THE NEW METHODS FOR SOLVING THE COMBINATION OF VARIATIONAL INEQUALITY PROBLEMS AND EQUILIBRIUM PROBLEMS IN HILBERT SPACE

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Abstract

The purpose of this thesis is to firstly introduce the combination of variational inequality problems and equilibrium problems in Hilbert space, and then to prove the strong convergence theorem for finding a common element of the set of all solutions of the combination of variational inequality problems and equilibrium problems. Next, we will apply the theorem to other problems, such as the general split feasibility problem. In addition, we have also given calculation examples to support such the theorem of this thesis.

Keywords : fixed point problem, equilibrium problem, variational inequality problem, κ-strictly pseudononspreading mapping, α-inverse strongly monotone mapping

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Introduction

1.1 Research Motivation

Currently, there are many studies, inventions and production of methods to be used to solve problems, create or develop various technologies to be able to be used or solve problems effectively, whether it is a computer problem, physics, engineering, economics and others, which the basis for inventing technology or various solutions are all based on mathematics but to invent or develop methods for solving various problems mentioned above, some methods are still complicated and difficult to solve that problem directly. Therefore, one of the mathematical methods is applied to make it easier and get more diverse results. That is, the fixed point theory by the fixed point theory is a study about existence and uniqueness that when applied to problems such as equilibrium problems, variational inequality problems and etc. By converting the problems of interest into the fixed point problems, it can create a theorem that can solve that problems and can also be applied in other fields such as mechanics, physics, optimization, finance, ecology, network, game theory, economics, engineering science, etc., which converting to be in the form of a fixed point problem will make it possible to solve the problem of interest in an easier way and also get a solution that can solve many problems. Interested problems of various branches mentioned above and more than that, sometimes can also bring the results to expand, collapse or apply additionally to be able to solve other problems as well. From the aforementioned reasons, the theory of fixed point is another important method and received widespread attention from researchers or many mathematicians.

In addition, there are many interesting problems in mathematics that can be reduced to equilibrium problems, such as variational inequality problems, optimization problems, fixed point problems, Nash equilibrium problems in game theory, saddle point problems, etc., which will talk about equilibrium problems in the next step.

Throughout in this section, let H be a real Hilbert space and let C be a nonempty closed and convex subset of H.

For a mapping T of C into itself, we denote $F(T)$ the set of all *fixed points* of T i.e.,

$$F(T) = \{x \in C : Tx = x\}.$$

Example 1.1. Let $\mathbb{R}$ be the set of a real numbers. We have

1. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = 2x^2 - 1$, then $F(T) = \{1\}$.
2. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = \frac{1}{x}$, then $F(T) = \{1\}$.

3. If $T:\mathbb{R}\to\mathbb{R}$ and $Tx=x^2$, then $F(T)=\{0,1\}$.
4. If $T:\mathbb{R}\to\mathbb{R}$ and $Tx=3x-4$, then $F(T)=\{2\}$.
5. If $T:\mathbb{R}\to\mathbb{R}$ and $Tx=\frac{2x+5}{3}$, then $F(T)=\{5\}$.

Let C be a nonempty closed convex subset of a real Hilbert space H. The mapping $T:C\to C$ is said to be *nonexpansive* if

$$\|Tx-Ty\|\leq\|x-y\| \tag{1.1}$$

for all $x,y\in C$.

In 2008, Kohsaka and Takahashi [1] introduced the mapping $T:C\to C$ called *nonspreading mapping* in Hilbert spaces H which is defined as follows:

$$2\|Tx-Ty\|^2\leq\|Tx-y\|^2+\|x-Ty\|^2 \tag{1.2}$$

for all $x,y\in C$.

In 2011, Osilike and Isiogugu [2] introduced that the mapping $T:C\to C$ is called *κ-strictly pseudononspreading mapping* if there exists $\kappa\in[0,1)$ such that

$$\|Tx-Ty\|^2\leq\|x-y\|^2+\kappa\|(I-T)x-(I-T)y\|^2+2\langle x-Tx,y-Ty\rangle \tag{1.3}$$

for all $x,y\in C$. Clearly every nonspreading mapping is κ-strictly pseudononspreading mapping.

A mapping A of C into H is called *α-inverse strongly monotone*, if there exists a positive real number α such that

$$\langle x-y,Ax-Ay\rangle\geq\alpha\|Ax-Ay\|^2 \tag{1.4}$$

for all $x,y\in C$.

In 1967, Halpern [3] introduced the Halpern iteration to find a fixed point of nonexpansive mapping $T:C\to C$ as follows:

$$x_{n+1}=\alpha_n u+(1-\alpha_n)Tx_n \tag{1.5}$$

for each $n\in\mathbb{N}\cup\{0\}$ and $x_0=u\in C$ where $\{\alpha_n\}\subset(0,1)$. He proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T.

In 1964, Lions and Stampacchia [4] first introduced the *variational inequality problem* for find a point $u\in C$ such that

$$\langle Au,v-u\rangle\geq 0 \tag{1.6}$$

where $A:C\to H$ is a mapping and $v\in C$. The set of solutions of (1.6) is denoted by $VI(C,A)$.

The variational inequality problem has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, pure, applied sciences and etc., see [5, 6, 7, 8].

In 2008, Ceng et al. [9] introduced a problem for finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda_1 D_1 z^* + x^* - z^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_2 D_2 x^* + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.7}$$

which is called a *system of variational inequalities problem* where $D_1, D_2 : C \to H$ are mappings and parameters $\lambda_1, \lambda_2 > 0.$ In the case of $\lambda_1 = \lambda_2, D_1 = D_2, x^* = z^*$, system of variational inequalities problem is reduced to variational inequalities problem.

After that, Kangtunyakarn [10] modified (1.7) for finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a) z^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle z^* - (I - \lambda_2 D_2) x^*, x - z^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.8}$$

which is called a *modification of system of variational inequalities problem*, for every $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1]$. It easy to see that if $a = 0$, then problem (1.8) reduces to system of variational inequalities problem. He introduced the relation between solutions of (1.8) and fixed point of the mapping G as follows:

Lemma 1.2. [11] Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be mappings. For every $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1]$, the following statements are equivalent:

1. $(x^*, z^*) \in C \times C$ is a solution of problem (1.8),
2. x^* is a fixed point of the mapping $G : C \to C$, i.e., $x^* \in F(G)$, defined by

$$G(x) = P_C (I - \lambda_1 D_1)(ax + (1-a) P_C (I - \lambda_2 D_2) x),$$

where $z^* = P_C (I - \lambda_2 D_2) x^*$.

Moreover, he proved the following strong convergence theorem of the variational inequality problem and fixed point problem for κ-strictly pseudononspreading mapping which modified Halpern iterative method generated by (1.9) .

Theorem 1.3. [11] Let H be a real Hilbert space and C be a nonempty closed convex subset of H. For every $i = 1, 2, ..., N$ let $B_i : C \to H$ be δ_i-inverse strongly monotone mappings and let $T : C \to C$ be κ-strictly pseudononspreading mapping for some $\kappa \in [0, 1)$. Let $G_i : C \to C$ be defined by $G_i x = P_C (I - \eta B_i) x$ for every $x \in C$ and $\eta \in (0, 2\delta_i)$ for every $i = 1, 2, ..., N$, and let $\delta_j = \left(\alpha_1^j, \alpha_2^j, \alpha_3^j\right) \in I \times I \times I, j = 1, 2, 3, ..., N$, where $I = [0, 1], \alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j \in (0, 1)$ for all $j = 1, 2, ..., N-1, \alpha_1^N \in (0, 1], \alpha_2^j, \alpha_3^j \in (0, 1]$ for all $j = 1, 2, ..., N$. Let $S : C \to C$ be the S-mappings generated by $G_1, G_2, ..., G_N$ and $\delta_1, \delta_2, ..., \delta_N$. Assume that $\mathfrak{F} = F(T) \cap \bigcap_{i=1}^{N} VI(C, B_i) \neq \varnothing$. For every $n \in \mathbb{N}$, $i = 1, 2, ..., N$, let $x_1, u \in C$ and $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C (I - \lambda_n (I - T)) x_n + \gamma_n S x_n \tag{1.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\} \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1, \beta_n \in [c, d] \subset (0, 1), \{\lambda_n\} \subset (0, 1 - \kappa)$ and suppose the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $\sum_{n=1}^{\infty}\lambda_n < \infty$,

(iii) $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty,\ \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n| < \infty,\ \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty,$
$\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}}u$.

In 1994, Blum and Oettli [12] introduced the equilibrium problem which has had a great impact and influence in the development of several branches of pure and applied sciences. Many problems in optimization, economics and physics are related to seeking the elements of $EP(F)$, see [12, 13]. Let $F : C \times\ C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points and denoted

$$EP(F) = \{x \in C : F(x, y) \geq 0,\ \forall y \in C\} \tag{1.10}$$

is the set of all solutions of equilibrium problem.

Numerous problems in economics, physics and optimization reduce to find a solution of $EP(F)$; see [12, 13, 14].

Recently, many authors consider the iterative scheme for finding a common element of the set of solution of equilibrium problem and the set of solutions of fixed point problem; see [13, 15, 16, 17].

Furthermore, from (1.6) and (1.10), we have the following *generalized equilibrium problem*, i.e., find $z \in C$ such that

$$F(z, y) + \langle Az, y - z\rangle \geq 0 \tag{1.11}$$

for all $y \in C$. The set of such $z \in C$ is denoted by $EP(F, A)$, i.e.,

$$EP(F, A) = \{z \in C : F(z, y) + \langle Az, y - z\rangle \geq 0,\ \forall y \in C\}.$$

In addition, in the case of $A \equiv 0$, $EP(F, A)$ is denoted by $EP(F)$ and in the case of $F \equiv 0$, $EP(F, A)$ is also denoted by $VI(C, A)$.

Let $A, B : C \to H$ be two mappings. By modification of (1.6), Kangtunyakarn [18] introduce the *combination of variational inequality problem* and denote

$$VI(C, aA + (1-a)B) = \{x \in C : \langle y - x, (aA + (1-a)B)x\rangle \geq 0,\ \forall y \in C,\ a \in (0, 1)\} \tag{1.12}$$

is the set of all solutions of the combination of variational inequality problem.

In particular, from (1.11) and (1.12), we introduce a problem with relatively between the combination of variational inequality problem and equilibrium problem, i.e., find $z \in C$ such that

$$F(z, y) + \langle (aA + (1-a)B)z, y - z\rangle \geq 0 \tag{1.13}$$

for all $y \in C$, and $a \in (0,1)$. The set of all solutions of the *combination of variational inequality problem and equilibrium problem* is denoted by $EP\left(F, (aA + (1-a)\,B)\right)$.

Throughout this section, let H_1 and H_2 be real Hilbert spaces and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* are adjoint of A and B, respectively.

In 1994, Censor and Elfving [19] introduced the *split feasibility problem* (SFP) which is to find a point $x \in C$ such that $Ax \in Q$. The set of all solutions of split feasibility problem is denoted by $\varphi = \{x \in C : Ax \in Q\}$.

The split feasibility problem was studied extensively as an extremely powerful tool in various fields such as signal processing, sensor networks, medical image reconstruction, instance in radiation therapy treatment planning, resolution enhancement, intensity-modulated radiation therapy problems and computer tomograph; see [20].

In 2012, Ceng, Ansari and Yao [21] introduced the lemma to solve SFP and many authors use this lemma to prove their results; see [22].

After that Kangtunyakarn [23] modified split feasibility problem, he introduce the *general split feasibility problem* (GSFP) which is to find a point $x^* \in C$ such that $Ax^*, Bx^* \in Q$. The set of all solutions of general split feasibility problem is denoted by $\Gamma = \{x \in C : Ax, Bx \in Q\}$. When $A \equiv B$, GSFP can be reduced to SFP.

In this thesis, the concepts of (1.9) and (1.13) are adapted to find a new methods for solving the problem. In addition, the related theorems are further studied as guidelines for create new theories covering the combination of variational inequality problems and equilibrium problems in Hilbert space.

1.2 Objectives of the study

1) To propose new methods of the combination of variational inequality problems and equilibrium problems that can solve complicated problems more than the original problem including can be applied to the original problem.
2) To establish fixed point theory can be solved new combination of variational inequality problems and equilibrium problems.
3) Create new knowledge related to the theory of the combination of the variational inequality problems and equilibrium problems in Hilbert space.
4) To give numerical examples to support our main results.

1.3 Scopes of the study

1) Study of combination of variational inequality and equilibrium problems in Hilbert space.

2) A study of definitions, theorems and lemma related to the combination of variational inequality and equilibrium problems in Hilbert space.

1.4 Benefits of the study

1) Obtain strong convergence theorems for finding a common element of the solution of the combination of variational inequality problems and equilibrium problems in Hilbert space.
2) Obtain mathematical tools for solving the combination of variational inequality problems, equilibrium problems and the fixed point problems in Hilbert space.
3) Obtain the knowledge and new mathematical tools for solving mathematical problems.

1.5 Research methodology

1) Study advanced topics in fixed point theory for nonexpansive mapping, κ-strictly pseudononspreading mapping and α-inverse strongly monotone mapping.
2) Study background in a real Hilbert space.
3) Study the basic khowledge of variational inequality problem.
4) Study the basic khowledge of equilibrium problem.
5) Study research papers and textbooks concerning fixed point theorem, variational inequality and equilibrium problems.
6) Define the scope and objectives of the research.
7) Create tools and methods for proving the strong convergence theorem of the combination of variational inequality, equilibrium and fixed point problems.
8) Prove the strong convergence theorem of the combination of variational inequality, equilibrium and fixed point problems in Hilbert space.
9) Provide applications and numerical examples.
10) Verify the accuracy of all content and write the thesis.

Chapter 2

Theory and Literature Reviews

In this chapter, we will introduce knowledge that is important for this thesis. We will also introduce necessary definitions, theorems, lemmas and remarks for our thesis. Throughout this chapter, we will let $\mathbb{R}$ represent the set of all real numbers, $\mathbb{C}$ represent the set of all complex numbers and $\mathbb{F}$ represent the set of all real or complex numbers.

2.1 Banach spaces and Hilbert spaces

Definition 2.1. [24] Let E be a vector space over the field $\mathbb{F}$. A *norm* is a function $\|\cdot\| : E \to [0, +\infty)$, which satisfies the following properties: For every $x, y \in E$

(E1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;

(E2) $\|\delta x\| = |\delta|\|x\|$ for all scalar δ;

(E3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

The ordered pair $(E, \|\cdot\|)$ or E is called a *normed space*.

Definition 2.2. [25] A sequence of vectors $\{x_n\}$ in a normed space E is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n \geq M$.

Definition 2.3. [25] A normed space E is called *complete* if every Cauchy sequence in E converges to an element $x^* \in E$.

Definition 2.4. [26] A complete normed linear space is called a *Banach space.*

Theorem 2.1. [25] A subset S of a normed space E is *closed* if and only if every sequence of elements of S convergent in E has its limit in S i.e.,

$$\{x_n\} \subseteq S \text{ and } x_n \to x \text{ implies } x \in S.$$

Definition 2.5. [27] Let X be a linear space over field $\mathbb{F}$. An *inner product* on X is a functional $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$ satisfying:

(N1) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(N2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ where the bar denotes complex conjugation;

(N3) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y \in X$ and $\alpha, \beta \in \mathbb{F}$.

The ordered pair $(X, \langle \cdot, \cdot \rangle)$ is called an *inner product space*.

Theorem 2.2. [28] For an inner product space X, $x, y, z \in X$ and $\alpha \in \mathbb{F}$, Then the following properties holds:

(P1) $\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$;

(P2) $\langle x, \alpha y\rangle = \overline{\alpha}\,\langle x,y\rangle$;

(P3) $\langle x, 0\rangle = \langle 0,x\rangle = 0$;

(P4) $\langle x, x\rangle = 0$ if and only if $x = 0$;

(P5) $\langle x, y\rangle = \langle x,z\rangle$, then $y = z$.

Theorem 2.3 (Schwarz inequality [26]). Let X be an inner product space, then

$$|\langle x,y\rangle| \leq \|x\|\|y\|$$

for all $x, y \in X$.

Definition 2.6 (Strong convergence [25]). A sequence $\{x_n\}$ of vectors in an inner product space X is called *strongly convergent* to x in X if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.7 (Weak convergence [25]). A sequence $\{x_n\}$ of vectors in an inner product space X is called *weakly convergent* to x in X if

$$\langle x_n, y\rangle \to \langle x,y\rangle \text{ as } n \to \infty,$$

for all $y \in X$.

Remark 2.4. In this thesis, strong convergence and weak convergence are represented by " $\to$ " and " $\rightharpoonup$ " respectively.

Theorem 2.5. [26] Let $\{x_n\}$ be a Cauchy sequence of an inner product space X such that $x_n \rightharpoonup x$. Then $x_n \to x$.

Definition 2.8. [26] A complete inner product space is called a *Hilbert space.*

Theorem 2.6. [26] Let H be a Hilbert space and let C be a nonempty closed convex subset of H with $\{x_n\} \subset C$ and $x_n \rightharpoonup x$, then $x \in C$.

Theorem 2.7. [26] Let $\{a_n\}$ be a bounded of real numbers. Then, there exists subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$a = \limsup_{n\to\infty} a_n = \lim_{k\to\infty} a_{n_k}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$b = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Theorem 2.8. [26] Let H be an inner product space. Then we know that the following (i) and (ii) are equivalent:

(i) H is complete,

(ii) each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

2.2 Fundamental properties in Hilbert spaces

Lemma 2.9. [26, 29] Let H be a Hilbert space. Then the following properties hold:

(1) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$,

(2) $\langle x+y, x-y\rangle = \|x\|^2 - \|y\|^2$,

(3) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$,

(4) $\|\delta x + (1-\delta)y\|^2 = \delta\|x\|^2 + (1-\delta)\|y\|^2 - \delta(1-\delta)\|x-y\|^2$,

(5) $|\,\|x\| - \|y\|\,| \leq \|x+y\| \leq \|x\| + \|y\|$,

(6) $\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2$,

for all $x, y, z \in H$ and $\delta, \alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Definition 2.9. [26] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\} \text{ is closed.}$$

Furthermore, f is called *convex* if for any $x, y \in C$ and $t \in [0,1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Theorem 2.10. [26] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightharpoonup x_0$. Then

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

Lemma 2.11. [26] Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

Definition 2.10. [26] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. The *metric projection or nearest point projection* of H onto C, denoted by P_C, is defined, for any $x \in H$, as the only point in C with the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.12. [26] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0$$

for all $z \in C$.

Lemma 2.13. [30] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then the following holds:

(i) $\|P_C x - P_C y\| \leq \|x - y\|$, for all $x, y \in H$,

(ii) $\langle y - x, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$, for all $x, y \in H$.

2.3 Variational inequality problems and Equilibrium problems

Lemma 2.14. [30] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.15. [31] Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is *demi-closed at zero.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0 \ \forall x \in C$;

(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0, \ \forall x, y \in C$;

(A3) $\forall x, y, z \in C, \ \lim_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) $\forall x \in C, \ y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.16. [12] Let C be a nonempty closed convex subset of H, let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1) - (A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$$

for all $y \in C$.

Lemma 2.17. [13] Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1) - (A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive i.e.,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle \quad \forall x, y \in H;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

2.4 Some useful lemmas

Lemma 2.18. [4] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.19. [4] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfying the conditions

(1) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \beta_n \leq 0$ or $\sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.20. [11] Let $T : C \to C$ be a κ-strictly pseudononspreading mapping with $F(T) \neq \varnothing$. Then $F(T) = VI(C, (I-T))$.

Remark 2.21. From Lemmas 2.14 and 2.20, we have $F(T) = F(P_C(I - \lambda(I-T)))$ for all $\lambda > 0$.

Lemma 2.22. [33] Let A be a bounded linear operator on a Hilbert space H. The operator $A^* : H \to H$ defined by

$$\langle Ax, y \rangle = \langle x, A^*y \rangle,$$

for all $x, y \in H$, is called the *adjoint operator* of A.

Lemma 2.23. [34] Let T be a bounded linear operator on a Hilbert space H. The spectal radius of T, denoted by $r_\sigma(T)$, is the number defined by

$$r_\sigma(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

where $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)(x) = 0, \text{ for some } 0 \neq x \in H\}$.

Lemma 2.24. [23] Let H_1 and H_2 be real Hilbert spaces and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* are adjoint of A and B, respectively. Let $\Gamma = \{x \in C : Ax, Bx \in Q\} \neq \varnothing$. Then the followings are equivalent:

(i) $x^* \in \Gamma$,

(ii) $P_C\left(I - a\left(\frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}\right)\right)x^* = x^*$,

for all $a > 0$ and L_A, L_B are spectal radius of A^*A and B^*B, respectively with $a \in (0, \frac{2}{L})$ and $L = \max\{L_A, L_B\}$.

Lemma 2.25. [32] Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \varnothing$. Then $F(T) = VI\left(C, (I - T)\right)$.

Lemma 2.26. [10] Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively with $VI(C, D_1) \cap VI(C, D_2) \neq \varnothing$. Define $G : C \to C$ by

$$G(x) = P_C\left(I - \lambda_1 D_1\right)\left(ax + (1-a)\,P_C\left(I - \lambda_2 D_2\right)x\right),$$

for all $\lambda_1 \in (0, 2d_1), \lambda_2 \in (0, 2d_2)$ and $a \in (0, 1)$. Then $F(G) = VI(C, D_1) \cap VI(C, D_2)$.

Lemma 2.27. [18] Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively with $\alpha, \beta > 0$ and $VI(C, A) \cap VI(C, B) \neq \varnothing$. Then

$$VI\left(C, aA + (1-a)\,B\right) = VI(C, A) \cap VI(C, B),$$

for all $a \in (0, 1)$.

Chapter 3

Main Results and Discussion

In this chapter, we prove a strong convergence theorem for approximating the solution of the modification of system variational inequality and generalized equilibrium problems and fixed point problems of κ-strictly pseudononspreading mapping, by modify Halpern iterative method.

3.1 An aproximation method of nonlinear mapping for a modified general equilibrium and system of variational inequality problems

Theorem 3.1. Let C be a closed convex subset of Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), let $A, B, A', B' : C \to H$ be $\alpha, \beta, \alpha', \beta'$-inverse strongly monotone, respectively. Define $G : C \to C$ by $Gx = P_C(I - \lambda_1 A')(ax + (1 - a)P_C(I - \lambda_2 B')x)$ for all $x \in C$ with $\lambda_1 \in (0, 2\alpha')$ and $\lambda_2 \in (0, 2\beta')$. Let $T : C \to C$ be κ-strictly pseudononspreading mapping for some $\kappa \in [0, 1)$ with $\mathfrak{F} = F(T) \cap F(G) \cap EP(F, aA+(1-a)B) \neq \varnothing$ for all $a \in (0, 1)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and

$$\begin{cases} F(u_n, y) + \langle (aA + (1-a)B)x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n G x_n + \gamma_n P_C(I - \lambda_n(I - T))u_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1], \lambda_n \in (0, 1-\kappa), \alpha_n + \beta_n + \gamma_n = 1, \forall n \in \mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma], \gamma = \min\{\alpha, \beta\}$ satisfy;

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \lambda_n < \infty;$

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1,\ 0 < e \leq\ r_n \leq f < 2\gamma;$

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0;$

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$.

Proof. We divide the proof into seven steps:

Step 1. We will show that $aA + (1 - a)B$ is γ-inverse strongly monotone. Put $D = aA + (1 - a)B$ for all $a \in (0, 1)$. Let $x, y \in C$, we have

$$\begin{aligned} &\langle Dx - Dy, x - y \rangle \\ &\quad = \langle (aA + (1-a)B)x - (aA + (1-a)B)y, x - y \rangle \\ &\quad = \langle a(Ax - Ay) + (1-a)(Bx - By), x - y \rangle \\ &\quad = a \langle Ax - Ay, x - y \rangle + (1-a) \langle Bx - By, x - y \rangle \end{aligned}$$

$$
\begin{aligned}
&\geq a\alpha \|Ax-Ay\|^2+(1-a)\beta\|Bx-By\|^2\\
&\geq a\gamma \|Ax-Ay\|^2+(1-a)\gamma\|Bx-By\|^2\\
&= \gamma\left(a\|Ax-Ay\|^2+(1-a)\|Bx-By\|^2\right)\\
&\geq \gamma\|a(Ax-Ay)+(1-a)(Bx-By)\|^2\\
&= \gamma\|aAx-aAy+(1-a)Bx-(1-a)By\|^2\\
&= \gamma\|(aA+(1-a)B)x-(aA+(1-a)B)y\|^2\\
&= \gamma\|Dx-Dy\|^2.
\end{aligned}
$$

Then

$$\langle Dx-Dy, x-y\rangle \geq \gamma\|Dx-Dy\|^2. \tag{3.2}$$

Hence $aA+(1-a)B$ is γ-inverse strongly monotone mapping where $D=aA+(1-a)B$. Next, we will show that $I-r_nD$ is a nonexpansive mapping for every $n\in\mathbb{N}$. Let $x,y\in C$ and $n\in\mathbb{N}$. From (3.2), we have

$$
\begin{aligned}
\|(I-r_nD)x-(I-r_nD)y\|^2 &= \|(x-y)-r_n(Dx-Dy)\|^2\\
&= \|x-y\|^2-2r_n\langle x-y, Dx-Dy\rangle+r_n^2\|Dx-Dy\|^2\\
&\leq \|x-y\|^2-2r_n\gamma\|Dx-Dy\|^2+r_n^2\|Dx-Dy\|^2\\
&= \|x-y\|^2+r_n(r_n-2\gamma)\|Dx-Dy\|^2\\
&\leq \|x-y\|^2.
\end{aligned}
$$

Then

$$\|(I-r_nD)x_n-(I-r_nD)y\| \leq \|x-y\|. \tag{3.3}$$

Therefore, $I-r_nD$ is a nonexpansive mapping for every $n\in\mathbb{N}$.
Next, we will show that G is a nonexpansive mapping. Let $x,y\in C$ and $a\in(0,1)$. Since A' is α'-inverse strongly monotone mapping and $\lambda_1\in(0,2\alpha')$, we have

$$
\begin{aligned}
\|P_C(I-\lambda_1A')x-P_C(I-\lambda_1A')y\|^2 &\leq \|(I-\lambda_1A')x-(I-\lambda_1A')y\|^2\\
&= \|(x-y)-\lambda_1(A'x-A'y)\|^2\\
&= \|x-y\|^2-2\lambda_1\langle x-y, A'x-A'y\rangle+\lambda_1^2\|A'x-A'y\|^2\\
&\leq \|x-y\|^2-2\alpha'\lambda_1\|A'x-A'y\|^2+\lambda_1^2\|A'x-A'y\|^2\\
&= \|x-y\|^2+\lambda_1(\lambda_1-2\alpha')\|A'x-A'y\|^2\\
&\leq \|x-y\|^2.
\end{aligned}
$$

Then

$$\|P_C(I-\lambda_1A')x-P_C(I-\lambda_1A')y\| \leq \|x-y\|.$$

Therefore, $P_C(I-\lambda_1A')$ is a nonexpansive mapping. By using the same method above, we have $P_C(I-\lambda_2B')$ is a nonexpansive mapping with $\lambda_2\in(0,2\beta')$ and B' is β'-inverse strongly monotone mapping. From definition of G, we have

$$\begin{aligned}\|Gx-Gy\| &= \|P_C\left(I-\lambda_1 A'\right)\left(ax+(1-a)P_C\left(I-\lambda_2 B'\right)x\right) \\ &\quad - P_C\left(I-\lambda_1 A'\right)\left(ay+(1-a)P_C\left(I-\lambda_2 B'\right)y\right)\| \\ &\leq \|\left(ax+(1-a)P_C\left(I-\lambda_2 B'\right)x\right)-\left(ay+(1-a)P_C\left(I-\lambda_2 B'\right)y\right)\| \\ &= \|a\left(x-y\right)+(1-a)\left(P_C\left(I-\lambda_2 B'\right)x-P_C\left(I-\lambda_2 B'\right)y\right)\| \\ &\leq a\left\|x-y\right\|+(1-a)\left\|P_C\left(I-\lambda_2 B'\right)x-P_C\left(I-\lambda_2 B'\right)y\right\| \\ &\leq a\left\|x-y\right\|+(1-a)\left\|x-y\right\| \\ &= \left\|x-y\right\|.\end{aligned}$$

Then

$$\|Gx-Gy\| \leq \|x-y\|,$$

for all $x, y \in C$. Therefore, G is a nonexpansive mapping.

Step 2. For every $a \in (0,1)$, we will show that $\{x_n\}$ is bounded. Let $z \in \mathfrak{F}$. From (3.1), we have

$$F(u_n, y) + \langle (aA+(1-a)B)x_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0,\ \forall y \in C.$$

From Lemma 2.17 , we have

$$\begin{aligned}T_{r_n}(I-r_nD)x_n &= \big\{F(u_n,y)+\langle (aA+(1-a)B)x_n, y-u_n\rangle \\ &\qquad + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \forall y \in C\big\}.\end{aligned}$$

Then $u_n \in T_{r_n}(I-r_nD)x_n$.

It implies that

$$u_n = T_{r_n}(I-r_nD)x_n. \tag{3.4}$$

Since $z \in \mathfrak{F}$.

Then $z \in F(T)$.

From Lemma 2.20, we have $F(T) = VI\left(C, (I-T)\right)$.

It follows that $z \in VI\left(C, (I-T)\right)$.

From Remark 2.21, we have $z \in F\left(P_C\left(I-\lambda_n\left(I-T\right)\right)\right)$.

It implies that

$$z = P_C\left(I-\lambda_n\left(I-T\right)\right)z \tag{3.5}$$

for all $n \in \mathbb{N}$. Since $z \in EP(F,D)$ where $D = aA+(1-a)B$, we have

$$F(z,y)+\langle y-z, Dz\rangle \geq 0,\ \forall y \in C.$$

So, we have

$$F(z,y)+\frac{1}{r_n}\left\langle y-z, z-z+r_nDz\right\rangle \geq 0,\ \forall n \in \mathbb{N}, y \in C.$$

From Lemma 2.17, we have

$$z = T_{r_n}(I - r_n D)z \tag{3.6}$$

for all $n \in \mathbb{N}$. Next, we show that $\|P_C (I - \lambda_n (I - T)) u_n - z\| \leq \|u_n - z\|$.
By the nonexpansiveness of P_C and (3.5), we have

$$\begin{array}{rcl} \|P_C (I - \lambda_n (I - T)) u_n - z\| & = & \|P_C (I - \lambda_n (I - T)) u_n - P_C (I - \lambda_n (I - T)) z\| \\ & \leq & \|(I - \lambda_n (I - T)) u_n - (I - \lambda_n (I - T)) z\|, \end{array}$$

which implies that

$$\|P_C (I - \lambda_n (I - T)) u_n - z\| \leq \|(I - \lambda_n (I - T)) u_n - (I - \lambda_n (I - T)) z\|. \tag{3.7}$$

Since T is κ-strictly pseudononspreading mapping and let $E = I - T$, we have

$$\begin{array}{rcl} \|Tu_n - Tz\|^2 & = & \|(I - E) u_n - (I - E) z\|^2 \\ & = & \|(u_n - z) - (Eu_n - Ez)\|^2 \\ & = & \|u_n - z\|^2 - 2 \langle u_n - z, Eu_n - Ez\rangle + \|Eu_n - Ez\|^2 \\ & = & \|u_n - z\|^2 - 2 \langle u_n - z, Eu_n\rangle + \|Eu_n\|^2 \\ & \leq & \|u_n - z\|^2 + \kappa \|Eu_n - Ez\|^2 + 2 \langle Eu_n, Ez\rangle \\ & = & \|u_n - z\|^2 + \kappa \|Eu_n\|^2 + 2 \langle Eu_n, 0\rangle \\ & = & \|u_n - z\|^2 + \kappa \|Eu_n\|^2, \end{array}$$

which implies that

$$\|u_n - z\|^2 - 2 \langle u_n - z, Eu_n\rangle + \|Eu_n\|^2 \leq \|u_n - z\|^2 + \kappa \|Eu_n\|^2.$$

Thus,

$$(1 - \kappa) \|Eu_n\|^2 \leq 2 \langle u_n - z, Eu_n\rangle. \tag{3.8}$$

From (3.8), we have

$$\begin{array}{rcl} \|(I - \lambda_n E) u_n - (I - \lambda_n E) z\|^2 & = & \|(u_n - z) - \lambda_n (Eu_n - Ez)\|^2 \\ & = & \|u_n - z\|^2 - 2 \langle u_n - z, \lambda_n (Eu_n - Ez)\rangle \\ & & + \|\lambda_n (Eu_n - Ez)\|^2 \\ & = & \|u_n - z\|^2 - 2\lambda_n \langle u_n - z, Eu_n\rangle + \lambda_n^2 \|Eu_n\|^2 \\ & \leq & \|u_n - z\|^2 - \lambda_n (1 - \kappa) \|Eu_n\|^2 + \lambda_n^2 \|Eu_n\|^2 \\ & = & \|u_n - z\|^2 - \lambda_n ((1 - \kappa) - \lambda_n) \|Eu_n\|^2 \\ & \leq & \|u_n - z\|^2. \end{array}$$

Then

$$\|(I - \lambda_n E) u_n - (I - \lambda_n E) z\|^2 \leq \|u_n - z\|^2.$$

It implies that

$$\left\|\left(I-\lambda_n E\right) u_n-\left(I-\lambda_n E\right) z\right\| \leq\left\|u_n-z\right\| . \tag{3.9}$$

From (3.7) and (3.9), we can imply that

$$\left\|P_C\left(I-\lambda_n(I-T)\right) u_n-z\right\| \leq\left\|u_n-z\right\| . \tag{3.10}$$

Since $z \in \mathfrak{F}$, we have $z \in F(G)$.
It implies that

$$z=G(z)=P_C\left(I-\lambda_1 A'\right)\left(a z+(1-a) P_C\left(I-\lambda_2 B'\right) z\right) .$$

Put $M_n=a x_n+(1-a) P_C\left(I-\lambda_2 B'\right) x_n$. Then, we have $G_n=P_C\left(I-\lambda_1 A'\right) M_n$.
From definition of x_n, (3.4), (3.6), (3.10), and nonexpansiveness of G, we have

$$\begin{aligned}
\left\|x_{n+1}-z\right\| & =\left\|\alpha_n(u-z)+\beta_n\left(G x_n-z\right)+\gamma_n\left(P_C\left(I-\lambda_n(I-T)\right) u_n-z\right)\right\| \\
& \leq \alpha_n\|u-z\|+\beta_n\left\|G x_n-z\right\|+\gamma_n\left\|P_C\left(I-\lambda_n(I-T)\right) u_n-z\right\| \\
& =\alpha_n\|u-z\|+\beta_n\left\|G x_n-G z\right\|+\gamma_n\left\|P_C\left(I-\lambda_n(I-T)\right) u_n-z\right\| \\
& \leq \alpha_n\|u-z\|+\beta_n\left\|x_n-z\right\|+\gamma_n\left\|u_n-z\right\| \\
& =\alpha_n\|u-z\|+\beta_n\left\|x_n-z\right\|+\gamma_n\left\|T_{r_n}\left(I-r_n D\right) x_n-T_{r_n}\left(I-r_n D\right) z\right\| \\
& \leq \alpha_n\|u-z\|+\beta_n\left\|x_n-z\right\|+\gamma_n\left\|x_n-z\right\| \\
& =\alpha_n\|u-z\|+\left(\beta_n+\gamma_n\right)\left\|x_n-z\right\| \\
& =\alpha_n\|u-z\|+\left(1-\alpha_n\right)\left\|x_n-z\right\| \\
& \leq \max\{\left\|x_1-z\right\|,\|u-z\|\}.
\end{aligned}$$

By induction, we can prove that $\{x_n\}$ is bounded and so is $\{u_n\}$.
Step 3. We will show that $\lim_{n \rightarrow \infty}\left\|x_{n+1}-x_n\right\|=0$. Putting $v_n=x_n-r_n D x_n$, we have $u_n=T_{r_n}(I-r_n D) x_n=T_{r_n}(x_n-r_n D x_n)=T_{r_n} v_n$. From definition of u_n, we have

$$F(u_n, y)+\frac{1}{r_n}\left\langle y-u_n, u_n-v_n\right\rangle \geq 0, \ \forall y \in C \tag{3.11}$$

and

$$F\left(u_{n+1}, y\right)+\frac{1}{r_{n+1}}\left\langle y-u_{n+1}, u_{n+1}-v_{n+1}\right\rangle \geq 0, \ \forall y \in C. \tag{3.12}$$

Instead of y by u_{n+1} and u_n in (3.11) and (3.12), respectively, we have

$$F\left(u_n, u_{n+1}\right)+\frac{1}{r_n}\left\langle u_{n+1}-u_n, u_n-v_n\right\rangle \geq 0 \tag{3.13}$$

and

$$F\left(u_{n+1}, u_n\right)+\frac{1}{r_{n+1}}\left\langle u_n-u_{n+1}, u_{n+1}-v_{n+1}\right\rangle \geq 0. \tag{3.14}$$

Adding (3.13) and (3.14), we have

$$F\left(u_n, u_{n+1}\right)+F\left(u_{n+1}, u_n\right)+\frac{1}{r_n}\left\langle u_{n+1}-u_n, u_n-v_n\right\rangle+\frac{1}{r_{n+1}}\left\langle u_n-u_{n+1}, u_{n+1}-v_{n+1}\right\rangle \geq 0,$$

and using (A2) which implies that

$$\begin{aligned}
0 &\le \frac{1}{r_n}\langle u_{n+1}-u_n, u_n-v_n\rangle + \frac{1}{r_{n+1}}\langle u_n-u_{n+1}, u_{n+1}-v_{n+1}\rangle\\
&= \left\langle u_{n+1}-u_n, \frac{u_n-v_n}{r_n}\right\rangle + \left\langle u_n-u_{n+1}, \frac{u_{n+1}-v_{n+1}}{r_{n+1}}\right\rangle\\
&= \left\langle u_{n+1}-u_n, \frac{u_n-v_n}{r_n}\right\rangle - \left\langle u_{n+1}-u_n, \frac{u_{n+1}-v_{n+1}}{r_{n+1}}\right\rangle\\
&= \left\langle u_{n+1}-u_n, \frac{u_n-v_n}{r_n} - \frac{u_{n+1}-v_{n+1}}{r_{n+1}}\right\rangle.
\end{aligned}$$

It implies that

$$0 \le \left\langle u_{n+1}-u_n, \frac{u_n-v_n}{r_n} - \frac{u_{n+1}-v_{n+1}}{r_{n+1}}\right\rangle.$$

Multiplying r_n, we have

$$\begin{aligned}
0 &\le \left\langle u_{n+1}-u_n, r_n\left(\frac{u_n-v_n}{r_n}\right) - r_n\left(\frac{u_{n+1}-v_{n+1}}{r_{n+1}}\right)\right\rangle\\
&= \left\langle u_{n+1}-u_n, u_n-v_n-\frac{r_n}{r_{n+1}}(u_{n+1}-v_{n+1})\right\rangle\\
&= \left\langle u_{n+1}-u_n, u_n-u_{n+1}+u_{n+1}-v_n-\frac{r_n}{r_{n+1}}(u_{n+1}-v_{n+1})\right\rangle\\
&= \langle u_{n+1}-u_n, u_n-u_{n+1}\rangle + \left\langle u_{n+1}-u_n, u_{n+1}-v_n-\frac{r_n}{r_{n+1}}(u_{n+1}-v_{n+1})\right\rangle\\
&= -\langle u_{n+1}-u_n, u_{n+1}-u_n\rangle + \left\langle u_{n+1}-u_n, u_{n+1}-v_n-\frac{r_n}{r_{n+1}}(u_{n+1}-v_{n+1})\right\rangle\\
&= -\|u_{n+1}-u_n\|^2 + \left\langle u_{n+1}-u_n, u_{n+1}-v_n-\frac{r_n}{r_{n+1}}(u_{n+1}-v_{n+1})\right\rangle. \qquad (3.15)
\end{aligned}$$

From (3.15), we have

$$\begin{aligned}
\|u_{n+1}-u_n\|^2 &\le \left\langle u_{n+1}-u_n, u_{n+1}-v_n-\frac{r_n}{r_{n+1}}(u_{n+1}-v_{n+1})\right\rangle\\
&= \left\langle u_{n+1}-u_n, u_{n+1}-v_{n+1}+v_{n+1}-v_n-\frac{r_n}{r_{n+1}}(u_{n+1}-v_{n+1})\right\rangle\\
&= \left\langle u_{n+1}-u_n, v_{n+1}-v_n+\left(1-\frac{r_n}{r_{n+1}}\right)(u_{n+1}-v_{n+1})\right\rangle\\
&\le \|u_{n+1}-u_n\|\left\|v_{n+1}-v_n+\left(1-\frac{r_n}{r_{n+1}}\right)(u_{n+1}-v_{n+1})\right\|\\
&\le \|u_{n+1}-u_n\|\left(\|v_{n+1}-v_n\|+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\,\|u_{n+1}-v_{n+1}\|\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|u_{n+1}-u_n\| &\le \|v_{n+1}-v_n\|+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\,\|u_{n+1}-v_{n+1}\|\\
&\le \|v_{n+1}-v_n\|+\frac{1}{e}|r_{n+1}-r_n|\,\|u_{n+1}-v_{n+1}\|.
\end{aligned}$$

Hence

$$\|u_{n+1}-u_n\| \le \|v_{n+1}-v_n\|+\frac{1}{e}|r_{n+1}-r_n|\,\|u_{n+1}-v_{n+1}\|. \qquad (3.16)$$

Since $v_n = x_n - r_n D x_n$, we have

$$\begin{aligned}
\|v_{n+1} - v_n\| &= \|(x_{n+1} - r_{n+1} D x_{n+1}) - (x_n - r_n D x_n)\| \\
&= \|(I - r_{n+1} D)\, x_{n+1} - (I - r_n D)\, x_n\| \\
&= \|(I - r_{n+1} D)\, x_{n+1} - (I - r_{n+1} D)\, x_n + (I - r_{n+1} D)\, x_n - (I - r_n D)\, x_n\| \\
&\leq \|(I - r_{n+1} D)\, x_{n+1} - (I - r_{n+1} D)\, x_n\| + \|(I - r_{n+1} D)\, x_n - (I - r_n D)\, x_n\| \\
&= \|(I - r_{n+1} D)\, x_{n+1} - (I - r_{n+1} D)\, x_n\| + \|(r_n - r_{n+1})\, D x_n\| \\
&\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\, \|D x_n\|. \qquad (3.17)
\end{aligned}$$

Substitute (3.17) into (3.16), we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|v_{n+1} - v_n\| + \frac{1}{e}|r_{n+1} - r_n|\, \|u_{n+1} - v_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\, \|D x_n\| + \frac{1}{e}|r_{n+1} - r_n|\, \|u_{n+1} - v_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\, L + \frac{1}{e}|r_{n+1} - r_n|\, L, \qquad (3.18)
\end{aligned}$$

where $L = \max_{n\in\mathbb{N}} \{\|D x_n\|, \|u_n - v_n\|\}$.
From definition of x_n and (3.18), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|\alpha_n u + \beta_n G x_n + \gamma_n P_C\,(I - \lambda_n\,(I - T))\, u_n \\
&\quad - \alpha_{n-1} u - \beta_{n-1} G x_{n-1} - \gamma_{n-1} P_C\,(I - \lambda_{n-1}\,(I - T))\, u_{n-1}\| \\
&= \|\alpha_n u + \beta_n G x_n - \beta_n G x_{n-1} + \beta_n G x_{n-1} + \gamma_n P_C\,(I - \lambda_n\,(I - T))\, u_n \\
&\quad - \gamma_n P_C\,(I - \lambda_{n-1}\,(I - T))\, u_{n-1} + \gamma_n P_C\,(I - \lambda_{n-1}\,(I - T))\, u_{n-1} - \alpha_{n-1} u \\
&\quad - \beta_{n-1} G x_{n-1} - \gamma_{n-1} P_C\,(I - \lambda_{n-1}\,(I - T))\, u_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}|\, \|u\| + \beta_n \|G x_n - G x_{n-1}\| + |\beta_n - \beta_{n-1}|\, \|G x_{n-1}\| \\
&\quad + \gamma_n \|P_C\,(I - \lambda_n\,(I - T))\, u_n - P_C\,(I - \lambda_{n-1}\,(I - T))\, u_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\, \|P_C\,(I - \lambda_{n-1}\,(I - T))\, u_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}|\, \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\, \|G x_{n-1}\| \\
&\quad + \gamma_n \|(I - \lambda_n\,(I - T))\, u_n - (I - \lambda_{n-1}\,(I - T))\, u_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\, \|P_C\,(I - \lambda_{n-1}\,(I - T))\, u_{n-1}\| \\
&= |\alpha_n - \alpha_{n-1}|\, \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\, \|G x_{n-1}\| \\
&\quad + \gamma_n \|\,(u_n - u_{n-1}) - \lambda_n\,(I - T)\, u_n + \lambda_n\,(I - T)\, u_{n-1} \\
&\quad - \lambda_n\,(I - T)\, u_{n-1} + \lambda_{n-1}\,(I - T)\, u_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\, \|P_C\,(I - \lambda_{n-1}\,(I - T))\, u_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}|\, \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\, \|G x_{n-1}\| \\
&\quad + \gamma_n \Big(\, \|u_n - u_{n-1}\| + \lambda_n \|(I - T)\, u_n - (I - T)\, u_{n-1}\| \\
&\quad + |\lambda_n - \lambda_{n-1}|\, \|(I - T)\, u_{n-1}\| \Big)
\end{aligned}$$

$$
\begin{aligned}
&\quad + |\gamma_n - \gamma_{n-1}| \, \| P_C (I - \lambda_{n-1} (I - T)) u_{n-1} \| \\
&\leq |\alpha_n - \alpha_{n-1}| \, \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \, \|G x_{n-1}\| \\
&\quad + \gamma_n \Big(\|x_n - x_{n-1}\| + |r_{n-1} - r_n| \, L + \frac{1}{e} |r_{n-1} - r_n| \, L \\
&\quad + \lambda_n \|(I - T) u_n - (I - T) u_{n-1}\| + |\lambda_n - \lambda_{n-1}| \, \|(I - T) u_{n-1}\| \Big) \\
&\quad + |\gamma_n - \gamma_{n-1}| \, \| P_C (I - \lambda_{n-1} (I - T)) u_{n-1} \| \\
&\leq |\alpha_n - \alpha_{n-1}| \, \|u\| + (1 - \alpha_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \, \|G x_{n-1}\| \\
&\quad + |r_{n-1} - r_n| \, L + \frac{1}{e} |r_{n-1} - r_n| \, L + \lambda_n \|(I - T) u_n - (I - T) u_{n-1}\| \\
&\quad + |\lambda_n - \lambda_{n-1}| \, \|(I - T) u_{n-1}\| + |\gamma_n - \gamma_{n-1}| \, \| P_C (I - \lambda_{n-1} (I - T)) u_{n-1} \| \\
&\leq |\alpha_n - \alpha_{n-1}| \, K + (1 - \alpha_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \, K + |r_{n-1} - r_n| \, K \\
&\quad + \frac{1}{e} |r_{n-1} - r_n| \, K + \lambda_n K + |\lambda_n - \lambda_{n-1}| \, K + |\gamma_n - \gamma_{n-1}| \, K.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|x_{n+1} - x_n\| \leq |\alpha_n - \alpha_{n-1}| \, K + (1 - \alpha_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \, K + |r_{n-1} - r_n| \, K \\
+ \frac{1}{e} |r_{n-1} - r_n| \, K + \lambda_n K + |\lambda_n - \lambda_{n-1}| \, K + |\gamma_n - \gamma_{n-1}| \, K
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{aligned}
K = \max_{n \in \mathbb{N}} \Big\{ &\|u\|, \|G x_{n-1}\|, \|(I - T) u_n - (I - T) u_{n-1}\|, \|(I - T) u_{n-1}\|, \\
&\| P_C (I - \lambda_{n-1} (I - T)) u_{n-1} \|, \|D x_n\|, \|u_n - v_n\| \Big\}.
\end{aligned}
$$

From Lemma 2.18, (3.19), conditions (i), (iii), and (iv), we have

$$
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.20}
$$

Step 4. We will show that $\lim_{n \to \infty} \|x_n - G x_n\| = 0$. Since $u_n = T_{r_n} (x_n - r_n D x_n)$, we have

$$
\begin{aligned}
\|u_n - z\|^2 &= \|T_{r_n} (I - r_n D) x_n - T_{r_n} (I - r_n D) z\|^2 \\
&\leq \langle (I - r_n D) x_n - (I - r_n D) z, T_{r_n} (I - r_n D) x_n - T_{r_n} (I - r_n D) z \rangle \\
&= \langle (I - r_n D) x_n - (I - r_n D) z, u_n - z \rangle \\
&= \frac{1}{2} \Big(\|(I - r_n D) x_n - (I - r_n D) z\|^2 + \|u_n - z\|^2 \\
&\quad - \|(I - r_n D) x_n - (I - r_n D) z - u_n + z\|^2 \Big) \\
&\leq \frac{1}{2} \left(\|x_n - z\|^2 + \|u_n - z\|^2 - \|(x_n - u_n) - r_n (D x_n - D z)\|^2 \right) \\
&= \frac{1}{2} \Big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|x_n - u_n\|^2 - r_n^2 \|D x_n - D z\|^2 \\
&\quad + 2 r_n \langle x_n - u_n, D x_n - D z \rangle \Big).
\end{aligned}
$$

It implies that

$$
\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|x_n - u_n\|^2 - r_n^2 \|D x_n - D z\|^2 + 2 r_n \langle x_n - u_n, D x_n - D z \rangle . \tag{3.21}
$$

By nonexpansiveness of T_{r_n}, we have

$$\begin{aligned}\|u_n - z\|^2 &= \|T_{r_n}(I - r_nD)x_n - T_{r_n}(I - r_nD)z\|^2 \\ &\leq \|(I - r_nD)x_n - (I - r_nD)z\|^2 \\ &= \|(x - z) - r_n(Dx_n - Dz)\|^2 \\ &= \|x - z\|^2 - 2r_n\langle x_n - z, Dx_n - Dz\rangle + r_n^2\|Dx_n - Dz\|^2 \\ &\leq \|x_n - z\|^2 - 2r_n\gamma\|Dx_n - Dz\|^2 + r_n^2\|Dx_n - Dz\|^2 \\ &= \|x_n - z\|^2 + r_n(r_n - 2\gamma)\|Dx_n - Dz\|^2 \\ &\leq \|x_n - z\|^2.\end{aligned}$$

Then

$$\|u_n - z\|^2 \leq \|x_n - z\|^2. \tag{3.22}$$

From definition of x_n, (3.10) and (3.22), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n(u - z) + \beta_n(Gx_n - z) + \gamma_n(P_C(I - \lambda_n(I - T))u_n - z)\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \beta_n\|Gx_n - z\|^2 + \gamma_n\|P_C(I - \lambda_n(I - T))u_n - z\|^2 \\ &\quad - \beta_n\gamma_n\|P_C(I - \lambda_n(I - T))u_n - Gx_n\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \beta_n\|Gx_n - z\|^2 + \gamma_n\|u_n - z\|^2 \\ &\quad - \beta_n\gamma_n\|P_C(I - \lambda_n(I - T))u_n - Gx_n\|^2 \\ &= \alpha_n\|u - z\|^2 + \beta_n\|Gx_n - Gz\|^2 + \gamma_n\|u_n - z\|^2 \\ &\quad - \beta_n\gamma_n\|P_C(I - \lambda_n(I - T))u_n - Gx_n\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\|x_n - z\|^2 \\ &\quad - \beta_n\gamma_n\|P_C(I - \lambda_n(I - T))u_n - Gx_n\|^2 \\ &= \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 \\ &\quad - \beta_n\gamma_n\|P_C(I - \lambda_n(I - T))u_n - Gx_n\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \beta_n\gamma_n\|P_C(I - \lambda_n(I - T))u_n - Gx_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}&\beta_n\gamma_n\|P_C(I - \lambda_n(I - T))u_n - Gx_n\|^2 \\ &\quad\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\quad= \alpha_n\|u - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)(\|x_n - z\| - \|x_{n+1} - z\|) \\ &\quad\leq \alpha_n\|u - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\|.\end{aligned} \tag{3.23}$$

From (3.20), (3.23), conditions (i) and (ii), we have

$$\lim_{n\to\infty}\|P_C(I - \lambda_n(I - T))u_n - Gx_n\| = 0. \tag{3.24}$$

By definition of x_n, we obtain

$$\begin{aligned} x_{n+1} - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n &= \alpha_n\left(u - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right) \\ &\quad + \beta_n\left(Gx_n - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right) \\ &\quad + \gamma_n\left(P_C\left(I - \lambda_n\left(I - T\right)\right)u_n - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right). \end{aligned}$$

It follows that

$$\begin{aligned} x_{n+1} - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n &= \alpha_n\left(u - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right) \\ &\quad + \beta_n\left(Gx_n - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right), \end{aligned}$$

which implies that

$$\begin{aligned} \left\|x_{n+1} - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right\| &= \|\alpha_n\left(u - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right) \\ &\quad + \beta_n\left(Gx_n - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right)\| \\ &\leq \alpha_n\left\|u - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right\| \\ &\quad + \beta_n\left\|Gx_n - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right\|. \end{aligned}$$

From (3.24) and condition (i), we have

$$\lim_{n\to\infty}\left\|x_{n+1} - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right\| = 0. \tag{3.25}$$

Since

$$\begin{aligned} \left\|x_n - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right\| &= \left\|x_n - x_{n+1} + x_{n+1} - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right\| \\ &\leq \left\|x_n - x_{n+1}\right\| + \left\|x_{n+1} - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right\|, \end{aligned}$$

(3.20) and (3.25), we have

$$\lim_{n\to\infty}\left\|x_n - P_C\left(I - \lambda_n\left(I - T\right)\right)u_n\right\| = 0. \tag{3.26}$$

By definition of x_n, we obtain

$$x_{n+1} - x_n = \alpha_n\left(u - x_n\right) + \beta_n\left(Gx_n - x_n\right) + \gamma_n\left(P_C\left(I - \lambda_n\left(I - T\right)\right)u_n - x_n\right).$$

So, we get

$$\begin{aligned} \beta_n\left\|Gx_n - x_n\right\| &= \left\|x_{n+1} - x_n - \alpha_n\left(u - x_n\right) - \gamma_n\left(P_C\left(I - \lambda_n\left(I - T\right)\right)u_n - x_n\right)\right\| \\ &\leq \left\|x_{n+1} - x_n\right\| + \alpha_n\left\|u - x_n\right\| + \gamma_n\left\|P_C\left(I - \lambda_n\left(I - T\right)\right)u_n - x_n\right\|. \end{aligned}$$

From (3.20), (3.26), conditions (i) and (ii), we have

$$\lim_{n\to\infty}\left\|Gx_n - x_n\right\| = 0. \tag{3.27}$$

Step 5. We will show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. By nonexpansiveness of T_{r_n}, we have

$$\begin{aligned}
\|u_n - z\|^2 &= \|T_{r_n}(I - r_nD)x_n - T_{r_n}(I - r_nD)z\|^2\\
&\le \|(I - r_nD)x_n - (I - r_nD)z\|^2\\
&= \|(x_n - z) - r_n(Dx_n - Dz)\|^2\\
&= \|x_n - z\|^2 - 2r_n\langle x_n - z, Dx_n - Dz\rangle + r_n^2\|Dx_n - Dz\|^2\\
&\le \|x_n - z\|^2 - 2r_n\gamma\|Dx_n - Dz\|^2 + r_n^2\|Dx_n - Dz\|^2\\
&= \|x_n - z\|^2 - r_n(2\gamma - r_n)\|Dx_n - Dz\|^2.
\end{aligned}$$

Then

$$\|u_n - z\|^2 \le \|x_n - z\|^2 - r_n(2\gamma - r_n)\|Dx_n - Dz\|^2. \tag{3.28}$$

From definition of x_n, (3.10) and (3.28), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n(u - z) + \beta_n(Gx_n - z) + \gamma_n(P_C(I - \lambda_n(I - T))u_n - z)\|^2\\
&\le \alpha_n\|u - z\|^2 + \beta_n\|Gx_n - z\|^2 + \gamma_n\|P_C(I - \lambda_n(I - T))u_n - z\|^2\\
&= \alpha_n\|u - z\|^2 + \beta_n\|Gx_n - Gz\|^2 + \gamma_n\|P_C(I - \lambda_n(I - T))u_n - z\|^2\\
&\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\|u_n - z\|^2\\
&\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left(\|x_n - z\|^2 - r_n(2\gamma - r_n)\|Dx_n - Dz\|^2\right)\\
&= \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\|x_n - z\|^2 - r_n\gamma_n(2\gamma - r_n)\|Dx_n - Dz\|^2\\
&= \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - r_n\gamma_n(2\gamma - r_n)\|Dx_n - Dz\|^2\\
&\le \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - r_n\gamma_n(2\gamma - r_n)\|Dx_n - Dz\|^2.
\end{aligned} \tag{3.29}$$

It implies that

$$\begin{aligned}
r_n\gamma_n(2\gamma - r_n)\|Dx_n - Dz\|^2 &\le \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2\\
&= \alpha_n\|u - z\|^2\\
&\quad + (\|x_n - z\| + \|x_{n+1} - z\|)(\|x_n - z\| - \|x_{n+1} - z\|)\\
&\le \alpha_n\|u - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\|.
\end{aligned} \tag{3.30}$$

From (3.20), (3.30), conditions (i) and (ii), we have

$$\lim_{n\to\infty}\|Dx_n - Dz\| = 0. \tag{3.31}$$

From definition of x_n, (3.10) and (3.21), we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2\\
&\quad= \|\alpha_n(u - z) + \beta_n(Gx_n - z) + \gamma_n(P_C(I - \lambda_n(I - T))u_n - z)\|^2\\
&\quad\le \alpha_n\|u - z\|^2 + \beta_n\|Gx_n - z\|^2 + \gamma_n\|P_C(I - \lambda_n(I - T))u_n - z\|^2\\
&\quad= \alpha_n\|u - z\|^2 + \beta_n\|Gx_n - Gz\|^2 + \gamma_n\|P_C(I - \lambda_n(I - T))u_n - z\|^2\\
&\quad\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\|u_n - z\|^2\\
&\quad\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2
\end{aligned}$$

$$\begin{aligned}
&+\gamma_n\Big(\|x_n-z\|^2-\|x_n-u_n\|^2-r_n^2\|Dx_n-Dz\|^2\\
&+2r_n\langle x_n-u_n,Dx_n-Dz\rangle\Big)\\
\leq\ &\alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\|x_n-z\|^2-\gamma_n\|x_n-u_n\|^2\\
&+2r_n\gamma_n\langle x_n-u_n,Dx_n-Dz\rangle\\
\leq\ &\alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\|x_n-z\|^2-\gamma_n\|x_n-u_n\|^2\\
&+2r_n\gamma_n\|x_n-u_n\|\|Dx_n-Dz\|\\
=\ &\alpha_n\|u-z\|^2+(1-\alpha_n)\|x_n-z\|^2-\gamma_n\|x_n-u_n\|^2\\
&+2r_n\gamma_n\|x_n-u_n\|\|Dx_n-Dz\|\\
\leq\ &\alpha_n\|u-z\|^2+\|x_n-z\|^2-\gamma_n\|x_n-u_n\|^2\\
&+2r_n\gamma_n\|x_n-u_n\|\|Dx_n-Dz\|
\end{aligned}$$

which implies that

$$\begin{aligned}
\gamma_n\|x_n-u_n\|^2 \leq\ &\alpha_n\|u-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2\\
&+2r_n\gamma_n\|x_n-u_n\|\|Dx_n-Dz\|\\
=\ &\alpha_n\|u-z\|^2+(\|x_n-z\|+\|x_{n+1}-z\|)(\|x_n-z\|-\|x_{n+1}-z\|)\\
&+2r_n\gamma_n\|x_n-u_n\|\|Dx_n-Dz\|\\
\leq\ &\alpha_n\|u-z\|^2+(\|x_n-z\|+\|x_{n+1}-z\|)\|x_{n+1}-x_n\|\\
&+2r_n\gamma_n\|x_n-u_n\|\|Dx_n-Dz\|. \qquad (3.32)
\end{aligned}$$

From (3.20), (3.31), (3.32), conditions (i) and (ii), we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \qquad (3.33)$$

Step 6. We will show that $\limsup_{n\to\infty}\langle u-z_0,x_n-z_0\rangle\leq 0$, where $z_0=P_{\mathfrak{F}}u$.
To show this equality, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-z_0,x_n-z_0\rangle=\lim_{k\to\infty}\langle u-z_0,x_{n_k}-z_0\rangle. \qquad (3.34)$$

Without loss of generality, we may assume that $x_{n_k}\rightharpoonup\omega$ as $k\to\infty$ where $\omega\in C$.
We first show that $\omega\in EP(F,D)$, where $D=aA+(1-a)B$ for all $a\in[0,1]$.
From (3.33), we have $u_{n_k}\rightharpoonup\omega$ as $k\to\infty$.
From (3.4), we obtain

$$F(u_n,y)+\langle Dx_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\ \forall y\in C.$$

Adding $F(y,u_n)$, we have

$$F(u_n,y)+F(y,u_n)+\langle Dx_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq F(y,u_n),\ \forall y\in C.$$

Using (A2), we obtain

$$\langle Dx_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq F(y,u_n),\ \forall y\in C.$$

It follows that

$$\langle Dx_{n_k}, y-u_{n_k}\rangle + \frac{1}{r_{n_k}}\langle y-u_{n_k}, u_{n_k}-x_{n_k}\rangle \geq F(y,u_{n_k}),\ \forall y\in C. \tag{3.35}$$

Put $z_t := ty+(1-t)\omega$ for all $t\in(0,1]$ and $y\in C$. Then, we have $z_t\in C$. By (3.35) substitute $y=z_t\in C$, we have

$$\langle Dx_{n_k}, z_t-u_{n_k}\rangle + \frac{1}{r_{n_k}}\langle z_t-u_{n_k}, u_{n_k}-x_{n_k}\rangle \geq F(z_t,u_{n_k}),$$

and adding $\langle z_t-u_{n_k}, Dz_t\rangle$, we obtain

$$\langle z_t-u_{n_k}, Dz_t\rangle + \langle Dx_{n_k}, z_t-u_{n_k}\rangle + \frac{1}{r_{n_k}}\langle z_t-u_{n_k}, u_{n_k}-x_{n_k}\rangle \geq \langle z_t-u_{n_k}, Dz_t\rangle + F(z_t,u_{n_k}).$$

It implies that

$$\begin{aligned}
\langle z_t-u_{n_k}, Dz_t\rangle \geq\ & \langle z_t-u_{n_k}, Dz_t\rangle - \langle z_t-u_{n_k}, Dx_{n_k}\rangle \\
& - \frac{1}{r_{n_k}}\langle z_t-u_{n_k}, u_{n_k}-x_{n_k}\rangle + F(z_t,u_{n_k}) \\
=\ & \langle z_t-u_{n_k}, Dz_t\rangle - \langle z_t-u_{n_k}, Dx_{n_k}\rangle \\
& - \left\langle z_t-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle + F(z_t,u_{n_k}) \\
=\ & \langle z_t-u_{n_k}, Dz_t-Dx_{n_k}\rangle - \left\langle z_t-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle + F(z_t,u_{n_k}) \\
=\ & \langle z_t-u_{n_k}, Dz_t-Du_{n_k}+Du_{n_k}-Dx_{n_k}\rangle \\
& - \left\langle z_t-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle + F(z_t,u_{n_k}) \\
=\ & \langle z_t-u_{n_k}, Dz_t-Du_{n_k}\rangle + \langle z_t-u_{n_k}, Du_{n_k}-Dx_{n_k}\rangle \\
& - \left\langle z_t-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle + F(z_t,u_{n_k}).
\end{aligned}$$

Then

$$\begin{aligned}
\langle z_t-u_{n_k}, Dz_t\rangle \geq\ & \langle z_t-u_{n_k}, Dz_t-Du_{n_k}\rangle + \langle z_t-u_{n_k}, Du_{n_k}-Dx_{n_k}\rangle \\
& - \left\langle z_t-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle + F(z_t,u_{n_k}).
\end{aligned} \tag{3.36}$$

Since D is γ-inverse strongly monotone, we get

$$\begin{aligned}
\gamma\|Du_{n_k}-Dx_{n_k}\|^2 \leq\ & \langle u_{n_k}-x_{n_k}, Du_{n_k}-Dx_{n_k}\rangle \\
\leq\ & \|u_{n_k}-x_{n_k}\|\,\|Du_{n_k}-Dx_{n_k}\|.
\end{aligned}$$

Therefore,

$$\gamma\|Du_{n_k}-Dx_{n_k}\|^2 \leq \|u_{n_k}-x_{n_k}\|\,\|Du_{n_k}-Dx_{n_k}\|. \tag{3.37}$$

By (3.33), we have

$$\lim_{k\to\infty}\|u_{n_k}-x_{n_k}\| = 0. \tag{3.38}$$

From (3.37) and (3.38), we obtain

$$\lim_{k\to\infty}\|Du_{n_k}-Dx_{n_k}\| = 0. \tag{3.39}$$

Further, from monotonicity of D, we have

$$\langle z_t - u_{n_k}, Dz_t - Du_{n_k} \rangle \geq 0.$$

From (3.36) and monotonicity of D, we get

$$\begin{aligned}\lim_{k\to\infty} \langle z_t - u_{n_k}, Dz_t \rangle &\geq \lim_{k\to\infty} \langle z_t - u_{n_k}, Dz_t - Du_{n_k} \rangle + \lim_{k\to\infty} \langle z_t - u_{n_k}, Du_{n_k} - Dx_{n_k} \rangle \\ &\quad - \lim_{k\to\infty} \left\langle z_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle + \lim_{k\to\infty} F(z_t, u_{n_k}) \\ &\geq \lim_{k\to\infty} F(z_t, u_{n_k}).\end{aligned}$$

Then

$$\lim_{k\to\infty} \langle z_t - u_{n_k}, Dz_t \rangle \geq \lim_{k\to\infty} F(z_t, u_{n_k}). \tag{3.40}$$

From $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$, (3.40) and (A4), we have

$$\langle z_t - \omega, Dz_t \rangle \geq F(z_t, \omega). \tag{3.41}$$

From (A1), (A4) and (3.41), we also have

$$\begin{aligned}0 &= F(z_t, z_t) \\ &= F(z_t, ty + (1-t)\omega) \\ &\leq tF(z_t, y) + (1-t) F(z_t, \omega) \\ &\leq tF(z_t, y) + (1-t) \langle z_t - \omega, Dz_t \rangle \\ &= tF(z_t, y) + (1-t) \langle ty + (1-t)\omega - \omega, Dz_t \rangle \\ &= tF(z_t, y) + (1-t) \langle ty - t\omega, Dz_t \rangle \\ &= tF(z_t, y) + (1-t) t \langle y - \omega, Dz_t \rangle \\ &= t(F(z_t, y) + (1-t) \langle y - \omega, Dz_t \rangle).\end{aligned}$$

It implies that

$$0 \leq F(z_t, y) + (1-t) \langle y - \omega, Dz_t \rangle,$$

for all $t \in (0, 1]$ and $y \in C$.

Since $z_t \rightharpoonup \omega$ as $t \to 0^+$, then we have $Dz_t \rightharpoonup D\omega$ as $z_t \rightharpoonup \omega$ and $t \to 0^+$.

Letting $t \to 0^+$ and using (A3), we have

$$\begin{aligned}0 &\leq \lim_{t\to 0^+} (F(z_t, y) + (1-t) \langle y - \omega, Dz_t \rangle) \\ &= \lim_{t\to 0^+} F(z_t, y) + \lim_{t\to 0^+} (1-t) \langle y - \omega, Dz_t \rangle \\ &= \lim_{t\to 0^+} F(ty + (1-t)\omega, y) + \lim_{t\to 0^+} (1-t) \langle y - \omega, Dz_t \rangle \\ &\leq F(\omega, y) + \lim_{t\to 0^+} (1-t) \langle y - \omega, Dz_t \rangle \\ &= F(\omega, y) + \langle y - \omega, D\omega \rangle.\end{aligned}$$

Then

$$0 \leq F(\omega, y) + \langle y - \omega, D\omega \rangle \ \ \forall y \in C. \tag{3.42}$$

Therefore

$$\omega \in EP(F, D), \tag{3.43}$$

where $D = aA + (1-a)B$ for all $a \in [0,1]$.

Next, we will show that $\omega \in F(T)$. Since

$$\begin{aligned} \|P_C(I - \lambda_n(I-T))u_n - u_n\| &= \|P_C(I - \lambda_n(I-T))u_n - x_n + x_n - u_n\| \\ &\leq \|P_C(I - \lambda_n(I-T))u_n - x_n\| + \|x_n - u_n\|, \end{aligned}$$

by using (3.26) and (3.33), we have

$$\lim_{n\to\infty} \|P_C(I - \lambda_n(I-T))u_n - u_n\| = 0. \tag{3.44}$$

From Remark 2.21, we have $F(T) = F(P_C(I - \lambda_{n_k}(I-T)))$.

Assume that $\omega \neq P_C(I - \lambda_{n_k}(I-T))\omega$.

Since $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$, Opial's property, (3.44) and condition (i), we have

$$\begin{aligned} \liminf_{k\to\infty} \|u_{n_k} - \omega\| &< \liminf_{k\to\infty} \|u_{n_k} - P_C(I - \lambda_{n_k}(I-T))\omega\| \\ &= \liminf_{k\to\infty} \Big(\|u_{n_k} - P_C(I - \lambda_{n_k}(I-T))u_{n_k} \\ &\quad + P_C(I - \lambda_{n_k}(I-T))u_{n_k} - P_C(I - \lambda_{n_k}(I-T))\omega\|\Big) \\ &\leq \liminf_{k\to\infty} \Big(\|u_{n_k} - P_C(I - \lambda_{n_k}(I-T))u_{n_k}\| \\ &\quad + \|P_C(I - \lambda_{n_k}(I-T))u_{n_k} - P_C(I - \lambda_{n_k}(I-T))\omega\|\Big) \\ &\leq \liminf_{k\to\infty} \Big(\|u_{n_k} - P_C(I - \lambda_{n_k}(I-T))u_{n_k}\| \\ &\quad + \|(I - \lambda_{n_k}(I-T))u_{n_k} - (I - \lambda_{n_k}(I-T))\omega\|\Big) \\ &= \liminf_{k\to\infty} \Big(\|u_{n_k} - P_C(I - \lambda_{n_k}(I-T))u_{n_k}\| \\ &\quad + \|u_{n_k} - \omega + \lambda_{n_k}(I-T)\omega - \lambda_{n_k}(I-T)u_{n_k}\|\Big) \\ &\leq \liminf_{k\to\infty} \Big(\|u_{n_k} - P_C(I - \lambda_{n_k}(I-T))u_{n_k}\| \\ &\quad + \|u_{n_k} - \omega\| + \lambda_{n_k}\|(I-T)u_{n_k} - (I-T)\omega\|\Big) \\ &= \liminf_{k\to\infty} \|u_{n_k} - \omega\|. \end{aligned} \tag{3.45}$$

This is a contradiction. Then

$$\omega \in F(T). \tag{3.46}$$

Next, we will show that $\omega \in F(G)$. From (3.27), we have

$$\lim_{k\to\infty} \|Gx_{n_k} - x_{n_k}\| = 0.$$

From the nonexpansiveness of $G, x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and Lemma 2.15, we have

$$\omega \in F(G). \tag{3.47}$$

From (3.43), (3.46) and (3.47), we can deduce that $\omega \in \mathfrak{F}$.
Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in \mathfrak{F}$, then, by Lemma 2.12, we can conclude that

$$\limsup_{n\to\infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{k\to\infty} \langle u - z_0, x_{n_k} - z_0 \rangle = \langle u - z_0, \omega - z_0 \rangle \leq 0. \tag{3.48}$$

Step 7. Finally, we will show that $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$.
From definition of x_n and (3.10), we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n (u - z_0) + \beta_n (Gx_n - z_0) + \gamma_n (P_C (I - \lambda_n (I - T)) u_n - z_0)\|^2 \\
&\leq \|\beta_n (Gx_n - z_0) + \gamma_n (P_C (I - \lambda_n (I - T)) u_n - z_0)\|^2 \\
&\quad + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&\leq \beta_n \|Gx_n - z_0\|^2 + \gamma_n \|P_C (I - \lambda_n (I - T)) u_n - z_0\|^2 \\
&\quad + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&= \beta_n \|Gx_n - Gz_0\|^2 + \gamma_n \|P_C (I - \lambda_n (I - T)) u_n - z_0\|^2 \\
&\quad + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&\leq \beta_n \|x_n - z_0\|^2 + \gamma_n \|P_C (I - \lambda_n (I - T)) u_n - z_0\|^2 \\
&\quad + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&\leq \beta_n \|x_n - z_0\|^2 + \gamma_n \|u_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&= \beta_n \|x_n - z_0\|^2 + \gamma_n \|T_{r_n} (I - r_n D) x_n - T_{r_n} (I - r_n D) z_0\|^2 \\
&\quad + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&\leq \beta_n \|x_n - z_0\|^2 + \gamma_n \|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&= (\beta_n + \gamma_n) \|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&= (1 - \alpha_n) \|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle .
\end{aligned}$$

From (3.48), the condition (i) and Lemma 2.19, we can conclude that $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$. By (3.33), we have $\{u_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$. This completes the proof. □

Remark 3.2. By using Theorem 3.1 and Lemma 2.26, putting $F(G) = VI(C, A') \cap VI(C, B')$, we have $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$.

Chapter 4

Applications

In this chapter, we apply the main results to solve the variational inequality, equilibrium, nonexpansive mappings of α,β-inverse strongly monotone mappings and the general split feasibility problems in Hilbert space.

4.1 Approximation theorem for fixed point problem and modification of variational inequality problem

Theorem 4.1. Let C be a closed convex subset of Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), let $A, B, A', B' : C \to H$ be $\alpha, \beta, \alpha', \beta'$-inverse strongly monotone, respectively. Define $G : C \to C$ by $Gx = P_C\left(I - \lambda_1 A'\right)\left(ax + (1-a)\, P_C\left(I - \lambda_2 B'\right)x\right)$ for all $x \in C$ with $\lambda_1 \in (0, 2\alpha')$ and $\lambda_2 \in (0, 2\beta')$. Let $T : C \to C$ be κ-strictly pseudononspreading mapping for some $\kappa \in [0,1)$ with $\mathfrak{F} = F(T) \cap F(G) \cap VI(C, A) \cap VI(C, B) \neq \varnothing$ for all $a \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and

$$\begin{cases} u_n = P_C\left(I - r_n\left(aA + (1-a)\,B\right)\right)x_n, \\ x_{n+1} = \alpha_n u + \beta_n G x_n + \gamma_n P_C\left(I - \lambda_n\left(I - T\right)\right)u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subset [0,1]$, $\lambda_n \in (0, 1-\kappa)$, $\alpha_n + \beta_n + \gamma_n = 1, \forall n \in \mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma]$, $\gamma = \min\{\alpha, \beta\}$ satisfy;

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \lambda_n < \infty;$

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1,\ 0 < e \leq\ r_n \leq f < 2\gamma;$

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0;$

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$.

Proof. From (3.1) putting $F \equiv 0$ in Theorem 3.1, we have

$$0 \leq \left\langle \left(aA + (1-a)\,B\right)x_n, y - u_n \right\rangle + \frac{1}{r_n}\left\langle y - u_n, u_n - x_n \right\rangle.$$

Multiplying r_n, we get that

$$\begin{aligned} 0 &\leq \left\langle r_n\left(aA + (1-a)\,B\right)x_n, y - u_n \right\rangle + \left\langle y - u_n, u_n - x_n \right\rangle \\ &= \left\langle r_n\left(aA + (1-a)\,B\right)x_n + u_n - x_n, y - u_n \right\rangle \\ &= \left\langle u_n - \left(x_n - r_n\left(aA + (1-a)\,B\right)x_n\right), y - u_n \right\rangle \\ &= \left\langle u_n - \left(I - r_n\left(aA + (1-a)\,B\right)\right)x_n, y - u_n \right\rangle. \end{aligned}$$

It implies that

$$\langle u_n - (I - r_n (aA + (1-a) B)) x_n, y - u_n \rangle \geq 0, \ \forall y \in C.$$

From Lemma 2.12, we have

$$u_n = P_C (I - r_n (aA + (1-a) B)) x_n. \tag{4.2}$$

Then, we have (4.1). Since $F \equiv 0$, we obtain

$$\begin{aligned} EP(F, aA + (1-a) B) &= VI(C, aA + (1-a) B) \\ &= VI(C, A) \cap VI(C, B). \end{aligned}$$

From Theorem 3.1, we can conclude the desired conclusion. □

4.2 Application for system of variational inequality problem and equilibrium problem

Theorem 4.2. Let C be a closed convex subset of Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), let $A, B, A', B' : C \to H$ be $\alpha, \beta, \alpha', \beta'$-inverse strongly monotone, respectively. Define $G : C \to C$ by $Gx = P_C (I - \lambda_1 A') (ax + (1-a) P_C (I - \lambda_2 B') x)$ for all $x \in C$ with $\lambda_1 \in (0, 2\alpha')$ and $\lambda_2 \in (0, 2\beta')$. Let $T : C \to C$ be κ-strictly pseudononspreading mapping for some $\kappa \in [0,1)$ with $\mathfrak{F} = F(T) \cap F(G) \cap EP(F, A) \neq \varnothing$ for all $a \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and

$$\begin{cases} F(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n G x_n + \gamma_n P_C (I - \lambda_n (I - T)) u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subset [0,1]$, $\lambda_n \in (0, 1-\kappa)$, $\alpha_n + \beta_n + \gamma_n = 1, \forall n \in \mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma], \gamma = \min\{\alpha, \beta\}$ satisfy;

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \ \lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \lambda_n < \infty;$

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1, \ 0 < e \leq \ r_n \leq f < 2\gamma;$

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0;$

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$.

Proof. From Theorem 3.1, putting $A \equiv B$, we have

$$\begin{aligned} 0 &\leq F(u_n, y) + \langle (aA + (1-a) A) x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \\ &= F(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle. \end{aligned}$$

It implies that

$$F(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C.$$

Then, we have (4.3). From Theorem 3.1, we obtain the desired conclusion. □

4.3 Application for nonexpansive mappings of α, β-inverse strongly monotone mappings

Theorem 4.3. Let C be a closed convex subset of Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), let $S, S' : C \to C$ be nonexpansive mappings and let $A', B' : C \to H$ be α', β'-inverse strongly monotone, respectively. Define $G : C \to C$ by $Gx = P_C (I - \lambda_1 A') (ax + (1-a) P_C (I - \lambda_2 B') x)$ for all $x \in C$ with $\lambda_1 \in (0, 2\alpha')$ and $\lambda_2 \in (0, 2\beta')$. Let $T : C \to C$ be κ-strictly pseudononspreading mapping for some $\kappa \in [0,1)$ with $\mathfrak{F} = F(T) \cap F(G) \cap F(S) \cap F(S') \neq \varnothing$ for all $a \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and

$$\begin{cases} u_n = P_C \left(I - r_n \left(a (I - S) + (1-a)(I - S')\right)\right) x_n, \\ x_{n+1} = \alpha_n u + \beta_n G x_n + \gamma_n P_C \left(I - \lambda_n (I - T)\right) u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.4}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subset [0,1]$, $\lambda_n \in (0, 1-\kappa)$, $\alpha_n + \beta_n + \gamma_n = 1, \forall n \in \mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma]$, $0 < \gamma < \frac{1}{2}$ satisfy;

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$, $0 < e \leq r_n \leq f < 2\gamma$;

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$.

Proof. If we let $S, S' : C \to C$ are nonexpansive mappings, we have $A = I - S$ and $B = I - S'$ are $\frac{1}{2}$-inverse strongly monotone. By using Theorem 4.1 and Lemma 2.25, we obtain the conclusion. ☐

4.4 Application for the general split feasibility problem

Theorem 4.4. Let C, Q be a closed convex subset of Hilbert space H_1, H_2 respectively and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4) , let $A', B' : C \to H_1$ be α', β'-inverse strongly monotone, respectively. Let $A_i, B_i : H_1 \to H_2$ be bounded linear operator with A_i^*, B_i^* are adjoint of A_i and B_i, respectively and $L = \max\{L_{A_i}, L_{B_i}\}$ where L_{A_i} and L_{B_i} are spectal radius of $A_i^* A_i$ and $B_i^* B_i$ with $i = 1, 2$. Define $G : C \to C$ by $Gx = P_C (I - \lambda_1 A') (ax + (1-a) P_C (I - \lambda_2 B') x)$ for all $x \in C$ with $\lambda_1 \in (0, 2\alpha')$ and $\lambda_2 \in (0, 2\beta')$. Let $T : C \to C$ be κ-strictly pseudononspreading mapping for some $\kappa \in [0,1)$. Assume that $\mathfrak{F} = F(T) \cap F(G) \cap \Gamma_1 \cap \Gamma_2 \neq \varnothing$, where $\Gamma_i = \{x \in C : A_i x, B_i x \in Q\}$ for all $i = 1, 2$ and $a \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and

$$\begin{cases} u_n = P_C \left(I - r_n \left(a (I - W_1) + (1-a)(I - W_2)\right)\right) x_n, \\ x_{n+1} = \alpha_n u + \beta_n G x_n + \gamma_n P_C \left(I - \lambda_n (I - T)\right) u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.5}$$

where

$$W_1 = P_C\left(I - a\left(\frac{A_1^*\left(I-P_Q\right)A_1}{2} + \frac{B_1^*\left(I-P_Q\right)B_1}{2}\right)\right),$$
$$W_2 = P_C\left(I - a\left(\frac{A_2^*\left(I-P_Q\right)A_2}{2} + \frac{B_2^*\left(I-P_Q\right)B_2}{2}\right)\right),$$

and $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subset [0,1],\ \lambda_n \in (0, 1-\kappa),\ \alpha_n+\beta_n+\gamma_n = 1, \forall n \in \mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma],\ \gamma = \min\{\alpha, \beta\}$ satisfy;

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \lambda_n < \infty;$

(ii) $0 < c \le \beta_n, \gamma_n \le d < 1,\ 0 < e \le\ r_n \le f < 2\gamma;$

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0;$

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}}u$.

Proof. By using Lemma 2.24 and Theorem 4.3, we have the desired conclusion. $\square$

Chapter 5

Examples and Numerical Results

In this chapter, Example 5.1 and Example 5.2 are given for supporting Theorem 3.1 and Theorem 4.1, respectively.

Example 5.1. Let $\mathbb{R}$ be the set of real numbers, $C = [-50, 50]$, and $H = \mathbb{R}$. Let $A, B, A', B' : C \to H$ defined by

$$Ax = x + \frac{2}{3},\ Bx = x - \frac{4}{6},\ A'x = \frac{2x+1}{2} \ \text{ and } \ B'x = \frac{3x-7}{3} \ \text{ for all } \ x \in C.$$

It is easy to see that A, B, A', B' are 1-inverse strongly monotone with $\alpha, \beta, \alpha', \beta' = 1$. Then, we can choose $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{7} \in (0, 2)$ and $a = \frac{1}{2} \in (0, 1)$. Let the mapping $G : C \to C$ be defined by

$$Gx = P_C\left(I - \frac{1}{2}A'\right)\left(\frac{1}{2}x + \frac{1}{2}P_C\left(I - \frac{3}{7}B'\right)x\right),\ \forall x \in C.$$

Let T be a mapping from C into itself defined by $Tx = x$ for all $x \in C$. It is easy to see that T is $\frac{1}{5}$-strictly pseudononspreading. Let $F : C \times\ C \to \mathbb{R}$ defined by

$$F(x, y) = -5x^2 + xy - 4y^2, \forall x, y \in C.$$

By the definition of F, we have

$$\begin{aligned}
0 &\leq F\left(u_n, y\right) + \left\langle\left(aA + (1-a)\,B\right)x_n, y - u_n\right\rangle + \frac{1}{r_n}\left\langle y - u_n, u_n - x_n\right\rangle \\
&= \left(-5u_n^2 + u_n y + 4y^2\right) + \left(x_n\right)\left(y - u_n\right) + \frac{1}{r_n}\left(y - u_n\right)\left(u_n - x_n\right) \\
&= \left(-5u_n^2 + u_n y + 4y^2\right) + \left(x_n y - x_n u_n\right) + \frac{1}{r_n}\left(u_n y - x_n y - u_n^2 + u_n x_n\right)
\end{aligned}$$

$$\Longleftrightarrow$$

$$\begin{aligned}
0 &\leq r_n(-5u_n^2 + u_n y + 4y^2) + r_n(x_n y - x_n u_n) + (u_n y - x_n y - u_n^2 + u_n x_n) \\
&= -5r_n u_n^2 + r_n u_n y + 4r_n y^2 + r_n x_n y - r_n x_n u_n + u_n y - x_n y - u_n^2 + u_n x_n \\
&= 4r_n y^2 + (r_n u_n + r_n x_n + u_n - x_n)y - 5r_n u_n^2 - r_n x_n u_n - u_n^2 + u_n x_n.
\end{aligned}$$

Let $Q(y) = 4r_n y^2 + (r_n u_n + r_n x_n + u_n - x_n)y - 5r_n u_n^2 - r_n x_n u_n - u_n^2 + u_n x_n$ which is a quadratic function of y with coefficient $a = 4r_n, b = r_n u_n + r_n x_n + u_n - x_n, c = -5r_n u_n^2 - r_n x_n u_n - u_n^2 + u_n x_n$.

Determine the discriminant Δ of Q as follow:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= \left(r_n u_n + r_n x_n + u_n - x_n\right)^2 - 4\left(4r_n\right)\left(-5r_n u_n^2 - r_n x_n u_n - u_n^2 + u_n x_n\right) \\
&= r_n^2 u_n^2 + r_n^2 x_n u_n + r_n u_n^2 - r_n x_n u_n + r_n^2 x_n u_n + r_n^2 x_n^2 + r_n x_n u_n - r_n x_n^2 \\
&\quad + r_n u_n^2 + r_n x_n u_n + u_n^2 - u_n x_n - r_n x_n u_n - r_n x_n^2 - x_n u_n + x_n^2 \\
&\quad - 16r_n\left(-5r_n u_n^2 - r_n x_n u_n - u_n^2 + u_n x_n\right)
\end{aligned}$$

$$
\begin{aligned}
&= r_n^2u_n^2 + r_n^2x_nu_n + r_nu_n^2 + r_n^2x_nu_n + r_n^2x_n^2 - r_nx_n^2 + r_nu_n^2 + u_n^2 - u_nx_n \\
&\quad - r_nx_n^2 - x_nu_n + x_n^2 + 80r_n^2u_n^2 + 16r_n^2x_nu_n + 16r_nu_n^2 - 16r_nu_nx_n \\
&= u_n^2 + 18r_nu_n^2 + 81r_n^2u_n^2 + 18r_n^2x_nu_n - 2u_nx_n - 16r_nu_nx_n + r_n^2x_n^2 \\
&\quad - 2r_nx_n^2 + x_n^2 \\
&= (u_n + 9r_nu_n)^2 + 2\left(u_n + 9r_nu_n\right)\left(r_n - 1\right)\left(x_n\right) + \left(\left(r_n - 1\right)x_n\right)^2 \\
&= \left(u_n + 9r_nu_n + \left(r_n - 1\right)x_n\right)^2 \\
&= \left(u_n + 9r_nu_n + r_nx_n - x_n\right)^2.
\end{aligned}
$$

We know that $Q(y) \geq 0, \forall y \in C$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$. So we obtain

$$u_n = \frac{1 - r_n}{1 + 9r_n}x_n. \tag{5.1}$$

Let $x_1 \in C$ and $\{x_n\}$ generated by (3.1) with $\alpha_n = \frac{1}{3n}, \beta_n = \frac{3n-2}{3n}, \gamma_n = \frac{1}{3n}, \lambda_n = \frac{1}{n(n+1)}$ and $r_n = \frac{n}{n+1}$ for all $n \in \mathbb{N}$. It is clear that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}$ and $\{r_n\}$ satisfy all the conditions of Theorem 3.1. From Theorem 3.1, we have $0 \in F(T) \cap F(G) \cap EP(F, aA + (1-a)B)$. Therefore, the sequences $\{x_n\}$ and $\{u_n\}$ converge to 0. They are rewritten as follows:

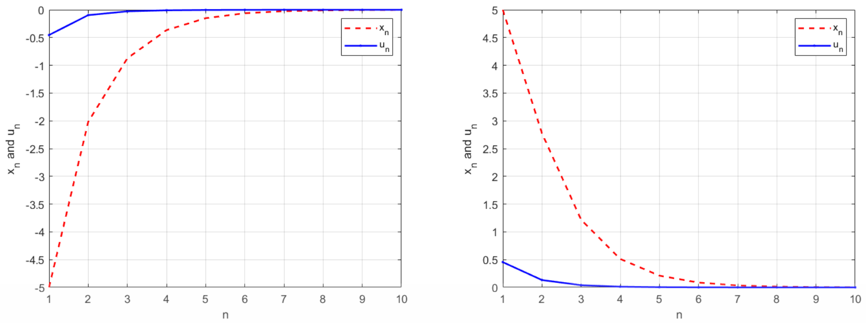
$$
\begin{cases}
u_n = \frac{1-r_n}{1+9r_n}x_n, \\
x_{n+1} = \frac{1}{3n}u + \frac{3n-2}{3n}Gx_n + \frac{1}{3n}P_{[-50,50]}\left(I - \frac{1}{n(n+1)}(I - T)\right)u_n, \ \forall n \in \mathbb{N}.
\end{cases}
$$

The following table shows the values of sequences $\{x_n\}$ and $\{u_n\}$ with initial values $u = x_1 = -5$, $u = x_1 = 5$ and $n = 10$.

Table 5.1: The values of $\{u_n\}$ and $\{x_n\}$ with $n = 10$.

	$u = x_1 = -5$		$u = x_1 = 5$	
n	u_n	x_n	u_n	x_n
1	-0.454545	-5.000000	0.454545	5.000000
2	-0.096114	-2.018398	0.132189	2.775974
3	-0.028161	-0.872988	0.039299	1.218256
4	-0.008936	-0.366386	0.012489	0.512053
5	-0.002964	-0.151174	0.004144	0.211323
6	-0.001012	-0.061739	0.001415	0.086307
7	-0.000353	-0.025045	0.000493	0.035011
8	-0.000125	-0.010111	0.000175	0.014135
9	-0.000045	-0.004068	0.000062	0.005686
10	-0.000016	-0.001632	0.000023	0.002281

Figure 5.1 is the values of the sequences $\{x_n\}$ and $\{u_n\}$ derived from Table 5.1.



(a) $u = x_1 = -5$ (b) $u = x_1 = 5$

Figure 5.1: The convergence of the sequences $\{x_n\}$ and $\{u_n\}$ with different initial value $u = x_1$ and $n = 10$.

Example 5.2. Let $H = \mathbb{R}$ be the set of real numbers and $C = [-60, 60]$. Let $A, B, A', B' : [-60, 60] \to \mathbb{R}$ defined by

$$Ax = x + \frac{9}{2},\ Bx = \frac{5x+6}{5},\ A'x = x - \frac{17}{3} \text{ and } B'x = \frac{8x-15}{8} \text{ for all } x \in C.$$

It is easy to see that A, B, A', B' are 1-inverse strongly monotone with $\alpha, \beta, \alpha', \beta' = 1$. Then, we can choose $\lambda_1 = \frac{3}{8}, \lambda_2 = \frac{16}{17} \in (0,2)$ and $a = \frac{1}{2} \in (0,1)$. Let the mapping $G : C \to C$ be defined by

$$Gx = P_C\left(I - \frac{3}{8}A'\right)\left(\frac{1}{2}x + \frac{1}{2}P_C\left(I - \frac{16}{17}B'\right)x\right),\ \forall x \in C.$$

Let T be a mapping from C into itself defined by

$$Tx = \begin{cases} -3x & \text{if } x \in [-60, 4), \\ x & \text{if } x \in [4, 60], \end{cases}$$

for all $x \in C$. It is easy to see that T is $\frac{1}{2}$-strictly pseudononspreading. Let $x_1 \in C$ and $\{x_n\}$ generated by (4.1) with $\alpha_n = \frac{1}{4n}, \beta_n = \frac{4n-2}{4n}, \gamma_n = \frac{1}{4n}, \lambda_n = \frac{1}{n(n+1)}$ and $r_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$. It is clear that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}$ and $\{r_n\}$ satisfy all the conditions of Theorem 4.1. From Theorem 4.1, we have $4 \in F(T) \cap F(G) \cap VI(C, A) \cap VI(C, B)$. Therefore, the sequences $\{x_n\}$ converge to 4. They are rewritten as follows:

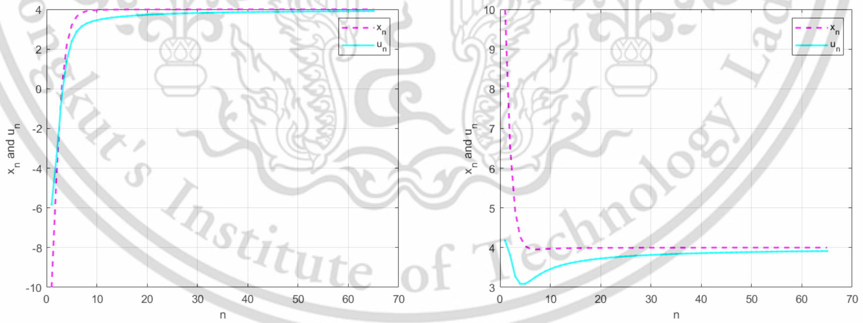
$$\begin{cases} u_n = P_{[-60,60]}\left(I - \frac{1}{n+1}\left(\frac{1}{2}A + \frac{1}{2}B\right)\right)x_n, \\ x_{n+1} = \frac{1}{4n}u + \frac{4n-2}{4n}Gx_n + \frac{1}{4n}P_{[-60,60]}\left(I - \frac{1}{n(n+1)}(I - T)\right)u_n,\ \forall n \in \mathbb{N}. \end{cases}$$

The following table shows the values of sequences $\{x_n\}$ and $\{u_n\}$ with initial values $u = x_1 = -10$, $u = x_1 = 10$ and $n = 65$.

Table 5.2: The values of $\{u_n\}$ and $\{x_n\}$ with $n = 65$.

	$u = x_1 = -10$		$u = x_1 = 10$	
n	u_n	x_n	u_n	x_n
1	-5.825000	-10.000000	4.175000	10.000000
2	-3.398284	-4.272426	3.807598	6.536397
3	-0.421304	-0.011739	3.279327	4.922436
4	1.422855	2.191068	3.086929	4.271162
5	2.393454	3.202145	3.090323	4.038388
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
30	3.815258	3.997434	3.815258	3.997434
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
61	3.908289	3.999408	3.908289	3.999408
62	3.909754	3.999427	3.909754	3.999427
63	3.911173	3.999446	3.911173	3.999446
64	3.912549	3.999463	3.912549	3.999463
65	3.912549	3.999480	3.912549	3.999480

Figure 5.2 is the values of the sequences $\{x_n\}$ and $\{u_n\}$ derived from Table 5.2.



(a) $u = x_1 = -10$ (b) $u = x_1 = 10$

Figure 5.2: The convergence of the sequences $\{x_n\}$ and $\{u_n\}$ with different initial value $u = x_1$ and $n = 65$.

Chapter 6

Conclusions and Suggestions

In the last chapter, we summarize all the theorems and applications obtained from the thesis.

6.1 Conclusions

(1) Let C be a closed convex subset of Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), let $A, B, A', B' : C \to H$ be $\alpha, \beta, \alpha', \beta'$-inverse strongly monotone, respectively. Define $G : C \to C$ by $Gx = P_C(I - \lambda_1 A')(ax + (1-a)P_C(I - \lambda_2 B')x)$ for all $x \in C$ with $\lambda_1 \in (0, 2\alpha')$ and $\lambda_2 \in (0, 2\beta')$. Let $T : C \to C$ be κ-strictly pseudononspreading mapping for some $\kappa \in [0,1)$ with $\mathfrak{F} = F(T) \cap F(G) \cap EP(F, aA + (1-a)B) \neq \varnothing$ for all $a \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and

$$\begin{cases} F(u_n, y) + \langle (aA + (1-a)B)x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n G x_n + \gamma_n P_C(I - \lambda_n (I - T))u_n, \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0,1], \lambda_n \in (0, 1-\kappa), \alpha_n + \beta_n + \gamma_n = 1, \forall n \in \mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma], \gamma = \min\{\alpha, \beta\}$ satisfy;

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty,\ \lim_{n \to \infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \lambda_n < \infty;$

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1,\ 0 < e \leq r_n \leq f < 2\gamma;$

(iii) $\lim_{n \to \infty} |r_{n+1} - r_n| = 0;$

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$.

(2) Let C be a closed convex subset of Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), let $A, B, A', B' : C \to H$ be $\alpha, \beta, \alpha', \beta'$-inverse strongly monotone, respectively. Define $G : C \to C$ by $Gx = P_C (I - \lambda_1 A') \big(ax + (1-a)\ P_C (I - \lambda_2 B') x\big)$ for all $x \in C$ with $\lambda_1 \in (0, 2\alpha')$ and $\lambda_2 \in (0, 2\beta')$. Let $T : C \to C$ be κ-strictly pseudononspreading mapping for some $\kappa \in [0,1)$ with $\mathfrak{F} = F(T) \cap F(G) \cap VI(C, A) \cap VI(C, B) \neq \varnothing$ for all $a \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and

$$\begin{cases} u_n = P_C (I - r_n (aA + (1-a) B)) x_n, \\ x_{n+1} = \alpha_n u + \beta_n G x_n + \gamma_n P_C (I - \lambda_n (I - T)) u_n,\ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subset [0,1],\ \lambda_n \in (0, 1-\kappa),\ \alpha_n + \beta_n + \gamma_n = 1, \forall n \in \mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma],\ \gamma = \min\{\alpha, \beta\}$ satisfy;

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \lambda_n < \infty;$

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1,\ 0 < e \leq\ r_n \leq f < 2\gamma;$

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0;$

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}}u$.

(3) Let C be a closed convex subset of Hilbert space H and let $F : C \times\ C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), let $A, B, A', B' :\ C \to H$ be $\alpha, \beta, \alpha', \beta'$-inverse strongly monotone, respectively. Define $G : C \to C$ by $Gx = P_C\,(I - \lambda_1 A')\big(ax + (1-a)\,P_C\,(I - \lambda_2 B')\,x\big)$ for all $x \in C$ with $\lambda_1 \in (0, 2\alpha')$ and $\lambda_2 \in (0, 2\beta')$. Let T : $C \to C$ be κ-strictly pseudononspreading mapping for some $\kappa \in [0, 1)$ with $\mathfrak{F} = F(T) \cap F(G) \cap EP(F, A) \neq \varnothing$ for all $a \in (0, 1)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and

$$\begin{cases} F(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0,\ \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n Gx_n + \gamma_n P_C\,(I - \lambda_n\,(I - T))\,u_n,\ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subset [0, 1], \lambda_n \in (0, 1-\kappa), \alpha_n + \beta_n + \gamma_n = 1, \forall n \in \mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma], \gamma = \min\{\alpha, \beta\}$ satisfy;

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \lambda_n < \infty;$

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1,\ 0 < e \leq\ r_n \leq f < 2\gamma;$

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0;$

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}}u$.

(4) Let C be a closed convex subset of Hilbert space H and let $F : C \times\ C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4), let $S, S' : C \to C$ be nonexpansive mappings and let $A', B' : C \to H$ be α', β'-inverse strongly monotone, respectively. Define $G : C \to C$ by $Gx = P_C\,(I - \lambda_1 A')\,(ax + (1-a)\,P_C\,(I - \lambda_2 B')\,x)$ for all $x \in C$ with $\lambda_1 \in (0, 2\alpha')$ and $\lambda_2 \in (0, 2\beta')$. Let $T : C \to C$ be κ-strictly pseudononspreading mapping for some $\kappa \in [0, 1)$ with $\mathfrak{F} = F(T) \cap F(G) \cap F(S) \cap F(S') \neq \varnothing$ for all $a \in (0, 1)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and

$$\begin{cases} u_n = P_C\,(I - r_n\,(a\,(I - S) + (1-a)\,(I - S')))\,x_n, \\ x_{n+1} = \alpha_n u + \beta_n Gx_n + \gamma_n P_C\,(I - \lambda_n\,(I - T))\,u_n,\ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subset [0, 1],\ \lambda_n \in (0, 1-\kappa),\ \alpha_n + \beta_n + \gamma_n = 1, \forall n \in \mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma], 0 < \gamma < \frac{1}{2}$ satisfy;

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \lambda_n < \infty;$

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1,\ 0 < e \leq\ r_n \leq f < 2\gamma;$

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0;$

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}}u.$

(5) Let C, Q be a closed convex subset of Hilbert space H_1, H_2 respectively and let $F : C \times\ C \to \mathbb{R}$ be a bifunction satisfying (A1) - (A4) , let $A', B' : C \to H_1$ be α', β'-inverse strongly monotone, respectively. Let $A_i, B_i : H_1 \to H_2$ be bounded linear operator with A_i^*, B_i^* are adjoint of A_i and B_i, respectively and $L = \max\{L_{A_i}, L_{B_i}\}$ where L_{A_i} and L_{B_i} are spectal radius of $A_i^*A_i$ and $B_i^*B_i$ with $i = 1, 2$. Define $G : C \to C$ by $Gx = P_C\left(I - \lambda_1 A'\right)\left(ax + (1-a)\, P_C\left(I - \lambda_2 B'\right)x\right)$ for all $x \in C$ with $\lambda_1 \in (0, 2\alpha')$ and $\lambda_2 \in (0, 2\beta')$. Let $T : C \to C$ be κ-strictly pseudononspreading mapping for some $\kappa \in [0, 1)$. Assume that $\mathfrak{F} = F(T) \cap F(G) \cap \Gamma_1 \cap \Gamma_2 \neq \varnothing$, where $\Gamma_i = \{x \in C : A_i x, B_i x \in Q\}$ for all $i = 1, 2$ and $a \in (0, 1)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and

$$\begin{cases} u_n = P_C\left(I - r_n\left(a\left(I - W_1\right) + (1-a)\left(I - W_2\right)\right)\right)x_n, \\ x_{n+1} = \alpha_n u + \beta_n G x_n + \gamma_n P_C\left(I - \lambda_n\left(I - T\right)\right)u_n,\ \forall n \in \mathbb{N}, \end{cases}$$

where

$$W_1 = P_C\left(I - a\left(\frac{A_1^*\left(I - P_Q\right)A_1}{2} + \frac{B_1^*\left(I - P_Q\right)B_1}{2}\right)\right),$$
$$W_2 = P_C\left(I - a\left(\frac{A_2^*\left(I - P_Q\right)A_2}{2} + \frac{B_2^*\left(I - P_Q\right)B_2}{2}\right)\right),$$

and $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subset [0, 1]$, $\lambda_n \in (0, 1-\kappa)$, $\alpha_n + \beta_n + \gamma_n = 1, \forall n \in \mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma], \gamma = \min\{\alpha, \beta\}$ satisfy;

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \lambda_n < \infty;$

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1,\ 0 < e \leq\ r_n \leq f < 2\gamma;$

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0;$

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}}u.$

6.2 Suggestions

In this thesis, we obtain the results of the variational inequality problems, fixed point problems and equilibrium problems in Hilbert space. To extend these results, it should also be studied in other spaces, such as Banach space, which cover the original space to build upon the knowledge.

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Appendix

Appendix A
The research paper

ทฤษฎีบทการประมาณค่าสำหรับปัญหาจุดตรึงและการดัดแปลงปัญหาอสมการการแปรผัน

พิมพ์ปวีณ์ ขุมทรัพย์[1] และ อาทิตย์ แข็งธัญการ[1,*]

บทคัดย่อ

บทความวิจัยฉบับนี้มีจุดประสงค์เพื่อแนะนำทฤษฎีบทการประมาณค่าสำหรับแก้ปัญหาการดัดแปลงอสมการการแปรผัน ปัญหาจุดตรึงของ $\kappa-$ไม่กระจายเทียมโดยแท้และปัญหาอสมการการแปรผัน จากนั้นได้ทำการพิสูจน์ทฤษฎีการลู่เข้าแบบเข้มของลำดับที่เกิดจากการประมาณค่าที่สร้างขึ้นเพื่อหาผลเฉลยของปัญหาจุดตรึงและการส่งแบบ $\kappa-$ไม่กระจายเทียมโดยแท้และการดัดแปลงปัญหาอสมการการแปรผันที่เกี่ยวข้องกับตัวดำเนินการผกผันทางเดียวแบบเข้ม ท้ายที่สุดมีการยกตัวอย่างเชิงตัวเลขเพื่อสนับสนุนบทความวิจัยนี้

คำสำคัญ: การส่งแบบ $\kappa-$ไม่กระจายเทียมโดยแท้, ปัญหาการรวมของอสมการการแปรผัน, ปัญหาจุดตรึง, ระบบปัญหาอสมการการแปรผัน

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Approximation theorem for fixed point problem and modification of variational inequality problem

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ABSTRACT

The purpose of this research article is to introduce the approximation theorem for solving the modification of variational inequality problem, the fixed point problem of $\kappa-$strictly pseudononspreading mapping and the variational inequality problem. Then, we prove our result give a method to solve the solution of fixed point problem of $\kappa-$ strictly pseudononspreading mapping and modification of variational inequality problem associated with inverse strongly monotone operator. In support our result, a numerical example is also presented.

Keywords: Combination of variational inequality problem, Fixed point problem, $\kappa-$strictly pseudononspreading mapping, System of variational inequality problem

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Introduction

Let H be a real Hilbert space. Let C be a nonempty closed convex subset of H and let $T:C\to C$. We use $F(T)$ to denote **the set of fixed point** of T i.e. $F(T)=\{x\in C: Tx=x\}$.

Recall that the mapping T is said to be **nonexpansive** if $\|Tx-Ty\| \leq \|x-y\|$ for all $x,y\in C$.

A mapping T is called $\kappa-$ **strictly pseudononspreading** if there exists $\kappa\in[0,1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2+\kappa\|(I-T)x-(I-T)y\|^2+2\langle x-Tx, y-Ty\rangle \text{ for all } x,y\in C.$$

A mapping A of C into H is called $\alpha-$ **inverse strongly monotone** (ism), if there exists a positive real number α such that $\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2$ for all $x,y\in C$.

Let $A:C\to H$. The **variational inequality problem** (VIP) is to find a point $u\in C$ such that

$$\langle Au, v-u\rangle \geq 0 \tag{1}$$

for all $v\in C$. The set of solutions of (1) is denoted by $VI(C,A)$. The variational inequality problem as a strong and important tool, has already been studied for a wide range of applications in industry, economics and optimization, physic [1]. In 2008, Ceng et al. [2] modify VIP for finding $(x^*,z^*)\in C\times C$ such that

$$\langle \lambda_1 D_1 z^* + x^* - z^*, x-x^*\rangle \geq 0 \text{ and } \langle \lambda_2 D_2 x^* + z^* - x^*, x-z^*\rangle \geq 0,\ \forall x\in C. \tag{2}$$

which is called a **system of variational inequalities problem** $(SVIP)$ where $D_1, D_2: C\to H$ are mappings and parameters $\lambda_1,\lambda_2>0$. In particular, $SVIP$ is VIP if and only if λ_1,λ_2, $D_1=D_2$, $x^*=z^*$.

After that, Kangtunyakarn [3] modified $SVIP$ for finding $(x^*,z^*)\in C\times C$ such that

$$\langle x^*-(I-\lambda_1 D_1)(ax^*+(1-a)z^*), x-x^*\rangle \geq 0 \text{ and } \langle z^*-(I-\lambda_2 D_2)x^*, x-z^*\rangle \geq 0, \forall x\in C. \tag{3}$$

which is called a **modification of system of variational inequalities problem** $(MSVIP)$, for every $\lambda_1,\lambda_2>0$ and $a\in[0,1]$. In the case $a=0$, $MSVIP$ reduces to $SVIP$. He was introduced the relation between solutions of $MSVIP$ and fixed point of the mapping G as follows:

Lemma 1. [3] Let $D_1, D_2: C\to H$ be mappings. For every $\lambda_1,\lambda_2>0$ and $a\in[0,1]$, the following statements are equivalent: 1) $(x^*,z^*)\in C\times C$ is a solution of problem (3),

2) x^* is a fixed point of the mapping $G:C\to C$ defined by

$$G(x)=P_C(I-\lambda_1 D_1)(ax+(1-a)P_C(I-\lambda_2 D_2)x), \text{ where } z^*=P_C(I-\lambda_2 D_2)x^*.$$

In 2012, Kangtunyakarn [4] has modified (1) and introduced the **combination of variational inequality problem** ($CVIP$) by let $A, B : C \to H$ such that

$$\left\langle y - x, \left(aA + (1-a)B \right) x \right\rangle \geq 0 \qquad (4)$$

for all $x, y \in C,$ and $a \in (0,1)$. The set of $CVIP$ is denoted by $VI\left(C, aA + (1-a)B\right)$. If $A = B$, $CVIP$ can be reduced to VIP. It is clear that (3) and (4) can be reduced to (1), but (4) cannot be reduced to (3). In addition, the strong convergence theorem of the variational inequality problem and fixed point problem for $\kappa-$strictly pseudononspreading mapping was proved by modifying Halpern iterative method and $S-$mappings [3] which generated by (5).

Theorem 2. [3] For every $i = 1, 2, ..., N$ let $B_i : C \to H$ be $\delta_i - ism$ and let $T : C \to C$ be $\kappa-$strictly pseudo-nonspreading mapping for some $\kappa \in [0,1)$. Let $G_i : C \to C$ be defined by $G_i x = P_C(I - \eta B_i)x$ for all $x \in C$ and $\eta \in (0, 2\delta_i)$ for every $i = 1, 2, ..., N$, suppose G_i and let $\delta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, j = 1, 2, ..., N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j \in (0,1)$ for all $j = 1, 2, ..., N-1, \alpha_1^N \in (0,1], \alpha_2^j, \alpha_3^j \in (0,1]$ for all $j = 1, 2, ..., N$. Let $S : C \to C$ be the $S-$mappings generated by $G_1, G_2, ..., G_N$ and $\delta_1, \delta_2, ..., \delta_N$. Assume that $\Im = F(T) \cap \bigcap_{i=1}^{N} VI(C, B_i) \neq \emptyset$. For every $n \in \mathbb{N}, i = 1, 2, ..., N,$ let $x_1, u \in C$ and $\left\{x_n\right\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C \left(I - \lambda_n (I - T) \right) x_n + \gamma_n S x_n \qquad (5)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1, \beta_n \in [c,d] \subset (0,1)$, $\{\lambda_n\} \subset (0, 1-\kappa)$ and suppose the following conditions hold: (i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, ($ii$) $\sum_{n=1}^{\infty} \lambda_n < \infty$,

$$(iii)\ \sum_{n=1}^{\infty} \left|\lambda_{n+1} - \lambda_n\right| < \infty,\ \sum_{n=1}^{\infty} \left|\gamma_{n+1} - \gamma_n\right| < \infty,\ \sum_{n=1}^{\infty} \left|\alpha_{n+1} - \alpha_n\right| < \infty,\ \sum_{n=1}^{\infty} \left|\beta_{n+1} - \beta_n\right| < \infty.$$

Then $\left\{x_n\right\}$ converges strongly to $z_0 = P_{\Im} u$.

By using Lemma 1, Theorem 2 and the concept (4), we give a method for solving the solution of fixed point problem of $\kappa-$strictly pseudononspreading mapping and modification of variational inequality problem and system of variational inequality problem.

Preliminaries

Lemma 3. [5] Let $A : C \to H$. Let $u \in C$. Then for $\lambda > 0$, $u = P_C(I - \lambda A)u \iff u \in VI\left(C, A\right)$, where P_C is the metric projection of H onto C.

Lemma 4. [3] Let $T : C \to C$ be $\kappa-$ strictly pseudononspreading mapping with $F(T) \neq \emptyset$. Then $F\left(T\right) = VI\left(C, (I - T)\right)$.

Lemma 5. [3] Let $\left\{s_n\right\}$ be a sequence of nonnegative real number satisfying $s_{n+1} = (1-\alpha_n)s_n + \delta_n,\ \forall n \geq 0,$ where $\left\{\alpha_n\right\}$ is a sequence in $(0,1)$ and $\left\{\delta_n\right\}$ is a sequence such that

$$1)\ \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad 2)\ \limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0 \ \text{or}\ \sum_{n=1}^{\infty} \left|\delta_n\right| < \infty.$$

Then $\lim\limits_{n\to\infty} s_n = 0.$

Remark 6. It is well-known that metric projection P_C has the following properties:

1) P_C is firmly nonexpansive, i.e., $\left\|P_C x - P_C y\right\|^2 \leq \left\langle P_C x - P_C y, x - y\right\rangle,\ \ \forall x, y \in H.$

2) For each $\left\|P_C x - P_C y\right\|^2 \leq \left\langle P_C x - P_C y, x - y\right\rangle,\ \ \forall x, y \in H.$

Lemma 7. [4] Let C be a closed convex subset of Hilbert space H and let $A, B : C \to H$ be and $\alpha, \beta-$ inverse strongly monotone, respectively, with $\alpha, \beta > 0$ and $VI(C,A) \cap VI(C,B) \neq \emptyset$. Then

$$VI\big(C, aA + (1-a)B\big) = VI(C,A) \cap VI(C,B),\ \ \forall a \in (0,1).$$

Furthermore if $0 < \gamma < \min\left\{2\alpha, 2\beta\right\}$, we have $I - \gamma\big(aA + (1-a)B\big)$ is a nonexpansive mapping.

Lemma 8. [6] Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $S : C \to C$ be a nonexpansive mapping. Then, $I - S$ is demiclosed at zero.

Remark 9. From Lemmas 3 and 4, we have $F(T) = F\Big(P_C(I - \lambda(I-T))\Big),\ \forall \lambda > 0.$

Lemma 10. [3] Let $\left\{s_n\right\}$ be a sequence of nonnegative real number satisfying

$$s_{n+1} = (1-\alpha_n)s_n + \alpha_n\beta_n, \forall n \geq 0,$$

where $\left\{\alpha_n\right\}, \left\{\beta_n\right\}$ satisfy the conditions 1) $\left\{\alpha_n\right\} \subset [0,1],\ \sum\limits_{n=1}^{\infty} \alpha_n = \infty$ 2) $\limsup\limits_{n\to\infty} \beta_n \leq 0$ or $\sum\limits_{n=1}^{\infty} \left|\alpha_n\beta_n\right| < \infty.$

Then $\lim\limits_{n\to\infty} s_n = 0.$

Methodology

1. To study and give basic knowledge about the fixed point problem, variational inequality problem, the strong convergence theorems of nonlinear mappings and other, which we can use in our research article.

2. To give a method for solving the solution of fixed point problem of $\kappa-$ strictly pseudononspreading mapping and modification of variational inequality problem and system of variational inequality problem.

3. Our theorem are proved in framework of Hilbert space for solving fixed point problem of $\kappa-$ strictly pseudononspreading mapping and modification of variational inequality problem and system of variational inequality problem.

4. To give numerical example for supporting our theorem in $\mathbb{R}$.

Results and Discussion

Theorem 11. Let C be a closed convex subset of Hilbert space H and let $A, B, \overline{A}, \overline{B} : C \rightarrow H$ be $\alpha, \beta, \overline{\alpha}, \overline{\beta}-$ inverse strongly monotone, respectively with $\alpha, \beta, \overline{\alpha}, \overline{\beta} \in (0, \infty)$. Define $G : C \rightarrow C$ by

$Gx = P_C\left(I - \lambda_1 \overline{A}\right)\left(ax + (1-a)P_C(I - \lambda_2 \overline{B})x\right)$ for all $x \in C$ with $\lambda_1 \in (0, 2\overline{\alpha})$ and $\lambda_2 \in (0, 2\overline{\beta})$. Let $T : C \rightarrow C$ be $\kappa-$ strictly pseudononspreading mapping with $\Im = F(T) \cap F(G) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$ for all $a \in (0, 1)$. Let $\left\{x_n\right\}$ be the sequence generated by $x_1, u \in C$ and

$$x_{n+1} = \alpha_n u + \beta_n G x_n + \gamma_n P_C\left(I - \lambda_n (I - T)\right) P_C\left(I - r_n (aA + (1-a)B)\right) x_n, \quad \text{for all } n \geq 1 \tag{6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1], \lambda_n \in (0, 1-\kappa), \alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $r_n \in [0, 2\gamma]$, $\gamma = \min\{\alpha, \beta\}$ satisfy; (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \lambda_n < \infty$; (ii) $0 < c \leq \beta_n \leq d < 1$, $0 < e \leq r_n \leq f < 2\gamma$;

(iii) $\lim_{n\to\infty} \left|r_{n+1} - r_n\right| = 0$; (iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\left\{x_n\right\}$ converges strongly to $z_0 = P_{\Im} u$.

Proof We divide the proof into seven steps:

Step 1. We will show that $D = aA + (1-a)B$ is $\gamma-$ inverse strongly monotone. Let $x, y \in C$, we have

$$\begin{aligned} \left\langle Dx - Dy, x - y \right\rangle &= \left\langle \left(aA + (1-a)B\right)x - \left(aA + (1-a)B\right)y, x - y \right\rangle \\ &\geq a\alpha \left\|Ax - Ay\right\|^2 + (1-a)\beta \left\|Bx - By\right\|^2 \qquad (7) \\ &\geq \gamma \left\|a\left(Ax - Ay\right) + (1-a)\left(Bx - By\right)\right\|^2 = \gamma \left\|Dx - Dy\right\|^2. \end{aligned}$$

By using the same method as [5], we have $I - r_n D$ is a nonexpansive mapping for every $n \in \mathbb{N}$.

Step 2. For every $n \in \mathbb{N}$. we will show that $\left\{x_n\right\}$ is bounded. Let $u_n = P_C\left(I - r_n D\right)x_n$ and $z \in \mathfrak{I}$. From Lemma 3 and Lemma 4, we have

$$z = P_C\left(I - \lambda_n(I-T)\right) z. \tag{8}$$

From the nonexpansiveness of P_C and (8), we have

$$\left\|P_C\left(I - \lambda_n(I-T)\right)u_n - z\right\| \leq \left\|\left(I - \lambda_n(I-T)\right)u_n - \left(I - \lambda_n(I-T)\right)z\right\|. \tag{9}$$

Since T is $\kappa-$strictly pseudononspreading mapping and let $E = I - T$, we have

$$\left\|Tu_n - Tz\right\|^2 = \left\|(I-E)u_n - (I-E)z\right\|^2 - \left\|u_n - z\right\|^2 - 2\left\langle u_n - z, Eu_n\right\rangle + \left\|Eu_n\right\|^2 \leq \left\|u_n - z\right\|^2 + \kappa\left\|Eu_n\right\|^2,$$

which implies that

$$(1-\kappa)\left\|Eu_n\right\|^2 \leq 2\left\langle u_n - z, Eu_n\right\rangle. \tag{10}$$

From (10), we have

$$\left\|\left(I - \lambda_n E\right)u_n - \left(I - \lambda_n E\right)z\right\|^2 \leq \left\|u_n - z\right\|^2. \tag{11}$$

From (9) and (11), we can imply that

$$\left\|P_C\left(I - \lambda_n(I-T)\right)u_n - z\right\| \leq \left\|u_n - z\right\|. \tag{12}$$

From definition of x_n, (12), and nonexpansiveness of G [3] ,we have

$$\begin{aligned}\left\|x_{n+1} - z\right\| &\leq \alpha_n\left\|u - z\right\| + \beta_n\left\|x_n - z\right\| + \gamma_n\left\|P_C\left(I - \lambda_n(I-T)\right)u_n - z\right\| = \alpha_n\left\|u - z\right\| + (1-\alpha_n)\left\|x_n - z\right\| \\ &\leq \max\left\{\left\|x_1 - z\right\|, \left\|u - z\right\|\right\}.\end{aligned}$$

By induction, we can prove that $\left\{x_n\right\}$ is bounded.

Step 3. We will show that $\lim_{n\to\infty}\left\|x_{n+1} - x_n\right\| = 0$. From definition of x_n and nonexpansiveness of P_C we have

$$\begin{aligned}\left\|x_{n+1} - x_n\right\| \leq\ & \left|\alpha_n - \alpha_{n-1}\right|\left\|u\right\| + (1-\alpha_n)\left\|x_n - x_{n-1}\right\| + \left|\beta_n - \beta_{n-1}\right|\left\|Gx_{n-1}\right\| + \left|r_{n-1} - r_n\right|\left\|Dx_{n-1}\right\| \\ & + \lambda_{n-1}\left\|(I-T)P_C\left(I - r_{n-1}D\right)x_{n-1} - (I-T)P_C\left(I - r_{n-1}D\right)x_n\right\| \\ & + \left|\lambda_{n-1} - \lambda_n\right|\left\|(I-T)P_C\left(I - r_{n-1}D\right)x_n - (I-T)P_C\left(I - r_{n-1}D\right)x_n\right\| \\ & + \left|\gamma_n - \gamma_{n-1}\right|\left\|P_C\left(I - \lambda_{n-1}(I-T)\right)P_C\left(I - r_{n-1}D\right)x_{n-1}\right\| \\ \leq\ & \left|\alpha_n - \alpha_{n-1}\right|M + (1-\alpha_n)\left\|x_n - x_{n-1}\right\| + \left|\beta_n - \beta_{n-1}\right|M + \left|r_{n-1} - r_n\right|M + \lambda_{n-1}M + \left|\lambda_{n-1} - \lambda_n\right|M + \left|\gamma_n - \gamma_{n-1}\right|M,\end{aligned}$$

where $M = \max\Big\{\|u\|, \|Gx_{n-1}\|, \|Dx_{n-1}\|, \Big\|(I-T)P_C\big(I-r_{n-1}D\big)x_{n-1} - (I-T)P_C\big(I-r_{n-1}D\big)x_n\Big\|,$

$$\Big\|(I-T)P_C\big(I-r_{n-1}D\big)x_n - (I-T)P_C\big(I-r_{n-1}D\big)x_n\Big\|, \Big\|P_C\big(I-\lambda_{n-1}(I-T)\big)P_C\big(I-r_{n-1}D\big)x_{n-1}\Big\|\Big\}.$$

From Lemma 5 and conditions $(i),(ii),(iv)$, we have $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$ (13)

Step 4. We will show that $\lim_{n\to\infty}\|x_n - Gx_n\| = 0$. From definition of u_n and P_C is nonexpansiveness, we have

$$\|u_n - z\|^2 \le \|x_n - z\|^2 + r_n^2\|Dx_n - Dz\|^2 - 2r_n\gamma\|Dx_n - Dz\|^2 \le \|x_n - z\|^2. \tag{14}$$

From definition of x_n ,(12) and (14), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \alpha_n\|u-z\|^2 + \beta_n\|Gx_n - z\|^2 + \gamma_n\Big\|P_C\big(I-\lambda_n(I-T)\big)u_n - z\Big\|^2 - \beta_n\gamma_n\Big\|P_C\big(I-\lambda_n(I-T)\big)u_n - Gx_n\Big\|^2\\ &\le \alpha_n\|u-z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \beta_n\gamma_n\Big\|P_C\big(I-\lambda_n(I-T)\big)u_n - Gx_n\Big\|^2\end{aligned}$$

which implies that

$$\beta_n\gamma_n\Big\|P_C\big(I-\lambda_n(I-T)\big)u_n - Gx_n\Big\|^2 \le \alpha_n\|u-z\|^2 + \big(\|x_n - z\| + \|x_{n+1}-z\|\big)\|x_{n+1}-x_n\|. \tag{15}$$

From (13), (15) and conditions $(i),(ii)$, we have

$$\lim_{n\to\infty}\Big\|P_C\big(I-\lambda_n(I-T)\big)u_n - Gx_n\Big\| = 0. \tag{16}$$

From definition of x_n and triangle inequality, we have

$$\Big\|x_{n+1} - P_C\big(I-\lambda_n(I-T)\big)u_n\Big\| \le \alpha_n\Big\|u - P_C\big(I-\lambda_n(I-T)\big)u_n\Big\| + \beta_n\Big\|Gx_n - P_C\big(I-\lambda_n(I-T)\big)u_n\Big\|,$$

(16) and condition (i), we have

$$\lim_{n\to\infty}\Big\|x_{n+1} - P_C\big(I-\lambda_n(I-T)\big)u_n\Big\| = \lim_{n\to\infty}\Big\|x_{n+1} - P_C\big(I-\lambda_n(I-T)\big)P_C\big(I-r_nD\big)x_n\Big\| = 0. \tag{17}$$

From definition of x_n, we have $\Big\|x_n - P_C\big(I-\lambda_n(I-T)\big)u_n\Big\| \le \|x_{n+1}-x_n\| + \Big\|x_{n+1} - P_C\big(I-\lambda_n(I-T)\big)u_n\Big\|,$

(13) and (17), we have $\lim_{n\to\infty}\Big\|x_n - P_C\big(I-\lambda_n(I-T)\big)u_n\Big\| = \lim_{n\to\infty}\Big\|x_n - P_C\big(I-\lambda_n(I-T)\big)P_C\big(I-r_nD\big)x_n\Big\| = 0.$ (18)

Since $\beta_n\|Gx_n - x_n\| \le \|x_{n+1}-x_n\| + \alpha_n\|u - x_n\| + \gamma_n\Big\|P_C\big(I-\lambda_n(I-T)\big)u_n - x_n\Big\|$, (13), (18) and conditions $(i),(ii)$, we have $\lim_{n\to\infty}\|x_n - Gx_n\| = 0.$ (19)

Step 5. We will show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$ where $u_n = P_C\big(I-r_nD\big)x_n$.

From Remark 6, we have

$$
\begin{aligned}
\left\|u_n - z\right\|^2 \le\ & \left\langle \left(I - r_n D\right) x_n - \left(I - r_n D\right) z, u_n - z \right\rangle \\
\le\ & \frac{1}{2}\left(\left\|x_n - z\right\|^2 + \left\|u_n - z\right\|^2 - \left\|x_n - u_n\right\|^2 - r_n^2 \left\|Dx_n - Dz\right\|^2 + 2r_n \left\langle x_n - u_n, Dx_n - Dz \right\rangle \right),
\end{aligned}
$$

which implies that

$$
\left\|u_n - z\right\|^2 \le \left\|x_n - z\right\|^2 - \left\|x_n - u_n\right\|^2 - r_n^2 \left\|Dx_n - Dz\right\|^2 + 2r_n \left\langle x_n - u_n, Dx_n - Dz \right\rangle. \qquad (20)
$$

By nonexpansiveness of P_C we have

$$
\left\|u_n - z\right\|^2 \le \left\|\left(I - r_n D\right) x_n - \left(I - r_n D\right) z\right\|^2 \le \left\|x_n - z\right\|^2 - r_n\left(2\gamma - r_n\right)\left\|Dx_n - Dz\right\|^2. \qquad (21)
$$

From definition of x_n and (21), we have

$$
\begin{aligned}
\left\|x_{n+1} - z\right\|^2 \le\ & \alpha_n \left\|u - z\right\|^2 + \beta_n \left\|Gx_n - z\right\|^2 + \gamma_n \left\|P_C\left(I - \lambda_n (I - T)\right) u_n - z\right\|^2 \\
\le\ & \alpha_n \left\|u - z\right\|^2 + \left\|x_n - z\right\|^2 - r_n \gamma_n \left(2\gamma - r_n\right)\left\|Dx_n - Dz\right\|^2,
\end{aligned} \qquad (22)
$$

which implies that

$$
r_n \gamma_n \left(2\gamma - r_n\right)\left\|Dx_n - Dz\right\|^2 \le \alpha_n \left\|u - z\right\|^2 + \left(\left\|x_n - z\right\| + \left\|x_{n+1} - z\right\|\right)\left\|x_{n+1} - x_n\right\|. \qquad (23)
$$

From (13), (23) and conditions $(i), (ii)$, we have $\lim_{n\to\infty} \left\|Dx_n - Dz\right\| = 0.$ (24)

From definition of x_n and (12), (20), we have

$$
\begin{aligned}
\left\|x_{n+1} - z\right\|^2 \le\ & \alpha_n \left\|u - z\right\|^2 + \beta_n \left\|x_n - z\right\|^2 + \gamma_n \left\|u_n - z\right\|^2 \\
\le\ & \alpha_n \left\|u - z\right\|^2 + \beta_n \left\|x_n - z\right\|^2 + \gamma_n \left(\left\|x_n - z\right\|^2 - \left\|x_n - u_n\right\|^2 - r_n^2 \left\|Dx_n - Dz\right\|^2 + 2r_n \left\langle x_n - u_n, Dx_n - Dz \right\rangle \right) \\
\le\ & \alpha_n \left\|u - z\right\|^2 + \left\|x_n - z\right\|^2 - \gamma_n \left\|x_n - u_n\right\|^2 + 2r_n \gamma_n \left\|x_n - u_n\right\| \left\|Dx_n - Dz\right\|,
\end{aligned}
$$

which implies that

$$
\gamma_n \left\|x_n - u_n\right\|^2 \le \alpha_n \left\|u - z\right\|^2 + \left(\left\|x_n - z\right\| + \left\|x_{n+1} - z\right\|\right)\left\|x_{n+1} - x_n\right\| + 2r_n \gamma_n \left\|x_n - u_n\right\| \left\|Dx_n - Dz\right\|. \qquad (25)
$$

From (13), (24), (25) and conditions $(i), (ii)$, we have

$$
\lim_{n\to\infty} \left\|u_n - x_n\right\| = \lim_{n\to\infty} \left\|P_C\left(I - r_n D\right) x_n - x_n\right\| = 0. \qquad (26)
$$

Step 6. We will show that $\limsup_{n\to\infty} \left\langle u - z_0, x_n - z_0 \right\rangle \le 0$ where $z_0 = P_{\mathfrak{I}} u$. To show this equality, take a subsequence $\left\{x_{n_k}\right\}$ of $\left\{x_n\right\}$ such that $\limsup_{n\to\infty} \left\langle u - z_0, x_n - z_0 \right\rangle = \lim_{k\to\infty} \left\langle u - z_0, x_{n_k} - z_0 \right\rangle.$ (27)

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ for some $\omega \in C$. We first show that $\omega \in VI(C,A) \cap VI(C,B)$. From Lemma 7, we have $VI\left(C, aA + (1-a)B\right) = VI(C,A) \cap VI(C,B)$. Assume that $\omega \neq P_C\left(I - r_n D\right)\omega$ where $D = aA + (1-a)B$. From Opial's property, nonexpansiveness of $P_C\left(I - r_n D\right)$ and (26), we have

$$\begin{aligned}
\liminf_{k\to\infty}\left\|x_{n_k} - \omega\right\| &< \liminf_{k\to\infty}\left\|x_{n_k} - P_C\left(I - r_n D\right)\omega\right\| \\
&\le \liminf_{k\to\infty}\left(\left\|x_{n_k} - P_C\left(I - r_n D\right)x_{n_k}\right\| + \left\|P_C\left(I - r_n D\right)x_{n_k} - P_C\left(I - r_n D\right)\omega\right\|\right) \\
&= \liminf_{k\to\infty}\left\|P_C\left(I - r_n D\right)x_{n_k} - P_C\left(I - r_n D\right)\omega\right\| \le \liminf_{k\to\infty}\left\|x_{n_k} - \omega\right\|.
\end{aligned}$$

This is a contradiction. Then $\omega \in VI(C,A) \cap VI(C,B)$. From (19), we have $\lim_{k\to\infty}\left\|Gx_{n_k} - x_{n_k}\right\| = 0$. From the nonexpansiveness of G [3] , $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and Lemma 8, we have $\omega \in F(G)$. Since $\left\|P_C\left(I - \lambda_n(I-T)\right)u_n - u_n\right\| \le \left\|P_C\left(I - \lambda_n(I-T)\right)u_n - x_n\right\| + \left\|x_n - u_n\right\|$, (18) and (26), we have

$$\lim_{n\to\infty}\left\|P_C\left(I - \lambda_n(I-T)\right)u_n - u_n\right\| = 0. \tag{28}$$

From Remark 9, we have $F(T) = F\left(P_C\left(I - \lambda_{n_k}(I-T)\right)\right)$. Assume that $\omega \neq P_C\left(I - \lambda_{n_k}(I-T)\right)\omega$. Since $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$, Opial's property, (31) and condition (i), we have

$$\begin{aligned}
\liminf_{k\to\infty}\left\|u_{n_k} - \omega\right\| &< \liminf_{k\to\infty}\left\|u_{n_k} - P_C\left(I - \lambda_{n_k}(I-T)\right)\omega\right\| \\
&\le \liminf_{k\to\infty}\left(\left\|u_{n_k} - P_C\left(I - \lambda_{n_k}(I-T)\right)u_{n_k}\right\| + \left\|P_C\left(I - \lambda_{n_k}(I-T)\right)u_{n_k} - P_C\left(I - \lambda_{n_k}(I-T)\right)\omega\right\|\right) \\
&\le \liminf_{k\to\infty}\left(\left\|u_{n_k} - P_C\left(I - \lambda_{n_k}(I-T)\right)u_{n_k}\right\| + \left\|\left(I - \lambda_{n_k}(I-T)\right)u_{n_k} - \left(I - \lambda_{n_k}(I-T)\right)\omega\right\|\right) \\
&\le \liminf_{k\to\infty}\left(\left\|u_{n_k} - P_C\left(I - \lambda_{n_k}(I-T)\right)u_{n_k}\right\| + \left\|u_{n_k} - \omega\right\| + \lambda_{n_k}\left\|(I-T)u_{n_k} - (I-T)\omega\right\|\right) \\
&= \liminf_{k\to\infty}\left\|u_{n_k} - \omega\right\|.
\end{aligned}$$

This is a contradiction. Then $\omega \in F(T)$. Therefore $\omega \in \Im$. Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in \Im$, we have

$$\limsup_{n\to\infty}\left\langle u - z_0, x_n - z_0\right\rangle = \lim_{k\to\infty}\left\langle u - z_0, x_{n_k} - z_0\right\rangle = \left\langle u - z_0, \omega - z_0\right\rangle \le 0. \tag{29}$$

Step 7. Finally, we show that $\{x_n\}$ converse strongly to $z_0 = P_{\Im}u$. From definition of x_n and (12), we have

$$\begin{aligned}
\left\|x_{n+1} - z_0\right\|^2 &\le \beta_n\left\|\left(Gx_n - z_0\right) + \gamma_n\left(P_C\left(I - \lambda_n(I-T)\right)P_C\left(I - r_n D\right)x_n - z_0\right)\right\|^2 + 2\alpha_n\left\langle u - z_0, x_{n+1} - z_0\right\rangle \\
&\le (1-\alpha_n)\left\|x_n - z_0\right\|^2 + 2\alpha_n\left\langle u - z_0, x_{n+1} - z_0\right\rangle.
\end{aligned} \tag{30}$$

From (29) and Lemma 10, we have $\left\{x_n\right\}$ converse strongly to $z_0 = P_{\Im}u$. This completes the prove. □

Example and numerical results

Example 1. Let $\mathbb{R}$ be the set of real numbers, $C=[-60,60]$, and $H=\mathbb{R}$. Let $A,B,\overline{A},\overline{B}:C\to H$ defined by $Ax=x+\frac{9}{2}, Bx=\frac{5x+6}{5}, \overline{A}x=x-\frac{17}{3}, \overline{B}x=\frac{8x-15}{8}$ for all $x\in C$. Let $T:C\to C$ defined by $Tx=-3x$ for all $x\in C$. Define $G:C\to C$ by $Gx=P_C\left(I-\frac{3}{8}\overline{A}\right)\left(\frac{1}{2}x+\frac{1}{2}P_C(I-\frac{16}{17}\overline{B})x\right)$ for all $x\in C$. The mappings $A,B,\overline{A},\overline{B},T$ and G satisfy all the conditions of Theorem 11. Let the x_n sequence be generated by $x_1,u\in C$ and by

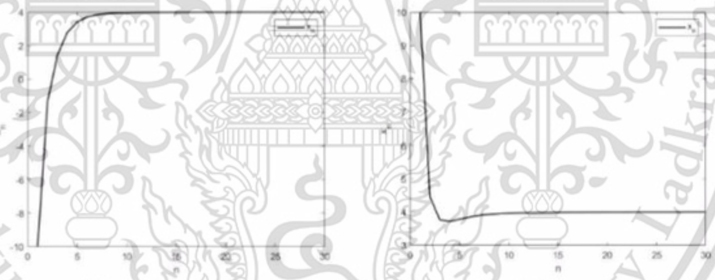
$$x_{n+1}=\frac{1}{4n}u+\frac{4n-2}{4n}Gx_n+\frac{1}{4n}P_C\left(I-\frac{1}{n(n+1)}(I-T)\right)P_C\left(I-\left(\frac{1}{n+1}\right)\left(\frac{1}{2}A+\frac{1}{2}B\right)\right)x_n \text{ for all } n\in\mathbb{N}.$$

From Theorem 11, we can conclude that the sequence $\left\{x_n\right\}$ converges strongly to 4.

Table 1. The value of $\left\{x_n\right\}$ with initial values $u=x_1=-10, u=x_1=10$ and $n=N=30$.

n		1	2	$\cdots$	15	$\cdots$	29	30
$u=x_1=-10$	x_n	-10.000000	-1.359926	$\cdots$	3.986304	$\cdots$	3.996974	3.997190
$u=x_1=10$	x_n	10.000000	4.448897	$\cdots$	3.986320	$\cdots$	3.996974	3.997190

Figure 1. The convergence of $\left\{x_n\right\}$ with initial values $u=x_1=-10, u=x_1=10$ and $n=N=30$.



Conclusions

1. We get a method for solve the fixed point problem of $\kappa-$ strictly pseudononspreading mapping and modification of variational inequality problem and system of variational inequality problem.

2. Table 1. and Figure 1. show that the sequence $\left\{x_n\right\}$ converge to $\left\{4\right\}\in F(T)\cap F(G)\cap VI(C,A)\cap VI(C,B)$.

3. Theorem 11. guarantee the convergence of $\left\{x_n\right\}$ in Example 1.

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Academic Publications

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