

APPROXIMATION METHOD FOR VARIATIONAL INCLUSION  
PROBLEM IN BANACH SPACE AND HILBERT SPACE



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### Abstract

For the purpose of this thesis, we first introduced split combination of variational inclusion problem in Hilbert space and used Halpern method for finding the common solution of split combination of variational inclusion problem and hierarchical problem. And also, we proved a strong convergence theorem of the intermixed algorithm. Secondly, we introduced Banach generalized system of variational inequality problems and modify viscosity iterative method by using the concept of the modification of a system of variational inequality problems introduced. Then, we established and proved strong convergence theorem for finding a common element of the set of solutions of the Banach generalized system of variational inequality problems and fixed point problems. Thirdly, we applied our main theorem to prove strong convergence theorems to solve variational inequality problem. Finally, we gave numerical examples to support our main results.

**Keywords :** Variational inclusion problem, Variational inequality problem, Hierarchical problem, fixed point problem

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# Chapter 1

## Introduction

Throughout this thesis, let  $E$  be a real Banach with its dual space  $E^*$  and  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . Let  $C_E$  be a nonempty closed convex subset of  $E$  and  $C_H$  be a nonempty closed convex subset of  $H$ . We denote the norm of  $E$  and  $E^*$  by the same symbol  $\| \cdot \|$ . We denote weak and strong convergence by the notations  $\rightharpoonup$  and  $\rightarrow$ , respectively.

### 1.1 Banach space and Hilbert space

In this section, we present the notation, definition on Banach space and Hilbert space, which will be used in further parts of the thesis.

Firstly, we introduced definition and important properties of *metric space* and *vector space*.

**Definition 1.1.** ([42]) Let  $X$  be a non-empty set. A mapping  $d : X \times X \rightarrow \mathbb{R}^+$  is called a *metric* or a *distance* on  $X$  provided that

- (i)  $d(x, y) = 0 \Leftrightarrow x = y$ ;
- (ii)  $d(x, y) = d(y, x)$ , for all  $x, y \in X$ ;
- (iii)  $d(x, z) \leq d(x, y) + d(y, z)$ , for all  $x, y, z \in X$ .

A set  $X$  endowed with a metric  $d$  is called *metric space* and is denoted by  $(X, d)$ .

**Definition 1.2.** ([42]) Let  $\{x_n\}$ , for all  $n = 1, 2, \dots$  be a sequence in a metric space  $(X, d)$ . We say that the sequence  $\{x_n\}$  is *Cauchy sequence* if, for any  $\varepsilon > 0$ , there exists  $n_0 = n_0(\varepsilon)$  such that  $d(x_n, x_{n+p}) < \varepsilon$ , for all  $n \in \mathbb{N}$ ,  $n \geq n_0$  and any  $p \in \mathbb{N}$ .

**Definition 1.3.** ([42]) A metric space  $(X, d)$  is called *complete* if any Cauchy sequence in  $X$  is convergent.

**Definition 1.4.** ([41]) A vector space over  $\mathbb{F}$  is a set  $V$  together with the operations of addition  $V \times V \rightarrow V$  and scalar multiplication  $\mathbb{F} \times V \rightarrow V$  satisfying the following properties:

- 1) Commutativity:  $u + v = v + u$  for all  $u, v \in V$ ;
- 2) Associativity:  $(u + v) + w = u + (v + w)$  and  $(ab)v = a(bv)$  for all  $u, v, w \in V$  and  $a, b \in \mathbb{F}$ ;
- 3) Additive identity: There exists an element  $0 \in V$  such that  $0 + v = v$  for all  $v \in V$ ;

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- 4) Additive inverse: For every  $v \in V$ , there exists an element  $w \in V$  such that  $v + w = 0$ ;
- 5) Multiplicative identity:  $1v = v$  for all  $v \in V$ ;
- 6) Distributivity:  $a(u + v) = au + av$  and  $(a + b)u = au + bu$  for all  $u, v \in V$  and  $a, b \in \mathbb{F}$ .

Usually, a vector space over  $\mathbb{R}$  is called a real vector space and a vector space over  $\mathbb{C}$  is called a complex vector space. The element  $v \in V$  of a vector space are called vectors.

**Definition 1.5.** ([41]) A set  $E$  in a vector space is called *convex* if for any  $x, y \in E$  and  $\alpha \in (0, 1)$  we have

$$\alpha x + (1 - \alpha)y \in E.$$

### 1.1.1 Banach space

Banach spaces are important in our study of iterative algorithm for nonlinear operators. Now, we introduce *Banach space, uniformly convex*.

**Definition 1.6.** ([42]) Let  $E$  be a real (complex) vector space. A *norm* on  $E$  is a mapping  $\|\cdot\| : E \times E \rightarrow \mathbb{R}^+$  having the following properties

- (i)  $\|x\| = 0 \Leftrightarrow x = 0$ , the null element of  $E$ ;
- (ii)  $\|\alpha x\| = |\alpha|\|x\|$ , for any  $x \in E$  and any scalar  $\alpha$ ;
- (iii)  $\|x + y\| \leq \|x\| + \|y\|$ , for all  $x, y \in E$  (the triangle inequality).

The pair  $(E, \|\cdot\|)$  is called *normed (linear) space*. A complete normed space is called a *Banach space*.

**Definition 1.7.** ([42]) A Banach space  $E$  is said to be *uniformly convex* iff for any  $\varepsilon$ ,  $0 < \varepsilon \leq 2$ , the inequalities  $\|x\| \leq 1$ ,  $\|y\| \leq 1$  and  $\|x - y\| \geq \varepsilon$  imply there exists a  $\delta > 0$  such that  $\|\frac{x+y}{2}\| \leq 1 - \delta$ .

### 1.1.2 Hilbert space

Hilbert spaces are the most important examples of uniformly convex Banach space that serve as very natural ambient spaces for various fixed point iteration procedures.

A linear space  $H$  with an inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$  introduced by inner product is called a *Hilbert space*.

**Definition 1.8.** ([42]) Let  $H$  be a real vector space. An *inner product* is a functional  $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{R}$  satisfying :

- (i)  $\langle x, x \rangle \geq 0$ , for all  $x \in H$  and  $\langle x, x \rangle = 0$  if and only if  $x = 0$ , the null vector in  $H$ ;
- (ii)  $\langle x, y \rangle = \langle y, x \rangle$ , for all  $x, y \in H$ ;
- (iii)  $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ , for each  $a, y, z \in H$  and all  $a, b \in \mathbb{R}$ .

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**Remark 1.1.** ([42])

- (i) Any Hilbert space is a uniformly convex Banach space;
- (ii) It is then clear that all notions introduced in Banach space can be reformulated by replacing the duality pairing by the inner product.

**Definition 1.9.** ([40]) A mapping  $J$  from  $E$  onto  $E^*$  satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\}$$

is called the normalized duality mapping of  $E$ . The duality pair  $\langle x, f \rangle$  represents  $f(x)$  for  $f \in E^*$  and  $x \in E$ .

It is well known that if  $E$  is smooth, then  $J$  is a single value, which we denote by  $j$ .

## 1.2 Iterative methods of nonlinear mappings

The fixed point theorem based on the contraction principle has studied the existence and uniqueness of the solutions. In the past few decades, many mathematicians have created the convergence theorem to solve the fixed point problem. This makes the fixed point theory developed extensively and applied in various fields such as mathematics, economics, physics, engineering, optimizations, computer sciences, etc. For an interesting example, equilibrium problems or variational inequality problems are fundamental concepts in economics, engineering, and physics, which can be transformed into fixed points.

This thesis we denote  $F(T)$  is the set of fixed points of  $T$  (i.e.,  $F(T) = \{x \in E : Tx = x\}$ ).

Many researchers have studied the iterative scheme to approximate the fixed point problem of nonlinear mapping as follows;

In 1953, Mann [34] introduced the sequence  $\{x_n\}$  generated by  $x_0 \in E$  and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad \forall n \geq 0, \quad (1.1)$$

where  $C$  is a nonempty closed convex subset of a normed space,  $T : C_E \rightarrow C_E$  is a mapping and the sequence  $\{\alpha_n\}$  is in the interval  $(0, 1)$ , iteration (1.1) is called *Mann iteration*. If  $T$  is a nonexpansive mapping under some suitable condition  $\alpha_n$  satisfying  $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ , then the sequence  $\{x_n\}$  generated by (1.1) converge weakly to element of the set of fixed points of  $T$ . Many authors have been trying to modify Mann's iteration for solving solution of various problems such as the fixed point problem, split feasibility problem, equilibrium problem, monotone inclusion and image restoration problem; see more detail in [35, 36, 37].

In 1967, Halpern [8] studied the following recursive formula:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \quad (1.2)$$

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where  $\{\alpha_n\}$  is sequence of numbers in  $(0, 1)$  and  $T : C_H \rightarrow C_H$  is a nonexpansive mapping. He proved strong convergence of  $\{x_n\}$  to a fixed point of  $T$ , in the framework of Hilbert spaces. Halpern's iterative (1.2) has been studied extensively by many authors; see more detail in [9, 10].

In 2000, Moudafi [35] proposed *the viscosity approximation method* for nonexpansive mapping  $S$  and the sequence  $\{x_n\}$  generated by  $x_1 \in C_E$  and

$$x_{n+1} = \frac{1}{1 + \epsilon_n} Sx_n + \frac{\epsilon_n}{1 + \epsilon_n} f(x_n), \quad \forall n \in \mathbb{N}, \quad (1.3)$$

where  $\{\epsilon_n\} \subset (0, 1)$  satisfies certain conditions,  $S : C_E \rightarrow C_E$  is a nonexpansive mapping and  $f : C_E \rightarrow C_E$  is a contraction. Then he proved the sequence  $\{x_n\}$  converges strongly to  $z = P_{F(S)}f(z)$ . Moreover, the viscosity approximation method has been studied and developed in many pieces of research to approximate the convex feasibility problem, hierarchical fixed point problem, variational inequality problem, split common null point problem, see previous studies in [18]. Notice that the sum of coefficients  $\frac{1}{1 + \epsilon_n}$  and  $\frac{\epsilon_n}{1 + \epsilon_n}$  in (1.3) is equal to 1.

### 1.3 Variational inequality problems

In this section, we present Hierarchical problem and Generalized system of variational inequalities problem in Banach space.

#### 1.3.1 Hierarchical problem

Now, we introduce modification of variational inequality problem, that is called hierarchical problem.

Variational inequality problem, which was firstly introduced by Lions and Stampacchia [29], is famous in many fields of pure and applied sciences. The variational inequality problems can apply to operation research, economic equilibrium, physics, game theory, finance, optimization, and mechanics as special cases. Let  $A : H \rightarrow H$  be a single-valued mapping. *Variational inequality problems* is to find a point  $u \in C_H$  such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C_H. \quad (1.4)$$

The set of all solutions of (1.4) is denoted by  $VI(C_H, A)$ . Various iterative method for solving variational inequality (1.4) have been proposed and many authors in Hilbert spaces or Banach space; see more detail in [29, 30, 31, 32].

Note that the set of solutions of fixed points is related to the set of solutions to variational inequalities, that is  $VI(C_H, I - T) = F(T)$  under suitable condition of nonlinear mapping  $T : C_H \rightarrow C_H$  such as  $T$  is nonexpansive mapping or  $T$  is strictly pseudo-contractive mapping with  $F(T) \neq \emptyset$ .

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Moudafi and Mainge [14] firstly introduced *hierarchical fixed point problem* for a nonexpansive mapping  $T$  with respect to another nonexpansive mapping  $S$  on  $H$ : Find  $x^* \in F(T)$  such that

$$\langle Sx^* - x^*, x - x^* \rangle \leq 0, \forall x \in F(T), \quad (1.5)$$

where  $S : H \rightarrow H$  is a nonexpansive mapping. It is easy to see that (1.5) is equivalent to the following fixed point problem: Find  $x^* \in H$  such that

$$x^* = P_{F(T)} \circ Sx^*, \quad (1.6)$$

where  $P_{F(T)}$  is the metric projection of  $H$  onto  $F(T)$ . The solution set of (1.5) is denoted by  $\Phi = \{x^* \in H : \langle Sx^* - x^*, x - x^* \rangle \leq 0, \forall x \in F(T)\}$ . It is obvious that  $\Phi = VI(F(T), I - S)$ . Note that (1.5) covers monotone variational inequality on fixed point sets, minimization problem over equilibrium constraints, hierarchical minimization problems, etc. Many iterative methods have been developed for solving the hierarchical fixed point problem (1.5), see example [14]-[19].

### 1.3.2 Generalized system of variational inequalities problem in Banach space

Now, we introduce modification of generalized variational inequality problem in Banach space, that is called Banach generalized system of variational inequalities problem.

Aoyama et al. [11] first introduced the following *generalized variational inequality* problem in Banach spaces. Let  $A : C_E \rightarrow E$  be an accretive operator. Find a point  $x^* \in C_E$  such that for some  $j(x - x^*) \in J(x - x^*)$ ,

$$\langle Ax^*, j(x - x^*) \rangle \geq 0, \forall x \in C_E. \quad (1.7)$$

The set of solutions of the generalized variational inequality in a Banach space is denoted by  $S(C_E, A)$ , that is,

$$S(C_E, A) = \{u \in C_E : \langle Au, J(v - u) \rangle \geq 0, \forall v \in C_E\}.$$

**Remark 1.2.** Let  $H$  be real Hilbert spaces. If  $E \equiv H$ , then generalized variational inequality problem reduce to *variational inequality* problem in Hilbert space, that is, find a point  $u \in C_H$  such that

$$\langle v - u, Du \rangle \geq 0, \text{ for all } v \in C_H. \quad (1.8)$$

The set of solutions of the variational inequalities is denoted  $VI(C_H, D)$ .

Yao et al. [12] who is based on generalized variational inequality in a Banach space introduced *the system of general variational inequalities* problem. Let  $A, B : C_E \rightarrow E$  be two nonlinear mappings. Finding a point  $(x^*, y^*) \in C_E \times C_E$  such that

$$\begin{cases} \langle Ay^* + x^* - y^*, j(x - x^*) \rangle \geq 0, \forall x \in C_E, \\ \langle Bx^* + y^* - x^*, j(x - y^*) \rangle \geq 0, \forall x \in C_E. \end{cases} \quad (1.9)$$

They first gave associated between the system of variational inequalities problem (1.9) and some fixed point problem of nonexpansive mapping. Furthermore, They proved strong convergence theorem of sequence generated by the modification of extragradient methods for solving the system of general variational inequalities problem (1.9) under suitable condition; see more detail in [12].

In 2014, Kangtunyakarn [1] introduced *the modification of a system of variational inequalities* problems in Banach space for finding  $(x^*, y^*) \in C_E \times C_E$  such that

$$\begin{cases} \langle x^* - (I - \lambda_A A)(ax^* + (1-a)y^*), j(x - x^*) \rangle \geq 0, \forall x \in C_E, \\ \langle y^* - (I - \lambda_B B)x^*, j(x - y^*) \rangle \geq 0, \forall x \in C_E. \end{cases} \quad (1.10)$$

for all  $\lambda_A, \lambda_B > 0$  and  $a \in [0, 1]$ . The set of solution of (1.10) we denote by  $\Omega'$ .

If  $\lambda_A = \lambda_B = 1$  and  $a = 0$ , then problem (1.10) reduce to (1.9). Kangtunyakarn [1] proved strong convergence theorem for the set of fixed points of two finite families of nonexpansive and strictly pseudo-contractive mappings and the set of solutions of modification of system of variational inequalities problems under suitable condition; see more detail in [1].

Inspired and motivated by the modification of a system of variational inequalities problems of Kangtunyakarn [1], we define *Banach generalized system of variational inequalities problem* which is find  $(x^*, y^*, z^*) \in C_E \times C_E \times C_E$  such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), j(x - x^*) \rangle \geq 0, \forall x \in C_E, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), j(x - y^*) \rangle \geq 0, \forall x \in C_E, \\ \langle z^* - (I - \lambda_3 D_3)x^*, j(x - z^*) \rangle \geq 0, \forall x \in C_E. \end{cases} \quad (1.11)$$

for all  $\lambda_1, \lambda_2, \lambda_3 > 0$  and  $a \in [0, 1]$ . In Hilbert space (1.11) reduce to

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, \forall x \in C_H, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, \forall x \in C_H, \\ \langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, \forall x \in C_H. \end{cases} \quad (1.12)$$

for all  $\lambda_1, \lambda_2, \lambda_3 > 0$  and  $a \in [0, 1]$ , which is modified generalized system of variational inequalities in Hilbert space introduced by Siriyan and Kantunyakarn [13]. If  $D_3 \equiv 0$  and  $x^* = z^*$ , then problem (1.11) reduce to (1.10).

#### 1.4 Variational inclusion problems

In this section, we introduce split combination of variational inclusion problem.

The variational inclusion problems are extensively studied in mathematical programming, complementarity problems, optimal control, mathematical economics,

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and game theory, etc. In this thesis, we give a new iteration for solving split combination of variational inclusion problems.

The *variational inclusion problem* is to find  $x \in H$  such that

$$\theta \in Bx + Mx, \quad (1.13)$$

where  $\theta$  is a zero vector in  $H$ ,  $B : H \rightarrow H$  be a mapping and  $M : H \rightarrow 2^H$  be a multi-valued mapping. The set of solution of (1.13) is denoted by  $VI(H, B, M)$ .

In recent years, considerable interest has been shown in developing various extensions and generalization of variational inclusion problem, see for instance [2]-[7] and reference therein.

For  $i = 1, 2, \dots, N$ , let  $A_i : H \rightarrow H$  be a single-valued mapping, from the concept of (1.13), Khuangsatung and Kangtunyakarn [3] introduced the problem of finding  $x \in H$  such that

$$\theta \in \sum_{i=1}^N a_i A_i x + Mx, \quad (1.14)$$

for all  $a_i \in (0, 1)$  with  $\sum_{i=1}^N a_i = 1$  and  $\theta$  is a zero vector. This problem is called *the modified variational inclusion*. The set of solutions of (1.14) is denoted by  $VI(H, \sum_{i=1}^N a_i A_i, M)$ . If we set  $A_i = B$  for  $i = 1, 2, \dots, N$  then  $VI(H, \sum_{i=1}^N a_i A_i, M) = VI(H, B, M)$ .

Let  $H_1$  and  $H_2$  be two real Hilbert spaces and  $D, Q$  be a nonempty closed convex subset of a real Hilbert spaces  $H_1$  and  $H_2$ , respectively. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator. Censor and Elfving [24] introduced *the split feasibility problem (SEP)* which is to find a point  $x \in D$  and  $Ax \in Q$ . Many authors have studied this concept of SEP to modified their problem, see example [20]-[25].

In 2010, Censor, Gibali and Reich [20] introduced split variational inequality problem which relies on split feasibility problem and created iterative algorithm for solving a strong convergence theorem of split variational inclusion problem, see more detail [20].

Moudafi[28] who relies on split feasibility problem and split variational inequality problem introduced *the split monotone variational inclusion (SMVI)*:

$$\text{find } x^* \in H_1 \text{ such that } 0 \in f(x^*) + B_1(x^*), \quad (1.15)$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in g(y^*) + B_2(y^*), \quad (1.16)$$

where  $B_1 : H_1 \rightarrow 2^{H_1}$  is a multi-valued mapping on a Hilbert space  $H_1$ ,  $B_2 : H_2 \rightarrow 2^{H_2}$  is a multi-valued mapping on a Hilbert space  $H_2$ ,  $A : H_1 \rightarrow H_2$  is a bounded linear operator,  $f : H_1 \rightarrow H_1$  and  $g : H_2 \rightarrow H_2$  are two given single-valued operators. The set of all this solution (1.15)-(1.16) is denoted by  $\Theta = \{x^* \in VI(H_1, f, B_2) : Ax^* \in VI(H_2, g, B_2)\}$ . If we set  $H_1 = H_2$ ,  $f = g$ ,  $B_1 = B_2$  and  $A = I$  then  $\Theta = VI(H, B, M)$ .

The split monotone variational inclusion problem which consists as special cases, which being used in practice as a model in the intensity-modulated radiation

therapy treatment planing; see [21]. This formulation is also at the main point of the modeling of many inverse problems arising and other problems; for instance, in sensor networks and data compression; see [26]-[27].

For every  $i = 1, 2, \dots, N$ . Let  $A_i : H_1 \rightarrow H_1$ ,  $B_i : H_2 \rightarrow H_2$  be mapping and  $M_A : H_1 \rightarrow 2^{H_1}$  and  $M_B : H_2 \rightarrow 2^{H_2}$  be multivalued mapping. Inspired and motivated by Moudafi[28] and Khuangsatung and Kangtunyakarn [3], we define *the split combination of variational inclusion problem (SCVIP)* which is find  $x^* \in H_1$  such that

$$\theta_{H_1} \in \sum_{i=1}^N a_i A_i x^* + M_A x^*, \quad (1.17)$$

and

$$y^* = Ax^* \text{ such that } \theta_{H_2} \in \sum_{i=1}^N b_i B_i y^* + M_B y^*, \quad (1.18)$$

where  $A : H_1 \rightarrow H_2$  is a bounded linear operator and  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$ .

The set of all this solution (1.17)-(1.18) is denoted by  $\Omega = \{x \in VI(H_1, \sum_{i=1}^N a_i A_i, M_A) : Ax \in VI(H_2, \sum_{i=1}^N b_i B_i, M_B)\}$ . If we set  $A_i = A$  and  $B_i = B$  for  $i = 1, 2, \dots, N$  then  $\Omega = \Theta$ .

## 1.5 Objectives of the study

- 1) To give a lemma, an essential tool for proof strong convergence of our main theorems in Banach space.
- 2) To give a lemma, an essential tool for proof strong convergence of our main theorems in Hilbert space.
- 3) To propose the Banach generalized system of variational inequalities problem in Banach space.
- 4) To propose the split combination of variational inclusion problem (SCVIP) in Hilbert space.
- 5) To prove strong convergence theorem in Banach space for finding a common solution of Banach generalized system of variational inequalities problem and fixed point problem of nonexpansive mapping.
- 6) To prove strong convergence theorem in Hilbert space for finding a common solution of hierarchical fixed point problem and split combination of variational inclusion problem.
- 7) To prove strong convergence theorem in Banach space to solve generalized variational inequalities problem and fixed point problem of nonexpansive mapping.
- 8) To prove strong convergence theorem in Hilbert space to solve variational inequality problem and zero point problem.
- 9) To give numerical examples for our results to support our main results.

## 1.6 Scopes of the study

- 1) Study the definitions and properties of variational inequality problems, variational inclusion problem, hierarchical problems, and fixed point problems in real Hilbert space.
- 2) Study the definitions and properties of generalized variational inequality problems and fixed point problems in Banach space.
- 3) Investigate the fixed point problems of nonlinear mappings, including nonexpansive mapping, maximal monotone and  $\alpha$ -inverse strongly monotone mapping in real Hilbert space.
- 4) Investigate the fixed point problems of nonlinear mappings, including nonexpansive mapping, sunny nonexpansive, accretive mapping and  $\alpha$ -inverse strongly accretive mapping in Banach space.
- 5) All strong convergence theorems are considered and proved in a real Hilbert space.
- 6) All strong convergence theorems are considered and proved in Banach space.
- 7) Give numerical examples for supporting our main results in  $\mathbb{R}$ ,  $\mathbb{R}^2$  and  $L_2$  spaces.

## 1.7 Benefits of the study

- 1) Obtain new tools for fixed point problems on real Hilbert space.
- 2) Obtain new tools for fixed point problems on Banach space.
- 3) Obtain a strong convergence theorem in Hilbert space of a Halpern algorithm to find the common solution of hierarchical fixed point problem and split combination of variational inclusion problem.
- 4) Obtain a strong convergence theorem in Banach space of a modified viscosity algorithm to find the common solution of Banach generalized system of variational inequalities problem and fixed point problem of nonexpansive mapping.
- 5) Obtain a strong convergence theorem in Banach space to solve generalized variational inequalities problem and fixed point problem of nonexpansive mapping.
- 6) Obtain a strong convergence theorem in Hilbert space to solve variational inequality problem and zero point problem.

## 1.8 Research methodology

- 1) Study advanced topics in fixed point theory for sunny nonexpansive, accretive mapping and  $\alpha$ -inverse strongly accretive mapping in Banach space.
- 2) Study advanced topics in fixed point theory for nonexpansive mapping, maximal monotone,  $\alpha$ -inverse strongly monotone mapping in Hilbert space.
- 3) Study background in a real Hilbert space and Banach space.
- 4) Study the Halpern algorithm in Hilbert space and modified viscosity algorithm in Banach space.
- 5) Collect and study research papers and textbooks concerning fixed point theorem.
- 6) Determine the objectives and scope of the research.
- 7) Produce tools for a strong convergence theorem of fixed point problem in Banach space.
- 8) Produce tools for a strong convergence theorem of fixed point problem in Hilbert space.
- 9) Prove strong convergence theorem for fixed point problems in a Hilbert space.
- 10) Prove strong convergence theorem for fixed point problems in Banach space.
- 11) Provide examples and applications in Banach space.
- 12) Provide examples and applications in Hilbert space.
- 13) Conclude the results, make suggestions for further works and write the thesis.

## Chapter 2

### Preliminaries and Literature Reviews

The purpose of this chapter is to collect lemmas, definitions, theorems and terminology for used throughout of the thesis.

#### 2.1 Fundamental properties in Banach spaces

Now, we now give examples of Banach spaces.

**Example 2.1.** ([40]) Let  $E = \mathbb{R}^n$ ,  $n > 1$  be a normed space with the following norms:

$$\begin{aligned}\|x\|_1 &= \sum_{i=1}^n |x_i|, \forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n; \\ \|x\|_p &= \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}, \forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \text{ and } p \in (1, \infty); \\ \|x\|_\infty &= \max_{1 \leq i \leq n} |x_i|, \forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n.\end{aligned}$$

Then  $\mathbb{R}^n$  equipped with the norm defined by  $\|x\|_p$  and  $\mathbb{R}^n$  equipped with the norm defined by  $\|x\|_\infty$  are Banach spaces.

**Remark 2.2.**

- (i)  $\mathbb{R}^n$  equipped with the norm defined by  $\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$  is denoted by  $l_p^n$  for all  $1 \leq p < \infty$ ;
- (ii)  $\mathbb{R}^n$  equipped with the norm defined by  $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$  is denoted by  $l_\infty^n$ .

**Example 2.3.** ([40]) Let  $E = l_1$  be a normed space with the norm defined by

$$l_1 = \{x : x = (x_1, x_2, \dots, x_i, \dots) \text{ and } \sum_{i=1}^{\infty} |x_i| < \infty\}. \quad (2.1)$$

Then  $l_1$  is a Banach space with the norm defined by  $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ .

**Example 2.4.** ([40]) Let  $E = l_p (1 < p < \infty)$  be a normed space with the norm defined by

$$l_p = \{x : x = (x_1, x_2, \dots, x_i, \dots) \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty\}. \quad (2.2)$$

Then  $l_p$  is a Banach space with the norm defined by  $\|x\|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}$ .

**Example 2.5.** ([40]) Let  $E = L_p[0,1] (1 \leq p < \infty)$  be a normed space with the norm defined by

$$\|f\|_p = \left( \int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}} < \infty. \quad (2.3)$$

Then  $L_p$  is a Banach space.

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**Example 2.6.** ([40])  $l_p^n$  is a finite-dimensional Banach space that is not a Hilbert space for  $p \neq 2$ . Indeed, for  $x = (1, 1, 0, 0, \dots)$  and  $y = (1, -1, 0, 0, \dots)$ , we have  $x+y = (2, 0, 0, 0, \dots)$  and  $x-y = (0, 2, 0, 0, \dots)$ . Hence

$$\begin{aligned}\|x\| &= \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}, \\ \|y\| &= (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}, \\ \|x+y\| &= (2^p)^{\frac{1}{p}} = 2, \\ \|x-y\| &= (2^p)^{\frac{1}{p}} = 2.\end{aligned}$$

If  $p = 2$ , then the parallelogram law:  $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$  is satisfied, which show that  $l_2^n$  is a Hilbert space. If  $p \neq 2$ , the parallelogram law is not satisfied. Therefore,  $l_p^n$  is not a Hilbert space for  $p \neq 2$ .

**Remark 2.7.**  $l_p^n$ ,  $l_p$  and  $L_p[a, b]$  ( $p \neq 2$ ) are Banach spaces but not Hilbert spaces.

**Definition 2.1.** ([40]) A sequence  $\{x_n\}$  in a normed space  $X$  is said to be *convergent* to  $x$  if  $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ . In this case, we write  $x_n \rightarrow x$  or  $\lim_{n \rightarrow \infty} x_n = x$ .

**Definition 2.2.** ([42]) Let  $E$  be a Banach space. Then a function  $\rho_E : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is said to be *the modulus of smoothness of  $E$*  if

$$\rho_E(t) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = t \right\}.$$

A Banach space  $E$  is said to be *uniformly smooth* if

$$\lim_{t \rightarrow 0} \frac{\rho_E(t)}{t} = 0.$$

Let  $q > 1$ . A Banach space  $E$  is said to be  *$q$ -uniformly smooth* if there exists a fixed point constant  $c > 0$  such that  $\rho_E \leq ct^q$ . It is easy to see that, if  $E$  is  $q$ -uniformly smooth, then  $q \leq 2$  and  $E$  is uniformly smooth. It is well known that  $L^p$  is  $\min\{p, 2\}$ -uniformly smooth for  $p > 1$ .

**Definition 2.3.** ([40]) The space of all bounded linear functionals on a normed space  $X$  is called *the dual* of  $X$  and is denoted by  $X^*$ . Clearly,  $X^* = B(X, \mathbb{R})$  and is a normed space with norm denoted and defined by

$$\|f\|_* = \sup\{|f(x)|\}.$$

**Corollary 2.8.** ([40]) The dual space  $(X^*, \|\cdot\|_*)$  of a normed space  $X$  is always a Banach space.

**Proposition 2.9.** ([40]) Let  $E^*$  be the dual of  $E$ . Then we have the following:

(i) The duality pairing is a bilinear functional on  $E \times E^*$ ;

$$(1) \langle ax + bx, j \rangle = a\langle x, j \rangle + b\langle y, j \rangle \text{ for all } x, y \in E, j \in E^* \text{ and } a, b \in \mathbb{R};$$

$$(2) \langle x, \alpha j_1 + \beta j_2 \rangle = \alpha \langle x, j_1 \rangle + \beta \langle x, j_2 \rangle \text{ for all } x \in E, j_1, j_2 \in E^*, \alpha, \beta \in \mathbb{R};$$

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- (ii)  $\langle x, j \rangle = 0$  for all  $x \in E$  implies  $j = 0$ ;  
 (iii)  $\langle x, j \rangle = 0$  for all  $j \in E^*$  implies  $x = 0$ .

**Lemma 2.10.** ([23]) Let  $E$  be a real 2-uniformly smooth Banach space with the best smooth constant  $K$ . Then the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x) \rangle + 2\|Ky\|^2,$$

for any  $x, y \in E$ .

**Lemma 2.11.** ([49]) Let  $E$  be a uniformly convex Banach space and  $B_r = \{x \in E : \|x\| \leq r\}$ ,  $r > 0$ . Then there exists a continuous, strictly increasing and convex function  $g : [0, \infty) \rightarrow [0, \infty)$ ,  $g(0) = 0$  such that

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g(\|x - y\|),$$

for all  $x, y, z \in B_r$  and  $\alpha, \beta, \gamma \in [0, 1]$  with  $\alpha + \beta + \gamma = 1$ .

**Lemma 2.12.** ([23]) Let  $r > 0$ . If  $E$  is uniformly convex, then there exists a continuous, strictly increasing and convex function  $g : [0, \infty) \rightarrow [0, \infty)$ ,  $g(0) = 0$  such that for all  $x, y \in B_r(0) = \{x \in E : \|x\| \leq r\}$  and for all  $\alpha \in [0, 1]$ , we have

$$\|\alpha x + (1 - \alpha)y\|^2 \leq \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)g(\|x - y\|).$$

## 2.2 Fundamental properties in Hilbert spaces

Now, we give example space of Hilbert space.

**Example 2.13.** ([42]) The Euclidean space  $\mathbb{R}^n := \{x = (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \dots, n\}$ . An element  $x = (x_1, x_2, \dots, x_n)$  of  $\mathbb{R}^n$  can be identified with matrix of type  $n \times 1$ , i.e., we write  $x = [x_1, x_2, \dots, x_n]^T$ .

The Euclidean space is equipped with the standard inner product defined by

$$\langle x, y \rangle := x^T y = \sum_{i=1}^n x_i y_i,$$

where  $x = (x_1, x_2, \dots, x_n)$ ,  $y = (y_1, y_2, \dots, y_n)$ , and with the norm  $\|x\| := (x^T x)^{\frac{1}{2}}$ .

It follows that the Euclidean space satisfies Definition 1.8. Then the Euclidean space is a Hilbert space.

**Example 2.14.** ([42]) The space  $l_2$  of real sequence  $x = (x_1, x_2, \dots)$  which are square summable, i.e.,

$$l_2 = \{x : x = (x_1, x_2, \dots, x_i, \dots) \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$$

with the inner product  $\langle x, y \rangle := \sum_{i=1}^{\infty} x_i y_i$ , and with the norm  $\|x\| := (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$ . Then  $l_2$  is Hilbert space.

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**Example 2.15.** ([42]) The space  $L_2[a, b]$ , where  $a, b \in (-\infty, +\infty]$ ,  $a < b$ , let Lebesgue measurable functional  $f : \mathbb{R} \rightarrow \mathbb{R}$  which are square integrable, i.e.,

$$L_2[a, b] = \int_a^b f^2(t)dt < \infty,$$

with the inner product

$$\langle f, g \rangle = \int_a^b f(t)g(t)dt < \infty,$$

and with the norm

$$\|f\| = \left( \int_a^b f^2(t)dt \right)^{\frac{1}{2}} < \infty.$$

Then  $L_2$  is a Hilbert space.

**Definition 2.4.** ([41]) A sequence  $\{x_n\}$  of vectors in an inner product space  $K$  is called *strongly convergent* to a vector  $x$  in  $K$  if

$$\|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Definition 2.5.** ([41]) A sequence  $\{x_n\}$  of vectors in an inner product space  $K$  is called *weakly convergent* to a vector  $x$  in  $K$  if

$$\langle x_n, y \rangle \rightarrow \langle x, y \rangle \text{ as } n \rightarrow \infty \text{ for every } y \in K.$$

**Theorem 2.16.** ([41]) A strongly convergence sequence is weakly convergence (to the same limit), that is,  $x_n \rightarrow x$  implies  $x_n \rightharpoonup x$ .

**Lemma 2.17.** ([43]) In a real Hilbert spaces  $H$ , the following results hold:

- (i)  $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \forall x, y \in H;$
- (ii)  $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y \rangle + \|y\|^2, \forall x, y \in H;$
- (iii) for all  $x, y \in H$  and  $\alpha \in [0, 1]$ ,

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$$

**Definition 2.6.** ([44]) Let  $H$  be a Hilbert space and let  $C_H$  be a nonempty closed convex subset of  $H$ . Let  $f$  be a function of  $C_H$  into  $(-\infty, \infty]$ , where  $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$ . Then,  $f$  is called *lower semicontinuous* if for any  $a \in \mathbb{R}$ , the set

$$\{x \in C_H : f(x) \leq a\} \text{ is closed.}$$

Moreover,  $f$  is called *convex* if for any  $x_1, x_2 \in C_H$  and  $t \in (0, 1)$ ,

$$f(tx_1 + (1 - t)x_2) \leq tf(x_1) + (1 - t)f(x_2).$$

Similarly,  $f$  is said to be *concave* if for any  $x_1, x_2 \in C_H$  and  $t \in (0, 1)$ ,

$$f(tx_1 + (1 - t)x_2) \geq tf(x_1) + (1 - t)f(x_2).$$

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**Theorem 2.18.** ([44]) Let  $H$  be a Hilbert space, let  $C_H$  be a nonempty closed convex subset of  $H$  and let  $f$  be a proper convex lower semicontinuous function of  $C_H$  into  $(-\infty, \infty]$ . Let  $\{x_n\}$  be a bounded sequence in  $C_H$  such that  $x_n \rightharpoonup x_0$ . Then

$$f(x_0) \leq \liminf_{n \rightarrow \infty} f(x_n).$$

**Theorem 2.19.** ([44]) Let  $\{a_n\}$  be a bounded of real numbers. Then, there exists subsequence  $\{a_{n_i}\}$  of  $\{a_n\}$  such that

$$\alpha = \limsup_{n \rightarrow \infty} a_n = \lim_{i \rightarrow \infty} a_{n_i}.$$

Similarly, there exists a subsequence  $\{a_{n_j}\}$  of  $\{a_n\}$  such that

$$\beta = \liminf_{n \rightarrow \infty} a_n = \lim_{j \rightarrow \infty} a_{n_j}.$$

### 2.3 Fixed point set of nonexpansive mappings in Banach space

**Lemma 2.20.** ([46]) Let  $E$  be a uniformly convex Banach space,  $C_E$  a nonempty closed convex subset of  $E$ , and  $S : C_E \rightarrow C_E$  a nonexpansive mapping with  $F(S) \neq \emptyset$ . Then  $I - S$  is demiclosed at zero.

**Definition 2.7.** Let  $C_E$  be a nonempty subset of a Banach space  $E$  and  $T : C_E \rightarrow C_E$  be a self-mapping.

(i)  $T$  is called a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C_E.$$

(ii)  $T$  is called a contraction mapping if there exists a constant  $\alpha \in (0, 1)$  such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \forall x, y \in C_E.$$

A mapping  $A : C_E \rightarrow E$  is said to be *accretive* if there exists  $j(x - y) \in J(x - y)$  such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0, \forall x, y \in C_E.$$

A mapping  $A : C_E \rightarrow E$  is said to be  *$\alpha$ -inverse strongly accretive* if there exists  $\alpha > 0$  and  $j(x - y) \in J(x - y)$  such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|x - y\|^2, \forall x, y \in C_E.$$

**Definition 2.8.** Let  $C_E \subseteq E$  be closed convex and  $Q_{C_E}$  be a mapping of  $E$  onto  $C_E$ . The mapping  $Q_{C_E}$  is said to be *sunny* if  $Q_{C_E}(x + t(x - Q_{C_E}x)) = Q_{C_E}x$  for all  $x \in E$  and  $t \geq 0$ . A mapping  $Q_{C_E}$  is called *retraction* if  $Q_{C_E}^2 = Q_{C_E}$ . A subset  $C_E$  of  $E$  is called a *sunny nonexpansive retract* of  $E$  if there exists a sunny nonexpansive retraction of  $E$  onto  $C_E$ .

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**Lemma 2.21.** ([10]) Let  $C_E$  be a nonempty closed convex subset of a smooth Banach space and  $Q_{C_E}$  be a retraction from  $E$  onto  $C_E$ . Then the following are equivalent:

- (i)  $Q_{C_E}$  is both sunny and nonexpansive;
- (ii)  $\langle x - Q_{C_E}x, J(y - Q_{C_E}x) \rangle \leq 0$  for all  $x \in E$  and  $y \in C_E$ .

From Lemma 2.21, let  $x \in E$  and  $x_0 \in C_E$ . Then we have  $x_0 = Q_{C_E}x$  if and only if  $\langle x - x_0, J(y - x_0) \rangle \leq 0$ , for all  $y \in C_E$ , where  $Q_{C_E}$  is a sunny nonexpansive retraction from  $E$  onto  $C_E$ .

**Lemma 2.22.** ([22]) Let  $\{s_n\}$  be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  and  $\{\delta_n\}$  is a sequence such that

- (i)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0$  or  $\sum_{n=1}^{\infty} |\delta_n| < \infty$ .

Then  $\lim_{n \rightarrow \infty} s_n = 0$ .

**Lemma 2.23.** ([48]) Let  $C_E$  be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space  $E$  and let  $T$  be a nonexpansive mapping of  $C_E$  into itself with  $F(T) \neq \emptyset$ . Then  $F(T)$  is a sunny nonexpansive retract of  $C_E$ .

**Lemma 2.24.** ([50]) Let  $C_E$  be a closed and convex subset of a real uniformly smooth Banach space  $E$  and let  $T : C_E \rightarrow C_E$  be a nonexpansive mapping with a nonempty fixed point  $F(T)$ . If  $\{x_n\} \subset C_E$  is a bounded sequence such that  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ . Then there exists a unique sunny nonexpansive retraction  $Q_{F(T)} : C_E \rightarrow F(T)$  such that

$$\limsup_{n \rightarrow \infty} \langle u - Q_{F(T)}u, J(x_n - Q_{F(T)}u) \rangle \leq 0,$$

for any given  $u \in C_E$ .

**Lemma 2.25.** ([51]) Let  $C_E$  be a closed convex subset of a strictly convex Banach space  $E$ . Let  $T_1$  and  $T_2$  be two nonexpansive mappings from  $C_E$  into itself with  $F(T_1) \cap F(T_2) \neq \emptyset$ . Define a mapping  $S$  by

$$Sx = \lambda T_1x + (1 - \lambda)T_2x, \forall x \in C_E,$$

where  $\lambda$  is a constant in  $(0, 1)$ . Then  $S$  is nonexpansive and  $F(S) = F(T_1) \cap F(T_2)$ .

**Lemma 2.26.** ([1]) Let  $C_E$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$ . Let  $T, S : C_E \rightarrow C_E$  be nonexpansive mappings with  $F(T) \cap F(S) \neq \emptyset$ . Define the mapping  $T_a : C_E \rightarrow C_E$  by  $T_ax = T(ax + (1 - a)Sx)$  for all  $x \in C_E$  and  $a \in (0, 1)$ . Then  $F(T_a) = F(T) \cap F(S)$  and  $T_a$  is a nonexpansive mapping.

## 2.4 Fixed point set of nonexpansive mappings and resolvent operator in Hilbert space

**Definition 2.9.** Let  $C_H$  be a nonempty subset of a real Hilbert spaces  $H$  and  $T : C_H \rightarrow C_H$  be a self mapping.  $T$  is called a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C_H.$$

$T$  is called a firmly nonexpansive mapping if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \text{ for all } x, y \in C_H.$$

**Definition 2.10.** ([44]) The (nearest point) projection  $P_{C_H}$  from  $H$  onto  $C_H$  assigns to each  $x \in H$  the unique point  $P_{C_H}x \in C_H$  satisfying the following properties:

- (i)  $P_{C_H}x \in C_H$ ,
- (ii)  $\|x - P_{C_H}x\| = \min_{y \in C_H} \|x - y\|$ ,

**Lemma 2.27.** ([44]) For a given  $z \in H$  and  $u \in C_H$ ,

$$u = P_{C_H}z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \text{ for all } v \in C_H.$$

It is well-known that  $P_{C_H}$  is a firmly nonexpansive mapping of  $H$  onto  $C_H$ , that is

$$\|P_{C_H}x - P_{C_H}y\|^2 \leq \langle P_{C_H}x - P_{C_H}y, x - y \rangle, \text{ for all } x, y \in H.$$

**Lemma 2.28.** ([44]) Let  $H$  be a Hilbert space, let  $C_H$  be a nonempty closed convex subset of  $H$  and let  $A$  be a mapping of  $C_H$  into  $H$ . Let  $u \in C_H$ . Then, for  $\lambda > 0$ ,

$$u = P_{C_H}(I - \lambda A)u \Leftrightarrow u \in VI(C_H, A),$$

where  $P_{C_H}$  is the metric projection of  $H$  onto  $C_H$ .

**Definition 2.11.** ([42]) We say that an operator  $T : H \rightarrow H$  is *monotone* if

$$\langle Tx - Ty, x - y \rangle \geq \eta \|x - y\|^2, \tag{2.4}$$

for a constant  $\eta \geq 0$  and for all  $x, y \in X$ . If  $\eta > 0$ , then we say that  $T$  is strongly monotone or  $\eta$ -strongly monotone.

**Definition 2.12.** ([45]) Let  $M : H \rightarrow 2^H$  be a multi-valued operator on  $H$ . Then

(i) the graph  $G(M)$  of  $M$  is defined by

$$G(M) := \{(x, u) \in H \times H : u \in M(x)\},$$

(ii) the operator  $M$  is called a maximal monotone operator if  $M$  is monotone, i.e.

$$\langle u - v, x - y \rangle \geq 0, \text{ wherever } u \in M(x), v \in M(y),$$

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and the graph  $G(M)$  of  $M$  is not property contained in the graph of any other monotone operator.

It is clear that a monotone mapping  $M$  is maximal if and only if for any  $(x, u) \in H \times H$ , if  $\langle u - v, x - y \rangle \geq 0$  for all  $(v, y) \in M(x)$ .

**Remark 2.29.** If  $\eta = 1$  in (2.4), then  $T$  is firmly nonexpansive mapping.

**Definition 2.13.** Let  $M : H \rightarrow 2^H$  be a multi-valued maximal monotone mapping, then the single-valued mapping  $J_\lambda^M : H \rightarrow H$  defined by

$$J_\lambda^M(x) = (I + \lambda M)^{-1}(x), \forall x \in H,$$

is called the *resolvent operator* associated with  $M$ , where  $\lambda$  is any positive number and  $I$  is an identity mapping.

**Theorem 2.30.** ([44]) Let  $H$  be a Hilbert space and let  $C_H$  be a nonempty bounded closed convex subset of  $H$ . Let  $T$  be a nonexpansive mapping of  $C_H$  into itself. Then  $T$  has a fixed point in  $C_H$ .

**Lemma 2.31.** ([52]) Let  $C_H \subseteq H$  be a nonempty closed and convex set and let  $T : C_H \rightarrow H$  be a nonexpansive mapping. Then  $F(T)$  is closed and convex.

**Lemma 2.32.** ([6]) Let  $\{a_n\}, \{c_n\} \subset \mathbb{R}^+, \{\alpha_n\} \subset (0, 1)$  and  $\{b_n\} \subset \mathbb{R}$  be sequences such that

$$a_{n+1} = (1 - \alpha_n)a_n + b_n + c_n, \text{ for all } n \geq 0.$$

Assume  $\sum_{n=0}^{\infty} c_n < \infty$ . Then the following results hold:

- (i) if  $b_n \leq \alpha_n C$  where  $C \geq 0$ , then  $\{a_n\}$  is a bounded sequence,
- (ii) if  $\sum_{n=0}^{\infty} \alpha_n = \infty$  and  $\limsup_{n \rightarrow \infty} \frac{b_n}{\alpha_n} \leq 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Lemma 2.33.** ([6]) Let  $u \in H$  is a solution of variational inclusion if and only if  $u = J_\lambda^M(u - \lambda B u)$ ,  $\forall \lambda > 0$ , i.e.,

$$VI(H, B, M) = F(J_\lambda^M(I - \lambda B)), \forall \lambda > 0.$$

Further, if  $\lambda \in (0, 2\alpha]$ , then  $VI(H, B, M)$  is a closed convex subset in  $H$ .

**Lemma 2.34.** ([3]) Let  $H$  be a real Hilbert space and let  $M : H \rightarrow 2^H$  be a multi-valued maximal monotone mapping. For every  $i = 1, 2, \dots, N$ , let  $A_i : H \rightarrow H$  be  $\alpha_i$ -inverse strongly monotone mapping with  $\eta = \min_{i=1,2,\dots,N} \{\alpha_i\}$  and  $\bigcap_{i=1}^N VI(H, A_i, M) \neq \emptyset$ . Then

$$VI\left(H, \sum_{i=1}^N a_i A_i, M\right) = \bigcap_{i=1}^N VI(H, A_i, M),$$

where  $\sum_{i=1}^N a_i = 1$  and  $0 < a_i < 1$  for every  $i = 1, 2, \dots, N$ . Moreover,  $J_\lambda^M(I - \lambda \sum_{i=1}^N a_i A_i)$  is a nonexpansive mapping, for all  $0 < \lambda < 2\eta$ .

**Example 2.35.** Let  $H = \mathbb{R}$ . For every  $i = 1, 2, \dots, N$ , let  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $A_i x = \frac{ix}{4} + (i + 1)$  for all  $x \in H$  and  $M : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $Mx = \{\frac{x}{4}\}$  for all  $x \in \mathbb{R}$ . Let  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$  for all  $i = 1, 2, \dots, N$ . Then  $VI\left(H, \sum_{i=1}^N a_i A_i, M\right) = \bigcap_{i=1}^N VI(H, A_i, M)$ .

*Solution.* Since  $A_i x = \frac{ix}{4} + (i + 1)$ , we have  $A_i$  is  $\frac{4}{i}$ -inverse strongly monotone mapping. By definition of  $a_i$  and  $A_i$ , we have

$$\sum_{i=1}^N a_i A_i x = \sum_{i=1}^N \left(\frac{3}{4^i} + \frac{1}{N4^N}\right) A_i x = \sum_{i=1}^N \left(\frac{3}{4^i} + \frac{1}{N4^N}\right) \left(\frac{ix}{4} + (i + 1)\right).$$

From Lemma 2.34, we have  $VI(H, \sum_{i=1}^N a_i A_i, M) = \bigcap_{i=1}^N VI(H, A_i, M) = \{-4\}$ .  $\square$

**Example 2.36.** Let  $H = \mathbb{R}$ . For every  $i = 1, 2, \dots, N$ , let  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $A_i x = \frac{ix}{4} + (i + 1)$  for all  $x \in H$  and  $M : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $Mx = \{\frac{x}{4}\}$  for all  $x \in \mathbb{R}$ . Let  $a_i = \frac{3}{4^i} + \frac{1}{N}(\frac{1}{4^N} + 1)$  for all  $i = 1, 2, \dots, N$ . Then  $VI\left(H, \sum_{i=1}^N a_i A_i, M\right) \neq \bigcap_{i=1}^N VI(H, A_i, M)$ .

*Solution.* Since  $A_i x = \frac{ix}{4} + (i + 1)$ , we have  $A_i$  is  $\frac{4}{i}$ -inverse strongly monotone mapping. By definition of  $a_i$  and  $A_i$ , we have

$$\sum_{i=1}^N a_i A_i x = \sum_{i=1}^N \left(\frac{3}{4^i} + \frac{1}{N}(\frac{1}{4^N} + 1)\right) A_i x = \sum_{i=1}^N \left(\frac{3}{4^i} + \frac{1}{N}(\frac{1}{4^N} + 1)\right) \left(\frac{ix}{4} + (i + 1)\right).$$

Then  $\bigcap_{i=1}^N VI(H, A_i, M) = \{-4\}$  and  $VI(H, \sum_{i=1}^N a_i A_i, M) \neq \{-4\}$ . It implies that  $VI(H, \sum_{i=1}^N a_i A_i, M) \neq \bigcap_{i=1}^N VI(H, A_i, M)$  because  $\sum_{i=1}^N a_i = 2$ .  $\square$

**Remark 2.37.** Example 2.35 show that Lemma 2.34 is true where  $\sum_{i=1}^N a_i = 1$  and Example 2.36 shows that Lemma 2.34 is not true if some condition fails, that is  $\sum_{i=1}^N a_i \neq 1$ .

Let  $h$  be a proper lower semicontinuous convex function of  $H$  into  $(-\infty, +\infty]$ . The subdifferential  $\partial h$  is defined by

$$\partial h(x) = \{z \in H : h(x) + \langle z, u - x \rangle \leq h(u), \forall u \in H\},$$

for all  $x \in H$ . For Rochafellar [39], we get that  $\partial h$  is a maximal monotone operator.

Let  $i_{C_H}$  be the indicator function of  $C_H$ , i.e.,

$$i_{C_H} = \begin{cases} 0; & \text{if } x \in C_H, \\ +\infty; & \text{if } x \notin C_H. \end{cases} \quad (2.5)$$

Then  $i_{C_H}$  is a proper, lower semicontinuous and convex function on  $H$  and so the subdifferential  $\partial i_{C_H}$  of  $i_{C_H}$  is a maximal monotone operator. The resolvent operator  $J_r^{\partial i_{C_H}}$  of  $\partial i_{C_H}$  for  $\lambda > 0$  defined by  $J_r^{\partial i_{C_H}}(x) = (I + \lambda \partial i_{C_H})^{-1}(x)$ ,  $x \in H$ , then we have  $J_r^{\partial i_{C_H}}(x) = P_{C_H} x$  for all  $x \in H$  and  $\lambda > 0$ ; see more detail [47]. Moreover, let  $h : H \rightarrow H$  is the single valued operator, we have  $x \in VI(H, h, \partial i_{C_H}) = VI(C_H, h)$ .

## Chapter 3

### Strong Convergence Theorems

#### 3.1 A Strong convergence theorem in Banach space to approximate a common solution of Banach generalized system of variational inequalities problem

In this section, we prove the lemma in Banach space which show the equivalence between Banach generalized system of variational inequalities problem (1.11) and some fixed point problem involving the nonexpansive mapping under suitable conditions on our parameters. Utilizing our lemma we prove the strong convergence theorem in Banach space of the sequences generated by modified viscosity method for finding a common solution of Banach generalized system of variational inequalities problem (1.11) and fixed point of nonexpansive mapping.

The following lemma is needed to prove the main theorem.

**Lemma 3.1.** Let  $C_E$  be a nonempty closed convex subset of a smooth Banach space  $E$  and let  $D_1, D_2, D_3 : C_E \rightarrow E$  be mappings. Let  $Q_{C_E}$  be a sunny nonexpansive retraction of  $E$  onto  $C_E$ . For every  $\lambda_1, \lambda_2, \lambda_3 > 0$  and  $a \in [0, 1]$ . The following are equivalent  
(i)  $(x^*, y^*, z^*)$  is a solution of (1.11),  
(ii)  $x^*$  is a fixed point of the mapping  $Z$ , i.e.  $x^* \in F(Z)$ , defined the mapping  $Z : C_E \rightarrow C_E$  by  $Z(x) = Q_{C_E}(I - \lambda_1 D_1)(ax + (1-a)Q_{C_E}(I - \lambda_2 D_2)(ax + (1-a)Q_{C_E}(I - \lambda_3 D_3)x))$ ,  $\forall x \in C_E$  where  $y^* = Q_{C_E}(I - \lambda_2 D_2)(ax^* + (1-a)z^*)$  and  $z^* = Q_{C_E}(I - \lambda_3 D_3)x^*$ .

**Proof.** Let conditions hold.

(i)  $\Rightarrow$  (ii). Suppose that  $(x^*, y^*, z^*) \in C_E \times C_E \times C_E$  is a solution of (1.11). For every  $x \in C_E$ , we have

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), j(x - x^*) \rangle \geq 0, \forall x \in C_E, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), j(x - y^*) \rangle \geq 0, \forall x \in C_E, \\ \langle z^* - (I - \lambda_3 D_3)x^*, j(x - z^*) \rangle \geq 0, \forall x \in C_E. \end{cases}$$

From Lemma 2.21, we have

$$\begin{aligned} x^* &= Q_{C_E}(I - \lambda_1 D_1)(ax^* + (1-a)y^*), \\ y^* &= Q_{C_E}(I - \lambda_2 D_2)(ax^* + (1-a)z^*), \\ z^* &= Q_{C_E}(I - \lambda_3 D_3)x^*. \end{aligned}$$

It follows that

$$x^* = Q_{C_E}(I - \lambda_1 D_1)(ax + (1-a)Q_{C_E}(I - \lambda_2 D_2)(ax + (1-a)Q_{C_E}(I - \lambda_3 D_3)x)) = Zx^*.$$

Then  $x^* \in F(Z)$ , where  $y^* = Q_{C_E}(I - \lambda_2 D_2)(ax^* + (1 - a)z^*)$  and  $z^* = Q_{C_E}(I - \lambda_3 D_3)x^*$ .

(ii)  $\Rightarrow$  (i). Let  $x^* \in F(Z)$ ,  $y^* = Q_{C_E}(I - \lambda_2 D_2)(ax^* + (1 - a)z^*)$  and  $z^* = Q_{C_E}(I - \lambda_3 D_3)x^*$ . Then

$$\begin{aligned} x^* &= Zx^* \\ &= Q_{C_E}(I - \lambda_1 D_1)(ax^* + (1 - a)Q_{C_E}(I - \lambda_2 D_2)(ax^* + (1 - a)Q_{C_E}(I - \lambda_3 D_3)x^*)) \\ &= Q_{C_E}(I - \lambda_1 D_1)(ax^* + (1 - a)y^*). \end{aligned}$$

From Lemma 2.21, we have

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1 - a)y^*), j(x - x^*) \rangle \geq 0, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1 - a)z^*), j(x - y^*) \rangle \geq 0, \\ \langle z^* - (I - \lambda_3 D_3)x^*, j(x - z^*) \rangle \geq 0. \end{cases}$$

for all  $x \in C_E$ . Then we find that  $(x^*, y^*, z^*)$  is solution of (1.11).  $\square$

We prove a strong convergence theorem in Banach space to approximate a common solution of Banach generalized system of variational inequalities problem (1.11) and fixed point problem of nonexpansive mapping.

**Theorem 3.2.** Let  $C_E$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  and let  $Q_{C_E}$  be a sunny nonexpansive retraction of  $C_E$ . Let  $D_1, D_2, D_3 : C_E \rightarrow E$  be  $d_1, d_2, d_3$ -inverse strongly accretive operators, respectively. Define the mapping  $Z : C_E \rightarrow C_E$  by  $Z(x) = Q_{C_E}(I - \lambda_1 D_1)(ax + (1 - a)Q_{C_E}(I - \lambda_2 D_2)(ax + (1 - a)Q_{C_E}(I - \lambda_3 D_3)x))$  for all  $x \in C_E$ ,  $\lambda_1 \in (0, \frac{d_1}{K^2})$ ,  $\lambda_2 \in (0, \frac{d_2}{K^2})$ ,  $\lambda_3 \in (0, \frac{d_3}{K^2})$ ,  $a \in [0, 1]$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . Let  $T, S : C_E \rightarrow C_E$  be nonexpansive mappings. Assume that  $\mathbb{F} = F(Z) \cap F(S) \cap F(T) \neq \emptyset$ . Let the sequence  $\{x_n\}$  be generated by  $x_1 \in C_E$  and

$$\begin{cases} u_n = T(\delta_n x_n + (1 - \delta_n)Sx_n), \\ x_{n+1} = Z(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n), \end{cases} \quad (3.1)$$

where  $f : C_E \rightarrow C_E$  be a contraction mapping with a constant  $\xi \in (0, 1)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$  with  $\alpha_n + \beta_n + \gamma_n = 1$ . Suppose that the following condition are satisfied

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$  for all  $c, d > 0$  and for all  $n \geq 1$ ,
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ ,
- (iv)  $\lim_{n \rightarrow \infty} \delta_n = \delta$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}}f(x_0)$  and  $(x_0, y_0, z_0)$  is a solution of (1.11), where  $y_0 = Q_{C_E}(I - \lambda_2 D_2)(ax_0 + (1 - a)z_0)$  and  $z_0 = Q_{C_E}(I - \lambda_3 D_3)x_0$ .

**Proof.** Step1. First, we prove that  $\{x_n\}$  and  $\{u_n\}$  are bounded.

We show that  $Q_{C_E}(I - \lambda_1 D_1)$ ,  $Q_{C_E}(I - \lambda_2 D_2)$  and  $Q_{C_E}(I - \lambda_3 D_3)$  are nonexpansive mappings. Let  $x, y \in C_E$ , we have

$$\begin{aligned}
& \|Q_{C_E}(I - \lambda_1 D_1)x - Q_{C_E}(I - \lambda_1 D_1)y\|^2 \\
& \leq \|x - y - \lambda_1 D_1 x + \lambda_1 D_1 y\|^2 \\
& \leq \|x - y\|^2 - 2\lambda_1 \langle D_1 x - D_1 y, j(x - y) \rangle + 2K^2 \lambda_1^2 \|D_1 x - D_1 y\|^2 \\
& \leq \|x - y\|^2 - 2\lambda_1 d_1 \|D_1 x - D_1 y\|^2 + 2K^2 \lambda_1^2 \|D_1 x - D_1 y\|^2 \\
& = \|x - y\|^2 - 2\lambda_1 (d_1 - K^2 \lambda_1) \|D_1 x - D_1 y\|^2 \\
& \leq \|x - y\|^2.
\end{aligned}$$

Then  $Q_{C_E}(I - \lambda_1 D_1)$  is a nonexpansive mapping. By using the same method, we obtain that  $Q_{C_E}(I - \lambda_2 D_2)$  and  $Q_{C_E}(I - \lambda_3 D_3)$  are nonexpansive mappings. From the definition of  $Z$ , we see that  $Z$  is nonexpansive mapping.

Let  $x^* \in \mathbb{F}$ , then  $x^* \in F(S)$  and  $x^* \in F(T)$  which implies  $Sx^* = x^*$  and  $Tx^* = x^*$ . From the definition of  $\{u_n\}$  and Lemma 2.26, we have

$$\begin{aligned}
\|u_n - x^*\| &= \|T(\delta_n x_n + (1 - \delta_n)Sx_n) - x^*\| \\
&\leq \|\delta_n x_n + (1 - \delta_n)Sx_n - x^*\| \\
&= \|\delta_n(x_n - x^*) + (1 - \delta_n)(Sx_n - x^*)\| \\
&\leq \delta_n \|x_n - x^*\| + (1 - \delta_n) \|Sx_n - x^*\| \\
&\leq \delta_n \|x_n - x^*\| + (1 - \delta_n) \|x_n - x^*\| \\
&= \|x_n - x^*\|.
\end{aligned} \tag{3.2}$$

Since  $x^* \in \mathbb{F}$ , we have  $x^* \in F(Z)$  which implies  $Zx^* = x^*$ . Put  $y_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n$ . From the definition of  $\{x_n\}$ , (3.2) and Lemma 3.1, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|Zy_n - x^*\| \\
&\leq \|y_n - x^*\| \\
&= \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(u_n - x^*)\| \\
&\leq \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|u_n - x^*\| \\
&\leq \alpha_n (\|f(x_n) - f(x^*)\| + \|f(x^*) - x^*\|) + \beta_n \|x_n - x^*\| \\
&\quad + \gamma_n \|x_n - x^*\| \\
&\leq \alpha_n \xi \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n) \|x_n - x^*\| \\
&= \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n(1 - \xi)) \|x_n - x^*\| \\
&\leq \max \left\{ \frac{\|f(x^*) - x^*\|}{1 - \xi}, \|x_1 - x^*\| \right\}.
\end{aligned} \tag{3.3}$$

Applying mathematical induction, we can conclude that the sequence  $\{x_n\}$  is bounded and so is  $\{u_n\}$ .

Step 2. Show that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0, \lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$  and  $\lim_{n \rightarrow \infty} \|Zy_n - y_n\| = 0$ .

From the definition of  $\{x_n\}$ , we have

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
&= \|Z(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n) - Z(\alpha_{n-1} f(x_{n-1}) \\
&\quad + \beta_{n-1} x_{n-1} + \gamma_{n-1} u_{n-1})\| \\
&\leq \|(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n) - (\alpha_{n-1} f(x_{n-1}) + \beta_{n-1} x_{n-1} + \gamma_{n-1} u_{n-1})\| \\
&= \|\alpha_n f(x_n) - \alpha_{n-1} f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_n f(x_{n-1}) \\
&\quad + \beta_n x_n - \beta_{n-1} x_{n-1} + \beta_n x_{n-1} - \beta_n x_{n-1} \\
&\quad + \gamma_n u_n - \gamma_{n-1} u_{n-1} + \gamma_n u_{n-1} - \gamma_n u_{n-1}\| \\
&\leq \alpha_n \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \gamma_n \|u_n - u_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|u_{n-1}\| \\
&\leq \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \gamma_n \|u_n - u_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|u_{n-1}\|.
\end{aligned} \tag{3.4}$$

By the definition of  $\{u_n\}$ , we have

$$\begin{aligned}
& \|u_n - u_{n-1}\| \\
&= \|T(\delta_n x_n + (1 - \delta_n) Sx_n) - T(\delta_{n-1} x_{n-1} + (1 - \delta_{n-1}) Sx_{n-1})\| \\
&\leq \|(\delta_n x_n + (1 - \delta_n) Sx_n) - (\delta_{n-1} x_{n-1} + (1 - \delta_{n-1}) Sx_{n-1})\| \\
&= \|\delta_n x_n - \delta_{n-1} x_{n-1} + \delta_n x_{n-1} - \delta_n x_{n-1} + (Sx_n - Sx_{n-1}) \\
&\quad - \delta_n Sx_n + \delta_{n-1} Sx_{n-1} + \delta_n Sx_{n-1} - \delta_n Sx_{n-1}\| \\
&\leq \delta_n \|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| \|x_{n-1}\| + (1 - \delta_n) \|Sx_n - Sx_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}| \|Sx_{n-1}\| \\
&\leq \delta_n \|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| \|x_{n-1}\| + (1 - \delta_n) \|x_n - x_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}| \|Sx_{n-1}\| \\
&= \|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| (\|x_{n-1}\| + \|Sx_{n-1}\|).
\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we have

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
&\leq \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \gamma_n \|u_n - u_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|u_{n-1}\| \\
&\leq \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|u_{n-1}\| \\
&\quad + \gamma_n \left[ \|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| (\|x_{n-1}\| + \|Sx_{n-1}\|) \right] \\
&= \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\|
\end{aligned}$$

$$\begin{aligned}
& + (1 - \alpha_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| \\
& + \gamma_n|\delta_n - \delta_{n-1}|(\|x_{n-1}\| + \|Sx_{n-1}\|) + |\gamma_n - \gamma_{n-1}|\|u_{n-1}\| \\
= & (1 - \alpha_n(1 - \xi))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
& + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n|\delta_n - \delta_{n-1}|(\|x_{n-1}\| + \|Sx_{n-1}\|) \\
& + |\gamma_n - \gamma_{n-1}|\|u_{n-1}\|.
\end{aligned} \tag{3.6}$$

Applying (3.6), condition (i), (iii) and Lemma 2.22, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

We will show that  $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$ . From the definition of  $\{x_n\}$  and (3.2), we have

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
& = \|Z(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n) - x^*\|^2 \\
& \leq \|(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n) - x^*\|^2 \\
& = \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(u_n - x^*)\|^2 \\
& \leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|u_n - x^*\|^2 - \beta_n \gamma_n g(\|x_n - u_n\|) \\
& \leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 - \beta_n \gamma_n g(\|x_n - u_n\|) \\
& \leq \alpha_n \|f(x_n) - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 - \beta_n \gamma_n g(\|x_n - u_n\|) \\
& \leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \beta_n \gamma_n g(\|x_n - u_n\|).
\end{aligned}$$

It follows that

$$\begin{aligned}
\beta_n \gamma_n g(\|x_n - u_n\|) & \leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
& \leq \alpha_n \|f(x_n) - x^*\|^2 \\
& \quad + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|.
\end{aligned}$$

From (3.7) and condition (i) and (ii), we have

$$\lim_{n \rightarrow \infty} g(\|x_n - u_n\|) = 0.$$

From the property of  $g$ , we have

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \tag{3.8}$$

From the definition of  $\{y_n\}$ , we have

$$\begin{aligned}
\|y_n - x_n\| & = \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n - x_n\| \\
& = \|\alpha_n(f(x_n) - x_n) + \gamma_n(u_n - x_n)\| \\
& \leq \alpha_n \|f(x_n) - x_n\| + \gamma_n \|u_n - x_n\|.
\end{aligned}$$

From (3.8) and a condition (i), we have

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \tag{3.9}$$

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From the definition of  $\{y_n\}$ , we have

$$\begin{aligned}\|y_n - u_n\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n - u_n\| \\ &= \|\alpha_n(f(x_n) - u_n) + \beta_n(x_n - u_n)\| \\ &\leq \alpha_n \|f(x_n) - u_n\| + \beta_n \|x_n - u_n\|.\end{aligned}$$

From (3.8) and a condition (i), we have

$$\lim_{n \rightarrow \infty} \|y_n - u_n\| = 0. \quad (3.10)$$

Since  $\lim_{n \rightarrow \infty} \delta_n = \delta \in (0, 1)$ , we define the mapping  $T_\delta : C_E \rightarrow C_E$  by  $T_\delta x = T(\delta x + (1 - \delta)Sx)$  for all  $x \in C_E$  and  $\delta \in (0, 1)$ , then we have  $u_n = T_{\delta_n} x_n$ . By Lemma 2.26, we have  $F(T_\delta) = F(T) \cap F(S)$  and  $T_\delta$  is a nonexpansive mapping.

From the nonexpansive of  $T_\delta$ , we have

$$\begin{aligned}\|T_\delta y_n - y_n\| &= \|T_\delta y_n - T_{\delta_n} y_n + T_{\delta_n} y_n - y_n\| \\ &\leq \|T_\delta y_n - T_{\delta_n} y_n\| + \|T_{\delta_n} y_n - y_n\|.\end{aligned} \quad (3.11)$$

Considers,

$$\begin{aligned}\|T_\delta y_n - T_{\delta_n} y_n\| &= \|T(\delta y_n + (1 - \delta)S y_n) - T(\delta_n y_n + (1 - \delta_n)S y_n)\| \\ &\leq \|(\delta y_n + (1 - \delta)S y_n) - (\delta_n y_n + (1 - \delta_n)S y_n)\| \\ &= \|(\delta - \delta_n)y_n + (\delta_n - \delta)S y_n\| \\ &\leq |\delta - \delta_n| \|y_n\| + |\delta_n - \delta| \|S y_n\|.\end{aligned}$$

From a condition (iv), we have

$$\lim_{n \rightarrow \infty} \|T_\delta y_n - T_{\delta_n} y_n\| = 0. \quad (3.12)$$

Since the nonexpansiveness of  $T_{\delta_n}$ , we have

$$\begin{aligned}\|T_{\delta_n} y_n - y_n\| &= \|T_{\delta_n} y_n - T_{\delta_n} x_n + T_{\delta_n} x_n - y_n\| \\ &\leq \|T_{\delta_n} y_n - T_{\delta_n} x_n\| + \|T_{\delta_n} x_n - y_n\| \\ &\leq \|y_n - x_n\| + \|u_n - y_n\|.\end{aligned}$$

From (3.9) and (3.10), we have

$$\lim_{n \rightarrow \infty} \|T_{\delta_n} y_n - y_n\| = 0. \quad (3.13)$$

From (3.11), (3.12) and (3.13), we have

$$\lim_{n \rightarrow \infty} \|T_\delta y_n - y_n\| = 0. \quad (3.14)$$

From the definition of  $\{x_n\}$ , we have

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$$\begin{aligned}
&\leq \|Zy_n - x_n\| + \|x_n - y_n\| \\
&= \|x_{n+1} - x_n\| + \|x_n - y_n\|.
\end{aligned}$$

From (3.7) and (3.9), we have

$$\lim_{n \rightarrow \infty} \|Zy_n - y_n\| = 0. \quad (3.15)$$

Step 3. We will show that  $\limsup_{n \rightarrow \infty} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \leq 0$  where  $x_0 = Q_{\mathbb{F}}f(x_0)$ .

Define the mapping  $B : C_E \rightarrow C_E$  by  $Bx = \varepsilon Zx + (1 - \varepsilon)T_\delta x$  for all  $x \in C_E$  and  $\varepsilon \in (0, 1)$ . From Lemma 2.25 and Lemma 2.26, we have  $F(B) = F(Z) \cap F(T_\delta) = F(Z) \cap F(T) \cap F(S) = \mathbb{F}$ .

From the definition of  $B$ , we have

$$\begin{aligned}
\|y_n - By_n\| &= \|y_n - (\varepsilon Zy_n + (1 - \varepsilon)T_\delta y_n)\| \\
&= \|\varepsilon(y_n - Zy_n) + (1 - \varepsilon)(y_n - T_\delta y_n)\| \\
&\leq \varepsilon\|y_n - Zy_n\| + (1 - \varepsilon)\|y_n - T_\delta y_n\|
\end{aligned}$$

From (3.14) and (3.15), we have

$$\lim_{n \rightarrow \infty} \|y_n - By_n\| = 0. \quad (3.16)$$

Since  $Z$  and  $T_\delta$  are nonexpansive mappings, we have  $B$  is a nonexpansive mapping. From Lemma 2.24, we have

$$\limsup_{n \rightarrow \infty} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \leq 0, \quad (3.17)$$

where  $x_0 = Q_{\mathbb{F}}f(x_0)$ .

Step4. We show that the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}}f(x_0)$ .

From the definition of  $\{y_n\}$ , we have

$$\begin{aligned}
&\|y_n - x_0\|^2 \\
&= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n - x_0\|^2 \\
&= \|\alpha_n (f(x_n) - x_0) + \beta_n (x_n - x_0) + \gamma_n (u_n - x_0)\|^2 \\
&\leq \|\beta_n (x_n - x_0) + \gamma_n (u_n - x_0)\|^2 + 2\alpha_n \langle f(x_n) - x_0, j(y_n - x_0) \rangle \\
&\leq \left( \beta_n \|x_n - x_0\| + \gamma_n \|u_n - x_0\| \right)^2 \\
&\quad + 2\alpha_n \langle f(x_n) - f(x_0) + f(x_0) - x_0, j(y_n - x_0) \rangle \\
&\leq \left( \beta_n \|x_n - x_0\| + \gamma_n \|x_n - x_0\| \right)^2 + 2\alpha_n \langle f(x_n) - f(x_0), j(y_n - x_0) \rangle \\
&\quad + 2\alpha_n \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
&\leq (1 - \alpha_n)^2 \|x_n - x_0\|^2 + 2\alpha_n \|f(x_n) - f(x_0)\| \|y_n - x_0\| \\
&\quad + 2\alpha_n \langle f(x_0) - x_0, j(y_n - x_0) \rangle
\end{aligned}$$

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$$\begin{aligned}
&\leq (1 - \alpha_n)^2 \|x_n - x_0\|^2 + 2\alpha_n \xi \|x_n - x_0\| \|y_n - x_0\| \\
&\quad + 2\alpha_n \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
&\leq (1 - \alpha_n)^2 \|x_n - x_0\|^2 + 2\alpha_n \xi \left[ \|x_n - x_0\|^2 + \|y_n - x_0\|^2 \right] \\
&\quad + 2\alpha_n \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
&= ((1 - \alpha_n)^2 + \alpha_n \xi) \|x_n - x_0\|^2 + \alpha_n \xi \|y_n - x_0\|^2 \\
&\quad + 2\alpha_n \langle f(x_0) - x_0, j(y_n - x_0) \rangle.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\|y_n - x_0\|^2 \\
&\leq \frac{(1 - \alpha_n)^2 + \alpha_n \xi}{(1 - \alpha_n \xi)} \|x_n - x_0\|^2 + \frac{2\alpha_n}{(1 - \alpha_n \xi)} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
&= \frac{1 - \alpha_n \xi - 2\alpha_n(1 - \xi)}{(1 - \alpha_n \xi)} \|x_n - x_0\|^2 + \frac{\alpha_n^2}{(1 - \alpha_n \xi)} \|x_n - x_0\|^2 \\
&\quad + \frac{2\alpha_n}{(1 - \alpha_n \xi)} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
&= \left(1 - \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi}\right) \|x_n - x_0\|^2 + \frac{\alpha_n^2}{1 - \alpha_n \xi} \|x_n - x_0\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n \xi} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
&= \left(1 - \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi}\right) \|x_n - x_0\|^2 \\
&\quad + \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi} \left( \frac{\alpha_n}{2(1 - \xi)} \|x_n - x_0\|^2 + \frac{1}{1 - \xi} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \right). \tag{3.18}
\end{aligned}$$

By the definition of  $\{x_{n+1}\}$  and (3.18), we have

$$\begin{aligned}
&\|x_{n+1} - x_0\|^2 \\
&= \|Zy_n - x_0\|^2 \\
&\leq \|y_n - x_0\|^2 \\
&\leq \left(1 - \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi}\right) \|x_n - x_0\|^2 + \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi} \left( \frac{\alpha_n}{2(1 - \xi)} \|x_n - x_0\|^2 \right. \\
&\quad \left. + \frac{1}{1 - \xi} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \right). \tag{3.19}
\end{aligned}$$

Applying (3.19), a condition (i) and Lemma 2.22, we can conclude that the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}}f(x_0)$  and  $(x_0, y_0, z_0)$  is a solution of (1.11), where  $y_0 = Q_{C_E}(I - \lambda_2 D_2)(ax_0 + (1 - a)z_0)$  and  $z_0 = Q_{C_E}(I - \lambda_3 D_3)x_0$ . This complete the proof.  $\square$

The following corollary is consequences which are applied by Theorem 3.2. Therefore, we obtain the following results.

**Corollary 3.3.** Let  $C_E$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  and let  $Q_{C_E}$  be a sunny nonexpansive retraction of  $C_E$ . Let  $D_1, D_2, D_3 : C_E \rightarrow E$  be  $d_1, d_2, d_3$ -inverse strongly accretive operators, respectively. Define the mapping  $Z : C_E \rightarrow C_E$  by  $Z(x) = Q_{C_E}(I - \lambda_1 D_1)(Q_{C_E}(I - \lambda_2 D_2)(Q_{C_E}(I -$

$\lambda_3 D_3)x)$  for all  $x \in C_E$ ,  $\lambda_1 \in (0, \frac{d_1}{K^2})$ ,  $\lambda_2 \in (0, \frac{d_2}{K^2})$ ,  $\lambda_3 \in (0, \frac{d_3}{K^2})$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . Let  $T, S : C_E \rightarrow C_E$  be nonexpansive mappings. Assume that  $\mathbb{F} = F(Z) \cap F(S) \cap F(T) \neq \emptyset$ . Let the sequence  $\{x_n\}$  be generated by  $x_1 \in C_E$  and

$$\begin{cases} u_n = T(\delta_n x_n + (1 - \delta_n)Sx_n), \\ x_{n+1} = Z(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n), \end{cases} \quad (3.20)$$

where  $f : C_E \rightarrow C_E$  be a contraction mapping with a constant  $\xi \in (0, 1)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$  with  $\alpha_n + \beta_n + \gamma_n = 1$ . Suppose that the following condition are satisfied

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$  for all  $c, d > 0$  and for all  $n \geq 1$ ,
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ ,
- (iv)  $\lim_{n \rightarrow \infty} \delta_n = \delta$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}}f(x_0)$  and  $(x_0, y_0, z_0)$  is a solution of (1.11), where  $y_0 = Q_{C_E}(I - \lambda_2 D_2)z_0$  and  $z_0 = Q_{C_E}(I - \lambda_3 D_3)x_0$ .

### 3.2 Application of strong convergence theorem in Banach space to approximate a common solution of Banach generalized system of variational inequalities problem

In this section, we prove strong convergence to approximation a common element of solutions of generalized variational inequalities in Banach space and fixed point problem of nonexpansive mapping. We give some useful lemmas to prove Theorem 3.6.

**Lemma 3.4.** ([11]) Let  $C_E$  be a nonempty closed convex subset of a smooth Banach space  $E$ . Let  $Q_{C_E}$  be a sunny nonexpansive retraction from  $E$  onto  $C_E$  and let  $A$  be an accretive operator of  $C_E$  into  $E$ . Then, for all  $\lambda > 0$ ,

$$S(C_E, A) = F(Q_{C_E}(I - \lambda A)).$$

**Lemma 3.5.** Let  $C_E$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach  $E$  and let  $Q_{C_E}$  be a sunny nonexpansive retraction from  $E$  onto  $C_E$ . Let  $D_1, D_2, D_3 : C_E \rightarrow E$  be  $d_1, d_2, d_3$ -inverse strongly accretive operators, receptively. Define a mapping  $Z$  as in Lemma 3.1 and for every  $\lambda_1 \in (0, \frac{d_1}{K^2})$ ,  $\lambda_2 \in (0, \frac{d_2}{K^2})$ ,  $\lambda_3 \in (0, \frac{d_3}{K^2})$ ,  $a \in [0, 1]$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . If  $S(C_E, A) \cap S(C_E, B) \neq \emptyset$ , then  $F(Z) = S(C_E, A) \cap S(C_E, B)$ .

**Proof.** From Lemma 3.4, we have

$$S(C_E, D_1) = F(Q_{C_E}(I - \lambda_1 D_1)), S(C_E, D_2) = F(Q_{C_E}(I - \lambda_2 D_2)),$$

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and

$$S(C_E, D_3) = F(Q_{C_E}(I - \lambda_3 D_3)).$$

Using the same method as Theorem 3.2, we find that  $Q_{C_E}(I - \lambda_1 D_1)$ ,  $Q_{C_E}(I - \lambda_2 D_2)$  and  $Q_{C_E}(I - \lambda_3 D_3)$  are nonexpansive mappings.

From the definition of  $Z$  and Lemma 2.26, we have

$$\begin{aligned} F(Z) &= F(Q_{C_E}(I - \lambda_1 D_1)) \cap F(Q_{C_E}(I - \lambda_2 D_2)) \cap F(Q_{C_E}(I - \lambda_3 D_3)) \\ &= S(C_E, D_1) \cap S(C_E, D_2) \cap S(C_E, D_3). \end{aligned}$$

□

From Theorem 3.2 and Lemma 3.5, we have the following theorem.

**Theorem 3.6.** Let  $C_E$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  and let  $Q_{C_E}$  be a sunny nonexpansive retraction of  $C_E$ . Let  $D_1, D_2, D_3 : C_E \rightarrow E$  be  $d_1, d_2, d_3$ -inverse strongly accretive operators, respectively. Define the mapping  $Z : C_E \rightarrow C_E$  by  $Z(x) = Q_{C_E}(I - \lambda_1 D_1)(ax + (1 - a)Q_{C_E}(I - \lambda_2 D_2)(ax + (1 - a)Q_{C_E}(I - \lambda_3 D_3)x))$  for all  $x \in C_E$ ,  $\lambda_1 \in (0, \frac{d_1}{K^2})$ ,  $\lambda_2 \in (0, \frac{d_2}{K^2})$ ,  $\lambda_3 \in (0, \frac{d_3}{K^2})$ ,  $a \in [0, 1]$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . Let  $T, S : C_E \rightarrow C_E$  be nonexpansive mappings. Assume that  $\mathbb{F} = S(C_E, D_1) \cap S(C_E, D_2) \cap S(C_E, D_3) \cap F(S) \cap F(T) \neq \emptyset$ . Let the sequence  $\{x_n\}$  be generated by  $x_1 \in C_E$  and

$$\begin{cases} u_n = T(\delta_n x_n + (1 - \delta_n) S x_n), \\ y_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n, \\ x_{n+1} = Q_{C_E}(I - \lambda_1 D_1)(aI + (1 - a)Q_{C_E}(I - \lambda_2 D_2)(aI \\ + (1 - a)Q_{C_E}(I - \lambda_3 D_3)))y_n, \forall n \geq 1, \end{cases} \quad (3.21)$$

where  $f : C_E \rightarrow C_E$  be a contraction mapping with a constant  $\xi \in (0, 1)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$  with  $\alpha_n + \beta_n + \gamma_n = 1$ . Suppose that the following condition are satisfied

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$  for all  $c, d > 0$  and for all  $n \geq 1$ ,
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$   
 $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ ,
- (iv)  $\lim_{n \rightarrow \infty} \delta_n = \delta$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}} f(x_0)$  and  $(x_0, y_0, z_0)$  is a solution of (1.11), where  $y_0 = Q_{C_E}(I - \lambda_2 D_2)(ax_0 + (1 - a)z_0)$  and  $z_0 = Q_{C_E}(I - \lambda_3 D_3)x_0$ .

### 3.3 A Strong convergence theorem in Hilbert space to approximate a common solution of SCVIP and hierarchical fixed point problem of nonexpansive mapping

In this section, we prove lemmas related to SCVIP and fixed point problem of nonexpansive mapping under suitable conditions on our parameters. Utilizing our lemmas we prove the strong convergence theorem in Hilbert space of the sequences generated by the modified Halpern iterative method for finding a common solution of hierarchical fixed point problem for a nonexpansive mapping and SCVIP.

The following lemmas are needed to prove the main theorem.

**Lemma 3.7.** Let  $H_1$  and  $H_2$  be Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator. For every  $i = 1, 2, \dots, N$ , let  $A_i : H_1 \rightarrow H_1$  be  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1,2,\dots,N}\{\alpha_i\}$  and  $B_i : H_2 \rightarrow H_2$  be  $\beta_i$ -inverse strongly monotone with  $\eta_B = \min_{i=1,2,\dots,N}\{\beta_i\}$ . For each  $x, y \in H_1$ , then

$$\begin{aligned} & \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax) \\ & - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(y - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay)\|^2 \\ & \leq \|x - y\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2, \end{aligned}$$

where  $\lambda_A \in (0, 2\eta_A)$ ,  $\lambda_B \in (0, 2\eta_B)$ ,  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ .

**Proof.** Let  $x, y \in H_1$ . Consider

$$\begin{aligned} & \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax) \\ & - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(y - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay)\|^2 \\ & \leq \|(x - y) - \gamma(A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay)\|^2 \\ & = \|x - y\|^2 - 2\gamma\langle x - y, A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay \rangle \\ & \quad + \gamma^2 \|A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \\ & \leq \|x - y\|^2 + 2\gamma\langle Ay - Ax, (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay \rangle \\ & \quad + \gamma^2 L\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \end{aligned}$$

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$$\begin{aligned}
&= \|x - y\|^2 + 2\gamma \langle Ay - Ax + J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax \\
&\quad + J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ay - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ay, (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax \\
&\quad - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay \rangle \\
&\quad + \gamma^2 L \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \\
&= \|x - y\|^2 + 2\gamma \left[ \langle J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ay - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax, \right. \\
&\quad (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay \rangle \\
&\quad - \langle (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay, \\
&\quad (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay \rangle \Big] \\
&\quad + \gamma^2 L \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \\
&\leq \|x - y\|^2 + 2\gamma \left[ \frac{1}{2} \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \right. \\
&\quad \left. - \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \right] \\
&\quad + \gamma^2 L \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \\
&= \|x - y\|^2 - \gamma(1 - \gamma L) \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax) \\
&\quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(y - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay)\|^2 \\
&\leq \|x - y\|^2 - \gamma(1 - \gamma L) \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - \\
&\quad (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2.
\end{aligned}$$

□

**Lemma 3.8.** Let  $H_1$  and  $H_2$  be Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator. For every  $i = 1, 2, \dots, N$ ,

let  $A_i : H_1 \rightarrow H_1$  be  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1,2,\dots,N}\{\alpha_i\}$  and  $B_i : H_2 \rightarrow H_2$  be  $\beta_i$ -inverse strongly monotone with  $\eta_B = \min_{i=1,2,\dots,N}\{\beta_i\}$ . Suppose that  $\Omega \neq \emptyset$ . Then the following are equivalent:

(i)  $x^* \in \Omega$

(ii)  $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*),$

where  $\lambda_A \in (0, 2\eta_A)$ ,  $\lambda_B \in (0, 2\eta_B)$ ,  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ .

**Proof.** Let the condition holds.

(i)  $\Rightarrow$  (ii) Let  $x^* \in \Omega$ , we have  $x^* \in VI(H_1, \sum_{i=1}^N a_i A_i, M_A)$  and

$Ax^* \in VI(H_2, \sum_{i=1}^N b_i B_i, M_B)$ .

From Lemma 2.33, we have  $x^* \in F(J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i))$  and  $Ax^* \in F(J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))$ , which implies that  $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^*$  and  $Ax^* = J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax^*$ .

By  $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^*$  and  $Ax^* = J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax^*$ , we have

$$\begin{aligned} & J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) \\ &= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(Ax^* - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax^*)) \\ &= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^* \\ &= x^*. \end{aligned}$$

It implies that

$$J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = x^*. \quad (3.22)$$

(ii)  $\Rightarrow$  (i) Let  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = x^*$  and let  $w \in \Omega$ .

We will show that  $I - \lambda_A \sum_{i=1}^N a_i A_i$  and  $I - \lambda_B \sum_{i=1}^N b_i B_i$  are nonexpansive, for all  $i = 1, 2, \dots, N$ .

Since  $A_i : H_1 \rightarrow H_1$  be  $\alpha_i$ -inverse strongly monotone mapping with  $\eta_A = \min_{i=1,2,\dots,N}\{\alpha_i\}$  and  $\lambda_A \in (0, 2\eta_A)$ , we have

$$\begin{aligned} & \|(I - \lambda_A \sum_{i=1}^N a_i A_i)x - (I - \lambda_A \sum_{i=1}^N a_i A_i)y\|^2 \\ &= \|x - y\|^2 - 2\lambda_A \sum_{i=1}^N a_i \langle x - y, A_i x - A_i y \rangle + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_A \sum_{i=1}^N a_i \alpha_i \|A_i x - A_i y\|^2 + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \end{aligned}$$

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$$\begin{aligned}
&\leq \|x - y\|^2 - 2\lambda_A \eta_A \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \\
&= \|x - y\|^2 + \lambda_A \sum_{i=1}^N a_i (\lambda_A - 2\eta_A) \|A_i x - A_i y\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Thus  $I - \lambda_A \sum_{i=1}^N a_i A_i$  is a nonexpansive mapping, for all  $i = 1, 2, \dots, N$ . By using the same proof, we obtain that  $I - \lambda_B \sum_{i=1}^N b_i B_i$ , for all  $i = 1, 2, \dots, N$  is a nonexpansive mapping. Then  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)$  and  $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)$  are nonexpansive mappings.

From  $w \in \Omega$  and (i)  $\Rightarrow$  (ii), we have  $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Aw = Aw$  and  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(w - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Aw) = w$ .

From Lemma 3.7 and  $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Aw = w$ , we have

$$\begin{aligned}
\|x^* - w\|^2 &= \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) \\
&\quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(w - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Aw)\|^2 \\
&\leq \|x^* - w\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^* \\
&\quad - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Aw\|^2 \\
&= \|x^* - w\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*\|^2. \tag{3.23}
\end{aligned}$$

Applying (3.23), we have

$$Ax^* \in F(J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)). \tag{3.24}$$

From Lemma 2.34, we have

$$Ax^* \in VI\left(H_1, \sum_{i=1}^N b_i B_i, M_B\right). \tag{3.25}$$

From the definition of  $x^*$  and (3.24), we have

$$\begin{aligned}
x^* &= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) \\
&= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^*.
\end{aligned}$$

From Lemma 2.34, we have

$$x^* \in VI\left(H_2, \sum_{i=1}^N a_i A_i, M_A\right). \tag{3.26}$$

From (3.25) and (3.26), we have  $x^* \in \Omega$ .  $\square$

**Example 3.9.** Let  $H_1 = H_2 = \mathbb{R}$ . For every  $i = 1, 2, \dots, N$ , let  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $A_i x = \frac{ix}{4} + (i+1)$  for all  $x \in H_1$  and  $B_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $B_i y = \frac{iy}{2} + (i+1)$  for all  $y \in H_2$ . Let  $M_A : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_A x = \{\frac{x}{4}\}$  for all  $x \in \mathbb{R}$  and  $M_B : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_B x = \{\frac{y}{2}\}$  for all  $y \in \mathbb{R}$ . Let  $Ax = x$ , for all  $x \in \mathbb{R}$ . Let  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$  and  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$  for all  $i = 1, 2, \dots, N$ . Then  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = -4$ .

*Solution.* It is easy to observe  $A_i$  is  $\frac{4}{i}$ -inverse strongly monotone mapping and  $B_i$  is  $\frac{2}{i}$ -inverse strongly monotone mapping. By definition of  $A_i$ ,  $B_i$  and  $a_i$ ,  $b_i$ , we have

$$\sum_{i=1}^N a_i A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N4^N} \right) A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N4^N} \right) \left( \frac{ix}{4} + (i+1) \right),$$

and

$$\sum_{i=1}^N b_i B_i y = \sum_{i=1}^N \left( \frac{2}{3^i} + \frac{1}{N3^N} \right) B_i y = \sum_{i=1}^N \left( \frac{2}{3^i} + \frac{1}{N3^N} \right) \left( \frac{iy}{2} + (i+1) \right).$$

Then  $\Omega = \{-4\}$ . From definition of  $A$ , we have  $L = 1$ . Choose  $\lambda_A = \frac{1}{N}$ ,  $\lambda_B = \frac{1}{N}$  and  $\gamma = \frac{1}{10}$ . From Lemma 3.8, we have  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = -4$ .  $\square$

**Example 3.10.** Let  $H_1 = H_2 = \mathbb{R}$ . For every  $i = 1, 2, \dots, N$ , let  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $A_i x = \frac{ix}{4} + (i+1)$  for all  $x \in H_1$  and  $B_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $B_i y = \frac{iy}{2} + (i+1)$  for all  $y \in H_2$ . Let  $M_A : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_A x = \{\frac{x}{4}\}$  for all  $x \in \mathbb{R}$  and  $M_B : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_B x = \{\frac{y}{2}\}$  for all  $y \in \mathbb{R}$ . Let  $Ax = x$ , for all  $x \in \mathbb{R}$ . Let  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$  and  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$  for all  $i = 1, 2, \dots, N$ . Then  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = x^*$  for all  $x^* \in \mathbb{R}$ .

*Solution.* It is easy to observe  $A_i$  is  $\frac{4}{i}$ -inverse strongly monotone mapping and  $B_i$  is  $\frac{2}{i}$ -inverse strongly monotone mapping. By definition of  $A_i$ ,  $B_i$  and  $a_i$ ,  $b_i$ , we have

$$\sum_{i=1}^N a_i A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N4^N} \right) A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N4^N} \right) \left( \frac{ix}{4} + (i+1) \right),$$

and

$$\sum_{i=1}^N b_i B_i y = \sum_{i=1}^N \left( \frac{2}{3^i} + \frac{1}{N3^N} \right) B_i y = \sum_{i=1}^N \left( \frac{2}{3^i} + \frac{1}{N3^N} \right) \left( \frac{iy}{2} + (i+1) \right).$$

Then  $\Omega = \{-4\}$ . From definition of  $A$ , we have  $L = 1$ . Choose  $\lambda_A = 0$ ,  $\lambda_B = 0$  and  $\gamma = \frac{1}{10}$ , we have  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = x^*$  for all  $x^* \in \mathbb{R}$ .

So Example 3.10 shows that Lemma 3.8 is not true because choose  $\lambda_A = 0$  and  $\lambda_B = 0$ .  $\square$

We prove a strong convergence theorem in Hilbert space to approximate a common solution of SCVIP and hierarchical fixed point problem of nonexpansive mapping.

**Theorem 3.11.** Let  $H_1, H_2$  be real Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $A^*$ . Let  $A_i : H_1 \rightarrow H_1$  be  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1} \{\alpha_i\}$  and  $B_i : H_2 \rightarrow H_2$  be  $\beta_i$ -inverse strongly monotone with  $\eta_B = \min_{i=1} \{\beta_i\}$ . Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{F} = \Phi \cap \Omega \neq \emptyset$ . Let the sequence  $\{x_n\}$  generated by  $u, x_1 \in H_1$  and

$$\begin{cases} u_n = J_{\lambda_A}^{M_A} (I - \lambda_A \sum_{i=1}^N a_i A_i) (x_n - \gamma A^* (I - J_{\lambda_B}^{M_B} (I - \lambda_B \sum_{i=1}^N b_i B_i)) A x_n), \\ y_n = (1 - \alpha_n) x_n + \alpha_n (\sigma_n S x_n + (1 - \sigma_n) T x_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n, \end{cases} \quad (3.27)$$

where  $\{\mu_n\}, \{\varphi_n\}, \{\theta_n\}, \{\alpha_n\}, \{\sigma_n\} \subseteq [0, 1]$  with  $\mu_n + \varphi_n + \theta_n = 1$  for all  $n \geq 1$ ,  $\lambda_A \in (0, 2\eta_A)$ ,  $\lambda_B \in (0, 2\eta_B)$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ . Suppose the following conditions hold:

- (i)  $\lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty,$
- (ii)  $0 < c \leq \varphi_n, \theta_n, \alpha_n \leq d < 1, \exists c, d > 0,$
- (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty,$
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty,$
- (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0,$
- (vi)  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1, a_i > 0$  and  $b_i > 0$  for all  $i = 1, 2, \dots, N$ .

Then  $\{x_n\}$  converges strongly to  $z_0 \in \mathbb{F}$ , where  $z_0 = P_{\mathbb{F}} u$ .

**Proof.** Step1. First, we prove that  $\{x_n\}, \{y_n\}$  and  $\{u_n\}$  are bounded.

We will show that  $J_{\lambda_A}^{M_A} (I - \lambda_A \sum_{i=1}^N a_i A_i)$  and  $J_{\lambda_B}^{M_B} (I - \lambda_B \sum_{i=1}^N b_i B_i)$  are nonexpansive mappings.

Since  $A_i$  is  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1} \{\alpha_i\}$ , we have

$$\begin{aligned} \left\| (I - \lambda_A \sum_{i=1}^N a_i A_i) x - (I - \lambda_A \sum_{i=1}^N a_i A_i) y \right\|^2 &= \left\| (x - y) - \lambda_A \left( \sum_{i=1}^N a_i A_i x - \sum_{i=1}^N a_i A_i y \right) \right\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_A \sum_{i=1}^N a_i \langle x - y, A_i x - A_i y \rangle \\ &\quad + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_A \sum_{i=1}^N a_i \alpha_i \|A_i x - A_i y\|^2 \\ &\quad + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \end{aligned}$$

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$$\begin{aligned} &\leq \|x - y\|^2 + \lambda_A \sum_{i=1}^N a_i (\lambda_n - 2\eta_A) \|A_i x - A_i y\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Thus  $I - \lambda_A \sum_{i=1}^N a_i A_i$  is a nonexpansive mapping, for all  $i = 1, 2, \dots, N$ . By using the same proof, we obtain that  $I - \lambda_B \sum_{i=1}^N b_i B_i$  is a nonexpansive mapping. Since  $J_{\lambda_A}^{M_A}$  and  $J_{\lambda_B}^{M_B}$  are nonexpansive mappings, we have  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)$  and  $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)$  are nonexpansive mappings.

Let  $p \in \mathbb{F}$  then  $p \in H_1$  and  $p \in \Phi$  which  $Tp = p$ . Now, we estimate

$$\begin{aligned} \|y_n - p\|^2 &= \|(1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n) - p\|^2 \\ &= \|(1 - \alpha_n)(x_n - p) + \alpha_n(\sigma_n(Sx_n - p) + (1 - \sigma_n)(Tx_n - p))\|^2 \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|(\sigma_n(Sx_n - p) + (1 - \sigma_n)(Tx_n - p))\|^2 \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\sigma_n\|Sx_n - p\|^2 + \alpha_n(1 - \sigma_n)\|Tx_n - p\|^2 \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\sigma_n\|Sx_n - p\|^2 + \alpha_n(1 - \sigma_n)\|x_n - p\|^2 \\ &= (1 - \alpha_n\sigma_n)\|x_n - p\|^2 + \alpha_n\sigma_n\|Sx_n - p\|^2. \end{aligned} \quad (3.28)$$

Since  $p \in \mathbb{F}$ , then  $p \in \Omega$  and  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)p = p$  and  $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ap = Ap$ . By Lemma 3.8, we have

$$J_{\lambda_A}^{M_A}\left(I - \lambda_A \sum_{i=1}^N a_i A_i\right)\left(p - \gamma A^*\left(I - J_{\lambda_B}^{M_B}\left(I - \lambda_B \sum_{i=1}^N b_i B_i\right)\right)Ap\right) = p.$$

By Lemma 3.7, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|J_{\lambda_A}^{M_A}\left(I - \lambda_A \sum_{i=1}^N a_i A_i\right)(x_n - \gamma A^*\left(I - J_{\lambda_B}^{M_B}\left(I - \lambda_B \sum_{i=1}^N b_i B_i\right)\right)Ax_n) \\ &\quad - J_{\lambda_A}^{M_A}\left(I - \lambda_A \sum_{i=1}^N a_i A_i\right)(p - \gamma A^*\left(I - J_{\lambda_B}^{M_B}\left(I - \lambda_B \sum_{i=1}^N b_i B_i\right)\right)Ap)\|^2 \\ &\leq \|x_n - p\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ap\|^2 \\ &\leq \|x_n - p\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \quad (3.29)$$

By (3.28) and (3.29), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\mu_n u + \varphi_n y_n + \theta_n u_n - p\|^2 \\ &\leq \mu_n \|u - p\|^2 + \varphi_n \|y_n - p\|^2 + \theta_n \|u_n - p\|^2 \\ &\leq \mu_n \|u - p\|^2 + \varphi_n \left[ (1 - \alpha_n \sigma_n) \|x_n - p\|^2 + \alpha_n \sigma_n \|Sx_n - p\|^2 \right] + \theta_n \|x_n - p\|^2 \\ &= \mu_n \|u - p\|^2 + \varphi_n \|x_n - p\|^2 - \varphi_n \alpha_n \sigma_n \|x_n - p\|^2 + \varphi_n \alpha_n \sigma_n \|Sx_n - p\|^2 + \theta_n \|x_n - p\|^2 \end{aligned}$$

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$$\begin{aligned}
&= \mu_n \|u - p\|^2 + (1 - \mu_n) \|x_n - p\|^2 - \varphi_n \alpha_n \sigma_n \|x_n - p\|^2 + \varphi_n \alpha_n \sigma_n \|Sx_n - p\|^2 \\
&\leq (1 - \mu_n) \|x_n - p\|^2 + \mu_n \|u - p\|^2 + \mu_n \alpha_n \sigma_n \|Sx_n - p\|^2.
\end{aligned} \tag{3.30}$$

From Lemma 2.32(i), therefore  $\{x_n\}$  is bounded. So are  $\{u_n\}, \{y_n\}$ .

Step2. Show that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ ,  $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$  and  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\mu_n u + \varphi_n y_n + \theta_n u_n - \mu_{n-1} u - \varphi_{n-1} y_{n-1} - \theta_{n-1} u_{n-1}\| \\
&= \|(\mu_n - \mu_{n-1})u + (\varphi_n - \varphi_{n-1})y_{n-1} + \varphi_n(y_n - y_{n-1}) \\
&\quad + (\theta_n - \theta_{n-1})u_{n-1} + \theta_n(u_n - u_{n-1})\| \\
&\leq |\mu_n - \mu_{n-1}| \|u\| + |\varphi_n - \varphi_{n-1}| \|y_{n-1}\| + \varphi_n \|y_n - y_{n-1}\| \\
&\quad + |\theta_n - \theta_{n-1}| \|u_{n-1}\| + \theta_n \|u_n - u_{n-1}\|.
\end{aligned} \tag{3.31}$$

From definition of  $\{u_n\}$ , Lemma 3.7 and  $\gamma \in (0, \frac{1}{L})$ , we have

$$\begin{aligned}
\|u_n - u_{n-1}\|^2 &= \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n) \\
&\quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_{n-1} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n-1})\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 - \gamma(1 - \gamma L) \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n \\
&\quad - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n-1}\|^2 \\
&\leq \|x_n - x_{n-1}\|^2.
\end{aligned}$$

It implies that

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\|. \tag{3.32}$$

From definition of  $\{y_n\}$ , we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|(1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n) \\
&\quad - [(1 - \alpha_{n-1})x_{n-1} + \alpha_{n-1}(\sigma_{n-1}Sx_{n-1} + (1 - \sigma_{n-1})Tx_{n-1})]\| \\
&= \|(x_n - x_{n-1}) - \alpha_n x_n + \alpha_{n-1} x_{n-1} + \alpha_n x_{n-1} - \alpha_n x_{n-1} \\
&\quad + \alpha_n \sigma_n Sx_n - \alpha_{n-1} \sigma_{n-1} Sx_{n-1} + \alpha_n \sigma_n Sx_{n-1} - \alpha_n \sigma_n Sx_{n-1} \\
&\quad + \alpha_n Tx_n - \alpha_{n-1} Tx_{n-1} + \alpha_n Tx_{n-1} - \alpha_n Tx_{n-1} \\
&\quad - \alpha_n \sigma_n Tx_n + \alpha_{n-1} \sigma_{n-1} Tx_{n-1} + \alpha_n \sigma_n Tx_{n-1} - \alpha_n \sigma_n Tx_{n-1}\| \\
&= \|(1 - \alpha_n)(x_n - x_{n-1}) + (\alpha_{n-1} - \alpha_n)x_{n-1} \\
&\quad + (\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1})Sx_{n-1} + \alpha_n \sigma_n (Sx_n - Sx_{n-1}) \\
&\quad + \alpha_n (1 - \sigma_n)(Tx_n - Tx_{n-1}) + (\alpha_n - \alpha_{n-1})Tx_{n-1} \\
&\quad + (\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n)Tx_{n-1}\| \\
&\leq (1 - \alpha_n) \|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n| \|x_{n-1}\|
\end{aligned}$$

$$\begin{aligned}
& + |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}| \|Sx_{n-1}\| + \alpha_n \sigma_n \|Sx_n - Sx_{n-1}\| \\
& + \alpha_n (1 - \sigma_n) \|Tx_n - Tx_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|Tx_{n-1}\| \\
& + |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n| \|Tx_{n-1}\| \\
\leq & (1 - \alpha_n) \|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n| \|x_{n-1}\| \\
& + |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}| \|Sx_{n-1}\| + \alpha_n \sigma_n \|x_n - x_{n-1}\| \\
& + \alpha_n (1 - \sigma_n) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|Tx_{n-1}\| \\
& + |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n| \|Tx_{n-1}\| \\
= & \|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n| \|x_{n-1}\| \\
& + |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}| \|Sx_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|Tx_{n-1}\| \\
& + |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n| \|Tx_{n-1}\|. \tag{3.33}
\end{aligned}$$

From (3.31)–(3.33), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| & \leq |\mu_n - \mu_{n-1}| \|u\| + |\varphi_n - \varphi_{n-1}| \|y_{n-1}\| + \varphi_n \|y_n - y_{n-1}\| \\
& + |\theta_n - \theta_{n-1}| \|u_{n-1}\| + \theta_n \|u_n - u_{n-1}\| \\
& \leq |\mu_n - \mu_{n-1}| \|u\| + |\varphi_n - \varphi_{n-1}| \|y_{n-1}\| \\
& + \varphi_n [\|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n| \|x_{n-1}\| \\
& + |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}| \|Sx_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|Tx_{n-1}\| \\
& + |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n| \|Tx_{n-1}\|] \\
& + |\theta_n - \theta_{n-1}| \|u_{n-1}\| + \theta_n \|x_n - x_{n-1}\| \\
= & (\varphi_n + \theta_n) \|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}| \|u\| + |\varphi_n - \varphi_{n-1}| \|y_{n-1}\| \\
& + \varphi_n |\alpha_{n-1} - \alpha_n| \|x_{n-1}\| + \varphi_n |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}| \|Sx_{n-1}\| \\
& + \varphi_n |\alpha_n - \alpha_{n-1}| \|Tx_{n-1}\| + \varphi_n |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n| \|Tx_{n-1}\| \\
& + |\theta_n - \theta_{n-1}| \|u_{n-1}\| \\
\leq & (1 - \mu_n) \|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}| \|u\| + |\varphi_n - \varphi_{n-1}| \|y_{n-1}\| \\
& + |\alpha_{n-1} - \alpha_n| \|x_{n-1}\| + |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}| \|Sx_{n-1}\| \\
& + |\alpha_n - \alpha_{n-1}| \|Tx_{n-1}\| + |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n| \|Tx_{n-1}\| \\
& + |\theta_n - \theta_{n-1}| \|u_{n-1}\|.
\end{aligned}$$

By Lemma 2.32(i), conditions (i) and (iii), we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.34}$$

From definition of  $\{u_n\}$ , we have

$$\begin{aligned}
x_{n+1} - u_n & = \mu_n u + \varphi_n y_n + \theta_n u_n - u_n \\
& = \mu_n (u - u_n) + \varphi_n (y_n - u_n). \tag{3.35}
\end{aligned}$$

From (3.28) and (3.29), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\mu_n u + \varphi_n y_n + \theta_n u_n - p\|^2 \\
&= \mu_n \|u - p\|^2 + \varphi_n \|y_n - p\|^2 + \theta_n \|u_n - p\|^2 \\
&\quad - \mu_n \varphi_n \|u - y_n\|^2 - \mu_n \theta_n \|u - u_n\|^2 - \varphi_n \theta_n \|y_n - u_n\|^2 \\
&\leq \mu_n \|u - p\|^2 + \varphi_n \|y_n - p\|^2 + \theta_n \|u_n - p\|^2 - \varphi_n \theta_n \|y_n - u_n\|^2 \\
&\leq \mu_n \|u - p\|^2 + \varphi_n [(1 - \alpha_n \sigma_n) \|x_n - p\|^2 + \alpha_n \sigma_n \|Sx_n - p\|^2] \\
&\quad + \theta_n \|x_n - p\|^2 - \varphi_n \theta_n \|y_n - u_n\|^2 \\
&= \mu_n \|u - p\|^2 + \varphi_n \|x_n - p\|^2 - \varphi_n \alpha_n \sigma_n \|x_n - p\|^2 \\
&\quad + \varphi_n \alpha_n \sigma_n \|Sx_n - p\|^2 + \theta_n \|x_n - p\|^2 - \varphi_n \theta_n \|y_n - u_n\|^2 \\
&\leq \mu_n \|u - p\|^2 + (1 - \mu_n) \|x_n - p\|^2 + \varphi_n \alpha_n \sigma_n \|Sx_n - p\|^2 \\
&\quad - \varphi_n \theta_n \|y_n - u_n\|^2 \\
&\leq \mu_n \|u - p\|^2 + \|x_n - p\|^2 + \varphi_n \alpha_n \sigma_n K - \varphi_n \theta_n \|y_n - u_n\|^2,
\end{aligned}$$

where  $K = \sup_n \{\|Sx_n - p\|^2\}$ . It follow that

$$\begin{aligned}
\varphi_n \theta_n \|y_n - u_n\|^2 &\leq \mu_n \|u - p\|^2 + \varphi_n \alpha_n \sigma_n K + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\leq \mu_n \|u - p\|^2 + \varphi_n \alpha_n \sigma_n K + \|x_n - x_{n+1}\| (\|x_n - p\| + \|x_{n+1} - p\|) \\
&\leq \mu_n \|u - p\|^2 + \varphi_n \alpha_n \sigma_n K + \|x_n - x_{n+1}\| L_1,
\end{aligned}$$

where  $L_1 = \sup_n \{\|x_n - p\| + \|x_{n+1} - p\|\}$ . From (3.34), conditions (i), (ii) and (v), we have

$$\lim_{n \rightarrow \infty} \|y_n - u_n\| = 0. \quad (3.36)$$

From (3.35) and (3.36), we have

$$\begin{aligned}
\|x_{n+1} - u_n\| &= \|\mu_n (u - u_n) + \varphi_n (y_n - u_n)\| \\
&\leq \mu_n \|u - u_n\| + \varphi_n \|y_n - u_n\|.
\end{aligned}$$

From (3.36) and a condition (i), we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0. \quad (3.37)$$

Since

$$\begin{aligned}
\|x_n - u_n\| &= \|x_n - x_{n+1} + x_{n+1} - u_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|.
\end{aligned}$$

From (3.35) and (3.37), we have

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (3.38)$$

Since

$$\|x_n - y_n\| = \|x_n - u_n + u_n - y_n\|$$

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$$\leq \|x_n - u_n\| + \|u_n - y_n\|.$$

From (3.36) and (3.38), we have

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (3.39)$$

Step3.  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ .

We have

$$\|x_n - Tx_n\| \leq \|x_n - y_n\| + \|y_n - Tx_n\|. \quad (3.40)$$

Since  $\{x_n\}$  is bounded and the mappings  $S, T$  are nonexpansive then there exists a  $K_1 > 0$  such that  $\|Sx_n - Tx_n\| \leq K_1$ , for all  $n \geq 0$ . Now, we estimate

$$\begin{aligned} \|y_n - Tx_n\| &= \|(1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n) - Tx_n\| \\ &= \|(1 - \alpha_n)(x_n - Tx_n) + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n - Tx_n)\| \\ &= \|(1 - \alpha_n)(x_n - Tx_n) + \alpha_n(\sigma_n Sx_n - \sigma_n Tx_n)\| \\ &\leq (1 - \alpha_n)\|x_n - Tx_n\| + \alpha_n \sigma_n \|Sx_n - Tx_n\| \\ &\leq (1 - \alpha_n)[\|x_n - y_n\| + \|y_n - Tx_n\|] + \alpha_n \sigma_n \|Sx_n - Tx_n\| \\ &\leq (1 - \alpha_n)\|x_n - u_n\| + (1 - \alpha_n)\|y_n - Tx_n\| + \alpha_n \sigma_n \|Sx_n - Tx_n\|, \end{aligned}$$

which implies

$$\begin{aligned} \alpha_n \|y_n - Tx_n\| &\leq (1 - \alpha_n)\|x_n - y_n\| + \alpha_n \sigma_n \|Sx_n - Tx_n\| \\ &\leq \|x_n - y_n\| + \alpha_n \sigma_n K_1. \end{aligned}$$

It follow that

$$\|y_n - Tx_n\| \leq \frac{\|x_n - y_n\|}{\alpha_n} + \sigma_n K_1. \quad (3.41)$$

Since  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0$ , we have  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n} = \lim_{n \rightarrow \infty} \sigma_n \cdot \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0$ .

From  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n} = 0$ , (3.41) and a condition (v), we have

$$\lim_{n \rightarrow \infty} \|y_n - Tx_n\| = 0. \quad (3.42)$$

Thus, it follows from (3.39), (3.40) and (3.42), we have

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \quad (3.43)$$

Step4.  $x^* \in \mathbb{F}$

Since  $\{x_n\}$  is bounded, there exists a subsequence  $\{x_{n_k}\}$  which converges weakly to  $x^*$ . We may assume that

$$\liminf_{n \rightarrow \infty} \langle -x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle = \lim_{k \rightarrow \infty} \langle -x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \rangle,$$

and

$$\liminf_{n \rightarrow \infty} \langle Sx_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle = \lim_{k \rightarrow \infty} \langle Sx_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \rangle.$$

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We will show that  $x^* \in F(T)$ . Assume that  $x^* \notin F(T)$ , then  $x^* \neq Tx^*$  and using Opial's property of Hilbert space and (3.42), we have

$$\begin{aligned} \liminf_{k \rightarrow \infty} \|x_{n_k} - x^*\| &< \liminf_{k \rightarrow \infty} \|x_{n_k} - Tx^*\| \\ &\leq \liminf_{k \rightarrow \infty} (\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tx^*\|) \\ &\leq \liminf_{k \rightarrow \infty} \|x_{n_k} - x^*\|, \end{aligned}$$

which is a contradiction. Therefore,  $x^* \in F(T)$ .

Next, we show that  $x^* \in \Phi$ . Consider

$$\begin{aligned} y_n - x_n &= (1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n) - x_n \\ &= \alpha_n\sigma_n(Sx_n - x_n) + \alpha_n(1 - \sigma_n)(Tx_n - x_n), \end{aligned}$$

which implies

$$\begin{aligned} Sx_n - x_n &= \frac{y_n - x_n}{\alpha_n\sigma_n} - \frac{\alpha_n(1 - \sigma_n)(Tx_n - x_n)}{\alpha_n\sigma_n} \\ &= \frac{y_n - x_n}{\alpha_n\sigma_n} + \frac{(1 - \sigma_n)(I - T)x_n}{\sigma_n}. \end{aligned}$$

It follows that

$$Sx_n - x_n - \frac{y_n - x_n}{\alpha_n\sigma_n} = \frac{(1 - \sigma_n)(I - T)x_n}{\sigma_n}.$$

Since  $T$  is nonexpansive, we have  $I - T$  is monotone. Let  $x \in F(T)$ , we have

$$\begin{aligned} &\langle Sx_n - x_n - \frac{y_n - x_n}{\alpha_n\sigma_n}, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \\ &= \frac{(1 - \sigma_n)}{\sigma_n} \langle (I - T)x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \\ &= \frac{(1 - \sigma_n)}{\sigma_n} \langle (I - T)x_n - (I - T)(x - \frac{y_n - x_n}{\alpha_n}), (I - T)(x - \frac{y_n - x_n}{\alpha_n}) \rangle \\ &= \frac{(1 - \sigma_n)}{\sigma_n} \left[ \langle (I - T)x_n - (I - T)(x - \frac{y_n - x_n}{\alpha_n}), x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \right. \\ &\quad \left. + \langle (I - T)(x - \frac{y_n - x_n}{\alpha_n}), x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \right] \\ &\leq \frac{(1 - \sigma_n)}{\sigma_n} \langle (I - T)(x - \frac{y_n - x_n}{\alpha_n}), x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \\ &\leq \frac{(1 - \sigma_n)}{\sigma_n} \left\| (I - T)(x - \frac{y_n - x_n}{\alpha_n}) \right\| \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\| \\ &= \frac{(1 - \sigma_n)}{\sigma_n} \left\| (I - T)(x - \frac{y_n - x_n}{\alpha_n}) - (I - T)x \right\| \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\| \\ &\leq 2(1 - \sigma_n) \frac{\|y_n - x_n\|}{\alpha_n\sigma_n} \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\|, \end{aligned}$$

which implies

$$\begin{aligned} \langle Sx_n - x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle &\leq 2(1 - \sigma_n) \frac{\|y_n - x_n\|}{\alpha_n\sigma_n} \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\| \\ &\quad + \langle \frac{y_n - x_n}{\alpha_n\sigma_n}, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \end{aligned}$$

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$$\leq 3 \frac{\|y_n - x_n\|}{\alpha_n \sigma_n} \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\|. \quad (3.44)$$

Since  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n} = 0$ , we have

$$\lim_{k \rightarrow \infty} \left\langle \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle = 0. \quad (3.45)$$

From (3.45) and  $\frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} + x_{n_k} \rightharpoonup x^*$ , we have

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \left\langle -x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \right\rangle \\ &= \lim_{k \rightarrow \infty} \left\langle -x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle \\ &= \lim_{k \rightarrow \infty} \left\langle -x_{n_k} - \left(x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}\right) + \left(x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}\right), x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle \\ &= \lim_{k \rightarrow \infty} \left[ \left\langle x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle \right. \\ & \quad \left. - \left\langle x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle \right] \\ &= \lim_{k \rightarrow \infty} \left[ \left\langle x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle - \left\langle x, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle \right. \\ & \quad \left. + \left\langle \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle \right] \\ &= \|x - x^*\|^2 - \langle x, x - x^* \rangle. \end{aligned} \quad (3.46)$$

Since  $S$  is weakly continuous and  $\frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} + x_{n_k} \rightharpoonup x^*$ , we obtain

$$\liminf_{n \rightarrow \infty} \langle Sx_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle = \lim_{k \rightarrow \infty} \langle Sx_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \rangle = \langle Sx^*, x - x^* \rangle. \quad (3.47)$$

From (3.44), (3.46) and (3.47), we have

$$\begin{aligned} \langle Sx^* - x^*, x - x^* \rangle &= \langle Sx^*, x - x^* \rangle - \langle x^*, x - x^* \rangle \\ &= \langle Sx^*, x - x^* \rangle + \|x - x^*\|^2 - \langle x, x - x^* \rangle \\ &= \liminf_{n \rightarrow \infty} \left[ \langle Sx_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle - \left\langle x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \right\rangle \right] \\ &= \liminf_{n \rightarrow \infty} \langle Sx_n - x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \\ &\leq \liminf_{n \rightarrow \infty} 3 \frac{\|y_n - x_n\|}{\alpha_n \sigma_n} \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\| \\ &\leq 0. \end{aligned}$$

Hence  $x^*$  solve Hierarchical fixed point problem, i.e.,  $x^* \in \Phi$ .

Next, we show that  $x^* \in \Omega$ . Assume that  $x^* \neq J_{\lambda_A}^{M_A} (I - \lambda_A \sum_{i=1}^N a_i A_i) (x^* - \gamma A^* (I - J_{\lambda_B}^{M_B} (I - \lambda_B \sum_{i=1}^N b_i B_i)) A x^*)$ . Applying the Opial's property, (3.38) and Lemma 3.7, we have

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \|x_{n_k} - x^*\| \\ & < \liminf_{k \rightarrow \infty} \|x_{n_k} - J_{\lambda_A}^{M_A} (I - \lambda_A \sum_{i=1}^N a_i A_i) (x^* - \gamma A^* (I - J_{\lambda_B}^{M_B} (I - \lambda_B \sum_{i=1}^N b_i B_i)) A x^*)\| \end{aligned}$$

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$$\begin{aligned}
&= \liminf_{k \rightarrow \infty} \|x_{n_k} - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n_k}) \\
&\quad + J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n_k}) \\
&\quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*)\| \\
&\leq \liminf_{k \rightarrow \infty} \left[ \|x_{n_k} - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n_k})\| \right. \\
&\quad + \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n_k}) \\
&\quad \left. - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*)\right] \\
&\leq \liminf_{k \rightarrow \infty} \left[ \|x_{n_k} - u_{n_k}\| + \|x_{n_k} - x^*\| \right. \\
&\quad \left. - \gamma(1 - \gamma L) \left\| (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n_k} - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^* \right\| \right] \\
&\leq \liminf_{k \rightarrow \infty} \left[ \|x_{n_k} - u_{n_k}\| + \|x_{n_k} - x^*\| \right] \\
&= \liminf_{k \rightarrow \infty} \|x_{n_k} - x^*\|.
\end{aligned}$$

This is a contradiction. Then  $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*)$ . From Lemma 3.8, we have  $x^* \in \Omega$ . Therefore,  $x^* \in \mathbb{F}$ .

Step5. Finally, we will prove that  $\{x_n\}$  converges strongly to  $z_0 = P_{\mathbb{F}}u$ .

We show that  $\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle \leq 0$ , where  $z_0 = P_{\mathbb{F}}u$ . We may assume the subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  with

$$\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{k \rightarrow \infty} \langle u - z_0, x_{n_k} - z_0 \rangle. \quad (3.48)$$

Since  $x_{n_k} \rightarrow x^*$  as  $k \rightarrow \infty$  and  $x^* \in \mathbb{F}$ . By (3.48) and Lemma 2.27, we have

$$\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{k \rightarrow \infty} \langle u - z_0, x_{n_k} - z_0 \rangle \leq 0 \quad (3.49)$$

From (3.28) and (3.29), we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\mu_n u + \varphi_n y_n + \theta_n u_n - z_0\|^2 \\
&= \|\mu_n(u - z_0) + \varphi_n(y_n - z_0) + \theta_n(u_n - z_0)\|^2 \\
&\leq \|\varphi_n(y_n - z_0) + \theta_n(u_n - z_0)\|^2 + 2\langle \mu_n(u - z_0), x_{n+1} - z_0 \rangle \\
&\leq \varphi_n \|y_n - z_0\|^2 + \theta_n \|u_n - z_0\|^2 + 2\mu_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&\leq \varphi_n [(1 - \alpha_n \sigma_n) \|x_n - z_0\|^2 + \alpha_n \sigma_n \|Sx_n - z_0\|^2] \\
&\quad + \theta_n \|x_n - z_0\|^2 + 2\mu_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&\leq \varphi_n \|x_n - z_0\|^2 + \varphi_n \alpha_n \sigma_n \|Sx_n - z_0\|^2 \\
&\quad + \theta_n \|x_n - z_0\|^2 + 2\mu_n \langle u - z_0, x_{n+1} - z_0 \rangle
\end{aligned}$$

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$$\begin{aligned} &\leq (1 - \mu_n)\|x_n - z_0\|^2 + \varphi_n\alpha_n\sigma_n\|Sx_n - z_0\|^2 \\ &\quad + 2\mu_n\langle u - z_0, x_{n+1} - z_0 \rangle. \end{aligned}$$

Applying Lemma 2.32(ii), conditions (i), (iv) and (3.49), we can conclude that the  $\{x_n\}$  converges strongly to  $z_0 = P_{\mathbb{F}}u$ . This completes the proof.  $\square$

Next, we have the following strong convergence to approximation a common element of solution the set of SMVI and hierarchical fixed point problem of nonexpansive mapping.

**Corollary 3.12.** Let  $H_1, H_2$  be real Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $F : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $F^*$ . Let  $A : H_1 \rightarrow H_1$  be  $\alpha$ -inverse strongly monotone and  $B : H_2 \rightarrow H_2$  be  $\beta$ -inverse strongly monotone. Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{F} = \Phi \cap \Theta \neq \emptyset$ . Let the sequence  $\{x_n\}$  generated by  $u, x_1 \in H_1$  and

$$\begin{cases} u_n = J_{\lambda_A}^{M_A}(I - \lambda_A A)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B B))F x_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1 - \sigma_n)T x_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n, \end{cases} \quad (3.50)$$

where  $\{\mu_n\}, \{\varphi_n\}, \{\theta_n\}, \{\alpha_n\}, \{\sigma_n\} \subseteq [0, 1]$  with  $\mu_n + \varphi_n + \theta_n = 1$  for all  $n \geq 1$ ,  $\lambda_A \in (0, 2\alpha)$ ,  $\lambda_B \in (0, 2\beta)$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $F^*F$ . Suppose the following conditions hold:

- (i)  $\lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty,$
- (ii)  $0 < c \leq \varphi_n, \theta_n \leq d < 1, \exists c, d > 0,$
- (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty,$
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty,$
- (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0,$

Then  $\{x_n\}$  converges strongly to  $z_0 \in \mathbb{F}$ , where  $z_0 = P_{\mathbb{F}}u$ .

**Proof.** Put  $A_i \equiv A$  and  $B_i \equiv B$  for all  $i = 1, 2, \dots, N$  in Theorem 3.11. From Theorem 3.11, we obtain the desired result.  $\square$

### 3.4 Application of strong convergence theorem in Hilbert space to approximate a common solution of SCVIP and hierarchical fixed point problem of nonexpansive mapping

#### 3.4.1 Split Zero Point Problem

Let  $H$  be a real Hilbert space. Let  $M : H \rightarrow 2^H$  be a maximal monotone operator. Then *the zero point problem* is to find  $x^* \in H$  such that

$$0 \in Mx^*, \quad (3.51)$$

such an  $x^* \in H$  is called a zero point of  $M$ . The set of zero point of  $M$  is denoted by  $M^{-1}(0)$ .

Let  $H_1$  and  $H_2$  be two real Hilbert spaces. Setting  $A_i \equiv 0$  and  $B_i \equiv 0$  for all  $i = 1, 2, \dots, N$  in SCVIP, then SCVIP reduce to *the split zero point problem*: Find  $x^* \in H_1$  such that

$$0 \in M_A x^*, \quad (3.52)$$

and

$$y^* \in Ax^* \text{ such that } 0 \in M_B y^*, \quad (3.53)$$

where  $A : H_1 \rightarrow H_2$  is bounded linear operator,  $M_A : H_1 \rightarrow 2^{H_1}$  and  $M_B : H_2 \rightarrow 2^{H_2}$  are multi-valued mapping. The set of all solution of this problem is denoted by  $\Omega_2 = \{x \in M_A^{-1}(0) : Ax \in M_B^{-1}(0)\}$ .

The split zero point problem which consists of the special cases, split feasibility problem, variational inequalities, etc., which is used in practice as a model in machine learning, image processing and linear inverse problem.

Next, we give the strong convergence theorem for solving the split zero point problem and the hierarchical fixed point problem of nonexpansive mapping.

**Corollary 3.13.** Let  $H_1, H_2$  be real Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $A^*$ . Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{F} = \Phi \cap \Omega_2 \neq \emptyset$ . Let the iterative sequence generated by hybrid iterative algorithm:

$$\begin{cases} u_n = J_{\lambda_A}^{M_A}(x_n - \gamma A^* J_{\lambda_B}^{M_B} Ax_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n, \end{cases} \quad (3.54)$$

where  $\{\delta_n\}, \{\varphi_n\}, \{\eta_n\}, \{\alpha_n\}, \{\sigma_n\} \subseteq [0, 1]$  with  $\delta_n + \varphi_n + \eta_n = 1$  for all  $n \geq 1$ , and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ . Suppose the following conditions hold:

(i)  $\lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty,$

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- (ii)  $0 < c \leq \varphi_n, \theta_n, \alpha_n \leq d < 1, \exists c, d > 0,$
- (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty,$
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty,$
- (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0,$

Then  $\{x_n\}$  converges strongly to  $z \in \mathbb{F}$ , where  $z = P_{\mathbb{F}}u$ .

**Proof.** Put  $A_i \equiv 0$  and  $B_i \equiv 0$  for all  $i = 1, 2, \dots, N$  in Theorem 3.11. From Theorem 3.11, we obtain the desired conclusion.  $\square$

### 3.4.2 Split Combination of Variational Inequalities Problem

Setting  $M_A = \partial i_{H_1}$  and  $M_B = \partial i_{H_2}$  in (1.17) and (1.18), then SCVIP reduce to the split combination of variational inequality problem, that is find  $x^* \in H_1$  such that

$$\left\langle \sum_{i=1}^N a_i A_i x^*, x - x^* \right\rangle \geq 0, \forall x \in H_1, \quad (3.55)$$

and

$$y^* = Ax^* \in H_2 \text{ such that } \left\langle \sum_{i=1}^N b_i B_i y^*, y - y^* \right\rangle \geq 0, \forall y \in H_2, \quad (3.56)$$

where  $A : H_1 \rightarrow H_2$  is bounded linear operator and  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$ . The set of all this is denoted by  $\Omega_3 = \{x \in VI(H_1, \sum_{i=1}^N a_i A_i) : Ax \in VI(H_2, \sum_{i=1}^N b_i B_i)\}$ .

**Remark 3.14.** If  $M_A = \partial i_{H_1}$  and  $M_B = \partial i_{H_2}$  in (1.17) and (1.18), then we have  $\Omega$  reduce to  $\Omega_3$ .

**Proof.** We will show that  $VI(H_1, \sum_{i=1}^N a_i A_i, M_A) = VI(H_1, \sum_{i=1}^N a_i A_i)$ . We have for  $x^* \in H_1$ .

Consider,

$$\begin{aligned} x^* \in VI(H_1, \sum_{i=1}^N a_i A_i, M_A) &\Leftrightarrow \theta_{H_1} \in \sum_{i=1}^N a_i A_i x^* + M_A x^* \\ &\Leftrightarrow \theta_{H_1} \in \sum_{i=1}^N a_i A_i x^* + \partial i_{H_1}(x^*) \\ &\Leftrightarrow - \sum_{i=1}^N a_i A_i x^* \in \partial i_{H_1}(x^*) \\ &\Leftrightarrow \left\langle \sum_{i=1}^N a_i A_i x^*, x - x^* \right\rangle \geq 0, \forall x \in H_1, \\ &\Leftrightarrow x^* \in VI(H_1, \sum_{i=1}^N a_i A_i). \end{aligned}$$

Similarly, we also have  $VI(H_2, \sum_{i=1}^N b_i B_i, M_B) = VI(H_2, \sum_{i=1}^N b_i B_i)$ . Then  $\Omega \equiv \Omega_3$  where  $M_A = \partial i_{H_1}$  and  $M_B = \partial i_{H_2}$ .  $\square$

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The split combination of variational inequality problem has played an essential role for concrete problems in dynamic emission tomographic image reconstruction, signal recovery problems, beam-forming problems, power-control problems, bandwidth allocation problems and optimal control problems.

Next, we establish a strong convergence theorem for solving the split combination of variational inequality problem and hierarchical fixed point problem of non-expansive mapping by using a modified Halpern iterative method as follows:

**Theorem 3.15.** Let  $H_1, H_2$  be real Hilbert spaces. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $A^*$ . Let  $A_i : H_1 \rightarrow H_1$  be  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1} \{\alpha_i\}$  and  $B_i : H_2 \rightarrow H_2$  be  $\beta_i$ -inverse strongly monotone with  $\eta_B = \min_{i=1} \{\beta_i\}$ . Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{F} = \Phi \cap \Omega_3 \neq \emptyset$ . Let the sequence  $\{x_n\}$  generated by  $u, x_1 \in H_1$  and

$$\begin{cases} u_n = P_{H_1}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_n - \gamma A^*(I - P_{H_2}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n, \end{cases} \quad (3.57)$$

where  $\{\mu_n\}, \{\varphi_n\}, \{\theta_n\}, \{\alpha_n\}, \{\sigma_n\} \subseteq [0, 1]$  with  $\mu_n + \varphi_n + \theta_n = 1$  for all  $n \geq 1$ ,  $\lambda_A \in (0, 2\eta_A)$ ,  $\lambda_B \in (0, 2\eta_B)$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ . Suppose the following conditions hold:

- (i)  $\lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty,$
- (ii)  $0 < c \leq \varphi_n, \theta_n, \alpha_n \leq d < 1, \exists c, d > 0,$
- (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty,$
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty,$
- (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0,$
- (vi)  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1, a_i > 0$  and  $b_i \geq 0$  for all  $i = 1, 2, \dots, N.$

Then  $\{x_n\}$  converges strongly to  $z_0 \in \mathbb{F}$ , where  $z_0 = P_{\mathbb{F}}u$ .

**Proof.** Put  $M_A = \partial i_{H_1}$  and  $M_B = \partial i_{H_2}$  in Theorem 3.11. Using the same method in Theorem 3.11, we have the desired conclusion. □

## Chapter 4

### Example and numerical

#### 4.1 Example and numerical in Banach space

In this section, we give numerical examples to support our some main results. The following examples are given for supporting Theorem 3.2.

**Example 4.1.** Let  $\mathbb{R}$  be the set of real numbers and

$$E = L_2[-5, 5] = \{e : \mathbb{R} \rightarrow \mathbb{R} \mid \int_{-5}^5 e^2(x)dx < \infty\},$$

with the inner product defined by

$$\langle e, g \rangle = \int_{-5}^5 e(x)g(x)dx,$$

and

$$\|e\| = \left( \int_{-5}^5 e^2(x)dx \right)^{\frac{1}{2}},$$

for all  $e, g \in E$ . Let  $C_E = H(g, 500) = \{h \in L_2[-5, 5] \mid \int_{-5}^5 g(x)h(x)dx = 500\}$  where  $g \in L_2[-5, 5]$  with  $g(x) = 2x - 3$ , for all  $x \in [-5, 5]$ . Define mappings  $D_1, D_2, D_3, T, S, f : C_E \rightarrow C_E$  as follows  $D_1(h) = \frac{h-3I}{2}, D_2(h) = \frac{h-3I}{5}, D_3(h) = \frac{h-3I}{7}, T(h) = \frac{h+3I}{2}, S(h) = \frac{h+6I}{3}$  and  $f(h) = \frac{h}{5}$ , for all  $h \in C_E$ . Define the mapping  $Z : C_E \rightarrow C_E$  by  $Zx = Q_{C_E}(I - \lambda_1 D_1)(ah + (1-a)Q_{C_E}(I - \lambda_2 D_2)(ah + (1-a)Q_{C_E}(I - \lambda_3 D_3)h))$  for all  $h \in C_E$ . Setting  $\{\alpha_n\} = \{\frac{1}{7n}\}, \{\beta_n\} = \{\frac{31n-4}{49n}\}, \{\gamma_n\} = \{\frac{18n-3}{49n}\}$  and  $\{\delta_n\} = \{\frac{2n-1}{7n}\}$ , for all  $n \in \mathbb{N}$ . Then the sequence  $\{x_n\}$  generated by (5.1) converges strongly to a point  $x^* = 3I$ .

*Solution.* By definition of  $D_1, D_2$  and  $D_3$ , we have  $D_1, D_2$  and  $D_3$  are 1-inverse strongly accretive operators. It is easy to check that  $f$  is  $\frac{1}{5}$ -contraction mappings and  $T, S$  are nonexpansive mapping with  $F(S) \cap F(T) = \{3I\}$ .

By definition of  $Q_{C_E}$  and  $C_E$ , we have

$$Q_{C_E}f = f - \frac{\langle f, g \rangle - 500}{\|g\|^2}g.$$

Let  $K = \frac{1}{3}$ , we have  $K$  is 2-uniformly smooth constant of  $\mathbb{R}$ . We choose  $\lambda_1 = 2 \in (0, 9), \lambda_2 = 3 \in (0, 9)$  and  $\lambda_3 = 4 \in (0, 9)$  and  $a = \frac{1}{2} \in [0, 1]$ .

For every  $n \in \mathbb{N}, \{\alpha_n\} = \{\frac{1}{7n}\}, \{\beta_n\} = \{\frac{31n-4}{49n}\}, \{\gamma_n\} = \{\frac{18n-3}{49n}\}$  and  $\{\delta_n\} = \{\frac{2n-1}{7n}\}$ , then the sequence  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  and  $\{\delta_n\}$  satisfy all the conditions of Theorem 3.2.

From the definition of  $Z$ , that is  $Z(h) = Q_{C_E}(I - 2D_1)(\frac{1}{2}h + \frac{1}{2}Q_{C_E}(I - 3D_2)(\frac{1}{2}h + \frac{1}{2}Q_{C_E}(I - 4D_3)h))$  for all  $h \in C$ , we have  $F(Z) = \{3I\}$ . Then  $\mathbb{F} = F(Z) \cap F(S) \cap F(T) = \{3I\}$ .

From Theorem 3.2, we obtain the sequence  $\{x_n\}$  converges strongly to  $3I$ . □

**Example 4.2.** Let  $\mathbb{R}$  be the real line with Euclidean norm and let  $C_E = [0, 3]$  and  $D_1, D_2, D_3 : C_E \rightarrow \mathbb{R}$  be mappings defined by  $D_1x = \frac{x}{3}, D_2x = \frac{x}{5}$  and  $D_3x = \frac{x}{7}$  for all  $x \in C_E$ . Let  $T, S : C_E \rightarrow C_E$  be defined by  $Tx = \max\{0, -x\}$  and  $Sx = \min\{0, \frac{x}{2}\}$  for all  $x \in C_E$ . Let  $f : C_E \rightarrow C_E$  be mapping defined by  $f(x) = \frac{x}{4}$  for all  $x \in C_E$ . Define the mapping  $Z : C_E \rightarrow C_E$  by  $Zx = Q_{C_E}(I - \lambda_1 D_1)(ax + (1 - a)Q_{C_E}(I - \lambda_2 D_2)(ax + (1 - a)Q_{C_E}(I - \lambda_3 D_3)x))$  for all  $x \in C_E$ . Setting  $\{\alpha_n\} = \{\frac{1}{5n}\}, \{\beta_n\} = \{\frac{9n-4}{25n}\}, \{\gamma_n\} = \{\frac{16n-1}{25n}\}$  and  $\{\delta_n\} = \{\frac{2n-1}{5n}\}$ , for all  $n \in \mathbb{N}$ . Then the sequence  $\{x_n\}$  generated by (5.1) converges strongly to a point  $x^* = 0$ .

*Solution.* Since  $f(x) = \frac{x}{4}$  for all  $x \in C_E$ , we have  $f$  is  $\frac{1}{4}$ -contraction mapping. It is easy to see that  $T$  and  $S$  are nonexpansive mapping with  $F(S) \cap F(T) = \{0\}$ . Since  $D_1x = \frac{x}{3}, D_2x = \frac{x}{5}$  and  $D_3x = \frac{x}{7}$  for all  $x \in C$ , we have  $D_1, D_2$  and  $D_3$  are 1-inverse strongly accretive operator.

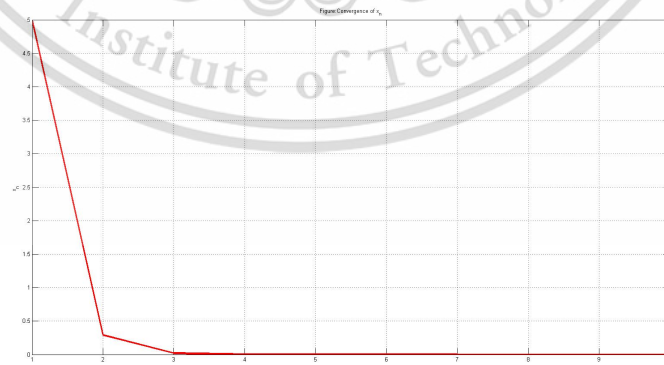
Let  $K = \frac{1}{3}$ , we have  $K$  is 2-uniformly smooth constant of  $\mathbb{R}$ . We choose  $\lambda_1 = 2 \in (0, 9), \lambda_2 = 3 \in (0, 9)$  and  $\lambda_3 = 4 \in (0, 9)$  and  $a = \frac{1}{2} \in [0, 1]$ .

For every  $n \in \mathbb{N}$ ,  $\{\alpha_n\} = \{\frac{1}{5n}\}, \{\beta_n\} = \{\frac{9n-4}{25n}\}, \{\gamma_n\} = \{\frac{16n-1}{25n}\}$  and  $\{\delta_n\} = \{\frac{2n-1}{5n}\}$ , then the sequence  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  and  $\{\delta_n\}$  satisfy all the conditions of Theorem 3.2.

From the definition of  $Z$ , that is  $Zx = Q_{C_E}(I - 2D_1)(\frac{1}{2}x + \frac{1}{2}Q_{C_E}(I - 3D_2)(\frac{1}{2}x + \frac{1}{2}Q_{C_E}(I - 4D_3)x))$ , we have  $F(Z) = \{0\}$  and  $(0,0,0)$  is a solution of (1.11). Then  $\mathbb{F} = F(Z) \cap F(S) \cap F(T) = \{0\}$ . We rewrite (5.1) as follows:

$$\begin{cases} u_n = T(\frac{2n-1}{5n}x_n + (1 - \frac{2n-1}{5n})Sx_n), \\ x_{n+1} = Z(\frac{1}{5n}f(x_n) + \frac{9n-4}{25n}x_n + \frac{16n-1}{25n}u_n), \end{cases} \quad (4.1)$$

Choose  $x_1 = 5$  and  $n = 10$ . The numerical for the sequence  $x_n$  are shown in Tabel 1 and Figure 1.



**Figure 4.1:** The convergence of  $\{x_n\}$  with initial values  $x_1 = 5$  and  $n = 10$

□

**Table 4.1:** The values of  $\{x_n\}$  with  $x_1 = 5, n = 10$ 

n	$x_n$	error (%)
1	5.0000000000000000	-
2	0.291707821059663	1614.043861048105
3	0.020107004951208	1350.773363101550
4	0.001004015300175	1902.689243018201
5	0.000000200184010	1021.334912011851
6	0.000000000140005	978.1581001421004
7	0.0000000000000134	904.1854100014710
8	0.0000000000000124	8.064516129017012
9	0.0000000000000117	5.982905983102004
10	0.0000000000000116	0.862068966001242

## 4.2 Example and numerical in Hilbert space

The purpose of this section is to give numerical examples to support some of our. The following example given for supporting Theorem 3.11 and example show that Theorem 3.11 is not true if condition (iv) fails, but conditions (i), (ii), (iii), (v) and (vi) are satisfied.

**Example 4.3.** Let  $H_1 = H_2 = \mathbb{R}$ , the set of all real numbers, with the inner product defined by  $\langle x, y \rangle = xy$ , for all  $x, y \in \mathbb{R}$  and induced usual norm  $|\cdot|$ . For every  $i = 1, 2, \dots, N$ , let the mapping  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $A_i x = \frac{x^i}{4} + (i + 1)$  for all  $x \in H_1$  and  $B_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $B_i y = \frac{y^i}{2} + (i + 1)$  for all  $y \in H_2$ , respectively, let  $M_A, M_B : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_A(x) = \{\frac{x}{4}\}$ , for all  $x \in \mathbb{R}$  and  $M_B(y) = \{\frac{y}{2}\}$ , for all  $y \in \mathbb{R}$ . Let the mapping  $A : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $A(x) = -2x$ , for all  $x \in \mathbb{R}$  and let  $\gamma \in (0, \frac{1}{4})$ , so we choose  $\gamma = \frac{1}{10}$ . Let the mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $Tx = \frac{x-12}{4}$ , for all  $x \in \mathbb{R}$  and let the mapping  $S : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $Sx = \frac{x-16}{5}$ , for all  $x \in \mathbb{R}$ . Setting  $\{\mu_n\} = \{\frac{1}{5n}\}$ ,  $\{\varphi_n\} = \{\frac{7n+1}{15n}\}$ ,  $\{\theta_n\} = \{\frac{8n-4}{15n}\}$ ,  $\{\alpha_n\} = \{\frac{1}{n}\}$  and  $\{\sigma_n\} = \{\frac{1}{4n^2}\}$ ,  $\forall n \in \mathbb{N}$ . For every  $i = 1, 2, \dots, N$ , suppose that  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$  and  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$ . Then  $\{x_n\}$  converges strongly to a point  $x^* = -4 \in \mathbb{F}$ .

*Solution.* It is easy to check that  $a_i$  and  $b_i$  satisfies all the conditions of Theorem 3.11 and  $A_i$  is  $\frac{1}{4^i}$ -inverse strongly monotone and  $B_i$  is  $\frac{1}{3^i}$ -inverse strongly monotone for all  $i = 1, 2, \dots, N$ . We choose  $\lambda_A = \frac{1}{4N}$ ,  $\lambda_B = \frac{1}{3N}$ . Since  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$ , we obtain

$$\sum_{i=1}^N a_i A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N4^N} \right) \frac{x}{4^i}.$$

Then  $0 \in VI(H_1, \sum_{i=1}^N a_i A_i, M_A)$ . Since  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$ , we have

$$\sum_{i=1}^N b_i B_i y = \sum_{i=1}^N \left( \frac{2}{3^i} + \frac{1}{N3^N} \right) \frac{y}{3^i}.$$

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Then  $0 \in VI(H_2, \sum_{i=1}^N b_i B_i, M_B)$ . Thus  $\{-4\} = \Omega$ .

It is easy to observe that  $T, S$  are nonexpansive mappings with  $F(T) = \{-4\}$ ,  $F(S) = \{-4\}$ . Hence  $\Phi = \{-4\}$ . Therefore  $\mathbb{F} = \Phi \cap \Omega = \{0\}$ .

For every  $n \in \mathbb{N}$ ,  $\{\mu_n\} = \{\frac{1}{5n}\}$ ,  $\{\varphi_n\} = \{\frac{7n+1}{15n}\}$ ,  $\{\theta_n\} = \{\frac{8n-4}{15n}\}$ ,  $\{\alpha_n\} = \{\frac{1}{n}\}$  and  $\{\sigma_n\} = \{\frac{1}{4n^2}\}$ , then the sequence  $\{\mu_n\}$ ,  $\{\varphi_n\}$ ,  $\{\theta_n\}$ ,  $\{\alpha_n\}$  and  $\{\sigma_n\}$  satisfy all the conditions of Theorem 3.11. We rewrite (5.4) as follows:

$$\begin{cases} u_n = J_{\lambda_A}^{M_A}(I - \frac{1}{4N} \sum_{i=1}^N a_i A_i)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \frac{1}{3N} \sum_{i=1}^N b_i B_i))Ax_n), \\ y_n = (1 - \frac{1}{n})x_n + \frac{1}{n}(\frac{1}{4n^2}Sx_n + (1 - \frac{1}{4n^2})Tx_n), \\ x_{n+1} = \frac{1}{5n}u + \frac{7n+1}{15n}y_n + \frac{8n-4}{15n}u_n, \end{cases} \quad (4.2)$$

Choose  $u = -1$ ,  $x_1 = 1$ ,  $N = 100$  and  $n = 20$ . The numerical for the sequence  $\{x_n\}$  are shown Table 4.2 and Figure 4.2.  $\square$

Table 4.2: The values of  $\{x_n\}$  with  $N = 100$ ,  $n = 10$

n	$x_n$	error (%)
1	1.0000000000000000	-
2	-2.078490048215079	148.1118504001824
3	-3.117800841919048	33.33473047153201
4	-3.558708584007431	12.38954329001782
5	-3.870081853071857	8.045650734017523
6	-3.981400713810520	2.795972273301401
7	-3.984752801862711	0.084122839010014
8	-3.989180408119522	0.082161991114027
9	-3.992001745821323	0.077987542611015
10	-3.995101701418320	0.077593930101514

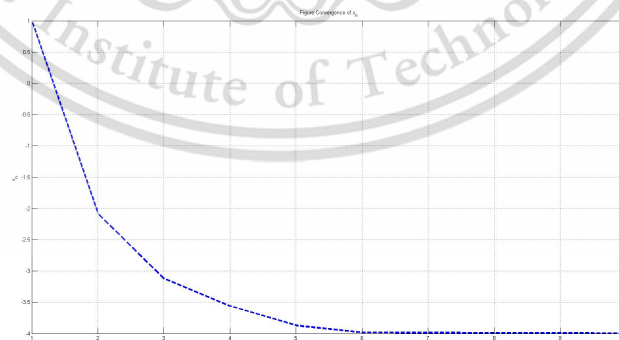


Figure 4.2: The sequence  $\{x_n\}$  converges strongly to 0 with initial values  $x_1 = 1$ ,  $N = 100$  and  $n = 10$ .

**Example 4.4.** Let  $H_1 = H_2 = \mathbb{R}$ , the set of all real numbers, with the inner product defined by  $\langle x, y \rangle = xy$ , for all  $x, y \in \mathbb{R}$  and induced usual norm  $\|\cdot\|$ . For every  $i = 1, 2, \dots, N$ ,

let the mapping  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $A_i x = \frac{x}{4^i}$  for all  $x \in H_1$  and  $B_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $B_i y = \frac{y}{3^i}$  for all  $y \in H_2$ , respectively, let  $M_A, M_B : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_A(x) = \{2x\}$ , for all  $x \in \mathbb{R}$  and  $M_B(y) = \{2y\}$ , for all  $y \in \mathbb{R}$ . Let the mapping  $A : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $A(x) = -2x$ , for all  $x \in \mathbb{R}$  and let  $\gamma \in (0, \frac{1}{4})$ , so we choose  $\gamma = \frac{1}{10}$ . Let the mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $Tx = \max\{0, -x\}$ , for all  $x \in \mathbb{R}$  and let the mapping  $S : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $Sx = \min\{0, \frac{x}{2}\}$ , for all  $x \in \mathbb{R}$ . Setting  $\{\mu_n\} = \{\frac{1}{5n}\}$ ,  $\{\varphi_n\} = \{\frac{7n+1}{15n}\}$ ,  $\{\theta_n\} = \{\frac{8n-4}{15n}\}$ ,  $\{\alpha_n\} = \{\frac{1}{n}\}$  and  $\{\sigma_n\} = \{n\}$ ,  $\forall n \in \mathbb{N}$ . For every  $i = 1, 2, \dots, N$ , suppose that  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$  and  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$ . Then  $\{x_n\}$  is divergence.

*Solution.* Note that the sequence  $\{\mu_n\}$ ,  $\{\varphi_n\}$ ,  $\{\theta_n\}$ ,  $\{\alpha_n\}$ ,  $a_i$  and  $b_i$  satisfies the conditions (i), (ii), (iii), (v) and (vi) from Theorem 3.11, while assumption (iv) does not converge to 0 since

$$\lim_{n \rightarrow \infty} n = \infty.$$

Choose  $u = -1$ ,  $x_1 = 1$ ,  $N = 100$  and  $n = 25$ . The numerical for the sequence  $\{x_n\}$  are shown in Table 4.3 and Figure 4.3. Therefore,  $\{x_n\}$  is divergence.  $\square$

**Table 4.3:** The values of  $\{x_n\}$  with  $N = 100, n = 25$

n	$x_n$	error (%)
1	1.0000000000000000	-
2	-0.076707182681007	1402.531278017896
3	-0.171707951035812	89.30283449101785
4	-0.352507819321075	51.28932136201489
5	-0.473648900753108	25.57613348820185
⋮	⋮	⋮
17	-12.01089004833105	96.05650457815 23
⋮	⋮	⋮
21	-41.72920179001821	15.65510793512841
22	-57.20560179500482	27.05421056891218
23	-78.52977955410336	26.70918355210252
24	-107.9388078991472	27.24605807520021
25	-148.5342689941007	27.00346112821053

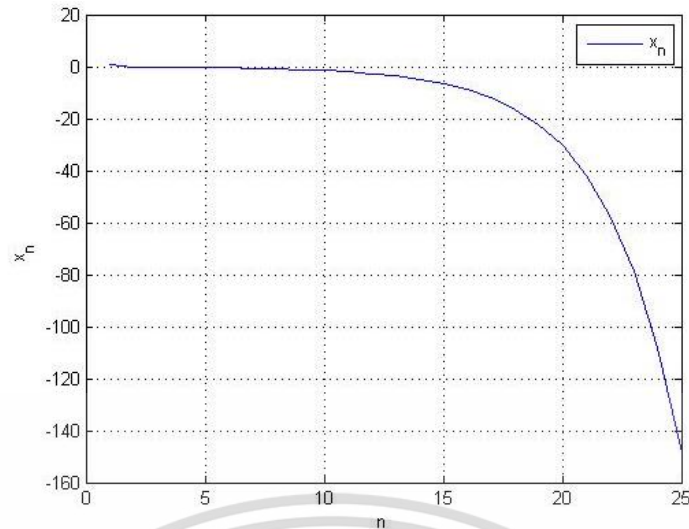


Figure 4.3: The sequence  $\{x_n\}$  is divergence with initial values  $x_1 = 1$ ,  $N = 100$  and  $n = 25$ .

Next, we give example to support out some result in a two dimensional space of real numbers.

**Example 4.5.** Let  $H_1 = H_2 = \mathbb{R}^2$ , with the inner product defined by  $\langle x, y \rangle = xy = x_1 \cdot y_1 + x_2 \cdot y_2$ , for all  $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$  and induced usual norm  $\|\cdot\|$  defined by  $\|x\| = \sqrt{x_1^2 + x_2^2}$  for all  $x = (x_1, x_2) \in \mathbb{R}^2$ . For every  $i = 1, 2, \dots, N$ , let the mapping  $A_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  define by  $A_i x = \frac{x}{3^i}$  for all  $x = (x_1, x_2) \in H_1$  and  $B_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  define by  $B_i y = \frac{y}{4^i}$  for all  $y = (y_1, y_2) \in H_2$ , respectively, let  $M_A, M_B : \mathbb{R}^2 \rightarrow 2^{\mathbb{R}^2}$  be defined by  $M_A(x) = \{x\}$ , for all  $x = (x_1, x_2) \in \mathbb{R}^2$  and  $M_B(y) = \{3y\}$ , for all  $y = (y_1, y_2) \in \mathbb{R}^2$ . Let the mapping  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by  $A(x) = 3x$ , for all  $x = (x_1, x_2) \in \mathbb{R}^2$  and let  $\gamma \in (0, \frac{1}{9})$ , so we choose  $\gamma = \frac{1}{15}$ . Let the mapping  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by  $Tx = \frac{x}{3}$ , for all  $x = (x_1, x_2) \in \mathbb{R}^2$  and let the mapping  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by  $Sx = \min\{0, \frac{x}{3}\}$ , for all  $x = (x_1, x_2) \in \mathbb{R}^2$ . Setting  $\{\mu_n\} = \{\frac{1}{4^n}\}$ ,  $\{\varphi_n\} = \{\frac{7n+1}{12n}\}$ ,  $\{\theta_n\} = \{\frac{5n-4}{12n}\}$ ,  $\{\alpha_n\} = \{\frac{1}{n}\}$  and  $\{\sigma_n\} = \{\frac{1}{4n^2}\}$ ,  $\forall n \in \mathbb{N}$ . For every  $i = 1, 2, \dots, N$ , suppose that  $a_i = \frac{9}{10^i} + \frac{1}{N10^N}$  and  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$ . Then  $\{x_n\}$  converges strongly to a point  $x^* = (0, 0) \in \mathbb{F}$ .

*Solution.* It is easy to check that  $a_i$  and  $b_i$  satisfies all the conditions of Theorem 3.11 and  $A_i$  is  $\frac{1}{3^i}$ -inverse strongly monotone and  $B_i$  is  $\frac{1}{4^i}$ -inverse strongly monotone for all  $i = 1, 2, \dots, N$ . We choose  $\lambda_A = \frac{1}{5N}$ ,  $\lambda_B = \frac{1}{7N}$ . Thus  $\{(0, 0)\} = \Omega$ .

For definition of  $T$  and  $S$ , then  $T$  and  $S$  are nonexpansive mapping with  $F(T) = \{(0, 0)\}$ . Hence  $\Phi = \{(0, 0)\}$ . Therefore  $\mathbb{F} = \Phi \cap \Omega = \{(0, 0)\}$ .

For every  $n \in \mathbb{N}$ ,  $\{\mu_n\} = \{\frac{1}{4^n}\}$ ,  $\{\varphi_n\} = \{\frac{7n+1}{12n}\}$ ,  $\{\theta_n\} = \{\frac{5n-4}{12n}\}$ ,  $\{\alpha_n\} = \{\frac{1}{n}\}$  and  $\{\sigma_n\} = \{\frac{1}{4n^2}\}$ , then the sequence  $\{\mu_n\}$ ,  $\{\varphi_n\}$ ,  $\{\theta_n\}$ ,  $\{\alpha_n\}$  and  $\{\sigma_n\}$  satisfy all the conditions of Theorem 3.11.

From Theorem 3.11, we can conclude that the sequence  $\{x_n\}$  converges to  $(0, 0)$ .  $\square$

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## Chapter 5

### Conclusion

In this chapter, we summarize all main theorems and applications obtained in this thesis.

#### 5.1 Strong convergence theorem to approximate a common solution of Banach generalized system of variational inequalities problem in Banach space

(1) Let  $C_E$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  and let  $Q_{C_E}$  be a sunny nonexpansive retraction of  $C_E$ . Let  $D_1, D_2, D_3 : C_E \rightarrow E$  be  $d_1, d_2, d_3$ -inverse strongly accretive operators, respectively. Define the mapping  $Z : C_E \rightarrow C_E$  by  $Z(x) = Q_{C_E}(I - \lambda_1 D_1)(ax + (1-a)Q_{C_E}(I - \lambda_2 D_2)(ax + (1-a)Q_{C_E}(I - \lambda_3 D_3)x))$  for all  $x \in C_E$ ,  $\lambda_1 \in (0, \frac{d_1}{K^2})$ ,  $\lambda_2 \in (0, \frac{d_2}{K^2})$ ,  $\lambda_3 \in (0, \frac{d_3}{K^2})$ ,  $a \in [0, 1]$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . Let  $T, S : C_E \rightarrow C_E$  be nonexpansive mappings. Assume that  $\mathbb{F} = F(Z) \cap F(S) \cap F(T) \neq \emptyset$ . Let the sequence  $\{x_n\}$  be generated by  $x_1 \in C_E$  and

$$\begin{cases} u_n = T(\delta_n x_n + (1 - \delta_n) S x_n), \\ x_{n+1} = Z(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n), \end{cases} \quad (5.1)$$

where  $f : C_E \rightarrow C_E$  be a contraction mapping with a constant  $\xi \in (0, 1)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$  with  $\alpha_n + \beta_n + \gamma_n = 1$ . Suppose that the following conditions are satisfied

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$  for all  $c, d > 0$  and for all  $n \geq 1$ ,
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ ,
- (iv)  $\lim_{n \rightarrow \infty} \delta_n = \delta$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}} f(x_0)$  and  $(x_0, y_0, z_0)$  is a solution of (1.11), where  $y_0 = Q_{C_E}(I - \lambda_2 D_2)(ax_0 + (1-a)z_0)$  and  $z_0 = Q_{C_E}(I - \lambda_3 D_3)x_0$ .

(2) Let  $C_E$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  and let  $Q_{C_E}$  be a sunny nonexpansive retraction of  $C_E$ . Let  $D_1, D_2, D_3 : C_E \rightarrow E$  be  $d_1, d_2, d_3$ -inverse strongly accretive operators, respectively. Define the mapping  $Z : C_E \rightarrow C_E$  by  $Z(x) = Q_{C_E}(I - \lambda_1 D_1)(Q_{C_E}(I - \lambda_2 D_2)(Q_{C_E}(I - \lambda_3 D_3)x))$  for all  $x \in C_E$ ,  $\lambda_1 \in (0, \frac{d_1}{K^2})$ ,  $\lambda_2 \in (0, \frac{d_2}{K^2})$ ,  $\lambda_3 \in (0, \frac{d_3}{K^2})$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . Let  $T, S : C_E \rightarrow C_E$  be nonexpansive mappings. Assume that

$\mathbb{F} = F(Z) \cap F(S) \cap F(T) \neq \emptyset$ . Let the sequence  $\{x_n\}$  be generated by  $x_1 \in C_E$  and

$$\begin{cases} u_n = T(\delta_n x_n + (1 - \delta_n) S x_n), \\ x_{n+1} = Z(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n), \end{cases} \quad (5.2)$$

where  $f : C_E \rightarrow C_E$  be a contraction mapping with a constant  $\xi \in (0, 1)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$  with  $\alpha_n + \beta_n + \gamma_n = 1$ . Suppose that the following condition are satisfied

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$  for all  $c, d > 0$  and for all  $n \geq 1$ ,
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ ,
- (iv)  $\lim_{n \rightarrow \infty} \delta_n = \delta$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}} f(x_0)$  and  $(x_0, y_0, z_0)$  is a solution of (1.11), where  $y_0 = Q_{C_E}(I - \lambda_2 D_2)z_0$  and  $z_0 = Q_{C_E}(I - \lambda_3 D_3)x_0$ .

(3) Let  $C_E$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  and let  $Q_{C_E}$  be a sunny nonexpansive retraction of  $C_E$ . Let  $D_1, D_2, D_3 : C_E \rightarrow E$  be  $d_1, d_2, d_3$ -inverse strongly accretive operators, respectively. Define the mapping  $Z : C_E \rightarrow C_E$  by  $Z(x) = Q_{C_E}(I - \lambda_1 D_1)(ax + (1 - a)Q_{C_E}(I - \lambda_2 D_2)(ax + (1 - a)Q_{C_E}(I - \lambda_3 D_3)x))$  for all  $x \in C_E, \lambda_1 \in (0, \frac{d_1}{K^2}), \lambda_2 \in (0, \frac{d_2}{K^2}), \lambda_3 \in (0, \frac{d_3}{K^2}), a \in [0, 1]$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . Let  $T, S : C_E \rightarrow C_E$  be nonexpansive mappings. Assume that  $\mathbb{F} = S(C_E, D_1) \cap S(C_E, D_2) \cap S(C_E, D_3) \cap F(S) \cap F(T) \neq \emptyset$ . Let the sequence  $\{x_n\}$  be generated by  $x_1 \in C_E$  and

$$\begin{cases} u_n = T(\delta_n x_n + (1 - \delta_n) S x_n), \\ y_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n, \\ x_{n+1} = Q_{C_E}(I - \lambda_1 D_1)(aI + (1 - a)Q_{C_E}(I - \lambda_2 D_2)(aI \\ \quad + (1 - a)Q_{C_E}(I - \lambda_3 D_3)))y_n, \forall n \geq 1, \end{cases} \quad (5.3)$$

where  $f : C_E \rightarrow C_E$  be a contraction mapping with a constant  $\xi \in (0, 1)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$  with  $\alpha_n + \beta_n + \gamma_n = 1$ . Suppose that the following condition are satisfied

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$  for all  $c, d > 0$  and for all  $n \geq 1$ ,
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$   
 $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ ,
- (iv)  $\lim_{n \rightarrow \infty} \delta_n = \delta$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}} f(x_0)$  and  $(x_0, y_0, z_0)$  is a solution of (1.11), where  $y_0 = Q_{C_E}(I - \lambda_2 D_2)(ax_0 + (1 - a)z_0)$  and  $z_0 = Q_{C_E}(I - \lambda_3 D_3)x_0$ .

## 5.2 Strong convergence theorem to approximate a common solution of SCVIP and hierarchical fixed point problem of nonexpansive mapping in Hilbert space

(1) Let  $H_1, H_2$  be real Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $A^*$ . Let  $A_i : H_1 \rightarrow H_1$  be  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1}^N \{\alpha_i\}$  and  $B_i : H_2 \rightarrow H_2$  be  $\beta_i$ -inverse strongly monotone with  $\eta_B = \min_{i=1}^N \{\beta_i\}$ . Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{F} = \Phi \cap \Omega \neq \emptyset$ . Let the sequence  $\{x_n\}$  generated by  $u, x_1 \in H_1$  and

$$\begin{cases} u_n = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n x_n, \end{cases} \quad (5.4)$$

where  $\{\mu_n\}, \{\varphi_n\}, \{\theta_n\}, \{\alpha_n\}, \{\sigma_n\} \subseteq [0, 1]$  with  $\mu_n + \varphi_n + \theta_n = 1$  for all  $n \geq 1$ ,  $\lambda_A \in (0, 2\eta_A)$ ,  $\lambda_B \in (0, 2\eta_B)$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ . Suppose the following conditions hold:

- (i)  $\lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty,$
- (ii)  $0 < c \leq \varphi_n, \theta_n, \alpha_n \leq d < 1, \exists c, d > 0,$
- (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty,$
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty,$
- (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0,$
- (vi)  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1, a_i > 0$  and  $b_i > 0$  for all  $i = 1, 2, \dots, N.$

Then  $\{x_n\}$  converges strongly to  $z_0 \in \mathbb{F}$ , where  $z_0 = P_{\mathbb{F}}u$ .

(2) Let  $H_1, H_2$  be real Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $F : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $F^*$ . Let  $A : H_1 \rightarrow H_1$  be  $\alpha$ -inverse strongly monotone and  $B : H_2 \rightarrow H_2$  be  $\beta$ -inverse strongly monotone. Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{F} = \Phi \cap \Theta \neq \emptyset$ . Let the sequence  $\{x_n\}$  generated by  $u, x_1 \in H_1$  and

$$\begin{cases} u_n = J_{\lambda_A}^{M_A}(I - \lambda_A A)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B B))Fx_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n x_n, \end{cases} \quad (5.5)$$

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where  $\{\mu_n\}, \{\varphi_n\}, \{\theta_n\}, \{\alpha_n\}, \{\sigma_n\} \subseteq [0, 1]$  with  $\mu_n + \varphi_n + \theta_n = 1$  for all  $n \geq 1$ ,  $\lambda_A \in (0, 2\alpha)$ ,  $\lambda_B \in (0, 2\beta)$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $F^*F$ . Suppose the following conditions hold:

- (i)  $\lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty,$
- (ii)  $0 < c \leq \varphi_n, \theta_n, \alpha_n \leq d < 1, \exists c, d > 0,$
- (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty,$
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty,$
- (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0,$

Then  $\{x_n\}$  converges strongly to  $z_0 \in \mathbb{F}$ , where  $z_0 = P_{\mathbb{F}}u$ .

(3) Let  $H_1, H_2$  be real Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $A^*$ . Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{F} = \Phi \cap \Omega_2 \neq \emptyset$ . Let the iterative sequence generated by hybrid iterative algorithm:

$$\begin{cases} u_n = J_{\lambda_A}^{M_A}(x_n - \gamma A^* J_{\lambda_B}^{M_B} A x_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1 - \sigma_n)T x_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n, \end{cases} \quad (5.6)$$

where  $\{\delta_n\}, \{\varphi_n\}, \{\eta_n\}, \{\alpha_n\}, \{\sigma_n\} \subseteq [0, 1]$  with  $\delta_n + \varphi_n + \eta_n = 1$  for all  $n \geq 1$ , and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ . Suppose the following conditions hold:

- (i)  $\lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty,$
- (ii)  $0 < c \leq \varphi_n, \theta_n, \alpha_n \leq d < 1, \exists c, d > 0,$
- (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty,$
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty,$
- (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0,$

Then  $\{x_n\}$  converges strongly to  $z \in \mathbb{F}$ , where  $z = P_{\mathbb{F}}u$ .

(4) Let  $H_1, H_2$  be real Hilbert spaces. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $A^*$ . Let  $A_i : H_1 \rightarrow H_1$  be  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1} \{\alpha_i\}$  and  $B_i : H_2 \rightarrow H_2$  be  $\beta_i$ -inverse strongly monotone with  $\eta_B = \min_{i=1} \{\beta_i\}$ . Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{F} = \Phi \cap \Omega_3 \neq \emptyset$ . Let the sequence  $\{x_n\}$  generated by  $u, x_1 \in H_1$  and

$$\begin{cases} u_n = P_{H_1}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_n - \gamma A^*(I - P_{H_2}(I - \lambda_B \sum_{i=1}^N b_i B_i))A x_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1 - \sigma_n)T x_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n, \end{cases} \quad (5.7)$$

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where  $\{\mu_n\}, \{\varphi_n\}, \{\theta_n\}, \{\alpha_n\}, \{\sigma_n\} \subseteq [0, 1]$  with  $\mu_n + \varphi_n + \theta_n = 1$  for all  $n \geq 1$ ,  $\lambda_A \in (0, 2\eta_A)$ ,  $\lambda_B \in (0, 2\eta_B)$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ . Suppose the following conditions hold:

- (i)  $\lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty,$
- (ii)  $0 < c \leq \varphi_n, \theta_n, \alpha_n \leq d < 1, \exists c, d > 0,$
- (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty,$
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty,$
- (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0,$
- (vi)  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1, a_i > 0$  and  $b_i > 0$  for all  $i = 1, 2, \dots, N.$

Then  $\{x_n\}$  converges strongly to  $z_0 \in \mathbb{F}$ , where  $z_0 = P_{\mathbb{F}}u.$

### 5.3 Example and numerical

In the section, we give the concousion for Examples.

#### 5.3.1 Example and numerical in Banach space

(1) Example 4.2 show that the sequence  $\{x_n\}$  converges to  $3I$ , where  $\{3I\} = F(S) \cap F(T)$ . The convergence of  $\{x_n\}$  of Example 4.2 can be guaranteed by Theorem 3.2.

(2) Table 4.1 and Figure 4.1 show that the sequence  $\{x_n\}$  converges to 0, where  $\{0\} = F(S) \cap F(T)$ . The convergence of  $\{x_n\}$  of Example 4.2 can be guaranteed by Theorem 3.2.

#### 5.3.2 Example and numerical in Hilbert space

(1) Table 4.2 and Figure 4.2 show that the sequence  $\{x_n\}$  converges to 0, where  $\{0\} = \Phi \cap \Omega$ . The convergence of  $\{x_n\}$  of Example 4.3 can be guaranteed by Theorem 3.11.

(2) Table 4.3 and Figure 4.3 show that the sequence  $\{x_n\}$  diverge, where condition (iv) is violated since  $\lim_{n \rightarrow \infty} \sigma_n \neq 0.$

(3) Example 4.5 show that the sequence  $\{x_n\}$  converges to  $(0, 0)$ , where  $\{(0, 0)\} = \Phi \cap \Omega$ . The convergence of  $\{x_n\}$  of Example 4.5 can be guaranteed by Theorem 3.11.

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# Appendix A

## The research papers



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Article

# Modified Halpern Iterative Method for Solving Hierarchical Problem and Split Combination of Variational Inclusion Problem in Hilbert Space

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**Abstract:** The purpose of this paper is to introduce the split combination of variational inclusion problem which combines the concept of the modified variational inclusion problem introduced by Khuangsatung and Kangtunyakarn and the split variational inclusion problem introduced by Moudafi. Using a modified Halpern iterative method, we prove the strong convergence theorem for finding a common solution for the hierarchical fixed point problem and the split combination of variational inclusion problem. The result presented in this paper demonstrates the corresponding result for the split zero point problem and the split combination of variation inequality problem. Moreover, we discuss a numerical example for supporting our result and the numerical example shows that our result is not true if some conditions fail.

**Keywords:** split variational inclusion problem; hierarchical fixed point problem; fixed point problem

**MSC:** 47H09; 47J25; 49J40; 90C99

## 1. Introduction

Throughout this article, we let  $H$  be a real Hilbert spaces with inner products  $\langle \cdot, \cdot \rangle$  and norms  $\| \cdot \|$  and let  $C$  be a nonempty closed convex subset of a real Hilbert spaces  $H$ .

**Definition 1.** Let  $C$  be a nonempty subset of a real Hilbert spaces  $H$  and  $Z : C \rightarrow C$  be a self mapping.  $Z$  is called a nonexpansive mapping if

$$\|Zx - Zy\| \leq \|x - y\|, \text{ for all } x, y \in C.$$

$Z$  is called a firmly nonexpansive mapping if

$$\|Zx - Zy\|^2 \leq \langle x - y, Zx - Zy \rangle, \text{ for all } x, y \in C.$$

A mapping  $W : C \rightarrow H$  is called  $\alpha$ -inverse strongly monotone [1], if there exists a positive real number  $\alpha$  such that

$$\langle x - y, Wx - Wy \rangle \geq \alpha \|Wx - Wy\|^2, \forall x, y \in C. \quad (1)$$

If  $W : C \rightarrow H$  is  $\alpha$ -inverse strongly monotone, then  $W$  is monotone mapping, that is,

$$\langle Wx - Wy, x - y \rangle \geq 0, \forall x, y \in H.$$

**Remark 1.** (i) If  $\alpha = 1$  in Equation (1), then  $W$  is firmly nonexpansive mapping.

For  $i = 1, 2, \dots, N$ , let  $A_i : H \rightarrow H$  be a single-valued mapping and  $M : H \rightarrow 2^H$  be a multi-valued mapping, from the concept of variational inclusion problems, Khuangsatung and Kangtunyakarn [2] introduced the problem of finding  $x \in H$  such that

$$\theta \in \sum_{i=1}^N a_i A_i x + Mx, \quad (2)$$

for all  $a_i \in (0, 1)$  with  $\sum_{i=1}^N a_i = 1$  and  $\theta$  is a zero vector. This problem is called the *modified variational inclusion*. The set of solutions of Equation (2) is denoted by  $VI(H, \sum_{i=1}^N a_i A_i, M)$ . If we set  $A_i = B$  for  $i = 1, 2, \dots, N$  then Equation (2) reduces to  $\theta \in Bx + Mx$ , which is the *variational inclusion* problem. The set of solution of variational inclusion problem is denoted by  $VI(H, B, M)$ .

The variational inclusion problems are extensively studied in mathematical programming, optimal control, mathematical economics, etc. In recent years, considerable interest has been shown in developing various extensions and generalization of the variational inclusion problem; for instance [3,4] and reference therein.

The operator  $M$  is called a maximal monotone [5], if  $M$  is monotone, i.e.,  $\langle u - v, x - y \rangle \geq 0$ , wherever  $u \in M(x)$ ,  $v \in M(y)$  and the graph  $G(M)$  of  $M$  (that is,  $G(M) := \{(x, u) \in H \times H : u \in M(x)\}$ ) is not properly contained in the graph of any other monotone operator.

Let *resolvent operator*  $J_\lambda^M : H \rightarrow H$  be defined by  $J_\lambda^M(x) = (I + \lambda M)^{-1}(x)$ , for all  $x \in H$ , where  $M$  is a multi-valued maximal monotone mapping,  $\lambda > 0$  and  $I$  is an identity mapping.

Let  $T : C \rightarrow C$  be a mapping. A point  $x \in C$  is called a fixed point of  $T$  if  $Tx = x$ . The set of fixed points of  $T$  is denoted  $Fix(T) = \{x \in C : Tx = x\}$ . Fixed point problem is an important area of mathematical analysis. This problem applies about the solution in many problem in Hilbert space such as nonlinear operator equation, variational inclusion problem, etc.; for instance [2–18].

Khuangsatung and Kangtunyakarn [2] proposed the following iterative algorithm:

$$\begin{cases} w_1, \mu \in H, \\ \sum_{i=1}^N b_i \Psi_i(z_n, y) + \frac{1}{r_n} \langle y - z_n, z_n - w_n \rangle \geq 0, \forall y \in C, \\ w_{n+1} = \alpha_n \mu + \beta_n w_n + \gamma_n J_\lambda^M(I - \lambda \sum_{i=1}^N a_i A_i) w_n + \eta_n (I - \rho_n (I - S)) w_n + \delta_n z_n, \forall n \geq 1, \end{cases}$$

where  $S : H \rightarrow H$  is a  $\kappa$ -strictly pseudononspreading mapping (i.e., if there exists  $\kappa \in [0, 1)$  such that  $\|Su - Sv\|^2 \leq \|u - v\|^2 + \kappa \|(I - S)u - (I - S)v\|^2 + 2\langle u - Su, v - Sv \rangle$ ,  $\forall u, v \in H$ ) and under certain assumptions of  $\Psi_i : C \times C \rightarrow \mathbb{R}$  is a bifunction for all  $i = 1, 2, \dots, N$ , they proved strong convergence theorem for solving the modified variational inclusion problem under some suitable conditions of  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{\eta_n\}$ ,  $\{\delta_n\}$  and  $\{\rho_n\}$ .

Over the decades, there are many mathematicians interested in studying the variational inequality problem, which is one of the important problems. The methods used to solve this problem can be applied for other solutions such as physics, economics, finance, optimization, network analysis, medical images, water resourced and structural analysis. The set of solution of the *variational inequality problem* is denoted by

$$VI(C, A) = \{u \in C : \langle v - u, Au \rangle \geq 0\},$$

for all  $v \in C$  and  $A : C \rightarrow H$  is a mapping.

Many iterative methods have been developed for solving variational inequality problem, see, for instance [7,8].

By using the concept of the variational inequality problem, Moudafi and Mainge [9] firstly introduced *hierarchical fixed point problem* for a nonexpansive mapping  $T$  with respect to another nonexpansive mapping  $S$  on  $H$ : Find  $x^* \in Fix(T)$  such that

$$\langle Sx^* - x^*, x - x^* \rangle \leq 0, \forall x \in Fix(T), \quad (3)$$

where  $S : H \rightarrow H$  is a nonexpansive mapping. It is easy to see that Equation (3) is equivalent to the following fixed point problem: Find  $x^* \in H$  such that

$$x^* = P_{\text{Fix}(T)} \circ Sx^*, \quad (4)$$

where  $P_{\text{Fix}(T)}$  is the metric projection of  $H$  onto  $\text{Fix}(T)$ . The solution set of Equation (3) is denoted by  $\Phi = \{x^* \in H : \langle Sx^* - x^*, x - x^* \rangle \leq 0, \forall x \in \text{Fix}(T)\}$ . It is obvious that  $\Phi = VI(\text{Fix}(T), I - S)$ . Note that Equation (3) covers monotone variational inequality on fixed point sets, minimization problem, etc. Many iterative methods have been developed for solving the hierarchical fixed point problem in Equation (3), see example [9–11].

By using the concept of Krasnoselski–Mann iterative algorithm, Moudafi [10] introduced iterative scheme (5) for nonexpansive mapping  $S, T$  on a subset  $C$  of Hilbert space:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(\sigma_n P x_n + (1 - \sigma_n)T x_n), \forall n \geq 0. \end{cases} \quad (5)$$

He proved the weak convergence theorem of the sequence  $\{x_n\}$ , where  $\{\alpha_n\}, \{\sigma_n\} \subset (0, 1)$  satisfies

- (i)  $\sum_{n=0}^{+\infty} \sigma_n < +\infty$ ,
- (ii)  $\sum_{n=0}^{+\infty} \alpha_n(1 - \alpha_n) = +\infty$ ,
- (iii)  $\lim_{n \rightarrow +\infty} \frac{\|x_{n+1} - x_n\|}{(1 - \alpha_n)\sigma_n} = 0$ .

Let  $H_1$  and  $H_2$  be two real Hilbert spaces and  $C, Q$  be a nonempty closed convex subset of a real Hilbert spaces  $H_1$  and  $H_2$ , respectively. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator. Censor and Elfving [14] introduced the *split feasibility problem (SEP)* which is to find a point  $x \in C$  and  $Ax \in Q$ . Many authors have studied this concept of SEP to modified their problem, see example [12–15].

In 2010, Censor, Gibali and Reich [13] introduced the split variational inequality problem which relies on the split feasibility problem and thus created the iterative algorithm for solving a strong convergence theorem of the split variational inclusion problem; more detail [13].

The split monotone variational inclusion problem, which consists of special cases, which is being used in practice as a model in the intensity-modulated radiation therapy treatment planning, the modeling of many inverse problems, and other problems; see for instance [11–15].

For every  $i = 1, 2, \dots, N$ . Let  $A_i : H_1 \rightarrow H_1, B_i : H_2 \rightarrow H_2$  be mappings and  $M_A : H_1 \rightarrow 2^{H_1}$  and  $M_B : H_2 \rightarrow 2^{H_2}$  be multi-value mappings. Inspired and motivated by Moudafi [12] and Khuangsatung and Kangtunyakarn [2], we define the *split combination of the variational inclusion problem (SCVIP)* which is find  $x^* \in H_1$  such that

$$\theta_{H_1} \in \sum_{i=1}^N a_i A_i x^* + M_A x^*, \quad (6)$$

and

$$y^* = Ax^* \text{ such that } \theta_{H_2} \in \sum_{i=1}^N b_i B_i y^* + M_B y^*, \quad (7)$$

where  $A : H_1 \rightarrow H_2$  is a bounded linear operator and  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$ .

The set of all the solutions for Equations (6) and (7) are denoted by  $\Omega = \{x \in VI(H_1, \sum_{i=1}^N a_i A_i, M_A) : Ax \in VI(H_2, \sum_{i=1}^N b_i B_i, M_B)\}$ .

If we set  $A_i = A$  and  $B_i = B$  for all  $i = 1, 2, \dots, N$  then SCVIP reduces to the *split monotone variational inclusion problem (SMVI)*, which is,

$$\text{find } x^* \in H_1 \text{ such that } 0 \in A(x^*) + M_A(x^*), \quad (8)$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in B(y^*) + M_B(y^*), \quad (9)$$

introduced by Moudafi [12]. The set of all these solutions for Equations (8) and (9) are denoted by  $\Theta = \{x^* \in VI(H_1, f, B_2) : Ax^* \in VI(H_2, g, B_2)\}$ .

Very recently, Kazmi et al. [11] proved the strong convergence theorem under suitable condition of parameters for solving the hierarchical fixed point problem and SMVI by using hybrid iterative method as follows:

$$\begin{cases} x_0 \in C, C_0 = C; \\ u_n = (1 - \alpha_n)x_n + \alpha_n P_C(\sigma_n Sx_n + (1 - \sigma_n)W_n x_n); \\ z_n = J_\lambda^{M_1}(I - \lambda f)(u_n); \\ w_n = J_\lambda^{M_2}(I - \lambda g)(Az_n); \\ y_n = z_n + \gamma A^*(w_n - Az_n); \\ C_n = \{z \in C : \|y_n - z\|^2 \leq (1 - \alpha_n \sigma_n)\|x_n - z\|^2 + \alpha_n \sigma_n \|Sx_n - z\|^2\}; \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}; \\ x_{n+1} = P_{C_n \cap Q_n} x_0, n \geq 0. \end{cases} \quad (10)$$

where  $M_1 : H_1 \rightarrow 2^{H_1}$ ,  $M_2 : H_2 \rightarrow 2^{H_2}$  are multi-valued maximal monotone operators,  $f : C \rightarrow H_1$  is  $\theta_1$ -inverse strongly monotone mapping,  $g : Q \rightarrow H_2$  is  $\theta_2$ -inverse strongly monotone mapping,  $\{T_i\}_{i=1}^N : C \rightarrow C$  is a finite family of nonexpansive mappings and  $W_n$  is a  $W$ -mapping generated by  $T_1, T_2, \dots, T_N$  and  $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$  for all  $n \in \mathbb{N} \cup \{0\}$ .

Based on the results mentioned above, we give our theorem for SCVIP and some important results as follows:

- (i) We first establish Lemma 8 which shows the equivalence between SCVIP and fixed point problem of nonexpansive mapping under suitable conditions on our parameters. Further, we give some example to support Lemma 8 and the example shows that Lemma 8 is not true if some condition fails.
- (ii) We establish a strong convergence theorem of the sequences generated by the modified Halpern iterative method for finding a common solution of hierarchical fixed point problem for a nonexpansive mapping and SCVIP.
- (iii) We apply our main result to obtain a strong convergence theorem of the sequences generated by the modified Halpern iterative method for finding a common solution of hierarchical fixed point problem for a nonexpansive mapping and split combination of variational inequality problem and a strong convergence theorem for finding a common solution of hierarchical fixed point problem for nonexpansive mapping and split zero point problem.
- (iv) We give some illustrative numerical examples to support our main result and our examples show that our main result is not true if some conditions fail.

## 2. Preliminaries

In this paper, we denote weak and strong convergence by the notations ' $\rightharpoonup$ ' and ' $\rightarrow$ ', respectively. We recall some concepts and results needed in the sequel.

Let  $H$  be a real Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . Then for any  $x \in H$ , there exists a unique nearest point in  $C$ , denoted by  $P_C x$ , such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C. \quad (11)$$

The mapping  $P_C$  is called the metric projection of  $H$  onto  $C$ . It is well known that  $P_C$  is nonexpansive and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \forall x, y \in H. \quad (12)$$

Moreover,  $P_Cx$  is characterized by the fact  $P_Cx \in C$  and

$$\langle x - P_Cx, y - P_Cx \rangle \leq 0, \forall y \in C, \quad (13)$$

which implies that

$$\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2, \forall x \in H, y \in C. \quad (14)$$

**Lemma 1** ([4]). Let  $\{a_n\}, \{c_n\} \subset \mathbb{R}^+, \{\alpha_n\} \subset (0, 1)$  and  $\{b_n\} \subset \mathbb{R}$  be sequences such that

$$a_{n+1} = (1 - \alpha_n)a_n + b_n + c_n, \text{ for all } n \geq 0.$$

Assume  $\sum_{n=0}^{\infty} c_n < \infty$ . Then the following results hold:

- (i) if  $b_n \leq \alpha_n C$  where  $C \geq 0$ , then  $\{a_n\}$  is a bounded sequence,
- (ii) if  $\sum_{n=0}^{\infty} \alpha_n = \infty$  and  $\limsup_{n \rightarrow \infty} \frac{b_n}{\alpha_n} \leq 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Lemma 2** ([19]). Let  $E$  be a uniformly convex Banach space,  $C$  a nonempty closed convex subset of  $E$ , and  $S : C \rightarrow C$  a nonexpansive mapping with  $\text{Fix}(S) \neq \emptyset$ . Then  $I - S$  is demiclosed at zero.

**Lemma 3** ([4]). Let  $u \in H$  be a solution of variational inclusion if and only if  $u = J_{\lambda}^M(u - \lambda Bu), \forall \lambda > 0$ , i.e.,

$$\text{VI}(H, B, M) = \text{Fix}(J_{\lambda}^M(I - \lambda B)), \forall \lambda > 0.$$

where  $B : H \rightarrow H$  is a single-valued mapping. Further, if  $\lambda \in (0, 2\alpha]$ , then  $\text{VI}(H, B, M)$  is a closed convex subset in  $H$ .

**Lemma 4** ([4]). The resolvent operator  $J_{\lambda}^M$  associated with  $M$  is single-valued, nonexpansive for all  $\lambda > 0$  and 1-inverse strongly monotone.

**Lemma 5** ([2]). Let  $H$  be a real Hilbert space and let  $M : H \rightarrow 2^H$  be a multi-valued maximal monotone mapping. For every  $i = 1, 2, \dots, N$ , let  $A_i : H \rightarrow H$  be  $\alpha_i$ -inverse strongly monotone mapping with  $\eta = \min_{i=1, 2, \dots, N} \{\alpha_i\}$  and  $\bigcap_{i=1}^N \text{VI}(H, A_i, M) \neq \emptyset$ . Then

$$\text{VI}\left(H, \sum_{i=1}^N a_i A_i, M\right) = \bigcap_{i=1}^N \text{VI}(H, A_i, M),$$

where  $\sum_{i=1}^N a_i = 1$  and  $0 < a_i < 1$  for every  $i = 1, 2, \dots, N$ . Moreover,  $J_{\lambda}^M(I - \lambda \sum_{i=1}^N a_i A_i)$  is a nonexpansive mapping, for all  $0 < \lambda < 2\eta$ .

**Example 1.** Let  $H = \mathbb{R}$ . For every  $i = 1, 2, \dots, N$ , let  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $A_i x = \frac{ix}{4} + (i + 1)$  for all  $x \in H$  and  $M : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $Mx = \{\frac{x}{4}\}$  for all  $x \in \mathbb{R}$ . Let  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$  for all  $i = 1, 2, \dots, N$ . Then  $\text{VI}(H, \sum_{i=1}^N a_i A_i, M) = \bigcap_{i=1}^N \text{VI}(H, A_i, M)$ .

**Proof of Solution.** Since  $A_i x = \frac{ix}{4} + (i + 1)$ , we have  $A_i$  is  $\frac{1}{4}$ -inverse strongly monotone mapping. By definition of  $a_i$  and  $A_i$ , we have

$$\sum_{i=1}^N a_i A_i x = \sum_{i=1}^N \left(\frac{3}{4^i} + \frac{1}{N4^N}\right) A_i x = \sum_{i=1}^N \left(\frac{3}{4^i} + \frac{1}{N4^N}\right) \left(\frac{ix}{4} + (i + 1)\right).$$

From Lemma 5, we have  $\text{VI}(H, \sum_{i=1}^N a_i A_i, M) = \bigcap_{i=1}^N \text{VI}(H, A_i, M) = \{-4\}$ .  $\square$

**Example 2.** Let  $H = \mathbb{R}$ . For every  $i = 1, 2, \dots, N$ , let  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $A_i x = \frac{ix}{4} + (i + 1)$  for all  $x \in H$  and  $M : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $Mx = \{\frac{x}{4}\}$  for all  $x \in \mathbb{R}$ . Let  $a_i = \frac{3}{4} + \frac{1}{N}(\frac{1}{4^N} + 1)$  for all  $i = 1, 2, \dots, N$ . Then  $VI(H, \sum_{i=1}^N a_i A_i, M) \neq \bigcap_{i=1}^N VI(H, A_i, M)$ .

**Proof of Solution.** Since  $A_i x = \frac{ix}{4} + (i + 1)$ , we have  $A_i$  is  $\frac{4}{i}$ -inverse strongly monotone mapping. By definition of  $a_i$  and  $A_i$ , we have

$$\sum_{i=1}^N a_i A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N} \left( \frac{1}{4^N} + 1 \right) \right) A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N} \left( \frac{1}{4^N} + 1 \right) \right) \left( \frac{ix}{4} + (i + 1) \right).$$

Then  $\bigcap_{i=1}^N VI(H, A_i, M) = \{-4\}$  and  $VI(H, \sum_{i=1}^N a_i A_i, M) \neq \{-4\}$ . It implies that  $VI(H, \sum_{i=1}^N a_i A_i, M) \neq \bigcap_{i=1}^N VI(H, A_i, M)$  because  $\sum_{i=1}^N a_i = 2$ .  $\square$

**Remark 2.** Example 1 shows that Lemma 5 is true where  $\sum_{i=1}^N a_i = 1$  and Example 2 shows that Lemma 5 is not true if a condition fails, that is  $\sum_{i=1}^N a_i \neq 1$ .

**Lemma 6 ([17]).** Let  $C \subseteq H$  be a nonempty closed and convex set and let  $T : C \rightarrow H$  be a nonexpansive mapping. Then  $\text{Fix}(T)$  is closed and convex.

**Lemma 7.** Let  $H_1$  and  $H_2$  be Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator. For every  $i = 1, 2, \dots, N$ , let  $A_i : H_1 \rightarrow H_1$  be  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1,2,\dots,N} \{\alpha_i\}$  and  $B_i : H_2 \rightarrow H_2$  be  $\beta_i$ -inverse strongly monotone with  $\eta_B = \min_{i=1,2,\dots,N} \{\beta_i\}$ . For each  $x, y \in H_1$ , then

$$\begin{aligned} & \left\| J_{\lambda_A}^{M_A} \left( I - \lambda_A \sum_{i=1}^N a_i A_i \right) \left( x - \gamma A^* \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A x \right) \right. \\ & \left. - J_{\lambda_A}^{M_A} \left( I - \lambda_A \sum_{i=1}^N a_i A_i \right) \left( y - \gamma A^* \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A y \right) \right\|^2 \\ & \leq \|x - y\|^2 - \gamma(1 - \gamma L) \left\| \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A x - \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A y \right\|^2, \end{aligned}$$

where  $\lambda_A \in (0, 2\eta_A)$ ,  $\lambda_B \in (0, 2\eta_B)$ ,  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^* A$ .

**Proof.** Let  $x, y \in H_1$ . Consider

$$\begin{aligned} & \left\| J_{\lambda_A}^{M_A} \left( I - \lambda_A \sum_{i=1}^N a_i A_i \right) \left( x - \gamma A^* \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A x \right) \right. \\ & \left. - J_{\lambda_A}^{M_A} \left( I - \lambda_A \sum_{i=1}^N a_i A_i \right) \left( y - \gamma A^* \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A y \right) \right\|^2 \\ & \leq \left\| (x - y) - \gamma \left( A^* \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A x - A^* \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A y \right) \right\|^2 \\ & = \|x - y\|^2 - 2\gamma \langle x - y, A^* \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A x - A^* \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A y \rangle \\ & \quad + \gamma^2 \left\| A^* \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A x - A^* \left( I - J_{\lambda_B}^{M_B} \left( I - \lambda_B \sum_{i=1}^N b_i B_i \right) \right) A y \right\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|x - y\|^2 + 2\gamma \langle Ay - Ax, (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay \rangle \\
&\quad + \gamma^2 L \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \\
&= \|x - y\|^2 + 2\gamma \langle Ay - Ax + J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax \\
&\quad + J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ay - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ay, (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax \\
&\quad - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay \rangle \\
&\quad + \gamma^2 L \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \\
&= \|x - y\|^2 + 2\gamma \left[ \langle J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ay - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax, \right. \\
&\quad \left. (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay \rangle \right. \\
&\quad \left. - \langle (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay, \right. \\
&\quad \left. (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay \rangle \right] \\
&\quad + \gamma^2 L \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \\
&\leq \|x - y\|^2 + 2\gamma \left[ \frac{1}{2} \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \right. \\
&\quad \left. - \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \right] \\
&\quad + \gamma^2 L \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2 \\
&= \|x - y\|^2 - \gamma(1 - \gamma L) \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax) \\
&\quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(y - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay)\|^2 \\
&\leq \|x - y\|^2 - \gamma(1 - \gamma L) \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2. \quad \square
\end{aligned}$$

We introduce Lemma 8 which shows an association between the SCVIP and the fixed point problem of nonexpansive mapping under suitable conditions on our parameters. Furthermore, we give examples for supporting Lemma 8 and the examples shows that Lemma 8 is not true if parameters are not satisfied.

**Lemma 8.** Let  $H_1$  and  $H_2$  be Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator. For every  $i = 1, 2, \dots, N$ , let  $A_i : H_1 \rightarrow H_1$  be  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1,2,\dots,N} \{\alpha_i\}$  and  $B_i : H_2 \rightarrow H_2$  be  $\beta_i$ -inverse strongly monotone with  $\eta_B = \min_{i=1,2,\dots,N} \{\beta_i\}$ . Suppose that  $\Omega \neq \emptyset$ . Then the following are equivalent:

- (i)  $x^* \in \Omega$   
(ii)  $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*)$ ,

where  $\lambda_A \in (0, 2\eta_A)$ ,  $\lambda_B \in (0, 2\eta_B)$ ,  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ .

**Proof.** Let the condition holds.

(i)  $\Rightarrow$  (ii) Let  $x^* \in \Omega$ , we have  $x^* \in VI(H_1, \sum_{i=1}^N a_i A_i, M_A)$  and  $Ax^* \in VI(H_2, \sum_{i=1}^N b_i B_i, M_B)$ .

From Lemma 3, we have  $x^* \in \text{Fix}(J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i))$  and  $Ax^* \in \text{Fix}(J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))$ ,

which implies that  $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^*$  and  $Ax^* = J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax^*$ .

By  $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^*$  and  $Ax^* = J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax^*$ , we have

$$\begin{aligned} & J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) \\ &= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(Ax^* - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax^*)) \\ &= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^* \\ &= x^*. \end{aligned}$$

It implies that

$$J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = x^*. \quad (15)$$

(ii)  $\Rightarrow$  (i) Let  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = x^*$  and let  $w \in \Omega$ .

We will show that  $I - \lambda_A \sum_{i=1}^N a_i A_i$  and  $I - \lambda_B \sum_{i=1}^N b_i B_i$  are nonexpansive, for all  $i = 1, 2, \dots, N$ .

Since  $A_i : C \rightarrow H$  be  $\alpha_i$ -inverse strongly monotone mapping with  $\eta_A = \min_{i=1,2,\dots,N} \{\alpha_i\}$  and  $\lambda_A \in (0, 2\eta_A)$ , we have

$$\begin{aligned} & \|(I - \lambda_A \sum_{i=1}^N a_i A_i)x - (I - \lambda_A \sum_{i=1}^N a_i A_i)y\|^2 \\ &= \|x - y\|^2 - 2\lambda_A \sum_{i=1}^N a_i \langle x - y, A_i x - A_i y \rangle + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_A \sum_{i=1}^N a_i \alpha_i \|A_i x - A_i y\|^2 + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_A \eta_A \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \\ &= \|x - y\|^2 + \lambda_A \sum_{i=1}^N a_i (\lambda_A - 2\eta_A) \|A_i x - A_i y\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Thus  $I - \lambda_A \sum_{i=1}^N a_i A_i$  is a nonexpansive mapping, for all  $i = 1, 2, \dots, N$ . By using the same proof, we obtain that  $I - \lambda_B \sum_{i=1}^N b_i B_i$ , for all  $i = 1, 2, \dots, N$  is a nonexpansive mapping and  $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)$  is nonexpansive mapping.

From  $w \in \Omega$  and (i)  $\Rightarrow$  (ii), we have  $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Aw = Aw$  and  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(w - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Aw) = w$ .

From Lemma 7 and  $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Aw = w$ , we have

$$\begin{aligned} \|x^* - w\|^2 &= \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) \\ &\quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(w - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Aw)\|^2 \\ &\leq \|x^* - w\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^* \\ &\quad - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Aw\|^2 \\ &= \|x^* - w\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*\|^2. \end{aligned} \quad (16)$$

Applying Equation (16), we have

$$Ax^* \in \text{Fix}(J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)). \quad (17)$$

From Lemma 5, we have

$$Ax^* \in VI\left(H_1, \sum_{i=1}^N b_i B_i, M_B\right). \quad (18)$$

From the definition of  $x^*$  and Equation (17), we have

$$\begin{aligned} x^* &= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) \\ &= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^*. \end{aligned}$$

From Lemma 5, we have

$$x^* \in VI\left(H_2, \sum_{i=1}^N a_i A_i, M_A\right). \quad (19)$$

From Equations (18) and (19), we have  $x^* \in \Omega$ .  $\square$

**Example 3.** Let  $H_1 = H_2 = \mathbb{R}$ . For every  $i = 1, 2, \dots, N$ , let  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $A_i x = \frac{ix}{4} + (i + 1)$  for all  $x \in H_1$  and  $B_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $B_i y = \frac{iy}{2} + (i + 1)$  for all  $y \in H_2$ . Let  $M_A : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_A x = \{\frac{x}{4}\}$  for all  $x \in \mathbb{R}$  and  $M_B : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_B y = \{\frac{y}{2}\}$  for all  $y \in \mathbb{R}$ . Let  $Ax = x$ , for all  $x \in \mathbb{R}$ . Let  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$  and  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$  for all  $i = 1, 2, \dots, N$ . Then  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = -4$ .

**Proof of Solution.** It is easy to observe  $A_i$  is  $\frac{4}{7}$ -inverse strongly monotone mapping and  $B_i$  is  $\frac{2}{7}$ -inverse strongly monotone mapping. By definition of  $A_i$ ,  $B_i$  and  $a_i$ ,  $b_i$ , we have

$$\sum_{i=1}^N a_i A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N4^N} \right) A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N4^N} \right) \left( \frac{ix}{4} + (i+1) \right),$$

and

$$\sum_{i=1}^N b_i B_i y = \sum_{i=1}^N \left( \frac{2}{3^i} + \frac{1}{N3^N} \right) B_i y = \sum_{i=1}^N \left( \frac{2}{3^i} + \frac{1}{N3^N} \right) \left( \frac{iy}{2} + (i+1) \right).$$

Then  $\Omega = \{-4\}$ . From definition of  $A$ , we have  $L = 1$ . Choose  $\lambda_A = \frac{1}{N}$ ,  $\lambda_B = \frac{1}{N}$  and  $\gamma = \frac{1}{10}$ . From Lemma 8, we have  $J_{\lambda_A}^{M_A} (I - \lambda_A \sum_{i=1}^N a_i A_i) (x^* - \gamma A^* (I - J_{\lambda_B}^{M_B} (I - \lambda_B \sum_{i=1}^N b_i B_i)) A x^*) = -4$ .  $\square$

**Example 4.** Let  $H_1 = H_2 = \mathbb{R}$ . For every  $i = 1, 2, \dots, N$ , let  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $A_i x = \frac{ix}{4} + (i+1)$  for all  $x \in H_1$  and  $B_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $B_i y = \frac{iy}{2} + (i+1)$  for all  $y \in H_2$ . Let  $M_A : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_A x = \{\frac{x}{4}\}$  for all  $x \in \mathbb{R}$  and  $M_B : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_B y = \{\frac{y}{2}\}$  for all  $y \in \mathbb{R}$ . Let  $Ax = x$ , for all  $x \in \mathbb{R}$ . Let  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$  and  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$  for all  $i = 1, 2, \dots, N$ . Then  $J_{\lambda_A}^{M_A} (I - \lambda_A \sum_{i=1}^N a_i A_i) (x^* - \gamma A^* (I - J_{\lambda_B}^{M_B} (I - \lambda_B \sum_{i=1}^N b_i B_i)) A x^*) = x^*$  for all  $x^* \in \mathbb{R}$ .

**Proof of Solution.** It is easy to observe  $A_i$  is  $\frac{4}{7}$ -inverse strongly monotone mapping and  $B_i$  is  $\frac{2}{7}$ -inverse strongly monotone mapping. By definition of  $A_i$ ,  $B_i$  and  $a_i$ ,  $b_i$ , we have

$$\sum_{i=1}^N a_i A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N4^N} \right) A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N4^N} \right) \left( \frac{ix}{4} + (i+1) \right),$$

and

$$\sum_{i=1}^N b_i B_i y = \sum_{i=1}^N \left( \frac{2}{3^i} + \frac{1}{N3^N} \right) B_i y = \sum_{i=1}^N \left( \frac{2}{3^i} + \frac{1}{N3^N} \right) \left( \frac{iy}{2} + (i+1) \right).$$

Then  $\Omega = \{-4\}$ . From definition of  $A$ , we have  $L = 1$ . Choose  $\lambda_A = 0$ ,  $\lambda_B = 0$  and  $\gamma = \frac{1}{10}$ , we have  $J_{\lambda_A}^{M_A} (I - \lambda_A \sum_{i=1}^N a_i A_i) (x^* - \gamma A^* (I - J_{\lambda_B}^{M_B} (I - \lambda_B \sum_{i=1}^N b_i B_i)) A x^*) = x^*$  for all  $x^* \in \mathbb{R}$ .

So Example 4 shows that Lemma 8 is not true because  $\lambda_A = 0$  and  $\lambda_B = 0$ .  $\square$

### 3. Main Result

We prove a strong convergence theorem to approximate a common solution of SCVIP and hierarchical fixed point problem of nonexpansive mapping.

**Theorem 1.** Let  $H_1, H_2$  be real Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $A^*$ . Let  $A_i : H_1 \rightarrow H_1$  be  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1}^N \{\alpha_i\}$  and  $B_i : H_2 \rightarrow H_2$  be  $\beta_i$ -inverse strongly monotone with  $\eta_B = \min_{i=1}^N \{\beta_i\}$ . Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{R} = \Phi \cap \Omega \neq \emptyset$ . Let the sequence  $\{x_n\}$  generated by  $u, x_1 \in H_1$  and

$$\begin{cases} u_n = J_{\lambda_A}^{M_A} (I - \lambda_A \sum_{i=1}^N a_i A_i) (x_n - \gamma A^* (I - J_{\lambda_B}^{M_B} (I - \lambda_B \sum_{i=1}^N b_i B_i)) A x_n), \\ y_n = (1 - \alpha_n) x_n + \alpha_n (\sigma_n S x_n + (1 - \sigma_n) T x_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n, \end{cases} \quad (20)$$

where  $\{\mu_n\}$ ,  $\{\varphi_n\}$ ,  $\{\theta_n\}$ ,  $\{\alpha_n\}$ ,  $\{\sigma_n\} \subseteq [0, 1]$  with  $\mu_n + \varphi_n + \theta_n = 1$  for all  $n \geq 1$ ,  $\lambda_A \in (0, 2\eta_A)$ ,  $\lambda_B \in (0, 2\eta_B)$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ . Suppose the following conditions hold:

(i)  $\lim_{n \rightarrow \infty} \mu_n = 0$ ,  $\sum_{n=1}^{\infty} \mu_n = \infty$ ,

- (ii)  $0 < c \leq \varphi_n, \theta_n \leq d < 1, \exists c, d > 0$ ,  
 (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$   
 (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty$ ,  
 (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0$ ,  
 (vi)  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1, a_i > 0$  and  $b_i > 0$  for all  $i = 1, 2, \dots, N$ .

Then  $\{x_n\}$  converges strongly to  $z_0 \in \mathbb{F}$ , where  $z_0 = P_{\mathbb{F}}u$ .

**Proof.** Step1. First, we prove that  $\{x_n\}$ ,  $\{y_n\}$  and  $\{u_n\}$  are bounded.

We will show that  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)$  and  $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)$  are nonexpansive mapping. Since  $A_i$  is  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1} \{\alpha_i\}$ , we have

$$\begin{aligned} \left\| (I - \lambda_A \sum_{i=1}^N a_i A_i)x - (I - \lambda_A \sum_{i=1}^N a_i A_i)y \right\|^2 &= \left\| (x - y) - \lambda_A \left( \sum_{i=1}^N a_i A_i x - \sum_{i=1}^N a_i A_i y \right) \right\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_A \sum_{i=1}^N a_i \langle x - y, A_i x - A_i y \rangle \\ &\quad + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_A \sum_{i=1}^N a_i \alpha_i \|A_i x - A_i y\|^2 \\ &\quad + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \\ &\leq \|x - y\|^2 + \lambda_A \sum_{i=1}^N a_i (\lambda_n - 2\eta_A) \|A_i x - A_i y\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Thus  $I - \lambda_A \sum_{i=1}^N a_i A_i$  is a nonexpansive mapping, for all  $i = 1, 2, \dots, N$ . By using the same proof, we obtain that  $I - \lambda_B \sum_{i=1}^N b_i B_i$  is a nonexpansive mapping. Since  $J_{\lambda_A}^{M_A}$  and  $J_{\lambda_B}^{M_B}$  are nonexpansive mapping, we have  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)$  and  $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)$  are nonexpansive mapping.

Let  $p \in \mathbb{F}$  then  $p \in H_1$  and  $p \in \Phi$  which  $Tp = p$ . Now, we estimate

$$\begin{aligned} \|y_n - p\|^2 &= \|(1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n) - p\|^2 \\ &= \|(1 - \alpha_n)(x_n - p) + \alpha_n(\sigma_n(Sx_n - p) + (1 - \sigma_n)(Tx_n - p))\|^2 \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|\sigma_n(Sx_n - p) + (1 - \sigma_n)(Tx_n - p)\|^2 \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\sigma_n\|Sx_n - p\|^2 + \alpha_n(1 - \sigma_n)\|Tx_n - p\|^2 \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\sigma_n\|Sx_n - p\|^2 + \alpha_n(1 - \sigma_n)\|x_n - p\|^2 \\ &= (1 - \alpha_n\sigma_n)\|x_n - p\|^2 + \alpha_n\sigma_n\|Sx_n - p\|^2. \end{aligned} \quad (21)$$

Since  $p \in \mathbb{F}$ , then  $p \in \Omega$  and  $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)p = p$  and  $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ap = Ap$ . By Lemma 8, we have

$$J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(p - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ap) = p.$$

By Lemma 7, we have

$$\begin{aligned}
 \|u_n - p\|^2 &= \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n) \\
 &\quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(p - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ap)\|^2 \\
 &\leq \|x_n - p\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ap\|^2 \\
 &\leq \|x_n - p\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n\|^2 \\
 &\leq \|x_n - p\|^2.
 \end{aligned} \tag{22}$$

By Equations (21) and (22), we have

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &= \|\mu_n u + \varphi_n y_n + \theta_n u_n - p\|^2 \\
 &\leq \mu_n \|u - p\|^2 + \varphi_n \|y_n - p\|^2 + \theta_n \|u_n - p\|^2 \\
 &\leq \mu_n \|u - p\|^2 + \varphi_n [(1 - \alpha_n \sigma_n) \|x_n - p\|^2 + \alpha_n \sigma_n \|Sx_n - p\|^2] + \theta_n \|x_n - p\|^2 \\
 &= \mu_n \|u - p\|^2 + \varphi_n \|x_n - p\|^2 - \varphi_n \alpha_n \sigma_n \|x_n - p\|^2 + \varphi_n \alpha_n \sigma_n \|Sx_n - p\|^2 + \theta_n \|x_n - p\|^2 \\
 &= \mu_n \|u - p\|^2 + (1 - \mu_n) \|x_n - p\|^2 - \varphi_n \alpha_n \sigma_n \|x_n - p\|^2 + \varphi_n \alpha_n \sigma_n \|Sx_n - p\|^2 \\
 &\leq (1 - \mu_n) \|x_n - p\|^2 + \mu_n \|u - p\|^2 + \mu_n \alpha_n \sigma_n \|Sx_n - p\|^2.
 \end{aligned} \tag{23}$$

From Lemma 1(i), therefore  $\{x_n\}$  is bounded. So are  $\{u_n\}, \{y_n\}$ .

Step2. Show that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ ,  $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$  and  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .

$$\begin{aligned}
 \|x_{n+1} - x_n\| &= \|\mu_n u + \varphi_n y_n + \theta_n u_n - \mu_{n-1} u - \varphi_{n-1} y_{n-1} - \theta_{n-1} u_{n-1}\| \\
 &= \|(\mu_n - \mu_{n-1})u + (\varphi_n - \varphi_{n-1})y_{n-1} + \varphi_n(y_n - y_{n-1}) \\
 &\quad + (\theta_n - \theta_{n-1})u_{n-1} + \theta_n(u_n - u_{n-1})\| \\
 &\leq |\mu_n - \mu_{n-1}| \|u\| + |\varphi_n - \varphi_{n-1}| \|y_{n-1}\| + \varphi_n \|y_n - y_{n-1}\| \\
 &\quad + |\theta_n - \theta_{n-1}| \|u_{n-1}\| + \theta_n \|u_n - u_{n-1}\|.
 \end{aligned} \tag{24}$$

From definition of  $u_n$ , Lemma 7 and  $\gamma \in (0, \frac{1}{L})$ , we have

$$\begin{aligned}
 \|u_n - u_{n-1}\|^2 &= \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n) \\
 &\quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_{n-1} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n-1})\|^2 \\
 &\leq \|x_n - x_{n-1}\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n \\
 &\quad - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n-1}\|^2 \\
 &\leq \|x_n - x_{n-1}\|^2.
 \end{aligned}$$

It implies that

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\|. \tag{25}$$

From definition of  $y_n$ , we have

$$\begin{aligned}
 \|y_n - y_{n-1}\| &= \|(1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n) \\
 &\quad - [(1 - \alpha_{n-1})x_{n-1} + \alpha_{n-1}(\sigma_{n-1}Sx_{n-1} + (1 - \sigma_{n-1})Tx_{n-1})]\| \\
 &= \|(x_n - x_{n-1}) - \alpha_n x_n + \alpha_{n-1} x_{n-1} + \alpha_n x_{n-1} - \alpha_n x_{n-1} \\
 &\quad + \alpha_n \sigma_n Sx_n - \alpha_{n-1} \sigma_{n-1} Sx_{n-1} + \alpha_n \sigma_n Sx_{n-1} - \alpha_n \sigma_n Sx_{n-1} \\
 &\quad + \alpha_n Tx_n - \alpha_{n-1} Tx_{n-1} + \alpha_n Tx_{n-1} - \alpha_n Tx_{n-1} \\
 &\quad - \alpha_n \sigma_n Tx_n + \alpha_{n-1} \sigma_{n-1} Tx_{n-1} + \alpha_n \sigma_n Tx_{n-1} - \alpha_n \sigma_n Tx_{n-1}\| \\
 &= \|(1 - \alpha_n)(x_n - x_{n-1}) + (\alpha_{n-1} - \alpha_n)x_{n-1} \\
 &\quad + (\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1})Sx_{n-1} + \alpha_n \sigma_n (Sx_n - Sx_{n-1}) \\
 &\quad + \alpha_n (1 - \sigma_n)(Tx_n - Tx_{n-1}) + (\alpha_n - \alpha_{n-1})Tx_{n-1} \\
 &\quad + (\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n)Tx_{n-1}\| \\
 &\leq (1 - \alpha_n)\|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
 &\quad + |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}|\|Sx_{n-1}\| + \alpha_n \sigma_n \|Sx_n - Sx_{n-1}\| \\
 &\quad + \alpha_n (1 - \sigma_n)\|Tx_n - Tx_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Tx_{n-1}\| \\
 &\quad + |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n|\|Tx_{n-1}\| \\
 &\leq (1 - \alpha_n)\|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
 &\quad + |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}|\|Sx_{n-1}\| + \alpha_n \sigma_n \|x_n - x_{n-1}\| \\
 &\quad + \alpha_n (1 - \sigma_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Tx_{n-1}\| \\
 &\quad + |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n|\|Tx_{n-1}\| \\
 &= \|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
 &\quad + |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}|\|Sx_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Tx_{n-1}\| \\
 &\quad + |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n|\|Tx_{n-1}\|. \tag{26}
 \end{aligned}$$

From Equations (24)–(26), we have

$$\begin{aligned}
 \|x_{n+1} - x_n\| &\leq |\mu_n - \mu_{n-1}|\|u\| + |\varphi_n - \varphi_{n-1}|\|y_{n-1}\| + \varphi_n \|y_n - y_{n-1}\| \\
 &\quad + |\theta_n - \theta_{n-1}|\|u_{n-1}\| + \theta_n \|u_n - u_{n-1}\| \\
 &\leq |\mu_n - \mu_{n-1}|\|u\| + |\varphi_n - \varphi_{n-1}|\|y_{n-1}\| \\
 &\quad + \varphi_n [\|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
 &\quad + |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}|\|Sx_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Tx_{n-1}\| \\
 &\quad + |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n|\|Tx_{n-1}\|] \\
 &\quad + |\theta_n - \theta_{n-1}|\|u_{n-1}\| + \theta_n \|x_n - x_{n-1}\| \\
 &= (\varphi_n + \theta_n)\|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}|\|u\| + |\varphi_n - \varphi_{n-1}|\|y_{n-1}\| \\
 &\quad + \varphi_n |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| + \varphi_n |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}|\|Sx_{n-1}\| \\
 &\quad + \varphi_n |\alpha_n - \alpha_{n-1}|\|Tx_{n-1}\| + \varphi_n |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n|\|Tx_{n-1}\| \\
 &\quad + |\theta_n - \theta_{n-1}|\|u_{n-1}\| \\
 &\leq (1 - \mu_n)\|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}|\|u\| + |\varphi_n - \varphi_{n-1}|\|y_{n-1}\| \\
 &\quad + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| + |\alpha_n \sigma_n - \alpha_{n-1} \sigma_{n-1}|\|Sx_{n-1}\| \\
 &\quad + |\alpha_n - \alpha_{n-1}|\|Tx_{n-1}\| + |\alpha_{n-1} \sigma_{n-1} - \alpha_n \sigma_n|\|Tx_{n-1}\| \\
 &\quad + |\theta_n - \theta_{n-1}|\|u_{n-1}\|.
 \end{aligned}$$

By Lemma 1(i), conditions (i) and (iii), we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (27)$$

From definition of  $u_n$ , we have

$$\begin{aligned} x_{n+1} - u_n &= \mu_n u + \varphi_n y_n + \theta_n u_n - u_n \\ &= \mu_n (u - u_n) + \varphi_n (y_n - u_n). \end{aligned} \quad (28)$$

From Equations (21) and (22), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\mu_n u + \varphi_n y_n + \theta_n u_n - p\|^2 \\ &= \mu_n \|u - p\|^2 + \varphi_n \|y_n - p\|^2 + \theta_n \|u_n - p\|^2 \\ &\quad - \mu_n \varphi_n \|u - y_n\|^2 - \mu_n \theta_n \|u - u_n\|^2 - \varphi_n \theta_n \|y_n - u_n\|^2 \\ &\leq \mu_n \|u - p\|^2 + \varphi_n \|y_n - p\|^2 + \theta_n \|u_n - p\|^2 - \varphi_n \theta_n \|y_n - u_n\|^2 \\ &\leq \mu_n \|u - p\|^2 + \varphi_n [(1 - \alpha_n \sigma_n) \|x_n - p\|^2 + \alpha_n \sigma_n \|Sx_n - p\|^2] \\ &\quad + \theta_n \|x_n - p\|^2 - \varphi_n \theta_n \|y_n - u_n\|^2 \\ &= \mu_n \|u - p\|^2 + \varphi_n \|x_n - p\|^2 - \varphi_n \alpha_n \sigma_n \|x_n - p\|^2 \\ &\quad + \varphi_n \alpha_n \sigma_n \|Sx_n - p\|^2 + \theta_n \|x_n - p\|^2 - \varphi_n \theta_n \|y_n - u_n\|^2 \\ &\leq \mu_n \|u - p\|^2 + (1 - \mu_n) \|x_n - p\|^2 + \varphi_n \alpha_n \sigma_n \|Sx_n - p\|^2 \\ &\quad - \varphi_n \theta_n \|y_n - u_n\|^2 \\ &\leq \mu_n \|u - p\|^2 + \|x_n - p\|^2 + \varphi_n \alpha_n \sigma_n K - \varphi_n \theta_n \|y_n - u_n\|^2, \end{aligned}$$

where  $K = \sup_n \{\|Sx_n - p\|^2\}$ . It follows that

$$\begin{aligned} \varphi_n \theta_n \|y_n - u_n\|^2 &\leq \mu_n \|u - p\|^2 + \varphi_n \alpha_n \sigma_n K + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq \mu_n \|u - p\|^2 + \varphi_n \alpha_n \sigma_n K + \|x_n - x_{n+1}\| (\|x_n - p\| + \|x_{n+1} - p\|) \\ &\leq \mu_n \|u - p\|^2 + \varphi_n \alpha_n \sigma_n K + \|x_n - x_{n+1}\| L_1, \end{aligned}$$

where  $L_1 = \sup_n \{\|x_n - p\| + \|x_{n+1} - p\|\}$ . From Equation (27), conditions (i), (ii) and (v), we have

$$\lim_{n \rightarrow \infty} \|y_n - u_n\| = 0. \quad (29)$$

From Equations (28) and (29), we have

$$\begin{aligned} \|x_{n+1} - u_n\| &= \|\mu_n (u - u_n) + \varphi_n (y_n - u_n)\| \\ &\leq \mu_n \|u - u_n\| + \varphi_n \|y_n - u_n\|. \end{aligned}$$

From Equation (29) and a condition (i), we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0. \quad (30)$$

Since

$$\begin{aligned} \|x_n - u_n\| &= \|x_n - x_{n+1} + x_{n+1} - u_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|. \end{aligned}$$

From Equations (28) and (30), we have

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (31)$$

Since

$$\begin{aligned} \|x_n - y_n\| &= \|x_n - u_n + u_n - y_n\| \\ &\leq \|x_n - u_n\| + \|u_n - y_n\|. \end{aligned}$$

From Equations (29) and (31), we have

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (32)$$

Step3.  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ .

We have

$$\|x_n - Tx_n\| \leq \|x_n - y_n\| + \|y_n - Tx_n\|. \quad (33)$$

Since  $\{x_n\}$  is bounded and the mappings  $S, T$  are nonexpansive then there exists a  $K_1 > 0$  such that  $\|Sx_n - Tx_n\| \leq K_1$ , for all  $n \geq 0$ . Now, we estimate

$$\begin{aligned} \|y_n - Tx_n\| &= \|(1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n) - Tx_n\| \\ &= \|(1 - \alpha_n)(x_n - Tx_n) + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n - Tx_n)\| \\ &= \|(1 - \alpha_n)(x_n - Tx_n) + \alpha_n(\sigma_n Sx_n - \sigma_n Tx_n)\| \\ &\leq (1 - \alpha_n)\|x_n - Tx_n\| + \alpha_n \sigma_n \|Sx_n - Tx_n\| \\ &\leq (1 - \alpha_n)(\|x_n - y_n\| + \|y_n - Tx_n\|) + \alpha_n \sigma_n \|Sx_n - Tx_n\| \\ &\leq (1 - \alpha_n)\|x_n - u_n\| + (1 - \alpha_n)\|y_n - Tx_n\| + \alpha_n \sigma_n \|Sx_n - Tx_n\|, \end{aligned}$$

which implies

$$\begin{aligned} \alpha_n \|y_n - Tx_n\| &\leq (1 - \alpha_n)\|x_n - y_n\| + \alpha_n \sigma_n \|Sx_n - Tx_n\| \\ &\leq \|x_n - y_n\| + \alpha_n \sigma_n K_1. \end{aligned}$$

It follow that

$$\|y_n - Tx_n\| \leq \frac{\|x_n - y_n\|}{\alpha_n} + \sigma_n K_1. \quad (34)$$

Since  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0$ , we have  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n} = \lim_{n \rightarrow \infty} \sigma_n \cdot \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0$ .

From  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n} = 0$ , Equation (34) and a condition (v), we have

$$\lim_{n \rightarrow \infty} \|y_n - Tx_n\| = 0. \quad (35)$$

Thus, it follows from Equations (32), (33) and (35), we have

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \quad (36)$$

Step4.  $x^* \in \mathbb{F}$

Since  $\{x_n\}$  is bounded, there exists a subsequence  $\{x_{n_k}\}$  which converges weakly to  $x^*$ . We may assume that

$$\liminf_{n \rightarrow \infty} \langle -x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle = \lim_{k \rightarrow \infty} \langle -x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \rangle,$$

and

$$\liminf_{n \rightarrow \infty} \langle Sx_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle = \lim_{k \rightarrow \infty} \langle Sx_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \rangle.$$

We will show that  $x^* \in \text{Fix}(T)$ . Assume that  $x^* \notin \text{Fix}(T)$ , then  $x^* \neq Tx^*$  and using Opial's property of Hilbert space and Equation (35), we have

$$\begin{aligned} \liminf_{k \rightarrow \infty} \|x_{n_k} - x^*\| &< \liminf_{k \rightarrow \infty} \|x_{n_k} - Tx_{n_k}\| \\ &\leq \liminf_{k \rightarrow \infty} (\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tx^*\|) \\ &\leq \liminf_{k \rightarrow \infty} \|x_{n_k} - x^*\|, \end{aligned}$$

which is a contradiction. Therefore,  $x^* \in \text{Fix}(T)$ .

Next, we show that  $x^* \in \Phi$ . Consider

$$\begin{aligned} y_n - x_n &= (1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n) - x_n \\ &= \alpha_n \sigma_n (Sx_n - x_n) + \alpha_n (1 - \sigma_n)(Tx_n - x_n), \end{aligned}$$

which implies

$$\begin{aligned} Sx_n - x_n &= \frac{y_n - x_n}{\alpha_n \sigma_n} - \frac{\alpha_n (1 - \sigma_n)(Tx_n - x_n)}{\alpha_n \sigma_n} \\ &= \frac{y_n - x_n}{\alpha_n \sigma_n} + \frac{(1 - \sigma_n)(I - T)x_n}{\sigma_n}. \end{aligned}$$

It follows that

$$Sx_n - x_n - \frac{y_n - x_n}{\alpha_n \sigma_n} = \frac{(1 - \sigma_n)(I - T)x_n}{\sigma_n}.$$

Since  $T$  is nonexpansive, we have  $I - T$  is monotone. Let  $x \in \text{Fix}(T)$ , we have

$$\begin{aligned} &\langle Sx_n - x_n - \frac{y_n - x_n}{\alpha_n \sigma_n}, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \\ &= \frac{(1 - \sigma_n)}{\sigma_n} \langle (I - T)x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \\ &= \frac{(1 - \sigma_n)}{\sigma_n} \langle (I - T)x_n - (I - T)(x - \frac{y_n - x_n}{\alpha_n}) + (I - T)(x - \frac{y_n - x_n}{\alpha_n}), \\ &\quad x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \\ &= \frac{(1 - \sigma_n)}{\sigma_n} \left[ \langle (I - T)x_n - (I - T)(x - \frac{y_n - x_n}{\alpha_n}), x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \right. \\ &\quad \left. + \langle (I - T)(x - \frac{y_n - x_n}{\alpha_n}), x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \right] \\ &\leq \frac{(1 - \sigma_n)}{\sigma_n} \langle (I - T)(x - \frac{y_n - x_n}{\alpha_n}), x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \\ &\leq \frac{(1 - \sigma_n)}{\sigma_n} \left\| (I - T)(x - \frac{y_n - x_n}{\alpha_n}) \right\| \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\| \\ &= \frac{(1 - \sigma_n)}{\sigma_n} \left\| (I - T)(x - \frac{y_n - x_n}{\alpha_n}) - (I - T)x \right\| \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\| \\ &\leq 2(1 - \sigma_n) \frac{\|y_n - x_n\|}{\alpha_n \sigma_n} \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\|, \end{aligned}$$

which implies

$$\begin{aligned} \langle Sx_n - x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle &\leq 2(1 - \sigma_n) \frac{\|y_n - x_n\|}{\alpha_n \sigma_n} \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\| \\ &\quad + \left\langle \frac{y_n - x_n}{\alpha_n \sigma_n}, x - \frac{y_n - x_n}{\alpha_n} - x_n \right\rangle \\ &\leq 3 \frac{\|y_n - x_n\|}{\alpha_n \sigma_n} \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\|. \end{aligned} \quad (37)$$

Since  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n} = 0$ , we have

$$\lim_{k \rightarrow \infty} \left\langle \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle = 0. \quad (38)$$

From Equation (38) and  $\frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} + x_{n_k} \rightarrow x^*$ , we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle -x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle &= \lim_{k \rightarrow \infty} \langle -x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \rangle \\ &= \lim_{k \rightarrow \infty} \left[ \langle -x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} \rangle + \left\langle x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle \right] \\ &= \lim_{k \rightarrow \infty} \left[ \left\langle x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle \right. \\ &\quad \left. - \left\langle x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle \right] \\ &= \lim_{k \rightarrow \infty} \left[ \left\langle x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle - \left\langle x, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle \right. \\ &\quad \left. + \left\langle \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \right\rangle \right] \\ &= \|x - x^*\|^2 - \langle x, x - x^* \rangle. \end{aligned} \quad (39)$$

Since  $S$  is weakly continuous and  $\frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} + x_{n_k} \rightarrow x^*$ , we obtain

$$\liminf_{n \rightarrow \infty} \langle Sx_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle = \lim_{k \rightarrow \infty} \langle Sx_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k} \rangle = \langle Sx^*, x - x^* \rangle. \quad (40)$$

From Equations (37), (39) and (40), we have

$$\begin{aligned} \langle Sx^* - x^*, x - x^* \rangle &= \langle Sx^*, x - x^* \rangle - \langle x^*, x - x^* \rangle \\ &= \langle Sx^*, x - x^* \rangle + \|x - x^*\|^2 - \langle x, x - x^* \rangle \\ &= \liminf_{n \rightarrow \infty} \left[ \langle Sx_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle - \langle x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \right] \\ &= \liminf_{n \rightarrow \infty} \langle Sx_n - x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n \rangle \\ &\leq \liminf_{n \rightarrow \infty} 3 \frac{\|y_n - x_n\|}{\alpha_n \sigma_n} \left\| x - \frac{y_n - x_n}{\alpha_n} - x_n \right\| \\ &\leq 0. \end{aligned}$$

Hence  $x^*$  solve Hierarchical fixed point problem, i.e.,  $x^* \in \Phi$ .

Next, we show that  $x^* \in \Omega$ . Assume that  $x^* \neq J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*)$ . Applying the Opial's property, Equation (31) and Lemma 7, we have

$$\begin{aligned}
 & \liminf_{k \rightarrow \infty} \|x_{n_k} - x^*\| \\
 & < \liminf_{k \rightarrow \infty} \|x_{n_k} - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*)\| \\
 & = \liminf_{k \rightarrow \infty} \|x_{n_k} - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n_k}) \\
 & \quad + J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n_k}) \\
 & \quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*)\| \\
 & \leq \liminf_{k \rightarrow \infty} \left[ \|x_{n_k} - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n_k})\| \right. \\
 & \quad \left. + \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n_k}) \right. \\
 & \quad \left. - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*)\| \right] \\
 & \leq \liminf_{k \rightarrow \infty} \left[ \|x_{n_k} - u_{n_k}\| + \|x_{n_k} - x^*\| \right. \\
 & \quad \left. - \gamma(1 - \gamma L) \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_{n_k} - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*\| \right] \\
 & \leq \liminf_{k \rightarrow \infty} \left[ \|x_{n_k} - u_{n_k}\| + \|x_{n_k} - x^*\| \right] \\
 & = \liminf_{k \rightarrow \infty} \|x_{n_k} - x^*\|.
 \end{aligned}$$

This is a contradiction. Then  $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*)$ . From Lemma 8, we have  $x^* \in \Omega$ . Therefore,  $x^* \in \mathbb{F}$ .

Step5. Finally, we will prove that  $\{x_n\}$  converges strongly to  $z_0 = P_{\mathbb{F}}u$ .

We show that  $\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle \leq 0$ , where  $z_0 = P_{\mathbb{F}}u$ . We may assume the subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  with

$$\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{k \rightarrow \infty} \langle u - z_0, x_{n_k} - z_0 \rangle. \quad (41)$$

Since  $x_{n_k} \rightarrow x^*$  as  $k \rightarrow \infty$  and  $x^* \in \mathbb{F}$ . By Equations (13) and (41), we have

$$\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{k \rightarrow \infty} \langle u - z_0, x_{n_k} - z_0 \rangle \leq 0 \quad (42)$$

From Equations (21) and (22), we have

$$\begin{aligned}
 \|x_{n+1} - z_0\|^2 &= \|\mu_n u + \varphi_n y_n + \theta u_n - z_0\|^2 \\
 &= \|\mu_n(u - z_0) + \varphi_n(y_n - z_0) + \theta(u_n - z_0)\|^2 \\
 &\leq \|\varphi_n(y_n - z_0) + \theta(u_n - z_0)\|^2 + 2\langle \mu_n(u - z_0), x_{n+1} - z_0 \rangle \\
 &\leq \varphi_n \|y_n - z_0\|^2 + \theta \|u_n - z_0\|^2 + 2\mu_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
 &\leq \varphi_n [(1 - \alpha_n \sigma_n) \|x_n - z_0\|^2 + \alpha_n \sigma_n \|Sx_n - z_0\|^2] \\
 &\quad + \theta \|x_n - z_0\|^2 + 2\mu_n \langle u - z_0, x_{n+1} - z_0 \rangle
 \end{aligned}$$

$$\begin{aligned}
&\leq \varphi_n \|x_n - z_0\|^2 + \varphi_n \alpha_n \sigma_n \|Sx_n - z_0\|^2 \\
&\quad + \theta \|x_n - z_0\|^2 + 2\mu_n \langle u - z_0, x_{n+1} - z_0 \rangle \\
&\leq (1 - \mu_n) \|x_n - z_0\|^2 + \varphi_n \alpha_n \sigma_n \|Sx_n - z_0\|^2 \\
&\quad + 2\mu_n \langle u - z_0, x_{n+1} - z_0 \rangle.
\end{aligned}$$

Applying Lemma 1(ii), conditions (i), (iv) and Equation (42), we can conclude that the  $\{x_n\}$  converges strongly to  $z_0 = P_{\mathbb{F}}u$ . This completes the proof.  $\square$

Next, we have the following strong convergence to approximation a common element of solution the set of SMVI and hierarchical fixed point problem of nonexpansive mapping.

**Corollary 1.** Let  $H_1, H_2$  be real Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $F : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $F^*$ . Let  $A : H_1 \rightarrow H_1$  be  $\alpha$ -inverse strongly monotone and  $B : H_2 \rightarrow H_2$  be  $\beta$ -inverse strongly monotone. Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{F} = \Phi \cap \Theta \neq \emptyset$ . Let the sequence  $\{x_n\}$  generated by  $u, x_1 \in H_1$  and

$$\begin{cases}
u_n = J_{\lambda_A}^{M_A}(I - \lambda_A A)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B B))Fx_n), \\
y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n), \\
x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n,
\end{cases} \quad (43)$$

where  $\{\mu_n\}, \{\varphi_n\}, \{\theta_n\}, \{\alpha_n\}, \{\sigma_n\} \subseteq [0, 1]$  with  $\mu_n + \varphi_n + \theta_n = 1$  for all  $n \geq 1$ ,  $\lambda_A \in (0, 2\alpha)$ ,  $\lambda_B \in (0, 2\beta)$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $F^*F$ . Suppose the following conditions hold:

- (i)  $\lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty$ ,
- (ii)  $0 < c \leq \varphi_n, \theta_n \leq d < 1, \exists c, d > 0$ ,
- (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty$ ,
- (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0$ ,

Then  $\{x_n\}$  converges strongly to  $z_0 \in \mathbb{F}$ , where  $z_0 = P_{\mathbb{F}}u$ .

**Proof.** Put  $A_i \equiv A$  and  $B_i \equiv B$  for all  $i = 1, 2, \dots, N$  in Theorem 1. From Theorem 1, we obtain the desired result.  $\square$

## 4. Application

### 4.1. Split Zero Point Problem

Let  $H$  be a real Hilbert space. Let  $M : H \rightarrow 2^H$  be a maximal monotone operator. Then the zero problem is to find  $x^* \in H$  such that

$$0 \in Mx^*, \quad (44)$$

such an  $x^* \in H$  is called a zero point of  $M$ . The set of zero point of  $M$  is denoted by  $M^{-1}(0)$ .

Let  $H_1$  and  $H_2$  be two real Hilbert spaces. Setting  $A_i \equiv 0$  and  $B_i \equiv 0$  for all  $i = 1, 2, \dots, N$  in SCVIP, then SCVIP reduce to the split zero point problem: Find  $x^* \in H_1$  such that

$$0 \in M_A x^*, \quad (45)$$

and

$$y^* \in Ax^* \text{ such that } 0 \in M_B y^*, \quad (46)$$

where  $A : H_1 \rightarrow H_2$  is bounded linear operator,  $M_A : H_1 \rightarrow 2^{H_1}$  and  $M_B : H_2 \rightarrow 2^{H_2}$  are multi-valued mapping. The set of all solution of this problem is denoted by  $\Omega_2 = \{x \in M_A^{-1}(0) : Ax \in M_B^{-1}(0)\}$ .

The split zero point problem which consists of the special cases, split feasibility problem, variational inequalities, etc., which is used in practice as a model in machine learning, image processing and linear inverse problem.

Next, we give the strong convergence theorem for solving the split zero point problem and the hierarchical fixed point problem of nonexpansive mapping.

**Corollary 2.** Let  $H_1, H_2$  be real Hilbert spaces. Let  $M_A : H_1 \rightarrow 2^{H_1}$  be a multi-valued maximal monotone mapping and  $M_B : H_2 \rightarrow 2^{H_2}$  be a multi-valued maximal monotone mapping. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $A^*$ . Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{F} = \Phi \cap \Omega_2 \neq \emptyset$ . Let the iterative sequence generated by hybrid iterative algorithm:

$$\begin{cases} u_n = J_{\lambda A}^{M_A}(x_n - \gamma A^* J_{\lambda B}^{M_B} A x_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1 - \sigma_n)T x_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n, \end{cases} \quad (47)$$

where  $\{\delta_n\}, \{\varphi_n\}, \{\eta_n\}, \{\alpha_n\}, \{\sigma_n\} \subseteq [0, 1]$  with  $\delta_n + \varphi_n + \eta_n = 1$  for all  $n \geq 1$ , and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ . Suppose the following conditions hold:

- (i)  $\lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty,$
- (ii)  $0 < c \leq \varphi_n, \theta_n \leq d < 1, \exists c, d > 0,$
- (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty,$
- (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0,$

Then  $\{x_n\}$  converges strongly to  $z \in \mathbb{F}$ , where  $z = P_{\mathbb{F}}u$ .

**Proof.** Put  $A_i \equiv 0$  and  $B_i \equiv 0$  for all  $i = 1, 2, \dots, N$  in Theorem 1. From Theorem 1, we obtain the desired conclusion.  $\square$

#### 4.2. Split Combination of Variational Inequalities Problem

Let  $H$  be a real Hilbert space, let  $C$  be a nonempty closed convex subset of  $H$  and let  $h$  be a proper lower semicontinuous convex function of  $H$  into  $(-\infty, +\infty]$ . The subdifferential  $\partial h$  of  $h$  is defined by

$$\partial h(x) = \{z \in H : h(x) + \langle z, u - x \rangle \leq h(u), \forall u \in H\},$$

for all  $x \in H$ . From Rockafellar [16], we get that  $\partial h$  is a maximal monotone operator. Let  $i_C$  be the indicator function of  $C$ , i.e.,

$$i_C = \begin{cases} 0; & \text{if } x \in C, \\ +\infty; & \text{if } x \notin C. \end{cases}$$

Then  $i_C$  is a proper, lower semicontinuous and convex function on  $H$  and so the subdifferential  $\partial i_C$  of  $i_C$  is a maximal monotone operator. The resolvent operator  $J_r^{\partial i_C}$  of  $\partial i_C$  for  $\lambda > 0$  defined by  $J_r^{\partial i_C}(x) = (I + \lambda \partial i_C)^{-1}(x)$ ,  $x \in H$ , then we have  $J_r^{\partial i_C}(x) = P_C x$  for all  $x \in H$  and  $\lambda > 0$ ; see more detail [18]. Moreover, let  $h : H \rightarrow H$  be a single valued operator, we have  $x \in VI(H, h, \partial i_C) = VI(C, h)$ .

Setting  $M_A = \partial i_{H_1}$  and  $M_B = \partial i_{H_2}$  in Equations (6) and (7), then SCVIP reduce to the split combination of variational inequality problem, that is find  $x^* \in H_1$  such that

$$\left\langle \sum_{i=1}^N a_i A_i x^*, x - x^* \right\rangle \geq 0, \forall x \in H_1, \quad (48)$$

and

$$y^* = A x^* \in H_2 \text{ such that } \left\langle \sum_{i=1}^N b_i B_i y^*, y - y^* \right\rangle \geq 0, \forall y \in H_2, \quad (49)$$

where  $A : H_1 \rightarrow H_2$  is bounded linear operator and  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$ . The set of all this is denoted by  $\Omega_3 = \{x \in VI(H_1, \sum_{i=1}^N a_i A_i) : Ax \in VI(H_2, \sum_{i=1}^N b_i B_i)\}$ .

**Remark 3.** If  $M_A = \partial i_{H_1}$  and  $M_B = \partial i_{H_2}$ , then we have  $\Omega$  reduce to  $\Omega_3$ .

**Proof.** We will show that  $VI(H_1, \sum_{i=1}^N a_i A_i, M_A) = VI(H_1, \sum_{i=1}^N a_i A_i)$ . We have for  $x^* \in H_1$ . Consider,

$$\begin{aligned} x^* \in VI(H_1, \sum_{i=1}^N a_i A_i, M_A) &\Leftrightarrow \theta_{H_1} \in \sum_{i=1}^N a_i A_i x^* + M_A x^* \\ &\Leftrightarrow \theta_{H_1} \in \sum_{i=1}^N a_i A_i x^* + \partial i_{H_1}(x^*) \\ &\Leftrightarrow -\sum_{i=1}^N a_i A_i x^* \in \partial i_{H_1}(x^*) \\ &\Leftrightarrow \langle \sum_{i=1}^N a_i A_i x^*, x - x^* \rangle \geq 0, \forall x \in H_1, \\ &\Leftrightarrow x^* \in VI(H_1, \sum_{i=1}^N a_i A_i). \end{aligned}$$

Similarly, we also have  $VI(H_2, \sum_{i=1}^N b_i B_i, M_B) = VI(H_2, \sum_{i=1}^N b_i B_i)$ . Then  $\Omega \equiv \Omega_3$  where  $M_A = \partial i_{H_1}$  and  $M_B = \partial i_{H_2}$ .  $\square$

The split combination of variational inequality problem has played an essential role for concrete problems in dynamic emission tomographic image reconstruction, signal recovery problems, beam-forming problems, power-control problems, bandwidth allocation problems and optimal control problems.

Next, we establish a strong convergence theorem for solving the split combination of variational inequality problem and hierarchical fixed point problem of nonexpansive mapping by using a modified Halpern iterative method as follows:

**Theorem 2.** Let  $H_1, H_2$  be real Hilbert spaces. Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator  $A^*$ . Let  $A_i : H_1 \rightarrow H_1$  be  $\alpha_i$ -inverse strongly monotone with  $\eta_A = \min_{i=1} \{\alpha_i\}$  and  $B_i : H_2 \rightarrow H_2$  be  $\beta_i$ -inverse strongly monotone with  $\eta_B = \min_{i=1} \{\beta_i\}$ . Let  $S, T : H_1 \rightarrow H_1$  be two nonexpansive mappings. Assume that  $\mathbb{F} = \Phi \cap \Omega_3 \neq \emptyset$ . Let the sequence  $\{x_n\}$  generated by  $u, x_1 \in H_1$  and

$$\begin{cases} u_n = P_{H_1}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_n - \gamma A^*(I - P_{H_2}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n, \end{cases} \quad (50)$$

where  $\{\mu_n\}, \{\varphi_n\}, \{\theta_n\}, \{\alpha_n\}, \{\sigma_n\} \subseteq [0, 1]$  with  $\mu_n + \varphi_n + \theta_n = 1$  for all  $n \geq 1$ ,  $\lambda_A \in (0, 2\eta_A)$ ,  $\lambda_B \in (0, 2\eta_B)$  and  $\gamma \in (0, \frac{1}{L})$  with  $L$  is the spectral radius of  $A^*A$ . Suppose the following conditions hold:

- (i)  $\lim_{n \rightarrow \infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty$ ,
- (ii)  $0 < c \leq \varphi_n, \theta_n \leq d < 1, \exists c, d > 0$ ,
- (iii)  $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$
- (iv)  $\lim_{n \rightarrow \infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty$ ,
- (v)  $\lim_{n \rightarrow \infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0$ ,
- (vi)  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1, a_i > 0$  and  $b_i > 0$  for all  $i = 1, 2, \dots, N$ .

Then  $\{x_n\}$  converges strongly to  $z_0 \in \mathbb{F}$ , where  $z_0 = P_{\mathbb{F}}u$ .

**Proof.** Put  $M_A = \partial i_{H_1}$  and  $M_B = \partial i_{H_2}$  in Theorem 1. Using the same method in Theorem 1, we have the desired conclusion.  $\square$

## 5. Numerical

The purpose of this section is to give a numerical example to support some of our. The following example given for supporting Theorem 1 and example show that Theorem 1 is not true if condition (iv) fails, but conditions (i), (ii), (iii), (v) and (vi) are satisfied.

Since Theorem 1 can solve hierarchical fixed point problem for a nonexpansive mapping and SCVIP which our problems can modify for concrete problem in signal processing, image reconstruction, intensity-modulated radiationtherapy treatment planning and sensor networks in computerized tomography. So, we give a numerical example as follows:

**Example 5.** Let  $H_1 = H_2 = \mathbb{R}$ , the set of all real numbers, with the inner product defined by  $\langle x, y \rangle = xy$ , for all  $x, y \in \mathbb{R}$  and induced usual norm  $|\cdot|$ . For every  $i = 1, 2, \dots, N$ , let the mapping  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $A_i x = \frac{x}{4^i}$  for all  $x \in H_1$  and  $B_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $B_i y = \frac{y}{3^i}$  for all  $y \in H_2$ , respectively, let  $M_A, M_B : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_A(x) = \{2x\}$ , for all  $x \in \mathbb{R}$  and  $M_B(y) = \{2y\}$ , for all  $y \in \mathbb{R}$ . Let the mapping  $A : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $A(x) = -2x$ , for all  $x \in \mathbb{R}$  and let  $\gamma \in (0, \frac{1}{4})$ , so we choose  $\gamma = \frac{1}{10}$ . Let the mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $Tx = \max\{0, -x\}$ , for all  $x \in \mathbb{R}$  and let the mapping  $S : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $Sx = \min\{0, \frac{x}{2}\}$ , for all  $x \in \mathbb{R}$ . Setting  $\{\mu_n\} = \{\frac{1}{5n}\}$ ,  $\{\varphi_n\} = \{\frac{7n+1}{15n}\}$ ,  $\{\theta_n\} = \{\frac{8n-4}{15n}\}$ ,  $\{\alpha_n\} = \{\frac{1}{n}\}$  and  $\{\sigma_n\} = \{\frac{1}{4n^2}\}$ ,  $\forall n \in \mathbb{N}$ . For every  $i = 1, 2, \dots, N$ , suppose that  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$  and  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$ . Then  $\{x_n\}$  converges strongly to a point  $x^* = 0 \in \mathbb{F}$ .

**Proof of Solution.** It is easy to check that  $a_i$  and  $b_i$  satisfies all the conditions of Theorem 1 and  $A_i$  is  $\frac{1}{4^i}$ -inverse strongly monotone and  $B_i$  is  $\frac{1}{3^i}$ -inverse strongly monotone for all  $i = 1, 2, \dots, N$ . We choose  $\lambda_A = \frac{1}{4N}$ ,  $\lambda_B = \frac{1}{3N}$ . Since  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$ , we obtain

$$\sum_{i=1}^N a_i A_i x = \sum_{i=1}^N \left( \frac{3}{4^i} + \frac{1}{N4^N} \right) \frac{x}{4^i}.$$

Then  $0 \in VI(H_1, \sum_{i=1}^N a_i A_i, M_A)$ . Since  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$ , we have

$$\sum_{i=1}^N b_i B_i y = \sum_{i=1}^N \left( \frac{2}{3^i} + \frac{1}{N3^N} \right) \frac{y}{3^i}.$$

Then  $0 \in VI(H_2, \sum_{i=1}^N b_i B_i, M_B)$ . Thus  $\{0\} = \Omega$ .

It is easy to observe that  $T, S$  are nonexpansive mappings with  $Fix(T) = \{0\}$ ,  $Fix(S) = \{0\}$ . Hence  $\Phi = \{0\}$ . Therefore  $\mathbb{F} = \Phi \cap \Omega = \{0\}$ .

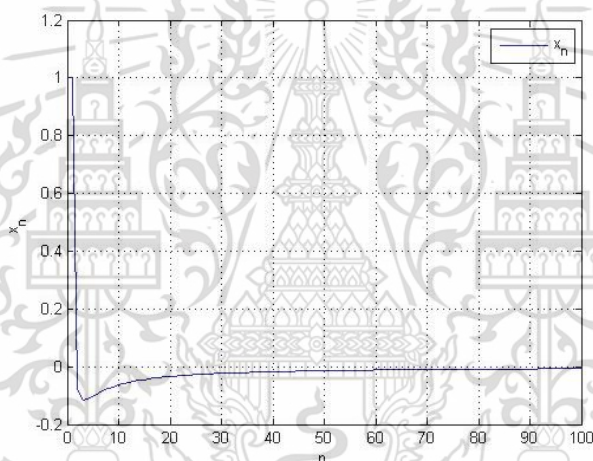
For every  $n \in \mathbb{N}$ ,  $\{\mu_n\} = \{\frac{1}{5n}\}$ ,  $\{\varphi_n\} = \{\frac{7n+1}{15n}\}$ ,  $\{\theta_n\} = \{\frac{8n-4}{15n}\}$ ,  $\{\alpha_n\} = \{\frac{1}{n}\}$  and  $\{\sigma_n\} = \{\frac{1}{4n^2}\}$ , then the sequence  $\{\mu_n\}$ ,  $\{\varphi_n\}$ ,  $\{\theta_n\}$ ,  $\{\alpha_n\}$  and  $\{\sigma_n\}$  satisfy all the conditions of Theorem 1. We rewrite (20) as follows:

$$\begin{cases} u_n = J_{\lambda_A}^{M_A} \left( I - \frac{1}{4N} \sum_{i=1}^N a_i A_i \right) (x_n - \gamma A^* (I - J_{\lambda_B}^{M_B} (I - \frac{1}{3N} \sum_{i=1}^N b_i B_i)) A x_n), \\ y_n = (1 - \frac{1}{n}) x_n + \frac{1}{0} \left( \frac{1}{4n^2} S x_n + (1 - \frac{1}{4n^2}) T x_n \right), \\ x_{n+1} = \frac{1}{5n} u + \frac{7n+1}{15n} y_n + \frac{8n-4}{15n} u_n, \end{cases} \quad (51)$$

Choose  $u = -1$ ,  $x_1 = 1$ ,  $N = 100$  and  $n = 100$ . The numerical for the sequence  $\{x_n\}$  are shown Table 1 and Figure 1.  $\square$

**Table 1.** The values of  $\{x_n\}$  with  $N = 100, n = 100$ .

n	$x_n$
1	1.0000
2	-0.0767
3	-0.1178
4	-0.1125
5	-0.1027
⋮	⋮
50	-0.0139
⋮	⋮
96	-0.0073
97	-0.0072
98	-0.0071
99	-0.0070
100	-0.0070

**Figure 1.** The sequence  $\{x_n\}$  converges strongly to 0 with initial values  $x_1 = 1, N = 100$  and  $n = 100$ .

**Example 6.** Let  $H_1 = H_2 = \mathbb{R}$ , the set of all real numbers, with the inner product defined by  $\langle x, y \rangle = xy$ , for all  $x, y \in \mathbb{R}$  and induced usual norm  $|\cdot|$ . For every  $i = 1, 2, \dots, N$ , let the mapping  $A_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $A_i x = \frac{x}{4^i}$  for all  $x \in H_1$  and  $B_i : \mathbb{R} \rightarrow \mathbb{R}$  define by  $B_i y = \frac{y}{3^i}$  for all  $y \in H_2$ , respectively, let  $M_A, M_B : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be defined by  $M_A(x) = \{2x\}$ , for all  $x \in \mathbb{R}$  and  $M_B(y) = \{2y\}$ , for all  $y \in \mathbb{R}$ . Let the mapping  $A : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $A(x) = -2x$ , for all  $x \in \mathbb{R}$  and let  $\gamma \in (0, \frac{1}{4})$ , so we choose  $\gamma = \frac{1}{10}$ . Let the mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $Tx = \max\{0, -x\}$ , for all  $x \in \mathbb{R}$  and let the mapping  $S : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $Sx = \min\{0, \frac{x}{2}\}$ , for all  $x \in \mathbb{R}$ . Setting  $\{\mu_n\} = \{\frac{1}{5n}\}$ ,  $\{\varphi_n\} = \{\frac{7n+1}{15n}\}$ ,  $\{\theta_n\} = \{\frac{8n-4}{15n}\}$ ,  $\{\alpha_n\} = \{\frac{1}{n}\}$  and  $\{\sigma_n\} = \{n\}$ ,  $\forall n \in \mathbb{N}$ . For every  $i = 1, 2, \dots, N$ , suppose that  $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$  and  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$ . Then  $\{x_n\}$  is divergent.

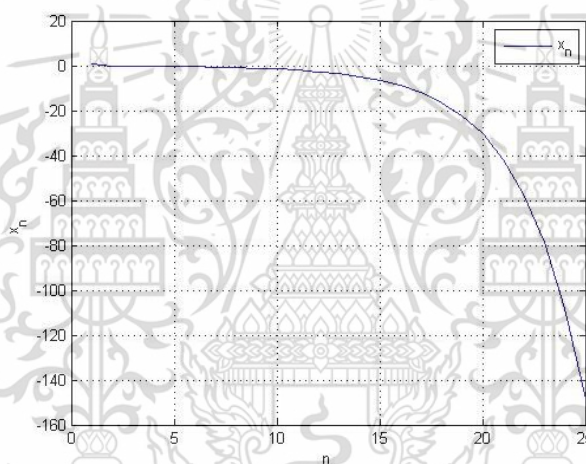
**Proof of Solution.** Note that the sequence  $\{\mu_n\}, \{\varphi_n\}, \{\theta_n\}, \{\alpha_n\}, a_i$  and  $b_i$  satisfies the conditions (i), (ii), (iii), (v) and (vi) from Theorem 1, while assumption (iv) does not converge to 0 since

$$\lim_{n \rightarrow \infty} n = \infty.$$

Choose  $u = -1, x_1 = 1, N = 100$  and  $n = 25$ . The numerical for the sequence  $\{x_n\}$  are shown in Table 2 and Figure 2. Therefore,  $\{x_n\}$  does not converge to 0.  $\square$

**Table 2.** The values of  $\{x_n\}$  with  $N = 100, n = 25$ .

n	$x_n$
1	1.0000
2	-0.0767
3	-0.1717
4	-0.3525
5	-0.4736
⋮	⋮
17	-12.0108
⋮	⋮
21	-41.7292
22	-57.2056
23	-78.5297
24	-107.9388
25	-148.5342

**Figure 2.** The sequence  $\{x_n\}$  is divergence with initial values  $x_1 = 1, N = 100$  and  $n = 25$ .

Next, we give example to support out some result in a two dimensional space of real numbers.

**Example 7.** Let  $H_1 = H_2 = \mathbb{R}^2$ , with the inner product defined by  $\langle x, y \rangle = xy = x_1 \cdot y_1 + x_2 \cdot y_2$ , for all  $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$  and induced usual norm  $\|\cdot\|$  defined by  $\|x\| = \sqrt{x_1^2 + x_2^2}$  for all  $x = (x_1, x_2) \in \mathbb{R}^2$ . For every  $i = 1, 2, \dots, N$ , let the mapping  $A_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  define by  $A_i x = \frac{x}{3^i}$  for all  $x = (x_1, x_2) \in H_1$  and  $B_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  define by  $B_i y = \frac{y}{4^i}$  for all  $y = (y_1, y_2) \in H_2$ , respectively, let  $M_A, M_B : \mathbb{R}^2 \rightarrow 2^{\mathbb{R}^2}$  be defined by  $M_A(x) = \{x\}$ , for all  $x = (x_1, x_2) \in \mathbb{R}^2$  and  $M_B(y) = \{3y\}$ , for all  $y = (y_1, y_2) \in \mathbb{R}^2$ . Let the mapping  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by  $A(x) = 3x$ , for all  $x = (x_1, x_2) \in \mathbb{R}^2$  and let  $\gamma \in (0, \frac{1}{9})$ , so we choose  $\gamma = \frac{1}{15}$ . Let the mapping  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by  $Tx = \frac{x}{3}$ , for all  $x = (x_1, x_2) \in \mathbb{R}^2$  and let the mapping  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by  $Sx = \min\{0, \frac{x}{3}\}$ , for all  $x = (x_1, x_2) \in \mathbb{R}^2$ . Setting  $\{\mu_n\} = \{\frac{1}{4^n}\}$ ,  $\{\varphi_n\} = \{\frac{7n+1}{12n}\}$ ,  $\{\theta_n\} = \{\frac{5n-4}{12n}\}$ ,  $\{\alpha_n\} = \{\frac{1}{n}\}$  and  $\{\sigma_n\} = \{\frac{1}{4n^2}\}$ ,  $\forall n \in \mathbb{N}$ . For every  $i = 1, 2, \dots, N$ , suppose that  $a_i = \frac{9}{10^i} + \frac{1}{N10^N}$  and  $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$ . Then  $\{x_n\}$  converges strongly to a point  $x^* = (0, 0) \in \mathbb{F}$ .

**Proof of Solution.** It is easy to check that  $a_i$  and  $b_i$  satisfies all the conditions of Theorem 1 and  $A_i$  is  $\frac{1}{3^i}$ -inverse strongly monotone and  $B_i$  is  $\frac{1}{4^i}$ -inverse strongly monotone for all  $i = 1, 2, \dots, N$ . We choose  $\lambda_A = \frac{1}{5N}, \lambda_B = \frac{1}{7N}$ . Thus  $\{(0, 0)\} = \Omega$ .

For definition of  $T$  and  $S$ , then  $T$  and  $S$  are nonexpansive mapping with  $\text{Fix}(T) = \{(0,0)\}$ . Hence  $\Phi = \{(0,0)\}$ . Therefore  $\mathbb{F} = \Phi \cap \Omega = \{(0,0)\}$ .

For every  $n \in \mathbb{N}$ ,  $\{\mu_n\} = \{\frac{1}{4n}\}$ ,  $\{\varphi_n\} = \{\frac{7n+1}{12n}\}$ ,  $\{\theta_n\} = \{\frac{5n-4}{12n}\}$ ,  $\{\alpha_n\} = \{\frac{1}{n}\}$  and  $\{\sigma_n\} = \{\frac{1}{4n^2}\}$ , then the sequence  $\{\mu_n\}$ ,  $\{\varphi_n\}$ ,  $\{\theta_n\}$ ,  $\{\alpha_n\}$  and  $\{\sigma_n\}$  satisfy all the conditions of Theorem 1.

From Theorem 1, we can conclude that the sequence  $\{x_n\}$  converges to  $(0,0)$ .  $\square$

## 6. Conclusions

- (i) Table 1 and Figure 1 show that the sequence  $\{x_n\}$  converges to 0, where  $\{0\} = \Phi \cap \Omega$ .
- (ii) Table 2 and Figure 2 show that the sequence  $\{x_n\}$  diverge, where condition (iv) is violated since  $\lim_{n \rightarrow \infty} \sigma_n \neq 0$ .

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ORIGINAL PAPER



## A theorem for solving Banach generalized system of variational inequality problems and fixed point problem in uniformly convex and 2-uniformly smooth Banach space

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### Abstract

In this paper, we consider a Banach generalized system of variational inequality problems by using the concept of Kangtunyakarn (Fixed Point Theory Appl 2014:123, 2014) and showed the equivalence between a Banach generalized system of variational inequality problems and fixed point problems. And also, using modified viscosity iterative method, we prove a strong convergence theorem for finding a common solution of a Banach generalized system of variational inequality problems and fixed point problems for a nonexpansive mapping. The main theorem presented in this paper extend the corresponding result of variational inequality problems introduced by Aoyama et al. (Fixed Point Theory Appl 2006:35390, <https://doi.org/10.1155/FPTA/2006/35390>, 2006). Moreover, we give some numerical examples for supporting our main theorem.

**Keywords** Fixed point problem · Nonexpansive mapping · Variational inequality problem

**Mathematics Subject Classification** 47H10 · 47H07 · 47J30

### 1 Introduction

Let  $E$  be a real Banach space with its dual space  $E^*$  and let  $C$  be a nonempty closed convex subset of  $E$ . We denote the norm of  $E$  and  $E^*$  by the same symbol  $\|\cdot\|$ . We denote weak and strong convergence by the notations  $'\rightharpoonup'$  and  $'\rightarrow'$ , respectively.

Let  $T : C \rightarrow C$  be a mapping. A point  $x \in C$  is called a *fixed point* of  $T$  if  $Tx = x$ . The set of fixed points of  $T$  is denoted by  $F(T) = \{x \in C : Tx = x\}$ .

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Fixed point problem is an important area of mathematical analysis. This problem applies about the solution in many problem in a Banach space such as nonlinear operator equation, variational inequality problems, equilibrium problem, etc., see for instance [1–22].

**Definition 1** A Banach space  $E$  is said to be *uniformly convex* if for any  $\varepsilon$ ,  $0 < \varepsilon \leq 2$ , the inequalities  $\|x\| \leq 1$ ,  $\|y\| \leq 1$  and  $\|x - y\| \geq \varepsilon$  imply there exists a  $\delta > 0$  such that  $\|\frac{x+y}{2}\| \leq 1 - \delta$ .

**Definition 2** Let  $E$  be a Banach space. Then a function  $\rho_E : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is said to be the *modulus of smoothness of  $E$*  if

$$\rho_E(t) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = t \right\}.$$

A Banach space  $E$  is said to be *uniformly smooth* if

$$\lim_{t \rightarrow 0} \frac{\rho_E(t)}{t} = 0.$$

Let  $q > 1$ . A Banach space  $E$  is said to be  *$q$ -uniformly smooth* if there exists a constant  $c > 0$  such that  $\rho_E \leq ct^q$ . It is easy to see that, if  $E$  is  $q$ -uniformly smooth, then  $q \leq 2$  and  $E$  is uniformly smooth. It is well known that  $L^p$  is  $\min\{p, 2\}$ -uniformly smooth for  $p > 1$ .

**Definition 3** A mapping  $J$  from  $E$  onto  $E^*$  satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\}$$

is called the *normalized duality mapping of  $E$* . The duality pair  $(x, f)$  represents  $f(x)$  for  $f \in E^*$  and  $x \in E$ .

It is well known that if  $E$  is smooth, then  $J$  is a single value, which denote by  $j$ .

**Definition 4** Let  $C$  be a nonempty subset of a Banach space  $E$  and  $T : C \rightarrow C$  be a self-mapping.

(i)  $T$  is called a *nonexpansive mapping* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C,$$

(ii)  $T$  is called a *contraction mapping* if there exists a constant  $\alpha \in (0, 1)$  such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

A mapping  $A : C \rightarrow E$  is said to be *accretive* if there exists  $j(x - y) \in J(x - y)$  such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0, \quad \forall x, y \in C.$$

A mapping  $A : C \rightarrow E$  is said to be  *$\alpha$ -inverse strongly accretive* if there exists  $\alpha > 0$  and  $j(x - y) \in J(x - y)$  such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

Many authors have studied concept of accretive mapping in Banach space to modified their problem, see, for instance [1, 2].

In 2019, Suantai et al. [1] introduced implicit and explicit iteration processes with perturbations. Furthermore, they proved a strong convergence theorem for fixed point problem of nonexpansive mappings and the quasi variational inclusion problem considering  $m$ -accretive operator and extended an improving previous results.

Very recently, Chang et al. [2] proved strong convergence theorem for finding a common element of the set of solutions to a system of quasi variational inclusion with accretive mappings and the set of fixed points for a  $\eta$ -strict pseudo-contractive mapping (that is,  $T$  is called an  $\eta$ -strictly pseudo-contractive mapping if there exists a constant  $\eta \in (0, 1)$  such that  $\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \eta\|(I - T)x - (I - T)y\|^2$ , for every  $x, y \in C$  and for some  $j(x - y) \in J(x - y)$ ) in Banach spaces. Moreover, they extended their main result corresponding with a system of variational inequalities, monotone variational inequalities, convex minimization problem and convexly constrained linear inverse problem.

**Definition 5** Let  $C \subseteq E$  be a closed convex and  $Q_C$  be a mapping of  $E$  onto  $C$ . The mapping  $Q_C$  is said to be *sunny* if  $Q_C(x + t(x - Q_Cx)) = Q_Cx$ , for all  $x \in E$  and  $t \geq 0$ . A mapping  $Q_C$  is called *retraction* if  $Q_C^2 = Q_C$ . A subset  $C$  of  $E$  is called a *sunny nonexpansive retract* of  $E$  if there exists a sunny nonexpansive retraction of  $E$  onto  $C$ .

Aoyama et al. [3] first introduced the following *generalized variational inequality problem* in Banach space. Let  $A : C \rightarrow E$  be an accretive operator. Find a point  $x^* \in C$  such that for some  $j(x - x^*) \in J(x - x^*)$ ,

$$\langle Ax^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C. \quad (1)$$

The set of solutions of the variational inequality in a Banach space is denoted by  $S(C, A)$ , that is,

$$S(C, A) = \{u \in C : \langle Au, J(v - u) \rangle \geq 0, \quad \forall v \in C\}.$$

**Remark 1** Let  $H$  be a real Hilbert space. If  $E \equiv H$ , then generalized variational inequality problem reduce to *variational inequality problem* in Hilbert space, that is, find a point  $u \in C$  such that

$$\langle v - u, Du \rangle \geq 0, \quad \text{for all } v \in C. \quad (2)$$

The set of solutions of the variational inequality in a Hilbert space is denoted by  $VI(C, D)$ .

Many authors have studied concept of variational inequality problem in Hilbert space to modified their problem, see, for instance [7–11].

Cai et al. [9] proved a strong convergence theorem of the sequences generated by modified viscosity implicit method for finding a common solution of variational inequality problems for two *inverse strongly monotone operators* (that is,  $B$  is called a  $\beta$ -inverse strongly monotone operator if there exists a constant  $\beta > 0$  such that  $\langle Bx - By, x - y \rangle \geq \beta\|Bx - By\|^2$ , for all  $x, y \in H$ ) and fixed point of nonexpansive mapping. Moreover, they applied their main results to solving fixed point problem for strict pseudo-contractive mappings and equilibrium problems in Hilbert space.

By using a hybrid extragradient-like implicit method, Ceng et al. [10] proved a strong convergence theorem for finding a common solution of a general system of variational inequalities and a common fixed point problem of a countable family of uniformly Lipschitzian pseudo-contractive mappings and an *asymptotically nonexpansive mapping* (that is,  $T$  is called an asymptotically nonexpansive mapping if there exists a sequence  $\{k_n\}$  of positive numbers with  $\lim_{n \rightarrow \infty} k_n = 1$  such that  $\|T^n x - T^n y\| \leq k_n \|x - y\|$ , for all  $x, y \in H$ ,  $n \in \mathbb{N}$ ) and extended their main results corresponding with a common fixed point of nonexpansive and pseudo-contractive mappings, variational inequality problems and generalized mixed equilibrium problems in Hilbert spaces.

The variational inequality problem in Banach space is profusely studied in physics, economic, engineering and applied science etc. Many researcher have studied this concept (1) to their problem, see example [3–6].

In 2019, Ariza-Ruiz et al. [6] showed that the existence and uniqueness results for a variational inequality in a real smooth Banach space.

By using the concept of Krasnoselski–Mann iterative algorithm, Aoyama et al. [3] proved a weak convergence theorem for a finding the solution of generalized variational inequality problem as follows:

**Theorem 1** *Let  $E$  be a uniformly convex and 2-uniformly smooth Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $Q_C$  be a sunny nonexpansive retraction from  $E$  onto  $C$ , let  $\alpha > 0$  and let  $A$  be an  $\alpha$ -inverse strongly accretive operator of  $C$  into  $E$  with  $S(C, A) \neq \emptyset$ . Suppose  $x_1 = x \in C$  and  $\{x_n\}$  is given by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \alpha_n A x_n),$$

for every  $n = 1, 2, \dots$ , where  $\{\lambda_n\}$  is a sequence of positive real numbers and  $\{\alpha_n\}$  is a sequence in  $[0, 1]$ . If  $\{\lambda_n\}$  and  $\{\alpha_n\}$  are chosen so that  $\lambda_n \in [a, \frac{a}{K^2}]$  for some  $a > 0$  and  $\alpha_n \in [b, c]$  for some  $b, c$  with  $0 < b < c < 1$ , then  $\{x_n\}$  converges weakly to some element  $z$  of  $S(C, A)$ , where  $K$  is the 2-uniformly smoothness constant of  $E$ .

Yao et al. [4] who is based on generalized variational inequality in a Banach space introduced the system of general variational inequality problems. Let  $A, B : C \rightarrow E$  be two nonlinear mappings. Finding a point  $(x^*, y^*) \in C \times C$  such that

$$\begin{cases} (Ay^* + x^* - y^*, j(x - x^*)) \geq 0, & \forall x \in C, \\ (Bx^* + y^* - x^*, j(x - y^*)) \geq 0, & \forall x \in C. \end{cases} \quad (3)$$

They first gave associated between the system of variational inequality problems (3) and some fixed point problem of nonexpansive mapping. Furthermore, they proved a strong convergence theorem of sequence generated by the modification of extragradient method for solving the system of general variational inequality problems (3) as follows:

**Theorem 2** (See [4]) *Let  $C$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  which admits a weakly sequentially continuous duality mapping. Let  $Q_C$  be the sunny nonexpansive retraction from  $E$  into  $C$ . Let the mappings  $A, B : C \rightarrow E$  be  $\alpha$ -inverse strongly accretive with  $\alpha \geq K^2$  and  $\beta$ -inverse strongly accretive with  $\beta \geq K^2$ , respectively. Define the mapping by  $Gx = Q_C(Q_C(x - Bx) - \lambda A Q_C(x - Bx))$  for all  $x \in C$  and the set of fixed points of  $G$  denoted by  $\Omega \neq \emptyset$ . For given  $x_n \in C$ , let the sequence  $\{x_n\}$  be generated by*

$$\begin{cases} y_n = Q_C(x_n - Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - Ay_n), & n \geq 0, \end{cases}$$

where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  in  $(0, 1)$ . Suppose the sequences  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  satisfy the following conditions:

- (i)  $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 0,$
- (ii)  $\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$
- (iii)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1.$

Then  $\{x_n\}$  converges strongly to  $Q_\Omega u$ , where  $Q_\Omega u$  is the sunny nonexpansive retraction of  $C$  onto  $\Omega$ .

By using the concept of a general variational inequality problems (3), Kangtunyakarn [5] introduced the modification of a system of variational inequality problems in Banach space

for finding  $(x^*, y^*) \in C \times C$  such that

$$\begin{cases} \langle x^* - (I - \lambda_A A)(ax^* + (1-a)y^*), j(x - x^*) \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \lambda_B B)x^*, j(x - y^*) \rangle \geq 0, & \forall x \in C. \end{cases} \quad (4)$$

for all  $\lambda_A, \lambda_B > 0$  and  $a \in [0, 1]$ . The set of solution of (4) is denote by  $\Omega'$ . If  $\lambda_A = \lambda_B = 1$  and  $a = 0$ , then problem (4) reduce to (3).

Kangtunyakarn [5] proved strong convergence theorem for the set of fixed points of two finite families of nonexpansive and strictly pseudo-contractive mappings and the set of solutions of modification of system of variational inequality problems as follows:

**Theorem 3** *Let  $C$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  and let  $Q_C$  be a sunny nonexpansive retraction of  $E$  onto  $C$ . Let  $A, B : C \rightarrow E$  be  $\alpha$ - and  $\beta$ -inverse strongly accretive operators, respectively. Define the mapping  $G : C \rightarrow C$  by  $Gx = Q_C(I - \lambda_A A)(aI + (1-a)Q_C(I - \lambda_B B))x$  for all  $x \in C$ ,  $\lambda_A \in (0, \frac{a}{K^2})$ ,  $\lambda_B \in (0, \frac{\beta}{K^2})$  and  $a \in [0, 1]$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . Let  $\{S_i\}_{i=1}^N$  be a finite family of  $\kappa_i$ -strict pseudo-contractions of  $C$  into itself and let  $\{T_i\}_{i=1}^N$  be a finite family of nonexpansive mappings of  $C$  into itself and  $\kappa = \min\{\kappa_i : i = 1, 2, \dots, N\}$  with  $K^2 \leq \kappa$ . Let  $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ , where  $I = [0, 1]$ ,  $\alpha_1^j \alpha_2^j \alpha_3^j = 1$ ,  $\alpha_1^j \in (0, 1]$ ,  $\alpha_2^j \in [0, 1]$  and  $\alpha_3^j \in (0, 1)$  for all  $j = 1, 2, \dots, N$ . Let  $S^A$  be the  $S^A$ -mapping generated by  $S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N$  and  $\alpha_1, \alpha_2, \dots, \alpha_N$ . Assume that  $\mathbb{F} = F(G) \cap \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$ . Let the sequence  $\{x_n\}$  be generated by  $u, x_1 \in C$  and*

$$x_{n+1} = G(\alpha_n u + \beta_n x_n + \gamma_n S^A x_n), \quad \forall n \geq 1,$$

where  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$  with  $\alpha_n + \beta_n + \gamma_n = 1$ . Suppose that the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < c \leq \beta_n \leq d < 1$  for all  $c, d > 0$  and  $n \geq 1$ ,
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ,  $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}}u$  and  $(x_0, z_0)$  is a solution of (4), where  $z_0 = Q_C(I - \lambda_B B)x_0$ .

Inspired and motivated by the modification of a system of variational inequality problems of Kangtunyakarn [5], we define Banach generalized system of variational inequality problems which is find  $(x^*, y^*, z^*) \in C \times C \times C$  such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), j(x - x^*) \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), j(x - y^*) \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \lambda_3 D_3)x^*, j(x - z^*) \rangle \geq 0, & \forall x \in C. \end{cases} \quad (5)$$

for all  $\lambda_1, \lambda_2, \lambda_3 > 0$  and  $a \in [0, 1]$ . In Hilbert space (5) reduce to

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, & \forall x \in C. \end{cases} \quad (6)$$

for all  $\lambda_1, \lambda_2, \lambda_3 > 0$  and  $a \in [0, 1]$ , which is modified generalized system of variational inequalities in Hilbert space introduced by Siriyan and Kantunyakarn [11]. If  $D_3 \equiv 0$  and  $x^* = z^*$ , then problem (5) reduce to (4).

Based on the results mentioned above, we give our theorem for Banach generalized system of variational inequality problems and some important result as follows:

- (i) We first establish Lemma 9 which show the equivalence a between Banach generalized system of variational inequality problems (5) and some fixed point problem involving the nonexpansive mapping under suitable conditions on our parameters.
- (ii) We establish a strong convergence theorem of the sequences generated by modified viscosity method for finding a common solution of Banach generalized system of variational inequality problems (5) and fixed point of nonexpansive mapping.
- (iii) We apply our main result to obtain a strong convergence theorem the sequences generated by modified viscosity method for finding a common solution of generalized variational inequality problems (1) by Aoyama et al. [3] and fixed point of nonexpansive mapping.
- (iv) We give a some illustrative numerical examples to support our main result.

## 2 Preliminaries

In this paper, we need the following lemmas and definition to prove our main result.

**Lemma 1** (See [14]) *Let  $\{s_n\}$  be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

*where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  and  $\{\delta_n\}$  is a sequence such that*

- (i)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0$  or  $\sum_{n=1}^{\infty} |\delta_n| < \infty$ .

*Then  $\lim_{n \rightarrow \infty} s_n = 0$ .*

**Lemma 2** (See [15]) *Let  $E$  be a real 2-uniformly smooth Banach space with the best smooth constant  $K$ . Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x) \rangle + 2\|Ky\|^2,$$

*for any  $x, y \in E$ .*

**Lemma 3** (See [16]) *Let  $C$  be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space  $E$  and let  $T$  be a nonexpansive mapping of  $C$  into itself with  $F(T) \neq \emptyset$ . Then  $F(T)$  is a sunny nonexpansive retract of  $C$ .*

**Lemma 4** (See [17]) *Let  $C$  be a nonempty closed convex subset of a smooth Banach space  $E$  and  $Q_C$  be a retraction from  $E$  onto  $C$ . Then the following are equivalent:*

- (i)  $Q_C$  is both sunny and nonexpansive;
- (ii)  $\langle x - Q_Cx, J(y - Q_Cx) \rangle \leq 0$  for all  $x \in E$  and  $y \in C$ .

From Lemma 4, let  $x \in E$  and  $x_0 \in C$ . Then we have  $x_0 = Q_Cx$  if and only if  $\langle x - x_0, J(y - x_0) \rangle \leq 0$  for all  $y \in C$ , where  $Q_C$  is a sunny nonexpansive retraction from  $E$  onto  $C$ .

**Lemma 5** (See [19]) *Let  $E$  be a uniformly convex Banach space and  $B_r = \{z \in E : \|z\| \leq r\}$ ,  $r > 0$ . Then there exists a continuous, strictly increasing and convex function  $g : [0, \infty] \rightarrow [0, \infty]$ ,  $g(0) = 0$  such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta g(\|x - y\|),$$

for all  $x, y, z \in B_r$  and  $\alpha, \beta, \gamma \in [0, 1]$  with  $\alpha + \beta + \gamma = 1$ .

**Lemma 6** (See [15]) *Let  $r > 0$ . If  $E$  is uniformly convex, then there exists a continuous, strictly increasing and convex function  $g : [0, \infty] \rightarrow [0, \infty]$ ,  $g(0) = 0$  such that for all  $x, y \in B_r(0) = \{z \in E : \|z\| \leq r\}$  and for all  $\alpha \in [0, 1]$ , we have*

$$\|\alpha x + (1 - \alpha)y\|^2 \leq \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)g(\|x - y\|).$$

**Lemma 7** (See [20]) *Let  $C$  be a closed and convex subset of a real uniformly smooth Banach space  $E$  and let  $T : C \rightarrow C$  be a nonexpansive mapping with a nonempty fixed point  $F(T)$ . If  $\{x_n\} \subset C$  is a bounded sequence such that  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ . Then there exists a unique sunny nonexpansive retraction  $Q_{F(T)} : C \rightarrow F(T)$  such that*

$$\limsup_{n \rightarrow \infty} \langle u - Q_{F(T)}u, J(x_n - Q_{F(T)}u) \rangle \leq 0,$$

for any given  $u \in C$ .

**Lemma 8** (See [22]) *Let  $C$  be a closed convex subset of a strictly convex Banach space  $E$ . Let  $T_1$  and  $T_2$  be two nonexpansive mappings from  $C$  into itself with  $F(T_1) \cap F(T_2) \neq \emptyset$ . Define a mapping  $S$  by*

$$Sx = \lambda T_1x + (1 - \lambda)T_2x, \quad \forall x \in C,$$

where  $\lambda$  is a constant in  $(0, 1)$ . Then  $S$  is nonexpansive and  $F(S) = F(T_1) \cap F(T_2)$ .

We introduce the following lemma which show associated between Banach generalized system of variational inequality problems and fixed point problem of nonexpansive mapping under suitable conditions on our parameters.

**Lemma 9** *Let  $C$  be a nonempty closed convex subset of a smooth Banach space  $E$  and let  $D_1, D_2, D_3 : C \rightarrow E$  be mappings. Let  $Q_C$  be a sunny nonexpansive retraction of  $E$  onto  $C$ . For every  $\lambda_1, \lambda_2, \lambda_3 > 0$  and  $a \in [0, 1]$ . The following are equivalent*

- (i)  $(x^*, y^*, z^*)$  is a solution of (5);
- (ii)  $x^*$  is a fixed point of the mapping  $Z$ , i.e.  $x^* \in F(Z)$ , defined the mapping  $Z : C \rightarrow C$  by  $Zx = Q_C(I - \lambda_1 D_1)(ax + (1 - a)Q_C(I - \lambda_2 D_2)(ax + (1 - a)Q_C(I - \lambda_3 D_3)x))$ , for all  $x \in C$  where  $y^* = Q_C(I - \lambda_2 D_2)(ax^* + (1 - a)z^*)$  and  $z^* = Q_C(I - \lambda_3 D_3)x^*$ .

**Proof** Let conditions hold.

(i)  $\Rightarrow$  (ii). Suppose that  $(x^*, y^*, z^*) \in C \times C \times C$  is a solution of (5). For every  $x \in C$ , we have

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1 - a)y^*), j(x - x^*) \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1 - a)z^*), j(x - y^*) \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \lambda_3 D_3)x^*, j(x - z^*) \rangle \geq 0, & \forall x \in C. \end{cases}$$

From Lemma 4, we have

$$\begin{aligned} x^* &= Q_C(I - \lambda_1 D_1)(ax^* + (1 - a)y^*), \\ y^* &= Q_C(I - \lambda_2 D_2)(ax^* + (1 - a)z^*), \\ z^* &= Q_C(I - \lambda_3 D_3)x^*. \end{aligned}$$

It follows that

$$x^* = Q_C(I - \lambda_1 D_1)(ax + (1 - a)Q_C(I - \lambda_2 D_2)(ax + (1 - a)Q_C(I - \lambda_3 D_3)x))$$

$$= Zx^*.$$

Then  $x^* \in F(Z)$ , where  $y^* = Q_C(I - \lambda_2 D_2)(ax^* + (1-a)z^*)$  and  $z^* = Q_C(I - \lambda_3 D_3)x^*$ .

(ii)  $\Rightarrow$  (i). Let  $x^* \in F(Z)$ ,  $y^* = Q_C(I - \lambda_2 D_2)(ax^* + (1-a)z^*)$  and  $z^* = Q_C(I - \lambda_3 D_3)x^*$ . Then

$$\begin{aligned} x^* &= Zx^* \\ &= Q_C(I - \lambda_1 D_1)(ax + (1-a)Q_C(I - \lambda_2 D_2)(ax + (1-a)Q_C(I - \lambda_3 D_3)x)) \\ &= Q_C(I - \lambda_1 D_1)(ax^* + (1-a)y^*). \end{aligned}$$

From Lemma 4, we have

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), j(x - x^*) \rangle \geq 0, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), j(x - y^*) \rangle \geq 0, \\ \langle z^* - (I - \lambda_3 D_3)x^*, j(x - z^*) \rangle \geq 0. \end{cases}$$

for all  $x \in C$ . Then we find that  $(x^*, y^*, z^*)$  is solution of (5).  $\square$

**Lemma 10** (See [5]) *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$ . Let  $T, S : C \rightarrow C$  be nonexpansive mappings with  $F(T) \cap F(S) \neq \emptyset$ . Define the mapping  $T_a : C \rightarrow C$  by  $T_a x = T(ax + (1-a)Sx)$  for all  $x \in C$  and  $a \in (0, 1)$ . Then  $F(T_a) = F(T) \cap F(S)$  and  $T_a$  is a nonexpansive mapping.*

### 3 Main result

We prove a strong convergence theorem to approximate a common solution of Banach generalized system of variational inequality problems (5) and fixed point problem of nonexpansive mapping.

**Theorem 4** *Let  $C$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  and let  $Q_C$  be a sunny nonexpansive retraction of  $C$ . Let  $D_1, D_2, D_3 : C \rightarrow E$  be  $d_1, d_2, d_3$ -inverse strongly accretive operators, respectively. Define the mapping  $Z : C \rightarrow C$  by  $Zx = Q_C(I - \lambda_1 D_1)(ax + (1-a)Q_C(I - \lambda_2 D_2)(ax + (1-a)Q_C(I - \lambda_3 D_3)x))$  for all  $x \in C$ ,  $\lambda_1 \in (0, \frac{d_1}{K^2})$ ,  $\lambda_2 \in (0, \frac{d_2}{K^2})$ ,  $\lambda_3 \in (0, \frac{d_3}{K^2})$ ,  $a \in [0, 1]$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . Let  $T, S : C \rightarrow C$  be nonexpansive mappings. Assume that  $\mathbb{F} = F(Z) \cap F(S) \cap F(T) \neq \emptyset$ . Let the sequence  $\{x_n\}$  be generated by  $x_1 \in C$  and*

$$\begin{cases} u_n = T(\delta_n x_n + (1 - \delta_n) Sx_n), \\ x_{n+1} = Z(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n), \end{cases} \quad (7)$$

where  $f : C \rightarrow C$  is a contraction mapping with a constant  $\xi \in (0, 1)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$  with  $\alpha_n + \beta_n + \gamma_n = 1$ . Suppose that the following conditions are satisfied

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$  for all  $c, d > 0$  and  $n \geq 1$ ,
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ,  $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ ,  $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ ,  
 $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ ,
- (iv)  $\lim_{n \rightarrow \infty} \delta_n = \delta$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}} f(x_0)$  and  $(x_0, y_0, z_0)$  is a solution of (5), where  $y_0 = Q_C(I - \lambda_2 D_2)(ax_0 + (1-a)z_0)$  and  $z_0 = Q_C(I - \lambda_3 D_3)x_0$ .

**Proof** Step1. First, we prove that  $\{x_n\}$  and  $\{u_n\}$  are bounded.

We show that  $Q_C(I - \lambda_1 D_1)$ ,  $Q_C(I - \lambda_2 D_2)$  and  $Q_C(I - \lambda_3 D_3)$  are nonexpansive mappings. Let  $x, y \in C$ , we have

$$\begin{aligned} & \|Q_C(I - \lambda_1 D_1)x - Q_C(I - \lambda_1 D_1)y\|^2 \\ & \leq \|x - y - \lambda_1 D_1 x + \lambda_1 D_1 y\|^2 \\ & \leq \|x - y\|^2 - 2\lambda_1(D_1 x - D_1 y, j(x - y)) + 2K^2 \lambda_1^2 \|D_1 x - D_1 y\|^2 \\ & \leq \|x - y\|^2 - 2\lambda_1 d_1 \|D_1 x - D_1 y\|^2 + 2K^2 \lambda_1^2 \|D_1 x - D_1 y\|^2 \\ & = \|x - y\|^2 - 2\lambda_1(d_1 - K^2 \lambda_1) \|D_1 x - D_1 y\|^2 \\ & \leq \|x - y\|^2. \end{aligned}$$

Then  $Q_C(I - \lambda_1 D_1)$  is a nonexpansive mapping. By using the same method, we obtain that  $Q_C(I - \lambda_2 D_2)$  and  $Q_C(I - \lambda_3 D_3)$  are nonexpansive mappings. From the definition of  $Z$ , we see that  $Z$  is nonexpansive mapping.

Let  $x^* \in \mathbb{F}$ , then  $x^* \in F(S)$  and  $x^* \in F(T)$  which implies  $Sx^* = x^*$  and  $Tx^* = x^*$ . From the definition of  $\{u_n\}$  and Lemma 10, we have

$$\begin{aligned} \|u_n - x^*\| & = \|T(\delta_n x_n + (1 - \delta_n)Sx_n) - x^*\| \\ & \leq \|\delta_n x_n + (1 - \delta_n)Sx_n - x^*\| \\ & = \|\delta_n(x_n - x^*) + (1 - \delta_n)(Sx_n - x^*)\| \\ & \leq \delta_n \|x_n - x^*\| + (1 - \delta_n) \|Sx_n - x^*\| \\ & \leq \delta_n \|x_n - x^*\| + (1 - \delta_n) \|x_n - x^*\| \\ & = \|x_n - x^*\|. \end{aligned} \quad (8)$$

Since  $x^* \in \mathbb{F}$ , we have  $x^* \in F(Z)$  which implies  $Zx^* = x^*$ . Put  $y_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n$ . From the definition of  $\{x_n\}$ , (8) and Lemma 9, we have

$$\begin{aligned} \|x_{n+1} - x^*\| & = \|Zy_n - x^*\| \\ & \leq \|y_n - x^*\| \\ & = \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(u_n - x^*)\| \\ & \leq \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|u_n - x^*\| \\ & \leq \alpha_n [\|f(x_n) - f(x^*)\| + \|f(x^*) - x^*\|] + \beta_n \|x_n - x^*\| \\ & \quad + \gamma_n \|x_n - x^*\| \\ & \leq \alpha_n \xi \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n) \|x_n - x^*\| \\ & = \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n(1 - \xi)) \|x_n - x^*\| \\ & \leq \max \left\{ \frac{\|f(x^*) - x^*\|}{1 - \xi}, \|x_1 - x^*\| \right\}. \end{aligned} \quad (9)$$

Applying mathematical induction, we can conclude that the sequence  $\{x_n\}$  is bounded and so is  $\{u_n\}$ .

Step 2. Show that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ ,  $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$  and  $\lim_{n \rightarrow \infty} \|Zy_n - y_n\| = 0$ .

From the definition of  $\{x_n\}$ , we have

$$\begin{aligned} & \|x_{n+1} - x_n\| \\ & = \|Z(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n) - Z(\alpha_{n-1} f(x_{n-1}))\| \end{aligned}$$

$$\begin{aligned}
& + \beta_{n-1}x_{n-1} + \gamma_{n-1}u_{n-1})\| \\
\leq & \|(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n) - (\alpha_{n-1} f(x_{n-1}) + \beta_{n-1}x_{n-1} + \gamma_{n-1}u_{n-1})\| \\
= & \|(\alpha_n f(x_n) - \alpha_{n-1} f(x_{n-1})) + \alpha_n f(x_{n-1}) - \alpha_n f(x_{n-1}) \\
& + \beta_n x_n - \beta_{n-1}x_{n-1} + \beta_n x_{n-1} - \beta_n x_{n-1} \\
& + \gamma_n u_n - \gamma_{n-1}u_{n-1} + \gamma_n u_{n-1} - \gamma_n u_{n-1})\| \\
\leq & \alpha_n \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \|x_n - x_{n-1}\| \\
& + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \gamma_n \|u_n - u_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|u_{n-1}\| \\
\leq & \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \|x_n - x_{n-1}\| \\
& + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \gamma_n \|u_n - u_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|u_{n-1}\|. \tag{10}
\end{aligned}$$

By the definition of  $\{u_n\}$ , we have

$$\begin{aligned}
& \|u_n - u_{n-1}\| \\
= & \|T(\delta_n x_n + (1 - \delta_n)Sx_n) - T(\delta_{n-1}x_{n-1} + (1 - \delta_{n-1})Sx_{n-1})\| \\
\leq & \|(\delta_n x_n + (1 - \delta_n)Sx_n) - (\delta_{n-1}x_{n-1} + (1 - \delta_{n-1})Sx_{n-1})\| \\
= & \|\delta_n x_n - \delta_{n-1}x_{n-1} + \delta_n x_{n-1} - \delta_n x_{n-1} + (Sx_n - Sx_{n-1}) \\
& - \delta_n Sx_n + \delta_{n-1}Sx_{n-1} + \delta_n Sx_{n-1} - \delta_n Sx_{n-1}\| \\
\leq & \delta_n \|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| \|x_{n-1}\| + (1 - \delta_n) \|Sx_n - Sx_{n-1}\| \\
& + |\delta_n - \delta_{n-1}| \|Sx_{n-1}\| \\
\leq & \delta_n \|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| \|x_{n-1}\| + (1 - \delta_n) \|x_n - x_{n-1}\| \\
& + |\delta_n - \delta_{n-1}| \|Sx_{n-1}\| \\
= & \|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| (\|x_{n-1}\| + \|Sx_{n-1}\|). \tag{11}
\end{aligned}$$

From (10) and (11), we have

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
\leq & \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \|x_n - x_{n-1}\| \\
& + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \gamma_n \|u_n - u_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|u_{n-1}\| \\
\leq & \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \|x_n - x_{n-1}\| \\
& + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|u_{n-1}\| \\
& + \gamma_n [\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}| (\|x_{n-1}\| + \|Sx_{n-1}\|)] \\
= & \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
& + (1 - \alpha_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| \\
& + \gamma_n |\delta_n - \delta_{n-1}| (\|x_{n-1}\| + \|Sx_{n-1}\|) + |\gamma_n - \gamma_{n-1}| \|u_{n-1}\| \\
= & (1 - \alpha_n (1 - \xi)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
& + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \gamma_n |\delta_n - \delta_{n-1}| (\|x_{n-1}\| + \|Sx_{n-1}\|) \\
& + |\gamma_n - \gamma_{n-1}| \|u_{n-1}\|. \tag{12}
\end{aligned}$$

Applying (12), conditions (i), (iii) and Lemma 1, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{13}$$

We will show that  $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$ . From the definition of  $\{x_n\}$  and (8), we have

$$\|x_{n+1} - x_n\|^2$$

$$\begin{aligned}
 &= \|Z(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n) - x^*\|^2 \\
 &\leq \|(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n) - x^*\|^2 \\
 &= \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(u_n - x^*)\|^2 \\
 &\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|u_n - x^*\|^2 - \beta_n \gamma_n g(\|x_n - u_n\|) \\
 &\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 - \beta_n \gamma_n g(\|x_n - u_n\|) \\
 &\leq \alpha_n \|f(x_n) - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 - \beta_n \gamma_n g(\|x_n - u_n\|) \\
 &\leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \beta_n \gamma_n g(\|x_n - u_n\|).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \beta_n \gamma_n g(\|x_n - u_n\|) &\leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x_n\|^2 \\
 &\leq \alpha_n \|f(x_n) - x^*\|^2 \\
 &\quad + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.
 \end{aligned}$$

From (13) and conditions (i) and (ii), we have

$$\lim_{n \rightarrow \infty} g(\|x_n - u_n\|) = 0.$$

From the property of  $g$ , we have

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \tag{14}$$

From the definition of  $\{y_n\}$ , we have

$$\begin{aligned}
 \|y_n - x_n\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n - x_n\| \\
 &= \|\alpha_n(f(x_n) - x_n) + \gamma_n(u_n - x_n)\| \\
 &\leq \alpha_n \|f(x_n) - x_n\| + \gamma_n \|u_n - x_n\|.
 \end{aligned}$$

From (14) and a condition (i), we have

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \tag{15}$$

From the definition of  $\{y_n\}$ , we have

$$\begin{aligned}
 \|y_n - u_n\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n - u_n\| \\
 &= \|\alpha_n(f(x_n) - u_n) + \beta_n(x_n - u_n)\| \\
 &\leq \alpha_n \|f(x_n) - u_n\| + \beta_n \|x_n - u_n\|.
 \end{aligned}$$

From (14) and a condition (i), we have

$$\lim_{n \rightarrow \infty} \|y_n - u_n\| = 0. \tag{16}$$

Since  $\lim_{n \rightarrow \infty} \delta_n = \delta \in (0, 1)$ , we define the mapping  $T_\delta : C \rightarrow C$  by  $T_\delta x = T(\delta x + (1 - \delta)Sx)$  for all  $x \in C$  and  $\delta \in (0, 1)$ , then we have  $u_n = T_\delta x_n$ . By Lemma 10, we have  $F(T_\delta) = F(T) \cap F(S)$  and  $T_\delta$  is a nonexpansive mapping.

From the nonexpansive of  $T_\delta$ , we have

$$\begin{aligned}
 \|T_\delta y_n - y_n\| &= \|T_\delta y_n - T_{\delta_n} y_n + T_{\delta_n} y_n - y_n\| \\
 &\leq \|T_\delta y_n - T_{\delta_n} y_n\| + \|T_{\delta_n} y_n - y_n\|.
 \end{aligned} \tag{17}$$

Considers,

$$\begin{aligned}\|T_\delta y_n - T_{\delta_n} y_n\| &= \|T(\delta y_n + (1 - \delta)S y_n) - T(\delta_n y_n + (1 - \delta_n)S y_n)\| \\ &\leq \|(\delta y_n + (1 - \delta)S y_n) - (\delta_n y_n + (1 - \delta_n)S y_n)\| \\ &= \|(\delta - \delta_n)y_n + (\delta_n - \delta)S y_n\| \\ &\leq |\delta - \delta_n|\|y_n\| + |\delta_n - \delta|\|S y_n\|.\end{aligned}$$

From a condition (iv), we have

$$\lim_{n \rightarrow \infty} \|T_\delta y_n - T_{\delta_n} y_n\| = 0. \quad (18)$$

Since the nonexpansiveness of  $T_{\delta_n}$ , we have

$$\begin{aligned}\|T_{\delta_n} y_n - y_n\| &= \|T_{\delta_n} y_n - T_{\delta_n} x_n + T_{\delta_n} x_n - y_n\| \\ &\leq \|T_{\delta_n} y_n - T_{\delta_n} x_n\| + \|T_{\delta_n} x_n - y_n\| \\ &\leq \|y_n - x_n\| + \|u_n - y_n\|.\end{aligned}$$

From (15) and (16), we have

$$\lim_{n \rightarrow \infty} \|T_{\delta_n} y_n - y_n\| = 0. \quad (19)$$

From (17), (18) and (19), we have

$$\lim_{n \rightarrow \infty} \|T_\delta y_n - y_n\| = 0. \quad (20)$$

From the definition of  $\{x_n\}$ , we have

$$\begin{aligned}\|Z y_n - y_n\| &= \|Z y_n - x_n + x_n - y_n\| \\ &\leq \|Z y_n - x_n\| + \|x_n - y_n\| \\ &= \|x_{n+1} - x_n\| + \|x_n - y_n\|.\end{aligned}$$

From (13) and (15), we have

$$\lim_{n \rightarrow \infty} \|Z y_n - y_n\| = 0. \quad (21)$$

Step3. We will show that  $\limsup_{n \rightarrow \infty} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \leq 0$  where  $x_0 = Q_{\mathbb{F}} f(x_0)$ .

Define the mapping  $B : C \rightarrow C$  by  $Bx = \varepsilon Zx + (1 - \varepsilon)T_\delta x$  for all  $x \in C$  and  $\varepsilon \in (0, 1)$ .

From Lemmas 8 and 10, we have  $F(B) = F(Z) \cap F(T_\delta) = F(Z) \cap F(T) \cap F(S) = \mathbb{F}$ .

From the definition of  $B$ , we have

$$\begin{aligned}\|y_n - B y_n\| &= \|y_n - (\varepsilon Z y_n + (1 - \varepsilon)T_\delta y_n)\| \\ &= \|\varepsilon(y_n - Z y_n) + (1 - \varepsilon)(y_n - T_\delta y_n)\| \\ &\leq \varepsilon\|y_n - Z y_n\| + (1 - \varepsilon)\|y_n - T_\delta y_n\|.\end{aligned}$$

From (20) and (21), we have

$$\lim_{n \rightarrow \infty} \|y_n - B y_n\| = 0. \quad (22)$$

Since  $Z$  and  $T_\delta$  are nonexpansive mappings, we have  $B$  is a nonexpansive mapping. From Lemma 7, we have

$$\limsup_{n \rightarrow \infty} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \leq 0, \quad (23)$$

where  $x_0 = Q_{\mathbb{F}} f(x_0)$ .

Step4. We show that the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}} f(x_0)$ .

From the definition of  $\{y_n\}$ , we have

$$\begin{aligned}
 & \|y_n - x_0\|^2 \\
 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n - x_0\|^2 \\
 &= \|\alpha_n (f(x_n) - x_0) + \beta_n (x_n - x_0) + \gamma_n (u_n - x_0)\|^2 \\
 &\leq \|\beta_n (x_n - x_0) + \gamma_n (u_n - x_0)\|^2 + 2\alpha_n \langle f(x_n) - x_0, j(y_n - x_0) \rangle \\
 &\leq (\beta_n \|x_n - x_0\| + \gamma_n \|u_n - x_0\|)^2 \\
 &\quad + 2\alpha_n \langle f(x_n) - f(x_0) + f(x_0) - x_0, j(y_n - x_0) \rangle \\
 &\leq (\beta_n \|x_n - x_0\| + \gamma_n \|x_n - x_0\|)^2 + 2\alpha_n \langle f(x_n) - f(x_0), j(y_n - x_0) \rangle \\
 &\quad + 2\alpha_n \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
 &\leq (1 - \alpha_n)^2 \|x_n - x_0\|^2 + 2\alpha_n \|f(x_n) - f(x_0)\| \|y_n - x_0\| \\
 &\quad + 2\alpha_n \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
 &\leq (1 - \alpha_n)^2 \|x_n - x_0\|^2 + 2\alpha_n \xi \|x_n - x_0\| \|y_n - x_0\| \\
 &\quad + 2\alpha_n \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
 &\leq (1 - \alpha_n)^2 \|x_n - x_0\|^2 + 2\alpha_n \xi [\|x_n - x_0\|^2 + \|y_n - x_0\|^2] \\
 &\quad + 2\alpha_n \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
 &= ((1 - \alpha_n)^2 + \alpha_n \xi) \|x_n - x_0\|^2 + \alpha_n \xi \|y_n - x_0\|^2 \\
 &\quad + 2\alpha_n \langle f(x_0) - x_0, j(y_n - x_0) \rangle.
 \end{aligned}$$

It implies that

$$\begin{aligned}
 & \|y_n - x_0\|^2 \\
 &\leq \frac{(1 - \alpha_n)^2 + \alpha_n \xi}{(1 - \alpha_n \xi)} \|x_n - x_0\|^2 + \frac{2\alpha_n}{(1 - \alpha_n \xi)} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
 &= \frac{1 - \alpha_n \xi - 2\alpha_n(1 - \xi)}{(1 - \alpha_n \xi)} \|x_n - x_0\|^2 + \frac{\alpha_n^2}{(1 - \alpha_n \xi)} \|x_n - x_0\|^2 \\
 &\quad + \frac{2\alpha_n}{(1 - \alpha_n \xi)} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
 &= \left(1 - \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi}\right) \|x_n - x_0\|^2 + \frac{\alpha_n^2}{1 - \alpha_n \xi} \|x_n - x_0\|^2 \\
 &\quad + \frac{2\alpha_n}{1 - \alpha_n \xi} \langle f(x_0) - x_0, j(y_n - x_0) \rangle \\
 &= \left(1 - \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi}\right) \|x_n - x_0\|^2 \\
 &\quad + \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi} \left(\frac{\alpha_n}{2(1 - \xi)} \|x_n - x_0\|^2 + \frac{1}{1 - \xi} \langle f(x_0) - x_0, j(y_n - x_0) \rangle\right). \quad (24)
 \end{aligned}$$

By the definition of  $\{x_{n+1}\}$  and (24), we have

$$\begin{aligned}
 & \|x_{n+1} - x_0\|^2 \\
 &= \|Zy_n - x_0\|^2 \\
 &\leq \|y_n - x_0\|^2
 \end{aligned}$$

$$\begin{aligned} &\leq \left(1 - \frac{2\alpha_n(1-\xi)}{1-\alpha_n\xi}\right) \|x_n - x_0\|^2 + \frac{2\alpha_n(1-\xi)}{1-\alpha_n\xi} \left(\frac{\alpha_n}{2(1-\xi)} \|x_n - x_0\|^2\right. \\ &\quad \left. + \frac{1}{1-\xi} \langle f(x_0) - x_0, j(y_n - x_0) \rangle\right). \end{aligned} \quad (25)$$

Applying (25), a condition (i) and Lemma 1, we can conclude that the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}}f(x_0)$  and  $(x_0, y_0, z_0)$  is a solution of (5), where  $y_0 = Q_C(I - \lambda_2 D_2)(ax_0 + (1-a)z_0)$  and  $z_0 = Q_C(I - \lambda_3 D_3)x_0$ . This completes the proof.  $\square$

The following corollary is consequence which is applied by Theorem 4. Therefore, we obtain the following results.

**Corollary 1** *Let  $C$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  and let  $Q_C$  be a sunny nonexpansive retraction of  $C$ . Let  $D_1, D_2, D_3 : C \rightarrow E$  be  $d_1, d_2, d_3$ -inverse strongly accretive operators, receptively. Define the mapping  $Z : C \rightarrow C$  by  $Zx = Q_C(I - \lambda_1 D_1)(Q_C(I - \lambda_2 D_2)(Q_C(I - \lambda_3 D_3)x))$  for all  $x \in C$ ,  $\lambda_1 \in (0, \frac{d_1}{K^2})$ ,  $\lambda_2 \in (0, \frac{d_2}{K^2})$ ,  $\lambda_3 \in (0, \frac{d_3}{K^2})$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . Let  $T, S : C \rightarrow C$  be nonexpansive mappings. Assume that  $\mathbb{F} = F(Z) \cap F(S) \cap F(T) \neq \emptyset$ . Let the sequence  $\{x_n\}$  be generated by  $x_1 \in C$  and*

$$\begin{cases} u_n = T(\delta_n x_n + (1 - \delta_n)Sx_n), \\ x_{n+1} = Z(\alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n), \end{cases} \quad (26)$$

where  $f : C \rightarrow C$  is a contraction mapping with a constant  $\xi \in (0, 1)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$  with  $\alpha_n + \beta_n + \gamma_n = 1$ . Suppose that the following conditions are satisfied

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$  for all  $c, d > 0$  and  $n \geq 1$ ,
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ,  $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ ,  $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ ,  $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ ,
- (iv)  $\lim_{n \rightarrow \infty} \delta_n = \delta$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}}f(x_0)$  and  $(x_0, y_0, z_0)$  is a solution of (5), where  $y_0 = Q_C(I - \lambda_2 D_2)z_0$  and  $z_0 = Q_C(I - \lambda_3 D_3)x_0$ .

#### 4 Application

In this section, we prove strong convergence to approximation a common element of solutions of generalized variational inequalities in Banach space and fixed point problem of nonexpansive mapping. We give some useful lemmas to prove Theorem 5.

**Lemma 11** (See [3]) *Let  $C$  be a nonempty closed convex subset of a smooth Banach space  $E$ . Let  $Q_C$  be a sunny nonexpansive retraction from  $E$  onto  $C$  and let  $A$  be an accretive operator of  $C$  into  $E$ . Then, for all  $\lambda > 0$ ,*

$$S(C, A) = F(Q_C(I - \lambda A)).$$

**Lemma 12** *Let  $C$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  and let  $Q_C$  be a sunny nonexpansive retraction from  $E$  onto  $C$ . Let  $D_1, D_2, D_3 : C \rightarrow E$  be  $d_1, d_2, d_3$ -inverse strongly accretive operators, receptively.*

Define a mapping  $Z$  as in Lemma 9 and for every  $\lambda_1 \in (0, \frac{d_1}{K^2})$ ,  $\lambda_2 \in (0, \frac{d_2}{K^2})$ ,  $\lambda_3 \in (0, \frac{d_3}{K^2})$ ,  $a \in [0, 1]$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . If  $S(C, A) \cap S(C, B) \neq \emptyset$ , then  $F(Z) = S(C, A) \cap S(C, B)$ .

**Proof** From Lemma 11, we have

$$S(C, D_1) = F(Q_C(I - \lambda_1 D_1)), S(C, D_2) = F(Q_C(I - \lambda_2 D_2)),$$

and

$$S(C, D_3) = F(Q_C(I - \lambda_3 D_3)).$$

Using the same method as Theorem 4, we find that  $Q_C(I - \lambda_1 D_1)$ ,  $Q_C(I - \lambda_2 D_2)$  and  $Q_C(I - \lambda_3 D_3)$  are nonexpansive mappings.

From the definition of  $Z$  and Lemma 10, we have

$$\begin{aligned} F(Z) &= F(Q_C(I - \lambda_1 D_1)) \cap F(Q_C(I - \lambda_2 D_2)) \cap F(Q_C(I - \lambda_3 D_3)) \\ &= S(C, D_1) \cap S(C, D_2) \cap S(C, D_3). \end{aligned}$$

□

From Theorem 4 and Lemma 12, we have the following theorem.

**Theorem 5** Let  $C$  be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space  $E$  and let  $Q_C$  be a sunny nonexpansive retraction of  $C$ . Let  $D_1, D_2, D_3 : C \rightarrow E$  be  $d_1, d_2, d_3$ -inverse strongly accretive operators, respectively. Define the mapping  $Z : C \rightarrow C$  by  $Zx = Q_C(I - \lambda_1 D_1)(ax + (1-a)Q_C(I - \lambda_2 D_2)(ax + (1-a)Q_C(I - \lambda_3 D_3)x))$  for all  $x \in C$ ,  $\lambda_1 \in (0, \frac{d_1}{K^2})$ ,  $\lambda_2 \in (0, \frac{d_2}{K^2})$ ,  $\lambda_3 \in (0, \frac{d_3}{K^2})$ ,  $a \in [0, 1]$ , where  $K$  is the 2-uniformly smooth constant of  $E$ . Let  $T, S : C \rightarrow C$  be nonexpansive mappings. Assume that  $\mathbb{F} = S(C, D_1) \cap S(C, D_2) \cap S(C, D_3) \cap F(S) \cap F(T) \neq \emptyset$ . Let the sequence  $\{x_n\}$  be generated by  $x_1 \in C$  and

$$\begin{cases} u_n = T(\delta_n x_n + (1 - \delta_n) Sx_n), \\ y_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n, \\ x_{n+1} = Q_C(I - \lambda_1 D_1)(aI + (1-a)Q_C(I - \lambda_2 D_2)(aI \\ \quad + (1-a)Q_C(I - \lambda_3 D_3)))y_n, \quad \forall n \geq 1, \end{cases} \quad (27)$$

where  $f : C \rightarrow C$  is a contraction mapping with a constant  $\xi \in (0, 1)$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$  with  $\alpha_n + \beta_n + \gamma_n = 1$ . Suppose that the following conditions are satisfied

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$  for all  $c, d > 0$  and  $n \geq 1$ ,
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ,  $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ ,  
 $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ ,  $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$ ,
- (iv)  $\lim_{n \rightarrow \infty} \delta_n = \delta$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x_0 = Q_{\mathbb{F}} f(x_0)$  and  $(x_0, y_0, z_0)$  is a solution of (5), where  $y_0 = Q_C(I - \lambda_2 D_2)(ax_0 + (1-a)z_0)$  and  $z_0 = Q_C(I - \lambda_3 D_3)x_0$ .

## 5 Numerical

The purpose of this section we give a numerical examples to support our some report. The following example given for supporting Theorem 4.

**Example 1** Let  $\mathbb{R}$  be the set of real number and

$$E = L_2([-5, 5]) = \left\{ e : \mathbb{R} \rightarrow \mathbb{R} \mid \int_{-5}^5 e^2(x) dx < \infty \right\},$$

with the inner product defined by

$$\langle e, g \rangle = \int_{-5}^5 e(x)g(x)dx,$$

and

$$\|e\| = \left( \int_{-5}^5 e^2(x)dx \right)^{\frac{1}{2}},$$

for all  $e, g \in E$ . Let  $C = H(g, 500) = \{h \in L_2([-5, 5]) \mid \int_{-5}^5 g(x)h(x)dx = 500\}$  where  $g \in L_2([-5, 5])$  with  $g(x) = 2x - 3$ , for all  $x \in [-5, 5]$ . Define mappings  $D_1, D_2, D_3, T, S, f : C \rightarrow C$  as follows  $D_1(h) = \frac{h-3I}{2}, D_2(h) = \frac{h-3I}{5}, D_3(h) = \frac{h-3I}{7}, T(h) = \frac{h+3I}{2}, S(h) = \frac{h+6I}{3}$  and  $f(h) = \frac{h}{5}$ , for all  $h \in C$ . Define the mapping  $Z : C \rightarrow C$  by  $Zx = Q_C(I - \lambda_1 D_1)(ah + (1-a)Q_C(I - \lambda_2 D_2)(ah + (1-a)Q_C(I - \lambda_3 D_3)h))$  for all  $h \in C$ . Setting  $\{\alpha_n\} = \{\frac{1}{7n}\}, \{\beta_n\} = \{\frac{31n-4}{49n}\}, \{\gamma_n\} = \{\frac{18n-3}{49n}\}$  and  $\{\delta_n\} = \{\frac{2n-1}{7n}\}$ , for all  $n \in \mathbb{N}$ . Then the sequence  $\{x_n\}$  generated by (7) converges strongly to a point  $x^* = 3I$ .

**Solution 1** By definition of  $D_1, D_2$  and  $D_3$ , we have  $D_1, D_2$  and  $D_3$  are 1-inverse strongly accretive operator. It is easy to check that  $f$  is  $\frac{1}{5}$ -contraction mapping and  $T, S$  are nonexpansive mapping with  $F(S) \cap F(T) = \{3I\}$ .

By definition of  $Q_C$  and  $C$ , we have

$$Q_C f = f - \frac{\langle f, g \rangle - 500}{\|g\|^2} g.$$

Let  $K = \frac{1}{3}$ , we have  $K$  is 2-uniformly smooth constant of  $\mathbb{R}$ . We choose  $\lambda_1 = 2 \in (0, 9), \lambda_2 = 3 \in (0, 9)$  and  $\lambda_3 = 4 \in (0, 9)$  and  $a = \frac{1}{2} \in [0, 1]$ .

For every  $n \in \mathbb{N}$ ,  $\{\alpha_n\} = \{\frac{1}{7n}\}, \{\beta_n\} = \{\frac{31n-4}{49n}\}, \{\gamma_n\} = \{\frac{18n-3}{49n}\}$  and  $\{\delta_n\} = \{\frac{2n-1}{7n}\}$ , then the sequence  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  and  $\{\delta_n\}$  satisfy all the conditions of Theorem 4.

From the definition of  $Z$ , that is  $Zh = Q_C(I - 2D_1)(\frac{1}{2}h + \frac{1}{2}Q_C(I - 3D_2)(\frac{1}{2}h + \frac{1}{2}Q_C(I - 4D_3)h))$  for all  $h \in C$ , we have  $F(Z) = \{3I\}$ . Then  $\mathbb{F} = F(Z) \cap F(S) \cap F(T) = \{3I\}$ .

From Theorem 4, we obtain the sequence  $\{x_n\}$  converges strongly to  $3I$ .

**Example 2** Let  $\mathbb{R}$  be the real line with Euclidean norm and let  $C = [0, 3]$  and  $D_1, D_2, D_3 : C \rightarrow \mathbb{R}$  be mappings defined by  $D_1x = \frac{x}{3}, D_2x = \frac{x}{5}$  and  $D_3x = \frac{x}{7}$  for all  $x \in C$ . Let  $T, S : C \rightarrow C$  be defined by  $Tx = \max\{0, -x\}$  and  $Sx = \min\{0, \frac{x}{2}\}$  for all  $x \in C$ . Let  $f : C \rightarrow C$  be a mapping defined by  $f(x) = \frac{x}{4}$  for all  $x \in C$ . Define the mapping  $Z : C \rightarrow C$  by  $Zx = Q_C(I - \lambda_1 D_1)(ax + (1-a)Q_C(I - \lambda_2 D_2)(ax + (1-a)Q_C(I - \lambda_3 D_3)x))$  for all  $x \in C$ . Setting  $\{\alpha_n\} = \{\frac{1}{5n}\}, \{\beta_n\} = \{\frac{9n-4}{25n}\}, \{\gamma_n\} = \{\frac{16n-1}{25n}\}$  and  $\{\delta_n\} = \{\frac{2n-1}{5n}\}$ , for all  $n \in \mathbb{N}$ . Then the sequence  $\{x_n\}$  generated by (7) converges strongly to a point  $x^* = 0$ .

**Solution 2** Since  $f(x) = \frac{x}{4}$  for all  $x \in C$ , we have  $f$  is  $\frac{1}{4}$ -contraction mapping. It is easy to see that  $T$  and  $S$  are nonexpansive mappings with  $F(S) \cap F(T) = \{0\}$ . Since  $D_1x = \frac{x}{3}, D_2x = \frac{x}{5}$  and  $D_3x = \frac{x}{7}$  for all  $x \in C$ , we have  $D_1, D_2$  and  $D_3$  are 1-inverse strongly accretive operators.

**Table 1** The values of  $\{x_n\}$  with  $x_1 = 5, n = 25$

n	$x_n$
1	5.0000
2	0.2917
3	0.0208
4	0.0016
5	0.0001
⋮	⋮
13	0.0000
⋮	⋮
21	0.0000
22	0.0000
23	0.0000
24	0.0000
25	0.0000

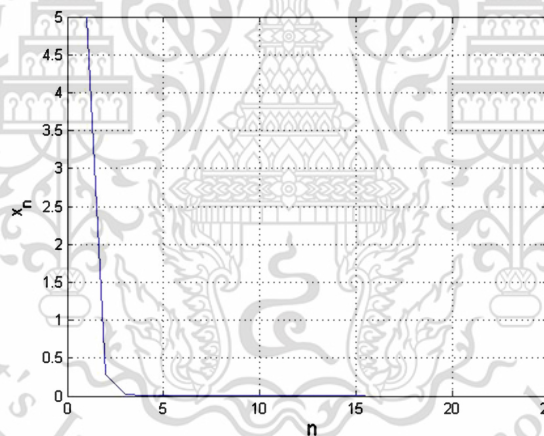


Fig. 1 The convergence of  $\{x_n\}$  with initial values  $x_1 = 5$  and  $n = 25$

Let  $K = \frac{1}{3}$ , we have  $K$  is 2-uniformly smooth constant of  $\mathbb{R}$ . We choose  $\lambda_1 = 2 \in (0, 9)$ ,  $\lambda_2 = 3 \in (0, 9)$  and  $\lambda_3 = 4 \in (0, 9)$  and  $a = \frac{1}{2} \in [0, 1]$ .

For every  $n \in \mathbb{N}$ ,  $\{\alpha_n\} = \{\frac{1}{5n}\}$ ,  $\{\beta_n\} = \{\frac{9n-4}{25n}\}$ ,  $\{\gamma_n\} = \{\frac{16n-1}{25n}\}$  and  $\{\delta_n\} = \{\frac{2n-1}{5n}\}$ , then the sequence  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$  and  $\{\delta_n\}$  satisfy all the conditions of Theorem 4.

From the definition of  $Z$ , that is  $Zx = Q_C(I - 2D_1)(\frac{1}{2}x + \frac{1}{2}Q_C(I - 3D_2)(\frac{1}{2}x + \frac{1}{2}Q_C(I - 4D_3)x))$ , we have  $F(Z) = \{0\}$  and  $(0,0,0)$  is a solution of (5). Then  $\mathbb{F} = F(Z) \cap F(S) \cap F(T) = \{0\}$ . We rewrite (7) as follows:

$$\begin{cases} u_n = T \left( \frac{2n-1}{5n} x_n + \left(1 - \frac{2n-1}{5n}\right) Sx_n \right), \\ x_{n+1} = Z \left( \frac{1}{5n} f(x_n) + \frac{9n-4}{25n} x_n + \frac{16n-1}{25n} u_n \right), \end{cases} \quad (28)$$

Choose  $x_1 = 5$  and  $n = 25$ . The numerical computations for the sequence  $\{x_n\}$  are shown in Table 1 and Fig. 1.

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