THE METHOD TO SOLVE VARIATIONAL INEQUALITY PROBLEM WITHOUT SOME CONDITION ON THE SET OF SOLUTIONS OF VARIATIONAL INEQUALITY PROBLEMS

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Abstract

In this research, we introduce some method for approximating the result of variational inequality problem and fixed-point problem by using the new condition and method. That is, we omit the use of some popular lemmas, which is the most widely prove strong convergence theorem as basis. Finally, we give a numerical example to support our some results.

Keywords : Fixed-point problem, Variational inequality problem, Strong convergence

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Chapter 1

Introduction

Throughout this thesis, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. We use "$\rightharpoonup$" for weak convergence and "$\to$" for strong convergence.

1.1 Research motivation

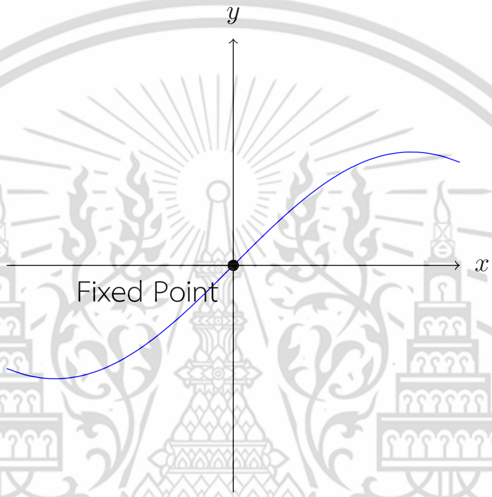

Figure 1.1: Graph with a fixed point

The Fixed-point theory is widely recognized as an exceptionally potent and fundamental tool in the study of nonlinear phenomena, spanning applications in diverse fields such as biology, chemistry, economics, engineering, game theory, computer science, physics, geometry, and image processing. Fixed point techniques have emerged as a vital cornerstone of modern scientific research. For example, [1] [2] [3] [4] [5] have all utilized these techniques.

The concept of fixed points and the related problem of finding them have been studied extensively in various mathematical disciplines. The origins of the fixed point problem can be traced back to the early 19th century, with significant contributions from mathematicians such as Augustin-Louis Cauchy and Karl Weierstrass [6].

Cauchy's formulation of the fixed point problem in 1823 [7] established the idea of finding points that remain unchanged under a given transformation. Although Cauchy's fixed point theorem did not include a proof, it laid the foundation for further investigations into the problem.

A comprehensive proof of the existence of fixed points in a more general setting was provided by Stefan Banach in 1922 [35]. Banach's fixed point theorem, also known as the contraction mapping theorem, has become a cornerstone of fixed point theory. The theorem ensures the existence and uniqueness of fixed points in complete metric spaces and has found wide application in various mathematical fields.

The development of fixed point theory continued with significant contributions from mathematicians such as Maurice Fréchet [36], Shizuo Kakutani [37] , and Leray-Schauder [38]. Their work expanded the theory to include more general spaces and non-linear mappings.

Over the past few years, mathematicians have made significant advancements in the development and extensive study of fixed point theory. A fixed point of a mapping T refers to a point x belonging to a set X that satisfies the condition $x = Tx$, where $T : X \to X$ represents a nonlinear mapping.

In this thesis, the notation $F(T)$ is employed to represent the fixed point set of a mapping T. Consequently, fixed point theory revolves around establishing conditions for the set X and the mapping $T : X \to X$ that ensure the existence and uniqueness of fixed points.

Moreover, researchers have dedicated their efforts to investigating the structure of the fixed point set and exploring methods for approximating fixed points.

Variational Inequality problems have a rich and significant history, originating from the fields of mathematical optimization and game theory. The concept was first introduced by Stampacchia in 1964 [40], and since then, researchers have made continuous refinements to the problem formulation.

In 1976, Lions and Stampacchia laid the foundation for the fundamental theory of Variational Inequality, providing a comprehensive framework for studying and solving these problems [41]. Their pioneering work set the stage for subsequent advancements in the field.

Over the years, numerous researchers have made noteworthy contributions to the theory of Variational Inequality and its practical applications. Pang and Gabriel (1978) [42] introduced the concept of complementarity problems as a special case of Variational Inequality, thereby expanding its applicability.

Facchinei and Pang (2003) [43] developed the theory of finite-dimensional Variational Inequality problems, which offered valuable insights into solving such problems in finite-dimensional spaces.

In recent decades, Variational Inequality has found extensive usage in various disciplines, including economics, engineering, and computer science. Researchers have devoted their efforts to developing efficient numerical algorithms to tackle Variational Inequality problems, and notable contributions in this area have been made by scholars such as Bendersky and Guler (2010) [44].

Today, Variational Inequality problems continue to be an active and thriving area of research. Ongoing endeavors are aimed at advancing theoretical understanding, devising innovative algorithms, and exploring new applications for this versatile framework.

The continuous progress in this field holds the promise of enhancing our comprehension of complex systems and contributing to practical solutions across diverse domains.

Let $B : C \to H$ be a mapping. The variational inequality problem is to find a point $u \in C$ such that

$$\langle Bu, v - u\rangle \geq 0,$$

for all $v \in C$. And $VI(C, B)$ denoted the set of solutions of the variational inequality problem. Moreover, if B is strongly monotone and Lipschitzian on C, then it is well known that the $VI(C, B)$ has a unique solution.

The concept of Lipschitz continuity is named after the German mathematician Rudolf Lipschitz, who introduced it in the 19th century. However, the idea of boundedness of derivatives, which forms the basis of Lipschitz continuity, can be traced back to earlier works.

The study of functions with bounded derivatives began with the work of mathematicians such as Joseph-Louis Lagrange in the 18th century. Lagrange developed the mean value theorem, which states that if a function is differentiable on an interval, then there exists at least one point within that interval where the instantaneous rate of change (derivative) of the function is equal to the average rate of change over the interval.

The investigation of functions with bounded derivatives continued with mathematicians like Augustin-Louis Cauchy and Karl Weierstrass. Cauchy explored the concept of uniform continuity, which is a stronger form of continuity than Lipschitz continuity. Meanwhile, Weierstrass made significant contributions to the theory of functions by studying functions that satisfy certain growth conditions.

In 1872, Rudolf Lipschitz published a seminal paper titled "Zur Theorie der linearen Differentialgleichungen" [34], where he formalized the concept of Lipschitz continuity. Lipschitz's work focused on ordinary differential equations, and he showed that if the derivative of a function is bounded on an interval, then the function itself is Lipschitz continuous on that interval. This breakthrough provided a valuable tool for analyzing differential equations and studying the behavior of functions.

Since Lipschitz's original work, Lipschitz continuity has been extensively studied and applied in various branches of mathematics, including analysis, functional analysis, optimization theory, and numerical analysis. The concept plays a fundamental role in understanding the behavior of functions and has found numerous applications in diverse fields.

A mapping $T : C \to C$ is defined as *β-Lipschitz continuous* if there exists $\beta > 0$

which

$$\|Tx - Ty\| \leq \beta\|x - y\|,$$

for all $x, y \in C$ with Lipschitz constant β. We said T is also *β-contractive mapping* if $\beta \in (0,1)$ and T is called nonexpansive if $\beta = 1$.

Strongly monotone mappings are valuable in mathematics and related fields due to their ability to ensure the existence, uniqueness, and convergence of solutions. They facilitate efficient algorithm development, aid in variational inequalities, play a role in convex analysis, and help determine equilibria in game theory. In summary, strongly monotone mappings provide essential properties and conditions, advancing research in various mathematical domains.

It is well known that a mapping $A : C \to H$ is called *α-strongly montone* if there exists $\alpha \geq 0$ satisfies

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2,$$

for all $x, y \in C$. A must be called *monotone* if $\alpha = 0$.

A mapping A of C into H is called *γ-inverse strongly monotone* [8], if there exists a positive real number γ such that

$$\langle x - y, Ax - Ay\rangle \geq \gamma \|Ax - Ay\|^2,$$

for all $x, y \in C$.

The Mann iteration, introduced by Mann in 1953 [39], is a fixed-point iteration method commonly used to approximate fixed points of a mapping T. Given an initial point x_0, the Mann iteration generates a sequence $\{x_n\}$ defined by the recurrence relation:

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n, \tag{1.1}$$

where $\{\lambda_n\} \subseteq (0,1)$ is a parameter.

The convergence behavior of the Mann iteration depends on the properties of the mapping T and the choice of $\{\lambda_n\}$. Under suitable conditions, the Mann iteration converges to a fixed point of the mapping T.

Specifically, if T is a contractive mapping then the Mann iteration converges to a unique fixed point of T for any choice of $\{\lambda_n\} \subseteq (0,1)$. This result is known as the Mann iteration convergence theorem.

It is worth noting that the convergence rate of the Mann iteration can vary depending on the choice of $\{\lambda_n\}$ and the properties of the mapping T. In some cases, a larger value of $\{\lambda_n\}$ can lead to faster convergence, while in other cases, a smaller value of $\{\lambda_n\}$ may be more appropriate. The optimal choice of $\{\lambda_n\}$ often depends on the specific problem at hand and may require some experimentation or analysis.

The Mann iteration has been extensively studied and applied in various fields, including optimization, numerical analysis, and nonlinear equations. It provides a simple yet powerful tool for approximating fixed points and has found numerous applications in practical problems.

Xu [21] introduced one of the most fundamental and popular methods for proving strong convergence theorems to approximate nonlinear and inverse problems. For example, see [27], [28], and [30]. The details are shown below.

Lemma 1.1. Let $\{S_n\}$ be a sequence of real number satisfying

$$S_{n+1} \leq (1-\alpha_n)S_n + \xi_n,$$

for all $n \geq 0$ with a sequence $\{\alpha_n\}$ in (0,1) and a sequence $\{\xi_n\}$ such that
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \xi_n/\alpha_n \leq 0$ or $\sum_{n=1}^{\infty} \xi_n < \infty$.
Then, we have

$$\lim_{n\to\infty} S_n = 0.$$

However, there are many researchers who try to prove strong convergence theorems without using Lemma 1.1. For example, see [24], [25], and [26].

Throughout this research article, we will prove the strong convergence theorem by modifying the Mann iteration of our proposed sequence without using Lemma 1.1. In addition, we give an application to solve the variational inequality problem by applying our main result.

1.2 Objectives of the study

The primary objectives of this thesis can be summarized as follows:

1) Introduce a new method for approximating the results of variational inequality problems and fixed-point problems.
2) Omit the use of a popular lemma typically used as the basis for proving strong convergence theorems
3) Demonstrate the effectiveness of this new method by comparing its results with those obtained through the popular lemma
4) Provide a numerical example to support the study's findings

1.3 Scopes of the study

1) Study the definitions and properties of variational inequality problems and fixed point problems in real Hilbert space.
2) Investigate the fixed point problems of nonlinear mappings, including nonexpansive mapping, L-Lipschitzian mapping, and α-inverse strongly monotone mapping.

1.4 Benefits of the study

1) Improved understanding of variational inequality problems and fixed-point problem of nonexpansive mappings, which are important concepts in optimization and mathematical analysis.

2) Developing of a new method for solving variational inequality problems, which may be useful in practical applications by omitting the solution of such problem.

3) A new techniqe to prove strong convergence theorem for fixed-point problem and variational inequality problem.

4) Application of the new method to the fixed-point problem of nonexpansive mappings, which has important applications in many areas of science and engineering.

5) Demonstration of the effectiveness of the new method through numerical experiments, which may encourage its adoption and use in practice.

6) Identification of areas for future research and development in the field of variational inequality problems and nonexpansive mappings, potentially leading to further improvements and discoveries in this area of mathematics.

Chapter 2

Theory and literature reviews

In this chapter, we give important lemmas, definitions, and theorems for used throughout of this thesis. Moreover, we use the letter $\mathbb{R}$ for the set of all real numbers, $\mathbb{C}$ for the set of all complex numbers and $\mathbb{F}$ for the set of all real or complex numbers.

2.1 Inner product space

Definition 2.1 (Inner product space [13]). An *inner product space* is a vector space X with an inner product defined on X. Here, an inner product on X is a mapping $\langle \cdot, \cdot \rangle$ of $X \times X$ into the scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; that is, with every pair of vector x and y there is associated a scalar which is written and is called *the inner product of x and y*, such that for all vectors x, y, z and scalar $\alpha \in \mathbb{F}$ we have:

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$;

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$;

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;

(iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, x \rangle$.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x \rangle}$.

A complete inner product space, also known as a complete pre-Hilbert space, is a vector space equipped with an inner product that possesses the property of completeness. Completeness refers to the property that every Cauchy sequence in the space converges to a limit that also belongs to the same space.

Definition 2.2 (Complete Inner product space [45]). Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$. V is said to be complete if, for every Cauchy sequence $\{x_n\}$ in V, there exists a vector x in V such that:

$$\lim_{n \to \infty} x_n = x,$$

where the limit is taken with respect to the norm induced by the inner product:

$$\|x - x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

In simpler terms, in a complete inner product space, any sequence of vectors that becomes arbitrarily close to each other as the sequence progresses will have a limit within the space.

Theorem 2.1 ([9]). Let $\{a_n\}$ be a bounded of real numbers. Then, there exists subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Remark 2.2 ([9]). Let H be an inner product space. Then we know that the following (1) and (2) are equivalent:

1) H is complete,

2) Each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

2.2 Norm space

In mathematics, a *normed space*, or *norm space*, is a vector space equipped with a norm, which is a function that assigns a non-negative length or magnitude to each vector in the space.

Definition 2.3 (Normed space [12]). Let X be a vector space. A norm on X is a real-valued function on X such that the following conditions are satisfied:

(i) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$, $\forall \alpha \in \mathbb{F}$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$ (the triangle inequality),

for all $x, y \in X$ and $\alpha \in \mathbb{R}$ or $\mathbb{C}$. The ordered pair $(X, \|\cdot\|)$ is called *normed space*. A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\|,$$

for all $x, y \in X$, and is called the *metric induced by the norm*.

Definition 2.4 (Complete norm [46]). Let X be a norm space. If any Cauchy sequence in X converges to $x \in X$, then X is a complete norm space or called Banach space.

Definition 2.5 (Convex set [9]). Let X be a normed space and let C be a subset of X. Then the set C is called *convex* if

$$\alpha x + (1 - \alpha)y \in C,$$

for all $x, y \in C$ and $\alpha \in [0, 1]$.

2.3 Metric projection

Metric projection refers to finding the point in a metric space that is closest to a given point or set. It involves minimizing the distance between the given point and the set. Various techniques like orthogonal projection, nearest neighbor search, optimization-based approaches, and approximation methods can be used for metric projection. The choice of method depends on the problem and computational requirements.

Definition 2.6 (Metric projection)**.** Let C be closed convex subset of real Hilbert space H, let P_C be the metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Definition 2.7. (Property of metric projection) Let H be a Hilbert space and let a nonempty closed convex subset of H. Then the metric project P of H onto C has the following properties (1) and (2):

(1) P is a mapping of H onto C and $P^2 = P$,

(2) $\|Px - Py\| \leq \|x - y\|$ for all $x, y \in H$.

The following lemmas concern the properties of P_C and the set of inequality problems.

Lemma 2.3. (See [21]) Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and A be a mapping of C into H. Let $u \in C$, then for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \text{ if and only if } u \in VI(C, A).$$

2.4 Opial's condition

Zdzisław Opial introduced the Opial property, which deals with weak convergence in Banach spaces. If a sequence satisfies this property and weakly converges to a point, a specific function associated with the sequence will also converge to a predetermined value.

It involves the following steps: identifying the sequence and derived function, verifying weak convergence, checking if the Opial condition holds, analyzing the convergence properties of the derived function, and applying the findings to solve the problem at hand.

By following these steps, Opial's condition provides valuable insights into convergence properties and facilitates progress in diverse mathematical fields.

Lemma 2.4 (See [21])**.** Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{u_n\} \subset H$ with $u_n \rightharpoonup u$, the inequality

$$\liminf_{n \to \infty} \|u_n - u\| < \liminf_{n \to \infty} \|u_n - v\|$$

holds for every $v \in H$ with $v \neq u$.

2.5 Other lemmas to solve strong convergence

Lemma 2.5. (See [21]) Let C be a closed convex subset of Hilbert space H, and let A and B be operators from C to H that are α and β-inverse strongly monotone, respectively. If $VI(C,A) \cap VI(C,B) \neq \emptyset$. Then

$$VI(C, aA + (1-a)B) = VI(C,A) \cap VI(C,B),$$

for all $a \in (0,1)$. Furthermore if $0 < \gamma < \min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1-a)B)$ is a nonexpansive mapping.

Lemma 2.6. (See [22]) Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequences of non-negative numbers satisfying

$$a_{n+1} \leq a_n + b_n,$$

for all $n \geq 0$.

(i) If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

(ii) If $\sum_{n=0}^{\infty} b_n < \infty$ and $\{a_n\}_{n=0}^{\infty}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 ([21]). Let H be a Hilbert space. Then, for all $x, y, x \in H$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Chapter 3

Research methodology

The methodology employed in this study follows a step-by-step process to address the research objectives pertaining to approximating solutions for variational inequality and fixed-point problems. The sequential approach outlined below forms the basis for conducting our investigation:

1) Review of Fixed-Point Theorems: Collect relevant research papers and authoritative textbooks that cover fixed-point theorems and their applications. Analyze and synthesize the information to gain a comprehensive understanding of existing approaches and techniques.

2) Objective Definition: Clearly define the objectives and scope of the research, delineating the specific problems that will be addressed, and the expected outcomes.

3) Theoretical Framework: Develop the necessary theoretical tools and concepts required for proving a strong convergence theorem for both the fixed-point problem and variational inequality problem. This involves formulating appropriate mathematical models and frameworks to support the subsequent analysis.

4) Strong Convergence Theorem Proof: Rigorously prove the strong convergence theorem for the fixed-point problem and variational inequality problem. Clearly present the mathematical derivations, logical steps, and justifications for the theorems.

5) Numerical Examples and Applications: Provide illustrative examples that demonstrate the practical implementation of the proposed methods. Utilize numerical methods and simulations to showcase the effectiveness and efficiency of the proposed solutions. Additionally, explore real-world applications of the developed theorems in relevant contexts.

6) Conclusion: Summarize and discuss the obtained results, emphasizing their significance and implications. Highlight the contributions of the research and identify potential areas for future exploration and improvement.

7) References: Compile a comprehensive list of all the references used throughout the thesis, adhering to the appropriate citation style.

Chapter 4

Main results and discussion

4.1 Strong convergences theorems

In this section, we introduce the theoretical framework related to strong convergence, which represents the core achievement of our research. This framework presents adaptable applications for tackling various problem domains. Additionally, we outline the corresponding proof methods as follows.

Theorem 4.1. Let C be nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Let $A : C \to H$ be α-strongly monotone and L-Lipschitzian. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$x_{n+1} = ax_n + bTx_n + cP_C(I - \lambda A)x_n,$$

for all $n \geq 1$ with $0 < L \leq \alpha < 1$, $\lambda \in (0, 1)$, and $a + b + c = 1$. Then the followings are equivalent:

(i) The sequence $\{x_n\}$ converges strongly to $x^\star \in F(T) \cap VI(C, A)$.

(ii) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. Assuming that condition (i) holds, we show that (i) $\to$ (ii). Let x_n be the sequence generated as defined in Theorem 4.1. Then we have

$$\begin{aligned} \|x_n - Tx_n\| &= \|x_n - x^\star + x^\star - Tx_n\| \\ &\leq \|x_n - x^\star\| + \|x^\star - Tx_n\| \\ &= \|x_n - x^\star\| + \|Tx^\star - Tx_n\| \\ &\leq 2\|x_n - x^\star\|. \end{aligned}$$

Since $\{x_n\}$ converges strongly to $x^\star$, then

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Conversely, suppose condition (ii) holds and employ the nonexpansiveness of P_C. We get

$$\begin{aligned} \|P_C(I - \lambda A)x - P_C(I - \lambda A)y\|^2 &\leq \|(I - \lambda A)x - (I - \lambda A)y\|^2 \\ &= \|(x - y) - (\lambda Ax - \lambda Ay)\|^2 \\ &= \|x - y\|^2 - 2\langle x - y, \lambda Ax - \lambda Ay\rangle + \|\lambda Ax - \lambda Ay\|^2 \\ &\leq \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\alpha^2\|x - y\|^2 \\ &\leq (1 - 2\lambda\alpha + \lambda^2\alpha^2)\|x - y\|^2 \\ &= (1 - \lambda\alpha)^2\|x - y\|^2 \\ &= \omega^2\|x - y\|^2. \end{aligned}$$

with $\omega = 1 - \lambda\alpha \in (0,1)$.

Consequently, $P_C(I-\lambda A)$ is an ω-contractive mapping. From T is a nonexpansive mapping, getting

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|(ax_n + bTx_n + cP_C(I-\lambda A)x_n) \\
&\qquad - (ax_{n-1} + bTx_{n-1} + cP_C(I-\lambda A)x_{n-1})\| \\
&= \|a(x_n - x_{n-1}) + b(Tx_n - Tx_{n-1}) + c(P_C(I-\lambda A)x_n - P_C(I-\lambda A)x_{n-1})\| \\
&\le a\|x_n - x_{n-1}\| + b\|Tx_n - Tx_{n-1}\| + c\|P_C(I-\lambda A)x_n - P_C(I-\lambda A)x_{n-1}\| \\
&\le a\|x_n - x_{n-1}\| + b\|x_n - x_{n-1}\| + c\omega\|x_n - x_{n-1}\| \\
&\le (1-(1-\omega)c)\|x_n - x_{n-1}\| \\
&\le (1-(1-\omega)c)^2\|x_{n-1} - x_{n-2}\| \\
&\le (1-(1-\omega)c)^3\|x_{n-2} - x_{n-3}\| \\
&\vdots \\
&\le (1-(1-\omega)c)^n\|x_1 - x_0\|. \qquad (4.1)
\end{aligned}$$

For any positive integers n and κ with using equation (4.1), we have

$$\begin{aligned}
\|x_{n+\kappa}-x_n\| &= \|x_{n+\kappa} - x_{n+\kappa-1} + x_{n+\kappa-1} - x_{n+\kappa-2} + x_{n+\kappa-2} - \ldots - x_n\| \\
&\le \|x_{n+\kappa} - x_{n+\kappa-1}\| + \|x_{n+\kappa-1} - x_{n+\kappa-2}\| + \|x_{n+\kappa-2} - x_{n+\kappa-3}\| \\
&\qquad + \ldots + \|x_{n+1} - x_n\| \\
&= \sum_{j=n}^{n+\kappa-1} \|x_{j+1} - x_j\| \\
&\le \sum_{j=n}^{n+\kappa-1} (1-(1-\omega)c)^j\|x_1 - x_0\| \\
&\le \frac{(1-(1-\omega)c)^n}{(1-\omega)c}\|x_1 - x_0\|. \qquad (4.2)
\end{aligned}$$

From (4.2) and $\lim_{n\to\infty}(1-(1-\omega)c)^n = 0$, we have $\{x_n\}$ is a Cauchy sequence. From the completeness of H, there exists $x^\star \in H$ such that

$$\lim_{x\to\infty} x_n = x^\star. \qquad (4.3)$$

Since C is closed, we obtain $x^\star \in C$. Assume that $x^\star \neq Tx^\star$, using the fact that $\lim_{x\to\infty}\|x_n - Tx_n\| = 0$, Lemma 2.4, and nonexpansiveness of T. Getting,

$$\begin{aligned}
\liminf_{n\to\infty}\|x_n - x^\star\| &< \liminf_{n\to\infty}\|x_n - Tx^\star\| \\
&= \liminf_{n\to\infty}(\|x_n - Tx_n + Tx_n - Tx^\star\|) \\
&\le \liminf_{n\to\infty}(\|x_n - Tx_n\| + \|Tx_n - Tx^\star\|) \\
&\le \liminf_{n\to\infty}\|x_n - Tx_n\| + \liminf_{n\to\infty}\|x_n - x^\star\| \\
&= \liminf_{n\to\infty}\|x_n - x^\star\|.
\end{aligned}$$

This is a contradiction, we obtain that $x^\star = Tx^\star$. It means that $x^\star \in F(T)$. From definition of x_{n+1}, then

$$c\|P_C(I-\lambda A)x_n - x_n\| \leq \|x_{n+1}-x_n\| + b\|Tx_n - x_n\|.$$

From $\lim_{n\to\infty} x_n = x^\star$ and condition (ii), we have

$$\lim_{n\to\infty} \|P_C(I-\lambda A)x_n - x_n\| = 0. \tag{4.4}$$

Assume that $x^\star \neq P(I-\lambda A)x^\star$. From (4.4), Lemma 2.4, and the nonexpansiveness of $P_C(I-\lambda A)$. Therefore, we have shown that

$$\begin{aligned}\liminf_{n\to\infty}\|x_n - x^\star\| &< \liminf_{n\to\infty}\|x_n - P_C(I-\lambda A)x^\star\| \\ &= \liminf_{n\to\infty}\|x_n - P_C(I-\lambda A)x_n + P_C(I-\lambda A)x_n - P_C(I-\lambda A)x^\star\| \\ &\leq \liminf_{n\to\infty}(\|x_n - P_C(I-\lambda A)x_n\| + \|P_C(I-\lambda A)x_n - P_C(I-\lambda A)x^\star)\| \\ &\leq \liminf_{n\to\infty}\|x_n - P_C(I-\lambda A)x_n\| + \liminf_{n\to\infty}\|x_n - x^\star\| \\ &= \liminf_{n\to\infty}\|x_n - x^\star\|.\end{aligned}$$

This is a contradiction, we obtain $x^\star = P_C(I-\lambda A)x^\star$. From Lemma 2.3, then $x^\star \in VI(C,A)$.

Hence, $\{x_n\}$ converges strongly to $x^\star \in F(T)\cap VI(C,A)$.

□

From the aforementioned theorem, it is evident that it leads to the subsequent theorem discussed directly and succinctly.

Theorem 4.2. Let C be nonempty closed convex subset of a Hilbert space H and let $T: C\to C$ be a nonexpansive mapping. Let $A: C\to H$ be α-strongly monotone and L-Lipschitzian with $F(T)\cap VI(C,A)\neq\emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0\in C$ and

$$x_{n+1} = ax_n + bTx_n + cP_C(I-\lambda A)x_n,$$

for all $n\geq 1$ with $0 < L \leq \alpha < 1$, $\lambda\in(0,1)$, and $a+b+c=1$. Then the sequence $\{x_n\}$ converges strongly to $x^\star \in F(T)\cap VI(C,A)$.

Proof. Let $\psi \in F(T) \cap VI(C,A)$, then

$$
\begin{aligned}
\|x_{n+1}-\psi\|^2 &= \|ax_n + bTx_n + cP_C(I-\lambda A)x_n - \psi\|^2 \\
&= \|ax_n + bTx_n + cP_C(I-\lambda A)x_n - (a+b+c)\psi\|^2 \\
&= \|a(x_n-\psi) + b(Tx_n-\psi) + c(P_C(I-\lambda A)x_n - \psi)\|^2 \\
&= a\|x_n-\psi\|^2 + b\|Tx_n-\psi\|^2 + c\|P_C(I-\lambda A)x_n-\psi\|^2 - ab\|(x_n-\psi)-(Tx_n-\psi)\|^2 \\
&\quad - bc\|Tx_n - P_C(I-\lambda A)x_n\|^2 \\
&= a\|x_n-\psi\|^2 + b\|Tx_n-\psi\|^2 + c\|P_C(I-\lambda A)x_n-\psi\|^2 - ab\|x_n-Tx_n\|^2 \\
&\quad - bc\|Tx_n - P_C(I-\lambda A)x_n\|^2 \\
&\le a\|x_n-\psi\|^2 + b\|Tx_n-T\psi\|^2 + c\|P_C(I-\lambda A)x_n - P_C(I-\lambda A)\psi\|^2 - ab\|x_n-Tx_n\|^2 \\
&\le a\|x_n-\psi\|^2 + b\|x_n-\psi\|^2 + c\omega^2\|x_n-\psi\|^2 - ab\|x_n-Tx_n\|^2 \\
&\le a\|x_n-\psi\|^2 + b\|x_n-\psi\|^2 + c\|x_n-\psi\|^2 - ab\|x_n-Tx_n\|^2 \\
&= \|x_n-\psi\|^2 - ab\|x_n-Tx_n\|^2. \qquad (4.5)
\end{aligned}
$$

From (4.5), it implies that

$$\|x_{n+1}-\psi\| \le \|x_n-\psi\|. \qquad (4.6)$$

Employing Lemma 2.6, we have $\lim_{n\to\infty}\|x_n-\psi\|$ exists for all $\psi \in F(T) \cap VI(C,A)$.

From (4.5), we get

$$ab\|x_n-Tx_n\|^2 \le \|x_n-\psi\|^2 - \|x_{n+1}-\psi\|^2.$$

Accordingly,

$$\lim_{n\to\infty}\|x_n-Tx_n\| = 0. \qquad (4.7)$$

Using (4.7) and Theorem 4.1, it follows that the sequence $\{x_n\}$ converges strongly to $x^\star \in F(T)\cap VI(C,A)$.

□

4.2 Applications

The combination of the variational inequality problem [33] is to find $x \in C$ such that

$$\left\langle y-x, \sum_{i=1}^{N} a_i A_i x \right\rangle \ge 0,$$

where $A_i : C \to H$ is a nonlinear mapping, and $a_i \in (0,1)$ with $\sum_{i=1}^{N} a_i = 1$, for all $i = 1,2,\dots,N$.

The set of solutions of combination of this variational inequality problem is given by

$$VI(C, \sum_{i=1}^{N} a_i A_i) = \left\{ u \in C : \left\langle y-x, \sum_{i=1}^{N} a_i A_i x \right\rangle \ge 0, \forall y \in C \right\}.$$

This problem is called the variational inequality problem if $A_i = A$ for all $i = 1, 2, \ldots, N$.

Kangtunyakarn [33] proved a strong convergence theorem for finding a common element of the sets of fixed points of a finite family of nonspreading mappings and the sets of solutions of such a problem. The convergence theorem involves two split variational inequality problems and is proved by applying the results of Lemma 2.5. This leads to the proof of the following theorem.

Theorem 4.3. Let C be nonempty closed convex subset of a real Hilbert space H, and let $A^{\heartsuit}, B^{\heartsuit} : C \to H$ be α, β-inverse strongly monotone, respectively. Let $A : C \to H$ be α-strongly monotone and L-Lipschitz operator, with $VI(C, A) \cap VI(C, A^{\heartsuit}) \cap VI(C, B^{\heartsuit}) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$x_{n+1} = ax_n + bP_C(I - \gamma(aA^{\heartsuit} + (1-a)B^{\heartsuit}))x_n + cP_C(I - \lambda A)x_n,$$

for all $n \geq 1$ with $0 < L \leq \alpha < 1$, $\lambda \in (0, 1)$, $0 < \gamma < \min\{2\alpha, 2\beta\}$, and $a + b + c = 1$. Then, the sequence $\{x_n\}$ converges strongly to $x^\star \in VI(C, A) \cap VI(C, A^{\heartsuit}) \cap VI(C, B^{\heartsuit})$.

Proof. From Lemma 2.6, we have $P_C(I - \gamma(aA^{\heartsuit} + (1-a)B^{\heartsuit})$ is a nonexpansive mapping. From Theorem 4.2 and Lemma 2.5, we obtain that $\{x_n\}$ converges strongly to $x^\star \in VI(C, A) \cap VI(C, A^{\heartsuit}) \cap VI(C, B^{\heartsuit})$.

□

Corollary 4.4. Let C be nonempty closed convex subset of a real Hilbert space H. Let $A^{\heartsuit} : C \to H$ be α-inverse strongly monotone, and $A : C \to H$ be α-strongly monotone, and L-Lipschitz operator with $VI(C, A) \cap VI(C, A^{\heartsuit}) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$x_{n+1} = ax_n + bP_C(I - \gamma A^{\heartsuit})x_n + cP_C(I - \lambda A)x_n,$$

for all $n \geq 1$ with $0 < L \leq \alpha < 1$, $\lambda \in (0, 1)$, $0 < \gamma < 1$, and $a + b + c = 1$. Then, the sequence $\{x_n\}$ converges strongly to $x^\star \in VI(C, A) \cap VI(C, A^{\heartsuit})$.

Remark 4.5.

1. If $S : C \to H$ is a nonexpansive mapping with $F(S) \neq \emptyset$, then $VI(C, I - S) = F(S)$.

2. If S is a nonexpansive mapping, then $I - S$ is $\frac{1}{2}$-inverse strongly monotone.

We use the above two remarks and Theorem 4.2 to prove the following results.

Corollary 4.6. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-strongly monotone, and L-Lipschitz. Let $\bar{S}, \hat{S} : C \to C$ be nonexpansive mapping with $VI(C, A) \cap F(\bar{S}) \cap F(\hat{S}) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$x_{n+1} = ax_n + bP_C(I - \gamma((I - \bar{S})a + (1-a)(I - \hat{S}))x_n + cP_C(I - \lambda A)x_n,$$

for all $n \geq 1$ with $0 < L \leq \alpha < 1$, $\lambda \in (0,1)$, $0 < \gamma < 1$, and $a + b + c = 1$. Then the sequence $\{x_n\}$ converges strongly to $x^\star \in VI(C, A) \cap F(\bar{S}) \cap F(\hat{S})$.

4.3 Numerical Method

In the field of physics, Apéry's constant is particularly intriguing and essential to the computation of the electron's gyromagnetic ratio using quantum electrodynamics. It is defined as the sum of the reciprocals of the positive cubes and represented by the Euler-Riemann zeta function:

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

This constant holds an irrational value, approximately equal to 1.2020 5690 3159 5942 8539 9738 1615 1144 ..., making it all the more fascinating. Furthermore, we have presented evidence of the newly proven theorem's application to support our findings, that is, the convergence trend of sample sequences.

Example 4.7. Let $C = [-100, 100]$, $Tx = \frac{1}{2}x$, and $Ax = \frac{\zeta(3)}{4}x$. Let x_n be a sequence generated by $x_0 \in C$, and

$$x_{n+1} = 0.2x_n + 0.3Tx_n + 0.5P_C\left(I - \frac{\zeta(3)\pi}{20}\right)x_n,$$

for all $n \geq 1$. We know that A is both $\frac{\zeta(3)}{4}$-strongly monotone and $\frac{\zeta(3)}{4}$-Lipschitz. Moreover, T is nonexpansive with $0 \in VI(C, A) \cap F(T)$. Applying Theorem 4.2, we can conclude that the sequence x_n converges strongly to 0.

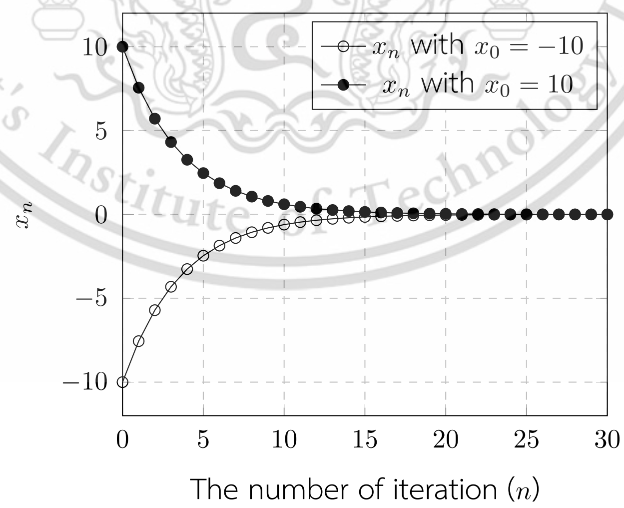

Figure 4.1: The convergence numbers of iterative calculations with an initial value $x_0 = 10, -10$.

Table 4.1: The values of $\{x_n\}$ with $x_0 = -10$ and $x_0 = 10$.

n	x_n with $x_0 = -10$	x_n with $x_0 = 10$
1	-7.5559067159592303	7.5559067159592303
2	-5.709172630027780	5.709172630027780
3	-4.3137975817797525	4.3137975817797525
4	-3.2594652119458319	3.2594652119458319
5	-2.4628215085376987	2.4628215085376987
$\vdots$	$\vdots$	$\vdots$
15	-0.1493821869718095	0.1493821869718095
16	-0.1128717869784972	0.1128717869784972
17	-0.0852848693273147	0.0852848693273147
18	-0.0644404516919962	0.0644404516919962
19	-0.0486906041719000	0.0486906041719000
20	-0.0367901663066572	0.0367901663066572
$\vdots$	$\vdots$	$\vdots$
50	-0.0000082097395123	0.0000082097395123

After examining table 4.1 and figure 4.1, it is observed that this iterative process causes x_n to converge towards the fixed point values of T and $VI(C, A)$, which aligns clearly with our theorem.

Chapter 5

Conclusions and Suggestions

5.1 Conclusions

In this thesis, we have presented a modified Mann iteration that achieves strong convergence without relying on Lemma 1.1 and without assuming $F(T) \cap VI(C, A) \neq \emptyset$. The modified Mann iteration is applicable to various scenarios of interest, and we have explored its convergence properties in different settings. Below, we summarize the main conclusions drawn from our research:

1) Let C be nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Let $A : C \to H$ be α-strongly monotone and L-Lipschitzian. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$x_{n+1} = ax_n + bTx_n + cP_C(I - \lambda A)x_n,$$

for all $n \geq 1$ with $0 < L \leq \alpha < 1$, $\lambda \in (0, 1)$, and $a + b + c = 1$. Then the followings are equivalent:

(i) The sequence $\{x_n\}$ converges strongly to $x^\star \in F(T) \cap VI(C, A)$.

(ii) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

2) In the context of a Hilbert space H and a nonempty closed convex subset C with $F(T) \cap VI(C, A) \neq \emptyset$, the sequence $\{x_n\}$ generated by $x_0 \in C$ and

$$x_{n+1} = ax_n + bTx_n + cP_C(I - \lambda A)x_n$$

converges strongly to $x^\star \in F(T) \cap VI(C, A)$ with $0 < L \leq \alpha < 1$, $\lambda \in (0, 1)$, and $a + b + c = 1$, and $A : C \to H$ is α-strongly monotone and L-Lipschitzian.

3) We extended the convergence analysis to the case when there are two inverse strongly monotone mappings $A^\heartsuit$ and $B^\heartsuit$, along with a strongly monotone operator $A : C \to H$. Under the conditions: $0 < L \leq \alpha < 1$, $\lambda \in (0, 1)$, $0 < \gamma < \min\{2\alpha, 2\beta\}$, and $a + b + c = 1$, the sequence $\{x_n\}$ generated by $x_0 \in C$ and

$$x_{n+1} = ax_n + bP_C(I - \gamma(aA^\heartsuit + (1 - a)B^\heartsuit))x_n + cP_C(I - \lambda A)x_n$$

converges strongly to $x^\star \in VI(C, A) \cap VI(C, A^\heartsuit) \cap VI(C, B^\heartsuit)$.

4) We further investigated the case where $A^\heartsuit$ is an inverse strongly monotone mapping. For $0 < L \leq \alpha < 1$, $\lambda \in (0, 1)$, $0 < \gamma < 1$, and $a + b + c = 1$, the sequence $\{x_n\}$ generated by $x_0 \in C$ and

$$x_{n+1} = ax_n + bP_C(I - \gamma A^\heartsuit)x_n + cP_C(I - \lambda A)x_n,$$

converges strongly to $x^\star \in VI(C, A) \cap VI(C, A^\heartsuit)$.

5) Finally, we explored the situation with two nonexpansive mappings $\bar{S}$ and $\hat{S}$, along with a strongly monotone operator $A : C \to H$, and showed that the sequence $\{x_n\}$ generated by $x_0 \in C$ and

$$x_{n+1} = ax_n + bP_C(I - \gamma((I - \bar{S})a + (1-a)(I - \hat{S}))x_n + cP_C(I - \lambda A)x_n$$

converges strongly to $x^\star \in VI(C, A) \cap F(\bar{S}) \cap F(\hat{S})$ for $0 < L \leq \alpha < 1$, $\lambda \in (0, 1)$, $0 < \gamma < 1$, and $a + b + c = 1$.

Our proposed modified Mann iteration provides a versatile approach for solving variational inequality problems in different settings. We successfully utilized this method to prove Corollary 3.2 and approximate the solution of the combined variational inequality problem. Additionally, we supported our theoretical findings with numerical evidence, demonstrating the practical effectiveness of the proposed approach.

In conclusion, the modified Mann iteration presented in this thesis offers a robust and efficient algorithm for tackling variational inequality problems, and it extends the applicability of the classical Mann iteration method. The results obtained in this research contribute to the advancement of optimization algorithms and provide valuable insights into the study of variational inequalities in Hilbert spaces.

5.2 Suggestions

In this thesis, we put forth innovative methods to approximate solutions for variational inequality problems and fixed-point problems. The introduction of our proof holds promising prospects for establishing robust strong convergence theorems in expansive spaces like Banach spaces.

Through strategic modifications to the mapping properties and a transformation of constants α, β, and γ into sequences, we deviate from conventional approaches and successfully demonstrate the versatility of our methodology.

The inclusion of numerical examples serves to validate our findings, effectively showcasing the efficacy and applicability of our novel approach, thereby paving the way for potential breakthroughs in diverse mathematical contexts.

References

[1] M. Abdellaoui, Z. Dahmani, and N. Bedjaoui, “Applications of Fixed Point Theorems for Coupled Systems of Fractional Integro-Differential Equations Involving Convergent Series,” *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 173-278, 2015.

[2] I. Batiha, L. Aoua, T. Oussaeif, A. Ouannas, S. Ishorman, I. Jebril, and S. Momani, “Common Fixed Point Theorem in Non-Archimedean Menger PM-Spaces Using CLR Property with Application to Functional Equations,” *IAENG International Journal of Applied Mathematics*, vol. 53, no. 1, pp. 360-368, 2023.

[3] S. Premjitpraphan and A. Kangtunyakarn, “Strong Convergence Theorem for the Split Equality Fixed Point Problem for Quasi-Nonexpansive Mapping and Application,” *IAENG International Journal of Applied Mathematics*, vol. 49, no. 3, pp. 318-325, 2019.

[4] H. Scarf, “Fixed-Point Theorems and Economic Analysis,” *American Scientist*, vol. 71, no. 3, pp. 289-296, 1983.

[5] M. Patriche, “Bayesian Abstract Fuzzy Economies, Random Quasi-Variational Inequalities with Random Fuzzy Mappings and Random Fixed Point Theorems,” *Fuzzy Sets and Systems*, vol. 245, pp. 125-136, 2014.

[6] W. A. Kirk, ”The concept of fixed points and the related problem of finding them,” *International Journal of Mathematics and Mathematical Sciences*, vol. 1981, no. 4, pp. 729-736, 1981.

[7] A. Cauchy, ”Cauchy’s formulation of the fixed point problem in 1823,” *Historia Mathematica*, vol. 3, no. 2, pp. 167-169, 1976.

[8] S. Antman, “The Influence of Elasticity in Analysis: Modern Developments,” *Bulletin of the American Mathematical Society*, vol. 9, no. 3, pp. 267–291, November 1983.

[9] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, 2009.

[10] F. E. Browder, “Convergence of Approximants to Fixed Points of Nonexpansive Nonlinear Mappings in Banach Space,” *Archive for Rational Mechanics and Analysis*, vol. 24, pp. 82-89, 1967.

[11] Agarwal, R.P., O’Regan, D., and Sahu, D.R. 2009. “Fixed Point Theory for Lipschitzian-type Mappings with Applications.” *Topological Fixed Point Theory and Its Applications*, 6.

[12] Megginson, R.E. 1998. “An Introduction to Banach Space.” New York: J. Springer.

[13] Kreyszig, E. 1978. “Introductory Functional Analysis with Applications.” New York: John Wiley and Sons.

[14] Jolaoso, L.O., Ogbuisi, F., and Mewomo, O. 2019. “An Intermixed Algorithm for Two Strict Pseudocontractions in q-Uniformly Smooth Banach Space.” *Nonlinear Studies*, 26(1): 27–41.

[15] Suwannaut, S. 2018. “The s-Intermixed Iterative Method for Equilibrium Problems.” *Thai Journal of Mathematics*, 60–74.

[16] Suwannaut, S. 2020. “The General Intermixed Iteration for Equilibrium Problems and Variational Inequality Problems in Hilbert Spaces.” *Thai Journal of Mathematics*, 18(3): 1497-1518.

[17] Nazeer, W., Munir, M., and Kang, S.M. 2018. “An Intermixed Algorithm for Three Strict Pseudocontractions in Hilbert Spaces.” *Journal of Computational Analysis and Applications*, 24: 1322-1333.

[18] Sripattanet, A., and Kangtunyakarn, A. 2019. “Convergence Theorem of Two Sequences for Solving the Modified Generalized System of Variational Inequalities and Numerical Analysis.” *Mathematics*, 7(10).

[19] Sriprad, W., and Srisawat, S. 2021. “An Iterative Method for Solving Split Monotone Variational Inclusion Problems and Finite Family of Variational Inequality Problems in Hilbert Spaces.” *International Journal of Mathematics and Mathematical Sciences*, 5: 1-17.

[20] Jain, P.K., Ahuja, O.P., and Ahmad, K. 1995. *Functional Analysis.* New Delhi: New Age International (P) Ltd., Publisher.

[21] Kangtunyakarn, A. 2021. *Iterative Algorithms and Solution Method for Variational Inequalities Problems.* 2nd ed. Bangkok, Thailand: Mean Service Supply Limited Partnership.

[22] Osilike, M., and Igbokwe, D. 2000. “Weak and Strong Convergence Theorems for Fixed Points of Pseudocontractions and Solutions of Monotone Type Operator Equations.” *Computers and Mathematics with Applications*, 40: 559–567.

[23] Thongpaen, P., Kaewkhao, A., Phudolsitthiphat, N., Suantai, S., and Inthakon, W. 2021. “Weak and Strong Convergence Theorems for Common Attractive Points of Widely More Generalized Hybrid Mappings in Hilbert Spaces.” *Mathematics 2021*, 9: 1–12.

[24] Kim, T., and Xu, H. 2005. "Strong Convergence of Modified Mann Iterations." *Nonlinear Analysis: Theory, Methods and Applications*, 61: 51–60.

[25] Martinez, C., and Xu, H. 2006. "Strong Convergence of the CQ Method for Fixed Point Iteration Processes." *Nonlinear Analysis: Theory, Methods and Applications*, 64(11): 2400–2411.

[26] Matsushita, S., and Takahashi, W. 2005. "A Strong Convergence Theorem for Relatively Nonexpansive Mappings in a Banach Space." *Journal of Approximation Theory*, 134: 257–266.

[27] Marino, G., Xiaolong, V.C., and Min, Q. 2009. "Strong Convergence of the Modified Mann Iterative Method for Strict Pseudo-contractions." *Computers and Mathematics with Applications*, 57: 455–465.

[28] Liu, H., and Chen, R. 2011. "Strong Convergence Theorems of Modified Mann Iterative Process for Nonexpansive Mapping in Hilbert Spaces." *Journal of Mathematical Inequalities*, 5(1): 141–153.

[29] Yao, Z., Liou, Y., Zhu, L., Lo, C., และ Wu, C. (2015) "Strong convergence theorems for fixed point problems, variational inequality problems, and equilibrium problems." *SpringerOpen Journal*, no. 198, June.

[30] Yao, Y., และ Cho, Y. (2010) "A strong convergence of a modified Krasnoselskii-Mann method for nonexpansive mappings in Hilbert spaces." *Mathematical Modelling and Analysis*, vol. 15, no. 2, pp. 265–274.

[31] Wang, S., Wang, S., และ Liou, Y. (2012) "Strong convergence theorems for variational inequality, equilibrium and fixed point problems with applications." *Journal of Global Optimization*, vol. 54, pp. 155–171.

[32] Shehu, Y. (2011) "Strong convergence theorems for fixed point problems, variational inequality problems, and system of generalized mixed equilibrium problems." *Mathematical and Computer Modelling*, vol. 54, no. 5–6, pp. 1510–1522.

[33] Kangtunyakarn, A., และ Suwannaut, S. (2021) "On approximation of the combination of variational inequality problem and equilibrium problem for nonlinear mappings." *Thai Journal of Mathematics*, vol.

[34] Lipschitz, R. (1872). Zur Theorie der linearen Differentialgleichungen. Journal für die Reine und Angewandte Mathematik, vol. 74, pp. 1-22.

[35] Banach, S. (1922). "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales" [On operations on abstract sets and their application to integral equations]. Fundamenta Mathematicae, vol. 3, pp. 133–181.

[36] Fréchet, M. (1922). “Sur quelques points du calcul fonctionnel” [On some points of functional calculus]. Rendiconti del Circolo Matematico di Palermo, vol. 45, pp. 189–201.

[37] Kakutani, S. (1941). “A generalization of Brouwer’s fixed point theorem”. Duke Mathematical Journal, vol. 8(3), pp. 457–459.

[38] Leray, J., and Schauder, J. (1934). “Topologie et équations fonctionnelles” [Topology and functional equations]. Journal de Mathématiques Pures et Appliquées, vol. 13, pp. 331–367.

[39] Mann, W. R. (1953). “Mean value methods in iteration”. Proceedings of the American Mathematical Society, vol. 4(3), pp. 506-510.

[40] Stampacchia, G. (1964). “Variazionali (convesse) con vincoli unilaterali: il problema di Signorini” [Convex variational problems with unilateral constraints: the Signorini problem]. Annali della Scuola Normale Superiore di Pisa, vol. 3(2), pp. 31-82.

[41] Lions, J. L., and Stampacchia, G. (1976). “Variational inequalities”. Communications on Pure and Applied Mathematics, vol. 29(4), pp. 539-579.

[42] Pang, J.-S., and Gabriel, S. (1978). “Complementarity problems”. Mathematical Programming, vol. 13(1), pp. 345-376.

[43] Facchinei, F., and Pang, J.-S. (2003). “Finite-Dimensional Variational Inequalities and Complementarity Problems”. Springer Science Business Media.

[44] Bendersky, D., and Güler, O. (2010). “On the efficiency of a cutting plane method for solving variational inequalities”. Computational Optimization and Applications, vol. 46(1), pp. 23-46.

[45] Debnath, L., & Mikusinski, P. (2005). Introduction to Hilbert Spaces with Applications. Academic Press.

[46] Sanusi, M. R., & Ekariani, S. (2019). Proceedings of the International Conference on Industrial Engineering and Operations Management Bangkok, Thailand, March 5-7, 2019.

Appendix

วิธีการสำหรับการแก้ปัญหาอสมการการแปรผันและการประยุกต์ สำหรับปัญหาจุดตรึงของการส่งไม่ขยาย

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บทคัดย่อ

ในบทความวิจัยนี้ เราพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มสำหรับการแก้ปัญหาอสมการการแปรผันที่เกี่ยวข้องกับตัวดำเนินการทางเดียวโดยปราศจากสมมติฐานการมีอยู่ของปัญหาดังกล่าว เราใช้ทฤษฎีหลักของเราเพื่อประมาณค่าปัญหาจุดตรึงของตัวดำเนินการไม่ขยาย นอกจากนี้เรายังให้ตัวอย่างวิธีการเชิงตัวเลขเพื่อสนับสนุนบางทฤษฎีของเรา

คำสำคัญ: ปัญหาจุดตรึง, ปัญหาอสมการแปรผัน, กระบวนการทำซ้ำ

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THE METHOD FOR SOLVING VARIATIONAL INEQUALITY PROBLEM AND APPLYING TO FIXED-POINT PROBLEM OF NONEXPANSIVE MAPPING

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ABSTRACT

In this article, we prove a strong convergence theorem for solving variational inequality problems associated with the monotone operator without assuming the existence of such problem. We applied our main result to approximate the fixed-point problem of a nonexpansive mapping. Moreover, we give a numerical example to guarantee some results.

Keywords: Fixed point problem, Variational inequality problem, Iteration

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Introduction

The variational inequality problem was defined by G. Stampacchia in the late 1960s [5]. The theory of the variational inequality problem demonstrates the general nature of boundary value theory, and it allows us to solve other applied mathematical problems, such as mechanics, engineering, and control theory.

More recently, many different formulas have been proposed for various problems. It is well-known that the variational inequality problem is equivalent to the fixed-point problem, which allows us to calculate the solution using an iterative algorithm.

Several studies have solved the variational inequality problem by stipulating that $VI(C,A)$, the set of all solutions of the variational inequality problem, is not an empty set, such as [7] [8] [9].

The fixed-point theory began to appear in articles demonstrating the existence of solutions to differential equations in the second quarter of the 18th century [6]. Subsequently, this technique has been refined into a continuous approximation method that is extracted and summarized as a fixed-point theorem in the framework of complete normed space which shows the existence as well as the uniqueness of fixed-points and provides techniques for estimating the fixed-point position and preliminary calculating the convergence rate. After that, it was said that the theory of constant point was initiated by Stefan Banach.

There are numerous problems in sciences, economics, physics, biology, game theory, computer science, image processing and engineering defined by nonlinear functional equations can be solved by reducing them to an equivalent fixed-point problem. Absolutely, an operator equation $Gx=0$ is expressed as a fixed-point equation $Tx_0 = x_0$, where T is a self-mapping with $x_0 \in D(T)$ and $F(T)$ is used for representing the set of solution of fixed-point of T.

Moreover, the fixed-point theory provides the necessary tools for solving problems in various fields of mathematical analysis. For example, variational inequality problems, split feasibility problems, nonlinear optimization problems, complementarity problems, equilibrium problems, selection and matching problems, and problems of proving the existence of solution of integral, differential equations and more.

Throughout this paper, let H be a real Hilbert space and let C be a nonempty closed convex subset of H and inner product $\langle\cdot,\cdot\rangle$ with norm $\|\cdot\|$.

Let B be a mapping of C into H and the variational inequality problem is to find a point $u \in C$ such that

$$\langle Bu, v-u\rangle \geq 0,$$

for all $v \in C$. The set of solutions of the variational inequality problem is represented by $VI(C,B)$. Additionally, $VI(C,B)$ has a unique solution if B is strongly monotone and Lipschitzian on C.

Let T be a mapping of C *into itself* a and T is $\beta-$*Lipschitz continuous* if there exists β >0 with

$$\|Tx-Ty\| \leq \beta\|x-y\|,$$

for all $x,y \in C$. β is called a *Lipschitz constant*. If $\beta \in (0,1)$, then T is also $\beta-$*contractive mapping*. Furthermore, we said T is *nonexpansive* if β = 1.

A mapping $A: C \to H$ is called $\alpha-$*strongly montone* if there exists $\alpha \geq 0$ satisfies

$$\langle Ax-Ay, x-y\rangle \geq \alpha\|x-y\|^2,$$

for all $x,y \in C$. In addition, if $\alpha = 0$ then A is called *monotone.*

Let A be a mapping of C into H. A is called $\gamma-$*inverse strongly monotone* if there exists a positive real number γ such that

$$\langle x-y, Ax-Ay\rangle \geq \gamma\|Ax-Ay\|^2,$$

for all $x, y \in C$.

This paper considers the theorem related to the variational inequality problem without the premise that $VI(C,A)\cap VI(C,B)\neq\varnothing$. In addition, we will further extend the obtained results to prove a new theorem by conditioning the existence of elements in the set of the problem.

Methodology

In this research, we have a research methodology as shown below.

1. Study essential topics involving fixed-point theory such as nonexpansive mapping, Lipschitzian mapping, monotone mapping, and inverse strongly monotone mapping.
2. Study properties and definitions of Hilbert space and Banach space.
3. Collect research papers and textbooks concerning fixed-point theorem.
4. Determine the objectives and scope of the research.
5. Produce tools for a strong convergence theorem of fixed-point problem and variational inequality problem.
6. Prove a strong convergence theorem for fixed-point problem and variational inequality problem.
7. Provide examples with numerical method for support our main result, and applications.
8. Conclude the results.

Preliminaries

In this section, we give some important lemmas for proving our main result.

Let C be a closed convex subset of real Hilbert space H, let P_C be the metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x-P_C x\| = \min_{y\in C}\|x-y\|.$$

The following lemmas are properties of P_C and the set of all solutions of variational inequality problem.

Lemma 1. (See [2]) Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and A be a mapping of C into H. Let $u \in C$, then for $\lambda > 0$,

$$u = P_C(I-\lambda A)u \text{ if and only if } u \in VI(C,A).$$

Lemma 2. (See [2]) Let C be a closed convex set in Hilbert space. Let $A, B : C \longrightarrow H$ be an α and $\beta-$inverse strongly monotone, respectively. If $VI(C,A)\cap VI(C,B)\neq\varnothing$, then

$$VI(C, aA+(1-a)B) = VI(C,A)\cap VI(C,B),$$

for all $a \in (0,1)$. Moreover, if $0<\gamma<\min\{2\alpha, 2\beta\}$ then $I-\gamma(aA+(1-a)B)$ is a nonexpansive mapping.

Lemma 3. (See [3]) Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequences of non-negative numbers satisfying

$$a_{n+1} \leq a_n + b_n, \text{ for all } n \geq 0.$$

(i) If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

(ii) If $\sum_{n=0}^{\infty} b_n < \infty$ and $\{a_n\}_{n=0}^{\infty}$ has subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0.$

Lemma 4. (See [4]) Let H be a Hilbert space and suppose $x_n \rightharpoonup x$. Then,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

for all $y \in H$ with $x \neq y$.

Lemma 5. (See [3]) Let H be a Hilbert space. Then, for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Results and Discussion

Theorem 1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A: C \to H$ be an $\alpha-$strongly monotone, and L-Lipschitz mapping. Let $B: C \to H$ be a $\beta-$inverse strongly monotone and $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$x_{n+1} = ax_n + bP_C(I - \gamma B)x_n + cP_C(I - \lambda A)x_n,$$

for all $n \geq 1$ with $0 < L \leq \alpha < 1$, $\lambda \in (0,1), \gamma \in (0, 2\beta)$ and $a + b + c = 1$. Then the followings are equivalent:

(i) The sequence $\{x_n\}$ converges strongly to $x^* \in VI(C, A) \cap VI(C, B)$.

(ii) $\lim_{n\to\infty} \|x_n - P_C(I - \gamma B)x_n\| = 0.$

Proof (i) $\to$ (ii). Let condition (i) holds. From nonexpansiveness of P_C, and B is a $\beta-$inverse strongly monotone, we obtain

$$\begin{aligned}
\|P_C(I-\gamma B)x - P_C(I-\gamma B)y\|^2 &\leq \|(x-y) - \gamma(Bx - By)\|^2 \\
&= \|x-y\|^2 - 2\gamma\langle x-y, Bx-By\rangle + \gamma^2\|Bx-By\|^2 \\
&\leq \|x-y\|^2 - 2\gamma\beta\|Bx-By\|^2 + \gamma^2\|Bx-By\|^2 \\
&= \|x-y\|^2 + \gamma(\gamma - 2\beta)\|Bx-By\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

Therefore, $P_C(I - \gamma B)$ is a nonexpansive mapping, getting

$$\|x_n - P_C(I-\gamma B)x_n\| \leq \|x_n - x^*\| + \|x^* - P_C(I-\gamma B)x_n\| \leq 2\|x_n - x^*\|.$$

Since $\{x_n\}$ converges strongly to x^*, then $\lim_{n\to\infty}\|x_n - P_C(I-\gamma B)x_n\| = 0.$

Conversely, we assume conditions (ii) holds. It is well-known that P_C is a nonexpansive mapping, we have

$$\begin{aligned}
\|P_C(I-\lambda A)x - P_C(I-\lambda A)y\|^2 &\leq \|(I-\lambda A)x - (I-\lambda A)y\|^2 \\
&= \|x-y\|^2 - 2\langle x-y, \lambda Ax - \lambda Ay\rangle + \|\lambda Ax - \lambda Ay\|^2 \\
&\leq \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\alpha^2\|x-y\|^2 \\
&\leq \eta^2\|x-y\|^2
\end{aligned}$$

with $\eta = 1 - \lambda\alpha \in (0,1)$. Consequently, $P_C(I - \lambda A)$ is a $\eta-$contractive mapping. From $P_C(I - \gamma B)$ is a nonexpansive mapping, getting

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\le a\|x_n-x_{n-1}\|+b\|P_C(I-\gamma B)x_n-P_C(I-\gamma B)x_{n-1}\|+c\eta\|x_n-x_{n-1}\| \\
&\le (1-(1-\eta)c)\|x_n-x_{n-1}\| \\
&\le (1-(1-\eta)c)^2\|x_{n-1}-x_{n-2}\| \\
&\le (1-(1-\eta)c)^3\|x_{n-2}-x_{n-3}\| \\
&\vdots \\
&\le (1-(1-\eta)c)^n\|x_1-x_0\|. \qquad (1)
\end{aligned}
$$

For any number $n,\kappa\in\mathbb{N}$ with $\kappa>0$ and (1), obtaining

$$
\|x_{n+\kappa}-x_n\|\le\sum_{j=n}^{n+\kappa-1}\|x_{j+1}-x_j\|\le\sum_{j=n}^{n+\kappa-1}(1-(1-\eta)c)^j\|x_1-x_0\|\le\frac{(1-(1-\eta)c)^n}{(1-\eta)c}\|x_1-x_0\|. \qquad (2)
$$

From (2) and $\lim_{n\to\infty}(1-(1-\eta)c)^n=0$, we have that $\{x_n\}$ is a Cauchy sequence. From completeness of H, there exists $x^\star\in H$ such that

$$
\lim_{n\to\infty}x_n=x^\star. \qquad (3)
$$

Since C is a close set, $x^\star\in C$.

Assume that $x^\star\ne P_C(I-\gamma B)x^\star$. Employing $\lim_{n\to\infty}\|x_n-P_C(I-\gamma B)x_n\|=0$, Lemma 4, and nonexpansiveness of $P_C(I-\gamma B)$, getting

$$
\begin{aligned}
\liminf_{n\to\infty}\|x_n-x^\star\| &< \liminf_{n\to\infty}\|x_n-P_C(I-\gamma B)x^\star\| \\
&\le \liminf_{n\to\infty}\left(\|x_n-P_C(I-\gamma B)x_n\|+\|P_C(I-\gamma B)x_n-P_C(I-\gamma B)x^\star\|\right) \\
&\le \liminf_{n\to\infty}\|x_n-x^\star\|.
\end{aligned}
$$

This is a contradiction, we obtain $x^\star=P_C(I-\gamma B)x^\star$. It means that $x^\star\in VI(C,B)$. From the definition of x_{n+1}, then

$$
c\|P_C(I-\lambda A)x_n-x_n\|\le\|x_{n+1}-x_n\|+b\|P_C(I-\gamma B)x_n-x_n\|.
$$

From (3) and condition (ii), we have

$$
\lim_{n\to\infty}\|P_C(I-\lambda A)x_n-x_n\|=0 \qquad (4)
$$

Assume that $x^\star\ne P(I-\lambda A)x^\star$. From (4), Lemma 4, and nonexpansiveness of $P_C(I-\lambda A)$, obtaining

$$
\begin{aligned}
\liminf_{n\to\infty}\|x_n-x^\star\| &= \liminf_{n\to\infty}\|x_n-P_C(I-\lambda A)x^\star\| \\
&< \liminf_{n\to\infty}\|x_n-P_C(I-\lambda A)x_n\|+\|P_C(I-\lambda A)x_n-P_C(I-\lambda A)x^\star\| \\
&\le \liminf_{n\to\infty}\|x_n-x^\star\|.
\end{aligned}
$$

This is a contradiction, we obtain $x^\star=P_C(I-\lambda A)x^\star$. From Lemma 3, then $x^\star\in VI(C,A)$. Hence, $\{x_n\}$ converges strongly to $x^\star\in VI(C,A)\cap VI(C,B)$.

□

From Theorem 1, we prove strong convergence theorem by using condition $F(T)\cap VI(C,A)\neq\emptyset$ as follows.

Theorem 2. Let C be a nonempty closed convex subset of a Hilbert space H. Let $A:C\longrightarrow H$ be an $\alpha-$strongly monotone, and L-Lipschitz mapping. Let $B:C\longrightarrow H$ be a $\beta-$inverse strongly monotone mapping with $VI(C,A)\cap VI(C,B)\neq\emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0\in C$ and

$$x_{n+1}=ax_n+bP_C(I-\gamma B)x_n+cP_C(I-\lambda A)x_n,$$

for all $n\geq 1$ with $0<L\leq\alpha<1$, $\lambda\in(0,1)$, $\gamma\in(0,2\beta)$, and $a+b+c=1$. Then, the sequence $\{x_n\}$ converges strongly to $x^\star\in VI(C,A)\cap VI(C,B)$.

Proof Let conditions (i) - (ii) hold. Let $\varphi\in VI(C,A)\cap VI(C,B)$, using Lemma 5, and definition of x_{n+1}. Then,

$$\begin{aligned}\left\|x_{n+1}-\varphi\right\|^2 &=\left\|ax_n+bP_C(I-\gamma B)x_n+cP_C(I-\lambda A)x_n-\varphi\right\|^2\\ &=a\left\|x_n-\varphi\right\|^2+b\left\|P_C(I-\gamma B)x_n-\varphi\right\|^2+c\left\|P_C(I-\lambda A)x_n-\varphi\right\|^2\\ &\quad -ab\left\|x_n-P_C(I-\gamma B)x_n\right\|^2-bc\left\|P_C(I-\gamma B)x_n-P_C(I-\lambda A)x_n\right\|^2\\ &\leq a\left\|x_n-\varphi\right\|^2+b\left\|P_C(I-\gamma B)x_n-P_C(I-\gamma B)\varphi\right\|^2+c\left\|P_C(I-\lambda A)x_n-P_C(I-\lambda A)\varphi\right\|^2\\ &\quad -ab\left\|x_n-P_C(I-\gamma B)x_n\right\|^2\\ &\leq\left\|x_n-\varphi\right\|^2-ab\left\|x_n-P_C(I-\gamma B)x_n\right\|^2. \end{aligned}\tag{5}$$

From (5), we have $\left\|x_{n+1}-\varphi\right\|\leq\left\|x_n-\varphi\right\|$. Employing Lemma 3, we have $\lim_{n\to\infty}\left\|x_n-\varphi\right\|$ exists for all $\varphi\in VI(C,A)\cap VI(C,B)$. From (5), we get that $ab\left\|x_n-P_C(I-\gamma B)x_n\right\|^2\leq\left\|x_n-\varphi\right\|^2-\left\|x_{n+1}-\varphi\right\|^2$.

Consequently, $\lim_{n\to\infty}\left\|x_n-P_C(I-\gamma B)x_n\right\|=0$. Using this condition and Theorem 1, the sequence $\{x_n\}$ converges strongly to $x^\star\in VI(C,A)\cap VI(C,B)$.

□

Application and Numerical method

We use the below two remarks and Theorem 2 to prove Theorem 3.

Remark (1) If $S:C\longrightarrow H$ is a nonexpansive mapping with $F(S)\neq\emptyset$, then $VI(C,I-S)=F(S)$.

(2) If S is a nonexpansive mapping, then I - S is $\frac{1}{2}-$inverse strongly monotone.

Theorem 3. Let C be a nonempty closed convex subset of a Hilbert space H. Let $A:C\longrightarrow H$ be an $\alpha-$strongly monotone, and L-Lipschitz mapping. Let $T:C\longrightarrow C$ be a nonexpansive mapping with $F(T)\cap VI(C,A)\neq\emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0\in C$, and

$$x_{n+1}=ax_n+bP_C(I-\gamma(I-T))x_n+cP_C(I-\lambda A)x_n,$$

with $0<L\leq\alpha<1$, $a+b+c=1$, and $\lambda,\gamma\in(0,1)$. Then the sequence $\{x_n\}$ converges strongly to $x^\star\in F(T)\cap VI(C,A)$.

Proof From remarks and Theorem 2, we get that $\{x_n\}$ converges strongly to $x^\star\in F(T)\cap VI(C,A)$.

□

We have provided an example of the application of this newly proven theorem to support our results and to point out the convergence trend of the sample sequences, as shown below.

Example: Let $A, B : [-99,99] \to \mathbb{R}$. Let $Ax = \sin(2)x$ and $Bx = \cos(7)x$ for all $x \in [-99,99]$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$, and

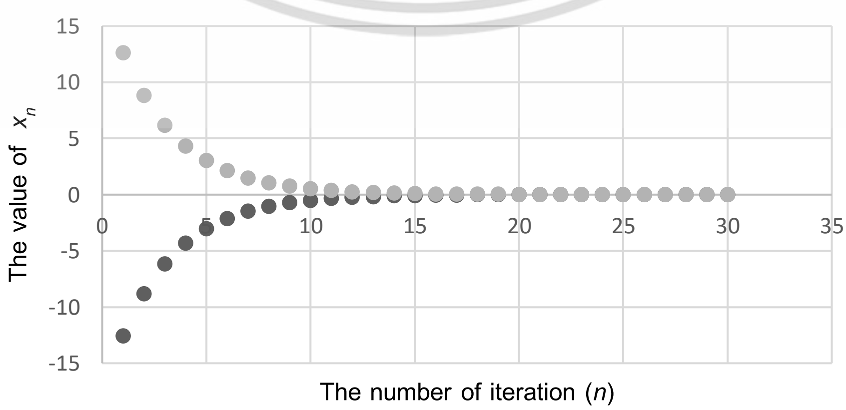
$$x_{n+1} = 0.1x_n + 0.2P_C(I - 0.3\cos(7)x)x_n + 0.7P_C(I - 0.4\sin(2)x)x_n,$$

for all $n \geq 1$. Getting, A is $\sin(2)-$strongly monotone and $\sin(2)-$Lipschitz mapping. B is $\cos(7)-$inverse strongly monotone with $0 \in VI(C,A) \cap F(T)$. From Theorem 2, we get that $\{x_n\}$ converges strongly to 0.

The table 1 and Figure 1 show the value of $\{x_n\}$ with $x_0 = -18$, 18 and $n = 30$.

n	x_n with $x_0 = -18$	x_n with $x_0 = 18$
1	-12.60292653410779521561	12.60292653410779521561
2	-8.8240976235621290845	8.8240976235621290845
3	-6.1783030044193719857	6.1783030044193719857
4	-4.32581660389748966484	4.32581660389748966484
5	-3.02877493660798570258	3.02877493660798570258
⋮		
15	-0.08575431816903479991	0.08575431816903479991
16	-0.06004196510371393 79	0.06004196510371393 79
17	-0.04203913750919837147	0.04203913750919837147
18	-0.02943423119920458015	0.02943423119920458015
19	-0.02060874741063993956	0.02060874741063993956
20	-0.01442947386534885625	0.01442947386534885625
⋮		
30	-0.00040854461581311399	-0.00040854461581311399

Figure 1



Conclusion

We have implemented modify Mann iteration to prove a strong convergence theorem without assumption $VI(C,A)\cap VI(C,B)\neq\emptyset$. Moreover, we applied our main result to prove Theorem 3. In addition, we have provided an example of implementation, along with numerical method, to support effective convergence.

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References

[1] S. Antman. (1983). The influence of elasticity in analysis: modern developments. *Bulletin of the American Mathematical Society*, 9(3), 267–291.

[2] A. Kangtunyakarn. (2021). Iterative algorithms and solution method for variational inequalities problems. *Mean service supply limited partnership*.

[3] M. Osilike and D. Igbokwe. (2000). Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations. Computers and Mathematics with Applications, 40, 559–567.

[4] P. Thongpaen, A. Kaewkhao, N. Phudolsitthiphat, S. Suantai, and W. Inthakon. (2021). Weak and strong convergence theorems for common attractive points of widely more generalized hybrid mappings in Hilbert spaces. *Mathematics 2021*, 9:1–12.

[5] Stampacchia, Guido (1964). Formes bilineaires coercitives sur les ensembles convexes, Comptes rendus hebdomadaires des séances de l'Académie des sciences, 258: 4413–4416

[6] A. Tomar, and M. C. Joshi. (2021). Fixed Point Theory and its Applications to Real World Problems. (1). *The United States of America*: Nova Science Publishers.

[7] D. Chan, and J. S. Pang. (1982) The Generalized Quasi-Variational Inequality Problem. *Mathematics of Operations Research*. 7(2):211-222.

[8] L. Zeng, and J. Yao. (2006). Strong Convergence Theorem by An Extragradient Method for Fixed Point Problems and Variational Inequality Problems. Taiwanese Journal of Mathematics.10(5): 1293-1303.

[9] H. Zegeye1 and n. Shahzad. (2012). Strong convergence theorem for a common point of solution of variational inequality and fixed-point problem. Advances in fixed point theory, 2(4): 374-397.

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