SOLVING SYLVESTER-TYPE MATRIX EQUATIONS OVER GENERALIZED QUATERNIONS

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Thesis Title	Solving Sylvester-Type Matrix Equations over Generalized Quaternions
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Abstract

In this research, we investigate Sylvester-type matrix equations over a generalized quaternion skew-field. Indeed, we consider a generalized Sylvester matrix equation and a generalized Sylvester-transpose one. We utillize the real representation of generalized quaternions and vectorization techniques to transform matrices overgeneralized quaternions into linear system. Then, we derive an equivalent condition that characterizes the consistency of the matrix equation. Thus, we can investigate a solvability condition, the general exact/least-squares solutions, the imaginary part exact/least-squares solutions, the real exact/least-squares solutions, the minimal-norm exact/least-squares solution, and the exact/least-squares solution closest to a given matrix. This work encompasses the Sylvester equation, the Sylvester-transposemmatrix equation, the Stein equation, linear systems over a generalized quaternion, and quaternionic matrix equations.

Keywords : Sylvester-type matrix equation, matrix over a generalized quaternion, least-squares solution, minimal-norm solution, vector operator, Kronecker product

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Table of Contents

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King Mongkut's Institute of Technology Ladkrabang

This material is reserved for educational use only, not allowed for commercial use.
Forbidden to modify the content, and cite the document when use.

List of Tables

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List of Figures

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Chapter 1

Introduction

1.1 Research motivation

Linear matrix equations over the field $\mathbb{R}$ play a fundamental role in both theorectical and applied mathematics. They arise in a wide range of disciplines, such as differential equations, and control and system theory; see e.g. [1, 2, 3]. A well-known instance is the Sylvester matrix equation:

$$AX + XD = E \tag{1.1.1}$$

among the most popular and these equations play important roles in model reduction [4], numerical methods for differential equations [5, 6] and involved in linear algebra [7, 8, 9]. Besides, there are many researchers consider a generalized Sylvester equation:

$$AXB + CXD = E \tag{1.1.2}$$

see e.g. [10, 11, 12, 13, 14]. Morever, under the condition that matrices C and D are identity matrices, Eq. (1.1.2) can be futher simplified and rewritten in the form of the Stein matrix equation. The Sylvester-transpose matrix equation

$$AX + X^T D = E, \tag{1.1.3}$$

has been widely examined in various mathermatical and engineering domains, notably in the context of pole assignment, eigenstructure assignment, and fault detection in dynamical systems (see, e.g., [3, 15, 16, 17]). In the past decade, considerable attention has been given to another prominent matrix equation, namely the generalized Sylvester-transpose equation:

$$AXB + CX^T D = E. \tag{1.1.4}$$

Both theoretical analyses and computational approaches to Eq. (1.1.4) have been extensively investigated by numerous researchers (e.g. [18, 19, 20, 21, 22, 23, 24]).

Rather than restricting the analysis to the field of real numbers, it is possible to extend theory of matrix equations to more general algebraic structures, such as the quaternion skew-field or other skew-fields. Let $\mathbb{Q}$ be a 4-dimensional vector space over $\mathbb{R}$ with basis $\{1, i, j, k\}$ and multiplication rules:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k, \tag{1.1.5}$$
$$jk = -kj = i,\ ki = -ik = j.$$

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A quaternion q (a Hamilton quaternion) can be explicitly represented as follows:

$$q \ = \ q_1 + q_2 i + q_3 j + q_4 k \ \in \ \mathbb{Q}$$

with real coefficients q_1, q_2, q_3, q_4.

The quaternions are widely used in quantum physics [25, 26], computer graphics [27], robot trajectory planning [28], and modeling [29], etc. A detailed discussion on quaternions can be found in the survey paper [30]. The generalization of the rule in (1.1.5) naturally gives rise to a generalized quaternion algebra. This structure denoted by $\mathbb{Q}_{u,v}$, is defined as a 4-dimensional real vector space with an ordered basis $\{1, i, j, k\}$, i.e.,

$$q \ = \ q_1 + q_2 i + q_3 j + q_4 k \ \in \mathbb{Q}_{u,v},$$

where u and v are nonzeros real numbers. The operations of addition and scalar multiplication on $\mathbb{Q}_{u,v}$ are carried out according to the standard definitions commonly used in a vector space theory. The multiplication of any two of $1, i, j, k$ is different from $\mathbb{Q}$ because u and v are involved. The multiplication operation is specified by the following rules:

$$i^2 \ = \ u, \ j^2 \ = \ v, \ k^2 \ = \ ijk \ = \ -uv,$$
$$ij \ = \ -ji \ = \ k, \ jk \ = \ -kj \ = \ -vi, \ ik \ = \ -ki \ = \ uj.$$

Consequently, $\mathbb{Q}_{u,v}$ exhibits the structure of a non commutative division ring. Specifically, the generalized quaternion algebra $\mathbb{Q}_{u,v}$ reduces to the Hamilton quaternions when $(u, v) = (-1, -1)$. When $(u, v) = (-1, 1)$, the algebra is referred to as the split quaternion ring. The cases $(u, v) = (1, -1)$ and $(u, v) = (1, 1)$ correspond to the nectarine quaternion ring, and the conacterine quaternion ring, respectively.

Matrices defined over quaternions constitute an interesting area of study within linear algebra; see e.g. [30]. Matrix equations defined over $\mathbb{Q}$ or $\mathbb{Q}_{u,v}$ play a crucial role in various domains of study, e.g. computer platform [31], signal processing [32, 33], quantum mechanics [34, 35], and image processing [36, 37, 38, 39]. In color image processing, a color image can be represented as a vector or matrix. According to RGB color model, let a color pixel be denoted as

$$(q_r, q_g, q_b) \ = \ \text{(red, green, blue)}.$$

The color information of a single pixel can be represented using a quaternion in a form

$$q \ = \ 0 + q_r i + q_g j + q_b k \in \mathbb{Q}.$$

This formulation allows all three color components to be integrated into a single structure, and the addition operation reflects the physical behavior of light combination

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through spectral summation. For an RGB image consisting of many array pixels, the color data can be modeled as a quaternion-valued vector:

$$\dot{y} \;=\; 0 + y_r i + y_g j + y_b k \in \mathbb{Q}^m,$$

where $y_r, y_g, y_b \in \mathbb{R}^m$. Each of these real-valued vectors stores the intensity valued for the corresponding color channel across all pixels in the image. A quaternion-based sparse representaion model [40] says that $\dot{y} = \dot{D}\dot{a}$, where

$$\dot{D} \;=\; D_s + D_r i + D_g j + D_b k \in \mathbb{Q}^{m\times n} \;\text{ and }\; \dot{a} \;=\; a_1 + a_2 i + a_2 j + a_4 k \in \mathbb{Q}^n.$$

The matrix $\dot{D}$ and the vector $\dot{a}$ are called a dictionary matrix and a sparse coefficient vector, respectively.

During the last ten years, conciderable attention has been devoted to the theoretical analysis of such matrix equations by various authors. In [41], Shi-Fang Yuan presented explicit analytical expression for solving Eq. (1.1.2) with the least norm, including the least-squares solution, the least-squares pure-imaginary solution, and the real solution. F. Zhang et al. [42] studied special least-squares solutions of Eq. (1.1.2), so that they obtained the expressions of the minimal-norm least-squares solution, the pure-imaginary least-squares solution, and the real least-squares solution. Recently, Y. Tian et al. [43] considered Hermitian solutions of Eq. (1.1.2). Indeed, they proposed necessary and sufficient conditions for the existence of a Hermitian solution and provided the explicit general expression of the solution when it is solvable.

Given generalized quaternion matrices A, B, C, D, and E of compatible dimensions, we investigate the Sylvester matix equation (1.1.2), where X is an unknown matrix. We analyze several types of solutions, including the general exact/least-squares solutions, the least-squares solutions that consist only of the imaginary part, the least-squares real solutions, and the exact/least-squares solution closest to a given matrix. We further extend our study to the Sylvester-transpose matix equation (1.1.4), where we examine the general exact/least-squares solutions, the minimal-norm solutions, and the exact/least-squares solution closest to a given matrix. In addition, special cases of Eq. (1.1.2) and Eq. (1.1.4) are considered, such as Eq. (1.1.1), Eq. (1.1.3), the Stein equation, linear systems over generalized quaternions, and quaternionic matrix equations.

1.2 Objectives of the study

1) Study a generalized Sylvester matrix equation $AXB + CXD = E$ over generalized quaternions. We would like to find various kinds of solutions as follows:

   - general exact/least-squares solutions,
   - least-squares solutions that consist only of the imaginary part,

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- least-squares real solutions,
- exact/least-squares solution closest to a given matrix,

Moreover, we provide numerical examples to illustrate the results.

2) Study the generalized Sylvester-transpose matrix equation $AXB + CX^TD = E$ over generalized quaternions. We would like to find the following solutions:

- general exact/least-squares solutions,
- minimal-norm solutions,
- exact/least-squares solution closest to a given matrix.

Moreover, we provide numerical examples to illustrate the results.

1.3 Scope of the study

We consider a generalized Sylvester matrix equation $AXB + CXD = E$, where A, B, C, D, E are given rectangular matrices over a generalized quaternion skew-field, and X is an unknown matrix. Indeed, we discuss the following problems.

Problem 1.1. Find all least-squares solutions $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{n\times p}$ of Eq. (1.1.2). Let $G_{\mathcal{L}}$ be the set of least-squares solutions of (1.1.2). In addition, find a matrix X_G such that

$$\|X_G\| = \min_{X\in G_{\mathcal{L}}} \|X\|.$$

Problem 1.2. Find all least-squares solutions of Eq. (1.1.2) that consist only of the imaginary part, i.e., $X = X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{n\times p}$. Let $I_{\mathcal{L}}$ be the set of such solutions. In addition, find a matrix X_I such that

$$\|X_I\| = \min_{X\in I_{\mathcal{L}}} \|X\|.$$

Problem 1.3. Find all least-squares real solutions of Eq. (1.1.2). Let $R_{\mathcal{L}}$ be the set of such solutions. In addition, find a matrix X_R such that

$$\|X_R\| = \min_{X\in R_{\mathcal{L}}} \|X\|.$$

Problem 1.4. Let $Y \in \mathbb{Q}_{u,v}^{n\times p}$ be given.

1.4.1 Find the matrix X' such that

$$\left\|X' - Y\right\| = \min_{X\in G_{\mathcal{L}}} \|X - Y\|.$$

1.4.2 Find the matrix X'' such that

$$\left\|X'' - Y\right\| = \min_{X\in I_{\mathcal{L}}} \|X - Y\|.$$

1.4.3 Find the matrix X''' such that

$$\left\|X''' - Y\right\| = \min_{X\in R_{\mathcal{L}}} \|X - Y\|.$$

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Next, we consider the Sylvester-transpose matrix equation $AXB + CX^TD = E$, where A, B, C, D, E are given generalized quaternion matrices with compatible size and X is an unknown. Indeed, we discuss the following problems.

Problem 2.1. Find the solution set $\mathcal{S}$ of exact solutions to Eq. (1.1.4). In addition, find the minimal-norm element of $\mathcal{S}$, i.e., find a matrix X^* such that

$$\|X^*\| = \min_{X \in \mathcal{S}} \|X\|.$$

Problem 2.2. Find a solution $\bar{X} \in \mathcal{S}$ closest to a given matrix $Y \in \mathbb{Q}_{u,v}^{n\times p}$, i.e., find $\bar{X}$ such that

$$\left\|\bar{X} - Y\right\| = \min_{X \in \mathcal{S}} \|X - Y\|.$$

Problem 2.3. Find the set $\mathcal{L}$ of LS solutions to Eq. (1.1.4). In addition, find $\tilde{X}$ such that

$$\left\|\tilde{X}\right\| = \min_{X \in \mathcal{L}} \|X\|.$$

Problem 2.4. Find an LS solution of Eq. (1.1.4) closest to a given matrix $Y \in \mathbb{Q}_{u,v}^{n\times p}$. That is, find the matrix $\acute{X}$ such that

$$\left\|\acute{X} - Y\right\| = \min_{X \in \mathcal{L}} \|X - Y\|.$$

1.4 Benefits of the study

1) To obtain criteria for the existence and the uniqueness of various kinds of solutions for Sylvester-type matrix equations over generalized quaternions.
2) To obtain practical methods to solving Eqs. (1.1.2) and (1.1.4) over generalized quaternions.

1.5 Research methodology

1) Study linear algebra, matrix analysis and multilinear algebra.
2) Study quaternions and matrices of quaternions from textbooks and research papers.
3) Study topics about solving quaternion matrix equations and generalized quaternion matrix equations from research papers
4) Determine the objectives and scope of the research.
5) Transform the matrix equation $AXB + CXD = E$ over generalized quaternions into a linear systems with respect to the usual matrix product.

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6) Investigate criteria for the equation $AXB+CXD = E$ over generalized quaternions to have general least-squares solutions, least-squares solutions that consist only of the imaginary part, least-squares real solutions, and the least-squares solution closest to a given matrix.

7) Provide numerical examples for the equation $AXB + CXD = E$.

8) Transform $AXB + CX^TD = E$ over generalized quaternions into a linear system with respect to the usual matrix product.

9) Investigate criteria for the equation $AXB + CX^TD = E$ over generalized quaternions to have general exact/least-squares solutions, minimal-norm solutions, and the exact/least-squares solution closet to a given matrix.

10) Provide numerical examples for the equation $AXB + CX^TD = E$.

11) Write the thesis.

Table 1.1: The research schedule

<table>
<tr><th rowspan="3">Activity</th><th colspan="12">Time frame</th></tr>
<tr><th>2022</th><th colspan="4">2023</th><th colspan="5">2024</th><th colspan="2">2025</th></tr>
<tr><th>Aug. - Dec.</th><th>Jan. - Mar.</th><th>Apr. - Jul</th><th>Aug. - Sep.</th><th>Oct. - Dec.</th><th>Jan. - Mar.</th><th>Apr. - May</th><th>Jun. - Aug.</th><th>Sep. - Nov.</th><th>Dec.</th><th>Jan.</th><th>Feb. - Mar.</th></tr>
<tr><td>Step 1</td><td>$\longleftrightarrow$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 2</td><td colspan="2">$\longleftrightarrow$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 3</td><td></td><td colspan="2">$\longleftrightarrow$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 4</td><td></td><td></td><td></td><td>$\longleftrightarrow$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 5</td><td></td><td></td><td></td><td></td><td colspan="2">$\longleftrightarrow$</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 6</td><td></td><td></td><td></td><td></td><td></td><td colspan="2">$\longleftrightarrow$</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 7</td><td></td><td></td><td></td><td></td><td></td><td></td><td>$\longleftrightarrow$</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Step 8</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td colspan="2">$\longleftrightarrow$</td><td></td><td></td><td></td></tr>
<tr><td>Step 9</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td colspan="2">$\longleftrightarrow$</td><td></td><td></td></tr>
<tr><td>Step 10</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td colspan="2">$\longleftrightarrow$</td><td></td></tr>
<tr><td>Step 11</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>$\longleftrightarrow$</td></tr>
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Chapter 2

Preliminaries

In this chapter, we set up basic notation and recall fundamental tools for solving linear matix equations. Let $\mathbb{R}$ and $\mathbb{C}$ be the set of real numbers and the set of complex numbers, respectively. We denote the set of $m \times n$ real matrices and complex matrices, real symmetric matrices, and real antisymmetric matrices by $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$, $\mathbb{SR}^{m\times n}$, and $\mathbb{ASR}^{m\times n}$ where m and n are positive integers, respectively. The identity matrix of order n is denoted by I_n and we define the ith column of matrix A by $\text{col}_i(A)$. We denote the conjugate transpose of a matrix A by A^*.

2.1 Linear systems and Moore-Penrose inverses

Recall the following definition:

Definition 2.1. Let $A \in \mathbb{R}^{m\times n}$.

(i) If rank $A = m$, then A is said to have a full-row rank.

(ii) If rank $A = n$, then A is said to have a full-column rank.

(iii) If rank $A = \min\{m, n\}$, then A is said to have a full rank.

Theorem 2.2. (e.g. [44]) Let $A \in \mathbb{C}^{m\times n}$ be a nonzero matrix of rank r, where $1 \leq r \leq \min\{m, n\}$. Then there exist positive real numbers $s_1 \geq s_2 \geq \cdots \geq s_r > 0$, an orthonormal set $\{u_1, \ldots, u_r\} \subseteq \mathbb{C}^m$, and an orthonormal set $\{v_1, \ldots, v_r\} \subseteq \mathbb{C}^n$ such that

$$A = \sum_{j=1}^{r} s_j u_j v_j^*. \tag{2.1.1}$$

Definition 2.3. (e.g. [44]) Let $A \in \mathbb{C}^{m\times n}$ be a matrix of rank r with the decomposition (2.1.1). Then the Moore-Penrose pseudoinverse of A is defined by

$$A^\dagger = \sum_{j=1}^{r} \frac{1}{s_j} v_j u_j^*.$$

The matrix $A^\dagger$ is uniquely determined and does not depend on the choice of the vectors $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$.

Example 2.4. Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2}$. Then

$$A = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix},$$

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and the MoorePenrose pseudoinverse of A is

$$\begin{aligned} A^{\dagger} &= \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix} + 1\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix} \\ &= \begin{bmatrix}0 & 1 & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}. \end{aligned}$$

Definition 2.5. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. We say that the linear system

$$Ax = b$$

is consistent if and only if it has a solution $x \in \mathbb{R}^n$. Otherwise, the system is said to be inconsistent.

Lemma 2.6. (e.g. [44]) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The real linear system

$$Ax = b \tag{2.1.2}$$

is consistent if and only if

$$\text{rank } [A \;\; b] = \text{rank } A.$$

In this case, we have the following:

(i) The general solutions of Eq. (2.1.2) can be expressed as

$$x = A^{\dagger}b + (I_n - A^{\dagger}A)y, \tag{2.1.3}$$

where $y \in \mathbb{R}^n$ is arbitrary.

(ii) The minimal-norm solution of Eq. (2.1.3) is given by

$$x = A^{\dagger}b. \tag{2.1.4}$$

(iii) If A is of full-column rank, then Eq. (2.1.2) has a unique solution (2.1.4).

Lemma 2.7. (e.g. [44]) Consider the linear system (2.1.2) when it is inconsistent. We have the following:

(i) The general least-squares solutions of Eq. (2.1.2) are given by (2.1.3), where $y \in \mathbb{R}^n$ is arbitrary.

(ii) The minimal-norm least-squares solution of Eq. (2.1.3) is given by (2.1.4).

(iii) If A is of full-column rank, then Eq. (2.1.2) has a unique least-squares solution (2.1.4).

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2.2 Vectorizations and the Kronecker product

Definition 2.8. (e.g. [45]) The operator vec$(\cdot)$ transforms any matrix $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ into a column vector

$$\text{vec}(A) \;=\; \begin{bmatrix} a_{11} & \dots & a_{m1} & a_{12} & \dots & a_{m2} & \dots & a_{1n} & \dots & a_{mn}\end{bmatrix}^T \in \mathbb{R}^{mn}.$$

Example 2.9. Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \in \mathbb{R}^{3\times 2}$. Then

$$\text{Vec}(A) = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 2 \\ 4 \\ 6 \end{bmatrix} \in \mathbb{R}^6.$$

Definition 2.10. The Kronecker product of matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{s\times t}$ is defined as

$$\begin{aligned} A\otimes B \;&=\; [a_{ij}B] \\ &= \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1j}B \\ a_{21}B & a_{22}B & \dots & a_{2j}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1}B & a_{i2}B & \dots & a_{ij}B \end{bmatrix} \in \mathbb{R}^{ms\times nt}. \end{aligned}$$

Example 2.11. Consider $A = \begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3}$ and $B = \begin{bmatrix} -5 & 1 \end{bmatrix} \in \mathbb{R}^{1\times 2}$. Then

$$\begin{aligned} A\otimes B &= \begin{bmatrix} 0\begin{bmatrix} -5 & 1 \end{bmatrix} & 2\begin{bmatrix} -5 & 1 \end{bmatrix} & 4\begin{bmatrix} -5 & 1 \end{bmatrix} \\ 3\begin{bmatrix} -5 & 1 \end{bmatrix} & 1\begin{bmatrix} -5 & 1 \end{bmatrix} & 6\begin{bmatrix} -5 & 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -10 & 2 & -20 & 4 \\ -15 & 3 & -5 & 1 & -30 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 6}. \end{aligned}$$

Theorem 2.12. (e.g. [46]) Let $A \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{p\times q}$. Then

$$\text{vec}(AXB) \;=\; (B^T \otimes A)\,\text{vec}(X).$$

For any positive integers n and p, let us denote

$$\text{P}(n,p) \;=\; \sum_{i=1}^{n}\sum_{j=1}^{p} E_{ij}\otimes E_{ij}^T,$$

where each $E_{ij} \in \mathbb{R}^{n\times p}$ has a 1 in the (i,j)–th position, and zeros elsewhere.

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Lemma 2.13. (e.g. [46]) For any $X \in \mathbb{R}^{n\times p}$, we have

$$\mathsf{vec}(X^T) \;=\; \mathsf{P}(n,p)\,\mathsf{vec}(X).$$

Lemma 2.14. [47] Suppose $X \in \mathbb{R}^{n\times n}$.

(i) If $X \in \mathbb{SR}^{n\times n}$, then

$$\mathsf{vec}(X) \;=\; K_S\,\mathsf{vec}_S(X),$$

where $\mathsf{vec}_S(X)$ is represented as Eq. (2.4.1), and the matrix $K_S \in \mathbb{R}^{n^2\times(n(n+1)/2)}$ is of the following form:

$$K_S \;=\; \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}e_1 & e_2 & \dots & e_n & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & e_1 & \dots & 0 & \sqrt{2}e_2 & \dots & e_n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & e_n & 0 \\ 0 & 0 & \dots & e_1 & 0 & \dots & e_2 & \dots & e_{n-1} & \sqrt{2}e_n \end{bmatrix}.$$

(ii) If $X \in \mathbb{ASR}^{n\times n}$, then

$$\mathsf{vec}(X) \;=\; K_A\,\mathsf{vec}_A(X),$$

where $\mathsf{vec}_A(X)$ is represented as Eq. (2.4.2), and the matrix $K_A \in \mathbb{R}^{n^2\times(n(n+1)/2)}$ is of the following form:

$$K_A \;=\; \frac{1}{\sqrt{2}}\begin{bmatrix} e_2 & e_3 & \dots & e_n & 0 & \dots & 0 & 0 & \dots & 0 \\ -e_1 & 0 & \dots & 0 & e_3 & \dots & e_{n-1} & e_n & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & -e_2 & 0 & \dots & e_n \\ 0 & 0 & \dots & -e_1 & 0 & \dots & 0 & -e_2 & \dots & -e_{n-1} \end{bmatrix}.$$

2.3 Generalized quaternions

A concept of quaternions was first proposed by Sir William Rowan Hamilton in 1843 in order to explain motions in the three-dimensional space. The algebra $\mathbb{Q}$ of quaternions is formed by its ordered basis $\{1, i, j, k\}$ where 1 acts as the multiplicative identity. A quaternion q can be uniquely expressed as

$$q \;=\; q_1 + q_2 i + q_3 j + q_4 k \;\in \mathbb{Q},$$

where the real numbers q_1, q_2, q_3, q_4 are called the real part, the i- or imaginary part, the j-part and k-part of q, respectively. The addition on $\mathbb{Q}$ and the scalar multiplication with real constants are defined coordinatewise. The zero real number 0 becomes

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the additive identity. For each $q \in \mathbb{Q}$, we denote its additive inverse by $-q$. The multiplication on $\mathbb{Q}$ is defined on the basis $\{1, i, j, k\}$ by

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k,$$
$$jk = -kj = i, \ ki = -ik = j.$$

Let a and b be real numbers, and let c and d be element chosen from the set $\{1, i, j, k\}$. The product $(ab)(cd)$ is defined to be $(ac)(bd)$. Together with the distributive law, this definition extends to determine the multiplication of abitrary quaternions. The algebra of quaternions constitutes a skew field, as its multiplication is inherently non-commutative. Moreover, we define the conjugate of a quaternion q by

$$\bar{q} = q_1 - q_2 i - q_3 j - q_4 k \in \mathbb{Q},$$

and the norm of q is defined as

$$|q| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}.$$

Example 2.15. Let $a, b \in \mathbb{R}$. We consider

$$(ai)(bi) = aibi = abii = abi^2 = -(ab),$$
$$(ai)(bk) = aibk = abik = (ab)(ik) = -(ab)j,$$
$$(aj)(bi) = ajbi = abji = (ab)(ji) = -(ab)k,$$
$$(ai)(bj) = aibj = abij = (ab)(ij) = (ab)k.$$

We can see that $(aj)(bi) \neq (ai)(bj)$.

Theorem 2.16. [30] Let $k, k_1, k_2, x, y, z \in \mathbb{Q}$. Then

1) $(x + y) + z = x + (y + z)$.

2) $0 + x = x = x + 0$.

3) $x + (-x) = 0 = (-x) + x$.

4) $x + y = y + x$.

5) $k(x + y) = kx + ky$.

6 $k_1(k_2 x) = (k_1 k_2)x$.

7) $(k_1 + k_2)x = k_1 x + k_2 x$.

8) $1x = x$.

9) $\frac{\bar{x}}{|x|^2}$ is the multiplicative inverse of x.

10) every quaternion q can be uniquely written in the form $q = c_1 + c_2 j$, where c_1 and c_2 are complex numbers.

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Next, we consider the generalized quaternions $\mathbb{Q}_{u,v}$ according to the following definitions.

Definition 2.17. Let u and v be nonzero real numbers. Let $\mathbb{Q}_{u,v}$ be a 4-dimensional vector space over $\mathbb{R}$ with an ordered basis $\{1, i, j, k\}$. That is, any element $q \in \mathbb{Q}_{u,v}$ can be written as

$$q \ = \ q_1 + q_2 i + q_3 j + q_4 k \ \in \mathbb{Q}_{u,v}$$

with real coefficients q_1, q_2, q_3, q_4. The addition on $\mathbb{Q}$ is defined coordinatewise. Each element $q \in \mathbb{Q}_{u,v}$ has its additive inverse, denoted by $-q$. The multiplication of the basis elements $1, i, j, k$ is defined by assigning 1 as the identity element, while the products involving i, j and k follow the specific rules outlined below:

$$i^2 \ = \ u, \ j^2 \ = \ v, \ k^2 \ = \ ijk \ = \ -uv,$$
$$ij \ = \ -ji \ = \ k, \ jk \ = \ -kj \ = \ -vi, \ ik \ = \ -ki \ = \ uj.$$

It turns out that $\mathbb{Q}_{u,v}$ is a skew field.

Example 2.18. Let $a, b \in \mathbb{R}$. We consider

$$(ak)(bk) \ = \ akbk \ = \ abkk \ = \ (ab)k^2 \ = \ -(ab)(uv),$$
$$(ak)(bj) \ = \ akbj \ = \ abkj \ = \ (ab)(kj) \ = \ (ab)(vi),$$
$$(aj)(bk) \ = \ ajbk \ = \ abjk \ = \ (ab)(jk) \ = \ -(ab)(vi),$$
$$(ai)(bk) \ = \ aibk \ = \ abik \ = \ (ab)(ik) \ = \ (ab)(uj).$$

We can see that $(ak)(bj) \neq (aj)(bk)$.

Theorem 2.19. Let $k_1, k_2, x, y, z \in \mathbb{Q}_{u,v}$. Then

1) $(x + y) + z = x + (y + z)$.

2) $0 + x = x = x + 0$.

3) $x + (-x) = 0 = (-x) + x$.

4) $x + y = y + x$.

5) $k_1(x + y) = k_1 x + k_1 y$.

6 $k_1(k_2 x) = (k_1 k_2)x$.

7) $(k_1 + k_2)x = k_1 x + k_2 x$.

8) $1x = x$.

Specially,

- if $(u, v) = (-1, -1)$ then we call $\mathbb{Q}_{-1,-1}$ the Hamilton quaternions.

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- if $(u, v) = (-1, 1)$, then we call $\mathbb{Q}_{-1,1}$ the split quaternion ring.
- if $(u, v) = (1, -1)$, then we call $\mathbb{Q}_{1,-1}$ the nectarine quaternion ring.
- if $(u, v) = (1, 1)$, then we call $\mathbb{Q}_{1,1}$ the conectarine quaternion ring.

2.4 Matrices of generalized quaternions

The notation $\mathbb{Q}_{u,v}^{m\times n}$ represents the set of all $m \times n$ matrices over generalized quaternion, for any positive integers m and n. We define the addition, the scalar multiplication and the usual multiplication for matrices over $\mathbb{Q}_{u,v}$ in a similar manner as those for matrices over $\mathbb{R}$.

Remark 2.20. For any $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbb{Q}_{u,v}^{m\times n}$, $B = B_1 + B_2 i + B_3 j + B_4 k \in \mathbb{Q}_{u,v}^{m\times n}$, and $C = C_1 + C_2 i + C_3 j + C_4 k \in \mathbb{Q}_{u,v}^{n\times p}$ where $A_r, B_r \in \mathbb{R}^{m\times n}$ and $C_r \in \mathbb{R}^{n\times p}$ $(r = 1, 2, 3, 4.)$ Then

1) $A + B = (A_1 + B_1) + (A_2 + B_2)i + (A_3 + B_3)j + (A_4 + B_4)k \in \mathbb{Q}_{u,v}^{m\times n}$.
2) $AC = (A_1C_1 + uA_2C_2 + vA_3C_3 - uvA_4C_4) + (A_1C_2 + A_2C_1 - vA_3C_4 + vA_4C_3)i$ $+ (A_1C_3 + uA_2C_4 + A_3C_1 - uA_4C_2)j + (A_1C_4 + A_2C_3 - A_3C_2 + A_4C_1)k \in \mathbb{Q}_{u,v}^{m\times p}$.
3) $A^T = (A_1 + A_2 i + A_3 j + A_4 k)^T = A_1^T + A_2^T i + A_3^T j + A_4^T k \in \mathbb{Q}_{u,v}^{n\times m}$.

Definition 2.21. For a matrix $A = [a_{ij}] \in \mathbb{Q}_{u,v}^{n\times n}$, let

$$\begin{aligned} a_1 &= [a_{11}, \sqrt{2}a_{21}, \ldots, \sqrt{2}a_{n1}], \\ a_2 &= [a_{22}, \sqrt{2}a_{32}, \ldots, \sqrt{2}a_{n2}], \\ &\vdots \\ a_{n-1} &= [a_{(n-1)(n-1)}, \sqrt{2}a_{n(n-1)}], \\ a_n &= a_{nn}. \end{aligned}$$

Then, we define

$$\text{vec}_S(A) = \begin{bmatrix} a_1 & a_2 & \ldots & a_{n-1} & a_n \end{bmatrix}^T \in \mathbb{Q}_{u,v}^{n(n+1)/2}. \tag{2.4.1}$$

Definition 2.22. For a matrix $B = [b_{ij}] \in \mathbb{Q}_{u,v}^{n\times n}$, let

$$\begin{aligned} b_1 &= [b_{21}, b_{31}, \ldots, b_{n1}], \\ b_2 &= [b_{32}, b_{42}, \ldots, b_{n2}], \\ &\vdots \\ b_{n-2} &= [b_{(n-1)(n-2)}, a_{n(n-2)}], \\ b_{n-1} &= b_{n(n-1)}. \end{aligned}$$

Then we defined

$$\text{vec}_A(B) = \sqrt{2}\begin{bmatrix} b_1 & b_2 & \ldots & b_{n-2} & b_{n-1} \end{bmatrix}^T \in \mathbb{Q}_{u,v}^{n(n+1)/2}. \tag{2.4.2}$$

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2.5 Real representations of generalized quaternion matrices

For each $A \in \mathbb{Q}_{u,v}^{m\times n}$, we can write

$$A \;=\; A_1 + A_2 i + A_3 j + A_4 k,$$

where $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$. We define

$$\Gamma_1(A) \;=\; \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \in \mathbb{R}^{4m\times n}.$$

Now, consider $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{n\times p}$, where $X_1, X_2, X_3, X_4 \in \mathbb{R}^{n\times p}$. We have

$$\begin{aligned} AX &= (A_1 + A_2 i + A_3 j + A_4 k)(X_1 + X_2 i + X_3 j + X_4 k) \\ &= A_1(X_1 + X_2 i + X_3 j + X_4 k) + A_2 i(X_1 + X_2 i + X_3 j + X_4 k) \\ &\quad + A_3 j(X_1 + X_2 i + X_3 j + X_4 k) + A_4 k(X_1 + X_2 i + X_3 j + X_4 k) \\ &= (A_1X_1 + uA_2X_2 + vA_3X_3 - uvA_4X_4) + (A_1X_2 + A_2X_1 - vA_3X_4 + vA_4X_3)i \\ &\quad + (A_1X_3 + uA_2X_4 + A_3X_1 - uA_4X_2)j + (A_1X_4 + A_2X_3 - A_3X_2 + A_4X_1)k. \end{aligned}$$

Thus,

$$\Gamma_1(AX) \;=\; \begin{bmatrix} A_1X_1 + uA_2X_2 + vA_3X_3 - uvA_4X_4 \\ A_1X_2 + A_2X_1 - vA_3X_4 + vA_4X_3 \\ A_1X_3 + uA_2X_4 + A_3X_1 - uA_4X_2 \\ A_1X_4 + A_2X_3 - A_3X_2 + A_4X_1 \end{bmatrix} \;=\; \mathcal{R}(A) \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}, \tag{2.5.1}$$

where

$$\mathcal{R}(A) \;=\; \begin{bmatrix} A_1 & uA_2 & vA_3 & -uvA_4 \\ A_2 & A_1 & vA_4 & -vA_3 \\ A_3 & -uA_4 & A_1 & uA_2 \\ A_4 & -A_3 & A_2 & A_1 \end{bmatrix}$$

is called a real matrix representation of A. From the structure of the block columns in A, we define the following:

$$\Gamma_2(A) \;=\; \begin{bmatrix} uA_2 \\ A_1 \\ -uA_4 \\ -A_3 \end{bmatrix}, \quad \Gamma_3(A) \;=\; \begin{bmatrix} vA_3 \\ vA_4 \\ A_1 \\ A_2 \end{bmatrix}, \quad \Gamma_4(A) \;=\; \begin{bmatrix} -uvA_4 \\ -vA_3 \\ uA_2 \\ A_1 \end{bmatrix} \in \mathbb{R}^{4m\times n}.$$

Clearly, the transformations vec, $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 are injective. It is easy to see that

$$\|A\| \;=\; \sqrt{\|A_1\|^2 + \|A_2\|^2 + \|A_3\|^2 + \|A_4\|^2} \;=\; \|\Gamma_1(A)\|. \tag{2.5.2}$$

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Proposition 2.23. (e.g. [43]) The following properties hold:

(i) $\Gamma_1(A+B) = \Gamma_1(A) + \Gamma_1(B)$, $\Gamma_1(kA) = k\Gamma_1(A)$ for any $A, B \in \mathbb{Q}_{u,v}^{m\times n}$ and $k \in \mathbb{R}$.

(ii) $\mathcal{R}(AB) = \mathcal{R}(A)\mathcal{R}(B)$ for any $A \in \mathbb{Q}_{u,v}^{m\times n}$ and $B \in \mathbb{Q}_{u,v}^{n\times p}$.

(iii) $\mathcal{R}(I_m) = I_{4m}$.

2.6 Least-squares solutions of the quaternion matrix equation $AXB + CXD = E$

The work of F. Zhang et al. [42] focused on special least-squares solutions of $AXB + CXD = E$. They derived formulas for the minimal-norm least-squares solution, the pure-imaginary least-squares solution, and the real least-squares solution. To achieve this, they employed vectorization techniques and real representations to transform the quaternion matrix equation into an equivalent equation over the real numbers.

Lemma 2.24. [48] Let $X \in \mathbb{Q}^{n\times k}$. Then vec$(\mathcal{R}(X)) = \mathcal{F}vec(\Gamma(X))$, where

$$\mathcal{F} = \begin{bmatrix} I_k \otimes I_{4n} \\ I_k \otimes \bar{O}_n \\ I_k \otimes \bar{L}_n \\ I_k \otimes \bar{S}_n \end{bmatrix} \in \mathbb{R}^{16nk\times 4nk},$$

and

$$\bar{O}_n = \begin{bmatrix} 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \end{bmatrix}, \bar{L}_n = \begin{bmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & -I_n \\ I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \end{bmatrix}, \bar{S}_n = \begin{bmatrix} 0 & 0 & 0 & -I_n \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{bmatrix}.$$

Lemma 2.25. [42] Suppose $X = 0 + X_2 i + X_3 j + X_4 k \in \mathbb{Q}^{n\times k}$. Then

$$\text{vec}(\Gamma(X)) = \mathcal{H}\,\text{vec}\begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix},$$

where

$$\mathcal{H} = \begin{bmatrix} h & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & h \end{bmatrix} \in \mathbb{R}^{4nk\times 3nk}, \quad h = \begin{bmatrix} 0 & 0 & 0 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix} \in \mathbb{R}^{4n\times 3n}.$$

Lemma 2.26. [42] Suppose $X = X_1 + 0i + 0j + 0k \in \mathbb{Q}^{n\times k}$. Then

$$\text{vec}(\Gamma(X)) = \tilde{\mathcal{H}}\,\text{vec}(X_1),$$

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where

$$\tilde{\mathcal{H}} = \begin{bmatrix} \tilde{h} & 0 & 0 & 0 \\ 0 & \tilde{h} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \tilde{h} \end{bmatrix} \in \mathbb{R}^{4nk \times nk}, \quad \tilde{h} = \begin{bmatrix} I_n \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{4n \times n}.$$

Therefore, they obtained the following theorem.

Theorem 2.27. [42] Let $A, C \in \mathbb{Q}^{m\times n}, B, D \in \mathbb{Q}^{k\times s}, E \in \mathbb{Q}^{m\times s}$, and denote $\mathcal{M} = (\Gamma(B)^T \otimes \mathcal{R}(A)) + (\Gamma(D)^T \otimes \mathcal{R}(C))$. Then the set G_L can be expressed as

$$G_L = \left\{X \mid \text{vec}(\Gamma(X)) = (\mathcal{MF})^\dagger \text{vec}(\Gamma(E)) + \left[I_{4nk} - (\mathcal{MF})^\dagger(\mathcal{MF})\right] y, y \in \mathbb{R}^{4nk}\right\}.$$

Therefore, the unique solution $X_Q \in Q_L$ satisfies

$$\text{vec}(\Gamma(X_G)) = (\mathcal{MF})^\dagger \text{vec}(\Gamma(E)).$$

Eq. (1.1.2) has a solution $X \in \mathbb{Q}^{n\times k}$ if and only if

$$\left[I_{4ms} - (\mathcal{MF})(\mathcal{MF})^\dagger\right] \text{vec}(\Gamma(E)) = 0.$$

Theorem 2.28. [42] Let $A, C \in \mathbb{Q}^{m\times n}, B, D \in \mathbb{Q}^{k\times s}, E \in \mathbb{Q}^{m\times s}, X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{Q}^{n\times k}$ and $\mathcal{M}$ is the same as that of Theorem 2.27. Then the set I_L can be expressed as

$$I_L = \left\{ X \mid X_1 = 0, \text{vec}\begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix} = (\mathcal{MFH})^\dagger \text{vec}(\Gamma(E)) + \left[I_{3nk} - (\mathcal{MFH})^\dagger(\mathcal{MFH})\right] y, y \in \mathbb{R}^{3nk} \right\}.$$

Therefore, the unique pure imaginary solution $X_L = X_2 i + X_3 j + X_4 k \in I_L$ satisfies

$$\text{vec}\begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix} = (\mathcal{MFH})^\dagger \text{vec}(\Gamma(E)).$$

Eq. (1.1.2) has a solution $X \in \mathbb{IQ}^{n\times k}$ if and only if

$$\left[I_{4ms} - (\mathcal{MFH})(\mathcal{MFH})^\dagger\right] \text{vec}(\Gamma(E)) = 0.$$

Theorem 2.29. [42] Let $A, C \in \mathbb{Q}^{m\times n}, B, D \in \mathbb{Q}^{k\times s}, E \in \mathbb{Q}^{m\times s}$, and $\mathcal{M}$ the same as that of Theorem 2.27. Then the set R_L can be expressed as

$$R_L = \left\{X \mid \text{vec}(X) = (\mathcal{MF}\tilde{\mathcal{H}})^\dagger \text{vec}(\Gamma(E)) + \left[I_{nk} - (\mathcal{MF}\tilde{\mathcal{H}})^\dagger(\mathcal{MF}\tilde{\mathcal{H}})\right] y, y \in \mathbb{R}^{3nk}\right\}.$$

Therefore, the unique real solution satisfies

$$\text{vec}(X_R) = (\mathcal{MF}\tilde{\mathcal{H}})^\dagger \text{vec}(\Gamma(E)).$$

Eq. (1.1.2) has a solution $X \in \mathbb{R}^{n\times k}$ if and only if

$$\left[I_{4ms} - (\mathcal{MF}\tilde{\mathcal{H}})(\mathcal{MF}\tilde{\mathcal{H}})^\dagger\right] \text{vec}(\Gamma(E)) = 0.$$

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2.7 Hermitian solutions of the generalized quaternion matrix equation $AXB + CXD = E$

Y. Tian et al. [43] considered Hermitian solutions of $AXB + CXD = E$. They transform the equation over generalized quaternions to the real representations of generalized quaternion matrices and used Lamma 2.14. Then they obtained the following theorem.

Theorem 2.30. [43] Let $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbb{Q}_{u,v}^{m\times n}$, $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{HQ}_{u,v}^{n\times n}$ and $B \in \mathbb{Q}_{u,v}^{n\times s}$. Then

$$\begin{aligned}\text{vec}(\Gamma(AXB)) \;=\; & \Big[\left(N_s \mathcal{R}(B) N_t^{-1}\right)^T \otimes A_1 + \left(N_s \mathcal{R}(B) O_t\right)^T \otimes A_2 \\ & + \left(N_s \mathcal{R}(B) L_t\right)^T \otimes A_3 + \left(N_s \mathcal{R}(B) S_t\right)^T \otimes A_4 \Big] \cdot W \begin{bmatrix} \text{vec}_S(X_1) \\ \text{vec}_A(X_2) \\ \text{vec}_A(X_3) \\ \text{vec}_A(X_4) \end{bmatrix},\end{aligned}$$

in which

$$N_s = \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & uI_s & 0 & 0 \\ 0 & 0 & vI_s & 0 \\ 0 & 0 & 0 & -uvI_s \end{bmatrix}, \quad O_s = \begin{bmatrix} 0 & I_s & 0 & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{v} I_s \\ 0 & 0 & -\frac{1}{v} I_s & 0 \end{bmatrix},$$

$$L_s = \begin{bmatrix} 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & -\frac{1}{u} I_s \\ I_t & 0 & 0 & 0 \\ 0 & \frac{1}{u} I_s & 0 & 0 \end{bmatrix}, \quad S_s = \begin{bmatrix} 0 & 0 & 0 & I_s \\ 0 & 0 & -I_s & 0 \\ 0 & I_t & 0 & 0 \\ I_s & 0 & 0 & 0 \end{bmatrix},$$

$$W = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_A \end{bmatrix}.$$

Theorem 2.31. [43] Let $A, C \in \mathbb{Q}_{u,v}^{m\times n}$, $B, D \in \mathbb{Q}_{u,v}^{n\times s}$, and $E \in \mathbb{Q}_{u,v}^{m\times s}$. Then, the Eq. 1.1.2 has a solution $X \in H_E$ if and only if

$$PP^{\dagger} e \;=\; e. \tag{2.7.1}$$

If this condition satisfies, then

$$H_E \;=\; \left\{ X \mid \text{vec}(\Gamma(X)) = W \left[P^{\dagger} e + \left(I_{2n^2-n} - P^{\dagger} P \right) y \right] \right\} \tag{2.7.2}$$

where $y \in \mathbb{R}^{2n^2-n}$ is an arbitrary vector. Furthermore, if Eq. (2.7.1) holds, then the generalized quaternion matrix equation (1.1.2) has a unique solution $X \in H_E$ if an

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only if

$$\text{rank}(P) = 2n^2 - n.$$

In this case,

$$H_E = \left\{ X \mid \text{Vec}(\Phi_X) = WP^{\dagger}e \right\}.$$

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Chapter 3

Generalized Sylvester matrix equation over generalized quaternions

In this chapter, we study on solving the generalized Sylvester matrix equation (1.1.2) where $A \in \mathbb{Q}_{u,v}^{m\times a}, B \in \mathbb{Q}_{u,v}^{b\times p}, C \in \mathbb{Q}_{u,v}^{m\times a}, D \in \mathbb{Q}_{u,v}^{b\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}$ are given, and $X \in \mathbb{Q}_{u,v}^{a\times b}$ is an unknown matrix. We apply a real representation of generalized quaternion matrices and certain vectorizations to transform the matrix equation into a real linear system. We divide the consideration of Eq. (1.1.2) into four cases. By considering the general least-squares solution, the pure-imaginary least-squares solution, the real least-squares solution and the least-squares solution closest to a given matrix, respectively. We first start from the solution of Problem 1.1.

3.1 The general solution

In this section, we investigate Problem 1.1. Indeed, we would like to find the general solutions of Eq. (1.1.2). The following lemma will be used in a calculation.

Lemma 3.1. Let $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{a\times b}$ where $X_1, X_2, X_3, X_4 \in \mathbb{R}^{a\times b}$. Then

$$\begin{bmatrix} \operatorname{vec}(\Gamma_1(X)) \\ \operatorname{vec}(\Gamma_2(X)) \\ \operatorname{vec}(\Gamma_3(X)) \\ \operatorname{vec}(\Gamma_4(X)) \end{bmatrix} = \mathcal{M}_{u,v} \operatorname{vec}(\Gamma_1(X)), \text{ where } \mathcal{M}_{u,v} = \begin{bmatrix} I_{4ab} \\ I_b \otimes N \\ I_b \otimes K \\ I_b \otimes T \end{bmatrix} \in \mathbb{R}^{16ab\times 4ab}. \tag{3.1.1}$$

Here,

$$N = \begin{bmatrix} 0 & uI_a & 0 & 0 \\ I_a & 0 & 0 & 0 \\ 0 & 0 & 0 & -uI_a \\ 0 & 0 & -I_a & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & vI_a & 0 \\ 0 & 0 & 0 & vI_a \\ I_a & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 & 0 & -uvI_a \\ 0 & 0 & -vI_a & 0 \\ 0 & uI_a & 0 & 0 \\ I_a & 0 & 0 & 0 \end{bmatrix}.$$

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Proof. A direct computation reveals that

$$\operatorname{vec}(\Gamma_2(X)) = \operatorname{vec}\left(\begin{bmatrix} uX_2 \\ X_1 \\ -uX_4 \\ -X_3 \end{bmatrix}\right) = \begin{bmatrix} u\,\mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_1) \\ -u\,\mathrm{col}_1(X_4) \\ -\mathrm{col}_1(X_3) \\ \vdots \\ u\,\mathrm{col}_b(X_2) \\ \mathrm{col}_b(X_1) \\ -u\,\mathrm{col}_b(X_4) \\ -\mathrm{col}_b(X_3) \end{bmatrix} \tag{3.1.2}$$

$$= \begin{bmatrix} 0 & uI_a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ I_a & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -uI_a & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_a & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & uI_a & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & I_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -uI_a \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -I_a & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{col}_1(X_1) \\ \mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_3) \\ \mathrm{col}_1(X_4) \\ \vdots \\ \mathrm{col}_b(X_1) \\ \mathrm{col}_b(X_2) \\ \mathrm{col}_b(X_3) \\ \mathrm{col}_b(X_4) \end{bmatrix}$$

$$= \left(I_b \otimes \begin{bmatrix} 0 & uI_a & 0 & 0 \\ I_a & 0 & 0 & 0 \\ 0 & 0 & 0 & -uI_a \\ 0 & 0 & -I_a & 0 \end{bmatrix}\right) \cdot \operatorname{vec}\left(\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}\right)$$

$$= (I_b \otimes N)\operatorname{vec}(\Gamma_1(X)). \tag{3.1.3}$$

Next, we consider

$$\operatorname{vec}(\Gamma_3(X)) = \operatorname{vec}\left(\begin{bmatrix} vX_3 \\ vX_4 \\ X_1 \\ X_2 \end{bmatrix}\right) = \begin{bmatrix} v\,\mathrm{col}_1(X_3) \\ v\,\mathrm{col}_1(X_4) \\ \mathrm{col}_1(X_1) \\ \mathrm{col}_1(X_2) \\ \vdots \\ v\,\mathrm{col}_b(X_3) \\ v\,\mathrm{col}_b(X_4) \\ \mathrm{col}_b(X_1) \\ \mathrm{col}_b(X_2) \end{bmatrix}$$

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$$
\begin{aligned}
&= \begin{bmatrix}
0 & 0 & vI_a & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & vI_a & \dots & 0 & 0 & 0 & 0 \\
I_a & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & I_a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & vI_a & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & vI_a \\
0 & 0 & 0 & 0 & \dots & I_a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & I_a & 0 & 0
\end{bmatrix} \cdot
\begin{bmatrix}
\mathrm{col}_1(X_1) \\ \mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_3) \\ \mathrm{col}_1(X_4) \\ \vdots \\ \mathrm{col}_b(X_1) \\ \mathrm{col}_b(X_2) \\ \mathrm{col}_b(X_3) \\ \mathrm{col}_b(X_4)
\end{bmatrix} \\
&= \left(I_b \otimes \begin{bmatrix}
0 & 0 & vI_a & 0 \\
0 & 0 & 0 & vI_a \\
I_a & 0 & 0 & 0 \\
0 & I_a & 0 & 0
\end{bmatrix} \right) \cdot \mathrm{vec} \left(\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \right) \\
&= (I_b \otimes K)\,\mathrm{vec}(\Gamma_1(X)). \qquad (3.1.4)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{vec}(\Gamma_4(X)) &= \mathrm{vec}\left(\begin{bmatrix} -uvX_4 \\ -vX_3 \\ uX_2 \\ X_1 \end{bmatrix} \right) = \begin{bmatrix}
-uv\,\mathrm{col}_1(X_4) \\ -v\,\mathrm{col}_1(X_3) \\ u\,\mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_1) \\ \vdots \\ -uv\,\mathrm{col}_b(X_4) \\ -v\,\mathrm{col}_b(X_3) \\ u\,\mathrm{col}_b(X_2) \\ \mathrm{col}_b(X_1)
\end{bmatrix} \\
&= \begin{bmatrix}
0 & 0 & 0 & -uvI_a & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & -vI_a & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & uI_a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
I_a & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -uvI_a \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & -vI_a & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & uI_a & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & I_a & 0 & 0 & 0
\end{bmatrix} \cdot
\begin{bmatrix}
\mathrm{col}_1(X_1) \\ \mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_3) \\ \mathrm{col}_1(X_4) \\ \vdots \\ \mathrm{col}_b(X_1) \\ \mathrm{col}_b(X_2) \\ \mathrm{col}_b(X_3) \\ \mathrm{col}_b(X_4)
\end{bmatrix}
\end{aligned}
$$

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$$= \left(I_b \otimes \begin{bmatrix} 0 & 0 & 0 & -uvI_a \\ 0 & 0 & -vI_a & 0 \\ 0 & uI_a & 0 & 0 \\ I_a & 0 & 0 & 0 \end{bmatrix} \right) \cdot \mathrm{vec} \left(\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \right)$$

$$= (I_b \otimes N)\,\mathrm{vec}(\Gamma_1(X)). \tag{3.1.5}$$

From Eqs. (3.1.3), (3.1.4) and (3.1.5), we obtain

$$\begin{bmatrix} \mathrm{vec}(\Gamma_1(X)) \\ \mathrm{vec}(\Gamma_2(X)) \\ \mathrm{vec}(\Gamma_3(X)) \\ \mathrm{vec}(\Gamma_4(X)) \end{bmatrix} = \begin{bmatrix} I_{4ab} \\ I_b \otimes N \\ I_b \otimes K \\ I_b \otimes T \end{bmatrix} \cdot \mathrm{vec}(\Gamma_1(X)) = \mathcal{M}_{u,v}\,\mathrm{vec}(\Gamma_1(X)).$$

□

Thus, we can derive the following theorem.

Theorem 3.2. Consider Eq. (1.1.2) for given matrices $A, C \in \mathbb{Q}_{u,v}^{m\times a}, B, D \in \mathbb{Q}_{u,v}^{b\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}$ and unknown matrix $X \in \mathbb{Q}_{u,v}^{a\times b}$. Let us denote

$$\mathcal{F} = \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(C)\right). \tag{3.1.6}$$

Then Eq. (1.1.2) is consistent if and only if the following rank condition holds:

$$\mathrm{rank}\ [\mathcal{F}\mathcal{M}_{u,v}\ \ \mathrm{vec}(\Gamma_1(E))] = \mathrm{rank}\ [\mathcal{F}\mathcal{M}_{u,v}]. \tag{3.1.7}$$

In both consistent and inconsistent cases, we have the following:

(i) The general exact/least-squares solutions of Problem 1.1 can be represented by

$$\mathrm{vec}(\Gamma_1(X)) = (\mathcal{F}\mathcal{M}_{u,v})^\dagger\,\mathrm{vec}(\Gamma_1(E)) + \left[I_{4ab} - (\mathcal{F}\mathcal{M}_{u,v})^\dagger(\mathcal{F}\mathcal{M}_{u,v})\right] w, \tag{3.1.8}$$

where $w \in \mathbb{R}^{4ab}$ is arbitrary.

(ii) The minimal-norm exact/least-squares solution is given by

$$\mathrm{vec}(\Gamma_1(X)) = (\mathcal{F}\mathcal{M}_{u,v})^\dagger\,\mathrm{vec}(\Gamma_1(E)). \tag{3.1.9}$$

(iii) If $\mathcal{F}\mathcal{M}_{u,v}$ is of full-column rank (rank $[\mathcal{F}\mathcal{M}_{u,v}] = 4ab$), then Problem 1.1 has a unique solution given by (3.1.9).

Proof. From Eqs. (1.1.2) and consider the associated norm-error $\|AXB + CXD - E\|$ by using Eq. (2.5.2),

$$\|AXB + CXD - E\| = \|\Gamma_1(AXB + CXD - E)\|.$$

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Applying Definition 2.8, Theorem 2.12 and Proposition 2.23, we have

$$\begin{aligned}
\Gamma_1(AXB+CXD-E) &= \Gamma_1(AXB)+\Gamma_1(CXD)-\Gamma_1(E) \\
&= \mathcal{R}(A)\mathcal{R}(X)\Gamma_1(B)+\mathcal{R}(C)\mathcal{R}(X)\Gamma_1(D)-\Gamma_1(E) \\
&= \mathrm{vec}\left[\mathcal{R}(A)\mathcal{R}(X)\Gamma_1(B)+\mathcal{R}(C)\mathcal{R}(X)\Gamma_1(D)-\Gamma_1(E)\right] \\
&= \left[\left(\Gamma_1(B)^T\otimes\mathcal{R}(A)\right)+\left(\Gamma_1(D)^T\otimes\mathcal{R}(C)\right)\right]\mathrm{vec}\left(\mathcal{R}(X)\right)-\mathrm{vec}(\Gamma_1(E)) \\
&= \mathcal{F}\,\mathrm{vec}\left(\mathcal{R}(X)\right)-\mathrm{vec}(\Gamma_1(E)) \\
&= \mathcal{F}\,\mathrm{vec}\left(\begin{bmatrix}\Gamma_1(X) & \Gamma_2(X) & \Gamma_3(X) & \Gamma_4(X)\end{bmatrix}\right)-\mathrm{vec}(\Gamma_1(E)) \\
&= \mathcal{F}\begin{bmatrix}\mathrm{vec}(\Gamma_1(X)) \\ \mathrm{vec}(\Gamma_2(X)) \\ \mathrm{vec}(\Gamma_3(X)) \\ \mathrm{vec}(\Gamma_4(X))\end{bmatrix}-\mathrm{vec}(\Gamma_1(E)).
\end{aligned}$$

Using Lemma 3.1, we obtain

$$\|AXB+CXD-E\| = \|\mathcal{F}\mathcal{M}_{u,v}\,\mathrm{vec}(\Gamma_1(X))-\mathrm{vec}(\Gamma_1(E))\|. \tag{3.1.10}$$

Note that, the generalized quaternion matrix equation (1.1.2) is consistent if and only if the associated norm-error equals to zero. Thus, Eq. (1.1.2) is equivalent to a real linear system

$$\mathcal{F}\mathcal{M}_{u,v}\,\mathrm{vec}(\Gamma_1(X)) = \mathrm{vec}(\Gamma_1(E)). \tag{3.1.11}$$

According to Lemma 2.6, the system (3.1.11) is consistent if and only if the rank condition (3.1.7) holds. By Lemma 2.6, the system (3.1.11) has the general solution

$$\mathrm{vec}(\Gamma_1(X)) = (\mathcal{F}\mathcal{M}_{u,v})^{\dagger}\,\mathrm{vec}(\Gamma_1(E))+\left[I_{4ab}-(\mathcal{F}\mathcal{M}_{u,v})^{\dagger}(\mathcal{F}\mathcal{M}_{u,v})\right]w,$$

where $w\in\mathbb{R}^{4ab}$ is arbitrary. From Lemmas 2.6 and 2.7, it can be concluded that, in both the consistent and inconsistent cases, the general exact/least-squares solution can be expressed by Eq. (3.1.8). According to Lemmas 2.6 and 2.7, Eq. (3.1.9) provides the minimal-norm exact/least-squares solution. If $\mathcal{F}\mathcal{M}_{u,v}$ is of full-column rank, then Problem 1.1 has a unique solution, which is also given by (3.1.9).

□

3.2 Specific solutions

In this section, we investigate Problems 1.2-1.3. In order to find imaginary least-squares solutions, real least-squares solutions. The next lemma provides a real-vector representation of a generalized quaternion matrix.

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Lemma 3.3. Suppose $X = X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{a\times b}$ where $X_2, X_3, X_4 \in \mathbb{R}^{a\times b}$. Then

$$\mathrm{vec}(\Gamma_1(X)) \;=\; \mathcal{K}\,\mathrm{vec}\left(\begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix}\right),$$

where

$$\mathcal{K} \;=\; I_b \otimes \begin{bmatrix} 0 & 0 & 0 \\ I_a & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & I_a \end{bmatrix} \in \mathbb{R}^{4ab\times 3ab}.$$

Proof. A direct computation reveals that

$$\begin{aligned}
\mathrm{vec}(\Gamma_1(X)) \;&=\; \mathrm{vec}\left(\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}\right) = \begin{bmatrix} \mathrm{col}_1(0) \\ \mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_3) \\ \mathrm{col}_1(X_4) \\ \vdots \\ \mathrm{col}_b(0) \\ \mathrm{col}_b(X_2) \\ \mathrm{col}_b(X_3) \\ \mathrm{col}_b(X_4) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ I_a & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & I_a & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & I_a & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & I_a & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & I_a & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & I_a \end{bmatrix} \cdot \begin{bmatrix} \mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_3) \\ \mathrm{col}_1(X_4) \\ \vdots \\ \mathrm{col}_b(X_2) \\ \mathrm{col}_b(X_3) \\ \mathrm{col}_b(X_4) \end{bmatrix} \\
&= \left(I_b \otimes \begin{bmatrix} 0 & 0 & 0 \\ I_a & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & I_a \end{bmatrix} \right) \cdot \mathrm{vec}\left(\begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix}\right) \\
&= \mathcal{K}\,\mathrm{vec}\left(\begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix}\right).
\end{aligned}$$

□

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Lemma 3.4. For any $X \in \mathbb{R}^{a\times b}$, we have

$$\text{vec}(\Gamma_1(X)) \;=\; \tilde{\mathcal{K}}\,\text{vec}(X)$$

where

$$\tilde{\mathcal{K}} \;=\; I_b \otimes \begin{bmatrix} I_a \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{4ab\times ab}..$$

Proof. Since $X = X + 0i + 0j + 0k$, we have

$$\begin{aligned}
\text{vec}(\Gamma_1(X)) \;&=\; \text{vec}\left(\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}\right) \;=\; \begin{bmatrix} \text{col}_1(X) \\ \text{col}_1(0) \\ \text{col}_1(0) \\ \text{col}_1(0) \\ \vdots \\ \text{col}_b(X) \\ \text{col}_b(0) \\ \text{col}_b(0) \\ \text{col}_b(0) \end{bmatrix} \\
&= \begin{bmatrix} I_a & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_a \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{bmatrix} \cdot \begin{bmatrix} \text{col}_1(X) \\ \vdots \\ \text{col}_b(X) \end{bmatrix} \\
&= \left(I_b \otimes \begin{bmatrix} I_a \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \cdot \text{vec}(X) \\
&= \tilde{\mathcal{K}}\,\text{vec}(X).
\end{aligned}$$

□

Theorem 3.5. Consider Eq. (1.1.2) for given matrices $A, C \in \mathbb{Q}_{u,v}^{m\times a}, B, D \in \mathbb{Q}_{u,v}^{b\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}$ and unknown matrix $X = X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{a\times b}$. Let us denote $\mathcal{F}$ as in Eq. (3.1.6). Then Eq. (1.1.2) is consistent if and only if the following rank condition holds:

$$\text{rank}\;[\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}\;\text{vec}(\Gamma_1(E))] \;=\; \text{rank}\;[\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}]. \tag{3.2.1}$$

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In both consistent and inconsistent cases, we have the following:

(i) The imaginary-part exact/least-square solutions of Problem 1.2 can be represented by

$$\operatorname{vec}\left(\begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix}\right) = (\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})^{\dagger} \operatorname{vec}(\Gamma_1(E)) + \left[I_{4ab} - (\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})^{\dagger}(\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})\right] w, \quad (3.2.2)$$

where $w \in \mathbb{R}^{4ab}$ is arbitrary.

(ii) The minimal-norm imaginary-part exact/least-squares is given by

$$\operatorname{vec}\left(\begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix}\right) = (\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})^{\dagger} \operatorname{vec}(\Gamma_1(E)). \quad (3.2.3)$$

(iii) If $\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}$ is of full-column rank (rank $[\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}] = 4ab$), then Problem 1.2 has a unique solution given by (3.2.3).

Proof. Consider $X = X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{a\times b}$ where $X_2, X_3, X_4 \in \mathbb{R}^{a\times b}$. From Eq. (3.1.10) and using Lemma 3.3, we obtain

$$\begin{aligned} \|AXB + CXD - E\| &= \|\mathcal{F}\mathcal{M}_{u,v} \operatorname{vec}(\Gamma_1(X)) - \operatorname{vec}(\Gamma_1(E))\| \\ &= \left\| \mathcal{F}\mathcal{M}_{u,v} \operatorname{vec}\left(\begin{bmatrix} 0 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}\right) - \operatorname{vec}(\Gamma_1(E)) \right\| \\ &= \left\| \mathcal{F}\mathcal{M}_{u,v}\mathcal{K} \operatorname{vec}\left(\begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix}\right) - \operatorname{vec}(\Gamma_1(E)) \right\|. \end{aligned}$$

Note that, the generalized quaternion matrix equation (1.1.2) is consistent if and only if the associated norm-error equals to zero. Thus, Eq. (1.1.2) is equivalent to a real linear system

$$\mathcal{F}\mathcal{M}_{u,v}\mathcal{K} \operatorname{vec}\left(\begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix}\right) = \operatorname{vec}(\Gamma_1(E)). \quad (3.2.4)$$

According to Lemma 2.6, the system (3.2.4) is consistent if and only if the rank condition (3.2.1) holds. By Lemma 2.6, the system (3.2.4) has the general solutions

$$\operatorname{vec}\left(\begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix}\right) = (\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})^{\dagger} \operatorname{vec}(\Gamma_1(E)) + \left[I_{4ab} - (\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})^{\dagger}(\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})\right] w,$$

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where $w \in \mathbb{R}^{4ab}$ is arbitrary. From Lemmas 2.6 and 2.7, it can be concluded that, in both the consistent and inconsistent cases, the formula for the imaginary-part exact/least-squares solution can be expressed by Eq. (3.2.2). The minimal-norm imaginary-part exact/least-squares solution, as derived from Lemmas 2.6 and 2.7, is given by Eq. (3.2.3). If $\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}$ has full-column rank, then Problem 1.2 has a unique solution, which is given by Eq. (3.2.3). $\square$

Theorem 3.6. Consider Eq. (1.1.2) for given matrices $A, C \in \mathbb{Q}_{u,v}^{m\times a}, B, D \in \mathbb{Q}_{u,v}^{b\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}$ and unknown matrix $X \in \mathbb{R}^{a\times b}$. Let us denote $\mathcal{F}$ as in Eq. (3.1.6). Then Eq. (1.1.2) is consistent if and only if

$$\text{rank } [\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}} \ \text{vec}(\Gamma_1(E))] \; = \; \text{rank } [\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}]. \tag{3.2.5}$$

In both consistent and inconsistent cases, we have the following:

(i) Then the real exact/least-squares solutions of Problem 1.3 can be represented by

$$\text{vec}(\Gamma_1(X)) \; = \; (\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}})^{\dagger}\,\text{vec}(\Gamma_1(E)) + \left[I_{4ab} - (\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}})^{\dagger}(\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}})\right] w, \tag{3.2.6}$$

where $w \in \mathbb{R}^{4ab}$ is an arbitrary vector.

(ii) The minimal-norm real exact/least-squares is given by

$$\text{vec}(\Gamma_1(X)) \; = \; (\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}})^{\dagger}\,\text{vec}(\Gamma_1(E)). \tag{3.2.7}$$

(iii) If $\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}$ is of full-column rank (rank$[\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}] = 4ab$), then Problem 1.3 has a unique solution given by (3.2.7).

Proof. Consider $X = X + 0i + 0j + 0k$ and from Eq. (3.1.10) and Lemma 3.4, we obtain

$$\begin{aligned}
\|AXB + CXD - E\| \; &= \; \|\mathcal{F}\mathcal{M}_{u,v}\,\text{vec}(\Gamma_1(X)) - \text{vec}(\Gamma_1(E))\| \\
&= \left\| \mathcal{F}\mathcal{M}_{u,v}\,\text{vec}\left(\begin{bmatrix} X_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) - \text{vec}(\Gamma_1(E)) \right\| \\
&= \left\| \mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}\,\text{vec}(X) - \text{vec}(\Gamma_1(E)) \right\|.
\end{aligned}$$

Note that, the generalized quaternion matrix equation (1.1.2) is consistent if and only if the associated norm-error equals to zero. Thus, Eq. (1.1.2) is equivalent to a real linear system

$$\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}\,\text{vec}(\Gamma_1(X)) \; = \; \text{vec}(\Gamma_1(E)). \tag{3.2.8}$$

According to Lemma 2.6, the system (3.2.8) is consistent if and only if the rank condition (3.2.5) holds. By Lemma 2.6, the system (3.2.8) has the general solutions

$$\text{vec}(X) \; = \; (\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}})^{\dagger}\,\text{vec}(\Gamma_1(E)) + \left[I_{4ab} - (\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}})^{\dagger}(\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}})\right] w,$$

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where $w \in \mathbb{R}^{4ab}$ is arbitrary. From Lemmas 2.6 and 2.7, it can be concluded that, in both the consistent and inconsistent cases, the formula for the real exact/least-squares solution can be expressed by Eq. (3.2.6). The minimal-norm real exact/least-squares solution, as derived from Lemmas 2.6 and 2.7, is given by Eq. (3.2.7). If $\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}$ is of full-column rank, then Problem 1.3 has a unique solution, which is given by Eq. (3.2.7). □

3.3 The solution closest to a given matrix

In this section, we consider the exact/least squares solution X of Eq. (1.1.2) closest to a given matrix $Y = Y_1 + Y_2 i + Y_3 j + Y_4 k \in \mathbb{Q}_{u,v}^{a\times b}$ in Problem 1.4.

Theorem 3.7. Consider Eq. (1.1.2) for given matrices $A, C \in \mathbb{Q}_{u,v}^{m\times a}, B, D \in \mathbb{Q}_{u,v}^{b\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}, Y \in \mathbb{Q}_{u,v}^{a\times b}$ be given and $X \in \mathbb{Q}_{u,v}^{a\times b}$ is an unknown matrix. Then the general exact/least-squares solutions of X closest to given matrix Y is equivalent to finding the general exact/least-squares solution of the associated matrix equation

$$AZB + CZD \;=\; \check{E}, \tag{3.3.1}$$

where $\check{E} = E - (AYB + CYD)$.

Proof. Let us denote by $G_{\mathcal{L}}$ the set of general exact/least-squares solutions of the equation $AXB + CXD = E$. Consider the following error

$$\begin{aligned} AXB + CXD - E \;&=\; AXB + CXD - E - AYB - CYD + AYB + CYD \\ &=\; A(X-Y)B + C(X-Y)D - E + AYB + CYD \\ &=\; AZB + CZD - \check{E}. \end{aligned} \tag{3.3.2}$$

By letting $Z = X - Y$, we have that the Problem 1.4.1 is equivalent to the following minimization

$$\begin{aligned} \min_{X\in G_{\mathcal{L}}} \|X - Y\| \;&=\; \min_{\|AXB+CXD-E\| = \min} \|X - Y\| \\ &=\; \min_{\|AZB+CZD-\check{E}\| = \min} \|Z\|. \end{aligned}$$

Now, we see that the solution X' is equal to $Z_G + Y$ where Z_G is minimal norm least-squares solution of equation

$$AZB + CZD \;=\; \check{E}$$

in an unknown Z. By Theorem 3.2, the matrix Z can be solved by Eq. (3.1.9) where $\check{E} = E - (AYB + CYD)$. □

Theorem 3.8. Consider Eq. (1.1.2) for given matrices $A, C \in \mathbb{Q}_{u,v}^{m\times a}, B, D \in \mathbb{Q}_{u,v}^{b\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}, Y \in \mathbb{Q}_{u,v}^{a\times b}$ be given and $X \in \mathbb{Q}_{u,v}^{a\times b}$ is an unknown matrix. Then the imaginary-part exact/least-squares solutions of X closest to given matrix Y is equivalent to finding the imaginary exact/least-squares solution of the associated matrix equation (3.3.1).

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Proof. The proof of the theorem is similar to that of Theorem 3.7. By denote $I_{\mathcal{L}}$ be the set of exact/least-squares solutions of the equation $AXB + CXD = E$ that consist only of the imaginary part. The matrix Z can be solved by Eq. (3.2.3) in Theorem 3.5. □

Theorem 3.9. Consider Eq. (1.1.2) for given matrices $A, C \in \mathbb{Q}_{u,v}^{m\times a}, B, D \in \mathbb{Q}_{u,v}^{b\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}, Y \in \mathbb{Q}_{u,v}^{a\times b}$ be given and $X \in \mathbb{Q}_{u,v}^{a\times b}$ is an unknown matrix. Then the real exact/ least-squares solutions of X closest to given matrix Y is equivalent to finding the real exact/least-squares solution of the associated matrix equation (3.3.1).

Proof. The proof of the theorem is similar to that of Theorem 3.7. By denote $R_{\mathcal{L}}$ be the set of real exact/least-squares solutions of the equation $AXB + CXD = E$. The matrix Z can be solved by Eq. (3.2.7) in Theorem 3.6. □

3.4 Special cases of the generalized Sylvester matrix equation

This section, we investigate the special cases of the generalized Sylvester equation (1.1.2). First, we consider the Sylvester matrix equation.

Corollary 3.10. Let $A \in \mathbb{Q}_{u,v}^{m\times a}, D \in \mathbb{Q}_{u,v}^{b\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}$ be given matrix $X \in \mathbb{Q}_{u,v}^{a\times b}$ is an unknown matrix. Consider the Sylvester matrix equation

$$AX + XD = E.$$

Then the conclusions of Theorems 3.2 and 3.5 - 3.9 hold, where the matrix $\mathcal{F}$ is given by

$$\mathcal{F} = \left(\Gamma_1(I_b)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes I_{4a}\right).$$

Proof. Consider Eq. (1.1.2) and the Sylvester matrix equation

$$AX + XD = E,$$

, we obtain $B = I_b$ and $C = I_a$. Consider $\mathcal{F}$ in Theorems 3.2 and 3.5 - 3.9, then

$$\begin{aligned}
\mathcal{F} &= \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(C)\right) \\
&= \left(\Gamma_1(I_b)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(I_a)\right) \\
&= \left(\Gamma_1(I_b)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes I_{4a}\right).
\end{aligned}$$

□

The next special case is the Stein matrix equation.

Corollary 3.11. Let $A \in \mathbb{Q}_{u,v}^{m\times a}, B \in \mathbb{Q}_{u,v}^{b\times p}$, and $E \in \mathbb{Q}_{u,v}^{m\times p}$. Consider the Stein matrix equation

$$AXB + X = E$$

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in an unknown $X \in \mathbb{Q}_{u,v}^{n\times p}$. Then the conclusions of Theorems 3.2 and 3.5 - 3.9 hold, where the matrix $\mathcal{F}$ is given by

$$\mathcal{F} \;=\; \left(\Gamma_1(B) \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(I_b)^T \otimes \mathcal{R}(I_a)\right).$$

Proof. From the Stein matrix equation

$$AXB + X \;=\; E,$$

we obtain $C = I_a$ and $D = I_b$. Consider $\mathcal{F}$ in Theorems 3.2 and 3.5 - 3.9, then

$$\begin{aligned}\mathcal{F} \;&=\; \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(C)\right)\\ &=\; \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(I_b)^T \otimes \mathcal{R}(I_a)\right).\end{aligned}$$

□

In the next result, we consider Eq. (1.1.2) over the (Hamilton) quaternions. The conclusions of Theorems 3.2, 3.5, and 3.6 were investigated in [42].

Corollary 3.12. Let $A, C \in \mathbb{Q}_{-1,-1}^{m\times a}$, $B, D \in \mathbb{Q}_{-1,-1}^{p\times b}$, and $E \in \mathbb{Q}_{-1,-1}^{m\times p}$. Consider the generalized Sylvester matrix equation

$$AXB + CXD \;=\; E.$$

Then the conclusions of Theorems 3.2 and 3.5 - 3.9 hold, where the matrix $\mathcal{M}_{-1,-1}$ is given explicitly by

$$\mathcal{M}_{-1,-1} \;=\; \begin{bmatrix} I_{4ab} \\ I_b \otimes \check{N} \\ I_b \otimes \check{K} \\ I_b \otimes \check{T} \end{bmatrix} \in \mathbb{R}^{16ab\times 4ab} \tag{3.4.1}$$

and

$$\check{N} = \begin{bmatrix} 0 & -I_a & 0 & 0 \\ I_a & 0 & 0 & 0 \\ 0 & 0 & 0 & I_a \\ 0 & 0 & -I_a & 0 \end{bmatrix}, \quad \check{K} = \begin{bmatrix} 0 & 0 & -I_a & 0 \\ 0 & 0 & 0 & -I_a \\ I_a & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \end{bmatrix}, \quad \check{T} = \begin{bmatrix} 0 & 0 & 0 & -I_a \\ 0 & 0 & I_a & 0 \\ 0 & -I_a & 0 & 0 \\ I_a & 0 & 0 & 0 \end{bmatrix}.$$

Proof. From Lemma 3.1 we have

$$N = \begin{bmatrix} 0 & uI_a & 0 & 0 \\ I_a & 0 & 0 & 0 \\ 0 & 0 & 0 & -uI_a \\ 0 & 0 & -I_a & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & vI_a & 0 \\ 0 & 0 & 0 & vI_a \\ I_a & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 & 0 & -uvI_a \\ 0 & 0 & -vI_a & 0 \\ 0 & uI_a & 0 & 0 \\ I_a & 0 & 0 & 0 \end{bmatrix}.$$

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Now, substituting $u = v = -1$ in Theorems 3.2, 3.5 - 3.9, we have

$$\check{N} = N = \begin{bmatrix} 0 & -I_a & 0 & 0 \\ I_a & 0 & 0 & 0 \\ 0 & 0 & 0 & I_a \\ 0 & 0 & -I_a & 0 \end{bmatrix},$$

$$\check{K} = K = \begin{bmatrix} 0 & 0 & -I_a & 0 \\ 0 & 0 & 0 & -I_a \\ I_a & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \end{bmatrix},$$

$$\check{T} = T = \begin{bmatrix} 0 & 0 & 0 & -I_a \\ 0 & 0 & I_a & 0 \\ 0 & -I_a & 0 & 0 \\ I_a & 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$\mathcal{M}_{-1,-1} = \begin{bmatrix} I_{4ab} \\ I_b \otimes \check{N} \\ I_b \otimes \check{K} \\ I_b \otimes \check{T} \end{bmatrix} \in \mathbb{R}^{16ab \times 4ab}.$$

□

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Chapter 4

Generalized Sylvester-transpose matrix equation over generalized quaternions

In this chapter, we study on solving the generalized Sylvester-transpose matrix equation (1.1.4) where $A \in \mathbb{Q}_{u,v}^{m\times a}, B \in \mathbb{Q}_{u,v}^{b\times p}, C \in \mathbb{Q}_{u,v}^{m\times a}, D \in \mathbb{Q}_{u,v}^{b\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}$ are given, and $X \in \mathbb{Q}_{u,v}^{a\times b}$ is an unknown matrix. We divide the consideration of Eq. (1.1.4) into four cases. By considering the general exact solution, the general least-squares solution, the exact solution closest to a given matrix, and the least-squares solution closest to a given matrix, respectively.

4.1 The general solution

To solve Sylvester-transpose matrix equation 1.1.4, we study Problem 2.1 and 2.3, both defined in Chapter 1, utilizing the following lemmas for the required computations.

Lemma 4.1. For any $X_1, X_2, X_3, X_4 \in \mathbb{R}^{a\times b}$. Then

$$\operatorname{vec}\left(\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \\ X_4^T \end{bmatrix}\right) = \mathsf{P}(a,4b)(\mathsf{P}(4,b)\otimes I_b)\operatorname{vec}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}.$$

Proof. Using Lemma 2.13, we obtain

$$\begin{aligned}
\operatorname{vec}\left(\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \\ X_4^T \end{bmatrix}\right) &= \operatorname{vec}\left(\begin{bmatrix} X_1 & X_2 & X_3 & X_4 \end{bmatrix}\right)^T \\
&= \mathsf{P}(a,4b)\operatorname{vec}\begin{bmatrix} X_1 & X_2 & X_3 & X_4 \end{bmatrix} \\
&= \mathsf{P}(a,4b)\begin{bmatrix} \mathrm{col}_1(X_1) & \cdots & \mathrm{col}_b(X_1) & \cdots & \mathrm{col}_1(X_4) & \cdots & \mathrm{col}_b(X_4) \end{bmatrix}^T \\
&= \mathsf{P}(a,4b)[\mathsf{P}(4,b)\otimes I_a]\begin{bmatrix} \mathrm{col}_1(X_1)\cdots\mathrm{col}_b(X_1) \\ \mathrm{col}_1(X_2)\cdots\mathrm{col}_b(X_2) \\ \mathrm{col}_1(X_3)\cdots\mathrm{col}_b(X_3) \\ \mathrm{col}_1(X_4)\cdots\mathrm{col}_b(X_4) \end{bmatrix} \\
&= \mathsf{P}(a,4b)(\mathsf{P}(4,b)\otimes I_a)\operatorname{vec}\left(\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}\right).
\end{aligned}$$

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$\square$

Theorem 4.2. Consider Eq. (1.1.4) for given matrices $A \in \mathbb{Q}_{u,v}^{m\times a}, B \in \mathbb{Q}_{u,v}^{b\times p}, C \in \mathbb{Q}_{u,v}^{m\times b}, D \in \mathbb{Q}_{u,v}^{a\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}$ and unknown matrix $X \in \mathbb{Q}_{u,v}^{a\times b}$. Let us denote

$$\hat{\mathcal{F}} \;=\; \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(C)\right)\left(I_4 \otimes \mathsf{P}(a,4b)(\mathsf{P}(4,b)\otimes I_a)\right). \tag{4.1.1}$$

Then Eq. (1.1.4) is consistent if and only if the following rank condition holds:

$$\text{rank }[\hat{\mathcal{F}}\mathcal{M}_{u,v} \;\; \text{vec}(\Gamma_1(E))] \;=\; \text{rank }[\hat{\mathcal{F}}\mathcal{M}_{u,v}]. \tag{4.1.2}$$

In this case, we have the following:

(i) The solution of Problem 2.1 can be represented by

$$\text{vec}(\Gamma_1(X)) \;=\; (\hat{\mathcal{F}}\mathcal{M}_{u,v})^{\dagger}\,\text{vec}(\Gamma_1(E)) + \left[I_{4ab} - (\hat{\mathcal{F}}\mathcal{M}_{u,v})^{\dagger}(\hat{\mathcal{F}}\mathcal{M}_{u,v})\right]\hat{w}, \tag{4.1.3}$$

where $w \in \mathbb{R}^{4ab}$ is arbitrary.

(ii) The minimal-norm solution is given by

$$\text{vec}(\Gamma_1(X)) \;=\; (\hat{\mathcal{F}}\mathcal{M}_{u,v})^{\dagger}\,\text{vec}(\Gamma_1(E)). \tag{4.1.4}$$

(iii) If $\hat{\mathcal{F}}\mathcal{M}_{u,v}$ is of full-column rank (rank $[\hat{\mathcal{F}}\mathcal{M}_{u,v}] = 4ab$), then Problem 2.1 has a unique solution given by (4.1.4).

Proof. From Eqs. (1.1.4) and consider the associated norm-error $\|AXB + CX^TD - E\|$ by using Eq. (2.5.2),

$$\|AXB + CX^TD - E\| \;=\; \left\|\Gamma_1(AXB + CX^TD - E)\right\|.$$

Next, Theorem 2.12 and Proposition 2.23 imply that

$$\begin{aligned}
&\Gamma_1(AXB + CX^TD - E)\\
&\quad= \Gamma_1(AXB) + \Gamma_1(CX^TD) - \Gamma_1(E)\\
&\quad= \mathcal{R}(A)\mathcal{R}(X)\Gamma_1(B) + \mathcal{R}(C)\mathcal{R}(X^T)\Gamma_1(D) - \Gamma_1(E)\\
&\quad= \text{vec}\left[\mathcal{R}(A)\mathcal{R}(X)\Gamma_1(B) + \mathcal{R}(C)\mathcal{R}(X^T)\Gamma_1(D) - \Gamma_1(E)\right]\\
&\quad= \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right)\text{vec}\left(\mathcal{R}(X)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(C)\right)\text{vec}\left(\mathcal{R}(X^T)\right) - \text{vec}(\Gamma_1(E))
\end{aligned}$$

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Next, we compute vec $(\mathcal{R}(X^T))$ by using Lemma 4.1,

$$\begin{aligned}
\text{vec}\,(\mathcal{R}(X^T)) &= \text{vec}\left(\begin{bmatrix}\Gamma_1(X^T) & \Gamma_2(X^T) & \Gamma_3(X^T) & \Gamma_4(X^T)\end{bmatrix}\right) \\
&= \begin{bmatrix}\text{vec}(\Gamma_1(X^T)) \\ \text{vec}(\Gamma_2(X^T)) \\ \text{vec}(\Gamma_3(X^T)) \\ \text{vec}(\Gamma_4(X^T))\end{bmatrix} \\
&= \begin{bmatrix}\mathsf{P}(a,4b)(\mathsf{P}(4,b)\otimes I_a)\,\text{vec}(\Gamma_1(X)) \\ \mathsf{P}(a,4b)(\mathsf{P}(4,b)\otimes I_a)\,\text{vec}(\Gamma_2(X)) \\ \mathsf{P}(a,4b)(\mathsf{P}(4,b)\otimes I_a)\,\text{vec}(\Gamma_3(X)) \\ \mathsf{P}(a,4b)(\mathsf{P}(4,b)\otimes I_a)\,\text{vec}(\Gamma_4(X))\end{bmatrix} \\
&= \begin{bmatrix}I_4\otimes \mathsf{P}(a,4b)(\mathsf{P}(4,b)\end{bmatrix}\begin{bmatrix}\text{vec}(\Gamma_1(X)) \\ \text{vec}(\Gamma_2(X)) \\ \text{vec}(\Gamma_3(X)) \\ \text{vec}(\Gamma_4(X))\end{bmatrix} \\
&= \begin{bmatrix}I_4\otimes \mathsf{P}(a,4b)(\mathsf{P}(4,b)\end{bmatrix}\text{vec}\,(\mathcal{R}(X)).
\end{aligned}$$

Applying Lemma 3.1, we get

$$\begin{aligned}
\|AXB+CX^TD-E\| &= \Big\|\left[(\Gamma_1(B)^T\otimes\mathcal{R}(A)) + (\Gamma_1(D)^T\otimes\mathcal{R}(C))\,(I_4\otimes \mathsf{P}(a,4b)(\mathsf{P}(4,b))\right] \\
&\quad \cdot\text{vec}\,(\mathcal{R}(X)) - \text{vec}(\Gamma_1(E))\Big\| \\
&= \|\hat{\mathcal{F}}\,\text{vec}\,(\mathcal{R}(X)) - \text{vec}(\Gamma_1(E))\| \\
&= \|\hat{\mathcal{F}}\mathcal{M}_{u,v}\,\text{vec}(\Gamma_1(X)) - \text{vec}(\Gamma_1(E))\|.
\end{aligned}$$

Note that, the generalized quaternion matrix equation (1.1.4) is consistent if and only if the associated norm-error equals to zero. Thus, Eq. (1.1.4) is equivalent to a real linear system

$$\hat{\mathcal{F}}\mathcal{M}_{u,v}\,\text{vec}(\Gamma_1(X)) = \text{vec}(\Gamma_1(E)). \tag{4.1.5}$$

According to Lemma 2.6, the system (4.1.5) is consistent if and only if the rank condition (4.1.2) holds. By Lemma 2.6, the system (4.1.5) has the general solution

$$\text{vec}(\Gamma_1(X)) = (\hat{\mathcal{F}}\mathcal{M}_{u,v})^{\dagger}\,\text{vec}(\Gamma_1(E)) + \left[I_{4ab} - (\hat{\mathcal{F}}\mathcal{M}_{u,v})^{\dagger}(\hat{\mathcal{F}}\mathcal{M}_{u,v})\right]\hat{w},$$

where $\hat{w}\in\mathbb{R}^{4ab}$ is arbitrary. According to Lemma 2.6, Eq. (4.1.4) provides the minimal-norm solution. If $\hat{\mathcal{F}}\mathcal{M}_{u,v}$ is of full-column rank, then Problem 2.1 has a unique solution, which is also given by (4.1.4).

□

Theorem 4.3. Assume that Eq. (1.1.4) is inconsistent. In this case, we have the following:

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(i) The solution of Problem 2.3 can be represented by

$$\text{vec}(\Gamma_1(X)) \; = \; (\hat{\mathcal{F}}\mathcal{M}_{u,v})^{\dagger}\,\text{vec}(\Gamma_1(E)) + \left[I_{4ab} - (\hat{\mathcal{F}}\mathcal{M}_{u,v})^{\dagger}(\hat{\mathcal{F}}\mathcal{M}_{u,v})\right]\hat{w}, \qquad (4.1.6)$$

where $\hat{w} \in \mathbb{R}^{4ab}$ is arbitrary.

(ii) The minimal-norm least-squares solution is given by

$$\text{vec}(\Gamma_1(X)) \; = \; (\hat{\mathcal{F}}\mathcal{M}_{u,v})^{\dagger}\,\text{vec}(\Gamma_1(E)). \qquad (4.1.7)$$

(iii) If $\hat{\mathcal{F}}\mathcal{M}_{u,v}$ is of full-column rank (rank $[\hat{\mathcal{F}}\mathcal{M}_{u,v}] = 4ab$), then Problem 2.3 has a unique least-squares solution given by (4.1.7).

Proof. From the proof of Theorem 4.2, we see that Eq. (1.1.4) is equivalent to the real linear system (4.1.5). In the inconsistent case, Lemma 2.7 shows that the general solution is given by Eq. (4.1.6), while the minimal-norm least-squares solution is provided by Eq. (4.1.7). If $\hat{\mathcal{F}}\mathcal{M}_{u,v}$ is of full-column rank, then Problem 2.3 has a unique least-squares solution, which is given by (4.1.7).

□

4.2 The solution closest to a given matrix

In this section, we consider the solution X of Eq. (1.1.4) closest to a given matrix $Y = Y_1 + Y_2 i + Y_3 j + Y_4 k \in \mathbb{Q}_{u,v}^{a\times b}$ in Problem 2.2 and Problem 2.4.

Theorem 4.4. Consider Eq. (1.1.4) for given matrices $A \in \mathbb{Q}_{u,v}^{m\times a}, B \in \mathbb{Q}_{u,v}^{b\times p}, C \in \mathbb{Q}_{u,v}^{m\times b}, D \in \mathbb{Q}_{u,v}^{a\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}, Y \in \mathbb{Q}_{u,v}^{a\times b}$ be given and $X \in \mathbb{Q}_{u,v}^{a\times b}$ is an unknown matrix. Then Problem 2.2 is equivalent to finding the minimal-norm solution $Z \in \mathbb{Q}_{u,v}^{n\times p}$ of a matrix equation

$$AZB + CZ^T D \; = \; \hat{E},$$

where $\hat{E} = E - (AYB + CY^T D)$.

Proof. Let us denote by $\mathcal{S}$ the set of exact solutions of the equation $AXB + CX^T D = E$. Consider the following error

$$\begin{aligned} AXB + CX^T D - E \; &= \; AXB + CX^T D - E - AYB - CY^T D + AYB + CY^T D \\ &= \; A(X-Y)B + C(X^T - Y^T)D - E + AYB + CY^T D \\ &= \; AZB + CZ^T D - \hat{E}. \end{aligned}$$

By letting $Z = X - Y$, we have that the Problem 2.2 is equivalent to the following minimization

$$\begin{aligned} \min_{X\in\mathcal{S}} \|X - Y\| \; &= \; \min_{AXB+CX^T D\,=\,E} \|X - Y\| \\ &= \; \min_{AZB+CZ^T D\,=\,\hat{E}} \|Z\|, \end{aligned}$$

Now, we see that the solution $\acute{X}$ is equal to $Z_{\mathcal{S}} + Y$ where $Z_{\mathcal{S}}$ is minimal-norm solution of equation. □

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Theorem 4.5. Consider Eq. (1.1.4) for given matrices $A \in \mathbb{Q}_{u,v}^{m\times a}, B \in \mathbb{Q}_{u,v}^{b\times p}, C \in \mathbb{Q}_{u,v}^{m\times b}, D \in \mathbb{Q}_{u,v}^{a\times p}, E \in \mathbb{Q}_{u,v}^{m\times p}, Y \in \mathbb{Q}_{u,v}^{a\times b}$ be given and $X \in \mathbb{Q}_{u,v}^{a\times b}$ is an unknown matrix. Then Problem 2.4 is equivalent to finding the minimal-norm least-squares solution $Z \in \mathbb{Q}_{u,v}^{n\times p}$ of a matrix equation

$$AZB + CZ^TD \;=\; \hat{E},$$

where $\hat{E} = E - (AYB + CY^TD)$.

Proof. Let us denote by $\mathcal{L}$ the set of least-squares solutions of the equation $AXB + CX^TD = E$. Consider the following error

$$\left\|AXB + CX^TD - E\right\| \;=\; \left\|AZB + CZ^TD - \hat{E}\right\|,$$

By letting $Z = X - Y$, we have that the Problem 2.4 is equivalent to the following minimization

$$\begin{aligned}\min_{X\in\mathcal{L}} \|X - Y\| &= \min_{\|AXB+CX^TD-E\| = \min} \|X - Y\| \\ &= \min_{\|AZB+CZ^TD-\hat{E}\| = \min} \|Z\|,\end{aligned}$$

Now, we see that the solution $\bar{X}$ is equal to $Z_{\mathcal{L}} + Y$ where $Z_{\mathcal{L}}$ is minimal-norm solution of equation. □

4.3 Special cases of the generalized Sylvester-transpose matrix equation

Through the analysis of the Sylvester-transpose equation (1.1.4), we have identified the following significant cases.

Corollary 4.6. Let $A \in \mathbb{Q}_{u,v}^{b\times a}, D \in \mathbb{Q}_{u,v}^{a\times b}$, and $E \in \mathbb{Q}_{u,v}^{b\times b}$ be given and $X \in \mathbb{Q}_{u,v}^{a\times b}$ is an unknown matrix. Consider the Sylvester-transpose matrix equation

$$AX + X^TD \;=\; E. \tag{4.3.1}$$

Then the conclusions of Theorems 4.2 - 4.4 hold, where

$$\hat{\mathcal{F}} \;=\; \left(\Gamma_1(I_b)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes I_{4b}\right)\left(I_4 \otimes \mathsf{P}(a,4b)(\mathsf{P}(4,b) \otimes I_a)\right).$$

Proof. Consider Eq. (1.1.4) and Eq. (4.3.1), we obtain $B = C = I_b$. Consider $\hat{\mathcal{F}}$ in Theorems 4.2 - 4.4, then

$$\begin{aligned}\hat{\mathcal{F}} &= \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(C)\right)\left(I_4 \otimes \mathsf{P}(a,4b)(\mathsf{P}(4,b) \otimes I_a)\right) \\ &= \left(\Gamma_1(I_b)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(I_b)\right)\left(I_4 \otimes \mathsf{P}(a,4b)(\mathsf{P}(4,b) \otimes I_a)\right) \\ &= \left(\Gamma_1(I_b)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes I_{4b}\right)\left(I_4 \otimes \mathsf{P}(a,4b)(\mathsf{P}(4,b) \otimes I_a)\right).\end{aligned}$$

□

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The next case involves considering the equation

$$AXB \;=\; E, \tag{4.3.2}$$

which demonstrates that this case is a special instance of Eq. (1.1.2) and Eq. (1.1.4).

Corollary 4.7. Let $A \in \mathbb{Q}_{u,v}^{m\times n}, B \in \mathbb{Q}_{u,v}^{p\times q}, E \in \mathbb{Q}_{u,v}^{m\times q}$ be given and $X \in \mathbb{Q}_{u,v}^{n\times p}$ is an unknown matrix. Consider Eq. (4.3.2) Then the conclusions of Theorems 4.2 - 4.5 hold, where the matrix $\hat{\mathcal{F}}$ is given by

$$\hat{\mathcal{F}} \;=\; \Gamma_1(B)^T \otimes \mathcal{R}(A).$$

Proof. Consider Eq. (1.1.2), Eq. (1.1.4) and Eq. (4.3.2), we obtain $C = 0$ and $D = 0$. Consider $\hat{\mathcal{F}}$ in Theorems 4.2 - 4.5, then

$$\begin{aligned}\hat{\mathcal{F}} &= (\Gamma_1(B)^T \otimes \mathcal{R}(A)) + (\Gamma_1(0)^T \otimes \mathcal{R}(0)) \\ &= \Gamma_1(B)^T \otimes \mathcal{R}(A).\end{aligned}$$

□

Moreover, the linear system over a generalized quaternion is a special case of Eq. (1.1.2) and Eq. (1.1.4) where $B = I_1$ and $C = 0$.

Corollary 4.8. Let $A \in \mathbb{Q}_{u,v}^{m\times a}$ and $b \in \mathbb{Q}_{u,v}^{a}$ are given, and $x \in \mathbb{Q}_{u,v}^{n}$ is an unknown. Consider the linear system

$$Ax \;=\; b \tag{4.3.3}$$

Then Eq. (4.3.3) is consistent if and only if the following rank condition holds:

$$\text{rank }[\mathcal{R}(A)\mathcal{M}_{u,v} \;\; \text{vec}(\Gamma_1(b))] \;=\; \text{rank }[(\mathcal{R}(A)\mathcal{M}_{u,v}]. \tag{4.3.4}$$

(i) The general exact/least solution of Eq. (4.3.3) can be expressed as

$$\Gamma_1(x) \;=\; \Big(\mathcal{R}(A)\mathcal{M}_{u,v}\Big)^{\dagger}\Gamma_1(b) + \Big[I_{4a} - \Big(\mathcal{R}(A)\mathcal{M}_{u,v}\Big)^{\dagger}\Big(\mathcal{R}(A)\mathcal{M}_{u,v}\Big)\Big]\, y, \tag{4.3.5}$$

where $y \in \mathbb{R}^{4a}$ is an arbitrary vector.

(ii) The minimal-norm exact/least-squares is given by

$$\Gamma_1(x) \;=\; \Big(\mathcal{R}(A)\mathcal{M}\Big)^{\dagger}\Gamma_1(b). \tag{4.3.6}$$

(iii) If $\mathcal{R}(A)\mathcal{M}_{u,v}$ is of full-column rank (rank $[(\mathcal{R}(A)\mathcal{M}_{u,v}] = 4a$), then Eq. (4.3.3) has a unique exact/least-squares solution given by (4.3.6).

Proof. The proof of is similar to that of Theorems 3.2, 4.2, and 4.3, set $a = 1, B = I_1$ and $C = 0$. □

In the next result, we consider Eq. (1.1.4) over the (Hamilton) quaternions.

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Corollary 4.9. Let $A \in \mathbb{Q}^{m\times a}_{-1,-1}, B \in \mathbb{Q}^{b\times p}_{-1,-1}, C \in \mathbb{Q}^{m\times b}_{-1,-1}, D \in \mathbb{Q}^{a\times p}_{-1,-1}$, and $E \in \mathbb{Q}^{m\times p}_{-1,-1}$. Consider the generalized Sylvester-transpose matrix equation

$$AXB + CX^T D \; = \; E.$$

Then the conclusions of Theorems 4.2 - 4.5 hold, where the matrix $\mathcal{M}_{-1,-1}$ is given explicitly by Eq. (3.4.1).

Proof. The proof of is similar to that of Corollary 3.12. By substituting $u = v = -1$ in Theorems 4.2 - 4.5. $\square$

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Chapter 5

Numerical examples and applications

This chapter presents numerical examples to demonstrate the practical applicability of the proposed theoretical framework. Moreover, we apply our theory to certain problems in color image processing. We examine cases with varying matrix sizes and element compositions to capture different solution behaviors. As solving matrix equations over quaternions necessitates conversion to matrix equations over the real numbers resulting in large-dimensional matrix, MATLAB is used to facilitate the computational process.

5.1 Numerical examples of the generalized Sylvester matrix equation

In the following examples, we present numerical results to illustrate the theoretical concepts discussed in Chapter 3.

Example 5.1. Consider a consistent generalized Sylvester matrix equation over the necterine quaternion (i.e. $(u,v)=(1,-1)$). Let the matrices be given as:

$$A = \begin{bmatrix} -3 & i+j \\ 1 & 0 \end{bmatrix}_{2\times 2}, C = \begin{bmatrix} -1+2j & k \\ -1 & i \end{bmatrix}_{2\times 2},$$

$$B = \begin{bmatrix} 2-i+3j \\ -i \end{bmatrix}_{2\times 1}, D = \begin{bmatrix} -2+i-3j \\ -k \end{bmatrix}_{2\times 1}, E = \begin{bmatrix} 1+i+3k \\ -i+j \end{bmatrix}_{2\times 1}.$$

We compute

$$\mathcal{F} = \left(\Gamma(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma(D)^T \otimes \mathcal{R}(C)\right),$$

where

$$\mathcal{R}(A) = \begin{bmatrix} -3 & 0 & 0 & 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\Gamma_1(B)^T = \begin{bmatrix} 2 & 0 & -1 & -1 & 3 & 0 & 0 & 0 \end{bmatrix},$$

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$$\mathcal{R}(C) = \begin{bmatrix} -1 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

$$\Gamma_1(D)^T = \begin{bmatrix} -2 & 0 & 1 & 0 & -3 & 0 & 0 & -1 \end{bmatrix}.$$

Then we obtain

$$\Gamma_1(E) = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 1 & 3 & 0 \end{bmatrix}^T,$$

and

$$\text{rank } [\mathcal{FM}_{1,-1} \ \text{vec}(\Gamma_1(E))] = \text{rank } [\mathcal{FM}_{1,-1}] = 8.$$

According to Theorem 3.2, the matrix equation has a minimal-norm solution. We calculated the results using MATLAB, and the outcomes are as follows:

$$\begin{aligned} X = & \begin{bmatrix} 0.0745 & -0.5973 \\ 0.2121 & -0.5511 \end{bmatrix} + \begin{bmatrix} 0.0147 & -0.0854 \\ -0.0578 & 0.0062 \end{bmatrix} i \\ & + \begin{bmatrix} 0.0980 & -0.0363 \\ 0.0059 & 0.0373 \end{bmatrix} j + \begin{bmatrix} -0.2272 & 0.8503 \\ -0.1434 & 0.4616 \end{bmatrix} k. \end{aligned}$$

Example 5.2. As an example, we can look at a generalized Sylvester matrix equation over the split quaternions, where the multiplication rule is defined by $(u, v) = (-1, 1)$. Denote:

$$A = \begin{bmatrix} 1 & -j+2k \end{bmatrix}_{1\times 2}, \quad C = \begin{bmatrix} i-j-k & 2 \end{bmatrix}_{1\times 2},$$

$$B = \begin{bmatrix} 2+j-k \\ 1-4i \end{bmatrix}_{2\times 1}, \quad D = \begin{bmatrix} 1+3j-k \\ j \end{bmatrix}_{2\times 1}, \quad E = \begin{bmatrix} -2+3i-j+k \end{bmatrix}_{1\times 1}.$$

We can calculated $\mathcal{F}$ from

$$\mathcal{F} = \left(\Gamma(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma(D)^T \otimes \mathcal{R}(C)\right).$$

Then

$$\mathcal{R}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & -1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathcal{R}(C) = \begin{bmatrix} 0 & 2 & -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 2 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \end{bmatrix},$$

$$\Gamma_1(B)^T = \begin{bmatrix} 2 & 1 & 0 & -4 & 1 & 0 & -1 & 0 \end{bmatrix}, \quad \Gamma_1(D)^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 3 & 1 & -1 & 0 \end{bmatrix}.$$

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We obtain

$$\Gamma_1(E) \;=\; \begin{bmatrix}-2 & 3 & -1 & 1\end{bmatrix}^T,$$

and

$$\text{rank }[\mathcal{FM}_{-1,1}\ \text{vec}(\Gamma_1(E))] \;=\; \text{rank }[\mathcal{FM}_{-1,1}] \;=\; 4.$$

By Theorem 3.5, the matrix equation has a minimal-norm solution. The solution can be derived by using MATLAB, resulting in the following outcome:

$$X \;=\; \begin{bmatrix}0.1794 & 0.2679\\ 0.0476 & 0.0045\end{bmatrix} i + \begin{bmatrix}-0.0972 & -0.0392\\ 0.1360 & 0.1396\end{bmatrix} j + \begin{bmatrix}0.1794 & 0.0173\\ -0.0899 & 0.1152\end{bmatrix} k.$$

Example 5.3. Consider an inconsistent generalized Sylvester matrix equation over the Hamilton quaternion (i.e. $(u, v) = (-1, -1)$) where A, B, C, D, and D are given randomly as follows:

$$\begin{aligned} A \;=\; & \begin{bmatrix}0.8147 & 0.9134 & 0.2785 & 0.9649 & 0.9572\\ 0.9058 & 0.6324 & 0.5469 & 0.1576 & 0.4854\\ 0.1270 & 0.0975 & 0.9575 & 0.9706 & 0.8003\end{bmatrix}\\ & + \begin{bmatrix}0.1419 & 0.7922 & 0.0357 & 0.6787 & 0.3922\\ 0.4218 & 0.9595 & 0.8491 & 0.7577 & 0.6555\\ 0.9157 & 0.6557 & 0.3940 & 0.7431 & 0.1712\end{bmatrix} i\\ & + \begin{bmatrix}0.7060 & 0.0462 & 0.6948 & 0.0344 & 0.7655\\ 0.0318 & 0.0971 & 0.3171 & 0.4387 & 0.7952\\ 0.2769 & 0.8235 & 0.9502 & 0.3816 & 0.1869\end{bmatrix} j\\ & + \begin{bmatrix}0.4898 & 0.7094 & 0.6797 & 0.1190 & 0.3404\\ 0.4456 & 0.7547 & 0.6551 & 0.4984 & 0.5853\\ 0.6463 & 0.2760 & 0.1626 & 0.9597 & 0.2238\end{bmatrix} k,\\ B \;=\; & \begin{bmatrix}0.1067 & 0.0046 & 0.8173\\ 0.9619 & 0.7749 & 0.8687\end{bmatrix} + \begin{bmatrix}0.0844 & 0.2599 & 0.4314\\ 0.3998 & 0.8001 & 0.9106\end{bmatrix} i\\ & + \begin{bmatrix}0.1818 & 0.1455 & 0.8693\\ 0.2638 & 0.1361 & 0.5797\end{bmatrix} j + \begin{bmatrix}0.5499 & 0.8530 & 0.3510\\ 0.1450 & 0.6221 & 0.5132\end{bmatrix} k,\\ C \;=\; & \begin{bmatrix}0.7513 & 0.6991 & 0.5472 & 0.2575 & 0.8143\\ 0.2551 & 0.8909 & 0.1386 & 0.8407 & 0.2435\\ 0.5060 & 0.9593 & 0.1493 & 0.2543 & 0.9293\end{bmatrix}\\ & + \begin{bmatrix}0.3500 & 0.6160 & 0.8308 & 0.9172 & 0.7537\\ 0.1966 & 0.4733 & 0.5853 & 0.2858 & 0.3804\\ 0.2511 & 0.3517 & 0.5497 & 0.7572 & 0.5678\end{bmatrix} i \end{aligned}$$

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$$+\begin{bmatrix}0.0759 & 0.7792 & 0.5688 & 0.3371 & 0.3112\\ 0.0540 & 0.9340 & 0.4694 & 0.1622 & 0.5285\\ 0.5308 & 0.1299 & 0.0119 & 0.7943 & 0.1656\end{bmatrix}j$$

$$+\begin{bmatrix}0.6020 & 0.6892 & 0.0838 & 0.1524 & 0.9961\\ 0.2630 & 0.7482 & 0.2290 & 0.8258 & 0.0782\\ 0.6541 & 0.4505 & 0.9133 & 0.5383 & 0.4427\end{bmatrix}k,$$

$$D = \begin{bmatrix}0.4018 & 0.2399 & 0.1839\\ 0.0760 & 0.1233 & 0.2400\end{bmatrix} + \begin{bmatrix}0.4173 & 0.9027 & 0.4909\\ 0.0497 & 0.9448 & 0.4893\end{bmatrix}i$$

$$+\begin{bmatrix}0.3377 & 0.3692 & 0.7803\\ 0.9001 & 0.112 & 0.3897\end{bmatrix}j + \begin{bmatrix}0.2417 & 0.0965 & 0.9421\\ 0.4039 & 0.1320 & 009561\end{bmatrix}k,$$

$$E = \begin{bmatrix}0.5752 & 0.3532 & 0.0430\\ 0.0598 & 0.8212 & 0.1690\\ 0.2348 & 0.0154 & 0.6491\end{bmatrix} + \begin{bmatrix}0.7317 & 0.5470 & 0.1890\\ 0.6477 & 0.2963 & 0.6868\\ 0.4509 & 0.7447 & 0.1835\end{bmatrix}i$$

$$+\begin{bmatrix}0.3685 & 0.0811 & 0.4868\\ 0.6256 & 0.9294 & 0.4359\\ 0.7802 & 0.7757 & 0.4468\end{bmatrix}j + \begin{bmatrix}0.3063 & 0.8176 & 0.3786\\ 0.5085 & 0.7948 & 0.8116\\ 0.5108 & 0.6443 & 0.5328\end{bmatrix}k.$$

We can calculated $\mathcal{F}$ from

$$\mathcal{F} = (\Gamma(B)^T \otimes \mathcal{R}(A)) + (\Gamma(D)^T \otimes \mathcal{R}(C)).$$

We have

$$\text{rank } [\mathcal{F}\mathcal{M}_{-1,-1} \ \text{vec}(\Gamma_1(E))] = 11 \ \text{ and } \ \text{rank } [\mathcal{F}\mathcal{M}_{-1,-1}] = 10.$$

Accoring to Theorem 3.6, the matrix equation has a minimal-norm least-squares solution. We calculated the results using MATLAB, and the outcomes are as follows:

$$X = \begin{bmatrix}-0.2017 & 0.4510\\ 0.4176 & 0.0403\\ 0.1673 & 0.3264\\ -0.2968 & 0.0086\\ -0.1501 & 0.2995\end{bmatrix}.$$

Example 5.4. Consider a consistent generalized Sylvester matrix equation over the conecterine quaternion (i.e. $(u, v) = (1, 1)$). To illustrate, we denote:

$$A = \begin{bmatrix}k & 0\\ 1+2j & -1\end{bmatrix}_{2\times 2}, C = \begin{bmatrix}-1+j & i\\ -j & k\end{bmatrix}_{2\times 2},$$

$$B = \begin{bmatrix}2+4j\\ -i\end{bmatrix}_{2\times 1}, D = \begin{bmatrix}j\\ -3i\end{bmatrix}_{2\times 1}, E = \begin{bmatrix}1-2i-2j+3k\\ -i-3j-3k\end{bmatrix}_{2\times 1},$$

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. We aim to find a solution for X that is closest to a given matrix

$$Y \;=\; \begin{bmatrix} -i & j \\ k & 1 \end{bmatrix}_{2\times 2}.$$

We obtain that

$$\check{E} \;=\; E-(AYB+CYD) \;=\; \begin{bmatrix} 5-4i+2j+k \\ -2-8j+2k \end{bmatrix}.$$

Then the solution of X closest to given matrix Y is equivalent to finding the solution of the associated matrix equation

$$AZB+CZD \;=\; \check{E}.$$

We compute

$$\mathcal{F} \;=\; \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right)+\left(\Gamma_1(D)^T \otimes \mathcal{R}(C)\right),$$

where

$$\mathcal{R}(A) \;=\; \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 & 0 & -1 \end{bmatrix},$$

$$\Gamma_1(B)^T \;=\; \begin{bmatrix} 2 & 0 & 0 & -1 & 4 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{R}(C) \;=\; \begin{bmatrix} -1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_1(D)^T \;=\; \begin{bmatrix} 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then we obtain

$$\Gamma(\check{E}) \;=\; \begin{bmatrix} 5 & -2 & -4 & 0 & 2 & -8 & 1 & 2 \end{bmatrix}^T,$$

and

$$\text{rank } [\mathcal{F}\mathcal{M}_{1,1} \;\; \text{vec}(\Gamma_1(\check{E}))] \;=\; \text{rank } [\mathcal{F}\mathcal{M}_{1,1}] \;=\; 8.$$

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According to Theorem 3.7, the matrix equation has a minimal-norm solution. We calculated the results using MATLAB, and the outcomes are as follows:

$$\begin{aligned} Z &= \begin{bmatrix} -0.9098 & 0.2978 \\ -1.8750 & 0.5315 \end{bmatrix} + \begin{bmatrix} -2.3504 & 1.3139 \\ 2.0482 & 0.8205 \end{bmatrix} i \\ &+ \begin{bmatrix} 0.1862 & -0.1787 \\ 2.1223 & -0.3712 \end{bmatrix} j + \begin{bmatrix} 0.5962 & -1.1648 \\ -2.4433 & 0.1197 \end{bmatrix} k. \end{aligned}$$

Thus, the minimal-norm solution of X closest to given matrix Y is

$$\begin{aligned} X &= Z + Y \\ &= \begin{bmatrix} -0.9098 & 0.2978 \\ -1.8750 & 0.5315 \end{bmatrix} + \begin{bmatrix} -3.3504 & 1.3139 \\ 2.0482 & 0.8205 \end{bmatrix} i \\ &+ \begin{bmatrix} 0.1862 & 0.8213 \\ 2.1223 & -0.3712 \end{bmatrix} j + \begin{bmatrix} 0.5962 & -1.1648 \\ -1.4433 & 0.1197 \end{bmatrix} k. \end{aligned}$$

5.2 Numerical examples of the generalized Sylvester-transpose matrix equation

The following examples are provided to demonstrate the practical implications of the theory discussed in Chapter 4.

Example 5.5. As an example, we can look at a generalized Sylvester-transpose matrix equation over the conecterine quaternions, where the multiplication rule is defined by $(u, v) = (1, 1)$. Denote:

$$A = \begin{bmatrix} -1+i & 2 \\ 1 & i-k \end{bmatrix}_{2\times 2}, C = \begin{bmatrix} 4i & j+k \\ -2-2k & 3j \end{bmatrix}_{2\times 2},$$

$$B = \begin{bmatrix} 5+2i+3j+k \\ -3i+j \end{bmatrix}_{2\times 1}, D = \begin{bmatrix} -1+i+j \\ 1-j \end{bmatrix}_{2\times 1}, E = \begin{bmatrix} 2-i+j-5k \\ -3i+k \end{bmatrix}_{2\times 1}.$$

We compute

$$\hat{\mathcal{F}} = \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(C)\right)(I_4 \otimes \mathsf{P}(2,8)(\mathsf{P}(4,2) \otimes I_1)),$$

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where

$$\mathcal{R}(A) = \begin{bmatrix} -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 2 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

$$\Gamma_1(B)^T = \begin{bmatrix} 5 & 0 & 2 & -3 & 3 & 1 & 1 & 0 \end{bmatrix},$$

$$\mathcal{R}(C) = \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 1 & 0 & -1 \\ -2 & 0 & 0 & 0 & 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & -2 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 & 0 & 0 & 4 & 0 \\ 0 & 3 & 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 4 & 0 & 0 & 0 \\ -2 & 0 & 0 & -3 & 0 & 0 & -2 & 0 \end{bmatrix},$$

$$\Gamma_1(D)^T = \begin{bmatrix} -1 & 1 & 1 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}.$$

Then we obtain

$$\Gamma_1(E) = \begin{bmatrix} 2 & 0 & -1 & -3 & 1 & 0 & -5 & 1 \end{bmatrix}^T,$$

and

$$\text{rank } [\hat{\mathcal{F}}\mathcal{M}_{1,1} \ \text{vec}(\Gamma_1(E))] = \text{rank } [\hat{\mathcal{F}}\mathcal{M}_{1,1}] = 8.$$

According to Theorem 4.2, the matrix equation has a minimal-norm solution. We calculated the results using MATLAB, and the outcomes are as follows:

$$X = \begin{bmatrix} 0.1418 & 0.1301 \\ 0.1011 & -0.0199 \end{bmatrix} + \begin{bmatrix} 0.0192 & -0.1242 \\ 0.0156 & -0.3033 \end{bmatrix} i + \begin{bmatrix} 0.2629 & 0.0252 \\ 0.1731 & 0.1096 \end{bmatrix} j + \begin{bmatrix} 0.2778 & -0.0790 \\ -0.2334 & -0.3098 \end{bmatrix} k.$$

Example 5.6. Consider an inconsistent generalized Sylvester-transpose matrix equation over the necterine quaternion (i.e. $(u, v) = (1, -1)$). Let the matrices be given

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randomly as follows:

$$A = \begin{bmatrix} 0.2545 \\ 0.3242 \end{bmatrix} + \begin{bmatrix} 0.4018 \\ 0.4064 \end{bmatrix} i + \begin{bmatrix} 0.3862 \\ 0.6098 \end{bmatrix} j + \begin{bmatrix} 0.1669 \\ 0.1881 \end{bmatrix} k,$$

$$B = \begin{bmatrix} 0.6806 & 0.4564 \\ 0.2327 & 0.3846 \end{bmatrix} + \begin{bmatrix} 0.5386 & 0.7552 \\ 0.9913 & 0.9805 \end{bmatrix} i + \begin{bmatrix} 0.2348 & 0.0514 \\ 0.5286 & 0.7569 \end{bmatrix} j + \begin{bmatrix} 0.6020 & 0.9883 \\ 0.8572 & 0.3295 \end{bmatrix} k,$$

$$C = \begin{bmatrix} 0.0946 & 0.7696 \\ 0.3232 & 0.2341 \end{bmatrix} + \begin{bmatrix} 0.7404 & 0.8241 \\ 0.6928 & 0.8280 \end{bmatrix} i + \begin{bmatrix} 0.2934 & 0.5230 \\ 0.3094 & 0.3253 \end{bmatrix} j + \begin{bmatrix} 0.8318 & 0.5570 \\ 0.8103 & 0.2630 \end{bmatrix} k,$$

$$D = \begin{bmatrix} 0.4095 & 0.0000 \end{bmatrix} + \begin{bmatrix} 0.5409 & 0.2077 \end{bmatrix} i + \begin{bmatrix} 0.2193 & 0.3258 \end{bmatrix} j + \begin{bmatrix} 0.0959 & 0.7475 \end{bmatrix} k,$$

$$E = \begin{bmatrix} 0.7485 & 0.3381 \\ 0.5433 & 0.8323 \end{bmatrix} + \begin{bmatrix} 0.5526 & 0.8928 \\ 0.9575 & 0.3565 \end{bmatrix} i + \begin{bmatrix} 0.5464 & 0.6228 \\ 0.3467 & 0.7966 \end{bmatrix} j + \begin{bmatrix} 0.7459 & 0.8224 \\ 0.1255 & 0.0252 \end{bmatrix} k.$$

We compute

$$\hat{\mathcal{F}} = \left(\Gamma(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma(D)^T \otimes \mathcal{R}(C)\right) \left(I_4 \otimes \mathsf{P}(1,8)(\mathsf{P}(4,2) \otimes I_1)\right),$$

and

$$\text{rank}[\hat{\mathcal{F}}\mathcal{M}_{1,-1} \ \text{vec}(\Gamma_1(E))] = 9 \ \text{ and } \ \text{rank}[\hat{\mathcal{F}}\mathcal{M}_{1,-1}] = 8.$$

By Theorem 4.3, the matrix equation has a minimal-norm least-squares solution. We calculated the results using MATLAB, and the outcomes are as follows:

$$X = \begin{bmatrix} -0.8944 & 1.2187 \end{bmatrix} + \begin{bmatrix} 2.0153 & -1.7340 \end{bmatrix} i + \begin{bmatrix} 0.7580 & 0.0109 \end{bmatrix} j + \begin{bmatrix} -1.8497 & 0.5119 \end{bmatrix} k.$$

Example 5.7. Consider a consistent generalized Sylvester-transpose matrix equation over the Hamiltion quaternion (i.e. $(u, v) = (-1, -1)$). Let the matrices be given as:

$$A = \begin{bmatrix} i+j & 0 \\ -k & 2j \\ -1 & 3i \end{bmatrix}_{2\times 2}, C = \begin{bmatrix} -1 & 3+j \\ -k & i \\ -2i & 0 \end{bmatrix}_{2\times 2},$$

$$B = \begin{bmatrix} -1+3k \\ i \end{bmatrix}_{2\times 1}, D = \begin{bmatrix} 3j \\ 2i-j \end{bmatrix}_{2\times 1}, E = \begin{bmatrix} i-j+2k \\ -1+k \\ 2i-5k \end{bmatrix}_{3\times 1}.$$

We aim to find a solution for X that is closest to a given matrix

$$Y = \begin{bmatrix} i & 2k \\ -j & 0 \end{bmatrix}_{2\times 2}.$$

We obtain that

$$\hat{E} = E - (AYB + CY^T D) = \begin{bmatrix} 1+17i-7j+5k \\ -5+i-j-7k \\ -9-i+3j-3k \end{bmatrix}_{3\times 1}.$$

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Then the solution of X closest to given matrix Y is equivalent to finding the solution of the associated matrix equation

$$AZB + CZ^TD \;=\; \hat{E}.$$

We compute

$$\hat{\mathcal{F}} \;=\; \left(\Gamma(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma(D)^T \otimes \mathcal{R}(C)\right),$$

where

$$\mathcal{R}(A) \;=\; \begin{bmatrix} 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -2 & 1 & 0 \\ -1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & -1 & 0 \end{bmatrix},$$

$$\Gamma_1(B)^T \;=\; \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 3 & 0 \end{bmatrix},$$

$$\mathcal{R}(C) \;=\; \begin{bmatrix} -1 & 3 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_1(D)^T \;=\; \begin{bmatrix} 0 & 0 & 0 & 2 & 3 & -1 & 0 & 0 \end{bmatrix}.$$

Then we obtain

$$\Gamma_1(\hat{E}) \;=\; \begin{bmatrix} 1 & -5 & -9 & 17 & 1 & -1 & -7 & 1 & 3 & 5 & -7 & 3 \end{bmatrix}^T.$$

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and

$$\text{rank } [\hat{\mathcal{F}}\mathcal{M}_{-1,-1} \text{ vec}(\Gamma_1(\hat{E}))] = \text{rank } [\hat{\mathcal{F}}\mathcal{M}_{-1,-1}] = 12.$$

According to Theorem 4.4, the matrix equation has a minimal-norm solution. We calculated the results using MATLAB, and the outcomes are as follows:

$$\begin{aligned} Z = & \begin{bmatrix} -0.7189 & -0.4134 \\ -1.5984 & 0.4925 \end{bmatrix} + \begin{bmatrix} 0.4864 & 0.4886 \\ -0.7031 & -0.8860 \end{bmatrix} i \\ & + \begin{bmatrix} -0.0210 & -0.6318 \\ -0.1386 & 0.2723 \end{bmatrix} j + \begin{bmatrix} 0.0926 & -0.0834 \\ -0.4943 & -0.1117 \end{bmatrix} k. \end{aligned}$$

Thus, the minimal-norm solution of X closest to given matrix Y is

$$\begin{aligned} X = & Z + Y \\ = & \begin{bmatrix} -0.7189 & -0.4134 \\ -0.5984 & 0.4925 \end{bmatrix} + \begin{bmatrix} 1.4864 & 0.4886 \\ -0.7031 & -0.8860 \end{bmatrix} i \\ & + \begin{bmatrix} -0.0210 & -1.6318 \\ -0.1386 & 0.2723 \end{bmatrix} j + \begin{bmatrix} 2.0926 & -0.0834 \\ -0.4943 & -0.1117 \end{bmatrix} k. \end{aligned}$$

5.3 Application to image processing model

In color image processing, a color image is commonly represented as a vector or matrix, serving as an extension of grayscale image processing. Image degradation refers to the deterioration of image quality, which often manifests as blurring and overall quality reduction caused by various factors. According to [39], the general image degradation model is described as follows:

$$f = Aw + b, \tag{5.3.1}$$

where f is the observation, w is the desired image, b is additive noise, and A is a linear operator. We can restore the desired image w from f and we can applied to Theorem 3.5. We get $E = f - b, B = I_1, C = 0$ and $X = w$. By Theorem 3.5, Eq. (5.3.1) can be expressed as

$$\begin{aligned} \text{vec}\begin{pmatrix} w_r \\ w_g \\ w_b \end{pmatrix} &= (\mathcal{F}\mathcal{M}_{-1,-1}\mathcal{K})^{\dagger} \text{vec}(\Gamma_1(f-b)) \\ &= \left[\left(\Gamma_1(I_1)^T \otimes \mathcal{R}(A)\right)\mathcal{M}_{-1,-1}\mathcal{K}\right]^{\dagger} \\ &\quad \times \text{vec}(\Gamma_1(f-b)) \\ &= \left[\left((e_4^1)^T \otimes \mathcal{R}(A)\right)\mathcal{M}_{-1,-1}\mathcal{J}_a\right]^{\dagger} \\ &\quad \times \text{vec}(\Gamma_1(f-b)). \end{aligned} \tag{5.3.2}$$

We then get w_r, w_g and w_b due to the injectivity of the operator vec.

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We summarize the process of image deblurring as in the following algorithm.

Algorithm 1: Algorithm for image deblurring

(1) Input a blurred image.

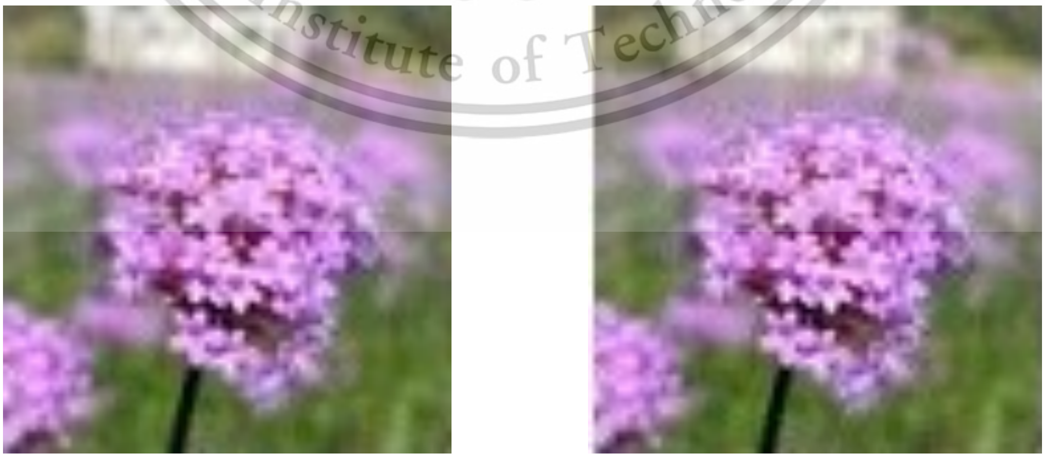
(2) Determine the value of A.

(3) Calculate w according to Eq. (5.3.2).

Note that to simplify computations on a small-scale computer, an original image could be resized to a smaller dimension.

Example 5.8. Consider a blurred image of size 80×80 pixels, as in Fig. 5.1. We can represent the image as a 3-tuples of vector $w = (w_r, w_g, w_b)$. The image can be recovered as the minimal-norm least-squares solution of the model (5.3.1). Indeed, by using (5.3.2), we get the restored image as shown in Fig. 5.2a.

Figure 5.1*:* The blurred image.



(a) *(b)*

Figure 5.2*: (a)* The restored image. *(b)* The orignal image.

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The least-squares error of each component are given by

$$\begin{aligned} \|\hat{w}_r - w_r\|^2 &= 6.2234 \times 10^{-13}, \\ \|\hat{w}_g - w_b\|^2 &= 7.7719 \times 10^{-13}, \quad \text{and} \\ \|\hat{w}_b - w_b\|^2 &= 5.1310 \times 10^{-13}, \end{aligned}$$

where $\hat{w}_r, \hat{w}_g$, and $\hat{w}_b$ are the red, green, and blue components of the original color pixel, respectively.

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Chapter 6

Conclusion and suggestion

6.1 Conclusion

We investigate a generalized Sylvester matrix equation $AXB + CXD = E$ and a generalized Sylvester-transpose matrix equation $AXB+CX^TD = E$ where A, B, C, D, E and X are rectangular compatible matrices over a generalized quaternion. For both matrix equations, we obtain their equivalent linear systems by transforming the quaternion matrix equation using the real representation of a generalized quaternions in conjuction with vectorization techniques. Thus, the solution of matrix equation over a generalized quaternion can be reformulated and addressed within the framework of real linear system. The resulting solutions are expressed in terms of Kronecker product and Moore-Penrose inverses. For the generalized Sylvester matrix equation (1.1.2), we can derive formulas of the (minimal-norm) general exact/least-squares solutions, the (minimal-norm) least-squares solutions that consist only of the imaginary part, the (minimal-norm) least-squares real solutions, and the (minimal-norm) exact/least-squares solution closest to a given matrix. Similarly, in the case of the generalized Sylvester-transpose matrix equation (1.1.4), we arrived at formulas of general exact/least-squares solutions, minimal-norm exact/least-squares solutions and exact/least-squares solution closest to a given matrix. Our work includes the study of the matrix equation $AXB = E$, the Sylvester matrix equation (1.1.1), the Sylvester-transpose matrix equation (1.1.1), the Stein matrix equation, linear systems over a generalized quaternion, and quaternionic matrix equations.

6.2 Suggestion

This thesis discusses generalized Sylvester-type matrix equations (1.1.2) and (1.1.4) under the usual matrix product, where the dimensions of A, B, C, D, E and X must be compatible. For future work, we would study generalized Sylvester-type matrix equations (1.1.2) and (1.1.4) under the general semi-tensor matrix product where coefficient matrices and the unknown have arbitrary dimensions over a generalized quaternion matrices.

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Appendices

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Appendix A

The research paper

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Research article

Exact and least-squares solutions of a generalized Sylvester-transpose matrix equation over generalized quaternions

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Abstract: We have considered a generalized Sylvester-transpose matrix equation $AXB + CX^T D = E$, where A, B, C, D, and E are given rectangular matrices over a generalized quaternion skew-field, and X is an unknown matrix. We have applied certain vectorizations and real representations to transform the matrix equation into a matrix equation over the real numbers. Thus, we have investigated a solvability condition, general exact/least-squares solutions, minimal-norm solutions, and the exact/least-squares solution closest to a given matrix. The main equation included the equation $AXB = E$ and the Sylvester-transpose equation. Our results also covered such matrix equations over the quaternions, and quaternionic linear systems.

Keywords: generalized Sylvester-transpose matrix equation; generalized quaternion matrix; minimal norm solution, least-squares solution; vector operator; Kronecker product

1. Introduction

Linear matrix equations over the field $\mathbb{R}$ of real numbers have a strong connection to certain problems in differential equations, and control and system theory [1–3]. Indeed, the Sylvester-transpose matrix equation

$$AX + X^T D = E, \tag{1.1}$$

is closely related to eigenstructure assignment [4], pole assignment [3], and fault detection in dynamical systems [5]. More generally, many authors investigated a generalized Sylvester-transpose equation:

$$AXB + CX^T D = E, \tag{1.2}$$

and a generalized Sylvester one

$$AXB + CXD = E. \tag{1.3}$$

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In the last decade, theory and computational aspects for such equations were investigated for Eq (1.1) [6] and Eq (1.2) [7–13].

Instead of the real number field, we can develop a theory for matrix equations over suitable algebraic structures, e.g., the quaternion skew-field or other skew-fields. Recall that the (Hamilton) quaternions

$$\mathbb{Q} = \left\{ q_1 + q_2 i + q_3 j + q_4 k \,\middle|\, q_1, q_2, q_3, q_4 \in \mathbb{R} \right\},$$

is a non-comutative division ring with respect to the coordinatewise addition and the Hamilton multiplication defined by

$$\begin{aligned} i^2 = j^2 = k^2 = -1,\ ij = -ji = k, \\ jk = -kj = i,\ ki = -ik = j. \end{aligned} \tag{1.4}$$

The quaternions are widely used in quantum physics [14, 15], computer graphics [16], robot trajectory planning [17], and modeling [18], etc., [19–21]. The reader can find more information about quaternions in the survey paper [22]. Moreover, if we generalize the rule (1.4), then we get a generalized quaternion [23]. Let $u, v \in \mathbb{R} - \{0\}$. Let $\mathbb{Q}_{u,v}$ be a four-dimensionl vector space over $\mathbb{R}$ with an ordered basis $\{1, i, j, k\}$, i.e.,

$$\mathbb{Q}_{u,v} = \left\{ x_1 + x_2 i + x_3 j + x_4 k \,\middle|\, x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}.$$

The additon and the scalar multiplication on $\mathbb{Q}_{u,v}$ are defined in usual ways. The multiplication of any two of $1, i, j,$ or k is defined so that 1 acts as an identity, and the following rules apply:

$$\begin{aligned} i^2 = u,\ j^2 = v,\ k^2 = ijk = -uv, \\ ij = -ji = k,\ jk = -kj = -vi,\ ik = -ki = uj. \end{aligned}$$

It turns out that $\mathbb{Q}_{u,v}$ becomes a non-commutative division ring. A famous special case $(u, v) = (-1, -1)$ of $\mathbb{Q}_{u,v}$ is known as the Hamilton quaternions. The case $(u, v) = (-1, 1)$, the case $(u, v) = (1, -1)$, and the case $(u, v) = (1, 1)$ are called the split quaternion ring, the nectarine quaternion ring, and the conectarine quaternion ring, respectively.

Matrices over quaternions are one of the main interest topics in linear algebra [22]. Matrix equations over $\mathbb{Q}$ or $\mathbb{Q}_{u,v}$ turn out to be important in various fields, e.g., computer platforms [24], image processing [25, 26], color image restoration [27], image and video inpainting [28, 29], signal processing [30] and quantum mechanics [31]. In the last decade, various authors investigated such matrix equations from theoretical points of view. The work in [32] introduced fast and robust algorithms for the eigenproblem and the QR factorization of matrices over $\mathbb{Q}$. Yuan et al. [33] proposed an explicit expression of the least-squares (LS) solution, the LS pure-imaginary solution, and the real solution of Eq (1.3) with the least norm. Zhang et al. [34] studied special LS solutions of Eq (1.3), and obtained the expressions of the minimal-norm LS solution, the pure-imaginary LS solution, and the real LS solution. Recently, Tian et al. [35] considered Hermitian solutions of Eq (1.3). Indeed, they proposed necessary and sufficient conditions for the existence of a Hermitian solution and provided the explicit general expression of the solution when it was solvable.

In this paper, we investigated the Sylvester-transpose matix Eq (1.2) where A, B, C, D, and E are given generalized quaternion matices with compatible size and X is an unknown. We have measured the associated error of a matrix by the Frobenius norm $\|\cdot\|$. Indeed, we have discussed the following problems.

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Problem 1.1. Find the solution set $\mathcal{S}$ of exact solutions to Eq (1.2). In addition, find the minimal-norm element of $\mathcal{S}$, i.e., find a matrix X^* such that

$$\|X^*\| = \min_{X \in \mathcal{S}} \|X\|.$$

Problem 1.2. Find a solution $\bar{X} \in \mathcal{S}$ closest to a given matrix $Y \in \mathbb{Q}_{u,v}^{n\times p}$, i.e., find $\bar{X}$ such that

$$\left\|\bar{X} - Y\right\| = \min_{X \in \mathcal{S}} \|X - Y\|.$$

Problem 1.3. Find the set $\mathcal{L}$ of LS solutions to Eq (1.2). In addition, find $\tilde{X}$ such that

$$\left\|\tilde{X}\right\| = \min_{X \in \mathcal{L}} \|X\|.$$

Problem 1.4. Find an LS solution of Eq (1.2) closest to a given matrix $Y \in \mathbb{Q}_{u,v}^{n\times p}$. That is, find the matrix $\acute{X}$ such that

$$\left\|\acute{X} - Y\right\| = \min_{X \in \mathcal{L}} \|X - Y\|.$$

Moreover, we have discussed certain special cases of Eq (1.2), namely Eq (1.1), the equation $AXB = E$, and the case when $u = v = -1$.

The rest of this paper is structured as follows. In Section 2, we set up basic notations and provide auxiliary tools from matrix theory in order to study matrix equations. In Section 3, we investigate Problems 1.1 and 1.2. In Section 4, we investigate Problems 1.3 and 1.4. In Section 5, we take a look at certain special cases of the main Eq (1.2). In Section 6, we provide numerical examples to illustrate our theory. Finally, we summarize the whole work in the last section.

2. Preliminaries

Let us denote by $\mathbb{R}^{m\times n}$ the set of all $m \times n$ real matrices. The set of n-dimensional real vectors is written by $\mathbb{R}^n := \mathbb{R}^{n\times 1}$. The transpose, the conjugate, the Moor-Penrose inverse, and the Frobenius norm of a matrix A are written by $A^T, \bar{A}, A^\dagger$ and $\|A\|$, respectively. The identity matrix of order n is denoted by I_n. The ith column of a matrix A is denoted by $\mathrm{col}_i(A)$.

2.1. Vectorization and Kronecker products

With each matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{s\times t}$, the (column) vector $\mathrm{V_c}(A)$ is defined as

$$\mathrm{V_c}(A) = \begin{pmatrix} a_{11} & \dots & a_{m1} & a_{12} & \dots & a_{m2} & \dots & a_{1n} & \dots & a_{mn} \end{pmatrix}^T \in \mathbb{R}^{mn},$$

and the Kronecker product of A and B is defined as

$$A \otimes B = (a_{ij}B) = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1j}B \\ a_{21}B & a_{22}B & \dots & a_{2j}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1}B & a_{i2}B & \dots & a_{ij}B \end{pmatrix} \in \mathbb{R}^{ms\times nt}.$$

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Lemma 2.1. [36] *For any $A \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{p\times q}$, we have*

$$\mathrm{V_c}(AXC) \;=\; (C^T \otimes A)\,\mathrm{V_c}(X).$$

Lemma 2.2. [36] *For any $X \in \mathbb{R}^{n\times p}$, we have*

$$\mathrm{V_c}(X^T) \;=\; \mathrm{P}(n,p)\,\mathrm{V_c}(X).$$

Here, $\mathrm{P}(n,p)$ *is a permutation matrix defined by*

$$\mathrm{P}(n,p) \;=\; \sum_{i=1}^{n}\sum_{j=1}^{p} E_{ij} \otimes E_{ij}^T,$$

where each $E_{ij} \in \mathbb{R}^{n\times p}$ has entry 1 in position (i, j) and all other entries are zero.

2.2. Real representations of generalized quaternion matrices

For any positive integers m and n, we denote the set of all $m\times n$ generalized quaternion matrices by $\mathbb{Q}_{u,v}^{m\times n}$. For each $A \in \mathbb{Q}_{u,v}^{m\times n}$, we can write

$$A \;=\; A_1 + A_2 i + A_3 j + A_4 k,$$

where $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$. We define

$$\Gamma(A) \;=\; \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \in \mathbb{R}^{4m\times n}.$$

Now, consider $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{n\times p}$, where $X_1, X_2, X_3, X_4 \in \mathbb{R}^{n\times p}$. We have

$$\begin{aligned} AX &= (A_1 + A_2 i + A_3 j + A_4 k)(X_1 + X_2 i + X_3 j + X_4 k) \\ &= A_1(X_1 + X_2 i + X_3 j + X_4 k) + A_2 i(X_1 + X_2 i + X_3 j + X_4 k) \\ &\quad + A_3 j(X_1 + X_2 i + X_3 j + X_4 k) + A_4 k(X_1 + X_2 i + X_3 j + X_4 k) \\ &= (A_1X_1 + uA_2X_2 + vA_3X_3 - uvA_4X_4) + (A_1X_2 + A_2X_1 - vA_3X_4 + vA_4X_3)i \\ &\quad + (A_1X_3 + uA_2X_4 + A_3X_1 - uA_4X_2)j + (A_1X_4 + A_2X_3 - A_3X_2 + A_4X_1)k. \end{aligned}$$

Thus,

$$\Gamma(AX) \;=\; \begin{pmatrix} A_1X_1 + uA_2X_2 + vA_3X_3 - uvA_4X_4 \\ A_1X_2 + A_2X_1 - vA_3X_4 + vA_4X_3 \\ A_1X_3 + uA_2X_4 + A_3X_1 - uA_4X_2 \\ A_1X_4 + A_2X_3 - A_3X_2 + A_4X_1 \end{pmatrix} \;=\; \mathcal{R}(A)\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}, \tag{2.1}$$

where

$$\mathcal{R}(A) \;=\; \begin{pmatrix} A_1 & uA_2 & vA_3 & -uvA_4 \\ A_2 & A_1 & vA_4 & -vA_3 \\ A_3 & -uA_4 & A_1 & uA_2 \\ A_4 & -A_3 & A_2 & A_1 \end{pmatrix},$$

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is called a real matrix representation of A. From the block columns of A, it is useful to define the following:

$$\Theta(A) = \begin{pmatrix} uA_2 \\ A_1 \\ -uA_4 \\ -A_3 \end{pmatrix}, \quad \Delta(A) = \begin{pmatrix} vA_3 \\ vA_4 \\ A_1 \\ A_2 \end{pmatrix}, \quad \Phi(A) = \begin{pmatrix} -uvA_4 \\ -vA_3 \\ uA_2 \\ A_1 \end{pmatrix} \in \mathbb{R}^{4m\times n}.$$

Clearly, the transformations $\mathrm{V_c}, \Gamma, \Theta, \Delta$, and Φ are injective. It is easy to see that

$$\|A\| = \sqrt{\|A_1\|^2 + \|A_2\|^2 + \|A_3\|^2 + \|A_4\|^2} = \|\Gamma(A)\|. \tag{2.2}$$

Proposition 2.3. [35] *Let $A, B \in \mathbb{Q}_{u,v}^{m\times n}$ and $k \in \mathbb{R}$. Then the following properties hold.*

(i) $\Gamma(A + B) = \Gamma(A) + \Gamma(B)$, $\Gamma(kA) = k(A)$.

(ii) $\mathcal{R}(AB) = \mathcal{R}(A)\mathcal{R}(B)$.

(iii) $\mathcal{R}(I_m) = I_{4m}$.

3. Consistent generalized Sylvester-transpose matrix equation

In this section, we discuss how to solve the Sylvester-transpose matrix equation

$$AXB + CX^T D = E, \tag{3.1}$$

where $A \in \mathbb{Q}_{u,v}^{m\times n}, B \in \mathbb{Q}_{u,v}^{p\times q}, C \in \mathbb{Q}_{u,v}^{m\times p}, D \in \mathbb{Q}_{u,v}^{n\times q}$, and $E \in \mathbb{Q}_{u,v}^{m\times q}$ are given matrices and $X \in \mathbb{Q}_{u,v}^{n\times p}$ is an unknown. Our idea is to transform Eq (3.1) into a real linear system. So, let us recall the following result.

Lemma 3.1. [37] *Given $K \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, we consider the linear system*

$$Kx = b. \tag{3.2}$$

Then the system (3.2) *has a solution $x \in \mathbb{R}^n$ if and only if $KK^\dagger b = b$, where $K^\dagger$ is the Moore-Penrose inverse of K. For the consistent case, we have the following:*

(i) The general solution of Eq (3.2) *is given by*

$$x = K^\dagger b + (I_n - K^\dagger K)y, \tag{3.3}$$

where $y \in \mathbb{R}^n$ is an arbitrary vector.

(ii) Among the general solution (3.3), *the minimal-norm solution is given by*

$$x = K^\dagger b. \tag{3.4}$$

(iii) If $\mathrm{rank}(K) = n$, *then the system* (3.2) *has an unique solution given by* (3.4).

The next lemmas are utilized to transform Eq (3.1) into a linear system.

Lemma 3.2. *Let A, B, C, and $D \in \mathbb{R}^{m\times n}$. Then*

$$\mathrm{V_c}\begin{pmatrix} A^T \\ B^T \\ C^T \\ D^T \end{pmatrix} = \mathrm{P}(m, 4n)(\mathrm{P}(4, n) \otimes I_m)\,\mathrm{V_c}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}.$$

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Proof. Using Lemma 2.2, we obtain

$$\mathrm{V_c}\begin{pmatrix} A^T \\ B^T \\ C^T \\ D^T \end{pmatrix} = \mathrm{V_c}\begin{pmatrix} A & B & C & D \end{pmatrix}^T = \mathrm{P}(m,4n)\,\mathrm{V_c}\begin{pmatrix} A & B & C & D \end{pmatrix}$$

$$= \mathrm{P}(m,4n)\begin{pmatrix} \mathrm{V_c}(A) \\ \mathrm{V_c}(B) \\ \mathrm{V_c}(C) \\ \mathrm{V_c}(D) \end{pmatrix} = \mathrm{P}(m,4n)(\mathrm{P}(4,n)\otimes I_m)\,\mathrm{V_c}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}.$$

Lemma 3.3. *Let $X \in \mathbb{Q}_{u,v}^{n\times p}$. Then*

$$\begin{pmatrix} \mathrm{V_c}(\Gamma(X)) \\ \mathrm{V_c}(\Theta(X)) \\ \mathrm{V_c}(\Delta(X)) \\ \mathrm{V_c}(\Phi(X)) \end{pmatrix} = \mathcal{M}\,\mathrm{V_c}(\Gamma(X)), \quad where \quad \mathcal{M} = \begin{pmatrix} I_{4np} \\ I_p \otimes R_p \otimes I_n \\ I_p \otimes S_p \otimes I_n \\ I_p \otimes T_p \otimes I_n \end{pmatrix} \in \mathbb{R}^{16np\times 4np}, \tag{3.5}$$

and

$$\begin{aligned} R_p &= \begin{pmatrix} e_2^4 & ue_1^4 & -e_4^4 & -ue_3^4 \end{pmatrix} \in \mathbb{R}^{4\times 4}, \\ S_p &= \begin{pmatrix} e_3^4 & e_4^4 & ve_1^4 & ve_2^4 \end{pmatrix} \in \mathbb{R}^{4\times 4}, \\ T_p &= \begin{pmatrix} e_4^4 & ue_3^4 & -ve_2^4 & -uve_1^4 \end{pmatrix} \in \mathbb{R}^{4\times 4}, \end{aligned}$$

where $e_i^4 = \mathrm{col}_i(I_4)$.

Proof. We compute

$$\begin{aligned} \mathrm{V_c}(\Theta(X)) &= \begin{pmatrix} u\,\mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_1) \\ -u\,\mathrm{col}_1(X_4) \\ -\mathrm{col}_1(X_3) \\ \vdots \\ u\,\mathrm{col}_p(X_2) \\ \mathrm{col}_p(X_1) \\ -u\,\mathrm{col}_p(X_4) \\ -\mathrm{col}_p(X_3) \end{pmatrix} = \begin{pmatrix} 0 & uI_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -uI_n & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_n & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & uI_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -uI_n \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -I_n & 0 \end{pmatrix} \begin{pmatrix} \mathrm{col}_1(X_1) \\ \mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_3) \\ \mathrm{col}_1(X_4) \\ \vdots \\ \mathrm{col}_p(X_1) \\ \mathrm{col}_p(X_2) \\ \mathrm{col}_p(X_3) \\ \mathrm{col}_p(X_4) \end{pmatrix} \\ &= I_p \otimes \left[\begin{pmatrix} e_2^4 & ue_1^4 & -e_4^4 & -ue_3^4 \end{pmatrix} \otimes I_n\right] \mathrm{V_c}(\Gamma(X)) \\ &= (I_p \otimes R_p \otimes I_n)\,\mathrm{V_c}(\Gamma(X)). \end{aligned}$$

With a similar process, we obtain

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$$\begin{aligned}
V_c(\Delta(X)) &= I_p \otimes \left[\left(e_3^4 \quad e_4^4 \quad ve_1^4 \quad ve_2^4\right) \otimes I_n\right] V_c(\Gamma(X)) \\
&= (I_p \otimes S_p \otimes I_n)\, V_c(\Gamma(X)),
\end{aligned}$$

and

$$\begin{aligned}
V_c(\Phi(X)) &= I_p \otimes \left[\left(e_4^4 \quad ue_3^4 \quad -ve_2^4 \quad -uve_1^4\right) \otimes I_n\right] V_c(\Gamma(X)) \\
&= (I_p \otimes T_p \otimes I_n)\, V_c(\Gamma(X)).
\end{aligned}$$

Thus, we obtain Eq (3.5).

Theorem 3.4. *Consider Eq* (3.1). *Let us denote*

$$\mathcal{W} = \left(\Gamma(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma(D)^T \otimes \mathcal{R}(C)\right)(I_4 \otimes P(n,4p)(P(4,p) \otimes I_n)). \tag{3.6}$$

(i) The matrix Eq (3.1) *has a solution if and only if*

$$(\mathcal{W}\mathcal{M})(\mathcal{W}\mathcal{M})^{\dagger} V_c(\Gamma(E)) = V_c(\Gamma(E)).$$

(ii) Then the solution set $\mathcal{S}$ of Problem 1.1 can be expressed as

$$\mathcal{S} = \left\{X \,\middle|\, V_c(\Gamma(X)) = (\mathcal{W}\mathcal{M})^{\dagger} V_c(\Gamma(E)) + \left[I_{4np} - (\mathcal{W}\mathcal{M})^{\dagger}(\mathcal{W}\mathcal{M})\right] y\right\}, \tag{3.7}$$

where $y \in \mathbb{R}^{4np}$ is an arbitrary vector.

(iii) Among all solutions (3.7), *the minimal-norm solution is given by*

$$V_c(\Gamma(X)) = (\mathcal{W}\mathcal{M})^{\dagger} V_c(\Gamma(E)). \tag{3.8}$$

(iv) When $\mathcal{W}\mathcal{M}$ is of full-column rank, Eq (3.1) *has a unique solution given by* (3.8).

Proof. From Eq (3.1), we consider the associated norm-error $\|AXB + CX^T D - E\|$. Using Eq (2.2), Proposition 2.3 and Lemma 2.1, we obtain

$$\begin{aligned}
\left\|AXB + CX^T D - E\right\| &= \left\|\Gamma(AXB + CX^T D - E)\right\| \\
&= \left\|\Gamma(AXB) + \Gamma(CX^T D) - \Gamma(E)\right\| \\
&= \left\|\mathcal{R}(A)\mathcal{R}(X)\Gamma(B) + \mathcal{R}(C)\mathcal{R}(X^T)\Gamma(D) - \Gamma(E)\right\| \\
&= \left\|V_c\left[\mathcal{R}(A)\mathcal{R}(X)\Gamma(B) + \mathcal{R}(C)\mathcal{R}(X^T)\Gamma(D) - \Gamma(E)\right]\right\| \\
&= \left\|\left(\Gamma(B)^T \otimes \mathcal{R}(A)\right) V_c(\mathcal{R}(X)) + \left(\Gamma(D)^T \otimes \mathcal{R}(C)\right) V_c\left(\mathcal{R}(X^T)\right) - V_c(\Gamma(E))\right\|.
\end{aligned}$$

By Lemma 3.2, we have

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$$\begin{aligned}
\mathrm{V_c}(\Gamma(X^T)) &= \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Gamma(X)),\\
\mathrm{V_c}(\Theta(X^T)) &= \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Theta(X)),\\
\mathrm{V_c}(\Delta(X^T)) &= \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Delta(X)),\\
\mathrm{V_c}(\Phi(X^T)) &= \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Phi(X)).
\end{aligned}$$

Using Lemma 3.3, we compute

$$\begin{aligned}
&\left(\Gamma(B)^T\otimes\mathcal{R}(A)\right)\,\mathrm{V_c}(\mathcal{R}(X))+\left(\Gamma(D)^T\otimes\mathcal{R}(C)\right)\,\mathrm{V_c}\left(\mathcal{R}(X^T)\right)-\mathrm{V_c}(\Gamma(E))\\
&= \left(\Gamma(B)^T\otimes\mathcal{R}(A)\right)\,\mathrm{V_c}(\mathcal{R}(X))+\left(\Gamma(D)^T\otimes\mathcal{R}(C)\right)\begin{pmatrix}\mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Gamma(X))\\ \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Theta(X))\\ \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Delta(X))\\ \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Phi(X))\end{pmatrix}\\
&\quad -\mathrm{V_c}(\Gamma(E))\\
&= (\Gamma(B)^T\otimes\mathcal{R}(A))\begin{pmatrix}\mathrm{V_c}(\Gamma(X))\\ \mathrm{V_c}(\Theta(X))\\ \mathrm{V_c}(\Delta(X))\\ \mathrm{V_c}(\Phi(X))\end{pmatrix}\\
&\quad +(\Gamma(D)^T\otimes\mathcal{R}(C))(I_4\otimes\mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n))\begin{pmatrix}\mathrm{V_c}(\Gamma(X))\\ \mathrm{V_c}(\Theta(X))\\ \mathrm{V_c}(\Delta(X))\\ \mathrm{V_c}(\Phi(X))\end{pmatrix}-\mathrm{V_c}(\Gamma(E))\\
&= \left[\left(\Gamma(B)^T\otimes\mathcal{R}(A)\right)+\left(\Gamma(D)^T\otimes R(C)\right)(I_4\otimes P(n,4p)(P(4,p)\otimes I_n))\right]\begin{pmatrix}V_c(\Gamma(X))\\ V_c(\Theta(X))\\ V_c(\Delta(X))\\ V_c(\Phi(X))\end{pmatrix}-V_c(\Gamma(E))\\
&= \left[\left(\Gamma(B)^T\otimes\mathcal{R}(A)\right)+\left(\Gamma(D)^T\otimes\mathcal{R}(C)\right)(I_4\otimes\mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n))\right]\mathcal{M}\,\mathrm{V_c}(\Gamma(X))-\mathrm{V_c}(\Gamma(E))\\
&= \mathcal{WM}\,\mathrm{V_c}(\Gamma(X))-\mathrm{V_c}(\Gamma(E)).
\end{aligned}$$

So, the generalized quaternion matrix Eq (3.1) is equivalent to a real linear system

$$\mathcal{WM}\,\mathrm{V_c}(\Gamma(X)) = \mathrm{V_c}(\Gamma(E)). \tag{3.9}$$

By Lemma 3.1, the system (3.9) has the general solution

$$\mathrm{V_c}(\Gamma(X)) = (\mathcal{WM})^\dagger\,\mathrm{V_c}(\Gamma(E))+[I_{4np}-(\mathcal{WM})^\dagger(\mathcal{WM})]y,$$

where $y\in\mathbb{R}^{4np}$ is an arbitrary vector. The assertions (*iii*) and (*iv*) now follow from Lemma 3.1.

Theorem 3.5. *Consider Eq* (3.1). *Let $Y\in\mathbb{Q}_{u,v}^{n\times p}$ be given. Then Problem 1.2 is equivalent to finding the minimal-norm solution $Z\in\mathbb{Q}_{u,v}^{n\times p}$ of a matrix equation*

$$AZB+CZ^TD = \hat{E},$$

where $\hat{E}=E-(AYB+CY^TD)$.

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Proof. Letting $Z = X - Y$, we consider the following error

$$\begin{aligned} AXB + CX^T D - E &= AXB + CX^T D - E - AYB - CY^T D + AYB + C\mathcal{Y}^T D \\ &= A(X - Y)B + C(X^T - Y^T)D - E + AYB + CY^T D \\ &= AZB + CZ^T D - \hat{E}. \end{aligned}$$

Thus, Problem 1.2 is equivalent to the following minimization:

$$\begin{aligned} \min_{AXB+CX^T D = E} \|X - Y\| &= \min_{AXB+CX^T D = E} \|Z\| \\ &= \min_{AZB+CZ^T D = \hat{E}} \|Z\|, \end{aligned}$$

as desired.

4. Inconsistent generalized Sylvester-transpose matrix equation

In this section, we investigate Eq (3.2) when it is inconsistant. We seek for least-squares (LS) solutions with minimal-norm or the closest solution to a given matrix. Recall the following result:

Lemma 4.1. [37] *Consider the linear system* (3.2) *in the inconsistent case. We have the following:*

(i) The general LS solutions of Eq (3.2) *are given by* (3.3)*, where* $y \in \mathbb{R}^{4np}$ *is an arbitrary vector.*
(ii) Among such LS solutions, the minimal-norm solution is given by (3.4).
(iii) If $\operatorname{rank}(K) = 4np$, *then the system* (3.2) *has a unique LS solution given by* (3.4).

Theorem 4.2. *Suppose that Eq* (3.2) *is inconsistent. Denote* $\mathcal{W}$ *as in* (3.6).

(i) Then the solution set $\mathcal{L}$ *of Problem 1.3 can be expressed as*

$$\mathcal{L} = \Big\{X \;\Big|\; \mathrm{V_c}(\Gamma(X)) = (\mathcal{W}\mathcal{M})^{\dagger}\, \mathrm{V_c}(\Gamma(E)) + \big[I_{4np} - (\mathcal{W}\mathcal{M})^{\dagger}(\mathcal{W}\mathcal{M})\big] y\Big\}, \tag{4.1}$$

where $y \in \mathbb{R}^{4np}$ *is an arbitrary vector.*
(ii) Among such solutions (4.1)*, the minimal-norm solution is given by* (3.8).
(iii) Moreover, if $\operatorname{rank}(\mathcal{W}\mathcal{S}) = 4np$, *Eq* (3.1) *has a unique LS solution given by* (3.8).

Proof. From the proof of Theorem 3.4, we see that Eq (3.1) is equivalent to the real linear system (3.9). Lemma 4.1 now implies that the LS solutions of Eq (3.1) are given by

$$\mathrm{V_c}(\Gamma(X)) = (\mathcal{W}\mathcal{M})^{\dagger}\, \mathrm{V_c}(\Gamma(E)) + [I_{4np} - (\mathcal{W}\mathcal{M})^{\dagger}(\mathcal{W}\mathcal{M})]y,$$

where $y \in \mathbb{R}^{4np}$ is an arbitrary vector. The assertions (*ii*) and (*iii*) also follow from Lemma 4.1.

Theorem 4.3. *Consider Eq* (3.1)*. Let* $Y \in \mathbb{Q}_{u,v}^{n\times p}$ *be given. Then Problem 1.4 is equivalent to finding the minimal-norm least-squares solution* $Z \in \mathbb{Q}_{u,v}^{n\times p}G$ *of a matrix equation*

$$AZB + CZ^T D = \hat{E},$$

where $\hat{E} = E - (AYB + CY^T D)$.

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Proof. From the proof of Theorem 3.5, we have

$$\left\|AXB + CX^T D - E\right\| = \left\|AZB + CZ^T D - \hat{E}\right\|,$$

where $Z = X - Y$. Thus, Problem 1.4 is equivalent to the following:

$$\begin{aligned} \min_{X \in \mathcal{L}} \|X - Y\| &= \min_{\left\|AXB + CX^T D - E\right\| = \min} \|X - Y\| \\ &= \min_{\left\|AXB + CX^T D - E\right\| = \min} \|Z\| \\ &= \min_{\left\|AZB + CZ^T D - \hat{E}\right\| = \min} \|Z\|, \end{aligned}$$

as desired.

5. Special cases

From the Sylvester-transpose Eq (1.2), we can investigate its certain special cases.

5.1. Matrix equations

Corollary 5.1. *Let $A \in \mathbb{Q}_{u,v}^{m\times n}$, $B \in \mathbb{Q}_{u,v}^{p\times q}$, and $E \in \mathbb{Q}_{u,v}^{m\times q}$. Consider the matrix equation*

$$AXB = E$$

in an unknown $X \in \mathbb{Q}_{u,v}^{n\times p}$. Then the conclusions of Theorems 3.4, 3.5, 4.2 and 4.3 hold, where the matrix $\mathcal{W}$ is given by

$$\mathcal{W} = \Gamma(B)^T \otimes \mathcal{R}(A).$$

Proof. We set $C = 0$ and $D = 0$ in those theorems.

The next special case is the Sylvester-transpose matrix equation

$$AX + X^T D = E. \tag{5.1}$$

Corollary 5.2. *Let $A \in \mathbb{Q}_{u,v}^{p\times n}$, $D \in \mathbb{Q}_{u,v}^{n\times p}$, and $E \in \mathbb{Q}_{u,v}^{p\times p}$. Consider Eq (5.1) in an unknown $X \in \mathbb{Q}_{u,v}^{n\times p}$. Then the conclusions of Theorems 3.4, 3.5, 4.2 and 4.3 hold, where*

$$\mathcal{W} = \left(\Gamma(I_p) \otimes \mathcal{R}(A)\right) + \left(\Gamma(D)^T \otimes I_{4p}\right)(I_4 \otimes \mathrm{P}(n, 4p)(\mathrm{P}(4, p) \otimes I_n)).$$

Proof. We set $B = C = I_p$ in those theorems.

In the next result, we consider Eq (1.2) over the quaternions.

Corollary 5.3. *Let $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{p\times q}$, $C \in \mathbb{Q}^{m\times p}$, $D \in \mathbb{Q}^{n\times q}$, and $E \in \mathbb{Q}^{m\times q}$. Consider the matrix equation*

$$AXB + CX^T D = E.$$

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Then the conclusions of Theorems 3.4, 3.5, 4.2 and 4.3 hold, where the matrix $\mathcal{M}$ is given explicitly by

$$\mathcal{M} = \begin{pmatrix} I_{4np} \\ I_p \otimes \acute{R}_P \\ I_p \otimes \acute{S}_P \\ I_p \otimes \acute{T}_P \end{pmatrix} \in \mathbb{R}^{16np \times 4np}, \tag{5.2}$$

and

$$\acute{R}_p = \begin{pmatrix} 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \end{pmatrix}, \quad \acute{S}_p = \begin{pmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & -I_n \\ I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \end{pmatrix}, \quad \acute{T}_p = \begin{pmatrix} 0 & 0 & 0 & -I_n \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Set $u = v = -1$ in those theorems.

5.2. Linear systems over the quaternions

In this subsection, we consider a quaternion linear system

$$Ax = b, \tag{5.3}$$

where $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$ are given, and $x \in \mathbb{Q}^n$ is an unknown.

Corollary 5.4. *Consider the linear system* (5.3). *Denote $\mathcal{M}$ as in* (5.2) *where* $p = 1$.

(i) Then the system (5.3) *has a solution if and only if*

$$\big(\mathcal{R}(A)\mathcal{M}\big)\big(\mathcal{R}(A)\mathcal{M}\big)^{\dagger}\Gamma(b) = \Gamma(b).$$

(ii) The general exact/LS solution of Eq (5.3) *can be expressed as*

$$\Gamma(x) = \big(\mathcal{R}(A)\mathcal{M}\big)^{\dagger}\Gamma(b) + \Big[I_{4n} - \big(\mathcal{R}(A)\mathcal{M}\big)^{\dagger}\big(\mathcal{R}(A)\mathcal{M}\big)\Big]y, \tag{5.4}$$

where $y \in \mathbb{R}^{4n}$ is an arbitrary vector.

(iii) Among all solutions (5.4), *the minimal-norm solution is given by*

$$\Gamma(x) = \big(\mathcal{R}(A)\mathcal{M}\big)^{\dagger}\Gamma(b). \tag{5.5}$$

(iv) When $\mathcal{R}(A)\mathcal{M}$ is of full-column rank, Eq (5.3) *has a unique exact/LS solution given by* (5.5).

Proof. From Theorems 3.4 and 4.2, set $p = 1$, $B = I_1$, and $C = 0$.

A conjugate gradient type to solve the quaternion linear system (5.3) is the quaternion generalized minimal residual method (QGMRES) [38]. Now, we discuss the following problem.

Problem 5.5. Let $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$ be given. Find an LS solution of Eq (5.3) closest to a given vector $h \in \mathbb{Q}^n$. That is, find the vector $\tilde{x}$ such that

$$\|\tilde{x} - h\| = \min_{\|Ax-b\| = \min} \|x - h\|.$$

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Corollary 5.6. *Consider Eq* (5.3)*. Let $h \in \mathbb{Q}^n$ be given. Then the solution of Problem 5.5 is given by $x = h + z$ where*

$$\Gamma(z) \;=\; \left(\mathcal{R}(A)\mathcal{M}\right)^{\dagger}\left[\Gamma(b) - \mathcal{R}(A)\Gamma(h)\right].$$

Here, the matrix $\mathcal{M}$ is given by (5.2) *where $p = 1$.*

Proof. From the case $B = 1$ and $C = 0$ in Theorem 4.3, we see that Problem 5.5 is equivalent to finding a minimal-norm LS solution z of the linear system

$$Az \;=\; b - Ah.$$

Indeed, the desired solution is $x = h + z$. From Corollary 5.4 and Eq (2.1), we obtain

$$\begin{aligned}\Gamma(z) &= \left(\mathcal{R}(A)\mathcal{M}\right)^{\dagger}\Gamma(b - Ah) \\ &= \left(\mathcal{R}(A)\mathcal{M}\right)^{\dagger}\left[\Gamma(b) - \mathcal{R}(A)\Gamma(h)\right].\end{aligned}$$

6. Numerical examples

In this section, we provide numerical examples to illustrate our results.

Example 6.1. Consider the generalized Sylvester-transpose matrix equation $AXB + CX^T D = E$ over the split quaternions (i.e., $(u, v) = (-1, 1)$),

$$A = \begin{pmatrix}1 & i + 2j\end{pmatrix}_{1\times 2}, \quad C = \begin{pmatrix}-1 & -i + j + k\end{pmatrix}_{1\times 2},$$
$$B = \begin{pmatrix}i + k \\ 2 + 3j\end{pmatrix}_{2\times 1}, \quad D = \begin{pmatrix}2i \\ 3 - k\end{pmatrix}_{2\times 1}, \quad E = \begin{pmatrix}-1 + 4i + 3j + k\end{pmatrix}_{1\times 1}.$$

Then we have

$$\mathcal{R}(A) = \begin{pmatrix}1 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0\end{pmatrix}, \quad \Gamma(B)^T = \begin{pmatrix}0 & 2 & 1 & 0 & 0 & -3 & 1 & 0\end{pmatrix},$$

$$\mathcal{R}(C) = \begin{pmatrix}-1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0\end{pmatrix}, \quad \Gamma(D)^T = \begin{pmatrix}0 & 3 & 2 & 0 & 0 & 0 & 0 & -1\end{pmatrix},$$

and

$$\begin{aligned}\mathcal{W} &= \left(\Gamma(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma(D)^T \otimes \mathcal{R}(C)\right), \\ \mathcal{M} &= I_4 \otimes \mathrm{P}(2, 8)\left(\mathrm{P}(4, 2) \otimes I_2\right), \quad \Gamma(E) = \begin{pmatrix}-1 & 4 & 3 & 1\end{pmatrix}^T.\end{aligned}$$

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According to Theorem 3.4, the matrix equation has a unique solution, computed via MATLAB as follows:

$$X = \begin{pmatrix} 0 & 0 \\ 0 & -0.2709 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0.0739 \end{pmatrix} i + \begin{pmatrix} 0 & 0 \\ 0 & -0.8079 \end{pmatrix} j + \begin{pmatrix} 0 & 0 \\ 0 & -0.8079 \end{pmatrix} k.$$

Example 6.2. Consider the matrix equation $AXB + CX^T D = E$ over the split quaternions, i.e., $(u, v) = (-1, 1)$. Here, we are given the matrices A, B, C, D, and E as in Example 6.1, and we will find a solution X closest to a given matrix

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

We obtain

$$\hat{E} = E - (AYB + CY^T D) \text{ and } \Gamma(\hat{E}) = \begin{pmatrix} -2 & 4 & 2 & -8 \end{pmatrix}^T.$$

Using Theorem 3.5 and MATLAB, we obtain:

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0.3481 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0.6404 \end{pmatrix} i + \begin{pmatrix} 0 & 0 \\ 0 & 0.3350 \end{pmatrix} j + \begin{pmatrix} 0 & 0 \\ 0 & -0.1938 \end{pmatrix} k.$$

Thus, we get the desired solution:

$$X = Z + Y = \begin{pmatrix} 1 & 0 \\ 0 & 0.3481 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -0.3596 \end{pmatrix} i + \begin{pmatrix} 0 & 0 \\ 0 & 0.3350 \end{pmatrix} j + \begin{pmatrix} 0 & 0 \\ 0 & -0.1938 \end{pmatrix} k.$$

7. Conclusions

We investigated a generalized Sylvester-transpose matrix equation $AXB + CX^T D = E$, where A, B, C, D, E, and X are matrices over a generalized quaternion skew-field. When all matrix dimensions were compatible, we provided a criterion for the equation to have a solution, involving Moore-Penrose inverses of associated matrices. Applying vectorizations and real representations of generalized quaternion matrices, we derived formulas of general exact/least-squares solutions, the minimal-norm solution, and the solution closest to a given matrix. Our results included the equation $AXB = E$ and the Sylvester-transpose equation, quaternionic matrix equations, and quaternionic linear systems.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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Appendix B

The research paper

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IAENG International Journal of Applied Mathematics

Sylvester-Type Matrix Equations over Generalized Quaternions with Applications to Color Image Processing

Janthip Jaiprasert and Pattrawut Chansangiam

***Abstract*—This paper focuses on solving a generalized Sylvester matrix equation over a generalized quaternion skew-field. We apply a real representation of generalized quaternion matrices and certain vectorizations to transform the matrix equation into a real linear system. Then, we obtain an equivalent condition for the consistency of the matrix equation. We can derive vector representations of the (minimal-norm) least-squares (LS) solution, the (minimal-norm) pure-imaginary LS solution, the (minimal-norm) real LS solution, and the (minimal-norm) LS solution closet to a given matrix. Such solutions are expressed in terms of Kronecker products and Moore-Penrose inverses. When the matrix equation is consistent, such LS solutions become exact solutions. This work includes Sylvester and Stein matrix equations over generalized quaternions, and quaternionic matrix equations. We also propose a gradient-descent iterative (GDI) algorithm to solve the transformed linear system. Moreover, the theory can be applied to a color image processing model.**

***Index Terms*—Sylvester-type matrix equation, matrix over a generalized quaternion, least-squares solution, Kronecker product, iterative algorithm, RGB color model**

I. Introduction

THE paper focuses on matrix equations over quaternion-like structures. Let us recall that for any pair (u, v) of nonzero real numbers, we can associate with a four-dimensional algebra $\mathbb{Q}_{u,v}$ over the real number field $\mathbb{R}$. The algebra $\mathbb{Q}_{u,v}$ of generalized quaternions is formed by its ordered basis $\{1, i, j, k\}$ where 1 acts as the multiplicative identity. Besides the addition and the scalar multiplication on $\mathbb{Q}_{u,v}$, the product of any two of i, j, k is defined by the following multiplication rules:

$$i^2 = u,\ j^2 = v,\ k^2 = ijk = -uv,$$
$$ij = -ji = k,\ jk = -kj = -vi,\ ik = -ki = uj.$$

Thus, every element of $\mathbb{Q}_{u,v}$ can be represented as

$$q = q_1 + q_2 i + q_3 j + q_4 k \in \mathbb{Q}_{u,v}$$

where $q_1, q_2, q_3, q_4 \in \mathbb{R}$. We call q_1 the real part of q, while the vector (q_2, q_3, q_4) is called the imaginary part of q. The set $\mathbb{Q}_{u,v}$, together with the addition and the above multiplication, forms a skew field. The famous particular case of $\mathbb{Q}_{u,v}$ is when $(u, v) = (-1, -1)$, known as the (Hamilton) quaternions $\mathbb{Q}$. Other interesting cases of $\mathbb{Q}_{u,v}$ are the split quaternion ring $(u = -1, v = 1)$, the nectarine quaternion ring $(u = 1, v = -1)$, and the conectarine quaternion ring $(u = v = 1)$.

Quaternionic matrix theory is an attractive area in linear algebra; see e.g. [1]. Matrix equations over quaternion-like structures play an important role in computer platform [2], signal processing [3], [4], quantum mechanics [5], [6], and image processing [7], [8], [9]. In color image processing, a color image can be represented as a vector or matrix. According to RGB color model, the color information of a pixel can be represented as

$$q = 0 + q_r i + q_g j + q_b k \in \mathbb{Q},$$

where q_r, q_g, q_b are the red/green/blue component of the color pixel, respectively. The addition on $\mathbb{Q}$ represents the addition between different colors, so that their light spectra adds up. An RGB color image consists of many array pixels, and can be represented as a vector

$$\dot{y} = 0 + y_r i + y_g j + y_b k \in \mathbb{Q}^a,$$

where $y_r, y_g, y_b \in \mathbb{R}^a$. A quaternion-based sparse representation model [10] says that $\dot{y} = \dot{D}\dot{a}$, where

$$\dot{D} = D_s + D_r i + D_g j + D_b k \in \mathbb{Q}^{a \times b} \text{ and}$$
$$\dot{a} = a_1 + a_2 i + a_2 j + a_4 k \in \mathbb{Q}^b.$$

The matrix $\dot{D}$ and the vector $\dot{a}$ are called a dictionary matrix and a sparse coefficient vector.

Linear matrix equations over the field $\mathbb{R}$ arise naturally in pure and applied mathematics, e.g. differential equations, and mathematical control theory. The famous Sylvester matrix equation:

$$AX + XD = E \tag{1}$$

plays an important role in model reduction [11], numerical methods for differential equations [12], [13] and control systems [14], [15]. Besides, there are many researchers studied a generalized Sylvester equation:

$$AXB + CXD = E \tag{2}$$

see e.g. [16], [17], [18], [19], [20]. In particular, if C and D are identity matrices with suitable size, Eq. (2) is reduced to the Stein matrix equation. In the last decent, many author investigated Sylvester-type matrix equation over $\mathbb{Q}$ or $\mathbb{Q}_{u,v}$. In 2014, Shi-Fang Yuan [21] derived explicit forms of the least-squares (LS) solution, the imaginary LS solution, and the real solution of Eq. (2). Later, F. Zhang et al. [22] solved Eq. (2) over $\mathbb{Q}$ for the minimal-norm LS solution, the

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imaginary LS solution, and the real LS solution. Furthermore, Tian et al. [23] investigated Eq. (2) over $\mathbb{Q}_{u,v}$. Indeed, they provided criterion for an existence of a Hermitian solution, and derived an explicit formula of the solution. Recently, the exact and least-squares solutions of a generalized Sylvester-transpose matrix equation over $\mathbb{Q}_{u,v}$ were studied in [24].

This paper is a continuation of the work [22]. We consider the generalized Sylvester matrix equation (2) where the given coefficients and the unknown are compatible rectangular matrices over $\mathbb{Q}_{u,v}$. We will discuss the following general/specific types of solutions.

Problem 1. *Find the general exact/least-squares solutions $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{a\times b}$ of Eq. (2). In addition, among the general solutions, find the minimal-norm one.*

Problem 2. *Find all exact/least-squares solutions of Eq. (2) that consist only of the imaginary part, i.e., $X = X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{a\times b}$. Among such solutions, find the minimal-norm one.*

Problem 3. *Find all exact/least-squares solutions of Eq. (2) that consist only of the real part. Among such solutions, find the minimal-norm one.*

Problem 4. *Let $Y \in \mathbb{Q}_{u,v}^{a\times b}$ be given.*

1.4.1 *Find the general exact/least-squares solution of Eq. (2) closet to Y.*
1.4.2 *Find the imaginary-part exact/least-squares solution of Eq. (2) closet to Y.*
1.4.3 *Find the real exact/least-squares solution of Eq. (2) closet to Y.*

Problems 1-4 include both consistent and inconsistent cases. When the associated least-squares error is zero, such least-squares solutions become an exact solution. We use real representations of generalized quaternion matrices and vectorizations of real matrices to transform Eq. (2) into a real linear system. So, we can derive the desired solutions of Problems 1-4 in terms of the Kronecker product and Moore-Penrose inverses; see Sections III and IV. We discuss certain special cases of Eq. (2), namely, Eq. (1), the Stein equation $AXB + X = E$, and the split quaternions case; see Section V. In Section VI, we provide numerical examples to illustrate the theory. In Section VII, we propose an iterative algorithm to solve the linear system associated with Eq. (2). In Section VIII, we apply the theory to a color image processing model. Finally, we summarize the whole work in the last section.

Next, we prepare basic notations and recall prerequisite results from classical and quaternionic matrix theory in Section II. These results involve real linear systems, vectorizations, the Kronecker product, and real representations for the generalized quaternion matrices.

II. Preliminaries from Classical and Quaternionic Matrix Theory

Throughout this paper, let $u, v \in \mathbb{R} - \{0\}$. Denote the set of all m-by-n real matrices and generalized quaternion matrices by $\mathbb{R}^{m\times n}$ and $\mathbb{Q}_{u,v}^{m\times n}$, respectively. For any matrix A, its transpose, its Moore-Penrose inverse, and its Frobenius norm are denoted by $A^T, A^\dagger$ and $\|A\|$, respectively. Let us denote the i-th column of matrix A by $\mathrm{col}_i(A)$. The identity matrix of order n is denoted by I_n, and we define $e_i^n = \mathrm{col}_i(A)$. Recall the following results.

Lemma 5. *(e.g. [25]) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then the linear system*

$$Ax = b, \tag{3}$$

has a solution $x \in \mathbb{R}^n$ if and only if $AA^\dagger b = b$, or equivalently, $\mathrm{rank}[A \;\; b] = \mathrm{rank}\; A$. *In both consistent and inconsistent cases, the following statements hold:*

(i) *The general exact/LS solutions of Eq. (3) can be expressed by the formula*

$$x = A^\dagger b + (I_n - A^\dagger A)w, \tag{4}$$

where $w \in \mathbb{R}^n$ is arbitrary.

(ii) *The minimal-norm exact/LS solution of Eq. (4) is given by the formula*

$$x = A^\dagger b. \tag{5}$$

(iii) *Eq. (3) has a unique exact/LS solution given by the formula (5) if A is of full-column rank (i. e. $\mathrm{rank}[A] = n$).*

Recall that the operator $\mathrm{V_c}(\cdot)$ transforms any matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ to be a column vector

$$\mathrm{V_c}(A) = \begin{pmatrix}\mathrm{col}_1(A) & \mathrm{col}_2(A) & \ldots & \mathrm{col}_n(A)\end{pmatrix}^T \in \mathbb{R}^{mn}.$$

The Kronecker product of $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{s\times t}$ is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \ldots & a_{1j}B \\ a_{21}B & a_{22}B & \ldots & a_{2j}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \ldots & a_{mn}B \end{pmatrix} \in \mathbb{R}^{ms\times nt}.$$

Lemma 6. *(e.g. [26]) For any $A \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{p\times q}$, we have*

$$\mathrm{V_c}(AXB) = (B^T \otimes A)\,\mathrm{V_c}(X).$$

For any generalized quaternion matrix $A \in \mathbb{Q}_{u,v}^{m\times n}$, we can write

$$A = A_1 + A_2 i + A_3 j + A_4 k$$

with real coefficients $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$. We denote the column block of real coefficients by

$$\Gamma_1(A) = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \in \mathbb{R}^{4m\times n}.$$

Now, consider $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{n\times p}$, where $X_1, X_2, X_3, X_4 \in \mathbb{R}^{n\times p}$. A direct computation reveals that

$$\begin{aligned}\Gamma_1(AX) &= \begin{pmatrix} A_1X_1 + uA_2X_2 + vA_3X_3 - uvA_4X_4 \\ A_1X_2 + A_2X_1 - vA_3X_4 + vA_4X_3 \\ A_1X_3 + uA_2X_4 + A_3X_1 - uA_4X_2 \\ A_1X_4 + A_2X_3 - A_3X_2 + A_4X_1 \end{pmatrix} \\ &= \mathcal{R}(A)\Gamma_1(X), \end{aligned} \tag{6}$$

where

$$\mathcal{R}(A) = \begin{pmatrix} A_1 & uA_2 & vA_3 & -uvA_4 \\ A_2 & A_1 & vA_4 & -vA_3 \\ A_3 & -uA_4 & A_1 & uA_2 \\ A_4 & -A_3 & A_2 & A_1 \end{pmatrix}.$$

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We call $\mathcal{R}(A)$ a real-matrix representation of A. We define the following representations of A :

$$\Gamma_2(A) \;=\; \begin{pmatrix} uA_2 \\ A_1 \\ -uA_4 \\ -A_3 \end{pmatrix},$$

$$\Gamma_3(A) \;=\; \begin{pmatrix} vA_3 \\ vA_4 \\ A_1 \\ A_2 \end{pmatrix}, \text{ and}$$

$$\Gamma_4(A) \;=\; \begin{pmatrix} -uvA_4 \\ -vA_3 \\ uA_2 \\ A_1 \end{pmatrix} \in \mathbb{R}^{4m\times n}.$$

The representations $\mathrm{V_c}, \Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 are clearly one-to-one. Moreover, A and $\Gamma_1(A)$ have the same (Frobenius) norm:

$$\begin{aligned} \|A\| \;&=\; \sqrt{\|A_1\|^2 + \|A_2\|^2 + \|A_3\|^2 + \|A_4\|^2} \\ &=\; \|\Gamma_1(A)\|. \end{aligned} \tag{7}$$

Proposition 7. *([1]) The following properties hold:*

(i) $\Gamma_1(A+B) = \Gamma_1(A) + \Gamma_1(B)$, $\Gamma_1(kA) = k\Gamma_1(A)$ *for any* $A, B \in \mathbb{Q}_{u,v}^{m\times n}$ *and* $k \in \mathbb{R}$.

(ii) $\mathcal{R}(A+B) = \mathcal{R}(A) + \mathcal{R}(B)$, $\mathcal{R}(kA) = k\mathcal{R}(A)$ *for any* $A, B \in \mathbb{Q}_{u,v}^{m\times n}$ *and* $k \in \mathbb{R}$.

(iii) $\mathcal{R}(AB) = \mathcal{R}(A)\mathcal{R}(B)$ *for any* $A \in \mathbb{Q}_{u,v}^{m\times n}$ *and* $B \in \mathbb{Q}_{u,v}^{n\times p}$.

III. General solutions of the generalized Sylvester matrix equation

In this section, we investigate Problem 1. From now on, we are given $A \in \mathbb{Q}_{u,v}^{m\times a}, B \in \mathbb{Q}_{u,v}^{b\times p}, C \in \mathbb{Q}_{u,v}^{m\times a}, D \in \mathbb{Q}_{u,v}^{b\times p}$, and $E \in \mathbb{Q}_{u,v}^{m\times p}$. In order to find the general solutions of Eq. (2), the following lemma will be used in a calculation.

Lemma 8. *For any* $X \in \mathbb{Q}_{u,v}^{a\times b}$, *we have*

$$\begin{pmatrix} \mathrm{V_c}(\Gamma_1(X)) \\ \mathrm{V_c}(\Gamma_2(X)) \\ \mathrm{V_c}(\Gamma_3(X)) \\ \mathrm{V_c}(\Gamma_4(X)) \end{pmatrix} \;=\; \mathcal{M}_{u,v}\,\mathrm{V_c}(\Gamma_1(X)), \tag{8}$$

where $\mathcal{M}_{u,v} \;=\; \begin{pmatrix} I_{4ab} \\ I_b \otimes N \otimes I_a \\ I_b \otimes K \otimes I_a \\ I_b \otimes T \otimes I_a \end{pmatrix} \in \mathbb{R}^{16ab\times 4ab}.$

Here, $N = \left(e_2^4 \;\; ue_1^4 \;\, -e_4^4 \;\, -ue_3^4\right)$,
$K = \left(e_3^4 \;\; e_4^4 \;\; ve_1^4 \;\; ve_2^4\right)$, *and*
$T = \left(e_4^4 \;\; ue_3^4 \;\, -ve_2^4 \;\, -uve_1^4\right), \in \mathbb{R}^{4\times 4}$.

Proof: A direct computations reveals that

$$\begin{aligned} \mathrm{V_c}(\Gamma_2(X)) \;&=\; \begin{pmatrix} u\,\mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_1) \\ -u\,\mathrm{col}_1(X_4) \\ -\,\mathrm{col}_1(X_3) \\ \vdots \\ u\,\mathrm{col}_b(X_2) \\ \mathrm{col}_b(X_1) \\ -u\,\mathrm{col}_b(X_4) \\ -\,\mathrm{col}_b(X_3) \end{pmatrix} \\ &= \begin{pmatrix} 0 & uI_a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ I_a & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -uI_a & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_a & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & uI_a & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & I_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -uI_a \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -I_a & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} \mathrm{col}_1(X_1) \\ \mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_3) \\ \mathrm{col}_1(X_4) \\ \vdots \\ \mathrm{col}_b(X_1) \\ \mathrm{col}_b(X_2) \\ \mathrm{col}_b(X_3) \\ \mathrm{col}_b(X_4) \end{pmatrix} \\ &=\; I_b \otimes \left[\left(e_2^4 \quad ue_1^4 \quad -e_4^4 \quad -ue_3^4\right) \otimes I_a\right] \mathrm{V_c}(\Gamma_1(X)) \\ &=\; (I_b \otimes N \otimes I_a)\,\mathrm{V_c}(\Gamma_1(X)). \end{aligned} \tag{9}$$

Similarly, we obtain

$$\begin{aligned} &\mathrm{V_c}(\Gamma_3(X)) \\ &\quad=\; I_b \otimes \left[\left(e_3^4 \quad e_4^4 \quad ve_1^4 \quad ve_2^4\right) \otimes I_a\right] \mathrm{V_c}(\Gamma_1(X)) \\ &\quad=\; (I_b \otimes K \otimes I_a)\,\mathrm{V_c}(\Gamma_1(X)) \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\mathrm{V_c}(\Gamma_4(X)) \\ &\quad=\; I_b \otimes \left[\left(e_4^4 \quad ue_3^4 \quad -ve_2^4 \quad -uve_1^4\right) \otimes I_n\right] \mathrm{V_c}(\Gamma_1(X)) \\ &\quad=\; (I_b \otimes T \otimes I_a)\,\mathrm{V_c}(\Gamma_1(X)). \end{aligned} \tag{11}$$

From Eqs. (9), (10) and (11), we arrive at Eq. (8). ■

Theorem 9. *Consider Eq. (2). Let us denote*

$$\mathcal{F} \;=\; \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(C)\right). \tag{12}$$

Then Eq. (2) is consistent if and only if the following rank condition holds:

$$\operatorname{rank}[\mathcal{F}\mathcal{M}_{u,v} \;\; \mathrm{V_c}(\Gamma_1(E))] \;=\; \operatorname{rank}[\mathcal{F}\mathcal{M}_{u,v}]. \tag{13}$$

In both consistent and inconsistent cases, we have the following:

(i) *Problem 1 has the general exact/LS solutions represented by*

$$\begin{aligned} &\mathrm{V_c}(\Gamma_1(X)) \\ &\quad= (\mathcal{F}\mathcal{M}_{u,v})^{\dagger}\,\mathrm{V_c}(\Gamma_1(E)) \\ &\quad\quad + \left[I_{4mp} - (\mathcal{F}\mathcal{M}_{u,v})^{\dagger}(\mathcal{F}\mathcal{M}_{u,v})\right] w, \end{aligned} \tag{14}$$

where $w \in \mathbb{R}^{4mp}$ *is arbitrary.*

(ii) *Among the general solutions* (14)*, the minimal-norm one is given by*

$$\mathrm{V_c}(\Gamma_1(X)) \;=\; (\mathcal{F}\mathcal{M}_{u,v})^{\dagger}\,\mathrm{V_c}(\Gamma_1(E)). \tag{15}$$

(iii) *Problem 1 has a unique exact/LS solution given by* (15) *if* $\mathcal{F}\mathcal{M}_{u,v}$ *is of full-column rank.*

Proof: From Eqs. (2) and (7), we consider the associated norm-error

$$\|AXB + CXD - E\| \;=\; \|\Gamma_1(AXB + CXD - E)\|.$$

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Now, Lemma 6 and Proposition 7 imply that

$$\begin{aligned}
&\Gamma_1(AXB+CXD-E)\\
&\quad= \Gamma_1(AXB)+\Gamma_1(CXD)-\Gamma_1(E)\\
&\quad= \mathcal{R}(A)\mathcal{R}(X)\Gamma_1(B)+\mathcal{R}(C)\mathcal{R}(X)\Gamma_1(D)-\Gamma_1(E)\\
&\quad= \mathrm{V_c}\left[\mathcal{R}(A)\mathcal{R}(X)\Gamma_1(B)+\mathcal{R}(C)\mathcal{R}(X)\Gamma_1(D)-\Gamma_1(E)\right]\\
&\quad= \left[\big(\Gamma_1(B)^T\otimes\mathcal{R}(A)\big)+\big(\Gamma_1(D)^T\otimes\mathcal{R}(C)\big)\right]\mathrm{V_c}\left(\mathcal{R}(X)\right)\\
&\qquad -\mathrm{V_c}(\Gamma_1(E)).
\end{aligned}$$

Using Lemma 8, we obtain

$$\begin{aligned}
&\|AXB+CXD-E\|\\
&\qquad= \|\mathcal{F}\,\mathrm{V_c}\left(\mathcal{R}(X)\right)-\mathrm{V_c}(\Gamma_1(E))\|\\
&\qquad= \|\mathcal{F}\mathcal{M}_{u,v}\,\mathrm{V_c}(\Gamma_1(X))-\mathrm{V_c}(\Gamma_1(E))\|.
\end{aligned}$$

Thus, the matrix equation (2) is equivalent to a real linear system

$$\mathcal{F}\mathcal{M}_{u,v}\,\mathrm{V_c}(\Gamma_1(X)) \;=\; \mathrm{V_c}(\Gamma_1(E)). \tag{16}$$

According to Lemma 5, the system (16) is consistent if and only if the rank condition (13) holds. In both consistent and inconsistent cases, the same lemma allows us to express the formula of the general exact/LS solution to be Eq. (14). The statements (ii) and (iii) now follow from Lemma 5. ∎

IV. Specific solutions of the generalized Sylvester matrix equation

In this section, we investigate Problems 2-4. Indeed, we would like to find imaginary (LS) solutions, real (LS) solutions, and solutions closet to a given matrix. The next lemma provides a real-vector representation of a generalized quaternion matrix.

Lemma 10. *Suppose $X=X_2i+X_3j+X_4k\in\mathbb{Q}_{u,v}^{a\times b}$ where $X_2,X_3,X_4\in\mathbb{R}^{a\times b}$. Then*

$$\mathrm{V_c}(\Gamma_1(X)) \;=\; \mathcal{K}\,\mathrm{V_c}\begin{pmatrix}X_2\\X_3\\X_4\end{pmatrix},$$

where

$$\begin{aligned}
&\mathcal{K}=I_b\otimes\mathcal{J}_a\in\mathbb{R}^{4ab\times 3ab}\ \textit{and}\\
&\mathcal{J}_a=\begin{pmatrix}e_2^4 & e_3^4 & e_4^4\end{pmatrix}\otimes I_a\in\mathbb{R}^{4a\times 3a}.
\end{aligned}$$

Proof: A direct computations reveals that

$$\mathrm{V_c}(\Gamma_1(X)) \;=\; \begin{pmatrix}\mathrm{col}_1(0)\\ \mathrm{col}_1(X_2)\\ \mathrm{col}_1(X_3)\\ \mathrm{col}_1(X_4)\\ \vdots\\ \mathrm{col}_b(0)\\ \mathrm{col}_b(X_2)\\ \mathrm{col}_b(X_3)\\ \mathrm{col}_b(X_4)\end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix}
0&0&0&\dots&0&0&0\\
I_a&0&0&\dots&0&0&0\\
0&I_a&0&\dots&0&0&0\\
0&0&I_a&\dots&0&0&0\\
\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\
0&0&0&\dots&0&0&0\\
0&0&0&\dots&I_a&0&0\\
0&0&0&\dots&0&I_a&0\\
0&0&0&\dots&0&0&I_a
\end{pmatrix}
\times\begin{pmatrix}\mathrm{col}_1(X_2)\\ \mathrm{col}_1(X_3)\\ \mathrm{col}_1(X_4)\\ \vdots\\ \mathrm{col}_b(X_2)\\ \mathrm{col}_b(X_3)\\ \mathrm{col}_b(X_4)\end{pmatrix}\\
&= I_b\otimes\left[\begin{pmatrix}e_2^4 & e_3^4 & e_4^4\end{pmatrix}\otimes I_a\right]\mathrm{V_c}\begin{pmatrix}X_2\\X_3\\X_4\end{pmatrix}\\
&= (I_b\otimes\mathcal{J}_a)\,\mathrm{V_c}\begin{pmatrix}X_2\\X_3\\X_4\end{pmatrix} = \mathcal{K}\,\mathrm{V_c}\begin{pmatrix}X_2\\X_3\\X_4\end{pmatrix}.
\end{aligned}$$

∎

Theorem 11. *Consider Eq.* (2)*. Let us denote $\mathcal{F}$ as in Eq.* (12)*. Then Eq.* (2) *is consistent if and only if*

$$\mathrm{rank}[\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}\ \mathrm{V_c}(\Gamma_1(E))] \;=\; \mathrm{rank}[\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}].$$

Moreover,

(i) *Problem 2 has the imaginary-part exact/LS solutions $X=X_2i+X_3j+X_4k$ expressed as*

$$\begin{aligned}
&\mathrm{V_c}\begin{pmatrix}X_2\\X_3\\X_4\end{pmatrix}\\
&\quad= (\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})^{\dagger}\,\mathrm{V_c}(\Gamma_1(E))\\
&\qquad+\left[I_{4mp}-(\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})^{\dagger}(\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})\right]w,
\end{aligned}\tag{17}$$

where $w\in\mathbb{R}^{4mp}$ is arbitrary.

(ii) *The minimal-norm exact/LS solution* (17) *is given by same formula:*

$$\mathrm{V_c}\begin{pmatrix}X_2\\X_3\\X_4\end{pmatrix} \;=\; (\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})^{\dagger}\,\mathrm{V_c}(\Gamma_1(E)). \tag{18}$$

(iii) *Problem 2 has a unique exact/LS solution given by the formula* (18) *if $\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}$ is of full-column rank.*

Proof: From the proof Theorem 9 and Lemma 10, we obtain

$$\begin{aligned}
&\|AXB+CXD-E\|\\
&\qquad= \|\mathcal{F}\mathcal{M}_{u,v}\,\mathrm{V_c}(\Gamma_1(X))-\mathrm{V_c}(\Gamma_1(E))\|\\
&\qquad= \left\|\mathcal{F}\mathcal{M}_{u,v}\,\mathrm{V_c}\begin{pmatrix}0\\X_2\\X_3\\X_4\end{pmatrix}-\mathrm{V_c}(\Gamma_1(E))\right\|\\
&\qquad= \left\|\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}\,\mathrm{V_c}\begin{pmatrix}X_2\\X_3\\X_4\end{pmatrix}-\mathrm{V_c}(\Gamma_1(E))\right\|.
\end{aligned}$$

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Thus, Eq. (2) is equivalent to a real linear system

$$\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}\,\mathrm{V_c}\begin{pmatrix}X_2\\X_3\\X_4\end{pmatrix} = \mathrm{V_c}(\Gamma_1(E)). \tag{19}$$

By Lemma 5, the Eq. (19) is consistent if and only if

$$\mathrm{rank}[\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}\ \mathrm{V_c}(\Gamma_1(E))] = \mathrm{rank}[\mathcal{F}\mathcal{M}_{u,v}\mathcal{K}].$$

The same lemma implies that the matrix equation (19) has the general exact/LS solutions

$$\begin{aligned}\mathrm{V_c}&\begin{pmatrix}X_2\\X_3\\X_4\end{pmatrix}\\ &= (\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})^\dagger\,\mathrm{V_c}(\Gamma_1(E))\\ &\quad + \left[I_{4mp} - (\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})^\dagger(\mathcal{F}\mathcal{M}_{u,v}\mathcal{K})\right]w,\end{aligned}$$

where $w \in \mathbb{R}^{4mp}$ is arbitrary. The statements (ii) and (iii) now follow from Lemma 5. ■

The next lemma will be used in a calculation involving Problem 3.

Lemma 12. *For any $X \in \mathbb{R}^{a\times b}$, we have*

$$\mathrm{V_c}(\Gamma_1(X)) = \tilde{\mathcal{K}}\,\mathrm{V_c}(X)$$

where $\tilde{\mathcal{K}} = I_b \otimes e_1^4 \otimes I_a \in \mathbb{R}^{4ab\times ab}$.

Proof: Since $X = X + 0i + 0j + 0k$, we have

$$\begin{aligned}\mathrm{V_c}(\Gamma_1(X)) &= \begin{pmatrix}\mathrm{col}_1(X)\\ \mathrm{col}_1(0)\\ \mathrm{col}_1(0)\\ \mathrm{col}_1(0)\\ \vdots\\ \mathrm{col}_b(X)\\ \mathrm{col}_b(0)\\ \mathrm{col}_b(0)\\ \mathrm{col}_b(0)\end{pmatrix}\\ &= \begin{pmatrix}I_a & \dots & 0\\ 0 & \dots & 0\\ 0 & \dots & 0\\ 0 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & I_a\\ 0 & \dots & 0\\ 0 & \dots & 0\\ 0 & \dots & 0\end{pmatrix}\begin{pmatrix}\mathrm{col}_1(X)\\ \vdots\\ \mathrm{col}_b(X)\end{pmatrix}\\ &= (I_b \otimes e_1^4 \otimes I_a)\,\mathrm{V_c}(X)\\ &= \tilde{\mathcal{K}}\,\mathrm{V_c}(X).\end{aligned}$$

■

Theorem 13. *Consider Eq.* (2). *Let us denote $\mathcal{F}$ as in Eq.* (12). *Then Eq.* (2) *is consistent if and only if*

$$\mathrm{rank}[\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}\ \mathrm{V_c}(\Gamma_1(E))] = \mathrm{rank}[\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}].$$

Morever,

(i) *Problem 3 has the real exact/LS solutions $X \in \mathbb{R}^{a\times b}$ expressed as*

$$\begin{aligned}\mathrm{V_c}&(X)\\ &= (\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}})^\dagger\,\mathrm{V_c}(\Gamma_1(E))\\ &\quad + \left[I_{4mp} - (\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}})^\dagger(\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}})\right]y,\end{aligned} \tag{20}$$

where $y \in \mathbb{R}^{16bp}$ is an arbitrary vector.

(ii) *The minimal-norm exact/LS solution* (20) *is given by the same formula:*

$$\mathrm{V_c}(X) = (\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}})^\dagger\,\mathrm{V_c}(\Gamma_1(E)). \tag{21}$$

(iii) *Problem 3 has a unique exact/LS solution given by the formula* (21) *if $\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}$ is of full-column rank.*

Proof: Since $X = X + 0i + 0j + 0k$ and from the proof Theorem 9 and Lemma 12, we obtain

$$\begin{aligned}&\|AXB + CXD - E\|\\ &\quad= \|\mathcal{F}\mathcal{M}_{u,v}\,\mathrm{V_c}(\Gamma_1(X)) - \mathrm{V_c}(\Gamma_1(E))\|\\ &\quad= \left\|\mathcal{F}\mathcal{M}_{u,v}\,\mathrm{V_c}\begin{pmatrix}X_1\\0\\0\\0\end{pmatrix} - \mathrm{V_c}(\Gamma_1(E))\right\|\\ &\quad= \left\|\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}\,\mathrm{V_c}(X) - \mathrm{V_c}(\Gamma_1(E))\right\|.\end{aligned}$$

Thus, Eq. (2) is equivalent to a real linear system

$$\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}\,\mathrm{V_c}(\Gamma_1(X)) = \mathrm{V_c}(\Gamma_1(E)). \tag{22}$$

By Lemma 5, Eq. (22) is consistent if and only if

$$\mathrm{rank}[\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}\ \mathrm{V_c}(\Gamma_1(E))] = \mathrm{rank}[\mathcal{F}\mathcal{M}_{u,v}\tilde{\mathcal{K}}].$$

The assertions $(i) - (iii)$ now follow from Lemma 5. ■

Now, we investigate Problem 4.

Theorem 14. *Consider Eq.* (2). *Let $Y \in \mathbb{Q}_{u,v}^{a\times b}$ be given.*

(i) *Problem 4.1 has the solution $X = Y + Z \in \mathbb{Q}_{u,v}^{n\times p}$, where Z is the general exact/LS solution of the associated matrix equation*

$$AZB + CZD = E - (AYB + CYD). \tag{23}$$

(ii) *Problem 4.2 has the solution $X = Y + Z$, where Z is the imaginary-part exact/LS solution of Eq.* (23).

(iii) *Problem 4.3 has the solution $X = Y + Z$, where Z is the real exact/LS solution of Eq.* (23).

Proof: Denote $\check{E} = E - (AYB + CYD)$. Let us denote by $G_{\mathcal{S}}$ the set of general exact/LS solutions of the equation $AXB + CXD = E$. Consider the following error

$$\begin{aligned}AXB &+ CXD - E\\ &= AXB + CXD - E - AYB\\ &\quad - CYD + AYB + CYD\\ &= A(X-Y)B + C(X-Y)D\\ &\quad - E + AYB + CYD\\ &= AZB + CZD - \check{E}.\end{aligned} \tag{24}$$

By letting $Z = X - Y$, we have that the Problem 4.1 is equivalent to the following minimization

$$\begin{aligned}\min_{X\in G_{\mathcal{S}}}\|X - Y\| &= \min_{\|AXB+CXD-E\| = \min}\|X - Y\|\\ &= \min_{\|AXB+CXD-E\| = \min}\|Z\|\\ &= \min_{\|AZB+CZD-\check{E}\| = \min}\|Z\|.\end{aligned}$$

Similarly, we obtain the statements (ii) and (iii). ■

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V. Sylvester and Stein matrix equations, and quaternionic matrix equations

The generalized Sylvester equation (2) includes the following special cases.

Corollary 15. *Consider the Sylvester matrix equation*

$$AX + XD \;=\; E \tag{25}$$

in an unknown $X \in \mathbb{Q}_{u,v}^{a\times b}$. Here, the matrices $A \in \mathbb{Q}_{u,v}^{m\times a}, D \in \mathbb{Q}_{u,v}^{b\times p}$, and $E \in \mathbb{Q}_{u,v}^{m\times p}$ are given. Then the conclusions of Theorems 9, 11, 13, and 14 hold, where the matrix $\mathcal{F}$ is given by

$$\mathcal{F} \;=\; \left((e_1^4)^T \otimes I_b \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes I_{4a}\right). \tag{26}$$

Proof: We set $B = I_b$ and $C = I_a$ in those theorems. Note that $\Gamma_1(I_b) = e_1^4 \otimes I_b$. So, the matrix $\mathcal{F}$ in (12) is reduced to (26). ■

In the next corollary, we consider the Stein matrix equation.

Corollary 16. *Consider the Stein matrix equation*

$$AXB + X \;=\; E$$

in an unknown $X \in \mathbb{Q}_{u,v}^{n\times p}$. Here, $A \in \mathbb{Q}_{u,v}^{m\times a}, B \in \mathbb{Q}_{u,v}^{b\times p}$, and $E \in \mathbb{Q}_{u,v}^{m\times p}$ are given. Then the conclusions of Theorems 9, 11, 13, and 14 hold, where the matrix $\mathcal{F}$ is given by

$$\mathcal{F} \;=\; \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right) + \left((e_1^4)^T \otimes I_{4ab}\right).$$

Proof: We set $C = I_a$ and $D = I_b$ in those theorems. ■

In particular when $u = v = -1$, the previous results in Sections III and IV become those for matrices over the Hamilton quaternions $\mathbb{Q}$.

Corollary 17. *Consider the matrix equation*

$$AXB + CXD \;=\; E,$$

where $A, C \in \mathbb{Q}^{m\times a}, B, D \in \mathbb{Q}^{p\times b}$, and $E \in \mathbb{Q}^{m\times p}$ are given. Then the conclusions of Theorems 9, 11, 13, and 14 hold, where the matrix $\mathcal{M}_{u,v}$ is given explicitly by

$$\mathcal{M}_{-1,-1} \;=\; \begin{pmatrix} I_{4ab} \\ I_b \otimes \hat{N} \\ I_b \otimes \hat{K} \\ I_b \otimes \hat{T} \end{pmatrix} \in \mathbb{R}^{16ab\times 4ab}$$

and

$$\hat{N} = \begin{pmatrix} 0 & -I_a & 0 & 0 \\ I_a & 0 & 0 & 0 \\ 0 & 0 & 0 & I_a \\ 0 & 0 & -I_a & 0 \end{pmatrix},$$

$$\hat{K} = \begin{pmatrix} 0 & 0 & -I_a & 0 \\ 0 & 0 & 0 & -I_a \\ I_a & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \end{pmatrix},$$

$$\hat{T} = \begin{pmatrix} 0 & 0 & 0 & -I_a \\ 0 & 0 & I_a & 0 \\ 0 & -I_a & 0 & 0 \\ I_a & 0 & 0 & 0 \end{pmatrix}.$$

Proof: Set $u = v = -1$ in those theorems. The conclusions of Theorems 9, 11, and 13 were investigated in [22]. ■

VI. Numerical examples

In this section, we provide numerical examples to illustrate our results.

Example 18. *Consider the generalized Sylvester matrix equation $AXB + CXD = E$ over the split quaternions (i.e., $(u, v) = (-1, 1)$), where*

$$A \;=\; \begin{pmatrix} 1 & -j+2k \end{pmatrix}_{1\times 2}, \quad C \;=\; \begin{pmatrix} i-j-k & 2 \end{pmatrix}_{1\times 2},$$
$$B \;=\; \begin{pmatrix} j-k \\ 3-i \end{pmatrix}_{2\times 1}, \quad D \;=\; \begin{pmatrix} 3k \\ 2-j \end{pmatrix}_{2\times 1},$$
$$E \;=\; \begin{pmatrix} -2+i-4j+k \end{pmatrix}_{1\times 1}.$$

Then we have

$$\mathcal{R}(A) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & -1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$
$$\mathcal{R}(C) \;=\; \begin{pmatrix} 2 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 2 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\Gamma_1(B)^T \;=\; \begin{pmatrix} 0 & 3 & 0 & -1 & 1 & 0 & -1 & 0 \end{pmatrix},$$
$$\Gamma_1(D)^T \;=\; \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & -1 & 3 & 0 \end{pmatrix},$$
$$\Gamma_1(E) \;=\; \begin{pmatrix} -2 & 1 & -4 & 1 \end{pmatrix}^T,$$

and

$$\mathcal{F} \;=\; \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(C)\right).$$

We see that

$$\operatorname{rank}\left[\mathcal{F}\mathcal{M}_{-1,1} \;\; \mathrm{V_c}(\Gamma_1(E))\right] \;=\; \operatorname{rank}\left[\mathcal{F}\mathcal{M}_{-1,1}\right] \;=\; 4,$$

thus Eq. (2) is consistent. According to Theorem 9, the matrix equation has a minimal-norm solution, computed via MATLAB as follows:

$$X = \begin{pmatrix} 0.0919 & -0.0157 \\ 0.0815 & 0.1384 \end{pmatrix} + \begin{pmatrix} -0.0595 & 0.0449 \\ 0.1372 & -0.0747 \end{pmatrix} i + \begin{pmatrix} -0.1335 & -0.0537 \\ -0.0325 & -0.0181 \end{pmatrix} j + \begin{pmatrix} 0.0809 & 0.0804 \\ -0.1514 & -0.0696 \end{pmatrix} k.$$

Example 19. *Consider the generalized Sylvester matrix equation $AXB + CXD = E$ over the split quaternions (i.e., $(u, v) = (-1, 1)$), where*

$$A \;=\; \begin{pmatrix} 2i & 1+j \\ i-k & 0 \end{pmatrix}_{2\times 2}, \quad C \;=\; \begin{pmatrix} i+3k & -k \\ 1+j & 0 \end{pmatrix}_{2\times 2},$$
$$B \;=\; \begin{pmatrix} 1 \\ -2j \end{pmatrix}_{2\times 1}, \quad D \;=\; \begin{pmatrix} -1 \\ -k \end{pmatrix}_{2\times 1}, \quad E \;=\; \begin{pmatrix} -1 \\ -2i \end{pmatrix}_{2\times 1}.$$

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Then we have

$$\mathcal{R}(A) = \begin{pmatrix} 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{R}(C) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 3 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\Gamma_1(B)^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \end{pmatrix},$$
$$\Gamma_1(D)^T = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$
$$\Gamma(E) = \begin{pmatrix} -1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \end{pmatrix}^T,$$

and

$$\mathcal{F} = \left(\Gamma_1(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma_1(D)^T \otimes \mathcal{R}(C)\right).$$

We see that

$$\operatorname{rank}[\mathcal{FM}_{-1,1}\mathcal{K}\ \mathrm{V_c}(\Gamma_1(E))] = 7 \quad \textit{and} \quad \operatorname{rank}[\mathcal{FM}_{-1,1}\mathcal{S}] = 6,$$

thus Eq. (2) *is inconsistent. According to Theorem 11, the matrix equation has a minimal-norm LS solution, computed via MATLAB as follows:*

$$X = \begin{pmatrix} 0.0139 & -0.0765 \\ -0.1781 & 0.0084 \end{pmatrix} i + \begin{pmatrix} 0.0551 & 0.0575 \\ 0.0007 & -0.3225 \end{pmatrix} j + \begin{pmatrix} -0.2938 & -0.3269 \\ 0.2865 & -0.0352 \end{pmatrix} k.$$

Example 20. *Consider the matrix equation $AXB+CXD = E$ over the split quaternions, i.e., $(u,v) = (-1,1)$. Here, we are given the matrices A, B, C, D, E as in Example 6.1, and we will find a solution X closest to a given matrix*

$$Y = \begin{pmatrix} -i & 1 \\ 0 & j \end{pmatrix}.$$

We obtain

$$\check{E} = E - (AYB + CYD) = \left(1 + i + -3j\right).$$

and

$$\operatorname{rank}[\mathcal{FM}_{-1,1}\ \mathrm{V_c}(\Gamma_1(\check{E}))] = \operatorname{rank}[\mathcal{FM}_{-1,1}] = 4.$$

Using Theorem 14 and MATLAB, we obtain:

$$Z = \begin{pmatrix} 0.0329 & 0.0573 \\ -0.0002 & 0.1340 \end{pmatrix} + \begin{pmatrix} 0.0132 & 0.0298 \\ 0.1151 & -0.0503 \end{pmatrix} i + \begin{pmatrix} -0.0260 & -0.0834 \\ -0.0462 & -0.0356 \end{pmatrix} j + \begin{pmatrix} 0.0609 & 0.0406 \\ -0.0197 & 0.0325 \end{pmatrix} k.$$

Thus, we get the desire exact solution $X = Z + Y$:

$$X = \begin{pmatrix} 0.0329 & 1.0573 \\ -0.0002 & 0.1340 \end{pmatrix} + \begin{pmatrix} -0.9868 & 0.0298 \\ 0.1151 & -0.0503 \end{pmatrix} i + \begin{pmatrix} -0.0260 & -0.0834 \\ -0.0462 & 0.9644 \end{pmatrix} j + \begin{pmatrix} 0.0609 & 0.0406 \\ -0.0197 & 0.0325 \end{pmatrix} k.$$

VII. Gradient-descent iterative (GDI) algorithm for the associated linear system

From the discussion in Section VII, we see that the LS solution of Eq. (2) is equivalent to a real linear system (16). In order to resolve this system, we utilize a technique for enhancing the gradient descent optimization, as described in [27]. The core concept is to reduce the residual error $\|\mathcal{FM}_{u,v}\,\mathrm{V_c}(\Gamma_1(X)) - \mathrm{V_c}(\Gamma_1(E))\|$ at each interation. Consequently, we derive the following gradient-descent iterative (GDI) algorithm:

Algorithm 1: GDI Algorithm for system (16)

$A \in \mathbb{Q}_{u,v}^{m\times a}$, $B \in \mathbb{Q}_{u,v}^{b\times p}$, $C \in \mathbb{Q}_{u,v}^{m\times a}$, $D \in \mathbb{Q}_{u,v}^{b\times p}$, and $E \in \mathbb{Q}_{u,v}^{m\times p}$. ;
Set $i = 0$. Choose $x^{(0)} \in \mathbb{R}^{4ab}$. Compute
  $\hat{A} = \mathcal{FM}_{u,v}$, $\hat{f} = \mathrm{V_c}(\Gamma_1(E))$, $\hat{P} = \hat{A}\hat{A}^T$.
for $i = 0, 1, 2, 3, \ldots$ **do**
  $r^{(i)} = \hat{f} - \hat{A}\mathrm{x}^{(i)}$;
  if $\|r^{(i)}\| \leqslant \epsilon$ **then**
    $\mathrm{x}^{(i)}$ is a solution; break;
  else
    $m_i = \hat{P}r^{(i)}$;
    $\alpha_{i+1} = m_i^T r^{(i)}/(2m_i^T m_i)$;
    $\mathrm{x}^{(i+1)} = \mathrm{x}^{(i)} + \alpha_{i+1}\hat{A}^T r^{(i)}$;
  end
  update i;
end

We implement all simulations using MATLAB R2017a on the same PC environment: AMD A9-9425 RADEON R5 @3.10GHz with RAM 4 GB.

Example 21. *Let $(u,v) = (-1,-1)$. Consider the generalized Sylvester matrix equation $AXB + CXD = E$, where $A, B, C, D, E \in \mathbb{Q}_{u,v}^{2\times 2}$ are given randomly as follows:*

$$A = \begin{pmatrix} 0.8147 & 0.1270 \\ 0.9058 & 0.9134 \end{pmatrix} + \begin{pmatrix} 0.6324 & 0.2785 \\ 0.0975 & 0.5469 \end{pmatrix} i + \begin{pmatrix} 0.9575 & 0.1576 \\ 0.9649 & 0.9706 \end{pmatrix} j + \begin{pmatrix} 0.9572 & 0.8003 \\ 0.4854 & 0.1419 \end{pmatrix} k,$$

$$B = \begin{pmatrix} 0.4218 & 0.7922 \\ 0.9157 & 0.9595 \end{pmatrix} + \begin{pmatrix} 0.6557 & 0.8491 \\ 0.0357 & 0.9340 \end{pmatrix} i + \begin{pmatrix} 0.6787 & 0.7431 \\ 0.7577 & 0.3922 \end{pmatrix} j + \begin{pmatrix} 0.6555 & 0.7060 \\ 0.1712 & 0.0318 \end{pmatrix} k,$$

$$C = \begin{pmatrix} 0.2769 & 0.0971 \\ 0.0462 & 0.8235 \end{pmatrix} + \begin{pmatrix} 0.6948 & 0.9502 \\ 0.3171 & 0.0344 \end{pmatrix} i + \begin{pmatrix} 0.4387 & 0.7655 \\ 0.3816 & 0.7952 \end{pmatrix} j + \begin{pmatrix} 0.1869 & 0.4456 \\ 0.4898 & 0.6463 \end{pmatrix} k,$$

$$D = \begin{pmatrix} 0.7094 & 0.2760 \\ 0.7547 & 0.6797 \end{pmatrix} + \begin{pmatrix} 0.6551 & 0.1190 \\ 0.1626 & 0.4984 \end{pmatrix} i + \begin{pmatrix} 0.9597 & 0.5853 \\ 0.3404 & 0.2238 \end{pmatrix} j + \begin{pmatrix} 0.7513 & 0.5060 \\ 0.2551 & 0.6991 \end{pmatrix} k,$$

$$E = \begin{pmatrix} 0.8909 & 0.5472 \\ 0.9593 & 0.1386 \end{pmatrix} + \begin{pmatrix} 0.1493 & 0.8407 \\ 0.2575 & 0.2543 \end{pmatrix} i + \begin{pmatrix} 0.8143 & 0.9293 \\ 0.2435 & 0.3500 \end{pmatrix} j + \begin{pmatrix} 0.1966 & 0.6160 \\ 0.2511 & 0.4733 \end{pmatrix} k.$$

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We would like to find an LS solution

$$X = X_1 + X_2 i + X_3 j + X_4 k.$$

We apply Algorithm 1 with an intial guess $X^{(0)}$ is a zero matrix and a tolerance error $\epsilon = 0.005$. The relative error $\|r^{(i)}\|$ at each iteration is illustrate in Figure 1.

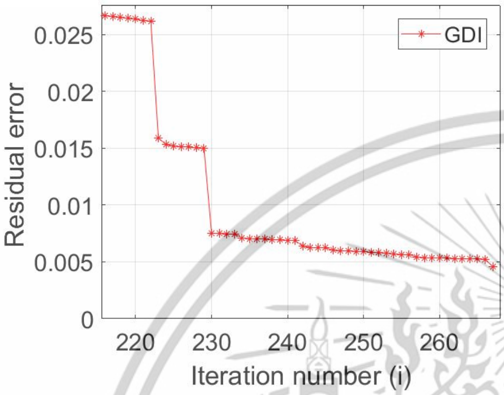

Fig. 1: The relative error at each iteration for Example 20.

It turns out that Algorithm 1 requires 267 iterations and an approximate times of 0.161798 seconds to arrive a desire solution.

$$X^{(267)} = \begin{pmatrix} 0.4155 & -0.0476 \\ -0.2244 & -0.1754 \end{pmatrix} + \begin{pmatrix} -0.0331 & -0.7911 \\ 0.1011 & 0.3502 \end{pmatrix} i + \begin{pmatrix} -0.3315 & 0.5111 \\ 0.0976 & -0.6186 \end{pmatrix} j + \begin{pmatrix} -0.0077 & 0.5213 \\ -0.0355 & -0.3490 \end{pmatrix} k.$$

VIII. Application to Color Image Processing

In color image processing, a color image can be represented by a vector. According to RGB color model, the color information of a pixel can be represented as

$$w \;=\; 0 + w_r i + w_g j + w_b k \in \mathbb{Q},$$

where w_r, w_g, w_b are the red/green/blue component of the color pixel, respectively. We can see that w_r, w_g and w_b are the imaginary part of w. In [28], the general image degradation model is provided by

$$Aw + b \;=\; f, \tag{27}$$

where f is the observation, w is the desired image, b is the additive noise. The matrix A acts as a linear operator related to the task. For example, A is the an identity matrix for the image denoising task, A is a projection matrix for image inpainting, A is the blur matrix related to the blur kernel for the image deblurring task. Our task is to restore the desired image w from the observation f.

Note that Eq. (27) can be written as $Aw = f - b$. Thus, Eq. (27) is a special case of Eq. (2) when $E = f - b, B = I_1, C = 0$ and $X = w$. We can solve for a minimal-norm least-squares solution solution by using Theorem 11 as follows:

$$\begin{aligned} \mathrm{V_c}\begin{pmatrix} w_r \\ w_g \\ w_b \end{pmatrix} &= (\mathcal{F}\mathcal{M}_{-1,-1}\mathcal{K})^{\dagger}\, \mathrm{V_c}(\Gamma_1(f-b)) \\ &= \left[\left(\Gamma_1(I_1)^T \otimes \mathcal{R}(A)\right)\mathcal{M}_{-1,-1}\mathcal{K}\right]^{\dagger} \\ &\quad \times \mathrm{V_c}(\Gamma_1(f-b)) \\ &= \left[\left((e_4^1)^T \otimes \mathcal{R}(A)\right)\mathcal{M}_{-1,-1}\mathcal{J}_a\right]^{\dagger} \\ &\quad \times \mathrm{V_c}(\Gamma_1(f-b)). \end{aligned} \tag{28}$$

We then get w_r, w_g and w_b due to the injectivity of the operator $\mathrm{V_c}$.

We summarize the process of image deblurring as in the following algorithm.

Algorithm 2: Algorithm for image deblurring

(1) Import an original image.
(2) Blur the image using the option in MATLAB R2019b, namely,
f = *fspecial('motion', len, theta)*
where the parameters *len* and *theta* indicate the length and the angle of motion in degrees in a counter-clockwise direction, respectively.
(3) Determine the value of A from image bluring.
(4) Calculate w according to Eq. (28).

Note that to simplify computations on a small-scale computer, an original image could be resized to a smaller dimension.

Example 22. *Given an original color image of size 100×100 pixels. In order to reduce time and memory for computations, we modify the pixels of the image to a smaller dimension 80×80, as in Fig. 2. We represent the image Fig. 2 as a 3-tuples of vector $w = (w_r, w_g, w_b)$. Then, we blur this image using len = 10 and theta = 40 to get the image $f = (f_r, f_g, f_b)$ as in Fig. 3. The image Fig. 2 can be recovered as the minimal-norm least-squares solution of the model* (27). *Indeed, by using* (28)*, we get the restored image as shown in Fig. 4.*

Fig. 2: The (80×80)-pixels image.

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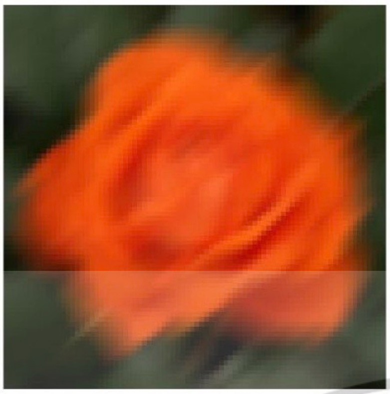

Fig. 3: The blurred image.

Fig. 4: The restored image.

Example 23. *Given an original color image of size* 350×350 *pixels. In order to reduce time and memory for computations, we modify the pixels of the image to a smaller dimension* 80×80*, as in Fig. 5. We apply the command fspecial('motion', len, theta) where len = 30 and theta = 11 to disturb Fig. 5. So, we get the blurred image as in Fig. 6. From the color model* (27)*, we can restore the image Fig. 7 as the minimal-norm least-squares solution given by* (28)*. Indeed, the restored image is shown in Fig. 7.*

Fig. 5: The (80×80)-pixels image.

Fig. 6: The blurred image.

Fig. 7: The restored image.

Example 24. *Given an original color image of size* 3000×3000 *pixels. In order to reduce time and memory for computations, we modify the pixels of the image to a smaller dimension* 80×80*, as in Fig. 8. We apply the command fspecial('motion', len, theta) to disturb Fig. 8. So, we get the blurred image as in Fig. 9 and Fig. 10. The image Fig. 2 can be recovered as the minimal-norm least-squares solution of the model* (27).

Fig. 8: The (80×80)-pixels image.

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Fig. 9: The blurred image using *len = 10* and *theta = 40*.

Fig. 10: The restored image.

Fig. 11: The blurred image using *len = 30* and *theta = 11*.

Fig. 12: The restored image.

TABLE I: The LS errors of the restored the image and CPU times

	$\|f_r - w_r\|^2$	$\|f_r - w_r\|^2$	$\|f_r - w_r\|^2$	CPU times
Fig. 4	6.3029e-13	6.2912e-13	2.0018e-13	8549.952s
Fig. 7	2.2437e-12	3.5017e-13	4.2708e-13	7937.218s
Fig. 10	3.7830e-13	5.2965e-13	5.4408e-13	8584.374s
Fig. 12	4.7741e-13	7.0398e-13	5.3691e-13	8404.744s

IX. Conclusion

We investigate Problems 1-4 to find general and specific exact/LS solutions of the generalized Sylvester matrix equation (2). All matrices considered here are rectangular compatible matrices over a generalized quaternion. We apply the techniques of real representations and vectorizations of generalized quaternion matrices to reduce the matrix equation (2) to a real linear system. Thus, we can extract a solvability criterion for the matrix equation. Moreover, we can derive formulas of the (minimal-norm) exact/LS solution, the (minimal-norm) pure-imaginary exact/LS solution, the (minimal-norm) real exact/LS solution, and the (minimal-norm) exact/LS solution closet to a given matrix. Such solutions are expressed in terms of Kronecker products and Moore-Penrose inverses. This work includes the studies of Sylvester and Stein equations over generalized quaternions, quaternionic matrix equations, and particularly the work [22]. Moreover, we propose gradient-descent iterative (GDI) algorithm to solve the linear system associated with Eq. (2). In color image processing, we can apply our theory to get an algorithm for image deblurring.

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