THEOREMS FOR VARIATIONAL INEQUALITY PROBLEM AND GENERALIZED SYSTEM OF MODIFIED VARIATIONAL INCLUSION PROBLEM TO SOLVE SOME ECONOMICS PROBLEMS

ARAYA KHEAWBORISUT

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY IN APPLIED MATHEMATICS
DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG
2023
KMITL-2023-SC-D-001-055

Thesis Title Theorems for Variational Inequality Problem and Generalized System of Modified Variational Inclusion Problem to Solve Some Economics Problems

Student Name Miss Araya Kheawborisut

Student ID 61605122

Degree Doctor of Philosophy (Applied Mathematics)

Department Mathematics

Year 2023

Thesis Advisor Assoc. Prof. Dr. Atid Kangtunyakarn

Abstract

The aim of this thesis was to introduce a modified form of generalized system of variational inclusion that is called generalized system of modified variational inclusion problem(GSMVIP). Motivated by several recent results related to the subgradient extragradient method, we proposed a new subgradient extragradient method for finding a common element of the set of solutions of GSMVIP and the set of a finite family of variational inequalities problem. Secondly, we introduced G-subgradient extragradient method for solving G-variational inequality problem in Hilbert space endowed with graph. Under suitable assumptions, strong convergence theorem had been proved in framework of a Hilbert space. Finally, some numerical results indicated that the proposed method is effective. Moreover, several computational tests and applications for market equilibrium problems in economics were given to illustrate the advantages of the proposed iterative schemes.

Keywords : System of variational inclusion problem, Variational inequality problem, Subgradient extragradient algorithm, G-Variational inequality problem, G-Subgradient extragradient algorithm, Market equilibrium problem

Acknowledgements

Firstly, I would like to express my sincere gratitude to my advisor, Associate Professor Dr. Atid Kangtunyakarn for the advice, supervision, and encouragement throughout this research helped me to complete this thesis successfully.

In addition, I am grateful to all committees, Assistant Professor Dr. Wongvisarut Khuangsatung, Associate Professor Dr. Nopparat Pochai, Associate Professor Dr. Pattrawut Chansangiam, and Assistant Professor Dr. Kanchana Kumnungkit for suggestions, useful comments, and all their help.

I would like to express my deep gratitude to my parents and family for their encouragement throughout the study. Moreover, I would like to thank my seniors and my friends in the Department of Mathematics, King Mongkut's Institute of Technology Ladkrabang for pieces knowledge, suggestions, and assistance with studying.

Finally, I am also thankful to the Department of Mathematics, King Mongkut's Institute of Technology Ladkrabang for providing me with the necessary facilities. Furthermore, I would like to thank King Mongkut's Institute of Technology Ladkrabang for the opportunity to grant a scholarship for a Doctoral Degree.

ARARYA KHEAWBORISUT

Table of Contents

List of Tables

Table | Page

List of Figures

Figure **Page**

Chapter 1

Introduction

1.1 Iterative methods of nonlinear mappings

Mathematics is one of the important tools for developing new technologies. The original of new technologies was happened from observing, suspicion and questioning. Normally, the most problems are nonlinear problems. We transformed the problem into many mathematical models and used different methods to solve the problem. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear mapping. The fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, and physics.Finding the answer of the equation by using the fixed point theory may have the answer or no answer. Therefore, fixed point theory is involved with finding conditions on the set X and the mapping $T : X \to X$ to guarantee the existence and uniqueness of fixed points. Moreover, researchers have been studying about the structure of fixed point set and the approximation of fixed points. Iterative schemes for finding the solution set of nonlinear mappings such as nonexpansive mappings, quasi-nonexpansive mappings have been increasingly studied by many mathematicians. They have introduced various types of iterative methods to approximate fixed points.

Throughout this paper, let H be a real Hilbert space and C be a nonempty closed convex subset of H with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $T : C \to C$ be a mapping. Then, T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We denote $F(T)$ by the set of fixed points of T, that is $F(T) = \{x \in C : Tx = x\}$. It is well known that $F(T)$ is closed convex and also nonempty.

Many researchers have studied the iterative scheme to approximate the fixed point problem of nonlinear mapping as follows;

In 1953, Mann [1] introduced the sequence $\{x_n\}$ generated by $x_0 \in H$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \qquad \forall n \geq 0, \tag{1.1}$$

where C is a nonempty closed convex subset of a normed space, $T : C \to C$ is a mapping and the sequence $\{\alpha_n\}$ is in the interval $(0, 1)$, iteration (1.1) is called *Mann iteration*. The Mann iteration algorithm is extremely useful for finding the fixed point problem of nonexpansive mappings, and provides a unified framework for different algorithms. However, it should be pointed out that even in a Hilbert space, the iterative sequence $\{x_n\}$ defined by (1.1) has only weak convergence under certain conditions. Many authors have been trying to modify Mann's iteration to solve various problems such as the fixed point problem, split feasibility problem, equilibrium problem, monotone inclusion and image restoration problem; see more detail in [2, 28, 29].

In 1967, Halpern [20] introduced the Halpern iteration to find a fixed point of nonexpansive mapping $T : C \to C$ as follow:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n \tag{1.2}$$

for each $n \leq 0$ and $x_0 = u \in C$ where $\{\alpha_n\} \subset (0, 1)$. He proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T.

In 2000, Moudafi [2] extended the Halpern iteration to the following process, which is called *the viscosity approximation method* for nonexpansive mapping T and the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \qquad \forall n \geq 0, \tag{1.3}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies certain conditions, $T : C \to C$ is a nonexpansive mapping and $f : C \to C$ is a contraction. Then he proved the sequence $\{x_n\}$ converges strongly to $z = P_{F(S)}f(z)$. Moreover, the viscosity approximation method has been studied and developed in many pieces of research to approximate the convex feasibility problem, hierarchical fixed point problem, variational inequality problem, split common null point problem, see previous studies in [32, 31].

1.2 Variational Inequality Problem

The variational inequality problem (VIP) is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \qquad \forall v \in C. \tag{1.4}$$

The set of the solutions of the variational inequality problem is denoted by $VI(C, A)$. This problem is an important tool in economics, engineering and mathematics. It includes, as special cases, many problems of nonlinear analysis such as optimization, optimal control problems, saddle point problems and mathematical programming, see for example, [3, 33, 19, 21].

In 2003, Takahashi and Toyoda [10] introduced an iterative scheme for finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequalities problem in a Hilbert space as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)TP_C(I - \lambda_n A)x_n, \forall n \geq 0,$$

and proved weak convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$.

In 2005, Iiduka and Takahashi [11] introduced an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem in a Hilbert space as follows:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)TP_C(I-\lambda_n A)x_n, \forall n \geq 0,$$

and proved strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$.

A well known that one method is most popular for solving the problem (VIP) is the extragradient method proposed by Korpelevich [4]. The extragradient method is need to calculate two projections onto the feasible set C in each iteration. So, in case that the set C is not simple to project on to it, As in some remarks of the authors in [5], when the subset is a closed expression as a ball or a half-space, the projection onto the feasible subset C can be computed easily. This can affect the efficiency of the used method. In recent years, the extragradient method has received great attention by many authors, who improved it in various ways, see, e.g. [7, 8, 12, 13, 34, 15, 16] and the many references therein.

In 2011, Censor et al. [15] proposed the subgradient extragradient method for solving variational ineuality problems as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \leq 0\} \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n) \end{cases} \tag{1.5}$$

for each $n \geq 1$, where $\lambda \in (0, 1/L)$. In this method, they have replaced the second projection in Korpelevich's extragradient method by a projection on to a half-space, which is computed explicitly.

In 2012, Kangtunyakarn [26] modified the set of variational inequality problems as follows:

$$\begin{aligned} VI(C, aA + (1-a)B) &= \{x \in C : \langle y - x, (aA + (1-a)B)x \rangle \\ &\geq 0, \forall y \in C, a \in (0,1)\}, \end{aligned} \tag{1.6}$$

where A and B are the mappings of C into H.
In order to develop efficient algorithms for finding solution of a finite family variational inequalities problem, inspired by problem (1.6), we define the new half space $Q_n = \{z \in H : \left\langle (I - \lambda \sum_{i=1}^{N} a_i A_i)x_n - y_n, y_n - z \rangle \geq 0 \right\}$, which as a tool to prove strong convergence theorem. In particular,if we put $i = 1$,then Q_n reduces to T_n in subgradient extragradient method (1.5). However, the sequence $\{x_n\}$ generated by (1.5) converges weakly to a solution of the variational inequality problem.

1.3 Variational Inclusion Problem

Let $B : H \to H$ be a mapping and $M : H \to 2^H$ be a multi-valued mapping. The variational inclusion problem is to find $x \in H$ such that

$$\theta \in Bx + Mx, \tag{1.7}$$

where θ is zero vector in H. The set of the solution of (1.7) is denoted by $VI(H, B, M)$. This problem has received much attention due to its applications in large variety of problems arising in convex programming, variational inequalities, split feasibility problem, and minimization problem. To be more precise, some concrete problems in machine learning, image processing, and linear inverse problem can be modelled mathematically as this formulation. A multi-valued mapping $M : H \to 2^H$ is called monotone, if for all $x, y \in H$, $u \in Mx$ and $v \in My$ implies that $< u - v, x - y > \geq 0$. A multi-valued mapping $M : H \to 2^H$ is called maximal monotone, if it is monotone and if for any $(x, u) \in H \times H$, $< u - v, x - y > \geq 0$ for every $(y, v) \in Graph(M)(Graph(M) := \{(x, u) \in H \times H : u \in Mx\})$ implies that $u \in Mx$.

Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad \forall u \in H,$$

is called the resolvent operator associated with M where λ is positive number and I is an identity mapping, see [25]. Note that $J_{M,\lambda}$ is nonexpansive mapping.

In 2008, Zhang et al. [25] proved a strong convergence theorem for finding a common element of the set of solutions of variational inclusion problem and the set of fixed points of nonexpansive mappings in Hilbert space. They introduced the iterative scheme as follows:

$$\begin{aligned} y_n &= J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} &= \alpha x + (1 - \alpha_n) S y_n, \quad \forall n \geq 0, \end{aligned}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$ and λ.

Motivated by problem (1.7), we introduce a new problem of the system of variational inclusion in a real Hilbert space as follows:

Let H be a real Hilbert space and let $A : H \to H$ be mapping and $M_A, M_B : H \to 2^H$ be set value mapping. We consider the problem for finding $x^* \in H$ such that

$$\theta \in Ax^* + M_A x^* \ \ and \ \ \theta \in Ax^* + M_B x^* \tag{1.8}$$

where θ is zero mapping in H, which is called generalized system of modified variational inclusion problem(in short, GSMVIP) . The set of solution of (1.8) is denoted by Ω ,i.e., $\Omega = \{x^* \in H : \theta \in Ax^* + M_A x^* \ \ and \ \ \theta \in Ax^* + M_B x^*\}$. In particular, if $M_A = M_B$, then the problem (1.8) reduces to the problem (1.7) and if $J_{\lambda_A}^{M_A} = J_{\lambda_B}^{M_B} = P_C$, then the problem (1.8) reduces to VIP.

1.4 G-Variational Inequality Problem with graph

The following symbols will be used throughout this research:

i) $G = (Eed(G), Ver(G))$ is a *directed graph* where $Ver(G)$ is *vertices set* and $Eed(G)$ is set of its edges with $\{(x,x) : x \in Ver(G)\} \subseteq Eed(G)$

ii) $Eed(G^{-1}) = \{(y,x) : (x,y) \in Eed(G)\}$.

Jachymski [41] was the first to analyze the fixed point problem in metric space endowed with graph and introduce the crucial conclusion in this space by integrating fixed point properties and graph theory, see more detail in [41].

Let $C = Ver(G)$ and the mapping $T : C \to C$ is called G-nonexpansive if the following conditions hold:

1) T is edge-preserving i.e., for each $x, y \in D$ such that $(x,y) \in Eed(G) \Rightarrow (Tx, Ty) \in Eed(G)$,

2) $\|Tx - Ty\| \leqslant \|x - y\|$, whenever $(x,y) \in Eed(G)$ for all $x, y \in C$.

Tiammee et al. were the first to prove the strong convergence theorem of a sequence generated by Halpern iteration for approximating fixed point problem of G-nonexpansive mapping in Hilbert space endowed with a directed graph. See more detail [42].

Using concepts related to the variational inequality problem and graph theory, Kangtunyakarn [43] introduced the G-variational inequality problem, which is to find a point $x^* \in C$ such that

$$\langle y - x^*, Bx^* \rangle \geqslant 0,$$

for all $y \in C$ with $(x^*, y) \in Eed(G)$ and $B : C \to H$ is a mapping, where $C = Ver(G)$. The set of all solution of such problem denoted by $G - Var(C, B)$. He proved strong convergence theorem to solve G-variational inequality problem.

By combining the concepts of subgradient extragadient method and graph theory in this research, we introduce G-subgradient extragadient method for approximating the solution of G-variational inequality problem. To use such a method, we introduce G-Half space by

$$T_G = \{w \in C : \langle (I - \lambda B)x - y, y - w \rangle \geqslant 0\},$$

where $\lambda > 0$, $B : C \to H$ is a mapping and $y = P_C(I - \lambda B)x$ for all $x \in H$ with $(w, x) \in Eed(G)$.

Example 1.1. Let $H = \mathbb{R}^2$ and $C = [-100, 100] \times [-100, 100]$ and metric projection $P_C : H \to C$ define by

$$P_C(z_1, z_2) = (\max\{\min\{z_1, 100\}, -100\}, \max\{\min\{z_2, 100\}, -100\}),$$

for all $z = (z_1, z_2) \in H$.

Let $B : C \to H$ define by $Bx = (\frac{v_1}{3}, \frac{v_2}{3})$ for all $x = (v_1, v_2) \in C$ and $Ver(G) = C$, $Eed(G) =$

$\{(u,v) : u = (u_1,u_2) \in [0,100] \times [0,100]$ and $v = (v_1,v_2) \in (300,\infty) \times (300,\infty)\}$. Putting $\lambda = 2$. From definitions of P_C and B, we have $P_C(1-\lambda B)x = P_C\left(\frac{v_1}{3}, \frac{v_2}{3}\right)$ for all $x = (v_1,v_2) \in H$.

Let $(w,x) \in Eed(G)$, where $w = (w_1,w_2), x = (v_1,v_2)$. From definition of P_C, we have $P_C(I-\lambda B)x = (100,100)$ and $T_G = [0,100] \times [0,100]$.

1.5 Objectives of the study

1) To propose the generalized system of modified variational inclusion problem and a new iterative schemes for solving the GSMVIP and the variational inequalities problem in a framework of Hilbert space.
2) To propose a new iterative schemes for solving G-variational inequality problem in a framework of Hilbert space endowed with a directed graph.
3) To give a lemma, an essential tool for proof strong convergence of our main theorems.
4) To prove a strong convergence theorem for finding a common solution of the GSMVIP and the variational inequalities problem .
5) To prove a strong convergence theorem for finding G-variational inequality problem in a framework of Hilbert space endowed with a directed graph.
6) To give numerical examples for our results to support our main results.
7) To give numerical examples for finding the solution of market equilibrium problem in economics.

1.6 Scopes of the study

1) Study the definitions and properties of variational inclusion problem, variational inequality problems, minimization problems, and fixed point problems in real Hilbert space.
2) Study the definitions and properties of G-variational inequality problem in Hilbert space endowed with a directed graph.
3) Investigate the fixed point problems of nonlinear mappings, including nonexpansive mapping, and α-inverse strongly monotone mapping.
4) Give numerical examples for supporting our main results in $\mathbb{R}$, $\mathbb{R}^2$ and L_2 spaces.

1.7 Benefits of the study

1) Obtain new tools for fixed point problems on real Hilbert space.

2) Obtain strong convergence theorem of the sequence $\{x_n\}$ generated by the new subgradient extragradient for finding a common element of the set of solutions of GSMVIP and the set of a finite family of variational inequalities problem.

3) Obtain strong convergence theorem of the sequence $\{x_n\}$ generated by the new subgradient extragradient for finding solutions of G-variational inequality problem in Hilbert space endowed with a directed graph.

4) Obtain new tools for market equilibrium problem in economics.

1.8 Research methodology

1) Study advanced topics in fixed point theory for nonexpansive mapping, α-inverse strongly monotone mapping and $G-\alpha$-inverse strongly monotone mapping.

2) Study background in a real Hilbert space.

3) Study background in a directed graph.

4) Study the subgradient extragradient algorithm and half space.

5) Collect and study research papers and textbooks concerning fixed point theorem.

6) Determine the objectives and scope of the research.

7) Produce tools for a strong convergence theorem of fixed point problem.

8) Prove strong convergence theorem for fixed point problems in a real Hilbert space.

9) Provide examples and applications.

10) Conclude the results, make suggestions for further works and write the thesis.

Chapter 2

Preliminaries

The objective of this chapter is to provide an understanding of the fundamental concepts and definitions that will be utilized throughout this thesis. Additionally, we present some lemmas, remarks, and useful results that will be referenced in the subsequent chapters.

2.1 Basic Concepts

Definition 2.1 (**Cauchy sequence** [48]). A sequence of vectors $\{x_n\}$ in a normed space X is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n > N$.

Definition 2.2 (**complete** [48]). A normed space X is called *complete* if every Cauchy sequence in X converges to an element of X.

Definition 2.3 (**Convex set** [54]). Let X be a normed space and let C be a subset of X. Then the set C is called *convex* if

$$\alpha x + (1 - \alpha)y \in C,$$

for all $x, y \in C$ and $\alpha \in [0, 1]$.

Remark 2.1 ([54]). An inner product space is called a *real inner product space* for the case when the scalars are the real numbers and $\langle x, y\rangle$ is a real number. For the case, (3) means

$$\langle x, y\rangle = \langle y, x\rangle.$$

Theorem 2.2 (**Schwarz inequality**[54]). Let X be an inner product space and let x and y be element in X. Then the following holds:

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Theorem 2.3 (**Parallelogram Law**[48]). For any two elements x and y of inner product space X we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Definition 2.4 (**Strong convergence** [48]). A sequence $\{x_n\}$ of vectors in an inner product space X is called *strongly convergent* to x in X if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.5 (**Weak convergence** [48]). A sequence $\{x_n\}$ of vectors in an inner product space X is called *weakly convergent* to x in X if

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ as } n \to \infty \text{ for all } y \in X.$$

Theorem 2.4 ([54]). The inner product of an inner product space X is jointly continuous:

$$x_n \to x \text{ and } y_n \to y \Rightarrow \langle x_n, y_n\rangle \to \langle x, y\rangle.$$

In this thesis, we denote weak and strong convergence by the notations " $\rightharpoonup$" and " $\to$", respectively.

Remark 2.5 ([54]). We of course obtain from Theorem 2.4 that if $x_n \to x$, then for a fixed $y \in X$

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ and } \langle y, x_n\rangle \to \langle x, y\rangle.$$

Theorem 2.6 ([48]). A strongly convergence sequence is weakly convergence (to the same limit), that is, $x_n \to x$ implies $x_n \rightharpoonup x$.

Remark 2.7 ([54]). If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Lemma 2.8 ([54]). Let $\{x_n\}$ be a Cauchy sequence of an inner product space C such that $x_n \rightharpoonup x$. Then $x_n \to x$.

2.2 Some properties in Hilbert space

In this section, we indicate the following theorems in Hilbert space which are very useful for our results.

Let C be a nonempty closed convex subset of a real Hilbert space H. For every $x \in H$, there exists a unique nearest point $P_C x \in C$ such that

$$||x - P_C x|| \leq ||x - y||, \qquad for\ all \quad y \in C.$$

P_C is called metric projection of H onto C. It follows that

$$||x - y||^2 \geq ||x - P_C x||^2 + ||y - P_C x||^2, \qquad for\ all \quad x \in H, y \in C. \tag{2.1}$$

For each $x \in H$ and $y \in C$. It follows that

$$||x - y||^2 \geq ||x - P_C x||^2 + ||y - P_C x||^2. \tag{2.2}$$

Lemma 2.9. Let P_C be the metric projection from H onto C. Then P_C is a nonexpansive mapping, i.e.

$$\|P_C x - P_C y\| \leqslant \|x - y\|, \ \forall\ x, y \in H.$$

Lemma 2.10. [18] Given $x \in H$ and $y \in C$. Then, $y = P_C x$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0, \ \forall z \in C.$$

Definition 2.6 (**Hilbert space** [48]). Let X be an inner product space and X is called *Hilbert space* if X is complete inner product space.

Example 2.11 ([51]). The n-dimensional Euclidean space $\mathbb{R}^n$ is Hilbert spaces with inner product defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 + \cdots + x_ny_n \text{ and } \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$$

where $\mathbf{x} = (x_1, x_2, x_3, ..., x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, y_2, y_3, ..., y_n) \in \mathbb{R}^n$.

Lemma 2.12 ([54]). Let H be a real Hilbert space. Then the following results hold:

1) $\|\lambda x + (1-\lambda)y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$,

2) $\|x+y\|^2 \leq \|y\|^2 + 2\langle y, x+y\rangle$,

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

Remark 2.13 ([54]). Let H be an inner product space. Then we know that the following (1) and (2) are equivalent:

1) H is complete,

2) each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Theorem 2.14 ([54]). A Hilbert space H is complete.

Theorem 2.15 ([54]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H with $\{x_n\} \subset C$ and $x_n \to x$, then $x \in C$.

Definition 2.7. Let $M : H \to 2^H$ be a multi-valued mapping.
(i) the graph $G(M)$ of M is defined by

$$G(M) := \{(x, u) \in H \times H : u \in M(x)\},$$

(ii) the operator M is called *a maximal monotone operator* if M is monotone, i.e.

$$\langle u - v, x - y\rangle \geq 0 \qquad \forall u \in M(x), v \in M(y),$$

and the graph $G(M)$ of M is not properly contained in the graph of any other monotone operator. It is clear that a monotone mapping M is maximal if and only if for any $(x, u) \in H \times H$, if $\langle u - v, x - y\rangle \geq 0$ for every $(y, v) \in G(M)$ implies that $u \in M(x)$.

Definition 2.8. Let $A : C \to H$ be a mapping.
(i) A is called *monotone* if

$$\langle x - y, Ax - Ay\rangle \geq 0, \forall x, y \in C.$$

(ii) A is called *μ-Lipschitz continuous* if there exists a nonnegative real number $\mu \geq 0$ such that

$$\|Ax - Ay\| \leq \mu \|x - y\|, \forall x, y \in C.$$

(iii) A is called *α-inverse-strongly-monotone* if there exists a nonnegative real number $\alpha \geq 0$ such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha \left\|Ax-Ay\right\|^2, \forall x,y \in C.$$

Proposition 2.16. Let T be an operator from H to itself. Then T is nonexpansive if and only if $I-T$ is $\frac{1}{2}$-inverse strongly monotone.

Lemma 2.17. [45] Let C be a nonempty closed and convex subset of a real Hilbert space H. If $T: C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, then the mapping $I-T$ is demiclosed at 0, i.e., if $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{x_n - Tx_n\}$ converges strongly to 0, then $x \in F(T)$.

Lemma 2.18. [45] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \qquad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that
(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(2) $\limsup_{n\to 0} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| = \infty$;
Then, $\lim_{n\to 0} s_n = 0$.

Lemma 2.19. [44] Let $\{a_n\}$ and $\{b_n\}$ be subset of $[0,\infty)$ satisfying

$$a_{n+1} \leqslant a_n + b_n,$$

for all $n \in \mathbb{N}$.

i) if $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists

ii) if $\sum_{n=1}^{\infty} b_n < \infty$ and there exist a subsequence of $\{a_n\}$ converging to zero , then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.20. [44] Let $\{v_n\}$ be a sequence in H. Suppose that, for all $u \in C$,

$$\|v_{n+1} - u\| \leqslant \|v_n - u\| + b_n,$$

for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then $\{P_C v_n\}$ converges strongly to some $z \in C$.

Lemma 2.21. [45] Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$$

holds for every $y \in H$ with $x \neq y$.

2.3 Variational Inequalities Problem

In this section, we give a lemma and remark for variational inequalities problem to prove the main results.

Lemma 2.22. [26] Let C be a nonempty closed convex subset of a real Hilbert space H and let A,B : C $\to H$ be α and β - inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C,A) \cap VI(C,B) \neq \emptyset$. Then

$$VI(C, aA + (1-a)B) = VI(C,A) \cap VI(C,B), \qquad \forall a \in (0,1).$$

Furthermore if $0 < \gamma < min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1-a)B)$ is a nonexpansive mapping.

Remark 2.23. For every $i = 1, 2, ..., N$ the mapping $A_i : C \to H$ be α_i- inverse strongly monotone mappings with $\eta = min_{1,2,...,N}\{\alpha_i\}$ and $\bigcap_{i=1}^{N} VI(C, A_i) \neq \emptyset$. Then

$$VI(C, \sum_{i=1}^{N} a_i A_i) = \bigcap_{i=1}^{N} VI(C, A_i), \tag{2.3}$$

where $\sum_{i=1}^{N} a_i = 1$ and $0 < a_i < 1$ for every $i = 1, 2, ..., N$. Moreover, we have $\sum_{i=1}^{N} a_i A_i$ is monotone and μ-Lipschitz continuous mapping.

Proof. It easy to see that $\sum_{i=k+1}^{N} \frac{a_i}{\prod_{j=1}^{k}(1-a_j)} A_i$ is η - inverse strongly monotone mappings with $\eta = min\{\beta_i\}$ for each $i = 2, ..., N$ and $k = 1, 2, ..., N-1$.

If $N = 3$ and let $VI(C, A_1) \cap VI(C, A_2) \cap VI(C, A_3) \neq \emptyset$. By using the lemma 2.22, we have

$$\begin{aligned} VI(C, a_1A_1 + a_2A_2 + a_3A_3) &= VI(C, a_1A_1 + (1-a_1)(\frac{a_2}{1-a_1}A_2 + \frac{a_3}{1-a_1}A_3)) \\ &= VI(C, A_1) \cap VI(C, \frac{a_2}{1-a_1}A_2 + \frac{a_3}{1-a_1}A_3) \\ &= VI(C, A_1) \cap VI(C, A_2) \cap VI(C, A_3), \end{aligned} \tag{2.4}$$

where $a_1, a_2, a_3 \in (0,1)$ and $\sum_{i=1}^{3} a_i = 1$.

If $N = 4$ and let $\bigcap_{i=1}^{4} VI(C, A_i) \neq \emptyset$. By using the lemma 2.22 and (2.4), we have

$$\begin{aligned} VI(C, a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4) &= VI(C, (1-a_4)(\frac{a_1}{1-a_4}A_1 \\ &\quad + \frac{a_2}{1-a_4}A_2 + \frac{a_3}{1-a_4}A_3) + a_4A_4) \\ &= VI(C, \frac{a_1}{1-a_4}A_1 + \frac{a_2}{1-a_4}A_2 \\ &\quad + \frac{a_3}{1-a_4}A_3) \cap VI(C, A_4) \\ &= VI(C, A_1) \cap VI(C, A_2) \\ &\quad \cap VI(C, A_3) \cap VI(C, A_4), \end{aligned} \tag{2.5}$$

where $a_1, a_2, a_3, a_4 \in (0,1)$ and $\sum_{i=1}^{4} a_i = 1$.

By the same way, if $\bigcap_{i=1}^{N} VI(C, A_i) \neq \emptyset$, we obtain

$$VI(C, \sum_{i=1}^{N} a_i A_i) = \bigcap_{i=1}^{N} VI(C, A_i), \tag{2.6}$$

where $a_i \in (0,1)$, for each $i = 1, 2, ..., N$, and $\sum_{i=1}^{N} a_i = 1$. □

2.4 Variational Inclusion Problem

In this section, we give some useful lemma for variational inclusion problem to prove the main results.

Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad \forall u \in H,$$

is called the resolvent operator associated with M where λ is positive number and I is an identity mapping, see [25]. Note that $J_{M,\lambda}$ is nonexpansive mapping.

Lemma 2.24. [25] $u \in H$ is a solution of variational inclusion (1.7) if and only if $u = J_{M,\lambda}(u - \lambda Bu)$, $\forall \lambda > 0$, i.e.,

$$VI(H, B, M) = F(J_{M,\lambda}(I - \lambda B)), \forall \lambda > 0.$$

if $\lambda \in (0, 2\alpha]$, then $VI(H, B, M)$ is closed convex subset in H.

The next lemma presents associate between fixed point of nonlinear mapping and solution of GSMVIP under suitable conditions on parameters.

Lemma 2.25. Let H be a real Hilbert space and let $A_G : H \to H$ be α-inverse strongly monotone mappings. Let $M_A, M_B : H \to 2^H$ be multivalue maximum monotone mapping with $\Omega \neq \emptyset$. If $x^* \in \Omega$ if and only if $x^* = Gx^*$, where $G : H \to H$ be a mapping defined by

$$G(x) = J_{M_A,\lambda_A}(I - \lambda_A A_G)(bx + (1 - b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x)$$

for all $x \in H$, $b \in (0, 1)$ and $\lambda_A, \lambda_B \in (0, 2\alpha)$. Moreover, we have G is a nonexpansive mapping.

Proof. Let conditions hold.
($\Rightarrow$) Let $x^* \in \Omega$, we have $x \in H$ such that $\theta \in A_G x^* + M_A x^*$ and $\theta \in A_G x^* + M_B x^*$, that is $x^* \in VI(H, A_G, M_A)$ and $x^* \in VI(H, A_G, M_B)$.
From lemma 2.24, we have
$x^* \in F(J_{M_A,\lambda_A}(I - \lambda_A A_G))$ and $x^* \in F(J_{M_B,\lambda_B}(I - \lambda_B A_G))$. It implies that

$$x^* = J_{M_A,\lambda_A}(I - \lambda_A A_G)x^* \tag{2.7}$$

and

$$x^* = J_{M_B,\lambda_B}(I - \lambda_B A_G)x^* \tag{2.8}$$

By definition of G ,(2.7) and (2.8) we have

$$\begin{aligned} G(x^*) &= J_{M_A,\lambda_A}(I-\lambda_A A_G)(bx^*+(1-b)J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*) \\ &= x^* \end{aligned}$$

($\Leftarrow$) Let $x^*=G(x^*)$.
We will show that $J_{M_A,\lambda_A}(I-\lambda_A A_G)$ and $J_{M_B,\lambda_B}(I-\lambda_B A_G)$ are nonexpansive mapping.
Since $A_G : H \to H$ is a α-inverse strongly monotone mappings,
we have

$$\begin{aligned} ||J_{M_A,\lambda_A}(I &- \lambda_A A_G)x - J_{M_A,\lambda_A}(I-\lambda_A A_G)y||^2 \\ &\le ||(I-\lambda_A A_G)x-(I-\lambda_A A_G)y||^2 \\ &= ||x-y||^2-2\lambda_A < x-y, A_Gx-A_Gy > +\lambda_A^2||A_Gx-A_Gy||^2 \\ &\le ||x-y||^2-2\lambda_A\alpha||A_Gx-A_Gy||^2+\lambda_A^2||A_Gx-A_Gy||^2 \\ &\le ||x-y||^2-\lambda_A(\lambda_A-2\eta)||A_Gx-A_Gy||^2 \\ &\le ||x-y||^2. \end{aligned}$$

Hence, we obtain $J_{M_A,\lambda_A}(I-\lambda_A A_G)$ is nonexpansive mapping.
Similarly, we can show that $J_{M_B,\lambda_B}(I-\lambda_B A_G)$ is also nonexpansive mapping.
Since $x^*=G(x^*)$, we have

$$x^*=G(x^*)=J_{M_A,\lambda_A}(I-\lambda_A A_G)(bx^*+(1-b)J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*).$$

Let $y\in\Omega$, we have $\theta\in A_Gy+M_Ay$ and $\theta\in A_Gy+M_By$.
From Lemma 2.24, it implies that
$y\in F(J_{M_A,\lambda_A}(I-\lambda_A A_G))\cap F(J_{M_B,\lambda_B}(I-\lambda_B A_G))$. Then

$$\begin{aligned} ||x^*-y||^2 &= ||J_{M_A,\lambda_A}(I-\lambda_A A_G)(bx^*+(1-b)J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*)-y||^2 \\ &= ||J_{M_A,\lambda_A}(I-\lambda_A A_G)(bx^*+(1-b)J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*) \\ &\quad -J_{M_A,\lambda_A}(I-\lambda_A A_G)y||^2 \\ &\le ||(bx^*+(1-b)J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*)-y||^2 \\ &= ||b(x^*-y)+(1-b)(J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*-y)||^2 \\ &= b||x^*-y||^2+(1-b)||J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*-y||^2 \\ &\quad -b(1-b)||x^*-J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*||^2 \\ &\le b||x^*-y||^2+(1-b)||x^*-y||^2-b(1-b)||x^* \\ &\quad -J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*||^2 \\ &= ||x^*-y||^2-b(1-b)||x^*-J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*||^2. \end{aligned} \tag{2.9}$$

It impiles that $||x^*-J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*||=0$.
That is $x^*\in F(J_{M_B,\lambda_B}(I-\lambda_B A_G))$.

Since $x^* = G(x^*)$ and $x^* \in F(J_{M_B,\lambda_B}(I - \lambda_B A_G))$.
We have

$$\begin{aligned} x^* &= J_{M_A,\lambda_A}(I - \lambda_A A_G)(bx^* + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x^*) \\ &= J_{M_A,\lambda_A}(I - \lambda_A A_G)(bx^* + (1-b)x^*) \\ &= J_{M_A,\lambda_A}(I - \lambda_A A_G)x^*. \end{aligned}$$

Therefore $x^* \in F(J_{M_A,\lambda_A}(I - \lambda_A A_G))$.
From Lemma 2.24, $x^* \in F(J_{M_A,\lambda_A}(I - \lambda_A A_G))$ and $x^* \in F(J_{M_B,\lambda_B}(I - \lambda_B A_G))$, we have $\theta \in A_G x^* + M_A x^*$ and $\theta \in A_G x^* + M_B x^*$. Then $x^* \in \Omega$.
Applying (2.9), we can conclude that G is nonexpansive mapping.

We give some examples to support Lemma 2.25 and show that Lemma 2.25 is not true if some condition fails.

□

Example 2.26. Let $H = \mathbb{R}^2$ be the two dimensional space of real numbers with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\langle \mathtt{x}, \mathtt{y} \rangle = x \cdot y = x_1 y_1 + x_2 y_2$, for all $\mathtt{x} = (x_1, x_2) \in \mathbb{R}^2, \mathtt{y} = (y_1, y_2) \in \mathbb{R}^2$ and a usual norm $|| \cdot || : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ give by $||\mathtt{x}|| = \sqrt{x_1^2 + x_2^2}$ for all $\mathtt{x} = (x_1, x_2) \in \mathbb{R}^2$ and A_G: $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $A_G((x_1, x_2)) = (x_1 - 5, x_2 - 5)$. Let $M_A : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by $\{(2x_1 - 1, 2x_2 - 1)\}$ and $M_B : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by $\{(\frac{x_1}{2} + 2, \frac{x_2}{2} + 2)\}$. Show that $(2, 2) \in F(G)$.
Solution Since A_G is a 1-inverse strongly monotone mapping, then we have $\Omega = (2, 2)$. Choose $\lambda_A = \frac{1}{2} \in (0, 2\alpha)$, $\lambda_B = 1 \in (0, 2\alpha)$ and $a = 0.25$, we obtain

$$G(x) = J_{M_A,\frac{1}{2}}(I - \frac{1}{2}A_G)(0.25x + 0.25J_{M_B,1}(I - 1A_G)x).$$

By Lemma 2.25, we have $(2, 2) \in F(G)$.

Example 2.27. Let $H = \mathbb{R}^2$ be the two dimensional space of real numbers with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\langle \mathtt{x}, \mathtt{y} \rangle = x \cdot y = x_1 y_1 + x_2 y_2$, for all $\mathtt{x} = (x_1, x_2) \in \mathbb{R}^2, \mathtt{y} = (y_1, y_2) \in \mathbb{R}^2$ and a usual norm $|| \cdot || : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ give by $||\mathtt{x}|| = \sqrt{x_1^2 + x_2^2}$ for all $\mathtt{x} = (x_1, x_2) \in \mathbb{R}^2$ and A_G: $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $A_G((x_1, x_2)) = (x_1 - 5, x_2 - 5)$. Let $M_A : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by $\{(2x_1 - 1, 2x_2 - 1)\}$ and $M_B : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by $\{(\frac{x_1}{2} + 2, \frac{x_2}{2} + 2)\}$. Show that $(2, 2) \notin F(G)$.
Solution Since A_G is 1-inverse strongly monotone mapping, then we have $\Omega = (2, 2)$. Choose $\lambda_A = 2 \notin (0, 2\alpha)$, $\lambda_B = 4 \notin (0, 2\alpha)$ and $a = 0.25$, we obtain

$$G(x) = J_{M_A,2}(I - 2A_G)(0.25x + 0.25J_{M_B,4}(I - 4A_G)x).$$

By Lemma 2.25, we have $(2, 2) \notin F(G)$.

Lemma 2.28. [27] Let $\{\Gamma_n\}$ be a sequence of real numbers that not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\tau(n)\}_{n \geq n_0}$ defined by

$$\tau(n) = max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n \geq n_0}$ is nondecreasing sequence verifying $\lim_{n\to\infty} \tau(n) = \infty$ and , for all $n \geq n_0$,

$$max\left\{\Gamma_{\tau(n)}, \Gamma_n\right\} \leq \Gamma_{\tau(n)+1}.$$

Lemma 2.29. Let H be a real Hilbert space, for every $i = 1, 2, ..., N$, let $A_i : H \to H$ be α_i - inverse strongly monotone mappings with $\eta = min\{\alpha_i\}$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequence generated by

$$\begin{aligned} y_n &= P_C(I - \lambda \sum_{i=1}^{N} a_i A_i) x_n, \\ Q_n &= \left\{ z \in H : \left\langle (I - \lambda \sum_{i=1}^{N} a_i A_i) x_n - y_n, y_n - z \right\rangle \geq 0 \right\} \end{aligned}$$

and $x^* \in \bigcap_{i=1}^{N} VI(C, A_i)$ for all $i = 1, 2, ..., N$. Then the following inequality is fulfilled.

$$\begin{aligned} ||P_{Q_n}(x_n &- \lambda \sum_{i=1}^{N} a_i A_i y_n) - x^*||^2 \\ &\leq ||x_n - x^*||^2 - (1 - \frac{\lambda}{\eta})||P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n) - y_n||^2 \\ &\quad -(1 - \frac{\lambda}{\eta})||x_n - y_n||^2 \end{aligned}$$

where $\sum_{i=1}^{N} a_i = 1, 0 < a_i < 1$ and $\lambda \in (0, \eta)$ with $\eta = min_{i=1,2,...,N}\{\alpha_i\}$ for every $i = 1, 2, ..., N$.

Proof. Since $x^* \in \bigcap_{i=1}^{N} VI(C, A_i)$, we have $x^* \in VI(C, A_i)$ for every $i = 1, 2, ..., N$ and (2.2), we obtain

$$\begin{aligned} ||P_{Q_n}(x_n &- \lambda \sum_{i=1}^{N} a_i A_i y_n) - x^*||^2 \leq ||x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n - x^*||^2 \\ &\quad -||P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n) - (x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n)||^2 \\ &= ||x_n - x^*||^2 \\ &\quad -2\lambda \left\langle P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n) - x^*, \sum_{i=1}^{N} a_i A_i y_n \right\rangle \\ &\quad -||P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n) - x_n||^2. \end{aligned} \tag{2.10}$$

From monotonicity of $\sum_{i=1}^{N} a_iA_i$, we have

$$\begin{aligned} 0 &\leq \left\langle \sum_{i=1}^{N} a_iA_iy_n - \sum_{i=1}^{N} a_iA_ix^*, y_n - x^* \right\rangle \\ &= \left\langle \sum_{i=1}^{N} a_iA_iy_n, y_n - x^* \right\rangle - \left\langle \sum_{i=1}^{N} a_iA_ix^*, y_n - x^* \right\rangle \\ &\leq \left\langle \sum_{i=1}^{N} a_iA_iy_n, y_n - x^* \right\rangle \\ &= \left\langle \sum_{i=1}^{N} a_iA_iy_n, y_n - P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n) \right\rangle \\ &\quad + \left\langle \sum_{i=1}^{N} a_iA_iy_n, P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n) - x^* \right\rangle. \end{aligned}$$

It implies that

$$\begin{aligned} \langle x^* - P_{Q_n}(x_n &- \lambda \sum_{i=1}^{N} a_iA_iy_n), \sum_{i=1}^{N} a_iA_iy_n \rangle \\ &\leq \left\langle \sum_{i=1}^{N} a_iA_iy_n, y_n - P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n) \right\rangle. \end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we have

$$\begin{aligned} ||P_{Q_n}(x_n &- \lambda \sum_{i=1}^{N} a_iA_iy_n) - x^*||^2 \\ &\leq ||x_n - x^*||^2 + 2\lambda \left\langle \sum_{i=1}^{N} a_iA_iy_n, y_n - P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n) \right\rangle \\ &\quad -||P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n) - x_n||^2 \\ &= ||x_n - x^*||^2 - ||P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n) - y_n||^2 - ||y_n - x_n||^2 \\ &\quad -2 \left\langle P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n) - y_n, y_n - x_n \right\rangle \\ &\quad +2\lambda \left\langle \sum_{i=1}^{N} a_iA_iy_n, y_n - P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n) \right\rangle \\ &= ||x_n - x^*||^2 - ||P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n) - y_n||^2 - ||y_n - x_n||^2 \\ &\quad +2 \left\langle x_n - y_n - \lambda \sum_{i=1}^{N} a_iA_iy_n, P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n) - y_n \right\rangle \\ &= ||x_n - x^*||^2 - ||P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n) - y_n||^2 - ||y_n - x_n||^2 \end{aligned}$$

$$
\begin{aligned}
& +2\left\langle (I-\lambda\sum_{i=1}^{N}a_iA_i)x_n-y_n, P_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)-y_n\right\rangle \\
& +2\left\langle \lambda\sum_{i=1}^{N}a_iA_ix_n-\lambda\sum_{i=1}^{N}a_iA_iy_n, P_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)-y_n\right\rangle \\
\leq\; & ||x_n-x^*||^2-||P_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)-y_n||^2-||y_n-x_n||^2 \\
& +2\lambda||\sum_{i=1}^{N}a_iA_ix_n-\sum_{i=1}^{N}a_iA_iy_n||||P_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)-y_n|| \\
\leq\; & ||x_n-x^*||^2-||P_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)-y_n||^2-||y_n-x_n||^2 \\
& +2\frac{\lambda}{\eta}||x_n-y_n||||P_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)-y_n|| \\
=\; & ||x_n-x^*||^2-||P_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)-y_n||^2-||y_n-x_n||^2 \\
& +\frac{\lambda}{\eta}(||x_n-y_n||^2+||P_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)-y_n||^2) \\
=\; & ||x_n-x^*||^2-(1-\frac{\lambda}{\eta})||P_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)-y_n||^2 \\
& -(1-\frac{\lambda}{\eta})||y_n-x_n||^2. \qquad (2.12)
\end{aligned}
$$

□

2.5 G-Variational Inequalities Problem endowed with graph

This section present lemmas as an essential tool for solving the solution of G-variational inequality problem in Hilbert space endowed with graph.

Definition 2.9. [46] A subset X of $Ver(G)$ is called a *dominating set* if for every v belong to $Ver(G) - X$ there exists a point x belong to X such that (x, v) belong to $Eed(G)$ and we said that x *dominates* v or v *is dominated by* x. A subset Z of $Ver(G)$ is *dominated by* $v \in Ver(G)$ if $(v, z) \in Eed(G), \forall z \in Z$ and we said that X *dominates* v if $(x, v) \in Eed(G), \forall x \in X$.

Definition 2.10. [46] A graph G is called *transitive* if for every $x, y \in Ver(G)$ with$(x, y), (y, z) \in Eed(G)$, then $(x, z) \in Eed(G)$.

Property G [43] Vertices set $Ver(G) = C$ is said to have *Property G* if every sequence $\{a_n\}$ in C converging weakly to $x \in C$, there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $(a_{n_k}, x) \in Eed(G)\ \forall\ k \in \mathbb{N}$.

Definition 2.11. [43] Let $Ver(G) = C$. The mapping $B : C \to H$ is called *G-α-inverse* *strongly monotone* (G-α-ism) if there is $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geqslant \alpha \|Bx - By\|^2$$

$\forall x, y \in C$ with $(x, y) \in Eed(G)$.

The difference between G-α-ism and α-inverse strongly monotone is found in the reference [43].

Lemma 2.30. [43] Let $Eed(G)$ be a convex and $Ver(G) = C$. Let $G = (Ver(G), Eed(G))$ be a direct graph and G be transitive with $Eed(G) = Eed(G^{-1})$. Let $B : C \to H$ is G-α-ism operator with $B^{-1}(0) \neq \emptyset$. Then $G - Var(C, B) = B^{-1}(0) = F(P_C(I - \lambda B))$, for all $\lambda > 0$.

Lemma 2.31. [43] Let $Eed(G)$ be a convex and $Ver(G) = C$. Let $G = (Ver(G), Eed(G))$ be a direct graph and let $B : C \to H$ is G-α-ism operator. For every $\forall\ \lambda \in (0, 2\alpha)$, if $F(P_C(I - \lambda B)) \times F(P_C(I - \lambda B)) \subseteq Eed(G)$, then $F(P_C(I - \lambda B))$ is closed and convex.

2.6 Demand, Supply and Equilibrium price

In economics, supply and demand refers to the relationship between the quantity of a good that producers want to sell at different prices and the quantity that buyers want to purchase. It is the primary price determination model employed in economic theory. A market's relationship between supply and demand affects a commodity's price. The price that results from this is known as the equilibrium price, and it shows an agreement between the good's producers and customers. In an equilibrium, the amount of a good that producers supply and the amount that consumers desire are equal.

2.6.1 Demand curve

The quantity of a product demanded depends on its price and other factors, such as consumer preferences, incomes, and prices of related goods. When analyzing the relationship between price and quantity, economists often hold all factors constant except for the commodity's price. This allows them to create a demand curve, where price is shown on the vertical axis and quantity on the horizontal axis (See Figure 2.1). The demand curve is typically downward-sloping, indicating that consumers are willing to purchase more of the commodity at lower prices. Changes in non-price factors lead to shifts in the demand curve, while changes in price are represented by movements along the fixed demand curve.

2.6.2 Supply curve

The quantity supplied of a commodity in the market is determined not only by the price of the commodity but also by other factors such as the prices of substitute

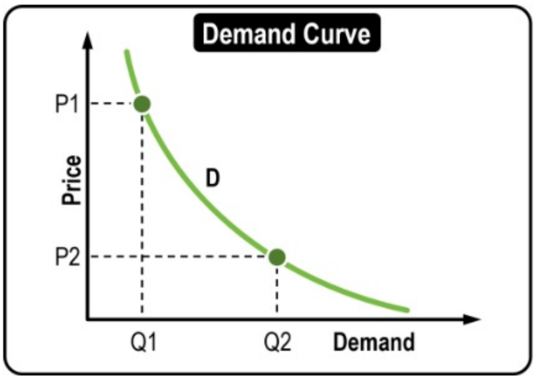

Figure 2.1: The demand curve. [57]

products, production technology, availability and cost of resources, and labor. In basic economic analysis, the focus is on examining the relationship between different prices and the quantity potentially offered by producers, while holding all other factors constant.

These price-quantity combinations can be graphically represented on a supply curve, with price depicted on the vertical axis and quantity on the horizontal axis (See Figure 2.2). The supply curve typically has an upward slope, indicating that producers are willing to supply more of the commodity at higher prices. Changes in non-price factors result in shifts in the supply curve, while changes in the price of the commodity are shown by movements along the fixed supply curve.

Essentially, analyzing supply involves considering the influence of various factors on the quantity producers are willing to supply at different prices, represented by the supply curve. Non-price factors cause shifts in the curve, while changes in the price of the commodity lead to movements along the curve.

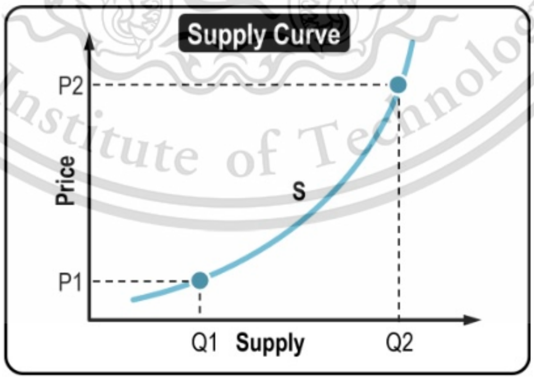

Figure 2.2: The supply curve.[57]

2.6.3 Equilibrium Price

The equilibrium price is the price at which the quantity demanded by consumers equals the quantity supplied by producers in a market. It is the point where the forces of demand and supply are balanced, and there is no inherent tendency for the price to change. At the equilibrium price, the market clears, meaning that there are no shortages or surpluses of the product or service.

The equilibrium price is determined by the intersection of the demand and supply curves (See Figure 2.3). When the market price is below the equilibrium price, it creates excess demand or a shortage, leading to upward pressure on prices. On the other hand, when the market price is above the equilibrium price, it creates excess supply or a surplus, leading to downward pressure on prices. The market adjusts through price changes until the equilibrium price is reached, where the quantity demanded equals the quantity supplied.

Understanding demand, supply, and equilibrium price is essential for businesses, investors, policymakers, and economists to analyze market dynamics, make pricing decisions, allocate resources efficiently, and assess the overall health of an economy.

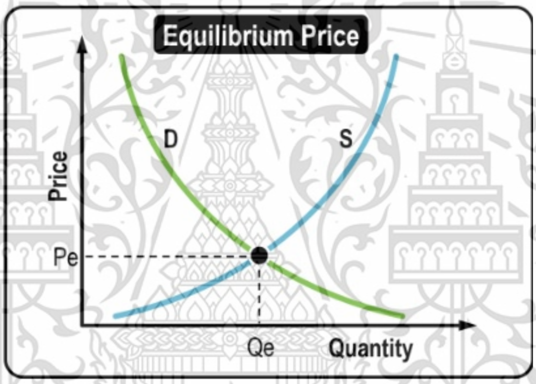

Figure 2.3: The relationship of price to supply and demand curve. [57]

2.7 The Correlation Coefficient

The correlation coefficient[56] is a statistical concept which helps in establishing a relation between predicted and actual values obtained in a statistical experiment. The calculated value of the correlation coefficient explains the exactness between the predicted and actual values.

The correlation coefficient defines the degree of relation between two variables and is denoted by r. It is also called a cross-correlation coefficient, as it predicts the relation between two quantities. Now, let us proceed to a statistical way of calculating the correlation coefficient.

Given n data pairs $(x_1, y_1), ..., (x_n, y_n)$, then the correlation coefficient can be calculated using the formula

$$r = \frac{n(\sum_{i=0}^{n} x_i y_i) - \sum_{i=0}^{n} x_i \sum_{i=0}^{n} y_i}{[n \sum_{i=0}^{n} x_i^2 - (\sum_{i=0}^{n} x_i)^2][n \sum_{i=0}^{n} y_i^2 - (\sum_{i=0}^{n} y_i)^2]}. \tag{2.13}$$

The correlation coefficient, r, can take on values between -1 and 1. The further away r is from zero, the stronger the linear relationship between the two variables. The sign of r corresponds to the direction of the relationship. If r is positive, then as one variable increases, the other tends to increase. If r is negative, then as one variable increases, the other tends to decrease. A perfect linear relationship ($r = -1$ or $r = 1$) means that one of the variables can be perfectly explained by a linear function of the other.

2.8 Least Square Method

The least squares method [58] is a form of mathematical regression analysis used to determine the line of best fit for a set of data, providing a visual demonstration of the relationship between the data points. Each point of data represents the relationship between a known independent variable and an unknown dependent variable.

2.8.1 Linear Regression

A line of best fit is a straight line that is the best approximation of the given set of data. A more accurate way of finding the line of best fit is the least square method. Given n data pairs $(x_1, y_1), ..., (x_n, y_n)$, the best fit for the straight-line regression model as follows:

$$y = a_0 + a_1 x + e_i, \tag{2.14}$$

where

$$a_1 = \frac{n\sum_{i=0}^{n}(x_i y_i) - \sum_{i=0}^{n} x_i \sum_{i=0}^{n} y_i}{n\sum_{i=0}^{n} x_i^2 - (\sum_{i=0}^{n} x_i)^2}$$

,

$$a_0 = \frac{\sum_{i=0}^{n} y_i}{n} - a_1 \frac{\sum_{i=0}^{n} x_i}{n}.$$

and

$$e_i = y_i - (a_0 + a_1 x_i).$$

2.8.2 Polynomial Regression

Another alternative is to fit polynomials to the data using polynomial regression. The least-squares procedure can be readily extended to fit the data to a higher-order polynomial. For example, suppose that we fit a second-order polynomial or quadratic:

$$y = a_0 + a_1 x + a_2 x^2. \tag{2.15}$$

Normal equations to find a least-squares parabola are

$$\begin{bmatrix} n & \sum_{i=0}^{n} x_i & \sum_{i=0}^{n} x_i^2 \\ \sum_{i=0}^{n} x_i & \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i^3 \\ \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i^3 & \sum_{i=0}^{n} x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n} y_i \\ \sum_{i=0}^{n} x_i y_i \\ \sum_{i=0}^{n} x_i^2 y_i \end{bmatrix}$$

Chapter 3

Convergences theorem

3.1 The Modified subgradient extragradient method for system of variational inclusion problem and variational inequalities problem

In this section, we prove a strong convergence of the sequence acquired from the proposed iterative methods for finding a common element of the set of finite family variational inequalities problem and the set of solution of the proposed problem.

Theorem 3.1. Let H be a real Hilbert space. For $i = 1, 2, ..., N$, let $A_i : H \to H$ be α_i - inverse strongly monotone mappings and let $A_G : H \to H$ be α_G - inverse strongly monotone mappings. Let $M_A, M_B : H \to 2^H$ be multivalue maximum monotone mapping. Define the mapping $G : H \to H$ by $G(x) = J_{M_A,\lambda_A}(I - \lambda_A A_G)(bx + (1 - b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x)$ for all $x \in H$, $b \in (0, 1)$ and $\lambda_A, \lambda_B \in (0, 2\alpha_G)$. Assume that $\Gamma = \bigcap_{i=1}^{N} VI(C, A_i) \cap F(G) \neq \emptyset$. Let the sequence $\{y_n\}$ and $\{x_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} y_n = P_C(I - \lambda \sum_{i=1}^{N} a_i A_i)x_n \\ Q_n = \left\{ z \in H : \left\langle (I - \lambda \sum_{i=1}^{N} a_i A_i)x_n - y_n, y_n - z \right\rangle \geq 0 \right\} \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n) + \gamma_n G x_n \end{cases} \tag{3.1}$$

where $\sum_{i=1}^{N} a_i = 1, 0 < a_i < 1, \{\alpha_n\},\ \{\beta_n\},\ \{\gamma_n\} \subset [0,\ 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\lambda \in (0, \eta)$ with $\eta = min_{i=1,2,...,N}\ \{\alpha_i\}$.

Suppose the following conditions hold:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

(ii) $0 < c < \beta_n, \gamma_n \leq d < 1$

Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$ where $x^* = P_\Gamma u$.

Proof. We must to show that $\{x_n\}$ is bounded. Let $z_n = P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n)$.

We consider

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n z_n + \gamma_n G x_n \\ &= \alpha_n u + (1 - \alpha_n)(\frac{\beta_n z_n + \gamma_n G x_n}{1 - \alpha_n}) \\ &= \alpha_n u + (1 - \alpha_n)t_n, \end{aligned}$$

where $t_n = \frac{\beta_n z_n + \gamma_n G x_n}{1-\alpha_n}$. Let $x^* \in \Gamma = \bigcap_{i=1}^N VI(C, A_i) \cap F(G)$, we have

$$\begin{aligned} ||t_n - x^*||^2 &= ||\frac{\beta_n z_n + \gamma_n G x_n}{1-\alpha_n} - x^*||^2 \\ &= ||\frac{\beta_n z_n + \gamma_n G x_n - (1-\alpha_n)x^*}{1-\alpha_n}||^2 \\ &= \frac{\beta_n}{1-\alpha_n}||z_n - x^*||^2 + \frac{\gamma_n}{1-\alpha_n}||Gx_n - x^*||^2 \\ & \quad -\frac{\beta_n\gamma_n}{(1-\alpha_n)^2}||z_n - Gx_n||^2 \end{aligned} \tag{3.2}$$

From definition of x_{n+1} and (3.2), we consider

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||\alpha_n u + (1-\alpha_n)t_n - x^*||^2 \\ &= ||\alpha_n(u - x^*) - (1-\alpha_n)(t_n - x^*)||^2 \\ &= \alpha_n||u - x^*||^2 + (1-\alpha_n)||t_n - x^*||^2 - \alpha_n(1-\alpha_n)||u - t_n||^2 \\ &= \alpha_n||u - x^*||^2 + (1-\alpha_n)[\frac{\beta_n}{1-\alpha_n}||z_n - x^*||^2 \\ & \quad +\frac{\gamma_n}{1-\alpha_n}||Gx_n - x^*||^2 - \frac{\beta_n\gamma_n}{(1-\alpha_n)^2}||z_n - Gx_n||^2] \\ & \quad -\alpha_n(1-\alpha_n)||u - t_n||^2 \\ &= \alpha_n||u - x^*||^2 + \beta_n||z_n - x^*||^2 + \gamma_n||Gx_n - x^*||^2 \\ & \quad -\frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2 - \alpha_n(1-\alpha_n)||u - t_n||^2 \\ &\leq \alpha_n||u - x^*||^2 + \beta_n||z_n - x^*||^2 + \gamma_n||x_n - x^*||^2 \\ & \quad -\frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2 \\ & \quad -\alpha_n(1-\alpha_n)||u - t_n||^2. \end{aligned} \tag{3.3}$$

By Lemma 2.29 and $\lambda \in (0,1)$, we have

$$||z_n - x^*||^2 \leq ||x_n - x^*||^2 \tag{3.4}$$

From (3.3) and (3.4), we get

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq \alpha_n||u - x^*||^2 + \beta_n||z_n - x^*||^2 + \gamma_n||x_n - x^*||^2 \\ & \quad -\frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2 - \alpha_n(1-\alpha_n)||u - t_n||^2 \\ &\leq \alpha_n||u - x^*||^2 + \beta_n||x_n - x^*||^2 + \gamma_n||x_n - x^*||^2 \\ & \quad -\frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2 - \alpha_n(1-\alpha_n)||u - t_n||^2 \\ &= \alpha_n||u - x^*||^2 + (1-\beta_n)||x_n - x^*||^2 \\ & \quad -\frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2 - \alpha_n(1-\alpha_n)||u - t_n||^2 \\ &\leq \alpha_n||u - x^*||^2 + (1-\beta_n)||x_n - x^*||^2 \\ & \quad \vdots \\ &\leq \max\{||u - x^*||^2, ||x_1 - x^*||^2\}. \end{aligned} \tag{3.5}$$

By induction,

$$||x_{n+1} - x^*||^2 \leq \max\left\{||u - x^*||^2, ||x_1 - x^*||^2\right\},$$

then $\{x_n\}$ is a bounded sequence.
Since

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq \alpha_n||u - x^*||^2 + \beta_n||z_n - x^*||^2 + \gamma_n||x_n - x^*||^2 \\
&\quad -\frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2 \\
&\leq \alpha_n||u - x^*||^2 + \beta_n[||x_n - x^*||^2 - (1 - \frac{\lambda}{\eta})||z_n - y_n||^2 \\
&\quad -(1 - \frac{\lambda}{\eta})||x_n - y_n||^2] + \gamma_n||x_n - x^*||^2 - \frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2 \\
&= \alpha_n||u - x^*||^2 + (1-\alpha_n)||x_n - x^*||^2 - \beta_n(1 - \frac{\lambda}{\eta})||z_n - y_n||^2 \\
&\quad -\beta_n(1 - \frac{\lambda}{\eta})||x_n - y_n||^2 - \frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2 \\
&\leq \alpha_n||u - x^*||^2 + ||x_n - x^*||^2 - \beta_n(1 - \frac{\lambda}{\eta})||z_n - y_n||^2 \\
&\quad -\beta_n(1 - \frac{\lambda}{\eta})||x_n - y_n||^2 - \frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2.
\end{aligned}$$

It implies that

$$\begin{aligned}
\beta_n(1 - \frac{\lambda}{\eta})||z_n &- y_n||^2 + \beta_n(1 - \frac{\lambda}{\eta})||x_n - y_n||^2 + \frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2 \\
&\leq \alpha_n||u - x^*||^2 + ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2. \qquad (3.6)
\end{aligned}$$

Let $S_n := \beta_n(1 - \frac{\lambda}{\eta})||z_n - y_n||^2 + \beta_n(1 - \frac{\lambda}{\eta})||x_n - y_n||^2 + \frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2$.
Then, we have

$$S_n \leq \alpha_n||u - x^*||^2 + ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2. \qquad (3.7)$$

Now, we consider two possible cases:
Case 1. Put $\Gamma_n := ||x_n - x^*||^2$ for all $n \in \mathbb{N}$.
Assume that there is $n_0 \geq 0$ such that for each $n \geq n_0$, $\Gamma_{n+1} \leq \Gamma_n$.
In this case, $\lim_{n\to\infty} \Gamma_n$ exists and $\lim_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) = 0$.
Since $\lim_{n\to\infty} \alpha_n = 0$, it follows from (3.7) that $\lim_{n\to\infty} S_n = 0$.
Therefore , we have $\lim_{n\to\infty} \beta_n(1 - \frac{\lambda}{\eta})||z_n - y_n||^2 = 0$,
$\lim_{n\to\infty} \beta_n(1 - \frac{\lambda}{\eta})||x_n - y_n||^2 = 0$ and $\lim_{n\to\infty} \frac{\beta_n\gamma_n}{1-\alpha_n}||z_n - Gx_n||^2 = 0$.
From the assumption i), ii), we obtain

$$\lim_{n\to\infty} ||z_n - y_n|| = \lim_{n\to\infty} ||x_n - y_n|| = \lim_{n\to\infty} ||z_n - Gx_n|| = 0. \qquad (3.8)$$

Hence, we obtain

$$||x_n - Gx_n|| \leq ||x_n - y_n|| + ||y_n - z_n|| + ||z_n - Gx_n||.$$

From (3.8), we have

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{3.9}$$

We now show that $\limsup_{n\to\infty} \langle u - x^*, x_n - x^*\rangle \leq 0$.
We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^*\rangle = \lim_{i\to\infty} \langle u - x^*, x_{n_i} - x^*\rangle. \tag{3.10}$$

Because $\{x_n\}$ is a bounded sequence in H, there exists a subsequence of $\{x_n\}$ converges weakly to element in H. Without loss of geneality, we can assume that $x_{n_i} \rightharpoonup w$ where $w \in H$. Since $\lim_{n\to\infty} ||x_n - z_n|| = 0$, we have $z_{n_i} \rightharpoonup w$.
Since $\lim_{n\to\infty} ||x_n - y_n|| = 0$, therefore $y_{n_i} \rightharpoonup w$.
Assume that $w \notin \bigcap_{i=1}^N VI(C, A_i)$. So, we have $w \notin F(P_C(I - \lambda\sum_{i=1}^N a_iA_i))$.
Then , we have $w \neq P_C(I - \lambda\sum_{i=1}^N a_iA_i)w$. By nonexpansiveness of $P_C(I - \lambda\sum_{i=1}^N a_iA_i)$, (3.8) and Opial's property, we have

$$\begin{aligned}
\liminf_{i\to\infty} ||x_{n_i} \; - \; & w|| \\
< \; & \liminf_{i\to\infty} ||x_{n_i} - P_C(I - \lambda\sum_{i=1}^N a_iA_i)w|| \\
\leq \; & \liminf_{i\to\infty}(||x_{n_i} - y_{n_i}|| + ||y_{n_i} - P_C(I - \lambda\sum_{i=1}^N a_iA_i)w||) \\
\leq \; & \liminf_{i\to\infty}(||x_{n_i} - y_{n_i}|| \\
& + ||P_C(I - \lambda\sum_{i=1}^N a_iA_i)x_{n_i} - P_C(I - \lambda\sum_{i=1}^N a_iA_i)w||) \\
\leq \; & \liminf_{i\to\infty} ||x_{n_i} - w||.
\end{aligned}$$

This is a contradiction, this is, $w \in VI(C, \sum_{i=1}^N a_iA_i)$. From Remark 2.23, we have

$$w \in \bigcap_{i=1}^N VI(C, A_i). \tag{3.11}$$

Assume that $w \notin F(G)$. Then , we have $w \neq Gw$. From (3.9) and Opial's property, we have

$$\begin{aligned}
\liminf_{i\to\infty} ||x_{n_i} - w|| &< \liminf_{n\to\infty} ||x_{n_i} - Gw|| \\
&\leq \liminf_{i\to\infty}(||x_{n_i} - Gx_{n_i}|| + ||Gx_{n_i} - Gw||) \\
&\leq \liminf_{i\to\infty}(||x_{n_i} - Gx_{n_i}|| + ||x_{n_i} - w||) \\
&\leq \liminf_{i\to\infty} ||x_{n_i} - w||.
\end{aligned}$$

This is a contradiction, this is,

$$w \in F(G). \tag{3.12}$$

From (3.11) and (3.12) , we have $w \in \bigcap_{i=1}^{N} VI(C, A_i) \cap F(G)$.
Therefore , we get

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle = \lim_{i\to\infty} \langle u - x^*, x_{n_i} - x^* \rangle = \langle u - x^*, w - x^* \rangle \leq 0, \tag{3.13}$$

where $x^* = P_\Gamma u$. Next, We show that $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Gamma u$. From the definition of x_n and $x^* = P_\Gamma u$, we have

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||\alpha_n(u - x^*) - (1-\alpha_n)(t_n - x^*)||^2 \\ &\leq (1-\alpha_n)||t_n - x^*||^2 + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\ &\leq (1-\alpha_n)||x_n - x^*||^2 + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \end{aligned} \tag{3.14}$$

By applying Lemma 2.18 to (3.14), we have the sequence $\{x_n\}$ converges strongly to x^*.

Case 2. Assume that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} \leq \Gamma_{n_i+1}$ for all $i \in \mathcal{N}$. In this case, we can define $\tau : \mathcal{N} \to \mathcal{N}$ by $\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}$. Then we have $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. So, we have from (3.6) that

$$\begin{aligned} \beta_{\tau(n)}(1-\frac{\lambda}{\eta})||z_{\tau(n)} &- y_{\tau(n)}||^2 + \beta_{\tau(n)}(1-\frac{\lambda}{\eta})||x_{\tau(n)} - y_{\tau(n)}||^2 \\ &+\frac{\beta_{\tau(n)}\gamma_{\tau(n)}}{1-\alpha_{\tau(n)}}||z_{\tau(n)} - Gx_{\tau(n)}||^2 \\ &\leq \alpha_{\tau(n)}||u - x^*||^2 + ||x_{\tau(n)} - x^*||^2 - ||x_{\tau(n)+1} - x^*||^2. \end{aligned}$$

Arguing as in case 1, we have

$$\lim_{n\to\infty} ||z_{\tau(n)} - y_{\tau(n)}|| = \lim_{n\to\infty} ||x_{\tau(n)} - y_{\tau(n)}|| = \lim_{n\to\infty} ||z_{\tau(n)} - Gx_{\tau(n)}|| = 0. \tag{3.15}$$

Because $\{x_{\tau(n)}\}$ is a bounded sequence, there exists a subsequence $\{x_{\tau(n_j)}\}$ such that

$$\limsup_{n\to\infty} \langle u - x^*, x_{\tau(n)} - x^* \rangle = \lim_{j\to\infty} \langle u - x^*, x_{\tau(n_j)} - x^* \rangle .$$

Following the same argument as the proof of Case 1 for $\{x_{\tau(n_j)}\}$, we have that

$$\limsup_{n\to\infty} \langle u - x^*, x_{\tau(n)} - x^* \rangle \leq 0,$$

and

$$||x_{\tau(n)+1} - x^*||^2 \leq (1-\alpha_{\tau(n)})||x_{\tau(n)} - x^*||^2 + 2\alpha_{\tau(n)} \langle u - x^*, x_{\tau(n)+1} - x^* \rangle$$

where $\alpha_{\tau(n)} \to 0$, $\sum_{n=1}^{\infty} \alpha_{\tau(n)} = \infty$ and $\limsup_{n\to\infty} \langle u - x^*, x_{\tau(n)+1} - x^* \rangle \leq 0$.
Hence , by lemma 2.18, we have that
$\lim_{n\to\infty} ||x_{\tau(n)} - x^*|| = 0$ and $\lim_{n\to\infty} ||x_{\tau(n)+1} - x^*|| = 0$
Therefore, by Lemma 2.28, we have

$$0 \leq ||x_n - x^*|| \leq \max\{||x_{\tau(n)} - x^*||, ||x_n - x^*||\} \leq ||x_{\tau(n)+1} - x^*||.$$

Hence, $\{x_n\}$ converge strongly to $x^* = P_\Gamma u$.
This completes the proof of main theorem. □

3.2 The G-subgradient extragradient method for G-variational inequalities problem endowed with graph

In this section, we prove weak and strong convergence of the sequence acquired from the proposed iterative methods for solving the G-variational inequalities problem in Hilbert space endowed with a direct graph.

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G, Ver(G), Eed(G), B$ as in Lemma 2.30. Assume that $G - Var(C,B) \neq \emptyset$ with $G - Var(C,B) \times G - Var(C,B) \subseteq Eed(G)$. Let $\{v_n\}$ be a sequence defined by $v_0 \in C$ and

$$\begin{cases} w_n = P_C(I - \lambda B)v_n \\ T_G^n = \{w \in C : \langle (I-\lambda B)v_n - w_n, w_n - w\rangle \geqslant 0\} \\ v_{n+1} = P_{T_G^n}(v_n - \lambda B w_n), \end{cases}$$

for all $n \in \mathbb{N}$ where $\lambda \in (0,\alpha)$ and T_G^n is G- Half space. Then sequence $\{v_n\}$ converges weakly to an element $\bar{x} \in G - Var(C,B)$ and the sequence $\{P_{G-Var(C,B)}v_n\}$ converges strongly to $\bar{x}$, where $G - Var(C,B)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0.

Proof. Let $v^* \in G - Var(C,B)$. Since $G - Var(C,B)$ dominates by v_n, we have $(v_{n,}, v^*) \in Eed(G)$ for all $n \in \mathbb{N}$. From Lemma 2.30, we have $v^* = P_C(I - \lambda B)v^*$.
Utilizing Definition 2.11, we have

$$\begin{aligned} \|w_n - v^*\|^2 &\leq \|v_n - v^*\|^2 - 2\lambda \langle Bv_n - Bv^*, v_n - \nu^*\rangle + \lambda^2 \|Bv_n - Bv^*\|^2 \\ &\leq \|v_n - \nu^*\|^2 - 2\lambda\alpha \|Bv_n - Bv^*\|^2 + \lambda^2 \|Bv_n - Bv^*\|^2 \\ &= \|v_n - v^*\|^2 - \lambda(2\alpha - \lambda)\|Bv_n - Bv^*\|^2 \\ &\leq \|v_n - v^*\|^2. \end{aligned}$$

Due to $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0, we have $(v_n, v_0), (v_0, w_n) \in Eed(G)$.
Exploiting of G is transitive, we get $(v_n, w_n) \in Eed(G)$.
From the assumption that $Eed(G) = Eed(G^{-1})$, we deduce that $(w_n, v_n) \in Eed(G)$.
From $(w_n, v_n), (v_{n,}, v^*) \in Eed(G)$ and the assumption that G is transitive, we get $(w_n, v^*) \in Eed(G)$.

From iteration of sequence $\{v_n\}$, we have

$$\begin{aligned}
\|v_{n+1}-v^*\|^2 &\leqslant \|v_n-\lambda Bw_n-v^*\|^2 - \left\|v_n-\lambda Bw_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 \\
&= \|v_n-v^*\|^2 - 2\lambda\langle Bw_n, v_n-v^*\rangle + \|\lambda Bw_n\|^2 \\
&\quad - \left\|v_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 + 2\lambda\left\langle Bw_n, v_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle \\
&\quad - \|\lambda Bw_n\|^2 \\
&= \|v_n-v^*\|^2 - 2\lambda\langle Bw_n, P_{T_G}(v_n-\lambda Bw_n)-v^*\rangle \\
&\quad - \left\|v_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2
\end{aligned} \tag{3.16}$$

From $(w_n, v^*) \in Eed(G)$ and monotonicity of B, we have

$$\begin{aligned}
0 &\leqslant \langle Bw_n-Bv^*, w_n-v^*\rangle \\
&= \langle Bw_n, w_n-v^*\rangle - \langle Bv^*, w_n-v^*\rangle \\
&\leqslant \langle Bw_n, w_n-v^*\rangle \\
&= \left\langle Bw_n, w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle + \left\langle Bw_n, P_{T_G^n}(v_n-\lambda Bw_n)-v^*\right\rangle.
\end{aligned}$$

It implies that

$$-2\lambda\left\langle Bw_n, P_{T_G^n}(v_n-\lambda Bw_n)-v^*\right\rangle \leqslant 2\lambda\left\langle Bw_n, w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle \tag{3.17}$$

From (3.16) and (3.17), we have

$$\begin{aligned}
\|v_{n+1}-v^*\|^2 &\leq \|v_n-v^*\|^2 - 2\lambda\left\langle Bw_n, P_{T_G^n}(v_n-\lambda Bw_n)-v^*\right\rangle \\
&\quad - \left\|v_n-P_{T_G^n}(v_n-\lambda B_{v_n})\right\|^2 \\
&\leq \|v_n-v^*\|^2 + 2\lambda\left\langle Bw_n, w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle \\
&\quad - \left\|v_n-P_{T_G^n}(v_n-\lambda B_{v_n})\right\|^2 \\
&= \|v_n-v^*\|^2 + 2\lambda\left\langle Bw_n, w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle \\
&\quad - \|v_n-w_n\|^2 - 2\left\langle v_n-w_n, w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle \\
&\quad - \left\|w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 \\
&= \|v_n-v^*\|^2 - \|v_n-w_n\|^2 - \left\|w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 \\
&\quad +2\left\langle \lambda Bw_n-v_n+w_n, w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle \\
&= \|v_n-v^*\|^2 - \|v_n-w_n\|^2 - \left\|w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 \\
&\quad +2\left\langle (I-\lambda B)v_n-w_n, P_{T_G^n}(v_n-\lambda Bw_n)-w_n\right\rangle \\
&\quad +2\lambda\left\langle Bv_n-Bw_n, P_{T_G^n}(v_n-\lambda Bw_n)-w_n\right\rangle \\
&\leq \|v_n-v^*\|^2 - \|v_n-w_n\|^2 - \left\|w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 \\
&\quad +2\lambda\left\langle Bv_n-Bw_n, P_{T_G^n}(v_n-\lambda Bw_n)-w_n\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \|v_n - v^*\|^2 - \|v_n - w_n\|^2 - \left\|w_n - P_{T_G^n}(v_n - \lambda B w_n)\right\|^2 \\
&\quad +2\frac{\lambda}{\alpha}\|Bv_n - Bw_n\| \cdot \|P_{T_G^n}(v_n - \lambda B w_n) - w_n\| \\
&\leq \|v_n - v^*\|^2 - \|v_n - w_n\|^2 - \left\|w_n - P_{T_G^n}(v_n - \lambda B w_n)\right\|^2 \\
&\quad +\frac{\lambda}{\alpha}\left(\|v_n - w_n\|^2 + \|P_{T_G^n}(v_n - \lambda B w_n) - w_n\|^2\right) \\
&= \|v_n - v^*\|^2 - \left(1 - \frac{\lambda}{\alpha}\right)\|v_n - w_n\|^2 \\
&\quad -\left(1 - \frac{\lambda}{\alpha}\right)\left\|P_{T_G^n}(v_n - \lambda B w_n) - w_n\right\|^2. \qquad (3.18)
\end{aligned}
$$

From Lemma 2.19, we have $\lim_{n\to\infty}\|v_n - v^*\|^2$ exists for all $v^* \in G - Var(C, B)$ and $\{v_n\}$ is a bounded sequence.
From (3.18) and $\lim_{n\to\infty}\|v_n - v^*\|^2$ exists, we have

$$\lim_{n\to 2}\|P_C(I - \lambda B)v_n - v_n\| = \lim_{n\to\infty}\|v_n - w_n\| = 0.$$

Since $\{v_n\}$ is a bounded sequence, there is a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ converses weakly to $\bar{x}$.
Since C have a property G, we have $(v_{n_k}, \bar{x}) \in Eed(G)$.
Assume that $P_C(I - \lambda B)\bar{x} \neq \bar{x}$. By opial property and using the same method as (3.16), we have

$$
\begin{aligned}
\limsup_{k\to\infty}\|v_{n_k} - \bar{x}\| &< \limsup_{k\to\infty}\|v_{n_k} - P_C(I - \lambda B)\bar{x}\| \\
&\leq \limsup_{k\to\infty}\left(\|v_{n_k} - P_C(I - \lambda B)v_{n_k}\| + \|P_C(I - \lambda B)v_{n_k} - P_C(I - \lambda B)\bar{x}\|\right) \\
&\leq \limsup_{k\to\infty}\|v_{n_k} - \bar{x}\|.
\end{aligned}
$$

Contradiction. So, we have $P_C(I - \lambda B)\bar{x} = \bar{x}$.
Let $y \in C$ with $(\bar{x}, y) \in C$, then

$$\langle (I - \lambda B)\tilde{x} - \tilde{x}, \bar{x} - y\rangle \geqslant 0.$$

It follows that

$$\langle y - \bar{x}, B\tilde{x}\rangle \geqslant 0,$$

for all $y \in C$ with $(\bar{x}, y) \in C$. Then, we have$\bar{x} \in G - Var(C, B)$.
Therefore $v_{n_k} \rightharpoonup \bar{x} \in G - Var(C, B)$ as $k \to \infty$.
Since$(v_{n_k}, \bar{x}) \in Eed(G)$ and using the same method as $\lim_{n\to\infty}\|v_n - v^*\|$ exists, we have $\lim_{k\to\infty}\|v_{n_k} - \bar{x}\|$ exists.
At the end of this theorem we demonstrate that $\{v_n\}$ converses weakly to $\bar{x}$. Assume

that $v_{n_n} \rightharpoonup \hat{x}$ as $k \to \infty$ and $\bar{x} \neq \hat{x}$. Thank to the Opial's condition, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|v_n - \bar{x}\| &= \limsup_{k\to\infty} \|v_{n_k} - \bar{x}\| \\
&< \limsup_{k\to\infty} \|v_{n_k} - \hat{x}\| \\
&< \limsup_{k\to\infty} \|v_{n_k} - \bar{x}\| \\
&= \lim_{n\to\infty} \|v_n - \bar{x}\|.
\end{aligned}$$

Contradiction. So, we get $\bar{x} = \hat{x}$. We can conclude that a sequence $\{v_n\}$ converges weakly to $\bar{x} \in G - Var(C, B)$.

Due to (3.18) and exploiting of Lemma 2.20, we have $\{P_{G-Var(C,B)}v_n\}$ converges strongly to $z \in G - Var(C, B)$.

From property of $P_{G-Var(C,B)}$, we have

$$\langle v_n - P_{G-Var(C,B)}v_n, P_{G-Var(C,B)}v_n - \bar{x}\rangle \geqslant 0.$$

Take $n \to \infty$, we have $\|z - \bar{x}\| = 0$. So, we have $z = \bar{x}$. Therefore we can conclude that$\{P_{G-Var(C,B)}v_n\}$ converges strongly to $\bar{x} \in G - Var(C, B)$.This is ultimately the prove.

□

Chapter 4

Examples and Numerical Results

4.1 Applications of the G-subgradient extragradient method for G-variational inequalities problem endowed with graph

To resolved a fixed point problem in Hilbert space endowed with a direct graph by using G-subgradient extragradient method, we required the following lemma;

Lemma 4.1. [43] Let C be a nonempty closed convex subset of a real Hilbert space H and let $G = (Ver(G), Eed(G))$ be a directed graph with $C = Ver(G)$ having property G. Let $Eed(G)$ be a convex set with $Eed(G) = Eed(G^{-1})$. Let $T : C \to C$ be G-nonexpansive mapping with $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq Eed(G)$. Then
i) $I - T$ is $G - \frac{1}{2}$- inverse strongly monotone,
ii) $G - Var(C, I - T) = F(T)$.

The following theorem is an immediate result of Theorem 3.2 and Lemma 4.1.

Theorem 4.2. Let C be a nonempty closed convex subset of a real Hilbert space H and let $G = (Ver(G), Eed(G))$ be a directed graph with $C = Ver(G)$ having property G. Let $Eed(G)$ be a convex set and G be transitive with $Eed(G) = Eed\left(G^{-1}\right)$ and let $T : C \to C$ be G-nonexpansive mapping with $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq Eed(G)$. Let $\{v_n\}$ be a sequence defined by $v_0 \in C$ and

$$\begin{cases} w_n = P_C(I - \lambda(I - T)v_n \\ T_G^n = \{w \in C : \langle (I - \lambda(I - T)v_n - w_n, w_n - v\rangle \geq 0\} \\ v_{n+1} = P_{T_G^n}\left(v_n - \lambda(I - T)w_n\right), \end{cases}$$

for all $n \in N$ where $\lambda \in (0, \alpha)$ and T_G^n is G- Half space. Then sequence $\{v_n\}$ converses weakly to an element $\bar{v} \in F(T)$ and the sequence $\{P_{F(T)}v_n\}$ converges strongly to $\bar{v}$, where $F(T)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0.

4.2 Example for The G-subgradient extragradient method for G-variational inequalities problem endowed with graph

Following that, we provide an example to support our main result.

Example 4.3. Let $C = [-1, 1]$ and $G = (C, Eed(G))$ be a directed graph, where $Eed(G) = \{(x, y) : x, y \in [0, 1]\}$. Let the mappings $B : C \to \mathbb{R}$ define by $Bx = x - \frac{x^3}{4} - \frac{15}{32}$, and $S : C \to \mathbb{R}$ define by $Sx = \frac{x^3}{4} + \frac{15}{32}$, for all $x \in C$.

Suppose that the sequence $\{v_n\}$ is generated by $v_0 = 1$ and

$$\begin{cases} w_n = P_C(I-\lambda B)v_n \\ T_G^n = \{w \in C : \langle (I-\lambda B)v_n - w_n, w_n - w\rangle \geq 0\} \\ v_{n+1} = P_{T_G^n}(v_n - \lambda B w_n), \end{cases} \tag{4.1}$$

for all $n \in N$ where $\lambda \in (0, \alpha)$ and T_G^n is G- Half space. Then sequence $\{v_n\}$ converses weakly to an element of $\bar{v} \in G - Var(C, B)$ and the sequence $\{P_{G-Var(C,B)}v_n\}$ converges strongly to $\bar{v}$, where $G - Var(C, B)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0.

Solution It is obvious that $\frac{1}{2} \in F(S)$, and $Eed(G) = Eed(G^{-1})$.

First, we show that S is a G-nonexpansive mapping.

Let $x, y \in C$ with $(x, y) \in Eed(G)$. Then, we have $x, y \in [0, 1]$.

Since x^3, y^3 , $\frac{5}{8} \in [0, 1]$ and $[0, 1]$ is a convex set, we have

$$Sx = \frac{1}{4}x^3 + \frac{3}{4}(\frac{5}{8}) \quad \in \quad [0,1]$$

and

$$Sy = \frac{1}{4}y^3 + \frac{3}{4}(\frac{5}{8}) \quad \in \quad [0,1].$$

From definition of S, we have

$$\begin{aligned} |Sx - Sy| &= |(\frac{x^3}{4} + \frac{15}{32}) - (\frac{y^3}{4} + \frac{15}{32})| = |\frac{x^3}{4} - \frac{y^3}{4}| \\ &= \frac{1}{4}|x^2 + xy + y^2||x-y| \leq \frac{1}{4}(3)|x-y| \\ &\leq |x-y|. \end{aligned}$$

Then $(Sx, Sy) \in Eed(G)$. Therefore S is a G-nonexpansive mapping.

Since $Bx = (I - S)x$, S is a G-nonexpansive mapping and Lemma 4.1, we have B is G-$\frac{1}{2}$-inverse strongly monotone. It is obvious that $G - Var(C, B) = \{\frac{1}{2}\}$.

Putting $\lambda = \frac{1}{4}$. From convexity of $[0, 1]$, we have

$$(I - \frac{1}{4}B)z \quad = \quad \frac{3}{4}z + \frac{1}{4}(\frac{z^3}{4} + \frac{15}{32}) \in [0,1],$$

for all $z \in [0, 1]$.

From definition of P_C, it follows that

$$P_C(I - \frac{1}{4}B)z \in [0,1], \tag{4.2}$$

for all $z \in [0, 1]$.

Let $(w, z) \in Eed(G)$. From definition of T_G^n and (4.2), we have $T_G^n \subseteq [0, 1]$.

Since $v_0 \in [0, 1]$ and (4.2), we have

$$w_0 = P_C(I - \frac{1}{4}B)v_0 \in [0,1], \tag{4.3}$$

From $T_G^n \subseteq [0,1]$, we have

$$v_1 = P_{T_G^n}(v_0 - \frac{1}{4}Bw_0) \in [0,1], \tag{4.4}$$

Continue the method of (4.3) and (4.4), we have $w_n, v_n \in [0,1]$ for all $n \in \mathbb{N}$.
Since $v_0, \frac{1}{2}, v_n$ and $w_n \in [0,1]$, it follows that $(\frac{1}{2}, v_n)$, (v_n, v_0) and $(w_n, v_0) \in Eed(G)$.
We can conclude that $G - Var(C, B)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0. All conditions of Example 4.3 satisfies Theorem 3.2, so we can conclude that sequence $\{v_n\}$ converses weakly to an element of $\frac{1}{2} \in G - Var(C, B)$ and the sequence $\{P_{G-Var(C,B)}v_n\}$ converges strongly to $\frac{1}{2}$.

Table 4.1*:* Detailed analysis of computational methods (4.1) for Example 4.1 with $\mathsf{v}_0 = 1$, $N = 20$.

n	v_n
1	1.0000000
2	0.9349873
3	0.8699746
4	0.8108711
⋮	⋮
19	0.5159698
20	0.5101756

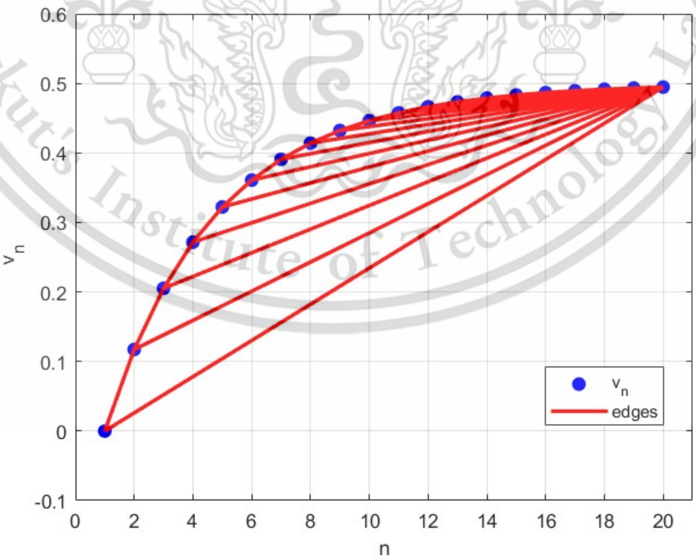

Figure 4.1*:* The convergence behavious of $\{\mathsf{v}_\mathsf{n}\}$ with $v_0 = 1$ and $N = 20$.

Example 4.4. Let $C = [-5, 5]$ and $G = (C \times C, Eed(G))$ be a directed graph, where $Eed(G) = \{(x, y) : x = (x_1, x_2), y = (y_1, y_2) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]\}$. Let the mappings B :

$C \times C \to \mathbb{R}^2$ define by $B(x_1, x_2) = (\frac{4x_1}{5} - \frac{8}{5}, \frac{x_2}{4})$, for all $x_1, x_2 \in C$.
Let metric projection $P_C : H \times H \to C \times C$ define by

$$P_C(z_1, z_2) = (\max\{\min\{z_1, 5\}, -5\}, \max\{\min\{z_2, 5\}, -5\}),$$

for all $z = (z_1, z_2) \in H \times H$.
Suppose that the sequence $\{v^n\}$ is generated by $v^0 = (v_1^0, v_2^0) = (1, 1)$ and

$$\begin{cases} w^n = P_C(I - \lambda B)v^n \\ T_G^n = \{w \in C \times C \, \langle (I - \lambda B)v^n - w^n, w^n - w\rangle \geq 0\} \\ v^{n+1} = P_{T_G^n}(v^n - \lambda B w^n), \end{cases} \tag{4.5}$$

for all $n \in N$ where $v^n = (v_1^n, v_2^n), w^n = (w_1^n, w_2^n)$,$\lambda \in (0, \alpha)$ and T_G^n is G- Half space. Then sequence $\{v^n\}$ converses weakly to an element of $\bar{v} \in G - Var(C, B)$ and the sequence $\{P_{G-Var(C,B)}v^n\}$ converges strongly to $\bar{v}$, where $G - Var(C, B)$ dominates v^n, $\{v^n\}$ dominates v^0 and $\{w^n\}$ is dominated by v^0.

Solution It is easy to see that $(2, 0) \in G - Var(C, B)$, and $Eed(G) = Eed(G^{-1})$. It is obvious that B is G-$\frac{1}{3}$-inverse strongly monotone.
Putting $\lambda = \frac{1}{4}$. From the definition of B, we have

$$(I - \frac{1}{4}B)z \;\; = \;\; (\frac{4z_1}{5} + \frac{2}{5}, \frac{15z_2}{16}) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2], \tag{4.6}$$

for all $z = (z_1, z_2) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.

From definition of P_C, it follows that

$$P_C(I - \frac{1}{4}B)z \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}0, 2], \tag{4.7}$$

for all $z = (z_1, z_2) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.
Let $(w, z) \in Eed(G)$. From definition of T_G^n and (4.7), we have $T_G^n \subseteq [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.
Since $v^0 \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$ and (4.7), we have

$$w^0 = P_C(I - \frac{1}{4}B)v^0 \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2], \tag{4.8}$$

where $w^0 = (w_1^0, w_2^0) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$. From $T_G^n \subseteq [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$, we have

$$v^1 = P_{T_G^n}(v^0 - \frac{1}{4}Bw^0) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2], \tag{4.9}$$

where $v^1 = (v_1^1, v_2^1) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.
Continue the method of (4.8) and (4.9), we have $w^n, v^n \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$ for all $n \in \mathbb{N}$,$v^n = (v_1^n, v_2^n), w^n = (w_1^n, w_2^n)$.
Since $v^0, (2, 0), v^n$ and $w^n \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$, it follows that $((2, 0), v^n)$, (v^n, v^0) and $(w^n, v^0) \in Eed(G)$.
We can conclude that $G - Var(C, B)$ dominates v^n, $\{v^n\}$ dominates v^0 and $\{w^n\}$ is dominated by v^0. All conditions of Example 4.4 satisfies Theorem 3.2, so we can conclude that sequence $\{v^n\}$ converses weakly to an element of $(2, 0) \in G - Var(C, B)$ and the sequence $\{P_{G-Var(C,B)}v^n\}$ converges strongly to $(2, 0)$.

Table 4.2: Detailed analysis of computational methods (4.5) for Example 4.2 with $\mathbf{v}^0 = (1,1)$, $N = 100$.

n	$\mathbf{v}_1^n$	$\mathbf{v}_2^n$
1	1.0000000	1.0000000
2	1.1600000	0.9414062
3	1.2944000	0.8862457
4	1.4072960	0.8343173
⋮	⋮	⋮
99	2.0000000	0.0028601
100	2.0000000	0.0026925

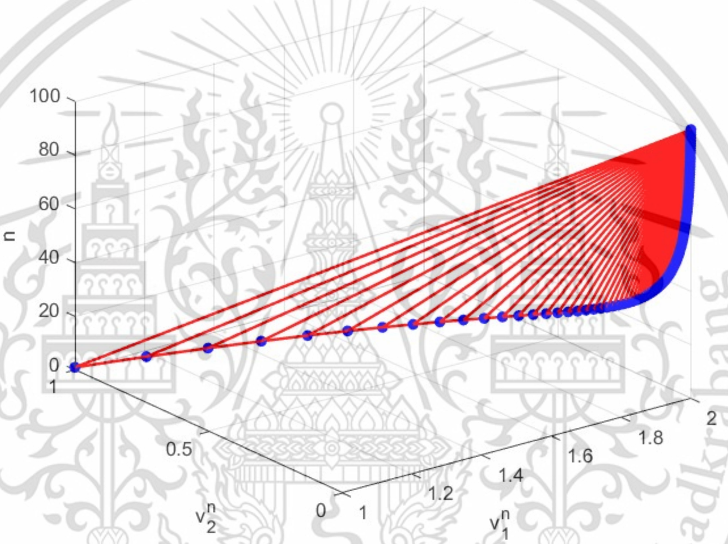

Figure 4.2: The convergence behavious of $\{\mathbf{v}^n\}$ with $v^0 = (1,1)$ and $N = 100$.

4.3 Example for the Modified subgradient extragradient method for system of variational inclusion problem and variational inequalities problem

In this section, we give an example supporting Theorem 3.1.

Example 4.5. Let $H = \mathbb{R}^2$ be the two dimensional space of real numbers with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\langle x, y \rangle = x \cdot y = x_1y_1 + x_2y_2$ and a usual norm $|| \cdot || : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ give by $||x|| = \sqrt{x_1^2 + x_2^2}$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let $C_1 = \{(x_1, x_2) \in H| - 2x_1 + x_2 \leq 1\}$ and $C_2 = \{(x_1, x_2) \in H|4x_1 - 2x_2 \leq 3\}$. Define the mapping $A_1 : C_1 \to \mathbb{R}^2$ by $A_1(x_1, x_2) = (\frac{3x_1}{2}, \frac{3x_2}{2})$. Define the mapping $A_2 : C_2 \to \mathbb{R}^2$ by $A_2(x_1, x_2) = (2x_1, 2x_2)$. Let the mapping $A_G : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A_G(x_1, x_2) = (x_1 + 1, x_2 + 1)$. Let $C = C_1 \cap C_2$. Also, it is well known that

$$P_C(x_1,x_2)=\begin{cases}(-1999x_1+1000x_2+750,4000x_1-1999x_2-1500) & ;if-40x_1+20x_2<-15\\ (x_1,x_2) & ;if-15\leq -40x_1+20x_2\leq 5\\ (-1999x_1+1000x_2-250,4000x_1-1999x_2-500) & ;if-40x_1+20x_2>5\end{cases}$$

Let $x_1,u\in\mathbb{R}^2$, $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be generated by

$$\begin{cases} y_n=P_C(I-\lambda\sum_{i=1}^2 a_iA_i)x_n\\ Q_n=\{z\in H:\langle (I-\lambda\sum_{i=1}^2 a_iA_i)x_n-y_n,y_n-z\rangle\geq 0\}\\ x_{n+1}=\alpha_n u+\beta_n P_{Q_n}(x_n-\lambda\sum_{i=1}^2 a_iA_iy_n)+\gamma_n Gx_n\end{cases}\quad (4.10)$$

where $\{\alpha_n\}=\frac{1}{12n}$, $\{\beta_n\}=\frac{5n-2}{12n}$, $\{\gamma_n\}=\frac{7n+1}{12n}\subset[0,\ 1]$ and $a=0.5\in(0,1)$. Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0,0)$.

Solution Since A_1,A_2 and A_G are $\frac{2}{3},\frac{1}{2}$ and 1 - inverse strongly monotone mappings, respectively, then $\eta=\frac{1}{2}$. Choose $\lambda_A=\frac{1}{2},\lambda_B=1\in(0,2\alpha_G)$ and $b=\frac{1}{4}$,we obtain $G(x_1,x_2)=(\frac{x_1}{16},\frac{x_2}{16})$. Choose $\lambda=\frac{1}{4}\in(0,\eta)$. It is easy to see that the sequences $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1 and $(0,0)\in VI(C,A_1)\cap VI(C,A_2)\cap F(G)$. From Theorem 3.1, we can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0,0)$.

Table 4.3: Detailed analysis of computational methods (5.1) and (1.5) for Example 4.1 with u $=(5,5)$, $N=15$, $E(x_1^n)=\|x_1^{n+1}-x_1^n\|$, $n\in N_0$ and $E(x_2^n)=\|x_2^{n+1}-x_2^n\|$, $n\in N_0$.

	Iterative (5.1)		Iterative (1.5)	
n	$E(\mathsf{x}_1^n)$	$E(\mathsf{x}_2^n)$	$E(\mathsf{x}_1^n)$	$E(\mathsf{x}_2^n)$
1	1.0000	2.0000	1.0000	2.0000
2	0.6468	0.8770	0.6497	0.8828
3	0.3961	0.4630	0.3995	0.4681
4	0.2619	0.2826	0.2646	0.2862
⋮	⋮	⋮	⋮	⋮
15	0.0472	0.0457	0.0476	0.0476

Example 4.6. Let $H=L_2([-1,1])$ with product $\langle f,g\rangle=\int_{-1}^1 f(t)g(t)dt$ and the associated norm given as $||f||:=\sqrt{\int_{-1}^1 f(t)g(t)dt}$ for all $f,g\in L_2([-1,1])$. Take $C=\{x\in H:||x||\leq 2\}$. Define the mapping $A_1:L_2([-1,1])\to L_2([-1,1])$ by $A_1(h(t))=h(t)-2t$ for all $t\in[-1,1]$. Define the mapping $A_2:L_2([-1,1])\to L_2([-1,1])$ by $A_2(h(t))=\frac{3}{2}h(t)-3t$ for all $t\in[-1,1]$. Let the mapping $A_G:L_2([-1,1])\to L_2([-1,1])$ defined by $A_G(h(t))=h(t)-5t$ for all $t\in[-1,1]$. Also, it is well known that

$$P_C(f(t))=\begin{cases} f(t) & ;if||f(t)||\leq 2\\ \frac{2f(t)}{||f(t)||} & ;if||f(t)||>2\end{cases}$$

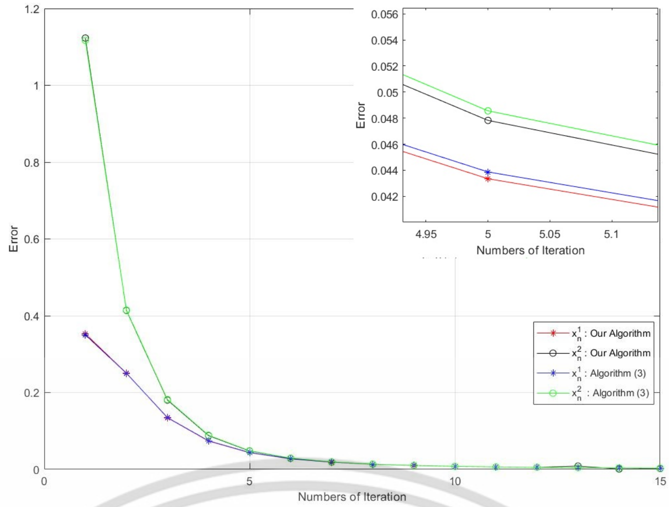

Figure 4.3*:* Comparison between algorithms (5.1) and (1.5) for Example 1 with $\mathbf{u} = (5, 5)$ and $N = 15$.

Let $i = 1, 2, x_1, u \in \mathbb{R}^2$, $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be generated by (5.1) where $\{\alpha_n\} = \frac{1}{12n}$, $\{\beta_n\} = \frac{5n-2}{12n}$, $\{\gamma_n\} = \frac{7n+1}{12n} \subset [0,\ 1]$ and $a = 0.4 \in (0, 1)$. Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to $2t$.

`Solution` Since A_1, A_2 and A_G are $\frac{1}{2}, \frac{1}{3}$ and 1 - inverse strongly monotone mappings, respectively, then $\eta = \frac{1}{2}$. Choose $\lambda_A = \frac{1}{2}, \lambda_B = 1 \in (0, 2\alpha_G)$ and $b = \frac{1}{4}$,we obtain $G(h(t)) = \frac{h(t)}{16}$. Choose $\lambda = \frac{1}{4} \in (0, \eta)$. It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1 and $2t \in VI(C, A_1) \cap VI(C, A_2) \cap F(G)$. From Theorem 3.1, we can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converge strongly to $2I$.

Table 4.4*:* Detailed analysis of computational methods (5.1) and (1.5) for Example 4.2 with $\mathbf{u} = 3t$, $N = 15$ and $E(x_n) = \|x_{n+1} - x_n\|$, $n \in N_0$.

n	$E(\mathbf{x}_n) : Algorithm(5.1)$	$E(\mathbf{x}_n) : Algorithm(1.5)$
1	0.7626	0.7626
2	0.1291	0.1221
3	0.0480	0.0492
4	0.0208	0.0226
$\vdots$	$\vdots$	$\vdots$
15	0.0006	0.0007

Example 4.7. Let $f : H \to \mathbb{R}$ be a convex function. Consider the following convex optimization problem

$$\min_{x \in H} f(x) \tag{4.11}$$

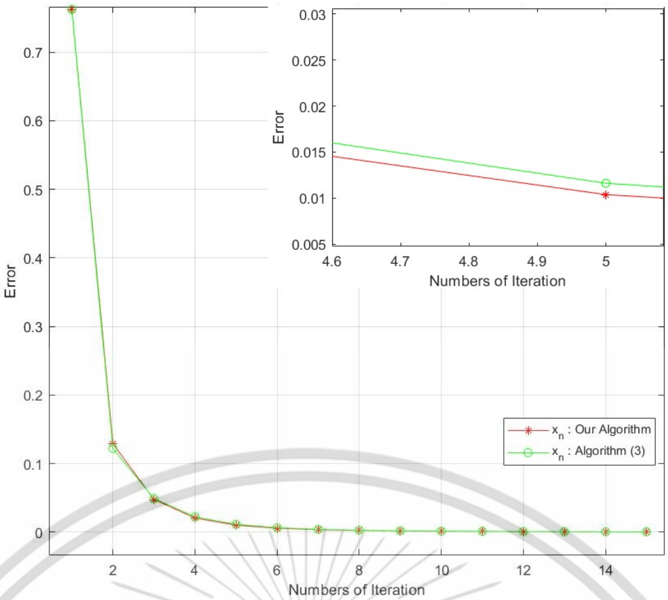

Figure 4.4: Comparison between algorithms (5.1) and (1.5) for Example 2 with $\mathbf{u} = 3t$ and $N = 15$.

and

$$\min_{x \in H} g(x) \tag{4.12}$$

It's well known that $x^* \in C$ solves (4.11) and (4.12) if and only if $x^* \in C$ satisfies following variational inequalities holds:

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C \tag{4.13}$$

and

$$\langle \nabla g(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{4.14}$$

that is, $x^* \in VI(C, \nabla f) \cap VI(C, \nabla g)$. Let $H = \mathbb{R}$. Take $C = [1, 10]$. Define the mapping $f : [1, 10] \to \mathbb{R}$ by $f(x) = \frac{(x-1)^2}{3} + 1$. Define the mapping $g : [1, 10] \to \mathbb{R}$ by $g(x) = \frac{x^2}{2} - \ln x - \frac{1}{2}$. Let $x_1, u \in \mathbb{R}^2$. From (5.1), we have $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be generated by

$$\begin{cases} y_n = P_C(I - \lambda(a_1 \nabla f + a_2 \nabla g))x_n \\ Q_n = \{z \in H : \langle (I - \lambda(a_1 \nabla f + a_2 \nabla g))x_n - y_n, y_n - z \rangle \geq 0\} \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda(a_1 \nabla f + a_2 \nabla g)y_n) + \gamma_n G x_n \end{cases} \tag{4.15}$$

where $\{\alpha_n\} = \frac{1}{12n}$, $\{\beta_n\} = \frac{5n-2}{12n}$, $\{\gamma_n\} = \frac{7n+1}{12n} \subset [0, \ 1]$ and $a = 0.5 \in (0, 1)$. Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to 1.

Solution Since f and g are convex and differentiable with $f'(x) = \frac{2(x-1)}{3}$ and $g'(x) = x - \frac{1}{x}$.It implies that ∇f and ∇g are $\frac{2}{3}$ and 1 - inverse strongly monotone mappings, respectively. Choose $\eta = \frac{1}{2}$, $\lambda_A = \frac{1}{2}, \lambda_B = 1 \in (0, 2\alpha_G)$ and $b = \frac{1}{4}$,we obtain $G(x) = \frac{x}{12} + \frac{11}{12}$. Choose $\lambda = \frac{1}{4} \in (0, \eta)$. It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1 and $1 \in VI(C, \nabla f) \cap VI(C, \nabla g) \cap F(G)$. From Theorem 3.1, we can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converge strongly to 1.

Table 4.5*:* Detailed analysis of computational methods (5.1) and (1.5) for Example 4.3 with $\mathbf{u} = 3$, $N = 15$ and $E(x_n) = \|x_{n+1} - x_n\|$, $n \in N_0$.

n	$E(\mathbf{x}_n) : Algorithm(5.1)$	$E(\mathbf{x}_n) : Algorithm(1.5)$
1	2.9044	2.7500
2	0.7088	0.7428
3	0.2200	0.2681
4	0.0762	0.1082
⋮	⋮	⋮
15	0.0012	0.0015

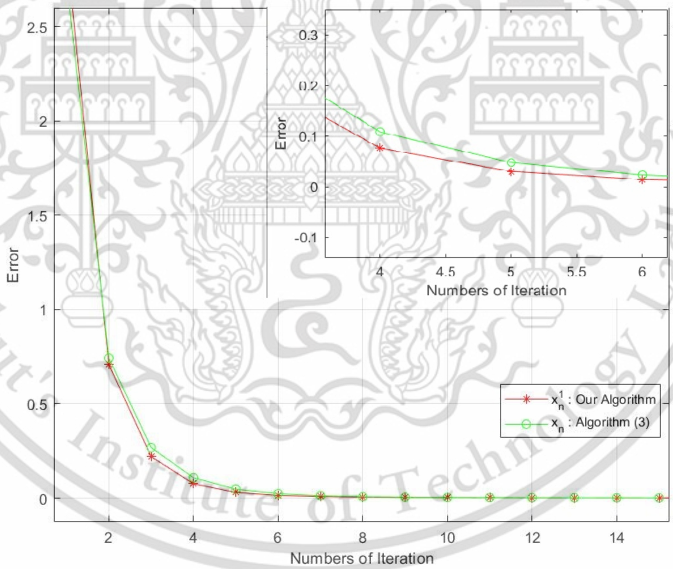

Figure 4.5*:* Comparison between algorithms (5.1) and (1.5) for Example 3 with $\mathbf{u} = 3$ and $N = 15$.

Remark 4.8. According to Table 4.3-4.7 and Figures 4.3-4.5 ,it is shown that the our algorithm (5.1) converges to an element of the set $\bigcap_{i=1}^{N} VI(C, A_i) \cap F(G)$ at a faster than that of algorithm (1.5). Therefore, our algorithm is more efficient.

4.4 Applying in Economic

In this part, we apply our main result to find equilibrium price.

Example 4.9. Let's consider the demand and supply for rice in a specific country:
Demand : The demand for rice is influenced by factors such as population, income levels, cultural preferences, dietary habits, and price. Let's assume that the demand for rice in the country is as follows:
Price: 1 dollar per kilogram - Quantity demanded: 4,800 kilograms per week
Price: 2 dollars per kilogram - Quantity demanded: 3,700 kilograms per week
Price: 3 dollars per kilogram - Quantity demanded: 2,800 kilograms per week
Price: 4 dollars per kilogram - Quantity demanded: 2,400 kilograms per week
Price: 5 dollars per kilogram - Quantity demanded: 1,900 kilograms per week
Price: 6 dollars per kilogram - Quantity demanded: 500 kilograms per week
Supply : The supply of rice is influenced by factors such as agricultural productivity, weather conditions, government policies, technology, and input costs. Let's assume that the supply of rice in the country is as follows:
Price: 1 dollar per kilogram - Quantity supplied: 1,500 kilograms per week
Price: 2 dollars per kilogram - Quantity supplied: 2,500 kilograms per week
Price: 3 dollars per kilogram - Quantity supplied: 3,500 kilograms per week
Price: 4 dollars per kilogram - Quantity supplied: 4,500 kilograms per week
Price: 5 dollars per kilogram - Quantity supplied: 5,500 kilograms per week
By using Polynomial Regression, we have the demand function as follows:

$$D(p) = 0.0179p^2 - 7.925p + 54.3, \tag{4.16}$$

where p is the price of rice (dollars per kilogram) and D is the quantity of units demanded (in hundred kilogram per week). From (2.13), we can calculate the correlation coefficient $r = -0.9437$.
By using Linear Regression, we have the supply function as follows:

$$S(p) = 9p + 7, \tag{4.17}$$

where p is the price of rice (dollars per kilogram) and S is the quantity of units supplies (in hundred kilogram per week). From (2.13), we can calculate the correlation coefficient $r = 0.9823$.

The demand curve for rice shows that as the price increases, the quantity demanded decreases. (See Figure 4.6) This indicates the inverse relationship between price and quantity demanded. The supply curve for rice shows that as the price increases, the quantity supplied increases. (See Figure 4.6) This indicates the direct relationship between price and quantity supplied.

To find the equilibrium price, we can set the demand and supply functions equal to each other:

$$S(p) = D(p). \tag{4.18}$$

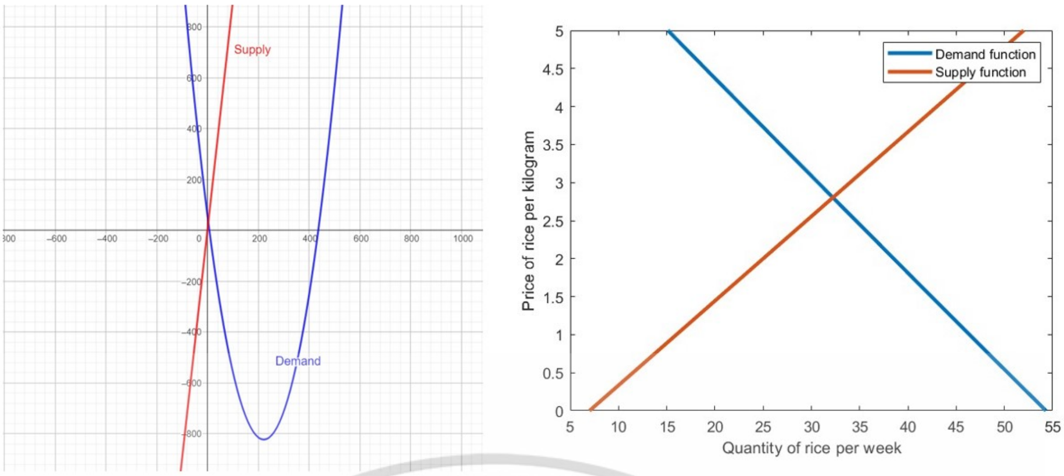

Figure 4.6: The demand and supply model of rice in a specific country.

From 4.24, we have

$$\begin{aligned} p &= S^{-1}(D(p)) \\ &= (S^{-1} \circ D)(p). \end{aligned} \tag{4.19}$$

Then

$$\begin{aligned} T(p) &= (S^{-1} \circ D)(p) \\ &= 0.001989p^2 - 0.88056p + 5.2556, \end{aligned} \tag{4.20}$$

for all $p \in \mathbb{R}$. Let $H = \mathbb{R}$. Take $C = [0, 25]$. Let the mappings $T : C \to \mathbb{R}$ define by $Tx = 0.001989x^2 - 0.88056x + 5.2556$, and $A_G : C \to \mathbb{R}$ define by $A_G x = x - 5$, for all $x \in C$. Let the mappings $A : C \to \mathbb{R}$ define by $Ax = -0.001989x^2 + 1.88056x - 5.2556$,for all $x \in C$. Let $x_1, u \in \mathbb{R}^2$. From (5.1), we have $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be generated by

$$\begin{cases} y_n = P_C(I - \lambda A x_n) \\ Q_n = \{z \in H : \langle (I - \lambda A)x_n - y_n, y_n - z\rangle \geq 0\} \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda A y_n) + \gamma_n G x_n \end{cases} \tag{4.21}$$

where $\{\alpha_n\} = \frac{1}{12n}$, $\{\beta_n\} = \frac{5n-2}{12n}$, $\{\gamma_n\} = \frac{7n+1}{12n} \subset [0,\ 1]$ and $a = 0.5 \in (0, 1)$. Show that $\{x_n\}$ converges strongly to 2.8029.

Solution First, we show that T is a nonexpansive mapping.

Let $x, y \in C$. From the definition of T, we have

$$\begin{aligned}
\|Tx - Ty\| &= \|0.001989x^2 - 0.88056x + 5.2556 - (0.001989y^2 - 0.88056y + 5.2556)\| \\
&= \|0.001989(x^2 - y^2) - 0.88056(x - y)\| \\
&= \|0.001989(x + y)(x - y) - 0.88056(x - y)\| \\
&\leq 0.001989\|(x + y)(x - y)\| + 0.88056\|x - y\| \\
&\leq 0.001989(50)\|x - y\| + 0.88056\|x - y\| \\
&= 0.98001\|x - y\| \\
&\leq \|x - y\|.
\end{aligned}$$

Therefore T is a nonexpansive mapping.
Since $Ax = (I - T)x$, T is a nonexpansive mapping and Proposition 2.16, we have A is $\frac{1}{2}$-inverse strongly monotone. Since A_G is 1 - inverse strongly monotone mappings, then $\eta = 1$. Choose $\lambda_A = \frac{1}{2}, \lambda_B = 1 \in (0, 2\alpha_G)$ and $b = \frac{1}{4}$,we obtain $G(x) = 0.0625x + 2.62771875$. It is obvious that $VI(C, A) = F(G) = 2.8032$.
Choose $\lambda = \frac{1}{4} \in (0, \eta)$. It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1 and $2.8032 \in VI(C, A) \cap F(G)$. From Theorem 3.1, we can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converge strongly to 2.8032.

Thus, at the market price of 2.8032 dollars per kilogram and the equilibrium quantity is 32.2288 hundred kilogram per week, the quantity demanded and supplied are in balance. If the price were higher than 2.8032 dollars per kilogram, the quantity supplied would exceed the quantity demanded, resulting in a surplus of rice. If the price were lower than 2.8032 dollars per kilogram, the quantity demanded would exceed the quantity supplied, resulting in a shortage of rice.

The market price of 2.8032 dollars per kilogram represents the point where the interests of buyers (demand) and sellers (supply) align, and the rice market reaches equilibrium.

Table 4.6*:* Detailed analysis of computational methods (5.1) for Example 4.9 with $\mathbf{u} = 3$, $N = 15$.

n	$\mathbf{x}_n : Algorithm(4.21)$
1	3.2000
2	2.9457
3	2.8697
4	2.8391
⋮	⋮
15	2.8032

Example 4.10. Let's consider the demand and supply for the Thai Baht (THB), the unit of currency in Thailand.

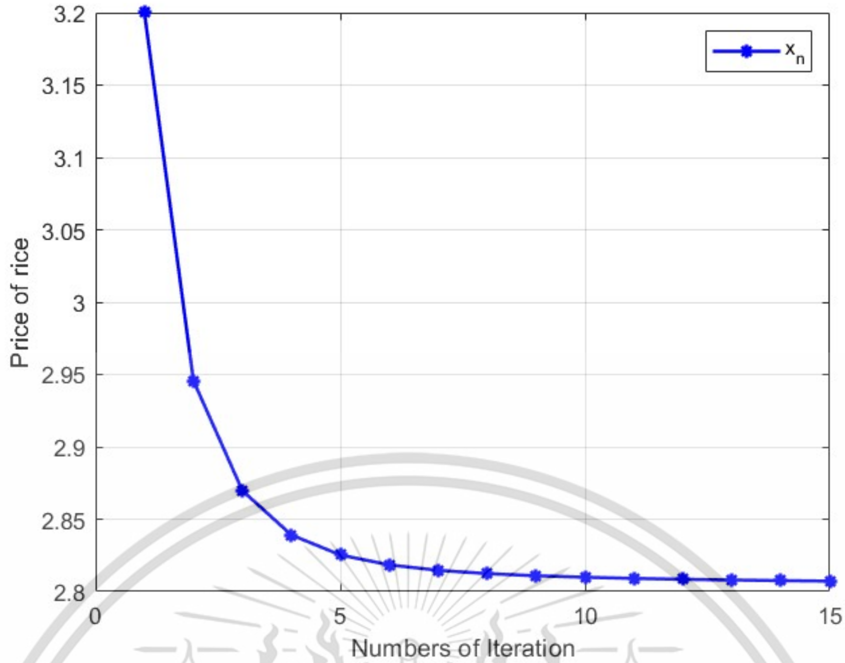

Figure 4.7: The convergence behavious of $\{x_n\}$ with $x_0 = 3.2$ and $N = 15$.

Demand : The demand for Thai Baht is influenced by factors such as international trade, tourism, foreign investment, and speculation. Let's assume that the demand for Thai Baht is as follows: Exchange rate: 1 USD = 26 THB
Quantity demanded: 34.89 million dollars
Exchange rate: 1 USD/ = 29 THB
Quantity demanded: 30.96 million dollars
Exchange rate: 1 USD = 30 THB
Quantity demanded: 29.69 million dollars
Exchange rate: 1 USD = 32 THB
Quantity demanded: 27.21 million dollars
Exchange rate: 1 USD = 35 THB
Quantity demanded: 23.64 million dollars

Supply : The supply of Thai Baht is influenced by factors such as government policies, central bank interventions, foreign exchange reserves, and economic conditions. Let's assume that the supply of Thai Baht is as follows:
Exchange rate: 1 USD = 27 THB
Quantity supplied: 6.5 million dollars
Exchange rate: 1 USD = 30 THB
Quantity supplied: 20 million dollars
Exchange rate: 1 USD = 34 THB
Quantity supplied: 29 million dollars

Exchange rate: 1 USD = 36 THB
Quantity supplied: 47 million dollars
Exchange rate: 1 USD = 37 THB
Quantity supplied: 51.5 million dollars

By using Polynomial Regression, we have the demand function as follows:

$$D(p) = 0.01p^2 - 1.86p + 76.49, \tag{4.22}$$

where p is the price of exchange rates (1 USD to Thai baht) and D is the quantity of units demanded (million dollars). From (2.13), we can calculate the correlation coefficient $r = -0.9563$.
By using Linear Regression, we have the supply function as follows:

$$S(p) = 4.5p - 115, \tag{4.23}$$

where p is the price of exchange rates (1 USD to Thai baht) and S is the quantity of units supplies (million dollars). From (2.13), we can calculate the correlation coefficient $r = 0.9716$.
The demand curve for Thai Baht shows that as the exchange rate increases (USD appreciates), the quantity of Thai Baht demanded decreases. (See Figure 4.8) This indicates the inverse relationship between the exchange rate and the quantity demanded of Thai Baht.
The supply curve for Thai Baht shows that as the exchange rate increases (USD appreciates), the quantity of Thai Baht supplied increases.(See Figure 4.8) This indicates the direct relationship between the exchange rate and the quantity supplied of Thai Baht.

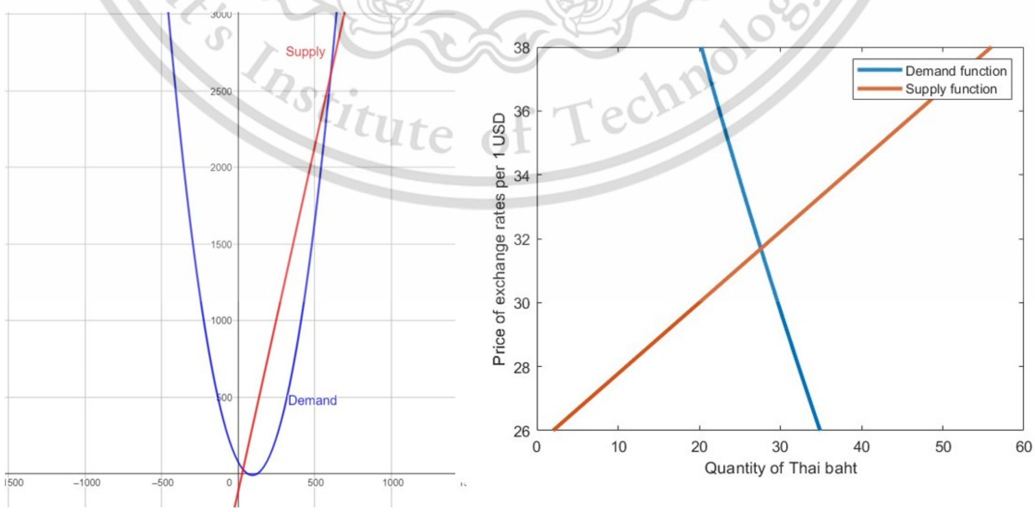

Figure 4.8: The demand and supply model for the Thai Baht (THB), the unit of currency in Thailand.

To find the equilibrium price, we can set the demand and supply functions equal to each other:

$$S(p) = D(p). \tag{4.24}$$

From 4.24, we have

$$\begin{aligned} p &= S^{-1}(D(p)) \\ &= (S^{-1} \circ D)(p). \end{aligned} \tag{4.25}$$

Then

$$\begin{aligned} T(p) &= (S^{-1} \circ D)(p) \\ &= \frac{1}{450}p^2 - \frac{31}{75}p + \frac{6383}{150}, \end{aligned} \tag{4.26}$$

for all $p \in \mathbb{R}$. Let $H = \mathbb{R}$. Take $C = [10, 45]$. Let the mappings $T : C \to \mathbb{R}$ define by $Tx = \frac{1}{450}x^2 - \frac{31}{75}x + \frac{6383}{150}$, and $A_G : C \to \mathbb{R}$ define by $A_G x = x - 25$, for all $x \in C$. Let the mappings $A : C \to \mathbb{R}$ define by $Ax = -\frac{1}{450}x^2 + \frac{106}{75}x - \frac{6383}{150}$,for all $x \in C$. Let $x_1, u \in \mathbb{R}^2$. From (5.1), we have $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be generated by

$$\begin{cases} y_n = P_C(I - \lambda A x_n) \\ Q_n = \{z \in H : \langle (I - \lambda A)x_n - y_n, y_n - z \rangle \geq 0\} \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda A y_n) + \gamma_n G x_n \end{cases} \tag{4.27}$$

where $\{\alpha_n\} = \frac{1}{12n}$, $\{\beta_n\} = \frac{5n-2}{12n}$, $\{\gamma_n\} = \frac{7n+1}{12n} \subset [0,\ 1]$ and $a = 0.5 \in (0, 1)$. Show that $\{x_n\}$ converges strongly to 31.68494.

Solution First, we show that T is a nonexpansive mapping.

Let $x, y \in C$. From the definition of T, we have

$$\begin{aligned} \|Tx - Ty\| &= \|\frac{1}{450}x^2 - \frac{31}{75}x + \frac{6383}{150} - (\frac{1}{450}y^2 - \frac{31}{75}y + \frac{6383}{150})\| \\ &= \|\frac{1}{450}(x^2 - y^2) - \frac{31}{75}(x - y)\| \\ &= \|\frac{1}{450}(x + y)(x - y) - \frac{31}{75}(x - y)\| \\ &\leq \frac{1}{450}\|(x + y)(x - y)\| + \frac{31}{75}\|x - y\| \\ &\leq \frac{1}{450}(90)\|x - y\| + \frac{31}{75}\|x - y\| \\ &= \frac{46}{75}\|x - y\| \\ &\leq \|x - y\|. \end{aligned}$$

Therefore T is a nonexpansive mapping.

Since $Ax = (I - T)x$, T is a nonexpansive mapping and Proposition 2.16, we have A is $\frac{1}{2}$-inverse strongly monotone. Since A_G is 1 - inverse strongly monotone mappings, then $\eta = 1$. Choose $\lambda_A = \frac{1}{2}, \lambda_B = 1 \in (0, 2\alpha_G)$ and $b = \frac{1}{4}$, we obtain $G(x) = 0.0625x + 29.7052125$.

It is obvious that $VI(C, A) = F(G) = 31.68494$.

Choose $\lambda = \frac{1}{4} \in (0, \eta)$. It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1 and $31.68494 \in VI(C, A) \cap F(G)$. From Theorem 3.1, we can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converge strongly to 31.68494.

Therefore, at the market exchange rate of 1 USD = 31.68494 THB and the equilibrium quantity is 27.58223 million dollars, the quantity demanded and supplied are in balance. If the exchange rate were higher than 1 USD = 31.68494 THB, the quantity supplied would exceed the quantity demanded, resulting in an excess supply of Thai Baht. If the exchange rate were lower than 1 USD = 31.68494 THB, the quantity demanded would exceed the quantity supplied, resulting in a shortage of Thai Baht.

Table 4.7*:* Detailed analysis of computational methods (5.1) for Example 4.10 with $\mathbf{u} = 28$, $N = 15$.

n	$\mathbf{x}_n : Algorithm(4.21)$
1	28.00000
2	30.51490
3	31.50527
4	31.65402
$\vdots$	$\vdots$
15	31.68494

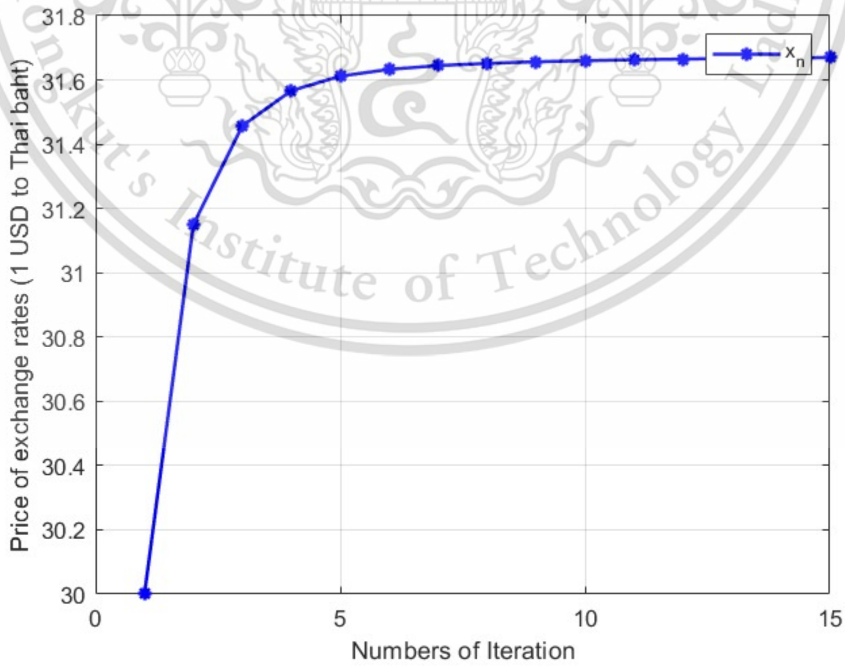

Figure 4.9*:* The convergence behavious of $\{\mathbf{x}_n\}$ with $x_0 = 28$ and $N = 15$.

Chapter 5

Conclusions and Suggestions

In this chapter, we summarize all theorems and corollaries given in this thesis.

(1) Let H be a real Hilbert space. For $i = 1, 2, ..., N$, let $A_i : H \to H$ be α_i - inverse strongly monotone mappings and let $A_G : H \to H$ be α_G - inverse strongly monotone mappings. Define the mapping $G : H \to H$ by $G(x) = J_{M_A,\lambda_A}(I - \lambda_A A_G)(bx + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x)$ for all $x \in H$, $b \in (0,1)$ and $\lambda_A, \lambda_B \in (0, 2\alpha_G)$. Assume that $\Gamma = \bigcap_{i=1}^{N} VI(C, A_i) \cap F(G) \neq \emptyset$. Let the sequence $\{y_n\}$ and $\{x_n\}$ be generated by $x_1, u \in H$ and

$$\begin{cases} y_n = P_C(I - \lambda \sum_{i=1}^{N} a_i A_i)x_n \\ Q_n = \left\{ z \in H : \left\langle (I - \lambda \sum_{i=1}^{N} a_i A_i)x_n - y_n, y_n - z \right\rangle \geq 0 \right\} \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n) + \gamma_n G x_n \end{cases} \quad (5.1)$$

where $\sum_{i=1}^{N} a_i = 1, 0 < a_i < 1, \{\alpha_n\}, \ \{\beta_n\}, \ \{\gamma_n\} \subset [0, \ 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\lambda \in (0, \eta)$ with $\eta = min_{i=1,2,\ldots,N} \{\alpha_i\}$.
Suppose the following conditions hold:
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.
(ii) $0 < c < \beta_n, \ \gamma_n \leq d < 1$
Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$ where $x^* = P_\Gamma u$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G, Ver(G), Eed(G), B$ as in Lemma 2.30. Assume that $G - Var(C, B) \neq \emptyset$ with $G - Var(C, B) \times G - Var(C, B) \subseteq Eed(G)$. Let $\{v_n\}$ be a sequence defined by $v_0 \in C$ and

$$\begin{cases} w_n = P_C(I - \lambda B)v_n \\ T_G^n = \{w \in C : \langle (I - \lambda B)v_n - w_n, w_n - w \rangle \geqslant 0\} \\ v_{n+1} = P_{T_G^n}(v_n - \lambda B w_n), \end{cases}$$

for all $n \in N$ where $\lambda \in (0, \alpha)$ and T_G^n is G- Half space. Then sequence $\{v_n\}$ converges weakly to an element $\bar{x} \in G - Var(C, B)$ and the sequence $\{P_{G-Var(C,B)} v_n\}$ converges strongly to $\bar{x}$, where $G - Var(C, B)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0.

References

[1] Mann WR (1953) Mean value methods in iteration. J Proc Am Math Soc 4:506-510

[2] Moudafi A (2000) Viscosity approximation methods for fixed-points problems. J Math Anal Appl 241:46-55

[3] Aubin,J.P., Ekeland,I, Applied Nonlinear Analysis, Wiley, New York, (1984)

[4] Korpelevich, G.M., The extragradient method for finding saddle points and other problens, Ekonomikai Matermaticheskie Metody, 12, 747-756 (1976)

[5] Censor,Y., Gibali, A., Reich, S., Algorithms for the split variational inequality problem, Numerical Algorithms, 59, 301-323 (2012)

[6] Zhang, S.S., Lee J.H.M. and Chan, C.K., Algorithms of common solutions for quasi variational inclusion and fixed point problems, Appl. Math. Mech., 29, 571-578 (2008)

[7] Hieu, D.V., Thong, D.V., New extragradient like algorithms for strongly pseudomonotone variational inequalities, J. Glob. Optim., 70, 385-399 (2018)

[8] Hieu, D.V., Thong, D.V., A new projection method for a class of variational inequalities, Appl. Anal., (2018)

[9] Blum, E., Oettli, W., From optimization and variational inequalities to equilibrium problems, Math. Program., 63, 123-145 (1994)

[10] Takahashi, W. and Toyoda, M. 2003. "Weak convergence theorems for nonexpansive mappings and monotone mappings." *J. Optim. Theory Appl.* 118: 417-428.

[11] Iiduka, H. and Takahashi, W. 2005. "Strong convergence theorems for nonexpansive mappings and inversestrongly monotone mappings." *Nonlinear Anal TMA*. 61(3):341–350.

[12] Malitsky, Y.V., Semenov, V.V., A hybrid method without extrapolation step for solving variational inequalities problems, J. Glob. Optimi., 61, 193-202 (2015)

[13] Malitsky, Y.V., Projected reflected gradient methods for monotone variational inequalities, SIAM J. Optim., 25, 502-520 (2015)

[14] Yao, Y., Marino, G., Muglia, L., A modified Korpelevich's method convergent to the minimum-norm solution of a variational inequality, Optimization, 63, 559-569 (2014)

[15] Censor,Y., Gibali, A., Reich, S., The subgradient extragradient method for solving variational inequalities in Hilbert space, J. Optim. Theory Appl., 148, 318-335 (2011)

[16] Solodov, M.V., Svaiter, B.F., A new projection method for variational inequalities problems, SIAM J. Control Optim., 37, 765-776 (1999)

[17] Xu, H.K., Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl., 298, 279-291 (2004)

[18] Brezis, H., Operateurs maximaux monotone et semi-groupes de contractions dans les espaces de Hilbert, Math. Stud., Amsterdam North Holand, 5, 759-775 (1973)

[19] Dafermos, S.C., Traffic equilibrium and variational inequalities, Transp. Sci.,14, 42-54 (1980)

[20] Halpern, B. Fixed points of nonexpanding maps. Bull. Am. Math. Soc. 73: 957-961(1967)

[21] Dafermos, S.C., Mckelvey, S.C., Partitionable variational inequalities with applications to network and economic equilibrium, J. Optim., Theory Appl., 73, 243-268 (1992)

[22] Xu, H.K., An iterative approach to quadric optimization, J. Optim., Theory Appl., 116, 659-678 (2003)

[23] Rockafellar, R.T., On the maximal monotonicity of subdifferential mappings, Pac.J.Math., 33, 209-216 (2005)

[24] Maing P.E., Approximation method for common fixed points of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl., 325, 469-479 (2007)

[25] S.S. Zhang, J.H.M. Lee, and C.K. Chan, Algorithms of common solutions for quasi variational inclusion and fixed point problems, Appl. Math. Mech., 29(2008), 571-578.

[26] Kangtunyakarn, A., An iterative algorithm to approximate a common element of the set of common fixed points for a finite family of strict pseudo-contractions and the set of solutions for a modified system of variational inequalities, Fixed Point Theory Appl., 143 (2013)

[27] Mainge P.E., A hybrid extragradient viscosity method for monotone operators and fixed point problems, SIAMJ. Control Optim., 49, 14499-1515 (2008)

[28] Kim T.H., Xu H.K. Strong convergence of modified Mann iterations for with asymptotically nonexpansive mappings and semigroups. Nonlinear Anal. 2006;**64**:1140-1152.

[29] Plubtieng S, Wangkeeree R. Strong convergence of modified Mann iterations for a countable family of nonexpansive mappings. Nonlinear Anal. 2009;**70**:3110-3118.

[30] Kanzow C, Shehu Y. Generalized Krasnoselskii-Mann-type iterations for nonexpansive mappings in Hilbert spaces. J. Comput Optim Appl. 2007;**67**:595. doi:10.1007/s10589-017-9902-0.

[31] Xu H. Viscosity method for hierarchical fixed point approach to variational inequalities. Taiwanese Journal of mathematics. 2000;**14**:463-478.

[32] Marino G, Xu H.K. Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. J. Math Anal Appl. 2007;**329**:336-346.

[33] Blum E, Oettli W. "From optimization and variational inequalities to equilibrium problems," The Mathematics Student. 1994;**63**:123-145.

[34] Yao Z, Kang S.M, Li H.J (2015) An intermixed algorithm for strict pseudo-contractions in Hilbert spaces. J Fixed Point Theory and Applications, 206 (2015).

[35] Bretarkas, D.P., Gafin, E.M.: Projection methods for variational inequalities with applications to the traffic assignment problem, Math. Program. Stud. 17 (1982), pp. 139-159.

[36] Korpelevich GM. The extragradient method for finding saddle points and other problems. Ekon Mat Metody. 1976 ; 12: 747-756.

[37] Censor,Y.,Gibali,A.,Reich,S.:The subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. 148, 318-335 (2011).

[38] Ceng LC, Wang CY, Yao JC. Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities. Math Meth Oper Res. 2008; 67: 375-390.

[39] Hieu,D.V.,Thong,D.V.:New extragradient like algorithms for strongly pseudomonotone variationalinequalities. J. Glob. Optim. 70, 385-399 (2018)

[40] Jouymandi Z, Moradlou F. Extragradient methods for solving equilibrium problems, variational inequalities and fixed point problems. Numer Funct Anal Optim. 2017; 38: 185-1409.

[41] Jachymski, J.: The contraction principle for mappings on a metric space with a graph. Proc. Am. Math. Soc. 136(4), 1359-1373 (2008).

[42] Tiammee, J., Kaewkhao, A., Suantai, S.: On Browder 's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs. Fixed Point Theory Appl. 187, 1-25 (2015).

[43] Kangtunyakarn, A., The variational inequality problem in Hilbert spaces endowed with graphs, J. Fixed Point Theory Appl. (2020) 22:4 DOI10.1007/s11784-019-0736-5

[44] Saechou, K., Kangtunyakarn, A., The subgradient extragradient method for approximation of fixed-point problem and modification of equilibrium problem, Optimization, Vol. 40, 264(8) DOI10.1007/s40314-021-01659-4

[45] Xu, H.K., An iterative approach to quadric optimization, J. Optim., Theory Appl., 116, 659-678 (2003)

[46] Pang, C, Zhang, R, Zhang, Q, Wang, J: Dominating sets in directed graphs. Inf. Sci. 180, 3647-3652 (2010)

[47] Megginson, R.E. **An Introduction to Banach Space**.New York: JSpringers. 1998.

[48] Debnath, L. and Mikusinski, P. **Introduction to Hilbert Spaces with Applications**. London:Elsevier Academic Press. 2005.

[49] Berinde, V. **Iterative Approximation of Fixed Points.** New York:Springer Berlin Heidelberg. 2007.

[50] Opial, Z. 1967. “Weak convergence of the sequence of successive approximation of nonexpansive mappings.” *Bull. Amer. Math. Soc.* 73:591-597.

[51] Zegeye, H. and Shahzadb, N. 2011. “Convergence of Mann’s type iteration method for generalized asymptotically nonexpansive mappings.” *Comput. Math. Appl.* 62:4007–4014.

[52] Bauschke, H.H. and CombettesP. L. **Convex Analysis and Monotone Operator Theory in Hilbert Spaces.** New York:Springer. 2011.

[53] Xu, H.K. 2003. “An iterative approach to quadric optimization.” *J. Optim. Theory Appl.* 116:659-678.

[54] Takahashi, W. **Introduction to Nonlinear and Convex Analysis.** Yokohama: Yokohama Publishers. 2009.

[55] Takahashi, W. **Nonlinear Functional Analysis**. Yokohama: Yokohama Publishers. 2000.

[56] Stephanie Glen. **Correlation Coefficient: Simple Definition, Formula, Easy Steps** From StatisticsHowTo.com: Elementary Statistics for the rest of us! https://www.statisticshowto.com/probability-and-statistics/correlation-coefficient-formula/

[57] Rodrigue Jean-Paul. **Market Area Analysis** From https://transportgeography.org/contents/methods/market-area-analysis/

[58] Esfandiari Ramin S.,**Numerical Methods for Engineers and Scientists using Matlab.** Boca Raton: Taylor and Francis Group. 2017

Appendix

The Paper Research

Kheawborisut and Kangtunyakarn *Journal of Inequalities and Applications* (2021) 2021:53
https://doi.org/10.1186/s13660-021-02583-1

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Modified subgradient extragradient method for system of variational inclusion problem and finite family of variational inequalities problem in real Hilbert space

Araya Kheawborisut[1] and Atid Kangtunyakarn[1*]

*Correspondence:
beawrock@hotmail.com
[1]Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand

Abstract

For the purpose of this article, we introduce a modified form of a generalized system of variational inclusions, called the generalized system of modified variational inclusion problems (GSMVIP). This problem reduces to the classical variational inclusion and variational inequalities problems. Motivated by several recent results related to the subgradient extragradient method, we propose a new subgradient extragradient method for finding a common element of the set of solutions of GSMVIP and the set of a finite family of variational inequalities problems. Under suitable assumptions, strong convergence theorems have been proved in the framework of a Hilbert space. In addition, some numerical results indicate that the proposed method is effective.

Keywords: System of variational inclusions problem; Variational inequalities problem; Half-space; Nonexpansive mapping

1 Introduction

Throughout this paper, let H be a real Hilbert space and C be a nonempty closed convex subset of H with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $T : C \to C$ be a mapping. Then T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. It is well known that $F(T)$ is closed convex and also nonempty.

Let $B : H \to H$ be a mapping and $M : H \to 2^H$ be a multi-valued mapping. The variational inclusion problem is to find $x \in H$ such that

$$\theta \in Bx + Mx, \tag{1}$$

where θ is the zero vector in H. The set of solutions of (1) is denoted by $VI(H, B, M)$. This problem has received much attention due to its applications in large variety of problems arising in convex programming, variational inequalities, split feasibility problems, and minimization problems. To be more precise, some concrete problems in machine learning, image processing, and linear inverse problems can be modeled mathematically by this formulation.

The variational inequality problem (VIP) is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{2}$$

The set of solutions of the variational inequality problem is denoted by $VI(C,A)$. This problem is an important tool in economics, engineering and mathematics. It includes, as special cases, many problems of nonlinear analysis such as optimization, optimal control problems, saddle point problems and mathematical programming; see, for example, [1–4].

It is well known that one of the most popular methods for solving the problem (VIP) is the extragradient method proposed by Korpelevich [5]. The extragradient method is needed to calculate two projections onto the feasible set C in each iteration. So, in the case that the set C is not simple to project on to it, as analyzed in some remarks of the authors in [6], when the subset is a closed expression as in the case of a ball or a half-space, the projection onto the feasible subset C can be computed easily. This can affect the efficiency of the used method. In recent years, the extragradient method has received great attention by many authors, who improved it in various ways; see, e.g. [7–13] and the references therein.

In 2011, Censor et al. [12] proposed the subgradient extragradient method for solving variational inequality problems as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ T_n = \{x \in H : \langle x_n - \lambda Ax_n - y_n, x - y_n\rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda Ay_n), \end{cases} \tag{3}$$

for each $n \geq 1$, where $\lambda \in (0, 1/L)$. In this method, they have replaced the second projection in Korpelevich's extragradient method by a projection on to a half-space, which is computed explicitly.

Motivated by the problem (1), in this paper, we introduce a new problem of the system of variational inclusions in a real Hilbert space as follows:

Let H be a real Hilbert space and let $A : H \to H$ be mapping and $M_A, M_B : H \to 2^H$ be set value mapping. We consider the problem for finding $x^* \in H$ such that

$$\theta \in Ax^* + M_Ax^* \quad \text{and} \quad \theta \in Ax^* + M_Bx^*, \tag{4}$$

where θ is the zero mapping in H, which is called a generalized system of modified variational inclusion problems (in short, GSMVIP). The set of solutions of (4) is denoted by Ω,i.e., $\Omega = \{x^* \in H : \theta \in Ax^* + M_Ax^* \text{ and } \theta \in Ax^* + M_Bx^*\}$. In particular, if $M_A = M_B$, then the problem (4) reduces to the problem (1) and if $J_{M_A,\lambda_A} = J_{M_B,\lambda_B} = P_C$, then the problem (4) reduces to VIP.

In 2012, Kangtunyakarn [14] modified the set of variational inequality problems as follows:

$$VI\big(C, aA + (1 - a)B\big)$$

$$= \{x \in C : \langle y - x, (aA + (1 - a)B)x\rangle \geq 0, \forall y \in C, a \in (0,1)\}, \quad (5)$$

where A and B are the mappings of C into H.

In order to develop efficient algorithms for finding solution of a finite family variational inequalities problem, inspired by problem (5), we define the new half-space $Q_n = \{z \in H : \langle (I - \lambda \sum_{i=1}^{N} a_i A_i)x_n - y_n, y_n - z\rangle \geq 0\}$, which as a tool to prove the strong convergence theorem. In particular, if we put $i = 1$, then Q_n reduces to T_n in subgradient extragradient method (3). However, the sequence $\{x_n\}$ generated by (3) converges weakly to a solution of the variational inequality problem.

In this paper, motivated by recent research [7, 12] and [14], we introduce a new problem (4) and the new iterative scheme for finding a common element of the set of a finite family of variational inequalities problems and the set of solutions of the proposed problem (4) in a real Hilbert space. Then we establish and prove the strong convergence theorem under some proper conditions. Furthermore, we also give some various examples to support our main result.

2 Preliminaries

In this section, we give some useful lemmas that will be needed to prove our main result.

Let C be a nonempty closed convex subset of a real Hilbert space H. We denote strong convergence and weak convergence by the notations $\rightarrow$ and $\rightharpoonup$, respectively. For every $x \in H$, there exists a unique nearest point $P_C x \in C$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called a metric projection of H onto C. It follows that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \text{for all } x \in H, y \in C. \quad (6)$$

Lemma 2.1 ([15]) *Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if we have the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Definition 2.2 Let $M : H \rightarrow 2^H$ be a multi-valued mapping.

(i) The graph $G(M)$ of M is defined by

$$G(M) := \{(x, u) \in H \times H : u \in M(x)\},$$

(ii) the operator M is called *a maximal monotone operator* if M is monotone, i.e.

$$\langle u - v, x - y\rangle \geq 0 \quad \forall u \in M(x), v \in M(y),$$

and the graph $G(M)$ of M is not properly contained in the graph of any other monotone operator. It is clear that a monotone mapping M is maximal if and only if for any $(x, u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0$ for every $(y, v) \in G(M)$ implies that $u \in M(x)$.

Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad \forall u \in H,$$

is called the resolvent operator associated with M where λ is positive number and I is an identity mapping; see [16]. Note that $J_{M,\lambda}$ is a nonexpansive mapping.

Definition 2.3 Let $A : C \to H$ be a mapping.

(i) A is called *μ-Lipschitz continuous* if there exists a nonnegative real number $\mu \geq 0$ such that

$$\|Ax - Ay\| \leq \mu\|x - y\|, \quad \forall x, y \in C.$$

(ii) A is called *α-inverse strongly monotone* if there exists a nonnegative real number $\alpha \geq 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Lemma 2.4 ([14]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let A,B: $C \to H$ be α- and β-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C, A) \cap VI(C, B) \neq \emptyset$. Then*

$$VI\big(C, aA + (1 - a)B\big) = VI(C, A) \cap VI(C, B), \quad \forall a \in (0, 1).$$

Furthermore, if $0 < \gamma < \min\{2\alpha, 2\beta\}$, we find that $I - \gamma(aA + (1 - a)B)$ is a nonexpansive mapping.

Remark 2.5 For every $i = 1, 2, \dots, N$ the mapping $A_i : C \to H$ be α_i-inverse strongly monotone mappings with $\eta = \min_{1,2,\dots,N}\{\alpha_i\}$ and $\bigcap_{i=1}^N VI(C, A_i) \neq \emptyset$. Then

$$VI\left(C, \sum_{i=1}^N a_i A_i\right) = \bigcap_{i=1}^N VI(C, A_i), \tag{7}$$

where $\sum_{i=1}^N a_i = 1$ and $0 < a_i < 1$ for every $i = 1, 2, \dots, N$. Moreover, we find that $\sum_{i=1}^N a_i A_i$ is monotone and is a μ-Lipschitz continuous mapping.

Proof It easy to see that $\sum_{i=k+1}^N \frac{a_i}{\prod_{j=1}^k (1-a_j)} A_i$ is η-inverse strongly monotone mappings with $\eta = \min\{\beta_i\}$ for each $i = 2, \dots, N$ and $k = 1, 2, \dots, N - 1$.

Take $N = 3$ and let $VI(C, A_1) \cap VI(C, A_2) \cap VI(C, A_3) \neq \emptyset$. By using Lemma 2.4, we have

$$\begin{aligned} VI(C, a_1A_1 + a_2A_2 + a_3A_3) &= VI\left(C, a_1A_1 + (1 - a_1)\left(\frac{a_2}{1 - a_1}A_2 + \frac{a_3}{1 - a_1}A_3\right)\right) \\ &= VI(C, A_1) \cap VI\left(C, \frac{a_2}{1 - a_1}A_2 + \frac{a_3}{1 - a_1}A_3\right) \\ &= VI(C, A_1) \cap VI(C, A_2) \cap VI(C, A_3), \end{aligned} \tag{8}$$

where $a_1, a_2, a_3 \in (0, 1)$ and $\sum_{i=1}^3 a_i = 1$.

Take $N = 4$ and let $\bigcap_{i=1}^{4} VI(C, A_i) \neq \emptyset$. By using Lemma 2.4 and (8), we have

$$\begin{aligned} VI(C, a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4) &= VI\left(C, (1-a_4)\left(\frac{a_1}{1-a_4}A_1 + \frac{a_2}{1-a_4}A_2 + \frac{a_3}{1-a_4}A_3\right) + a_4A_4\right) \\ &= VI\left(C, \frac{a_1}{1-a_4}A_1 + \frac{a_2}{1-a_4}A_2 + \frac{a_3}{1-a_4}A_3\right) \cap VI(C, A_4) \\ &= VI(C, A_1) \cap VI(C, A_2) \cap VI(C, A_3) \cap VI(C, A_4), \end{aligned} \tag{9}$$

where $a_1, a_2, a_3, a_4 \in (0,1)$ and $\sum_{i=1}^{4} a_i = 1$.
In the same way, if $\bigcap_{i=1}^{N} VI(C, A_i) \neq \emptyset$, we obtain

$$VI\left(C, \sum_{i=1}^{N} a_iA_i\right) = \bigcap_{i=1}^{N} VI(C, A_i), \tag{10}$$

where $a_i \in (0,1)$, for each $i = 1, 2, \ldots, N$, and $\sum_{i=1}^{N} a_i = 1$. □

Lemma 2.6 *In real Hilbert spaces H, the following well-known results hold:*
(i) For all $x, y \in H$ and $\alpha \in [0,1]$,

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,$$

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$ for all $x, y \in H$.

Lemma 2.7 ([17]) *Let C be a nonempty closed and convex subset of a real Hilbert space H. If $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, then the mapping $I - T$ is demiclosed at 0, i.e., if $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{x_n - Tx_n\}$ converges strongly to 0, then $x \in F(T)$.*

Lemma 2.8 ([17]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \quad \forall \geq 0,$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that
(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(2) $\limsup_{n\to 0} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| = \infty$.
Then $\lim_{n\to 0} s_n = 0$.

Lemma 2.9 ([17]) *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

Lemma 2.10 ([16]) *$u \in H$ is a solution of variational inclusion (1) if and only if $u = J_{M,\lambda}(u - \lambda Bu)$, $\forall \lambda > 0$, i.e.,*

$$VI(H,B,M) = F\big(J_{M,\lambda}(I - \lambda B)\big), \quad \forall \lambda > 0.$$

If $\lambda \in (0, 2\alpha]$, then $VI(H,B,M)$ is a closed convex subset in H.

The next lemma presents the association of the fixed point of a nonlinear mapping and the solution of GSMVIP under suitable conditions on the parameters.

Lemma 2.11 *Let H be a real Hilbert space and let $A_G : H \to H$ be an α-inverse strongly monotone mapping. Let $M_A, M_B : H \to 2^H$ be multi-value maximum monotone mappings with $\Omega \neq \emptyset$. $x^* \in \Omega$ if and only if $x^* = Gx^*$, where $G : H \to H$ is a mapping defined by*

$$G(x) = J_{M_A,\lambda_A}(I - \lambda_A A_G)\big(bx + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x\big),$$

for all $x \in H$, $b \in (0,1)$ and $\lambda_A, \lambda_B \in (0, 2\alpha)$. Moreover, we see that G is a nonexpansive mapping.

Proof Let the conditions hold.

($\Rightarrow$) Let $x^* \in \Omega$, we have $x \in H$ such that $\theta \in A_G x^* + M_A x^*$ and $\theta \in A_G x^* + M_B x^*$, that is, $x^* \in VI(H, A_G, M_A)$ and $x^* \in VI(H, A_G, M_B)$.

From Lemma 2.10, we have $x^* \in F(J_{M_A,\lambda_A}(I - \lambda_A A_G))$ and $x^* \in F(J_{M_B,\lambda_B}(I - \lambda_B A_G))$.

It implies that

$$x^* = J_{M_A,\lambda_A}(I - \lambda_A A_G)x^* \tag{11}$$

and

$$x^* = J_{M_B,\lambda_B}(I - \lambda_B A_G)x^*. \tag{12}$$

By the definition of G, (11) and (12), we have

$$\begin{aligned} G(x^*) &= J_{M_A,\lambda_A}(I - \lambda_A A_G)\big(bx^* + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x^*\big) \\ &= x^*. \end{aligned}$$

($\Leftarrow$) Let $x^* = G(x^*)$. Applying the same method of Lemma 2.1 (2) in [16], we find that $J_{M_A,\lambda_A}(I - \lambda_A A_G)$ and $J_{M_B,\lambda_B}(I - \lambda_B A_G)$ are nonexpansive mappings.

Since $x^* = G(x^*)$, we have

$$x^* = G(x^*) = J_{M_A,\lambda_A}(I - \lambda_A A_G)\big(bx^* + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x^*\big).$$

Let $y \in \Omega$, we have $\theta \in A_G y + M_A y$ and $\theta \in A_G y + M_B y$.

From Lemma 2.10, it implies that

$y \in F(J_{M_A,\lambda_A}(I - \lambda_A A_G)) \cap F(J_{M_B,\lambda_B}(I - \lambda_B A_G))$. Then

$$\|x^* - y\|^2 = \big\|J_{M_A,\lambda_A}(I - \lambda_A A_G)\big(bx^* + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x^*\big) - y\big\|^2$$

$$\begin{aligned}
&= \left\| J_{M_A,\lambda_A}(I-\lambda_A A_G)\big(bx^* + (1-b)J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\big) \right.\\
&\quad \left. - J_{M_A,\lambda_A}(I-\lambda_A A_G)y \right\|^2 \\
&\le \left\| \big(bx^* + (1-b)J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\big) - y \right\|^2 \\
&= \left\| b\big(x^*-y\big) + (1-b)\big(J_{M_B,\lambda_B}(I-\lambda_B A_G)x^* - y\big) \right\|^2 \\
&= b\left\|x^*-y\right\|^2 + (1-b)\left\|J_{M_B,\lambda_B}(I-\lambda_B A_G)x^* - y\right\|^2 \\
&\quad - b(1-b)\left\|x^* - J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\right\|^2 \\
&\le b\left\|x^*-y\right\|^2 + (1-b)\left\|x^*-y\right\|^2 - b(1-b)\left\|x^* \right.\\
&\quad \left. - J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\right\|^2 \\
&= \left\|x^*-y\right\|^2 - b(1-b)\left\|x^* - J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\right\|^2. \qquad (13)
\end{aligned}$$

It implies that $\|x^* - J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\| = 0$.

That is, $x^* \in F(J_{M_B,\lambda_B}(I-\lambda_B A_G))$.

Since $x^* = G(x^*)$ and $x^* \in F(J_{M_B,\lambda_B}(I-\lambda_B A_G))$, we have

$$\begin{aligned}
x^* &= J_{M_A,\lambda_A}(I-\lambda_A A_G)\big(bx^* + (1-b)J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\big) \\
&= J_{M_A,\lambda_A}(I-\lambda_A A_G)\big(bx^* + (1-b)x^*\big) \\
&= J_{M_A,\lambda_A}(I-\lambda_A A_G)x^*.
\end{aligned}$$

Therefore $x^* \in F(J_{M_A,\lambda_A}(I-\lambda_A A_G))$.

From Lemma 2.10, $x^* \in F(J_{M_A,\lambda_A}(I-\lambda_A A_G))$ and $x^* \in F(J_{M_B,\lambda_B}(I-\lambda_B A_G))$, we have $\theta \in A_G x^* + M_A x^*$ and $\theta \in A_G x^* + M_B x^*$. Then $x^* \in \Omega$.

Applying (13), we can conclude that G is a nonexpansive mapping. □

We give some examples to support Lemma 2.11 and show that Lemma 2.11 is not true if some condition fails.

Example 2.12 Let $H = \mathcal{R}^2$ be the two dimensional space of real numbers with an inner product $\langle\cdot,\cdot\rangle : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ defined by $\langle \mathbf{x},\mathbf{y}\rangle = x\cdot y = x_1y_1 + x_2y_2$, for all $\mathbf{x} = (x_1,x_2) \in \mathcal{R}^2$, $\mathbf{y} = (y_1,y_2) \in \mathcal{R}^2$ and a usual norm $\|\cdot\| : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ give by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ for all $\mathbf{x} = (x_1,x_2) \in \mathcal{R}^2$ and $A_G : \mathcal{R}^2 \to \mathcal{R}^2$ defined by $A_G((x_1,x_2)) = (x_1 - 5, x_2 - 5)$. Let $M_A : \mathcal{R}^2 \to 2^{\mathcal{R}^2}$ be defined by $\{(2x_1 - 1, 2x_2 - 1)\}$ and $M_B : \mathcal{R}^2 \to 2^{\mathcal{R}^2}$ be defined by $\{(\frac{x_1}{2} + 2, \frac{x_2}{2} + 2)\}$. Show that $(2,2) \in F(G)$.

Solution. It is obvious that $\Omega = \{(2,2)\}$. Choose $\lambda_A = \frac{1}{2}$. From $M_A(x_1,x_2) = \{(2x_1 - 1, 2x_2 - 1)\}$ and the resolvent of M_A, $J_{M_A,\lambda_A}x = (I + \lambda_A M_A)^{-1}x$ for all $x = (x_1,x_2) \in \mathcal{R}^2$, we have

$$J_{M_A,\lambda_A}(x) = \frac{x}{2} + \frac{1}{4}, \qquad (14)$$

for all $x = (x_1,x_2) \in \mathcal{R}^2$. Choose $\lambda_B = 1$. From $M_B(x_1,x_2) = \{(\frac{x_1}{2} + 2, \frac{x_2}{2} + 2)\}$ and the resolvent of M_B, $J_{M_B,\lambda_B}x = (I + \lambda_B M_B)^{-1}x$ for all $x = (x_1,x_2) \in \mathcal{R}^2$, we have

$$J_{M_B,\lambda_B}(x) = \frac{2x}{3} - \frac{4}{3}, \qquad (15)$$

for all $x = (x_1, x_2) \in \mathcal{R}^2$. It is easy to see that A_G is 1-inverse strongly monotone. Choose $b = \frac{1}{4}$. From (14) and (15), we have

$$\begin{aligned} G(x) &= J_{M_A,\frac{1}{2}}\left(I - \frac{1}{2}A_G\right)\left(\frac{1}{4}x + \frac{3}{4}J_{M_B,1}(I - 1A_G)x\right) \\ &= \frac{x}{16} + \frac{30}{16}, \end{aligned}$$

for all $x = (x_1, x_2) \in \mathcal{R}^2$. By Lemma 2.11, we have $(2,2) \in F(G)$.

Example 2.13 Let $H = \mathcal{R}^2$ be the two dimensional space of real numbers with an inner product $\langle \cdot,\cdot \rangle : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ defined by $\langle \mathbf{x}, \mathbf{y} \rangle = x \cdot y = x_1y_1 + x_2y_2$, for all $\mathbf{x} = (x_1, x_2) \in \mathcal{R}^2, \mathbf{y} = (y_1, y_2) \in \mathcal{R}^2$ and a usual norm $\|\cdot\| : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ give by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ for all $\mathbf{x} = (x_1, x_2) \in \mathcal{R}^2$ and $A_G: \mathcal{R}^2 \to \mathcal{R}^2$ defined by $A_G((x_1, x_2)) = (x_1 - 5, x_2 - 5)$. Let $M_A : \mathcal{R}^2 \to 2^{\mathcal{R}^2}$ be defined by $\{(2x_1 - 1, 2x_2 - 1)\}$ and $M_B : \mathcal{R}^2 \to 2^{\mathcal{R}^2}$ be defined by $\{(\frac{x_1}{2} + 2, \frac{x_2}{2} + 2)\}$. Show that $(2,2) \notin F(G)$.

Solution. It is obvious that $\Omega = \{(2,2)\}$. Choose $\lambda_A = 2$. From $M_A(x_1, x_2) = \{(2x_1 - 1, 2x_2 - 1)\}$ and the resolvent of M_A, $J_{M_A,\lambda_A}x = (I + \lambda_A M_A)^{-1}x$ for all $x = (x_1, x_2) \in \mathcal{R}^2$, we have

$$J_{M_A,\lambda_A}(x) = \frac{x}{5} + \frac{2}{5}, \tag{16}$$

for all $x = (x_1, x_2) \in \mathcal{R}^2$. Choose $\lambda_B = 4$. From $M_B(x_1, x_2) = \{(\frac{x_1}{2} + 2, \frac{x_2}{2} + 2)\}$ and the resolvent of M_B, $J_{M_B,\lambda_B}x = (I + \lambda_B M_B)^{-1}x$ for all $x = (x_1, x_2) \in \mathcal{R}^2$, we have

$$J_{M_B,\lambda_B}(x) = \frac{x}{3} - \frac{8}{3}, \tag{17}$$

for all $x = (x_1, x_2) \in \mathcal{R}^2$. Choose $b = \frac{1}{4}$. From (16), (17) and A_G being 1-inverse strongly monotone, we have

$$\begin{aligned} G(x) &= J_{M_A,2}(I - 2A_G)\left(\frac{1}{4}x + \frac{3}{4}J_{M_B,4}(I - 4A_G)x\right) \\ &= \frac{x}{10} + \frac{9}{5}, \end{aligned}$$

for all $x = (x_1, x_2) \in \mathcal{R}^2$. By Lemma 2.11, we have $(2,2) \notin F(G)$.

Lemma 2.14 ([18]) *Let $\{\Gamma_n\}$ be a sequence of real numbers that do not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also we consider the sequence of integers $\{\tau(n)\}_{n \geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \tau(n) = \infty$ and, for all $n \geq n_0$,

$$\max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}.$$

Lemma 2.15 *Let H be a real Hilbert space, for every $i = 1,2,\dots,N$, let $A_i : H \to H$ be α_i-inverse strongly monotone mappings with $\eta = \min\{\alpha_i\}$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be a sequence generated by $y_n = P_C(I - \lambda \sum_{i=1}^N a_iA_i)x_n$, $Q_n = \{z \in H : \langle (I - \lambda \sum_{i=1}^N a_iA_i)x_n - y_n, y_n - z\rangle \geq 0\}$ and $x^* \in \bigcap_{i=1}^N VI(C,A_i)$ for all $i = 1,2,\dots,N$. Then the following inequality is fulfilled*:

$$\begin{aligned}&\left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^N a_iA_iy_n\right) - x^*\right\|^2 \\ &\quad\leq \|x_n - x^*\|^2 - \left(1 - \frac{\lambda}{\eta}\right)\left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^N a_iA_iy_n\right) - y_n\right\|^2 \\ &\qquad - \left(1 - \frac{\lambda}{\eta}\right)\|x_n - y_n\|^2,\end{aligned}$$

where $\sum_{i=1}^N a_i = 1, 0 < a_i < 1$ and $\lambda \in (0,\eta)$ with $\eta = \min_{i=1,2,\dots,N}\{\alpha_i\}$ for every $i = 1,2,\dots,N$.

Proof Since $x^* \in \bigcap_{i=1}^N VI(C,A_i)$, we have $x^* \in VI(C,A_i)$ for every $i = 1,2,\dots,N$ and (6), we obtain

$$\begin{aligned}&\left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^N a_iA_iy_n\right) - x^*\right\|^2 \\ &\quad\leq \left\| x_n - \lambda \sum_{i=1}^N a_iA_iy_n - x^*\right\|^2 \\ &\qquad - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^N a_iA_iy_n\right) - \left(x_n - \lambda \sum_{i=1}^N a_iA_iy_n\right)\right\|^2 \\ &\quad= \|x_n - x^*\|^2 \\ &\qquad - 2\lambda\left\langle P_{Q_n}\left(x_n - \lambda \sum_{i=1}^N a_iA_iy_n\right) - x^*, \sum_{i=1}^N a_iA_iy_n\right\rangle \\ &\qquad - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^N a_iA_iy_n\right) - x_n\right\|^2.\end{aligned} \tag{18}$$

From the monotonicity of $\sum_{i=1}^N a_iA_i$, we have

$$\begin{aligned}0 &\leq \left\langle \sum_{i=1}^N a_iA_iy_n - \sum_{i=1}^N a_iA_ix^*, y_n - x^*\right\rangle \\ &= \left\langle \sum_{i=1}^N a_iA_iy_n, y_n - x^*\right\rangle - \left\langle \sum_{i=1}^N a_iA_ix^*, y_n - x^*\right\rangle \\ &\leq \left\langle \sum_{i=1}^N a_iA_iy_n, y_n - x^*\right\rangle \\ &= \left\langle \sum_{i=1}^N a_iA_iy_n, y_n - P_{Q_n}\left(x_n - \lambda \sum_{i=1}^N a_iA_iy_n\right)\right\rangle\end{aligned}$$

$$+\left\langle \sum_{i=1}^{N} a_i A_i y_n, P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - x^*\right\rangle.$$

It implies that

$$\begin{aligned}
&\left\langle x^* - P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right), \sum_{i=1}^{N} a_i A_i y_n\right\rangle \\
&\quad \le \left\langle \sum_{i=1}^{N} a_i A_i y_n, y_n - P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right)\right\rangle.
\end{aligned} \tag{19}$$

From (18) and (19), we have

$$\begin{aligned}
&\left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - x^*\right\|^2 \\
&\quad \le \|x_n - x^*\|^2 + 2\lambda \left\langle \sum_{i=1}^{N} a_i A_i y_n, y_n - P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right)\right\rangle \\
&\qquad - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - x_n\right\|^2 \\
&\quad = \|x_n - x^*\|^2 - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\|^2 - \|y_n - x_n\|^2 \\
&\qquad - 2\left\langle P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n, y_n - x_n\right\rangle \\
&\qquad + 2\lambda \left\langle \sum_{i=1}^{N} a_i A_i y_n, y_n - P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right)\right\rangle \\
&\quad = \|x_n - x^*\|^2 - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\|^2 - \|y_n - x_n\|^2 \\
&\qquad + 2\left\langle x_n - y_n - \lambda \sum_{i=1}^{N} a_i A_i y_n, P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\rangle \\
&\quad = \|x_n - x^*\|^2 - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\|^2 - \|y_n - x_n\|^2 \\
&\qquad + 2\left\langle \left(I - \lambda \sum_{i=1}^{N} a_i A_i\right) x_n - y_n, P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\rangle \\
&\qquad + 2\left\langle \lambda \sum_{i=1}^{N} a_i A_i x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n, P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\rangle \\
&\quad \le \|x_n - x^*\|^2 - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\|^2 - \|y_n - x_n\|^2
\end{aligned}$$

$$+2\lambda\left\|\sum_{i=1}^{N}a_iA_ix_n-\sum_{i=1}^{N}a_iA_iy_n\right\|\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\|$$

$$\leq\|x_n-x^*\|^2-\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\|^2-\|y_n-x_n\|^2$$

$$+2\frac{\lambda}{\eta}\|x_n-y_n\|\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\|$$

$$=\|x_n-x^*\|^2-\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\|^2-\|y_n-x_n\|^2$$

$$+\frac{\lambda}{\eta}\left(\|x_n-y_n\|^2+\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\|^2\right)$$

$$=\|x_n-x^*\|^2-\left(1-\frac{\lambda}{\eta}\right)\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\|^2$$

$$-\left(1-\frac{\lambda}{\eta}\right)\|y_n-x_n\|^2. \tag{20}$$

□

3 Main result

In this section, we prove the strong convergence of the sequence acquired from the proposed iterative methods for finding a common element of the set of finite family variational inequalities problems and the set of solutions of the proposed problem.

Theorem 3.1 *Let H be a real Hilbert space. For $i=1,2,\dots,N$, let $A_i:H\to H$ be α_i-inverse strongly monotone mappings and let $A_G:H\to H$ be α_G-inverse strongly monotone mappings. Define the mapping $G:H\to H$ by $G(x)=J_{M_A,\lambda_A}(I-\lambda_AA_G)(bx+(1-b)J_{M_B,\lambda_B}(I-\lambda_BA_G)x)$ for all $x\in H$, $b\in(0,1)$ and $\lambda_A,\lambda_B\in(0,2\alpha_G)$. Assume that $\Gamma=\bigcap_{i=1}^{N}VI(C,A_i)\cap F(G)\neq\emptyset$. Let the sequence $\{y_n\}$ and $\{x_n\}$ be generated by $x_1,u\in H$ and*

$$\begin{cases}y_n=P_C(I-\lambda\sum_{i=1}^{N}a_iA_i)x_n,\\ Q_n=\{z\in H:\langle(I-\lambda\sum_{i=1}^{N}a_iA_i)x_n-y_n,y_n-z\rangle\geq0\},\\ x_{n+1}=\alpha_nu+\beta_nP_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)+\gamma_nGx_n,\end{cases} \tag{21}$$

where $\sum_{i=1}^{N}a_i=1, 0<a_i<1, \{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset[0,1]$ with $\alpha_n+\beta_n+\gamma_n=1$, $\lambda\in(0,\eta)$ with $\eta=\min_{i=1,2,\dots,N}\{\alpha_i\}$.

Suppose the following conditions hold:

(i) $\sum_{n=0}^{\infty}\alpha_n=\infty$, $\lim_{n\to\infty}\alpha_n=0$,

(ii) $0<c<\beta_n,\gamma_n\leq d<1$.

Then $\{x_n\}$ converges strongly to $x^*\in\Gamma$ where $x^*=P_\Gamma u$.

Proof We must show that $\{x_n\}$ is bounded. Let $z_n=P_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)$.

We consider

$$x_{n+1}=\alpha_nu+\beta_nz_n+\gamma_nGx_n$$

$$= \alpha_n u + (1-\alpha_n)\left(\frac{\beta_n z_n + \gamma_n G x_n}{1-\alpha_n}\right)$$
$$= \alpha_n u + (1-\alpha_n) t_n,$$

where $t_n = \frac{\beta_n z_n + \gamma_n G x_n}{1-\alpha_n}$. Letting $x^* \in \Gamma = \bigcap_{i=1}^N VI(C, A_i) \cap F(G)$, we have

$$\begin{aligned} \|t_n - x^*\|^2 &= \left\| \frac{\beta_n z_n + \gamma_n G x_n}{1-\alpha_n} - x^* \right\|^2 \\ &= \left\| \frac{\beta_n z_n + \gamma_n G x_n - (1-\alpha_n) x^*}{1-\alpha_n} \right\|^2 \\ &= \frac{\beta_n}{1-\alpha_n} \|z_n - x^*\|^2 + \frac{\gamma_n}{1-\alpha_n} \|G x_n - x^*\|^2 \\ &\quad - \frac{\beta_n \gamma_n}{(1-\alpha_n)^2} \|z_n - G x_n\|^2. \end{aligned} \tag{22}$$

From definition of x_{n+1} and (22), we consider

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n u + (1-\alpha_n) t_n - x^*\|^2 \\ &= \|\alpha_n (u - x^*) - (1-\alpha_n)(t_n - x^*)\|^2 \\ &= \alpha_n \|u - x^*\|^2 + (1-\alpha_n) \|t_n - x^*\|^2 - \alpha_n (1-\alpha_n) \|u - t_n\|^2 \\ &= \alpha_n \|u - x^*\|^2 + (1-\alpha_n) \left[\frac{\beta_n}{1-\alpha_n} \|z_n - x^*\|^2 \right. \\ &\quad \left. + \frac{\gamma_n}{1-\alpha_n} \|G x_n - x^*\|^2 - \frac{\beta_n \gamma_n}{(1-\alpha_n)^2} \|z_n - G x_n\|^2 \right] \\ &\quad - \alpha_n (1-\alpha_n) \|u - t_n\|^2 \\ &= \alpha_n \|u - x^*\|^2 + \beta_n \|z_n - x^*\|^2 + \gamma_n \|G x_n - x^*\|^2 \\ &\quad - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - G x_n\|^2 - \alpha_n (1-\alpha_n) \|u - t_n\|^2 \\ &\le \alpha_n \|u - x^*\|^2 + \beta_n \|z_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 \\ &\quad - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - G x_n\|^2 - \alpha_n (1-\alpha_n) \|u - t_n\|^2. \end{aligned} \tag{23}$$

By Lemma 2.15 and $\lambda \in (0,1)$, we have

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2. \tag{24}$$

From (23) and (24), we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \alpha_n \|u - x^*\|^2 + \beta_n \|z_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 \\ &\quad - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - G x_n\|^2 - \alpha_n (1-\alpha_n) \|u - t_n\|^2 \\ &\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\beta_n\gamma_n}{1-\alpha_n}\|z_n - Gx_n\|^2 - \alpha_n(1-\alpha_n)\|u-t_n\|^2 \\
&= \alpha_n\left\|u-x^*\right\|^2 + (1-\beta_n)\left\|x_n - x^*\right\|^2 \\
&\quad - \frac{\beta_n\gamma_n}{1-\alpha_n}\|z_n - Gx_n\|^2 - \alpha_n(1-\alpha_n)\|u-t_n\|^2 \\
&\le \alpha_n\left\|u-x^*\right\|^2 + (1-\beta_n)\left\|x_n - x^*\right\|^2 \\
&\quad \vdots \\
&\le \max\left\{\left\|u-x^*\right\|^2 + \left\|x_1 - x^*\right\|^2\right\}.
\end{aligned} \tag{25}
$$

By induction,

$$
\left\|x_{n+1} - x^*\right\|^2 \le \max\left\{\left\|u-x^*\right\|^2 + \left\|x_1 - x^*\right\|^2\right\},
$$

then $\{x_n\}$ is a bounded sequence.

We use

$$
\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 &\le \alpha_n\left\|u-x^*\right\|^2 + \beta_n\left\|z_n - x^*\right\|^2 + \gamma_n\left\|x_n - x^*\right\|^2 \\
&\quad - \frac{\beta_n\gamma_n}{1-\alpha_n}\|z_n - Gx_n\|^2 \\
&\le \alpha_n\left\|u-x^*\right\|^2 + \beta_n\left[\left\|x_n - x^*\right\|^2 - \left(1-\frac{\lambda}{\eta}\right)\|z_n - y_n\|^2\right. \\
&\quad \left. - \left(1-\frac{\lambda}{\eta}\right)\|x_n - y_n\|^2\right] + \gamma_n\left\|x_n - x^*\right\|^2 - \frac{\beta_n\gamma_n}{1-\alpha_n}\|z_n - Gx_n\|^2 \\
&= \alpha_n\left\|u-x^*\right\|^2 + (1-\alpha_n)\left\|x_n - x^*\right\|^2 - \beta_n\left(1-\frac{\lambda}{\eta}\right)\|z_n - y_n\|^2 \\
&\quad - \beta_n\left(1-\frac{\lambda}{\eta}\right)\|x_n - y_n\|^2 - \frac{\beta_n\gamma_n}{1-\alpha_n}\|z_n - Gx_n\|^2 \\
&\le \alpha_n\left\|u-x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \beta_n\left(1-\frac{\lambda}{\eta}\right)\|z_n - y_n\|^2 \\
&\quad - \beta_n\left(1-\frac{\lambda}{\eta}\right)\|x_n - y_n\|^2 - \frac{\beta_n\gamma_n}{1-\alpha_n}\|z_n - Gx_n\|^2.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
&\beta_n\left(1-\frac{\lambda}{\eta}\right)\|z_n - y_n\|^2 + \beta_n\left(1-\frac{\lambda}{\eta}\right)\|x_n - y_n\|^2 + \frac{\beta_n\gamma_n}{1-\alpha_n}\|z_n - Gx_n\|^2 \\
&\quad \le \alpha_n\left\|u-x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2.
\end{aligned} \tag{26}
$$

Let $S_n := \beta_n(1-\frac{\lambda}{\eta})\|z_n - y_n\|^2 + \beta_n(1-\frac{\lambda}{\eta})\|x_n - y_n\|^2 + \frac{\beta_n\gamma_n}{1-\alpha_n}\|z_n - Gx_n\|^2$.

Then we have

$$
S_n \le \alpha_n\left\|u-x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2. \tag{27}
$$

Now, we consider two possible cases:

Case 1. Put $\Gamma_n := \|x_n - x^*\|^2$ for all $n \in \mathcal{N}$.

Assume that there is $n_0 \geq 0$ such that, for each $n \geq n_0$, $\Gamma_{n+1} \leq \Gamma_n$.

In this case, $\lim_{n\to\infty} \Gamma_n$ exists and $\lim_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) = 0$.

Since $\lim_{n\to\infty} \alpha_n = 0$, it follows from (27) that $\lim_{n\to\infty} S_n = 0$.

Therefore, we have $\lim_{n\to\infty} \beta_n(1 - \frac{\lambda}{\eta})\|z_n - y_n\|^2 = 0$, $\lim_{n\to\infty} \beta_n(1 - \frac{\lambda}{\eta})\|x_n - y_n\|^2 = 0$ and $\lim_{n\to\infty} \frac{\beta_n\gamma_n}{1-\alpha_n}\|z_n - Gx_n\|^2 = 0$.

From the assumptions i), ii), we obtain

$$\lim_{n\to\infty} \|z_n - y_n\| = \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|z_n - Gx_n\| = 0. \tag{28}$$

Hence, we obtain

$$\|x_n - Gx_n\| \leq \|x_n - y_n\| + \|y_n - z_n\| + \|z_n - Gx_n\|.$$

From (28), we have

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{29}$$

We now show that $\limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle \leq 0$.

We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle u - x^*, x_{n_i} - x^*\rangle. \tag{30}$$

Because $\{x_n\}$ is a bounded sequence in H, there exists a subsequence of $\{x_n\}$ that converges weakly to an element in H. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$ where $w \in H$. Since $\lim_{n\to\infty} \|x_n - z_n\| = 0$, we have $z_{n_i} \rightharpoonup w$.

Since $\lim_{n\to\infty} \|x_n - y_n\| = 0$, $y_{n_i} \rightharpoonup w$.

Assume that $w \notin \bigcap_{i=1}^N VI(C, A_i)$. So, we have $w \notin F(P_C(I - \lambda \sum_{i=1}^N a_iA_i))$.

Then we have $w \neq P_C(I - \lambda \sum_{i=1}^N a_iA_i)w$. By the nonexpansiveness of $P_C(I - \lambda \sum_{i=1}^N a_iA_i)$, (28) and Opial's property, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \|x_{n_i} - w\| \\
&\quad < \liminf_{n\to\infty} \left\| x_{n_i} - P_C\left(I - \lambda \sum_{i=1}^N a_iA_i\right)w \right\| \\
&\quad \leq \liminf_{n\to\infty} \left(\|x_{n_i} - y_{n_i}\| + \left\| y_{n_i} - P_C\left(I - \lambda \sum_{i=1}^N a_iA_i\right)w \right\| \right) \\
&\quad \leq \liminf_{n\to\infty} \left(\|x_{n_i} - y_{n_i}\| \right. \\
&\qquad \left. + \left\| P_C\left(I - \lambda \sum_{i=1}^N a_iA_i\right)x_{n_i} - P_C\left(I - \lambda \sum_{i=1}^N a_iA_i\right)w \right\| \right) \\
&\quad \leq \liminf_{n\to\infty} \|x_{n_i} - w\|.
\end{aligned}$$

This is a contradiction; we have $w \in VI(C, \sum_{i=1}^{N} a_i A_i)$. From Remark 2.5, we have

$$w \in \bigcap_{i=1}^{N} VI(C, A_i). \tag{31}$$

Assume that $w \notin F(G)$. Then we have $w \neq Gw$. From (29) and Opial's property, we have

$$\begin{aligned}\liminf_{n\to\infty} \|x_{n_i} - w\| &< \liminf_{n\to\infty} \|x_{n_i} - Gw\| \\ &\leq \liminf_{n\to\infty} \big(\|x_{n_i} - Gx_{n_i}\| + \|Gx_{n_i} - Gw\|\big) \\ &\leq \liminf_{n\to\infty} \big(\|x_{n_i} - Gx_{n_i}\| + \|x_{n_i} - w\|\big) \\ &\leq \liminf_{n\to\infty} \|x_{n_i} - w\|.\end{aligned}$$

This is a contradiction; we have

$$w \in F(G). \tag{32}$$

From (31) and (32), we have $w \in \bigcap_{i=1}^{N} VI(C, A_i) \cap F(G)$.

Therefore, we get

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle = \lim_{i\to\infty} \langle u - x^*, x_{n_i} - x^* \rangle = \langle u - x^*, w - x^* \rangle \leq 0, \tag{33}$$

where $x^* = P_\Gamma u$.

Next, we show that $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Gamma u$.

From the nonexpansiveness of G, (22) and (24), we have

$$\begin{aligned}\|t_n - x^*\|^2 &= \frac{\beta_n}{1-\alpha_n}\|z_n - x^*\|^2 + \frac{\gamma_n}{1-\alpha_n}\|Gx_n - x^*\|^2 \\ &\quad - \frac{\beta_n \gamma_n}{(1-\alpha_n)^2}\|z_n - Gx_n\|^2 \\ &\leq \frac{\beta_n}{1-\alpha_n}\|z_n - x^*\|^2 + \frac{\gamma_n}{1-\alpha_n}\|Gx_n - x^*\|^2 \\ &\leq \frac{\beta_n}{1-\alpha_n}\|x_n - x^*\|^2 + \frac{\gamma_n}{1-\alpha_n}\|x_n - x^*\|^2 \\ &= \|x_n - x^*\|^2.\end{aligned} \tag{34}$$

From the definition of x_n, (34) and $x^* = P_\Gamma u$, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(u - x^*) - (1-\alpha_n)(t_n - x^*)\|^2 \\ &\leq (1-\alpha_n)\|t_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle.\end{aligned} \tag{35}$$

By applying Lemma 2.8 to (35), we find that the sequence $\{x_n\}$ converges strongly to x^*.

Case 2. Assume that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} \leq \Gamma_{n_i+1}$ for all $i \in \mathcal{N}$. In this case, we can define $\tau : \mathcal{N} \to \mathcal{N}$ by $\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}$.

Then we have $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. So, we have from (26)

$$\begin{aligned} &\beta_{\tau(n)}\left(1 - \frac{\lambda}{\eta}\right)\|z_{\tau(n)} - y_{\tau(n)}\|^2 + \beta_{\tau(n)}\left(1 - \frac{\lambda}{\eta}\right)\|x_{\tau(n)} - y_{\tau(n)}\|^2 \\ &\quad + \frac{\beta_{\tau(n)}\gamma_{\tau(n)}}{1 - \alpha_{\tau(n)}}\|z_{\tau(n)} - Gx_{\tau(n)}\|^2 \\ &\leq \alpha_{\tau(n)}\|u - x^*\|^2 + \|x_{\tau(n)} - x^*\|^2 - \|x_{\tau(n)+1} - x^*\|^2. \end{aligned}$$

Arguing as in Case 1, we have

$$\lim_{n\to\infty}\|z_{\tau(n)} - y_{\tau(n)}\| = \lim_{n\to\infty}\|x_{\tau(n)} - y_{\tau(n)}\| = \lim_{n\to\infty}\|z_{\tau(n)} - Gx_{\tau(n)}\| = 0. \tag{36}$$

Because $\{x_{\tau(n)}\}$ is a bounded sequence, there exists a subsequence $\{x_{\tau(n_j)}\}$ such that

$$\limsup_{n\to\infty}\langle u - x^*, x_{\tau(n)} - x^*\rangle = \lim_{i\to\infty}\langle u - x^*, x_{\tau(n)+1} - x^*\rangle.$$

Following the same argument as the proof of Case 1 for $\{x_{\tau(n_j)}\}$, we have

$$\limsup_{n\to\infty}\langle u - x^*, x_{\tau(n)+1} - x^*\rangle \leq 0$$

and

$$\|x_{\tau(n)+1} - x^*\|^2 \leq (1 - \alpha_{\tau(n)})\|x_{\tau(n)} - x^*\|^2 + 2\alpha_{\tau(n)}\langle u - x^*, x_{\tau(n)+1} - x^*\rangle,$$

where $\alpha_{\tau(n)} \to 0$, $\sum_{n=1}^{\infty}\alpha_{\tau(n)} = \infty$ and $\limsup_{n\to\infty}\langle u - x^*, x_{\tau(n)+1} - x^*\rangle \leq 0$.

Hence, by Lemma 2.8, we have $\lim_{n\to\infty}\|x_{\tau(n)} - x^*\| = 0$ and $\lim_{n\to\infty}\|x_{\tau(n)+1} - x^*\| = 0$

Therefore, by Lemma 2.14, we have

$$0 \leq \|x_n - x^*\| \leq \max\{\|x_{\tau(n)} - x^*\|, \|x_n - x^*\|\} \leq \|x_{\tau(n)+1} - x^*\|.$$

Hence, $\{x_n\}$ converge strongly to $x^* = P_\Gamma u$. This completes the proof of the main theorem. □

4 Application

In 2013, Kangtunyakarn [14] introduced a modification of the system of variational inequalities as follows: finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)z^*), x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \lambda_2 D_2)x^*, x - z^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{37}$$

where $D_1, D_2 : C \to H$ be two mappings, for every $\lambda_1, \lambda_2 \geq 0$ and $a \in [0, 1]$.

Let h be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$. The subdifferential ∂h of h is defined by

$$\partial h(x) = \{z \in H : h(x) + \langle z, u - x\rangle \leq h(u), \forall u \in H\}$$

for all $x \in H$. From Rockafellar [19], we find that ∂h is a maximal monotone operator. Let C be a nonempty closed convex subset of H and i_C be the indicator function of C, i.e.,

$$i_C = \begin{cases} 0; & \text{if } x \in C, \\ +\infty; & \text{if } x \notin C, \end{cases}$$

Then i_C is a proper, lower semicontinuous and convex function on H and so the subdifferential ∂i_C of i_C is a maximal monotone operator. The resolvent operator $J_{\partial i_C,r}$ of i_C for $\lambda > 0$, can be defined by $J_{\partial i_C,r}(x) = (I + \lambda \partial i_C)^{-1}(x), x \in H$. We have $J_{\partial i_C,r}(x) = P_C x$, for all $x \in H$ and $\lambda > 0$. As a special case, if $M_A = M_B = \partial i_C$ in Lemma 2.11, we find that $J_{M_A,\lambda_A} = J_{M_B,\lambda_B} = P_C$. So we obtain the following result.

Lemma 4.1 ([14]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be mappings. For every $\lambda_1, \lambda_2 > 0$ and $b \in [0,1]$, the following statements are equivalent:*

(a) $(x^*, z^*) \in C \times C$ is a solution of problem (37),

(b) x^* is a fixed point of the mapping $\widehat{G} : C \to C$, i.e., $x^* \in F(T)$, defined by

$$\widehat{G}(x) = P_C(I - \lambda_1 D_1)\big(bx + (1-b)P_C(I - \lambda_2 D_2)x\big), \tag{38}$$

where $z^* = P_C(I - \lambda_2 D_2)x^*$.

Theorem 4.2 *Let H be a real Hilbert space. For $i = 1, 2, \ldots, N$, let $A_i : H \to H$ be α_i-inverse strongly monotone mappings and let $A_G : H \to H$ be α_G-inverse strongly monotone mappings. Define the mapping $\widehat{G} : H \to H$ by (38). Assume that $\Gamma = \bigcap_{i=1}^N VI(C, A_i) \cap F(T) \neq \emptyset$. Let the sequence $\{y_n\}$ and $\{x_n\}$ be generated by $x_1, u \in H$ and*

$$\begin{cases} y_n = P_C(I - \lambda \sum_{i=1}^N a_i A_i)x_n, \\ Q_n = \{z \in H : \langle (I - \lambda \sum_{i=1}^N a_i A_i)x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda \sum_{i=1}^N a_i A_i y_n) + \gamma_n T x_n, \end{cases} \tag{39}$$

where $\sum_{i=1}^N a_i = 1, 0 < a_i < 1, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\lambda \in (0, \eta)$ with $\eta = \min_{i=1,2,\ldots,N}\{\alpha_i\}$.

Suppose the following conditions hold:

(i) $\sum_{n=0}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

(ii) $0 < c < \beta_n, \gamma_n \leq d < 1$.

Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$ where $x^* = P_\Gamma u$.

Proof Taking $J_{M_A,\lambda_A} = J_{M_B,\lambda_B} = P_C$ in Theorem 3.1, we obtain the desired conclusion. □

In order to apply our main result, we give the following lemma.

Lemma 4.3 ([14]) *Let C be a nonempty closed convex subset of real Hilbert space H. Let $T, S : C \to C$ be nonexpansive mappings. Define a mapping $B^A : C \to C$ by $B^A xj = T(aI + (1a)S)x$ for every $x \in C$ and $a \in (0,1)$. Then $F(B^A) = F(T) \cap F(S)$ and B^A is a nonexpansive mapping.*

We apply our Theorem 3.1, by using with Lemma 4.3 ([14]), to find a solution of the variational inclusion problem.

Lemma 4.4 *Let H be a real Hilbert space and let $A_G : H \to H$ be α_G-inverse strongly monotone mappings. Let $M_A, M_B : H \to 2^H$ be a multi-value maximum monotone mapping with $VI(H, A_G, M_A) \cap VI(H, A_G, M_B) \neq \emptyset$. Define a mapping $G : H \to H$ as in Lemma 2.11 for all $x \in H$, $a \in (0,1)$ and $\lambda_A, \lambda_B \in (0, 2\alpha_G)$. Then $F(G) = VI(H, A_G, M_A) \cap VI(H, A_G, M_B)$.*

Proof Let $x, y \in C$. From Lemma 2.11, we find that G is nonexpansive and $J_{M_A,\lambda_A}(I - \lambda_A A_G)$ and $J_{M_B,\lambda_B}(I - \lambda_B A_G)$ are nonexpansive. Since

$$G(x) = J_{M_A,\lambda_A}(I - \lambda_A A_G)\big(bx + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x\big)$$

and Lemma 4.3, we have

$$F(G) = F\big(J_{M_A,\lambda_A}(I - \lambda_A A_G)\big) \cap F\big(J_{M_B,\lambda_B}(I - \lambda_B A_G)\big).$$

By Lemma 2.10, we have

$$F(G) = VI(H, A_G, M_A) \cap VI(H, A_G, M_B). \qquad \square$$

Theorem 4.5 *Let H be a real Hilbert space. For $i = 1, 2, \ldots, N$, let $A_i : H \to H$ be α_i-inverse strongly monotone mappings and let $A_G : H \to H$ be α_G-inverse strongly monotone mappings. Define the mapping $G : H \to H$ by $G(x) = J_{M_A,\lambda_A}(I - \lambda_A A_G)(bx + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x)$ for all $x \in H$, $b \in (0,1)$ and $\lambda_A, \lambda_B \in (0, 2\alpha_G)$. Assume that $\Gamma = \bigcap_{i=1}^N VI(C, A_i) \cap VI(H, A_G, M_A) \cap VI(H, A_G, M_B) \neq \emptyset$. Let the sequence $\{y_n\}$ and $\{x_n\}$ be generated by $x_1, u \in H$ and*

$$\begin{cases} y_n = P_C(I - \lambda \sum_{i=1}^N a_i A_i)x_n, \\ Q_n = \{z \in H : \langle (I - \lambda \sum_{i=1}^N a_i A_i)x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda \sum_{i=1}^N a_i A_i y_n) + \gamma_n G x_n, \end{cases} \tag{40}$$

where $\sum_{i=1}^N a_i = 1, 0 < a_i < 1, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\lambda \in (0, \eta)$ with $\eta = \min_{i=1,2,\ldots,N}\{\alpha_i\}$.

Suppose the following conditions hold:

(i) $\sum_{n=0}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

(ii) $0 < c < \beta_n, \gamma_n \leq d < 1$.

Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$ where $x^* = P_\Gamma u$.

Proof From Lemma 4.4, and Theorem 3.1, we obtain the desired conclusion. $\square$

Remark 4.6 if $VI(H, A_G, M_A) \cap VI(H, A_G, M_B) \neq \emptyset$, then observe that $VI(H, A_G, M_A) \cap VI(H, A_G, M_B) = \Omega$.

5 Example and numerical results

In this section, we give an example supporting Theorem 3.1.

Example 5.1 Let $H = \mathcal{R}^2$ be the two dimensional space of real numbers with an inner product $\langle\cdot,\cdot\rangle : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ defined by $\langle x,y\rangle = x\cdot y = x_1y_1 + x_2y_2$ and the usual norm $\|\cdot\| : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ given by $\|x\| = \sqrt{x_1^2 + x_2^2}$ for all $x = (x_1,x_2) \in \mathcal{R}^2$. Let $C_1 = \{(x_1,x_2) \in H | -2x_1 + x_2 \le 1\}$ and $C_2 = \{(x_1,x_2) \in H | 4x_1 - 2x_2 \le 3\}$. Define the mapping $A_1 : C_1 \to \mathcal{R}^2$ by $A_1(x_1,x_2) = (\frac{3x_1}{2}, \frac{3x_2}{2})$. Define the mapping $A_2 : C_2 \to \mathcal{R}^2$ by $A_2(x_1,x_2) = (2x_1, 2x_2)$. Let the mapping $A_G : \mathcal{R}^2 \to \mathcal{R}^2$ be defined by $A_G(x_1,x_2) = (x_1+1, x_2+1)$. Let $C = C_1 \cap C_2$. We have

$$P_C(x_1,x_2) = \begin{cases} (-1999x_1 + 1000x_2 + 750, 4000x_1 - 1999x_2 - 1500); \\ \quad \text{if } -40x_1 + 20x_2 < -15, \\ (x_1,x_2); \\ \quad \text{if } -15 \le -40x_1 + 20x_2 \le 5, \\ (-1999x_1 + 1000x_2 - 250, 4000x_1 - 1999x_2 - 500); \\ \quad \text{if } -40x_1 + 20x_2 > 5. \end{cases}$$

Let $x_1, u \in \mathcal{R}^2$, $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ be generated by

$$\begin{cases} y_n = P_C(I - \lambda \sum_{i=1}^2 a_iA_i)x_n, \\ Q_n = \{z \in H : \langle (I - \lambda \sum_{i=1}^2 a_iA_i)x_n - y_n, y_n - z\rangle \ge 0\}, \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda \sum_{i=1}^2 a_iA_iy_n) + \gamma_n Gx_n, \end{cases} \tag{41}$$

where $\{\alpha_n\} = \frac{1}{12n}, \{\beta_n\} = \frac{5n-2}{12n}, \{\gamma_n\} = \frac{7n+1}{12n} \subset [0,1]$ and $a = 0.5 \in (0,1)$. Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0,0)$.

Solution. Since A_1, A_2 and A_G are $\frac{2}{3}, \frac{1}{2}$ and 1-inverse strongly monotone mappings, respectively, $\eta = \frac{1}{2}$. Choose $\lambda_A = \frac{1}{2}, \lambda_B = 1 \in (0, 2\alpha_G)$ and $b = \frac{1}{4}$, we obtain $G(x_1,x_2) = (\frac{x_1}{16}, \frac{x_2}{16})$. Choose $\lambda = \frac{1}{4} \in (0,\eta)$. It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1 and $(0,0) \in VI(C,A_1) \cap VI(C,A_2) \cap F(G)$. From Theorem 3.1, we can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0,0)$.

Example 5.2 Let $H = L_2([-1,1])$ with product $\langle f,g\rangle = \int_{-1}^1 f(t)g(t)\,dt$ and the associated norm given as $\|f\| := \sqrt{\int_{-1}^1 f(t)g(t)\,dt}$ for all $f,g \in L_2([-1,1])$. Take $C = \{x \in H : \|x\| \le 2\}$. Define the mapping $A_1 : L_2([-1,1]) \to L_2([-1,1])$ by $A_1(h(t)) = h(t) - 2t$ for all $t \in [-1,1]$. Define the mapping $A_2 : L_2([-1,1]) \to L_2([-1,1])$ by $A_2(h(t)) = \frac{3}{2}h(t) - 3t$ for all $t \in [-1,1]$. Let the mapping $A_G : L_2([-1,1]) \to L_2([-1,1])$ be defined by $A_G(h(t)) = h(t) - 5t$ for all $t \in [-1,1]$. We have

$$P_C(f(t)) = \begin{cases} f(t); & \text{if } \|f(t)\| \le 2, \\ \frac{2f(t)}{\|f(t)\|}; & \text{if } \|f(t)\| > 2. \end{cases}$$

Let $i = 1,2, x_1, u \in \mathcal{R}^2$, $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ be generated by (21) where $\{\alpha_n\} = \frac{1}{12n}, \{\beta_n\} = \frac{5n-2}{12n}, \{\gamma_n\} = \frac{7n+1}{12n} \subset [0,1]$ and $a = 0.4 \in (0,1)$. Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to $2t$.

Table 1 Detailed analysis of computational methods (21) and (3) for Example 1 with $\mathbf{u} = (5,5)$, $N = 15$, $E(x_1^n) = \|x_1^{n+1} - x_1^n\|, n \in N_0$ and $E(x_2^n) = \|x_2^{n+1} - x_2^n\|, n \in N_0$

	Iterative (21)		Iterative (3)	
n	$E(\mathbf{x}_1^n)$	$E(\mathbf{x}_2^n)$	$E(\mathbf{x}_1^n)$	$E(\mathbf{x}_2^n)$
1	1.0000	2.0000	1.0000	2.0000
2	0.6468	0.8770	0.6497	0.8828
3	0.3961	0.4630	0.3995	0.4681
4	0.2619	0.2826	0.2646	0.2862
⋮	⋮	⋮	⋮	⋮
15	0.0472	0.0457	0.0476	0.0476

Table 2 Detailed analysis of computational methods (21) and (3) for Example 1 with $\mathbf{u} = 3t$, $N = 15$ and $E(x_n) = \|x_{n+1} - x_n\|, n \in N_0$

n	$E(\mathbf{x}_n)$: Algorithm (21)	$E(\mathbf{x}_n)$: Algorithm (3)
1	0.7626	0.7626
2	0.1291	0.1221
3	0.0480	0.0492
4	0.0208	0.0226
⋮	⋮	⋮
15	0.0006	0.0007

Solution. Since A_1, A_2 and A_G are $\frac{1}{2}, \frac{1}{3}$ and 1-inverse strongly monotone mappings, respectively, $\eta = \frac{1}{2}$. Choose $\lambda_A = \frac{1}{2}, \lambda_B = 1 \in (0, 2\alpha_G)$ and $b = \frac{1}{4}$, we obtain $G(h(t)) = \frac{h(t)}{16}$. Choose $\lambda = \frac{1}{4} \in (0,\eta)$. It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1 and $2t \in VI(C,A_1) \cap VI(C,A_2) \cap F(G)$. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $2I$.

Example 5.3 Let $f : H \to \mathcal{R}$ be a convex function. Consider the following convex optimization problem:

$$\min_{x\in H} f(x) \tag{42}$$

and

$$\min_{x\in H} g(x) \tag{43}$$

It is well known that $x^* \in C$ solves (42) and (43) if and only if $x^* \in C$ satisfies the following variational inequalities:

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{44}$$

and

$$\langle \nabla g(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{45}$$

that is, $x^* \in VI(C,\nabla f) \cap VI(C,\nabla g)$. Let $H = \mathcal{R}$. Take $C = [1,10]$. Define the mapping $f : [1,10] \to \mathcal{R}$ by $f(x) = \frac{(x-1)^2}{3} + 1$. Define the mapping $g : [1,10] \to \mathcal{R}$ by $g(x) = \frac{x^2}{2} - \ln x - \frac{1}{2}$.

Table 3 Detailed analysis of computational methods (21) and (3) for Example 1 with $\mathbf{u} = 3$, $N = 15$ and $E(x_n) = \|x_{n+1} - x_n\|$, $n \in N_0$

n	$E(\mathbf{x}_n)$: Algorithm (21)	$E(\mathbf{x}_n)$: Algorithm (3)
1	2.9044	2.7500
2	0.7088	0.7428
3	0.2200	0.2681
4	0.0762	0.1082
⋮	⋮	⋮
15	0.0012	0.0015

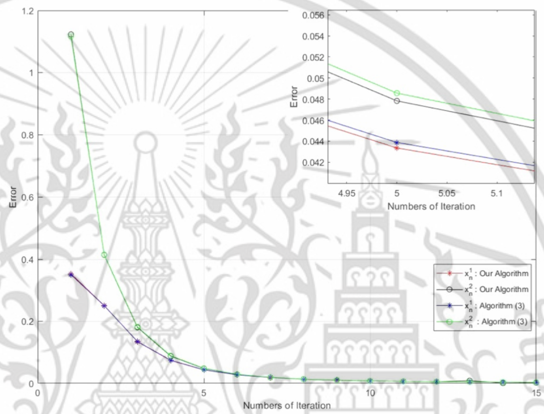

Figure 1 Comparison between Algorithms (21) and (3) for Example 1 with $\mathbf{u} = (5, 5)$ and $N = 15$

Let $x_1, u \in \mathcal{R}^2$. From (21), we find that $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are generated by

$$\begin{cases} y_n = P_C(I - \lambda(a_1\nabla f + a_2\nabla g))x_n, \\ Q_n = \{z \in H : \langle (I - \lambda(a_1\nabla f + a_2\nabla g))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda(a_1\nabla f + a_2\nabla g)y_n) + \gamma_n G x_n, \end{cases} \tag{46}$$

where $\{\alpha_n\} = \frac{1}{12n}, \{\beta_n\} = \frac{5n-2}{12n}, \{\gamma_n\} = \frac{7n+1}{12n} \subset [0,1]$ and $a = 0.5 \in (0,1)$. Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to 1.

Solution. Since f and g are convex and differentiable with $f'(x) = \frac{2(x-1)}{3}$ and $g'(x) = x - \frac{1}{x}$. It implies that ∇f and ∇g are $\frac{2}{3}$ and 1-inverse strongly monotone mappings, respectively. Choose $\eta = \frac{1}{2}$, $\lambda_A = \frac{1}{2}, \lambda_B = 1 \in (0, 2\alpha_G)$ and $b = \frac{1}{4}$, we obtain $G(x) = \frac{x}{12} + \frac{11}{12}$. Choose $\lambda = \frac{1}{4} \in (0, \eta)$. It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1 and $1 \in VI(C, \nabla f) \cap VI(C, \nabla g) \cap F(G)$. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 1.

Remark 5.4 According to Tables 1–3 and Figs. 1–3, it is shown that our Algorithm (21) converges to an element of the set $\bigcap_{i=1}^{N} VI(C, A_i) \cap F(G)$ at a faster rate than Algorithm (3). Therefore, our algorithm is more efficient.

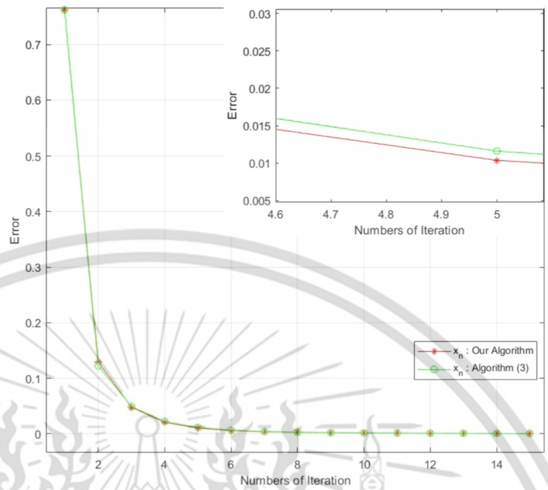

Figure 2 Comparison between Algorithms (21) and (3) for Example 2 with $\mathbf{u} = 3t$ and $N = 15$

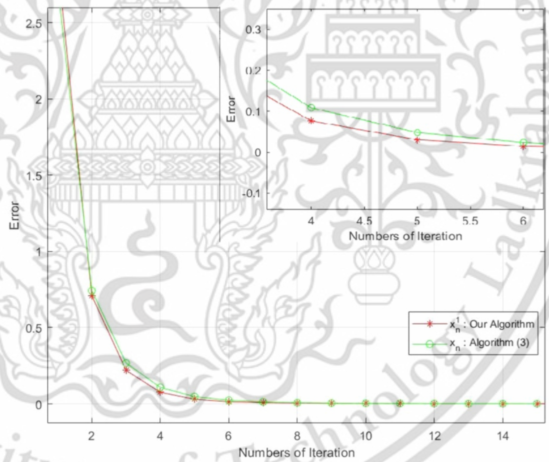

Figure 3 Comparison between Algorithms (21) and (3) for Example 3 with $\mathbf{u} = 3$ and $N = 15$

6 Conclusion

In this paper, we have proposed a new problem, called a generalized system of modified variational inclusion problems (GSMVIP). This problem can be reduced to a classical variational inclusion problem and a classical variational inequalities problem. Moreover, we study the half-space

$$Q_n = \left\{ z \in H : \left\langle \left(I - \lambda \sum_{i=1}^{N} a_i A_i \right) x_n - y_n, y_n - z \right\rangle \geq 0 \right\},$$

which can be reduced to T_n in Algorithm (3). In order to solve the GSMVIP and the set of a finite family of variational inequalities problem, we have presented a new subgradient extragradient algorithm which uses Q_n and show that it converges to a solution of the GSMVIP and the set of a finite family of variational inequalities problem under suitable conditions. Therefore, our algorithm improves the algorithm proposed by Censor et al. [12]. The efficiency of the proposed algorithm has also been illustrated by several numerical experiments.

Acknowledgements
This work is supported by King Mongkut's Institute of Technology Ladkrabang.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The two authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 16 September 2020 Accepted: 5 March 2021 Published online: 19 March 2021

References
1. Aubin, J.P., Ekeland, I.: Applied Nonlinear Analysis. Wiley, New York (1984)
2. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Program. **63**, 123–145 (1994)
3. Dafermos, S.C.: Traffic equilibrium and variational inequalities. Transp. Sci. **14**, 42–54 (1980)
4. Dafermos, S.C., Mckelvey, S.C.: Partitionable variational inequalities with applications to network and economic equilibrium. J. Optim. Theory Appl. **73**, 243–268 (1992)
5. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Ekon. Mat. Met. **12**, 747–756 (1976)
6. Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. Numer. Algorithms **59**, 301–323 (2012)
7. Hieu, D.V., Thong, D.V.: New extragradient like algorithms for strongly pseudomonotone variational inequalities. J. Glob. Optim. **70**, 385–399 (2018)
8. Hieu, D.V., Thong, D.V.: A new projection method for a class of variational inequalities. Appl. Anal. **98**(13), 2423–2439 (2019)
9. Malitsky, Y.V., Semenov, V.V.: A hybrid method without extrapolation step for solving variational inequalities problems. J. Glob. Optim. **61**, 193–202 (2015)
10. Malitsky, Y.V.: Projected reflected gradient methods for monotone variational inequalities. SIAM J. Optim. **25**, 502–520 (2015)
11. Yao, Y., Marino, G., Muglia, L.: A modified Korpelevich's method convergent to the minimum-norm solution of a variational inequality. Optimization **63**, 559–569 (2014)
12. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318–335 (2011)
13. Solodov, M.V., Svaiter, B.F.: A new projection method for variational inequalities problems. SIAM J. Control Optim. **37**, 765–776 (1999)
14. Kangtunyakarn, A.: An iterative algorithm to approximate a common element of the set of common fixed points for a finite family of strict pseudo-contractions and the set of solutions for a modified system of variational inequalities. Fixed Point Theory Appl. **2013**, 143 (2013)
15. Brezis, H.: Operateurs maximaux monotone et semi-groupes de contractions dans les espaces de Hilbert. Math. Stud. **5**, 759–775 (1973)
16. Zhang, S.S., Lee, J.H.M., Chan, C.K.: Algorithms of common solutions for quasi variational inclusion and fixed point problems. Appl. Math. Mech. **29**, 571–578 (2008)
17. Xu, H.K.: An iterative approach to quadric optimization. J. Optim. Theory Appl. **116**, 659–678 (2003)
18. Mainge, P.E.: A hybrid extragradient viscosity method for monotone operators and fixed point problems. SIAM J. Control Optim. **49**, 1499–1515 (2008)
19. Rockafellar, R.T.: On the maximal monotonicity of subdifferential mappings. Pac. J. Math. **33**, 209–216 (2005)
20. Xu, H.K.: Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. **298**, 279–291 (2004)
21. Maing, P.E.: Approximation method for common fixed points of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **325**, 469–479 (2007)

CARPATHIAN J. MATH.
Volume **39** (2023), No. 2,
Pages 359 - 369

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2023.02.02

Approximating G-variational inequality problem by G-subgradient extragradient method in Hilbert space endowed with graphs

AUTCHA ARAVEEPORN[1], ARAYA KHEAWBORISUT[2] and ATID KANGTUNYAKARN[2]

ABSTRACT. In this article, we introduce G-subgradient extragradient method for solving the G- variational inequality problem in Hilbert space endowed with a direct graph. Utilizing our mathematical tools, weak and strong convergence theorem are established for the proposed algorithm. In addition, we provide numerical experiments to illustrate the convergence behavior of our proposed algorithm.

1. INTRODUCTION

Let H be a real Hilbert space and D be a nonempty closed convex subset of a real Hilbert space H. The set of fixed points is denoted by $F(T) = \{x \in C : Tx = x\}$, where $T : D \to D$ is a mapping. The following symbols will be used throughout this research:

i) $G = (Eed(G), Ver(G))$ is a *directed graph* where $Ver(G)$ is *vertices set* and $Eed(G)$ is set of its edges with $\{(x, x) : x \in Ver(G)\} \subseteq Eed(G)$

ii) $Eed(G^{-1}) = \{(y, x) : (x, y) \in Eed(G)\}$.

The variational inequality problem (VIP) is to find a point $z^* \in D$ such that

$$\langle y - z^*, Bz^* \rangle \geqslant 0,$$

for all $y \in D$, where $B : D \to H$ is a mapping. The Variational inequality problems can be used to solve problems in engineering, economics, and physics; see more details in [2, 5, 9, 11].

The most famous technique for solving the problem (VIP) is the extragradient method suggested by Korpelevich [7]. This process must enumerate two projections onto the feasible set D in each iteration. If the set D is a half-space or a closed ball, effectiveness is completed in the result of the projection onto D. In the recent years, the extragradient method has approved meaningful awareness by numerous authors, who developed it in different ways, see, e.g. [2, 3, 5] and the several citations therein.

In [1], Censor et al. introduced a new extragradient method as follows:

$$\begin{cases} w_n = P_D(I - \lambda B)v_n \\ T^n = \{w \in D : \langle (I - \lambda B)v_n - w_n, w_n - w\rangle \geqslant 0\} \\ v_{n+1} = P_{T^n}(v_n - \lambda Bw_n), \end{cases} \tag{1.1}$$

for all $n \in \mathbb{N}$ and $\lambda > 0$. They proved that $\{v_n\}$ generated by (1.1) converges weakly to a solution of VIP. In this technique they have renovated the second projection in Korpelevich 's extragradient method with a projection onto a half-space, which is estimated explicitly. Such method is called *subgradient extragradient*.

Received: 14.03.2022. In revised form: 22.09.2022. Accepted: 27.09.2022
2010 *Mathematics Subject Classification.* 47H09, 47H10, 49J40, 90C99.
Key words and phrases. *G-subgradient extragradient method, G-variational inequality problem, Fixed point problem.*
Corresponding author: Atid Kangtunyakarn; beawrock@hotmail.com

Jachymski [4] was the first to analyze the fixed point problem in metric space endowed with graph and introduce the crucial conclusion in this space by integrating fixed point properties and graph theory, see more detail in [4].

Let $D = Ver(G)$ and the mapping $T : D \to D$ is called G-nonexpansive if the following conditions hold:

1) T is edge-preserving i.e., for each $x, y \in D$ such that $(x, y) \in Eed(G) \Rightarrow (Tx,Ty) \in Eed(G)$,

2) $\|Tx - Ty\| \leqslant \|x - y\|$, whenever $(x, y) \in Eed(G)$ for all $x, y \in D$.

Tiammee et al. were the first to prove the strong convergence theorem of a sequence generated by Halpern iteration for approximating fixed point problem of G-nonexpansive mapping in Hilbert space endowed with a directed graph. See more detail [10].

Using concepts related to the variational inequality problem and graph theory, Kangtunyakarn [6] introduced the G-variational inequality problem, which is to find a point $x^* \in D$ such that

$$\langle y - x^*, Bx^* \rangle \geqslant 0,$$

for all $y \in D$ with $(x^*, y) \in Eed(G)$ and $B : D \to H$ is a mapping, where $D = Ver(G)$. The set of all solution of such problem denoted by $G - Var(D, B)$. He proved strong convergence theorem to solve G-variational inequality problem.

By combining the concepts of subgradient extragadient method and graph theory in this research, we introduce G-subgradient extragadient method for approximating the solution of G-variational inequality problem. To use such a method, we introduce G-Half space by

$$T_G = \{w \in D : \langle (I - \lambda B)x - y, y - w \rangle \geqslant 0\},$$

where $\lambda > 0$, $B : D \to H$ is a mapping and $y = P_D(I - \lambda B)x$ for all $x \in H$ with $(w, x) \in Eed(G)$.

Example 1.1. Let $H = \mathbb{R}^2$ and $D = [-100, 100] \times [-100, 100]$ and metric projection $P_D : H \to D$ define by

$$P_D(z_1, z_2) = (\max\{\min\{z_1, 100\}, -100\}, \max\{\min\{z_2, 100\}, -100\}),$$

for all $z = (z_1, z_2) \in H$.

Let $B : D \to H$ define by $Bx = \left(\frac{v_1}{3}, \frac{v_2}{3}\right)$ for all $x = (v_1, v_2) \in D$ and $Ver(G) = D$, $Eed(G) = \{(u, v) : u = (u_1, u_2) \in [0, 100] \times [0, 100] \text{ and } v = (v_1, v_2) \in (300, \infty) \times (300, \infty)\}$. Putting $\lambda = 2$. From definitions of P_D and B, we have $P_D(1 - \lambda B)x = P_D\left(\frac{v_1}{3}, \frac{v_2}{3}\right)$ for all $x = (v_1, v_2) \in H$.

Let $(w, x) \in Eed(G)$, where $w = (w_1, w_2), x = (v_1, v_2)$. From definition of P_D, we have $P_D(I - \lambda B)x = (100, 100)$ and $T_G = [0, 100] \times [0, 100]$.

In this paper, motivated by the research [7, 1] and [6], we introduce a G-subgradient extragradient method for solving the G-variational inequality problem in Hilbert space endowed with a direct graph. Then we establish weak and strong convergence theorems under some proper conditions. Furthermore, we also give some examples to support our main result.

2. Preliminaries

This section collects well known definitions and lemmas as an essential tool for proving our main theorems.

Let D be a nonempty closed convex subset of a real Hilbert space H. We denote strong convergence and weak convergence by notations $\to$ and $\rightharpoonup$, respectively. For every $x \in H$, there exists a unique nearest point $P_D x \in D$ such that

$$||x - P_D x|| \leq ||x - y||, \quad \forall y \in D.$$

P_D is called metric projection of H onto D.
For each $x \in H$ and $y \in D$. It follows that

$$||x-y||^2 \geq ||x-P_Dx||^2 + ||y-P_Dx||^2. \tag{2.2}$$

Lemma 2.1. *Let P_D be the metric projection from H onto D. Then*
i) P_D is a nonexpansive mapping, i.e.

$$\|P_Dx - P_Dy\| \leqslant \|x-y\|, \ \forall\, x, y \in H.$$

ii) $y = P_Dx \Leftrightarrow \langle x-y, y-z\rangle \geqslant 0 \quad \forall x \in D$

Definition 2.1. [8] A subset X of $Ver(G)$ is called a *dominating set* if for every v belong to $Ver(G) - X$ there exists a point x belong to X such that (x, v) belong to $Eed(G)$ and we said that x *dominates* v or v *is dominated by* x. A subset Z of $Ver(G)$ is *dominated by* $v \in Ver(G)$ if $(v, z) \in Eed(G), \forall z \in Z$ and we said that X *dominates* v if $(x, v) \in Eed(G), \forall x \in X$.

Definition 2.2. [8] A graph G is called *transitive* if for every $x, y \in Ver(G)$ with$(x, y), (y, z) \in Eed(G)$, then $(x, z) \in Eed(G)$.

Property G [6] Vertices set $Ver(G) = D$ is said to have *Property G* if every sequence $\{a_n\}$ in D converging weakly to $x \in D$, there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $(a_{n_k}, x) \in Eed(G)\ \forall\, k \in \mathbb{N}$.

Definition 2.3. [6]Let $Ver(G) = D$. The mapping $B : D \to H$ is called *G-α-inverse strongly monotone*(G-α-ism) if there is $\alpha > 0$ such that

$$\langle Bx - By, x-y\rangle \geqslant \alpha \|Bx - By\|^2$$

$\forall x, y \in D$ with $(x, y) \in Eed(G)$.

The difference between G-α-ism and α-inverse strongly monotone is found in the reference [6].

Lemma 2.2. [6] *Let $Eed(G)$ be a convex and $Ver(G) = D$. Let $G = (Ver(G), Eed(G))$ be a direct graph and G be transitive with $Eed(G) = Eed(G^{-1})$. Let $B : D \to H$ is G-α-ism operator with $B^{-1}(0) \neq \emptyset$. Then $G - Var(D, B) = B^{-1}(0) = F\,(P_D(I - \lambda B))$, for all $\lambda > 0$.*

Lemma 2.3. [6] *Let $Eed(G)$ be a convex and $Ver(G) = D$. Let $G = (Ver(G), Eed(G))$ be a direct graph and let $B : D \to H$ is G-α-ism operator. For every $\forall\ \lambda \in (0, 2\alpha)$, if $F\,(P_D(I - \lambda B)) \times F\,(P_D(I - \lambda B)) \subseteq Eed(G)$, then $F\,(P_D(I - \lambda B))$ is closed and convex.*

Lemma 2.4. [9] *Let $\{a_n\}$ and $\{b_n\}$ be subset of $[0, \infty)$ satisfying*

$$a_{n+1} \leqslant a_n + b_n,$$

for all $n \in \mathbb{N}$.

i) if $\displaystyle\sum_{n=1}^{\infty} b_n < \infty$, *then* $\displaystyle\lim_{n\to\infty} a_n$ *exists*

ii) if $\displaystyle\sum_{n=1}^{\infty} b_n < \infty$ *and there exist a subsequence of $\{a_n\}$ converging to zero, then* $\displaystyle\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. [9] *Let $\{v_n\}$ be a sequence in H. Suppose that, for all $u \in D$,*

$$\|v_{n+1} - u\| \leqslant \|v_n - u\| + b_n,$$

for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then $\{P_Dv_n\}$ converges strongly to some $z \in D$.

Lemma 2.6. [11] *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$$

holds for every $y \in H$ with $x \neq y$.

3. MAIN RESULTS

Theorem 3.1. *Let $G, Ver(G), Eed(G), B$ as in Lemma 2.2. Assume that $G - Var(D,B) \neq \emptyset$ with $G - Var(D,B) \times G - Var(D,B) \subseteq Eed(G)$. Let $\{v_n\}$ be a sequence defined by $v_0 \in D$ and*

$$\begin{cases} w_n = P_D(I-\lambda B)v_n \\ T_G^n = \{w \in D : \langle (I-\lambda B)v_n - w_n, w_n - w\rangle \geqslant 0\} \\ v_{n+1} = P_{T_G^n}(v_n - \lambda B w_n), \end{cases}$$

for all $n \in N$ where $\lambda \in (0,\alpha)$ and T_G^n is G- Half space. Then sequence $\{v_n\}$ converges weakly to an element $\bar{x} \in G - Var(D,B)$ and the sequence $\{P_{G-Var(D,B)}v_n\}$ converges strongly to $\bar{x}$, where $G - Var(D,B)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0.

Proof. Let $v^* \in G - Var(D,B)$. Since $G - Var(D,B)$ dominates by v_n, we have $(v_{n,}, v^*) \in Eed(G)$ for all $n \in \mathbb{N}$. From Lemma 2.2, we have $v^* = P_D(I-\lambda B)v^*$.
Utilizing Definition 2.3, we have

$$\begin{aligned} \|w_n - v^*\|^2 &\leq \|v_n - v^*\|^2 - 2\lambda\langle Bv_n - Bv^*, v_n - \nu^*\rangle + \lambda^2\|Bv_n - Bv^*\|^2 \\ &\leq \|v_n - \nu^*\|^2 - 2\lambda\alpha\|Bv_n - Bv^*\|^2 + \lambda^2\|Bv_n - Bv^*\|^2 \\ &= \|v_n - v^*\|^2 - \lambda(2\alpha-\lambda)\|Bv_n - Bv^*\|^2 \\ &\leq \|v_n - v^*\|^2. \end{aligned}$$

Due to $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0, we have $(v_n, v_0), (v_0, w_n) \in Eed(G)$.
Exploiting of G is transitive, we get $(v_n, w_n) \in Eed(G)$.
From the assumption that $Eed(G) = Eed(G^{-1})$, we deduce that $(w_n, v_n) \in Eed(G)$.
From $(w_n, v_n), (v_{n,}, v^*) \in Eed(G)$ and the assumption that G is transitive, we get $(w_n, v^*) \in Eed(G)$.
From iteration of sequence $\{v_n\}$, we have

$$\begin{aligned} \|v_{n+1} - v^*\|^2 &\leqslant \|v_n - \lambda Bw_n - v^*\|^2 - \left\|v_n - \lambda Bw_n - P_{T_G^n}(v_n - \lambda Bw_n)\right\|^2 \\ &= \|v_n - v^*\|^2 - 2\lambda\langle Bw_n, v_n - v^*\rangle + \|\lambda Bw_n\|^2 \\ &\quad - \left\|v_n - P_{T_G^n}(v_n - \lambda Bw_n)\right\|^2 + 2\lambda\left\langle Bw_n, v_n - P_{T_G^n}(v_n - \lambda Bw_n)\right\rangle \\ &\quad - \|\lambda Bw_n\|^2 \\ &= \|v_n - v^*\|^2 - 2\lambda\langle Bw_n, P_{T_G}(v_n - \lambda Bw_n) - v^*\rangle \\ &\quad - \left\|v_n - P_{T_G^n}(v_n - \lambda Bw_n)\right\|^2 \end{aligned} \tag{3.3}$$

From $(w_n, v^*) \in Eed(G)$ and monotonicity of B, we have

$$\begin{aligned} 0 &\leqslant \langle Bw_n - Bv^*, w_n - v^*\rangle \\ &= \langle Bw_n, w_n - v^*\rangle - \langle Bv^*, w_n - v^*\rangle \\ &\leqslant \langle Bw_n, w_n - v^*\rangle \\ &= \left\langle Bw_n, w_n - P_{T_G^n}(v_n - \lambda Bw_n)\right\rangle + \left\langle Bw_n, P_{T_G^n}(v_n - \lambda Bw_n) - v^*\right\rangle. \end{aligned}$$

It implies that

$$-2\lambda\left\langle Bw_n, P_{T_G^n}(v_n-\lambda Bw_n)-v^*\right\rangle \leqslant 2\lambda\left\langle Bw_n, w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned}
\|v_{n+1}-v^*\|^2 &\leq \|v_n-v^*\|^2-2\lambda\left\langle Bw_n, P_{T_G^n}(v_n-\lambda Bw_n)-v^*\right\rangle \\
&\quad -\left\|v_n-P_{T_G^n}(v_n-\lambda B_{v_n})\right\|^2 \\
&\leq \|v_n-v^*\|^2+2\lambda\left\langle Bw_n, w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle \\
&\quad -\left\|v_n-P_{T_G^n}(v_n-\lambda B_{v_n})\right\|^2 \\
&= \|v_n-v^*\|^2+2\lambda\left\langle Bw_n, w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle \\
&\quad -\|v_n-w_n\|^2-2\left\langle v_n-w_n, w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle \\
&\quad -\left\|w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 \\
&= \|v_n-v^*\|^2-\|v_n-w_n\|^2-\left\|w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 \\
&\quad +2\left\langle \lambda Bw_n-v_n+w_n, w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\rangle \\
&= \|v_n-v^*\|^2-\|v_n-w_n\|^2-\left\|w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 \\
&\quad +2\left\langle (I-\lambda B)v_n-w_n, P_{T_G^n}(v_n-\lambda Bw_n)-w_n\right\rangle \\
&\quad +2\lambda\left\langle Bv_n-Bw_n, P_{T_G^n}(v_n-\lambda Bw_n)-w_n\right\rangle \\
&\leq \|v_n-v^*\|^2-\|v_n-w_n\|^2-\left\|w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 \\
&\quad +2\lambda\left\langle Bv_n-Bw_n, P_{T_G^n}(v_n-\lambda Bw_n)-w_n\right\rangle \\
&\leq \|v_n-v^*\|^2-\|v_n-w_n\|^2-\left\|w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 \\
&\quad +2\frac{\lambda}{\alpha}\|Bv_n-Bw_n\|\cdot\|P_{T_G^n}(v_n-\lambda Bw_n)-w_n\| \\
&\leq \|v_n-v^*\|^2-\|v_n-w_n\|^2-\left\|w_n-P_{T_G^n}(v_n-\lambda Bw_n)\right\|^2 \\
&\quad +\frac{\lambda}{\alpha}\left(\|v_n-w_n\|^2+\|P_{T_G^n}(v_n-\lambda Bw_n)-w_n\|^2\right) \\
&= \|v_n-v^*\|^2-\left(1-\frac{\lambda}{\alpha}\right)\|v_n-w_n\|^2 \\
&\quad -\left(1-\frac{\lambda}{\alpha}\right)\left\|P_{T_G^n}(v_n-\lambda Bw_n)-w_n\right\|^2. \qquad (3.5)
\end{aligned}$$

From Lemma 2.4, we have $\lim_{n\to\infty}\|v_n-v^*\|^2$ exists for all $v^*\in G-Var(D,B)$ and $\{v_n\}$ is a bounded sequence.
From (3.5) and $\lim_{n\to\infty}\|v_n-v^*\|^2$ exists, we have

$$\lim_{n\to 2}\|P_D(I-\lambda B)v_n-v_n\|=\lim_{n\to\infty}\|v_n-w_n\|=0.$$

Since $\{v_n\}$ is a bounded sequence, there is a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ converses weakly to $\bar{x}$.
Since D have a property G, we have $(v_{n_k},\bar{x})\in Eed(G)$.
Assume that $P_D(I-\lambda B)\bar{x}\neq\bar{x}$. By opial property and using the same method as (3.3), we

have

$$\begin{aligned}\limsup_{k\to\infty}\|v_{n_k}-\bar{x}\| &< \limsup_{k\to\infty}\|v_{n_k}-P_D(I-\lambda B)\bar{x}\| \\ &\leq \limsup_{k\to\infty}(\|v_{n_k}-P_D(I-\lambda B)v_{n_k}\|+\|P_D(I-\lambda B)v_{n_k}-P_D(I-\lambda B)\bar{x}\|) \\ &\leq \limsup_{k\to\infty}\|v_{n_k}-\bar{x}\|.\end{aligned}$$

Contradiction. So, we have $P_D(I-\lambda B)\bar{x}=\bar{x}$.
Let $y\in D$ with $(\bar{x},y)\in D$, then

$$\langle (I-\lambda B)\tilde{x}-\tilde{x},\bar{x}-y\rangle \geqslant 0.$$

It follows that

$$\langle y-\bar{x},B\tilde{x}\rangle \geqslant 0,$$

for all $y\in D$ with $(\bar{x},y)\in D$. Then, we have$\bar{x}\in G-Var(D,B)$.
Therefore $v_{n_k}\rightharpoonup \bar{x}\in G-Var(D,B)$ as $k\to\infty$.
Since$(v_{n_k},\bar{x})\in Eed(G)$ and using the same method as $\lim_{n\to\infty}\|v_n-v^*\|$ exists, we have $\lim_{k\to\infty}\|v_{n_k}-\bar{x}\|$ exists.
At the end of this theorem we demonstrate that $\{v_n\}$ converses weakly to $\bar{x}$. Assume that $v_{n_n}\rightharpoonup \hat{x}$ as $k\to\infty$ and $\bar{x}\neq\hat{x}$. Thank to the Opial's condition, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|v_n-\bar{x}\| &= \limsup_{k\to\infty}\|v_{n_k}-\bar{x}\| \\ &< \limsup_{k\to\infty}\|v_{n_k}-\hat{x}\| \\ &< \limsup_{k\to\infty}\|v_{n_k}-\bar{x}\| \\ &= \lim_{n\to\infty}\|v_n-\bar{x}\|.\end{aligned}$$

Contradiction. So, we get $\bar{x}=\hat{x}$. We can conclude that a sequence $\{v_n\}$ converges weakly to $\bar{x}\in G-Var(D,B)$.
Due to (3.5) and exploiting of Lemma 2.5, we have $\{P_{G-Var(D,B)}v_n\}$ converges strongly to $z\in G-Var(D,B)$.
From property of $P_{G-Var(D,B)}$, we have

$$\langle v_n-P_{G-Var(D,B)}v_n, P_{G-Var(D,B)}v_n-\bar{x}\rangle \geqslant 0.$$

Take $n\to\infty$, we have $\|z-\bar{x}\|=0$. So, we have $z=\bar{x}$. Therefore we can conclude that$\{P_{G-Var(D,B)}v_n\}$ converges strongly to $\bar{x}\in G-Var(D,B)$.This is ultimately the prove. □

4. Application

To resolved a fixed point problem in Hilbert space endowed with a direct graph by using G-subgradient extragradient method, we required the following lemma;

Lemma 4.7. [6] *Let D be a nonempty closed convex subset of a real Hilbert space H and let $G=(Ver(G),Eed(G))$ be a directed graph with $D=Ver(G)$ having property G. Let $Eed(G)$ be a convex set with $Eed(G)=Eed(G^{-1})$. Let $T:D\to D$ be G-nonexpansive mapping with $F(T)\neq\emptyset$ and $F(T)\times F(T)\subseteq Eed(G)$. Then*
i) $I-T$ is $G-\frac{1}{2}$- inverse strongly monotone,
ii) $G-Var(D,I-T)=F(T)$.

The following theorem is an immediate result of Theorem 3.1 and Lemma 4.7.

Theorem 4.2. *Let D be a nonempty closed convex subset of a real Hilbert space H and let $G = (Ver(G), Eed(G))$ be a directed graph with $D = Ver(G)$ having property G. Let $Eed(G)$ be a convex set and G be transitive with $Eed(G) = Eed\left(G^{-1}\right)$ and let $T : D \to D$ be G-nonexpansive mapping with $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq Eed(G)$. Let $\{v_n\}$ be a sequence defined by $v_0 \in D$ and*

$$\begin{cases} w_n = P_D(I - \lambda(I-T)v_n \\ T_G^n = \{w \in D : \langle (I - \lambda(I-T)v_n - w_n, w_n - v\rangle \geq 0\} \\ v_{n+1} = P_{T_G^n}\left(v_n - \lambda(I-T)w_n\right), \end{cases}$$

for all $n \in N$ where $\lambda \in (0, \alpha)$ and T_G^n is G- Half space. Then sequence $\{v_n\}$ converses weakly to an element $\bar{v} \in F(T)$ and the sequence $\left\{P_{F(T)}v_n\right\}$ converges strongly to $\bar{v}$, where $F(T)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0.

Following that, we provide an example to support our main result.

Example 4.2. Let $D = [-1, 1]$ and $G = (D, Eed(G))$ be a directed graph, where $Eed(G) = \{(x, y) : x, y \in [0, 1]\}$. Let the mappings $B : D \to \mathbb{R}$ define by $Bx = x - \frac{x^3}{4} - \frac{15}{32}$, and $S : D \to \mathbb{R}$ define by $Sx = \frac{x^3}{4} + \frac{15}{32}$, for all $x \in D$.
Suppose that the sequence $\{v_n\}$ is generated by $v_0 = 1$ and

$$\begin{cases} w_n = P_D(I - \lambda B)v_n \\ T_G^n = \{w \in D : \langle (I - \lambda B)v_n - w_n, w_n - w\rangle \geq 0\} \\ v_{n+1} = P_{T_G^n}\left(v_n - \lambda B w_n\right), \end{cases} \tag{4.6}$$

for all $n \in N$ where $\lambda \in (0, \alpha)$ and T_G^n is G- Half space. Then sequence $\{v_n\}$ converses weakly to an element of $\bar{v} \in G - Var(D, B)$ and the sequence $\left\{P_{G-Var(D,B)}v_n\right\}$ converges strongly to $\bar{v}$, where $G - Var(D, B)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0.
Solution. It is obvious that $\frac{1}{2} \in F(S)$, and $Eed(G) = Eed(G^{-1})$.
First, we show that S is a G-nonexpansive mapping. Let $x, y \in D$ with $(x, y) \in Eed(G)$. Then, we have $x, y \in [0, 1]$. Since x^3, y^3 , $\frac{5}{8} \in [0, 1]$ and $[0, 1]$ is a convex set, we have

$$Sx = \frac{1}{4}x^3 + \frac{3}{4}(\frac{5}{8}) \in [0, 1]$$

and

$$Sy = \frac{1}{4}y^3 + \frac{3}{4}(\frac{5}{8}) \in [0, 1].$$

From definition of S, we have

$$\begin{aligned} |Sx - Sy| &= |(\frac{x^3}{4} + \frac{15}{32}) - (\frac{y^3}{4} + \frac{15}{32})| = |\frac{x^3}{4} - \frac{y^3}{4}| \\ &= \frac{1}{4}|x^2 + xy + y^2||x - y| \leq \frac{1}{4}(3)|x - y| \\ &\leq |x - y|. \end{aligned}$$

Then $(Sx, Sy) \in Eed(G)$. Therefore S is a G-nonexpansive mapping.
Since $Bx = (I - S)x$, S is a G-nonexpansive mapping and Lemma 4.7, we have B is G-$\frac{1}{2}$-inverse strongly monotone. It is obvious that $G - Var(D, B) = \{\frac{1}{2}\}$.
Putting $\lambda = \frac{1}{4}$. From convexity of $[0, 1]$, we have

$$(I - \frac{1}{4}B)z \quad = \quad \frac{3}{4}z + \frac{1}{4}(\frac{z^3}{4} + \frac{15}{32}) \in [0, 1],$$

for all $z \in [0,1]$.
From definition of P_D, it follows that

$$P_D(I - \frac{1}{4}B)z \in [0,1], \tag{4.7}$$

for all $z \in [0,1]$.
Let $(w,z) \in Eed(G)$. From definition of T_G^n and (4.7), we have $T_G^n \subseteq [0,1]$.
Since $v_0 \in [0,1]$ and (4.7), we have

$$w_0 = P_D(I - \frac{1}{4}B)v_0 \in [0,1], \tag{4.8}$$

From $T_G^n \subseteq [0,1]$, we have

$$v_1 = P_{T_G^n}(v_0 - \frac{1}{4}Bw_0) \in [0,1], \tag{4.9}$$

Continue the method of (4.8) and (4.9), we have $w_n, v_n \in [0,1]$ for all $n \in \mathbb{N}$.
Since $v_0, \frac{1}{2}, v_n$ and $w_n \in [0,1]$, it follows that $(\frac{1}{2}, v_n)$, (v_n, v_0) and $(w_n, v_0) \in Eed(G)$.
We can conclude that $G - Var(D,B)$ dominates v_n, $\{v_n\}$ dominates v_0 and $\{w_n\}$ is dominated by v_0. All conditions of Example 4.2 satisfies Theorem 3.1, so we can conclude that sequence $\{v_n\}$ converses weakly to an element of $\frac{1}{2} \in G - Var(D,B)$ and the sequence $\{P_{G-Var(D,B)}v_n\}$ converges strongly to $\frac{1}{2}$.

n	$\mathbf{v}_n$
1	1.0000000
2	0.9349873
3	0.8699746
4	0.8108711
⋮	⋮
19	0.5159698
20	0.5101756

TABLE 1. Detailed analysis of computational methods (4.6) for Example 4.1 with $\mathbf{v_0} = 1$, $N = 20$.

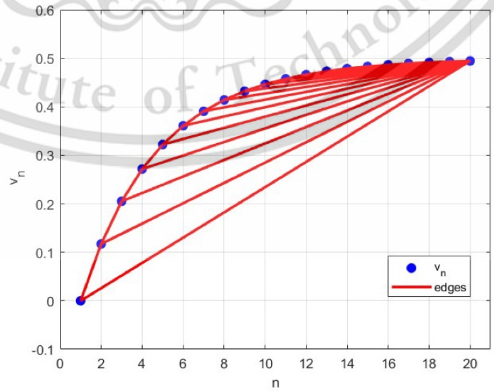

FIGURE 1. *The convergence behavious of* $\{\mathbf{v_n}\}$ with $v_0 = 1$ and $N = 20$.

Example 4.3. Let $D = [-5, 5]$ and $G = (D \times D, Eed(G))$ be a directed graph, where $Eed(G) = \{(x, y) : x = (x_1, x_2), y = (y_1, y_2) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]\}$. Let the mappings $B : D \times D \to \mathbb{R}^2$ define by $B(x_1, x_2) = (\frac{4x_1}{5} - \frac{8}{5}, \frac{x_2}{4})$, for all $x_1, x_2 \in D$.
Let metric projection $P_D : H \times H \to D \times D$ define by

$$P_D(z_1, z_2) = (\max\{\min\{z_1, 5\}, -5\}, \max\{\min\{z_2, 5\}, -5\}),$$

for all $z = (z_1, z_2) \in H \times H$.
Suppose that the sequence $\{v^n\}$ is generated by $v^0 = (v_1^0, v_2^0) = (1, 1)$ and

$$\text{(4.10)} \qquad \begin{cases} w^n = P_D(I - \lambda B)v^n \\ T_G^n = \{w \in D \times D \, \langle (I - \lambda B)v^n - w^n, w^n - w\rangle \geq 0\} \\ v^{n+1} = P_{T_G^n}(v^n - \lambda B w^n), \end{cases}$$

for all $n \in N$ where $v^n = (v_1^n, v_2^n), w^n = (w_1^n, w_2^n)$,$\lambda \in (0, \alpha)$ and T_G^n is G- Half space. Then sequence $\{v^n\}$ converses weakly to an element of $\bar{v} \in G - Var(D, B)$ and the sequence $\{P_{G-Var(D,B)}v^n\}$ converges strongly to $\bar{v}$, where $G - Var(D, B)$ dominates v^n, $\{v^n\}$ dominates v^0 and $\{w^n\}$ is dominated by v^0.

Solution. It is easy to see that $(2, 0) \in G - Var(D, B)$, and $Eed(G) = Eed(G^{-1})$. It is obvious that B is G-$\frac{1}{3}$-inverse strongly monotone.
Putting $\lambda = \frac{1}{4}$. From the definition of B, we have

$$\text{(4.11)} \qquad (I - \frac{1}{4}B)z = (\frac{4z_1}{5} + \frac{2}{5}, \frac{15z_2}{16}) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2],$$

for all $z = (z_1, z_2) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.

From definition of P_D, it follows that

$$\text{(4.12)} \qquad P_D(I - \frac{1}{4}B)z \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}0, 2],$$

for all $z = (z_1, z_2) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.
Let $(w, z) \in Eed(G)$. From definition of T_G^n and (4.12), we have $T_G^n \subseteq [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.
Since $v^0 \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$ and (4.12), we have

$$\text{(4.13)} \qquad w^0 = P_D(I - \frac{1}{4}B)v^0 \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2],$$

where $w^0 = (w_1^0, w_2^0) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$. From $T_G^n \subseteq [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$, we have

$$\text{(4.14)} \qquad v^1 = P_{T_G^n}(v^0 - \frac{1}{4}Bw^0) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2],$$

where $v^1 = (v_1^1, v_2^1) \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$.
Continue the method of (4.13) and (4.14), we have $w^n, v^n \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$ for all $n \in \mathbb{N}$, $v^n = (v_1^n, v_2^n), w^n = (w_1^n, w_2^n)$.
Since $v^0, (2, 0), v^n$ and $w^n \in [-\frac{1}{2}, 2] \times [-\frac{1}{2}, 2]$, it follows that $((2, 0), v^n)$, (v^n, v^0) and $(w^n, v^0) \in Eed(G)$.
We can conclude that $G - Var(D, B)$ dominates v^n, $\{v^n\}$ dominates v^0 and $\{w^n\}$ is dominated by v^0. All conditions of Example 4.3 satisfies Theorem 3.1, so we can conclude that sequence $\{v^n\}$ converses weakly to an element of $(2, 0) \in G - Var(D, B)$ and the sequence $\{P_{G-Var(D,B)}v^n\}$ converges strongly to $(2, 0)$.

n	$\mathbf{v}_1^n$	$\mathbf{v}_2^n$
1	1.0000000	1.0000000
2	1.1600000	0.9414062
3	1.2944000	0.8862457
4	1.4072960	0.8343173
$\vdots$	$\vdots$	$\vdots$
99	2.0000000	0.0028601
100	2.0000000	0.0026925

TABLE 2. Detailed analysis of computational methods (4.10) for Example 4.2 with $\mathbf{v^0} = (1, 1)$, $N = 100$.

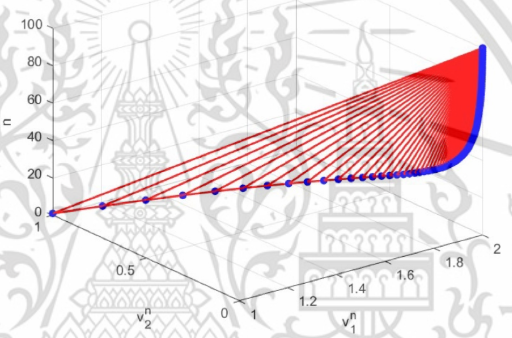

FIGURE 2. *The convergence behavious of* $\{\mathbf{v^n}\}$ with $v^0 = (1, 1)$ and $N = 100$.

Acknowledgments. This work is supported by King Mongkut's Institute of Technology Ladkrabang.

REFERENCES

[1] Censor, Y.; Gibali, A.; Reich, S. The subgradient extragradient method for solving variational inequalities in Hilbert spaces. *J. Optim. Theory Appl.* **148** (2011), 318–335.

[2] Ceng L. C.; Wang, C. Y.; Yao, J. C. Strong convergence theorems by a relaxed extragradi- ent method for a general system of variational inequalities. *Math Meth Oper Res.* **67** (2008), 375–390.

[3] Hieu, D. V.; Thong, D. V. New extragradient like algorithms for strongly pseudomonotone variation-alinequalities. *J. Glob. Optim.* **70** (2018), 385–399.

[4] Jachymski, J. The contraction principle for mappings on a metric space with a graph. *Proc. Am. Math. Soc.* **136** (2008), no. 4, 1359–1373.

[5] Jouymandi, Z.; Moradlou, F. Extragradient methods for solving equilibrium problems, variational inequalities and fixed point problems. *Numer Funct Anal Optim.* **38** (2017), 185–1409.

[6] Kangtunyakarn, A. The variational inequality problem in Hilbert spaces endowed with graphs. *J. Fixed Point Theory Appl.* **22** (2020), no. 4, DOI10.1007/s11784-019-0736-5

[7] Korpelevich, G. M. The extragradient method for finding saddle points and other problems. *Ekon Mat Metody.* **12** (1976), 747–756.

[8] Pang, C.; Zhang, R.; Zhang, Q.; Wang, J. Dominating sets in directed graphs. *Inf. Sci.* **180** (2020), 3647–3652.

[9] Saechou, K.; Kangtunyakarn, A. The subgradient extragradient method for approximation of fixed-point problem and modification of equilibrium problem. *Comput. Appl. Math.* **40** (2021), no. 246 DOI10.1007/s40314-021-01659-4

[10] Tiammee, J.; Kaewkhao, A.; Suantai, S. On Browder 's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs. Fixed Point Theory Appl. 187, 1-25 (2015).

[11] Xu, H. K. An iterative approach to quadric optimization. *J. Optim. Theory Appl.* **116** (2003), 659–678.

[1]DEPARTMENT OF STATISTICS, SCHOOL OF SCIENCE
KING MONGKUT's INSTITUTE OF TECHNOLOGY LADKRABANG
10520, BANGKOK, THAILAND
Email address: autcha.ar@kmitl.ac.th

[2]DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE
KING MONGKUT's INSTITUTE OF TECHNOLOGY LADKRABANG
10520, BANGKOK, THAILAND
Email address: araya.kheaw@gmail.com
Email address: beawrock@hotmail.com

Author Biography

Name	Miss Araya Kheawborisut
Date of Birth	9^{th} July 1992
Address	24 Moo. 2, Tambol Bangplagod, Ongkharak District, Nakhonnayok Province, Thailand, 26120.
Education	(2015) Bachelor of Science in Applied Mathematics GPA 3.37 (Second-Class Honors), King Mongkut's Institute of Technology Ladkrabang (2017) Master of Science in Applied Mathematics GPA 3.62 King Mongkut's Institute of Technology Ladkrabang (2016) Doctor of Philosophy in Applied Mathematics GPA 4.00 King Mongkut's Institute of Technology Ladkrabang
Scholarship	KMITL Doctoral Scholarships

Academic Publications

1. Kheawborisut, A., Suantai, S., Kangtunyakarn, A. 2017. The existence theorem of a new multi-valued mapping in metric space endowed with graph. *Carpathian Journal of Mathematics.* 33(2), 191-198.
2. Kheawborisut, A., Kangtunyakarn, A. 2021. Modified subgradient extragradient method for system of variational inclusion problem and finite family of variational inequalities problem in real Hilbert space. *Journal of Inequalities and Applications.* 2021(1).
3. Kheawborisut, A., Paokanta, S., Senasukh, J., Park, C. 2022. Ulam stability of hom-ders in fuzzy Banach algebras. *AIMS Mathematics*. 7(9), 16556–16568
4. Kheawborisut, A., Kangtunyakarn, A. 2022. Algorithms of common solutions to modified generalized system of variational inclusion problem and hierarchical fixed point problem. *Filomat*. 36(9), 3173-3188
5. Araveeporn, A., Kheawborisut, A., Kangtunyakarn, A. 2023. Approximating G-variational inequality problem by G-subgradient extragradient method in Hilbert space endowed with graphs. *Carpathian Journal of Mathematics.* 39(2), 359-369. https://doi.org/10.37193/CJM.2023.02.02

Conference proceeding

Kheawborisut, A., Kangtunyakarn, A. 2017. A new method for equilibrium problems and a finite family of pseudocontractive mapping. *The Nation and International Graduate Research Conference.* pmp4.