INTERMIXED METHODS FOR SOLVING SPLIT MODIFIED VARIATIONAL INEQUALITY PROBLEMS

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Abstract

The aim of this research was to propose the split of modified variational inequality problems (SMVIP), by using the concept of the modified generalized system of variational inequalities (MGSV) and to introduce an iterative algorithm of two sequences which depend on each other by using the intermixed method. Secondly, we proved a strong convergence theorem for solving fixed point problems of mappings and we treated two variational inequality problems which form an approximate the SMVIP. Then we obtained the additional results involving the split minimization problem and the split variational inequality problem. Thirdly, we introduced a new subgradient extragradient algorithm utilizing the concept of the set of solutions of the SMVIP and proved a weak convergence theorem for such an algorithm. Consequently, we also applied these results to approximate the split minimization problem. Finally, we gave numerical examples to support our main results.

Keywords : nonlinear mapping, the intermixed algorithm, fixed point, subgradient extragradient, strong convergence theorem, variational inequality problem

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Introduction

1.1 The intermixed method

Fixed point theory is essential to various theoretical and applied fields, such as the existence of solutions, the existence of orbits in dynamical systems, programming, economics, game theory, engineering, etc. Let C be a nonempty closed convex subset of a real Hilbert space H with its inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. A fixed point of a mapping T is a point x such that $x = Tx$, where $T : C \to C$ is a nonlinear mapping. The existence of a solution to a theoretical or real world problem is equivalent to the existence of a fixed point for a suitable map or operator. Therefore, fixed point theory is involved with finding conditions on set C and the mapping $T : C \to C$ to guarantee the existence and uniqueness of fixed points. Furthermore, many mathematicians have been studying about the structure of fixed point set.

Iterative methods for finding fixed points of nonexpansive mappings are an important topic in the theory of weak and strong convergence theorem, see for example [29, 32, 51] and the references therein.

Over recent decades, many authors have constructed various types of iterative methods to approximate fixed points. In 1953, Mann [27] introduced the Mann iteration which is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where $x_0 \in C$ is chosen arbitrarily and $\alpha_n \in [0, 1], T : C \to C$ is a mapping. If T is a nonexpansive mapping, the sequence $\{x_n\}$ be generated by (1.1) converges weakly to an element of $F(T)$.

It is well known that in an infinite-dimensional Hilbert space, the normal Mann's iterative algorithm [27] is only weakly convergent.

It is clear that strictly pseudo-contractions are more general than nonexpansive mappings, and therefore they have a wider range of applications. Therefore, it is important to develop the theory of iterative methods for strictly pseudo-contractions. Indeed, Browder and Petryshyn [1] proved that if the sequence $\{x_n\}$ is generated by (1.1) with a constant control parameter $\alpha_n \equiv \alpha$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges weakly to a fixed point of the strictly pseudo-contraction T. Moreover, many mathematicians proposed iterative algorithms and proved the strong convergence theorems for a nonexpansive mapping and a κ-strictly pseudo-contractive mapping in Hilbert space to find their fixed points, see for example [18, 21, 24, 45].

To prove the strong convergence of iterations determined by nonexpansive mapping, Moudafi [29] established a theorem for finding fixed points of nonexpansive mappings. More precisely, he established the following result, known as the *viscosity approximation method*.

Theorem 1.1. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let S be a nonexpansive mapping of C into itself such that $F(S)$ is nonempty. Let f be a contraction of C into itself and let $\{x_n\}$ be a sequence defined as follows:

$$\begin{cases} x_1 \in C \text{ is arbitrarily chosen}, \\ x_{n+1} = \frac{1}{1+\varepsilon_n} S x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n), \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\varepsilon_n\}$ is a sequence of positive real numbers having to go to zero. Then the sequence $\{x_n\}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$ and $P_{F(S)}$ is a metric projection of $\mathcal{H}$ onto $F(S)$.

The Moudafi viscosity approximation method can be applied to elliptic differential equations, linear programming, convex optimization and monotone inclusions, it has been widely studied in the literature (see [13, 41, 43]).

To construct an iterative algorithm such that it converges strongly to the fixed points of a finite family of strictly pseudo-contractions by using the concept of the viscosity approximation method (1.2) and Manns iteration (1.1), Yao et al. [48] proposed the intermixed algorithm for two strictly pseudo-contractions as follows:

Algorithm 1. *For arbitrarily given $x_0 \in C$, $y_0 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n],\ n \geq 0, \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_C[\alpha_n g(x_n) + (1-k-\alpha_n)y_n + kSy_n],\ n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of real number in (0,1), $T, S : C \to C$ are two strictly λ-pseudo-contractions, $f : C \to \mathcal{H}$ is a ρ_1-contraction and $g : C \to \mathcal{H}$ is a ρ_2-contraction, $k \in (0, 1-\lambda)$ is a constant.

Then they proved the strong convergence theorem of the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.3) as follows:

Theorem 1.2. Suppose that $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$. Assume the following conditions are satisfied:

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

$(C2)$ $\beta_n \in [\xi_1, \xi_2] \subset (0, 1)$ for all $n \geq 0$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.3) converge strongly to $P_{fix(T)} f(y^*)$ and $P_{fix(S)} g(x^*)$, respectively.

If putting $C = \mathcal{H}$ and $\beta_n = 1$ in (1.3), we have

$$\begin{cases} x_{n+1} = \alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n,\ n \geq 0, \\ y_{n+1} = \alpha_n g(x_n) + (1-k-\alpha_n)y_n + kSy_n,\ n \geq 0, \end{cases} \tag{1.4}$$

which is a modified version of viscosity approximation method. Observe that the sequence $\{x_n\}$ and $\{y_n\}$ are mutually dependent on each other.

1.2 Variational inequality problem

Let $B : C \to \mathcal{H}$. The *variational inequality problem* is to find a point $u^* \in C$ such that

$$\langle Bu^*, v - u^* \rangle \geq 0, \tag{1.5}$$

for all $v \in C$. The set of solutions of (1.5) is denoted by $VI(C, B)$. Historically the variational inequality was introduced by Stampachhia [36] in 1964. It is known that the variational inequality, as a strong and important tool, has already been studied for a wide class of optimization problems in economics, and equilibrium problems arising in physics and several other branches of pure and applied sciences, see for example [8, 30, 44, 47].

In 1999, Verma[40] considered *the new system of variational inequalities,* which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \ \forall x \in C, \\ &\langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, \ \ \forall x \in C, \end{aligned} \tag{1.6}$$

where $A, B : C \to H$ are two mappings and $\lambda, \mu > 0$ are two constants. If we add up the requirement that $x^* = y^*$, then the problem (1.6) reduces to the variational inequality problem[36].

In 2008, Ceng et al.[7] introduced the following *general system of variational inequalities,* which involves finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \ \forall x \in C, \\ &\langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \ \ \forall x \in C, \end{aligned} \tag{1.7}$$

where $A, B : C \to H$ are two different mappings and $\lambda, \mu > 0$ are two constants. In particular, if we put $A = B$, then the problem (1.7) reduces to system of variational[40].

In 2019, Siriyan and Kangtunyakarn[34] introduced the following *modified generalized system of variational inequalities(MGSV),* which involves finding (x^*, y^*, z^*) $\in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, \ \ \forall x \in C, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, \ \ \forall x \in C, \\ \langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, \ \ \forall x \in C, \end{cases} \tag{1.8}$$

where $D_1, D_2, D_3 : C \to H$, $\lambda_1, \lambda_2, \lambda_3 > 0$ and $a \in [0, 1]$.
If putting $a = 0$, in (1.8), we have

$$\begin{aligned} &\langle x^* - (I - \lambda_1 D_1)y^*, x - x^* \rangle \geq 0, \ \ \forall x \in C, \\ &\langle y^* - (I - \lambda_2 D_2)z^*, x - y^* \rangle \geq 0, \ \ \forall x \in C, \\ &\langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, \ \ \forall x \in C, \end{aligned} \tag{1.9}$$

which is generalized system of variational inequalities modified by Ceng et al. [7],

Then they prove the Lemma releted to this problem as follows.

Lemma 1.3. [34] Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2, D_3 : C \to H$ be three mappings. For every $\lambda_1, \lambda_2, \lambda_3 > 0$ and $a \in [0, 1]$. The following statement are equivalent

(i) $(x^*, y^*, z^*) \in C \times C \times C$ is a solution of problem (1.8)

(ii) x^* is a fixed point of the mapping G, i.e. $x^* \in F(G)$, defined the mapping $G : C \to C$ by

$G(x) = P_C(I - \lambda_1 D_1)(ax + (1 - a)P_C(I - \lambda_2 D_2)(ax + (1 - a)P_C(I - \lambda_3 D_3)x)),$
$\forall x \in C$, where $y^* = P_C(I - \lambda_2 D_2)\,(ax^* + (1 - a)z^*)$ and $z^* = P_C(I - \lambda_3 D_3)x^*$.

Remark 1.4. [34] If D_1, D_2, D_3, in Lemma 1.3, are d_1, d_2, d_3-inverse strongly monotone, respectively, then G is nonexpansive mapping, where $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2, d_3\}$.

The split feasibility problem(SFP) is to find a point $x \in C$ and $Ax \in Q$. This problem was introduced by Censor and Elfving[10]. The set of all solution(SFP) is denoted by $\Gamma = \{x \in C; Ax \in Q\}$. In fact, it has been extensively investigated in the literature (see [2]-[3],[12],[14],[23]).

In 2018, motivated by Ceng et al[10] and the concept of the SFP, Siriyan and Kangtunyakarn [33] introduced *the split general system of variational inequalities problem (SGSV),* which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, \ \ \forall x \in C, \\ &\langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, \ \ \forall x \in C, \end{aligned} \tag{1.10}$$

and find $(\bar{x^*} = Dx^*, \bar{y^*} = Dy^*) \in Q \times Q$ such that

$$\begin{aligned} &\left\langle \alpha \bar{A}\bar{y^*} + \bar{x^*} - \bar{y^*}, \bar{x} - \bar{x^*} \right\rangle \geq 0, \ \ \forall \bar{x} \in Q, \\ &\left\langle \gamma \bar{B}\bar{x^*} + \bar{y^*} - \bar{x^*}, \bar{x} - \bar{y^*} \right\rangle \geq 0, \ \ \forall \bar{x} \in Q, \end{aligned} \tag{1.11}$$

where $A, B : C \to H_1$ and $\bar{A}, \bar{B} : Q \to H_2$ are four different mappings,$\lambda, \mu, \alpha, \gamma > 0$, and $D : H_1 \to H_2$ is a bounded linear operator. The set of all solutions of (1.10) and (1.11) are denoted by $\Omega_{A,B}$ and $\Omega_{\bar{A},\bar{B}}$ respectively. The set of all solutions of the SGSV is denoted by $\Omega^{A,B}_{\bar{A},\bar{B}} = \{(x^*, y^*) \in \Omega_{A,B} : (\bar{x^*}, \bar{y^*}) \in \Omega_{\bar{A},\bar{B}}\}$, Moreover, if we put $A = B = \bar{A} = \bar{B} = 0, x^* = y^*$ and $\bar{x^*} = \bar{y^*}$ in (1.10) and (1.11), the SGSV reduces to the SFP.

Motivated by Siriyan and Kangtunyakarn[34] we introduce a new split problem as follows:

Problem 1 Let C, Q be nonempty subsets of real Hilbert H_1, H_2, respectively, Let $A : H_1 \to H_2$ be a bounded linear operator. A new split problem is to find $(x^*, y^*, z^*) \in$

$C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \zeta D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, \forall x \in C, \\ \langle y^* - (I - \zeta D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, \forall x \in C, \\ \langle z^* - (I - \zeta D_3)x^*, x - z^* \rangle \geq 0, \forall x \in C, \end{cases} \tag{1.12}$$

and find $(\bar{x^*} = Ax^*,\ \bar{y^*} = Ay^*,\ \bar{z^*} = Az^*) \in Q \times Q \times Q$ such that

$$\begin{cases} \langle \bar{x^*} - (I - \bar{\zeta}\bar{D_1})(a\bar{x^*} + (1-a)\bar{y^*}), \bar{x} - \bar{x^*} \rangle \geq 0, \forall \bar{x} \in Q, \\ \langle \bar{y^*} - (I - \bar{\zeta}\bar{D_2})(a\bar{x^*} + (1-a)\bar{z^*}), \bar{x} - \bar{y^*} \rangle \geq 0, \forall \bar{x} \in Q, \\ \langle \bar{z^*} - (I - \bar{\zeta}\bar{D_3})\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, \forall \bar{x} \in Q, \end{cases} \tag{1.13}$$

where $D_1, D_2, D_3 : C \to H_1$, $\bar{D_1}, \bar{D_2}, \bar{D_3} : Q \to H_2$ are six different mappings, $\zeta, \bar{\zeta} > 0$, and $a \in [0, 1]$ which is called *the split of modified variational inequality problems (SMVIP)*. The sets of all solutions of (1.12) and (1.13) are denoted by Ψ_{D_1,D_2,D_3} and $\Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}$ respectively. The set of all solutions of the SMVIP is denoted by $\Psi^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3},}$ that is

$$\Psi^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3}} = \{(x^*, y^*, z^*) \in \Psi_{D_1,D_2,D_3} : (\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}\}.$$

If we put $D_1 \equiv D_2 \equiv D_3 \equiv \bar{D_1} \equiv \bar{D_2} \equiv \bar{D_3} = 0$, $x^* = y^* = z^*$ and $\bar{x}^* = \bar{y}^* = \bar{z}^*$ in 1.12 and 1.13, we obtain the SMVIP is reduced to the SFP.

If putting $a = 0$ in (1.12) and (1.13) , we have

$$\begin{cases} \langle x^* - (I - \zeta D_1)y^*, x - x^* \rangle \geq 0, \forall x \in C, \\ \langle y^* - (I - \zeta D_2)z^*, x - y^* \rangle \geq 0, \forall x \in C, \\ \langle z^* - (I - \zeta D_3)x^*, x - z^* \rangle \geq 0, \forall x \in C, \end{cases}$$

and

$$\begin{cases} \langle \bar{x^*} - (I - \bar{\zeta}\bar{D_1})\bar{y^*}, \bar{x} - \bar{x^*} \rangle \geq 0, \forall \bar{x} \in Q, \\ \langle \bar{y^*} - (I - \bar{\zeta}\bar{D_2})\bar{z^*}, \bar{x} - \bar{y^*} \rangle \geq 0, \forall \bar{x} \in Q, \\ \langle \bar{z^*} - (I - \bar{\zeta}\bar{D_3})\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, \forall \bar{x} \in Q, \end{cases}$$

which is a modified the split general system of variational inequalities(SVIP)[33].

1.3 The subgradient extragradient algorithm

Several algorithms for solving the $VI(C, B)$ are projection algorithms that employ projections onto the feasible set C of the $VI(C, B)$, or onto some related set, in order to iteratively reach a solution. In 1976, Korpelevich [26] proposed an algorithm for solving the $VI(C, B)$ in Euclidean space, known as *the extragradient method*. In each iteration of her algorithm, in order to get the next iterate x^{k+1}, two orthogonal projections onto C are calculated, according to the following iterative step. Given the current iterate x^k, calculate

$$y^k = P_C(x^k - \tau f(x^k)), \tag{1.14}$$

$$x^{k+1} = P_C(x^k - \tau f(y^k)), \tag{1.15}$$

for all $k \in \mathbb{N}$, where τ is some positive number and P_C denotes the Euclidean least distance projection onto C.

The convergence was proved in [26] under the assumptions of Lipschitz continuity and pseudo-monotonicity. However there is still the need to calculate two projections onto C. If the set C is simple enough so that projections onto it can be easily computed, but if C is a general closed and convex set, a minimal distance problem has to be solved twice in order to obtain the next iterate. This might seriously affect the efficiency of the extragradient method. Korpelevich's extragradient method has been widely studied in the literature; see ([4, 5, 6, 9, 16, 19, 25, 35, 46, 49]) and the references therein.

In the past decade years, Censor et al. [11] developed the subgradient extragradient algorithm in Euclidean space, in which they replaced the (second) projection (1.15) onto C by a projection onto a specific constructible half-space as follows:

Algorithm 1 (The subgradient extragradient algorithm)

Step 0 : Select a starting point $x^0 \in H$ and $\tau > 0$, and set $k = 0$.

Step 1 : Given the current iterate x^k, compute

$$y^k = P_C(x^k - \tau f(x^k)),$$

construct the half-space T_k the bounding hyperplane of which supports C at y^k,

$$T_k := \{w \in H | \langle (x^k - \tau f(x^k)) - y^k, w - y^k \rangle \leq 0\}, \tag{1.16}$$

and calculate the next iterate

$$x^{k+1} = P_{T_k}(x^k - \tau f(y^k)).$$

Step 2 : If $x^k = y^k$ then stop. Otherwise, set $k \leftarrow (k+1)$ and return to step 1.

Furthermore, under some control conditions, they proved weak convergence theorems for their algorithms.

1.4 Objectives of the study

1) To propose the new split of modified variational inequality problems (SMVIP).
2) To propose some new mathematical tools and properties for the new split of modified variational inequality problems (SMVIP).
3) To define a new iterative method for approximating the solutions of two variational inequality problems which form an approximate the SMVIP.
4) To prove the strong convergence theorem for solving fixed point problems of nonlinear mappings and two variational inequality problems and solving the SMVIP.

5) To define a new subgradient extragradient algorithm utilizing the concept of the set of solutions of the SMVIP.

6) To prove a weak convergence theorem for such a new subgradient extragradient algorithm.

7) To give numerical examples to support our main theorems.

1.5 Scopes of the study

1) Study the variational inequality problems in a Hilbert space.

2) Investigate the fixed problems of nonlinear mappings including nonexpansive and α-inverse strongly monotone mappings.

3) All strong and weak convergence theorems are considered and proved in Hilbert spaces.

1.6 Research methodology

1) Study advanced topics in fixed point theory for nonexpansive and α-inverse strongly monotone mappings.

2) Study background in a real Hilbert space.

3) Study the variational inequality problems and the modified generalized system of variational inequalities (MGSV).

4) Collect and study research papers concerning fixed point theorem.

5) Determine the objectives and scope of the research.

6) Define a new split of modified variational inequality problems (SMVIP).

7) Define a new iterative method for approximating the solutions of two variational inequality problems which form an approximate the SMVIP.

8) Prove the strong convergence theorem for solving fixed point problems of non-linear mappings and two variational inequality problems and solving the SMVIP.

9) Produce tools and properties for the new split of modified variational inequality problems (SMVIP).

10) Define a new subgradient extragradient algorithm utilizing the concept of the set of solutions of the SMVIP.

11) Prove a weak convergence theorem for such a new subgradient extragradient algorithm.

12) Provide applications and examples to support our main theorems.
13) Conclude the result, make suggestions for further works and write the thesis.

1.7 Benefits of the study

1) Obtain the strong convergence theorem for solving fixed point problems of nonlinear mappings and two variational inequality problems and solving the new split of modified variational inequality problems (SMVIP).
2) Obtain new mathematical tools for the properties of the new split of modified variational inequality problems (SMVIP).
3) Obtain the weak convergence theorem for solving fixed point problems and solving the SMVIP.

This thesis consists of five chapters as follows:

In chapter 1, we show the background of this thesis, that is, iterative methods for fixed point theorems and some of definitions and properties of a finite family of nonlinear mappings and variational inequality problem.

In chapter 2, we give the definitions and properties of fixed point problem, the metric projection and about the strongly positive bounded linear operators in Hilbert spaces that are necessary to prove our main theorems in the next chapter.

In , firstly we introduce the split of modified variational inequality problems (SMVIP), by using the concept of the modified generalized system of variational inequalities (MGSV) and introduce an iterative algorithm of two sequences which depend on each other by using the intermixed method. Secondly, we prove a strong convergence theorem for solving fixed point problems of mappings and we treat two variational inequality problems which form an approximate the SMVIP. Then we obtain the additional results involving the split minimization problem and the split variational inequality problem. Thirdly, we introduce a new subgradient extragradient algorithm utilizing the concept of the set of solutions of the SMVIP and prove a weak convergence theorem for such an algorithm. Consequently, we also apply these results to approximate the split minimization problem.

In chapter 4, we give numerical examples for our theorems in the previous chapter.

In chapter 5, we describe the conclusion of the thesis.

Chapter 2

Preliminaries

We denote the weak convergence and the strong convergence by " $\rightharpoonup$" and " $\rightarrow$", respectively.

2.1 Fundamental properties in Hilbert spaces

Lemma 2.1. Let H be a real Hilbert space. Then the following identities hold:

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H;$

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H.$

Definition 2.1 (Weak convergence [15]). A sequence $\{x_n\}$ of vectors in an inner product space K is called weakly convergent to a vector x in K if

$$\langle x_n, y\rangle \rightarrow \langle x, y\rangle \text{ as } n \rightarrow \infty \text{ for every } y \in K.$$

Definition 2.2 (Strong convergence [15]). A sequence $\{x_n\}$ of vectors in an inner product space K is called strongly convergent to a vector x in K if

$$\|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Theorem 2.2 ([15]). A strongly convergence sequence is weakly convergence (to the same limit), that is, $x_n \rightarrow x$ implies $x_n \rightharpoonup x$.

Remark 2.3 ([39]). If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Lemma 2.4 ([39]). Let $\{x_n\}$ be a Cauchy sequence of an inner product space C such that $x_n \rightharpoonup x$. Then $x_n \rightarrow x$.

Definition 2.3 ([39]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called lower semicontinuous if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\} \text{ is closed.}$$

Moreover, f is called convex if for any $x_1, x_2 \in C$ and $t \in (0, 1)$,

$$f(tx_1 + (1 - t)x_2) \leq tf(x_1) + (1 - t)f(x_2).$$

Similarly, f is said to be concave if for any $x_1, x_2 \in C$ and $t \in (0, 1)$,

$$f(tx_1 + (1 - t)x_2) \geq tf(x_1) + (1 - t)f(x_2).$$

Theorem 2.5 ([39]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Suppose that $\{x_n\} \subset C$ and $x_n \rightharpoonup x$. Then $x \in C$.

Theorem 2.6 ([39]). Let $\{a_n\}$ be a bounded of real numbers. Then, there exists subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Lemma 2.7. [42] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{i=1}^{\infty} \alpha_n = \infty$,

(ii) lim $sup_{n\to\infty}\dfrac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.8. Let $\mathcal{H}$ be a real Hilbert space and let C be a nonempty closed convex subset of $\mathcal{H}$. Let the $\{x^k\}_{k=0}^{\infty} \subset \mathcal{H}$ be Fejer-monotone with respect to C, i.e., for every $u \in C$,

$$\|x^{k+1} - u\| \leq \|x^k - u\|, \ \forall k \geq 0.$$

Then $\{P_C x^k\}_{k=0}^{\infty}$ converges strongly to some $z \in C$.

Lemma 2.9 ([31]). Each Hilbert space $\mathcal{H}$ satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset \mathcal{H}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in \mathcal{H}$ with $y \neq x$.

2.2 Fixed point sets of nonexpansive mappings and strictly pseudo-contractive mappings

Let $\mathfrak{T}$ be a self-mapping of ζ. We use $F(\mathfrak{T})$ to denote the set of fixed points of $\mathfrak{T}$ (i.e., $F(\mathfrak{T}) = \{x \in \zeta : \mathfrak{T}x = x\}$).

Definition 2.4. Let the mapping $\mathfrak{T} : C \to C$. Then $\mathfrak{T}$ is called

1) a nonexpansive mapping if

$$\|\mathfrak{T}x - \mathfrak{T}y\| \leq \|x - y\|, \forall x, y \in C.$$

2) κ-strictly pseudo-contraction if there exists a constant $\kappa \in [0,1)$ such that

$$\|\mathfrak{T}x - \mathfrak{T}y\|^2 \leq \|x - y\|^2 + \kappa \|(I - \mathfrak{T})x - (I - \mathfrak{T})y\|^2, \forall x, y \in C.$$

Note that the class of κ-strictly pseudo-contractions includes the class of nonexpansive mappings, that is, nonexpansive mapping is a 0-strictly pseudo-contractive mapping and if $\kappa = 1$, then a mapping $\mathfrak{T}$ is called pseudo-contractive mapping.

Definition 2.5. Let the mapping $\mathfrak{T} : C \to C$. Then $\mathfrak{T}$ is called an α-contractive mapping if there exists $\alpha \in (0, 1)$ such that

$$\|\mathfrak{T}x - \mathfrak{T}y\| \leq \alpha\|x - y\|, \forall x, y \in C.$$

Obviously, if $\mathfrak{T}$ is contractive, then $\mathfrak{T}$ is nonexpansive. That is, the class of nonexpansive mappings includes the class of contractive mappings.

Theorem 2.10 ([39]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\mathfrak{T}$ be a nonexpansive mapping of C into itself. Then $Fix(\mathfrak{T})$ is closed and convex.

Theorem 2.11 ([39]). Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $\mathfrak{T}$ be a nonexpansive mapping of C into itself. Then $\mathfrak{T}$ has a fixed point in C.

Lemma 2.12 ([50]). Let H be a Hilbert space, C be a closed convex subset of H. If $\mathfrak{T}$ is a κ-strictly pseudo-contractive mapping on C, then the fixed point set $Fix(\mathfrak{T})$ is closed convex, so that the projection $P_{Fix(\mathfrak{T})}$ is well defined.

Lemma 2.13. [22] Let C be a nonempty closed convex subset of a real Hilbert space H and let $\mathfrak{T} : C \to C$ be a κ-strictly pseudo-contractive mapping with $Fix(\mathfrak{T}) \neq \emptyset$. Then, there hold the following statement:

(i) $Fix(\mathfrak{T}) = VI(C, I - \mathfrak{T})$;

(ii) For every $u \in C$ and $v \in Fix(\mathfrak{T})$,

$$\|P_C(I - \lambda(I - \mathfrak{T}))u - v\| \leq \|u - v\|,$$

for $u \in C$ and $v \in Fix(\mathfrak{T})$ and $\lambda \in (0, 1 - \kappa)$.

2.3 Properties of the metric projection

Definition 2.6 (Metric projection [39]). The (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the following properties:

(i) $P_C x \in C$,

(ii) $\|x - P_C x\| = \min_{y \in C} \|x - y\|$,

(iii) If C is a hyperplane, it follows that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.1}$$

$\forall x \in \mathcal{H}$, $y \in C$.

Lemma 2.14 ([38])**.** For a given $z \in H$ and $u \in C$,

$$u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \forall v \in C.$$

It is well-known that P_C is a firmly nonexpansive mapping of H onto C, that is

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \forall x, y \in H.$$

Lemma 2.15 ([38])**.** Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

2.4 Properties of strongly positive bounded linear operators in Hilbert spaces

Definition 2.7 ([39])**.** Let E and F be linear spaces with the same scalars, and let T be a mapping of E into F. Then T is called *linear* if for any $x, y \in E$ and any scalar $\alpha \in \mathbb{R}$,

$$T(x + y) = T(x) + T(y) \text{ and } T(\alpha x) = \alpha T(x).$$

In particular, for the case of $\mathbb{F} = \mathbb{R}$, T is called a *linear functional*.

Note that if $T : E \to F$ is a linear mapping, then

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \forall x, y \in E \text{ and } \alpha, \beta \in \mathbb{R}.$$

Definition 2.8 ([39])**.** Let E and F be normed linear spaces with the same scalars, and let T be a linear mapping of E into F. Then T is called *bounded* if there exists $K \geq 0$ such that

$$\|T(x)\| \leq K\|x\| \text{ for all } x \in E.$$

Let T be a bounded linear mapping of E into F. So, we have that for $x \in E$ with $\|x\| \leq 1$,

$$\|T(x)\| \leq K, \tag{2.2}$$

where $T(x)$ is often denoted by Tx.

For a bounded linear mapping T of E into F, we define its norm by

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|. \tag{2.3}$$

For such $\|T\|$, we have the following results.

Definition 2.9 ([39])**.** Let E and F be normed linear spaces and let T be a bounded linear mapping of E into F. Then the following hold:

1) $\|Tx\| \leq \|T\|\|x\|$ for all $x \in E$,

2) $\|T\| = \sup_{\|x\|=1} \|Tx\|$.

Definition 2.10 ([15]). Let A be a bounded operator on a Hilbert space H. If $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x \in H$, then A is called the *self-adjoint* operator.

Theorem 2.16 ([15]). Let T be a bounded linear self-adjoint operator on a Hilbert space H. Then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle| .$$

Definition 2.11 ([15]). A self-adjoint operator A is called *positive* if $\langle Ax, x\rangle \geq 0$ for all $x \in H$.

Definition 2.12 ([28]). A self-adjoint operator A is a strongly positive operator on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma} \|x\|^2, \text{ for all } x \in H.$$

Definition 2.13 ([17]). Let A be an operator of C into itself. Then, A is called α-inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2, \forall x, y \in C.$$

Definition 2.14 ([20]). Let T be a bounded linear operator on a Hilbert space H. The *spectral radius* of T, denoted by $r_\sigma(T)$, is the number defined by

$$r_\sigma(T) = sup\{|\lambda| : \lambda \in \sigma(T)\},$$

where $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)(x) = 0, \text{for some } 0 \neq x \in H\}$.

Chapter 3

Strong Convergence Theorems

3.1 Strong convergence theorem for solving the split of modified variational inequality problems (SMVIP)

In this section, we prove the lemmas related to the set of fixed points of nonlinear mapping with the SMVIP. Utilizing our lemma we prove the strong convergence theorem for solving the SMVIP and two variational inequality problems by using problem 1.

The following lemmas are needed to prove the main theorem.

Lemma 3.1. Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3-inverse strongly monotone respectively, where $\zeta \in (0, 2d^*)$ with d^* =min$\{d_1, d_2, d_3\}$. Let $\bar{D_1}, \bar{D_2}, \bar{D_3} : Q \to H_2$ be $\bar{d_1}, \bar{d_2}, \bar{d_3}$-inverse strongly monotone respectively, where $\bar{\zeta} \in (0, 2\hat{d})$ with $\hat{d}$ =min$\{\bar{d_1}, \bar{d_2}, \bar{d_3}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C : C \to C$ by

$$M_C(x) = P_C(I - \zeta D_1)(ax + (1-a)P_C(I - \zeta D_2)(ax + (1-a)P_C(I - \zeta D_3)x)),$$

$\forall x \in C$, and define $M_Q : Q \to Q$ by

$$M_Q(\hat{x}) = P_Q(I - \bar{\zeta}\bar{D_1})(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\bar{D_2})(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\bar{D_3})\hat{x})),$$

$\forall \hat{x} \in Q$. Define $M : C \to C$ by $M(x) = M_C(x - \eta A^*(I - M_Q)Ax)$ for all $x \in C$. Then M is a nonexpansive mapping for all $x \in C$.

Proof. Let $x, y \in C$. From the definition of M, we have

$$\begin{aligned}
\|Mx - My\|^2 &= \|M_C(x - \eta A^*(I - M_Q)Ax) - M_C(y - \eta A^*(I - M_Q)Ay)\|^2 \\
&\leq \|(x - \eta A^*(I - M_Q)Ax) - (y - \eta A^*(I - M_Q)Ay)\|^2 \\
&= \|x - y - \eta A^*[(I - M_Q)Ax - (I - M_Q)Ay]\|^2 \\
&= \|x - y\|^2 - 2\eta\langle x - y, A^*[(I - M_Q)Ax - (I - M_Q)Ay]\rangle \\
&\qquad + \eta^2\|A^*[(I - M_Q)Ax - (I - M_Q)Ay]\|^2 \\
&= \|x - y\|^2 - 2\eta\langle A(x - y), (I - M_Q)Ax - (I - M_Q)Ay\rangle \\
&\qquad + \eta^2\|A^*[(I - M_Q)Ax - (I - M_Q)Ay]\|^2 \\
&= \|x - y\|^2 - 2\eta\langle Ax - Ay + M_QAx - M_QAx + M_QAy \\
&\qquad - M_QAy, (I - M_Q)Ax - (I - M_Q)Ay\rangle
\end{aligned}$$

$$\begin{aligned}
&+\eta^2\|A^*[(I-M_Q)Ax-(I-M_Q)Ay]\|^2\\
&=\|x-y\|^2-2\eta[\langle Ax-Ay-M_QAx+M_QAy,(I-M_Q)Ax\\
&\quad-(I-M_Q)Ay\rangle-\langle M_QAy-M_QAx,(I-M_Q)Ax\\
&\quad-(I-M_Q)Ay\rangle]+\eta^2\|A^*[(I-M_Q)Ax-(I-M_Q)Ay]\|^2\\
&=\|x-y\|^2-2\eta[\langle(I-M_Q)Ax-(I-M_Q)Ay,(I-M_Q)Ax\\
&\quad-(I-M_Q)Ay\rangle-\langle M_QAy-M_QAx,(I-M_Q)Ax\\
&\quad-(I-M_Q)Ay\rangle]+\eta^2\|A^*[(I-M_Q)Ax-(I-M_Q)Ay]\|^2\\
&\leq\|x-y\|^2-2\eta[\|(I-M_Q)Ax-(I-M_Q)Ay\|^2\\
&\quad-\frac{1}{2}\|(I-M_Q)Ax-(I-M_Q)Ay\|^2]\\
&\quad+\eta^2\|A^*[(I-M_Q)Ax-(I-M_Q)Ay]\|^2\\
&=\|x-y\|^2-\eta\|(I-M_Q)Ax-(I-M_Q)Ay\|^2\\
&\quad+\eta^2\|A^*[(I-M_Q)Ax-(I-M_Q)Ay]\|^2\\
&\leq\|x-y\|^2-\eta\|(I-M_Q)Ax-(I-M_Q)Ay\|^2\\
&\quad+\eta^2L\|(I-M_Q)Ax-(I-M_Q)Ay\|^2\\
&=\|x-y\|^2-\eta(1-\eta L)\|(I-M_Q)Ax-(I-M_Q)Ay\|^2\\
&\leq\|x-y\|^2. \qquad (3.1)
\end{aligned}$$

It implies that $\|Mx-My\|\leq\|x-y\|$.
Hence, M is a nonexpansive mapping.

□

Remark 3.2. From the study of Lemma 3.1, we have

$$\|(x-\eta A^*(I-M_Q)Ax)-(y-\eta A^*(I-M_Q)Ay)\|^2$$

$$\leq\|x-y\|^2-\eta(1-\eta L)\|(I-M_Q)Ax-(I-M_Q)Ay\|^2.$$

for all $x,y\in H_1$.

Lemma 3.3. Let C,Q be nonempty closed convex subsets of H_1,H_2, respectively. Let $D_1,D_2,D_3:C\rightarrow H_1$ be d_1,d_2,d_3-inverse strongly monotone respectively, where $\zeta\in(0,2d^*)$ with $d^*=$min$\{d_1,d_2,d_3\}$. Let $\bar{D_1},\bar{D_2},\bar{D_3}:Q\rightarrow H_2$ be $\bar{d_1},\bar{d_2},\bar{d_3}$-inverse strongly monotone respectively, where $\bar{\zeta}\in(0,2\hat{d})$ with $\hat{d}=$ min$\{\bar{d_1},\bar{d_2},\bar{d_3}\}$. Let $A:H_1\rightarrow H_2$ be a bounded linear operator with adjoint A^* and $\eta\in(0,\frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C:C\rightarrow C$ by

$$M_C(x)=P_C(I-\zeta D_1)(ax+(1-a)P_C(I-\zeta D_2)(ax+(1-a)P_C(I-\zeta D_3)x)),$$

$\forall x\in C$, and define $M_Q:Q\rightarrow Q$ by

$$M_Q(\hat{x})=P_Q(I-\bar{\zeta}\bar{D_1})(a\hat{x}+(1-a)P_Q(I-\bar{\zeta}\bar{D_2})(a\hat{x}+(1-a)P_Q(I-\bar{\zeta}\bar{D_3})\hat{x})),$$

$\forall \hat{x} \in Q$. Assume

$$\Psi^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3}} = \{(x^*, y^*, z^*) \in \Psi_{D_1,D_2,D_3} : (\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}\} \neq \emptyset.$$

The following statements are equivalent

(i) $(x^*, y^*, z^*) \in \Psi^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3}}$,

(ii) $x^* = M_C(x^* - \eta A^*(I - M_Q)Ax^*)$,

where $y^* = P_C(I-\zeta D_2)(ax^*+(1-a)z^*)$, $z^* = P_C(I-\zeta D_3)x^*$, $\bar{x^*} = Ax^* = P_Q(I-\bar{\zeta}\bar{D_1})(a\bar{x^*}+(1-a)\bar{y^*})$, $\bar{y^*} = Ay^* = P_Q(I-\bar{\zeta}\bar{D_2})(a\bar{x^*}+(1-a)\bar{z^*})$ and $\bar{z^*} = Az^* = P_Q(I-\bar{\zeta}\bar{D_3})\bar{x^*}$.

Proof. Let the following condition hold.

$(i) \Rightarrow (ii)$ Let $(x^*, y^*, z^*) \in \Psi^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3}}$. We have $(x^*, y^*, z^*) \in \Psi_{D_1,D_2,D_3}$ and $(\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}$, where $\bar{x^*} = Ax^*$, $\bar{y^*} = Ay^*$ and $\bar{z^*} = Az^*$. Since $(x^*, y^*, z^*) \in \Psi_{D_1,D_2,D_3}$, we obtain that

$$\langle x^* - (I - \zeta D_1)(ax^* + (1-a)y^*), x - x^*\rangle \geq 0, \forall x \in C,$$

$$\langle y^* - (I - \zeta D_2)(ax^* + (1-a)z^*), x - y^*\rangle \geq 0, \forall x \in C,$$

$$\langle z^* - (I - \zeta D_3)x^*, x - z^*\rangle \geq 0, \forall x \in C.$$

By the property of P_C, we obtain

$$x^* = P_C(I - \zeta D_1)(ax^* + (1-a)y^*),$$

$$y^* = P_C(I - \zeta D_2)(ax^* + (1-a)z^*),$$

$$z^* = P_C(I - \zeta D_3)x^*.$$

It implies that

$$\begin{aligned} x^* &= P_C(I - \zeta D_1)(ax^* + (1-a)P_C(I - \zeta D_2)(ax^* \\ &\quad + (1-a)P_C(I - \zeta D_3)x^*)) \\ &= M_C x^*. \end{aligned} \tag{3.2}$$

Since $(\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}$, we obtain that

$$\langle \bar{x^*} - (I - \bar{\zeta}\bar{D_1})(a\bar{x^*} + (1-a)\bar{y^*}), \bar{x} - \bar{x^*}\rangle \geq 0, \forall \bar{x} \in Q,$$

$$\langle \bar{y^*} - (I - \bar{\zeta}\bar{D_2})(a\bar{x^*} + (1-a)\bar{z^*}), \bar{x} - \bar{y^*}\rangle \geq 0, \forall \bar{x} \in Q,$$

$$\langle \bar{z^*} - (I - \bar{\zeta}\bar{D_3})\bar{x^*}, \bar{x} - \bar{z^*}\rangle \geq 0, \forall \bar{x} \in Q.$$

Then, we obtain

$$\bar{x^*} = P_Q(I - \bar{\zeta}\bar{D_1})(a\bar{x^*} + (1-a)\bar{y^*}),$$

$$\bar{y^*} = P_Q(I - \bar{\zeta}\bar{D_2})(a\bar{x^*} + (1-a)\bar{z^*}),$$

$$\bar{z^*} = P_Q(I - \bar{\zeta}\bar{D_3})\bar{x^*}.$$

That is

$$\begin{aligned}\bar{x^*} &= P_Q(I-\bar{\zeta}\bar{D_1})(a\bar{x^*}+(1-a)P_Q(I-\bar{\zeta}\bar{D_2})(a\bar{x^*}\\ &\qquad +(1-a)P_Q(I-\bar{\zeta_3}\bar{D_3})\bar{x^*}))\\ &= M_Q\bar{x^*}\\ &= M_QAx^*. \end{aligned} \tag{3.3}$$

It implies that

$$x^* = M_C(x^*-\eta A^*(I-M_Q)Ax^*) \tag{3.4}$$

$(ii) \Rightarrow (i)$ Let $x^* = M_C(x^*-\eta A^*(I-M_Q)Ax^*)$ where $y^* = P_C(I-\zeta D_2)(ax^*+(1-a)z^*)$, $z^* = P_C(I-\zeta D_3)x^*$, $\bar{x^*} = Ax^* = P_Q(I-\bar{\zeta}\bar{D_1})(a\bar{x^*}+(1-a)\bar{y^*})$, $\bar{y^*} = Ay^* = P_Q(I-\bar{\zeta}\bar{D_2})(a\bar{x^*}+(1-a)\bar{z^*})$ and $\bar{z^*} = Az^* = P_Q(I-\bar{\zeta}\bar{D_3})\bar{x^*}$ and $(u^*,v^*,w^*) \in \Psi^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3}}$.

From Remark 1.4, we have M_C is nonexpansive mapping on C. It is obvious that M_Q is nonexpansive mapping on Q.

From $(u^*,v^*,w^*) \in \Psi^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3}}$, and $(i) \Rightarrow (ii)$, we have $u^* = M_C(u^*-\eta A^*(I-M_Q)Au^*)$. Utilizing the same method as (3.3) we have $Au^* = M_QAu^*$ and and applying (3.1), we have

$$\|x^*-u^*\|^2 \leq \|x^*-u^*\|^2-\eta(1-\eta L)\|(I-M_Q)Ax^*-(I-M_Q)Au^*\|^2$$

It implies that

$$\bar{x^*} = Ax^* \in F(M_Q).$$

That is,
$\bar{x^*} = M_Q\bar{x^*} = P_Q(I-\bar{\zeta}\bar{D_1})(a\bar{x^*}+(1-a)P_Q(I-\bar{\zeta}\bar{D_2})(a\bar{x^*}+(1-a)P_Q(I-\bar{\zeta}\bar{D_3})\bar{x^*}))$. It follows that

$$\bar{x^*} = P_Q(I-\bar{\zeta}\bar{D_1})(a\bar{x^*}+(1-a)\bar{y^*}),$$

where $\bar{y^*} = P_Q(I-\bar{\zeta}\bar{D_2}(a\bar{x^*}+(1-a)\bar{z^*})$ and $\bar{z^*} = P_Q(I-\bar{\zeta}\bar{D_3})\bar{x^*}$. From Lemma 1.3, we have

$$(\bar{x^*},\bar{y^*},\bar{z^*}) \in \Psi_{\bar{D_1},\bar{D_2},\bar{D_3}} \tag{3.5}$$

From $Ax^* \in F(M_Q)$, we have

$$x^* = M_C(x^*-\eta A^*(I-M_Q)Ax^*) = M_Cx^*.$$

That is $x^* \in F(M_C)$. Then, we have
$x^* = P_C(I-\zeta D_1)(ax^*+(1-a)P_C(I-\zeta D_2)(ax^*+(1-a)P_C(I-\zeta D_3)x^*)) = M_Cx^*$. It follows that
$x^* = P_C(I-\zeta D_1)(ax^*+(1-a)y^*)$, where $y^* = P_C(I-\zeta D_2)(ax^*+(1-a)z^*)$, and $z^* = P_C(I-\zeta D_3)x^*$. From Lemma 1.3, we have

$$(x^*,y^*,z^*) \in \Psi_{D_1,D_2,D_3} \tag{3.6}$$

From $(3.5)$and (3.6), we have

$$(x^*, y^*, z^*) \in \Psi^{D_1, D_2, D_3}_{\bar{D_1}, \bar{D_2}, \bar{D_3}}.$$

□

Example 3.4. Let $\mathbb{R}$ be the set of real numbers, $C = [-50, 50] \times [-50, 50]$, $Q = [-100, 100] \times [-100, 100]$, $H_1 = H_2 = \mathbb{R}$. Let $D_1, D_2, D_3 : [-50, 50] \to \mathbb{R}^2$ defined by $D_1(x_1, x_2) = (x_1 - 3, x_2 - 3)$, $D_2(x_1, x_2) = (\frac{x_1-3}{2}, \frac{x_2-3}{2})$, $D_3(x_1, x_2) = (\frac{x_1-3}{3}, \frac{x_2-3}{3})$, for all $(x_1, x_2) \in C$. Let $\bar{D_1}, \bar{D_2}, \bar{D_3} : [-100, 100] \times [-100, 100] \to \mathbb{R}^2$ defined by $\bar{D_1}(x_1, x_2) = (x_1 - 6, x_2 - 6)$, $\bar{D_2}(x_1, x_2) = (\frac{x_1-6}{2}, \frac{x_2-6}{2})$, $\bar{D_3}(x_1, x_2) = (\frac{x_1-6}{3}, \frac{x_2-6}{3})$, for all $(x_1, x_2) \in Q$ respectively. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A(x_1, x_2) = (2x, 2x)$ and $A^* : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A^*(x_1, x_2) = (2x, 2x)$. Define $M_C : [-50, 50] \times [-50, 50] \to [-50, 50] \times [-50, 50]$ by $M_C(x) = P_C(I - \frac{2}{5}D_1)(\frac{1}{2}x + \frac{1}{2}P_C(I - \frac{2}{5}D_2)(\frac{1}{2}x + \frac{1}{2}P_C(I - \frac{2}{5}D_3)x))$, $\forall x \in C$, and define $M_Q : [-100, 100] \times [-100, 100] \to [-100, 100] \times [-100, 100]$ by $M_Q(\hat{x}) = P_Q(I - \frac{1}{7}\bar{D_1})(\frac{1}{2}\hat{x} + \frac{1}{2}P_Q(I - \frac{1}{7}\bar{D_2})(\frac{1}{2}\hat{x} + \frac{1}{2}P_Q(I - \frac{1}{7}\bar{D_3})\hat{x}))$, $\forall \hat{x} \in Q$.
Then, we have $(x^*, y^*, z^*) \in \Psi^{D_1, D_2, D_3}_{\bar{D_1}, \bar{D_2}, \bar{D_3}}$, where $(x^*, y^*, z^*) = (3, 3) \in [-50, 50] \times [-50, 50]$. By Lemma 3.3, we have $(3, 3) = M_C((3, 3) - \frac{1}{5}A^*(I - M_Q)A(3, 3))$, where $y^* = P_C(I - \frac{2}{5}D_2)(3, 3)$, $z^* = P_C(I - \frac{2}{5}D_3)(3, 3)$, $\bar{x^*} = (6, 6)$, $\bar{y^*} = (6, 6)$ and $\bar{z^*} = (6, 6)$.

Next, we prove the strong convergence theorem, by using Lemma 3.3 as important tool for finding the solution of the SMVIP and the fixed point of nonexpansive mapping.

Theorem 3.5. Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. For every $i = 1, 2$, let $B_i : C \to H$ be α_i-inverse strongly monotone mapping with α= min$\{\alpha_1, \alpha_2\}$ and let $D_1^i, D_2^i, D_3^i : C \to H_1$ be d_1^i, d_2^i, d_3^i-inverse strongly monotone, respectively, with $d^* =$ min$\{d_1^i, d_2^i, d_3^i\}$. Let $\bar{D_1}^i, \bar{D_2}^i, \bar{D_3}^i : Q \to H_2$ be $\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i$-inverse strongly monotone, respectively, with $\hat{d} =$ min$\{\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Let $f, g : H \to H$ be a_f and a_g-contraction mapping with $a =$ max$\{a_f, a_g\}$. For every $i = 1, 2$ define $M_C^i : C \to C$ by $M_C^i(x) = P_C(I - \zeta^i D_1^i)(ax + (1-a)P_C(I - \zeta^i D_2^i)(ax + (1-a)P_C(I - \zeta^i D_3^i)x))$, $\forall x \in C$, where $\zeta^i \in (0, 2d^*)$ and define $M_Q^i : Q \to Q$ by $M_Q^i(\hat{x}) = P_Q(I - \bar{\zeta}^i \bar{D}_1^i)(a\hat{x} + (1 - a)P_Q(I - \bar{\zeta}^i \bar{D}_2^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i \bar{D}_3^i)\hat{x}))$, $\forall \hat{x} \in Q$, where $\bar{\zeta}^i \in (0, 2\hat{d})$. Define $M^i : C \to C$ by $M^i(x^*) = M_C^i(x^* - \eta A^*(I - M_Q^i)Ax^*)$ for all $x^* \in C$ and $i = 1, 2$. Let the sequences $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1 - \alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2 B_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1 - \alpha_n)M^2 w_n) \end{cases} \quad (3.7)$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0, 1]$ with $\delta_n + \eta_n + \mu_n = 1$ and $\gamma \in (0, 2\alpha)$ with γ=min$\{\gamma_1, \gamma_2\}$. Assume the following conditions hold;

(i) $\mathcal{F}_i = F(M^i) \cap VI(C, B_i) \neq \emptyset$ for all $i = 1, 2$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \to \infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ for all $n \in N$ and for some $\bar{\theta}, \theta > 0$,

(iv) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Then $\{x_n\}$ converges strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, where $y_1^* = P_C(I - \zeta^1 D_2^1)(ax_1^* + (1-a)z_1^*)$ and $z_1^* = P_C(I - \zeta^1 D_3^1)x_1^*$ with (x_1^*, y_1^*, z_1^*) is an element of $\Psi^{D_1^1, D_2^1, D_3^1}_{\bar{D}_1^1, \bar{D}_2^1, \bar{D}_3^1}$. and $\{w_n\}$ converges strongly to $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, where $y_2^* = P_C(I - \zeta^2 D_2^2)(ax_2^* + (1-a)z_2^*)$ and $z_2^* = P_C(I - \zeta^2 D_3^2)x_2^*$ with (x_2^*, y_2^*, z_2^*) is an element of $\Psi^{D_1^2, D_2^2, D_3^2}_{\bar{D}_1^2, \bar{D}_2^2, \bar{D}_3^2}$.

Proof. The proof of this theorem will be divided into five steps.

Step 1. We will show that $\{x_n\}$ is bounded.

First, we will prove that $I - \gamma B_i$ is nonexpansive with $\gamma = \min\{\gamma_1, \gamma_2\}$, for all $i = 1, 2$ we get

$$
\begin{aligned}
\|(I - \gamma B_i)x - (I - \gamma B_i)w\|^2 &= \|x - w - \gamma(B_i x - B_i w)\|^2 \\
&= \|x - w\|^2 - 2\gamma\langle x - w, B_i x - B_i w\rangle \\
&\quad + \gamma^2 \|B_i x - B_i w\|^2 \\
&\leq \|x - w\|^2 - 2\alpha\gamma\|B_i x - B_i w\|^2 \\
&\quad + \gamma^2 \|B_i x - B_i w\|^2 \\
&= \|x - w\|^2 - \gamma(2\alpha - \gamma)\|B_i x - B_i w\|^2 \\
&\leq \|x - w\|^2.
\end{aligned}
$$

Thus $I - \gamma B_i$ is a nonexpansive mapping, for all $i = 1, 2$. Then $P_C(I - \gamma B_i)$ is a nonexpansive mapping.

Let $\tilde{x} \in \mathcal{F}_1$ and $\tilde{w} \in \mathcal{F}_2$, from the definition of x_n, we have

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\| &= \|\delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1 - \alpha_n)M^1 x_n) \\
&\qquad - (\delta_n + \eta_n + \mu_n)\tilde{x}\| \\
&\leq \delta_n \|x_n - \tilde{x}\| + \eta_n \|P_C(I - \gamma_1 B_1)x_n - \tilde{x}\| + \mu_n \|P_C(\alpha_n f(w_n) \\
&\qquad + (1 - \alpha_n)M^1 x_n) - \tilde{x}\| \\
&\leq (1 - \mu_n)\|x_n - \tilde{x}\| + \mu_n \|\alpha_n(f(w_n) - \tilde{x}) + (1 - \alpha_n)(M^1 x_n - \tilde{x})\| \\
&\leq (1 - \mu_n)\|x_n - \tilde{x}\| + \mu_n \alpha_n \|f(w_n) - \tilde{x}\| + \mu_n(1 - \alpha_n)\|x_n - \tilde{x}\| \\
&\leq (1 - \mu_n)\|x_n - \tilde{x}\| + \mu_n \alpha_n a\|w_n - \tilde{w}\| + \mu_n \alpha_n \|f(\tilde{w}) - \tilde{x}\| \\
&\qquad + \mu_n(1 - \alpha_n)\|x_n - \tilde{x}\| \\
&= (1 - \mu_n \alpha_n)\|x_n - \tilde{x}\| + \mu_n \alpha_n a\|w_n - \tilde{w}\| + \mu_n \alpha_n \|f(\tilde{w}) - \tilde{x}\|. \qquad (3.8)
\end{aligned}
$$

Similarly, we get

$$
\|w_{n+1} - \tilde{w}\| \leq (1 - \mu_n \alpha_n)\|w_n - \tilde{w}\| + \mu_n \alpha_n a\|x_n - \tilde{x}\| + \mu_n \alpha_n \|g(\tilde{x}) - \tilde{w}\|. \qquad (3.9)
$$

Combining (3.39) and (3.40), we have

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|+\|w_{n+1}-\tilde{w}\| &\leq (1-\mu_n\alpha_n)\left[\|x_n-\tilde{x}\|+\|w_n-\tilde{w}\|\right]\\ &\quad+\mu_n\alpha_n a\left[\|x_n-\tilde{x}\|+\|w_n-\tilde{w}\|\right]\\ &\quad+\mu_n\alpha_n\left[\|g(\tilde{x})-\tilde{w}\|+\|f(\tilde{w})-\tilde{x}\|\right]\\ &=(1-\mu_n\alpha_n(1-a))\left[\|x_n-\tilde{x}\|+\|w_n-\tilde{w}\|\right]\\ &\quad+\mu_n\alpha_n\left[\|g(\tilde{x})-\tilde{w}\|+\|f(\tilde{w})-\tilde{x}\|\right].\end{aligned}$$

By induction, we can derive that

$$\|x_n-\tilde{x}\|+\|w_n-\tilde{w}\|\leq max\left\{\|x_1-\tilde{x}\|+\|w_1-\tilde{w}\|,\frac{\|g(\tilde{x})-\tilde{w}\|+\|f(\tilde{w})-\tilde{x}\|}{1-a}\right\},$$

for every $n\in\mathbb{N}$. This implies that $\{x_n\}$ and $\{w_n\}$ are bounded.

Step 2. Claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\|w_{n+1}-w_n\|=0$.
First, we let $U_n=P_C(\alpha_nf(w_n)+(1-\alpha_n)M^1x_n)$ and $V_n=P_C(\alpha_ng(x_n)+(1-\alpha_n)M^2w_n)$. Then, observe that

$$\begin{aligned}\|U_n-U_{n-1}\| &=\|P_C(\alpha_nf(w_n)+(1-\alpha_n)M^1x_n)-P_C(\alpha_{n-1}f(w_{n-1})\\ &\qquad+(1-\alpha_{n-1})M^1x_{n-1})\|\\ &\leq\alpha_n\|f(w_n)-f(w_{n-1})\|+|\alpha_n-\alpha_{n-1}|\|f(w_{n-1})\|\\ &\qquad+(1-\alpha_n)\|M^1x_n-M^1x_{n-1}\|\\ &\qquad+|\alpha_n-\alpha_{n-1}|\|M^1x_{n-1}\|\\ &\leq\alpha_na\|w_n-w_{n-1}\|+|\alpha_n-\alpha_{n-1}|\left[\|f(w_{n-1})\|+\|M^1x_{n-1}\|\right]\\ &\qquad+(1-\alpha_n)\|x_n-x_{n-1}\|. \end{aligned}\tag{3.10}$$

By the definition of x_n and (3.10) we obtain

$$\begin{aligned}\|x_{n+1}-x_n\| &=\|\delta_nx_n+\eta_nP_C(I-\gamma_1B_1)x_n+\mu_nU_n-\delta_{n-1}x_{n-1}\\ &\qquad-\eta_{n-1}P_C(I-\gamma_1B_1)x_{n-1}-\mu_{n-1}U_{n-1}\|\\ &\leq\delta_n\|x_n-x_{n-1}\|+|\delta_n-\delta_{n-1}|\|x_{n-1}\|+\eta_n\|P_C(I-\gamma_1B_1)x_n\\ &\qquad-P_C(I-\gamma_1B_1)x_{n-1}\|+|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_1B_1)x_{n-1}\|\\ &\qquad+\mu_n\|U_n-U_{n-1}\|+|\mu_n-\mu_{n-1}|\|U_{n-1}\|\\ &=(1-\mu_n)\|x_n-x_{n-1}\|+|\delta_n-\delta_{n-1}|\|x_{n-1}\|\\ &\qquad+|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_1B_1)x_{n-1}\|\\ &\qquad+\mu_n|\alpha_n-\alpha_{n-1}|\left[\|f(w_{n-1})\|+\|M^1x_{n-1}\|\right]\\ &\qquad+\mu_n(1-\alpha_n)\|x_n-x_{n-1}\|+|\mu_n-\mu_{n-1}|\|U_{n-1}\|\\ &\qquad+\mu_n\alpha_na\|w_n-w_{n-1}\|. \end{aligned}\tag{3.11}$$

Using the same method as derived in (3.11), we have

$$\|w_{n+1}-w_n\|\leq(1-\mu_n)\|w_n-w_{n-1}\|+|\delta_n-\delta_{n-1}|\|w_{n-1}\|$$

$$\begin{aligned}
&+ |\eta_n - \eta_{n-1}| \|P_C(I - \gamma_2 B_2) w_{n-1}\| \\
&+ \mu_n |\alpha_n - \alpha_{n-1}| \left[\|g(x_{n-1})\| + \|M^2 w_{n-1}\| \right] \\
&+ \mu_n (1 - \alpha_n) \|w_n - w_{n-1}\| + |\mu_n - \mu_{n-1}| \|V_{n-1}\| \\
&+ \mu_n \alpha_n a \|x_n - x_{n-1}\|. \qquad (3.12)
\end{aligned}$$

From (3.11) and (3.12), then we get

$$\begin{aligned}
\|x_{n+1} - x_n\| + \|w_{n+1} - w_n\| &\leq (1 - \mu_n) \left[\|x_n - x_{n-1}\| + \|w_n - w_{n-1}\| \right] \\
&\quad + |\delta_n - \delta_{n-1}| \left[\|x_{n-1}\| + \|w_{n-1}\| \right] \\
&\quad + |\eta_n - \eta_{n-1}| [\|P_C(I - \gamma_1 B_1) x_{n-1}\| \\
&\quad + \|P_C(I - \gamma_2 B_2) w_{n-1}\|] \\
&\quad + |\mu_n - \mu_{n-1}| \left[\|U_{n-1}\| + \|V_{n-1}\| \right] \\
&\quad + \mu_n \alpha_n a \left[\|w_n - w_{n-1}\| + \|x_n - x_{n-1}\| \right] \\
&\quad + \mu_n |\alpha_n - \alpha_{n-1}| [\|f(w_{n-1})\| + \|M^1 x_{n-1}\| \\
&\quad + \|g(x_{n-1})\| + \|M^2 w_{n-1}\|] \\
&\quad + \mu_n (1 - \alpha_n) \left[\|x_n - x_{n-1}\| + \|w_n - w_{n-1}\| \right] \\
&\leq (1 - \alpha_n \bar{\theta}(1 - a)) \left[\|x_n - x_{n-1}\| + \|w_n - w_{n-1}\| \right] \\
&\quad + |\delta_n - \delta_{n-1}| \left[\|x_{n-1}\| + \|w_{n-1}\| \right] \\
&\quad + |\eta_n - \eta_{n-1}| [\|P_C(I - \gamma_1 B_1) x_{n-1}\| \\
&\quad + \|P_C(I - \gamma_2 B_2) w_{n-1}\|] \\
&\quad + |\mu_n - \mu_{n-1}| \left[\|U_{n-1}\| + \|V_{n-1}\| \right] \\
&\quad + \theta |\alpha_n - \alpha_{n-1}| [\|f(w_{n-1})\| + \|M^1 x_{n-1}\| \\
&\quad + \|g(x_{n-1})\| + \|M^2 w_{n-1}\|].
\end{aligned}$$

Applying Lemma 2.7 and the condition (ii), (iii) and (iv) we can conclude that

$$\|x_{n+1} - x_n\| \to 0 \text{ and } \|w_{n+1} - w_n\| \to 0 \quad as \quad n \to \infty. \qquad (3.13)$$

Step 3. Prove that $\lim_{n\to\infty} \|U_n - P_C(I - \gamma_1 B_1) U_n\| = \lim_{n\to\infty} \|U_n - M_C^1 U_n\| = 0.$ To show this, take $U_n = P_C \tilde{u}_n, \ \ \forall n \in \mathbb{N}$. Then we derive that

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \|\delta_n (x_n - \tilde{x}) + \eta_n (P_C(I - \gamma_1 B_1) x_n - \tilde{x}) + \mu_n (U_n - \tilde{x})\|^2 \\
&\leq \delta_n \|x_n - \tilde{x}\|^2 + \eta_n \|P_C(I - \gamma_1 B_1) x_n - \tilde{x}\|^2 \\
&\qquad - \delta_n \eta_n \|x_n - P_C(I - \gamma_1 B_1) x_n\|^2 + \mu_n \|\tilde{u}_n - \tilde{x}\|^2 \\
&\leq (1 - \mu_n) \|x_n - \tilde{x}\|^2 - \delta_n \eta_n \|x_n - P_C(I - \gamma_1 B_1) x_n\|^2 \\
&\qquad + \mu_n \|\alpha_n (f(w_n) - M^1 x_n) + (M^1 x_n - \tilde{x})\|^2 \\
&\leq (1 - \mu_n) \|x_n - \tilde{x}\|^2 - \delta_n \eta_n \|x_n - P_C(I - \gamma_1 B_1) x_n\|^2 \\
&\qquad + \mu_n \left[\|M^1 x_n - \tilde{x}\|^2 + 2\alpha_n \langle f(w_n) - M^1 x_n, \tilde{u}_n - \tilde{x} \rangle \right]
\end{aligned}$$

$$\leq \|x_n - \tilde{x}\|^2 - \delta_n \eta_n \|x_n - P_C(I - \gamma_1 B_1)x_n\|^2$$
$$+ 2\mu_n \alpha_n \|f(w_n) - M^1 x_n\| \|\tilde{u}_n - \tilde{x}\|,$$

which implies that

$$\begin{aligned}\delta_n \eta_n \|x_n - P_C(I - \gamma_1 B_1)x_n\|^2 &\leq \|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 \\ &\quad + 2\mu_n \alpha_n \|f(w_n) - M^1 x_n\| \|\tilde{u}_n - \tilde{x}\| \\ &\leq \|x_n - x_{n+1}\| \left[\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|\right] \\ &\quad + 2\mu_n \alpha_n \|f(w_n) - M^1 x_n\| \|\tilde{u}_n - \tilde{x}\|.\end{aligned}$$

Then, we have

$$\|x_n - P_C(I - \gamma_1 B_1)x_n\| \to 0 \quad as \quad n \to \infty. \tag{3.14}$$

Observe that

$$x_{n+1} - x_n = \eta_n(P_C(I - \gamma_1 B_1)x_n - x_n) + \mu_n(U_n - x_n).$$

From (3.13) and (3.14), we obtain

$$\|U_n - x_n\| \to 0 \quad as \quad n \to \infty. \tag{3.15}$$

Observe that

$$\begin{aligned}\|U_n - P_C(I - \gamma_1 B_1)U_n\| &\leq \|U_n - x_n\| + \|x_n - P_C(I - \gamma_1 B_1)x_n\| \\ &\quad + \|P_C(I - \gamma_1 B_1)x_n - P_C(I - \gamma_1 B_1)U_n\| \\ &\leq \|U_n - x_n\| + \|x_n - P_C(I - \gamma_1 B_1)x_n\| + \|x_n - U_n\| \\ &= 2\|U_n - x_n\| + \|x_n - P_C(I - \gamma_1 B_1)x_n\|,\end{aligned}$$

by (3.14) and (3.15), we obtain

$$\|U_n - P_C(I - \gamma_1 B_1)U_n\| \to 0 \quad as \quad n \to \infty. \tag{3.16}$$

Applying the same argument as (3.16), we also obtain

$$\|V_n - P_C(I - \gamma_2 B_2)V_n\| \to 0 \quad as \quad n \to \infty.$$

Consider

$$\|x_{n+1} - U_n\| \leq \|x_{n+1} - x_n\| + \|x_n - U_n\|.$$

From (3.13) and (3.15), we have

$$\|x_{n+1} - U_n\| \to 0 \quad as \quad n \to \infty. \tag{3.17}$$

Since

$$\|x_n - M^1 x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \|U_n - M^1 x_n\|$$

$$\begin{aligned}
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \|\tilde{u}_n - M^1 x_n\| \\
&= \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| \\
&\qquad + \|\alpha_n f(w_n) + (1-\alpha_n) M^1 x_n - M^1 x_n\| \\
&= \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \alpha_n \|f(w_n) - M^1 x_n\|.
\end{aligned}$$

From (3.13), (3.17) and condition (ii), we get

$$\|x_n - M^1 x_n\| \to 0 \ \ as \ \ n \to \infty. \tag{3.18}$$

Consider

$$\begin{aligned}
\|U_n - M^1 U_n\| &\leq \|U_n - x_n\| + \|x_n - M^1 x_n\| + \|M^1 x_n - M^1 U_n\| \\
&\leq \|U_n - x_n\| + \|x_n - M^1 x_n\| + \|x_n - U_n\| \\
&\leq 2\|U_n - x_n\| + \|x_n - M^1 x_n\|.
\end{aligned}$$

From (3.15) and (3.18), we have

$$\|U_n - M^1 U_n\| \to 0 \ \ as \ \ n \to \infty. \tag{3.19}$$

Applying the same method as (3.18), we also have

$$\|V_n - M^2 V_n\| \to 0 \ \ as \ \ n \to \infty.$$

Step 4. Claim that $\limsup_{n\to\infty} \langle f(x_2^*) - x_1^*, U_n - x_1^* \rangle \leq 0$, where $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$.
First, take a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that

$$\limsup_{n\to\infty} \langle f(x_2^*) - x_1^*, U_n - x_1^* \rangle = \lim_{k\to\infty} \langle f(x_2^*) - x_1^*, U_{n_k} - x_1^* \rangle. \tag{3.20}$$

Since $\{x_n\}$ is bounded, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in C$ as $k \to \infty$. From (3.15), we obtain $U_{n_k} \rightharpoonup \hat{x} \ \ as \ \ k \to \infty$.
Next, we need to show that $\hat{x} \in \mathcal{F}_1 = F(M^1) \cap VI(C, B_1)$. Assume $\hat{x} \notin F(M^1)$. Then, we have $\hat{x} \neq M^1 \hat{x}$. By the Opial's condition, we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \|U_{n_k} - \hat{x}\| &< \liminf_{k\to\infty} \|U_{n_k} - M^1 \hat{x}\| \\
&\leq \liminf_{k\to\infty} \|U_{n_k} - M^1 U_{n_k}\| + \liminf_{k\to\infty} \|M^1 U_{n_k} - M^1 \hat{x}\| \\
&\leq \liminf_{k\to\infty} \|U_{n_k} - \hat{x}\|.
\end{aligned}$$

This is a contradiction.
Therefore

$$\hat{x} \in F(M^1). \tag{3.21}$$

Assume $\hat{x} \notin VI(C, B_1)$, then we get $\hat{x} \neq P_C(I - \gamma_1 B_1)\hat{x}$.
From the Opial's condition and (3.16), we have

$$\liminf_{k\to\infty} \|U_{n_k} - \hat{x}\| < \liminf_{k\to\infty} \|U_{n_k} - P_C(I - \gamma_1 B_1)\hat{x}\|$$

$$
\begin{aligned}
&\leq \liminf_{k\to\infty} \|U_{n_k} - P_C(I-\gamma_1 B_1)U_{n_k}\| \\
&\qquad\qquad + \liminf_{k\to\infty} \|P_C(I-\gamma_1 B_1)U_{n_k} - P_C(I-\gamma_1 B_1)\hat{x}\| \\
&\leq \liminf_{k\to\infty} \|U_{n_k} - \hat{x}\|.
\end{aligned}
$$

This is a contradiction.
Therefore

$$\hat{x} \in VI(C, B_1). \tag{3.22}$$

By (3.21) and (3.22), this yields that

$$\hat{x} \in \mathcal{F}_1 = F(M^1) \cap VI(C, B_1). \tag{3.23}$$

Since $U_{n_k} \rightharpoonup \hat{x}\ \ as\ \ k \to \infty$, from (3.23) and Lemma 2.14, we can derive that

$$
\begin{aligned}
\limsup_{n\to\infty} \langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle &= \lim_{k\to\infty} \langle f(x_2^*) - x_1^*, U_{n_k} - x_1^*\rangle \\
&= \langle f(x_2^*) - x_1^*, \hat{x} - x_1^*\rangle \\
&\leq 0.
\end{aligned} \tag{3.24}
$$

Following the same method as (3.24), we obtain that

$$\limsup_{n\to\infty} \langle g(x_1^*) - x_2^*, V_n - x_2^*\rangle \leq 0. \tag{3.25}$$

Step 5. Finally, prove that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$ and $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, respectively.
By firmly-nonexpansiveness of P_C, we derive that

$$
\begin{aligned}
\|U_n - x_1^*\|^2 &= \|P_C \tilde{u}_n - x_1^*\|^2 \\
&\leq \langle \tilde{u}_n - x_1^*, U_n - x_1^*\rangle \\
&= \langle \alpha_n(f(w_n) - x_1^*) + (1-\alpha_n)(M^1 x_n - x_1^*), U_n - x_1^*\rangle \\
&= \alpha_n \langle f(w_n) - x_1^*, U_n - x_1^*\rangle + (1-\alpha_n)\langle M^1 x_n - x_1^*, U_n - x_1^*\rangle \\
&= \alpha_n \langle f(w_n) - f(x_2^*), U_n - x_1^*\rangle + \alpha_n \langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\
&\qquad + (1-\alpha_n)\|M^1 x_n - x_1^*\|\|U_n - x_1^*\| \\
&\leq \alpha_n a \|w_n - x_2^*\|\|U_n - x_1^*\| + \alpha_n \langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\
&\qquad + (1-\alpha_n)\|x_n - x_1^*\|\|U_n - x_1^*\| \\
&\leq \frac{\alpha_n a}{2}\left\{\|w_n - x_2^*\|^2 + \|U_n - x_1^*\|^2\right\} + \alpha_n \langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\
&\qquad + \frac{(1-\alpha_n)}{2}\left\{\|x_n - x_1^*\|^2 + \|U_n - x_1^*\|^2\right\} \\
&= \frac{\alpha_n a}{2}\|w_n - x_2^*\|^2 + \frac{(1-\alpha_n)}{2}\|x_n - x_1^*\|^2 \\
&\qquad + \left(\frac{1-\alpha_n(1-a)}{2}\right)\|U_n - x_1^*\|^2 \\
&\qquad + \alpha_n \langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle,
\end{aligned}
$$

which hence yields

$$\|U_n - x_1^*\|^2 \leq \frac{\alpha_n a}{1+\alpha_n(1-a)}\|w_n - x_2^*\|^2 + \frac{(1-\alpha_n)}{1+\alpha_n(1-a)}\|x_n - x_1^*\|^2 \\ + \frac{\alpha_n}{1+\alpha_n(1-a)}\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle. \tag{3.26}$$

From the definition of x_n and (3.26), we get

$$\begin{aligned}\|x_{n+1} - x_1^*\|^2 &\leq \delta_n\|x_n - x_1^*\|^2 + \eta_n\|P_C(I-\gamma_1 B_1)x_n - x_1^*\|^2 + \mu_n\|U_n - x_1^*\|^2 \\ &\leq (1-\mu_n)\|x_n - x_1^*\|^2 + \frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\|w_n - x_2^*\|^2 \\ &\quad + \frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\ &\quad + \frac{\mu_n(1-\alpha_n)}{1+\alpha_n(1-a)}\|x_n - x_1^*\|^2 \\ &= \left(1 - \frac{\mu_n\alpha_n(2-a)}{1+\alpha_n(1-a)}\right)\|x_n - x_1^*\|^2 + \frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\|w_n - x_2^*\|^2 \\ &\quad + \frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle. \end{aligned} \tag{3.27}$$

Similarly, as derived above,we also have

$$\|w_{n+1} - x_2^*\|^2 \leq \left(1 - \frac{\mu_n\alpha_n(2-a)}{1+\alpha_n(1-a)}\right)\|w_n - x_2^*\|^2 + \frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\|x_n - x_1^*\|^2 \\ + \frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\langle g(x_1^*) - x_2^*, V_n - x_2^*\rangle. \tag{3.28}$$

From (3.27) and (3.28), we deduce that

$$\begin{aligned}&\|x_{n+1} - x_1^*\|^2 + \|w_{n+1} - x_2^*\|^2 \\ &\leq \left(1 - \frac{\mu_n\alpha_n(2-a)}{1+\alpha_n(1-a)}\right)\left(\|x_n - x_1^*\|^2 + \|w_n - x_2^*\|^2\right) \\ &\quad + \frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\left(\|x_n - x_1^*\|^2 + \|w_n - x_2^*\|^2\right) \\ &\quad + \frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left(\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle + \langle g(x_1^*) - x_2^*, V_n - x_2^*\rangle\right) \\ &= \left(1 - \frac{\mu_n\alpha_n(2-a)}{1+\alpha_n(1-a)} + \frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\right)\left(\|x_n - x_1^*\|^2 + \|w_n - x_2^*\|^2\right) \\ &\quad + \frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left(\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle + \langle g(x_1^*) - x_2^*, V_n - x_2^*\rangle\right) \\ &= \left(1 - \frac{2\mu_n\alpha_n(1-a)}{1+\alpha_n(1-a)}\right)\left\{\|x_n - x_1^*\|^2 + \|w_n - x_2^*\|^2\right\} \\ &\quad + \frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left(\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle + \langle g(x_1^*) - x_2^*, V_n - x_2^*\rangle\right).\end{aligned}$$

Applying the condition (ii), (3.24), (3.25), and Lemma 2.7, we can conclude that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $x_1^* = P_{\mathcal{F}_1}f(x_2^*)$ and $x_2^* = P_{\mathcal{F}_2}g(x_1^*)$, respectively. From Lemma 3.3, we have (x_1^*, y_1^*, z_1^*) is an element of $\Psi^{D_1^1,D_2^1,D_3^1}_{\bar{D}_1^1,\bar{D}_2^1,\bar{D}_3^1}$. and (x_2^*, y_2^*, z_2^*) is an element of $\Psi^{D_1^2,D_2^2,D_3^2}_{\bar{D}_1^2,\bar{D}_2^2,\bar{D}_3^2}$. This completes the proof.

□

As direct consequences of Theorem 3.5, we can obtain the following corollary.

Corollary 3.6. Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. For every $i = 1, 2$, let $T_i : C \to C$ be a κ_i-strictly pseudo-contractive mapping with $Fix(T_i) \neq \emptyset$, and let $D_1^i, D_2^i, D_3^i : C \to H_1$ be d_1^i, d_2^i, d_3^i-inverse strongly monotone, respectively, with $d^* = \min\{d_1^i, d_2^i, d_3^i\}$. Let $\bar{D_1}^i, \bar{D_2}^i, \bar{D_3}^i : Q \to H_2$ be $\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i$-inverse strongly monotone, respectively, with $\hat{d} = \min\{\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Let $f, g : H \to H$ be a_f and a_g-contraction mapping with $a = \max\{a_f, a_g\}$. For every $i = 1, 2$ define $M_C^i : C \to C$ by $M_C^i(x) = P_C(I - \zeta^i D_1^i)(ax + (1-a)P_C(I - \zeta^i D_2^i)(ax + (1-a)P_C(I - \zeta^i D_3^i)x))$, $\forall x \in C$, where $\zeta^i \in (0, 2d^*)$ and define $M_Q^i : Q \to Q$ by $M_Q^i(\hat{x}) = P_Q(I - \bar{\zeta}^i \bar{D}_1^i)(a\hat{x} + (1 - a)P_Q(I - \bar{\zeta}^i \bar{D}_2^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i \bar{D}_3^i)\hat{x}))$, $\forall \hat{x} \in Q$, where $\bar{\zeta}^i \in (0, 2\hat{d})$ Define $M^i : C \to C$ by $M^i(x^*) = M_C^i(x^* - \eta A^*(I - M_Q^i)Ax^*)$ for all $x^* \in C$, and $i = 1, 2$. Let the sequences $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1(I - T_1))x_n + \mu_n P_C(\alpha_n f(w_n) + (1 - \alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2(I - T_2))w_n + \mu_n P_C(\alpha_n g(x_n) + (1 - \alpha_n)M^2 w_n) \end{cases} \tag{3.29}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\delta_n + \eta_n + \mu_n = 1$, $\gamma \in (0, 2\alpha)$ with $\alpha = \min\{\frac{1-\kappa_1}{2}, \frac{1-\kappa_2}{2}\}$ and γ=min$\{\gamma_1, \gamma_2\}$. Assume the following conditions hold;

(i) $\mathcal{F}_i = F(M^i) \cap F(T_i) \neq \emptyset$ for all $i = 1, 2$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ for all $n \in N$ and for some $\bar{\theta}, \theta > 0$,

(iv) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Then $\{x_n\}$ converges strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, where $y_1^* = P_C(I - \zeta^1 D_2^1)(ax_1^* + (1-a)z_1^*)$ and $z_1^* = P_C(I - \zeta^1 D_3^1)x_1^*$ with (x_1^*, y_1^*, z_1^*) is an element of $\Psi_{\bar{D}_1^1, \bar{D}_2^1, \bar{D}_3^1}^{D_1^1, D_2^1, D_3^1}$. and $\{w_n\}$ converges strongly to $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, where $y_2^* = P_C(I - \zeta^2 D_2^2)(ax_2^* + (1-a)z_2^*)$ and $z_2^* = P_C(I - \zeta^2 D_3^2)x_2^*$ with (x_2^*, y_2^*, z_2^*) is an element of $\Psi_{\bar{D}_1^2, \bar{D}_2^2, \bar{D}_3^2}^{D_1^2, D_2^2, D_3^2}$.

Proof. From Theorem 3.5 and Lemma 2.13, we have the desired conclusion. ☐

3.2 Applications of strong convergence theorem for solving the split of modified variational inequality problems (SMVIP)

3.2.1 Strong convergence theorem for approximating the solution of the split variational inequality problem

In 2012, Censor et al. introduced SVIP, which is to find $\hat{x} \in C$ such that

$$\langle f_1 \hat{x}, x - \hat{x} \rangle \geq 0, \forall x \in C,$$

and find $\hat{y} = D\hat{x} \in Q$ such that

$$\langle f_2\hat{y}, x - \hat{y}\rangle \geq 0, \forall y \in Q,$$

where $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings and $D : H_1 \to H_2$ is a bounded linear operator. The set of all solution of the SVIP is denoted by

$$\Phi = \{\hat{x} \in VI(C, f_1) : \hat{y} \in VI(Q, f_2)\}. \tag{3.30}$$

The SVIP is reduced to the SFP if $f_1 \equiv f_2 \equiv 0$.

Before we prove the theorems, we need the following lemmas.

Lemma 3.7. In a strictly convex Banach space E, if

$$\|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\|,$$

for all $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$.

Lemma 3.8. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3-inverse strongly monotone respectively, where $\zeta \in (0, 2d^*)$ with d^* =min$\{d_1, d_2, d_3\}$. Let $\bar{D}_1, \bar{D}_2, \bar{D}_3 : Q \to H_2$ be $\bar{d}_1, \bar{d}_2, \bar{d}_3$-inverse strongly monotone respectively, where $\bar{\zeta} \in (0, 2\hat{d})$ with $\hat{d}$ =min$\{\bar{d}_1, \bar{d}_2, \bar{d}_3\}$. Defined the mapping M_C, M_Q as in Lemma 3.3. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Let $\bigcap_{i=1}^{3} \Phi_i \neq \emptyset$ and $\Phi_i = \{w \in VI(C, D_i) | Aw \in VI(Q, \bar{D}_i)\}$, for all $i = 1, 2, 3$. Then

$$\bigcap_{i=1}^{3} \Phi_i = F(M_C(I - \eta A^*(I - M_Q)A)).$$

Proof. First, we will prove that for every $i = 1, 2, 3$ if $T_i : C \to C$ is a nonexpansive mapping, $\bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and define the mapping $J : C \to C$ by

$$J(x) = T_1(ax + (1 - a)T_2(ax + (1 - a)T_3x))$$

Then $\bigcap_{i=1}^{3} F(T_i) = F(J)$.

It easy to see that $\bigcap_{i=1}^{3} F(T_i) \subseteq F(J)$.

Next, we claim that $F(J) \subseteq \bigcap_{i=1}^{3} F(T_i)$. To show this, let $x \in F(J)$ and $x^* \in \bigcap_{i=1}^{3} F(T_i)$.

By the definition of J, we get

$$\begin{aligned}
\|x-x^*\| &= \|T_1(ax+(1-a)T_2(ax+(1-a)T_3x))-x^*\| \\
&\leq \|a(x-x^*)+(1-a)[T_2(ax+(1-a)T_3x)-x^*]\| \\
&\leq a\|x-x^*\|+(1-a)\|ax+(1-a)T_3x-x^*\| \\
&= a\|x-x^*\|+(1-a)\|a(x-x^*)+(1-a)(T_3x-x^*)\| \\
&\leq a\|x-x^*\|+a(1-a)\|x-x^*\|+(1-a)^2\|T_3x-x^*\| \\
&= (2a-a^2)\|x-x^*\|+(1-a)^2\|T_3x-x^*\| \\
&\leq (2a-a^2)\|x-x^*\|+(1-a)^2\|x-x^*\| \\
&= \|x-x^*\|,
\end{aligned}$$

which implies that $\|x-x^*\| = \|T_3x-x^*\| = \|a(x-x^*)+(1-a)(T_3x-x^*)\|$. From Lemma 3.7, we have $T_3x = x$, that is $x \in F(T_3)$.

By the definition of J and $x \in F(T_3)$, we have

$$\begin{aligned}
\|x-x^*\| &= \|T_1(ax+(1-a)T_2(ax+(1-a)x))-x^*\| \\
&\leq \|ax+(1-a)T_2x-x^*\| \\
&= \|a(x-x^*)+(1-a)(T_2x-x^*)\| \\
&\leq a\|x-x^*\|+(1-a)\|T_2x-x^*\| \\
&\leq a\|x-x^*\|+(1-a)\|x-x^*\| \\
&= \|x-x^*\|,
\end{aligned}$$

which implies that $\|x-x^*\| = \|T_2x-x^*\| = a\|x-x^*\|+(1-a)\|T_2x-x^*\|$. From Lemma 3.7, we have $T_2x = x$, that is $x \in F(T_2)$.

By the definition of J, $x \in F(T_3)$ and $x \in F(T_2)$, we have

$$\begin{aligned}
x &= T_1(ax+(1-a)T_2(ax+(1-a)T_3x)) \\
&= T_1x
\end{aligned}$$

which implies that $T_1x = x$, that is $x \in F(T_1)$. Therefore

$$x \in \bigcap_{i=1}^{3} F(T_i).$$

Then, we have

$$F(J) \subseteq \bigcap_{i=1}^{3} F(T_i).$$

Hence

$$F(J) = \bigcap_{i=1}^{3} F(T_i). \tag{3.31}$$

Next, we claim that $\bigcap_{i=1}^{3}\Phi_i \subseteq F(M_C(I-\eta A^*(I-M_Q)A))$. To show this, let $x \in \bigcap_{i=1}^{3}\Phi_i$ then $x \in \Phi_i$, for all $i = 1,2,3$. Thus
$x \in VI(C, D_i)$ and $Ax \in VI(Q, \bar{D}_i)$, for all $i = 1,2,3$ and so
$x \in F(P_C(I-\zeta D_i))$ and $Ax \in F(P_Q(I-\bar{\zeta}\bar{D}_i))$, for all $i = 1,2,3$.
Then, we have $x \in \bigcap_{i=1}^{3} F(P_C(I-\zeta D_i))$ and $Ax \in \bigcap_{i=1}^{3} F(P_Q(I-\bar{\zeta}\bar{D}_i))$.
Utilizing the same method as (3.31), we have

$$x \in F(M_C),$$

that is
$x = M_C x = P_C(I-\zeta D_1)(ax + (1-a)P_C(I-\zeta D_2)(ax + (1-a)P_C(I-\zeta D_3)x))$,
where $y = P_C(I-\zeta D_2)(ax + (1-a)z)$, $z = P_C(I-\zeta D_3)x$ and

$$\bar{x} \in F(M_Q),$$

that is
$Ax = M_Q Ax = P_Q(I-\bar{\zeta}\bar{D}_1)(aAx + (1-a)P_Q(I-\bar{\zeta}\bar{D}_2)(aAx + (1-a)P_Q(I-\bar{\zeta}\bar{D}_3)Ax))$,
where $\bar{y} = P_Q(I-\zeta D_2)(a\bar{x} + (1-a)\bar{z})$, $\bar{z} = P_Q(I-\zeta D_3)\bar{x}$, $\bar{x} = Ax$, $\bar{y} = Ay$ and $\bar{z} = Az$.
From Lemma 1.3, we have
$(x,y,z) \in \Psi_{D_1,D_2,D_3}$ and $(\bar{x},\bar{y},\bar{z}) \in \Psi_{\bar{D}_1,\bar{D}_2,\bar{D}_3}$. Thus

$$(x,y,z) \in \Psi^{D_1,D_2,D_3}_{\bar{D}_1,\bar{D}_2,\bar{D}_3}.$$

From Lemma 3.3, we have

$$x = M_C(I-\eta A^*(I-M_Q)A)x$$

where $y = P_C(I-\zeta D_2)(ax + (1-a)z)$, $z = P_C(I-\zeta D_3)x$, $\bar{x} = Ax$, $\bar{y} = Ay$ and $\bar{z} = Az$. Thus

$$x \in F(M_C(I-\eta A^*(I-M_Q)A)).$$

Then, we have

$$\bigcap_{i=1}^{3}\Phi_i \subseteq F(M_C(I-\eta A^*(I-M_Q)A)).$$

Next, we claim that $F(M_C(I-\eta A^*(I-M_Q)A)) \subseteq \bigcap_{i=1}^{3}\Phi_i$. To show this, let $x \in F(M_C(I-\eta A^*(I-M_Q)A))$, then $x = M_C(I-\eta A^*(I-M_Q)A)x$.
From Lemma 3.3, we have

$$(x,y,z) \in \Psi^{D_1,D_2,D_3}_{\bar{D}_1,\bar{D}_2,\bar{D}_3},$$

$y = P_C(I-\zeta D_2)(ax + (1-a)z)$, $z = P_C(I-\zeta D_3)x$ and $\bar{y} = Ay = P_Q(I-\zeta D_2)(a\bar{x} + (1-a)\bar{z})$, $\bar{z} = Az = P_Q(I-\zeta D_3)\bar{x}$, $\bar{x} = Ax$. Then $(x,y,z) \in \Psi_{D_1,D_2,D_3}$ and $(\bar{x},\bar{y},\bar{z}) \in \Psi_{\bar{D}_1,\bar{D}_2,\bar{D}_3}$.
From Lemma 1.3, we have

$$x \in F(M_C),$$

and

$$\bar{x} \in F(M_Q),$$

that is

$x = M_C x = P_C(I-\zeta D_1)(ax + (1-a)P_C(I-\zeta D_2)(ax + (1-a)P_C(I-\zeta D_3)x))$
and $Ax = M_Q Ax = P_Q(I-\bar{\zeta}\bar{D}_1)(aAx + (1-a)P_Q(I-\bar{\zeta}\bar{D}_2)(aAx + (1-a)P_Q(I-\bar{\zeta}\bar{D}_3)Ax))$.
Utilizing the same method as (3.31), we have
$x \in \bigcap_{i=1}^{3} F(P_C(I-\zeta)D_i)$ and $Ax \in \bigcap_{i=1}^{3} F(P_Q(I-\bar{\zeta})\bar{D}_i)$.
Then $x \in F(P_C(I-\zeta)D_i)$ and $Ax \in F(P_Q(I-\bar{\zeta})\bar{D}_i)$, for all $i = 1,2,3$.
and so $x \in VI(C, D_i)$ and $Ax \in VI(Q, \bar{D}_i)$, for all $i = 1,2,3$
That is $x \in \bigcap_{i=1}^{3} \Phi_i$, then $x \in \Phi_i$, for all $i = 1,2,3$.
Thus

$$F(M_C(I - \eta A^*(I - M_Q)A)) \subseteq \bigcap_{i=1}^{3} \Phi_i.$$

Hence

$$\bigcap_{i=1}^{3} \Phi_i = F(M_C(I - \eta A^*(I - M_Q)A)).$$

□

Theorem 3.9. Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. For every $i = 1,2$, let $B_i : C \to H$ be α_i-inverse strongly monotone mapping with α= min$\{\alpha_1, \alpha_2\}$. For every $i = 1,2$, let $D_1^i, D_2^i, D_3^i : C \to H_1$ be d_1^i, d_2^i, d_3^i-inverse strongly monotone respectively, where $\zeta^i \in (0, 2d^*)$ with $d^* =$ min$\{d_1^i, d_2^i, d_3^i\}$ and let $\bar{D_1}^i, \bar{D_2}^i, \bar{D_3}^i : Q \to H_2$ be $\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i$-inverse strongly monotone respectively, where $\bar{\zeta}^i \in (0, 2\hat{d})$ with $\hat{d}$ =min$\{\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. let $f, g : H \to H$ be a_f and a_g-contraction mapping with $a =$ max$\{a_f, a_g\}$. For every $i = 1,2$ define $M_C^i : C \to C$ by $M_C^i(x) = P_C(I - \zeta^i D_1^i)(ax + (1-a)P_C(I - \zeta^i D_2^i)(ax + (1-a)P_C(I - \zeta^i D_3^i)x))$, $\forall x \in C$, and define $M_Q^i : Q \to Q$ by $M_Q^i(\hat{x}) = P_Q(I - \bar{\zeta}^i \bar{D}_1^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i \bar{D}_2^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i \bar{D}_3^i)\hat{x}))$, $\forall \hat{x} \in Q$. For every $i = 1,2$ define $M^i : C \to C$ by $M^i(x^*) = M_C^i(x^* - \eta A^*(I - M_Q^i)Ax^*)$ for all $x^* \in C$. Let the sequences $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2 B_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)M^2 w_n) \end{cases}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\delta_n + \eta_n + \mu_n = 1$ and $\gamma \in (0, 2\alpha)$ with γ=min$\{\gamma_1, \gamma_2\}$. Assume the following conditions hold;

(i) $\mathcal{F}_i = \bigcap_{j=1}^{3} \Phi_j^i \cap VI(C, B_i) \neq \emptyset$ for all $i = 1,2$ and Φ_j^i=$\{z \in VI(C, D_j^i) : Az \in \ VI(C, \bar{D}_j^i)\}$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ for all $n \in N$ and for some $\bar{\theta}, \theta > 0$,

(iv) $\sum\limits_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum\limits_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Then $\{x_n\}$ converges strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, where $y_1^* = P_C(I - \zeta^1 D_2^1)(ax_1^* + (1-a)z_1^*)$ and $z_1^* = P_C(I - \zeta^1 D_3^1)x_1^*$ with (x_1^*, y_1^*, z_1^*) is an element of $\Psi_{\bar{D}_1^1, \bar{D}_2^1, \bar{D}_3^1}^{D_1^1, D_2^1, D_3^1}$. and $\{w_n\}$ converges strongly to $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, where $y_2^* = P_C(I - \zeta^2 D_2^2)(ax_2^* + (1-a)z_2^*)$ and $z_2^* = P_C(I - \zeta^2 D_3^2)x_2^*$ with (x_2^*, y_2^*, z_2^*) is an element of $\Psi_{\bar{D}_1^2, \bar{D}_2^2, \bar{D}_3^2}^{D_1^2, D_2^2, D_3^2}$.

Proof. By using Theorem 3.5 and Lemma 3.8, we obtain the conclusion. ☐

3.2.2 Strong convergence theorem for approximating the solution of the split minimization problem

Let C be a closed convex subset of H. The standard constrained convex optimization problem is to find $x^* \in C$ such that

$$\Im(x^*) = \min_{x \in C} \Im(x), \tag{3.32}$$

where $\Im : C \to \mathbb{R}$ is a convex, Fr'echet differentiable function. The set of all solution of (3.32) is denoted by $\Phi_{\Im}$.

Lemma 3.10. [37] (Optimality condition) A necessary condition of Optimality for a point $x^* \in C$ to be a solution of the minimization problem (3.32) is that x^* solves the variational inequality

$$\langle \nabla \Im(x^*), x - x^* \rangle \geq 0, \tag{3.33}$$

for all $x \in C$. Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C(I - \zeta \nabla \Im)x^*,$$

for every $\zeta > 0$. If, in addition, $\Im$ is convex, then the optimality condition (3.33) is also sufficient.

By using the concept of the split of modified variational inequalities problem (SMVIP), we consider the problem for finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \zeta \nabla \Im_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, \forall x \in C, \\ \langle y^* - (I - \zeta \nabla \Im_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, \forall x \in C, \\ \langle z^* - (I - \zeta \nabla \Im_3)x^*, x - z^* \rangle \geq 0, \forall x \in C, \end{cases} \tag{3.34}$$

and find $(\bar{x^*} = Ax^*,\ \bar{y^*} = Ay^*,\ \bar{z^*} = Az^*) \in Q \times Q \times Q$ such that

$$\begin{cases} \langle \bar{x^*} - (I - \bar{\zeta} \nabla \bar{\Im}_1)(a\bar{x^*} + (1-a)\bar{y^*}), \bar{x} - \bar{x^*} \rangle \geq 0, \forall \bar{x} \in Q, \\ \langle \bar{y^*} - (I - \bar{\zeta} \nabla \bar{\Im}_2)(a\bar{x^*} + (1-a)\bar{z^*}), \bar{x} - \bar{y^*} \rangle \geq 0, \forall \bar{x} \in Q, \\ \langle \bar{z^*} - (I - \bar{\zeta} \nabla \bar{\Im}_3)\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, \forall \bar{x} \in Q, \end{cases} \tag{3.35}$$

where $\Im_1, \Im_2, \Im_3 : C \to \mathbb{R}$ with $\nabla\Im_1, \nabla\Im_2, \nabla\Im_3$ are the gradient of $\Im_1, \Im_2, \Im_3$, respectively and $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3 : Q \to \mathbb{R}$ with $\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3$ are the gradient of $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3$, respectively, $\zeta, \bar{\zeta} > 0$ and $a \in [0,1]$. The set of all solution of (3.34) and (3.35) are denoted by $\Psi_{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}$ and $\Psi_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}$ respectively. The set of all solution of the split of modified variational inequalities (SMVIP) is denoted by $\Psi^{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}$ that is

$$\Psi^{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3} = \{(x^*, y^*, z^*) \in \Psi_{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3} : (\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}\}$$

Before we prove the theorems, we need the following lemma.

Lemma 3.11. Let C, Q be nonempty closed convex subsets of H_1, H_2, respectively. For every $i = 1,2$ let $\Im_1^i, \Im_2^i, \Im_3^i : C \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\Im_1^i, \nabla\Im_2^i, \nabla\Im_3^i$ be $\frac{1}{L_{\Im_1^i}}, \frac{1}{L_{\Im_2^i}}, \frac{1}{L_{\Im_3^i}}$-inverse strongly monotone and continuous respectively, where $\zeta^i \in (0, \frac{2}{L_\Im})$ with $\frac{1}{L_\Im} = \min\{\frac{1}{L_{\Im_1^i}}, \frac{1}{L_{\Im_2^i}}, \frac{1}{L_{\Im_3^i}}\}$. Let $\bar{\Im}_1^i, \bar{\Im}_2^i, \bar{\Im}_3^i : Q \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\bar{\Im}_1^i, \nabla\bar{\Im}_2^i, \nabla\bar{\Im}_3^i$ be $\frac{1}{L_{\bar{\Im}_1^i}}, \frac{1}{L_{\bar{\Im}_2^i}}, \frac{1}{L_{\bar{\Im}_3^i}}$-inverse strongly monotone and continuous respectively, where $\bar{\zeta}^i \in (0, \frac{2}{L_{\bar{\Im}}})$ with $\frac{1}{L_{\bar{\Im}}} = \min\{\frac{1}{L_{\bar{\Im}_1^i}}, \frac{1}{L_{\bar{\Im}_2^i}}, \frac{1}{L_{\bar{\Im}_3^i}}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C : C \to C$ by $M_C(x) = P_C(I - \zeta\nabla\Im_1)(ax + (1-a)P_C(I - \zeta\nabla\Im_2)(ax + (1-a)P_C(I - \zeta\nabla\Im_3)x)), \forall x \in C$, and define $M_Q : Q \to Q$ by $M_Q(\hat{x}) = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_1)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\nabla\bar{\Im}_2)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\nabla\bar{\Im}_3)\hat{x})), \forall \hat{x} \in Q$. Assume $\Psi^{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3} = \{(x^*, y^*, z^*) \in \Psi_{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3} : (\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}\} \neq \emptyset$. The following statements are equivalent

(i) $(x^*, y^*, z^*) \in \Psi^{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}$,

(ii) $x^* = M_C(x^* - \eta A^*(I - M_Q)Ax^*)$,

where $y^* = P_C(I - \zeta\nabla\Im_2)(ax^* + (1-a)z^*)$, $z^* = P_C(I - \zeta\nabla\Im_3)x^*$, $\bar{x^*} = Ax^* = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_1)(a\bar{x^*} + (1-a)\bar{y^*})$, $\bar{y^*} = Ay^* = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_2)(a\bar{x^*} + (1-a)\bar{z^*})$ and $\bar{z^*} = Az^* = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_3)\bar{x^*}$.

Proof. By using Lemma 3.3, we have the desired conclusion. □

Example 3.12. Let $\mathbb{R}$ be the set of real numbers, $H_1 = H_2 = \mathbb{R}^2$, $C = [-50, 50] \times [-50, 50]$ and $Q = [-100, 100] \times [-100, 100]$. For every $i = 1, 2, 3$, let $\Im_i : C \to \mathbb{R}$ be defined by $\Im_1(x_1, x_2) = (\frac{x_1^2 - 4x}{6}, \frac{x_2^2 - 4x}{6})$, $\Im_2(x_1, x_2) = (\frac{x_1^2 - 4x}{10}, \frac{x_2^2 - 4x}{10})$, $\Im_2(x_1, x_2) = (\frac{x_1^2 - 4x}{14}, \frac{x_2^2 - 4x}{14})$, with $\nabla\Im_1(x_1, x_2) = (\frac{x_1 - 2}{3}, \frac{x_2 - 2}{3})$, $\nabla\Im_2(x_1, x_2) = (\frac{x_1 - 2}{5}, \frac{x_2 - 2}{5})$, $\nabla\Im_3(x_1, x_2) = (\frac{x_1 - 2}{7}, \frac{x_2 - 2}{7})$, for all $(x_1, x_2) \in C$ and let $\bar{\Im}_i : Q \to \mathbb{R}$ be defined by $\bar{\Im}_1(x_1, x_2) = (\frac{x_1^2 - 8x}{10}, \frac{x_2^2 - 8x}{10})$, $\bar{\Im}_2(x_1, x_2) = (\frac{x_1^2 - 8x}{12}, \frac{x_2^2 - 8x}{12})$, $\bar{\Im}_2(x_1, x_2) = (\frac{x_1^2 - 8x}{14}, \frac{x_2^2 - 8x}{14})$, with $\nabla\bar{\Im}_1(x_1, x_2) = (\frac{x_1 - 4}{5}, \frac{x_2 - 4}{5})$, $\nabla\bar{\Im}_2(x_1, x_2) = (\frac{x_1 - 4}{6}, \frac{x_2 - 4}{6})$, $\nabla\bar{\Im}_3(x_1, x_2) = (\frac{x_1 - 4}{7}, \frac{x_2 - 4}{7})$, for all $(x_1, x_2) \in Q$. Let $A : H_1 \to H_2$ be defined by $A(x_1, x_2) = (2x_1, 2x_2)$, for all $(x_1, x_2) \in H_1$ and let $A^* : H_2 \to H_1$ be defined by $A^*(\bar{x_1}, \bar{x_2}) = (2\bar{x_1}, 2\bar{x_2})$. For every $(\bar{x_1}, \bar{x_2}) \in H_2$. Define $M_C : [-50, 50] \times [-50, 50] \to [-50, 50] \times [-50, 50]$ by $M_C(x_1, x_2) = P_C(I - \frac{1}{3}\nabla\Im_1)(\frac{1}{2}(x_1, x_2) + \frac{1}{2}P_C(I - \frac{1}{3}\nabla\Im_2)(\frac{1}{2}(x_1, x_2) + \frac{1}{2}P_C(I - \frac{1}{3}\nabla\Im_3)(x_1, x_2)))$, $\forall (x_1, x_2) \in C$, and define $M_Q : [-100, 100] \times [-100, 100] \to$

$[-100,100]\times[-100,100]$ by $M_Q(\bar{x_1},\bar{x_2}) = P_Q(I-\frac{1}{5}\nabla\bar{\Im}_1)(\frac{1}{2}(\bar{x_1},\bar{x_2})+\frac{1}{2}P_Q(I-\frac{1}{5}\nabla\bar{\Im}_2)(\frac{1}{2}(\bar{x_1},\bar{x_2})+\frac{1}{2}P_Q(I-\frac{1}{5}\nabla\bar{\Im}_3)(\bar{x_1},\bar{x_2})), \forall(\bar{x_1},\bar{x_2}) \in Q$.
Then, we have $(x^*,y^*,z^*) \in \Psi^{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}$, where $x^*,y^*,z^* = (2,2) \in [-50,50]\times[-50,50]$.
By Lemma (3.11), we have $(2,2) = M_C((2,2)-\frac{1}{5}A^*(I-M_Q)A(2,2))$.

Lemma 3.13. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1,2$ let $\Im^i_1,\Im^i_2,\Im^i_3 : C \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\Im^i_1,\nabla\Im^i_2,\nabla\Im^i_3$ be $\frac{1}{L_{\Im^i_1}},\frac{1}{L_{\Im^i_2}},\frac{1}{L_{\Im^i_3}}$-inverse strongly monotone and continuous respectively, where $\zeta^i \in (0,\frac{2}{L_\Im})$ with $\frac{1}{L_\Im}$ = min$\{\frac{1}{L_{\Im^i_1}},\ \frac{1}{L_{\Im^i_2}},\frac{1}{L_{\Im^i_3}}\}$. Let $\bar{\Im}^i_1,\bar{\Im}^i_2,\bar{\Im}^i_3 : Q \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\bar{\Im}^i_1,\nabla\bar{\Im}^i_2,\nabla\bar{\Im}^i_3$ be $\frac{1}{L_{\bar{\Im}^i_1}},\frac{1}{L_{\bar{\Im}^i_2}},\frac{1}{L_{\bar{\Im}^i_3}}$-inverse strongly monotone and continuous respectively, where $\bar{\zeta}^i \in (0,\frac{2}{L_{\bar{\Im}}})$ with $\frac{1}{L_{\bar{\Im}}}$ = min$\{\frac{1}{L_{\bar{\Im}^i_1}},\frac{1}{L_{\bar{\Im}^i_2}},\frac{1}{L_{\bar{\Im}^i_3}}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0,\frac{1}{L})$ with L being the spectral radius of the operator A^*A. Defined the mapping $M_C,\ M_Q$ as in Lemma 3.11. Let $\bigcap_{i=1}^{3}\Phi_{\Im_i} \neq \emptyset$ and $\Phi_{\Im_i} = \{\Im_i(x) = \min_{x^*\in C}\Im_i(x^*) : \bar{\Im}_i(Ax) = \min_{Ax^*\in Q}\bar{\Im}_i(Ax^*)\}$, for all $i = 1,2,3$. then

$$\bigcap_{i=1}^{3}\Phi_{\Im_i} = F(M_C(I-\eta A^*(I-M_Q)A)).$$

Proof. We claim that $\bigcap_{i=1}^{3}\Phi_{\Im_i} \subseteq F(M_C(I-\eta A^*(I-M_Q)A))$. To show this, let $x \in \bigcap_{i=1}^{3}\Phi_{\Im_i}$ then $x \in \Phi_{\Im_i}$, for all $i = 1,2,3$. It implies that $\Im_i(x) = \min_{x^*\in C}\Im_i(x^*)$ and $\bar{\Im}_i(Ax) = \min_{Ax^*\in Q}\bar{\Im}_i(Ax^*)$, for all $i = 1,2,3$. By Lemma 3.10, we have
$x \in F(P_C(I-\zeta\nabla\Im_i))$ and $Ax \in F(P_Q(I-\bar{\zeta}\nabla\bar{\Im}_i))$, for all $i = 1,2,3$.
Then, we have $x \in \bigcap_{i=1}^{3}F(P_C(I-\zeta\nabla\Im_i))$ and $Ax \in \bigcap_{i=1}^{3}F(P_Q(I-\bar{\zeta}\nabla\bar{\Im}_i))$.
Utilizing the same method as (3.31), we have

$$x \in F(M_C),$$

that is

$$x = M_Cx = P_C(I-\zeta\nabla\Im_1)(ax+(1-a)y),$$

where $y = P_C(I-\zeta\nabla\Im_2)(ax+(1-a)z)$, $z = P_C(I-\zeta\nabla\Im_3)x$ and

$$Ax \in F(M_Q),$$

that is

$$Ax = M_QAx = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_1)(a\bar{x}+(1-a)\bar{y}),$$

where $\bar{y} = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_2)(a\bar{x}+(1-a)\bar{z})$, $\bar{z} = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_3)\bar{x}$ with $\bar{x} = Ax$, $\bar{y} = Ay$ and $\bar{z} = Az$. From Lemma 1.3, we have $(x,y,z) \in \Psi_{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}$ and $(\bar{x},\bar{y},\bar{z}) \in \Psi_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}$.
Thus $(x,y,z) \in \Psi^{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}$.
From Lemma 3.11, we have $x = M_C(I-\eta A^*(I-M_Q)A)x$.
Thus

$$x \in F(M_C(I-\eta A^*(I-M_Q)A)).$$

Then, we have

$$\bigcap_{i=1}^{3} \Phi_{\Im_i} \subseteq F(M_C(I-\eta A^*(I-M_Q)A)).$$

Next, we claim that $F(M_C(I-\eta A^*(I-M_Q)A)) \subseteq \bigcap_{i=1}^{3} \Phi_{\Im_i}$. To show this, let $x \in F(M_C(I-\eta A^*(I-M_Q)A))$, then

$$x = M_C(I-\eta A^*(I-M_Q)A)x.$$

From Lemma 3.11, we have

$$(x,y,z) \in \Psi^{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3},$$

where $y = P_C(I-\zeta\nabla\Im_2)(ax+(1-a)z)$, $z = P_C(I-\zeta\nabla\Im_3)x$, $\bar{y} = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_2)(a\bar{x}+(1-a)\bar{z})$, $\bar{z} = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_3)\bar{x}$ with $\bar{x} = Ax$, $\bar{y} = Ay$ and $\bar{z} = Az$.
Then $(x,y,z) \in \Psi_{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}$ and $(\bar{x},\bar{y},\bar{z}) \in \Psi_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}$.
From Lemma 1.3, we have

$$x \in F(M_C),$$

and

$$Ax \in F(M_Q),$$

That is

$$x = M_C x = P_C(I-\zeta\nabla\Im_1)(ax+(1-a)y),$$

where $y = P_C(I-\zeta\nabla\Im_2)(ax+(1-a)z)$, $z = P_C(I-\zeta\nabla\Im_3)x$ and

$$Ax = M_Q Ax = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_1)(a\bar{x}+(1-a)\bar{y}),$$

where $\bar{y} = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_2)(a\bar{x}+(1-a)\bar{z})$, $\bar{z} = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_3)\bar{x}$ with $\bar{x} = Ax$, $\bar{y} = Ay$ and $\bar{z} = Az$.
Utilizing the same method as (3.31), we have

$$x \in \bigcap_{i=1}^{3} F(P_C(I-\zeta_i\nabla\Im_i))$$

and

$$Ax \in \bigcap_{i=1}^{3} F(P_Q(I-\bar{\zeta}_i\nabla\bar{\Im}_i)).$$

Then $x \in F(P_C(I-\zeta_i\nabla\Im_i))$ and $Ax \in F(P_Q(I-\bar{\zeta}_i\nabla\bar{\Im}_i))$, for all $i = 1,2,3$.
By Lemma 3.10, we have
$\Im_i(x) = \min_{x^*\in C} \Im_i(x^*)$ and $\bar{\Im}_i(Ax) = \min_{Ax^*\in Q} \bar{\Im}_i(Ax^*)$, for all $i = 1,2,3$.
That is $x \in \Phi_{\Im_i}$, for all $i = 1,2,3$.
It follows that

$$x \in \bigcap_{i=1}^{3} \Phi_{\Im_i}.$$

Thus

$$F(M_C(I-\eta A^*(I-M_Q)A)) \subseteq \bigcap_{i=1}^{3} \Phi_{\Im_i}.$$

Hence

$$\bigcap_{i=1}^{3} \Phi_{\Im_i} = F(M_C(I - \eta A^*(I - M_Q)A)).$$

□

Theorem 3.14. Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. For every $i = 1, 2$, let $B_i : C \to H$ be α_i-inverse strongly monotone mapping with α= min$\{\alpha_1, \alpha_2\}$. For every $i = 1, 2$ let $\Im_1^i, \Im_2^i, \Im_3^i : C \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\Im_1^i, \nabla\Im_2^i, \nabla\Im_3^i$ be $\frac{1}{L_{\Im_1^i}}, \frac{1}{L_{\Im_2^i}}, \frac{1}{L_{\Im_3^i}}$-inverse strongly monotone and continuous respectively, where $\zeta^i \in (0, \frac{2}{L_{\Im}})$ with $\frac{1}{L_{\Im}}$ = min$\{\frac{1}{L_{\Im_1^i}}, \frac{1}{L_{\Im_2^i}}, \frac{1}{L_{\Im_3^i}}\}$. Let $\bar{\Im}_1^i, \bar{\Im}_2^i, \bar{\Im}_3^i : Q \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\bar{\Im}_1^i, \nabla\bar{\Im}_2^i, \nabla\bar{\Im}_3^i$ be $\frac{1}{L_{\bar{\Im}_1^i}}, \frac{1}{L_{\bar{\Im}_2^i}}, \frac{1}{L_{\bar{\Im}_3^i}}$-inverse strongly monotone and continuous respectively, where $\bar{\zeta}^i \in (0, \frac{2}{L_{\bar{\Im}}})$ with $\frac{1}{L_{\bar{\Im}}}$ = min$\{\frac{1}{L_{\bar{\Im}_1^i}}, \frac{1}{L_{\bar{\Im}_2^i}}, \frac{1}{L_{\bar{\Im}_3^i}}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. let $f, g : H \to H$ be a_f and a_g-contraction mapping with a = max$\{a_f, a_g\}$. For every $i = 1, 2$ define $M_C^i : C \to C$ by $M_C^i(x) = P_C(I - \zeta^i\nabla\Im_1^i)(ax + (1-a)P_C(I - \zeta^i\nabla\Im_2^i)(ax + (1-a)P_C(I - \zeta^i\nabla\Im_3^i)x)), \forall x \in C$, and define $M_Q^i : Q \to Q$ by $M_Q^i(\hat{x}) = P_Q(I - \bar{\zeta}^i\nabla\bar{\Im}_1^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i\nabla\bar{\Im}_2^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i\nabla\bar{\Im}_3^i)\hat{x})), \forall \hat{x} \in Q, \forall \hat{x} \in Q$. For every $i = 1, 2$ define $M^i : C \to C$ by $M^i(x^*) = M_C^i(x^* - \eta A^*(I - M_Q^i)Ax^*)$ for all $x^* \in C$. Let the sequences $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2 B_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)M^2 w_n) \end{cases} \tag{3.36}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0, 1]$ with $\delta_n + \eta_n + \mu_n = 1$, $\gamma \in (0, 2\alpha)$ and γ=min$\{\gamma_1, \gamma_2\}$. Assume the following conditions hold;

(i) $\mathcal{F}_i = \bigcap_{j=1}^{3} \Phi_{\Im_j}^i \cap VI(C, B_i) \neq \emptyset$ for all $i = 1, 2$ and $\Phi_{\Im_j}^i = \{\Im_j(x) = \min_{x^* \in C} \Im_j(x^*) : \bar{\Im}_j(Ax) = \min_{Ax^* \in Q} \bar{\Im}_j(Ax^*)\}$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ for all $n \in N$ and for some $\bar{\theta}, \theta > 0$,

(iv) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Then $\{x_n\}$ converges strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, where $y_1^* = P_C(I - \zeta^1\nabla\Im_2^1)(ax_1^* + (1-a)z_1^*)$ and $z_1^* = P_C(I - \zeta^1\nabla\Im_3^1)x_1^*$ with (x_1^*, y_1^*, z_1^*) is an element of $\Psi_{\nabla\bar{\Im}_1^1, \nabla\bar{\Im}_2^1, \nabla\bar{\Im}_3^1}^{\nabla\Im_1^1, \nabla\Im_2^1, \nabla\Im_3^1}$. and $\{w_n\}$ converges strongly to $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, where $y_2^* = P_C(I - \zeta^2\nabla\Im_2^2)(ax_2^* + (1-a)z_2^*)$ and $z_2^* = P_C(I - \zeta^2\nabla\Im_3^2)x_2^*$ with (x_2^*, y_2^*, z_2^*) is an element of $\Psi_{\nabla\bar{\Im}_1^2, \nabla\bar{\Im}_2^2, \nabla\bar{\Im}_3^2}^{\nabla\Im_1^2, \nabla\Im_2^2, \nabla\Im_3^2}$.

Proof. By using Theorem 3.5 and Lemma 3.13 , we obtain the conclusion. □

3.3 A new subgradient extragradient method for solving the split of modified variational inequality problems(SMVIP) and fixed point problem

In this section, we introduce a new half-space related to the SMVIP and prove weak convergence theorems of the sequence $\{x_n\}$ generated by our new half-space for approximating the solutions of the SMVIP.

In order to prove our main result we need to prove the lemmas involving the split variational inequality problem.

Lemma 3.15. Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1, H_2, respectively. Defined the mappings $D_1, D_2, D_3, \bar{D}_1, \bar{D}_2, \bar{D}_3, M_C$ and M_Q as in Lemma 3.1 where $\zeta \in (0, 2d^*)$ with d^* =min$\{d_1, d_2, d_3\}$, $\bar{\zeta} \in (0, 2\hat{d})$ with $\hat{d} =$ min$\{\bar{d_1}, \bar{d_2}, \bar{d_3}\}$ and $a \in [0, 1]$. Let $\{x_n\}$ be a sequence in H_1 and let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Let $\bigcap_{i=1}^{3} \Phi_i \neq \emptyset$ and $\Phi_i = \{w \in VI(C, D_i) | Aw \in VI(Q, \bar{D}_i)\}$, for all $i = 1, 2, 3$. For every $n \in \mathbb{N}$, let $T_n = aW_n + (1-a)P_C(I-\zeta D_2)(aW_n + (1-a)P_C(I-\zeta D_3)W_n))$ and $W_n = (I - \eta A^*(I - M_Q)A)x_n$. If $x^* \in \cap\bigcap_{i=1}^{3} \Phi_i$, then

$$\|T_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \eta(1 - \eta L)\|(I - M_Q)Ax_n\|^2,$$

for all $n \in \mathbb{N}$.

Proof. Let $x^* \in \bigcap_{i=1}^{3} \Phi_i$. From Lemma 3.8, we have

$$x^* \in F(M_C(I - \eta A^*(I - M_Q)A)).$$

It implies that $x^* = M_C(I - \eta A^*(I - M_Q)A)x^*$, $y^* = P_C(I - \zeta D_2)(ax^* + (1 - a)z^*)$ and $z^* = P_C(I - \zeta D_3)x^*$, where $\bar{x^*} = Ax^* = P_Q(I - \bar{\zeta}\bar{D}_1)(a\bar{x^*} + (1 - a)\bar{y^*})$, $\bar{y^*} = Ay^* = P_Q(I - \bar{\zeta}\bar{D}_2)(a\bar{x^*} + (1 - a)\bar{z^*})$ and $\bar{z^*} = Az^* = P_Q(I - \bar{\zeta}\bar{D}_3)\bar{x^*}$. From Lemma 3.3, we have $(x^*, y^*, z^*) \in \Psi^{D_1,D_2,D_3}_{\bar{D}_1,\bar{D}_2,\bar{D}_3}$. That is $(x^*, y^*, z^*) \in \Psi_{D_1,D_2,D_3}$ and $(\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\bar{D}_1,\bar{D}_2,\bar{D}_3}$. Form $(\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\bar{D}_1,\bar{D}_2,\bar{D}_3}$, we obtain that

$$\bar{x^*} = P_Q(I - \bar{\zeta}\bar{D}_1)(a\bar{x^*} + (1 - a)\bar{y^*}),$$

$$\bar{y^*} = P_Q(I - \bar{\zeta}\bar{D}_2)(a\bar{x^*} + (1 - a)\bar{z^*}),$$

$$\bar{z^*} = P_Q(I - \bar{\zeta}\bar{D}_3)\bar{x^*}.$$

It implies that

$$Ax^* = \bar{x^*} = P_Q(I - \bar{\zeta}\bar{D}_1)(a\bar{x^*} + (1 - a)P_Q(I - \bar{\zeta}\bar{D}_2)(a\bar{x^*} + (1 - a)P_Q(I-$$
$$\bar{\zeta}\bar{D}_3)\bar{x^*})) = M_Q\bar{x^*} = M_QAx^*.$$

From definition of x^* we get $x^* = P_C(I-\zeta D_1)T_x^*$, where $T_x^* = aW_x^* + (1-a)P_C(I-\zeta D_2)(aW_x^* + (1-a)P_C(I-\zeta D_3)W_x^*))$ and $W_x^* = (I-\eta A^*(I-M_Q)A)x^*) = x^*$.
From Lemma 3.8, we have that $P_C(I-\zeta D_1), P_C(I-\zeta D_2)$ and $P_C(I-\zeta D_3)$ are nonexpansive.
By the definition of T_n, Lemma 3.1 and Remark 3.2, we have

$$\begin{aligned}
\|T_n - x^*\|^2 &= \|aW_n + (1-a)P_C(I-\zeta D_2)(aW_n + (1-a)\times \\
&\qquad P_C(I-\zeta D_3)W_n)) - (aW_{x^*} + (1-a)P_C(I-\zeta D_2)(aW_{x^*} \\
&\qquad + (1-a)P_C(I-\zeta D_3)W_{x^*})))\|^2 \\
&= \|a(W_n - W_{x^*}) + (1-a)[P_C(I-\zeta D_2)(aW_n + (1-a)P_C(I- \\
&\qquad \zeta D_3)W_n)) - P_C(I-\zeta D_2)(aW_{x^*} + (1-a)P_C(I-\zeta D_3)W_{x^*}))]\|^2 \\
&\le a\|W_n - W_{x^*}\|^2 + (1-a)\|P_C(I-\zeta D_2)(aW_n + (1-a)P_C(I- \\
&\qquad \zeta D_3)W_n)) - P_C(I-\zeta D_2)(aW_{x^*} + (1-a)P_C(I-\zeta D_3)W_{x^*}))\|^2 \\
&\le a\|W_n - W_{x^*}\|^2 + (1-a)\|aW_n + (1-a)P_C(I-\zeta D_3)W_n \\
&\qquad - (aW_{x^*} + (1-a)P_C(I-\zeta D_3)W_{x^*})\|^2 \\
&= a\|W_n - W_{x^*}\|^2 + (1-a)\|a(W_n - W_{x^*}) + (1-a)\times \\
&\qquad [P_C(I-\zeta D_3)W_n - x^*]\|^2 \\
&\le a\|W_n - W_{x^*}\|^2 + a(1-a)\|W_n - W_{x^*}\|^2 + (1-a)^2\times \\
&\qquad \|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&= (2a-a^2)\|W_n - W_{x^*}\|^2 + (1-a)^2\|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\le \|W_n - x^*\|^2 \\
&= \|x_n - \eta A^*(I-M_Q)Ax_n - x^*\|^2 \\
&\le \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I-M_Q)Ax_n\|^2.
\end{aligned} \tag{3.37}$$

□

Theorem 3.16. Let C and Q be a nonempty closed convex subsets of H_1 and H_2, respectively and let $S: C \to C$ be a nonexpansive mapping. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3-inverse strongly monotone, respectively, with d^* =min$\{d_1, d_2, d_3\}$. Let $\bar{D_1}, \bar{D_2}, \bar{D_3} : Q \to H_2$ be $\bar{d_1}, \bar{d_2}, \bar{d_3}$-inverse strongly monotone, respectively, with $\hat{d}$ =min$\{\bar{d_1}, \bar{d_2}, \bar{d_3}\}$. Let $A: H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C : H_1 \to C$ by

$$M_C(x) = P_C(I-\zeta D_1)(ax + (1-a)P_C(I-\zeta D_2)(ax + (1-a)P_C(I-\zeta D_3)x)),$$

$\forall x \in H_1$, where $a \in [0,1)$, $\zeta \in (0, 2d^*)$ and define $M_Q : H_2 \to Q$ by

$$M_Q(x) = P_Q(I-\bar{\zeta}\bar{D_1})(a\hat{x} + (1-a)P_Q(I-\bar{\zeta}\bar{D_2})(a\hat{x} + (1-a)P_Q(I-\bar{\zeta}\bar{D_3})\hat{x})),$$

$\forall \hat{x} \in H_1$, where $a \in [0,1)$, $\bar{\zeta} \in (0, 2\hat{d})$. Let the sequences $\{x_n\}$ and $\{y_n\}$ generated by

$x_1 \in H_1$ and

$$y_n = M_C W_n = P_C(I-\zeta D_1)T_n,$$

where $W_n = (I-\eta A^*(I-M_Q)A)x_n$ and $T_n = aW_n + (1-a)P_C(I-\zeta D_2)(aW_n+(1-a)P_C(I-\zeta D_3)W_n))$.

$$Q_n = \{z \in H : \langle (I-\zeta D_1)T_n - y_n, y_n - z\rangle \geq 0\},$$

$$x_{n+1} = \alpha_n T_n + (1-\alpha_n)SP_{Q_n}(T_n - \zeta D_1(y_n)),$$

for all $n \in \mathbb{N}$.

Assume the following conditions hold:

(i) $\Im = F(S) \bigcap \bigcap\limits_{i=1}^{3} \Phi_i \neq \emptyset$, where $\Phi_i = \{w \in VI(C, D_i) | Aw \in VI(Q, \bar{D}_i)\}$, for all $i = 1, 2, 3$.

(ii) $\alpha_n \in [c, d] \subset (0, 1)$.

Then $\{x_n\}$ converges weakly to $x_0 = P_\Im x_n$, which $(x_0, y_0, z_0) \in \Psi^{D_1, D_2, D_3}_{\bar{D_1}, \bar{D_2}, \bar{D_3}}$, $y_0 = P_C(I-\zeta D_2)(ax_0 + (1-a)z_0)$ and $z_0 = P_C(I-\zeta D_3)x_0$ with $\bar{x_0} = Ax_0$, $\bar{y_0} = Ay_0$ and $\bar{z_0} = Az_0$.

Proof. Denote $k_n := P_{Q_n}(T_n - \zeta D_1(y_n))$ for all $n \geq 0$. Let $x^* \in \Im$. From the definition of P_{Q_n}, we have $y_n = P_{Q_n}(I-\zeta D_1)T_n$. Let $M_n = T_n - \zeta D_1(y_n)$. From $C \subseteq Q_n$, and applying (2.1), we have

$$\begin{aligned}
\|k_n - x^*\|^2 &= \|P_{Q_n} M_n - x^*\|^2 \\
&\leq \|M_n - x^*\|^2 - \|M_n - P_{Q_n} M_n\|^2 \\
&= \|T_n - \zeta D_1(y_n) - x^*\|^2 - \|T_n - \zeta D_1(y_n) - P_{Q_n} M_n\|^2 \\
&= \|T_n - x^*\|^2 - 2\zeta\langle T_n - x^*, D_1(y_n)\rangle + \zeta^2\|D_1(y_n)\|^2 \\
&\qquad - \|T_n - P_{Q_n} M_n\|^2 + 2\zeta\langle T_n - P_{Q_n} M_n, D_1(y_n)\rangle - \zeta^2\|D_1(y_n)\|^2 \\
&= \|T_n - x^*\|^2 - \|T_n - P_{Q_n} M_n\|^2 - 2\zeta\langle P_{Q_n} M_n - x^*, D_1(y_n)\rangle. \qquad (3.38)
\end{aligned}$$

From monotonicity of D_1, we have

$$\begin{aligned}
0 &\leq \langle D_1 y_n - D_1 x^*, y_n - x^*\rangle \\
&= \langle D_1 y_n, y_n - x^*\rangle - \langle D_1 x^*, y_n - x^*\rangle \\
&\leq \langle D_1 y_n, y_n - x^*\rangle \\
&= \langle D_1 y_n, y_n - P_{Q_n} M_n\rangle - \langle D_1 y_n, x^* - P_{Q_n} M_n\rangle,
\end{aligned}$$

which implies that

$$\langle D_1 y_n, x^* - P_{Q_n} M_n\rangle \leq \langle D_1 y_n, y_n - P_{Q_n} M_n\rangle. \qquad (3.39)$$

From (3.38) and (3.39), we have

$$\|k_n - x^*\|^2 \leq \|T_n - x^*\|^2 - \|T_n - P_{Q_n} M_n\|^2 + 2\zeta\langle D_1 y_n, y_n - P_{Q_n} M_n\rangle. \qquad (3.40)$$

From (3.40) and Lemma 3.15, we have

$$
\begin{aligned}
\|k_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I-M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - T_n\|^2 \\
&\quad + 2\zeta\langle D_1 y_n, y_n - P_{Q_n}M_n\rangle \\
&= \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I-M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - y_n\|^2 \\
&\quad - \|y_n - T_n\|^2 - 2\langle P_{Q_n}M_n - y_n, y_n - T_n\rangle \\
&\quad + 2\zeta\langle D_1 y_n, y_n - P_{Q_n}M_n\rangle \\
&= \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I-M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - y_n\|^2 \\
&\quad - \|y_n - T_n\|^2 + 2\langle P_{Q_n}M_n - y_n, T_n - y_n - \zeta D_1 y_n\rangle \\
&= \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I-M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - y_n\|^2 \\
&\quad - \|y_n - T_n\|^2 + 2\langle (I-\zeta D_1)T_n - y_n, P_{Q_n}M_n - y_n\rangle \\
&\quad + 2\langle \zeta D_1 T_n - \zeta D_1 y_n, P_{Q_n}M_n - y_n\rangle \\
&\leq \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I-M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - y_n\|^2 \\
&\quad - \|y_n - T_n\|^2 + 2\zeta\|D_1 T_n - D_1 y_n\|\|P_{Q_n}M_n - y_n\| \\
&\leq \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I-M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - y_n\|^2 \\
&\quad - \|y_n - T_n\|^2 + \frac{\zeta}{d_1}[\|T_n - y_n\|^2 + \|P_{Q_n}M_n - y_n\|^2] \\
&= \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I-M_Q)Ax_n\|^2 \\
&\quad - (1-\frac{\zeta}{d_1})\|P_{Q_n}M_n - y_n\|^2 - (1-\frac{\zeta}{d_1}))\|T_n - y_n\|^2. \qquad (3.41)
\end{aligned}
$$

By the definition of x_{n+1}, (3.41) and Lemma 3.15, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(T_n - x^*) + (1-\alpha_n)(Sk_n - x^*)\|^2 \\
&\leq \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|Sk_n - x^*\|^2 \\
&= \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|Sk_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|T_n - Sk_n\|^2 \qquad (3.42) \\
&= \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|k_n - x^*\|^2 \\
&\leq \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)[\|x_n - x^*\|^2 \\
&\quad - \eta(1-\eta L)\|(I-M_Q)Ax_n\|^2 \\
&\quad - (1-\frac{\zeta}{d_1})\|P_{Q_n}M_n - y_n\|^2 - (1-\frac{\zeta}{d_1}))\|T_n - y_n\|^2] \\
&\leq \alpha_n[\|x_n - x^*\|^2 - \alpha_n\eta(1-\eta L)\|(I-M_Q)Ax_n\|^2] \\
&\quad + (1-\alpha_n)[\|x_n - x^*\|^2 - \eta(1-\eta L)\|(I-M_Q)Ax_n\|^2 \\
&\quad - (1-\frac{\zeta}{d_1})\|P_{Q_n}M_n - y_n\|^2 - (1-\frac{\zeta}{d_1}))\|T_n - y_n\|^2] \\
&= \|x_n - x^*\|^2 - \eta(1-\eta L)(1+\alpha_n)\|(I-M_Q)Ax_n\|^2 \\
&\quad - (1-\alpha_n)(1-\frac{\zeta}{d_1})[\|T_n - y_n\|^2 + \|y_n - k_n\|^2] \\
&\qquad (3.43)
\end{aligned}
$$

So,

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2.$$

Therefore $\lim_{n\to\infty} \|x_{n+1} - x^*\|$ exists, $\forall x^* \in \Im$. So, we have $\{x_n\}_{n=0}^{\infty}$ and $\{k_n\}_{n=0}^{\infty}$ are bounded. From the last relations it follows that

$$\eta(1-\eta L)(1+\alpha_n)\|(I - M_Q)Ax_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

or

$$\|(I - M_Q)Ax_n\|^2 \leq \frac{\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2}{\eta(1-\eta L)(1+\alpha_n)}.$$

Thus

$$\lim_{n\to\infty} \|(I - M_Q)Ax_n\| = 0. \tag{3.44}$$

By using the same method as above, we have

$$\lim_{n\to\infty} \|T_n - y_n\| = 0. \tag{3.45}$$

From (3.42), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|Sk_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|T_n - Sk_n\|^2 \\
&\leq \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|T_n - Sk_n\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|T_n - Sk_n\|^2,
\end{aligned}$$

so

$$\|T_n - Sk_n\|^2 \leq \frac{\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2}{\alpha_n(1-\alpha_n)},$$

which implies that

$$\lim_{n\to\infty} \|T_n - Sk_n\| = 0. \tag{3.46}$$

Consider;

$$W_n - x_n = -\eta A^*(I - M_Q)Ax_n,$$

and by (3.44), we have

$$\lim_{n\to\infty} \|W_n - x_n\| = 0. \tag{3.47}$$

From the property of P_C, we have

$$
\begin{aligned}
\|P_C(I-\zeta D_3)W_n - x^*\|^2 &= \|P_C(I-\zeta D_3)W_n - P_C(I-\zeta D_3)x^*\|^2\\
&\leq \|(I-\zeta D_3)W_n - (I-\zeta D_3)x^*\|^2\\
&= \|(W_n - x^*) - \zeta(D_3W_n - D_3x^*)\|^2\\
&= \|W_n - x^*\|^2 - 2\zeta\langle W_n - x^*, D_3W_n - D_3x^*\rangle\\
&\qquad + \zeta^2\|D_3W_n - D_3x^*\|^2\\
&\leq \|W_n - x^*\|^2 - 2\zeta d_3\|D_3W_n - D_3x^*\|^2\\
&\qquad + \zeta^2\|D_3W_n - D_3x^*\|^2\\
&= \|W_n - x^*\|^2 - \zeta(2d_3-\zeta)\|D_3W_n - D_3x^*\|^2\\
&\leq \|x_n - x^*\|^2 - \zeta(2d_3-\zeta)\|D_3W_n - D_3x^*\|^2
\end{aligned}
\tag{3.48}
$$

By the definition of T_n, (3.37), Remark 3.2 and (3.48), we have

$$
\begin{aligned}
\|T_n - x^*\|^2 &\leq a\|W_n - W_{x^*}\|^2 + a(1-a)\|W_n - W_{x^*}\|^2\\
&\qquad + (1-a)^2\|P_C(I-\zeta D_3)W_n - x^*\|^2\\
&\leq a\|x_n - x^*\|^2 + a(1-a)\|x_n - x^*\|^2\\
&\qquad + (1-a)^2\|P_C(I-\zeta D_3)W_n - x^*\|^2\\
&\leq (2a-a^2)\|x_n - x^*\|^2 + (1-a)^2\|P_C(I-\zeta D_3)W_n - x^*\|^2\\
&\leq (2a-a^2)\|x_n - x^*\|^2 + (1-a)^2[\|x_n - x^*\|^2\\
&\qquad - \zeta(2d_3-\zeta)\|D_3W_n - D_3x^*\|^2]\\
&= \|x_n - x^*\|^2 - \zeta(2d_3-\zeta)(1-a)^2\|D_3W_n - D_3x^*\|^2
\end{aligned}
\tag{3.49}
$$

In addition, by the definition of x_{n+1} and (3.49), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|k_n - x^*\|^2\\
&\leq \alpha_n[\|x_n - x^*\|^2 - \zeta(2d_3-\zeta)(1-a)^2\|D_3W_n - D_3x^*\|^2]\\
&\qquad + (1-\alpha_n)\|k_n - x^*\|^2\\
&= \alpha_n\|x_n - x^*\|^2 - \alpha_n\zeta(2d_3-\zeta)(1-a)^2\|D_3W_n - D_3x^*\|^2\\
&\qquad + (1-\alpha_n)\|x_n - x^*\|^2\\
&= \|x_n - x^*\|^2 - \alpha_n\zeta(2d_3-\zeta)(1-a)^2\|D_3W_n - D_3x^*\|^2,
\end{aligned}
$$

so

$$
\|D_3W_n - D_3x^*\|^2 \leq \frac{\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2}{\alpha_n\zeta(2d_3-\zeta)(1-a)^2},
$$

which implies that

$$\lim_{n\to\infty} \|D_3W_n - D_3x^*\| = 0. \tag{3.50}$$

From the property of P_C, we have

$$\begin{aligned}
&\|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\le \langle (I-\zeta D_3)W_n - (I-\zeta D_3)x^*, P_C(I-\zeta D_3)W_n - x^*\rangle \\
&= \frac{1}{2}[\|(I-\zeta D_3)W_n - (I-\zeta D_3)x^*\|^2] + \|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\qquad - \|(I-\zeta D_3)W_n - (I-\zeta D_3)x^* - (P_C(I-\zeta D_3)W_n - x^*)\|^2] \\
&\le \frac{1}{2}[\|W_n - x^*\|^2 + \|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\qquad - \|(I-\zeta D_3)W_n - (I-\zeta D_3)x^* - (P_C(I-\zeta D_3)W_n - x^*)\|^2] \\
&= \frac{1}{2}[\|W_n - x^*\|^2 + \|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\qquad - \|(W_n - P_C(I-\zeta D_3)W_n) - \zeta(D_3W_n - D_3x^*)\|^2] \\
&= \frac{1}{2}[\|W_n - x^*\|^2 + \|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\qquad - \|W_n - P_C(I-\zeta D_3)W_n\|^2 - \zeta^2\|D_3W_n - D_3x^*\|^2 \\
&\qquad + 2\zeta\langle W_n - P_C(I-\zeta D_3)W_n, D_3W_n - D_3x^*\rangle],
\end{aligned}$$

so

$$\begin{aligned}
\|P_C(I-\zeta D_3)W_n - x^*\|^2 \le{}& \|W_n - x^*\|^2 - \|W_n - P_C(I-\zeta D_3)W_n\|^2 \\
&+ 2\zeta\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\|.
\end{aligned} \tag{3.51}$$

By the definition of T_n, (3.37), Remark 3.2 and (3.51), we have

$$\begin{aligned}
\|T_n - x^*\|^2 &\le a\|W_n - W_{x^*}\|^2 + a(1-a)\|W_n - W_{x^*}\|^2 \\
&\qquad + (1-a)^2\|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\le a\|x_n - x^*\|^2 + a(1-a)\|x_n - x^*\|^2 \\
&\qquad + (1-a)^2\|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\le (2a-a^2)\|x_n - x^*\|^2 + (1-a)^2\|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\le (2a-a^2)\|x_n - x^*\|^2 + (1-a)^2[\|W_n - x^*\|^2 - \|W_n - P_C(I- \\
&\qquad \zeta D_3)W_n\|^2 + 2\zeta\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\|] \\
&= (2a-a^2)\|x_n - x^*\|^2 + (1-a)^2\|x_n - x^*\|^2 \\
&\qquad - (1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|^2 \\
&\qquad + 2\zeta(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\| \\
&= \|x_n - x^*\|^2 - (1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|^2
\end{aligned}$$

$$+\,2\zeta(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\| \tag{3.52}$$

In addition, by the definition of x_{n+1}, (3.41) and (3.52), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|k_n - x^*\|^2\\
&\le \alpha_n[\|x_n - x^*\|^2 - (1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|^2\\
&\qquad + 2\zeta(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\|]\\
&\qquad + (1-\alpha_n)\|k_n - x^*\|^2\\
&\le \alpha_n\|x_n - x^*\|^2 - \alpha_n(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|^2\\
&\qquad + 2\alpha_n\zeta(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\|\\
&\qquad + (1-\alpha_n)\|x_n - x^*\|^2\\
&= \|x_n - x^*\|^2 - \alpha_n(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|^2\\
&\qquad + 2\alpha_n\zeta(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\|,
\end{aligned} \tag{3.53}$$

From (3.50) and (3.53), we get

$$\lim_{n\to\infty}\|W_n - P_C(I-\zeta D_3)W_n\| = 0. \tag{3.54}$$

Let $G_n = aW_n + (1-a)P_C(I-\lambda_3D_3)W_n$. From the property of P_C, we have

$$\begin{aligned}
\|P_C(I-\zeta D_2)G_n - x^*\|^2 &= \|P_C(I-\zeta D_2)G_n - P_C(I-\zeta D_2)x^*\|^2\\
&\le \|(I-\zeta D_2)G_n - (I-\zeta D_2)x^*\|^2\\
&= \|(G_n - x^*) - \zeta(D_2G_n - D_2x^*)\|^2\\
&= \|G_n - x^*\|^2 - 2\zeta\langle G_n - x^*, D_2G_n - D_2x^*\rangle\\
&\qquad + \zeta^2\|D_2G_n - D_2x^*\|^2\\
&\le \|x_n - x^*\|^2 - 2\zeta d_2\|D_2G_n - D_2x^*\|^2\\
&\qquad + \zeta^2\|D_2G_n - D_2x^*\|^2\\
&= \|x_n - x^*\|^2 - \zeta(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2,
\end{aligned} \tag{3.55}$$

By the definition of T_n and (3.55), we have

$$\begin{aligned}
\|T_n - x^*\|^2 &\le a\|W_n - W_{x^*}\|^2 + (1-a)\|P_C(I-\zeta D_2)G_n - x^*\|^2\\
&\le a\|x_n - x^*\|^2 + (1-a)[\|x_n - x^*\|^2\\
&\qquad - \zeta(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2]\\
&= \|x_n - x^*\|^2 - \zeta(1-a)(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2
\end{aligned} \tag{3.56}$$

In addition, by the definition of x_{n+1} and (3.56), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \alpha_n\|T_n-x^*\|^2+(1-\alpha_n)\|k_n-x^*\|^2\\ &\leq \alpha_n[\|x_n-x^*\|^2-\zeta(1-a)(2d_2-\zeta)\|D_2G_n-D_2x^*\|^2]\\ &\qquad +(1-\alpha_n)\|x_n-x^*\|^2\\ &= \|x_n-x^*\|^2-\zeta\alpha_n(1-\alpha_n)(2d_2-\zeta)\|D_2G_n-D_2x^*\|^2\end{aligned}$$

so

$$\|D_2G_n-D_2x^*\|^2 \leq \frac{\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2}{\zeta\alpha_n(1-\alpha_n)(2d_2-\zeta)}.$$

It implies that

$$\lim_{n\to\infty}\|D_2G_n-D_2x^*\|=0. \tag{3.57}$$

From the property of P_C, we have

$$\begin{aligned}&\|P_C(I-\zeta D_2)G_n-x^*\|^2\\ &\quad=\langle (I-\zeta D_2)G_n-(I-\zeta D_2)x^*, P_C(I-\zeta D_2)G_n-x^*\rangle\\ &\quad=\frac{1}{2}[\|(I-\zeta D_2)G_n-(I-\zeta D_2)x^*\|^2+\|P_C(I-\zeta D_2)G_n-x^*\|^2\\ &\qquad\quad -\|(I-\zeta D_2)G_n-(I-\zeta D_2)x^*-((I-\zeta D_2)G_n-x^*)\|^2]\\ &\quad\leq\frac{1}{2}[\|G_n-x^*\|^2+\|P_C(I-\zeta D_2)G_n-x^*\|^2\\ &\qquad\quad -\|(I-\zeta D_2)G_n-(I-\zeta D_2)x^*-((I-\zeta D_2)G_n-x^*)\|^2]\\ &\quad=\frac{1}{2}[\|G_n-x^*\|^2+\|P_C(I-\zeta D_2)G_n-x^*\|^2\\ &\qquad\quad -\|(G_n-P_C(I-\zeta D_2)G_n)-\zeta(D_2G_n-D_2x^*)\|^2]\\ &\quad=\frac{1}{2}[\|G_n-x^*\|^2+\|P_C(I-\zeta D_2)G_n-x^*\|^2\\ &\qquad\quad -\|G_n-P_C(I-\zeta D_2)G_n\|^2\\ &\qquad\quad +2\zeta\langle G_n-P_C(I-\zeta D_2)G_n, D_2G_n-D_2x^*\rangle\\ &\qquad\quad -\zeta^2\|D_2G_n-D_2x^*\|^2],\end{aligned}$$

It implies that

$$\begin{aligned}\|P_C(I-\zeta D_2)G_n-x^*\|^2 &\leq \|G_n-x^*\|^2-\|G_n-P_C(I-\zeta D_2)G_n\|^2\\ &\qquad +2\zeta\langle G_n-P_C(I-\zeta D_2)G_n, D_2G_n-D_2x^*\rangle\\ &\leq \|G_n-x^*\|^2-\|G_n-P_C(I-\zeta D_2)G_n\|^2\\ &\qquad +2\zeta\|G_n-P_C(I-\zeta D_2)G_n\|\|D_2G_n-D_2x^*\|.\end{aligned}$$

(3.58)

By the definition of T_n and (3.58), we have

$$\begin{aligned}\|T_n - x^*\|^2 &\leq a\|W_n - W_{x^*}\|^2 + (1-a)\|P_C(I-\zeta D_2)G_n - x^*\|^2 \\ &\leq a\|x_n - x^*\|^2 + (1-a)[\|G_n - x^*\|^2 - \|G_n - P_C(I- \\ &\qquad \zeta D_2)G_n\|^2 + 2\zeta\|G_n - P_C(I-\zeta D_2)G_n\|\|D_2G_n - D_2x^*\|] \\ &\leq a\|x_n - x^*\|^2 + (1-a)\|x_n - x^*\|^2 \\ &\qquad - (1-a)\|G_n - P_C(I-\zeta D_2)G_n\|^2 \\ &\qquad + 2\zeta\|G_n - P_C(I-\zeta D_2)G_n\|\|D_2G_n - D_2x^*\|]. \\ &= \|x_n - x^*\|^2 - (1-a)\|G_n - P_C(I-\zeta D_2)G_n\|^2 \\ &\qquad + 2\zeta\|G_n - P_C(I-\zeta D_2)G_n\|\|D_2G_n - D_2x^*\|].\end{aligned} \tag{3.59}$$

In addition, by the definition of x_{n+1} and (3.59), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|k_n - x^*\|^2 \\ &\leq \alpha_n[\|x_n - x^*\|^2 - (1-a)\|G_n - P_C(I-\zeta D_2)G_n\|^2 \\ &\qquad + 2\zeta\|G_n - P_C(I-\zeta D_2)G_n\|\|D_2G_n - D_2x^*\|] \\ &\qquad + (1-\alpha_n)\|x_n - x^*\|^2 \\ &= \|x_n - x^*\|^2 - \alpha_n(1-a)\|G_n - P_C(I-\zeta D_2)G_n\|^2 \\ &\qquad + 2\zeta\alpha_n(1-a)\|G_n - P_C(I-\zeta D_2)G_n\|\|D_2G_n - D_2x^*\|,\end{aligned} \tag{3.60}$$

by (3.60) and (3.57), we get

$$\lim_{n\to\infty}\|G_n - P_C(I-\zeta D_2)G_n\| = 0. \tag{3.61}$$

Since

$$T_n - W_n = (1-a)(P_C(I-\zeta D_2)(aW_n + (1-a)P_C(I-\zeta D_3)W_n) - W_n).$$

From the property of norm, we have

$$\begin{aligned}&\|P_C(I-\zeta D_2)(aW_n + (1-a)P_C(I-\zeta D_3)W_n) - W_n\| \\ &\quad \leq \|P_C(I-\zeta D_2)(aW_n + (1-a)P_C(I-\zeta D_3)W_n) \\ &\qquad - (aW_n + (1-a)P_C(I-\zeta D_3)W_n)\| \\ &\qquad + \|(aW_n + (1-a)P_C(I-\zeta D_3)W_n - W_n\| \\ &\quad = \|P_C(I-\zeta D_2)G_n - G_n\| + (1-a)\|P_C(I-\zeta D_3)W_n - W_n\|\end{aligned} \tag{3.62}$$

Then, we have

$$\|T_n - W_n\| \leq (1-a)[\|P_C(I - \zeta D_2)G_n - G_n\| \\ + (1-a)\|P_C(I - \zeta D_3)W_n - W_n\|].$$

From (3.54) and (3.61), it implies that

$$\lim_{n\to\infty} \|T_n - W_n\| = 0. \tag{3.63}$$

From (3.45), (3.47), (3.63) and

$$\|y_n - x_n\| \leq \|y_n - T_n\| + \|T_n - W_n\| + \|W_n - x_n\|$$

we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.64}$$

Moreover, from (3.46), (3.45), (3.64) and

$$\|x_n - Sk_n\| \leq \|x_n - y_n\| + \|y_n - T_n\| + \|T_n - Sk_n\|$$

we have

$$\lim_{n\to\infty} \|x_n - Sk_n\| = 0. \tag{3.65}$$

Since $\{x_n\}_{n=0}^{\infty}$ is bounded, it has a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ which weakly converges to some $\bar{x} \in C$.

Assume $\bar{x} \notin F(S)$. By nonexpansiveness of S and Opial's property and (3.65), we have

$$\begin{aligned}
\lim_{k\to\infty} \inf \|x_{n_k} - \bar{x}\| &< \lim_{k\to\infty} \inf \|x_{n_k} - S\bar{x}\| \\
&\leq \lim_{k\to\infty} \inf [\|x_{n_k} - Sk_{n_k}\| + \|Sk_{n_k} - S\bar{x}\|] \\
&\leq \lim_{k\to\infty} \inf [\|x_{n_k} - Sk_{n_k}\| + \|k_{n_k} - \bar{x}\|] \\
&= \lim_{k\to\infty} \inf \|k_{n_k} - \bar{x}\| \\
&\leq \lim_{k\to\infty} \inf \|x_{n_k} - \bar{x}\|.
\end{aligned}$$

This is a contradiction, then we have

$$\bar{x} \in F(S).$$

Assume $\bar{x} \notin \bigcap_{i=1}^{3} \Phi_i$. From Lemma 3.8, we have $\bar{x} \notin F(M_C(I - \eta A^*(I - M_Q)A))$. By the Opial's condition, (3.64) and Remark 3.2, we have

$$\begin{aligned}\liminf_{k\to\infty}\|x_{n_k}-\bar{x}\| &< \liminf_{k\to\infty}\|x_{n_k}-M_C(I-\eta A^*(I-M_Q)A)\bar{x}\| \\ &\leq \liminf_{k\to\infty}\|x_{n_k}-y_{n_k}\|+\liminf_{k\to\infty}\|M_C(x_{n_k}-\eta A^*(I- \\ &\qquad M_Q)Ax_{n_k})-M_C(I-\eta A^*(I-M_Q)A)\bar{x}\| \\ &\leq \liminf_{k\to\infty}(\|x_{n_k}-y_{n_k}\|+\|x_{n_k}-\bar{x}\|) \\ &= \liminf_{k\to\infty}\|x_{n_k}-\bar{x}\|.\end{aligned} \tag{3.66}$$

This is a contradiction, then we have

$$\bar{x}\in F(M_C(I-\eta A^*(I-M_Q)A)).$$

It implies that

$$\bar{x}\in\bigcap_{i=1}^{3}\Phi_i.$$

Hence

$$\bar{x}\in\Im.$$

In order to show that the entire sequence $\{x_n\}$ weakly converges to $\bar{x}$, assume $\{x\}_{n_k}\rightharpoonup\hat{x}$ as $k\to\infty$, with $\bar{x}\neq\hat{x}$ and $\hat{x}\in\Im$. By the Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-\bar{x}\| &= \liminf_{k\to\infty}\|x_{n_k}-\bar{x}\| \\ &< \liminf_{k\to\infty}\|x_{n_k}-\hat{x}\| \\ &= \lim_{n\to\infty}\|x_n-\hat{x}\| \\ &= \liminf_{n\to\infty}\|x_{n_k}-\hat{x}\| \\ &< \liminf_{n\to\infty}\|x_{n_k}-\bar{x}\| \\ &= \lim_{n\to\infty}\|x_n-\bar{x}\|,\end{aligned}$$

This is a contradiction, thus

$$\bar{x}=\hat{x}.$$

It implies that the sequence $\{x_n\}_{n=0}^{\infty}$ weakly converges to $\bar{x}\in\Im$.
From (3.64), we have $\{y_n\}_{n=0}^{\infty}$ weakly converges to $\bar{x}\in\Im$.
Finally, if we take

$$U_n=P_{\Im}x_n, \tag{3.67}$$

By Lemma 2.8, we see that $\{P_{\Im}x_n\}_{n=0}^{\infty}$ converges strongly to some $z \in \Im$. From (3.67), we get

$$\langle \bar{x} - U_n, U_n - x_n \rangle \geq 0, \ \forall \bar{x} \in \Im.$$

Take $n \to \infty$, we also have

$$\langle \bar{x} - z, z - \bar{x} \rangle \geq 0,$$

and hence $\bar{x} = z$. Therefore U_n converges strongly to $\bar{x} \in \Im$, this completes the proof. □

3.4 Application of a new subgradient extragradient method for solving the split of modified variational inequality problems(SMVIP) and fixed point problem

3.4.1 Weak and Strong convergence theorem for approximating the solution of the split minimization problem

Theorem 3.17. Let C and Q be a nonempty closed convex subsets of H_1 and H_2, respectively and let $S : C \to C$ be a nonexpansive mapping. Let $\Im_1, \Im_2, \Im_3 : C \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\Im_1, \nabla\Im_2, \nabla\Im_3$ be $\frac{1}{l_{\Im_1}}, \frac{1}{l_{\Im_2}}, \frac{1}{l_{\Im_3}}$-inverse strongly monotone and continuous respectively, where $\zeta \in (0, \frac{2}{l_{\Im}})$ with $\frac{1}{l_{\Im}}$= min$\{\frac{1}{l_{\Im_1}}, \frac{1}{l_{\Im_2}}, \frac{1}{l_{\Im_3}}\}$. Let $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3 : Q \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3$ be $\frac{1}{l_{\bar{\Im}_1}}, \frac{1}{l_{\bar{\Im}_2}}, \frac{1}{l_{\bar{\Im}_3}}$-inverse strongly monotone and continuous respectively, where $\bar{\zeta} \in (0, \frac{2}{l_{\bar{\Im}}})$ with $\frac{1}{l_{\bar{\Im}}}$= min$\{\frac{1}{l_{\bar{\Im}_1}}, \frac{1}{l_{\bar{\Im}_2}}, \frac{1}{l_{\bar{\Im}_3}}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{l})$ with L being the spectral radius of the operator A^*A. define $M_C : H_1 \to C$ by $M_C(x) = P_C(I - \zeta\nabla\Im_1)(ax + (1-a)P_C(I - \zeta\nabla\Im_2)(ax + (1-a)P_C(I - \zeta\nabla\Im_3)x))$, $\forall x \in H_1$, and define $M_Q : H_2 \to Q$ by $M_Q(\hat{x}) = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_1)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\nabla\bar{\Im}_2)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\nabla\bar{\Im}_3)\hat{x}))$, $\forall \hat{x} \in H_2$. Let the sequences $\{x_n\}$ and $\{y_n\}$ generated by $x_1 \in H_1$ and

$$y_n = M_C W_n = P_C(I - \zeta\nabla\Im_1)T_n,$$

where $W_n = (I - \eta A^*(I - M_Q)A)x_n$ and
$T_n = aW_n + (1-a)P_C(I - \zeta\nabla\Im_2)(aW_n + (1-a)P_C(I - \zeta\nabla\Im_3)W_n))$.

$$Q_n = \{z \in H : \langle (I - \zeta\nabla\Im_1)T_n - y_n, y_n - z \rangle \geq 0\},$$

$$x_{n+1} = \alpha_n T_n + (1-\alpha_n)SP_{Q_n}(T_n - \zeta\nabla\Im_1(y_n)), \ \forall n \in \mathbb{N}.$$

Assume the following conditions hold:

(i) $\Im = F(S) \bigcap \bigcap_{i=1}^{3} \phi_{\Im_i} \neq \emptyset$, where $\phi_{\Im_i} = \{\Im_i(x) = \min_{x^* \in C} \Im_i(x^*) : \bar{\Im}_i(Ax) = \min_{Ax^* \in Q} \bar{\Im}_i(Ax^*)\}$, for all $i = 1, 2, 3$.

(ii) $\alpha_n \in [c, d] \subset (0, 1)$.

Then $\{x_n\}$ converges weakly to $x_0 = P_{\Im} x_n$, which $(x_0, y_0, z_0) \in \Psi^{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3}_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3}$, where $y_0 = P_C(I - \zeta\nabla\Im_2)(ax_0 + (1-a)z_0)$ and $z_0 = P_C(I - \zeta\nabla\Im_3)x_0$ with $\bar{x_0} = Ax_0$, $\bar{y_0} = Ay_0$ and $\bar{z_0} = Az_0$.

Proof. By using Theorem 3.5 and Lemma 3.13 , we obtain the conclusion. $\square$

Chapter 4

Examples and numerical results

In this section, the examples 4.1-4.4 are given for supporting Theorem 3.5 and the example 4.5 is given for supporting Theorem 3.16.

Example 4.1. Let $\mathbb{R}$ be the set of real numbers, $H_1 = H_2 = \mathbb{R}^2$, $C = [-50, 50] \times [-50, 50]$, $Q = [-100, 100] \times [-100, 100]$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $D_j^i : C \to H_1$ be defined by $D_1^i(x_1, x_2) = (\frac{x_1-2}{3i}, 0)$, $D_2^i(x_1, x_2) = (\frac{x_1-2}{5i}, 0)$, $D_3^i(x_1, x_2) = (\frac{x_1-2}{7i}, 0)$, for all $x \in C$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $\bar{D}_j^i : Q \to H_2$ be defined by $\bar{D}_1^i(x_1, x_2) = (\frac{x_1-4}{3i}, 0)$, $\bar{D}_2^i(x_1, x_2) = (\frac{x_1-4}{5i}, 0)$, $\bar{D}_3^i(x_1, x_2) = (\frac{x_1-4}{7i}, 0)$, for all $x \in Q$. Let $A : H_1 \to H_2$ be defined by $A(x_1, x_2) = (2x_1, 2x_2)$ for all $(x_1, x_2) \in H_1$ and adjoint A^* of A defined by $A^*(\hat{x}_1, \hat{x}_2) = (2x_1, 2x_2)$ for all $(\hat{x}_1, \hat{x}_2) \in H_2$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $M_C^i : C \to C$ be defined by $M_C^i(x) = P_C(I - \frac{1}{3}(\frac{x_1-2}{3i}, 0))(0.5x + 0.5P_C(I - \frac{1}{3}(\frac{x_1-2}{5i}, 0))(0.5x + 0.5P_C(I - \frac{1}{3}(\frac{x_1-2}{7i}, 0))x))$, for all $x \in C$ and $M_Q^i : Q \to Q$ be defined by $M_Q^i(\hat{x}) = P_Q(I - \frac{1}{5}(\frac{x_1-4}{3i}, 0))(0.5\hat{x} + 0.5P_Q(I - \frac{1}{5}(\frac{x_1-4}{5i}, 0))(0.5\hat{x} + 0.5P_Q(I - \frac{1}{5}(\frac{x_1-4}{7i}, 0))\hat{x}))$, for all $\hat{x} \in Q$. For every $i = 1, 2$, let $M^i : C \to C$ be defined by $M^i(x^*) = M_C^i(x^* - \frac{1}{5}A^*(I - M_Q^i)Ax^*)$ for all $x \in C$. Let $T_1, T_2 : C \to C$ be defined by $T_1x = \{max(\frac{6-x_1}{2}, 0), max(\frac{6-x_2}{2}, 0)\}$, $T_2x = \{max(0, 4 - x_1), max(0, 4 - x_2)\}$, and let $B_i : C \to \mathbb{R}^2$ be defined by $B_i(x) = x - T_ix$, for every $x = (x_1, x_2) \in C$. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x) = (\frac{x_1}{2}, \frac{x_2}{2})$ and $g(x) = (\frac{x_1}{3}, \frac{x_2}{3})$, for all $x = (x_1, x_2) \in C$. Let the sequences $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \frac{n}{5n+2}x_n + \frac{2n+\frac{1}{2}}{5n+2}P_C(I - \frac{1}{2}B_1)x_n + \frac{2n+\frac{3}{2}}{5n+2}P_C(\frac{1}{4n}f(w_n) + \frac{4n-1}{4n}M^1x_n), \\ w_{n+1} = \frac{n}{5n+2}w_n + \frac{2n+\frac{1}{2}}{5n+2}P_C(I - \frac{7}{10}B_2)w_n + \frac{2n+\frac{3}{2}}{5n+2}P_C(\frac{1}{4n}g(x_n) + \frac{4n-1}{4n}M^2w_n). \end{cases} \tag{4.1}$$

By the definition of $T_i, B_i, f, g, D_j^i, M^i$ for every $i = 1, 2$, $j = 1, 2, 3$ we have $(2, 2) \in F(M^i) \cap VI(C, B_i)$. From Theorem 3.5, we can conclude that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $(2, 2)$.

The following Table 4.1 and Figure 4.1 show the numerical results of sequences $\{x_n\}$ and $\{w_n\}$ where $x_1 = (10, 10)$, $w_1 = (10, 10)$ and $n = N = 40$.

Table 4.1: The values of $\{x_n\}$ and $\{w_n\}$ with initial values $x_1 = (10, 10)$, $w_1 = (10, 10)$ and $n = N = 40$.

n	$x_n = (x_n^1, x_n^2)$	$w_n = (w_n^1, w_n^2)$
1	(10.000000, 10.000000)	(10.000000, 10.000000)
2	(8.083752, 9.166667)	(8.083752, 9.166667)
3	(6.274706, 8.068576)	(6.005248, 7.763021)
⋮	⋮	⋮
20	(1.981828, 2.039016)	(1.985800, 2.025690)
⋮	⋮	⋮
38	(1.990594, 1.985034)	(1.992727, 1.988405)
39	(1.990860, 1.985389)	(1.992933, 1.988685)
40	(1.991112, 1.985744)	(1.993127, 1.988964)

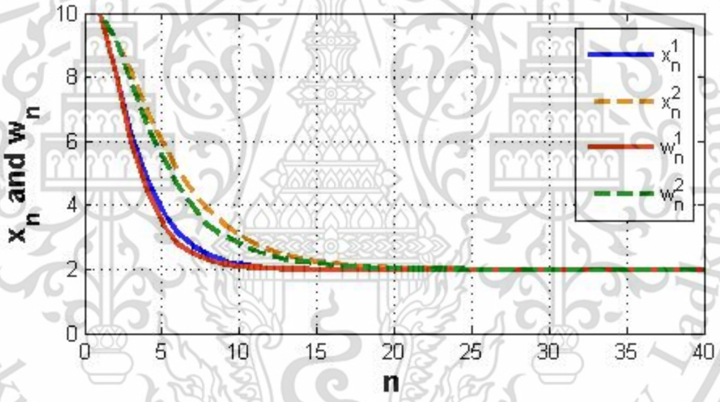

Figure 4.1: The convergence of $\{x_n\}$ and $\{w_n\}$ with initial values $x_1 = (10, 10)$, $w_1 = (10, 10)$ and $n = N = 40$.

Example 4.2. In this example, we use the same mappings and parameters as in Example 4.1. If we putting the sequence $\{x_n\} = \{w_n\}$, the mapping $M^1 = M^2$, $B_1 = B_2$, $\gamma_1 = \gamma_2 = 0.5$ and define $f(x) = g(x) = (\frac{x_1}{2}, \frac{x_2}{2})$, for all $x = (x_1, x_2) \in C$, we can rewrite (4.1) as follows:

$$x_{n+1} = \frac{n}{5n+2}x_n + \frac{2n+\frac{1}{2}}{5n+2}P_C(I - \tfrac{1}{2}B_1)x_n + \frac{2n+\frac{3}{2}}{5n+2}P_C(\tfrac{1}{4n}f(x_n) + (1 - \tfrac{1}{4n})M^1x_n) \tag{4.2}$$

By using the algorithm (4.2), the following Table 4.2 and Figure 4.2 show the numerical results of sequence $\{x_n\}$ where $x_1 = (10, 10)$ and $n = N = 40$.

Table 4.2: The value of $\{x_n\}$ with initial value $x_1 = (10, 10)$ and $n = N = 40$.

n	$x_n = (x_n^1, x_n^2)$
1	(10.000000, 10.000000)
2	(8.083752, 9.166667)
3	(6.274706, 8.068576)
⋮	⋮
20	(1.981869, 2.041259)
⋮	⋮
38	(1.990583, 1.985025)
39	(1.990850, 1.985376)
40	(1.991102, 1.985730)

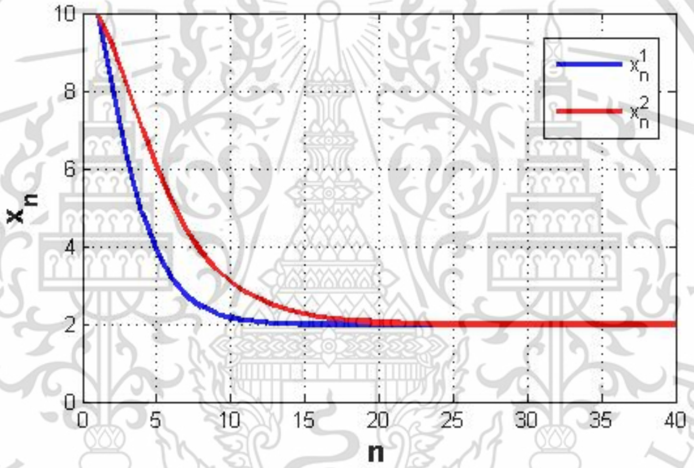

Figure 4.2: The convergence of $\{x_n\}$ with initial value $x_1 = (10, 10)$ and $n = N = 40$.

In the following example, we consider the metric projection onto a half-space $H(a, \beta) := \{z \in H : \langle a, z\rangle \leq \beta\}$, where $a \in H, a \neq 0$ and $\beta \in \mathbb{R}$. It is clear that $H(a, \beta)$ is closed and convex with

$$P_{H(a,\beta)}x = \begin{cases} x - \dfrac{\langle a, x\rangle - \beta}{\|a\|^2}a, & if \quad \langle a, x\rangle > \beta, \\ x, & if \quad \langle a, x\rangle \leq \beta. \end{cases}$$

Example 4.3. Let $H_1 = [-50, 50] \times [-50, 50]$ and $C = H\ (a, 20) := \{x \in H_1 : x_1 + 2x_2 \leq 20\}$, where $a = (1, 2)$. Then, we obtain

$$P_C(x_1, x_2) = \begin{cases} (x_1, x_2) - \dfrac{[x_1 + 2x_2 - 20](1, 2)}{5}, & if \quad x_1 + 2x_2 > 20, \\ (x_1, x_2), & if \quad x_1 + 2x_2 \leq 20, \end{cases}$$

for all $(x_1, x_2) \in C$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $D_j^i : C \to H_1$ defined by $D_1^i(x_1, x_2) = (\frac{x_1-4}{2i}, x_2)$, $D_2^i(x_1, x_2) = (x_1 - 4, \frac{x_2-3}{3i})$, $D_3^i(x_1, x_2) = (\frac{x_1-4}{4i}, x_2 - 8)$, for all

$(x_1, x_2) \in C$. For every $i = 1, 2$ let $M_C^i : C \to C$ defined by
$M_C^i(x) = P_C(I - \frac{1}{3}D_1^i(x)) \left[\frac{1}{2}(x) + \frac{1}{2}P_C(I - \frac{1}{3}D_2^i(x)) \left(\frac{1}{2}(x) + \frac{1}{2}P_C(I - \frac{1}{3}D_3^i(x))\right]\right.$ for all $x = (x_1, x_2) \in C$.

Let $H_2 = [-100, 100] \times [-100, 100]$ and $Q = H\ (b, 60) := \{\hat{x} \in H_2 : 3\hat{x}_1 + 4\hat{x}_2 \leq 60\}$, where $b = (3, 4)$. Then, we obtain

$$P_Q(\hat{x}_1, \hat{x}_2) = \begin{cases} (\hat{x}_1, \hat{x}_2) - \dfrac{[3\hat{x}_1 + 4\hat{x}_2 - 60](3,4)}{25}, & if \quad 3\hat{x}_1 + 4\hat{x}_2 > 60, \\ (\hat{x}_1, \hat{x}_2), & if \quad 3\hat{x}_1 + 4\hat{x}_2 \leq 60, \end{cases}$$

for all $(\hat{x}_1, \hat{x}_2) \in Q$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $\bar{D}_j^i : Q \to H_2$ defined by $\bar{D}_1^i(\hat{x}_1, \hat{x}_2) = (\frac{\hat{x}_1 - 8}{i}, \hat{x}_2)$, $\bar{D}_2^i(\hat{x}_1, \hat{x}_2) = (\frac{\hat{x}_1 - 8}{6i}, \hat{x}_2 - 6)$, $\bar{D}_3^i(\hat{x}_1, \hat{x}_2) = (\frac{x_1 - 8}{5i}, x_2 - 24)$, for all $(\hat{x}_1, \hat{x}_2) \in Q$. For every $i = 1, 2$ let $M_Q^i : Q \to Q$ defined by
$M_Q^i(\hat{x}) = P_Q(I - \frac{1}{5}\bar{D}_1^i(\hat{x})) \left[\frac{1}{2}(\hat{x}) + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D}_2^i(\hat{x})) \left(\frac{1}{2}(\hat{x}) + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D}_3^i(\hat{x}))\right]\right.$ for all $\hat{x} = (\hat{x}_1, \hat{x}_2) \in Q$.

Let $A : H_1 \to H_2$ defined by $A(x_1, x_2) = (2x_1, 2x_2)$, for all $(x_1, x_2) \in H_1$ and adjoint A^* of A defined by $A^*(\hat{x}_1, \hat{x}_2) = (2\hat{x}_1, 2\hat{x}_2)$, for all $(\hat{x}_1, \hat{x}_2) \in H_2$. For every $i = 1, 2$, let $M^i(x) = M_C^i(x - \frac{1}{10}A^*(I - M_Q^i)A(x))$, for all $x = (x_1, x_2) \in C$. Let $F, G : [-50, 50] \times [-50, 50] \to [-50, 50] \times [-50, 50]$ be defined by $F(x_1, x_2) = (\frac{x_1}{5}, \frac{x_2}{5})$ and $G(x_1, x_2) = (\frac{x_1}{7}, \frac{x_2}{7})$, for all $(x_1, x_2) \in [-50, 50] \times [-50, 50]$ and for $i = 1, 2$, let $B_i : C \to [-50, 50] \times [-50, 50]$ be defined by $B_1(x_1, x_2) = (\frac{x_1 - 4}{3}, \frac{x_2 - 2}{3})$ and $B_2(x_1, x_2) = (\frac{x_1 - 4}{9}, \frac{x_2 - 2}{9})$, for all $(x_1, x_2) \in C$. Let the sequences $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \frac{n}{3n+2}x_n + \frac{n+\frac{1}{2}}{3n+2}P_C(I - \frac{1}{2}B_1)x_n + \frac{n+\frac{3}{2}}{3n+2}P_C(\frac{1}{2n}F(w_n) + \frac{2n-1}{2n}M^1 x_n), \\ w_{n+1} = \frac{n}{3n+2}w_n + \frac{n+\frac{1}{2}}{3n+2}P_C(I - \frac{7}{10}B_2)w_n + \frac{n+\frac{3}{2}}{3n+2}P_C(\frac{1}{2n}G(x_n) + \frac{2n-1}{2n}M^2 w_n). \end{cases}$$

By the definition of B_i, F, G, D_j^i, M^i for every $i = 1, 2,\ j = 1, 2, 3$ we have $(4, 2) \in F(M^i) \cap VI(C, B_i)$. From Theorem 3.5, we can conclude that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $(4, 2)$.

The following Table 4.3 and Figure 4.3 show the numerical results of sequences $\{x_n\}$ and $\{w_n\}$ where $x_1 = (-10, 10)$, $w_1 = (-10, 10)$ and $n = N = 40$.

Table 4.3: The values of $\{x_n\}$ and $\{w_n\}$ with initial values $x_1 = (-10, 10), w_1 = (-10, 10)$ and $n = N = 40$.

n	$x_n = (x_n^1, x_n^2)$	$w_n = (w_n^1, w_n^2)$
1	(-10.000000, 10.000000)	(-10.000000, 10.000000)
2	(-5.978702, 6.632331)	(-6.563107, 6.729819)
3	(-3.712547, 5.055405)	(-4.723965, 5.230170)
⋮	⋮	⋮
20	(3.608551, 1.890749)	(2.974410, 1.887930)
⋮	⋮	⋮
38	(3.901467, 1.909353)	(3.768213, 1.889663)
39	(3.905313, 1.910485)	(3.781338, 1.891002)
40	(3.908821, 1.911562)	(3.793224, 1.892293)

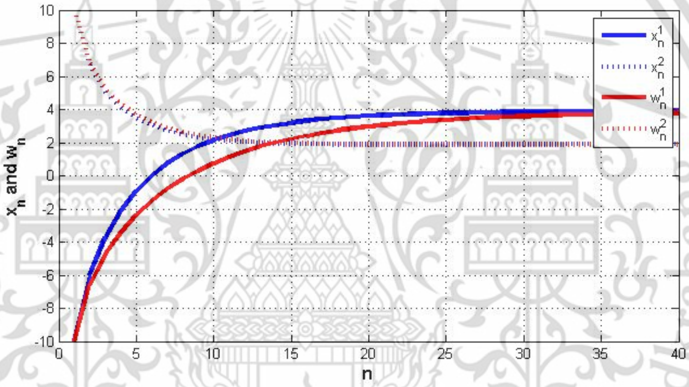

Figure 4.3: The convergence of $\{x_n\}$ and $\{w_n\}$ with initial values $x_1 = (-10, 10)$, $w_1 = (-10, 10)$ and $n = N = 40$.

Next, we give the following example in $H_1 = L_2([\alpha, \beta])$ with the inner product defined by $\langle f, g\rangle := \int_\alpha^\beta f(x)g(x)dx$, and with the norm $\|f\| := \left(\int_\alpha^\beta f^2(x)dx\right)^{\frac{1}{2}}$ for all $f, g \in H_1$. We consider the metric projection onto a hyperplane

$$H(g, \rho) := \{h \in L_2([\alpha, \beta]) : \int_\alpha^\beta g(x)h(x)dx = \rho\},$$

where $g \in L_2([\alpha, \beta]), g \neq 0$ and $\rho \in \mathbb{R}$. Then $H(g, \rho)$ is closed and convex with

$$P_{H(g,\rho)}f = f - \frac{\int_\alpha^\beta g(x)h(x)dx - \rho}{\int_\alpha^\beta g^2(x)dx} g,$$

and in $H_2 = \mathbb{R}^2$ which consider the metric projection onto a band

$$C := \{x \in H : \beta_1 \leq \langle a, x\rangle \leq \beta_2\},$$

where $a \in H, a \neq 0, \beta_1, \beta_2 \in \mathbb{R}$ and $\beta_1 < \beta_2$. It is clear that C is closed and convex with

$$P_C x = \begin{cases} x - \frac{\langle a,x\rangle - \beta_2}{\|a\|^2} a, & if \quad \langle a,x\rangle > \beta_2, \\ x & if \quad \beta_1 \leq \langle a,x\rangle \leq \beta_2, \\ x - \frac{\langle a,x\rangle - \beta_1}{\|a\|^2} a, & if \quad \langle a,x\rangle < \beta_1. \end{cases}$$

Example 4.4. Let $H_1 = L_2([-2,2])$ and

$$C := \left\{ h \in L_2([-2,2]) : \int_{-2}^{2} g(x)h(x)dx = \frac{32}{3} \right\},$$

we obtain $P_C f = f - \dfrac{\langle f,g\rangle - \frac{32}{3}}{\|g\|^2} g$, where $g \in L_2([-2,2])$ with $g(x) = x-1$, for all $x \in [-2,2]$. For every $i = 1,2$ and $j = 1,2,3$, let $D_j^i : C \to H_1$ defined by $D_1^i(h) = \frac{h-2I}{3i}$, $D_2^i(h) = \frac{h-2I}{5i}$, $D_3^i(h) = \frac{h-2I}{7i}$, for all $h \in C$. For every $i = 1,2$ let $M_C^i : C \to C$ defined by
$M_C^i(h) = P_C(I - \frac{1}{3}D_1^i(h))[\frac{1}{2}(h) + \frac{1}{2}P_C(I - \frac{1}{3}D_2^i(h))\left(\frac{1}{2}(h) + \frac{1}{2}P_C(I - \frac{1}{3}D_3^i(h))\right]$ for all $h \in C$.

Let $H_2 = \mathbb{R}^2$ and $Q := \{\bar{x} \in H_2 : 0 \leq 3\bar{x_1} - \bar{x_2} \leq 2\}$. Then, we obtain

$$P_Q \bar{x} = \begin{cases} (\bar{x_1}, \bar{x_2}) - \frac{[3\bar{x_1} - \bar{x_2} - 2](3,-1)}{10}, & if \quad 3\bar{x_1} - \bar{x_2} > 2, \\ (\bar{x_1}, \bar{x_2}) & if \quad 0 \leq 3\bar{x_1} - \bar{x_2} \leq 2, \\ (\bar{x_1}, \bar{x_2}) - \frac{[3\bar{x_1} - \bar{x_2}](3,-1)}{10}, & if \quad 3\bar{x_1} - \bar{x_2} < 0, \end{cases}$$

for all $\bar{x} = (\bar{x_1}, \bar{x_2}) \in H_2$. For every $i = 1,2$ and $j = 1,2,3$, let $\bar{D}_j^i : Q \to H_2$ defined by $\bar{D}_1^i(\bar{x_1}, \bar{x_2}) = (\frac{\bar{x_1}}{2i}, \frac{\bar{x_2}}{2i})$, $\bar{D}_2^i(\bar{x_1}, \bar{x_2}) = (\frac{\bar{x_1}}{4i}, \frac{\bar{x_2}}{4i})$, $\bar{D}_3^i(\bar{x_1}, \bar{x_2}) = (\frac{\bar{x_1}}{6i}, \frac{\bar{x_2}}{6i})$, for all $(\bar{x_1}, \bar{x_2}) \in Q$. For every $i = 1,2$, let $M_Q^i : Q \to Q$ defined by
$M_Q^i(\bar{x}) = P_Q(I - \frac{1}{5}\bar{D}_1^i(\bar{x}))[\frac{1}{2}(\bar{x}) + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D}_2^i(\bar{x}))\left(\frac{1}{2}(\bar{x}) + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D}_3^i(\bar{x}))\right]$ for all $\bar{x} = (\bar{x_1}, \bar{x_2}) \in Q$.

Let $A : H_1 \to H_2$ defined by

$$A(h) = \left(\int_{-2}^{2} h(x)dx, \int_{-2}^{2} h(x)dx \right),$$

for all $h \in H_1$ and adjoint A^* of A defined by $A^*(\bar{x_1}, \bar{x_2}) = \bar{x_1} + \bar{x_2}$, for all $(\bar{x_1}, \bar{x_2}) \in H_2$. For every $i = 1,2$, let $M^i(h) = M_C^i(h - \frac{1}{4}A^*(I - M_Q^i)A(h))$, for all $h \in C$. Let $F, G : L_2([-2,2]) \to L_2([-2,2])$ be defined by $F(h) = \frac{h}{2}$ and $G(h) = \frac{h}{3}$, for all $h \in L_2([-2,2])$ and for $i = 1,2$, let $B_i : C \to L_2([-2,2])$ be defined by $B_1(h) = \frac{3h-6I}{4}$ and $B_2(h) = \frac{4h-8I}{2}$, for all $h \in C$. Let the sequences $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \frac{2n}{7n+1}x_n + \frac{4n+1}{7n+1}P_C(I - \frac{1}{2}B_1)x_n + \frac{n}{7n+1}P_C(\frac{1}{8n}F(w_n) + \frac{8n-1}{8n})M^1 x_n), \\ w_{n+1} = \frac{2n}{7n+1}w_n + \frac{4n+1}{7n+1}P_C(I - \frac{7}{10}B_2)w_n + \frac{n}{7n+1}P_C(\frac{1}{8n}G(x_n) + \frac{8n-1}{8n})M^2 w_n). \end{cases}$$

By the definition of B_i, F, G, D_j^i, M^i for every $i = 1,2,\ j = 1,2,3$ we have $2I \in F(M^i) \cap VI(C, B_i)$. From Theorem 3.5, we can conclude that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $2I$.

Now, we compare the convergent behavior of sequences $\{x_n\}$ and $\{w_n\}$ with the same starting point and have the results in Table 4.4. In Figure 4.4, the values

Table 4.4: Comparison of the sequences $\{x_n\}$ and $\{w_n\}$ with the same starting point.

	starting point	Time taken (Secs)
the sequence $\{x_n\}$	$x_1 = 2t$	6.665
the sequence $\{w_n\}$	$w_1 = 2t$	7.430

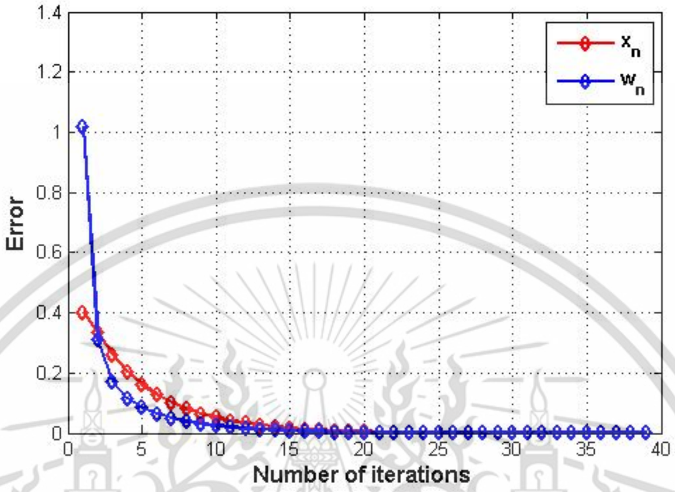

Figure 4.4: Comparison of the sequences $\{x_n\}$ and $\{w_n\}$ with $x_1 = 2t$ and $w_1 = 2t$.

of errors $\|x_{n+1} - x_n\|$ and $\|w_{n+1} - w_n\|$ are represented by the y-axis, the number of iterations are represented by the x-axis.

Example 4.5. Let $\mathbb{R}$ be the set of real numbers, $C := \{x \in H | 1 \leq 2x_1 + x_2 \leq 7\}$, $Q := \{x \in H | -10 \leq 3x_1 - x_2 \leq 20\}$, $H_1 = H_2 = \mathbb{R}^2$. Let $D_1, D_2, D_3 : C \to \mathbb{R}^2$ defined by $D_1(x_1, x_2) = (x_1 - 2, x_2 + 1)$, $D_2(x_1, x_2) = (x_1 - 3, x_2 - \frac{5}{2})$ and $D_3(x_1, x_2) = (x_1 + 2, x_2 - 6)$ for all $(x_1, x_2) \in C$. Let $\bar{D_1}, \bar{D_2}, \bar{D_3} : Q \to \mathbb{R}^2$ defined by $\bar{D_1}(\bar{x_1}, \bar{x_2}) = (\bar{x_1} - 4, \bar{x_2} + 8)$, $\bar{D_2}(\bar{x_1}, \bar{x_2}) = (\bar{x_1} - 12, \bar{x_2} - 8)$ and $\bar{D_3}(\bar{x_1}, \bar{x_2}) = (\bar{x_1} + 16, \bar{x_2} - 30)$ for all $(\bar{x_1}, \bar{x_2}) \in Q$. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A(x_1, x_2) = (2x_1, 2x_2)$ and $A^* : \mathbb{R}^2 \to \mathbb{R}$ defined by $A^*(x_1, x_2) = (2x_1, 2x_2)$. Define $M_C : H_1 \to C$ by $M_C(x) = P_C(I - \frac{1}{2}D_1)(\frac{1}{2}x + \frac{1}{2}P_C(I - \frac{1}{2}D_2)(\frac{1}{2}x + \frac{1}{2}P_C(I - \frac{1}{2}D_3)x))$, $\forall x = (x_1, x_2) \in H_1$, define $M_Q : H_2 \to Q$ by $M_Q(\hat{x}) = P_Q(I - \frac{1}{5}\bar{D_1})(\frac{1}{2}\hat{x} + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D_2})(\frac{1}{2}\hat{x} + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D_3})\hat{x}))$, $\forall \hat{x} = (\hat{x_1}, \hat{x_2}) \in H_2$, and define $S : C \to C$ by $S(x_1, x_2) = (\frac{x_1}{2} + 1, \frac{x_2}{2})$. Let the sequences $\{x_n\}$ and $\{y_n\}$ generated by $x_1 \in H_1$ and

$$y_n = M_C W_n = P_C(I - \frac{1}{2}(x_1 - 2, x_2 + 1))T_n,$$

where $W_n = (I - \frac{1}{8}A^*(I - M_Q)A)x_n$ and $T_n = \frac{1}{2}W_n + \frac{1}{2}P_C(I - \frac{1}{2}(x_1 - 3, x_2 - \frac{5}{2}))(\frac{1}{2}W_n + \frac{1}{2}P_C(I - \frac{1}{2}(x_1 + 2, x_2 - 6))W_n))$,

$$Q_n = \{z \in H : \langle (I - \frac{1}{2}(x_1 - 2, x_2 + 1))T_n - y_n, y_n - z \rangle \geq 0\},$$

and

$$x_{n+1} = \frac{n+1}{5n}T_n + (1 - \frac{n+1}{5n})SP_{Q_n}(T_n - \frac{1}{2}(x_1 - 2, x_2 + 1)(y_n)), \ \forall \, n \in \mathbb{N}$$

where

$$P_C x = \begin{cases} (x_1, x_2) - \frac{[2x_1 + x_2 - 7](2,1)}{5}, & if \quad 2x_1 + x_2 > 7, \\ (x_1, x_2) & if \quad 1 \leq 2x_1 + x_2 \leq 7, \\ (x_1, x_2) - \frac{[2x_1 + x_2 - 1](2,1)}{5}, & if \quad 2x_1 + x_2 < 1, \end{cases}$$

for every $x = (x_1, x_2) \in H_1$ and

$$P_Q \hat{x} = \begin{cases} (x_1, x_2) - \frac{[3x_1 - x_2 - 20](3,-1)}{10}, & if \quad 3x_1 - x_2 > 20, \\ (x_1, x_2) & if \quad -10 \leq 3x_1 - x_2 \leq 20, \\ (x_1, x_2) - \frac{[3x_1 - x_2 + 10](3,-1)}{10}, & if \quad 3x_1 - x_2 < -10, \end{cases}$$

for every $\hat{x} = (x_1, x_2) \in H_2$. By the definition of $S, D_i, \bar{D_i}, M_C, M_Q$ for every $i = 1, 2, 3$, we have $(2, 0) \in F(M_C(I - \frac{1}{8}A^*(I - M_Q)A))$. From Theorem 3.16, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $(2, 0)$.

The following Table 4.5 and Figure 4.5 show the numerical results of sequences $\{x_n\}$ and $\{y_n\}$ where $x_1 = (-5, 5)$ and $n = N = 30$.

Table 4.5*:* The values of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = (-5, 5)$ and $n = N = 30$.

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(-5.000000, 5.000000)	(0.034028, 1.404266)
2	(-0.457465, 2.305332)	(1.309813, 0.647460)
3	(1.223540, 1.203392)	(1.781929, 0.337977)
⋮	⋮	⋮
15	(2.000000, 0.000000)	(2.000000, 0.000000)
⋮	⋮	⋮
28	(2.000000, 0.000000)	(2.000000, 0.0000000)
29	(2.000000, 0.000000)	(2.000000, 0.000000)
30	(2.000000, 0.000000)	(2.000000, 0.000000)

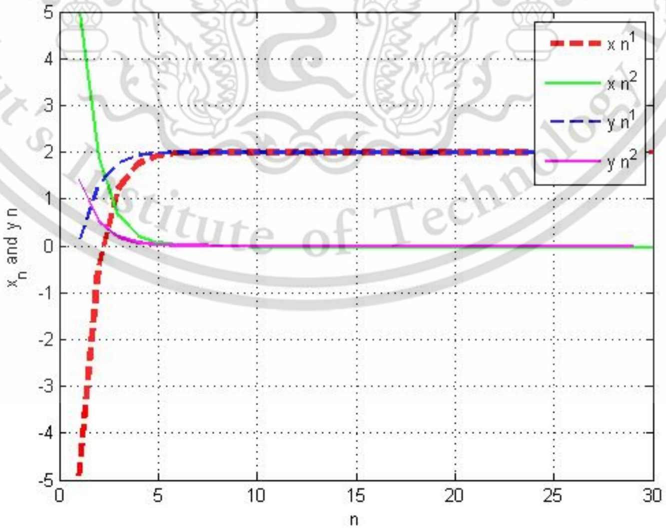

Figure 4.5*:* The convergence of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = (-5, 5)$ and $n = N = 30$.

Chapter 5

Conclusions

In this chapter, we conclude all theorems and corollary obtained in this thesis.

5.1 Strong convergence theorem for solving the split of modified variational inequality problems (SMVIP)

(1) Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. For every $i = 1, 2$, let $B_i : C \to H$ be α_i-inverse strongly monotone mapping with α= min$\{\alpha_1, \alpha_2\}$ and let $D_1^i, D_2^i, D_3^i : C \to H_1$ be d_1^i, d_2^i, d_3^i-inverse strongly monotone, respectively, with $d^* =$ min$\{d_1^i, d_2^i, d_3^i\}$. Let $\bar{D_1}^i, \bar{D_2}^i, \bar{D_3}^i : Q \to H_2$ be $\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i$-inverse strongly monotone, respectively, with $\hat{d} =$ min$\{\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Let $f, g : H \to H$ be a_f and a_g-contraction mapping with $a =$ max$\{a_f, a_g\}$. For every $i = 1, 2$ define $M_C^i : C \to C$ by $M_C^i(x) = P_C(I - \zeta^i D_1^i)(ax + (1-a)P_C(I - \zeta^i D_2^i)(ax + (1-a)P_C(I - \zeta^i D_3^i)x))$, $\forall x \in C$, where $\zeta^i \in (0, 2d^*)$ and define $M_Q^i : Q \to Q$ by $M_Q^i(\hat{x}) = P_Q(I - \bar{\zeta}^i \bar{D}_1^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i \bar{D}_2^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i \bar{D}_3^i)\hat{x}))$, $\forall \hat{x} \in Q$, where $\bar{\zeta}^i \in (0, 2\hat{d})$. Define $M^i : C \to C$ by $M^i(x^*) = M_C^i(x^* - \eta A^*(I - M_Q^i)Ax^*)$ for all $x^* \in C$ and $i = 1, 2$. Let the sequences $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2 B_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)M^2 w_n) \end{cases}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0, 1]$ with $\delta_n + \eta_n + \mu_n = 1$ and $\gamma \in (0, 2\alpha)$ with γ=min$\{\gamma_1, \gamma_2\}$. Assume the following conditions hold; (i) $\mathcal{F}_i = F(M^i) \cap VI(C, B_i) \neq \emptyset$ for all $i = 1, 2$,

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\lim\limits_{n\to\infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ for all $n \in N$ and for some $\bar{\theta}, \theta > 0$,

(iv) $\sum\limits_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum\limits_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Then $\{x_n\}$ converges strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, where $y_1^* = P_C(I - \zeta^1 D_2^1)(ax_1^* + (1-a)z_1^*)$ and $z_1^* = P_C(I - \zeta^1 D_3^1)x_1^*$ with (x_1^*, y_1^*, z_1^*) is an element of $\Psi^{D_1^1, D_2^1, D_3^1}_{\bar{D_1^1}, \bar{D_2^1}, \bar{D_3^1}}$. and $\{w_n\}$ converges strongly to $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, where $y_2^* = P_C(I - \zeta^2 D_2^2)(ax_2^* + (1-a)z_2^*)$ and $z_2^* = P_C(I - \zeta^2 D_3^2)x_2^*$ with (x_2^*, y_2^*, z_2^*) is an element of $\Psi^{D_1^2, D_2^2, D_3^2}_{\bar{D_1^2}, \bar{D_2^2}, \bar{D_3^2}}$.

(2) Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. For every $i = 1, 2$, let $T_i : C \to C$ be a κ_i-strictly pseudo-contractive mapping with $Fix(T_i) \neq \emptyset$, and let $D_1^i, D_2^i, D_3^i : C \to H_1$

be d_1^i, d_2^i, d_3^i-inverse strongly monotone, respectively, with $d^* = \min\{d_1^i, d_2^i, d_3^i\}$. Let $\bar{D_1}^i, \bar{D_2}^i, \bar{D_3}^i : Q \to H_2$ be $\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i$-inverse strongly monotone, respectively, with $\hat{d} = \min\{\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Let $f, g : H \to H$ be a_f and a_g-contraction mapping with $a = \max\{a_f, a_g\}$. For every $i = 1, 2$ define $M_C^i : C \to C$ by $M_C^i(x) = P_C(I - \zeta^i D_1^i)(ax + (1-a)P_C(I - \zeta^i D_2^i)(ax + (1-a)P_C(I - \zeta^i D_3^i)x))$, $\forall x \in C$, where $\zeta^i \in (0, 2d^*)$ and define $M_Q^i : Q \to Q$ by $M_Q^i(\hat{x}) = P_Q(I - \bar{\zeta^i}\bar{D_1^i})(a\hat{x} + (1-a)P_Q(I - \bar{\zeta^i}\bar{D_2^i})(a\hat{x} + (1-a)P_Q(I - \bar{\zeta^i}\bar{D_3^i})\hat{x}))$, $\forall \hat{x} \in Q$, where $\bar{\zeta^i} \in (0, 2\hat{d})$ Define $M^i : C \to C$ by $M^i(x^*) = M_C^i(x^* - \eta A^*(I - M_Q^i)Ax^*)$ for all $x^* \in C$, and $i = 1, 2$. Let the sequences $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1(I - T_1))x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2(I - T_2))w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)M^2 w_n) \end{cases}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\delta_n + \eta_n + \mu_n = 1$, $\gamma \in (0, 2\alpha)$ with $\alpha = \min\{\frac{1-\kappa_1}{2}, \frac{1-\kappa_2}{2}\}$ and γ=$\min\{\gamma_1, \gamma_2\}$. Assume the following conditions hold;

(i) $\mathcal{F}_i = F(M^i) \cap F(T_i) \neq \emptyset$ for all $i = 1, 2$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ for all $n \in N$ and for some $\bar{\theta}, \theta > 0$,

(iv) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Then $\{x_n\}$ converges strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, where $y_1^* = P_C(I - \zeta^1 D_2^1)(ax_1^* + (1-a)z_1^*)$ and $z_1^* = P_C(I - \zeta^1 D_3^1)x_1^*$ with (x_1^*, y_1^*, z_1^*) is an element of $\Psi^{D_1^1, D_2^1, D_3^1}_{\bar{D_1^1}, \bar{D_2^1}, \bar{D_3^1}}$. and $\{w_n\}$ converges strongly to $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, where $y_2^* = P_C(I - \zeta^2 D_2^2)(ax_2^* + (1-a)z_2^*)$ and $z_2^* = P_C(I - \zeta^2 D_3^2)x_2^*$ with (x_2^*, y_2^*, z_2^*) is an element of $\Psi^{D_1^2, D_2^2, D_3^2}_{\bar{D_1^2}, \bar{D_2^2}, \bar{D_3^2}}$.

(3) Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. For every $i = 1, 2$, let $B_i : C \to H$ be α_i-inverse strongly monotone mapping with α= $\min\{\alpha_1, \alpha_2\}$. For every $i = 1, 2$, let $D_1^i, D_2^i, D_3^i : C \to H_1$ be d_1^i, d_2^i, d_3^i-inverse strongly monotone respectively, where $\zeta^i \in (0, 2d^*)$ with $d^* = \min\{d_1^i, d_2^i, d_3^i\}$ and let $\bar{D_1}^i, \bar{D_2}^i, \bar{D_3}^i : Q \to H_2$ be $\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i$-inverse strongly monotone respectively, where $\bar{\zeta^i} \in (0, 2\hat{d})$ with $\hat{d}$ =$\min\{\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. let $f, g : H \to H$ be a_f and a_g-contraction mapping with $a = \max\{a_f, a_g\}$. For every $i = 1, 2$ define $M_C^i : C \to C$ by $M_C^i(x) = P_C(I - \zeta^i D_1^i)(ax + (1-a)P_C(I - \zeta^i D_2^i)(ax + (1-a)P_C(I - \zeta^i D_3^i)x))$, $\forall x \in C$, and define $M_Q^i : Q \to Q$ by $M_Q^i(\hat{x}) = P_Q(I - \bar{\zeta^i}\bar{D_1^i})(a\hat{x} + (1-a)P_Q(I - \bar{\zeta^i}\bar{D_2^i})(a\hat{x} + (1-a)P_Q(I - \bar{\zeta^i}\bar{D_3^i})\hat{x}))$, $\forall \hat{x} \in Q$. For every $i = 1, 2$ define $M^i : C \to C$ by $M^i(x^*) = M_C^i(x^* - \eta A^*(I - M_Q^i)Ax^*)$ for all $x^* \in C$. Let the sequences $\{x_n\}$ and

$\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2 B_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)M^2 w_n) \end{cases}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\delta_n + \eta_n + \mu_n = 1$ and $\gamma \in (0, 2\alpha)$ with γ=min$\{\gamma_1, \gamma_2\}$. Assume the following conditions hold;

(i) $\mathcal{F}_i = \bigcap_{j=1}^{3} \Phi_j^i \cap VI(C, B_i) \neq \emptyset$ for all $i = 1,2$ and Φ_j^i=$\{z \in VI(C, D_j^i) : Az \in VI(C, \bar{D}_j^i)\}$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ for all $n \in N$ and for some $\bar{\theta}, \theta > 0$,

(iv) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Then $\{x_n\}$ converges strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, where $y_1^* = P_C(I - \zeta^1 D_2^1)(ax_1^* + (1-a)z_1^*)$ and $z_1^* = P_C(I - \zeta^1 D_3^1)x_1^*$ with (x_1^*, y_1^*, z_1^*) is an element of $\Psi^{D_1^1, D_2^1, D_3^1}_{\bar{D}_1^1, \bar{D}_2^1, \bar{D}_3^1}$. and $\{w_n\}$ converges strongly to $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, where $y_2^* = P_C(I - \zeta^2 D_2^2)(ax_2^* + (1-a)z_2^*)$ and $z_2^* = P_C(I - \zeta^2 D_3^2)x_2^*$ with (x_2^*, y_2^*, z_2^*) is an element of $\Psi^{D_1^2, D_2^2, D_3^2}_{\bar{D}_1^2, \bar{D}_2^2, \bar{D}_3^2}$.

(4) Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. For every $i = 1,2$, let $B_i : C \to H$ be α_i-inverse strongly monotone mapping with α= min$\{\alpha_1, \alpha_2\}$. For every $i = 1,2$ let $\Im_1^i, \Im_2^i, \Im_3^i : C \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\Im_1^i, \nabla\Im_2^i, \nabla\Im_3^i$ be $\frac{1}{L_{\Im_1^i}}, \frac{1}{L_{\Im_2^i}}, \frac{1}{L_{\Im_3^i}}$-inverse strongly monotone and continuous respectively, where $\zeta^i \in (0, \frac{2}{L_\Im})$ with $\frac{1}{L_\Im}$ = min$\{\frac{1}{L_{\Im_1^i}}, \frac{1}{L_{\Im_2^i}}, \frac{1}{L_{\Im_3^i}}\}$. Let $\bar{\Im}_1^i, \bar{\Im}_2^i, \bar{\Im}_3^i : Q \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\bar{\Im}_1^i, \nabla\bar{\Im}_2^i, \nabla\bar{\Im}_3^i$ be $\frac{1}{L_{\bar{\Im}_1^i}}, \frac{1}{L_{\bar{\Im}_2^i}}, \frac{1}{L_{\bar{\Im}_3^i}}$-inverse strongly monotone and continuous respectively, where $\bar{\zeta}^i \in (0, \frac{2}{L_{\bar{\Im}}})$ with $\frac{1}{L_{\bar{\Im}}}$ = min$\{\frac{1}{L_{\bar{\Im}_1^i}}, \frac{1}{L_{\bar{\Im}_2^i}}, \frac{1}{L_{\bar{\Im}_3^i}}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. let $f, g : H \to H$ be a_f and a_g-contraction mapping with a = max$\{a_f, a_g\}$. For every $i = 1,2$ define $M_C^i : C \to C$ by $M_C^i(x) = P_C(I - \zeta^i\nabla\Im_1^i)(ax + (1-a)P_C(I - \zeta^i\nabla\Im_2^i)(ax + (1-a)P_C(I - \zeta^i\nabla\Im_3^i)x)), \forall x \in C$, and define $M_Q^i : Q \to Q$ by $M_Q^i(\hat{x}) = P_Q(I - \bar{\zeta}^i\nabla\bar{\Im}_1^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i\nabla\bar{\Im}_2^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i\nabla\bar{\Im}_3^i)\hat{x})), \forall \hat{x} \in Q$, $\forall \hat{x} \in Q$. For every $i = 1,2$ define $M^i : C \to C$ by $M^i(x^*) = M_C^i(x^* - \eta A^*(I - M_Q^i)Ax^*)$ for all $x^* \in C$. Let the sequences $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2 B_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)M^2 w_n) \end{cases}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\delta_n + \eta_n + \mu_n = 1$, $\gamma \in (0, 2\alpha)$ and γ=min$\{\gamma_1, \gamma_2\}$. Assume the following conditions hold;

(i) $\mathcal{F}_i = \bigcap_{j=1}^{3} \Phi_{\Im_j}^i \cap VI(C, B_i) \neq \emptyset$ for all $i = 1,2$ and $\Phi_{\Im_j}^i = \{\Im_j(x) = \min_{x^* \in C} \Im_j(x^*) :$

$\bar{\Im}_j(Ax) = \min\limits_{Ax^* \in Q} \Im_j(Ax^*)\}$,

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\lim\limits_{n \to \infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ for all $n \in N$ and for some $\bar{\theta}, \theta > 0$,

(iv) $\sum\limits_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum\limits_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Then $\{x_n\}$ converges strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, where $y_1^* = P_C(I - \zeta^1 \nabla \Im_2^1)(ax_1^* + (1 - a)z_1^*)$ and $z_1^* = P_C(I - \zeta^1 \nabla \Im_3^1)x_1^*$ with (x_1^*, y_1^*, z_1^*) is an element of $\Psi_{\nabla \bar{\Im}_1^1, \nabla \bar{\Im}_2^1, \nabla \bar{\Im}_3^1}^{\nabla \Im_1^1, \nabla \Im_2^1, \nabla \Im_3^1}$. and $\{w_n\}$ converges strongly to $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, where $y_2^* = P_C(I - \zeta^2 \nabla \Im_2^2)(ax_2^* + (1-a)z_2^*)$ and $z_2^* = P_C(I - \zeta^2 \nabla \Im_3^2)x_2^*$ with (x_2^*, y_2^*, z_2^*) is an element of $\Psi_{\nabla \bar{\Im}_1^2, \nabla \bar{\Im}_2^2, \nabla \bar{\Im}_3^2}^{\nabla \Im_1^2, \nabla \Im_2^2, \nabla \Im_3^2}$.

5.2 A new subgradient extragradient method for solving the split of modified variational inequality problems (SMVIP) and fixed point problem

(1) Let C and Q be a nonempty closed convex subsets of H_1 and H_2, respectively and let $S : C \to C$ be a nonexpansive mapping. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3-inverse strongly monotone, respectively, with d^* =min$\{d_1, d_2, d_3\}$. Let $\bar{D_1}, \bar{D_2}, \bar{D_3} : Q \to H_2$ be $\bar{d_1}, \bar{d_2}, \bar{d_3}$-inverse strongly monotone, respectively, with $\hat{d}$ =min$\{\bar{d_1}, \bar{d_2}, \bar{d_3}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C : H_1 \to C$ by

$$M_C(x) = P_C(I - \zeta D_1)(ax + (1-a)P_C(I - \zeta D_2)(ax + (1-a)P_C(I - \zeta D_3)x)),$$

$\forall x \in H_1$, where $a \in [0, 1)$, $\zeta \in (0, 2d^*)$ and define $M_Q : H_2 \to Q$ by

$$M_Q(x) = P_Q(I - \bar{\zeta}\bar{D_1})(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\bar{D_2})(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\bar{D_3})\hat{x})),$$

$\forall \hat{x} \in H_1$, where $a \in [0, 1)$, $\bar{\zeta} \in (0, 2\hat{d})$. Let the sequences $\{x_n\}$ and $\{y_n\}$ generated by $x_1 \in H_1$ and

$$y_n = M_C W_n = P_C(I - \zeta D_1)T_n,$$

where $W_n = (I - \eta A^*(I - M_Q)A)x_n$ and $T_n = aW_n + (1-a)P_C(I - \zeta D_2)(aW_n + (1 - a)P_C(I - \zeta D_3)W_n))$.

$$Q_n = \{z \in H : \langle (I - \zeta D_1)T_n - y_n, y_n - z \rangle \geq 0\},$$

$$x_{n+1} = \alpha_n T_n + (1 - \alpha_n)SP_{Q_n}(T_n - \zeta D_1(y_n)),$$

for all $n \in \mathbb{N}$.

Assume the following conditions hold:

(i) $\Im = F(S) \bigcap \bigcap\limits_{i=1}^{3} \Phi_i \neq \emptyset$, where $\Phi_i = \{w \in VI(C, D_i) | Aw \in VI(Q, \bar{D_i})\}$, for all $i = 1, 2, 3$.

(ii) $\alpha_n \in [c,d] \subset (0,1)$.
Then $\{x_n\}$ converges weakly to $x_0 = P_{\Im} x_n$, which $(x_0, y_0, z_0) \in \Psi^{D_1, D_2, D_3}_{\bar{D_1}, \bar{D_2}, \bar{D_3}}$, $y_0 = P_C(I - \zeta D_2)(ax_0 + (1-a)z_0)$ and $z_0 = P_C(I - \zeta D_3)x_0$ with $\bar{x_0} = Ax_0$, $\bar{y_0} = Ay_0$ and $\bar{z_0} = Az_0$.

(2) Let C and Q be a nonempty closed convex subsets of H_1 and H_2, respectively and let $S : C \to C$ be a nonexpansive mapping. Let $\Im_1, \Im_2, \Im_3 : C \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\Im_1, \nabla\Im_2, \nabla\Im_3$ be $\frac{1}{l_{\Im_1}}, \frac{1}{l_{\Im_2}}, \frac{1}{l_{\Im_3}}$-inverse strongly monotone and continuous respectively, where $\zeta \in (0, \frac{2}{l_{\Im}})$ with $\frac{1}{l_{\Im}}$= min$\{\frac{1}{l_{\Im_1}}, \frac{1}{l_{\Im_2}}, \frac{1}{l_{\Im_3}}\}$. Let $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3 : Q \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3$ be $\frac{1}{l_{\bar{\Im}_1}}, \frac{1}{l_{\bar{\Im}_2}}, \frac{1}{l_{\bar{\Im}_3}}$-inverse strongly monotone and continuous respectively, where $\bar{\zeta} \in (0, \frac{2}{l_{\bar{\Im}}})$ with $\frac{1}{l_{\bar{\Im}}}$= min$\{\frac{1}{l_{\bar{\Im}_1}}, \frac{1}{l_{\bar{\Im}_2}}, \frac{1}{l_{\bar{\Im}_3}}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{l})$ with L being the spectral radius of the operator A^*A. define $M_C : H_1 \to C$ by $M_C(x) = P_C(I - \zeta\nabla\Im_1)(ax + (1-a)P_C(I - \zeta\nabla\Im_2)(ax + (1-a)P_C(I - \zeta\nabla\Im_3)x))$, $\forall x \in H_1$, and define $M_Q : H_2 \to Q$ by $M_Q(\hat{x}) = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_1)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\nabla\bar{\Im}_2)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\nabla\bar{\Im}_3)\hat{x}))$, $\forall \hat{x} \in H_2$. Let the sequences $\{x_n\}$ and $\{y_n\}$ generated by $x_1 \in H_1$ and

$$y_n = M_C W_n = P_C(I - \zeta\nabla\Im_1)T_n,$$

where $W_n = (I - \eta A^*(I - M_Q)A)x_n$ and
$T_n = aW_n + (1-a)P_C(I - \zeta\nabla\Im_2)(aW_n + (1-a)P_C(I - \zeta\nabla\Im_3)W_n))$.

$$Q_n = \{z \in H : \langle (I - \zeta\nabla\Im_1)T_n - y_n, y_n - z\rangle \geq 0\},$$

$$x_{n+1} = \alpha_n T_n + (1-\alpha_n)SP_{Q_n}(T_n - \zeta\nabla\Im_1(y_n)),\ \forall n \in \mathbb{N}.$$

Assume the following conditions hold:
(i) $\Im = F(S) \bigcap \bigcap_{i=1}^{3} \phi_{\Im_i} \neq \emptyset$, where $\phi_{\Im_i} = \{\Im_i(x) = \min_{x^* \in C} \Im_i(x^*) : \bar{\Im}_i(Ax) = \min_{Ax^* \in Q} \bar{\Im}_i(Ax^*)\}$, for all $i = 1, 2, 3$.
(ii) $\alpha_n \in [c,d] \subset (0,1)$.
Then $\{x_n\}$ converges weakly to $x_0 = P_{\Im} x_n$, which $(x_0, y_0, z_0) \in \Psi^{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3}_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3}$, where $y_0 = P_C(I - \zeta\nabla\Im_2)(ax_0 + (1-a)z_0)$ and $z_0 = P_C(I - \zeta\nabla\Im_3)x_0$ with $\bar{x_0} = Ax_0$, $\bar{y_0} = Ay_0$ and $\bar{z_0} = Az_0$.

5.3 Examples and numerical results

In this section, we give the conclusions for Example 4.1-4.5.

(1) Table 4.1 and Figure 4.1 show that $\{x_n\}$ and $\{w_n\}$ converge to $(2,2)$, where $(2,2) \in F(M^i) \cap VI(C, B_i)$, for all $i = 1, 2$. The convergence of $\{x_n\}$ and $\{w_n\}$ of Example 4.1 can be guaranteed by Theorem 3.5.

(2) Table 4.2 and Figure 4.2 show that $\{x_n\}$ converges to $(2,2)$, where $(2,2) \in F(M^1) \cap VI(C,B_1)$. The convergence of $\{x_n\}$ of Example 4.2 can be guaranteed by Theorem 3.5.

(3) From these Examples, we obtain that the sequence $\{x_n\}$ in Example 4.1 converges faster than the sequence $\{x_n\}$ in Example 4.2 because the iterative sequence $\{x_n\}$ and $\{w_n\}$ converges dependently.

(4) Table 4.3 and Figure 4.3 show that the iterative sequence $\{x_n\}$ and $\{w_n\}$ converge to $(4,2)$, where $(4,2) \in F(M^i) \cap VI(C,B_i)$, for all $i = 1,2$. The convergence of $\{x_n\}$ and $\{w_n\}$ of Example 4.3 can be guaranteed by Theorem 3.5.

(5) The convergence of $\{x_n\}$ and $\{w_n\}$ of Example 4.4 can be guaranteed by Theorem 3.5. Furthermore, The table 4.4 and Figure 4.4 show that the sequence $\{x_n\}$ converge faster than the sequence $\{w_n\}$ with the same starting point.

(6) Table 4.5 and Figure 4.5 show that $\{x_n\}$ and $\{y_n\}$ converge to $(2,0)$, where $(2,0) \in F(M_C(I - \frac{1}{8}A^*(I - M_Q)A))$. The convergence of $\{x_n\}$ and $\{y_n\}$ of Example 4.5 can be guaranteed by Theorem 3.16.

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Appendix

The research papers

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ORIGINAL PAPER

Convergence theorem for solving a new concept of the split variational inequality problems and application

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Abstract
In this paper, we propose the split of modified variational inequality problems (SMVIP), by using the concept of the modified generalized system of variational inequalities (MGSV). Then, we prove the strong convergence theorem for solving fixed point problems of nonlinear mappings and two variational inequality problems and solving the SMVIP. Applying our main result, we prove strong convergence theorems of the split minimization problem and the split variational inequality problem. In support of our main result, a numerical example is also presented.

Keywords Variational inequality · The intermixed algorithm · Strong convergence theorem · Fixed point

Mathematics Subject Classification 47H09 · 47H10 · 90C33

1 Introduction

The variational inequality problem (VIP) is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0,$$

for all $v \in C$. The set of all solutions of the variational inequality is denoted by $VI(C, A)$. Historically the variational inequality was introduced by Stampachhia [34] in 1964. After that variational inequalities become interested in various topics such as optimization, physic and applied sciences (see, for instance, [5–15,17,22,29,31]).

In 2012, Kangtunyakarn [24] modified the set of variational inequality problems VI(C,A) as follows:

$$VI(C, aA + (1-a)B) = \{x \in C : \langle y - x, (aA + (1-a)B)x \rangle \geq 0, \forall y \in C, a \in (0, 1)\},$$

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where A and B are the mappings of C into H. He also proved the strong convergence theorem of a new iterative scheme for finding a common element of the set of fixed point problems of infinite family of κ_i-pseudo-contractive mappings and the set of solutions of equilibrium problem and two sets of solutions of variational inequality problem as follows:

Theorem 1 *Let C be a closed convex subset of Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A_1)–(A_4), let $A, B : C \to H$ be α and β-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-pseudo-contractive mappings of C into itself with $k = sup\ k_i$ and $\rho_i = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0, 1], \alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n and S be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\rho_n, \rho_{n-1}, \ldots$, respectively. Assume that $\mathcal{F} = \cap_{i=1}^{\infty} F(T_i) \cap EP(F) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$ and let $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) S_n P_C (I - \gamma(aA + (1 - a)B)) u_n, & \forall n \geq 1. \end{cases} \tag{1}$$

where $\alpha_n, a \in (0, 1), 0 < \gamma < min\{2\alpha, 2\beta\}$ and $\{r_n\} \subseteq [b, c] \subset (0, min\{2\alpha, 2\beta\})$, satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iii) $\sum_{n=1}^{\infty} \alpha_1^n < \infty$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \mathcal{F}$ where $z = P_{\mathcal{F}} u$.

The split feasibility problem (SFP) is to find a point $x \in C$ and $Ax \in Q$. This problem was introduced by Censor and Elfving [18]. The set of all solution (SFP) is denoted by $\Gamma = \{x \in C; Ax \in Q\}$. In fact, it has been extensively investigated in the literature (see [1–4,19,21,26,28]).

Motivated by Siriyan and Kangtunyakarn [32] we introduce a new iterative method as the following Problem:

Problem 1 Let C, Q be nonempty subsets of real Hilbert H_1, H_2, respectively, Let $A : H_1 \to H_2$ be a bounded linear operator. A new split problem is to find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \zeta D_1)(ax^* + (1 - a)y^*), x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \zeta D_2)(ax^* + (1 - a)z^*), x - y^*\rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \zeta D_3)x^*, x - z^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{2}$$

and find $(\bar{x}^* = Ax^*, \bar{y}^* = Ay^*, \bar{z}^* = Az^*) \in Q \times Q \times Q$ such that

$$\begin{cases} \langle \bar{x}^* - (I - \bar{\zeta} \bar{D}_1)(a\bar{x}^* + (1 - a)\bar{y}^*), \bar{x} - \bar{x}^*\rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{y}^* - (I - \bar{\zeta} \bar{D}_2)(a\bar{x}^* + (1 - a)\bar{z}^*), \bar{x} - \bar{y}^*\rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{z}^* - (I - \bar{\zeta} \bar{D}_3)\bar{x}^*, \bar{x} - \bar{z}^*\rangle \geq 0, & \forall \bar{x} \in Q, \end{cases} \tag{3}$$

where $D_1, D_2, D_3 : C \to H_1, \bar{D}_1, \bar{D}_2, \bar{D}_3 : Q \to H_2$ are six different mappings, $\zeta, \bar{\zeta} > 0$, and $a \in [0, 1]$ which is called *the split of modified variational inequality problems* (*SMVIP*). The sets of all solutions of (2) and (3) are denoted by Ψ_{D_1, D_2, D_3} and $\Psi_{\bar{D}_1, \bar{D}_2, \bar{D}_3}$ respectively. The set of all solutions of the SMVIP is denoted by $\Psi_{\bar{D}_1, \bar{D}_2, \bar{D}_3,}^{D_1, D_2, D_3}$ that is

$$\Psi_{\bar{D}_1, \bar{D}_2, \bar{D}_3}^{D_1, D_2, D_3} = \left\{(x^*, y^*, z^*) \in \Psi_{D_1, D_2, D_3} : (\bar{x}^*, \bar{y}^*, \bar{z}^*) \in \Psi_{\bar{D}_1, \bar{D}_2, \bar{D}_3}\right\}.$$

If we put $D_1 \equiv D_2 \equiv D_3 \equiv \bar{D}_1 \equiv \bar{D}_2 \equiv \bar{D}_3 = 0$, $x^* = y^* = z^*$ and $\bar{x}^* = \bar{y}^* = \bar{z}^*$ in (2) and (3), we obtain the SMVIP is reduced to the SFP.

If putting $a = 0$ in (2) and (3) , we have

$$\begin{cases} \langle x^* - (I - \zeta D_1)y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \zeta D_2)z^*, x - y^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \zeta D_3)x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases}$$

and

$$\begin{cases} \langle \bar{x^*} - (I - \bar{\zeta} \bar{D}_1)\bar{y^*}, \bar{x} - \bar{x^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{y^*} - (I - \bar{\zeta} \bar{D}_2)\bar{z^*}, \bar{x} - \bar{y^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{z^*} - (I - \bar{\zeta} \bar{D}_3)\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, & \forall \bar{x} \in Q, \end{cases}$$

which is a modified the split general system of variational inequalities (SVIP) [33].

Let H be a real Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$, $\|\cdot\|$ respectively. Let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonlinear mapping. A point $u \in C$ is called a *fixed point* of S if $Su = u$. The set of fixed point of S is denoted by $\mathrm{Fix}(S) := \{u \in C : Su = u\}$. A mapping S is called *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Su - Sv\| \leq L\|u - v\|, \quad \forall u, v \in C.$$

If the inequality holds for $L = 1$, then S is called *nonexpansive*.

One of the successful approximation methods to prove strong convergence for finding fixed points of nonexpansive mappings was introduced by Moudafi [30]. The following theorem is known as the *viscosity approximation method*.

Theorem 2 *Let C be a nonempty closed convex subset of a real Hilbert space H and let S be a nonexpansive mapping of C into itself such that $F(S)$ is nonempty. Let f be a contraction of C into itself and let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_1 \in C \text{ arbitrarychosen}, \\ x_{n+1} = \frac{1}{1+\varepsilon_n} S x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\varepsilon_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \varepsilon_n = 0$, $\sum_{i=1}^{\infty} \varepsilon_n = \infty$ and $\lim_{n\to\infty} \left| \frac{1}{1+\varepsilon_n} - \frac{1}{\varepsilon_n} \right| = 0$. Then, $\{x_n\}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$ and $P_F(S)$ is a metric projection of H onto $F(S)$.

Moreover, the Moudafi viscosity approximation method can be applied to elliptic differential equations, linear programming, convex optimization and monotone inclusions, it has been widely studied in the literature (see [20,37,39]). After that, many mathematicians proposed iterative algorithms and proved the strong convergence theorems for a nonexpansive mapping in Hilbert space to find their fixed points (see for example [23,27,40]).

Using the concept of the viscosity approximation method (4) Yao et al. [41] proposed the intermixed algorithm for two strict pseudo-contractions as follows:

Algorithm For arbitrarily given $x_0 \in C$, $y_0 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1 - k - \alpha_n)x_n + kTx_n], & n \geq 0, \\ y_{n+1} = (1 - \beta_n)y_n + \beta_n P_C[\alpha_n g(x_n) + (1 - k - \alpha_n)y_n + kSy_n], & n \geq 0, \end{cases} \tag{5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in (0,1), $T, S : C \to C$ are a λ-strict pseudo-contraction, $f : C \to H$ is a ρ_1-contraction and $g : C \to H$ is a ρ_2-contraction, $k \in (0, 1 - \lambda)$ is a constant.

Then they proved the strong convergence theorem of the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (5) as follows.

Theorem 3 *Suppose that $Fix(S) \neq \emptyset$ and $Fix(T) \neq \emptyset$. Assume the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C2) $\beta_n \in [\xi_1, \xi_2] \subset (0, 1)$ *for all* $n \geq 0$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (5) converge strongly to the fixed points $P_{fix(T)} f(y^*)$ and $P_{fix(S)} g(x^*)$ of T and S, respectively, where $x^* \in Fix(T)$ and $y^* \in Fix(S)$.

In 2019, Siriyan and Kangtunyakarn [32] introduced the following *modified generalized system of variational inequalities*, which involves finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{6}$$

where $D_1, D_2, D_3 : C \to H$, $\lambda_1, \lambda_2, \lambda_3 > 0$ and $a \in [0, 1]$.

If putting $a = 0$, in (6), we have

$$\begin{aligned} &\langle x^* - (I - \lambda_1 D_1)y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle y^* - (I - \lambda_2 D_2)z^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{7}$$

which is generalized system of variational inequalities modified by Ceng et al. [16],

In order to find an element of the set of solutions of modified generalized system of variational inequalities (MGSV) (6), Siriyan and Kangtunyakarn [32] introduced the following iterative scheme:

$$\begin{aligned} x_{n+1} &= \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n, \\ y_n &= \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n, \end{aligned} \tag{8}$$

where $D, D_1, D_2, D_3 : C \to H$ be d, d_1, d_2, d_3-inverse strongly monotone mappings, respectively, $G : C \to C$ is defined by $G(x) = P_C(I - \lambda_1 D_1)(ax + (1-a)P_C(I - \lambda_2 D_2)(ax + (1-a)P_C(I - \lambda_3 D_3)x)$, and $a \in [0, 1)$. Under some suitable conditions, see more detail [32], they proved that the sequence $\{x_n\}$ convergence strongly to $x_0 = P_\Psi(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (8) where $y_0 = P_C(I - \lambda_2 D_2)(ax_0 + (1-a)z_0)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$.

In this paper, we prove the lemmas related to the set of fixed points of nonlinear mapping with the SMVIP in the second part of this paper. Utilizing our lemma we prove the strong convergence theorem for solving the SMVIP and two variational inequality problems by using Problem 1. Moreover, using our main result, we obtain the additional results involving the split minimization problem and the split variational inequality problem. Finally, we give an example for the main theorem.

2 Preliminaries

We denote the weak convergence and the strong convergence by "$\rightharpoonup$" and "$\to$", respectively. For every $x \in H$, there exists a unique nearest point $P_C x$ in C such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C.

Lemma 1 [36] *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \geq 0, \quad \forall v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C, that is

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H.$$

Definition 1 Let $f : C \to C$ be contractive if there exists a constant $\xi \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \xi \|x - y\|,$$

for all $x, y \in C$.

A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$.

Lemma 2 [38] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{i=1}^{\infty} \alpha_n = \infty$,
(ii) $\lim sup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 3 [32] *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2, D_3 : C \to H$ be three mappings. For every $\lambda_1, \lambda_2, \lambda_3 > 0$ and $a \in [0, 1]$. The following statement are equivalent*

(i) *$(x^*, y^*, z^*) \in C \times C \times C$ is a solution of problem* (6)
(ii) *x^* is a fixed point of the mapping G, i.e. $x^* \in F(G)$, defined the mapping $G : C \to C$ by $G(x) = P_C(I - \lambda_1 D_1)(ax + (1 - a)P_C(I - \lambda_2 D_2)(ax + (1 - a)P_C(I - \lambda_3 D_3)x))$, $\forall x \in C$, where $y^* = P_C(I - \lambda_2 D_2)(ax^* + (1 - a)z^*)$ and $z^* = P_C(I - \lambda_3 D_3)x^*$.*

Remark 1 [32] If D_1, D_2, D_3, in Lemma 3, are d_1, d_2, d_3-inverse strongly monotone, respectively, then G is nonexpansive mapping, where $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\bar{d})$ with $\bar{d} = \min\{d_1, d_2, d_3\}$.

Lemma 4 *Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subset of a real Hilbert space H_1 and H_2, respectively. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3-inverse strongly monotone respectively, where $\zeta \in (0, 2d^*)$ with $d^* = min\{d_1, d_2, d_3\}$. Let $\bar{D}_1, \bar{D}_2, \bar{D}_3 : Q \to H_2$ be $\bar{d}_1, \bar{d}_2, \bar{d}_3$-inverse strongly monotone respectively, where $\bar{\zeta} \in (0, 2\hat{d})$ with $\hat{d} = min\{\bar{d}_1, \bar{d}_2, \bar{d}_3\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with* *adjoint* A^* *and* $\eta \in (0, \frac{1}{L})$ *with* L *being the spectral radius of the operator* A^*A. *Define* $M_C : C \to C$ *by*

$$M_C(x) = P_C(I - \zeta D_1)(ax + (1-a)P_C(I - \zeta D_2)(ax + (1-a)P_C(I - \zeta D_3)x)),$$

$\forall x \in C$, *and define* $M_Q : Q \to Q$ *by*

$$M_Q(\hat{x}) = P_Q(I - \bar{\zeta}\bar{D}_1)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\bar{D}_2)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\bar{D}_3)\hat{x})),$$

$\forall \hat{x} \in Q$. *Define* $M : C \to C$ *by* $M(x) = M_C(x - \eta A^*(I - M_Q)Ax)$ *for all* $x \in C$. *Then* M *is a nonexpansive mapping for all* $x \in C$.

Proof Let $x, y \in C$. From the definition of M, we have

$$\begin{aligned}
\|Mx - My\|^2 &= \|M_C(x - \eta A^*(I - M_Q)Ax) - M_C(y - \eta A^*(I - M_Q)Ay)\|^2 \\
&\leq \|(x - \eta A^*(I - M_Q)Ax) - (y - \eta A^*(I - M_Q)Ay)\|^2 \\
&= \|x - y - \eta A^*[(I - M_Q)Ax - (I - M_Q)Ay]\|^2 \\
&= \|x - y\|^2 - 2\eta\langle x - y, A^*[(I - M_Q)Ax - (I - M_Q)Ay]\rangle \\
&\quad + \eta^2\|A^*[(I - M_Q)Ax - (I - M_Q)Ay]\|^2 \\
&= \|x - y\|^2 - 2\eta\langle A(x - y), (I - M_Q)Ax - (I - M_Q)Ay\rangle \\
&\quad + \eta^2\|A^*[(I - M_Q)Ax - (I - M_Q)Ay]\|^2 \\
&= \|x - y\|^2 - 2\eta\langle Ax - Ay + M_QAx - M_QAx + M_QAy \\
&\quad - M_QAy, (I - M_Q)Ax - (I - M_Q)Ay\rangle \\
&\quad + \eta^2\|A^*[(I - M_Q)Ax - (I - M_Q)Ay]\|^2 \\
&= \|x - y\|^2 - 2\eta[\langle Ax - Ay - M_QAx + M_QAy, (I - M_Q)Ax \\
&\quad - (I - M_Q)Ay\rangle - \langle M_QAy - M_QAx, (I - M_Q)Ax \\
&\quad - (I - M_Q)Ay\rangle] + \eta^2\|A^*[(I - M_Q)Ax - (I - M_Q)Ay]\|^2 \\
&= \|x - y\|^2 - 2\eta[\langle (I - M_Q)Ax - (I - M_Q)Ay, (I - M_Q)Ax \\
&\quad - (I - M_Q)Ay\rangle - \langle M_QAy - M_QAx, (I - M_Q)Ax \\
&\quad - (I - M_Q)Ay\rangle] + \eta^2\|A^*[(I - M_Q)Ax - (I - M_Q)Ay]\|^2 \\
&\leq \|x - y\|^2 - 2\eta[\|(I - M_Q)Ax - (I - M_Q)Ay\|^2 \\
&\quad - \frac{1}{2}\|(I - M_Q)Ax - (I - M_Q)Ay\|^2] \\
&\quad + \eta^2\|A^*[(I - M_Q)Ax - (I - M_Q)Ay]\|^2 \\
&= \|x - y\|^2 - \eta\|(I - M_Q)Ax - (I - M_Q)Ay\|^2 \\
&\quad + \eta^2\|A^*[(I - M_Q)Ax - (I - M_Q)Ay]\|^2 \\
&\leq \|x - y\|^2 - \eta\|(I - M_Q)Ax - (I - M_Q)Ay\|^2 \\
&\quad + \eta^2 L\|(I - M_Q)Ax - (I - M_Q)Ay\|^2 \\
&= \|x - y\|^2 - \eta(1 - \eta L)\|(I - M_Q)Ax - (I - M_Q)Ay\|^2 \\
&\leq \|x - y\|^2. && (9)
\end{aligned}$$

It implies that $\|Mx - My\| \leq \|x - y\|$.

Hence, M is a nonexpansive mapping. □

Lemma 5 *Let C, Q be nonempty closed convex subsets of H_1, H_2, respectively. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3-inverse strongly monotone respectively, where $\zeta \in (0, 2d^*)$ with $d^* = min\{d_1, d_2, d_3\}$. Let $\bar{D_1}, \bar{D_2}, \bar{D_3} : Q \to H_2$ be $\bar{d_1}, \bar{d_2}, \bar{d_3}$-inverse strongly monotone respectively, where $\bar{\zeta} \in (0, 2\hat{d})$ with $\hat{d} = min\{\bar{d_1}, \bar{d_2}, \bar{d_3}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C : C \to C$ by*

$$M_C(x) = P_C(I - \zeta D_1)(ax + (1-a)P_C(I - \zeta D_2)(ax + (1-a)P_C(I - \zeta D_3)x)),$$

$\forall x \in C$, and define $M_Q : Q \to Q$ by

$$M_Q(\hat{x}) = P_Q(I - \bar{\zeta}\bar{D_1})(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\bar{D_2})(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\bar{D_3})\hat{x})),$$

$\forall \hat{x} \in Q$. Assume

$$\Psi^{D_1, D_2, D_3}_{\bar{D_1}, \bar{D_2}, \bar{D_3}} = \{(x^*, y^*, z^*) \in \Psi_{D_1, D_2, D_3} : (\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\bar{D_1}, \bar{D_2}, \bar{D_3}}\} \neq \emptyset.$$

The following statements are equivalent

(i) $(x^*, y^*, z^*) \in \Psi^{D_1, D_2, D_3}_{\bar{D_1}, \bar{D_2}, \bar{D_3}}$,
(ii) $x^* = M_C(x^* - \eta A^*(I - M_Q)Ax^*)$,

where $y^* = P_C(I - \zeta D_2)(ax^* + (1-a)z^*)$, $z^* = P_C(I - \zeta D_3)x^*$, $\bar{x^*} = Ax^* = P_Q(I - \bar{\zeta}\bar{D_1})(a\bar{x^*} + (1-a)\bar{y^*})$, $\bar{y^*} = Ay^* = P_Q(I - \bar{\zeta}\bar{D_2})(a\bar{x^*} + (1-a)\bar{z^*})$ and $\bar{z^*} = Az^* = P_Q(I - \bar{\zeta}\bar{D_3})\bar{x^*}$.

Proof Let the following condition hold.

$(i) \Rightarrow (ii)$ Let $(x^*, y^*, z^*) \in \Psi^{D_1, D_2, D_3}_{\bar{D_1}, \bar{D_2}, \bar{D_3}}$. We have $(x^*, y^*, z^*) \in \Psi_{D_1, D_2, D_3}$ and $(\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\bar{D_1}, \bar{D_2}, \bar{D_3}}$, where $\bar{x^*} = Ax^*$, $\bar{y^*} = Ay^*$ and $\bar{z^*} = Az^*$. Since $(x^*, y^*, z^*) \in \Psi_{D_1, D_2, D_3}$, we obtain that

$$\begin{aligned} \langle x^* - (I - \zeta D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle y^* - (I - \zeta D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, \quad \forall x \in C, \\ \langle z^* - (I - \zeta D_3)x^*, x - z^* \rangle \geq 0, \quad \forall x \in C. \end{aligned}$$

By the property of P_C, we obtain

$$\begin{aligned} x^* &= P_C(I - \zeta D_1)(ax^* + (1-a)y^*), \\ y^* &= P_C(I - \zeta D_2)(ax^* + (1-a)z^*), \\ z^* &= P_C(I - \zeta D_3)x^*. \end{aligned}$$

It implies that

$$\begin{aligned} x^* &= P_C(I - \zeta D_1)(ax^* + (1-a)P_C(I - \zeta D_2)(ax^* \\ &\quad + (1-a)P_C(I - \zeta D_3)x^*)) \\ &= M_C x^*. \end{aligned} \tag{10}$$

Since $(\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\bar{D_1}, \bar{D_2}, \bar{D_3}}$, we obtain that

$$\begin{aligned} &\langle \bar{x^*} - (I - \bar{\zeta}\bar{D_1})(a\bar{x^*} + (1-a)\bar{y^*}), \bar{x} - \bar{x^*} \rangle \geq 0, \quad \forall \bar{x} \in Q, \\ &\langle \bar{y^*} - (I - \bar{\zeta}\bar{D_2})(a\bar{x^*} + (1-a)\bar{z^*}), \bar{x} - \bar{y^*} \rangle \geq 0, \quad \forall \bar{x} \in Q, \\ &\langle \bar{z^*} - (I - \bar{\zeta}\bar{D_3})\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, \quad \forall \bar{x} \in Q. \end{aligned}$$

Then, we obtain

$$\begin{aligned}\bar{x}^* &= P_Q(I - \bar{\zeta}\bar{D}_1)(a\bar{x}^* + (1-a)\bar{y}^*),\\ \bar{y}^* &= P_Q(I - \bar{\zeta}\bar{D}_2)(a\bar{x}^* + (1-a)\bar{z}^*),\\ \bar{z}^* &= P_Q(I - \bar{\zeta}\bar{D}_3)\bar{x}^*.\end{aligned}$$

That is

$$\begin{aligned}\bar{x}^* &= P_Q(I - \bar{\zeta}\bar{D}_1)(a\bar{x}^* + (1-a)P_Q(I - \bar{\zeta}\bar{D}_2)(a\bar{x}^*\\ &\quad + (1-a)P_Q(I - \bar{\zeta_3}\bar{D}_3)\bar{x}^*))\\ &= M_Q\bar{x}^*\\ &= M_QAx^*. \end{aligned} \tag{11}$$

It implies that

$$x^* = M_C(x^* - \eta A^*(I - M_Q)Ax^*) \tag{12}$$

$(ii) \Rightarrow (i)$ Let $x^* = M_C(x^* - \eta A^*(I - M_Q)Ax^*)$ where $y^* = P_C(I - \zeta D_2)(ax^* + (1-a)z^*)$, $z^* = P_C(I - \zeta D_3)x^*$, $\bar{x}^* = Ax^* = P_Q(I - \bar{\zeta}\bar{D}_1)(a\bar{x}^* + (1-a)\bar{y}^*)$, $\bar{y}^* = Ay^* = P_Q(I - \bar{\zeta}\bar{D}_2)(a\bar{x}^* + (1-a)\bar{z}^*)$ and $\bar{z}^* = Az^* = P_Q(I - \bar{\zeta}\bar{D}_3)\bar{x}^*$ and $(u^*, v^*, w^*) \in \Psi^{D_1,D_2,D_3}_{\bar{D}_1,\bar{D}_2,\bar{D}_3}$.

From Remark 1, we have M_C is nonexpansive mapping on C. It is obvious that M_Q is nonexpansive mapping on Q.

From $(u^*, v^*, w^*) \in \Psi^{D_1,D_2,D_3}_{\bar{D}_1,\bar{D}_2,\bar{D}_3}$, and $(i) \Rightarrow (ii)$, we have $u^* = M_C(u^* - \eta A^*(I - M_Q)Au^*)$. Utilizing the same method as (11) we have $Au^* = M_QAu^*$ and applying (9), we have

$$\|x^* - u^*\|^2 \leq \|x^* - u^*\|^2 - \eta(1 - \eta L)\|(I - M_Q)Ax^* - (I - M_Q)Au^*\|^2$$

It implies that

$$\bar{x}^* = Ax^* \in F(M_Q).$$

That is,

$$\bar{x}^* = M_Q\bar{x}^* = P_Q(I - \bar{\zeta}\bar{D}_1)(a\bar{x}^* + (1-a)P_Q(I - \bar{\zeta}\bar{D}_2)(a\bar{x}^* + (1-a)P_Q(I - \bar{\zeta}\bar{D}_3)\bar{x}^*)).$$

It follows that

$$\bar{x}^* = P_Q(I - \bar{\zeta}\bar{D}_1)(a\bar{x}^* + (1-a)\bar{y}^*),$$

where $\bar{y}^* = P_Q(I - \bar{\zeta}\bar{D}_2(a\bar{x}^* + (1-a)\bar{z}^*)$ and $\bar{z}^* = P_Q(I - \bar{\zeta}\bar{D}_3)\bar{x}^*$. From Lemma 3, we have

$$(\bar{x}^*, \bar{y}^*, \bar{z}^*) \in \Psi_{\bar{D}_1,\bar{D}_2,\bar{D}_3} \tag{13}$$

From $Ax^* \in F(M_Q)$, we have

$$x^* = M_C(x^* - \eta A^*(I - M_Q)Ax^*) = M_Cx^*.$$

That is $x^* \in F(M_C)$. Then, we have $x^* = P_C(I - \zeta D_1)(ax^* + (1-a)P_C(I - \zeta D_2)(ax^* + (1-a)P_C(I - \zeta D_3)x^*)) = M_Cx^*$. It follows that $x^* = P_C(I - \zeta D_1)(ax^* + (1-a)y^*)$, where $y^* = P_C(I - \zeta D_2)(ax^* + (1-a)z^*)$, and $z^* = P_C(I - \zeta D_3)x^*$. From Lemma 3, we have

$$(x^*, y^*, z^*) \in \Psi_{D_1,D_2,D_3} \tag{14}$$

From (13)and (14), we have

$$(x^*, y^*, z^*) \in \Psi^{D_1, D_2, D_3}_{\bar{D_1}, \bar{D_2}, \bar{D_3}}.$$

□

Example 1 Let $\mathbb{R}$ be the set of real numbers, $C = [-50, 50] \times [-50, 50]$, $Q = [-100, 100] \times [-100, 100]$, $H_1 = H_2 = \mathbb{R}$. Let $D_1, D_2, D_3 : [-50, 50] \to \mathbb{R}^2$ defined by $D_1(x_1, x_2) = (x_1 - 3, x_2 - 3)$, $D_2(x_1, x_2) = (\frac{x_1-3}{2}, \frac{x_2-3}{2})$, $D_3(x_1, x_2) = (\frac{x_1-3}{3}, \frac{x_2-3}{3})$, for all $(x_1, x_2) \in C$. Let $\bar{D_1}, \bar{D_2}, \bar{D_3} : [-100, 100] \times [-100, 100] \to \mathbb{R}^2$ defined by $\bar{D_1}(x_1, x_2) = (x_1 - 6, x_2 - 6)$, $\bar{D_2}(x_1, x_2) = (\frac{x_1-6}{2}, \frac{x_2-6}{2})$, $\bar{D_3}(x_1, x_2) = (\frac{x_1-6}{3}, \frac{x_2-6}{3})$, for all $(x_1, x_2) \in Q$ respectively. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A(x_1, x_2) = (2x, 2x)$ and $A^* : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A^*(x_1, x_2) = (2x, 2x)$. Define $M_C : [-50, 50] \times [-50, 50] \to [-50, 50] \times [-50, 50]$ by $M_C(x) = P_C(I - \frac{2}{5}D_1)(\frac{1}{2}x + \frac{1}{2}P_C(I - \frac{2}{5}D_2)(\frac{1}{2}x + \frac{1}{2}P_C(I - \frac{2}{5}D_3)x))$, $\forall x \in C$, and define $M_Q : [-100, 100] \times [-100, 100] \to [-100, 100] \times [-100, 100]$ by $M_Q(\hat{x}) = P_Q(I - \frac{1}{7}\bar{D_1})(\frac{1}{2}\hat{x} + \frac{1}{2}P_Q(I - \frac{1}{7}\bar{D_2})(\frac{1}{2}\hat{x} + \frac{1}{2}P_Q(I - \frac{1}{7}\bar{D_3})\hat{x}))$, $\forall \hat{x} \in Q$.

Then, we have $(x^*, y^*, z^*) \in \Psi^{D_1, D_2, D_3}_{\bar{D_1}, \bar{D_2}, \bar{D_3}}$, where $(x^*, y^*, z^*) = (3, 3) \in [-50, 50] \times [-50, 50]$. By Lemma 5, we have $(3, 3) = M_C((3, 3) - \frac{1}{5}A^*(I - M_Q)A(3, 3))$, where $y^* = P_C(I - \frac{2}{5}D_2)(3, 3)$, $z^* = P_C(I - \frac{2}{5}D_3)(3, 3)$, $\bar{x^*} = (6, 6)$, $\bar{y^*} = (6, 6)$ and $\bar{z^*} = (6, 6)$.

Lemma 6 [25] *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a κ-strictly pseudo-contractive mapping with $Fix(T) \neq \emptyset$. Then, there hold the following statement:*

(i) *$Fix(T) = VI(C, I - T)$;*
(ii) *For every $u \in C$ and $v \in Fix(T)$,*

$$\|P_C(I - \lambda(I - T))u - v\| \leq \|u - v\|,$$

for $u \in C$ and $v \in Fix(T)$ and $\lambda \in (0, 1 - \kappa)$.

3 Main result

In this section, we prove the strong convergence theorem, by using Lemma 5 as important tool for finding the solution of the SMVIP and the fixed point of nonexpansive mapping.

Theorem 4 *Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subset of a real Hilbert space H_1 and H_2, respectively. For every $i = 1, 2$, let $B_i : C \to H$ be α_i-inverse strongly monotone mapping with $\alpha = min\{\alpha_1, \alpha_2\}$ and let $D^i_1, D^i_2, D^i_3 : C \to H_1$ be d^i_1, d^i_2, d^i_3-inverse strongly monotone, respectively, with $d^* = \min\{d^i_1, d^i_2, d^i_3\}$. Let $\bar{D_1}^i, \bar{D_2}^i, \bar{D_3}^i : Q \to H_2$ be $\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i$-inverse strongly monotone, respectively, with $\hat{d} = \min\{\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Let $f, g : H \to H$ be a_f and a_g-contraction mapping with $a = \max\{a_f, a_g\}$. For every $i = 1, 2$ define $M^i_C : C \to C$ by $M^i_C(x) = P_C(I - \zeta^i D^i_1)(ax + (1 - a)P_C(I - \zeta^i D^i_2)(ax + (1 - a)P_C(I - \zeta^i D^i_3)x))$, $\forall x \in C$, where $\zeta^i \in (0, 2d^*)$ and define $M^i_Q : Q \to Q$ by $M^i_Q(\hat{x}) = P_Q(I - \bar{\zeta}^i \bar{D}^i_1)(a\hat{x} + (1 - a)P_Q(I - \bar{\zeta}^i \bar{D}^i_2)(a\hat{x} + (1 - a)P_Q(I - \bar{\zeta}^i \bar{D}^i_3)\hat{x}))$, $\forall \hat{x} \in Q$, where $\bar{\zeta}^i \in (0, 2\hat{d})$. Define $M^i : C \to C$ by $M^i(x^*) = M^i_C(x^* - \eta A^*(I - M^i_Q)Ax^*)$ for all $x^* \in C$ and $i = 1, 2$. Let* *the sequence* $\{x_n\}$ *and* $\{w_n\}$ *generated by* $x_1, w_1 \in C$ *and*

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2 B_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)M^2 w_n) \end{cases} \tag{15}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$ *with* $\delta_n + \eta_n + \mu_n = 1$ *and* $\gamma \in (0, 2\alpha)$ *with* γ=*min*$\{\gamma_1, \gamma_2\}$. *Assume the following condition holds;*

(i) $\mathcal{F}_i = F(M^i) \cap VI(C, B_i) \neq \emptyset$ *for all* $i = 1, 2$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,
(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ *for all* $n \in N$ *and for some* $\bar{\theta}, \theta > 0$,
(iv) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, *where* $y_1^* = P_C(I - \zeta^1 D_2^1)(ax_1^* + (1-a)z_1^*)$ *and* $z_1^* = P_C(I - \zeta^1 D_3^1)x_1^*$ *with* (x_1^*, y_1^*, z_1^*) *is an element of* $\Psi^{D_1^1, D_2^1, D_3^1}_{\bar{D}_1^1, \bar{D}_2^1, \bar{D}_3^1}$. *and* $\{w_n\}$ *converges strongly to* $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, *where* $y_2^* = P_C(I - \zeta^2 D_2^2)(ax_2^* + (1-a)z_2^*)$ *and* $z_1^* = P_C(I - \zeta^2 D_3^2)x_1^*$ *with* (x_2^*, y_2^*, z_2^*) *is an element of* $\Psi^{D_1^2, D_2^2, D_3^2}_{\bar{D}_1^2, \bar{D}_2^2, \bar{D}_3^2}$.

Proof The proof of this theorem will be divided into five steps.

Step 1. We will show that $\{x_n\}$ is bounded.

First, we will prove that $I - \gamma B_i$ is nonexpansive with $\gamma = \min\{\gamma_1, \gamma_2\}$, for all $i = 1, 2$ we get

$$\begin{aligned} \|(I - \gamma B_i)x - (I - \gamma B_i)w\|^2 &= \|x - w - \gamma(B_i x - B_i w)\|^2 \\ &= \|x - w\|^2 - 2\gamma\langle x - w, B_i x - B_i w\rangle \\ &\quad + \gamma^2 \|B_i x - B_i w\|^2 \\ &\leq \|x - w\|^2 - 2\alpha\gamma \|B_i x - B_i w\|^2 \\ &\quad + \gamma^2 \|B_i x - B_i w\|^2 \\ &= \|x - w\|^2 - \gamma(2\alpha - \gamma)\|B_i x - B_i w\|^2 \\ &\leq \|x - w\|^2. \end{aligned}$$

Thus $I - \gamma B_i$ is a nonexpansive mapping, for all $i = 1, 2$. Then $P_C(I - \gamma B_i)$ is a nonexpansive mapping.

Let $\tilde{x} \in \mathcal{F}_1$ and $\tilde{w} \in \mathcal{F}_2$, from the definition of x_n, we have

$$\begin{aligned} \|x_{n+1} - \tilde{x}\| &= \|\delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)M^1 x_n) \\ &\quad - (\delta_n + \eta_n + \mu_n)\tilde{x}\| \\ &\leq \delta_n \|x_n - \tilde{x}\| + \eta_n \|P_C(I - \gamma_1 B_1)x_n - \tilde{x}\| + \mu_n \|P_C(\alpha_n f(w_n) \\ &\quad + (1-\alpha_n)M^1 x_n) - \tilde{x}\| \\ &\leq (1 - \mu_n)\|x_n - \tilde{x}\| + \mu_n \|\alpha_n(f(w_n) - \tilde{x}) + (1-\alpha_n)(M^1 x_n - \tilde{x})\| \\ &\leq (1 - \mu_n)\|x_n - \tilde{x}\| + \mu_n\alpha_n \|f(w_n) - \tilde{x}\| + \mu_n(1-\alpha_n)\|x_n - \tilde{x}\| \\ &\leq (1 - \mu_n)\|x_n - \tilde{x}\| + \mu_n\alpha_n a\|w_n - \tilde{w}\| + \mu_n\alpha_n \|f(\tilde{w}) - \tilde{x}\| \\ &\quad + \mu_n(1-\alpha_n)\|x_n - \tilde{x}\| \\ &= (1 - \mu_n\alpha_n)\|x_n - \tilde{x}\| + \mu_n\alpha_n a\|w_n - \tilde{w}\| + \mu_n\alpha_n \|f(\tilde{w}) - \tilde{x}\|. \end{aligned} \tag{16}$$

Similarly, we get

$$\|w_{n+1}-\tilde{w}\| \leq (1-\mu_n\alpha_n)\|w_n-\tilde{w}\| + \mu_n\alpha_n a\|x_n-\tilde{x}\| + \mu_n\alpha_n\|g(\tilde{x})-\tilde{w}\|. \tag{17}$$

Combining (16) and (17), we have

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\| + \|w_{n+1}-\tilde{w}\| &\leq (1-\mu_n\alpha_n)\left[\|x_n-\tilde{x}\| + \|w_n-\tilde{w}\|\right] \\
&\quad + \mu_n\alpha_n a\left[\|x_n-\tilde{x}\| + \|w_n-\tilde{w}\|\right] \\
&\quad + \mu_n\alpha_n\left[\|g(\tilde{x})-\tilde{w}\| + \|f(\tilde{w})-\tilde{x}\|\right] \\
&= (1-\mu_n\alpha_n(1-a))\left[\|x_n-\tilde{x}\| + \|w_n-\tilde{w}\|\right] \\
&\quad + \mu_n\alpha_n\left[\|g(\tilde{x})-\tilde{w}\| + \|f(\tilde{w})-\tilde{x}\|\right].
\end{aligned}$$

By induction, we can derive that

$$\|x_n-\tilde{x}\| + \|w_n-\tilde{w}\| \leq max\left\{\|x_1-\tilde{x}\| + \|w_1-\tilde{w}\|, \frac{\|g(\tilde{x})-\tilde{w}\| + \|f(\tilde{w})-\tilde{x}\|}{1-a}\right\},$$

for every $n\in\mathbb{N}$. This implies that $\{x_n\}$ and $\{w_n\}$ are bounded.

Step 2. Claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\|w_{n+1}-w_n\| = 0$.

First, we let $U_n = P_C(\alpha_n f(w_n)+(1-\alpha_n)M^1x_n)$ and $V_n = P_C(\alpha_n g(x_n)+(1-\alpha_n)M^2w_n)$. Then, observe that

$$\begin{aligned}
\|U_n-U_{n-1}\| &= \|P_C(\alpha_n f(w_n) + (1-\alpha_n)M^1x_n) - P_C(\alpha_{n-1}f(w_{n-1}) \\
&\quad + (1-\alpha_{n-1})M^1x_{n-1})\| \\
&\leq \alpha_n\|f(w_n)-f(w_{n-1})\| + |\alpha_n-\alpha_{n-1}|\|f(w_{n-1})\| \\
&\quad + (1-\alpha_n)\|M^1x_n - M^1x_{n-1}\| \\
&\quad + |\alpha_n-\alpha_{n-1}|\|M^1x_{n-1}\| \\
&\leq \alpha_n a\|w_n-w_{n-1}\| + |\alpha_n-\alpha_{n-1}|\left[\|f(w_{n-1})\| + \|M^1x_{n-1}\|\right] \\
&\quad + (1-\alpha_n)\|x_n-x_{n-1}\|.
\end{aligned} \tag{18}$$

By the definition of x_n and (18) we obtain

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\delta_n x_n + \eta_n P_C(I-\gamma_1B_1)x_n + \mu_n U_n - \delta_{n-1}x_{n-1} \\
&\quad - \eta_{n-1}P_C(I-\gamma_1B_1)x_{n-1} - \mu_{n-1}U_{n-1}\| \\
&\leq \delta_n\|x_n-x_{n-1}\| + |\delta_n-\delta_{n-1}|\|x_{n-1}\| + \eta_n\|P_C(I-\gamma_1B_1)x_n \\
&\quad - P_C(I-\gamma_1B_1)x_{n-1}\| + |\eta_n-\eta_{n-1}|\|P_C(I-\gamma_1B_1)x_{n-1}\| \\
&\quad + \mu_n\|U_n-U_{n-1}\| + |\mu_n-\mu_{n-1}|\|U_{n-1}\| \\
&= (1-\mu_n)\|x_n-x_{n-1}\| + |\delta_n-\delta_{n-1}|\|x_{n-1}\| \\
&\quad + |\eta_n-\eta_{n-1}|\|P_C(I-\gamma_1B_1)x_{n-1}\| \\
&\quad + \mu_n|\alpha_n-\alpha_{n-1}|\left[\|f(w_{n-1})\| + \|M^1x_{n-1}\|\right] \\
&\quad + \mu_n(1-\alpha_n)\|x_n-x_{n-1}\| + |\mu_n-\mu_{n-1}|\|U_{n-1}\| \\
&\quad + \mu_n\alpha_n a\|w_n-w_{n-1}\|.
\end{aligned} \tag{19}$$

Using the same method as derived in (19), we have

$$\begin{aligned}
\|w_{n+1}-w_n\| &\leq (1-\mu_n)\|w_n-w_{n-1}\| + |\delta_n-\delta_{n-1}|\|w_{n-1}\| \\
&\quad + |\eta_n-\eta_{n-1}|\|P_C(I-\gamma_2B_2)w_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&+ \mu_n|\alpha_n - \alpha_{n-1}|\left[\|g(x_{n-1})\| + \|M^2 w_{n-1}\|\right] \\
&+ \mu_n(1-\alpha_n)\|w_n - w_{n-1}\| + |\mu_n - \mu_{n-1}|\|V_{n-1}\| \\
&+ \mu_n\alpha_n a\|x_n - x_{n-1}\|. \qquad (20)
\end{aligned}
$$

From (19) and (20), then we get

$$
\begin{aligned}
\|x_{n+1} - x_n\| + \|w_{n+1} - w_n\| &\le (1-\mu_n)\left[\|x_n - x_{n-1}\| + \|w_n - w_{n-1}\|\right] \\
&+ |\delta_n - \delta_{n-1}|\left[\|x_{n-1}\| + \|w_{n-1}\|\right] \\
&+ |\eta_n - \eta_{n-1}|[\|P_C(I-\gamma_1 B_1)x_{n-1}\| \\
&+ \|P_C(I-\gamma_2 B_2)w_{n-1}\|] \\
&+ |\mu_n - \mu_{n-1}|\left[\|U_{n-1}\| + \|V_{n-1}\|\right] \\
&+ \mu_n\alpha_n a\left[\|w_n - w_{n-1}\| + \|x_n - x_{n-1}\|\right] \\
&+ \mu_n|\alpha_n - \alpha_{n-1}|[\|f(w_{n-1})\| + \|M^1 x_{n-1}\| \\
&+ \|g(x_{n-1})\| + \|M^2 w_{n-1}\|] \\
&+ \mu_n(1-\alpha_n)\left[\|x_n - x_{n-1}\| + \|w_n - w_{n-1}\|\right] \\
&\le (1-\alpha_n\bar{\theta}(1-a))\left[\|x_n - x_{n-1}\| + \|w_n - w_{n-1}\|\right] \\
&+ |\delta_n - \delta_{n-1}|\left[\|x_{n-1}\| + \|w_{n-1}\|\right] \\
&+ |\eta_n - \eta_{n-1}|[\|P_C(I-\gamma_1 B_1)x_{n-1}\| \\
&+ \|P_C(I-\gamma_2 B_2)w_{n-1}\|] \\
&+ |\mu_n - \mu_{n-1}|\left[\|U_{n-1}\| + \|V_{n-1}\|\right] \\
&+ \theta|\alpha_n - \alpha_{n-1}|[\|f(w_{n-1})\| + \|M^1 x_{n-1}\| \\
&+ \|g(x_{n-1})\| + \|M^2 w_{n-1}\|].
\end{aligned}
$$

Applying Lemma 2 and the condition (ii), (iii) and (iv) we can conclude that

$$
\|x_{n+1} - x_n\| \to 0 \quad \text{and} \quad \|w_{n+1} - w_n\| \to 0 \quad \text{as} \quad n \to \infty. \qquad (21)
$$

Step 3. Prove that $\lim_{n\to\infty}\|U_n - P_C(I-\gamma_1 B_1)U_n\| = \lim_{n\to\infty}\|U_n - M_C^1 U_n\| = 0$. To show this, take $U_n = P_C\tilde{u}_n$, $\forall n \in \mathbb{N}$. Then we derive that

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \|\delta_n(x_n - \tilde{x}) + \eta_n(P_C(I-\gamma_1 B_1)x_n - \tilde{x}) + \mu_n(U_n - \tilde{x})\|^2 \\
&\le \delta_n\|x_n - \tilde{x}\|^2 + \eta_n\|P_C(I-\gamma_1 B_1)x_n - \tilde{x}\|^2 \\
&\quad - \delta_n\eta_n\|x_n - P_C(I-\gamma_1 B_1)x_n\|^2 + \mu_n\|\tilde{u}_n - \tilde{x}\|^2 \\
&\le (1-\mu_n)\|x_n - \tilde{x}\|^2 - \delta_n\eta_n\|x_n - P_C(I-\gamma_1 B_1)x_n\|^2 \\
&\quad + \mu_n\|\alpha_n(f(w_n) - M^1 x_n) + (M^1 x_n - \tilde{x})\|^2 \\
&\le (1-\mu_n)\|x_n - \tilde{x}\|^2 - \delta_n\eta_n\|x_n - P_C(I-\gamma_1 B_1)x_n\|^2 \\
&\quad + \mu_n\left[\|M^1 x_n - \tilde{x}\|^2 + 2\alpha_n\langle f(w_n) - M^1 x_n, \tilde{u}_n - \tilde{x}\rangle\right] \\
&\le \|x_n - \tilde{x}\|^2 - \delta_n\eta_n\|x_n - P_C(I-\gamma_1 B_1)x_n\|^2 \\
&\quad + 2\mu_n\alpha_n\|f(w_n) - M^1 x_n\|\|\tilde{u}_n - \tilde{x}\|,
\end{aligned}
$$

which implies that

$$\begin{aligned}\delta_n\eta_n\|x_n - P_C(I-\gamma_1 B_1)x_n\|^2 &\le \|x_n-\tilde{x}\|^2 - \|x_{n+1}-\tilde{x}\|^2\\ &\quad + 2\mu_n\alpha_n\|f(w_n) - M^1x_n\|\|\tilde{u}_n - \tilde{x}\|\\ &\le \|x_n - x_{n+1}\|\left[\|x_n-\tilde{x}\| + \|x_{n+1}-\tilde{x}\|\right]\\ &\quad + 2\mu_n\alpha_n\|f(w_n) - M^1x_n\|\|\tilde{u}_n - \tilde{x}\|.\end{aligned}$$

Then, we have

$$\|x_n - P_C(I-\gamma_1 B_1)x_n\| \to 0 \ \text{ as } n\to\infty. \tag{22}$$

Observe that

$$x_{n+1} - x_n = \eta_n(P_C(I-\gamma_1 B_1)x_n - x_n) + \mu_n(U_n - x_n).$$

From (21) and (22), we obtain

$$\|U_n - x_n\| \to 0 \ \text{ as } \ n\to\infty. \tag{23}$$

Observe that

$$\begin{aligned}\|U_n - P_C(I-\gamma_1 B_1)U_n\| &\le \|U_n - x_n\| + \|x_n - P_C(I-\gamma_1 B_1)x_n\|\\ &\quad + \|P_C(I-\gamma_1 B_1)x_n - P_C(I-\gamma_1 B_1)U_n\|\\ &\le \|U_n - x_n\| + \|x_n - P_C(I-\gamma_1 B_1)x_n\| + \|x_n - U_n\|\\ &= 2\|U_n - x_n\| + \|x_n - P_C(I-\gamma_1 B_1)x_n\|,\end{aligned}$$

by (22) and (23), we obtain

$$\|U_n - P_C(I-\gamma_1 B_1)U_n\| \to 0 \ \text{ as } \ n\to\infty. \tag{24}$$

Applying the same argument as (24), we also obtain

$$\|V_n - P_C(I-\gamma_2 B_2)V_n\| \to 0 \ \text{ as } \ n\to\infty.$$

Consider

$$\|x_{n+1} - U_n\| \le \|x_{n+1} - x_n\| + \|x_n - U_n\|.$$

From (21) and (23), we have

$$\|x_{n+1} - U_n\| \to 0 \ \text{ as } \ n\to\infty. \tag{25}$$

Since

$$\begin{aligned}\|x_n - M^1x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \|U_n - M^1x_n\|\\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \|\tilde{u}_n - M^1x_n\|\\ &= \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\|\\ &\quad + \|\alpha_n f(w_n) + (1-\alpha_n)M^1x_n - M^1x_n\|\\ &= \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \alpha_n\|f(w_n) - M^1x_n\|.\end{aligned}$$

From (21), (25) and condition (ii), we get

$$\|x_n - M^1x_n\| \to 0 \ \text{ as } \ n\to\infty. \tag{26}$$

Consider

$$\begin{aligned}\|U_n - M^1 U_n\| &\le \|U_n - x_n\| + \|x_n - M^1 x_n\| + \|M^1 x_n - M^1 U_n\|\\ &\le \|U_n - x_n\| + \|x_n - M^1 x_n\| + \|x_n - U_n\|\\ &\le 2\|U_n - x_n\| + \|x_n - M^1 x_n\|.\end{aligned}$$

From (23) and (26), we have

$$\|U_n - M^1 U_n\| \to 0 \ \text{ as } \ n \to \infty. \tag{27}$$

Applying the same method as (26), we also have

$$\|V_n - M^2 V_n\| \to 0 \ \text{ as } \ n \to \infty.$$

Step 4. Claim that $\limsup_{n\to\infty}\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \le 0$, where $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$.

First, take a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle = \lim_{k\to\infty}\langle f(x_2^*) - x_1^*, U_{n_k} - x_1^*\rangle. \tag{28}$$

Since $\{x_n\}$ is bounded, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in C$ as $k \to \infty$. From (23), we obtain $U_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$.

Next, we need to show that $\hat{x} \in \mathcal{F}_1 = F(M^1) \cap VI(C, B_1)$. Assume $\hat{x} \notin F(M^1)$. Then, we have $\hat{x} \neq M^1\hat{x}$. By the Opial's condition, we obtain

$$\begin{aligned}\liminf_{k\to\infty}\|U_{n_k} - \hat{x}\| &< \liminf_{k\to\infty}\|U_{n_k} - M^1\hat{x}\|\\ &\le \liminf_{k\to\infty}\|U_{n_k} - M^1 U_{n_k}\| + \liminf_{k\to\infty}\|M^1 U_{n_k} - M^1\hat{x}\|\\ &\le \liminf_{k\to\infty}\|U_{n_k} - \hat{x}\|.\end{aligned}$$

This is a contradiction.

Therefore

$$\hat{x} \in F(M^1). \tag{29}$$

Assume $\hat{x} \notin VI(C, B_1)$, then we get $\hat{x} \neq P_C(I - \gamma_1 B_1)\hat{x}$.

From the Opial's condition and (24), we have

$$\begin{aligned}\liminf_{k\to\infty}\|U_{n_k} - \hat{x}\| &< \liminf_{k\to\infty}\|U_{n_k} - P_C(I - \gamma_1 B_1)\hat{x}\|\\ &\le \liminf_{k\to\infty}\|U_{n_k} - P_C(I - \gamma_1 B_1)U_{n_k}\|\\ &\quad + \liminf_{k\to\infty}\|P_C(I - \gamma_1 B_1)U_{n_k} - P_C(I - \gamma_1 B_1)\hat{x}\|\\ &\le \liminf_{k\to\infty}\|U_{n_k} - \hat{x}\|.\end{aligned}$$

This is a contradiction.

Therefore

$$\hat{x} \in VI(C, B_1). \tag{30}$$

By (29) and (30), this yields that

$$\hat{x} \in \mathcal{F}_1 = F(M^1) \cap VI(C, B_1). \tag{31}$$

Since $U_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$, from (31) and Lemma 1, we can derive that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle &= \lim_{k\to\infty}\langle f(x_2^*) - x_1^*, U_{n_k} - x_1^*\rangle \\ &= \langle f(x_2^*) - x_1^*, \hat{x} - x_1^*\rangle \\ &\leq 0. \end{aligned} \tag{32}$$

Following the same method as (32), we obtain that

$$\limsup_{n\to\infty}\langle g(x_1^*) - x_2^*, V_n - x_2^*\rangle \leq 0. \tag{33}$$

Step 5. Finally, Prove that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$ and $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, respectively.

By firmly-nonexpansiveness of P_C, we derive that

$$\begin{aligned}\|U_n - x_1^*\|^2 &= \|P_C\tilde{u}_n - x_1^*\|^2 \\ &\leq \langle \tilde{u}_n - x_1^*, U_n - x_1^*\rangle \\ &= \langle \alpha_n(f(w_n) - x_1^*) + (1-\alpha_n)(M^1x_n - x_1^*), U_n - x_1^*\rangle \\ &= \alpha_n\langle f(w_n) - x_1^*, U_n - x_1^*\rangle + (1-\alpha_n)\langle M^1x_n - x_1^*, U_n - x_1^*\rangle \\ &= \alpha_n\langle f(w_n) - f(x_2^*), U_n - x_1^*\rangle + \alpha_n\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\ &\quad + (1-\alpha_n)\|M^1x_n - x_1^*\|\|U_n - x_1^*\| \\ &\leq \alpha_n a\|w_n - x_2^*\|\|U_n - x_1^*\| + \alpha_n\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\ &\quad + (1-\alpha_n)\|x_n - x_1^*\|\|U_n - x_1^*\| \\ &\leq \frac{\alpha_n a}{2}\left\{\|w_n - x_2^*\|^2 + \|U_n - x_1^*\|^2\right\} + \alpha_n\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\ &\quad + \frac{(1-\alpha_n)}{2}\left\{\|x_n - x_1^*\|^2 + \|U_n - x_1^*\|^2\right\} \\ &= \frac{\alpha_n a}{2}\|w_n - x_2^*\|^2 + \frac{(1-\alpha_n)}{2}\|x_n - x_1^*\|^2 \\ &\quad + \left(\frac{1-\alpha_n(1-a)}{2}\right)\|U_n - x_1^*\|^2 \\ &\quad + \alpha_n\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle, \end{aligned}$$

which hence yields

$$\begin{aligned}\|U_n - x_1^*\|^2 &\leq \frac{\alpha_n a}{1+\alpha_n(1-a)}\|w_n - x_2^*\|^2 + \frac{(1-\alpha_n)}{1+\alpha_n(1-a)}\|x_n - x_1^*\|^2 \\ &\quad + \frac{\alpha_n}{1+\alpha_n(1-a)}\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle. \end{aligned} \tag{34}$$

From the definition of x_n and (34), we get

$$\begin{aligned}\|x_{n+1} - x_1^*\|^2 &\leq \delta_n\|x_n - x_1^*\|^2 + \eta_n\|P_C(I - \gamma_1 B_1)x_n - x_1^*\|^2 + \mu_n\|U_n - x_1^*\|^2 \\ &\leq (1-\mu_n)\|x_n - x_1^*\|^2 + \frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\|w_n - x_2^*\|^2 \\ &\quad + \frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\ &\quad + \frac{\mu_n(1-\alpha_n)}{1+\alpha_n(1-a)}\|x_n - x_1^*\|^2 \end{aligned}$$

$$= \left(1 - \frac{\mu_n \alpha_n (2-a)}{1+\alpha_n(1-a)}\right) \|x_n - x_1^*\|^2 + \frac{\mu_n \alpha_n a}{1+\alpha_n(1-a)} \|w_n - x_2^*\|^2 + \frac{\mu_n \alpha_n}{1+\alpha_n(1-a)} \langle f(x_2^*) - x_1^*, U_n - x_1^* \rangle. \tag{35}$$

Similarly, as derived above,we also have

$$\|w_{n+1} - x_2^*\|^2 \leq \left(1 - \frac{\mu_n \alpha_n (2-a)}{1+\alpha_n(1-a)}\right) \|w_n - x_2^*\|^2 + \frac{\mu_n \alpha_n a}{1+\alpha_n(1-a)} \|x_n - x_1^*\|^2 + \frac{\mu_n \alpha_n}{1+\alpha_n(1-a)} \langle g(x_1^*) - x_2^*, V_n - x_2^* \rangle. \tag{36}$$

From (35) and (36), we deduce that

$$\begin{aligned}
&\|x_{n+1} - x_1^*\|^2 + \|w_{n+1} - x_2^*\|^2 \\
&\leq \left(1 - \frac{\mu_n \alpha_n (2-a)}{1+\alpha_n(1-a)}\right) \left(\|x_n - x_1^*\|^2 + \|w_n - x_2^*\|^2\right) \\
&\quad + \frac{\mu_n \alpha_n a}{1+\alpha_n(1-a)} \left(\|x_n - x_1^*\|^2 + \|w_n - x_2^*\|^2\right) \\
&\quad + \frac{\mu_n \alpha_n}{1+\alpha_n(1-a)} \left(\langle f(x_2^*) - x_1^*, U_n - x_1^* \rangle + \langle g(x_1^*) - x_2^*, V_n - x_2^* \rangle\right) \\
&= \left(1 - \frac{\mu_n \alpha_n (2-a)}{1+\alpha_n(1-a)} + \frac{\mu_n \alpha_n a}{1+\alpha_n(1-a)}\right) \left(\|x_n - x_1^*\|^2 + \|w_n - x_2^*\|^2\right) \\
&\quad + \frac{\mu_n \alpha_n}{1+\alpha_n(1-a)} \left(\langle f(x_2^*) - x_1^*, U_n - x_1^* \rangle + \langle g(x_1^*) - x_2^*, V_n - x_2^* \rangle\right) \\
&= \left(1 - \frac{2\mu_n \alpha_n (1-a)}{1+\alpha_n(1-a)}\right) \left\{\|x_n - x_1^*\|^2 + \|w_n - x_2^*\|^2\right\} \\
&\quad + \frac{\mu_n \alpha_n}{1+\alpha_n(1-a)} \left(\langle f(x_2^*) - x_1^*, U_n - x_1^* \rangle + \langle g(x_1^*) - x_2^*, V_n - x_2^* \rangle\right).
\end{aligned}$$

Applying the condition (ii), (32), (33), and Lemma 2, we can conclude that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$ and $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, respectively. From Lemma 5, we have (x_1^*, y_1^*, z_1^*) is an element of $\Psi^{D_1^1, D_2^1, D_3^1}_{\bar{D}_1^1, \bar{D}_2^1, \bar{D}_3^1}$. and (x_2^*, y_2^*, z_2^*) is an element of $\Psi^{D_1^2, D_2^2, D_3^2}_{\bar{D}_1^2, \bar{D}_2^2, \bar{D}_3^2}$. This completes the proof. □

As direct consequences of Theorem 4, we can obtain the following corollary.

Corollary 1 *Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subset of a real Hilbert space H_1 and H_2, respectively. For every $i = 1, 2$, let $T_i : C \to C$ be a κ_i-strictly pseudo-contractive mapping with $Fix(T_i) \neq \emptyset$, and let $D_1^i, D_2^i, D_3^i : C \to H_1$ be d_1^i, d_2^i, d_3^i-inverse strongly monotone, respectively, with $d^* = \min\{d_1^i, d_2^i, d_3^i\}$. Let $\bar{D_1}^i, \bar{D_2}^i, \bar{D_3}^i : Q \to H_2$ be $\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i$-inverse strongly monotone, respectively, with $\hat{d} = \min\{\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Let $f, g : H \to H$ be a_f and a_g-contraction mapping with $a = \max\{a_f, a_g\}$. For every $i = 1, 2$ define $M_C^i : C \to C$ by $M_C^i(x) = P_C(I - \zeta^i D_1^i)(ax + (1-a)P_C(I - \zeta^i D_2^i)(ax + (1-a)P_C(I - \zeta^i D_3^i)x))$, $\forall x \in C$, where $\zeta^i \in (0, 2d^*)$ and define $M_Q^i : Q \to Q$ by $M_Q^i(\hat{x}) = P_Q(I - \bar{\zeta}^i \bar{D}_1^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i \bar{D}_2^i)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}^i \bar{D}_3^i)\hat{x}))$, $\forall \hat{x} \in Q$, where $\bar{\zeta}^i \in (0, 2\hat{d})$ Define*

$M^i : C \to C$ by $M^i(x^*) = M_C^i(x^* - \eta A^*(I - M_Q^i)Ax^*)$ for all $x^* \in C$, and $i = 1, 2$. Let the sequence $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1(I - T_1))x_n + \mu_n P_C(\alpha_n f(w_n) + (1 - \alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2(I - T_2))w_n + \mu_n P_C(\alpha_n g(x_n) + (1 - \alpha_n)M^2 w_n) \end{cases} \tag{37}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0, 1]$ with $\delta_n + \eta_n + \mu_n = 1$, $\gamma \in (0, 2\alpha)$ with $\alpha = min\{\frac{1-\kappa_1}{2}, \frac{1-\kappa_2}{2}\}$ and $\gamma = min\{\gamma_1, \gamma_2\}$. Assume the following condition holds;

- (i) $\mathcal{F}_i = F(M^i) \cap F(T_i) \neq \emptyset$ *for all* $i = 1, 2$,
- (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,
- (iii) $0 < \bar{\theta} \le \delta_n, \eta_n, \mu_n \le \theta$ *for all* $n \in N$ *and for some* $\bar{\theta}, \theta > 0$,
- (iv) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ converges strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, where $y_1^* = P_C(I - \zeta^1 D_2^1)(ax_1^* + (1 - a)z_1^*)$ and $z_1^* = P_C(I - \zeta^1 D_3^1)x_1^*$ with (x_1^*, y_1^*, z_1^*) is an element of $\Psi^{D_1^1, D_2^1, D_3^1}_{\bar{D}_1^1, \bar{D}_2^1, \bar{D}_3^1}$. and $\{w_n\}$ converges strongly to $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, where $y_2^* = P_C(I - \zeta^2 D_2^2)(ax_2^* + (1 - a)z_2^*)$ and $z_1^* = P_C(I - \zeta^2 D_3^2)x_1^*$ with (x_2^*, y_2^*, z_2^*) is an element of $\Psi^{D_1^2, D_2^2, D_3^2}_{\bar{D}_1^2, \bar{D}_2^2, \bar{D}_3^2}$.

Proof From Theorem 4 and Lemma 6, we have the desired conclusion. □

4 Application

In this section, applying our main result Theorem 4, we prove strong convergence theorems for approximating the solution of the split variational inequality and the split minimization problem.

4.1 The split variational inequality

In 2012, Censor et al. introduced SVIP, which is to find $\hat{x} \in C$ such that

$$\langle f_1\hat{x}, x - \hat{x}\rangle \ge 0, \quad \forall x \in C,$$

and find $\hat{y} = D\hat{x} \in Q$ such that

$$\langle f_2\hat{y}, x - \hat{y}\rangle \ge 0, \quad \forall y \in Q,$$

where $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings and $D : H_1 \to H_2$ is a bounded linear operator. The set of all solution of the SVIP is denoted by

$$\Phi = \{\hat{x} \in VI(C, f_1) : \hat{y} \in VI(Q, f_2)\}. \tag{38}$$

The SVIP is reduced to the SFP if $f_1 \equiv f_2 \equiv 0$.

Before we prove the theorems, we need the following lemmas.

Lemma 7 *In a strictly convex Banach space E, if*

$$\|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\|,$$

for all $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$.

Lemma 8 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3-inverse strongly monotone respectively, where $\zeta \in (0, 2d^*)$ with $d^* = \min\{d_1, d_2, d_3\}$. Let $\bar{D_1}, \bar{D_2}, \bar{D_3} : Q \to H_2$ be $\bar{d_1}, \bar{d_2}, \bar{d_3}$-inverse strongly monotone respectively, where $\bar{\zeta} \in (0, 2\hat{d})$ with $\hat{d} = \min\{\bar{d_1}, \bar{d_2}, \bar{d_3}\}$. Defined the mapping M_C, M_Q as in Lemma 5. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Let $\bigcap_{i=1}^{3} \Phi_i \neq \emptyset$ and $\Phi_i = \{w \in VI(C, D_i) | Aw \in VI(Q, \bar{D_i})\}$, for all $i = 1, 2, 3$. Then*

$$\bigcap_{i=1}^{3} \Phi_i = F(M_C(I - \eta A^*(I - M_Q)A)).$$

Proof First, we will prove that for every $i = 1, 2, 3$ if $T_i : C \to C$ is a nonexpansive mapping, $\bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and define the mapping $J : C \to C$ by

$$J(x) = T_1(ax + (1-a)T_2(ax + (1-a)T_3x))$$

Then $\bigcap_{i=1}^{3} F(T_i) = F(J)$.

It easy to see that $\bigcap_{i=1}^{3} F(T_i) \subseteq F(J)$.

Next, we claim that $F(J) \subseteq \bigcap_{i=1}^{3} F(T_i)$. To show this, let $x \in F(J)$ and $x^* \in \bigcap_{i=1}^{3} F(T_i)$.

By the definition of J, we get

$$\begin{aligned}
\|x - x^*\| &= \|T_1(ax + (1-a)T_2(ax + (1-a)T_3x)) - x^*\| \\
&\leq \|a(x - x^*) + (1-a)[T_2(ax + (1-a)T_3x) - x^*]\| \\
&\leq a\|x - x^*\| + (1-a)\|ax + (1-a)T_3x - x^*\| \\
&= a\|x - x^*\| + (1-a)\|a(x - x^*) + (1-a)(T_3x - x^*)\| \\
&\leq a\|x - x^*\| + a(1-a)\|x - x^*\| + (1-a)^2\|T_3x - x^*\| \\
&= (2a - a^2)\|x - x^*\| + (1-a)^2\|T_3x - x^*\| \\
&\leq (2a - a^2)\|x - x^*\| + (1-a)^2\|x - x^*\| \\
&= \|x - x^*\|,
\end{aligned}$$

which implies that $\|x - x^*\| = \|T_3x - x^*\| = \|a(x - x^*) + (1-a)(T_3x - x^*)\|$. From Lemma 7, we have $T_3x = x$, that is $x \in F(T_3)$.
By the definition of J and $x \in F(T_3)$, we have

$$\begin{aligned}
\|x - x^*\| &= \|T_1(ax + (1-a)T_2(ax + (1-a)x)) - x^*\| \\
&\leq \|ax + (1-a)T_2x - x^*\| \\
&= \|a(x - x^*) + (1-a)(T_2x - x^*)\| \\
&\leq a\|x - x^*\| + (1-a)\|T_2x - x^*\| \\
&\leq a\|x - x^*\| + (1-a)\|x - x^*\| \\
&= \|x - x^*\|,
\end{aligned}$$

which implies that $\|x - x^*\| = \|T_2x - x^*\| = a\|x - x^*\| + (1-a)\|T_2x - x^*\|$. From Lemma 7, we have $T_2x = x$, that is $x \in F(T_2)$.

By the definition of J, $x \in F(T_3)$ and $x \in F(T_2)$, we have

$$\begin{aligned}
x &= T_1(ax + (1-a)T_2(ax + (1-a)T_3x)) \\
&= T_1x
\end{aligned}$$

which implies that $T_1 x = x$, that is $x \in F(T_1)$. Therefore

$$x \in \bigcap_{i=1}^{3} F(T_i).$$

Then, we have

$$F(J) \subseteq \bigcap_{i=1}^{3} F(T_i).$$

Hence

$$F(J) = \bigcap_{i=1}^{3} F(T_i). \tag{39}$$

Next, we claim that $\bigcap_{i=1}^{3} \Phi_i \subseteq F(M_C(I - \eta A^*(I - M_Q)A))$. To show this, let $x \in \bigcap_{i=1}^{3} \Phi_i$ then $x \in \Phi_i$, for all $i = 1, 2, 3$. Thus $x \in VI(C, D_i)$ and $Ax \in VI(Q, \bar{D}_i)$, for all $i = 1, 2, 3$ and so $x \in F(P_C(I - \zeta D_i))$ and $Ax \in F(P_Q(I - \bar{\zeta} \bar{D}_i))$, for all $i = 1, 2, 3$.

Then, we have $x \in \bigcap_{i=1}^{3} F(P_C(I - \zeta D_i))$ and $Ax \in \bigcap_{i=1}^{3} F(P_Q(I - \bar{\zeta} \bar{D}_i))$.

Utilizing the same method as (39), we have

$$x \in F(M_C),$$

that is $x = M_C x = P_C(I - \zeta D_1)(ax + (1-a)P_C(I - \zeta D_2)(ax + (1-a)P_C(I - \zeta D_3)x))$, where $y = P_C(I - \zeta D_2)(ax + (1-a)z)$, $z = P_C(I - \zeta D_3)x$ and

$$\bar{x} \in F(M_Q),$$

that is $Ax = M_Q Ax = P_Q(I - \bar{\zeta} \bar{D}_1)(aAx + (1-a)P_Q(I - \bar{\zeta} \bar{D}_2)(aAx + (1-a)P_Q(I - \bar{\zeta} \bar{D}_3)Ax))$, where $\bar{y} = P_Q(I - \zeta D_2)(a\bar{x} + (1-a)\bar{z})$, $\bar{z} = P_Q(I - \zeta D_3)\bar{x}$, $\bar{x} = Ax$, $\bar{y} = Ay$ and $\bar{z} = Az$.

From Lemma 3, we have $(x, y, z) \in \Psi_{D_1, D_2, D_3}$ and $(\bar{x}, \bar{y}, \bar{z}) \in \Psi_{\bar{D}_1, \bar{D}_2, \bar{D}_3}$. Thus

$$(x, y, z) \in \Psi_{\bar{D}_1, \bar{D}_2, \bar{D}_3}^{D_1, D_2, D_3}.$$

From Lemma 5, we have

$$x = M_C(I - \eta A^*(I - M_Q)A)x$$

where $y = P_C(I - \zeta D_2)(ax + (1-a)z)$, $z = P_C(I - \zeta D_3)x$, $\bar{x} = Ax$, $\bar{y} = Ay$ and $\bar{z} = Az$. Thus

$$x \in F(M_C(I - \eta A^*(I - M_Q)A)).$$

Then, we have

$$\bigcap_{i=1}^{3} \Phi_i \subseteq F(M_C(I - \eta A^*(I - M_Q)A))$$

Next, we claim that $F(M_C(I - \eta A^*(I - M_Q)A)) \subseteq \bigcap_{i=1}^{3} \Phi_i$. To show this, let $x \in F(M_C(I - \eta A^*(I - M_Q)A))$, then $x = M_C(I - \eta A^*(I - M_Q)A)x$.

From Lemma 5, we have

$$(x, y, z) \in \Psi_{\bar{D}_1, \bar{D}_2, \bar{D}_3}^{D_1, D_2, D_3},$$

$y = P_C(I-\zeta D_2)(ax+(1-a)z)$, $z = P_C(I-\zeta D_3)x$ and $\bar{y} = Ay = P_Q(I-\zeta D_2)(a\bar{x} + (1-a)\bar{z})$, $\bar{z} = Az = P_Q(I-\zeta D_3)\bar{x}$, $\bar{x} = Ax$. Then $(x,y,z) \in \Psi_{D_1,D_2,D_3}$ and $(\bar{x},\bar{y},\bar{z}) \in \Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}$.

From Lemma 3, we have

$$x \in F(M_C),$$

and

$$\bar{x} \in F(M_Q),$$

that is $x = M_C x = P_C(I-\zeta D_1)(ax+(1-a)P_C(I-\zeta D_2)(ax+(1-a)P_C(I-\zeta D_3)x))$ and $Ax = M_Q Ax = P_Q(I-\bar{\zeta}\bar{D_1})(aAx+(1-a)P_Q(I-\bar{\zeta}\bar{D_2})(aAx+(1-a)P_Q(I-\bar{\zeta}\bar{D_3})Ax))$.

Utilizing the same method as (39), we have $x \in \bigcap_{i=1}^3 F(P_C(I-\zeta)D_i)$ and $Ax \in \bigcap_{i=1}^3 F(P_Q(I-\bar{\zeta})\bar{D}_i)$.

Then $x \in F(P_C(I-\zeta)D_i)$ and $Ax \in F(P_Q(I-\bar{\zeta})\bar{D}_i)$, for all $i = 1,2,3$. and so $x \in VI(C, D_i)$ and $Ax \in VI(Q, \bar{D}_i)$, for all $i = 1,2,3$ That is $x \in \bigcap_{i=1}^3 \Phi_i$, then $x \in \Phi_i$, for all $i = 1,2,3$.

Thus

$$F(M_C(I-\eta A^*(I-M_Q)A)) \subseteq \bigcap_{i=1}^{3} \Phi_i.$$

Hence

$$\bigcap_{i=1}^{3} \Phi_i = F(M_C(I-\eta A^*(I-M_Q)A)).$$

□

Theorem 5 *Let* H_1 *and* H_2 *be real Hilbert spaces and let* C, Q *be nonempty closed convex subset of a real Hilbert space* H_1 *and* H_2*, respectively. For every* $i = 1,2$*, let* $B_i : C \to H$ *be* α_i*-inverse strongly monotone mapping with* $\alpha = min\{\alpha_1, \alpha_2\}$*. For every* $i = 1,2$*, let* $D_1^i, D_2^i, D_3^i : C \to H_1$ *be* d_1^i, d_2^i, d_3^i*-inverse strongly monotone respectively, where* $\zeta^i \in (0, 2d^*)$ *with* $d^* = min\{d_1^i, d_2^i, d_3^i\}$ *and let* $\bar{D_1}^i, \bar{D_2}^i, \bar{D_3}^i : Q \to H_2$ *be* $\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i$*-inverse strongly monotone respectively, where* $\bar{\zeta}^i \in (0, 2\hat{d})$ *with* $\hat{d} = min\{\bar{d_1}^i, \bar{d_2}^i, \bar{d_3}^i\}$*. Let* $A : H_1 \to H_2$ *be a bounded linear operator with adjoint* A^* *and* $\eta \in (0, \frac{1}{L})$ *with* L *being the spectral radius of the operator* A^*A*. let* $f, g : H \to H$ *be* a_f *and* a_g*-contraction mapping with* $a = \max\{a_f, a_g\}$*. For every* $i = 1,2$ *define* $M_C^i : C \to C$ *by* $M_C^i(x) = P_C(I-\zeta^i D_1^i)(ax+(1-a)P_C(I-\zeta^i D_2^i)(ax+(1-a)P_C(I-\zeta^i D_3^i)x))$, $\forall x \in C$*, and define* $M_Q^i : Q \to Q$ *by* $M_Q^i(\hat{x}) = P_Q(I-\bar{\zeta^i}\bar{D_1^i})(a\hat{x}+(1-a)P_Q(I-\bar{\zeta^i}\bar{D_2^i})(a\hat{x} + (1-a)P_Q(I-\bar{\zeta^i}\bar{D_3^i})\hat{x}))$, $\forall \hat{x} \in Q$*. For every* $i = 1,2$ *define* $M^i : C \to C$ *by* $M^i(x^*) = M_C^i(x^* - \eta A^*(I-M_Q^i)Ax^*)$ *for all* $x^* \in C$*. Let the sequence* $\{x_n\}$ *and* $\{w_n\}$ *generated by* $x_1, w_1 \in C$ *and*

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I-\gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I-\gamma_2 B_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)M^2 w_n) \end{cases} \tag{40}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$ *with* $\delta_n + \eta_n + \mu_n = 1$ *and* $\gamma \in (0, 2\alpha)$ *with* $\gamma = min\{\gamma_1, \gamma_2\}$*. Assume the following condition holds;*

(i) $\mathcal{F}_i = \bigcap_{j=1}^{3} \Phi_j^i \cap VI(C, B_i) \neq \emptyset$ *for all* $i = 1, 2$ *and* $\Phi_j^i = \{z \in VI(C, D_j^i) : Az \in VI(C, \bar{D}_j^i)\}$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ *for all* $n \in N$ *and for some* $\bar{\theta}, \theta > 0$,

(iv) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, *where* $y_1^* = P_C(I - \zeta^1 D_2^1)(ax_1^* + (1 - a)z_1^*)$ *and* $z_1^* = P_C(I - \zeta^1 D_3^1)x_1^*$ *with* (x_1^*, y_1^*, z_1^*) *is an element of* $\Psi_{\bar{D}_1^1, \bar{D}_2^1, \bar{D}_3^1}^{D_1^1, D_2^1, D_3^1}$. *and* $\{w_n\}$ *converges strongly to* $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, *where* $y_2^* = P_C(I - \zeta^2 D_2^2)(ax_2^* + (1 - a)z_2^*)$ *and* $z_1^* = P_C(I - \zeta^2 D_3^2)x_1^*$ *with* (x_2^*, y_2^*, z_2^*) *is an element of* $\Psi_{\bar{D}_1^2, \bar{D}_2^2, \bar{D}_3^2}^{D_1^2, D_2^2, D_3^2}$.

Proof By using Theorem 4 and Lemma 8 , we obtain the conclusion. □

4.2 The split minimization problem

Let C be a closed convex subset of H. The standard constrained convex optimization problem is to find $x^* \in C$ such that

$$\Im(x^*) = \min_{x \in C} \Im(x), \tag{41}$$

where $\Im : C \to \mathbb{R}$ is a convex, Fr'echet differentiable function. The set of all solution of (41) is denoted by $\Phi_{\Im}$.

Lemma 9 [35] (Optimality condition) *A necessary condition of Optimality for a point* $x^* \in C$ *to be a solution of the minimization problem* (41) *is that* x^* *solves the variational inequality*

$$\langle \nabla \Im(x^*), x - x^* \rangle \geq 0, \tag{42}$$

for all $x \in C$. *Equivalently,* $x^* \in C$ *solves the fixed point equation*

$$x^* = P_C(I - \zeta \nabla \Im)x^*,$$

for every $\zeta > 0$. *If, in addition,* $\Im$ *is convex, then the optimality condition* (42) *is also sufficient.*

By using the concept of the split of modified variational inequalities problem (SMVIP), we consider the problem for finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \zeta \nabla \Im_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \zeta \nabla \Im_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \zeta \nabla \Im_3)x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{43}$$

and find $(\bar{x^*} = Ax^*, \bar{y^*} = Ay^*, \bar{z^*} = Az^*) \in Q \times Q \times Q$ such that

$$\begin{cases} \langle \bar{x^*} - (I - \bar{\zeta} \nabla \bar{\Im}_1)(a\bar{x^*} + (1-a)\bar{y^*}), \bar{x} - \bar{x^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{y^*} - (I - \bar{\zeta} \nabla \bar{\Im}_2)(a\bar{x^*} + (1-a)\bar{z^*}), \bar{x} - \bar{y^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{z^*} - (I - \bar{\zeta} \nabla \bar{\Im}_3)\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, & \forall \bar{x} \in Q, \end{cases} \tag{44}$$

where $\Im_1, \Im_2, \Im_3 : C \to \mathbb{R}$ with $\nabla \Im_1, \nabla \Im_2, \nabla \Im_3$ are the gradient of $\Im_1, \Im_2, \Im_3$, respectively and $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3 : Q \to \mathbb{R}$ with $\nabla \bar{\Im}_1, \nabla \bar{\Im}_2, \nabla \bar{\Im}_3$ are the gradient of $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3$, respectively,

$\zeta, \bar{\zeta} > 0$ and $a \in [0, 1]$. The set of all solution of (43) and (44) are denoted by $\Psi_{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3}$ and $\Psi_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3}$ respectively. The set of all solution of the split of modified variational inequalities (SMVIP) is denoted by $\Psi^{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3}_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3}$ that is

$$\Psi^{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3}_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3} = \{(x^*, y^*, z^*) \in \Psi_{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3} : (\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3}\}$$

Before we prove the theorems, we need the following lemma.

Lemma 10 *Let C, Q be nonempty closed convex subsets of H_1, H_2, respectively. For every $i = 1, 2$ let $\Im^i_1, \Im^i_2, \Im^i_3 : C \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\Im^i_1, \nabla\Im^i_2, \nabla\Im^i_3$ be $\frac{1}{L_{\Im^i_1}}, \frac{1}{L_{\Im^i_2}}, \frac{1}{L_{\Im^i_3}}$-inverse strongly monotone and continuous respectively, where $\zeta^i \in (0, \frac{2}{L_{\Im}})$ with $\frac{1}{L_{\Im}} = \min\{\frac{1}{L_{\Im^i_1}}, \frac{1}{L_{\Im^i_2}}, \frac{1}{L_{\Im^i_3}}\}$. Let $\bar{\Im}^i_1, \bar{\Im}^i_2, \bar{\Im}^i_3 : Q \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\bar{\Im}^i_1, \nabla\bar{\Im}^i_2, \nabla\bar{\Im}^i_3$ be $\frac{1}{L_{\bar{\Im}^i_1}}, \frac{1}{L_{\bar{\Im}^i_2}}, \frac{1}{L_{\bar{\Im}^i_3}}$-inverse strongly monotone and continuous respectively, where $\bar{\zeta}^i \in (0, \frac{2}{L_{\bar{\Im}}})$ with $\frac{1}{L_{\bar{\Im}}} = min\{\frac{1}{L_{\bar{\Im}^i_1}}, \frac{1}{L_{\bar{\Im}^i_2}}, \frac{1}{L_{\bar{\Im}^i_3}}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C : C \to C$ by $M_C(x) = P_C(I - \zeta\nabla\Im_1)(ax + (1-a)P_C(I - \zeta\nabla\Im_2)(ax + (1-a)P_C(I - \zeta\nabla\Im_3)x)), \forall x \in C$, and define $M_Q : Q \to Q$ by $M_Q(\hat{x}) = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_1)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\nabla\bar{\Im}_2)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\nabla\bar{\Im}_3)\hat{x})), \forall \hat{x} \in Q$. Assume $\Psi^{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3}_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3} = \{(x^*, y^*, z^*) \in \Psi_{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3} : (\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3}\} \neq \emptyset$. The following statements are equivalent*

- (i) $(x^*, y^*, z^*) \in \Psi^{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3}_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3}$,
- (ii) $x^* = M_C(x^* - \eta A^*(I - M_Q)Ax^*)$,

where $y^* = P_C(I - \zeta\nabla\Im_2)(ax^* + (1-a)z^*)$, $z^* = P_C(I - \zeta\nabla\Im_3)x^*$, $\bar{x^*} = Ax^* = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_1)(a\bar{x^*} + (1-a)\bar{y^*})$, $\bar{y^*} = Ay^* = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_2)(a\bar{x^*} + (1-a)\bar{z^*})$ and $\bar{z^*} = Az^* = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_3)\bar{x^*}$.

Proof By using Lemma 5 , we have the desired conclusion. □

Example 2 Let $\mathbb{R}$ be the set of real numbers, $H_1 = H_2 = \mathbb{R}^2$, $C = [-50, 50] \times [-50, 50]$ and $Q = [-100, 100] \times [-100, 100]$. For every $i = 1, 2, 3$, let $\Im_i : C \to \mathbb{R}$ be defined by $\Im_1(x_1, x_2) = (\frac{x_1^2 - 4x}{6}, \frac{x_2^2 - 4x}{6})$, $\Im_2(x_1, x_2) = (\frac{x_1^2 - 4x}{10}, \frac{x_2^2 - 4x}{10})$, $\Im_2(x_1, x_2) = (\frac{x_1^2 - 4x}{14}, \frac{x_2^2 - 4x}{14})$, with $\nabla\Im_1(x_1, x_2) = (\frac{x_1 - 2}{3}, \frac{x_2 - 2}{3})$, $\nabla\Im_2(x_1, x_2) = (\frac{x_1 - 2}{5}, \frac{x_2 - 2}{5})$, $\nabla\Im_3(x_1, x_2) = (\frac{x_1 - 2}{7}, \frac{x_2 - 2}{7})$, for all $(x_1, x_2) \in C$ and let $\bar{\Im}_i : Q \to \mathbb{R}$ be defined by $\bar{\Im}_1(x_1, x_2) = (\frac{x_1^2 - 8x}{10}, \frac{x_2^2 - 8x}{10})$, $\bar{\Im}_2(x_1, x_2) = (\frac{x_1^2 - 8x}{12}, \frac{x_2^2 - 8x}{12})$, $\bar{\Im}_2(x_1, x_2) = (\frac{x_1^2 - 8x}{14}, \frac{x_2^2 - 8x}{14})$, with $\nabla\bar{\Im}_1(x_1, x_2) = (\frac{x_1 - 4}{5}, \frac{x_2 - 4}{5})$, $\nabla\bar{\Im}_2(x_1, x_2) = (\frac{x_1 - 4}{6}, \frac{x_2 - 4}{6})$, $\nabla\bar{\Im}_3(x_1, x_2) = (\frac{x_1 - 4}{7}, \frac{x_2 - 4}{7})$, for all $(x_1, x_2) \in Q$. Let $A : H_1 \to H_2$ be defined by $A(x_1, x_2) = (2x_1, 2x_2)$, for all $(x_1, x_2) \in H_1$ and let $A^* : H_2 \to H_1$ be defined by $A^*(\bar{x_1}, \bar{x_2}) = (2\bar{x_1}, 2\bar{x_2})$. For every $(\bar{x_1}, \bar{x_2}) \in H_2$. Define $M_C : [-50, 50] \times [-50, 50] \to [-50, 50] \times [-50, 50]$ by $M_C(x_1, x_2) = P_C(I - \frac{1}{3}\nabla\Im_1)(\frac{1}{2}(x_1, x_2) + \frac{1}{2}P_C(I - \frac{1}{3}\nabla\Im_2)(\frac{1}{2}(x_1, x_2) + \frac{1}{2}P_C(I - \frac{1}{3}\nabla\Im_3)(x_1, x_2)))$, $\forall (x_1, x_2) \in C$, and define $M_Q : [-100, 100] \times [-100, 100] \to [-100, 100] \times [-100, 100]$ by $M_Q(\bar{x_1}, \bar{x_2}) = P_Q(I - \frac{1}{5}\nabla\bar{\Im}_1)(\frac{1}{2}(\bar{x_1}, \bar{x_2}) + \frac{1}{2}P_Q(I - \frac{1}{5}\nabla\bar{\Im}_2)(\frac{1}{2}(\bar{x_1}, \bar{x_2}) + \frac{1}{2}P_Q(I - \frac{1}{5}\nabla\bar{\Im}_3)(\bar{x_1}, \bar{x_2}))$, $\forall (\bar{x_1}, \bar{x_2}) \in Q$.

Then, we have $(x^*, y^*, z^*) \in \Psi^{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3}_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3}$, where $x^*, y^*, z^* = (2, 2) \in [-50, 50] \times [-50, 50]$. By Lemma 10, we have $(2, 2) = M_C((2, 2) - \frac{1}{5}A^*(I - M_Q)A(2, 2))$.

Lemma 11 *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2$ let $\Im_1^i, \Im_2^i, \Im_3^i : C \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\Im_1^i, \nabla\Im_2^i, \nabla\Im_3^i$ be $\frac{1}{L_{\Im_1^i}}, \frac{1}{L_{\Im_2^i}}, \frac{1}{L_{\Im_3^i}}$-inverse strongly monotone and continuous respectively, where $\zeta^i \in (0, \frac{2}{L_\Im})$ with $\frac{1}{L_\Im} = min\{\frac{1}{L_{\Im_1^i}}, \frac{1}{L_{\Im_2^i}}, \frac{1}{L_{\Im_3^i}}\}$. Let $\bar{\Im}_1^i, \bar{\Im}_2^i, \bar{\Im}_3^i : Q \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\bar{\Im}_1^i, \nabla\bar{\Im}_2^i, \nabla\bar{\Im}_3^i$ be $\frac{1}{L_{\bar{\Im}_1^i}}, \frac{1}{L_{\bar{\Im}_2^i}}, \frac{1}{L_{\bar{\Im}_3^i}}$-inverse strongly monotone and continuous respectively, where $\bar{\zeta}^i \in (0, \frac{2}{L_{\bar{\Im}}})$ with $\frac{1}{L_{\bar{\Im}}} = min\{\frac{1}{L_{\bar{\Im}_1^i}}, \frac{1}{L_{\bar{\Im}_2^i}}, \frac{1}{L_{\bar{\Im}_3^i}}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Defined the mapping M_C, M_Q as in Lemma 10. Let $\bigcap_{i=1}^3 \Phi_{\Im_i} \neq \emptyset$ and $\Phi_{\Im_i} = \{\Im_i(x) = \min_{x^* \in C} \Im_i(x^*) : \bar{\Im}_i(Ax) = \min_{Ax^* \in Q} \bar{\Im}_i(Ax^*)\}$, for all $i = 1, 2, 3$. then*

$$\bigcap_{i=1}^{3} \Phi_{\Im_i} = F(M_C(I - \eta A^*(I - M_Q)A)).$$

Proof We claim that $\bigcap_{i=1}^3 \Phi_{\Im_i} \subseteq F(M_C(I - \eta A^*(I - M_Q)A))$. To show this, let $x \in \bigcap_{i=1}^3 \Phi_{\Im_i}$ then $x \in \Phi_{\Im_i}$, for all $i = 1, 2, 3$. It implies that $\Im_i(x) = \min_{x^* \in C} \Im_i(x^*)$ and $\bar{\Im}_i(Ax) = \min_{Ax^* \in Q} \bar{\Im}_i(Ax^*)$, for all $i = 1, 2, 3$. By Lemma 9, we have $x \in F(P_C(I - \zeta\nabla\Im_i))$ and $Ax \in F(P_Q(I - \bar{\zeta}\nabla\bar{\Im}_i))$, for all $i = 1, 2, 3$.

Then, we have $x \in \bigcap_{i=1}^3 F(P_C(I - \zeta\nabla\Im_i))$ and $Ax \in \bigcap_{i=1}^3 F(P_Q(I - \bar{\zeta}\nabla\bar{\Im}_i))$.

Utilizing the same method as (39), we have

$$x \in F(M_C),$$

that is

$$x = M_C x = P_C(I - \zeta\nabla\Im_1)(ax + (1 - a)y),$$

where $y = P_C(I - \zeta\nabla\Im_2)(ax + (1 - a)z)$, $z = P_C(I - \zeta\nabla\Im_3)x$ and

$$Ax \in F(M_Q),$$

that is

$$Ax = M_Q Ax = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_1)(a\bar{x} + (1 - a)\bar{y}),$$

where $\bar{y} = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_2)(a\bar{x} + (1 - a)\bar{z})$, $\bar{z} = P_Q(I - \bar{\zeta}\nabla\bar{\Im}_3)\bar{x}$ with $\bar{x} = Ax$, $\bar{y} = Ay$ and $\bar{z} = Az$. From Lemma 3, we have $(x, y, z) \in \Psi_{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3}$ and $(\bar{x}, \bar{y}, \bar{z}) \in \Psi_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3}$. Thus $(x, y, z) \in \Psi^{\nabla\Im_1, \nabla\Im_2, \nabla\Im_3}_{\nabla\bar{\Im}_1, \nabla\bar{\Im}_2, \nabla\bar{\Im}_3}$.

From Lemma 10, we have $x = M_C(I - \eta A^*(I - M_Q)A)x$.

Thus

$$x \in F(M_C(I - \eta A^*(I - M_Q)A)).$$

Then, we have

$$\bigcap_{i=1}^{3} \Phi_{\Im_i} \subseteq F(M_C(I - \eta A^*(I - M_Q)A)).$$

Next, we claim that $F(M_C(I-\eta A^*(I-M_Q)A)) \subseteq \bigcap_{i=1}^{3} \Phi_{\Im_i}$. To show this, let $x \in F(M_C(I-\eta A^*(I-M_Q)A))$, then

$$x = M_C(I-\eta A^*(I-M_Q)A)x.$$

From Lemma 10, we have

$$(x, y, z) \in \Psi^{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3},$$

where $y = P_C(I-\zeta\nabla\Im_2)(ax+(1-a)z)$, $z = P_C(I-\zeta\nabla\Im_3)x$, $\bar{y} = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_2)(a\bar{x}+(1-a)\bar{z})$, $\bar{z} = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_3)\bar{x}$ with $\bar{x} = Ax$, $\bar{y} = Ay$ and $\bar{z} = Az$.

Then $(x, y, z) \in \Psi_{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}$ and $(\bar{x}, \bar{y}, \bar{z}) \in \Psi_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}$.

From Lemma 3, we have

$$x \in F(M_C),$$

and

$$Ax \in F(M_Q).$$

That is

$$x = M_C x = P_C(I-\zeta\nabla\Im_1)(ax+(1-a)y),$$

where $y = P_C(I-\zeta\nabla\Im_2)(ax+(1-a)z)$, $z = P_C(I-\zeta\nabla\Im_3)x$ and

$$Ax = M_Q Ax = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_1)(a\bar{x}+(1-a)\bar{y}),$$

where $\bar{y} = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_2)(a\bar{x}+(1-a)\bar{z})$, $\bar{z} = P_Q(I-\bar{\zeta}\nabla\bar{\Im}_3)\bar{x}$ with $\bar{x} = Ax$, $\bar{y} = Ay$ and $\bar{z} = Az$.

Utilizing the same method as (39), we have

$$x \in \bigcap_{i=1}^{3} F(P_C(I-\zeta_i\nabla\Im_i))$$

and

$$Ax \in \bigcap_{i=1}^{3} F(P_Q(I-\bar{\zeta}_i\nabla\bar{\Im}_i)).$$

Then $x \in F(P_C(I-\zeta_i\nabla\Im_i))$ and $Ax \in F(P_Q(I-\bar{\zeta}_i\nabla\bar{\Im}_i))$, for all $i = 1, 2, 3$.

By Lemma 9, we have $\Im_i(x) = \min_{x^*\in C} \Im_i(x^*)$ and $\bar{\Im}_i(Ax) = \min_{Ax^*\in Q} \bar{\Im}_i(Ax^*)$, for all $i = 1, 2, 3$.

That is $x \in \Phi_{\Im_i}$, for all $i = 1, 2, 3$.

It follows that

$$x \in \bigcap_{i=1}^{3} \Phi_{\Im_i}.$$

Thus

$$F(M_C(I-\eta A^*(I-M_Q)A)) \subseteq \bigcap_{i=1}^{3} \Phi_{\Im_i}.$$

Hence

$$\bigcap_{i=1}^{3}\Phi_{\Im_i} = F(M_C(I-\eta A^*(I-M_Q)A)).$$

□

Theorem 6 *Let* H_1 *and* H_2 *be real Hilbert spaces and let* C, Q *be nonempty closed convex subset of a real Hilbert space* H_1 *and* H_2*, respectively. For every* $i = 1, 2$*, let* $B_i : C \to H$ *be* α_i*-inverse strongly monotone mapping with* $\alpha = min\{\alpha_1, \alpha_2\}$*. For every* $i = 1, 2$ *let* $\Im_1^i, \Im_2^i, \Im_3^i : C \to \mathbb{R}$ *be a real-valued convex function with the gradient* $\nabla\Im_1^i, \nabla\Im_2^i, \nabla\Im_3^i$ *be* $\frac{1}{L_{\Im_1^i}}, \frac{1}{L_{\Im_2^i}}, \frac{1}{L_{\Im_3^i}}$*-inverse strongly monotone and continuous respectively, where* $\zeta^i \in (0, \frac{2}{L_\Im})$ *with* $\frac{1}{L_\Im} = \min\{\frac{1}{L_{\Im_1^i}}, \frac{1}{L_{\Im_2^i}}, \frac{1}{L_{\Im_3^i}}\}$*. Let* $\bar{\Im}_1^i, \bar{\Im}_2^i, \bar{\Im}_3^i : Q \to \mathbb{R}$ *be a real-valued convex function with the gradient* $\nabla\bar{\Im}_1^i, \nabla\bar{\Im}_2^i, \nabla\bar{\Im}_3^i$ *be* $\frac{1}{L_{\bar{\Im}_1^i}}, \frac{1}{L_{\bar{\Im}_2^i}}, \frac{1}{L_{\bar{\Im}_3^i}}$*-inverse strongly monotone and continuous respectively, where* $\bar{\zeta}^i \in (0, \frac{2}{L_{\bar{\Im}}})$ *with* $\frac{1}{L_{\bar{\Im}}} = \min\{\frac{1}{L_{\bar{\Im}_1^i}}, \frac{1}{L_{\bar{\Im}_2^i}}, \frac{1}{L_{\bar{\Im}_3^i}}\}$*. Let* $A : H_1 \to H_2$ *be a bounded linear operator with adjoint* A^* *and* $\eta \in (0, \frac{1}{L})$ *with* L *being the spectral radius of the operator* A^*A*. let* $f, g : H \to H$ *be* a_f *and* a_g*-contraction mapping with* $a = \max\{a_f, a_g\}$*. For every* $i = 1, 2$ *define* $M_C^i : C \to C$ *by* $M_C^i(x) = P_C(I-\zeta^i\nabla\Im_1^i)(ax+(1-a)P_C(I-\zeta^i\nabla\Im_2^i)(ax+(1-a)P_C(I-\zeta^i\nabla\Im_3^i)x))$*,* $\forall x \in C$*, and define* $M_Q^i : Q \to Q$ *by* $M_Q^i(\hat{x}) = P_Q(I-\bar{\zeta}^i\nabla\bar{\Im}_1^i)(a\hat{x}+(1-a)P_Q(I-\bar{\zeta}^i\nabla\bar{\Im}_2^i)(a\hat{x}+(1-a)P_Q(I-\bar{\zeta}^i\nabla\bar{\Im}_3^i)\hat{x}))$*,* $\forall\hat{x} \in Q, \forall\hat{x} \in Q$*. For every* $i = 1, 2$ *define* $M^i : C \to C$ *by* $M^i(x^*) = M_C^i(x^* - \eta A^*(I-M_Q^i)Ax^*)$ *for all* $x^* \in C$*. Let the sequence* $\{x_n\}$ *and* $\{w_n\}$ *generated by* $x_1, w_1 \in C$ *and*

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I-\gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)M^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I-\gamma_2 B_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)M^2 w_n) \end{cases} \tag{45}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0, 1]$ *with* $\delta_n + \eta_n + \mu_n = 1$*,* $\gamma \in (0, 2\alpha)$ *and* $\gamma = min\{\gamma_1, \gamma_2\}$*. Assume the following condition holds;*

(i) $\mathcal{F}_i = \bigcap_{j=1}^3 \Phi_{\Im_j}^i \cap VI(C, B_i) \neq \emptyset$ *for all* $i = 1, 2$ *and* $\Phi_{\Im_j}^i = \{\Im_j(x) = \min_{x^*\in C}\Im_j(x^*) : \bar{\Im}_j(Ax) = \min_{Ax^*\in Q}\bar{\Im}_j(Ax^*)\}$*,*
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$*,*
(iii) $0 < \bar{\theta} \le \delta_n, \eta_n, \mu_n \le \theta$ *for all* $n \in N$ *and for some* $\bar{\theta}, \theta > 0$*,*
(iv) $\sum_{n=1}^\infty |\delta_{n+1}-\delta_n| < \infty, \sum_{n=1}^\infty |\eta_{n+1}-\eta_n| < \infty, \sum_{n=1}^\infty |\alpha_{n+1}-\alpha_n| < \infty.$

Then $\{x_n\}$ *converges strongly to* $x_1^* = P_{\mathcal{F}_1}f(x_2^*)$*, where* $y_1^* = P_C(I-\zeta^1\nabla\Im_2^1)(ax_1^*+(1-a)z_1^*)$ *and* $z_1^* = P_C(I-\zeta^1\nabla\Im_3^1)x_1^*$ *with* (x_1^*, y_1^*, z_1^*) *is an element of* $\Psi_{\nabla\bar{\Im}_1^1, \nabla\bar{\Im}_2^1, \nabla\bar{\Im}_3^1}^{\nabla\Im_1^1, \nabla\Im_2^1, \nabla\Im_3^1}$*. and* $\{w_n\}$ *converges strongly to* $x_2^* = P_{\mathcal{F}_2}g(x_1^*)$*, where* $y_2^* = P_C(I-\zeta^2\nabla\Im_2^2)(ax_2^*+(1-a)z_2^*)$ *and* $z_1^* = P_C(I-\zeta^2\nabla\Im_3^2)x_1^*$ *with* (x_2^*, y_2^*, z_2^*) *is an element of* $\Psi_{\nabla\bar{\Im}_1^2, \nabla\bar{\Im}_2^2, \nabla\bar{\Im}_3^2}^{\nabla\Im_1^2, \nabla\Im_2^2, \nabla\Im_3^2}$*.*

Proof By using Theorem 4 and Lemma 11 , we obtain the conclusion. □

5 A numerical example

In this section, we give an example to support our main theorem.

Example 3 Let $\mathbb{R}$ be the set of real numbers, $H_1 = H_2 = \mathbb{R}^2$, $C = [-50, 50] \times [-50, 50]$, $Q = [-100, 100] \times [-100, 100]$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $D_j^i : C \to H_1$ be defined by $D_1^i(x_1, x_2) = (\frac{x_1-2}{3i}, 0)$, $D_2^i(x_1, x_2) = (\frac{x_1-2}{5i}, 0)$, $D_3^i(x_1, x_2) = (\frac{x_1-2}{7i}, 0)$, for all $x \in C$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $\bar{D}_j^i : Q \to H_2$ be defined by $\bar{D}_1^i(x_1, x_2) = (\frac{x_1-4}{3i}, 0)$, $\bar{D}_2^i(x_1, x_2) = (\frac{x_1-4}{5i}, 0)$, $\bar{D}_3^i(x_1, x_2) = (\frac{x_1-4}{7i}, 0)$, for all $x \in Q$. Let $A : H_1 \to H_2$ be defined by $A(x_1, x_2) = (2x_1, 2x_2)$ for all $(x_1, x_2) \in H_1$ and adjoint A^* of A defined by $A^*(\hat{x}_1, \hat{x}_2) = (2x_1, 2x_2)$ for all $(\hat{x}_1, \hat{x}_2) \in H_2$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $M_C^i : C \to C$ be defined by $M_C^i(x) = P_C(I - \frac{1}{3}(\frac{x_1-2}{3i}, 0))(0.5x + 0.5P_C(I - \frac{1}{3}(\frac{x_1-2}{5i}, 0))(0.5x + 0.5P_C(I - \frac{1}{3}(\frac{x_1-2}{7i}, 0))x))$, for all $x \in C$ and $M_Q^i : Q \to Q$ be defined by $M_Q^i(\hat{x}) = P_Q(I - \frac{1}{5}(\frac{x_1-4}{3i}, 0))(0.5\hat{x} + 0.5P_Q(I - \frac{1}{5}(\frac{x_1-4}{5i}, 0))(0.5\hat{x} + 0.5P_Q(I - \frac{1}{5}(\frac{x_1-4}{7i}, 0))\hat{x}))$, for all $\hat{x} \in Q$. For every $i = 1, 2$, let $M^i : C \to C$ be defined by $M^i(x^*) = M_C^i(x^* - \frac{1}{5}A^*(I - M_Q^i)Ax^*)$ for all $x \in C$. Let $T_1, T_2 : C \to C$ be defined by $T_1x = \{max(\frac{6-x_1}{2}, 0), max(\frac{6-x_2}{2}, 0)\}$, $T_2x = \{max(0, 4-x_1), max(0, 4-x_2)\}$, and let $B_i : C \to \mathbb{R}^2$ be defined by $B_i(x) = x - T_ix$, for every $x = (x_1, x_2) \in C$. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x) = (\frac{x_1}{2}, \frac{x_2}{2})$ and $g(x) = (\frac{x_1}{3}, \frac{x_2}{3})$, for all $x = (x_1, x_2) \in C$. Let the sequence $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \frac{n}{5n+2}x_n + \frac{2n+\frac{1}{2}}{5n+2}P_C(I - \frac{1}{2}B_1)x_n + \frac{2n+\frac{3}{2}}{5n+2}P_C(\frac{1}{4n}f(w_n) + \frac{4n-1}{4n}M^1x_n) \\ w_{n+1} = \frac{n}{5n+2}w_n + \frac{2n+\frac{1}{2}}{5n+2}P_C(I - \frac{7}{10}B_2)w_n + \frac{2n+\frac{3}{2}}{5n+2}P_C(\frac{1}{4n}g(x_n) + \frac{4n-1}{4n}M^2w_n) \end{cases} \tag{46}$$

By the definition of $T_i, B_i, f, g, D_j^i, M^i$ for every $i = 1, 2$, $j = 1, 2, 3$ we have $(2, 2) \in F(M^i) \cap VI(C, B_i)$. From Theorem 4, we can conclude that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $(2, 2)$.

The following Table 1 and Fig. 1 show the numerical results of sequences $\{x_n\}$ and $\{w_n\}$ where $x_1 = (10, 10)$, $w_1 = (10, 10)$ and $n = N = 40$.

Example 4 In this example, we use the same mappings and parameters as in Example 3. If we putting the sequence $\{x_n\} = \{w_n\}$, the mapping $M^1 = M^2$, $B_1 = B_2$, $\gamma_1 = \gamma_2 = 0.5$ and define $f(x) = g(x) = (\frac{x_1}{2}, \frac{x_2}{2})$, for all $x = (x_1, x_2) \in C$, we can rewrite (49) as follows:

$$x_{n+1} = \frac{n}{5n+2}x_n + \frac{2n+\frac{1}{2}}{5n+2}P_C\left(I - \frac{1}{2}B_1\right)x_n + \frac{2n+\frac{3}{2}}{5n+2}P_C\left(\frac{1}{4n}f(x_n) + \left(1 - \frac{1}{4n}\right)M^1x_n\right) \tag{47}$$

Table 1 The values of $\{x_n\}$ and $\{w_n\}$ with initial values $x_1 = (10, 10)$, $w_1 = (10, 10)$ and $n = N = 40$

n	$x_n = (x_n^1, x_n^2)$	$w_n = (w_n^1, w_n^2)$
1	(10.000000, 10.000000)	(10.000000, 10.000000)
2	(8.083752, 9.166667)	(8.083752, 9.166667)
3	(6.274706, 8.068576)	(6.005248, 7.763021)
⋮	⋮	⋮
20	(1.981828, 2.039016)	(1.985800, 2.025690)
⋮	⋮	⋮
38	(1.990594, 1.985034)	(1.992727, 1.988405)
39	(1.990860, 1.985389)	(1.992933, 1.988685)
40	(1.991112, 1.985744)	(1.993127, 1.988964)

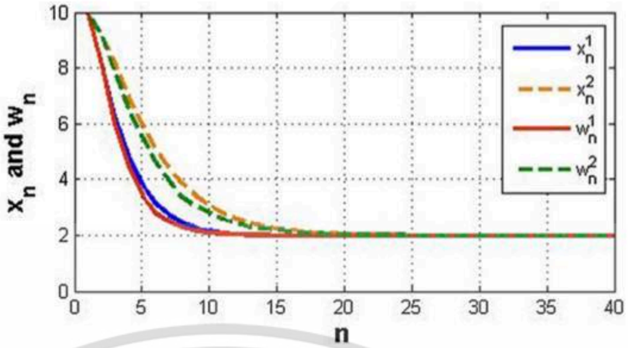

Fig. 1 The convergence of $\{x_n\}$ and $\{w_n\}$ with initial values $x_1 = (10, 10)$, $w_1 = (10, 10)$ and $n = N = 40$

By using the algorithm (47), the following Table 2 and Fig. 2 show the numerical results of sequence $\{x_n\}$ where $x_1 = (10, 10)$ and $n = N = 40$.

Table 2 The value of $\{x_n\}$ with initial value $x_1 = (10, 10)$ and $n = N = 40$

n	$x_n = (x_n^1, x_n^2)$
1	(10.000000, 10.000000)
2	(8.083752, 9.166667)
3	(6.274706, 8.068576)
⋮	⋮
20	(1.981869, 2.041259)
⋮	⋮
38	(1.990583, 1.985025)
39	(1.990850, 1.985376)
40	(1.991102, 1.985730)

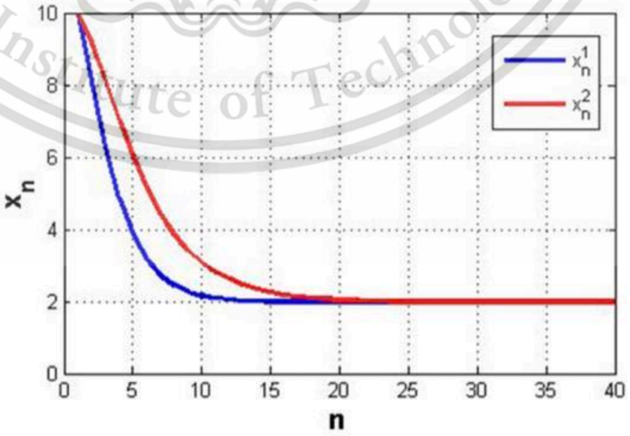

Fig. 2 The convergence of $\{x_n\}$ with initial value $x_1 = (10, 10)$ and $n = N = 40$

In the following example, we consider the metric projection onto a half-space $H\_(a, \beta) := \{z \in H : \langle a, z\rangle \le \beta\}$, where $a \in H, a \ne 0$ and $\beta \in \mathbb{R}$. It is clear that $H\_(a, \beta)$ is closed and convex with

$$P_{H\_(a,\beta)}x = \begin{cases} x - \frac{\langle a,x\rangle-\beta}{\|a\|^2}a, & \text{if} \quad \langle a, x\rangle > \beta, \\ x, & \text{if} \quad \langle a, x\rangle \le \beta. \end{cases}$$

Example 5 Let $H_1 = [-50, 50] \times [-50, 50]$ and $C = H\_(a, 20) := \{x \in H_1 : x_1 + 2x_2 \le 20\}$, where $a = (1, 2)$. Then, we obtain

$$P_C(x_1, x_2) = \begin{cases} (x_1, x_2) - \frac{[x_1+2x_2-20](1,2)}{5}, & \text{if} \quad x_1 + 2x_2 > 20, \\ (x_1, x_2), & \text{if} \quad x_1 + 2x_2 \le 20, \end{cases}$$

for all $(x_1, x_2) \in C$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $D^i_j : C \to H_1$ defined by $D^i_1(x_1, x_2) = (\frac{x_1-4}{2i}, x_2)$, $D^i_2(x_1, x_2) = (x_1 - 4, \frac{x_2-3}{3i})$, $D^i_3(x_1, x_2) = (\frac{x_1-4}{4i}, x_2 - 8)$, for all $(x_1, x_2) \in C$. For every $i = 1, 2$ let $M^i_C : C \to C$ defined by $M^i_C(x) = P_C(I - \frac{1}{3}D^i_1(x))\left[\frac{1}{2}(x) + \frac{1}{2}P_C(I - \frac{1}{3}D^i_2(x))\left(\frac{1}{2}(x) + \frac{1}{2}P_C(I - \frac{1}{3}D^i_3(x)\right)\right]$ for all $x = (x_1, x_2) \in C$.

Let $H_2 = [-100, 100]\times[-100, 100]$ and $Q = H\_(b, 60) := \{\hat{x} \in H_2 : 3\hat{x}_1+4\hat{x}_2 \le 60\}$, where $b = (3, 4)$. Then, we obtain

$$P_Q(\hat{x}_1, \hat{x}_2) = \begin{cases} (\hat{x}_1, \hat{x}_2) - \frac{[3\hat{x}_1+4\hat{x}_2-60](3,4)}{25}, & \text{if} \quad 3\hat{x}_1 + 4\hat{x}_2 > 60, \\ (\hat{x}_1, \hat{x}_2), & \text{if} \quad 3\hat{x}_1 + 4\hat{x}_2 \le 60, \end{cases}$$

for all $(\hat{x}_1, \hat{x}_2) \in Q$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $\bar{D}^i_j : Q \to H_2$ defined by $\bar{D}^i_1(\hat{x}_1, \hat{x}_2) = (\frac{\hat{x}_1-8}{i}, \hat{x}_2)$, $\bar{D}^i_2(\hat{x}_1, \hat{x}_2) = (\frac{\hat{x}_1-8}{6i}, \hat{x}_2 - 6)$, $\bar{D}^i_3(\hat{x}_1, \hat{x}_2) = (\frac{x_1-8}{5i}, x_2 - 24)$, for all $(\hat{x}_1, \hat{x}_2) \in Q$. For every $i = 1, 2$ let $M^i_Q : Q \to Q$ defined by $M^i_Q(\hat{x}) = P_Q(I - \frac{1}{5}\bar{D}^i_1(\hat{x}))\left[\frac{1}{2}(\hat{x}) + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D}^i_2(\hat{x}))\left(\frac{1}{2}(\hat{x}) + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D}^i_3(\hat{x})\right)\right]$ for all $\hat{x} = (\hat{x}_1, \hat{x}_2) \in Q$.

Let $A : H_1 \to H_2$ defined by $A(x_1, x_2) = (2x_1, 2x_2)$, for all $(x_1, x_2) \in H_1$ and adjoint A^* of A defined by $A^*(\hat{x}_1, \hat{x}_2) = (2\hat{x}_1, 2\hat{x}_2)$, for all $(\hat{x}_1, \hat{x}_2) \in H_2$. For every $i = 1, 2$, let $M^i(x) = M^i_C(x - \frac{1}{10}A^*(I - M^i_Q)A(x))$, for all $x = (x_1, x_2) \in C$. Let $F, G : [-50, 50] \times [-50, 50] \to [-50, 50] \times [-50, 50]$ be defined by $F(x_1, x_2) = (\frac{x_1}{5}, \frac{x_2}{5})$ and $G(x_1, x_2) = (\frac{x_1}{7}, \frac{x_2}{7})$, for all $(x_1, x_2) \in [-50, 50]\times[-50, 50]$ and for $i = 1, 2$, let $B_i : C \to [-50, 50]\times[-50, 50]$ be defined by $B_1(x_1, x_2) = (\frac{x_1-4}{3}, \frac{x_2-2}{3})$ and $B_2(x_1, x_2) = (\frac{x_1-4}{9}, \frac{x_2-2}{9})$, for all $(x_1, x_2) \in C$. Let the sequence $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \frac{n}{3n+2}x_n + \frac{n+\frac{1}{2}}{3n+2}P_C(I - \frac{1}{2}B_1)x_n + \frac{n+\frac{3}{2}}{3n+2}P_C(\frac{1}{2n}F(w_n) + \frac{2n-1}{2n}M^1x_n) \\ w_{n+1} = \frac{n}{3n+2}w_n + \frac{n+\frac{1}{2}}{3n+2}P_C(I - \frac{7}{10}B_2)w_n + \frac{n+\frac{3}{2}}{3n+2}P_C(\frac{1}{2n}G(x_n) + \frac{2n-1}{2n}M^2w_n) \end{cases} \tag{48}$$

By the definition of B_i, F, G, D^i_j, M^i for every $i = 1, 2,\ j = 1, 2, 3$ we have $(4, 2) \in F(M^i) \cap VI(C, B_i)$. From Theorem 4, we can conclude that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $(4, 2)$.

The following Table 3 and Fig. 3 show the numerical results of sequences $\{x_n\}$ and $\{w_n\}$ where $x_1 = (-10, 10)$, $w_1 = (-10, 10)$ and $n = N = 40$.

Next, we give the following example in $H_1 = L_2([\alpha, \beta])$ with the inner product defined by $\langle f, g\rangle := \int_\alpha^\beta f(x)g(x)dx$, and with the norm $\|f\| := \left(\int_\alpha^\beta f^2(x)dx\right)^{\frac{1}{2}}$ for all $f, g \in H_1$.

Table 3 The values of $\{x_n\}$ and $\{w_n\}$ with initial values $x_1 = (-10, 10)$, $w_1 = (-10, 10)$ and $n = N = 40$

n	$x_n = (x_n^1, x_n^2)$	$w_n = (w_n^1, w_n^2)$
1	(− 10.000000, 10.000000)	(− 10.000000, 10.000000)
2	(− 5.978702, 6.632331)	(− 6.563107, 6.729819)
3	(− 3.712547, 5.055405)	(− 4.723965, 5.230170)
⋮	⋮	⋮
20	(3.608551, 1.890749)	(2.974410, 1.887930)
⋮	⋮	⋮
38	(3.901467, 1.909353)	(3.768213, 1.889663)
39	(3.905313, 1.910485)	(3.781338, 1.891002)
40	(3.908821, 1.911562)	(3.793224, 1.892293)

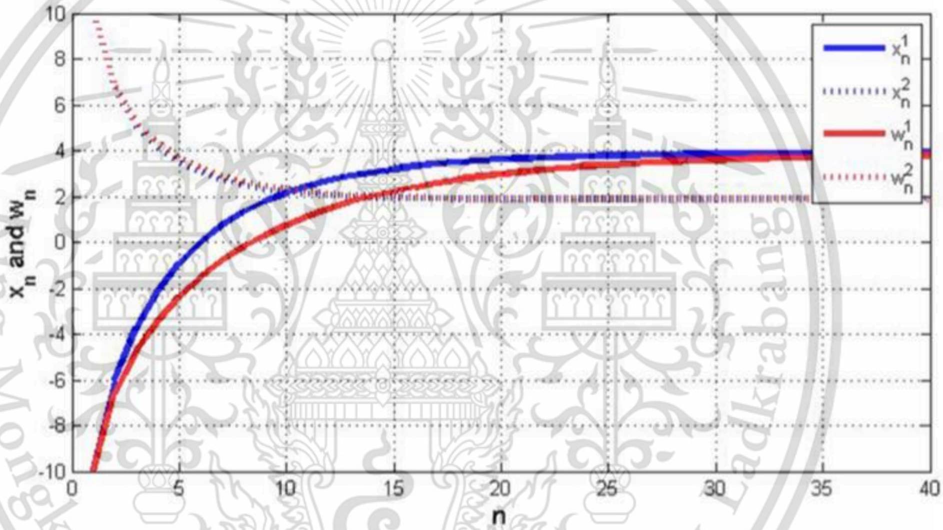

Fig. 3 The convergence of $\{x_n\}$ and $\{w_n\}$ with initial values $x_1 = (-10, 10)$, $w_1 = (-10, 10)$ and $n = N = 40$

We consider the metric projection onto a hyperplane

$$H\_(g, \rho) := \{h \in L_2([\alpha, \beta]) : \int_\alpha^\beta g(x)h(x)dx = \rho\},$$

where $g \in L_2([\alpha, \beta])$, $g \neq 0$ and $\rho \in \mathbb{R}$. Then $H\_(g, \rho)$ is closed and convex with

$$P_{H\_(g,\rho)} f = f - \frac{\int_\alpha^\beta g(x)h(x)dx - \rho}{\int_\alpha^\beta g^2(x)dx} g,$$

and in $H_2 = \mathbb{R}^2$ which consider the metric projection onto a band

$$C := \{x \in H : \beta_1 \leq \langle a, x\rangle \leq \beta_2\},$$

where $a \in H, a \neq 0, \beta_1, \beta_2 \in \mathbb{R}$ and $\beta_1 < \beta_2$. It is clear that C is closed and convex with

$$P_C x = \begin{cases} x - \frac{\langle a,x\rangle - \beta_2}{\|a\|^2} a, & \text{if} \quad \langle a, x\rangle > \beta_2, \\ x & if \quad \beta_1 \leq \langle a, x\rangle \leq \beta_2, \\ x - \frac{\langle a,x\rangle - \beta_1}{\|a\|^2} a, & \text{if} \quad \langle a, x\rangle < \beta_1. \end{cases}$$

Example 6 Let $H_1 = L_2([-2, 2])$ and $C := \{h \in L_2([-2, 2]) : \int_{-2}^{2} g(x)h(x)dx = \frac{32}{3}\}$, we obtain $P_C f = f - \frac{\langle f,g\rangle - \frac{32}{3}}{\|g\|^2} g$, where $g \in L_2([-2, 2])$ with $g(x) = x - 1$, for all $x \in [-2, 2]$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $D_j^i : C \to H_1$ defined by $D_1^i(h) = \frac{h-2I}{3i}$, $D_2^i(h) = \frac{h-2I}{5i}$, $D_3^i(h) = \frac{h-2I}{7i}$, for all $h \in C$. For every $i = 1, 2$ let $M_C^i : C \to C$ defined by $M_C^i(h) = P_C(I - \frac{1}{3}D_1^i(h))[\frac{1}{2}(h) + \frac{1}{2}P_C(I - \frac{1}{3}D_2^i(h))\left(\frac{1}{2}(h) + \frac{1}{2}P_C(I - \frac{1}{3}D_3^i(h))\right]$ for all $h \in C$.

Let $H_2 = \mathbb{R}^2$ and $Q := \{\bar{x} \in H_2 : 0 \leq 3\bar{x_1} - \bar{x_2} \leq 2\}$. Then, we obtain

$$P_Q\bar{x} = \begin{cases} (\bar{x_1}, \bar{x_2}) - \frac{[3\bar{x_1} - \bar{x_2} - 2](3, -1)}{10}, & \text{if} \quad 3\bar{x_1} - \bar{x_2} > 2, \\ (\bar{x_1}, \bar{x_2}) & \text{if} \quad 0 \leq 3\bar{x_1} - \bar{x_2} \leq 2, \\ (\bar{x_1}, \bar{x_2}) - \frac{[3\bar{x_1} - \bar{x_2}](3, -1)}{10}, & \text{if} \quad 3\bar{x_1} - \bar{x_2} < 0, \end{cases}$$

for all $\bar{x} = (\bar{x_1}, \bar{x_2}) \in H_2$. For every $i = 1, 2$ and $j = 1, 2, 3$, let $\bar{D}_j^i : Q \to H_2$ defined by $\bar{D}_1^i(\bar{x_1}, \bar{x_2}) = (\frac{\bar{x_1}}{2i}, \frac{\bar{x_2}}{2i})$, $\bar{D}_2^i(\bar{x_1}, \bar{x_2}) = (\frac{\bar{x_1}}{4i}, \frac{\bar{x_2}}{4i})$, $\bar{D}_3^i(\bar{x_1}, \bar{x_2}) = (\frac{\bar{x_1}}{6i}, \frac{\bar{x_2}}{6i})$, for all $(\bar{x_1}, \bar{x_2}) \in Q$. For every $i = 1, 2$, let $M_Q^i : Q \to Q$ defined by $M_Q^i(\bar{x}) = P_Q(I - \frac{1}{5}\bar{D}_1^i(\bar{x}))[\frac{1}{2}(\bar{x}) + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D}_2^i(\bar{x}))\left(\frac{1}{2}(\bar{x}) + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D}_3^i(\bar{x})\right)]$ for all $\bar{x} = (\bar{x_1}, \bar{x_2}) \in Q$.

Let $A : H_1 \to H_2$ defined by $A(h) = (\int_{-2}^{2} h(x)dx, \int_{-2}^{2} h(x)dx)$, for all $h \in H_1$ and adjoint A^* of A defined by $A^*(\bar{x_1}, \bar{x_2}) = \bar{x_1} + \bar{x_2}$, for all $(\bar{x_1}, \bar{x_2}) \in H_2$. For every $i = 1, 2$, let $M^i(h) = M_C^i(h - \frac{1}{4}A^*(I - M_Q^i)A(h))$, for all $h \in C$. Let $F, G : L_2([-2, 2]) \to L_2([-2, 2])$ be defined by $F(h) = \frac{h}{2}$ and $G(h) = \frac{h}{3}$, for all $h \in L_2([-2, 2])$ and for $i = 1, 2$, let $B_i : C \to L_2([-2, 2])$ be defined by $B_1(h) = \frac{3h-6I}{4}$ and $B_2(h) = \frac{4h-8I}{2}$, for all $h \in C$. Let the sequence $\{x_n\}$ and $\{w_n\}$ generated by $x_1, w_1 \in C$ and

$$\begin{cases} x_{n+1} = \frac{2n}{7n+1}x_n + \frac{4n+1}{7n+1}P_C(I - \frac{1}{2}B_1)x_n + \frac{n}{7n+1}P_C(\frac{1}{8n}F(w_n) + \frac{8n-1}{8n})M^1 x_n) \\ w_{n+1} = \frac{2n}{7n+1}w_n + \frac{4n+1}{7n+1}P_C(I - \frac{7}{10}B_2)w_n + \frac{n}{7n+1}P_C(\frac{1}{8n}G(x_n) + \frac{8n-1}{8n})M^2 w_n) \end{cases} \tag{49}$$

By the definition of B_i, F, G, D_j^i, M^i for every $i = 1, 2, j = 1, 2, 3$ we have $2I \in F(M^i) \cap VI(C, B_i)$. From Theorem 4, we can conclude that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $2I$.

Now, we compare the convergent behavior of sequences $\{x_n\}$ and $\{w_n\}$ with the same starting point and have the results in Table 4. In Fig. 4, the values of errors $\|x_{n+1} - x_n\|$ and $\|w_{n+1} - w_n\|$ are represented by the y-axis, the number of iterations are represented by the x-axis.

Table 4 Comparison of the sequences $\{x_n\}$ and $\{w_n\}$ with the same starting point

	Starting point	Time taken (s)
The sequence $\{x_n\}$	$x_1 = 2t$	6.665
The sequence $\{w_n\}$	$w_1 = 2t$	7.430

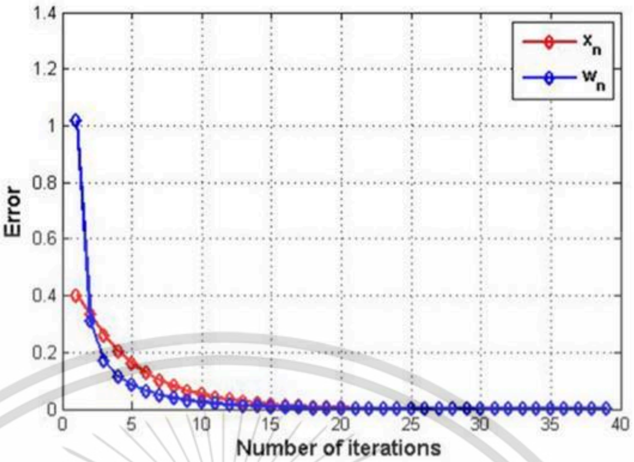

Fig. 4 Comparison of the sequences $\{x_n\}$ and $\{w_n\}$ with $x_1 = 2t$ and $w_1 = 2t$

6 Conclusion

1. Table 1 and Figure 1 show that $\{x_n\}$ and $\{w_n\}$ converge to $(2, 2)$, where $(2, 2) \in F(M^i) \cap VI(C, B_i)$, for all $i = 1, 2$. The convergence of $\{x_n\}$ and $\{w_n\}$ of Example 3 can be guaranteed by Theorem 4.
2. Table 2 and Figure 2 show that $\{x_n\}$ converges to $(2, 2)$, where $(2, 2) \in F(M^1) \cap VI(C, B_1)$. The convergence of $\{x_n\}$ of Example 4 can be guaranteed by Theorem 4.
3. From these Examples, we obtain that the sequence $\{x_n\}$ in Example 3 converges faster than the sequence $\{x_n\}$ in Example 4 because the iterative sequence $\{x_n\}$ and $\{w_n\}$ converges dependently.
4. Table 3 and Figure 3 show that the iterative sequence $\{x_n\}$ and $\{w_n\}$ converge to $(4, 2)$, where $(4, 2) \in F(M^i) \cap VI(C, B_i)$, for all $i = 1, 2$. The convergence of $\{x_n\}$ and $\{w_n\}$ of Example 5 can be guaranteed by Theorem 4.
5. The convergence of $\{x_n\}$ and $\{w_n\}$ of Example 6 can be guaranteed by Theorem 4. Furthermore, The Table 4 and Fig. 4 show that the sequence $\{x_n\}$ converge faster than the sequence $\{w_n\}$ with the same starting point.

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A new subgradient extragradient method for solving the split modified system of variational inequality problems and fixed point problem

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Abstract

We introduce a new subgradient extragradient algorithm utilizing the concept of the set of solutions of the split modified system of variational inequality problems (SMSVIP). Our main theorem is weak convergence theorem for such an algorithm for approximating the fixed point problem in a real Hilbert space. We also apply these results to approximate the split minimization problem. In the last section, we provide an example to illustrate the potential of our main theorem.

MSC: 47H09; 47H10; 90C33

Keywords: Fixed point; Subgradient extragradient; The split modified system of variational inequality problems; Variational inequality

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. The mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. An element $x \in C$ is said to be a fixed point of T if $Tx = x$ and $F(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of T. Fixed point problem has been widely studied and developed in the literature; see [5, 11, 26, 27, 29] and the references therein.

We now recall some well-known concepts and results in a real Hilbert space H.

The variational inequality problem (VIP) is to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0$$

for all $y \in C$. The set of all solutions of the variational inequality is denoted by $VI(C,A)$. Since its inception by Stampacchia [24] in 1964, the variational inequality problem has become interesting in several topics arising in structural analysis, physic, economics, optimization, and applied sciences; see [1, 3, 6, 8, 11–13, 15, 18, 20, 30, 32] and the references therein.

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Several algorithms for solving the VIP are projection algorithms that employ projections onto the feasible set C of the VIP, or onto some related set, in order to iteratively reach a solution. In 1976, Korpelevich [19] proposed an algorithm for solving the VIP in a Euclidean space, known as *the extragradient method*. In each iteration of her algorithm, in order to get the next iterate x^{k+1}, two orthogonal projections onto C are calculated, according to the following iterative step. Given the current iterate x^k, calculate

$$y^k = P_C\bigl(x^k - \tau f\bigl(x^k\bigr)\bigr), \tag{1}$$

$$x^{k+1} = P_C\bigl(x^k - \tau f\bigl(y^k\bigr)\bigr) \tag{2}$$

for all $k \in \mathbb{N}$, where τ is some positive number and P_C denotes the Euclidean least distance projection onto C.

The convergence was proved in [19] under the assumptions of Lipschitz continuity and pseudo-monotonicity. However, there is still the need to calculate two projections onto C. If the set C is simple enough so that projections onto it can be easily computed, but if C is a general closed and convex set, a minimal distance problem has to be solved twice in order to obtain the next iterate. This might seriously affect the efficiency of the extragradient method. Korpelevich's extragradient method has been widely studied in the literature; see [2, 4, 7, 9, 14, 16, 17, 22, 28, 31] and the references therein.

In the past decade years, Censor et al. [10] developed the subgradient extragradient algorithm in a Euclidean space, in which they replaced the (second) projection (2) onto C by a projection onto a specific constructible half-space as follows:

Algorithm 1 (The subgradient extragradient algorithm)

Step 0 : *Select a starting point* $x^0 \in H$ *and* $\tau > 0$, *and set* $k = 0$.

Step 1 : *Given the current iterate* x^k, *compute*

$$y^k = P_C\bigl(x^k - \tau f\bigl(x^k\bigr)\bigr),$$

construct the half-space T_k *the bounding hyperplane of which supports* C *at* y^k,

$$T_k := \bigl\{w \in H \mid \bigl\langle \bigl(x^k - \tau f\bigl(x^k\bigr)\bigr) - y^k, w - y^k\bigr\rangle \le 0\bigr\}, \tag{3}$$

and calculate the next iterate

$$x^{k+1} = P_{T_k}\bigl(x^k - \tau f\bigl(y^k\bigr)\bigr).$$

Step 2 : *If* $x^k = y^k$ *then stop. Otherwise, set* $k \leftarrow (k+1)$ *and return to step* 1.

Furthermore, under some control conditions, they proved weak convergence theorems for their algorithms.

Very recently, Sripattanet and Kangtunyakarn [23] introduced the following *split modified system of variational inequality problems (SMSVIP)*, which involves finding

$(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \zeta D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \zeta D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \zeta D_3)x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{4}$$

and finding $(\bar{x^*} = Ax^*, \bar{y^*} = Ay^*, \bar{z^*} = Az^*) \in Q \times Q \times Q$ such that

$$\begin{cases} \langle \bar{x^*} - (I - \bar{\zeta} \bar{D_1})(a\bar{x^*} + (1-a)\bar{y^*}), \bar{x} - \bar{x^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{y^*} - (I - \bar{\zeta} \bar{D_2})(a\bar{x^*} + (1-a)\bar{z^*}), \bar{x} - \bar{y^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{z^*} - (I - \bar{\zeta} \bar{D_3})\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, & \forall \bar{x} \in Q, \end{cases} \tag{5}$$

where $D_1, D_2, D_3 : C \to H_1$, $\bar{D_1}, \bar{D_2}, \bar{D_3} : Q \to H_2$ are six different mappings, $\zeta, \bar{\zeta} > 0$, and $a \in [0,1]$. The sets of all solutions of (4) and (5) are denoted by Ψ_{D_1,D_2,D_3} and $\Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}$, respectively. The set of all solutions of the SMSVIP is denoted by $\Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}^{D_1,D_2,D_3}$, that is,

$$\Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}^{D_1,D_2,D_3} = \{(x^*, y^*, z^*) \in \Psi_{D_1,D_2,D_3} : (\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}\}.$$

If we put $a = 0$ in (4) and (5), we have

$$\begin{cases} \langle x^* - (I - \zeta D_1)y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \zeta D_2)z^*, x - y^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \zeta D_3)x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases}$$

and

$$\begin{cases} \langle \bar{x^*} - (I - \bar{\zeta} \bar{D_1})\bar{y^*}, \bar{x} - \bar{x^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{y^*} - (I - \bar{\zeta} \bar{D_2})\bar{z^*}, \bar{x} - \bar{y^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{z^*} - (I - \bar{\zeta} \bar{D_3})\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, & \forall \bar{x} \in Q, \end{cases}$$

which is a modified the split general system of variational inequalities (SVIP) [21].

Based on the above works and observation of a half-space in Algorithm 1 related to the VIP, we introduce a new half-space related to the SMSVIP and prove weak convergence theorems of the sequence $\{x_n\}$ generated by our new half-space for approximating the solutions of the SMSVIP. Moreover, using our main result, we obtain the additional results involving the split minimization problem. Finally, we perform numerical examples to illustrate the computational performance of the proposed algorithms.

2 Preliminaries

We denote the weak convergence and the strong convergence by "$\rightharpoonup$" and "$\to$", respectively. For every $x \in \mathcal{H}$, there exists a unique nearest point $P_C x$ in C such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection of $\mathcal{H}$ onto C.

The metric projection P_C is characterized by the following two properties:

1. $P_C x \in C$,

2. $\langle x - P_C x, P_C x - y\rangle \geq 0, \forall x \in \mathcal{H}, y \in C$,
   and if C is a hyperplane, it follows that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{6}$$

$\forall x \in \mathcal{H}, y \in C$.

Definition 2.1 A mapping $A : C \to H$ is called *α-inversestronglymonotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$.

The following lemmas are needed to prove the main theorem.

Lemma 2.2 *Let $\mathcal{H}$ be a real Hilbert space, and let C be a nonempty closed convex subset of $\mathcal{H}$. Let $\{x^k\}_{k=0}^{\infty} \subset \mathcal{H}$ be Fejer-monotone with respect to C, i.e., for every $u \in C$,*

$$\|x^{k+1} - u\| \leq \|x^k - u\|, \quad \forall k \geq 0.$$

Then $\{P_C x^k\}_{k=0}^{\infty}$ converges strongly to some $z \in C$.

Lemma 2.3 *Each Hilbert space $\mathcal{H}$ satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset \mathcal{H}$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in \mathcal{H}$ with $y \neq x$.

Lemma 2.4 ([23]) *Let H_1 and H_2 be real Hilbert spaces, and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3-inverse strongly monotone, respectively, where $\zeta \in (0, 2d^*)$ with $d^* = \min\{d_1, d_2, d_3\}$. Let $\bar{D}_1, \bar{D}_2, \bar{D}_3 : Q \to H_2$ be $\bar{d}_1, \bar{d}_2, \bar{d}_3$-inverse strongly monotone, respectively, where $\bar{\zeta} \in (0, 2\hat{d})$ with $\hat{d} = \min\{\bar{d}_1, \bar{d}_2, \bar{d}_3\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C : C \to C$ by*

$$M_C(x) = P_C(I - \zeta D_1)\big(ax + (1 - a)P_C(I - \zeta D_2)\big(ax + (1 - a)P_C(I - \zeta D_3)x\big)\big),$$

$\forall x \in C$, and define $M_Q : Q \to Q$ by

$$M_Q(\hat{x}) = P_Q(I - \bar{\zeta}\bar{D}_1)\big(a\hat{x} + (1 - a)P_Q(I - \bar{\zeta}\bar{D}_2)\big(a\hat{x} + (1 - a)P_Q(I - \bar{\zeta}\bar{D}_3)\hat{x}\big)\big),$$

$\forall \hat{x} \in Q$. Define $M : C \to C$ by $M(x) = M_C(x - \eta A^*(I - M_Q)Ax)$ for all $x \in C$. Then M is a nonexpansive mapping for all $x \in C$.

Remark 1 From the study of Lemma 2.4, we have

$$\begin{aligned}&\left\|\left(x-\eta A^*(I-M_Q)Ax\right)-\left(y-\eta A^*(I-M_Q)Ay\right)\right\|^2\\&\quad\leq\|x-y\|^2-\eta(1-\eta L)\left\|(I-M_Q)Ax-(I-M_Q)Ay\right\|^2\end{aligned}$$

for all $x, y \in H_1$.

Lemma 2.5 ([23]) *Let H_1 and H_2 be real Hilbert spaces, and let C, Q be nonempty closed convex subsets of H_1, H_2, respectively. Define the mappings $D_1, D_2, D_3, \bar{D_1}, \bar{D_2}, \bar{D_3}, M_C$, and M_Q as in Lemma 2.4, where $\zeta \in (0, 2d^*)$ with $d^* = \min\{d_1, d_2, d_3\}$, $\bar{\zeta} \in (0, 2\hat{d})$ with $\hat{d} = \min\{\bar{d_1}, \bar{d_2}, \bar{d_3}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A.*

Assume

$$\Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}^{D_1,D_2,D_3} = \left\{\left(x^*, y^*, z^*\right) \in \Psi_{D_1,D_2,D_3} : \left(\bar{x^*}, \bar{y^*}, \bar{z^*}\right) \in \Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}\right\} \neq \emptyset.$$

The following statements are equivalent:

(i) $(x^*, y^*, z^*) \in \Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}^{D_1,D_2,D_3}$,

(ii) $x^* = M_C(x^* - \eta A^*(I - M_Q)Ax^*)$, *where* $y^* = P_C(I - \zeta D_2)(ax^* + (1 - a)z^*)$, $z^* = P_C(I - \zeta D_3)x^*$, $\bar{x^*} = Ax^* = P_Q(I - \bar{\zeta}\bar{D_1})(a\bar{x^*} + (1 - a)\bar{y^*})$, $\bar{y^*} = Ay^* = P_Q(I - \bar{\zeta}\bar{D_2})(a\bar{x^*} + (1 - a)\bar{z^*})$, *and* $\bar{z^*} = Az^* = P_Q(I - \bar{\zeta}\bar{D_3})\bar{x^*}$.

Lemma 2.6 ([23]) *Let H_1 and H_2 be real Hilbert spaces, and let C, Q be nonempty closed convex subsets of H_1, H_2, respectively. Define the mappings $D_1, D_2, D_3, \bar{D_1}, \bar{D_2}, \bar{D_3}, M_C$, and M_Q as in Lemma 2.4 where $\zeta \in (0, 2d^*)$ with $d^* = \min\{d_1, d_2, d_3\}$, $\bar{\zeta} \in (0, 2\hat{d})$ with $\hat{d} = \min\{\bar{d_1}, \bar{d_2}, \bar{d_3}\}$ and $a \in [0, 1]$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Let $\bigcap_{i=1}^3 \Phi_i \neq \emptyset$ and $\Phi_i = \{w \in VI(C, D_i) | Aw = \bar{w} \in VI(Q, \bar{D}_i)\}$ for all $i = 1, 2, 3$. Then*

$$\bigcap_{i=1}^{3} \Phi_i = F\left(M_C\left(I - \eta A^*(I - M_Q)A\right)\right).$$

In order to prove our main result, we need to prove the lemmas involving the split variational inequality problem.

Lemma 2.7 *Let H_1 and H_2 be real Hilbert spaces, and let C, Q be nonempty closed convex subsets of H_1, H_2, respectively. Define the mappings $D_1, D_2, D_3, \bar{D_1}, \bar{D_2}, \bar{D_3}, M_C$, and M_Q as in Lemma 2.4 where $\zeta \in (0, 2d^*)$ with $d^* = \min\{d_1, d_2, d_3\}$, $\bar{\zeta} \in (0, 2\hat{d})$ with $\hat{d} = \min\{\bar{d_1}, \bar{d_2}, \bar{d_3}\}$ and $a \in [0, 1]$. Let $\{x_n\}$ be a sequence in H_1, and let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. For every $n \in \mathbb{N}$, let $T_n = aW_n + (1 - a)P_C(I - \zeta D_2)(aW_n + (1 - a)P_C(I - \zeta D_3)W_n))$ and $W_n = (I - \eta A^*(I - M_Q)A)x_n$. If $x^* \in \bigcap_{i=1}^3 \Phi_i$, then*

$$\left\|T_n - x^*\right\|^2 \leq \left\|x_n - x^*\right\|^2 - \eta(1 - \eta L)\left\|(I - M_Q)Ax_n\right\|^2$$

for all $n \in \mathbb{N}$.

Proof Let $x^* \in \bigcap_{i=1}^{3} \Phi_i$. From Lemma 2.6, we have

$$x^* \in F\big(M_C\big(I - \eta A^*(I - M_Q)A\big)\big).$$

It implies that $x^* = M_C(I - \eta A^*(I - M_Q)A)x^*$, $y^* = P_C(I - \zeta D_2)(ax^* + (1-a)z^*)$, and $z^* = P_C(I - \zeta D_3)x^*$, where $\bar{x}^* = Ax^* = P_Q(I - \bar{\zeta}\bar{D}_1)(a\bar{x}^* + (1-a)\bar{y}^*)$, $\bar{y}^* = Ay^* = P_Q(I - \bar{\zeta}\bar{D}_2)(a\bar{x}^* + (1-a)\bar{z}^*)$, and $\bar{z}^* = Az^* = P_Q(I - \bar{\zeta}\bar{D}_3)\bar{x}^*$. From Lemma 2.5, we have $(x^*, y^*, z^*) \in \Omega_{\bar{D}_1,\bar{D}_2,\bar{D}_3}^{D_1,D_2,D_3}$. That is, $(x^*, y^*, z^*) \in \Omega_{D_1,D_2,D_3}$ and $(\bar{x}^*, \bar{y}^*, \bar{z}^*) \in \Omega_{\bar{D}_1,\bar{D}_2,\bar{D}_3}$. From $(\bar{x}^*, \bar{y}^*, \bar{z}^*) \in \Omega_{\bar{D}_1,\bar{D}_2,\bar{D}_3}$, we obtain that

$$\begin{aligned}
\bar{x}^* &= P_Q(I - \bar{\zeta}\bar{D}_1)\big(a\bar{x}^* + (1-a)\bar{y}^*\big),\\
\bar{y}^* &= P_Q(I - \bar{\zeta}\bar{D}_2)\big(a\bar{x}^* + (1-a)\bar{z}^*\big),\\
\bar{z}^* &= P_Q(I - \bar{\zeta}\bar{D}_3)\bar{x}^*.
\end{aligned}$$

It implies that

$$\begin{aligned}
Ax^* = \bar{x}^* &= P_Q(I - \bar{\zeta}\bar{D}_1)\big(a\bar{x}^* + (1-a)P_Q(I - \bar{\zeta}\bar{D}_2)\big(a\bar{x}^* + (1-a)P_Q(I - \bar{\zeta}\bar{D}_3)\bar{x}^*\big)\big)\\
&= M_Q\bar{x}^* = M_QAx^*.
\end{aligned}$$

From the definition of x^*, we get $x^* = P_C(I - \zeta D_1)T_x^*$, where $T_x^* = aW_x^* + (1-a)P_C(I - \zeta D_2)(aW_x^* + (1-a)P_C(I - \zeta D_3)W_x^*))$ and $W_x^* = (I - \eta A^*(I - M_Q)A)x^*) = x^*$.

From Lemma 2.6, we have that $P_C(I - \zeta D_1), P_C(I - \zeta D_2)$ and $P_C(I - \zeta D_3)$ are nonexpansive.

By the definition of T_n, Lemma 2.4, and Remark 1, we have

$$\begin{aligned}
\|T_n - x^*\|^2 &= \big\|aW_n + (1-a)P_C(I - \zeta D_2)\big(aW_n + (1-a)\\
&\qquad \times P_C(I - \zeta D_3)W_n\big) - \big(aW_{x^*} + (1-a)P_C(I - \zeta D_2)\big(aW_{x^*}\\
&\qquad + (1-a)P_C(I - \zeta D_3)W_{x^*}\big)\big)\big\|^2\\
&= \big\|a(W_n - W_{x^*}) + (1-a)\big[P_C(I - \zeta D_2)\big(aW_n + (1-a)P_C(I - \zeta D_3)W_n\big)\big)\\
&\qquad - P_C(I - \zeta D_2)\big(aW_{x^*} + (1-a)P_C(I - \zeta D_3)W_{x^*}\big)\big]\big\|^2\\
&\leq a\|W_n - W_{x^*}\|^2 + (1-a)\big\|P_C(I - \zeta D_2)\big(aW_n + (1-a)P_C(I - \zeta D_3)W_n\big)\big)\\
&\qquad - P_C(I - \zeta D_2)\big(aW_{x^*} + (1-a)P_C(I - \zeta D_3)W_{x^*}\big)\big)\big\|^2\\
&\leq a\|W_n - W_{x^*}\|^2 + (1-a)\big\|aW_n + (1-a)P_C(I - \zeta D_3)W_n\\
&\qquad - \big(aW_{x^*} + (1-a)P_C(I - \zeta D_3)W_{x^*}\big)\big\|^2\\
&= a\|W_n - W_{x^*}\|^2 + (1-a)\big\|a(W_n - W_{x^*}) + (1-a)\\
&\qquad \times \big[P_C(I - \zeta D_3)W_n - x^*\big]\big\|^2\\
&\leq a\|W_n - W_{x^*}\|^2 + a(1-a)\|W_n - W_{x^*}\|^2 + (1-a)^2\\
&\qquad \times \big\|P_C(I - \zeta D_3)W_n - x^*\big\|^2\\
&= \big(2a - a^2\big)\|W_n - W_{x^*}\|^2 + (1-a)^2\big\|P_C(I - \zeta D_3)W_n - x^*\big\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \left\| W_n - x^* \right\|^2 \\
&= \left\| x_n - \eta A^*(I - M_Q)Ax_n - x^* \right\|^2 \\
&\le \left\| x_n - x^* \right\|^2 - \eta(1-\eta L)\left\| (I - M_Q)Ax_n \right\|^2. \qquad (7)
\end{aligned}
$$

□

3 Main results

Theorem 3.1 *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $S : C \to C$ be a nonexpansive mapping. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3-inverse strongly monotone, respectively, with $d^* = \min\{d_1, d_2, d_3\}$. Let $\bar{D_1}, \bar{D_2}, \bar{D_3} : Q \to H_2$ be $\bar{d}_1, \bar{d}_2, \bar{d}_3$-inverse strongly monotone, respectively, with $\hat{d} = \min\{\bar{d}_1, \bar{d}_2, \bar{d}_3\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C : H_1 \to C$ by*

$$M_C(x) = P_C(I - \zeta D_1)\big(ax + (1-a)P_C(I - \zeta D_2)\big(ax + (1-a)P_C(I - \zeta D_3)x\big)\big),$$

$\forall x \in H_1$, where $a \in [0,1)$, $\zeta \in (0, 2d^*)$, and define $M_Q : H_2 \to Q$ by

$$M_Q(x) = P_Q(I - \bar{\zeta}\bar{D_1})\big(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\bar{D_2})\big(a\hat{x} + (1-a)P_Q(I - \bar{\zeta}\bar{D_3})\hat{x}\big)\big),$$

$\forall \hat{x} \in H_1$, where $a \in [0,1)$, $\bar{\zeta} \in (0, 2\hat{d})$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1 \in H_1$ and

$$y_n = M_C W_n = P_C(I - \zeta D_1)T_n,$$

where $W_n = (I - \eta A^*(I - M_Q)A)x_n$ and $T_n = aW_n + (1-a)P_C(I - \zeta D_2)(aW_n + (1-a)P_C(I - \zeta D_3)W_n))$.

$$
\begin{aligned}
&Q_n = \big\{ z \in H : \big\langle (I - \zeta D_1)T_n - y_n, y_n - z \big\rangle \ge 0 \big\}, \\
&x_{n+1} = \alpha_n T_n + (1-\alpha_n) S P_{Q_n}\big(T_n - \zeta D_1(y_n)\big)
\end{aligned}
$$

for all $n \in \mathbb{N}$.

Assume that the following conditions hold:

(i) *$\Im = F(S) \bigcap \bigcap_{i=1}^{3} \Phi_i \neq \emptyset$, where $\Phi_i = \{w \in VI(C, D_i) | Aw \in VI(Q, \bar{D}_i)\}$ for all $i = 1, 2, 3$.*

(ii) *$\alpha_n \in [c, d] \subset (0, 1)$.*

Then $\{x_n\}$ converges weakly to $x_0 = P_{\Im} x_n$, which $(x_0, y_0, z_0) \in \Omega_{\bar{D_1}, \bar{D_2}, \bar{D_3}}^{D_1, D_2, D_3}$, $y_0 = P_C(I - \zeta D_2)(ax_0 + (1-a)z_0)$, and $z_0 = P_C(I - \zeta D_3)x_0$ with $\bar{x_0} = Ax_0$, $\bar{y_0} = Ay_0$ and $\bar{z_0} = Az_0$.

Proof Denote $k_n := P_{Q_n}(T_n - \zeta D_1(y_n))$ for all $n \ge 0$. Let $x^* \in \Im$. From the definition of P_{Q_n}, we have $y_n = P_{Q_n}(I - \zeta D_1)T_n$. Let $M_n = T_n - \zeta D_1(y_n)$. From $C \subseteq Q_n$, and applying (6), we have

$$
\begin{aligned}
\left\| k_n - x^* \right\|^2 &= \left\| P_{Q_n} M_n - x^* \right\|^2 \\
&\le \left\| M_n - x^* \right\|^2 - \| M_n - P_{Q_n} M_n \|^2 \\
&= \left\| T_n - \zeta D_1(y_n) - x^* \right\|^2 - \left\| T_n - \zeta D_1(y_n) - P_{Q_n} M_n \right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| T_n - x^* \right\|^2 - 2\zeta \left\langle T_n - x^*, D_1(y_n) \right\rangle + \zeta^2 \left\| D_1(y_n) \right\|^2 \\
&\quad - \| T_n - P_{Q_n} M_n \|^2 + 2\zeta \left\langle T_n - P_{Q_n} M_n, D_1(y_n) \right\rangle - \zeta^2 \left\| D_1(y_n) \right\|^2 \\
&= \left\| T_n - x^* \right\|^2 - \| T_n - P_{Q_n} M_n \|^2 - 2\zeta \left\langle P_{Q_n} M_n - x^*, D_1(y_n) \right\rangle.
\end{aligned} \tag{8}
$$

From the monotonicity of D_1, we have

$$
\begin{aligned}
0 &\leq \left\langle D_1 y_n - D_1 x^*, y_n - x^* \right\rangle \\
&= \left\langle D_1 y_n, y_n - x^* \right\rangle - \left\langle D_1 x^*, y_n - x^* \right\rangle \\
&\leq \left\langle D_1 y_n, y_n - x^* \right\rangle \\
&= \langle D_1 y_n, y_n - P_{Q_n} M_n \rangle - \langle D_1 y_n, x^* - P_{Q_n} M_n \rangle,
\end{aligned}
$$

which implies that

$$
\langle D_1 y_n, x^* - P_{Q_n} M_n \rangle \leq \langle D_1 y_n, y_n - P_{Q_n} M_n \rangle. \tag{9}
$$

From (8) and (9), we have

$$
\left\| k_n - x^* \right\|^2 \leq \left\| T_n - x^* \right\|^2 - \| T_n - P_{Q_n} M_n \|^2 + 2\zeta \langle D_1 y_n, y_n - P_{Q_n} M_n \rangle. \tag{10}
$$

From (10) and Lemma 2.7, we have

$$
\begin{aligned}
\left\| k_n - x^* \right\|^2 &\leq \left\| x_n - x^* \right\|^2 - \eta(1 - \eta L) \left\| (I - M_Q) A x_n \right\|^2 - \| P_{Q_n} M_n - T_n \|^2 \\
&\quad + 2\zeta \langle D_1 y_n, y_n - P_{Q_n} M_n \rangle \\
&= \left\| x_n - x^* \right\|^2 - \eta(1 - \eta L) \left\| (I - M_Q) A x_n \right\|^2 - \| P_{Q_n} M_n - y_n \|^2 \\
&\quad - \| y_n - T_n \|^2 - 2 \langle P_{Q_n} M_n - y_n, y_n - T_n \rangle \\
&\quad + 2\zeta \langle D_1 y_n, y_n - P_{Q_n} M_n \rangle \\
&= \left\| x_n - x^* \right\|^2 - \eta(1 - \eta L) \left\| (I - M_Q) A x_n \right\|^2 - \| P_{Q_n} M_n - y_n \|^2 \\
&\quad - \| y_n - T_n \|^2 + 2 \langle P_{Q_n} M_n - y_n, T_n - y_n - \zeta D_1 y_n \rangle \\
&= \left\| x_n - x^* \right\|^2 - \eta(1 - \eta L) \left\| (I - M_Q) A x_n \right\|^2 - \| P_{Q_n} M_n - y_n \|^2 \\
&\quad - \| y_n - T_n \|^2 + 2 \left\langle (I - \zeta D_1) T_n - y_n, P_{Q_n} M_n - y_n \right\rangle \\
&\quad + 2 \langle \zeta D_1 T_n - \zeta D_1 y_n, P_{Q_n} M_n - y_n \rangle \\
&\leq \left\| x_n - x^* \right\|^2 - \eta(1 - \eta L) \left\| (I - M_Q) A x_n \right\|^2 - \| P_{Q_n} M_n - y_n \|^2 \\
&\quad - \| y_n - T_n \|^2 + 2\zeta \| D_1 T_n - D_1 y_n \| \| P_{Q_n} M_n - y_n \| \\
&\leq \left\| x_n - x^* \right\|^2 - \eta(1 - \eta L) \left\| (I - M_Q) A x_n \right\|^2 - \| P_{Q_n} M_n - y_n \|^2 \\
&\quad - \| y_n - T_n \|^2 + \frac{\zeta}{d_1} \left[\| T_n - y_n \|^2 + \| P_{Q_n} M_n - y_n \|^2 \right] \\
&= \left\| x_n - x^* \right\|^2 - \eta(1 - \eta L) \left\| (I - M_Q) A x_n \right\|^2
\end{aligned}
$$

$$-\left(1-\frac{\zeta}{d_1}\right)\|P_{Q_n}M_n-y_n\|^2-\left(1-\frac{\zeta}{d_1}\right))\|T_n-y_n\|^2. \tag{11}$$

By the definition of x_{n+1}, (11), and Lemma 2.7, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n(T_n-x^*)+(1-\alpha_n)(Sk_n-x^*)\|^2\\
&\le \alpha_n\|T_n-x^*\|^2+(1-\alpha_n)\|Sk_n-x^*\|^2\\
&= \alpha_n\|T_n-x^*\|^2+(1-\alpha_n)\|Sk_n-x^*\|^2\\
&\quad -\alpha_n(1-\alpha_n)\|T_n-Sk_n\|^2 \qquad (12)\\
&= \alpha_n\|T_n-x^*\|^2+(1-\alpha_n)\|k_n-x^*\|^2\\
&\le \alpha_n\|T_n-x^*\|^2+(1-\alpha_n)\Big[\|x_n-x^*\|^2\\
&\quad -\eta(1-\eta L)\|(I-M_Q)Ax_n\|^2\\
&\quad -\left(1-\frac{\zeta}{d_1}\right)\|P_{Q_n}M_n-y_n\|^2-\left(1-\frac{\zeta}{d_1}\right))\|T_n-y_n\|^2\Big]\\
&\le \alpha_n\big[\|x_n-x^*\|^2-\alpha_n\eta(1-\eta L)\|(I-M_Q)Ax_n\|^2\big]\\
&\quad +(1-\alpha_n)\Big[\|x_n-x^*\|^2-\eta(1-\eta L)\|(I-M_Q)Ax_n\|^2\\
&\quad -\left(1-\frac{\zeta}{d_1}\right)\|P_{Q_n}M_n-y_n\|^2-\left(1-\frac{\zeta}{d_1}\right))\|T_n-y_n\|^2\Big]\\
&= \|x_n-x^*\|^2-\eta(1-\eta L)(1+\alpha_n)\|(I-M_Q)Ax_n\|^2\\
&\quad -(1-\alpha_n)\left(1-\frac{\zeta}{d_1}\right)\big[\|T_n-y_n\|^2+\|y_n-k_n\|^2\big]. \qquad (13)
\end{aligned}$$

So,

$$\|x_{n+1}-x^*\|^2\le\|x_n-x^*\|^2.$$

Therefore $\lim_{n\to\infty}\|x_{n+1}-x^*\|$ exists, $\forall x^*\in\Im$. So, we have $\{x_n\}_{n=0}^{\infty}$ and $\{k_n\}_{n=0}^{\infty}$ are bounded. From the last relations it follows that

$$\eta(1-\eta L)(1+\alpha_n)\|(I-M_Q)Ax_n\|^2\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2$$

or

$$\|(I-M_Q)Ax_n\|^2\le\frac{\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2}{\eta(1-\eta L)(1+\alpha_n)}.$$

Thus

$$\lim_{n\to\infty}\|(I-M_Q)Ax_n\|=0. \tag{14}$$

By using the same method as above, we have

$$\lim_{n\to\infty}\|T_n-y_n\|=0. \tag{15}$$

From (12), we get

$$\begin{aligned}\left\|x_{n+1}-x^*\right\|^2 &\le \alpha_n\left\|T_n-x^*\right\|^2+(1-\alpha_n)\left\|Sk_n-x^*\right\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|T_n-Sk_n\|^2\\ &\le \alpha_n\left\|T_n-x^*\right\|^2+(1-\alpha_n)\left\|x_n-x^*\right\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|T_n-Sk_n\|^2\\ &\le \alpha_n\left\|x_n-x^*\right\|^2-\alpha_n(1-\alpha_n)\|T_n-Sk_n\|^2,\end{aligned}$$

so

$$\|T_n-Sk_n\|^2 \le \frac{\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2}{\alpha_n(1-\alpha_n)},$$

which implies that

$$\lim_{n\to\infty}\|T_n-Sk_n\|=0. \tag{16}$$

Consider

$$W_n-x_n=-\eta A^*(I-M_Q)Ax_n,$$

and by (14), we have

$$\lim_{n\to\infty}\|W_n-x_n\|=0. \tag{17}$$

From the property of P_C, we have

$$\begin{aligned}&\left\|P_C(I-\zeta D_3)W_n-x^*\right\|^2\\ &\quad=\left\|P_C(I-\zeta D_3)W_n-P_C(I-\zeta D_3)x^*\right\|^2\\ &\quad\le\left\|(I-\zeta D_3)W_n-(I-\zeta D_3)x^*\right\|^2\\ &\quad=\left\|(W_n-x^*)-\zeta(D_3W_n-D_3x^*)\right\|^2\\ &\quad=\left\|W_n-x^*\right\|^2-2\zeta\langle W_n-x^*,D_3W_n-D_3x^*\rangle\\ &\qquad+\zeta^2\left\|D_3W_n-D_3x^*\right\|^2\\ &\quad\le\left\|W_n-x^*\right\|^2-2\zeta d_3\left\|D_3W_n-D_3x^*\right\|^2\\ &\qquad+\zeta^2\left\|D_3W_n-D_3x^*\right\|^2\\ &\quad=\left\|W_n-x^*\right\|^2-\zeta(2d_3-\zeta)\left\|D_3W_n-D_3x^*\right\|^2\\ &\quad\le\left\|x_n-x^*\right\|^2-\zeta(2d_3-\zeta)\left\|D_3W_n-D_3x^*\right\|^2.\end{aligned} \tag{18}$$

By the definition of T_n, (7), Remark 1, and (18), we have

$$\left\|T_n-x^*\right\|^2\le a\|W_n-W_{x^*}\|^2+a(1-a)\|W_n-W_{x^*}\|^2$$

$$\begin{aligned}
&\quad + (1-a)^2 \|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\le a\|x_n - x^*\|^2 + a(1-a)\|x_n - x^*\|^2 \\
&\quad + (1-a)^2 \|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\le (2a - a^2)\|x_n - x^*\|^2 + (1-a)^2\|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\le (2a - a^2)\|x_n - x^*\|^2 + (1-a)^2\big[\|x_n - x^*\|^2 \\
&\quad - \zeta(2d_3 - \zeta)\|D_3W_n - D_3x^*\|^2\big] \\
&= \|x_n - x^*\|^2 - \zeta(2d_3 - \zeta)(1-a)^2\|D_3W_n - D_3x^*\|^2.
\end{aligned} \tag{19}$$

In addition, by the definition of x_{n+1} and (19), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|k_n - x^*\|^2 \\
&\le \alpha_n\big[\|x_n - x^*\|^2 - \zeta(2d_3 - \zeta)(1-a)^2\|D_3W_n - D_3x^*\|^2\big] \\
&\quad + (1-\alpha_n)\|k_n - x^*\|^2 \\
&= \alpha_n\|x_n - x^*\|^2 - \alpha_n\zeta(2d_3 - \zeta)(1-a)^2\|D_3W_n - D_3x^*\|^2 \\
&\quad + (1-\alpha_n)\|x_n - x^*\|^2 \\
&= \|x_n - x^*\|^2 - \alpha_n\zeta(2d_3 - \zeta)(1-a)^2\|D_3W_n - D_3x^*\|^2,
\end{aligned}$$

so

$$\|D_3W_n - D_3x^*\|^2 \le \frac{\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2}{\alpha_n\zeta(2d_3 - \zeta)(1-a)^2},$$

which implies that

$$\lim_{n\to\infty}\|D_3W_n - D_3x^*\| = 0. \tag{20}$$

From the property of P_C, we have

$$\begin{aligned}
&\|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\quad \le \big\langle (I-\zeta D_3)W_n - (I-\zeta D_3)x^*, P_C(I-\zeta D_3)W_n - x^*\big\rangle \\
&\quad = \frac{1}{2}\big[\|(I-\zeta D_3)W_n - (I-\zeta D_3)x^*\|^2\big] + \|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\qquad - \|(I-\zeta D_3)W_n - (I-\zeta D_3)x^* - (P_C(I-\zeta D_3)W_n - x^*)\|^2\big] \\
&\quad \le \frac{1}{2}\big[\|W_n - x^*\|^2 + \|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\qquad - \|(I-\zeta D_3)W_n - (I-\zeta D_3)x^* - (P_C(I-\zeta D_3)W_n - x^*)\|^2\big] \\
&\quad = \frac{1}{2}\big[\|W_n - x^*\|^2 + \|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\qquad - \|(W_n - P_C(I-\zeta D_3)W_n) - \zeta(D_3W_n - D_3x^*)\|^2\big]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\big[\|W_n - x^*\|^2 + \|P_C(I - \zeta D_3)W_n - x^*\|^2 \\
&\quad - \|W_n - P_C(I - \zeta D_3)W_n\|^2 - \zeta^2\|D_3W_n - D_3x^*\|^2 \\
&\quad + 2\zeta\langle W_n - P_C(I - \zeta D_3)W_n, D_3W_n - D_3x^*\rangle\big],
\end{aligned}$$

so

$$\begin{aligned}
\|P_C(I - \zeta D_3)W_n - x^*\|^2 \le{}& \|W_n - x^*\|^2 - \|W_n - P_C(I - \zeta D_3)W_n\|^2 \\
&+ 2\zeta\|W_n - P_C(I - \zeta D_3)W_n\|\|D_3W_n - D_3x^*\|.
\end{aligned} \tag{21}$$

By the definition of T_n, (7), Remark 1, and (21), we have

$$\begin{aligned}
&\|T_n - x^*\|^2 \\
&\quad \le a\|W_n - W_{x^*}\|^2 + a(1-a)\|W_n - W_{x^*}\|^2 \\
&\qquad + (1-a)^2\|P_C(I - \zeta D_3)W_n - x^*\|^2 \\
&\quad \le a\|x_n - x^*\|^2 + a(1-a)\|x_n - x^*\|^2 \\
&\qquad + (1-a)^2\|P_C(I - \zeta D_3)W_n - x^*\|^2 \\
&\quad \le (2a - a^2)\|x_n - x^*\|^2 + (1-a)^2\|P_C(I - \zeta D_3)W_n - x^*\|^2 \\
&\quad \le (2a - a^2)\|x_n - x^*\|^2 + (1-a)^2\big[\|W_n - x^*\|^2 - \|W_n - P_C \\
&\qquad \times (I - \zeta D_3)W_n\|^2 + 2\zeta\|W_n - P_C(I - \zeta D_3)W_n\|\|D_3W_n - D_3x^*\|\big] \\
&\quad = (2a - a^2)\|x_n - x^*\|^2 + (1-a)^2\|x_n - x^*\|^2 \\
&\qquad - (1-a)^2\|W_n - P_C(I - \zeta D_3)W_n\|^2 \\
&\qquad + 2\zeta(1-a)^2\|W_n - P_C(I - \zeta D_3)W_n\|\|D_3W_n - D_3x^*\| \\
&\quad = \|x_n - x^*\|^2 - (1-a)^2\|W_n - P_C(I - \zeta D_3)W_n\|^2 \\
&\qquad + 2\zeta(1-a)^2\|W_n - P_C(I - \zeta D_3)W_n\|\|D_3W_n - D_3x^*\|.
\end{aligned} \tag{22}$$

In addition, by the definition of x_{n+1}, (11), and (22), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|k_n - x^*\|^2 \\
&\le \alpha_n\big[\|x_n - x^*\|^2 - (1-a)^2\|W_n - P_C(I - \zeta D_3)W_n\|^2 \\
&\quad + 2\zeta(1-a)^2\|W_n - P_C(I - \zeta D_3)W_n\|\|D_3W_n - D_3x^*\|\big] \\
&\quad + (1-\alpha_n)\|k_n - x^*\|^2 \\
&\le \alpha_n\|x_n - x^*\|^2 - \alpha_n(1-a)^2\|W_n - P_C(I - \zeta D_3)W_n\|^2 \\
&\quad + 2\alpha_n\zeta(1-a)^2\|W_n - P_C(I - \zeta D_3)W_n\|\|D_3W_n - D_3x^*\| \\
&\quad + (1-\alpha_n)\|x_n - x^*\|^2 \\
&= \|x_n - x^*\|^2 - \alpha_n(1-a)^2\|W_n - P_C(I - \zeta D_3)W_n\|^2
\end{aligned}$$

$$+ 2\alpha_n\zeta(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\|. \tag{23}$$

From (20) and (23), we get

$$\lim_{n\to\infty}\|W_n - P_C(I-\zeta D_3)W_n\| = 0. \tag{24}$$

Let $G_n = aW_n + (1-a)P_C(I-\lambda_3D_3)W_n$. From the property of P_C, we have

$$\begin{aligned}
&\|P_C(I-\zeta D_2)G_n - x^*\|^2\\
&\quad = \|P_C(I-\zeta D_2)G_n - P_C(I-\zeta D_2)x^*\|^2\\
&\quad \le \|(I-\zeta D_2)G_n - (I-\zeta D_2)x^*\|^2\\
&\quad = \|(G_n - x^*) - \zeta(D_2G_n - D_2x^*)\|^2\\
&\quad = \|G_n - x^*\|^2 - 2\zeta\langle G_n - x^*, D_2G_n - D_2x^*\rangle\\
&\qquad + \zeta^2\|D_2G_n - D_2x^*\|^2\\
&\quad \le \|x_n - x^*\|^2 - 2\zeta d_2\|D_2G_n - D_2x^*\|^2\\
&\qquad + \zeta^2\|D_2G_n - D_2x^*\|^2\\
&\quad = \|x_n - x^*\|^2 - \zeta(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2.
\end{aligned} \tag{25}$$

By the definition of T_n and (25), we have

$$\begin{aligned}
\|T_n - x^*\|^2 &\le a\|W_n - W_{x^*}\|^2 + (1-a)\|P_C(I-\zeta D_2)G_n - x^*\|^2\\
&\le a\|x_n - x^*\|^2 + (1-a)\bigl[\|x_n - x^*\|^2\\
&\quad - \zeta(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2\bigr]\\
&= \|x_n - x^*\|^2 - \zeta(1-a)(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2.
\end{aligned} \tag{26}$$

In addition, by the definition of x_{n+1} and (26), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|k_n - x^*\|^2\\
&\le \alpha_n\bigl[\|x_n - x^*\|^2 - \zeta(1-a)(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2\bigr]\\
&\quad + (1-\alpha_n)\|x_n - x^*\|^2\\
&= \|x_n - x^*\|^2 - \zeta\alpha_n(1-\alpha_n)(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2,
\end{aligned}$$

so

$$\|D_2G_n - D_2x^*\|^2 \le \frac{\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2}{\zeta\alpha_n(1-\alpha_n)(2d_2-\zeta)}.$$

It implies that

$$\lim_{n\to\infty}\|D_2G_n - D_2x^*\| = 0. \tag{27}$$

From the property of P_C, we have

$$\begin{aligned}
&\|P_C(I-\zeta D_2)G_n - x^*\|^2\\
&\quad= \langle (I-\zeta D_2)G_n - (I-\zeta D_2)x^*, P_C(I-\zeta D_2)G_n - x^*\rangle\\
&\quad= \frac{1}{2}\big[\|(I-\zeta D_2)G_n - (I-\zeta D_2)x^*\|^2 + \|P_C(I-\zeta D_2)G_n - x^*\|^2\\
&\qquad - \|(I-\zeta D_2)G_n - (I-\zeta D_2)x^* - ((I-\zeta D_2)G_n - x^*)\|^2\big]\\
&\quad\le \frac{1}{2}\big[\|G_n - x^*\|^2 + \|P_C(I-\zeta D_2)G_n - x^*\|^2\\
&\qquad - \|(I-\zeta D_2)G_n - (I-\zeta D_2)x^* - ((I-\zeta D_2)G_n - x^*)\|^2\big]\\
&\quad= \frac{1}{2}\big[\|G_n - x^*\|^2 + \|P_C(I-\zeta D_2)G_n - x^*\|^2\\
&\qquad - \|(G_n - P_C(I-\zeta D_2)G_n) - \zeta(D_2G_n - D_2x^*)\|^2\big]\\
&\quad= \frac{1}{2}\big[\|G_n - x^*\|^2 + \|P_C(I-\zeta D_2)G_n - x^*\|^2\\
&\qquad - \|G_n - P_C(I-\zeta D_2)G_n\|^2\\
&\qquad + 2\zeta\langle G_n - P_C(I-\zeta D_2)G_n, D_2G_n - D_2x^*\rangle\\
&\qquad - \zeta^2\|D_2G_n - D_2x^*\|^2\big].
\end{aligned}$$

It implies that

$$\begin{aligned}
&\|P_C(I-\zeta D_2)G_n - x^*\|^2\\
&\quad\le \|G_n - x^*\|^2 - \|G_n - P_C(I-\zeta D_2)G_n\|^2\\
&\qquad + 2\zeta\langle G_n - P_C(I-\zeta D_2)G_n, D_2G_n - D_2x^*\rangle\\
&\quad\le \|G_n - x^*\|^2 - \|G_n - P_C(I-\zeta D_2)G_n\|^2\\
&\qquad + 2\zeta\|G_n - P_C(I-\zeta D_2)G_n\|\|D_2G_n - D_2x^*\|.
\end{aligned}\tag{28}$$

By the definition of T_n and (28), we have

$$\begin{aligned}
\|T_n - x^*\|^2 &\le a\|W_n - W_{x^*}\|^2 + (1-a)\|P_C(I-\zeta D_2)G_n - x^*\|^2\\
&\le a\|x_n - x^*\|^2 + (1-a)\big[\|G_n - x^*\|^2 - \|G_n - P_C\\
&\qquad \times (I-\zeta D_2)G_n\|^2 + 2\zeta\|G_n - P_C(I-\zeta D_2)G_n\|\|D_2G_n - D_2x^*\|\big]\\
&\le a\|x_n - x^*\|^2 + (1-a)\|x_n - x^*\|^2\\
&\qquad - (1-a)\|G_n - P_C(I-\zeta D_2)G_n\|^2\\
&\qquad + 2\zeta\|G_n - P_C(I-\zeta D_2)G_n\|\|D_2G_n - D_2x^*\|]\\
&= \|x_n - x^*\|^2 - (1-a)\|G_n - P_C(I-\zeta D_2)G_n\|^2\\
&\qquad + 2\zeta\|G_n - P_C(I-\zeta D_2)G_n\|\|D_2G_n - D_2x^*\|].
\end{aligned}\tag{29}$$

In addition, by the definition of x_{n+1} and (29), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|T_n-x^*\|^2+(1-\alpha_n)\|k_n-x^*\|^2\\ &\le \alpha_n\big[\|x_n-x^*\|^2-(1-a)\|G_n-P_C(I-\zeta D_2)G_n\|^2\\ &\quad +2\zeta\|G_n-P_C(I-\zeta D_2)G_n\|\|D_2G_n-D_2x^*\|\big]\\ &\quad +(1-\alpha_n)\|x_n-x^*\|^2\\ &=\|x_n-x^*\|^2-\alpha_n(1-a)\|G_n-P_C(I-\zeta D_2)G_n\|^2\\ &\quad +2\zeta\alpha_n(1-a)\|G_n-P_C(I-\zeta D_2)G_n\|\|D_2G_n-D_2x^*\|, \end{aligned}\tag{30}$$

by (30) and (27), we get

$$\lim_{n\to\infty}\|G_n-P_C(I-\zeta D_2)G_n\|=0.\tag{31}$$

Since

$$T_n-W_n=(1-a)\big(P_C(I-\zeta D_2)\big(aW_n+(1-a)P_C(I-\zeta D_3)W_n\big)-W_n\big).$$

From the property of norm, we have

$$\begin{aligned}&\|P_C(I-\zeta D_2)\big(aW_n+(1-a)P_C(I-\zeta D_3)W_n\big)-W_n\|\\ &\quad\le\|P_C(I-\zeta D_2)\big(aW_n+(1-a)P_C(I-\zeta D_3)W_n\big)\\ &\qquad-\big(aW_n+(1-a)P_C(I-\zeta D_3)W_n\big)\|\\ &\qquad+\|\big(aW_n+(1-a)P_C(I-\zeta D_3)W_n-W_n\|\\ &\quad=\|P_C(I-\zeta D_2)G_n-G_n\|+(1-a)\|P_C(I-\zeta D_3)W_n-W_n\|.\end{aligned}\tag{32}$$

Then we have

$$\begin{aligned}\|T_n-W_n\|&\le(1-a)\big[\|P_C(I-\zeta D_2)G_n-G_n\|\\ &\quad+(1-a)\|P_C(I-\zeta D_3)W_n-W_n\|\big].\end{aligned}$$

From (24) and (31), it implies that

$$\lim_{n\to\infty}\|T_n-W_n\|=0.\tag{33}$$

From (15), (17), (33), and

$$\|y_n-x_n\|\le\|y_n-T_n\|+\|T_n-W_n\|+\|W_n-x_n\|,$$

we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0.\tag{34}$$

Moreover, from (16), (15), (34), and

$$\|x_n - Sk_n\| \le \|x_n - y_n\| + \|y_n - T_n\| + \|T_n - Sk_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - Sk_n\| = 0. \tag{35}$$

Since $\{x_n\}_{n=0}^{\infty}$ is bounded, it has a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ which weakly converges to some $\bar{x} \in C$.

Assume $\bar{x} \notin F(S)$. By the nonexpansiveness of S and Opial's property and (35), we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| &< \liminf_{k\to\infty} \|x_{n_k} - S\bar{x}\| \\
&\le \liminf_{k\to\infty} \bigl[\|x_{n_k} - Sk_{n_k}\| + \|Sk_{n_k} - S\bar{x}\|\bigr] \\
&\le \liminf_{k\to\infty} \bigl[\|x_{n_k} - Sk_{n_k}\| + \|k_{n_k} - \bar{x}\|\bigr] \\
&= \liminf_{k\to\infty} \|k_{n_k} - \bar{x}\| \\
&\le \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\|.
\end{aligned}$$

This is a contradiction, then we have

$$\bar{x} \in F(S).$$

Assume $\bar{x} \notin \bigcap_{i=1}^{3} \Phi_i$. From Lemma 2.6, we have $\bar{x} \notin F(M_C(I - \eta A^*(I - M_Q)A))$. By Opial's condition, (34), and Remark 1, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| &< \liminf_{k\to\infty} \bigl\|x_{n_k} - M_C\bigl(I - \eta A^*(I - M_Q)A\bigr)\bar{x}\bigr\| \\
&\le \liminf_{k\to\infty} \|x_{n_k} - y_{n_k}\| + \liminf_{k\to\infty} \bigl\|M_C\bigl(x_{n_k} - \eta A^* \\
&\quad \times (I - M_Q)Ax_{n_k}\bigr) - M_C\bigl(I - \eta A^*(I - M_Q)A\bigr)\bar{x}\bigr\| \\
&\le \liminf_{k\to\infty} \bigl(\|x_{n_k} - y_{n_k}\| + \|x_{n_k} - \bar{x}\|\bigr) \\
&= \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\|.
\end{aligned} \tag{36}$$

This is a contradiction, then we have

$$\bar{x} \in F\bigl(M_C\bigl(I - \eta A^*(I - M_Q)A\bigr)\bigr).$$

It implies that

$$\bar{x} \in \bigcap_{i=1}^{3} \Phi_i.$$

Hence

$$\bar{x} \in \Im.$$

In order to show that the entire sequence $\{x_n\}$ weakly converges to $\bar{x}$, assume $\{x\}_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$, with $\bar{x} \neq \hat{x}$ and $\hat{x} \in \Im$. By Opial's condition, we have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - \bar{x}\| &= \lim_{k\to\infty} \inf \|x_{n_k} - \bar{x}\| \\ &< \lim_{k\to\infty} \inf \|x_{n_k} - \hat{x}\| \\ &= \lim_{n\to\infty} \|x_n - \hat{x}\| \\ &= \lim_{n\to\infty} \inf \|x_{n_k} - \hat{x}\| \\ &< \lim_{n\to\infty} \inf \|x_{n_k} - \bar{x}\| \\ &= \lim_{n\to\infty} \|x_n - \bar{x}\|. \end{aligned}$$

This is a contradiction, thus

$$\bar{x} = \hat{x}.$$

It implies that the sequence $\{x_n\}_{n=0}^{\infty}$ weakly converges to $\bar{x} \in \Im$.

From (34), we have $\{y_n\}_{n=0}^{\infty}$ weakly converges to $\bar{x} \in \Im$.

Finally, if we take

$$U_n = P_{\Im} x_n, \tag{37}$$

by Lemma 2.2, we see that $\{P_{\Im} x_n\}_{n=0}^{\infty}$ converges strongly to some $z \in \Im$. From (37), we get

$$\langle \bar{x} - U_n, U_n - x_n \rangle \geq 0, \quad \forall \bar{x} \in \Im.$$

Take $n \to \infty$, we also have

$$\langle \bar{x} - z, z - \bar{x} \rangle \geq 0,$$

and hence $\bar{x} = z$. Therefore U_n converges strongly to $\bar{x} \in \Im$, this completes the proof. □

4 Application

Let C be a closed convex subset of H. The standard constrained convex optimization problem is to find $x^* \in C$ such that

$$\Im(x^*) = \min_{x \in C} \Im(x), \tag{38}$$

where $\Im : C \to \mathbb{R}$ is a convex, Frechet differentiable function. The set of all solution of (38) is denoted by $\Phi_{\Im}$.

Lemma 4.1 ([25] Optimality condition) *A necessary condition of optimality for a point* $x^* \in C$ *to be a solution of the minimization problem* (38) *is that* x^* *solves the variational inequality*

$$\langle \nabla \Im(x^*), x - x^* \rangle \geq 0 \tag{39}$$

for all $x \in C$. *Equivalently,* $x^* \in C$ *solves the fixed point equation*

$$x^* = P_C(I - \zeta \nabla \Im)x^*$$

for every $\zeta > 0$. *If, in addition,* $\Im$ *is convex, then the optimality condition* (39) *is also sufficient.*

By using the concept of the split modified system of variational inequalities problem (SMSVIP), we consider the problem for finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \zeta \nabla \Im_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \zeta \nabla \Im_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \zeta \nabla \Im_3)x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{40}$$

and finding $(\bar{x^*} = Ax^*, \bar{y^*} = Ay^*, \bar{z^*} = Az^*) \in Q \times Q \times Q$ such that

$$\begin{cases} \langle \bar{x^*} - (I - \bar{\zeta} \nabla \bar{\Im}_1)(a\bar{x^*} + (1-a)\bar{y^*}), \bar{x} - \bar{x^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{y^*} - (I - \bar{\zeta} \nabla \bar{\Im}_2)(a\bar{x^*} + (1-a)\bar{z^*}), \bar{x} - \bar{y^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{z^*} - (I - \bar{\zeta} \nabla \bar{\Im}_3)\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, & \forall \bar{x} \in Q, \end{cases} \tag{41}$$

where $\Im_1, \Im_2, \Im_3 : C \to \mathbb{R}$ with $\nabla \Im_1, \nabla \Im_2, \nabla \Im_3$ are the gradients of $\Im_1, \Im_2, \Im_3$, respectively, and $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3 : Q \to \mathbb{R}$ with $\nabla \bar{\Im}_1, \nabla \bar{\Im}_2, \nabla \bar{\Im}_3$ are the gradients of $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3$, respectively, $\zeta, \bar{\zeta} > 0$ and $a \in [0,1]$. The sets of all solution of (40) and (41) are denoted by $\Psi_{\nabla \Im_1, \nabla \Im_2, \nabla \Im_3}$ and $\Psi_{\nabla \bar{\Im}_1, \nabla \bar{\Im}_2, \nabla \bar{\Im}_3}$, respectively. The set of all solutions of the split modified system of variational inequalities (SMSVIP) is denoted by $\Psi^{\nabla \Im_1, \nabla \Im_2, \nabla \Im_3}_{\nabla \bar{\Im}_1, \nabla \bar{\Im}_2, \nabla \bar{\Im}_3}$, that is,

$$\Psi^{\nabla \Im_1, \nabla \Im_2, \nabla \Im_3}_{\nabla \bar{\Im}_1, \nabla \bar{\Im}_2, \nabla \bar{\Im}_3} = \left\{ (x^*, y^*, z^*) \in \Psi_{\nabla \Im_1, \nabla \Im_2, \nabla \Im_3} : \left(\bar{x^*}, \bar{y^*}, \bar{z^*}\right) \in \Psi_{\nabla \bar{\Im}_1, \nabla \bar{\Im}_2, \nabla \bar{\Im}_3} \right\}.$$

Lemma 4.2 ([23]) *Let* C *and* Q *be nonempty closed convex subsets of real Hilbert spaces* H_1 *and* H_2, *respectively. Let* $\Im_1, \Im_2, \Im_3 : C \to \mathbb{R}$ *be real-valued convex functions with the gradients* $\nabla \Im_1, \nabla \Im_2, \nabla \Im_3$ *being* $\frac{1}{L_{\Im_1}}, \frac{1}{L_{\Im_2}}, \frac{1}{L_{\Im_3}}$*-inverse strongly monotone and continuous, respectively, where* $\zeta \in (0, \frac{2}{L_\Im})$ *with* $\frac{1}{L_\Im} = \min\{\frac{1}{L_{\Im_1}}, \frac{1}{L_{\Im_2}}, \frac{1}{L_{\Im_3}}\}$. *Let* $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3 : Q \to \mathbb{R}$ *be real-valued convex functions with the gradients* $\nabla \bar{\Im}_1$, $\nabla \bar{\Im}_2$, $\nabla \bar{\Im}_3$ *being* $\frac{1}{L_{\bar{\Im}_1}}, \frac{1}{L_{\bar{\Im}_2}}, \frac{1}{L_{\bar{\Im}_3}}$*-inverse strongly monotone and continuous, respectively, where* $\bar{\zeta} \in (0, \frac{2}{L_{\bar{\Im}}})$ *with* $\frac{1}{L_{\bar{\Im}}} = \min\{\frac{1}{L_{\bar{\Im}_1}}, \frac{1}{L_{\bar{\Im}_2}}, \frac{1}{L_{\bar{\Im}_3}}\}$. *Let* $A : H_1 \to H_2$ *be a bounded linear operator with adjoint* A^* *and* $\eta \in (0, \frac{1}{L})$ *with* L *being the spectral radius of the operator* A^*A. *Define* $M_C : H_1 \to C$ *by* $M_C(x) = P_C(I - \zeta \nabla \Im_1)(ax + (1-a)P_C(I - \zeta \nabla \Im_2)(ax + (1-a)P_C(I - \zeta \nabla \Im_3)x))$, $\forall x \in H_1$, *and define* $M_Q : H_2 \to Q$ *by* $M_Q(\hat{x}) = P_Q(I - \bar{\zeta} \nabla \bar{\Im}_1)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta} \nabla \bar{\Im}_2)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta} \nabla \bar{\Im}_3)\hat{x}))$, $\forall \hat{x} \in H_2$.

Let $\bigcap_{i=1}^{3}\Phi_{\Im_i}\neq\emptyset$ *and* $\Phi_{\Im_i}=\{\Im_i(x)=\min_{x^*\in C}\Im_i(x^*):\bar{\Im}_i(Ax)=\min_{Ax^*\in Q}\bar{\Im}_i(Ax^*)\}$ *for all* $i=1,2,3$. *Then*

$$\bigcap_{i=1}^{3}\Phi_{\Im_i}=F\bigl(M_C\bigl(I-\eta A^*(I-M_Q)A\bigr)\bigr).$$

Theorem 4.3 *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $S:C\to C$ be a nonexpansive mapping. Let $\Im_1,\Im_2,\Im_3:C\to\mathbb{R}$ be real-valued convex functions with the gradients $\nabla\Im_1,\nabla\Im_2,\nabla\Im_3$ being $\frac{1}{L_{\Im_1}},\frac{1}{L_{\Im_2}},\frac{1}{L_{\Im_3}}$-inverse strongly monotone and continuous, respectively, where $\zeta\in(0,\frac{2}{L_{\Im}})$ with $\frac{1}{L_{\Im}}=\min\{\frac{1}{L_{\Im_1}},\frac{1}{L_{\Im_2}},\frac{1}{L_{\Im_3}}\}$. Let $\bar{\Im}_1,\bar{\Im}_2,\bar{\Im}_3:Q\to\mathbb{R}$ be real-valued convex functions with the gradients $\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3$ being $\frac{1}{L_{\bar{\Im}_1}},\frac{1}{L_{\bar{\Im}_2}},\frac{1}{L_{\bar{\Im}_3}}$-inverse strongly monotone and continuous, respectively, where $\bar{\zeta}\in(0,\frac{2}{L_{\bar{\Im}}})$ with $\frac{1}{L_{\bar{\Im}}}=\min\{\frac{1}{L_{\bar{\Im}_1}},\frac{1}{L_{\bar{\Im}_2}},\frac{1}{L_{\bar{\Im}_3}}\}$. Let $A:H_1\to H_2$ be a bounded linear operator with adjoint A^* and $\eta\in(0,\frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C:H_1\to C$ by $M_C(x)=P_C(I-\zeta\nabla\Im_1)(ax+(1-a)P_C(I-\zeta\nabla\Im_2)(ax+(1-a)P_C(I-\zeta\nabla\Im_3)x))$, $\forall x\in H_1$, and define $M_Q:H_2\to Q$ by $M_Q(\hat{x})=P_Q(I-\bar{\zeta}\nabla\bar{\Im}_1)(a\hat{x}+(1-a)P_Q(I-\bar{\zeta}\nabla\bar{\Im}_2)(a\hat{x}+(1-a)P_Q(I-\bar{\zeta}\nabla\bar{\Im}_3)\hat{x}))$, $\forall\hat{x}\in H_2$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1\in H_1$ and*

$$y_n=M_CW_n=P_C(I-\zeta\nabla\Im_1)T_n,$$

where $W_n=(I-\eta A^*(I-M_Q)A)x_n$ and $T_n=aW_n+(1-a)P_C(I-\zeta\nabla\Im_2)(aW_n+(1-a)P_C(I-\zeta\nabla\Im_3)W_n))$.

$$\begin{aligned}&Q_n=\bigl\{z\in H:\bigl\langle(I-\zeta\nabla\Im_1)T_n-y_n,y_n-z\bigr\rangle\geq0\bigr\},\\&x_{n+1}=\alpha_nT_n+(1-\alpha_n)SP_{Q_n}\bigl(T_n-\zeta\nabla\Im_1(y_n)\bigr),\quad\forall n\in\mathbb{N}.\end{aligned}$$

Assume that the following conditions hold:

(i) $\Im=F(S)\bigcap\bigcap_{i=1}^{3}\Phi_{\Im_i}\neq\emptyset$, *where*
$\Phi_{\Im_i}=\{\Im_i(x)=\min_{x^*\in C}\Im_i(x^*):\bar{\Im}_i(Ax)=\min_{Ax^*\in Q}\bar{\Im}_i(Ax^*)\}$ *for all* $i=1,2,3$.

(ii) $\alpha_n\in[c,d]\subset(0,1)$.

Then $\{x_n\}$ converges weakly to $x_0=P_{\Im}x_n$, which $(x_0,y_0,z_0)\in\Omega^{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}$, where $y_0=P_C(I-\zeta\nabla\Im_2)(ax_0+(1-a)z_0)$ and $z_0=P_C(I-\zeta\nabla\Im_3)x_0$ with $\bar{x_0}=Ax_0$, $\bar{y_0}=Ay_0$, and $\bar{z_0}=Az_0$.

Proof By using Theorem 3.1 and Lemma 4.2, we obtain the conclusion. □

5 Example and numerical results

In this section, we give the following example to support our main theorem.

Example 5.1 Let $\mathbb{R}$ be the set of real numbers, $C:=\{x\in H|1\leq2x_1+x_2\leq7\}$, $Q:=\{x\in H|-10\leq3x_1-x_2\leq20\}$, $H_1=H_2=\mathbb{R}^2$. Let $D_1,D_2,D_3:C\to\mathbb{R}^2$ be defined by $D_1(x_1,x_2)=(x_1-2,x_2+1)$, $D_2(x_1,x_2)=(x_1-3,x_2-\frac{5}{2})$, and $D_3(x_1,x_2)=(x_1+2,x_2-6)$ for all $(x_1,x_2)\in C$. Let $\bar{D_1},\bar{D_2},\bar{D_3}:Q\to\mathbb{R}^2$ be defined by $\bar{D_1}(\bar{x_1},\bar{x_2})=(\bar{x_1}-4,\bar{x_2}+8)$, $\bar{D_2}(\bar{x_1},\bar{x_2})=(\bar{x_1}-12,\bar{x_2}-8)$, and $\bar{D_3}(\bar{x_1},\bar{x_2})=(\bar{x_1}+16,\bar{x_2}-30)$ for all $(\bar{x_1},\bar{x_2})\in Q$. Let $A:\mathbb{R}^2\to\mathbb{R}^2$ be defined by

$A(x_1,x_2) = (2x_1,2x_2)$ and $A^*:\mathbb{R}^2 \to \mathbb{R}$ be defined by $A^*(x_1,x_2) = (2x_1,2x_2)$. Define $M_C : H_1 \to C$ by $M_C(x) = P_C(I-\frac{1}{2}D_1)(\frac{1}{2}x+\frac{1}{2}P_C(I-\frac{1}{2}D_2)(\frac{1}{2}x+\frac{1}{2}P_C(I-\frac{1}{2}D_3)x)), \forall x = (x_1,x_2) \in H_1$, define $M_Q : H_2 \to Q$ by $M_Q(\hat{x}) = P_Q(I-\frac{1}{5}\bar{D}_1)(\frac{1}{2}\hat{x}+\frac{1}{2}P_Q(I-\frac{1}{5}\bar{D}_2)(\frac{1}{2}\hat{x}+\frac{1}{2}P_Q(I-\frac{1}{5}\bar{D}_3)\hat{x}))$, $\forall \hat{x} = (\hat{x_1},\hat{x_2}) \in H_2$, and define $S : C \to C$ by $S(x_1,x_2) = (\frac{x_1}{2}+1,\frac{x_2}{2})$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1 \in H_1$ and

$$y_n = M_C W_n = P_C\left(I - \frac{1}{2}(x_1-2,x_2+1)\right)T_n,$$

where $W_n = (I-\frac{1}{8}A^*(I-M_Q)A)x_n$ and $T_n = \frac{1}{2}W_n + \frac{1}{2}P_C(I-\frac{1}{2}(x_1-3,x_2-\frac{5}{2}))(\frac{1}{2}W_n+\frac{1}{2}P_C(I-\frac{1}{2}(x_1+2,x_2-6))W_n))$,

$$Q_n = \left\{z \in H : \left\langle \left(I - \frac{1}{2}(x_1-2,x_2+1)\right)T_n - y_n, y_n - z\right\rangle \geq 0\right\},$$

and

$$x_{n+1} = \frac{n+1}{5n}T_n + \left(1-\frac{n+1}{5n}\right)SP_{Q_n}\left(T_n - \frac{1}{2}(x_1-2,x_2+1)(y_n)\right), \quad \forall n \in \mathbb{N},$$

where

$$P_C x = \begin{cases} (x_1,x_2) - \frac{[2x_1+x_2-7](2,1)}{5} & \text{if } 2x_1+x_2 > 7, \\ (x_1,x_2) & \text{if } 1 \leq 2x_1+x_2 \leq 7, \\ (x_1,x_2) - \frac{[2x_1+x_2-1](2,1)}{5} & \text{if} 2x_1+x_2 < 1, \end{cases}$$

for every $x = (x_1,x_2) \in H_1$ and

$$P_Q \hat{x} = \begin{cases} (x_1,x_2) - \frac{[3x_1-x_2-20](3,-1)}{10} & \text{if } 3x_1-x_2 > 20, \\ (x_1,x_2) & \text{if } -10 \leq 3x_1-x_2 \leq 20, \\ (x_1,x_2) - \frac{[3x_1-x_2+10](3,-1)}{10} & \text{if } 3x_1-x_2 < -10, \end{cases}$$

for every $\hat{x} = (x_1,x_2) \in H_2$. By the definition of $S, D_i, \bar{D}_i, M_C, M_Q$ for every $i = 1,2,3$, we have $(2,0) \in F(M_C(I-\frac{1}{8}A^*(I-M_Q)A))$. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $(2,0)$.

Table 1 and Fig. 1 show the numerical results of sequences $\{x_n\}$ and $\{y_n\}$ where $x_1 = (-5,5)$ and $n = N = 30$.

Table 1 The values of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = (-5,5)$ and $n = N = 30$

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(−5.000000, 5.000000)	(0.034028, 1.404266)
2	(−0.457465, 2.305332)	(1.309813, 0.647460)
3	(1.223540, 1.203392)	(1.781929, 0.337977)
⋮	⋮	⋮
15	(2.000000, 0.000000)	(2.000000, 0.000000)
⋮	⋮	⋮
28	(2.000000, 0.000000)	(2.000000, 0.0000000)
29	(2.000000, 0.000000)	(2.000000, 0.000000)
30	(2.000000, 0.000000)	(2.000000, 0.000000)

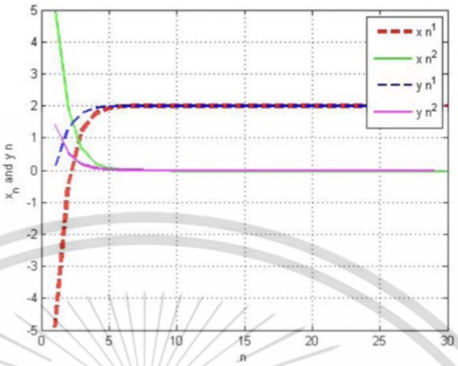

Figure 1 The convergence of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = (-5, 5)$ and $n = N = 30$

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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AK dealt with the conceptualization, formal analysis, supervision, writing—review and editing. AS writing—original draft, formal analysis, writing—review and editing. Both authors have read and approved the manuscript.

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