frac{d}{dt}E_2+\frac{d}{dt}I_2+\frac{d}{dt}Q_2\left(1-\frac{Q_2^*}{Q_2}\right) \\ &+\frac{d}{dt}S_3\left(1-\frac{S_3^*}{S_3}\right)+\frac{d}{dt}E_3+\frac{d}{dt}I_3+\frac{d}{dt}Q_3\left(1-\frac{Q_3^*}{Q_3}\right)\end{aligned}$$

$$=\left(bN_T-\left(\frac{a_1(E_1+I_1)}{N_T}+b+k\right)S_1\right)\left(1-\frac{S_1^*}{S_1}\right)+\left(\frac{a_1S_1(E_1+I_1)}{N_T}-(b+c+d+k)E_1\right)$$

$$+\left(cE_1-(b+e+f+k)I_1\right)+\left(eI_1-(b+g+k)Q_1\right)\left(1-\frac{Q_1^*}{Q_1}\right)$$

$$+\left(kS_1-\left(\frac{a_2(E_2+I_2)}{N_T}+b+k\right)S_2\right)\left(1-\frac{S_2^*}{S_2}\right)+\left(kE_1+\frac{a_2S_2(E_2+I_2)}{N_T}-(b+c+d+k)E_2\right)$$

$$+\left(kI_1+cE_2-(b+e+f+k)I_2\right)+\left(kQ_1+eI_2-(b+g+k)Q_2\right)\left(1-\frac{Q_2^*}{Q_2}\right)$$

$$+\left(kS_2-\left(\frac{a_3(E_3+I_3)}{N_T}+b\right)S_3\right)\left(1-\frac{S_3^*}{S_3}\right)+\left(kE_2+\frac{a_3S_3(E_3+I_3)}{N_T}-(b+c+d)E_3\right)$$

$$+\left(kI_2+cE_3-(b+e+f)I_3\right)+\left(kQ_2+eI_3-(b+g)Q_3\right)\left(1-\frac{Q_3^*}{Q_3}\right)$$

$$=bN_T\left(1-\frac{S_1^*}{S_1}\right)+(b+k)S_1^*-bS_1-k\frac{S_1}{S_2}S_2^*+(b+k)S_2^*-bS_2-k\frac{S_2}{S_3}S_3^*+bS_3^*-bS_3$$

$$+\left(\frac{a_1S_1^*}{N_T}-b-d\right)E_1+\left(\frac{a_1S_1^*}{N_T}-b-e-f\right)I_1+eI_1\left(1-\frac{Q_1^*}{Q_1}\right)+(b+g+k)Q_1^*\left(1-\frac{Q_1}{Q_1^*}\right)$$

$$+\left(\frac{a_2S_2^*}{N_T}-b-d\right)E_2+\left(\frac{a_2S_2^*}{N_T}-b-e-f\right)I_2+\left(kQ_1+eI_2\right)\left(1-\frac{Q_2^*}{Q_2}\right)$$

$$+(b+g+k)Q_2^*\left(1-\frac{Q_2}{Q_2^*}\right)+\left(\frac{a_3S_3^*}{N_T}-b-d\right)E_3+\left(\frac{a_3S_3^*}{N_T}-b-e-f\right)I_3$$

$$+\left(kQ_2+eI_3\right)\left(1-\frac{Q_3^*}{Q_3}\right)+(b+g)Q_3^*\left(1-\frac{Q_3}{Q_3^*}\right). \qquad (5.56)$$

Substituting four conditions of (5.55) into (5.56), we have

$$\frac{d}{dt}\kappa(t)=bN_T\left(1-\frac{S_1^*}{S_1}\right)+(b+k)S_1^*-bS_1-k\frac{S_1}{S_2}S_2^*+(b+k)S_2^*-bS_2-k\frac{S_2}{S_3}S_3^*$$

$$+bS_3^*-bS_3+eI_1\left(1-\frac{Q_1^*}{Q_1}\right)+(b+g+k)Q_1^*\left(1-\frac{Q_1}{Q_1^*}\right)+\left(kQ_1+eI_2\right)\left(1-\frac{Q_2^*}{Q_2}\right)$$

$$+(b+g+k)Q_2^*\left(1-\frac{Q_2}{Q_2^*}\right)+\left(kQ_2+eI_3\right)\left(1-\frac{Q_3^*}{Q_3}\right)+(b+g)Q_3^*\left(1-\frac{Q_3}{Q_3^*}\right)$$

$$= bN_T\left(1-\frac{S_1^*}{S_1}\right)+(b+k)R_0^*S_1^*\left(\frac{1}{R_0^*}-\frac{S_1}{R_0^*S_1^*}\right)$$

$$+kS_1^*\left(\frac{S_1}{S_1^*}-\frac{S_1}{S_1^*}\frac{S_2^*}{S_2}\right)+\left((b+k)S_2^*+\frac{a_2}{N_T}(E_2^*+I_2^*)S_2^*\right)\left(1-\frac{S_2}{S_2^*}\right)$$

$$-\frac{a_2}{N_T}(E_2^*+I_2^*)S_2^*\left(1-\frac{S_2}{S_2^*}\right)+kS_2^*\left(\frac{S_2}{S_2^*}-\frac{S_2}{S_2^*}\frac{S_3^*}{S_3}\right)+\left(bS_3^*+\frac{a_3}{N_T}(E_3^*+I_3^*)S_3^*\right)\left(1-\frac{S_3}{S_3^*}\right)$$

$$-\frac{a_3}{N_T}(E_3^*+I_3^*)S_3^*\left(1-\frac{S_3}{S_3^*}\right)+eI_1^*\left(\frac{I_1}{I_1^*}-\frac{I_1}{I_1^*}\frac{Q_1^*}{Q_1}\right)+(b+g+k)Q_1^*\left(1-\frac{Q_1}{Q_1^*}\right)$$

$$+(kQ_1^*+eI_2^*)\left(\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}-\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}\frac{Q_2^*}{Q_2}\right)+(b+g+k)Q_2^*\left(1-\frac{Q_2}{Q_2^*}\right)$$

$$+(kQ_2^*+eI_3^*)\left(\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}-\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}\frac{Q_3^*}{Q_3}\right)+(b+g)Q_3^*\left(1-\frac{Q_3}{Q_3^*}\right). \tag{5.57}$$

Next, using the endemic relations in the full endemic equilibrium state, we have $bN_T=(b+k)R_0^*S_1^*$, $(b+k)S_2^*+\frac{a_2}{N_T}(E_2^*+I_2^*)S_2^*=kS_1^*$, $bS_3^*+\frac{a_3}{N_T}(E_3^*+I_3^*)S_3^*=kS_2^*$, $eI_1^*=(b+g+k)Q_1^*$, $kQ_1^*+eI_2^*=(b+g+k)Q_2^*$ and $kQ_2^*+eI_3^*=(b+g)Q_3^*$, equation (5.57) becomes

$$\frac{d}{dt}\kappa(t)=bN_T\left(1-\frac{S_1^*}{S_1}\right)+bN_T\left(\frac{1}{R_0^*}-\frac{S_1}{R_0^*S_1^*}\right)+kS_1^*\left(\frac{S_1}{S_1^*}-\frac{S_1}{S_1^*}\frac{S_2^*}{S_2}\right)+kS_1^*\left(1-\frac{S_2}{S_2^*}\right)$$

$$-(kS_1^*-(b+k)S_2^*)\left(1-\frac{S_2}{S_2^*}\right)+kS_2^*\left(\frac{S_2}{S_2^*}-\frac{S_2}{S_2^*}\frac{S_3^*}{S_3}\right)+kS_2^*\left(1-\frac{S_3}{S_3^*}\right)$$

$$-(kS_2^*-bS_3^*)\left(1-\frac{S_3}{S_3^*}\right)+eI_1^*\left(\frac{I_1}{I_1^*}-\frac{I_1}{I_1^*}\frac{Q_1^*}{Q_1}\right)+eI_1^*\left(1-\frac{Q_1}{Q_1^*}\right)$$

$$+(b+g+k)Q_2^*\left(\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}-\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}\frac{Q_2^*}{Q_2}\right)+(b+g+k)Q_2^*\left(1-\frac{Q_2}{Q_2^*}\right)$$

$$+(b+g)Q_3^*\left(\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}-\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}\frac{Q_3^*}{Q_3}\right)+(b+g)Q_3^*\left(1-\frac{Q_3}{Q_3^*}\right),$$

$$\frac{d}{dt}\kappa(t)=bN_T\left(1-\frac{S_1^*}{S_1}+\frac{1}{R_0^*}-\frac{S_1}{R_0^*S_1^*}\right)+kS_1^*\left(\frac{S_1}{S_1^*}-\frac{S_1}{S_1^*}\frac{S_2^*}{S_2}+1-\frac{S_2}{S_2^*}\right)$$

$$-(b+k)\left(\frac{kS_1^*}{b+k}-S_2^*\right)\left(1-\frac{S_2}{S_2^*}\right)+kS_2^*\left(\frac{S_2}{S_2^*}-\frac{S_2}{S_2^*}\frac{S_3^*}{S_3}+1-\frac{S_3}{S_3^*}\right)$$

$$-b\left(\frac{kS_2^*}{b}-S_3^*\right)\left(1-\frac{S_3}{S_3^*}\right)+eI_1^*\left(\frac{I_1}{I_1^*}-\frac{I_1}{I_1^*}\frac{Q_1^*}{Q_1}+1-\frac{Q_1}{Q_1^*}\right)$$

$$+(b+g+k)Q_2^*\left(\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}-\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}\frac{Q_2^*}{Q_2}+1-\frac{Q_2}{Q_2^*}\right)$$

$$+(b+g)Q_3^*\left(\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}-\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}\frac{Q_3^*}{Q_3}+1-\frac{Q_3}{Q_3^*}\right),$$

$$\frac{d}{dt}\kappa(t)=-bN_T\left(1-\frac{S_1}{S_1^*}\right)\left(\frac{S_1^*}{S_1}-\frac{1}{R_0^*}\right)-kS_1^*\left(\left(1-\frac{S_2^*}{S_2}\right)\left(\frac{S_2}{S_2^*}-\frac{S_1}{S_1^*}\right)\right)$$

$$-(b+k)\left(\frac{kS_1^*}{b+k}-S_2^*\right)\left(1-\frac{S_2}{S_2^*}\right)-kS_2^*\left(\left(1-\frac{S_3^*}{S_3}\right)\left(\frac{S_3}{S_3^*}-\frac{S_2}{S_2^*}\right)\right)$$

$$-b\left(\frac{kS_2^*}{b}-S_3^*\right)\left(1-\frac{S_3}{S_3^*}\right)-eI_1^*\left(\left(1-\frac{Q_1^*}{Q_1}\right)\left(\frac{Q_1}{Q_1^*}-\frac{I_1}{I_1^*}\right)\right)$$

$$-(b+g+k)Q_2^*\left(\left(1-\frac{Q_2^*}{Q_2}\right)\left(\frac{Q_2}{Q_2^*}-\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}\right)\right)$$

$$-(b+g)Q_3^*\left(\left(1-\frac{Q_3^*}{Q_3}\right)\left(\frac{Q_3}{Q_3^*}-\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}\right)\right). \tag{5.58}$$

If $\frac{S_1^*}{S_1}\le\frac{1}{R_0^*}$ for all $S_1\ge S_1^*$ and $\frac{S_1^*}{S_1}\ge\frac{1}{R_0^*}$ for all $0<S_1\le S_1^*$,

therefore $-bN_T\left(1-\frac{S_1}{S_1^*}\right)\left(\frac{S_1^*}{S_1}-\frac{1}{R_0^*}\right)\le 0$.

If $\frac{kS_1^*}{b+k}\le S_2^*$ for all $S_2\ge S_2^*$ and $\frac{\mathrm{k}\mathrm{S}_1^*}{\mathrm{b}+\mathrm{k}}\ge \mathrm{S}_2^*$ for all $0<S_2\le S_2^*$,

thus $-(b+k)\left(\frac{kS_1^*}{b+k}-S_2^*\right)\left(1-\frac{S_2}{S_2^*}\right)\le 0$.

The remaining terms is same as before in Theorem 3. Therefore, all of terms in (5.58) are always non - positive and $\frac{d}{dt}\kappa(t) \le 0$. The limit set of each solution is contained in the largest invariant set for which $S_1 = S_1^*, S_2 = S_2^*, S_3 = S_3^*, Q_1 = Q_1^*, Q_2 = Q_2^*$ and $Q_3 = Q_3^*$ which is singleton $\{P_4\}$. Hence, by LaSalle's invariant principle, the second and third age group endemic equilibrium P_4 is globally asymptotically stable on Ω_1.

Chapter 6
Network Model

6.1 Transmission Model

The SEIQR network model contains people and public places, we suppose that there are P public places and N people. Each day, the people move to any public places by random process, i.e., random from the 1^{st} person to the N^{th} person visit between the 1^{st} place and P^{th} place (with uniformly distribution). The probability for each person to visit each place is equivalent.

The dynamics of human populations are given by

$$\Delta SF_{t,i} = \frac{-a_1 SF_{t,i}\left(EF_{t,i}+IF_{t,i}\right)}{N} - kSF_{t,i}, \tag{6.1}$$

$$\Delta EF_{t,i} = \frac{a_1 SF_{t,i}\left(EF_{t,i}+IF_{t,i}\right)}{N} - (c+d+k)EF_{t,i}, \tag{6.2}$$

$$\Delta IF_{t,i} = cEF_{t,i} - (e+f+k)IF_{t,i}, \tag{6.3}$$

$$\Delta QF_{t,i} = eIF_{t,i} - (g+k)QF_{t,i}, \tag{6.4}$$

$$\Delta RF_{t,i} = dEF_{t,i} + fIF_{t,i} + gQF_{t,i} - kRF_{t,i}, \tag{6.5}$$

$$\Delta SS_{t,i} = kSF_{t,i}\frac{-a_2 SS_{t,i}\left(ES_{t,i}+IS_{t,i}\right)}{N} - kSS_{t,i}, \tag{6.6}$$

$$\Delta ES_{t,i} = kEF_{t,i} + \frac{a_2 SS_{t,i}\left(ES_{t,i}+IS_{t,i}\right)}{N} - (c+d+k)ES_{t,i}, \tag{6.7}$$

$$\Delta IS_{t,i} = kIF_{t,i} + cES_{t,i} - (e+f+k)IS_{t,i}, \tag{6.8}$$

$$\Delta QS_{t,i} = kQF_{t,i} + eIS_{t,i} - (g+k)QS_{t,i}, \tag{6.9}$$

$$\Delta RS_{t,i} = kRF_{t,i} + dES_{t,i} + fIS_{t,i} + gQS_{t,i} - kRS_{t,i}, \tag{6.10}$$

$$\Delta ST_{t,i} = kSS_{t,i}\frac{-a_3 ST_{t,i}\left(ET_{t,i}+IT_{t,i}\right)}{N}, \tag{6.11}$$

$$\Delta ET_{t,i} = kES_{t,i} + \frac{a_3 ST_{t,i}\left(ET_{t,i} + IT_{t,i}\right)}{N} - \left(c+d\right)ET_{t,i}, \tag{6.12}$$

$$\Delta IT_{t,i} = kIS_{t,i} + cET_{t,i} - \left(e+f\right)IT_{t,i}, \tag{6.13}$$

$$\Delta QT_{t,i} = kQS_{t,i} + eIT_{t,i} - gQT_{t,i}, \tag{6.14}$$

$$\Delta RT_{t,i} = kRS_{t,i} + dET_{t,i} + fIT_{t,i} + gQT_{t,i}. \tag{6.15}$$

The variables in our model are defined as follows:

$SF_{t,i}$, $EF_{t,i}$, $IF_{t,i}$, $QF_{t,i}$, $RF_{t,i}$ represent the numbers of susceptible, exposed, infectious, quarantined and recovered persons of the first age group in place i^{th} after visited at day t, respectively.

$SS_{t,i}$, $ES_{t,i}$, $IS_{t,i}$, $QS_{t,i}$, $RS_{t,i}$ represent the numbers of susceptible, exposed, infectious, quarantined and recovered persons of the second age group in place i^{th} after visited at day t, respectively.

$ST_{t,i}$, $ET_{t,i}$, $IT_{t,i}$, $QT_{t,i}$, $RT_{t,i}$ represent the numbers of susceptible, exposed, infectious, quarantined and recovered persons of the third age group in place i^{th} after visited at day t, respectively.

N is the total number of persons, P is the total number of public places, and T is the ending time.

The other parameters are defined in the following table.

Parameters	Definitions
k	The rate at which the first age group pass into the second age group and also the second age group pass into the third age group
a_1	The average number of adequate contacts of the first age group which is equal to $\theta_1\sigma_n$
a_2	The average number of adequate contacts of the second age group which is equal to $\theta_2\tau_n$
a_3	The average number of adequate contacts of the third age group which is equal to $\theta_3\omega_n$
$\theta_1, \theta_2, \theta_3$	The probability of catching the disease per contact to the infected/ exposed person of the first age group, the second age group and the third age group
$\sigma_n, \tau_n, \omega_n$	The average number of people contacted by each person per day of the first age group, the second age group and the third age group
c	The rate at which the exposed individuals E become the infected individuals I
e	The rate at which the individuals leave the infective individuals I for the quarantined individuals Q
d, f, g	The rate at which individuals in the E, I and Q classes recover from the disease
P	Number of public places.

Table 6.1 List of parameters for our model.

6.2 Numerical Examples

The numerical simulation of the disease transmission is presented with assumptions that there is one exposed person and one infected person in each age group at the first day. The results show the time distribution of exposed, infected and quarantined human dealing with different situations. We show the maximum number of exposed, infected and quarantined individuals, the day at which each individual group reaches to the maximum and the day at which each group tend to the steady state for each situation.

First situation, we assume that there are 20 public places, the time distributions of exposed, infected and quarantined humans for the different number of persons in this situation are shown in the following figures.

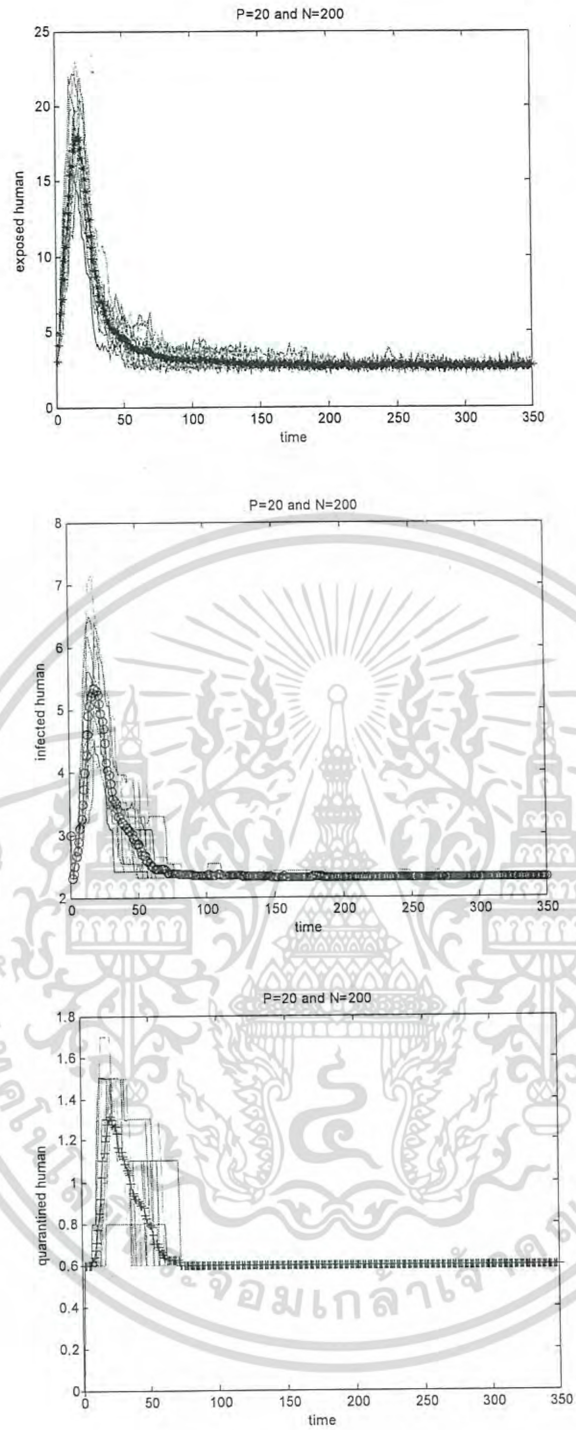

Figure 6.1 The time distribution of exposed, infected and quarantined human for $N = 200$ persons, respectively. The solid line represents the average outputs from 20 runs of simulation. The parameters in our model are $k = 1/(365\times10)$, $c = 1/9$, $d = 1/7$, $e = 1/5$, $f = 1/7$, $g = 0.7$, $a_1 = 20$, $a_2 = 25$, $a_3 = 15$, $P = 20$, $T = 350$.

From the above figure, we run more than 20 simulations to see the difference of average outputs. The average outputs for the different number of simulations are shown in the following figures.

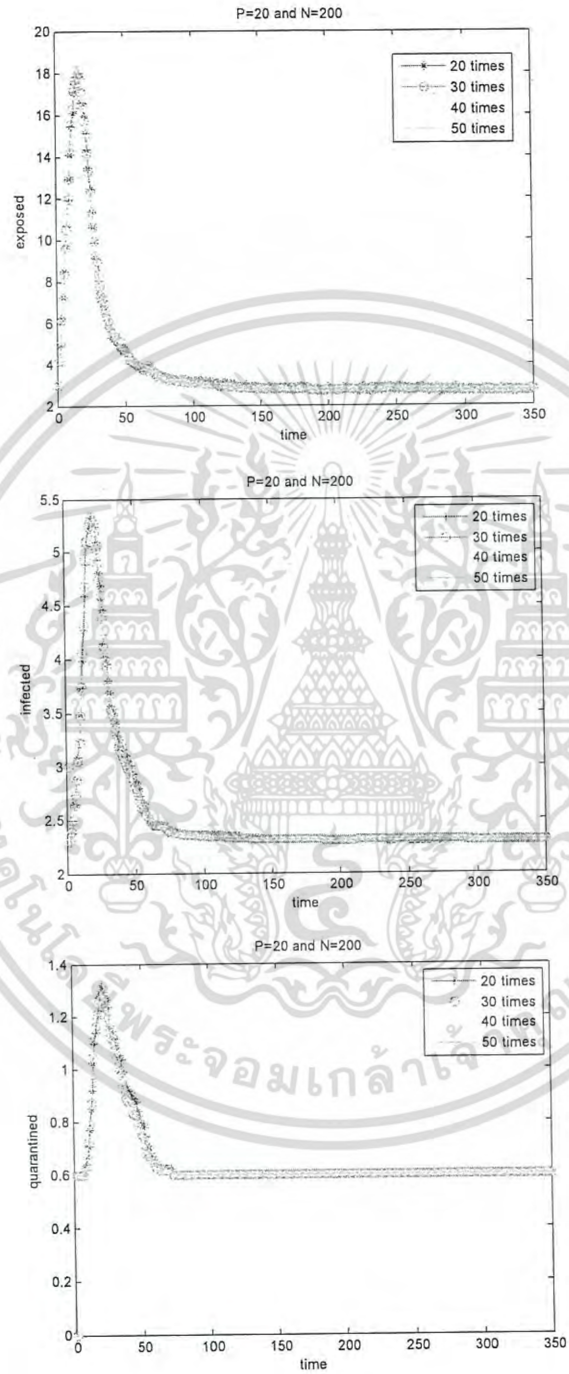

Figure 6.2 The average outputs for the different number of simulations.

From figure 6.2, we see that the average outputs are not different for run 20, 30, 40 or 50 simulations, therefore we run only 20 simulations for next situations.

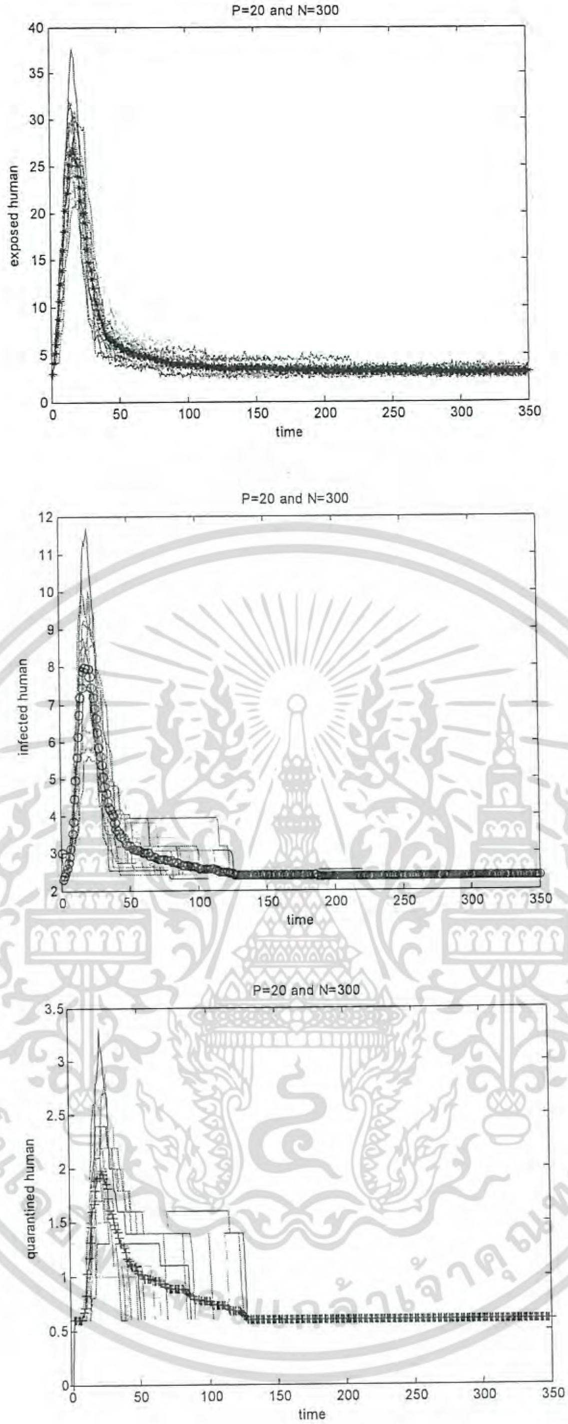

Figure 6.3 The time distribution of exposed, infected and quarantined human for $N = 300$ persons, respectively. The solid line represents the average output from 20 runs of simulation. The parameters are same as figure6.1.

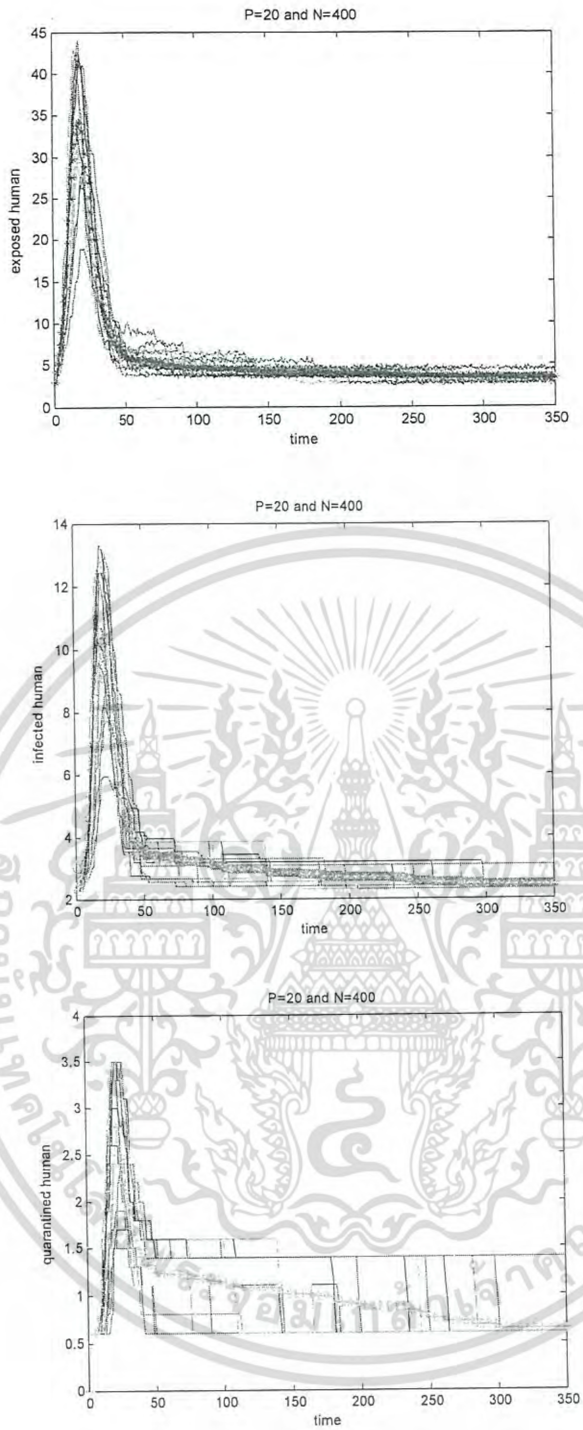

Figure 6.4 The time distribution of exposed, infected and quarantined human for $N = 400$ persons, respectively. The solid line represents the average output from 20 runs of simulation. The parameters are same as figure 6.1.

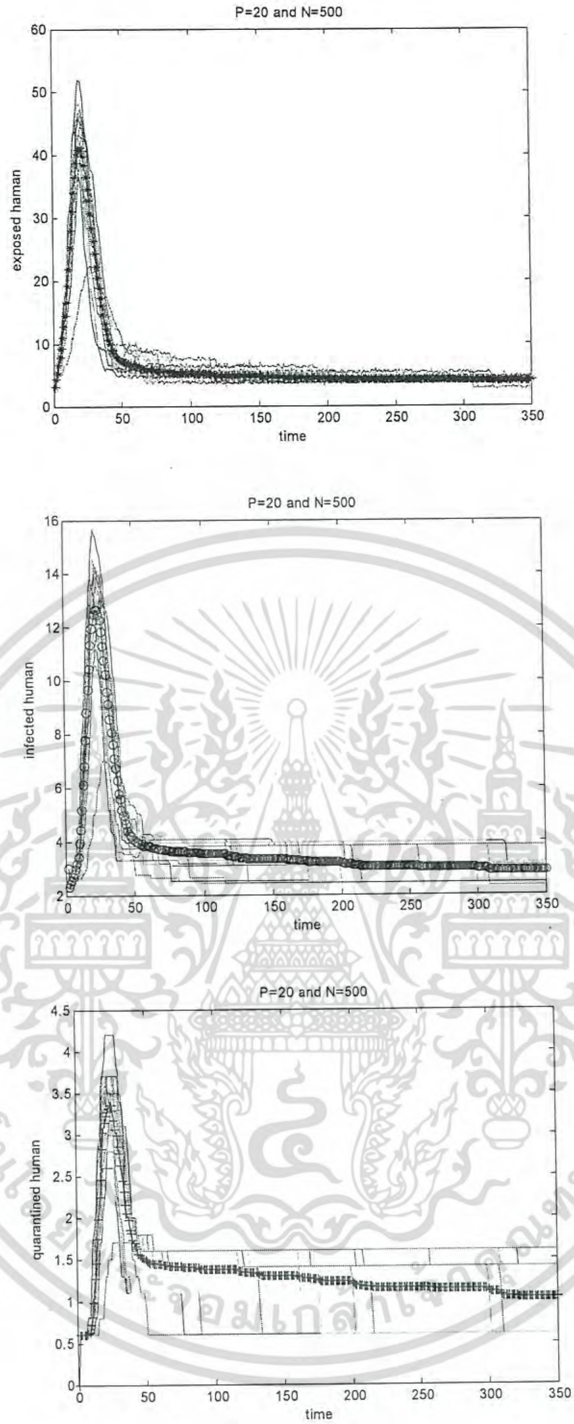

Figure 6.5 The time distribution of exposed, infected and quarantined human for $N = 500$ persons, respectively. The solid line represents the average output from 20 runs of simulation. The parameters are same as figure 6.1.

We simulate the time distribution of exposed, infected and quarantined human with the four different numbers of persons: $N = 200$, $N = 300$, $N = 400$ and $N = 500$. For each N, we run 20 times and find the average. The following figures show the average of time distributions of exposed, infected and quarantined humans, respectively.

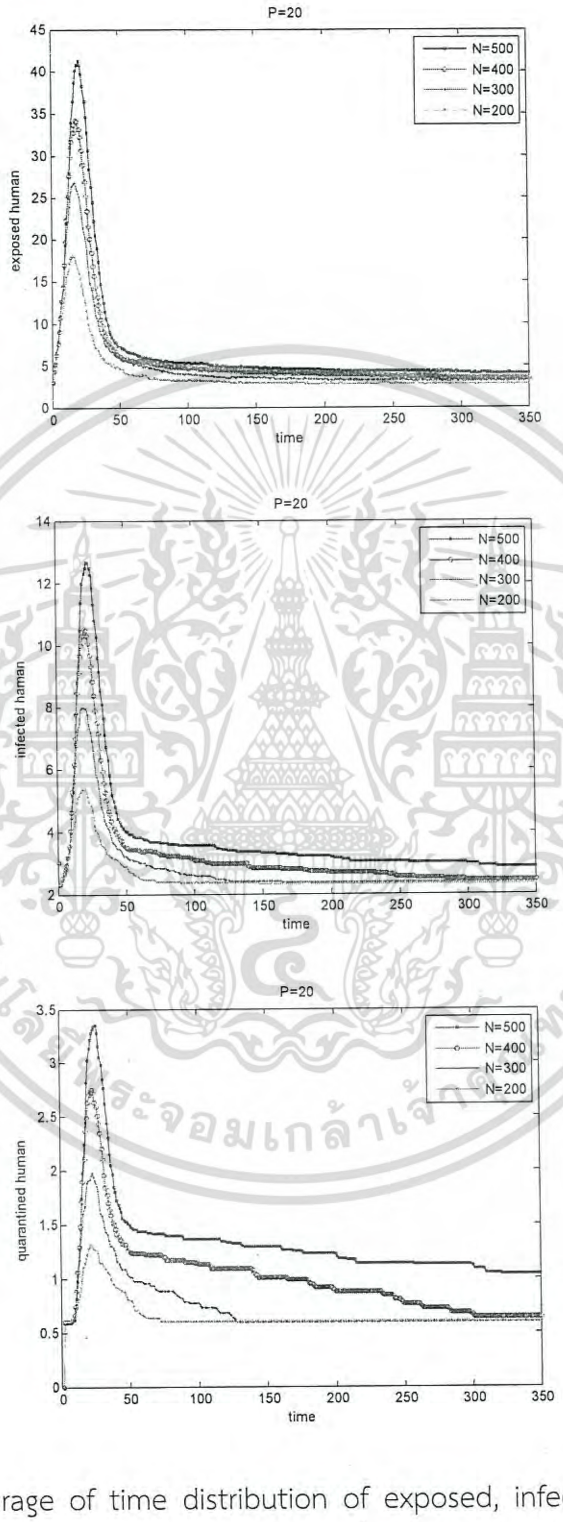

Figure 6.6 The average of time distribution of exposed, infected and quarantined human, respectively, for various number of persons: $N = 200$, $N = 300$, $N = 400$ and $N = 500$.

Figure 6.6 compares the average outputs of time distribution for exposed, infected and quarantined human population obtained from the model with four different numbers of persons. Figure 6.5(a) show the maximum number of exposed individuals and the day at which reaches maximum: $N = 200$, the maximum around 18 persons at 17 days and tend to steady state at 90 days; $N = 300$, the maximum around 27 persons at 18 days and tend to steady state at 140 days; $N = 400$, the maximum around 34 persons at 19 days and tend to steady state at 170 days; and $N = 500$, the maximum around 41 persons at 21 days and tend to steady state at 190 days. Figure 6.5(b) show the maximum number of infected individuals and the day at which reaches maximum: $N = 200$, the maximum around 5 persons at 19 days and tend to steady state at 85 days; $N = 300$, the maximum around 8 persons at 20 days and tend to steady state at 125 days; $N = 400$, the maximum around 10 persons at 21 days and tend to steady state at 145 days; and $N = 500$, the maximum around 13 persons at 24 days and tend to steady state at 215 days. Figure 6.5(c) show the maximum number of quarantined individuals and the day at which reaches maximum: $N = 200$, the maximum around 1 persons at 22 days and tend to steady state at 70 days; $N = 300$, the maximum around 2 persons at 23 days and tend to steady state at 130 days; $N = 400$, the maximum around 3 persons at 23 days and tend to steady state at 300 days; and $N = 500$, the maximum around 3 persons at 26 days and tend to steady state at 315 days.

Second situation, we assume that there are 500 persons, the time distributions of exposed, infected and quarantined humans for the different number of public places in this situation are shown in the following figures.

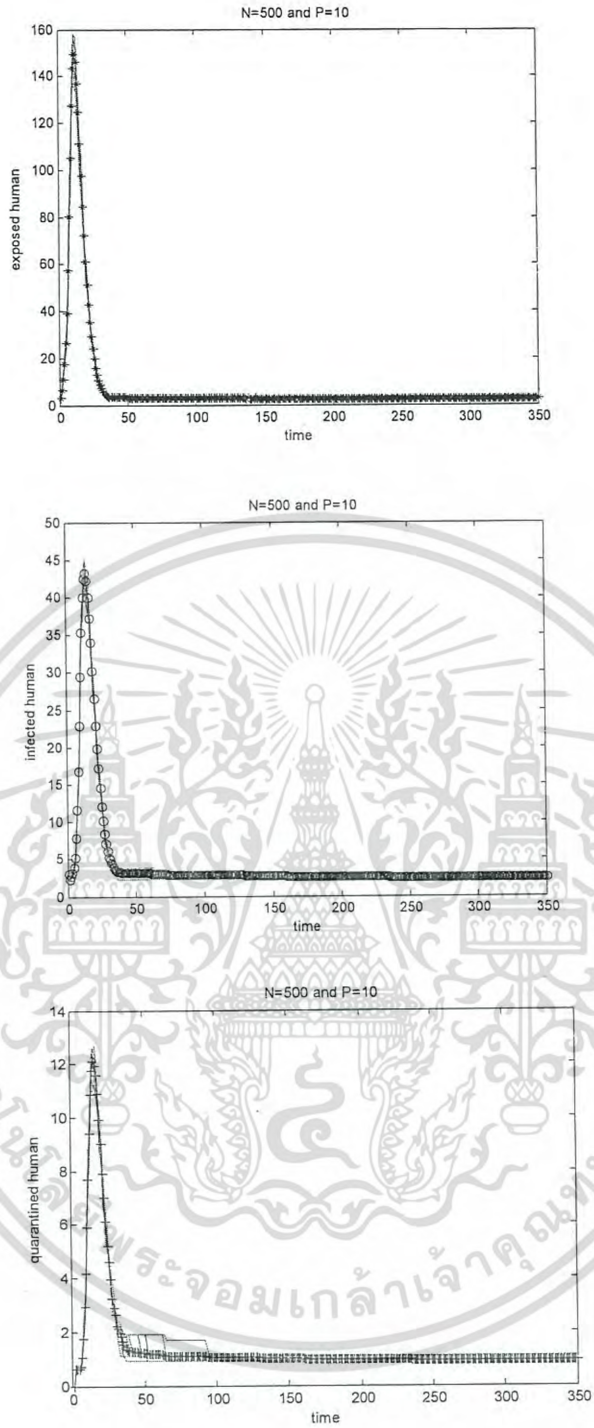

Figure 6.7 The time distribution of exposed, infected and quarantined human for $P=10$ public places, respectively. The solid line represents the average outputs from 20 runs of simulation. The parameters in our model are $k=1/(365\times10)$, $c=1/9$, $d=1/7$, $e=1/5$, $f=1/7$, $g=0.7$, $a_1=20$, $a_2=25$, $a_3=15$, $P=20$, $T=350$.

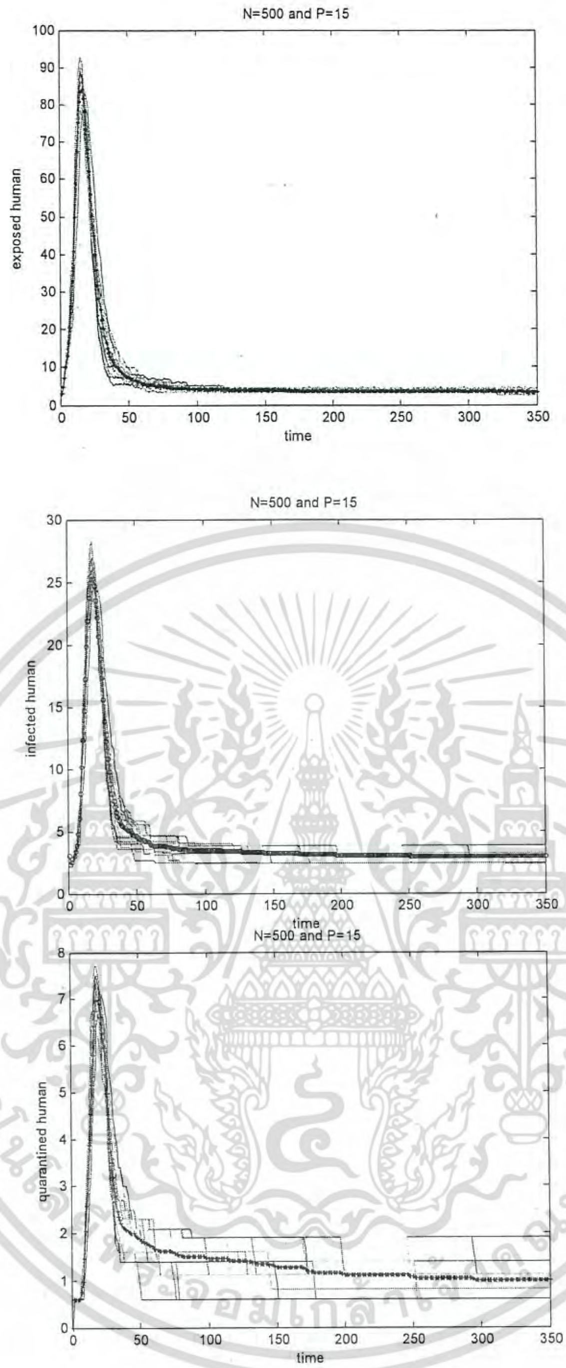

Figure 6.8 The time distribution of exposed, infected and quarantined human for $P = 15$ public places, respectively. The solid line represents the average outputs from 20 runs of simulation. The parameters are same as Figure 6.7.

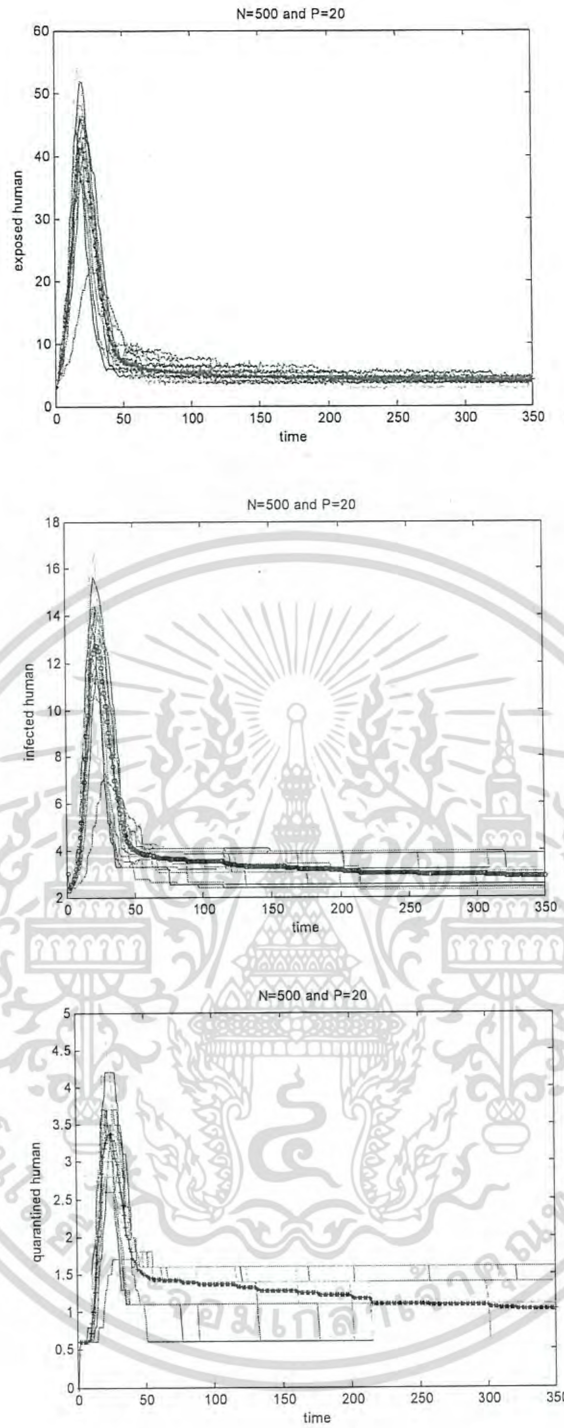

Figure 6.9 The time distribution of exposed, infected and quarantined human for $P=20$ public places, respectively. The solid line represents the average outputs from 20 runs of simulation. The parameters are same as Figure 6.7.

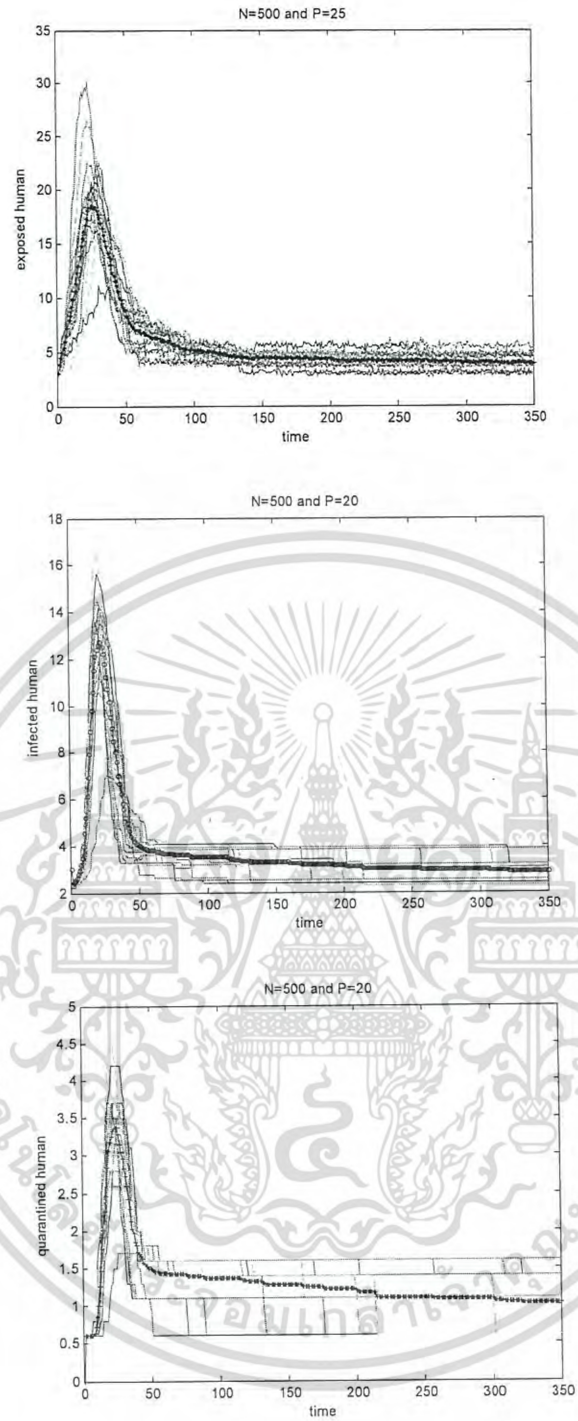

Figure 6.10 The time distribution of exposed, infected and quarantined human for $P = 25$ public places, respectively. The solid line represents the average outputs from 20 runs of simulation. The parameters are same as Figure 6.7.

We simulate the time distribution of exposed, infected and quarantined human with the four different numbers of public places: $P=10$, $P=15$, $P=20$ and $P=25$, respectively. For each P, we run 20 times and find the average outputs. The following figures show the average outputs of time distributions of exposed, infected and quarantined humans, respectively.

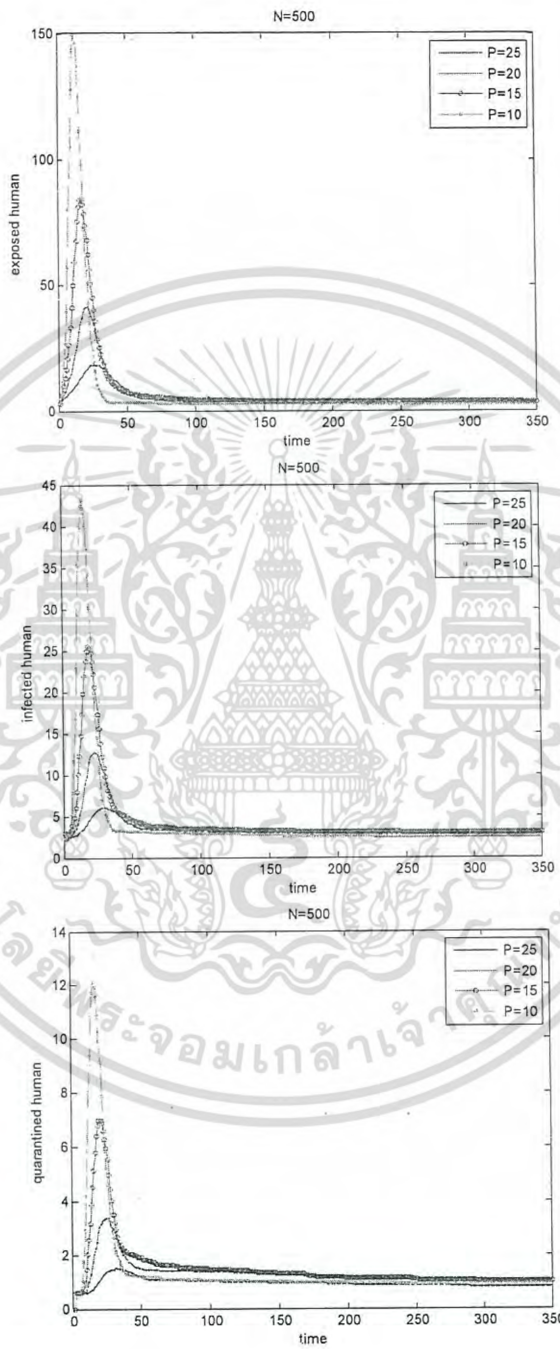

Figure 6.11 The average outputs of time distribution for exposed, infected and quarantined human, respectively, for various number of public places: $P=10$, $P=15$, $P=20$ and $P=25$.

Figure 6.11 compares the average outputs of time distribution of exposed, infected and quarantined human obtained from the model with four different numbers of public places. Figure 6.10(a) shows the maximum number of exposed individuals and the day at which reaches maximum: $P = 10$, the maximum around 150 persons at 12 days and tend to steady state at 80 days; $P = 15$, the maximum around 84 persons at 17 days and tend to steady state at 130 days; $P = 20$, the maximum around 41 persons at 21 days and tend to steady state at 175 days; and $P = 25$, the maximum around 18 persons at 26 days and tend to steady state at 200 days. Figure 6.10(b) shows the maximum number of infected individuals and the day at which reaches maximum: $P = 10$, the maximum around 43 persons at 15 days and tend to steady state at 180 days; $P = 15$, the maximum around 25 persons at 19 days and tend to steady state at 200 days; $P = 20$, the maximum around 13 persons at 23 days and tend to steady state at 215 days; and $P = 25$, the maximum around 6 persons at 31 days and tend to steady state at 265 days. Figure 6.10(c) shows the maximum number of quarantined individuals and the day at which reaches maximum: $P = 10$, the maximum around 12 persons at 16 days and tend to steady state at 150 days; $P = 15$, the maximum around 7 persons at 21 days and tend to steady state at 200 days; $P = 20$, the maximum around 3 persons at 25 days and tend to steady state at 215 days; and $P = 25$, the maximum around 2 persons at 32 days and tend to steady state at 275 days.

Chapter 7
Conclusion and Suggestion

In this research, we constructed the mathematical models based on the Susceptible-Exposed-Infected-Quarantined-Recovered (SEIQR) model by introducing age structure into the SEIQR model. We analyzed the reported data from the Bureau of Epidemiology, Ministry of Public Health, Thailand in year 2009 and formulated the system of differential equations for swine flu transmission. The human population is classified into three age groups such as groups of people 1–10 years, 11 – 20 years, and more than 20 years, respectively. We suppose that the total population and the total number for each age group remain constant. The transmission rates of the disease in all age groups are assumed to be different. We used the standard dynamical analysis to determine the conditions of parameters for local asymptotically stability. Numerical solutions are shown to confirm the analytical results. We constructed the Lyapunov functions with the conditions of parameters for global asymptotically stability.

In our model, there are four equilibrium points $P_0, \hat{P}, \tilde{P}$ and P^*. The first point is the disease free equilibrium P_0 represents the state in which swine flu are not endemic in the human. The second point is the third age group endemic equilibrium $\hat{P}$ which represents the state in which swine flu are endemic only in the third age group. The third point is the second and the third age group endemic equilibrium $\tilde{P}$ which represents the state in which swine flu are endemic in both the second age group and the third age group. The forth point is the full endemic equilibrium P^* which represents the state in which swine flu are endemic in all age groups. There exist three threshold number R_0^*, $\tilde{R}_0$ and $\hat{R}_0$. The disease free equilibrium P_0 is locally asymptotically stable when $R_0 = \max\left\{\hat{R}_0, \tilde{R}_0, R_0^*\right\} < 1$. The third age group endemic equilibrium $\hat{P}$ is locally asymptotically stable when $R_0 = \max\left\{\tilde{R}_0, R_0^*\right\} < 1$ and $\hat{R}_0 > 1$. The second and the third age group endemic equilibrium $\tilde{P}$ is locally asymptotically stable when $R_0^* < 1$ and $\tilde{R}_0 > 1$. The full endemic equilibrium P^* is locally asymptotically stable when $R_0^* > 1$.

The biological meaning of the basic reproductive number R_0^*, $\tilde{R}_0$ and $\hat{R}_0$ are explained as follows: $R_0^* = \left(\frac{a_1 b}{b+k}\right)\left[\left(\frac{c}{b+c+d+k}\right)\left(\frac{1}{b+e+f+k}\right)+\frac{1}{b+c+d+k}\right]$ is threshold number for swine flu in the first age group, where a_1 is the transmission rate per day of the first age group, $\frac{b}{b+k}$ is the ratio between the total number of the first age group and the total population, $\frac{c}{b+c+d+k}$ is the fraction of exposed members who move to the infective class, $\frac{1}{b+e+f+k}$ is the average time that an infective individual remains in the class I , and $\frac{1}{b+c+d+k}$ is the average time that an exposed member remains in that class.

The term $\tilde{R}_0 = \frac{a_2 bk}{(b+k)^2}\left[\left(\frac{c}{b+c+d+k}\right)\left(\frac{1}{b+e+f+k}\right)+\frac{1}{b+c+d+k}\right]$ is the threshold number for swine flu in the second age group, where a_2 is the transmission rate per day of the second age group, $\frac{bk}{(b+k)^2}$ is the ratio between the total number of second age group and the total population, the other terms are define as before.

The term $\hat{R}_0 = \frac{a_3 k^2}{(b+k)^2}\left[\left(\frac{c}{b+c+d}\right)\left(\frac{1}{b+e+f}\right)+\frac{1}{b+c+d}\right]$ is the threshold number for swine flu in the third age group, where a_3 is the transmission rate per day of the third age group, $\frac{k^2}{(b+k)^2}$ is the ratio between the total number of third age group and the total population, the other terms are define in the same manner as before.

The following numerical simulation confirmed these results.

Figure 4.4 shows the proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ approach to the disease free state when $R_0^* < 1$, $\tilde{R}_0 < 1$ and $\hat{R}_0 < 1$.

Figure 4.6 shows the proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ spiral into the third age group endemic equilibrium state (1, 0, 0, 0, 1, 0, 0, 0, 0.20045647, 0.00012320, 0.00003992, 0.00001141) when $R_0^* < 1$, $\tilde{R}_0 < 1$ and $\hat{R}_0 > 1$.

Figure 4.7 shows the proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ spiral into the second and the third age group endemic equilibrium state (1, 0, 0, 0, 0.70249166, 0.00036634, 0.00011861, 0.00003387, 0.17266629, 0.00008170, 0.00002649, 0.00000757) when $R_0^* < 1$ and $\tilde{R}_0 > 1$.

Figure 4.8 shows the proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ spiral into the full endemic equilibrium state (0.25611675, 0.00091599, 0.00029658, 0.00008470, 0.23726898, 0.00002434, 0.00000815, 0.00000237, 0.05010638, 0.00002884, 0.00000935, 0.00000267) when $R_0^* > 1$.

Figure 4.9, the bifurcation diagrams demonstrate the equilibrium solutions for the different values of $\hat{R}_0$. We can see that, when $\hat{R}_0 < 1$, T_1 will be stable and for $\hat{R}_0 > 1$, T_2 will be stable. If the threshold number of the third age group is greater than one, the third age group susceptible decrease. The third age group infectious, quarantined and recovered increase.

Figure 4.10, the bifurcation diagrams demonstrate the equilibrium solutions for the different values of $\tilde{R}_0$. We can see that, when $\tilde{R}_0 < 1$, T_1 will be stable and for $\tilde{R}_0 > 1$, T_2 will be stable. If the threshold number of the second age group is greater than one, the susceptible of the second age group and the third age group decrease. The second age group infectious, quarantined and recovered increase. The third age group infectious, quarantined and recovered increase.

Figure 4.11, the bifurcation diagrams demonstrate the equilibrium solutions for the different values of R_0^*. We can see that, when $R_0^* < 1$, T_1 will be stable and for $R_0^* > 1$, T_2 will be stable. If the threshold number of the first age group is greater than one, the susceptible of the first age group, the second age group and the third age group decrease. The first age group infectious, quarantined and recovered increase. The second age group infectious, quarantined and recovered increase. The third age group infectious, quarantined and recovered increase.

The global stability of our model has been resolved by using Lyapunov functions. If $R_0 = \max\left\{\hat{R}_0, \tilde{R}_0, R_0^*\right\} < 1$, then the disease free equilibrium state is globally asymptotically stable and the disease will die out. If $\hat{R}_0 > 1$ and $R_0 = \max\left\{\tilde{R}_0, R_0^*\right\} < 1$, then the third age group endemic equilibrium state is globally asymptotically stable and the disease is endemic only in the third age group. If $\tilde{R}_0 > 1$ and $R_0^* < 1$, then the second and the third age group endemic equilibrium state is globally asymptotically stable and the disease is endemic in both the second and the third age group . If $R_0^* > 1$ then the full endemic equilibrium state is globally asymptotically stable and the disease is endemic in all age groups.

The SEIQR network model contains people and public places, we suppose that the people move to any public places by random process. The numerical simulation of the disease transmission is presented with assumptions that there is one exposed person and one infected person in each age group at the first day. The results show the time distribution of exposed, infected and quarantined human dealing with different situations. We show the maximum number of exposed, infected and quarantined individuals, the day at which each individual group reaches to the maximum and the day at which each group tend to the steady state for each situation.

For the first situation, the number of public places is fixed and the number of persons are varied as shown in figure 6.6. We can see that the greater number of persons produce the higher epidemic peak and length of epidemic outburst. It can be interpreted that the higher number of persons can cause of the higher population density per unit area and greater contact transmission then the disease can be spread easier. For the time of the epidemic peak is not much difference: the time of exposed individual peak is between 17 and 21 days, the time of infected individual peak is between 19 and 24 days and time of the quarantined individual peak is between 22 and 26 days. For the second situation, the number of persons are fixed and the number of public places are varied as shown in figure 6.11. We can see that the higher number of public places produce the lower epidemic peak but the length of epidemic outburst and the time of the epidemic peak are higher. It caused by the population density per unit area and contact transmission of the higher number of public places are lower, therefore the disease can be gradually spread and take longer time to reach the epidemic peak which make longer of epidemic outburst. As we see in figure 6.6 and figure 6.11, the peaks of the exposed individuals for both situations are higher than the peaks of the infected individuals because some exposed individuals recover before they become infectious. Similarly, the peaks of the infected individuals for both situations are higher than the peaks of the quarantined individuals because some infected individuals recover before they become the quarantined individuals.

For the further work, other researchers will develop a SEIQR model for swine flu transmission by considering the side effects of the susceptible populations who were vaccinated against a disease or immunized.

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Appendices

Appendix A

Theoretical Background

Consider a system of two first-order differential equations as follows:

$$\frac{dx}{dt} = f_1(x, y) \tag{A.1}$$

$$\frac{dy}{dt} = f_2(x, y) \tag{A.2}$$

where f_1 and f_2 are nonlinear continuous functions having partial derivatives with respect to x and y. A solution (x_0, y_0) of this system is defined by $f_1(x_0, y_0) = f_2(x_0, y_0) = 0$ and (x_0, y_0) is called a steady state solution, stationary point, critical point or equilibrium point.

The Jacobian matrix J of equations (A.1) – (A.2) is stated as follows:

$$J(x_0, y_0) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_{(x_0, y_0)}.$$

Let $\beta = a_{11} + a_{22}$, $\gamma = a_{11}a_{22} - a_{12}a_{21}$ and $\delta = \beta^2 - 4\gamma$, then the characteristic equation is $\lambda^2 - \beta\lambda + \gamma = 0$ and the eigenvalues are $\lambda_{1,2} = \frac{\beta \pm \sqrt{\delta}}{2}$.

<u>Theorem A.1</u> (Classifying Stability Characteristics) [32] Let $\bar{x} = (x_0, y_0)$ be an equilibrium point of system (A.1) – (A.2). The behavior of the equilibrium point can be determined from the characteristic equation of eigenvalues from Jacobian matrix J.

For real eigenvalues, the equilibrium point can be classified into five cases as follow

a) If both eigenvalues are distinct positive $(\lambda_1 > 0, \lambda_2 > 0)$ then $\bar{x}$ will be an unstable two-tangent node.

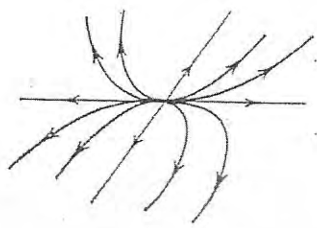
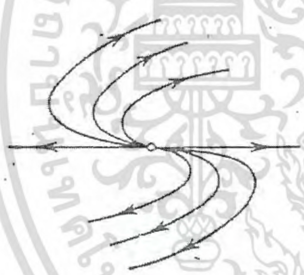


Figure A.1 An unstable two-tangent node.

b) If eigenvalues are repeated positive $(\lambda_1 = \lambda_2 > 0)$ then $\bar{x}$ will be an unstable one-tangent node or unstable stellar node.



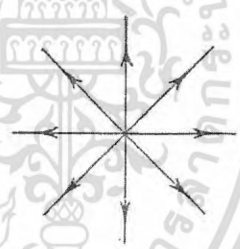


Figure A.2 An unstable one-tangent node. **Figure A.3** An unstable stellar node.

c) If both eigenvalues are distinct negative $(\lambda_1 < 0, \lambda_2 < 0)$ then $\bar{x}$ will be a stable two-tangent node.

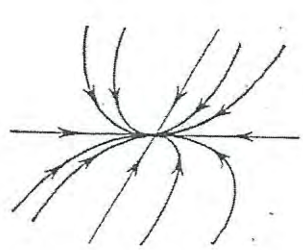


Figure A.4 A stable two-tangent node.

d) If eigenvalues are repeated negative $(\lambda_1 = \lambda_2 < 0)$ then $\overline{x}$ will be a stable one-tangent node or stable stellar node.

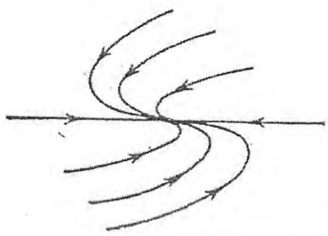


Figure A.5 A stable one-tangent node.

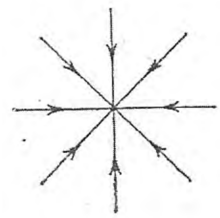


Figure A.6 A stable stellar node.

e) If eigenvalues are opposite signs $(\lambda_1 > 0, \lambda_2 < 0)$ then $\overline{x}$ will be a saddle point (unstable).

Figure A.7 A saddle point.

For complex eigenvalues, it is necessary and sufficient that $\delta = \beta^2 - 4\gamma$ be negative and then $\lambda_{1,2} = \dfrac{\beta \pm i\sqrt{-\delta}}{2}$. The equilibrium point can be classified into six cases as follow:

a) If $\beta > 0$ and $\gamma > 0$ then $\overline{x}$ will be an unstable node.
b) If $\beta < 0$ and $\gamma > 0$ then $\overline{x}$ will be a stable node.
c) If $\gamma < 0$ then $\overline{x}$ will be a saddle point.
d) If $\beta > 0$ then $\overline{x}$ will be an unstable spiral node.

Figure A.8 An unstable spiral node.

e) If $\beta = 0$ then $\overline{x}$ will be a neutral center.

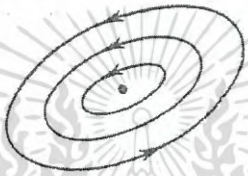


Figure A.9 A neutral center.

f) If $\beta < 0$ then $\overline{x}$ will be a stable spiral node.

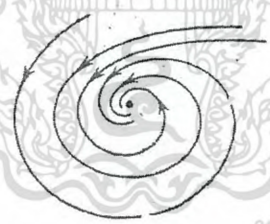


Figure A.10 A stable spiral node.

Next, consider the general form of a system of first-order differential equations

$$\frac{dx_i}{dt} = f_i\left(x_1, x_2, x_3, \ldots, x_n\right) \qquad \text{where } i = 1, 2, \ldots, n .$$

The equilibrium point $\bar{x}$ is obtained by solving $f_i\left(x_1, x_2, x_3, \ldots, x_n\right) = 0$ and the entries of Jacobian matrix is defined by

$$J(\bar{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\bar{x}}$$

where J is a $n \times n$ matrix. The eigenvalues λ of the Jacobian matrix which satisfy $\det(J - \lambda I) = 0$ is obtained from the characteristic equation

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n = 0 . \tag{A.3}$$

<u>Theorem A.2</u> (Routh-Hurwitz Criteria) [33] Consider the characteristic equation,

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n = 0$$

where the coefficients $a_i > 0$ for $i = 1, 2, 3, \ldots, n$ and define the n Hurwitz matrices as follows:

$$H_1 = \left[a_1\right], \quad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}, \quad H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix},$$

and

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

where the coefficients $a_j = 0$ for $j > n$. The roots of the characteristic equation have negative real parts if and only if the determinants of all Hurwitz matrices are positive:

$$\det H_i > 0, \; i = 1, 2, \ldots, n.$$

For $n=2$, the Routh-Hurwitz Criteria is

i) $\det H_1 = \det\left[a_1\right] = a_1 > 0$

ii) $\det H_2 = \det\begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix} = \det\begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix} = a_1 a_2 > 0$

The conditions of Routh-Hurwitz Criteria for local asymptotical stability in 2nd order characteristic polynomial equation are $a_1 > 0$ and $a_2 > 0$.

For $n=3$, the Routh-Hurwitz Criteria is

i) $\det H_1 = \det\left[a_1\right] = a_1 > 0$

ii) $\det H_2 = \det\begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix} = a_1 a_2 - a_3 > 0$

iii) $\det H_2 = \det\begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} = \det\begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{bmatrix} = a_3\left(a_1 a_2 - a_3\right) > 0$

The conditions of Routh-Hurwitz Criteria for local asymptotical stability in 3rd order characteristic polynomial equation are $a_1 > 0$, $a_1 a_2 - a_3 > 0$ and $a_3 > 0$.

The stability of the equilibrium point can be determined by applied the Routh-Hurwitz Criteria with equation (A.3). This method allows us to determine whether all roots have negative real parts and establish the local asymptotic stability of the system without solving the actual values of eigenvalues.

<u>Theorem A.3</u> (the fourth-order Runge-Kutta method) [34] The solution of a system of the form

$$\frac{dx}{dt} = f(t,x,y)$$

$$\frac{dy}{dt} = g(t,x,y)$$

with the initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$ is

$$x_{n+1} = x_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$m_1 = hf(t_n, x_n, y_n)$ $\quad$ $k_1 = hg(t_n, x_n, y_n)$

$m_2 = hf\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}m_1, y_n + \frac{1}{2}k_1\right)$ $\quad$ $k_2 = hg\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}m_1, y_n + \frac{1}{2}k_1\right)$

$m_3 = hf\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}m_2, y_n + \frac{1}{2}k_2\right)$ $\quad$ $k_3 = hg\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}m_2, y_n + \frac{1}{2}k_2\right)$

$m_4 = hf(t_n + h, x_n + m_3, y_n + k_3)$ $\quad$ $k_4 = hg(t_n + h, x_n + m_3, y_n + k_3)$.

<u>TheoremA.4</u> (Lyapunov's first method) [35, 36] Let $x = 0$ be an equilibrium point for point for $\dot{x} = f(x)$ and $D \subset R^n$. Let $V: D \to R$ be a continuously differentiable function such that

1. $V(0) = 0$ and $V(x) > 0 \qquad \forall x \in D \setminus \{0\}$

2. $\dot{V}(x) \le 0 \qquad \forall x \in D$.

   Then $x = 0$ is stable. Moreover,

3. If $\dot{V}(x) < 0$ $\forall x \in D \setminus \{0\}$ then $x = 0$ is asymtotically stable.

<u>Theorem A.5</u> (Lyapunov Theorem for Global Asymptotic Stability) Let $x=0$ be an equilibrium point for $\dot{x}=f(x)$ and $D\subset R^n$. Let $V:D\to R$ be a continuously differentiable function such that

1. $V(0)=0$ and $V(x)>0 \qquad \forall x\in D\setminus\{0\}$
2. $\dot{V}(x)<0 \qquad \forall x\in D\setminus\{0\}$
3. $V(x)\to\infty$ as $\|x\|\to\infty$,

then $x=0$ is globally asymptotically stable.

<u>Theorem A.6</u> (LaSalle's theorem) [35, 36] Let $D\subset R^n$ be a compact invariant set with respect to $\dot{x}=f(x)$. Let $V(x)$ be a continuously differentiable function defined over R^n such that $\dot{V}(x)\leq 0$ in D. Let E be a set of all points in D where $\dot{V}(x)=0$ and M be the largest invariant set in E. Then every solution starting in D approaches M as $t\to\infty$. Moreover, if the set E contains only one point $x=0$ then $x=0$ this point is asymptotically stable in D.

<u>Theorem A.7</u> (Krazovskii-Lasalle's Theorem) Let $x=0$ be an equilibrium point for $\dot{x}=f(x)$ and $D\subset R^n$. Let $V:D\to R$ be a continuously differentiable positive definite function over R^n such that $\dot{V}(x)\leq 0$ in D. Let $S=\{x\in D:\dot{V}(x)=0\}$ and suppose that no other solution can stay in S, other than $x=0$. Then $x=0$ is asymptotically stable. If, in addition, $V(x)$ is radially unbounded then $x=0$ is globally asymptotically stable.

Appendix B

Accepted Papers for Publication and Presented

For publication:

1. T. Changpuek, P. Pongsumpun and I-Ming Tang. 2013. "Analysis of Mathematical Model for Swine Flu Transmission by Age Group." Far East Journal of Mathematical Sciences (FJMS). 73(2) : 201 – 229.
2. T. Changpuek, P. Pongsumpun and I-Ming Tang. 2013. "Global Stability of the Age Structural Transmission Model for Swine Flu." Far East Journal of Mathematical Sciences (FJMS). 80(1) : 55 – 84.

For presentation:

1. T. Changpuek and P. Pongsumpun. 2010. "The Age Structural Transmission Model of Swine Flu." in **The 3rd Biomedical Engineering International Conference.** Kyoto, Japan.
2. T. Changpuek and P. Pongsumpun. 2014. "SEIQR Model with its Global Stability." in **The Forth TKU-KMITL Joint Symposium on Mathematics and Applied Mathematics (MAM 2014).** Tokai, Japan.

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ANALYSIS OF MATHEMATICAL MODEL FOR SWINE FLU TRANSMISSION BY AGE GROUP

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Abstract

Effect of age group in the human population is considered for the transmission of swine flu. Susceptible-Exposed-Infected-Quarantined-Recovered (SEIQR) model is used for describing the transmission of this disease. From the swine flu data in Thailand, age of patients is influenced to the transmission of this disease. The human population is separated into three groups such as 1-10 years, 11-20 years, and more than 20 years, respectively. The transmission rates of the disease in all

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age groups are assumed to be different. Local stability analysis of this model is given. Four equilibrium states are found and the conditions for stability of these four equilibrium states are established. Numerical solutions are shown to confirm the analytical results. The bifurcation diagrams are shown. The basic reproductive number is found and the alternative way to control the disease is discussed.

I. Introduction

The swine flu was an outbreak of a new strain of H1N1 influenza virus, derived originally from a strain which lived in pigs, and this origin gave rise to the common name of "swine flu". The outbreak began in the state of Veracruz, Mexico in April 2009, and it continued to spread globally. In June 2009, the World Health Organization (WHO) and US Centers for Disease Control (CDC) stopped counting cases and declared the outbreak as pandemic [1]. The total report of swine flu cases worldwide more than 213 countries was 622,482 by 27 November 2009 [2]. Updated data on swine flu deaths has reached a total of 16,931 deaths as on 21 March 2010 [3]. The virulence of swine flu virus is mild and the mortality rates are very low compared with bird flu virus. The H5N1 virus has a mortality rate between 60% and 70% [4], but the H1N1 virus has a mortality rate 3% [5].

The swine flu virus appeared to be a new strain of H1N1 which resulted when a previous triple reassortment of bird, swine and human flu viruses further combined with a Eurasian pig flu virus [6]. It has been determined that the strain contains genes from five different flu viruses: North American swine influenza, North American avian influenza, human influenza, and two swine influenza viruses typically found in Asia and Europe. Pigs have been termed the mixing vessel of flu because they can be infected both by avian flu viruses, which rarely directly infect people, and by human viruses. When pigs become simultaneously infected with more than one virus, the viruses can swap genes, producing new variants which can pass to humans and sometimes spread amongst them [7].

Despite being informally called "*swine flu*", the H1N1 flu virus cannot be spread by eating pork or pork products; [8, 9] similar to other influenza virus; it is typically contracted by person to person transmission through respiratory droplets [10]. Sometimes people may become infected by touching something such as a surface or object with flu viruses on it and then touching their face. The symptoms of H1N1 flu are similar to those of other influenzas, and may include a fever, cough (typically a "dry cough"), headache, muscle or joint pain, sore throat, chills, fatigue, and runny nose. Diarrhea, vomiting, and neurological problems have also been reported in some cases [11, 12]. People at higher risk of serious complications include those aged over 65, children younger than 5, children with neurodevelopment conditions, pregnant women [13] and those of any age with underlying medical conditions, such as asthma, diabetes, obesity, heart disease, or a weakened immune system (e.g., taking immunosuppressive medications or infected with HIV) [14].

The virus was found to be a novel strain of influenza for which extant vaccines against seasonal flu provided little protection. A study at CDC published in May 2009 found that children had no preexisting immunity to the new strain but that adults aged 18 to 64 had 6-9%, and older adults had some degree of immunity 33% [15, 16]. While it has been thought that these findings suggest the partial immunity in older adults may be due to previous exposure to similar seasonal influenza viruses. The H1N1 vaccine was initially in short supply and in the U.S., the CDC recommended that initial doses should go to priority groups such as pregnant women, people who live with or care for babies under six months old, children six months to four years old and health-care workers [17].

People in at-risk groups should be treated with antiviral (oseltamivir or zanamivir) as soon as possible when they first experience flu symptoms. The risk groups include pregnant and post partum women, children under two years old, and people with underlying conditions such as respiratory problems [18]. People who are not in an at-risk group who have persistent or rapidly worsening symptoms should also be treated with antiviral. Antiviral drugs are most useful if they are given within 48 hours of the start of symptoms and may improve outcomes in hospitalized patients [19]. Both medications have known side effects, including lightheadedness, chills, nausea, vomiting, loss of appetite and trouble breathing [20]. The CDC warned that the indiscriminate use of antiviral medications to prevent and treat influenza could ease the way for drug-resistant strains to emerge, which would make the fight against the pandemic that much harder. According to WHO, the reported 314 samples of the swine flu cases tested worldwide have shown resistance to oseltamivir (Tamiflu) [21], but no circulating flu has yet shown any resistance to zanamivir (Relenza) [22]. The epidemic models are used for predicting many emerging infectious diseases through human population. Zhou and Ma [23] analyzed an SEIQJR model for SARS transmission and control in China. Iwami et al. [24] considered an avian-human influenza epidemic model based on SI-SIR model. Dumont et al. [25] constructed an SEIR-LSEI model to investigate the spread of the chikungunya disease. In this paper, we analyze the mathematical model of swine flu in Thailand. The data of swine flu in Thailand is shown in Figure 1.

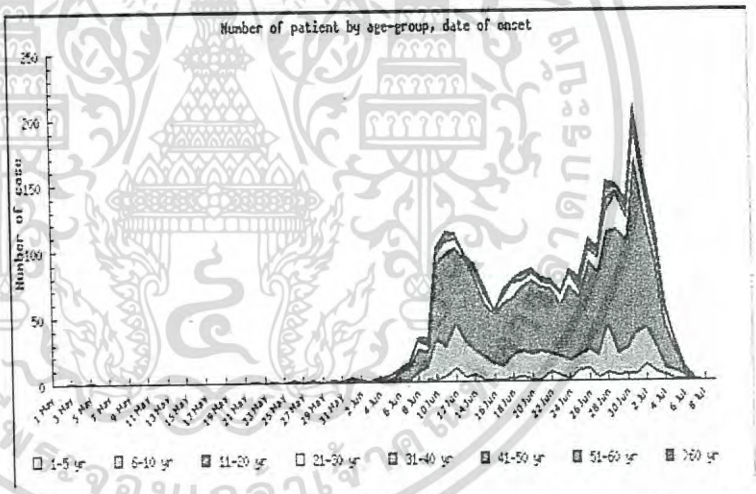

Figure 1. The data of swine flu in Thailand between April 28 and July 8, 2010.

From Figure 1, we can see that there is the different distribution of this disease in each age group. In this paper, we modified the model of Jumpen et al. [26] by incorporating the age structure of human population. The report of Ministry of Public Health, Thailand has showed that the most swine flu cases in Thailand occur in children under the age of 10, 11-20 and the people above the age of 21, respectively. The purpose of this paper is to study the age structural model of swine flu. In Section 2, we introduce a mathematical model to describe the transmission of this disease. In Section 3, we analyze our model and give the local stability analysis of the equilibriums states and also the numerical results. In the last section, we discuss the basic reproductive number of this disease and the bifurcation diagrams are shown.

II. Mathematical Model

We formulate a mathematical model to study the transmission of swine flu by introducing age structure into the SEIQR model. The human population is divided into three age groups such as groups of the people 1-10 years, 11-20 years, and more than 20 years, respectively. Each group is constant in size and is sub-divided into five classes, i.e., S', individuals susceptible to the disease; E', individuals who are latently infected (exposed); I', infectious individuals; Q', isolated or quarantined individuals; and R', individuals that have recovered and immune to the disease. The age structural SEIQR model is described by the following system of differential equations:

$$\frac{dS_1'}{dt} = \mu_b N_T - \frac{\delta_1 S_1'(E_1' + I_1')}{N_T} - (\mu + \kappa)S_1', \tag{1}$$

$$\frac{dE_1'}{dt} = \frac{\delta_1 S_1'(E_1' + I_1')}{N_T} - (\mu + \eta + \alpha + \kappa)E_1', \tag{2}$$

$$\frac{dI_1'}{dt} = \eta E_1' - (\mu + \varepsilon + \beta + \kappa)I_1', \tag{3}$$

$$\frac{dQ_1'}{dt} = \varepsilon I_1' - (\mu + \gamma + \kappa)Q_1', \tag{4}$$

$$\frac{dR_1'}{dt} = \alpha E_1' + \beta I_1' + \gamma Q_1' - (\mu + \kappa)R_1', \tag{5}$$

$$\frac{dS_2'}{dt} = \kappa S_1' - \frac{\delta_2 S_2'(E_2' + I_2')}{N_T} - (\mu + \kappa)S_2', \tag{6}$$

$$\frac{dE_2'}{dt} = \kappa E_1' + \frac{\delta_2 S_2'(E_2' + I_2')}{N_T} - (\mu + \eta + \alpha + \kappa)E_2', \tag{7}$$

$$\frac{dI_2'}{dt} = \kappa I_1' + \eta E_2' - (\mu + \varepsilon + \beta + \kappa)I_2', \tag{8}$$

$$\frac{dQ_2'}{dt} = \kappa Q_1' + \varepsilon I_2' - (\mu + \gamma + \kappa)Q_2', \tag{9}$$

$$\frac{dR_2'}{dt} = \kappa R_1' + \alpha E_2' + \beta I_2' + \gamma Q_2' - (\mu + \kappa)R_2', \tag{10}$$

$$\frac{dS_3'}{dt} = \kappa S_2' - \frac{\delta_3 S_3'(E_3' + I_3')}{N_T} - \mu S_3', \tag{11}$$

$$\frac{dE_3'}{dt} = \kappa E_2' + \frac{\delta_3 S_3'(E_3' + I_3')}{N_T} - (\mu + \eta + \alpha)E_3', \tag{12}$$

$$\frac{dI_3'}{dt} = \kappa I_2' + \eta E_3' - (\mu + \varepsilon + \beta)I_3', \tag{13}$$

$$\frac{dQ_3'}{dt} = \kappa Q_2' + \varepsilon I_3' - (\mu + \gamma)Q_3', \tag{14}$$

$$\frac{dR_3'}{dt} = \kappa R_2' + \alpha E_3' + \beta I_3' + \gamma Q_3' - \mu R_3', \tag{15}$$

with the conditions $N_T = N_{T_1} + N_{T_2} + N_{T_3}$, $N_{T_1} = S_1' + E_1' + I_1' + Q_1' + R_1'$, $N_{T_2} = S_2' + E_2' + I_2' + Q_2' + R_2'$ and $N_{T_3} = S_3' + E_3' + I_3' + Q_3' + R_3'$, where subscripts 1, 2 and 3 denote the first age group, the second age group and the third age group, respectively.

The parameters are defined as follows: N_T is the total population, N_{T_1} is the total number of first age group, N_{T_2} is the total number of second age group, N_{T_3} is the total number of third age group, μ_b is the natural birth rate, μ is the natural mortality rate, κ is the rate at which the first age group passes into the second age group and also the second age group passes into the third age group, δ_1 is equal to $\psi_1\phi_n$ in which ψ_1 is the probability of catching the disease per contact to the infected/exposed person and ϕ_n is the average number of people contacted by each person per day, δ_2 is equal to $\psi_2\varphi_n$ in which ψ_2 is the probability of catching the disease per contact to the infected/exposed person and φ_n is the average number of people contacted by each person per day, δ_3 is equal to $\psi_3\vartheta_n$ in which ψ_3 is the probability of catching the disease per contact to the infected/exposed person and ϑ_n is the average number of people contacted by each person per day, η is the rate at which the exposed individuals E become the infected individuals I, ε is the rate at which the individuals leave the infective individuals I for the quarantined individuals Q, α, β, γ are the rates at which individuals in the E, I and Q classes recover from the disease or die.

If we add (1)-(15), (1)-(5), (6)-(10), and (11)-(15), then we obtain:

$$\frac{dN_T}{dt} = \mu_b N_T - \mu N_T, \quad \frac{dN_{T_1}}{dt} = \mu_b N_T - (\mu + \kappa) N_{T_1},$$

$$\frac{dN_{T_2}}{dt} = \kappa N_{T_1} - (\mu + \kappa) N_{T_2}, \quad \frac{dN_{T_3}}{dt} = \kappa N_{T_2} - \mu N_{T_3}. \tag{16}$$

We assume that total population, total number of the first age group, total number of the second age group, and total number of the third age group remain constant. Therefore $\frac{dN_T}{dt} = 0$ and $\frac{dN_{T_1}}{dt} = \frac{dN_{T_2}}{dt} = \frac{dN_{T_3}}{dt} = 0$. Setting the right hand side of (16) to be zero, we obtain the following four relations: $\mu_b = \mu$ (birth rate equals to mortality rate), $\frac{N_{T_1}}{N_T} = \frac{\mu}{\mu + \kappa}$ (ratio between total number of the first age group and total population), $\frac{N_{T_2}}{N_T} = \frac{\mu\kappa}{(\mu + \kappa)^2}$ (ratio between total number of the second age group and total population) and $\frac{N_{T_3}}{N_T} = \frac{\kappa^2}{(\mu + \kappa)^2}$ (the ratio between total number of the third age group and total population).

We normalize (1)-(15) by letting

$$S_1 = \frac{S_1'}{N_{T_1}},\ E_1 = \frac{E_1'}{N_{T_1}},\ I_1 = \frac{I_1'}{N_{T_1}},\ Q_1 = \frac{Q_1'}{N_{T_1}},\ R_1 = \frac{R_1'}{N_{T_1}},$$

$$S_2 = \frac{S_2'}{N_{T_2}},\ E_2 = \frac{E_2'}{N_{T_2}},\ I_2 = \frac{I_2'}{N_{T_2}},\ Q_2 = \frac{Q_2'}{N_{T_2}},\ R_2 = \frac{R_2'}{N_{T_2}},$$

$$S_3 = \frac{S_3'}{N_{T_3}},\ E_3 = \frac{E_3'}{N_{T_3}},\ I_3 = \frac{I_3'}{N_{T_3}},\ Q_3 = \frac{Q_3'}{N_{T_3}},\ R_3 = \frac{R_3'}{N_{T_3}}.$$

The normalized variables satisfy the following new conditions: $S_1 + E_1 + I_1 + Q_1 + R_1 = 1$, $S_2 + E_2 + I_2 + Q_2 + R_2 = 1$, and $S_3 + E_3 + I_3 + Q_3 + R_3 = 1$. Hence equations (1)-(15) can be rewritten as

$$\frac{dS_1}{dt} = \theta_1(1 - S_1) - \theta_2 S_1(E_1 + I_1), \tag{17}$$

$$\frac{dE_1}{dt} = \theta_2 S_1(E_1 + I_1) - \theta_3 E_1, \tag{18}$$

$$\frac{dI_1}{dt} = \eta E_1 - \theta_4 I_1, \tag{19}$$

$$\frac{dQ_1}{dt} = \varepsilon I_1 - \theta_5 Q_1, \tag{20}$$

$$\frac{dS_2}{dt} = \theta_1(S_1 - S_2) - \theta_6 S_2(E_2 + I_2), \tag{21}$$

$$\frac{dE_2}{dt} = \theta_1 E_1 + \theta_6 S_2(E_2 + I_2) - \theta_3 E_2, \tag{22}$$

$$\frac{dI_2}{dt} = \theta_1 I_1 + \eta E_2 - \theta_4 I_2, \tag{23}$$

$$\frac{dQ_2}{dt} = \theta_1 Q_1 + \varepsilon I_2 - \theta_5 Q_2, \tag{24}$$

$$\frac{dS_3}{dt} = \mu(S_2 - S_3) - \theta_7 S_3(E_3 + I_3), \tag{25}$$

$$\frac{dE_3}{dt} = \mu E_2 + \theta_7 S_3(E_3 + I_3) - \theta_8 E_3, \tag{26}$$

$$\frac{dI_3}{dt} = \mu I_2 + \eta E_3 - \theta_9 I_3, \tag{27}$$

$$\frac{dQ_3}{dt} = \mu Q_2 + \varepsilon I_3 - \theta_{10} Q_3, \tag{28}$$

where $\theta_1 = \mu + \kappa$, $\theta_2 = \frac{\delta_1 \mu}{\mu + \kappa}$, $\theta_3 = \mu + \eta + \alpha + \kappa$, $\theta_4 = \mu + \varepsilon + \beta + \kappa$, $\theta_5 = \mu + \gamma + \kappa$, $\theta_6 = \frac{\delta_2 \mu \kappa}{(\mu + \kappa)^2}$, $\theta_7 = \frac{\delta_3 \kappa^2}{(\mu + \kappa)^2}$, $\theta_8 = \mu + \eta + \alpha$, $\theta_9 = \mu + \varepsilon + \beta$, and $\theta_{10} = \mu + \gamma$.

III. Analysis of the Mathematical Model

A. Analysis of models

The equilibrium points are obtained by setting the right hand side of equations (17)-(28) equal to zero. We get four equilibrium states:

(i) The disease free equilibrium state

$$P_0 = (1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0).$$

(ii) The third age group endemic equilibrium state

$$\hat{P} = (1, 0, 0, 0, 1, 0, 0, 0, \hat{S}_3, \hat{E}_3, \hat{I}_3, \hat{Q}_3),$$

where

$$\hat{S}_3 = \frac{1}{\hat{L}_0}, \quad \hat{E}_3 = \frac{\mu}{\theta_8 \hat{L}_0}[\hat{L}_0 - 1], \quad \hat{I}_3 = \frac{\mu\eta}{\theta_8\theta_9\hat{L}_0}[\hat{L}_0 - 1],$$

$$\hat{Q}_3 = \frac{\mu\eta\varepsilon}{\theta_8\theta_9\theta_{10}\hat{L}_0}[\hat{L}_0 - 1], \text{ with } \hat{L}_0 = \frac{\theta_7(\eta + \theta_9)}{\theta_8\theta_9}.$$

(iii) The second and third age group endemic equilibrium state $\widetilde{P} = (1, 0, 0, 0, \widetilde{S}_2, \widetilde{E}_2, \widetilde{I}_2, \widetilde{Q}_2, \widetilde{S}_3, \widetilde{E}_3, \widetilde{I}_3, \widetilde{Q}_3)$, where

$$\widetilde{S}_2 = \frac{1}{\widetilde{L}_0},\ \ \widetilde{E}_2 = \frac{\theta_1}{\theta_3\widetilde{L}_0}[\widetilde{L}_0 - 1],\ \ \widetilde{I}_2 = \frac{\theta_1\eta}{\theta_3\theta_4\widetilde{L}_0}[\widetilde{L}_0 - 1],$$

$$\widetilde{Q}_2 = \frac{\theta_1\eta\varepsilon}{\theta_3\theta_4\theta_5\widetilde{L}_0}[\widetilde{L}_0 - 1],\ \ \widetilde{S}_3 = \frac{\mu\widetilde{S}_2}{\theta_7(\widetilde{E}_3 + \widetilde{I}_3) + \mu},$$

$$\widetilde{E}_3 = \frac{-X_2 + \sqrt{X_2^2 - 4X_1X_3}}{2X_1},\ \ \widetilde{I}_3 = \frac{\mu\widetilde{I}_2 + \eta\widetilde{E}_3}{\theta_9},\ \ \widetilde{Q}_3 = \frac{\mu\widetilde{Q}_2 + \varepsilon\widetilde{I}_3}{\theta_{10}}$$

with

$$\widetilde{L}_0 = \frac{\theta_6(\eta + \theta_4)}{\theta_3\theta_4},\quad X_1 = \theta_7\theta_8(\eta + \theta_9),$$

$$X_2 = \mu\theta_8\theta_9 + \mu\theta_7(\theta_8\widetilde{I}_2 - (\eta + \theta_9)\widetilde{S}_2 - (\eta + \theta_9)\widetilde{E}_2),$$

and $X_3 = -[\mu^2\theta_7\widetilde{I}_2(\widetilde{S}_2 + \widetilde{E}_2) + \mu^2\theta_9\widetilde{E}_2]$.

(iv) The full endemic equilibrium state

$$P^* = (S_1^*, E_1^*, I_1^*, Q_1^*, S_2^*, E_2^*, I_2^*, Q_2^*, S_3^*, E_3^*, I_3^*, Q_3^*),$$

where

$$S_1^* = \frac{1}{L_0^*},\quad E_1^* = \frac{\theta_1}{\theta_3L_0^*}[L_0^* - 1],\quad I_1^* = \frac{\theta_1\eta}{\theta_3\theta_4L_0^*}[L_0^* - 1],$$

$$Q_1^* = \frac{\theta_1\eta\varepsilon}{\theta_3\theta_4\theta_5L_0^*}[L_0^* - 1],\quad S_2^* = \frac{\theta_1S_1^*}{\theta_6(E_2^* + I_2^*) + \theta_1},$$

$$E_2^* = \frac{-Y_2 + \sqrt{Y_2^2 - 4Y_1Y_3}}{2Y_1},\ \ I_2^* = \frac{\theta_1I_1^* + \eta E_2^*}{\theta_4},\ \ Q_2^* = \frac{\theta_1Q_1^* + \varepsilon I_2^*}{\theta_5},$$

$$S_3^* = \frac{\mu S_2^*}{\theta_7(E_3^* + I_3^*) + \mu},\quad E_3^* = -\frac{-Z_2 + \sqrt{Z_2^2 - 4Z_1Z_3}}{2Z_1},$$

$$I_3^* = \frac{\mu I_2^* + \eta E_3^*}{\theta_9},\quad Q_3^* = \frac{\mu Q_2^* + \varepsilon I_3^*}{\theta_{10}},$$

with

$$L_0^* = \frac{\theta_2(\eta + \theta_4)}{\theta_3\theta_4}, \quad Y_1 = \theta_3\theta_6(\eta + \theta_4),$$

$$Y_2 = \theta_1\theta_3\theta_4 + \theta_1\theta_6(\theta_3 I_1^* - (\eta + \theta_4)S_1^* - (\eta + \theta_4)E_1^*),$$

$$Y_3 = -[\theta_1^2\theta_6 I_1^*(S_1^* + E_1^*) + \theta_1^2\theta_4 E_1^*], \; Z_1 = \theta_7\theta_8(\eta + \theta_9),$$

$$Z_2 = \mu\theta_8\theta_9 + \mu\theta_7(\theta_8 I_2^* - (\eta + \theta_9)S_2^* - (\eta + \theta_9)E_2^*)$$

and

$$Z_3 = -[\mu^2\theta_7 I_2^*(S_2^* + E_2^*) + \mu^2\theta_9 E_2^*].$$

B. Analysis of stability

The local stability of the equilibrium solutions is determined from the Jacobian matrix of the RHS of the above set of differential equations evaluated at the equilibrium solutions. The eigenvalues are obtained by solving the characteristic equations; $\det(J(p) - \lambda I_{12}) = 0$, where $J(p)$ is the Jacobian matrix at equilibrium point p, I_{12} is the identity matrix dimension 12×12. If all eigenvalues have negative real parts, then the equilibrium solution is locally stable [27, 28]. The local stability analysis of each equilibrium state is given in the following propositions.

Proposition 1. *If* $L_0^* < 1$, $\widetilde{L}_0 < 1$ *and* $\hat{L}_0 < 1$, *then the equilibrium* P_0 *is locally asymptotically stable.*

Proof. For the disease free equilibrium state $P_0 = (1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0)$, we obtain the characteristic equation

$$(\lambda + \mu)(\lambda + \theta_1)^2(\lambda + \theta_5)^2(\lambda + \theta_{10})[\lambda^2 + a_1\lambda + a_2][\lambda^2 + b_1\lambda + b_2]$$

$$\cdot [\lambda^2 + c_1\lambda + c_2] = 0, \quad (29)$$

where

$$a_1 = \frac{(1-L_0^*)\theta_3\theta_4 + \eta\theta_3}{\eta+\theta_4} + \theta_4, \quad a_2 = \theta_3\theta_4(1-L_0^*),$$

$$b_1 = \frac{(1-\widetilde{L}_0)\theta_3\theta_4 + \eta\theta_3}{\eta+\theta_4} + \theta_4, \quad b_2 = \theta_3\theta_4(1-\widetilde{L}_0),$$

$$c_1 = \frac{(1-\hat{L}_0)\theta_8\theta_9 + \eta\theta_8}{\eta+\theta_9} + \theta_9 \text{ and } c_2 = \theta_8\theta_9(1-\hat{L}_0).$$

From the characteristic equation (29), we see that the first six eigenvalues are $\lambda_1 = -\mu$, $\lambda_{2,3} = -\theta_1$, $\lambda_{4,5} = -\theta_5$ and $\lambda_6 = -\theta_{10}$, all of these eigenvalues are negative. The remaining eigenvalues are found by solving $\lambda^2 + a_1\lambda + a_2 = 0$, $\lambda^2 + b_1\lambda + b_2 = 0$ and $\lambda^2 + c_1\lambda + c_2 = 0$. But for stability, we only need the sign of the eigenvalues to be negative. Thus, we apply the Routh-Hurwitz criteria to the equation in form $\lambda^2 + A\lambda + B = 0$, the stability holds if and only if $A > 0$ and $B > 0$.

For $\lambda^2 + a_1\lambda + a_2 = 0$, $a_1 > 0$ and $a_2 > 0$, if $L_0^* < 1$. For $\lambda^2 + b_1\lambda + b_2 = 0$, $b_1 > 0$ and $b_2 > 0$, if $\widetilde{L}_0 < 1$. For $\lambda^2 + c_1\lambda + c_2 = 0$, $c_1 > 0$ and $c_2 > 0$, if $\hat{L}_0 < 1$. Thus, all the roots of the three characteristic equations have negative real parts if $L_0^* < 1$, $\widetilde{L}_0 < 1$ and $\hat{L}_0 < 1$, respectively.

Proposition 2. *If* $L_0^* < 1$, $\widetilde{L}_0 < 1$ *and* $\hat{L}_0 > 1$, *then the equilibrium* $\hat{P}$ *is locally asymptotically stable.*

Proof. For the third age group endemic equilibrium state $\hat{P} = (1, 0, 0, 0, 1, 0, 0, 0, \hat{S}_3, \hat{E}_3, \hat{I}_3, \hat{Q}_3)$, we obtain the characteristic equation

$$(\lambda+\theta_1)^2(\lambda+\theta_5)^2(\lambda+\theta_{10})[\lambda^2 + a_1\lambda + a_2][\lambda^2 + b_1\lambda + b_2]$$
$$\cdot[\lambda^3 + d_1\lambda^2 + d_2\lambda + d_3] = 0, \tag{30}$$

where

$$d_1 = \mu + \theta_9 + \frac{\eta\theta_7}{\theta_9\hat{R}_0} + \theta_7(\hat{E}_3 + \hat{I}_3),$$

$$d_2 = \mu\theta_9 + \frac{\mu\eta\theta_7}{\theta_9\hat{R}_0} + \theta_7\theta_8(\hat{E}_3 + \hat{I}_3) + \theta_7\theta_9(\hat{E}_3 + \hat{I}_3),$$

$$d_3 = \theta_7\theta_8\theta_9(\hat{E}_3 + \hat{I}_3),$$

with $\hat{E}_3 = \frac{\mu}{\theta_8\hat{L}_0}[\hat{L}_0 - 1]$ and $\hat{I}_3 = \frac{\mu\eta}{\theta_8\theta_9\hat{L}_0}[\hat{L}_0 - 1]$.

From the characteristic equation (30), we see that the first five eigenvalues are $\lambda_{1,2} = -\theta_1$, $\lambda_{3,4} = -\theta_5$ and $\lambda_5 = -\theta_{10}$, all of these eigenvalues are negative. The remaining eigenvalues are found by solving $\lambda^2 + a_1\lambda + a_2 = 0$, $\lambda^2 + b_1\lambda + b_2 = 0$ and $\lambda^3 + d_1\lambda^2 + d_2\lambda + d_3 = 0$. For $\lambda^2 + a_1\lambda + a_2 = 0$ and $\lambda^2 + b_1\lambda + b_2 = 0$, all roots of these two characteristic equations have negative real parts if $L_0^* < 1$ and $\widetilde{L}_0 < 1$, respectively. Thus, we apply the Routh-Hurwitz criteria to the equation in the form $\lambda^3 + d_1\lambda^2 + d_2\lambda + d_3 = 0$, the stability holds iff $d_1 > 0$, $d_3 > 0$ and $d_1d_2 - d_3 > 0$. If $\hat{L}_0 > 1$, then we have $d_1 > 0$, and $d_3 > 0$, and it can be easily seen that $d_1d_2 - d_3 > 0$ as the term of $-d_3$ can be cleared with the product of the second term of d_1 and the third term of d_2. Hence the Routh-Hurwitz conditions are satisfied.

Proposition 3. *If $L_0^* < 1$ and $\widetilde{L}_0 > 1$, then the equilibrium $\widetilde{P}$ is locally asymptotically stable.*

Proof. For the second and third group endemic equilibrium point $\widetilde{P} = (1, 0, 0, 0, \widetilde{S}_2, \widetilde{E}_2, \widetilde{I}_2, \widetilde{Q}_2, \widetilde{S}_3, \widetilde{E}_3, \widetilde{I}_3, \widetilde{Q}_3)$, we obtain the characteristic equation

$$(\lambda + \theta_1)(\lambda + \theta_5)^2(\lambda + \theta_{10})[\lambda^2 + a_1\lambda + a_2][\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3] \cdot [\lambda^3 + f_1\lambda^2 + f_2\lambda + f_3] = 0, \quad (31)$$

where

$$e_1 = \theta_1 + \theta_4 + \frac{\eta\theta_6}{\theta_4 \widetilde{R}_0} + \theta_6(\widetilde{E}_2 + \widetilde{I}_2),$$

$$e_2 = \theta_1\theta_4 + \frac{\eta\theta_1\theta_6}{\theta_4 \widetilde{R}_0} + \theta_6\theta_3(\widetilde{E}_2 + \widetilde{I}_2) + \theta_6\theta_4(\widetilde{E}_2 + \widetilde{I}_2),$$

$$e_3 = \theta_6\theta_3\theta_4(\widetilde{E}_2 + \widetilde{I}_2),$$

$$f_1 = \mu + \theta_8 + \theta_9 + \theta_7(\widetilde{E}_3 + \widetilde{I}_3 - \widetilde{S}_3),$$

$$f_2 = \mu(\theta_8 + \theta_9) + \theta_8\theta_9 + \theta_7(\theta_8 + \theta_9)(\widetilde{E}_3 + \widetilde{I}_3) - \theta_7\widetilde{S}_3(\mu + \eta + \theta_9),$$

$$f_3 = \mu\theta_8\theta_9 + \theta_7\theta_8\theta_9(\widetilde{E}_3 + \widetilde{I}_3) - \mu\theta_7(\eta + \theta_9)\widetilde{S}_3,$$

with $\widetilde{E}_2 = \frac{\theta_1}{\theta_3 \widetilde{L}_0}[\widetilde{L}_0 - 1]$ and $\widetilde{I}_2 = \frac{\theta_1\eta}{\theta_3\theta_4\widetilde{L}_0}[\widetilde{L}_0 - 1]$.

From the characteristic equation (31), we see that the first four eigenvalues are $\lambda_1 = -\theta_1$, $\lambda_{2,3} = -\theta_5$ and $\lambda_4 = -\theta_{10}$, all of these eigenvalues are negative. The remaining eigenvalues are found by solving $\lambda^2 + a_1\lambda + a_2 = 0$, $\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0$ and $\lambda^3 + f_1\lambda^2 + f_2\lambda + f_3 = 0$.

For $\lambda^2 + a_1\lambda + a_2 = 0$, all roots of the characteristic equations have negative real parts if $L_0^* < 1$. For $\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0$, if $\widetilde{L}_0 > 1$, then we have $e_1 > 0$ and $e_3 > 0$, and it can be easily seen that $e_1e_2 - e_3 > 0$ as the term of $-e_3$ can be cleared with the product of the second term of e_1 and the third term of e_2. Hence the Routh-Hurwitz conditions are satisfied. For $\lambda^3 + f_1\lambda^2 + f_2\lambda + f_3 = 0$, since $\widetilde{E}_3$, $\widetilde{I}_3$ and $\widetilde{S}_3$ are defined in terms of $\widetilde{E}_2$, $\widetilde{I}_2$ and $\widetilde{S}_2$ as before, hence f_1, f_2 and f_3 are also in the forms of $\widetilde{E}_2$,

$\widetilde{I}_2$ and $\widetilde{S}_2$. It might be complicated to show by hand that equation $\lambda^3 + f_1\lambda^2 + f_2\lambda + f_3 = 0$ satisfies the Routh-Hurwitz conditions. We use *MATLAB* to show the conditions in the following figures by assigning various values of δ_2 in which $\widetilde{L}_0 > 1$ and the other parameters are fixed.

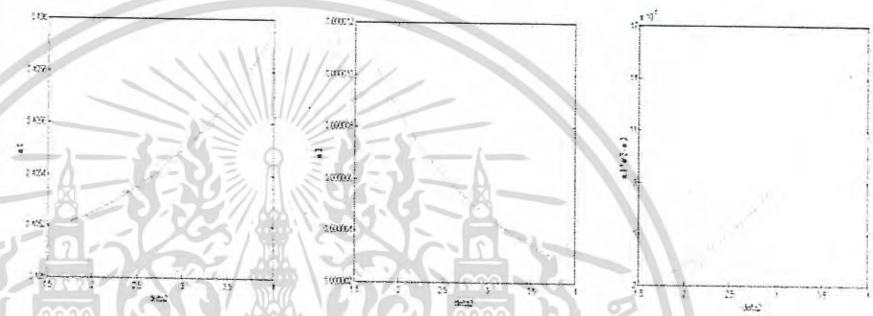

Figure 2. The parameter space for equation $\lambda^3 + f_1\lambda^2 + f_2\lambda + f_3 = 0$ which satisfies the Routh-Hurwitz conditions, plotted onto (f_1, δ_2), (f_3, δ_2), $(f_1 f_2 - f_3, \delta_2)$, respectively. The values of parameters are $\rho = 0.000273973$, $\mu = 0.000039139$, $\eta = 0.111111$, $\alpha = 0.142857$, $\varepsilon = 0.2$, $\beta = 0.142857$, $\gamma = 0.7$ and $\delta_3 = 1$. From the above figure, the Routh-Hurwitz conditions are satisfied for $\widetilde{L}_0 > 1$.

Proposition 4. *If* $L_0^* > 1$, *then the equilibrium* P^* *is locally asymptotically stable.*

Proof. For the full endemic equilibrium point $P^* = (S_1^*, E_1^*, I_1^*, Q_1^*, S_2^*, E_2^*, I_2^*, Q_2^*, S_3^*, E_3^*, I_3^*, Q_3^*)$, we obtain the characteristic equation

$$(\lambda + \theta_5)^2(\lambda + \theta_{10})[\lambda^3 + g_1\lambda^2 + g_2\lambda + g_3][\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3] \cdot [\lambda^3 + i_1\lambda^2 + i_2\lambda + i_3] = 0, \tag{32}$$

where

$$g_1 = \theta_1 + \theta_4 + \frac{\eta\theta_2}{\theta_4 L_0^*} + \theta_2(E_1^* + I_1^*),$$

$$g_2 = \theta_1\theta_4 + \frac{\eta\theta_1\theta_2}{\theta_4 L_0^*} + \theta_2\theta_3(E_1^* + I_1^*) + \theta_2\theta_4(E_1^* + I_1^*),$$

$$g_3 = \theta_2\theta_3\theta_4(E_1^* + I_1^*),$$

$$h_1 = \theta_1 + \theta_3 + \theta_4 + \theta_6(E_2^* + I_2^* - S_2^*),$$

$$h_2 = \theta_1\theta_3 + \theta_1\theta_4 + \theta_3\theta_4 + \theta_6(\theta_3 + \theta_4)(E_2^* + I_2^*) - \theta_6 S_2^*(\theta_1 + \eta + \theta_4),$$

$$h_3 = \theta_1\theta_3\theta_4 + \theta_6\theta_3\theta_4(E_2^* + I_2^*) - \theta_1\theta_6(\eta + \theta_4)S_2^*,$$

$$i_1 = \mu + \theta_8 + \theta_9 + \theta_7(E_3^* + I_3^* - S_3^*),$$

$$i_2 = \mu(\theta_8 + \theta_9) + \theta_8\theta_9 + \theta_7(\theta_8 + \theta_9)(E_3^* + I_3^*) - \theta_7 S_3^*(\mu + \eta + \theta_9),$$

$$i_3 = \mu\theta_8\theta_9 + \theta_7\theta_8\theta_9(E_3^* + I_3^*) - \mu\theta_7(\eta + \theta_9)S_3^*,$$

with $E_1^* = \dfrac{\theta_1}{\theta_3 L_0^*}[L_0^* - 1]$ and $I_1^* = \dfrac{\theta_1\eta}{\theta_3\theta_4 L_0^*}[L_0^* - 1]$.

From the characteristic equation (32), we see that the first three eigenvalues are $\lambda_{1,2} = -\theta_5$ and $\lambda_3 = -\theta_{10}$ all of these are negative. The remaining eigenvalues are found by solving $\lambda^3 + g_1\lambda^2 + g_2\lambda + g_3 = 0$, $\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = 0$ and $\lambda^3 + i_1\lambda^2 + i_2\lambda + i_3 = 0$.

For $\lambda^3 + g_1\lambda^2 + g_2\lambda + g_3 = 0$, if $L_0^* > 1$, then we have $g_1 > 0$ and $g_3 > 0$, and it can be easily seen that $g_1 g_2 - g_3 > 0$ as the term of $-g_3$ can be cleared with the product of the second term of g_1 and the third term of g_2. Hence, the Routh-Hurwitz conditions are satisfied. For $\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = 0$ and $\lambda^3 + i_1\lambda^2 + i_2\lambda + i_3 = 0$, since E_2^*, I_2^*, S_2^*, E_3^*, I_3^* and S_3^* are defined in terms of E_1^*, I_1^* and S_1^* as before, hence h_1, h_2, h_3, i_1, i_2 and i_3 are also in the forms of E_1^*, I_1^* and S_1^*. Proceeding with the same manner as above, use *MATLAB* to graph the conditions of the Routh-Hurwitz, by assigning various values of δ_1 in which $L_0^* > 1$ and the other parameters are fixed. The Routh-Hurwitz conditions are satisfied for $L_0^* > 1$.

C. Numerical results

We consider the numerical solutions for the transmission of swine flu in each age groups. The trajectories of the solutions when the parameter values will lead to each equilibrium state are shown in the figures. The values of parameters used in this study are determined by the real life observations. The values of the parameters are as follows: $\mu = 1/(365 \times 70)\,\text{day}^{-1}$, corresponding to a life expectancy of 70 years for human; each age group develops into another age group is 10 years, so $\kappa = 1/(365 \times 10)\,\text{day}^{-1}$. The other parameters are arbitrarily chosen. Numerical solutions of system of differential equations (17)-(28) are shown in the following figures.

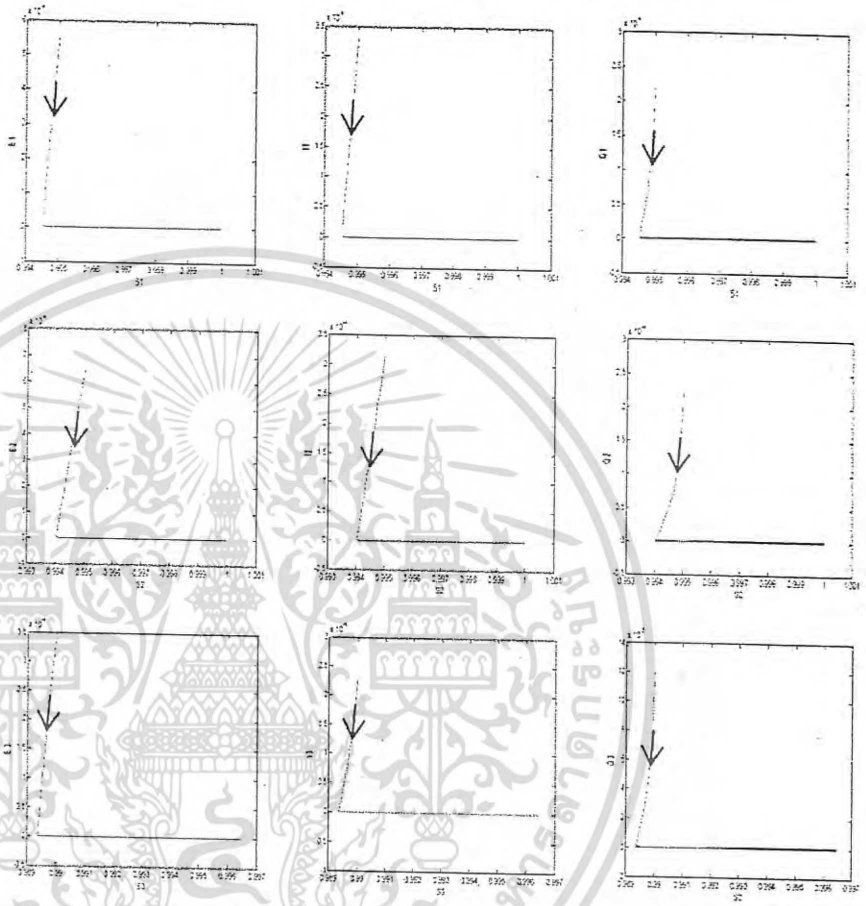

Figure 3. Numerical solutions demonstrate the solution trajectories, projected onto (S_1, E_1), (S_1, I_1), (S_1, Q_1), (S_2, E_2), (S_2, I_2), (S_2, Q_2), (S_3, E_3), (S_3, I_3), (S_3, Q_3), respectively. For $L_0^* < 1$, $\widetilde{L}_0 < 1$ and $\hat{L}_0 < 1$ with values of parameters are $\mu = 1/(365 \times 70)\,\text{day}^{-1}$, $\kappa = 1/(365 \times 10)\,\text{day}^{-1}$, $\eta = 1/9\,\text{day}^{-1}$, $\alpha = 1/7\,\text{day}^{-1}$, $\varepsilon = 1/5\,\text{day}^{-1}$, $\beta = 1/7\,\text{day}^{-1}$, $\gamma = 0.7\text{day}^{-1}$, $\delta_1 = 0.6$, $\delta_2 = 0.9$, $\delta_3 = 0.15$, $L_0^* = 0.39044693$, $\widetilde{L}_0 = 0.51246160$ and $\hat{L}_0 = 0.59863369$. The proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ approach to the disease free equilibrium state $(1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0)$.

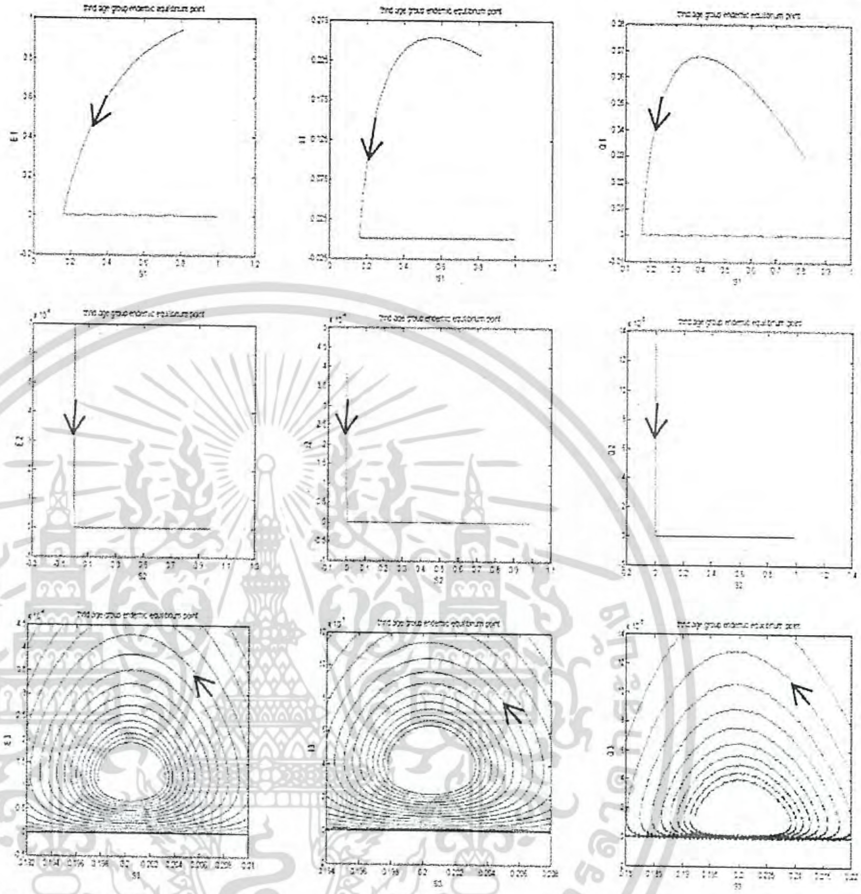

Figure 4. Numerical solutions demonstrate the solution trajectories, projected onto (S_1, E_1), (S_1, I_1), (S_1, Q_1), (S_2, E_2), (S_2, I_2), (S_2, Q_2), (S_3, E_3), (S_3, I_3), (S_3, Q_3), respectively. For $L_0^* < 1$, $\widetilde{L}_0 < 1$ and $\hat{L}_0 > 1$ with values of parameters are $\delta_1 = 1.45$, $\delta_2 = 1.65$, $\delta_3 = 1.25$, $L_0^* =$ 0.94358008, $\widetilde{L}_0 = 0.93951293$, and $\hat{L}_0 = 4.98861411$ but the other parameters are same as in Figure 3. The proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ spiral into the third age group endemic equilibrium state (1, 0, 0, 0, 1, 0, 0, 0, 0.20045647, 0.00012320, 0.00003992, 0.00001141).

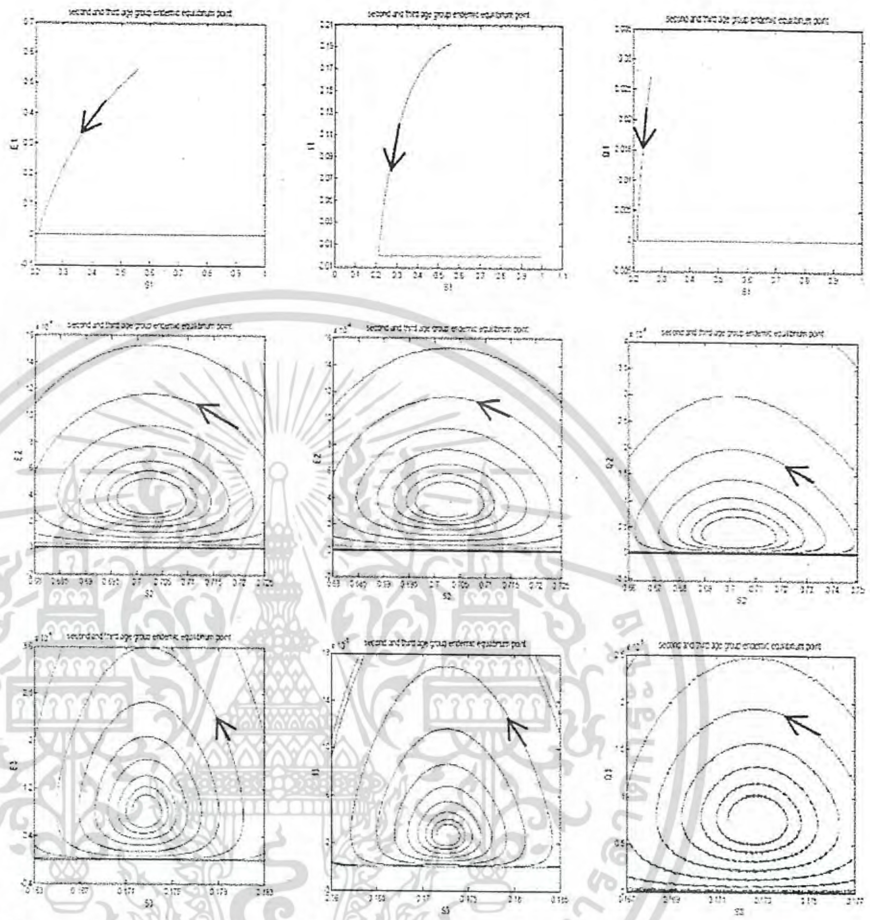

Figure 5. Numerical solutions demonstrate the solution trajectories, projected onto (S_1, E_1), (S_1, I_1), (S_1, Q_1), (S_2, E_2), (S_2, I_2), (S_2, Q_2), (S_3, E_3), (S_3, I_3), (S_3, Q_3), respectively. For $L_0^* < 1$, $\widetilde{L}_0 > 1$ with values of parameters are $\delta_1 = 1.52$, $\delta_2 = 2.5$, $\delta_3 = 1.45$, $L_0^* = 0.98913223$, $\widetilde{L}_0 = 1.42350444$ but the other parameters are same as in Figure 3. The proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ spiral into the second and the third age group endemic equilibrium state (1, 0 0, 0, 0.70249166, 0.00036634, 0.00011861, 0.00003387, 0.17266629, 0.00008170, 0.00002649, 0.00000757).

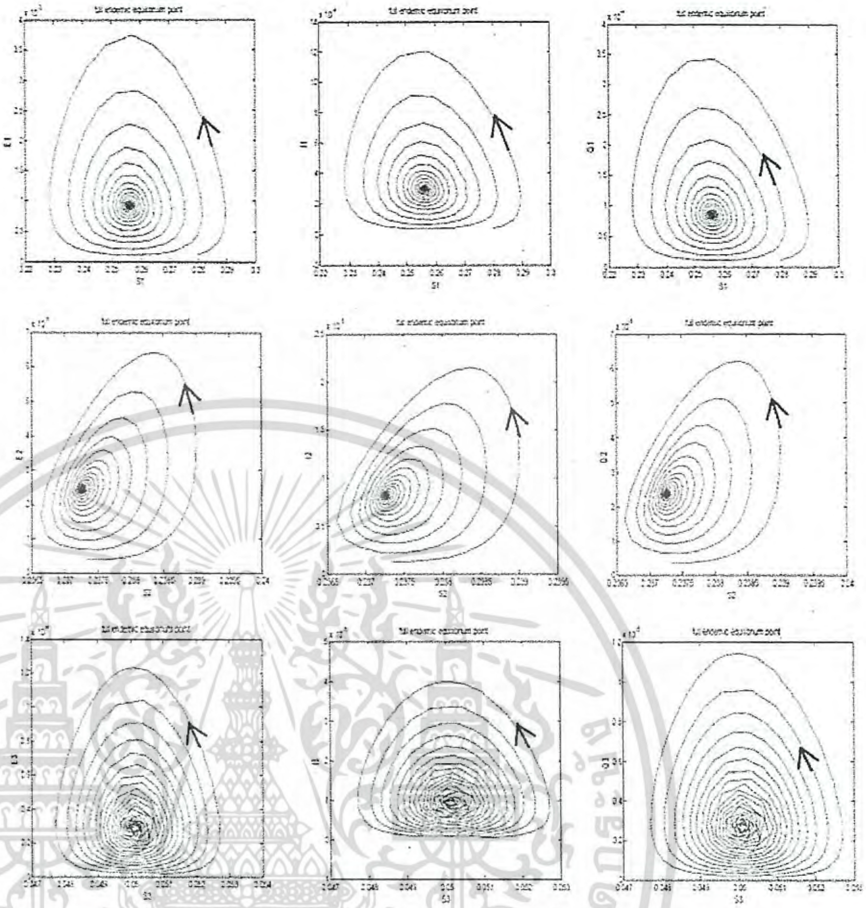

Figure 6. Numerical solutions demonstrate the solution trajectories, projected onto (S_1, E_1), (S_1, I_1), (S_1, Q_1), (S_2, E_2), (S_2, I_2), (S_2, Q_2), (S_3, E_3), (S_3, I_3), (S_3, Q_3), respectively. For $L_0^* > 1$ with values of parameters are $\delta_1 = 6$, $\delta_2 = 7$, $\delta_3 = 5$ and $L_0^* = 3.90446932$ but the other parameters are same as in Figure 3. The proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ spiral into the full endemic equilibrium state (0.25611675, 0.00091599, 0.00029658, 0.00008470, 0.23726898, 0.00002434, 0.00000815, 0.00000237, 0.05010638, 0.00002884, 0.00000935, 0.00000267).

IV. Discussion and Conclusion

In this study, we get four equilibrium points P_0, $\hat{P}$, $\widetilde{P}$ and P^*. The first point is the disease free equilibrium $P_0 = (1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0)$ represents the state in which swine flu is not endemic in the human and it is local stability for $L_0^* < 1$, $\widetilde{L}_0 < 1$ and $\hat{L}_0 < 1$. Figure 3 shows the proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ approach to the disease free state $(1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0)$ when $L_0^* < 1$, $\widetilde{L}_0 < 1$ and $\hat{L}_0 < 1$.

For $\hat{P} = (1, 0, 0, 0, 1, 0, 0, 0, \hat{S}_3, \hat{E}_3, \hat{I}_3, \hat{Q}_3)$, the second point is the third age group endemic equilibrium which represents the state in which swine flu is endemic only in the third age group and it is local stability for $L_0^* < 1$, $\widetilde{L}_0 < 1$ and $\hat{L}_0 > 1$. Figure 4 shows the proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ spirals into the third age group endemic equilibrium state (1, 0, 0, 0, 1, 0, 0, 0, 0.20045647, 0.00012320, 0.00003992, 0.00001141) when $L_0^* < 1$, $\widetilde{L}_0 < 1$ and $\hat{L}_0 > 1$.

For $\widetilde{P} = (1, 0, 0, 0, \widetilde{S}_2, \widetilde{E}_2, \widetilde{I}_2, \widetilde{Q}_2, \widetilde{S}_3, \widetilde{E}_3, \widetilde{I}_3, \widetilde{Q}_3)$, the third point is the second and the third age group endemic equilibrium which represents the state in which swine flu is endemic in both the second age group and the third age group and it is local stability for $L_0^* < 1$ and $\widetilde{L}_0 > 1$. Figure 5 shows the proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ spiral into the second and the third age group endemic equilibrium state (1, 0, 0, 0, 0.70249166, 0.00036634, 0.00011861, 0.00003387, 0.17266629, 0.00008170, 0.00002649, 0.00000757) when $L_0^* < 1$ and $\widetilde{L}_0 > 1$.

The fourth point is the full endemic equilibrium $P^* = (S_1^*, E_1^*, I_1^*, Q_1^*, S_2^*, E_2^*, I_2^*, Q_2^*, S_3^*, E_3^*, I_3^*, Q_3^*)$ which represents the state in which swine flu is endemic in all age groups and it is local stability for $L_0^* > 1$. Figure 6 shows the proportions of population $(S_1, E_1, I_1, Q_1, S_2, E_2, I_2, Q_2, S_3, E_3, I_3, Q_3)$ spiral into the full endemic equilibrium state (0.25611675, 0.00091599, 0.00029658, 0.00008470, 0.23726898, 0.00002434, 0.00000815, 0.00000237, 0.05010638, 0.00002884, 0.00000935, 0.00000267) when $L_0^* > 1$.

The biological meaning of the basic reproductive numbers L_0^*, $\widetilde{L}_0$ and $\hat{L}_0$ are explained as follows:

$$L_0^* = \left(\frac{\delta_1\mu}{\mu+\kappa}\right)\left[\left(\frac{\eta}{\mu+\eta+\alpha+\kappa}\right)\left(\frac{1}{\mu+\varepsilon+\beta+\kappa}\right)+\frac{1}{\mu+\eta+\alpha+\kappa}\right]$$

is the reproductive number for swine flu in the first age group, where δ_1 is the transmission rate per day of the first age group, $\frac{\mu}{\mu+\kappa}$ is the ratio between the total number of the first age group and the total population, $\frac{\eta}{\mu+\eta+\alpha+\kappa}$ is the fraction of exposed members who move to the infective class, $\frac{1}{\mu+\varepsilon+\beta+\kappa}$ is the average time that an infective individual remains in the class I, and $\frac{1}{\mu+\eta+\alpha+\kappa}$ is the average time that an exposed member remains in that class.

The term

$$\widetilde{L}_0 = \frac{\delta_2\mu\kappa}{(\mu+\kappa)^2}\left[\left(\frac{\eta}{\mu+\eta+\alpha+\kappa}\right)\left(\frac{1}{\mu+\varepsilon+\beta+\kappa}\right)+\frac{1}{\mu+\eta+\alpha+\kappa}\right]$$

is the reproductive number for swine flu in the second age group, where δ_2 is the transmission rate per day of the second age group, $\frac{\mu\kappa}{(\mu+\kappa)^2}$ is the ratio between the total number of second age group and the total population, the other terms are defined as before.

The term $\hat{L}_0 = \dfrac{\delta_3 \kappa^2}{(\mu + \kappa)^2}\left[\left(\dfrac{\eta}{\mu + \eta + \alpha}\right)\left(\dfrac{1}{\mu + \varepsilon + \beta}\right) + \dfrac{1}{\mu + \eta + \alpha}\right]$ is the reproductive number for swine flu in the third age group, where δ_3 is the transmission rate per day of the third age group, $\dfrac{\kappa^2}{(\mu + \kappa)^2}$ is the ratio between the total number of third age group and the total population, the other terms are defined in the same manner as before.

The bifurcation diagrams of (17)-(28) are shown in the following figures:

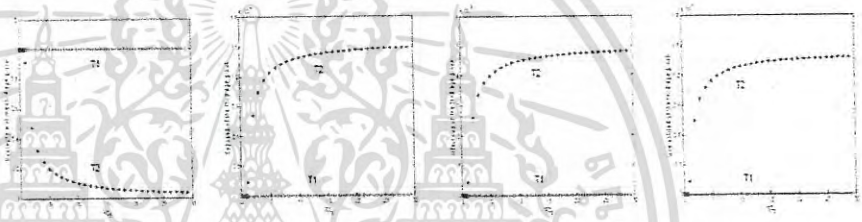

Figure 7. Bifurcation diagrams of the solutions of (17)-(28), plotted onto $(\hat{L}_0, S_3)$, $(\hat{L}_0, E_3)$, $(\hat{L}_0, I_3)$, $(\hat{L}_0, Q_3)$, respectively for the different values of $\hat{L}_0$. *-*-* denote the stable solutions and 0-0-0 denote the unstable solutions.

We can see that when $\hat{L}_0 < 1$, T_1 will be stable and for $\hat{L}_0 > 1$, T_2 will be stable. If the threshold number of the third age group is greater than one, then the third age group susceptible decreases. The third age group infectious, quarantined and recovered increase.

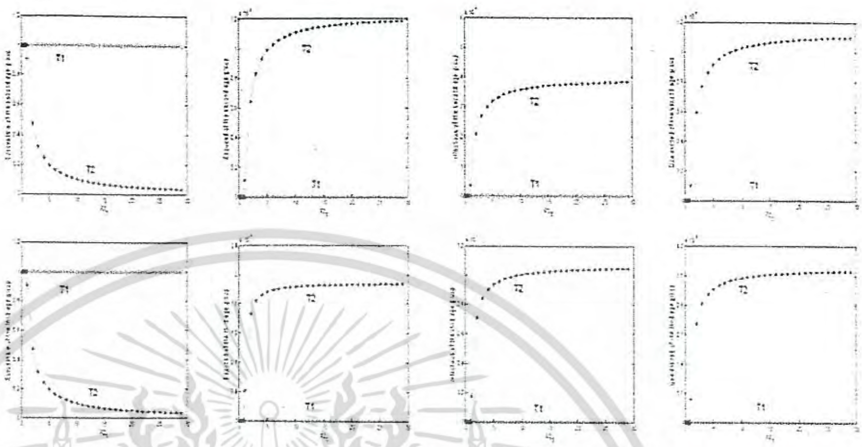

Figure 8. Bifurcation diagrams of the solutions of (17)-(28) plotted onto $(\widetilde{L}_0, S_2)$, $(\widetilde{L}_0, E_2)$, $(\widetilde{L}_0, I_2)$, $(\widetilde{L}_0, Q_2)$, $(\widetilde{L}_0, S_3)$, $(\widetilde{L}_0, E_3)$, $(\widetilde{L}_0, I_3)$, $(\widetilde{L}_0, Q_3)$, respectively, for the different values of $\widetilde{L}_0$. *-*-* denote the stable solutions and 0-0-0 denote the unstable solutions.

We can see that when $\widetilde{L}_0 < 1$, T_1 will be stable and for $\widetilde{L}_0 > 1$, T_2 will be stable. If the threshold number of the second age group is greater than one, the susceptible of the second age group and the third age group decreases. The second age group infectious, quarantined and recovered increase. The third age group infectious, quarantined and recovered increase.

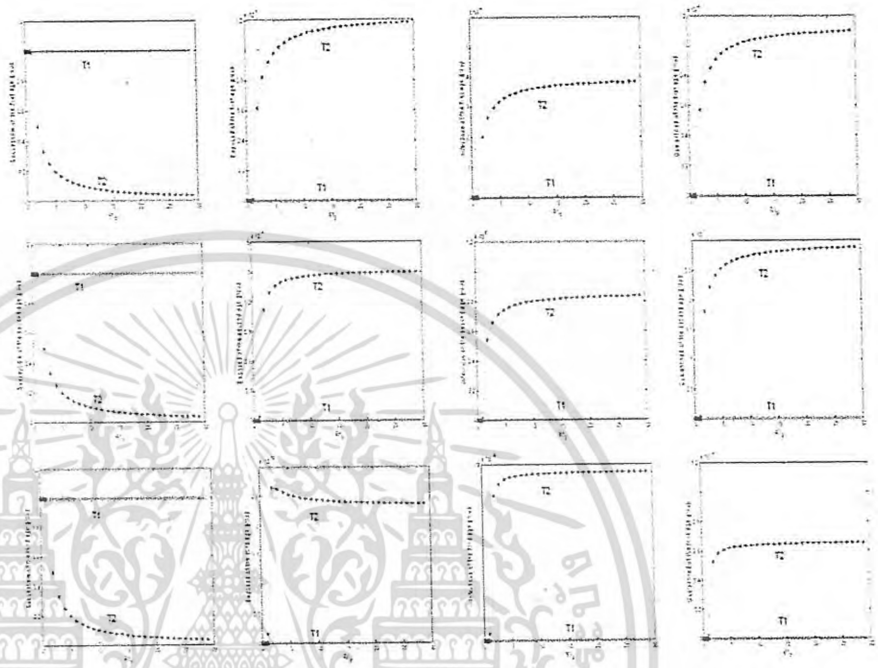

Figure 9. Bifurcation diagrams of the solutions of (17)-(28), plotted onto (L_0^*, S_1), (L_0^*, E_1), (L_0^*, I_1), (L_0^*, Q_1), (L_0^*, S_2), (L_0^*, E_2), (L_0^*, I_2), (L_0^*, Q_2), (L_0^*, S_3), (L_0^*, E_3), (L_0^*, I_3), (L_0^*, Q_3), respectively, for the different values of L_0^*. *-*-* denote the stable solutions and 0-0-0 denote the unstable solutions.

We can see that when $L_0^* < 1$, T_1 will be stable and for $L_0^* > 1$, T_2 will be stable. If the threshold number of the first age group is greater than one, the susceptible of the first age group, the second age group and the third age group decreases. The first age group infectious, quarantined and recovered increase. The second age group infectious, quarantined and recovered increase. The third age group infectious, quarantined and recovered increase.

The local stability of all equilibrium states is determined by the threshold numbers L_0^*, $\widetilde{L}_0$ and $\hat{L}_0$. To reduce the transmission of this disease, we should control the above threshold numbers. The threshold numbers are used for controlling many diseases [29-31]. The results of this study should be the alternative way for controlling the transmission of swine flu in Thailand.

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GLOBAL STABILITY OF THE AGE STRUCTURAL TRANSMISSION MODEL FOR SWINE FLU

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Abstract

In this paper, we examine the global stability of swine flu transmission model incorporated the age structure of human population based on SEIQR (susceptible-exposed-infected-quarantined-recovered) model. The human population is divided into three groups such as 1-10 years, 11-20 years and more than 20 years, respectively. We obtain four equilibrium states and then construct Lyapunov functions with the conditions for global stability of these four equilibrium states.

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is the rate at which the individuals leave the infective individuals I to become the quarantined individuals Q. The parameters d, f and g are the rate at which individuals in the E, I and Q classes recover from the disease.

B. Equations of the model

The age structural SEIQR model is described by the following system of differential equations:

$$S_1'(t) = bN_T - \left(\frac{a_1(E_1 + I_1)}{N_T} + \mu + k\right)S_1, \tag{1}$$

$$E_1'(t) = \frac{a_1 S_1(E_1 + I_1)}{N_T} - (\mu + c + d + k)E_1, \tag{2}$$

$$I_1'(t) = cE_1 - (\mu + e + f + k)I_1, \tag{3}$$

$$Q_1'(t) = eI_1 - (\mu + g + k)Q_1, \tag{4}$$

$$R_1'(t) = dE_1 + fI_1 + gQ_1 - (\mu + k)R_1, \tag{5}$$

$$S_2'(t) = kS_1 - \left(\frac{a_2(E_2 + I_2)}{N_T} + \mu + k\right)S_2, \tag{6}$$

$$E_2'(t) = kE_1 + \frac{a_2 S_2(E_2 + I_2)}{N_T} - (\mu + c + d + k)E_2, \tag{7}$$

$$I_2'(t) = kI_1 + cE_2 - (\mu + e + f + k)I_2, \tag{8}$$

$$Q_2'(t) = kQ_1 + eI_2 - (\mu + g + k)Q_2, \tag{9}$$

$$R_2'(t) = kR_1 + dE_2 + fI_2 + gQ_2 - (\mu + k)R_2, \tag{10}$$

$$S_3'(t) = kS_2 - \left(\frac{a_3(E_3 + I_3)}{N_T} + \mu\right)S_3, \tag{11}$$

$$E_3'(t) = kE_2 + \frac{a_3 S_3(E_3 + I_3)}{N_T} - (\mu + c + d)E_3, \tag{12}$$

$$I_3'(t) = kI_2 + cE_3 - (\mu + e + f)I_3, \tag{13}$$

$$Q_3'(t) = kQ_2 + eI_3 - (\mu + g)Q_3, \tag{14}$$

$$R_3'(t) = kR_2 + dE_3 + fI_3 + gQ_3 - \mu R_3. \tag{15}$$

If we add (1)-(15), (1)-(5), (6)-(10) and (11)-(15), then

$$N_T'(t) = bN_T - \mu N_T,\ N_1'(t) = bN_T - (\mu + k)N_1,$$

$$N_2'(t) = kN_1 - (\mu + k)N_2,\ N_3'(t) = kN_2 - \mu N_3. \tag{16}$$

In this paper, we assume that the total population, total number of the first age group, the second age group and the third age group remain constant. Therefore $N_T'(t) = N_1'(t) = N_2'(t) = N_3'(t) = 0$. Setting the right hand side of (16) to be zero, we obtain the following four relations: $b = \mu$, $\frac{N_1}{N_T} = \frac{b}{b+k}$, $\frac{N_2}{N_T} = \frac{bk}{(b+k)^2}$ and $\frac{N_3}{N_T} = \frac{k^2}{(b+k)^2}$.

C. The epidemic

We can see that all equations described by (1)-(15), the non-negative octant R_+^{15} is positively invariant (where R_+^{15} denotes the non-negative region). With respect to (1)-(15), we have the following results:

Proposition 1. *Let* $(S_1(t), E_1(t), I_1(t), Q_1(t), R_1(t), S_2(t), E_2(t), I_2(t), Q_2(t), R_2(t), S_3(t), E_3(t), I_3(t), Q_3(t), R_3(t))$ *be the solution of* (1)-(15) *with the initial condition* $(S_1(0), E_1(0), I_1(0), Q_1(0), R_1(0), S_2(0), E_2(0), I_2(0), Q_2(0), R_2(0), S_3(0), E_3(0), I_3(0), Q_3(0), R_3(0))$ *and the compact set*

$$\Omega_1 = \Big\{(S_1, E_1, I_1, Q_1, R_1, S_2, E_2, I_2, Q_2, R_2, S_3, E_3, I_3, Q_3, R_3) \in R_+^{15},$$

$$J_1 \le N_1 = \left(\frac{b}{b+k}\right)N_T,\ J_2 \le N_2 = \left(\frac{bk}{(b+k)^2}\right)N_T,$$

$$J_3 \le N_3 = \left(\frac{k^2}{(b+k)^2}\right)N_T\Big\}.$$

Then, under the flow described by (1)-(15), Ω_1 *is positively invariant set that attracts all solutions in* R_+^{15}.

Proof. We choose the Lyapunov function

$$J(t) = (J_1(t), J_2(t), J_3(t))$$
$$= (S_1 + E_1 + I_1 + Q_1 + R_1, S_2 + E_2 + I_2 + Q_2 + R_2, S_3 + E_3 + I_3 + Q_3 + R_3)$$

positive definite on R_+^{15} and we have

$$\frac{dJ}{dt} = \left(\frac{dJ_1}{dt}, \frac{dJ_2}{dt}, \frac{dJ_3}{dt}\right)$$
$$= \left(\frac{d}{dt}S_1 + \frac{d}{dt}E_1 + \frac{d}{dt}I_1 + \frac{d}{dt}Q_1 + \frac{d}{dt}R_1, \frac{d}{dt}S_2 + \frac{d}{dt}E_2 + \frac{d}{dt}I_2 + \frac{d}{dt}Q_2 + \frac{d}{dt}R_2, \frac{d}{dt}S_3 + \frac{d}{dt}E_3 + \frac{d}{dt}I_3 + \frac{d}{dt}Q_3 + \frac{d}{dt}R_3\right)$$
$$= (bN_T - (b+k)(S_1 + E_1 + I_1 + Q_1 + R_1),$$
$$k(S_1 + E_1 + I_1 + Q_1 + R_1) - (b+k)(S_2 + E_2 + I_2 + Q_2 + R_2),$$
$$k(S_2 + E_2 + I_2 + Q_2 + R_2) - b(S_3 + E_3 + I_3 + Q_3 + R_3))$$
$$= (bN_T - (b+k)N_1, kN_1 - (b+k)N_2, kN_2 - bN_3).$$

We use the fact that $N_1 = \frac{b}{b+k}N_T$, $N_2 = \frac{bk}{(b+k)^2}N_T$, and $N_3 = \frac{k^2}{(b+k)^2}N_T$, then we can prove that

$$\frac{dJ_1}{dt} = bN_T - (b+k)J_1 \le 0, \text{ for } J_1 \ge \frac{bN_T}{b+k}, \tag{17}$$

$$\frac{dJ_2}{dt} = \frac{bk}{b+k}N_T - (b+k)J_2 \le 0, \text{ for } J_2 \ge \frac{bkN_T}{(b+k)^2}, \tag{18}$$

$$\frac{dJ_3}{dt} = \frac{bk^2}{(b+k)^2}N_T - bJ_3 \le 0, \text{ for } J_3 \ge \frac{k^2N_T}{(b+k)^2}. \tag{19}$$

From (17)-(19), one has $\frac{dJ}{dt} \le 0$ which implies that Ω_1 is a positively invariant set. In other words, by solving (17)-(19), we obtain

$$0 \le (J_1(t), J_2(t), J_3(t)) \le (bN_T/(b+k)) + J_1(0)e^{-(b+k)t},$$
$$(bkN_T/(b+k)^2) + J_2(0)e^{-(b+k)t}, (k^2N_T/(b+k)^2) + J_3(0)e^{-bt}),$$

where $J_1(0)$, $J_2(0)$ and $J_3(0)$ are the initial conditions of $J_1(t)$, $J_2(t)$ and $J_3(t)$. Thus, as $t \to \infty$, $0 \le (J_1(t), J_2(t), J_3(t)) \le (bN_T/(b+k)$, $bkN_T/(b+k)^2, k^2N_T/(b+k)^2) = (N_1, N_2, N_3)$ and one can conclude that Ω_1 is an attractive set.

D. Equilibrium points

The equilibrium points are obtained by setting the right hand side of (1)-(15) to zero. We obtain four equilibrium points:

(i) The disease free equilibrium point

$$P_1(\overline{S}_1, \overline{E}_1, \overline{I}_1, \overline{Q}_1, \overline{R}_1, \overline{S}_2, \overline{E}_2, \overline{I}_2, \overline{Q}_2, \overline{R}_2, \overline{S}_3, \overline{E}_3, \overline{I}_3, \overline{Q}_3, \overline{R}_3)$$
$$= P_1\left(\frac{bN_T}{b+k}, 0, 0, 0, 0, \frac{bkN_T}{(b+k)^2}, 0, 0, 0, 0, \frac{k^2N_T}{(b+k)^2}, 0, 0, 0, 0\right).$$

(ii) The third age group endemic equilibrium point

$$P_2(\hat{S}_1, \hat{E}_1, \hat{I}_1, \hat{Q}_1, \hat{R}_1, \hat{S}_2, \hat{E}_2, \hat{I}_2, \hat{Q}_2, \hat{R}_2, \hat{S}_3, \hat{E}_3, \hat{I}_3, \hat{Q}_3, \hat{R}_3)$$
$$= P_2\left(\frac{bN_T}{b+k}, 0, 0, 0, 0, \frac{bkN_T}{(b+k)^2}, 0, 0, 0, 0, \hat{S}_3, \hat{E}_3, \hat{I}_3, \hat{Q}_3, \hat{R}_3\right),$$

where

$$\hat{S}_3 = \frac{k^2N_T}{(b+k)^2}\left(\frac{1}{\hat{R}_0}\right), \quad \hat{E}_3 = \frac{bk^2N_T}{(b+k)^2(b+c+d)}\left(1-\frac{1}{\hat{R}_0}\right),$$

$$\hat{I}_3 = \frac{bck^2N_T}{(b+k)^2(b+c+d)(b+e+f)}\left(1-\frac{1}{\hat{R}_0}\right),$$

$$\hat{Q}_3 = \frac{bcek^2 N_T}{(b+k)^2(b+c+d)(b+e+f)(b+g)}\left(1-\frac{1}{\hat{R}_0}\right),$$

$$\hat{R}_3 = \frac{(b+e+f)(b+g)d + cf(b+g) + ceg}{b(b+e+f)(b+g)}\left(1-\frac{1}{\hat{R}_0}\right)$$

with

$$\hat{R}_0 = \frac{a_3 k^2 (b+c+e+f)}{(b+k)^2(b+c+d)(b+e+f)}.$$

(iii) The second and third age group endemic equilibrium point

$$P_3(\widetilde{S}_1, \widetilde{E}_1, \widetilde{I}_1, \widetilde{Q}_1, \widetilde{R}_1, \widetilde{S}_2, \widetilde{E}_2, \widetilde{I}_2, \widetilde{Q}_2, \widetilde{R}_2, \widetilde{S}_3, \widetilde{E}_3, \widetilde{I}_3, \widetilde{Q}_3, \widetilde{R}_3)$$

$$= P_3\left(\frac{bN_T}{b+k}, 0, 0, 0, 0, \widetilde{S}_2, \widetilde{E}_2, \widetilde{I}_2, \widetilde{Q}_2, \widetilde{R}_2, \widetilde{S}_3, \widetilde{E}_3, \widetilde{I}_3, \widetilde{Q}_3, \widetilde{R}_3\right),$$

where

$$\widetilde{S}_2 = \frac{bkN_T}{(b+k)^2}\left(\frac{1}{\widetilde{R}_0}\right), \quad \widetilde{E}_2 = \frac{bkN_T}{(b+k)(b+c+d+k)}\left(1-\frac{1}{\widetilde{R}_0}\right),$$

$$\widetilde{I}_2 = \frac{bckN_T}{(b+k)(b+c+d+k)(b+e+f+k)}\left(1-\frac{1}{\widetilde{R}_0}\right),$$

$$\widetilde{Q}_3 = \frac{bcekN_T}{(b+k)(b+c+d+k)(b+e+f+k)(b+g+k)}\left(1-\frac{1}{\widetilde{R}_0}\right),$$

$$\widetilde{R}_2 = \frac{(b+e+f+k)(b+g+k)d + cf(b+g+k) + ceg}{(b+k)(b+e+f+k)(b+g+k)}\left(1-\frac{1}{\widetilde{R}_0}\right),$$

$$\widetilde{S}_3 = \frac{k\widetilde{S}_2}{\frac{a_3}{N_T}(\widetilde{E}_3 + \widetilde{I}_3) + b}, \quad \widetilde{E}_3 = \frac{-L_2 + \sqrt{L_2^2 - 4L_1L_3}}{2L_1},$$

$$\widetilde{I}_3 = \frac{k\widetilde{I}_2 + c\widetilde{E}_3}{b+e+f}, \quad \widetilde{Q}_3 = \frac{k\widetilde{Q}_2 + e\widetilde{I}_3}{b+g}, \quad \widetilde{R}_3 = \frac{k\widetilde{R}_2 + d\widetilde{E}_3 + f\widetilde{I}_3 + g\widetilde{Q}_3}{b}$$

with

$$\widetilde{R}_0 = \frac{a_2 bk(b+c+e+f+k)}{(b+k)^2(b+c+d+k)(b+e+f+k)},$$

$$L_1 = \frac{a_3}{N_T}(b+c+d)(b+c+e+f),$$

$$L_2 = b(b+c+d)(b+e+f) + \frac{a_3 k}{N_T}((b+c+d)\widetilde{I}_2 - (b+c+e+f)(\widetilde{S}_2 + \widetilde{E}_2)),$$

and

$$L_3 = -\left[\frac{k^2 a_3}{N_T}\widetilde{I}_2(\widetilde{S}_2 + \widetilde{E}_2) + bk(b+e+f)\widetilde{E}_2\right].$$

(iv) The full endemic equilibrium point $P_4(S_1^*, E_1^*, I_1^*, Q_1^*, S_2^*, E_2^*, I_2^*, Q_2^*, S_3^*, E_3^*, I_3^*, Q_3^*)$, where

$$S_1^* = \frac{bN_T}{b+k}\left(\frac{1}{R_0^*}\right), \quad E_1^* = \frac{bN_T}{(b+c+d+k)}\left(1 - \frac{1}{R_0^*}\right),$$

$$I_1^* = \frac{bcN_T}{(b+c+d+k)(b+e+f+k)}\left(1 - \frac{1}{R_0^*}\right),$$

$$Q_1^* = \frac{bceN_T}{(b+c+d+k)(b+e+f+k)(b+g+k)}\left(1 - \frac{1}{R_0^*}\right),$$

$$R_1^* = \frac{(b+e+f+k)(b+g+k)d + cf(b+g+k) + ceg}{(b+k)(b+e+f+k)(b+g+k)}\left(1 - \frac{1}{R_0^*}\right),$$

$$S_2^* = \frac{kS_1^*}{\frac{a_2}{N_T}(E_2^* + I_2^*) + (b+k)}, \quad E_2^* = \frac{-L_2 + \sqrt{L_2^2 - 4L_1L_3}}{2L_1},$$

$$I_2^* = \frac{kI_1^* + cE_2^*}{b+e+f+k}, \quad Q_2^* = \frac{kQ_1^* + eI_2^*}{b+g+k}, \quad R_2^* = \frac{kR_1^* + dE_2^* + fI_2^* + gQ_2^*}{b+k},$$

$$S_3^* = \frac{kS_2^*}{\frac{a_3}{N_T}(E_3^* + I_3^*) + b}, \quad E_3^* = \frac{-M_2 + \sqrt{M_2^2 - 4M_1M_3}}{2M_1},$$

$$I_3^* = \frac{kI_2^* + cE_3^*}{b+e+f},\ Q_3^* = \frac{kQ_2^* + eI_3^*}{b+g},\ R_3^* = \frac{kR_2^* + dE_3^* + fI_3^* + gQ_3^*}{b}$$

with

$$R_0^* = \frac{a_1 b(b+c+e+f+k)}{(b+k)(b+c+d+k)(b+e+f+k)},$$

$$L_1 = \frac{a_2}{N_T}(b+c+d+k)(b+c+e+f+k),$$

$$L_2 = (b+k)(b+c+d+k)(b+e+f+k) + \frac{a_2 k}{N_T}((b+c+d+k)I_1^* - (b+c+e+f+k)(S_1^* + E_1^*)),$$

$$L_3 = -\left[\frac{a_2 k^2}{N_T} I_1^*(S_1^* + E_1^*) + k(b+k)(b+e+f+k)E_1^*\right],$$

$$M_1 = \frac{a_3}{N_T}(b+c+d)(b+c+e+f),$$

$$M_2 = b(b+c+d)(b+e+f) + \frac{a_3 k}{N_T}((b+c+d)I_2^* - (b+c+e+f)(S_2^* + E_2^*)),\ \text{and}$$

$$M_3 = -\left[\frac{k^2 a_3}{N_T} I_2^*(S_2^* + E_2^*) + bk(b+e+f)E_2^*\right].$$

E. Global stability of the equilibrium states

The global behavior of the equilibrium states for equations (1)-(15) is determined from Lyapunov techniques.

Theorem 1. *Assume that*

$$\begin{cases} a_1 = \dfrac{bN_T}{\overline{S}_1} \\ a_2 = \dfrac{bN_T}{\overline{S}_2} \\ a_3 = \dfrac{bN_T}{\overline{S}_3}. \end{cases} \quad (20)$$

For $\hat{R}_0 < 1$, $\widetilde{R}_0 < 1$ *and* $R_0^* < 1$, *the disease free equilibrium point* P_1 *is globally asymptotically stable on* Ω_1.

Proof. Let us consider on Ω_1 the Lyapunov function

$$\rho(t) = (S_1 - \overline{S}_1 \ln S_1) + E_1 + I_1 + Q_1 + R_1 + (S_2 - \overline{S}_2 \ln S_2)$$
$$+ E_2 + I_2 + Q_2 + R_2 + (S_3 - \overline{S}_3 \ln S_3) + E_3 + I_3 + Q_3 + R_3.$$

The derivative with respect to time yields

$$\frac{d}{dt}\rho(t) = \frac{d}{dt}S_1\left(1 - \frac{\overline{S}_1}{S_1}\right) + \frac{d}{dt}E_1 + \frac{d}{dt}I_1 + \frac{d}{dt}Q_1 + \frac{d}{dt}R_1$$
$$+ \frac{d}{dt}S_2\left(1 - \frac{\overline{S}_2}{S_2}\right) + \frac{d}{dt}E_2 + \frac{d}{dt}I_2 + \frac{d}{dt}Q_2 + \frac{d}{dt}R_2$$
$$+ \frac{d}{dt}S_3\left(1 - \frac{\overline{S}_3}{S_3}\right) + \frac{d}{dt}E_3 + \frac{d}{dt}I_3 + \frac{d}{dt}Q_3 + \frac{d}{dt}R_3$$
$$= \left(bN_T - \left(\frac{a_1(E_1 + I_1)}{N_T} + b + k\right)S_1\right)\left(1 - \frac{\overline{S}_1}{S_1}\right)$$
$$+ \left(\frac{a_1 S_1(E_1 + I_1)}{N_T} - (b + c + d + k)E_1\right) + (cE_1 - (b + e + f + k)I_1)$$
$$+ (eI_1 - (b + g + k)Q_1) + (dE_1 + fI_1 + gQ_1 - (b + k)R_1)$$
$$+ \left(kS_1 - \left(\frac{a_2(E_2 + I_2)}{N_T} + b + k\right)S_2\right)\left(1 - \frac{\overline{S}_2}{S_2}\right)$$
$$+ \left(kE_1 + \frac{a_2 S_2(E_2 + I_2)}{N_T} - (b + c + d + k)E_2\right)$$
$$+ (kI_1 + cE_2 - (b + e + f + k)I_2) + (kQ_1 + eI_2 - (b + g + k)Q_2)$$
$$+ (kR_1 + dE_2 + fI_2 + gQ_2 - (b + k)R_2)$$
$$+ \left(kS_2 - \left(\frac{a_3(E_3 + I_3)}{N_T} + b\right)S_3\right)\left(1 - \frac{\overline{S}_3}{S_3}\right)$$

$$+\left(kE_2+\frac{a_3S_3(E_3+I_3)}{N_T}-(b+c+d)E_3\right)$$

$$+(kI_2+cE_3-(b+e+f)I_3)+(kQ_2+eI_3-(b+g)Q_3)$$

$$+(kR_2+dE_3+fI_3+gQ_3-bR_3)$$

$$=bN_T\left(1-\frac{\overline{S}_1}{S_1}\right)+(b+k)\overline{S}_1-bS_1-k\frac{S_1}{S_2}\overline{S}_2+(b+k)\overline{S}_2-bS_2$$

$$-k\frac{S_2}{S_3}\overline{S}_3+b\overline{S}_3-bS_3+(E_1+I_1)\left(\frac{a_1\overline{S}_1}{N_T}-b\right)$$

$$+(E_2+I_2)\left(\frac{a_2\overline{S}_2}{N_T}-b\right)+(E_3+I_3)\left(\frac{a_3\overline{S}_3}{N_T}-b\right)$$

$$-bQ_1-bR_1-bQ_2-bR_2-bQ_3-bR_3. \tag{21}$$

By using conditions in (20), we obtain

$$\frac{d}{dt}\rho(t)=bN_T\left(1-\frac{\overline{S}_1}{S_1}\right)+(b+k)\overline{S}_1\left(1-\frac{S_1}{\overline{S}_1}\right)+kS_1\left(1-\frac{\overline{S}_2}{S_2}\right)$$

$$+(b+k)\overline{S}_2\left(1-\frac{S_2}{\overline{S}_2}\right)+kS_2\left(1-\frac{\overline{S}_3}{S_3}\right)$$

$$+b\overline{S}_3\left(1-\frac{S_3}{\overline{S}_3}\right)-bQ_1-bR_1-bQ_2-bR_2-bQ_3-bR_3$$

$$=bN_T\left(1-\frac{\overline{S}_1}{S_1}\right)+(b+k)\overline{S}_1\left(1-\frac{S_1}{\overline{S}_1}\right)+k\overline{S}_1\left(\frac{S_1}{\overline{S}_1}-\frac{S_1}{\overline{S}_1}\frac{\overline{S}_2}{S_2}\right)$$

$$+(b+k)\overline{S}_2\left(1-\frac{S_2}{\overline{S}_2}\right)+k\overline{S}_2\left(\frac{S_2}{\overline{S}_2}-\frac{S_2}{\overline{S}_2}\frac{\overline{S}_3}{S_3}\right)$$

$$+b\overline{S}_3\left(1-\frac{S_3}{\overline{S}_3}\right)-bQ_1-bR_1-bQ_2-bR_2-bQ_3-bR_3. \tag{22}$$

Noting that on Ω_1, we have $\overline{S}_1=\frac{bN_T}{b+k}$, $\overline{S}_2=\frac{k}{b+k}\overline{S}_1$ and $\overline{S}_3=\frac{k}{b}\overline{S}_2$. The above equation becomes

$$\frac{d}{dt}\rho(t) = bN_T\left(1-\frac{\overline{S}_1}{S_1}\right) + bN_T\left(1-\frac{S_1}{\overline{S}_1}\right) + k\overline{S}_1\left(\frac{S_1}{\overline{S}_1} - \frac{S_1}{\overline{S}_1}\frac{\overline{S}_2}{S_2}\right) + k\overline{S}_1\left(1-\frac{S_2}{\overline{S}_2}\right)$$

$$+ k\overline{S}_2\left(\frac{S_2}{\overline{S}_2} - \frac{S_2}{\overline{S}_2}\frac{\overline{S}_3}{S_3}\right) + k\overline{S}_2\left(1-\frac{S_3}{\overline{S}_3}\right)$$

$$- bQ_1 - bR_1 - bQ_2 - bR_2 - bQ_3 - bR_3,$$

$$\frac{d}{dt}\rho(t) = bN_T\left(2-\frac{\overline{S}_1}{S_1}-\frac{S_1}{\overline{S}_1}\right) + k\overline{S}_1\left(\frac{S_1}{\overline{S}_1} - \frac{S_1}{\overline{S}_1}\frac{\overline{S}_2}{S_2} + 1 - \frac{S_2}{\overline{S}_2}\right)$$

$$+ k\overline{S}_2\left(\frac{S_2}{\overline{S}_2} - \frac{S_2}{\overline{S}_2}\frac{\overline{S}_3}{S_3} + 1 - \frac{S_3}{\overline{S}_3}\right) - bQ_1 - bR_1 - bQ_2 - bR_2$$

$$- bQ_3 - bR_3,$$

$$\frac{d}{dt}\rho(t) = -bN_T\frac{(\overline{S}_1 - S_1)^2}{S_1\overline{S}_1} - k\overline{S}_1\left(\left(1-\frac{\overline{S}_2}{S_2}\right)\left(\frac{S_2}{\overline{S}_2} - \frac{S_1}{\overline{S}_1}\right)\right)$$

$$- k\overline{S}_2\left(\left(1-\frac{\overline{S}_3}{S_3}\right)\left(\frac{S_3}{\overline{S}_3} - \frac{S_2}{\overline{S}_2}\right)\right) - bQ_1 - bR_1 - bQ_2 - bR_2$$

$$- bQ_3 - bR_3. \tag{23}$$

If $\frac{S_2}{\overline{S}_2} \geq \frac{S_1}{\overline{S}_1}$ for all $S_1 \geq \overline{S}_1$ and $\frac{S_2}{\overline{S}_2} \leq \frac{S_1}{\overline{S}_1}$ for all $0 < S_1 \leq \overline{S}_1$, then $-k\overline{S}_1\left(\left(1-\frac{\overline{S}_2}{S_2}\right)\left(\frac{S_2}{\overline{S}_2} - \frac{S_1}{\overline{S}_1}\right)\right) \leq 0.$

If $\frac{S_3}{\overline{S}_3} \geq \frac{S_2}{\overline{S}_2}$ for all $S_2 \geq \overline{S}_2$ and $\frac{S_3}{\overline{S}_3} \leq \frac{S_2}{\overline{S}_2}$ for all $0 < S_2 \leq \overline{S}_2$, then $-k\overline{S}_2\left(\left(1-\frac{\overline{S}_3}{S_3}\right)\left(\frac{S_3}{\overline{S}_3} - \frac{S_2}{\overline{S}_2}\right)\right) \leq 0.$

We can see that all of the terms in (23) are always non-positive. From equation (23), $\frac{d}{dt}\rho(t) \leq 0$, then the function $\frac{d}{dt}\rho(t)$ is negative definite. The limit set of each solution is contained in the largest invariant set for which $S_1 = \overline{S}_1$, $S_2 = \overline{S}_2$, $S_3 = \overline{S}_3$, $Q_1 = 0$, $R_1 = 0$, $Q_2 = 0$, $R_2 = 0$, $Q_3 = 0$ and $R_3 = 0$ which is singleton $\{P_1\}$. LaSalle's invariant principle [ref] implies that the disease free equilibrium P_1 is globally asymptotically stable on Ω_1.

To prove the global stability of the third age group endemic equilibrium point P_2, we consider the following theorem.

Theorem 2. *If* $\hat{R}_0 > 1$, $\widetilde{R}_0 < 1$ *and* $R_0^* < 1$, *then the third age group endemic equilibrium state* $P_2(\hat{S}_1, \hat{E}_1, \hat{I}_1, \hat{Q}_1, \hat{R}_1, \hat{S}_2, \hat{E}_2, \hat{I}_2, \hat{Q}_2, \hat{R}_2, \hat{S}_3, \hat{E}_3, \hat{I}_3, \hat{Q}_3, \hat{R}_3) \in \Omega_1$ *exists and is globally asymptotically stable on* Ω_1 *if*

$$\begin{cases} a_1 = \dfrac{bN_T}{\hat{S}_1} \\ a_2 = \dfrac{bN_T}{\hat{S}_2} \\ a_3 = \dfrac{(b+d)N_T}{\hat{S}_3} \\ d = e + f. \end{cases} \tag{24}$$

Proof. The Lyapunov function of the form

$$\psi(t) = (S_1 - \hat{S}_1 \ln S_1) + E_1 + I_1 + Q_1 + R_1 + (S_2 - \hat{S}_2 \ln S_2) + E_2 + I_2 + Q_2 + R_2 + (S_3 - \hat{S}_3 \ln S_3) + E_3 + I_3 + (Q_3 - \hat{Q}_3 \ln Q_3),$$

Its derivative along the trajectories of (1)-(15),

$$\frac{d}{dt}\psi(t) = \frac{d}{dt}S_1\left(1 - \frac{\hat{S}_1}{S_1}\right) + \frac{d}{dt}E_1 + \frac{d}{dt}I_1 + \frac{d}{dt}Q_1 + \frac{d}{dt}R_1$$
$$+ \frac{d}{dt}S_2\left(1 - \frac{\hat{S}_2}{S_2}\right) + \frac{d}{dt}E_2 + \frac{d}{dt}I_2 + \frac{d}{dt}Q_2 + \frac{d}{dt}R_2$$
$$+ \frac{d}{dt}S_3\left(1 - \frac{\hat{S}_3}{S_3}\right) + \frac{d}{dt}E_3 + \frac{d}{dt}I_3 + \frac{d}{dt}Q_3\left(1 - \frac{\hat{Q}_3}{Q_3}\right)$$

$$= \left(bN_T - \left(\frac{a_1(E_1+I_1)}{N_T} + b + k\right)S_1\right)\left(1 - \frac{\hat{S}_1}{S_1}\right)$$

$$+\left(\frac{a_1S_1(E_1+I_1)}{N_T} - (b+c+d+k)E_1\right) + (cE_1 - (b+e+f+k)I_1)$$

$$+ (eI_1 - (b+g+k)Q_1) + (dE_1 + fI_1 + gQ_1 - (b+k)R_1)$$

$$+\left(kS_1 - \left(\frac{a_2(E_2+I_2)}{N_T} + b + k\right)S_2\right)\left(1 - \frac{\hat{S}_2}{S_2}\right)$$

$$+\left(kE_1 + \frac{a_2S_2(E_2+I_2)}{N_T} - (b+c+d+k)E_2\right)$$

$$+ (kI_1 + cE_2 - (b+e+f+k)I_2)$$

$$+ (kQ_1 + eI_2 - (b+g+k)Q_2)$$

$$+ (kR_1 + dE_2 + fI_2 + gQ_2 - (b+k)R_2)$$

$$+\left(kS_2 - \left(\frac{a_3(E_3+I_3)}{N_T} + b\right)S_3\right)\left(1 - \frac{\hat{S}_3}{S_3}\right)$$

$$+\left(kE_2 + \frac{a_3S_3(E_3+I_3)}{N_T} - (b+c+d)E_3\right)$$

$$+ (kI_2 + cE_3 - (b+e+f)I_3)$$

$$+ (kQ_2 + eI_3 - (b+g)Q_3)\left(1 - \frac{\hat{Q}_3}{Q_3}\right)$$

$$= bN_T\left(1 - \frac{\hat{S}_1}{S_1}\right) + (b+k)\hat{S}_1 - bS_1 - k\frac{S_1}{S_2}\hat{S}_2 + (b+k)\hat{S}_2 - bS_2$$

$$- k\frac{S_2}{S_3}\hat{S}_3 + b\hat{S}_3 - bS_3 + (E_1+I_1)\left(\frac{a_1\hat{S}_1}{N_T} - b\right)$$

$$+ (E_2+I_2)\left(\frac{a_2\hat{S}_2}{N_T} - b\right) + \left(\frac{a_3\hat{S}_3}{N_T} - b - d\right)E_3$$

$$+\left(\frac{a_3\hat{S}_3}{N_T} - b - e - f\right)I_3$$

$$- bQ_1 - bR_1 - \left(b + k\frac{\hat{Q}_3}{Q_3}\right)Q_2 - (b+k)R_2$$

$$+ eI_3\left(1 - \frac{\hat{Q}_3}{Q_3}\right) + (b+g)\hat{Q}_3\left(1 - \frac{Q_3}{\hat{Q}_3}\right). \tag{25}$$

Substituting four conditions of (24) into (25), we have

$$\frac{d}{dt}\psi(t) = bN_T\left(1 - \frac{\hat{S}_1}{S_1}\right) + (b+k)\hat{S}_1 - bS_1 - k\frac{S_1}{S_2}\hat{S}_2 + (b+k)\hat{S}_2 - bS_2$$

$$- k\frac{S_2}{S_3}\hat{S}_3 + b\hat{S}_3 - bS_3 - bQ_1 - bR_1 - \left(b + k\frac{\hat{Q}_3}{Q_3}\right)Q_2$$

$$- (b+k)R_2$$

$$+ eI_3\left(1 - \frac{\hat{Q}_3}{Q_3}\right) + (b+g)\hat{Q}_3\left(1 - \frac{Q_3}{\hat{Q}_3}\right)$$

$$= bN_T\left(1 - \frac{\hat{S}_1}{S_1}\right) + (b+k)\hat{S}_1\left(1 - \frac{S_1}{\hat{S}_1}\right) + k\hat{S}_1\left(\frac{S_1}{\hat{S}_1} - \frac{S_1}{\hat{S}_1}\frac{\hat{S}_2}{S_2}\right)$$

$$+ (b+k)\hat{S}_2\left(1 - \frac{S_2}{\hat{S}_2}\right) + k\frac{\hat{S}_2}{\hat{R}_0}\left(\frac{S_2}{\hat{S}_2/\hat{R}_0} - \frac{S_2}{\hat{S}_2/\hat{R}_0}\frac{\hat{S}_3}{S_3}\right)$$

$$+ b\hat{S}_3\left(1 - \frac{S_3}{\hat{S}_3}\right)$$

$$- bQ_1 - bR_1 - \left(b + k\frac{\hat{Q}_3}{Q_3}\right)Q_2 - (b+k)R_2$$

$$+ e\hat{I}_3\left(\frac{I_3}{\hat{I}_3} - \frac{I_3}{\hat{I}_3}\frac{\hat{Q}_3}{Q_3}\right) + (b+g)\hat{Q}_3\left(1 - \frac{Q_3}{\hat{Q}_3}\right). \tag{26}$$

Next, using the endemic relations in the third age group endemic equilibrium state, we have $\hat{S}_1 = \dfrac{bN_T}{b+k}$, $\hat{S}_2 = \dfrac{k}{b+k}\hat{S}_1$, $b\hat{S}_3 = \dfrac{k\hat{S}_2}{\hat{R}_0}$ and $e\hat{I}_3 = (b+g)\hat{Q}_3$, equation (26) becomes

$$\frac{d}{dt}\psi(t) = bN_T\left(1-\frac{\hat{S}_1}{S_1}\right) + bN_T\left(1-\frac{S_1}{\hat{S}_1}\right) + k\hat{S}_1\left(\frac{S_1}{\hat{S}_1}-\frac{S_1}{\hat{S}_1}\frac{\hat{S}_2}{S_2}\right) + k\hat{S}_1\left(1-\frac{S_2}{\hat{S}_2}\right)$$
$$+ k\hat{S}_3\left(\frac{S_2}{\hat{S}_2/\hat{R}_0}-\frac{S_2}{\hat{S}_2/\hat{R}_0}\frac{\hat{S}_3}{S_3}\right) + b\hat{S}_3\left(1-\frac{S_3}{\hat{S}_3}\right) - bQ_1 - bR_1$$
$$-\left(b+k\frac{\hat{Q}_3}{Q_3}\right)Q_2 - (b+k)R_2 + e\hat{I}_3\left(\frac{I_3}{\hat{I}_3}-\frac{I_3}{\hat{I}_3}\frac{\hat{Q}_3}{Q_3}\right) + e\hat{I}_3\left(1-\frac{Q_3}{\hat{Q}_3}\right),$$

$$\frac{d}{dt}\psi(t) = bN_T\left(2-\frac{\hat{S}_1}{S_1}-\frac{S_1}{\hat{S}_1}\right) + k\hat{S}_1\left(\frac{S_1}{\hat{S}_1}-\frac{S_1}{\hat{S}_1}\frac{\hat{S}_2}{S_2}+1-\frac{S_2}{\hat{S}_2}\right)$$
$$+ b\hat{S}_3\left(\frac{S_2}{\hat{S}_2/\hat{R}_0}-\frac{S_2}{\hat{S}_2/\hat{R}_0}\frac{\hat{S}_3}{S_3}+1-\frac{S_3}{\hat{S}_3}\right) - bQ_1 - bR_1 - \left(b+k\frac{\hat{Q}_3}{Q_3}\right)Q_2$$
$$-(b+k)R_2 + e\hat{I}_3\left(\frac{I_3}{\hat{I}_3}-\frac{I_3}{\hat{I}_3}\frac{\hat{Q}_3}{Q_3}+1-\frac{Q_3}{\hat{Q}_3}\right),$$

$$\frac{d}{dt}\psi(t) = -bN_T\frac{(\hat{S}_1-S_1)^2}{S_1\hat{S}_1} - k\hat{S}_1\left(\left(1-\frac{\hat{S}_2}{S_2}\right)\left(\frac{S_2}{\hat{S}_2}-\frac{S_1}{\hat{S}_1}\right)\right)$$
$$- b\hat{S}_3\left(\left(1-\frac{\hat{S}_3}{S_3}\right)\left(\frac{S_3}{\hat{S}_3}-\frac{S_2}{\hat{S}_2/\hat{R}_0}\right)\right) - bQ_1 - bR_1 - \left(b+k\frac{\hat{Q}_3}{Q_3}\right)Q_2$$
$$-(b+k)R_2 - e\hat{I}_3\left(\left(1-\frac{\hat{Q}_3}{Q_3}\right)\left(\frac{Q_3}{\hat{Q}_3}-\frac{I_3}{\hat{I}_3}\right)\right). \tag{27}$$

Proceeding with the same manner as above for the second term of (27), for the third term, we see that, if $\dfrac{S_3}{\hat{S}_3} \ge \dfrac{S_2}{\hat{S}_2/\hat{R}_0}$ for all $S_2 \ge \hat{S}_2/\hat{R}_0$ and $\dfrac{S_3}{\hat{S}_3} \le \dfrac{S_2}{\hat{S}_2/\hat{R}_0}$ for all $0 < S_2 \le \hat{S}_2/\hat{R}_0$, therefore,

$$-b\hat{S}_3\left(\left(1-\frac{\hat{S}_3}{S_3}\right)\left(\frac{S_3}{\hat{S}_3}-\frac{S_2}{\hat{S}_2/\hat{R}_0}\right)\right) \le 0.$$

For the last term, if $\frac{Q_3}{\hat{Q}_3} \ge \frac{I_3}{\hat{I}_3}$ for all $I_3 \ge \hat{I}_3$ and $\frac{Q_3}{\hat{Q}_3} \le \frac{I_3}{\hat{I}_3}$ for all $0 < I_3 \le \hat{I}_3$ thus $-e\hat{I}_3\left(\left(1-\frac{\hat{Q}_3}{Q_3}\right)\left(\frac{Q_3}{\hat{Q}_3}-\frac{I_3}{\hat{I}_3}\right)\right) \le 0.$

Therefore, all of the terms in (27) are always non-positive and $\frac{d}{dt}\psi(t) \le 0$. The limit set of each solution is contained in the largest invariant set for which $S_1 = \hat{S}_1$, $S_2 = \hat{S}_2$, $S_3 = \hat{S}_3$, $Q_1 = 0$, $R_1 = 0$, $Q_2 = 0$, $R_2 = 0$ and $Q_3 = \hat{Q}_3$ which is singleton $\{P_2\}$. Hence, by LaSalle's invariant principle, the third age group endemic equilibrium P_2 is globally asymptotically stable on Ω_1.

Next, we consider the global stability of the second and third age group endemic equilibrium point P_3.

Theorem 3. *If $\widetilde{R}_0 > 1$ and $R_0^* < 1$, then the second and third age group endemic equilibrium state $P_3(\widetilde{S}_1, \widetilde{E}_1, \widetilde{I}_1, \widetilde{Q}_1, \widetilde{R}_1, \widetilde{S}_2, \widetilde{E}_2, \widetilde{I}_2, \widetilde{Q}_2, \widetilde{R}_2, \widetilde{S}_3, \widetilde{E}_3, \widetilde{I}_3, \widetilde{Q}_3, \widetilde{R}_3) \in \Omega_1$ exists and is globally asymptotically stable on Ω_1 if*

$$\begin{cases} a_1 = \dfrac{bN_T}{\widetilde{S}_1} \\ a_2 = \dfrac{(b+d)N_T}{\widetilde{S}_2} \\ a_3 = \dfrac{(b+d)N_T}{\widetilde{S}_3} \\ d = e + f. \end{cases} \tag{28}$$

Proof. The Lyapunov function of the form

$$\eta(t) = (S_1 - \widetilde{S}_1 \ln S_1) + E_1 + I_1 + Q_1 + R_1 + (S_2 - \widetilde{S}_2 \ln S_2) + E_2 + I_2$$
$$+ (Q_2 - \widetilde{Q}_2 \ln Q_2) + (S_3 - \widetilde{S}_3 \ln S_3) + E_3 + I_3 + (Q_3 - \widetilde{Q}_3 \ln Q_3),$$

$$\frac{d}{dt}\eta(t) = \frac{d}{dt}S_1\left(1-\frac{\widetilde{S}_1}{S_1}\right) + \frac{d}{dt}E_1 + \frac{d}{dt}I_1 + \frac{d}{dt}Q_1 + \frac{d}{dt}R_1$$

$$+\frac{d}{dt}S_2\left(1-\frac{\widetilde{S}_2}{S_2}\right) + \frac{d}{dt}E_2 + \frac{d}{dt}I_2 + \frac{d}{dt}Q_2\left(1-\frac{\widetilde{Q}_2}{Q_2}\right)$$

$$+\frac{d}{dt}S_3\left(1-\frac{\widetilde{S}_3}{S_3}\right) + \frac{d}{dt}E_3 + \frac{d}{dt}I_3 + \frac{d}{dt}Q_3\left(1-\frac{\widetilde{Q}_3}{Q_3}\right)$$

$$= \left(bN_T - \left(\frac{a_1(E_1+I_1)}{N_T} + b + k\right)S_1\right)\left(1-\frac{\widetilde{S}_1}{S_1}\right)$$

$$+\left(\frac{a_1S_1(E_1+I_1)}{N_T} - (b+c+d+k)E_1\right)$$

$$+(cE_1 - (b+e+f+k)I_1)$$

$$+(eI_1 - (b+g+k)Q_1) + (dE_1 + fI_1 + gQ_1 - (b+k)R_1)$$

$$+\left(kS_1 - \left(\frac{a_2(E_2+I_2)}{N_T} + b + k\right)S_2\right)\left(1-\frac{\widetilde{S}_2}{S_2}\right)$$

$$+\left(kE_1 + \frac{a_2S_2(E_2+I_2)}{N_T} - (b+c+d+k)E_2\right)$$

$$+(kI_1 + cE_2 - (b+e+f+k)I_2)$$

$$+(kQ_1 + eI_2 - (b+g+k)Q_2)\left(1-\frac{\widetilde{Q}_2}{Q_2}\right)$$

$$+\left(kS_2 - \left(\frac{a_3(E_3+I_3)}{N_T} + b\right)S_3\right)\left(1-\frac{\widetilde{S}_3}{S_3}\right)$$

$$+\left(kE_2 + \frac{a_3S_3(E_3+I_3)}{N_T} - (b+c+d)E_3\right)$$

$$+(kI_2 + cE_3 - (b+e+f)I_3)$$

$$+(kQ_2 + eI_3 - (b+g)Q_3)\left(1-\frac{\widetilde{Q}_3}{Q_3}\right)$$

$$= bN_T\left(1-\frac{\widetilde{S}_1}{S_1}\right)+(b+k)\widetilde{S}_1 - bS_1 - k\frac{S_1}{S_2}\widetilde{S}_2 + (b+k)\widetilde{S}_2 - bS_2$$

$$- k\frac{S_2}{S_3}\widetilde{S}_3 + b\widetilde{S}_3 - bS_3 + (E_1+I_1)\left(\frac{a_1\widetilde{S}_1}{N_T} - b\right) - \left(b + k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1$$

$$- (b+k)R_1 + E_2\left(\frac{a_2\widetilde{S}_2}{N_T} - b - d\right) + I_2\left(\frac{a_2\widetilde{S}_2}{N_T} - b - e - f\right)$$

$$+ (b+g+k)\widetilde{Q}_2\left(1-\frac{Q_2}{\widetilde{Q}_2}\right) + e\widetilde{I}_2\left(1-\frac{\widetilde{Q}_2}{Q_2}\right) + \left(\frac{a_3\widetilde{S}_3}{N_T} - b - d\right)E_3$$

$$+ \left(\frac{a_3\widetilde{S}_3}{N_T} - b - e - f\right)I_3 + (kQ_2 + eI_3)\left(1-\frac{\widetilde{Q}_3}{Q_3}\right)$$

$$+ (b+g)\widetilde{Q}_3\left(1-\frac{Q_3}{\widetilde{Q}_3}\right). \tag{29}$$

Substituting four conditions of (28) into (29), we have

$$\frac{d}{dt}\eta(t) = bN_T\left(1-\frac{\widetilde{S}_1}{S_1}\right)+(b+k)\widetilde{S}_1 - bS_1 - k\frac{S_1}{S_2}\widetilde{S}_2 + (b+k)\widetilde{S}_2 - bS_2$$

$$- k\frac{S_2}{S_3}\widetilde{S}_3 + b\widetilde{S}_3 - bS_3 - \left(b + k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1 - (b+k)R_1$$

$$+ (b+g+k)\widetilde{Q}_2\left(1-\frac{Q_2}{\widetilde{Q}_2}\right) + eI_2\left(1-\frac{\widetilde{Q}_2}{Q_2}\right)$$

$$+ (kQ_2 + eI_3)\left(1-\frac{\widetilde{Q}_3}{Q_3}\right) + (b+g)\widetilde{Q}_3\left(1-\frac{Q_3}{\widetilde{Q}_3}\right)$$

$$= bN_T\left(1-\frac{\widetilde{S}_1}{S_1}\right) + (b+k)\widetilde{S}_1\left(1-\frac{S_1}{\widetilde{S}_1}\right) + k\frac{\widetilde{S}_1}{\widetilde{R}_0}\left(\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0} - \frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}\frac{\widetilde{S}_2}{S_2}\right)$$

$$+ (b+k)\widetilde{S}_2\left(1-\frac{S_2}{\widetilde{S}_2}\right) + k\widetilde{S}_2\left(\frac{S_2}{\widetilde{S}_2} - \frac{S_2}{\widetilde{S}_2}\frac{\widetilde{S}_3}{S_3}\right)$$

$$+ \left(b\widetilde{S}_3 + \frac{a_3}{N_T}(\widetilde{E}_3 + \widetilde{I}_3)\widetilde{S}_3\right)\left(1-\frac{S_3}{\widetilde{S}_3}\right) - \frac{a_3}{N_T}(\widetilde{E}_3 + \widetilde{I}_3)\widetilde{S}_3\left(1-\frac{S_3}{\widetilde{S}_3}\right)$$

$$-\left(b+k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1-(b+k)R_1+(b+g+k)\widetilde{Q}_2\left(1-\frac{Q_2}{\widetilde{Q}_2}\right)$$

$$+e\widetilde{I}_2\left(\frac{I_2}{\widetilde{I}_2}-\frac{I_2}{\widetilde{I}_2}\frac{\widetilde{Q}_2}{Q_2}\right)+(k\widetilde{Q}_2+e\widetilde{I}_3)\left(\frac{kQ_2+eI_3}{k\widetilde{Q}_2+e\widetilde{I}_3}-\frac{kQ_2+eI_3}{k\widetilde{Q}_2+e\widetilde{I}_3}\frac{\widetilde{Q}_3}{Q_3}\right)$$

$$+(b+g)\widetilde{Q}_3\left(1-\frac{Q_3}{\widetilde{Q}_3}\right). \qquad (30)$$

Next, using the endemic relations in the second and third age group endemic equilibrium state, we have $\widetilde{S}_1=\frac{bN_T}{b+k}$, $(b+k)\widetilde{S}_2=\frac{k\widetilde{S}_1}{\widetilde{R}_0}$, $b\widetilde{S}_3+\frac{a_3}{N_T}(\widetilde{E}_3+\widetilde{I}_3)\widetilde{S}_3=k\widetilde{S}_2$, $e\widetilde{I}_2=(b+g+k)\widetilde{Q}_2$ and $k\widetilde{Q}_2+e\widetilde{I}_3=(b+g)\widetilde{Q}_3$, equation (30) becomes

$$\frac{d}{dt}\eta(t)=bN_T\left(1-\frac{\widetilde{S}_1}{S_1}\right)+bN_T\left(1-\frac{S_1}{\widetilde{S}_1}\right)+(b+k)\widetilde{S}_2\left(\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}-\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}\frac{\widetilde{S}_2}{S_2}\right)$$

$$+(b+k)\widetilde{S}_2\left(1-\frac{S_2}{\widetilde{S}_2}\right)+k\widetilde{S}_2\left(\frac{S_2}{\widetilde{S}_2}-\frac{S_2}{\widetilde{S}_2}\frac{\widetilde{S}_3}{S_3}\right)+k\widetilde{S}_2\left(1-\frac{S_3}{\widetilde{S}_3}\right)$$

$$-(k\widetilde{S}_2-b\widetilde{S}_3)\left(1-\frac{S_3}{\widetilde{S}_3}\right)-\left(b+k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1-(b+k)R_1$$

$$+e\widetilde{I}_2\left(1-\frac{Q_2}{\widetilde{Q}_2}\right)+e\widetilde{I}_2\left(\frac{I_2}{\widetilde{I}_2}-\frac{I_2}{\widetilde{I}_2}\frac{\widetilde{Q}_2}{Q_2}\right)$$

$$+(b+g)\widetilde{Q}_3\left(\frac{kQ_2+eI_3}{k\widetilde{Q}_2+e\widetilde{I}_3}-\frac{kQ_2+eI_3}{k\widetilde{Q}_2+e\widetilde{I}_3}\frac{\widetilde{Q}_3}{Q_3}\right)+(b+g)\widetilde{Q}_3\left(1-\frac{Q_3}{\widetilde{Q}_3}\right),$$

$$\frac{d}{dt}\eta(t)=bN_T\left(2-\frac{\widetilde{S}_1}{S_1}-\frac{S_1}{\widetilde{S}_1}\right)+(b+k)\widetilde{S}_2\left(\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}-\frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}\frac{\widetilde{S}_2}{S_2}+1-\frac{S_2}{\widetilde{S}_2}\right)$$

$$+k\widetilde{S}_2\left(\frac{S_2}{\widetilde{S}_2}-\frac{S_2}{\widetilde{S}_2}\frac{\widetilde{S}_3}{S_3}+1-\frac{S_3}{\widetilde{S}_3}\right)-b\left(\frac{k\widetilde{S}_2}{b}-\widetilde{S}_3\right)\left(1-\frac{S_3}{\widetilde{S}_3}\right)$$

$$-\left(b+k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1-(b+k)R_1+e\widetilde{I}_2\left(\frac{I_2}{\widetilde{I}_2}-\frac{I_2}{\widetilde{I}_2}\frac{\widetilde{Q}_2}{Q_2}+1-\frac{Q_2}{\widetilde{Q}_2}\right)$$

$$+ (b+g)\widetilde{Q}_3\left(\frac{kQ_2 + eI_3}{k\widetilde{Q}_2 + e\widetilde{I}_3} - \frac{kQ_2 + eI_3}{k\widetilde{Q}_2 + e\widetilde{I}_3}\frac{\widetilde{Q}_3}{Q_3} + 1 - \frac{Q_3}{\widetilde{Q}_3}\right),$$

$$\frac{d}{dt}\eta(t) = -bN_T\frac{(\widetilde{S}_1 - S_1)^2}{S_1\widetilde{S}_1} - (b+k)\widetilde{S}_2\left(\left(1 - \frac{\widetilde{S}_2}{S_2}\right)\left(\frac{S_2}{\widetilde{S}_2} - \frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}\right)\right)$$

$$- k\widetilde{S}_2\left(\left(1 - \frac{\widetilde{S}_3}{S_3}\right)\left(\frac{S_3}{\widetilde{S}_3} - \frac{S_2}{\widetilde{S}_2}\right)\right) - b\left(\frac{k\widetilde{S}_2}{b} - \widetilde{S}_3\right)\left(1 - \frac{S_3}{\widetilde{S}_3}\right)$$

$$- \left(b + k\frac{\widetilde{Q}_2}{Q_2}\right)Q_1 - (b+k)R_1 - e\widetilde{I}_2\left(\left(1 - \frac{\widetilde{Q}_2}{Q_2}\right)\left(\frac{Q_2}{\widetilde{Q}_2} - \frac{I_2}{\widetilde{I}_2}\right)\right)$$

$$- (b+g)\widetilde{Q}_3\left(\left(1 - \frac{\widetilde{Q}_3}{Q_3}\right)\left(\frac{Q_3}{\widetilde{Q}_3} - \frac{kQ_2 + eI_3}{k\widetilde{Q}_2 + e\widetilde{I}_3}\right)\right). \quad (31)$$

Proceeding with the same manner as Theorem 1 for the third term of (31), for the second term, if $\frac{S_2}{\widetilde{S}_2} \geq \frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}$ for all $S_1 \geq \widetilde{S}_1/\widetilde{R}_0$ and $\frac{S_2}{\widetilde{S}_2} \leq \frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}$ for all $0 < S_1 \leq \widetilde{S}_1/\widetilde{R}_0$, therefore

$$-(b+k)\widetilde{S}_2\left(\left(1 - \frac{\widetilde{S}_2}{S_2}\right)\left(\frac{S_2}{\widetilde{S}_2} - \frac{S_1}{\widetilde{S}_1/\widetilde{R}_0}\right)\right) \leq 0.$$

For the forth term, if $\frac{k\widetilde{S}_2}{b} \leq \widetilde{S}_3$ for all $S_3 \geq \widetilde{S}_3$ and $\frac{k\widetilde{S}_2}{b} \geq \widetilde{S}_3$ for all $0 < S_3 \leq \widetilde{S}_3$, then $-b\left(\frac{k\widetilde{S}_2}{b} - \widetilde{S}_3\right)\left(1 - \frac{S_3}{\widetilde{S}_3}\right) \leq 0$.

For the seventh term, if $\frac{Q_2}{\widetilde{Q}_2} \geq \frac{I_2}{\widetilde{I}_2}$ for all $I_2 \geq \widetilde{I}_2$ and $\frac{Q_2}{\widetilde{Q}_2} \leq \frac{I_2}{\widetilde{I}_2}$ for all $0 < I_2 \leq \widetilde{I}_2$, then $-e\widetilde{I}_2\left(\left(1 - \frac{\widetilde{Q}_2}{Q_2}\right)\left(\frac{Q_2}{\widetilde{Q}_2} - \frac{I_2}{\widetilde{I}_2}\right)\right) \leq 0$.

For the last term, if $\frac{Q_3}{\widetilde{Q}_3} \geq \frac{kQ_2 + eI_3}{k\widetilde{Q}_2 + e\widetilde{I}_3}$ for all $Q_2 \geq \widetilde{Q}_2$ and $I_3 \geq \widetilde{I}_3$,

$\frac{Q_3}{\widetilde{Q}_3} \leq \frac{kQ_2 + eI_3}{k\widetilde{Q}_2 + e\widetilde{I}_3}$ for all $0 < Q_2 \leq \widetilde{Q}_2$ and $0 < I_3 \leq \widetilde{I}_3$, then

$$-(b+g)\widetilde{Q}_3\left(\left(1 - \frac{\widetilde{Q}_3}{Q_3}\right)\left(\frac{Q_3}{\widetilde{Q}_3} - \frac{kQ_2 + eI_3}{k\widetilde{Q}_2 + e\widetilde{I}_3}\right)\right) \leq 0.$$

Therefore, all of terms in (31) are always non-positive and $\frac{d}{dt}\eta(t) \leq 0$. The limit set of each solution is contained in the largest invariant set for which $S_1 = \widetilde{S}_1$, $S_2 = \widetilde{S}_2$, $S_3 = \widetilde{S}_3$, $Q_1 = 0$, $R_1 = 0$, $Q_2 = \widetilde{Q}_2$, and $Q_3 = \widetilde{Q}_3$ which is singleton $\{P_3\}$. Hence, by LaSalle's invariant principle, the second and third age group endemic equilibrium P_3 is globally asymptotically stable on Ω_1.

Last, we consider the global stability of the full endemic equilibrium point P_4.

Theorem 4. *If $R_0^* > 1$, then the full endemic equilibrium state*

$P_4(S_1^*, E_1^*, I_1^*, Q_1^*, R_1^*, S_2^*, E_2^*, I_2^*, Q_2^*, R_2^*, S_3^*, E_3^*, I_3^*, Q_3^*, R_3^*) \in \Omega_1$

exists and is globally asymptotically stable on Ω_1 if

$$\begin{cases} a_1 = \dfrac{(b+d)N_T}{S_1^*} \\ a_2 = \dfrac{(b+d)N_T}{S_2^*} \\ a_3 = \dfrac{(b+d)N_T}{S_3^*} \\ d = e + f. \end{cases} \tag{32}$$

Proof. The Lyapunov function of the form

$$\begin{aligned}\kappa(t) = {} & (S_1 - S_1^* \ln S_1) + E_1 + I_1 + (Q_1 - Q_1^* \ln Q_1) + (S_2 - S_2^* \ln S_2) + E_2 \\ & + I_2 + (Q_2 - Q_2^* \ln Q_2) + (S_3 - S_3^* \ln S_3) + E_3 + I_3 \\ & + (Q_3 - Q_3^* \ln Q_3),\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\kappa(t) &= \frac{d}{dt}S_1\left(1-\frac{S_1^*}{S_1}\right)+\frac{d}{dt}E_1+\frac{d}{dt}I_1+\frac{d}{dt}Q_1\left(1-\frac{Q_1^*}{Q_1}\right)\\
&\quad+\frac{d}{dt}S_2\left(1-\frac{S_2^*}{S_2}\right)+\frac{d}{dt}E_2+\frac{d}{dt}I_2+\frac{d}{dt}Q_2\left(1-\frac{Q_2^*}{Q_2}\right)\\
&\quad+\frac{d}{dt}S_3\left(1-\frac{S_3^*}{S_3}\right)+\frac{d}{dt}E_3+\frac{d}{dt}I_3+\frac{d}{dt}Q_3\left(1-\frac{Q_3^*}{Q_3}\right)\\
&=\left(bN_T-\left(\frac{a_1(E_1+I_1)}{N_T}+b+k\right)S_1\right)\left(1-\frac{S_1^*}{S_1}\right)\\
&\quad+\left(\frac{a_1S_1(E_1+I_1)}{N_T}-(b+c+d+k)E_1\right)\\
&\quad+(cE_1-(b+e+f+k)I_1)+(eI_1-(b+g+k)Q_1)\left(1-\frac{Q_1^*}{Q_1}\right)\\
&\quad+\left(kS_1-\left(\frac{a_2(E_2+I_2)}{N_T}+b+k\right)S_2\right)\left(1-\frac{S_2^*}{S_2}\right)\\
&\quad+\left(kE_1+\frac{a_2S_2(E_2+I_2)}{N_T}-(b+c+d+k)E_2\right)\\
&\quad+(kI_1+cE_2-(b+e+f+k)I_2)\\
&\quad+(kQ_1+eI_2-(b+g+k)Q_2)\left(1-\frac{Q_2^*}{Q_2}\right)\\
&\quad+\left(kS_2-\left(\frac{a_3(E_3+I_3)}{N_T}+b\right)S_3\right)\left(1-\frac{S_3^*}{S_3}\right)\\
&\quad+\left(kE_2+\frac{a_3S_3(E_3+I_3)}{N_T}-(b+c+d)E_3\right)\\
&\quad+(kI_2+cE_3-(b+e+f)I_3)\\
&\quad+(kQ_2+eI_3-(b+g)Q_3)\left(1-\frac{Q_3^*}{Q_3}\right)
\end{aligned}
$$

$$= bN_T\left(1-\frac{S_1^*}{S_1}\right)+(b+k)S_1^*-bS_1-k\frac{S_1}{S_2}S_2^*+(b+k)S_2^*-bS_2$$

$$-k\frac{S_2}{S_3}S_3^*+bS_3^*-bS_3+\left(\frac{a_1S_1^*}{N_T}-b-d\right)E_1$$

$$+\left(\frac{a_1S_1^*}{N_T}-b-e-f\right)I_1$$

$$+eI_1\left(1-\frac{Q_1^*}{Q_1}\right)+(b+g+k)Q_1^*\left(1-\frac{Q_1}{Q_1^*}\right)$$

$$+\left(\frac{a_2S_2^*}{N_T}-b-d\right)E_2+\left(\frac{a_2S_2^*}{N_T}-b-e-f\right)I_2$$

$$+(kQ_1+eI_2)\left(1-\frac{Q_2^*}{Q_2}\right)+(b+g+k)Q_2^*\left(1-\frac{Q_2}{Q_2^*}\right)$$

$$+\left(\frac{a_3S_3^*}{N_T}-b-d\right)E_3+\left(\frac{a_3S_3^*}{N_T}-b-e-f\right)I_3$$

$$+(kQ_3+eI_3)\left(1-\frac{Q_3^*}{Q_3}\right)+(b+g)Q_3^*\left(1-\frac{Q_3}{Q_3^*}\right). \tag{33}$$

Substituting four conditions of (32) into (33), we have

$$\frac{d}{dt}\kappa(t)=bN_T\left(1-\frac{S_1^*}{S_1}\right)+(b+k)S_1^*-bS_1-k\frac{S_1}{S_2}S_2^*+(b+k)S_2^*-bS_2$$

$$-k\frac{S_2}{S_3}S_3^*+bS_3^*-bS_3+eI_1\left(1-\frac{Q_1^*}{Q_1}\right)+(b+g+k)Q_1^*\left(1-\frac{Q_1}{Q_1^*}\right)$$

$$+(kQ_1+eI_2)\left(1-\frac{Q_2^*}{Q_2}\right)+(b+g+k)Q_2^*\left(1-\frac{Q_2}{Q_2^*}\right)$$

$$+(kQ_2+eI_3)\left(1-\frac{Q_3^*}{Q_3}\right)+(b+g)Q_3^*\left(1-\frac{Q_3}{Q_3^*}\right)$$

$$= bN_T\left(1 - \frac{S_1^*}{S_1}\right) + (b+k)R_0^* S_1^*\left(\frac{1}{R_0^*} - \frac{S_1}{R_0^* S_1^*}\right)$$

$$+ kS_1^*\left(\frac{S_1}{S_1^*} - \frac{S_1}{S_1^*}\frac{S_2^*}{S_2}\right) + \left((b+k)S_2^* + \frac{a_2}{N_T}(E_2^* + I_2^*)S_2^*\right)\left(1 - \frac{S_2}{S_2^*}\right)$$

$$- \frac{a_2}{N_T}(E_2^* + I_2^*)S_2^*\left(1 - \frac{S_2}{S_2^*}\right) + kS_2^*\left(\frac{S_2}{S_2^*} - \frac{S_2}{S_2^*}\frac{S_3^*}{S_3}\right)$$

$$+ \left(bS_3^* + \frac{a_3}{N_T}(E_3^* + I_3^*)S_3^*\right)\left(1 - \frac{S_3}{S_3^*}\right) - \frac{a_3}{N_T}(E_3^* + I_3^*)S_3^*\left(1 - \frac{S_3}{S_3^*}\right)$$

$$+ eI_1^*\left(\frac{I_1}{I_1^*} - \frac{I_1}{I_1^*}\frac{Q_1^*}{Q_1}\right) + (b+g+k)Q_1^*\left(1 - \frac{Q_1}{Q_1^*}\right)$$

$$+ (kQ_1^* + eI_2^*)\left(\frac{kQ_2 + eI_2}{kQ_1^* + eI_2^*} - \frac{kQ_1 + eI_2}{kQ_1^* + eI_2^*}\frac{Q_2^*}{Q_2}\right)$$

$$+ (b+g+k)Q_2^*\left(1 - \frac{Q_2}{Q_2^*}\right)$$

$$+ (kQ_2^* + eI_3^*)\left(\frac{kQ_2 + eI_3}{kQ_2^* + eI_3^*} - \frac{kQ_2 + eI_3}{kQ_3^* + eI_3^*}\frac{Q_3^*}{Q_3}\right)$$

$$+ (b+g)Q_3^*\left(1 - \frac{Q_3}{Q_3^*}\right). \qquad (34)$$

Next, using the endemic relations in the full endemic equilibrium state, we have $bN_T = (b+k)R_0^* S_1^*$, $(b+k)S_2^* + \frac{a_2}{N_T}(E_2^* + I_2^*)S_2^* = kS_1^*$, $bS_3^* + \frac{a_3}{N_T}(E_3^* + I_3^*)S_3^* = kS_2^*$, $eI_1^* = (b+g+k)Q_1^*$, $kQ_1^* + eI_2^* = (b+g+k)Q_2^*$ and $kQ_2^* + eI_3^* = (b+g)Q_3^*$, equation (34) becomes

$$\frac{d}{dt}\kappa(t) = bN_T\left(1-\frac{S_1^*}{S_1}\right) + bN_T\left(\frac{1}{R_0^*}-\frac{S_1}{R_0^*S_1^*}\right) + kS_1^*\left(\frac{S_1}{S_1^*}-\frac{S_1}{S_1^*}\frac{S_2^*}{S_2}\right)$$

$$+ kS_1^*\left(1-\frac{S_2}{S_2^*}\right) - (kS_1^*-(b+k)S_2^*)\left(1-\frac{S_2}{S_2^*}\right) + kS_2^*\left(\frac{S_2}{S_2^*}-\frac{S_2}{S_2^*}\frac{S_3^*}{S_3}\right)$$

$$+ kS_2^*\left(1-\frac{S_3}{S_3^*}\right) - (kS_2^*-bS_3^*)\left(1-\frac{S_3}{S_3^*}\right) + eI_1^*\left(\frac{I_1}{I_1^*}-\frac{I_1}{I_1^*}\frac{Q_1^*}{Q_1}\right)$$

$$+ eI_1^*\left(1-\frac{Q_1}{Q_1^*}\right) + (b+g+k)Q_2^*\left(\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}-\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}\frac{Q_2^*}{Q_2}\right)$$

$$+ (b+g+k)Q_2^*\left(1-\frac{Q_2}{Q_2^*}\right)$$

$$+ (b+g)Q_3^*\left(\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}-\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}\frac{Q_3^*}{Q_3}\right)$$

$$+ (b+g)Q_3^*\left(1-\frac{Q_3}{Q_3^*}\right),$$

$$\frac{d}{dt}\kappa(t) = bN_T\left(1-\frac{S_1^*}{S_1}+\frac{1}{R_0^*}-\frac{S_1}{R_0^*S_1^*}\right) + kS_1^*\left(\frac{S_1}{S_1^*}-\frac{S_1}{S_1^*}\frac{S_2^*}{S_2}+1-\frac{S_2}{S_2^*}\right)$$

$$-(b+k)\left(\frac{kS_1^*}{b+k}-S_2^*\right)\left(1-\frac{S_2}{S_2^*}\right) + kS_2^*\left(\frac{S_2}{S_2^*}-\frac{S_2}{S_2^*}\frac{S_3^*}{S_3}+1-\frac{S_3}{S_3^*}\right)$$

$$-b\left(\frac{kS_2^*}{b}-S_3^*\right)\left(1-\frac{S_3}{S_3^*}\right) + eI_1^*\left(\frac{I_1}{I_1^*}-\frac{I_1}{I_1^*}\frac{Q_1^*}{Q_1}+1-\frac{Q_1}{Q_1^*}\right)$$

$$+ (b+g+k)Q_2^*\left(\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}-\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}\frac{Q_2^*}{Q_2}+1-\frac{Q_2}{Q_2^*}\right)$$

$$+ (b+g)Q_3^*\left(\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}-\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}\frac{Q_3^*}{Q_3}+1-\frac{Q_3}{Q_3^*}\right),$$

$$\frac{d}{dt}\kappa(t) = -bN_T\left(1-\frac{S_1}{S_1^*}\right)\left(\frac{S_1^*}{S_1}-\frac{1}{R_0^*}\right) - kS_1^*\left(\left(1-\frac{S_2^*}{S_2}\right)\left(\frac{S_2}{S_2^*}-\frac{S_1}{S_1^*}\right)\right)$$

$$-(b+k)\left(\frac{kS_1^*}{b+k}-S_2^*\right)\left(1-\frac{S_2}{S_2^*}\right) - kS_2^*\left(\left(1-\frac{S_3^*}{S_3}\right)\left(\frac{S_3}{S_3^*}-\frac{S_2}{S_2^*}\right)\right)$$

$$-b\left(\frac{kS_2^*}{b}-S_3^*\right)\left(1-\frac{S_3}{S_3^*}\right) - eI_1^*\left(\left(1-\frac{Q_1^*}{Q_1}\right)\left(\frac{Q_1}{Q_1^*}-\frac{I_1}{I_1^*}\right)\right)$$

$$-(b+g+k)Q_2^*\left(\left(1-\frac{Q_2^*}{Q_2}\right)\left(\frac{Q_2}{Q_2^*}-\frac{kQ_1+eI_2}{kQ_1^*+eI_2^*}\right)\right)$$

$$-(b+g)Q_3^*\left(\left(1-\frac{Q_3^*}{Q_3}\right)\left(\frac{Q_3}{Q_3^*}-\frac{kQ_2+eI_3}{kQ_2^*+eI_3^*}\right)\right). \tag{35}$$

If $\frac{S_1^*}{S_1} \le \frac{1}{R_0^*}$ for all $S_1 \ge S_1^*$ and $\frac{S_1^*}{S_1} \ge \frac{1}{R_0^*}$ for all $0 < S_1 \le S_1^*$, therefore $-bN_T\left(1-\frac{S_1}{S_1^*}\right)\left(\frac{S_1^*}{S_1}-\frac{1}{R_0^*}\right) \le 0.$

If $\frac{kS_1^*}{b+k} \le S_2^*$ for all $S_2 \ge S_2^*$ and $\frac{kS_1^*}{b+k} \ge S_2^*$ for all $0 < S_2 \le S_2^*$, thus $-(b+k)\left(\frac{kS_1^*}{b+k}-S_2^*\right)\left(1-\frac{S_2}{S_2^*}\right) \le 0.$

Proceeding with the same manner as above for remaining terms. Therefore, all of terms in (35) are always non-positive and $\frac{d}{dt}\kappa(t) \le 0$. The limit set of each solution is contained in the largest invariant set for which $S_1 = S_1^*$, $S_2 = S_2^*$, $S_3 = S_3^*$, $Q_1 = Q_1^*$, $Q_2 = Q_2^*$, and $Q_3 = Q_3^*$ which is singleton $\{P_4\}$. Hence, by LaSalle's invariant principle, the second and third age group endemic equilibrium P_4 is globally asymptotically stable on Ω_1.

III. Conclusion

We define

$$\hat{R}_0 = \frac{a_3 k^2 (b+c+e+f)}{(b+k)^2 (b+c+d)(b+e+f)},$$

$$\widetilde{R}_0 = \frac{a_2 bk(b+c+e+f+k)}{(b+k)^2 (b+c+d+k)(b+e+f+k)}$$

and $R_0^* \dfrac{a_1 b(b+c+e+f+k)}{(b+k)(b+c+d+k)(b+e+f+k)}$ as the threshold parameters. These parameters are called the *basic reproductive number*. It represents the average number of secondary cases caused by an infectious individual in a totally susceptible population. It depends on the transmissibility, contact rates and the expected duration of infection. It can determine the disease can spread through a population whether or not and also associates the persistence of endemic levels.

The global stability of our model has been resolved by using Lyapunov functions. If $\hat{R}_0 < 1$, $\widetilde{R}_0 < 1$ and $R_0^* < 1$, then the disease free equilibrium state is globally asymptotically stable and the disease will die out. If $\hat{R}_0 > 1$, $\widetilde{R}_0 < 1$ and $R_0^* < 1$, then the third age group endemic equilibrium state is globally asymptotically stable and the disease is endemic only in the third age group. If $\widetilde{R}_0 > 1$ and $R_0^* < 1$, then the second and the third age group endemic equilibrium state is globally asymptotically stable and the disease is endemic in both the second and the third age group. If $R_0^* > 1$ then the full endemic equilibrium state is globally asymptotically stable and the disease is endemic in all age groups.

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The Age Structural Transmission Model of Swine Flu

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ABSTRACT

The influence of age structure in the human population in the susceptible-exposed-infected-quarantined-recovered (SEIQR) model for describing the transmission of Swine flu is studied. The human population is divided into three groups such as 1 – 10 years, 11 – 20 years, and more than 20 years, respectively. The transmission of the disease is supposed to be difference in the three classes. Four equilibrium states are found and the conditions for stability of these four equilibrium states are established.

1. INTRODUCTION

It is believed that swine flu was first detected in factory farms in Mexico. After early outbreaks in North America in April 2009, the swine flu virus was spread rapidly around the world. The total report of swine flu cases worldwide more than 213 countries was 622,482 by 27 November, 2009[1]. Updated data on swine flu deaths has reached a total of 16,931 deaths as of 21 March, 2010 [2]. Swine flu is an Emerging Infectious Disease (EID) because the swine flu virus has not circulated previously in human; the virus is entirely new [3]. Genetic analyses of this virus showed that its gene segments were similar to influenza viruses that were circulated among pigs.

Generally, the three types of influenza viruses that caused human flu are influenza A, influenza B and influenza C. Influenza A viruses also infect both pigs and birds, influenza C viruses infect pigs but do not infect birds [4]. Swine flu is known to be caused by influenza A subtypes H1N1, H1N2, H2N3, H3N1 and H3N2 [5]–[7]. These strains resulted from reassortment, a process through two or more influenza viruses can swap genetic information by infecting a single human or animal host. When reassortment occurs, the emergent virus will have some gene segments from each of the infecting parent viruses and may have different characteristics than either of the parental viruses [8]. Pigs can be infected with influenza strains that usually found in pigs, birds and humans. Therefore, pigs represent a host where influenza viruses from different species can swap genes, produce new and dangerous strains [9]. Swine flu was a reassortment of at least four strains of influenza A virus, strains originated from humans, birds, North America pigs and Eurasian pigs [8].

The pandemic swine flu spreads between humans when infected people cough or sneeze, then other people breathe in the virus or touch something with the virus on it and then touch their own faces [9]. The transmission from swine to human is believed to occur mainly in swine farms where farmers are in closed contact with infected pigs. Although strains of swine influenza are usually not be able to infect humans but this may occasionally happen [10]. According to the Centers for Disease Control and Prevention (CDC), in humans, the symptoms of swine flu are similar to those of influenza and of influenza-like illness in general. Symptoms include fever, cough, sore throat, muscle pain, headache, runny nose, chills and fatigue [11].

Seasonal influenza occurs every year and the viruses change in each year, but many people have some immunity to the circulating virus that helps limit infections. By contrast, the pandemic swine flu virus was a new virus when it emerged and most people had no or little immunity to it. However, the studies have shown that a signification percentage of people age 65 and older have some immunity against the pandemic virus. Most of deaths caused by the pandemic swine flu have occurred among younger people, including those who were otherwise healthy. Pregnant women, younger children and people of any age with certain chronic lung or other medical conditions appear to be at higher risk of more complicated or severe illness [12]. If a person becomes sick with swine flu, antiviral drugs can make the illness milder and make the patient feel better faster. For treatment, antiviral drugs work best if started soon after getting sick (within 2 days of symptoms). CDC recommends the use of Tamiflu (oseltamivir) or Relenza (zanamivir) for the treatment and prevention of infection with swine flu viruses. However, the majority of people infected with the virus make a full recovery without requiring medical attention or antiviral drugs [13].

There are many epidemic models to predict the spread of an emerging infectious disease through human population. Zhou and Ma [14] studied an SEIQJR model for SARS transmission and control in China. Iwami and Takeuchi [15] considered an avian–human influenza epidemic model based on SI–SIR model. Dumont and Chiroleu [16] formulated an SEIR– LSEI model to investigate the spread of the chikungunya disease.

In this paper, we modified the model of Jumpen et al. [17] by incorporating the age structure of the human population. The reported of Ministry of Public Health, Thailand has showed that the most swine flu cases in Thailand occur in children under the age of 10, 11–20 and the people above the age of 21, respectively. The purpose of this paper is to study the transmission of swine flu in the three age groups. In section 2, we introduce a mathematical model to describe the transmission of swine flu in the three age groups. In section 3, we analyze our model and give the local stability results of the equilibriums states. In the last section, we explain the biological meaning of the reproductive number for swine flu in each age group.

2. MATHEMATICAL MODEL

We formulate a mathematical model to study the transmission of swine flu by introducing age structure into the SEIQR model. The human population is divided into three age groups such as group of the people 1 – 10 years, 11 – 20 years, and more than 20 years, respectively. Each group is constant in size and is sub-divided into five classes, i.e., S, individuals susceptible to the disease; E, individuals who are latently infected (exposed); I, infectious individuals; Q, isolated or quarantined individuals; and R, individuals that have recovered and immune to the disease. The SEIQR model in an age structured population is described by the following system of differential equations:

$$\frac{dS_1}{dt} = \mu_H N_T - \frac{a_1 S_1 (E_1 + I_1)}{N_T} - (b+k)S_1 , \qquad (1.1)$$

$$\frac{dE_1}{dt} = \frac{a_1 S_1 (E_1 + I_1)}{N_T} - (b+c+d+k)E_1 , \qquad (1.2)$$

$$\frac{dI_1}{dt} = cE_1 - (b+e+f+k)I_1 , \qquad (1.3)$$

$$\frac{dQ_1}{dt} = eI_1 - (b+g+k)Q_1 , \qquad (1.4)$$

$$\frac{dR_1}{dt} = dE_1 + fI_1 + gQ_1 - (b+k)R_1 , \qquad (1.5)$$

$$\frac{dS_2}{dt} = kS_1 - \frac{a_2 S_2 (E_2 + I_2)}{N_T} - (b+k)S_2 , \qquad (1.6)$$

$$\frac{dE_2}{dt} = kE_1 + \frac{a_2 S_2 (E_2 + I_2)}{N_T} - (b+c+d+k)E_2 , \qquad (1.7)$$

$$\frac{dI_2}{dt} = kI_1 + cE_2 - (b+e+f+k)I_2 , \qquad (1.8)$$

$$\frac{dQ_2}{dt} = kQ_1 + eI_2 - (b+g+k)Q_2 , \qquad (1.9)$$

$$\frac{dR_2}{dt} = kR_1 + dE_2 + fI_2 + gQ_2 - (b+k)R_2 , \qquad (1.10)$$

$$\frac{dS_3}{dt} = kS_2 - \frac{a_3 S_3 (E_3 + I_3)}{N_T} - bS_3 , \qquad (1.11)$$

$$\frac{dE_3}{dt} = kE_2 + \frac{a_3 S_3 (E_3 + I_3)}{N_T} - (b+c+d)E_3 , \qquad (1.12)$$

$$\frac{dI_3}{dt} = kI_2 + cE_3 - (b+e+f)I_3 , \qquad (1.13)$$

$$\frac{dQ_3}{dt} = kQ_2 + eI_3 - (b+g)Q_3 , \qquad (1.14)$$

$$\frac{dR_3}{dt} = kR_2 + dE_3 + fI_3 + gQ_3 - bR_3 , \qquad (1.15)$$

with the conditions $N_T = N_{T_1} + N_{T_2} + N_{T_3}$,

$$N_{T_1} = S_1 + E_1 + I_1 + Q_1 + R_1 ,$$

$$N_{T_2} = S_2 + E_2 + I_2 + Q_2 + R_2 ,$$

and $N_{T_3} = S_3 + E_3 + I_3 + Q_3 + R_3$,

where subscripts 1, 2 and 3 denote the first age group, the second age group and the third age group, respectively.

The parameters are defined as follows:

- N_T is the total population,
- N_{T_1} is the total number of first age group,
- N_{T_2} is the total number of second age group,
- N_{T_3} is the total number of third age group,
- b is the natural birth rate,
- μ_H is the natural mortality rate,
- k is the rate at which the first age group pass into the second age group and also the second age group pass into the third age group,
- a_1 is equal to $\varepsilon_1 \delta_n$ in which ε_1 is the probability of catching the disease per contact to the infected/exposed person and δ_n is the average number of people contacted by each person per day,
- a_2 is equal to $\varepsilon_2 \beta_n$ in which ε_2 is the probability of catching the disease per contact to the infected/exposed person and β_n is the average number of people contacted by each person per day,
- a_3 is equal to $\varepsilon_3 \alpha_n$ in which ε_3 is the probability of catching the disease per contact to the infected/exposed person and α_n is the average number of people contacted by each person per day,
- c is the rate at which the exposed individuals E become the infected individuals I,
- e is the rate at which the individuals leave the infective individuals I for the quarantined individuals Q,
- d, f, g are the rate at which individuals in the E, I classes recover from the disease or die.

If we add equations (1.1) – (1.15), (1.1) – (1.5), (1.6) – (1.10), and (1.11) – (1.15) then

$$\frac{dN_T}{dt} = bN_T - \mu_H N_T , \qquad (2.1)$$

$$\frac{dN_{T_1}}{dt} = bN_T - (b+k)N_{T_1} , \qquad (2.2)$$

$$\frac{dN_{T_2}}{dt} = kN_{T_1} - (b+k)N_{T_2} , \qquad (2.3)$$

$$\frac{dN_{T_3}}{dt} = kN_{T_2} - bN_{T_3} . \qquad (2.4)$$

We assume that the total population, total number of the first age group, total number of the second age group, and total number of the third age group remain constant. Therefore $\frac{dN_T}{dt} = 0$ and $\frac{dN_{T_1}}{dt} = \frac{dN_{T_2}}{dt} = \frac{dN_{T_3}}{dt} = 0$. Setting the right hand side of (2.1), (2.2), (2.3) and (2.4) to be zero, we obtain the following four relations:

$b=\mu_H$ (the birth rate equals to the mortality rate), $\frac{N_{T_1}}{N_T}=\frac{b}{b+k}$ (the ratio between the total number of first age group and the total population), $\frac{N_{T_2}}{N_T}=\frac{bk}{(b+k)^2}$ (the ratio between the total number of second age group and the total population) and $\frac{N_{T_3}}{N_T}=\frac{k^2}{(b+k)^2}$ (the ratio between the total number of third age group and the total population).

To normalize equations (1.1) – (1.15), let $\bar{S}_1=\frac{S_1}{N_{T_1}}$, $\bar{E}_1=\frac{E_1}{N_{T_1}}$, $\bar{I}_1=\frac{I_1}{N_{T_1}}$, $\bar{Q}_1=\frac{Q_1}{N_{T_1}}$, $\bar{R}_1=\frac{R_1}{N_{T_1}}$, $\bar{S}_2=\frac{S_2}{N_{T_2}}$, $\bar{E}_2=\frac{E_2}{N_{T_2}}$, $\bar{I}_2=\frac{I_2}{N_{T_2}}$, $\bar{Q}_2=\frac{Q_2}{N_{T_2}}$, $\bar{R}_2=\frac{R_2}{N_{T_2}}$, $\bar{S}_3=\frac{S_3}{N_{T_3}}$, $\bar{E}_3=\frac{E_3}{N_{T_3}}$, $\bar{I}_3=\frac{I_3}{N_{T_3}}$, $\bar{Q}_3=\frac{Q_3}{N_{T_3}}$, $\bar{R}_3=\frac{R_3}{N_{T_3}}$. We see that the proportions give the new three conditions: $\bar{S}_1+\bar{E}_1+\bar{I}_1+\bar{Q}_1+\bar{R}_1=1$, $\bar{S}_2+\bar{E}_2+\bar{I}_2+\bar{Q}_2+\bar{R}_2=1$, and $\bar{S}_3+\bar{E}_3+\bar{I}_3+\bar{Q}_3+\bar{R}_3=1$.

Hence the equations (1.1) – (1.15) can be written as

$$\frac{d\bar{S}_1}{dt}=m_1(1-\bar{S}_1)-m_2\bar{S}_1(\bar{E}_1+\bar{I}_1), \quad (3.1)$$

$$\frac{d\bar{E}_1}{dt}=m_2\bar{S}_1(\bar{E}_1+\bar{I}_1)-m_3\bar{E}_1, \quad (3.2)$$

$$\frac{d\bar{I}_1}{dt}=c\bar{E}_1-m_4\bar{I}_1, \quad (3.3)$$

$$\frac{d\bar{Q}_1}{dt}=e\bar{I}_1-m_5\bar{Q}_1, \quad (3.4)$$

$$\frac{d\bar{S}_2}{dt}=m_1(\bar{S}_1-\bar{S}_2)-m_6\bar{S}_2(\bar{E}_2+\bar{I}_2), \quad (3.5)$$

$$\frac{d\bar{E}_2}{dt}=m_1\bar{E}_1+m_6\bar{S}_2(\bar{E}_2+\bar{I}_2)-m_3\bar{E}_2, \quad (3.6)$$

$$\frac{d\bar{I}_2}{dt}=m_1\bar{I}_1+c\bar{E}_2-m_4\bar{I}_2, \quad (3.7)$$

$$\frac{d\bar{Q}_2}{dt}=m_1\bar{Q}_1+e\bar{I}_2-m_5\bar{Q}_2, \quad (3.8)$$

$$\frac{d\bar{S}_3}{dt}=b(\bar{S}_2-\bar{S}_3)-m_7\bar{S}_3(\bar{E}_3+\bar{I}_3), \quad (3.9)$$

$$\frac{d\bar{E}_3}{dt}=b\bar{E}_2+m_7\bar{S}_3(\bar{E}_3+\bar{I}_3)-m_8\bar{E}_3, \quad (3.10)$$

$$\frac{d\bar{I}_3}{dt}=b\bar{I}_2+c\bar{E}_3-m_9\bar{I}_3, \quad (3.11)$$

$$\frac{d\bar{Q}_3}{dt}=b\bar{Q}_2+e\bar{I}_3-m_{10}\bar{Q}_3, \quad (3.12)$$

where $m_1=b+k$, $m_2=\frac{a_1 b}{b+k}$, $m_3=b+c+d+k$, $m_4=b+e+f+k$, $m_5=b+g+k$, $m_6=\frac{a_2 bk}{(b+k)^2}$, $m_7=\frac{a_3 k^2}{(b+k)^2}$, $m_8=b+c+d$, $m_9=b+e+f$, and $m_{10}=b+g$.

3. ANALYSIS OF THE MATHEMATICAL MODEL

3.1. Equilibrium points

The equilibrium points are obtained by setting the right hand side of (3.1)-(3.12) equal to zero. We get four equilibrium points:

3.1.1. The disease free equilibrium point

$$P_0=(1,0,0,0,1,0,0,0,1,0,0,0)$$

3.1.2 The third age group endemic equilibrium point

$$\hat{P}=(1,0,0,0,1,0,0,0,\hat{S}_3,\hat{E}_3,\hat{I}_3,\hat{Q}_3)$$

where

$$\hat{S}_3=\frac{1}{\hat{R}_0},\ \hat{E}_3=\frac{b}{m_8\hat{R}_0}\left[\hat{R}_0-1\right],\ \hat{I}_3=\frac{bc}{m_8 m_9\hat{R}_0}\left[\hat{R}_0-1\right],$$

$$\hat{Q}_3=\frac{bce}{m_8 m_9 m_{10}\hat{R}_0}\left[\hat{R}_0-1\right], \text{ with } \hat{R}_0=\frac{m_7(c+m_9)}{m_8 m_9}.$$

3.1.3 The second and third age group endemic equilibrium point

$$\tilde{P}=(1,0,0,0,\tilde{S}_2,\tilde{E}_2,\tilde{I}_2,\tilde{Q}_2,\tilde{S}_3,\tilde{E}_3,\tilde{I}_3,\tilde{Q}_3)$$

where

$$\tilde{S}_2=\frac{1}{\tilde{R}_0},\ \tilde{E}_2=\frac{m_1}{m_3\tilde{R}_0}\left[\tilde{R}_0-1\right],\ \tilde{I}_2=\frac{m_1 c}{m_3 m_4\hat{R}_0}\left[\tilde{R}_0-1\right],$$

$$\tilde{Q}_2=\frac{m_1 ce}{m_3 m_4 m_5\tilde{R}_0}\left[\tilde{R}_0-1\right],\ \tilde{S}_3=\frac{b\tilde{S}_2}{m_7(\tilde{E}_3+\tilde{I}_3)+b},$$

$$\tilde{E}_3=\frac{-A_2+\sqrt{A_2^2-4A_1A_3}}{2A_1},\ \tilde{I}_3=\frac{b\tilde{I}_2+c\tilde{E}_3}{m_9},$$

$$\tilde{Q}_3=\frac{b\tilde{Q}_2+e\tilde{I}_3}{m_{10}}, \text{ with } \tilde{R}_0=\frac{m_6(c+m_4)}{m_3 m_4},$$

$A_1=m_7 m_8(c+m_9)$,

$A_2=bm_8m_9+bm_7\left(m_8\tilde{I}_2-(c+m_9)\tilde{S}_2-(c+m_9)\tilde{E}_2\right)$,

and $A_3=-[b^2m_7\tilde{I}_2(\tilde{S}_2+\tilde{E}_2)+b^2m_9\tilde{E}_2]$.

3.1.4 The full endemic equilibrium point

$$P^*=(S_1^*,E_1^*,I_1^*,Q_1^*,S_2^*,E_2^*,I_2^*,Q_2^*,S_3^*,E_3^*,I_3^*,Q_3^*)$$

where

$$S_1^*=\frac{1}{R_0^*},\ E_1^*=\frac{m_1}{m_3R_0^*}\left[R_0^*-1\right],\ I_1^*=\frac{m_1 c}{m_3 m_4 R_0^*}\left[R_0^*-1\right],$$

$$Q_1^*=\frac{m_1 ce}{m_3 m_4 m_5 R_0^*}\left[R_0^*-1\right],\ S_2^*=\frac{m_1 S_1^*}{m_6(E_2^*+I_2^*)+m_1},$$

$$E_2^*=\frac{-B_2+\sqrt{B_2^2-4B_1B_3}}{2B_1},\ I_2^*=\frac{m_1 I_1^*+cE_2^*}{m_4},$$

$$Q_2^*=\frac{m_1 Q_1^*+eI_2^*}{m_5},\ S_3^*=\frac{bS_2^*}{m_7(E_3^*+I_3^*)+b},$$

$$E_3^* = \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_1},\ I_3^* = \frac{bI_2^* + cE_3^*}{m_9},$$

$$Q_3^* = \frac{bQ_2^* + eI_3^*}{m_{10}},\ \text{with}\ R_0^* = \frac{m_2(c+m_4)}{m_3m_4},$$

$$B_1 = m_3m_6(c+m_4),$$

$$B_2 = m_1m_3m_4 + m_1m_6\left(m_3I_1^* - (c+m_4)S_1^* - (c+m_4)E_1^*\right),$$

$$B_3 = -\left[m_1^2m_6I_1^*(S_1^* + E_1^*) + m_1^2m_4E_1^*\right],$$

$$C_1 = m_7m_8(c+m_9),$$

$$C_2 = bm_8m_9 + bm_7\left(m_8I_2^* - (c+m_9)S_2^* - (c+m_9)E_2^*\right),$$

and $C_3 = -\left[b^2m_7I_2^*(S_2^* + E_2^*) + b^2m_9E_2^*\right]$.

3.2. Local stability

The local stability of the equilibrium solutions is determined from the Jacobian matrix of the RHS of the above set of differential equations evaluated at the equilibrium solutions. The eigenvalues are obtained by solving the characteristic equations; $\det(J(p) - \lambda I_{12}) = 0$ where $J(p)$ is the Jacobian matrix at equilibrium point p, I_{12} is the identity matrix dimension 12×12. If all eigenvalues have negative real parts, then the equilibrium solution is locally stable.

For the disease free equilibrium point $P_0 = (1,0,0,0,1,0,0,0,1,0,0,0)$, we obtain the characteristic equation

$$(\lambda+b)(\lambda+m_1)^2(\lambda+m_5)^2(\lambda+m_{10})\left[\lambda^2+r_1\lambda+r_2\right]$$
$$\left[\lambda^2+s_1\lambda+s_2\right]\left[\lambda^2+t_1\lambda+t_2\right]=0, \qquad (4.1)$$

where

$$r_1 = \frac{(1-R_0^*)m_3m_4 + cm_3}{c+m_4} + m_4,\ r_2 = m_3m_4\left(1-R_0^*\right),$$

$$s_1 = \frac{(1-\widetilde{R}_0)m_3m_4 + cm_3}{c+m_4} + m_4,\ s_2 = m_3m_4\left(1-\widetilde{R}_0\right),$$

$$t_1 = \frac{(1-\hat{R}_0)m_8m_9 + cm_8}{c+m_9} + m_9,\ \text{and}\ t_2 = m_8m_9\left(1-\hat{R}_0\right).$$

From the characteristic equation (4.1), we see that the first six eigenvalues are $\lambda_1 = -b$, $\lambda_{2,3} = -m_1$, $\lambda_{4,5} = -m_5$ and $\lambda_6 = -m_{10}$, all of these are negative. The remaining eigenvalues are found by solving $\lambda^2+r_1\lambda+r_2 = 0, \lambda^2+s_1\lambda+s_2 = 0$ and $\lambda^2+t_1\lambda+t_2 = 0$. But for stability, we only need the sign of the eigenvalues to be negative. Thus, we apply the Routh–Hurwitz criteria to the equation in form $\lambda^2+A\lambda+B=0$, the stability holds iff $A>0$ and $B>0$.

For $\lambda^2+r_1\lambda+r_2=0$, $r_1>0$ and $r_2>0$, if $R_0^*<1$.

For $\lambda^2+s_1\lambda+s_2=0$, $s_1>0$ and $s_2>0$, if $\widetilde{R}_0<1$.

For $\lambda^2+t_1\lambda+t_2=0$, $t_1>0$ and $t_2>0$, if $\hat{R}_0<1$.

Thus all the roots of the three characteristic equations have negative real parts if $R_0^*<1$, $\widetilde{R}_0<1$ and $\hat{R}_0<1$, respectively.

For the third age group endemic equilibrium point $\hat{P} = (1,0,0,0,1,0,0,0,\hat{S}_3,\hat{E}_3,\hat{I}_3,\hat{Q}_3)$, we obtain the characteristic equation

$$(\lambda+m_1)^2(\lambda+m_5)^2(\lambda+m_{10})\left[\lambda^2+r_1\lambda+r_2\right]$$
$$\left[\lambda^2+s_1\lambda+s_2\right]\left[\lambda^3+u_1\lambda^2+u_2\lambda+u_3\right]=0, \qquad (4.2)$$

where

$$u_1 = b+m_9+\frac{cm_7}{m_9\hat{R}_0}+m_7(\hat{E}_3+\hat{I}_3),$$

$$u_2 = bm_9+\frac{bcm_7}{m_9\hat{R}_0}+m_7m_8(\hat{E}_3+\hat{I}_3)+m_7m_9(\hat{E}_3+\hat{I}_3),$$

$$u_3 = m_7m_8m_9(\hat{E}_3+\hat{I}_3),\ \text{with}\ \hat{E}_3 = \frac{b}{m_8\hat{R}_0}\left[\hat{R}_0-1\right]\ \text{and}$$

$$\hat{I}_3 = \frac{bc}{m_8m_9\hat{R}_0}\left[\hat{R}_0-1\right].$$

From the characteristic equation (4.2), we see that the first five eigenvalues are $\lambda_{1,2}=-m_1, \lambda_{3,4}=-m_5$ and $\lambda_5=-m_{10}$, all of these are negative. The remaining eigenvalues are found by solving $\lambda^2+r_1\lambda+r_2=0$, $\lambda^2+s_1\lambda+s_2=0$ and $\lambda^3+u_1\lambda^2+u_2\lambda+u_3=0$. For $\lambda^2+r_1\lambda+r_2=0$ and $\lambda^2+s_1\lambda+s_2=0$, all roots of these two characteristic equations have negative real parts if $R_0^*<1$ and $\widetilde{R}_0<1$, respectively. Thus, we apply the Routh–Hurwitz criteria to the equation in the form $\lambda^3+u_1\lambda^2+u_2\lambda+u_3=0$, the stability holds iff $u_1>0$, $u_3>0$ and $u_1u_2-u_3>0$. If $\hat{R}_0>1$, we have $u_1>0$, and $u_3>0$, and it can be easily seen that $u_1u_2-u_3>0$ as the term of $-u_3$ can be cleared with the product of the second term of u_1 and the third term of u_2. Hence the Routh–Hurwitz conditions are satisfied.

For the second and third group endemic equilibrium point $\widetilde{P} = (1,0,0,0,\widetilde{S}_2,\widetilde{E}_2,\widetilde{I}_2,\widetilde{Q}_2,\widetilde{S}_3,\widetilde{E}_3,\widetilde{I}_3,\widetilde{Q}_3)$, we obtain the characteristic equation

$$(\lambda+m_1)(\lambda+m_5)^2(\lambda+m_{10})\left[\lambda^2+r_1\lambda+r_2\right]$$
$$\left[\lambda^3+v_1\lambda^2+v_2\lambda+v_3\right]\left[\lambda^3+w_1\lambda^2+w_2\lambda+w_3\right]=0, \qquad (4.3)$$

where

$$v_1 = m_1+m_4+\frac{cm_6}{m_4\widetilde{R}_0}+m_6(\widetilde{E}_2+\widetilde{I}_2),$$

$$v_2 = m\ m_4+\frac{m_1cm_6}{m_4\widetilde{R}_0}+m_6m_3(\widetilde{E}_2+\widetilde{I}_2)+m_6m_4(\widetilde{E}_2+\widetilde{I}_2),$$

$$v_3 = m_6m_3m_4(\widetilde{E}_2+\widetilde{I}_2),$$

$$w_1 = b+m_8+m_9+m_7(\widetilde{E}_3+\widetilde{I}_3-\widetilde{S}_3)$$

$$w_2 = bm_8+bm_9+m_8m_9+m_7(m_8+m_9)(\widetilde{E}_3+\widetilde{I}_3)$$
$$-m_7\widetilde{S}_3(b+c+m_9),$$

$$w_3 = bm_8m_9+m_7m_8m_9(\widetilde{E}_3+\widetilde{I}_3)-bm_7(c+m_9)\widetilde{S}_3,$$

with $\widetilde{E}_2 = \frac{m_1}{m_3\widetilde{R}_0}\left[\widetilde{R}_0-1\right]$ and $\widetilde{I}_2 = \frac{m_1c}{m_3m_4\hat{R}_0}\left[\widetilde{R}_0-1\right]$.

From the characteristic equation (4.3), we see that the first four eigenvalues are $\lambda_1=-m_1, \lambda_{2,3}=-m_5$ and $\lambda_4=-m_{10}$, all of these are negative. The remaining eigenvalues are found by solving $\lambda^2+r_1\lambda+r_2=0$, $\lambda^3+v_1\lambda^2+v_2\lambda+v_3=0$ and $\lambda^3+w_1\lambda^2+w_2\lambda+w_3=0$. For

$\lambda^2 + r_1\lambda + r_2 = 0$, similarly, all roots of the characteristic equations have negative real parts if $R_0^* < 1$.

For $\lambda^3 + v_1\lambda^2 + v_2\lambda + v_3 = 0$, if $\tilde{R}_0 > 1$, we have $v_1 > 0$ and $v_3 > 0$, and it can be easily seen that $v_1 v_2 - v_3 > 0$ as the term of $-v_3$ can be cleared with the product of the second term of v_1 and the third term of v_2. Hence the Routh–Hurwitz conditions are satisfied.

For $\lambda^3 + w_1\lambda^2 + w_2\lambda + w_3 = 0$, since $\tilde{E}_3, \tilde{I}_3$ and $\tilde{S}_3$ are defined in terms of $\tilde{E}_2, \tilde{I}_2$ and $\tilde{S}_2$ as before, hence w_1, w_2 and w_3 are also in the forms of $\tilde{E}_2, \tilde{I}_2$ and $\tilde{S}_2$. It might be complicate to show by hand that equation $\lambda^3 + w_1\lambda^2 + w_2\lambda + w_3 = 0$ satisfy the Routh–Hurwitz conditions. We use *MATLAB* to show the conditions in the following figures by assigning various values of a_2 in which $\tilde{R}_0 > 1$ and the other parameters are fixed.

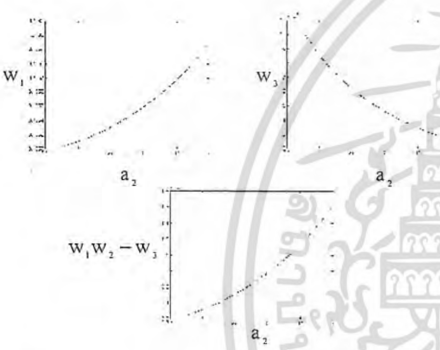

Figure 1. The parameter space for equation $\lambda^3 + w_1\lambda^2 + w_2\lambda + w_3 = 0$ which satisfies the Routh–Hurwitz conditions. The values of the other parameters are k = 0.000273973, b = 0.000039139, c = 0.111111, d = 0.142857, e = 0.2, f = 0.142857, g = 0.7 and a_3 = 1. From the above figure, the Routh–Hurwitz conditions are satisfied for $\tilde{R}_0 > 1$.

For the full endemic equilibrium point $P^* = (S_1^*, E_1^*, I_1^*, Q_1^*, S_2^*, E_2^*, I_2^*, Q_2^*, S_3^*, E_3^*, I_3^*, Q_3^*)$, we obtain the characteristic equation

$$(\lambda + m_5)^2(\lambda + m_{10})\left[\lambda^3 + x_1\lambda^2 + x_2\lambda + x_3\right]\left[\lambda^3 + y_1\lambda^2 + y_2\lambda + y_3\right]\left[\lambda^3 + z_1\lambda^2 + z_2\lambda + z_3\right] = 0, \quad (4.4)$$

where

$$x_1 = m_1 + m_4 + \frac{cm_2}{m_4 R_0^*} + m_2(E_1^* + I_1^*),$$

$$x_2 = m_1 m_4 + \frac{cm_1 m_2}{m_4 R_0^*} + m_2 m_3(E_1^* + I_1^*) + m_2 m_4(E_1^* + I_1^*),$$

$$x_3 = m_2 m_3 m_4 (E_1^* + I_1^*),$$

$$y_1 = m_1 + m_3 + m_4 + m_6(E_2^* + I_2^* - S_2^*),$$

$$y_2 = m_1 m_3 + m_1 m_4 + m_3 m_4 + m_6(m_3 + m_4)(E_2^* + I_2^*) - m_6 S_2^*(m_1 + c + m_4),$$

$$y_3 = m_1 m_3 m_4 + m_6 m_3 m_4 (E_2^* + I_2^*) - m_1 m_6 (c + m_4) S_2^*,$$

$$z_1 = b + m_8 + m_9 + m_7(E_3^* + I_3^* - S_3^*),$$

$$z_2 = bm_8 + bm_9 + m_8 m_9 + m_7(m_8 + m_9)(E_3^* + I_3^*) - m_7 S_3^*(b + c + m_9),$$

$$z_3 = bm_8 m_9 + m_7 m_8 m_9 (E_3^* + I_3^*) - bm_7(c + m_9) S_3^*,$$

with $E_1^* = \frac{m_1}{m_3 R_0^*}\left[R_0^* - 1\right]$ and $I_1^* = \frac{m_1 c}{m_3 m_4 R_0^*}\left[R_0^* - 1\right]$.

From the characteristic equation (4.4), we see that the first three eigenvalues are $\lambda_{1,2} = -m_5$ and $\lambda_3 = -m_{10}$ all of these are negative. The remaining eigenvalues are found by solving $\lambda^3 + x_1\lambda^2 + x_2\lambda + x_3 = 0$, $\lambda^3 + y_1\lambda^2 + y_2\lambda + y_3 = 0$ and $\lambda^3 + z_1\lambda^2 + z_2\lambda + z_3 = 0$.

For $\lambda^3 + x_1\lambda^2 + x_2\lambda + x_3 = 0$, if $R_0^* > 1$, we have $x_1 > 0$ and $x_3 > 0$, and it can be easily seen that $x_1 x_2 - x_3 > 0$ as the term of $-x_3$ can be cleared with the product of the second term of x_1 and the third term of x_2. Hence the Routh–Hurwitz conditions are satisfied.

For $\lambda^3 + y_1\lambda^2 + y_2\lambda + y_3 = 0$ and $\lambda^3 + z_1\lambda^2 + z_2\lambda + z_3 = 0$, since $E_2^*, I_2^*, S_2^*, E_3^*, I_3^*$ and S_3^* are defined in terms of E_1^*, I_1^* and S_1^* as before, hence y_1, y_2, y_3, z_1, z_2, and z_3 also are in the forms of E_1^*, I_1^* and S_1^*. Proceeding with the same manner as above, use *MATLAB* to graph the conditions of the Routh–Hurwitz, by assigning various values of a_1 in which $R_0^* > 1$ and the other parameters are fixed. The Routh–Hurwitz conditions are satisfied for $R_0^* > 1$.

4. DISCUSSION AND CONCLUSION

In this study, we get the four equilibrium points $P_0, \hat{P}, \tilde{P}$ and P^*. The first point is the disease free equilibrium $P_0 = (1,0,0,0,1,0,0,0,1,0,0,0)$ represents the state in which swine flu are not endemic in the human and this point is local stability for $R_0^* < 1$, $\tilde{R}_0 < 1$ and $\hat{R}_0 < 1$.

For $\hat{P} = (1,0,0,0,1,0,0,0,\hat{S}_3,\hat{E}_3,\hat{I}_3,\hat{Q}_3)$,the second point is the third age group endemic equilibrium which represents the state in which swine flu are endemic only in the third age group and this point is local stability for $R_0^* < 1$, $\tilde{R}_0 < 1$ and $\hat{R}_0 > 1$.

For $\tilde{P} = (1,0,0,0,\tilde{S}_2,\tilde{E}_2,\tilde{I}_2,\tilde{Q}_2,\tilde{S}_3,\tilde{E}_3,\tilde{I}_3,\tilde{Q}_3)$, the third point is the second and the third age group endemic equilibrium which represents the state in which swine flu are endemic in both the second age group and the third age group and this point is local stability for $R_0^* < 1$ and $\tilde{R}_0 > 1$.

The forth point is the full endemic equilibrium $P^* = (S_1^*, E_1^*, I_1^*, Q_1^*, S_2^*, E_2^*, I_2^*, Q_2^*, S_3^*, E_3^*, I_3^*, Q_3^*)$ which represents the state in which swine flu are endemic in all age groups and this point is local stability for $R_0^* > 1$.

The biological meaning of the basic reproductive number R_0^*, $\tilde{R}_0$ and $\hat{R}_0$ are explained as follows:

$R_n^* = \left(\frac{a_1 b}{b+k}\right)\left[\left(\frac{c}{b+c+d+k}\right)\left(\frac{1}{b+e+f+k}\right)+\frac{1}{b+c+d+k}\right]$ is the reproductive number for swine flu in the first age group, where a_1 is the transmission rate per day of the first age group, $\frac{b}{b+k}$ is the ratio between the total number of the first age group and the total population, $\frac{c}{b+c+d+k}$ is the fraction of exposed members who move to the infective class, $\frac{1}{b+e+f+k}$ is the average time that an infective individual remains in the class I, and $\frac{1}{b+c+d+k}$ is the average time that an exposed member remains in that class.

The term $\tilde{R}_0 = \frac{a_2 bk}{(b+k)^2}\left[\left(\frac{c}{b+c+d+k}\right)\left(\frac{1}{b+e+f+k}\right)+\frac{1}{b+c+d+k}\right]$ is the reproductive number for swine flu in the second age group, where a_2 is the transmission rate per day of the second age group, $\frac{bk}{(b+k)^2}$ is the ratio between the total number of second age group and the total population, the other terms are define as before.

The term $\hat{R}_0 = \frac{a_3 k^2}{(b+k)^2}\left[\left(\frac{c}{b+c+d}\right)\left(\frac{1}{b+e+f}\right)+\frac{1}{b+c+d}\right]$ is the reproductive number for swine flu in the third age group, where a_3 is the transmission rate per day of the third age group, $\frac{k^2}{(b+k)^2}$ is the ratio between the total number of third age group and the total population, the other terms are define in the same manner as before. We can see that the local stability of all equilibrium states are determined by the threshold number R_0^*, $\tilde{R}_0$ and $\hat{R}_0$. To reduce the transmission of this disease, we should control the above threshold numbers.

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SEIQR MODEL WITH ITS GLOBAL STABILITY

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Abstract

In this paper, we examine the global stability of SEIQR (susceptible-exposed-infected-quarantined-recovered) model. From analysis of SEIQR model, we obtain two equilibrium states, the disease free equilibrium state and endemic equilibrium state. Using Lyapunov functions with the conditions to show that if the basic reproductive number $R_0 \leq 1$, then the disease free equilibrium is globally asymptotically stable. If $R_0 > 1$, then there exists an unique endemic equilibrium which is asymptotically stable on the positive octant.

Keywords— *global stability, Lyapunov functions, SEIQR model*

I. INTRODUCTION

A SEIQR model is used for describing the transmission of many diseases. Gerberry and Milner extended an SIQR model to SEIQR model for Childhood diseases [1]. Jumpen et al. proposed a SEIQR model for pandemic influenza [2]. Changpuek and Pongsumpun modified SEIQR model for describing the transmission of swine flu [3]. In this paper, we analyze the SEIQR model and the human population is assumed to have constant size. The human population is separated into five classes, i.e., S, individuals susceptible to the disease; E, individuals who are latently infected (exposed); I, infectious individuals; Q, isolated or quarantined individuals; and R, individuals that have recovered and immune to the disease. A SEIQR model is described by the system of differential equations, from analysis of the fifth – dimensional SEIQR model, two equilibrium states are found. One is the disease free state and the another one is the endemic equilibrium state. There are different mathematical methods to study the global stability of the transmission models. The direct Lyapunov method is classically method used to establish global properties of the equilibrium states. However, this method requires a Lyapunov function which is function with specific properties and is not easy to find. Korobeinikov and Wake introduced Lyapunov function for classical SIR, SIRS and SIS epidemiological models [4]. Tewa et al. resolved the global dynamics of dengue disease transmission model through the use of Lyapunov functions and use LaSalle's extension to Lyapunov's method which is simplified method [5].

In this paper, we propose Lyapunov functions for SEIQR model which used for finding conditions to establish the global stability of the equilibrium states. These conditions are related with the basic reproductive number R_0, R_0 is defined as the number of secondary cases caused by a primary infectious individual in an entirely susceptible population. When R_0 is greater than one, the disease can spread to a totally susceptible population and the number of cases will increase, when R_0 is less than one, the disease will fail to spread. Therefore, in order to reduce the infection, it is necessary to cause the basic reproductive number below one and the smaller the basic reproductive number mean the more rapid the decrease of the disease.

II. FORMULATION OF THE MODEL AND GLOBAL STABILITY ANALYSIS

A. Parameter of the Model

Let N be the total population and formed the susceptible $(S(t))$, exposed $(E(t))$, infectious $(I(t))$, quarantined $(Q(t))$ and recovered $(R(t))$, where $N = S + E + I + Q + R$.

The other parameters are defined as follows: $K = \tau S_0$ is the constant recruitment in which the constant τ is the natural birth rate and S_0 is the initial value of susceptible individual , β is the natural mortality rate, α is equal to $\psi\phi$ in which ψ is the probability of catching the disease per each contact to the infected/exposed person and ϕ is the average number of people contacted by each person per day, γ is the rate at which the exposed individuals E become the infected individuals I, ε is the rate at which the individuals leave the infective individuals I to become the quarantined individuals Q. The parameters δ and ω are the rate at which the individuals leave the exposed individuals E and the quarantined individuals Q to become the recovered individuals R, respectively. The parameter λ is the rate at which the individuals in the I class recover from the disease.

B. Equations of the Model

The SEIQR model is described by the following system of differential equations:

$$S'(t) = K - \left(\frac{\alpha(E+I)}{N} + \beta \right) S , \quad (1)$$

$$E'(t) = \frac{\alpha S(E+I)}{N} - (\beta + \gamma + \delta)E , \quad (2)$$

$$I'(t) = \gamma E - (\beta + \varepsilon + \lambda)I , \quad (3)$$

$$Q'(t) = \varepsilon I - (\beta + \omega)Q , \quad (4)$$

$$R'(t) = \delta E + \lambda I + \omega Q - \beta R , \quad (5)$$

If we add (1) – (5) then $N'(t) = K - \beta N$. In this paper, we assume that the total population remain constant, therefore $N'(t) = 0$ and we obtain $N = \frac{K}{\beta}$.

C. The epidemic

We can see that all equations described by (1) – (5), the non-negative octant R_+^5 is positively invariant (where R_+^5 denotes the non-negative region). With respect to (1) – (5), we have the following results:

Proposition 1 *Let* $(S(t), E(t), I(t), Q(t), R(t))$ *be the solution of* (1) – (5) *with the initial condition* $(S(0), E(0), I(0), Q(0), R(0))$ *and the compact set* $\Omega_1 = \left\{ (S, E, I, Q, R) \in R_+^5, J \le N = \frac{K}{\beta} \right\}$

Then, under the flow described by (1) – (5), Ω_1 *is positively invariant set that attracts all solutions in* R_+^5.

Proof We choose the Lyapunov function

$$J(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$$

positive definite on R_+^5 and we have

$$\frac{dJ}{dt} = \frac{d}{dt}S + \frac{d}{dt}E + \frac{d}{dt}I + \frac{d}{dt}Q + \frac{d}{dt}R$$

$$= K - \beta(S + E + I + Q + R)$$

$$= K - \beta N.$$

We use the fact that $N = \frac{K}{\beta}$ and we obtain

$$\frac{dJ}{dt} = K - \beta J \leq 0, \quad \text{for } J \geq \frac{K}{\beta}. \tag{6}$$

From (6), we see that $\frac{dJ}{dt} \leq 0$ which implies that Ω_1 is a positively invariant set. By solving (6), we have $0 \leq J(t) \leq \frac{K}{\beta} + J(0)e^{-\beta t}$, where $J(0)$ is the initial condition of $J(t)$. Thus, as $t \to \infty$, $0 \leq J(t) \leq \frac{K}{\beta} = N$ and we can conclude that Ω_1 is an attractive set.

Proposition 2 *Equations* (1) – (5) *have two equilibrium states: for* $R_0 \leq 1$, *the only equilibrium state is the disease free equilibrium* $P_1(S^*, E^*, I^*, Q^*, R^*) = P_1\left(\frac{K}{\beta}, 0, 0, 0, 0\right) \in \Omega_1$. *For* $R_0 > 1$, *there is the endemic equilibrium state* $P_2(S^+, E^+, I^+, Q^+, R^+) \in \Omega_1$ *and it satisfies* $S^+, E^+, I^+, Q^+, R^+ > 0$, *where*

$$S^{+} = \frac{K}{\beta} R_0 \tag{7}$$

$$E^{+} = \frac{K}{\beta + \gamma + \delta}\left(1 - \frac{1}{R_0}\right) \tag{8}$$

$$I^{+} = \frac{K\gamma}{(\beta + \varepsilon + \lambda)(\beta + \gamma + \delta)}\left(1 - \frac{1}{R_0}\right) \tag{9}$$

$$Q^{+} = \frac{K\gamma\varepsilon}{(\beta + \omega)(\beta + \varepsilon + \lambda)(\beta + \gamma + \delta)}\left(1 - \frac{1}{R_0}\right) \tag{10}$$

$$R^{+} = \frac{K}{\beta(\beta + \gamma + \delta)}\left[\delta + \frac{\gamma\lambda}{\beta + \varepsilon + \lambda} + \frac{\gamma\varepsilon\omega}{(\beta + \omega)(\beta + \varepsilon + \lambda)}\right]\left(1 - \frac{1}{R_0}\right) \tag{11}$$

with $R_0 = \dfrac{\alpha(\beta + \gamma + \varepsilon + \lambda)}{(\beta + \varepsilon + \lambda)(\beta + \gamma + \delta)}$.

Proof. Let the right hand side of each differential equations (1) – (5) equal to zero. Then we obtain:

$$S^{+} = \frac{K}{\beta + \frac{\alpha}{N}\left(E^{+} + I^{+}\right)} \tag{12}$$

$$E^{+} = \frac{\frac{\alpha}{N} S^{+} I^{+}}{(\beta + \gamma + \delta) - \frac{\alpha}{N} S^{+}} \tag{13}$$

$$I^{+} = \frac{\gamma E^{+}}{\beta + \varepsilon + \lambda} \tag{14}$$

$$Q^{+} = \frac{\varepsilon I^{+}}{\beta + \omega} \tag{15}$$

$$R^{+} = \frac{\delta E^{+} + \lambda I^{+} + \omega Q^{+}}{\beta}. \tag{16}$$

Substituting equations (12) and (14) into equation (13), we obtain E^+ which is a solution of the equation $E^+(\chi_1 E^+ + \chi_2 - \chi_3 - \chi_4) = 0$,

where

$$\chi_1 = (\alpha/N)(\beta+\gamma+\delta)(\beta+\gamma+\varepsilon+\lambda),$$

$$\chi_2 = \beta(\beta+\gamma+\delta)(\beta+\varepsilon+\lambda),$$

$$\chi_3 = (\alpha/N)(\beta+\varepsilon+\lambda)K$$

and

$$\chi_4 = (\alpha/N)K\gamma.$$

We obtain two equilibrium states: for $R_0 \le 1$, the disease free equilibrium $E^* = E^+ = 0$, then we have $P_1(S^*, E^*, I^*, Q^*, R^*) = P_1\left(\frac{K}{\beta}, 0,0,0,0\right) \in \Omega_1$.

For $R_0 > 1$, there is the endemic equilibrium $E^+ = \frac{\chi_3 + \chi_4 - \chi_2}{\chi_1} = \frac{K}{\beta+\gamma+\delta}\left(1 - \frac{1}{R_0}\right)$, where $R_0 = \frac{\alpha(\beta+\gamma+\varepsilon+\lambda)}{(\beta+\varepsilon+\lambda)(\beta+\gamma+\delta)}$.

Substituting E^+ into equations (12), (14), (15) and (16), we obtain the endemic equilibrium state $P_2(S^+, E^+, I^+, Q^+, R^+) \in \Omega_1$, where S^+, E^+, I^+, Q^+, R^+ satisfy (7) – (11). This achieves proof.

D. Global stability of the equilibrium states

The global behavior of the equilibrium states for equations (1) – (5) is determined from Lyapunov techniques.

Theorem 1 *Assume that* $\alpha = \frac{\beta N}{S^*}$. *For* $R_0 \le 1$, *the disease free equilibrium point* P_1 *is globally asymptotically stable on* Ω_1.

Proof. Let us consider on Ω_1 the Lyapunov function

$$f(t)=\left(S-S^{*}\ln S\right)+E+I+Q+R$$

The derivative with respect to time yields

$$\frac{d}{dt}f(t)=\frac{d}{dt}S\left(1-\frac{S^{*}}{S}\right)+\frac{d}{dt}E+\frac{d}{dt}I+\frac{d}{dt}Q+\frac{d}{dt}R$$

$$=\left(K-\frac{\alpha(E+I)}{N}-\beta S\right)\left(1-\frac{S^{*}}{S}\right)+\left(\frac{\alpha S(E+I)}{N}-(\beta+\gamma+\delta)E\right)+\left(\gamma E-(\beta+\varepsilon+\lambda)I\right)$$

$$+\left(\varepsilon I-(\beta+\omega)Q\right)+\left(\delta E+\lambda I+\omega Q-\beta R\right)$$

$$=K\left(1-\frac{S^{*}}{S}\right)+\beta S^{*}-\beta S+\frac{\alpha(E+I)S^{*}}{N}-\beta E-\beta I-\beta Q-\beta R$$

$$=K\left(1-\frac{S^{*}}{S}\right)+\beta S^{*}\left(1-\frac{S}{S^{*}}\right)+(E+I)\left(\frac{\alpha S^{*}}{N}-\beta\right)-\beta Q-\beta R.$$

Note that on Ω_1, we have $S^{*}=\dfrac{K}{\beta}$ and from condition $\alpha=\dfrac{\beta N}{S^{*}}$. The above equation becomes

$$\frac{d}{dt}f(t)=K\left(2-\frac{S^{*}}{S}-\frac{S}{S^{*}}\right)-\beta Q-\beta R$$

$$=-K\frac{\left(S^{*}-S\right)^{2}}{SS^{*}}-\beta Q-\beta R. \qquad (17)$$

We can see that all of terms in (17) are always non – positive. From equation (17), $\frac{d}{dt}f(t)\le 0$, then the function $\frac{d}{dt}f(t)$ is negative definite. The limit set of each solution is contained in the largest invariant set for which $S=S^{*}$, $Q=0$ and $R=0$ which is singleton $\{P_1\}$. LaSalle's invariant principle implies that the disease free equilibrium P_1 is globally asymptotically stable on Ω_1.

To prove the global stability of the endemic equilibrium point P_2, we consider the following theorem.

Theorem 2 *If $R_0 > 1$, then the endemic equilibrium state $P_2(S^+, E^+, I^+, Q^+, R^+) \in \Omega_1$ exists and it is globally asymptotically stable on Ω_1 if*

$$\begin{cases} \alpha = \dfrac{(\beta+\delta)N}{S^+} \\ \delta = \varepsilon + \lambda \end{cases} \tag{18}$$

Proof. The Lyapunov function of the form

$$g(t) = (S - S^+ \ln S) + E + I + (Q - Q^+ \ln Q)$$

$$\frac{d}{dt}g(t) = \frac{d}{dt}S\left(1-\frac{S^+}{S}\right) + \frac{d}{dt}E + \frac{d}{dt}I + \frac{d}{dt}Q\left(1-\frac{Q^+}{Q}\right)$$

$$= \left(K - \frac{\alpha(E+I)}{N} - \beta S\right)\left(1-\frac{S^+}{S}\right) + \left(\frac{\alpha S(E+I)}{N} - (\beta+\gamma+\delta)E\right) + (\gamma E - (\beta+\varepsilon+\lambda)I)$$

$$+ (\varepsilon I - (\beta+\omega)Q)\left(1-\frac{Q^+}{Q}\right)$$

$$= K\left(1-\frac{S^+}{S}\right) + \beta S^+ - \beta S + \frac{\alpha(E+I)S^+}{N} - (\beta+\delta)E - (\beta+\lambda)I - \varepsilon\frac{I}{Q}Q^+$$

$$+ (\beta+\omega)Q^+ - (\beta+\omega)Q$$

$$= K\left(1-\frac{S^+}{S}\right) + \beta S^+\left(1-\frac{S}{S^+}\right) + E\left(\frac{\alpha S^+}{N} - \beta - \delta\right) + I\left(\frac{\alpha S^+}{N} - \beta - \varepsilon - \lambda\right)$$

$$+ \varepsilon I\left(1-\frac{Q^+}{Q}\right) + (\beta+\omega)Q^+\left(1-\frac{Q}{Q^+}\right). \tag{19}$$

Substituting two conditions of (18) into (19), we have

$$\frac{d}{dt}g(t)=K\left(1-\frac{S^+}{S}\right)+\beta S^+R_0\left(\frac{1}{R_0}-\frac{S}{R_0S^+}\right)+\varepsilon I^+\left(\frac{I}{I^+}-\frac{I}{I^+}\frac{Q^+}{Q}\right)+(\beta+\omega)Q^+\left(1-\frac{Q}{Q^+}\right). \tag{20}$$

Next, using the endemic relations in the endemic equilibrium state, we have $K=\beta R_0S^+$ and $\varepsilon I^+=(\beta+\omega)Q^+$, equation (20) becomes

$$\frac{d}{dt}g(t)=K\left(1-\frac{S^+}{S}+\frac{1}{R_0}-\frac{S}{R_0S^+}\right)+\varepsilon I^+\left(\frac{I}{I^+}-\frac{I}{I^+}\frac{Q^+}{Q}+1-\frac{Q}{Q^+}\right)$$

$$=-K\left(1-\frac{S}{S^+}\right)\left(\frac{S^+}{S}-\frac{1}{R_0}\right)-\varepsilon I^+\left(1-\frac{Q^+}{Q}\right)\left(\frac{Q}{Q^+}-\frac{I}{I^+}\right). \tag{21}$$

If $\frac{S^+}{S}\le\frac{1}{R_0}$ for all $S\ge S^+$ and $\frac{S^+}{S}\ge\frac{1}{R_0}$ for all $0<S\le S^+$, then $-K\left(1-\frac{S}{S^+}\right)\left(\frac{S^+}{S}-\frac{1}{R_0}\right)\le 0.$

If $\frac{Q}{Q^+}\ge\frac{I}{I^+}$ for all $I\ge I^+$ and $\frac{Q}{Q^+}\le\frac{I}{I^+}$ for all $0<I\le I^+$, then $-\varepsilon I^+\left(1-\frac{Q^+}{Q}\right)\left(\frac{Q}{Q^+}-\frac{I}{I^+}\right)\le 0$.

Therefore, all of terms in (21) are always non – positive and $\frac{d}{dt}g(t)\le 0$. The limit set of each solution is contained in the largest invariant set for which $S=S^+$ and $Q=Q^+$ which is singleton $\{P_2\}$. Hence, by LaSalle's invariant principle, the endemic equilibrium P_2 is globally asymptotically stable on Ω_1.

E. Conclusion

We define $R_0=\frac{\alpha(\beta+\gamma+\varepsilon+\lambda)}{(\beta+\varepsilon+\lambda)(\beta+\gamma+\delta)}$ as the threshold parameters and the quantity $R_0=\sqrt{R_0}$ is called the basic reproductive number of the disease. It represents the average number of secondary cases caused by an infectious individual in a totally susceptible population. It depends on the transmissibility, contact rates and the expected duration of infection. It can determine the disease can spread through a population whether or not and also associates the persistence of endemic levels.

The global stability of our model has been resolved by using Lyapunov functions. If $R_0 < 1$, then the disease free equilibrium state is globally asymptotically stable and the disease will die out. If $R_0 > 1$ then the endemic equilibrium state is globally asymptotically stable and the disease is endemic in population.

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Academic Publications

1. T. Changpuek, P. Pongsumpun and I-Ming Tang. 2013. "Analysis of Mathematical Model for Swine Flu Transmission by Age Group." *Far East Journal of Mathematical Sciences (FJMS).* 73(2) : 201 – 229.
2. T. Changpuek, P. Pongsumpun and I-Ming Tang. 2013. "Global Stability of the Age Structural Transmission Model for Swine Flu." *Far East Journal of Mathematical Sciences (FJMS).* 80(1) : 55 – 84.