APPROXIMATION METHODS FOR MODIFIED EQUILIBRIUM PROBLEM AND CONSTRAINED CONVEX MINIMIZATION PROBLEM IN A HILBERT SPACE

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Abstract

For the purpose of this thesis, we first introduced the two-step intermixed iteration for finding the common solution of the constrained convex minimization problem and also proved a strong convergence theorem of the intermixed algorithm. Secondly, we introduced the modification of equilibrium problem (MEP) and a new subgradient extragradient algorithm by using the concept of the set of solutions of the modified variational inequality problem introduced by [1]. Then, we establish and prove the weak and strong convergence theorem of the new subgradient extragradient algorithm for finding a common element of the set of solutions of the MEP and two sets of the variational inequality problems under some suitable conditions on α_n and β_n with $\alpha_n + \beta_n \leq 1$. Thirdly, we apply our main theorem to prove weak and strong convergence theorems to solve the generalized equilibrium problem, the system of equilibrium problems, the variational inequality problem, and the general system of variational inequalities problem. Finally, we give numerical examples to support our main results.

Keywords : Constrained convex minimization problem, Variational inequality, Subgradient extragradient algorithm, Equilibrium problem, Optimization problems, fixed point problem

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Chapter 1

Introduction

Throughout this thesis, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. We use "$\rightharpoonup$" for weak convergence and "$\rightarrow$" for strong convergence.

1.1 Iterative methods of nonlinear mappings

The fixed point theorem based on the contraction principle has studied the existence and uniqueness of the solutions. In the past few decades, many mathematicians have created the convergence theorem to solve the fixed point problem. This makes the fixed point theory developed extensively and applied in various fields such as mathematics, economics, physics, engineering, optimizations, computer sciences, etc. For an interesting example, equilibrium problems or variational inequality problems are fundamental concepts in economics, engineering, and physics, which can be transformed into fixed points.

Throughout this thesis we denote $F(T)$ is the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$).

Many researchers have studied the iterative scheme to approximate the fixed point problem of nonlinear mapping as follows;

In 1953, Mann [18] introduced the sequence $\{x_n\}$ generated by $x_1 \in H$ and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad \forall n \in \mathbb{N}, \tag{1.1}$$

where C is a nonempty closed convex subset of a normed space, $T : C \to C$ is a mapping and the sequence $\{\alpha_n\}$ is in the interval $(0,1)$, iteration (1.1) is called *Mann iteration*. If T is a nonexpansive mapping under some suitable condition α_n satisfying $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by (1.1) converges weakly to element of the set of fixed points of T. Many authors have been trying to modify Mann's iteration to solve various problems such as the fixed point problem, split feasibility problem, equilibrium problem, monotone inclusion, and image restoration problem; see more detail in [19, 20, 21].

In 2000, Moudafi [19] proposed *the viscosity approximation method* for nonexpansive mapping S by introduce the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$x_{n+1} = \frac{1}{1+\epsilon_n} S x_n + \frac{\epsilon_n}{1+\epsilon_n} f(x_n), \qquad \forall n \in \mathbb{N}, \tag{1.2}$$

where $\{\epsilon_n\} \subset (0,1)$ satisfies certain conditions, $S : C \to C$ is a nonexpansive mapping and $f : C \to C$ is a contraction. Then he proved the sequence $\{x_n\}$ converges strongly to $z = P_{F(S)} f(z)$. Moreover, the viscosity approximation method has been studied and

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developed in many pieces of research to approximate the convex feasibility problem, hierarchical fixed point problem, variational inequality problem, and split common null point problem, see previous studies in [24, 23]. Notice that the sum of coefficients $\frac{1}{1+\epsilon_n}$ and $\frac{\epsilon_n}{1+\epsilon_n}$ in (1.2) is equal to 1.

Recently, Kanzow and Shehu [22] modified inexact Krasnoselskii-Mann iteration and introduced the sequence $\{x_n\}$ generated by $x_1 \in H$, $u \in C$ and

$$x_{n+1} = \delta_n u + \alpha_n x_n + \beta_n T x_n + r_n, \qquad \forall n \in \mathbb{N}, \tag{1.3}$$

where r_n represents the residual, the nonnegative real numbers $\alpha_n, \beta_n, \delta_n$ are chosen such that $\alpha_n + \beta_n + \delta_n \leq 1$, for all $n \in \mathbb{N}$, and $T : H \to C$ is a nonexpansive mapping. The sequence $\{x_n\}$ generated by (1.3) converges strongly to a point in $F(T)$, which is the nearest point projection of u onto $F(T)$. Observe that iterative scheme (1.3) is a modified Halpern's iterative scheme. Notice that the coefficients of α_n, β_n, and δ_n do not necessarily sum up to one.

By concept Mann [18] and Moudafi [19], Yao et al. [26] introduced the following sequences $\{x_n\}$ and $\{y_n\}$ which is called *the intermixed algorithm* by the definition of the sequence $\{x_n\}$ is involved in the sequence $\{y_n\}$ and the definition of the sequence $\{y_n\}$ is involved in the sequence $\{x_n\}$ as following algorithm:

Algorithm 1.1. For arbitrarily given $x_1, y_1 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n], & \forall n \in \mathbb{N}, \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_C[\alpha_n g(x_n) + (1-k-\alpha_n)y_n + kSy_n], & \forall n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $S, T : C \to C$ are a λ-strictly pseudo-contraction, $f : C \to H$ is a ρ_1-contraction and $g : C \to H$ is a ρ_2-contraction, $k \in (0, 1-\lambda)$ is a constant and $\{\alpha_n\}$, $\{\beta_n\}$ are two real number sequences in $(0, 1)$. Furthermore, under some control conditions, they proved that the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.4) converge to $x^* = P_{F(T)} f(y^*)$ and $y^* = P_{F(S)} g(x^*)$, respectively. Moreover, if we put $C \equiv H$ and $\beta_n = 1$ in (1.4), then (1.4) reduced to new iteration which is modified viscosity x_n and y_n associate f, g are contraction mappings with $\{x_n\}$ depend on $\{y_n\}$ and $\{y_n\}$ depend on $\{x_n\}$ as follows:

$$\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n, \qquad \forall n \in \mathbb{N},$$
$$\alpha_n g(x_n) + (1-k-\alpha_n)y_n + kSy_n, \qquad \forall n \in \mathbb{N}.$$

Recently, many mathematicians have used (1.4) to solve the fixed point problems of many nonlinear operators in real Hilbert spaces, see more detail in [52, 53, 54, 55, 56, 57].

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1.2 Equilibrium problems and variational inequality problems

The equilibrium problem (EP) is famous in many fields of pure and applied sciences. The EP can apply to physics, finance, network, optimization problems, variational inequality problems, and Nash equilibrium problems as special cases. Let $F : C \times C \to \mathbb{R}$ be a bifunction. *The equilibrium problem* of F is to find a point $x \in C$ such that

$$F(x,y) \geq 0, \qquad \forall y \in C, \tag{1.5}$$

which is introduced by Blum and Oettli [25], in 1994.

Recently, many researchers have constructed their theorem to solve the equilibrium problem as follows;

Kim et al. [2] introduced a new extended extragradient iteration algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for a monotone and Lipschitz-type continuous mapping, and they proved a strong convergence theorem. Furthermore, Shang et al. [3] introduced a general iterative scheme using the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space; see more detail in [4, 1].

If $F(x,y) := \langle A(x), y - x\rangle$ for all $x, y \in C$, where $A : C \to H$, then problem (1.5) is equivalent to finding x^* such that

$$\langle A(x^*), y - x^*\rangle \geq 0, \qquad \forall y \in C. \tag{1.6}$$

The set of all solutions of (1.6) is denoted by $VI(C, A)$, which is known as *the variational inequality problem (VIP)*, introduced by Lions and Stampacchia [5] in 1964. Numerous problems in physic, game theory, finance, optimization, and mechanics reduce to find an element of (1.6); see more detail in [5, 6, 7, 8].

Many mathematicians have modified the VIP as follows;

In 1999, Verma [9] introduced *the new system of variational inequalities problem*, which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \beta A x^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.7}$$

Further, if we put $x^* = y^*$, then the problem (1.7) reduces to the problem (1.6).

In 2013, Kangtunyakarn [1] modified the set of variational inequality problem as follows;

$$VI(C, aA + (1-a)B) = \{x \in C : \langle y - x, (aA + (1-a)B)x\rangle \geq 0, \ \ \forall y \in C, \ a \in (0,1)\}, \tag{1.8}$$

where $A, B : C \to H$ are two mappings. In particular, if we put $A \equiv B$, then we have $VI(C, aA + (1-a)B) = VI(C, A)$, and he proved a strong convergence theorem to an

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element of $VI(C, aA+(1-a)B)$ under suitable condition; see more detail in [1].

Many authors introduced their algorithms for solving the VIP, see, for instance, [10, 11].

In 1976, Korpelevich [10] proposed an algorithm for solving the VIP in Euclidean space and introduced the sequences $\{x_n\}$ and $\{y_n\}$ generated by $x_0 \in C$ and

$$\begin{cases} y_n = P_C(x_n - \tau f(x_n)), \\ x_{n+1} = P_C(x_n - \tau f(y_n)), \end{cases} \tag{1.9}$$

where $f : H \to H$ is Lipschitz continuous on C with constant $L > 0$, $\tau \in (0, \frac{1}{L})$ and P_C denotes by the metric projection onto C, iteration (1.9) is called *the Extragradient Method*. If the solution set $VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.9) converges weakly to an element in $VI(C, A)$.

After that, using the concept of half-space, Censor et al. [11] modified Korpelevich [10] by replacing the second projection onto the closed and convex subset C of Hilbert space with the one onto the subgradient half-space T_n. The sequences $\{x_n\}$ and $\{y_n\}$ generated by $x_0 \in H$ and

$$\begin{cases} y_n = P_C(x_n - \tau f(x_n)), \\ T_n := \{w \in H | \langle (x_n - \tau f(x_n)) - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \tau f(y_n)), \end{cases} \tag{1.10}$$

where $f : H \to H$ is Lipschitz continuous on C with constant $L > 0$ and $\tau \in (0, \frac{1}{L})$, iteration (1.10) is called *the subgradient extragradient method*. Censor et al. [11] proved that the sequence $\{x_n\}$ generated by (1.10) converges weakly to a solution of the variational inequality and used Lemma 3.2 [12] for proved the strong convergence theorem of this sequence. A few years later, many mathematicians introduced the new problem and new iteration, which was developed and modified the subgradient extragradient method; see more detail in [13, 14, 15, 16, 17].

Based on the result mentioned above, inspired and a motivated by the concept of the equilibrium problem and the new system of variational inequalities problem, we introduce *modification of equilibrium problem (MEP)*, which is to find $(x^*, y^*) \in C \times C$, such that

$$\begin{cases} F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - y^* + \lambda A y^* \rangle \geq 0, & \forall y \in C, \\ Q(y^*, x) + \frac{1}{r}\langle x - y^*, y^* - x^* + \beta B x^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.11}$$

where $F, Q : C \times C \to \mathbb{R}$ are bifunctions and $A, B : C \to H$ are mappings, $\lambda, \beta, r > 0$ are three constants. If we put $F \equiv Q \equiv 0$ and $A \equiv B$ in (1.11), then the problem (1.7) is a special case of the problem (1.11). The MEP can apply across many disciplines in mathematics and sciences such as economics, finance, network analysis, transportation, elasticity, and optimization.

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1.3 Constrained convex minimization problem

In mathematics, conventional optimization problems make a trading system more effective and are usually stated in terms of minimization problems. In this thesis, we give a new iteration for solving two constrained convex minimization problems.

Convex constrains minimization problem is a popular and very important to various branches in physics, engineers and economics, e.g., to find the minimum travel distance and to find the lowest cost. Consider the constrained convex minimization problem as follows:

$$\text{minimize } \{f(x) : x \in C\}, \tag{1.12}$$

where $f : C \to \mathbb{R}$ is a real-valued convex function. If f is (Fréchet) differentiable, then the gradient-projection algorithm (GPA) generates a sequence $\{x_n\}$ using the following recursive formula:

$$x_{n+1} = P_C(x_n - \lambda \nabla f(x_n)), \qquad \forall n \in \mathbb{N}, \tag{1.13}$$

or more generally,

$$x_{n+1} = P_C(x_n - \lambda_n \nabla f(x_n)) \qquad \forall n \in \mathbb{N}, \tag{1.14}$$

where in both (1.13) and (1.14) the initial guess x_1 is taken from C arbitrarily, and the parameters, λ or λ_n, are positive real numbers satisfying certain conditions. The convergence of the algorithms (1.13) and (1.14) depend on the behavior of the gradient ∇f. In fact, it is known that if ∇f is α-strongly monotone and L-Lipschitzian with constants $\alpha, L \geq 0$, then the operator

$$T := P_C(I - \lambda \nabla f) \tag{1.15}$$

is a contraction; hence, the sequence $\{x_n\}$ defined by the algorithm (1.13) converges in norm to the unique minimizer of (1.12). However, if the gradient ∇f fails to be strongly monotone, the operator T defined by (1.15) would fail to be contractive; consequently, the sequence $\{x_n\}$ generated by the algorithm (1.13) may fail to converge strongly [23]. If ∇f is Lipschitzian, then the algorithms (1.13) and (1.14) can still converge in the weak topology under certain conditions [27, 28, 29].

Su and Xu [28] introduced the relation of solution of the minimization problem (1.12) and solutions of the variational inequality (1.6) as follows Lemma 1.2 and this lemma helps to prove the theorem about the minimization problem more effectively, see more detail in [30, 31, 32].

Lemma 1.2. (See [28]) (Optimality condition) A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem (1.12) is that x^* solves the variational inequality

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in C. \tag{1.16}$$

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Equivalently $x^* \in C$ solves the fixed point equation

$$x^* = P_C(x^* - \lambda \nabla f(x^*)),$$

for every constant $\lambda > 0$. If, in addition, f is convex, then the optimality condition (1.16) is also sufficient.

Note that U_f is denoted by the set of solutions of (1.12).

In 2011, Ceng et al. [33] introduced the following explicit iterative scheme that generates a sequence $\{x_n\}$ in an explicit way:

$$x_{n+1} = P_C[s_n \gamma V x_n + (I - s_n \mu F) T_n x_n], \qquad \forall n \in \mathbb{N},$$

where $s_n = \frac{2-\lambda_n L}{4}$ and $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n) T_n$ for each $n \in \mathbb{N}$. They proved that the sequence $\{x_n\}$ converges strongly to a minimizer of the constrained convex minimization problem.

In 2014, Tian and Liu [34] introduced explicit composite iterative methods for finding the common element of the set of solutions of an equilibrium problem and the solution set of a constrained convex minimization problem and proved a strong convergence theorem by using the following algorithm $\{x_n\}$ and parameters:

Algorithm 1.3. Given $x_1 \in C$, let the sequences $\{u_n\}$ and $\{x_n\}$ be generated iteratively by

$$\begin{cases} \phi(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n \gamma V u_n + (I - \alpha_n A) T_n u_n, & \forall n \in \mathbb{N}, \end{cases}$$

where T_n is nonexpansive mapping to get from $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n) T_n$ is $\frac{2+\lambda_n L}{4}$ averaged with $s_n = \frac{2-\lambda_n L}{4}$ and ∇f is an L-Lipschitzian mapping, for all $L \geq 0$, $V : C \to C$ is an l-Lipschitzian mapping with constant $l \geq 0$, $A : C \to C$ is a strongly positive bounded linear operator with coefficient $\bar{\gamma} \geq 0$ and $0 < \gamma < \frac{\bar{\gamma}}{l}$, $u_n = Q_{\beta_n} x_n$, $\{\lambda_n\} \subset (0, \frac{2}{L})$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, \infty)$ and $\{s_n\} \subset (0, \frac{1}{2})$.

Motivated by Yao et al. [26] and Tian et al. [34], we introduce the new iterative method as the following algorithm for approximating the common solution of a constrained convex minimization problem:

Algorithm 1.4. Given $x_1, y_1 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be defined by

$$\begin{cases} x_{n+1} = (1 - \mu_n) x_n + \mu_n P_C(\alpha_n f(y_n) + (1 - \alpha_n) T_n^{\widetilde{f}_1} x_n), \\ y_{n+1} = (1 - \mu_n) y_n + \mu_n P_C(\alpha_n g(x_n) + (1 - \alpha_n) T_n^{\widetilde{f}_2} y_n), \end{cases} \tag{1.17}$$

where $f, g : H \to H$ are a_f and a_g-contraction mapping with $a_f, a_g \in (0, 1)$ and $a = \max\{a_f, a_g\}$, $\nabla \widetilde{f}_i$ is an $\frac{1}{L_i}$-inverse strongly monotone with $L_i \geq 0$, for all $i = 1, 2$, $\{\mu_n\}$, $\{\alpha_n\} \subseteq [0, 1]$, $P_C(I - \lambda_n^i \nabla \widetilde{f}_i) = s_n^i I + (1 - s_n^i) T_n^{\widetilde{f}_i}$, $\forall i = 1, 2$ and $s_n^i = \frac{2-\lambda_n^i L_i}{4}$, $\{\lambda_n^i\} \subset (0, \frac{2}{L_i})$ and $0 < \overline{\theta} \leq \mu_n \leq \theta$ for all $n \in \mathbb{N}$ and for some $\overline{\theta}, \theta > 0$.

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1.4 Objectives of the study

1) To propose the modification of equilibrium problem (MEP) and a new subgradient extragradient algorithm for solving the MEP.
2) To prove the weak and strong convergence theorem of the sequence $\{x_n\}$ generated by the new subgradient extragradient for finding a common element of the set of solutions of the MEP and two sets of the variational inequality problems under some suitable conditions of our parameters α_n and β_n with $\alpha_n + \beta_n \leq 1$.
3) To prove the strong convergence theorem of an intermixed algorithm for finding the common solution of a constrained convex minimization problem.
4) To apply the obtained results on 2) and 3) to solve the generalized equilibrium problem, the system of equilibrium problems, the variational inequality problem, and the general system of variational inequalities problem.
5) To give numerical examples for our results to support our main results.

1.5 Scopes of the study

1) Study the definitions and properties of equilibrium problems, variational inequality problems, minimization problems, and fixed point problems in real Hilbert space.
2) Investigate the fixed point problems of nonlinear mappings, including nonexpansive mapping, L-Lipschitzian mapping, and α-inverse strongly monotone mapping.
3) All strong convergence theorems are considered and proved in a real Hilbert space.
4) Give numerical examples for supporting our main results in $\mathbb{R}$ and $\mathbb{R}^2$ spaces.

1.6 Benefits of the study

1) Obtain new tools for fixed point problems on real Hilbert space.
2) Obtain a strong convergence theorem of an intermixed algorithm to find the common solution of a constrained convex minimization problem.
3) Obtain weak and strong convergence theorem of the sequence $\{x_n\}$ generated by the new subgradient extragradient for finding a common element of the set of solutions of the MEP and two sets of the variational inequality problems.
4) Obtain weak and strong convergence theorem to solve the generalized equilibrium problem, the system of equilibrium problems, the variational inequality problem, and the general system of variational inequalities problem.

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1.7 Research methodology

1) Study advanced topics in fixed point theory for nonexpansive mapping, L-Lipschitzian mapping, and α-inverse strongly monotone mapping.
2) Study background in a real Hilbert space.
3) Study the intermixed algorithm and the subgradient extragradient algorithm.
4) Collect and study research papers and textbooks concerning fixed point theorem.
5) Determine the objectives and scope of the research.
6) Produce tools for a strong convergence theorem of fixed point problem.
7) Prove weak and strong convergence theorem for fixed point problems in a real Hilbert space.
8) Provide examples and applications.
9) Conclude the results, make suggestions for further works and write the thesis.

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Chapter 2

Preliminaries

In this chapter, we give important lemmas, definitions, and theorems for use throughout this thesis. Moreover, we use the letter $\mathbb{R}$ for the set of all real numbers, $\mathbb{C}$ for the set of all complex numbers, and $\mathbb{F}$ for the set of all real or complex numbers.

2.1 Basic Concepts

In this section, we provide some basic definitions, lemmas, and properties to apply throughout the thesis.

Definition 2.1 (Metric space [49]). Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^+ : [0, \infty)$ a function. Then d is called a *metric* on X if the following properties hold:

(i) $d(x, y) \geq 0$;

(ii) $d(x, y) = 0$ if and only if $x = y$;

(iii) $d(x, y) = d(y, x)$;

(iv) $d(x, y) \leq d(x, z) + d(z, y)$,

for all $x, y, z \in X$. The value of metric d at (x, y) is called *distance between x and y*, and the ordered pair (X, d) is called *metric space*.

Definition 2.2 (Normed space [50]). Let X be a vector space. A norm on X is a real-valued function on X such that the following conditions are satisfied:

(i) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$, $\forall \alpha \in \mathbb{F}$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$ (the triangle inequality),

for all $x, y \in X$ and $\alpha \in \mathbb{R}$ or $\mathbb{C}$. The ordered pair $(X, \|\cdot\|)$ is called *normed space*. A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\|, \qquad \text{for all } x, y \in X,$$

and is called the *metric induced by the norm*.

Definition 2.3 (Convex set [46]). Let X be a normed space and let C be a subset of X. Then the set C is called *convex* if

$$\alpha x + (1 - \alpha) y \in C,$$

for all $x, y \in C$ and $\alpha \in [0, 1]$.

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Definition 2.4 (Inner product space [51])**.** An *inner product space* is a vector space X with an inner product defined on X. Here, an inner product on X is a mapping $\langle \cdot, \cdot \rangle$ of $X \times X$ into the scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; that is, with every pair of vector x and y there is associated a scalar which is written and is called *the inner product of x and y*, such that for all vectors x, y, z and scalar $\alpha \in \mathbb{F}$ we have:

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$;

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$;

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;

(iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, x \rangle$.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x \rangle}$.

Definition 2.5 (Strong convergence [45])**.** A sequence $\{x_n\}$ of vectors in an inner product space K is called *strongly convergent* to a vector x in K if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.6 (Weak convergence [45])**.** A sequence $\{x_n\}$ of vectors in an inner product space K is called *weakly convergent* to a vector x in K if

$$\langle x_n, y \rangle \to \langle x, y \rangle \text{ as } n \to \infty, \qquad \text{for every } y \in K.$$

Theorem 2.1 ([45])**.** A strongly convergence sequence is weakly convergence (to the same limit), that is, $x_n \to x$ implies $x_n \rightharpoonup x$.

Remark 2.2 ([46])**.** If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Theorem 2.3 ([46])**.** Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Suppose that $\{x_n\} \subset C$ and $x_n \rightharpoonup x$. Then $x \in C$.

Definition 2.7 ([46])**.** Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\} \text{ is closed.}$$

Moreover, f is called *convex* if for any $x_1, x_2 \in C$ and $t \in (0, 1)$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Similarly, f is said to be *concave* if for any $x_1, x_2 \in C$ and $t \in (0, 1)$,

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2).$$

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Theorem 2.4 ([46])**.** Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightharpoonup x_0$. Then

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

Lemma 2.5 ([38])**.** Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{u_n\} \subset H$ with $u_n \rightharpoonup u$, the inequality

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

holds for every $v \in H$ with $v \neq u$.

Theorem 2.6 ([46])**.** Let $\{a_n\}$ be a bounded of real numbers. Then, there exists subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Lemma 2.7. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequences of non-negative numbers satisfying

$$a_{n+1} \leq a_n + b_n,$$

for all $n \in \mathbb{N}$.

(i) If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

(ii) If $\sum_{n=1}^{\infty} b_n < \infty$ and $\{a_n\}_{n=1}^{\infty}$ has a subsequence converging to zero, then

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 2.8 ([22])**.** Let X be a real inner product space. Then:

(a) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$,

(b) $\|tx+sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - st\|x-y\|^2$,

for all $x, y \in X$ and $s, t \in \mathbb{R}$.

2.2 Fundamental Properties in Hilbert spaces

In this section, we give important definitions, lemmas, and theorems in Hilbert space that are utilized in our main results.

Definition 2.8 (Hilbert space [45])**.** Let X be an inner product space and X is called *Hilbert space* if X is complete inner product space.

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Remark 2.9 ([46])**.** Let H be an inner product space. Then we know that the following conditions are equivalent:

i) H is complete,

ii) each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 2.10 ([37])**.** Let H be a real Hilbert space. Then the following results hold:

(i) For each $x, y \in H$ and $\alpha \in [0,1]$,

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,$$

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, for all $x, y \in H$.

Definition 2.9 (Metric projection [46])**.** The (nearest point) projection P_C from H onto C assigns to each $x \in H$, the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.11 ([36],[34])**.** Let $x \in H$ and $P_C : H \to C$ be a metric projection. Then

(a) $z = P_C x$ if and only if $\langle x - z, y - z\rangle \leq 0$, $\forall y \in C$.

(b) $z = P_C x$ if and only if $\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$.

(c) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$, $\forall x, y \in H$.

Consequently, P_C is nonexpansive and monotone.

Lemma 2.12 ([29])**.** Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n, \qquad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.13 ([39])**.** Assume $A : H \to H$ is a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < t \leq \|A\|^{-1}$. Then $\|I - tA\| \leq 1 - t\overline{\gamma}$.

Definition 2.10. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to H$ and $V : C \to C$ be the mappings. Then

(a) A is a *strongly positive* on H if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \qquad \forall x \in H.$$

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(b) A is a *monotone* if

$$\langle x - y, Ax - Ay\rangle \geq 0, \qquad \forall x, y \in H;$$

(c) A is a *β-strongly monotone* if there exists $\beta > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \beta\|x - y\|^2, \qquad \forall x, y \in H;$$

(d) A is a *v-inverse strongly monotone* (for short, v-ism) if there exists $v > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq v\|Ax - Ay\|^2, \qquad \forall x, y \in H.$$

(e) V is an *l-Lipschitzian* if there exists a constant $l \geq 0$, such that

$$\|Vx - Vy\| \leq l\|x - y\|, \qquad \forall x, y \in C.$$

If $l \in [0, 1)$, then V is called a contraction. Obviously, if $l = 1$, V is a nonexpansive mapping.

Remark 2.14. It is well-known that if $A, B : C \to H$ are α and β-inverse strongly monotone, then $(aA + (1 - a)B)$ is η-inverse strongly monotone, where $\eta = \min\{\alpha, \beta\}$ and $a \in [0, 1]$.

Definition 2.11. A mapping $T : H \to H$ is said to be firmly nonexpansive, if and only if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \qquad x, y \in H.$$

Alternatively, T is firmly nonexpansive, if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive.

Definition 2.12. A mapping $T : H \to H$ is said to be an *averaged mapping* if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T = (1 - \alpha)I + \alpha S, \tag{2.1}$$

where α is a number in $(0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when (2.1) holds, we say that T is α-averaged.

Clearly, a firmly nonexpansive mapping is $\frac{1}{2}$-averaged mapping.

Proposition 2.15. For given operators $S, T, V : H \to H$:

(i) If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.

(ii) T is firmly nonexpansive if and only if $I - T$ is firmly nonexpansive.

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(iii) If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$, S is firmly nonexpansive and V is nonexpansive, then T is averaged.

(iv) The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composit $T_1 \cdot T_2 \cdot, ..., T_N$. In particular, if T_1 is α_1-averaged, and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composit T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

Proposition 2.16. Let T be an operator from H to itself. Then

(a) T is nonexpansive if and only if $I - T$ is $\frac{1}{2}$-inverse strongly monotone;

(b) If T is v-inverse strongly monotone, then for $\gamma > 0$, γT is $\frac{v}{\gamma}$-inverse strongly monotone;

(c) T is averaged if and only if $I - T$ is v-inverse strongly monotone for some $v > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-inverse strongly monotone.

2.3 Bounded Linear Operators

Definition 2.13 ([45]). Let C be a nonempty closed convex subset of a real Hilbert space H. Then

(i) A is called a *linear operator* if

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y),$$

for all $x, y \in H$ and all scalars α, β.

(ii) A is called *bounded* if there is a number α such that

$$\|Ax\| \leq \alpha\|x\|,$$

for all $x \in H$.

Definition 2.14 (Adjoint Operator [45]). Let A be a bounded operator on a Hilbert space H. The operator $A^* : H \to H$ be defined by

$$\langle Ax, y\rangle = \langle x, A^*y\rangle, \qquad \text{for all } x, y \in H,$$

is called the *adjoint operator* of A.

Theorem 2.17 ([45]). The adjoint operator A^* of a bounded operator A is bounded. Furthermore, we have $\|A\| = \|A^*\|$ and $\|A^*A\| = \|A\|^2$.

Definition 2.15 (Self-Adjoint Operator [45]). Let A be a bounded operator on a Hilbert space H. If $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x \in H$, then A is called the *self-adjoint operator*.

Remark 2.18. If A is a bounded operator on a Hilbert space H, then A^*A is self-adjoint operator.

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Theorem 2.19 ([45])**.** Let T be a bounded linear self-adjoint operator on a Hilbert space H. Then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Definition 2.16 (Normal Operator [45])**.** A bounded operator T on a Hilbert space H is called a *normal operator* if $TT^* = T^*T$.

Theorem 2.20 ([45])**.** A bounded operator T on a Hilbert space H is normal if and only if $\|Tx\| = \|T^*x\|$, for all $x \in H$.

Definition 2.17 (Positive operator)**.** An operator A is called *positive* if it is self-adjoint and $\langle Ax, x\rangle \geq 0$, for all $x \in H$.

Definition 2.18 ([58])**.** Let T be a bounded linear operator on a Hilbert space H. The *spectral radius* of T, denoted by $r_\sigma(T)$, is the number defined by

$$r_\sigma(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

where $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)(x) = 0, \text{for some } 0 \neq x \in H\}$.

2.4 Fixed Point Theorems

Let X be a nonempty set and $T : X \to X$ a self-mapping. We say that $x \in X$ is a fixed point of T if and only if $Tx = x$ and $F(T)$ represents the set of all fixed points of T.

Example 2.21 ([47])**.** 1) If $X = \mathbb{R}$ and $T(x) = x^2 - 5x + 9$, then $F(T) = \{3\}$;

2) If $X = \mathbb{R}$ and $T(x) = 4x^2$, then $F(T) = \{0, \frac{1}{4}\}$;

3) If $X = \mathbb{R}$ and $T(x) = x - 5$, then $F(T) = \emptyset$;

4) If $X = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$;

Theorem 2.22 ([46])**.** Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Theorem 2.23 ([46])**.** Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.

Lemma 2.24 (Demiclosedness principle [48])**.** Assume that T is a nonexpansive self-mapping of closed convex C subset of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y it follows that $(I - T)x = y$, where I is the identity mapping of H.

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2.5 Equilibrium Problems and Variational inequality Problems

In this section, we provide definitions and lemmas for equilibrium problems and variational inequality problems that will use for our main results in the next chapter.

Lemma 2.25 ([36]). Let C be a nonempty closed convex subset of a real Hilbert space H, and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \;\Leftrightarrow\; u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.26 ([1]). Let C be a nonempty closed convex subset of a real Hilbert space H, and let $A, B : C \to H$ be α and β-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C, A) \bigcap VI(C, B) \neq \emptyset$. Then

$$VI(C, aA + (1 - a)B) = VI(C, A) \bigcap VI(C, B), \qquad \text{for all } a \in (0, 1).$$

Furthermore if $0 < \gamma < \min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1 - a)B)$ is a nonexpansive mapping.

Definition 2.19 ([25]). For solving the equilibrium problems for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

$(A1)$ $F(x, x) = 0, \qquad \forall x \in C,$

$(A2)$ F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \qquad \forall x, y \in C,$

$(A3)$ for all $x, y, z \in C$,

$$\lim_{t \to 0^+} F(tz + (1 - t)x, y) \leq F(x, y),$$

$(A4)$ for all $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

We give an example to illustrate the bifunction satisfying $(A1) - (A4)$.

Example 2.27. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x, y) = x^2 + 2xy - 3y^2, \forall x, y \in \mathbb{R}.$$

Then a bifunction F satisfies the condition (A1)-(A4) and $0 \in EP(F)$.

Solution. Let $x, y, z \in \mathbb{R}$. Since

$$F(x, x) = x^2 + 2x^2 - 3x^2 = 0,$$

thus we obtain (A1) holds. Next, observe that

$$F(x, y) + F(y, x) = \left(x^2 + 2xy - 3y^2\right) + \left(y^2 + 2xy - 3x^2\right) = -2x^2 + 4xy - 2y^2 = -2(x - y)^2 \leq 0.$$

This implies that F satisfies (A2). Let $t \in [0, 1]$. Consider

$$\lim_{t \to 0^+} F(tz + (1 - t)x, y) = \lim_{t \to 0^+} [(tz + (1 - t)x)^2 + 2(tz + (1 - t)x)y - 3y^2] = x^2 + 2xy - 3y^2 = F(x, y).$$

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Therefore, (A3) is true. To show (A4), first let $\alpha \in (0,1)$. Then we derive that

$$\begin{aligned}
&F(x, \alpha z + (1-\alpha)y) \\
&= x^2 + 2x(\alpha z + (1-\alpha)y) - 3(\alpha z + (1-\alpha)y)^2 \\
&= x^2 + 2\alpha xz + (1-\alpha)2xy - 3(\alpha^2 z^2 + 2\alpha(1-\alpha)zy + (1-\alpha)^2 y^2) \\
&\le x^2 + 2\alpha xz + (1-\alpha)2xy - 3(\alpha^2 z^2 + \alpha(1-\alpha)(z^2+y^2) + (1-\alpha)^2 y^2) \\
&= \alpha\left(x^2 + 2xz - 3(\alpha z^2 + (1-\alpha)z^2)\right) + (1-\alpha)\left(x^2 + 2xy - 3(\alpha y^2 + (1-\alpha)y^2)\right) \\
&= \alpha\left(x^2 + 2xz - 3z^2\right) + (1-\alpha)\left(x^2 + 2xy - 3y^2\right) \\
&= \alpha F(x,z) + (1-\alpha)F(x,y).
\end{aligned}$$

Hence F is a convex function. Let $\{y_n\} \subset \mathbb{R}$ with $y_n \to y$ as $n \to \infty$. Thus we get

$$\lim_{n\to\infty} F(x, y_n) = \lim_{n\to\infty} x^2 + 2xy_n - 3y_n^2 = x^2 + 2xy - 3y^2. \tag{2.2}$$

This yields that F is lower semicontinuous and (A4) holds.

The following lemma appears implicitly in [25].

Lemma 2.28 ([25])**.** Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \qquad \forall y \in C.$$

Lemma 2.29 ([40])**.** Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C\},$$

for all $z \in H$. Then, the following hold:

(1) T_r is single-valued,

(2) T_r is firmly nonexpansive i.e.,

$$\|T_r(x) - T_r(y)\|^2 \le \langle T_r(x) - T_r(y), x - y\rangle, \quad \forall x, y \in H,$$

(3) $F(T_r) = EP(F)$,

(4) $EP(F)$ is closed and convex.

The following Lemma 2.30 is a direct result of problem (1.11) and Lemma 2.29.

Lemma 2.30. Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $F, Q : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $r > 0$, then the following equivalent.

(i) (x^*, y^*) is a solution of (1.11),

(ii) x^* is a fixed point of a mapping $\varphi : C \to C$ defined by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$ for all $\beta, \lambda > 0$ and $x \in C$, where $y^* = T_r^Q(I - \beta B)x^*$.

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Proof. (i) $\Rightarrow$ (ii). Let (x^*, y^*) is a solution of (1.11). For every $x, y \in C$, we have

$$\begin{aligned} F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - (I - \lambda A)y^*\rangle \geq 0, \\ Q(y^*, x) + \frac{1}{r}\langle x - y^*, y^* - (I - \beta B)x^*\rangle \geq 0. \end{aligned}$$

From Lemma 2.29, we have

$$T_r(I - \lambda A)y^* = x^*, \tag{2.3}$$

and

$$T_r(I - \beta B)x^* = y^*. \tag{2.4}$$

From (2.3) and (2.4), we have

$$\begin{aligned} x^* =& T_r(I - \lambda A)T_r(I - \beta B)x^* \\ =& \varphi(x^*). \end{aligned}$$

It follows that $x^* \in F(\varphi)$, where $y^* = T_r(I - \beta B)x^*$.

(ii) $\Rightarrow$ (i). Let $x^* \in F(\varphi)$ and $y^* = T_r(I - \beta B)x^*$.

Since $x^* \in F(\varphi)$, we have

$$\begin{aligned} x^* =& T_r(I - \lambda A)T_r(I - \beta B)x^* \\ =& T_r(I - \lambda A)y^*. \end{aligned} \tag{2.5}$$

From (2.5) and $y^* = T_r(I - \beta B)x^*$, we get

$$\begin{aligned} F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - (I - \lambda A)y^*\rangle \geq 0, \qquad & \forall y \in C, \\ Q(y^*, x) + \frac{1}{r}\langle x - y^*, y^* - (I - \beta B)x^*\rangle \geq 0, \qquad & \forall x \in C. \end{aligned}$$

It follows that (x^*, y^*) is a solution of (1.11). $\square$

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Chapter 3

Convergence theorems in Hilbert space and its application

3.1 The modification of intermixed iteration for a constrained convex minimization problem

In this section, we introduce a strong convergence theorem for the two-step intermixed iteration for finding the common solution to a constrained convex minimization problem. We first recall some results from Xu [35] as follows:

Let $f : C \to \mathbb{R}$ be a real-valued convex function and assume that ∇f is an L-Lipschitzian mapping with $L \geq 0$. From Xu [35], we have $P_C(I - \lambda \nabla f)$ is $\frac{2+\lambda L}{4}$-averaged for $0 < \lambda < \frac{2}{L}$, and for each $n \in \mathbb{N}$, that is, we can write

$$P_C(I - \lambda_n \nabla f) = (1 - s_n)I + s_n T_n^f,$$

where T_n^f is nonexpansive and $s_n = \frac{2+\lambda_n L}{4}$.

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2$, $\widetilde{f}_i : C \to \mathbb{R}$ be real-valued convex functions and assume that $\nabla \widetilde{f}_i$ are $\frac{1}{L_i}$-inverse strongly monotones with $L_i \geq 0$ and $U_{\widetilde{f}_i} = VI(C, \nabla \widetilde{f}_i) \neq \emptyset$. Let $f, g : H \to H$ be a_f and a_g-contraction mapping, respectively, with $a_f, a_g \in (0, 1)$ and $a = \max\{a_f, a_g\}$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1, y_1 \in C$ and

$$\begin{cases} x_{n+1} = (1 - \mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1 - \alpha_n)T_n^{\widetilde{f}_1} x_n), \\ y_{n+1} = (1 - \mu_n)y_n + \mu_n P_C(\alpha_n g(x_n) + (1 - \alpha_n)T_n^{\widetilde{f}_2} y_n), \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $\{\mu_n\}, \{\alpha_n\} \subseteq [0, 1]$, $P_C(I - \lambda_n^i \nabla \widetilde{f}_i) = s_n^i I + (1 - s_n^i)T_n^{\widetilde{f}_i}$, $T_n^{\widetilde{f}_i}$ are nonexpansive, $s_n^i = \frac{2-\lambda_n^i L_i}{4}$, and $\{\lambda_n^i\} \subset (0, \frac{2}{L_i})$, for all $i = 1, 2$. Assume that the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \overline{\theta} \leq \mu_n \leq \theta$, for all $n \in \mathbb{N}$ and for some $\overline{\theta}, \theta > 0$,

(iii) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum\limits_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$,

(iv) $\lim\limits_{n\to\infty} \lambda_n^i = \frac{2}{L_i}$, for all $i = 1, 2$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_{U_{\tilde{f}_1}} f(y^*)$ and $y^* = P_{U_{\tilde{f}_2}} g(x^*)$, respectively.

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Proof. First, we show that $\{x_n\}$ and $\{y_n\}$ are bounded.
Assume that $\widetilde{x} \in U_{\widetilde{f}_1}$ and $\widetilde{y} \in U_{\widetilde{f}_2}$. Then, we have

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}\| &= \|(1-\mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n) - \widetilde{x}\| \\
&= \|(1-\mu_n)(x_n - \widetilde{x}) + \mu_n(P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n) - \widetilde{x})\| \\
&\le (1-\mu_n)\|x_n - \widetilde{x}\| + \mu_n\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n - \widetilde{x}\| \\
&\le (1-\mu_n)\|x_n - \widetilde{x}\| + \mu_n(\alpha_n\|f(y_n) - \widetilde{x}\| + (1-\alpha_n)\|T_n^{\widetilde{f}_1} x_n - \widetilde{x}\|) \\
&\le (1-\mu_n)\|x_n - \widetilde{x}\| + \mu_n(\alpha_n\|f(y_n) - \widetilde{x}\| + (1-\alpha_n)\|x_n - \widetilde{x}\|) \\
&= (1-\alpha_n\mu_n)\|x_n - \widetilde{x}\| + \alpha_n\mu_n\|f(y_n) - \widetilde{x}\| \\
&\le (1-\alpha_n\mu_n)\|x_n - \widetilde{x}\| + \alpha_n\mu_n(\|f(y_n) - f(\widetilde{y})\| + \|f(\widetilde{y}) - \widetilde{x}\|) \\
&\le (1-\alpha_n\mu_n)\|x_n - \widetilde{x}\| + \alpha_n\mu_n a\|y_n - \widetilde{y}\| + \alpha_n\mu_n\|f(\widetilde{y}) - \widetilde{x}\|. \qquad (3.2)
\end{aligned}$$

Similarly, we get

$$\|y_{n+1} - \widetilde{y}\| \le (1-\alpha_n\mu_n)\|y_n - \widetilde{y}\| + \alpha_n\mu_n a\|x_n - \widetilde{x}\| + \alpha_n\mu_n\|g(\widetilde{x}) - \widetilde{y}\|. \qquad (3.3)$$

Combining (3.2) and (3.3), we have

$$\|x_{n+1}-\widetilde{x}\|+\|y_{n+1}-\widetilde{y}\| \le (1-\alpha_n\mu_n(1-a))(\|x_n-\widetilde{x}\|+\|y_n-\widetilde{y}\|)+\alpha_n\mu_n(\|f(\widetilde{y})-\widetilde{x}\|+\|g(\widetilde{x})-\widetilde{y}\|).$$

By induction, we can derive that

$$\|x_n - \widetilde{x}\| + \|y_n - \widetilde{y}\| \le \max\{\|x_1 - \widetilde{x}\| + \|y_1 - \widetilde{y}\|, \|f(\widetilde{y}) - \widetilde{x}\| + \|g(\widetilde{x}) - \widetilde{y}\|\},$$

for every $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{y_n\}$ are bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$. Observe that

$$\begin{aligned}
&\|T_n^{\widetilde{f}_1} x_n - T_{n-1}^{\widetilde{f}_1} x_{n-1}\| \\
&\le \|T_n^{\widetilde{f}_1} x_n - T_n^{\widetilde{f}_1} x_{n-1}\| + \|T_n^{\widetilde{f}_1} x_{n-1} - T_{n-1}^{\widetilde{f}_1} x_{n-1}\| \\
&\le \|x_n - x_{n-1}\| \\
&\quad + \left\|\Big(\frac{4P_C(I-\lambda_n^1\nabla\widetilde{f}_1) - (2-\lambda_n^1 L_1)}{2+\lambda_n^1 L_1}\Big)x_{n-1} - \Big(\frac{4P_C(I-\lambda_{n-1}^1\nabla\widetilde{f}_1) - (2-\lambda_{n-1}^1 L_1)}{2+\lambda_{n-1}^1 L_1}\Big)x_{n-1}\right\| \\
&\le \|x_n - x_{n-1}\| + \left\|\Big(\frac{4P_C(I-\lambda_n^1\nabla\widetilde{f}_1)}{2+\lambda_n^1 L_1}\Big)x_{n-1} - \Big(\frac{4P_C(I-\lambda_{n-1}^1\nabla\widetilde{f}_1)}{2+\lambda_{n-1}^1 L_1}\Big)x_{n-1}\right\| \\
&\quad + \left\|\Big(\frac{2-\lambda_{n-1}^1 L_1}{2+\lambda_{n-1}^1 L_1}\Big)x_{n-1} - \Big(\frac{2-\lambda_n^1 L_1}{2+\lambda_n^1 L_1}\Big)x_{n-1}\right\| \\
&= \|x_n - x_{n-1}\| \\
&\quad + \left\|\frac{4(2+\lambda_{n-1}^1 L_1)P_C(I-\lambda_n^1\nabla\widetilde{f}_1)x_{n-1} - 4(2+\lambda_n^1 L_1)P_C(I-\lambda_{n-1}^1\nabla\widetilde{f}_1)x_{n-1}}{(2+\lambda_n^1 L_1)(2+\lambda_{n-1}^1 L_1)}\right\| \\
&\quad + \left\|\frac{(2-\lambda_{n-1}^1 L_1)(2+\lambda_n^1 L_1)x_{n-1} - (2-\lambda_n^1 L_1)(2+\lambda_{n-1}^1 L_1)x_{n-1}}{(2+\lambda_n^1 L_1)(2+\lambda_{n-1}^1 L_1)}\right\| \\
&= \|x_n - x_{n-1}\|
\end{aligned}$$

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$$
\begin{aligned}
&+ \Bigg\| \frac{4(2+\lambda_{n-1}^1 L_1)P_C(I-\lambda_n^1\nabla\widetilde{f}_1)x_{n-1} - 4(2+\lambda_n^1 L_1)P_C(I-\lambda_{n-1}^1\nabla\widetilde{f}_1)x_{n-1}}{(2+\lambda_n^1 L_1)(2+\lambda_{n-1}^1 L_1)} \Bigg\| \\
&+ \Big(\frac{4L_1|\lambda_n^1-\lambda_{n-1}^1|}{(2+\lambda_n^1 L_1)(2+\lambda_{n-1}^1 L_1)}\Big)\|x_{n-1}\| \\
=&\|x_n - x_{n-1}\| \\
&+ \Bigg\| \frac{4L_1(\lambda_{n-1}^1-\lambda_n^1)P_C(I-\lambda_n^1\nabla\widetilde{f}_1)x_{n-1}}{(2+\lambda_n^1 L_1)(2+\lambda_{n-1}^1 L_1)} \\
&+ \frac{4(2+\lambda_n^1 L_1)(P_C(I-\lambda_n^1\nabla\widetilde{f}_1)x_{n-1} - P_C(I-\lambda_{n-1}^1\nabla\widetilde{f}_1)x_{n-1})}{(2+\lambda_n^1 L_1)(2+\lambda_{n-1}^1 L_1)} \Bigg\| \\
&+ \Big(\frac{4L_1|\lambda_n^1-\lambda_{n-1}^1|}{(2+\lambda_n^1 L_1)(2+\lambda_{n-1}^1 L_1)}\Big)\|x_{n-1}\| \\
\leq&\|x_n - x_{n-1}\| \\
&+ \frac{4L_1|\lambda_{n-1}^1-\lambda_n^1|\|P_C(I-\lambda_n^1\nabla\widetilde{f}_1)x_{n-1}\|}{(2+\lambda_n^1 L_1)(2+\lambda_{n-1}^1 L_1)} + \frac{4\|\lambda_{n-1}^1\nabla\widetilde{f}_1 x_{n-1} - \lambda_n^1\nabla\widetilde{f}_1 x_{n-1}\|}{2+\lambda_{n-1}^1 L_1} \\
&+ \Big(\frac{4L_1|\lambda_n^1-\lambda_{n-1}^1|}{(2+\lambda_n^1 L_1)(2+\lambda_{n-1}^1 L_1)}\Big)\|x_{n-1}\| \\
\leq&\|x_n - x_{n-1}\| + L_1|\lambda_{n-1}^1-\lambda_n^1|\|P_C(I-\lambda_n^1\nabla\widetilde{f}_1)x_{n-1}\| \\
&+ 4|\lambda_{n-1}^1-\lambda_n^1|\|\nabla\widetilde{f}_1 x_{n-1}\| + L_1|\lambda_n^1-\lambda_{n-1}^1|\|x_{n-1}\| \\
\leq&\|x_n - x_{n-1}\| \\
&+ |\lambda_{n-1}^1-\lambda_n^1|(L_1\|P_C(I-\lambda_n^1\nabla\widetilde{f}_1)x_{n-1}\| + 4\|\nabla\widetilde{f}_1 x_{n-1}\| + L_1\|x_{n-1}\|) \\
\leq&\|x_n - x_{n-1}\| + M_1|\lambda_{n-1}^1-\lambda_n^1|, \qquad (3.4)
\end{aligned}
$$

for some $M_1 > 0$ such that $M_1 \geq L_1\|P_C(I-\lambda_n^1\nabla\widetilde{f}_1)x_{n-1}\| + 4\|\nabla\widetilde{f}_1 x_{n-1}\| + L_1\|x_{n-1}\|$, for all $n \in \mathbb{N}$.
From the definition of x_n and (3.4), we have

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
=&\|(1-\mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n) \\
&- ((1-\mu_{n-1})x_{n-1} + \mu_{n-1}P_C(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}))\| \\
\leq&(1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
&+ \mu_n\|P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n) - P_C(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1})\| \\
&+ |\mu_n-\mu_{n-1}|\|P_C(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1})\| \\
\leq&(1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
&+ \mu_n\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - (\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1})\| \\
&+ |\mu_n-\mu_{n-1}|\|P_C(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1})\| \\
\leq&(1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
&+ \mu_n(\|\alpha_n f(y_n) - \alpha_{n-1}f(y_{n-1})\| + \|(1-\alpha_n)T_n^{\widetilde{f}_1}x_n - (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\|) \\
&+ |\mu_n-\mu_{n-1}|\|P_C(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1})\| \\
\leq&(1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1}-\mu_n|\|x_{n-1}\|
\end{aligned}
$$

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$$
\begin{aligned}
&\quad + \mu_n(\alpha_n\|f(y_n) - f(y_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(y_{n-1})\| + (1-\alpha_n)\|T_n^{\widetilde{f}_1}x_n - T_{n-1}^{\widetilde{f}_1}x_{n-1}\| \\
&\quad + |\alpha_{n-1} - \alpha_n|\|T_{n-1}^{\widetilde{f}_1}x_{n-1}\|) + |\mu_n - \mu_{n-1}|\|P_C(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1})\| \\
\le&(1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1} - \mu_n|\|x_{n-1}\| \\
&\quad + \mu_n(\alpha_n\|f(y_n) - f(y_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(y_{n-1})\| \\
&\quad + (1-\alpha_n)(\|x_n - x_{n-1}\| + M_1|\lambda_{n-1}^1 - \lambda_n^1|) + |\alpha_{n-1} - \alpha_n|\|T_{n-1}^{\widetilde{f}_1}x_{n-1}\|) \\
&\quad + |\mu_n - \mu_{n-1}|\|P_C(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1})\| \\
\le&(1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1} - \mu_n|\|x_{n-1}\| \\
&\quad + \mu_n(\alpha_n\|f(y_n) - f(y_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(y_{n-1})\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\
&\quad + (1-\alpha_n)\frac{4M_1}{L_1}|s_n^1 - s_{n-1}^1| + |\alpha_{n-1} - \alpha_n|\|T_{n-1}^{\widetilde{f}_1}x_{n-1}\|) \\
&\quad + |\mu_n - \mu_{n-1}|\|P_C(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1})\| \\
\le&(1-\mu_n\alpha_n)\|x_n - x_{n-1}\| + |\mu_{n-1} - \mu_n|\|x_{n-1}\| \\
&\quad + |\mu_n - \mu_{n-1}|\|P_C(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1})\| \\
&\quad + \mu_n(\alpha_n a\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(y_{n-1})\| \\
&\quad + (1-\alpha_n)\frac{4M_1}{L_1}|s_n^1 - s_{n-1}^1| + |\alpha_{n-1} - \alpha_n|\|T_{n-1}^{\widetilde{f}_1}x_{n-1}\|). \qquad (3.5)
\end{aligned}
$$

Using the same method as derived in (3.5), we have

$$
\begin{aligned}
\|y_{n+1} - y_n\| \le&(1-\mu_n\alpha_n)\|y_n - y_{n-1}\| + |\mu_{n-1} - \mu_n|\|y_{n-1}\| \\
&+ |\mu_n - \mu_{n-1}|\|P_C(\alpha_{n-1}g(x_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_2}y_{n-1})\| \\
&+ \mu_n(\alpha_n a\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|g(x_{n-1})\| \\
&+ (1-\alpha_n)\frac{4M_2}{L_2}|s_n^2 - s_{n-1}^2| + |\alpha_{n-1} - \alpha_n|\|T_{n-1}^{\widetilde{f}_2}y_{n-1}\|), \qquad (3.6)
\end{aligned}
$$

for some $M_2 > 0$ such that $M_2 \ge L_2\|P_C(I - \lambda_n^2\nabla\widetilde{f}_2)y_{n-1}\| + 4\|\nabla\widetilde{f}_2 y_{n-1}\| + L_2\|y_{n-1}\|$, for all $n \in \mathbb{N}$.

From (3.5) and (3.6), we have

$$
\begin{aligned}
&\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\
\le&(1-(1-a)\mu_n\alpha_n)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) + |\mu_{n-1} - \mu_n|\Big(\|x_{n-1}\| + \|y_{n-1}\| \\
&+ \|P_C(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1})\| + \|P_C(\alpha_{n-1}g(x_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_2}y_{n-1})\|\Big) \\
&+ |\alpha_n - \alpha_{n-1}|(\|f(y_{n-1})\| + \|g(x_{n-1})\| + \|T_{n-1}^{\widetilde{f}_1}x_{n-1}\| + \|T_{n-1}^{\widetilde{f}_2}y_{n-1}\|) \\
&+ (1-\alpha_n)\Big(\frac{4M_1}{L_1}|s_n^1 - s_{n-1}^1| + \frac{4M_2}{L_2}|s_n^2 - s_{n-1}^2|\Big).
\end{aligned}
$$

Applying Lemma 2.12 and the condition (iii), we can conclude that

$$
\|x_{n+1} - x_n\| \to 0 \text{ and } \|y_{n+1} - y_n\| \to 0 \text{ as } n \to \infty. \qquad (3.7)
$$

Next, we show that $\|x_n - W_n\| \to 0$ where $W_n = \alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n$ and $\|y_n - V_n\| \to 0$ while $V_n = \alpha_n g(x_n) + (1-\alpha_n)T_n^{\widetilde{f}_2}y_n$.

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Let $\widetilde{x} \in U_{\widetilde{f}_1}$ and $\widetilde{y} \in U_{\widetilde{f}_2}$. Then, we derive that

$$
\begin{aligned}
&\|x_{n+1} - \widetilde{x}\|^2\\
=&\|(1-\mu_n)x_n + \mu_n P_C W_n - \widetilde{x}\|^2\\
=&\|(1-\mu_n)(x_n - \widetilde{x}) + \mu_n(P_C W_n - \widetilde{x})\|^2\\
=&(1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n\|P_C W_n - \widetilde{x}\|^2 - (1-\mu_n)\mu_n\|x_n - P_C W_n\|^2\\
\leq&(1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\|^2 - (1-\mu_n)\mu_n\|x_n - P_C W_n\|^2\\
=&(1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n\|\alpha_n(f(y_n) - T_n^{\widetilde{f}_1}x_n) + T_n^{\widetilde{f}_1}x_n - \widetilde{x}\|^2 - (1-\mu_n)\mu_n\|x_n - P_C W_n\|^2\\
\leq&(1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n(\|T_n^{\widetilde{f}_1}x_n - \widetilde{x}\|^2 + 2\alpha_n\langle f(y_n) - T_n^{\widetilde{f}_1}x_n, \alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\rangle)\\
&-(1-\mu_n)\mu_n\|x_n - P_C W_n\|^2\\
\leq&(1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n(\|T_n^{\widetilde{f}_1}x_n - \widetilde{x}\|^2 + 2\alpha_n\|f(y_n) - T_n^{\widetilde{f}_1}x_n\|\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\|)\\
&-(1-\mu_n)\mu_n\|x_n - P_C W_n\|^2\\
\leq&(1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n\|x_n - \widetilde{x}\|^2 + 2\alpha_n\mu_n\|f(y_n) - T_n^{\widetilde{f}_1}x_n\|\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\|\\
&-(1-\mu_n)\mu_n\|x_n - P_C W_n\|^2\\
=&\|x_n - \widetilde{x}\|^2 + 2\alpha_n\mu_n\|f(y_n) - T_n^{\widetilde{f}_1}x_n\|\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\|\\
&-(1-\mu_n)\mu_n\|x_n - P_C W_n\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(1-\mu_n)\mu_n\|x_n - P_C W_n\|^2\\
\leq&\|x_n - \widetilde{x}\|^2 - \|x_{n+1} - \widetilde{x}\|^2 + 2\alpha_n\mu_n\|f(y_n) - T_n^{\widetilde{f}_1}x_n\|\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\|\\
\leq&\|x_n - x_{n+1}\|(\|x_n - \widetilde{x}\| + \|x_{n+1} - \widetilde{x}\|) + 2\alpha_n\mu_n\|f(y_n) - T_n^{\widetilde{f}_1}x_n\|\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\|.
\end{aligned}
$$

By (3.7), the condition i) and ii), we get

$$\|P_C W_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.8}$$

From definition of y_n and applying the same method as (3.8), we have

$$\|P_C V_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.9}$$

Since,

$$
\begin{aligned}
\|P_C W_n - \widetilde{x}\|^2 =&\|P_C W_n - P_C\widetilde{x}\|^2\\
\leq&\langle W_n - \widetilde{x}, P_C W_n - \widetilde{x}\rangle\\
=&\frac{1}{2}(\|W_n - \widetilde{x}\|^2 + \|P_C W_n - \widetilde{x}\|^2 - \|W_n - P_C W_n\|^2),
\end{aligned}
$$

we have

$$\|P_C W_n - \widetilde{x}\| \leq \|W_n - \widetilde{x}\|^2 - \|W_n - P_C W_n\|^2. \tag{3.10}$$

Observe that

$$\|W_n - \widetilde{x}\|^2 =\|\alpha_n(f(y_n) - \widetilde{x}) + (1-\alpha_n)(T_n^{\widetilde{f}_1}x_n - \widetilde{x})\|^2$$

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$$\leq \alpha_n\|f(y_n)-\widetilde{x}\|^2+(1-\alpha_n)\|T_n^{\widetilde{f}_1}x_n-\widetilde{x}\|^2$$
$$\leq \alpha_n\|f(y_n)-\widetilde{x}\|^2+(1-\alpha_n)\|x_n-\widetilde{x}\|^2. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\begin{aligned}\|x_{n+1}-\widetilde{x}\|^2 =& \|(1-\mu_n)x_n+\mu_n P_C(\alpha_n f(y_n)+(1-\alpha_n)T_n^{\widetilde{f}_1}x_n)-\widetilde{x}\|^2\\ =& \|(1-\mu_n)(x_n-\widetilde{x})+\mu_n(P_CW_n-\widetilde{x})\|^2\\ \leq& (1-\mu_n)\|x_n-\widetilde{x}\|^2+\mu_n\|P_CW_n-\widetilde{x}\|^2\\ \leq& (1-\mu_n)\|x_n-\widetilde{x}\|^2+\mu_n(\|W_n-\widetilde{x}\|^2-\|W_n-P_CW_n\|^2)\\ \leq& (1-\mu_n)\|x_n-\widetilde{x}\|^2\\ &+\mu_n(\alpha_n\|f(y_n)-\widetilde{x}\|^2+(1-\alpha_n)\|x_n-\widetilde{x}\|^2-\|W_n-P_CW_n\|^2),\end{aligned}$$

which implies that

$$\begin{aligned}\mu_n\|W_n-P_CW_n\|^2 \leq& (1-\alpha_n\mu_n)\|x_n-\widetilde{x}\|^2-\|x_{n+1}-\widetilde{x}\|^2+\alpha_n\mu_n\|f(y_n)-\widetilde{x}\|^2\\ \leq& \|x_n-\widetilde{x}\|^2-\|x_{n+1}-\widetilde{x}\|^2+\alpha_n\mu_n\|f(y_n)-\widetilde{x}\|^2\\ \leq& \|x_n-x_{n+1}\|(\|x_n-\widetilde{x}\|+\|x_{n+1}-\widetilde{x}\|)+\alpha_n\mu_n\|f(y_n)-\widetilde{x}\|^2.\end{aligned}$$

From $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$ and condition i), we have

$$\|W_n-P_CW_n\|\to 0 \text{ as } n\to\infty. \tag{3.12}$$

From definition of V_n and applying the same argument as (3.12), we also obtain

$$\|V_n-P_CV_n\|\to 0 \text{ as } n\to\infty. \tag{3.13}$$

Consider

$$\begin{aligned}\|x_n-W_n\| =& \|x_n-P_CW_n+P_CW_n-W_n\|\\ \leq& \|x_n-P_CW_n\|+\|P_CW_n-W_n\|.\end{aligned}$$

From (3.8) and (3.12), we have

$$\|x_n-W_n\|\to 0 \text{ as } n\to\infty. \tag{3.14}$$

From definition of y_n and applying the same method as (3.14), we also have

$$\|y_n-V_n\|\to 0 \text{ as } n\to\infty. \tag{3.15}$$

Next, we show that $\|W_n-P_C(I-\frac{2}{L_1}\nabla\widetilde{f}_1)W_n\|\to 0$ as $n\to\infty$ and $\|V_n-P_C(I-\frac{2}{L_2}\nabla\widetilde{f}_2)V_n\|\to 0$ as $n\to\infty$.

Observe that

$$W_n-x_n=\alpha_n(f(y_n)-x_n)+(1-\alpha_n)(T_n^{\widetilde{f}_1}x_n-x_n),$$

it follows that

$$(1-\alpha_n)\|T_n^{\widetilde{f}_1}x_n-x_n\|\leq\|W_n-x_n\|+\alpha_n\|f(y_n)-x_n\|.$$

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From (3.14) and condition i), we have

$$\|T_n^{\widetilde{f}_1}x_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.16}$$

Consider

$$\begin{aligned}\|W_n - T_n^{\widetilde{f}_1}W_n\| &= \|W_n - x_n + x_n - T_n^{\widetilde{f}_1}x_n + T_n^{\widetilde{f}_1}x_n - T_n^{\widetilde{f}_1}W_n\| \\ &\le \|W_n - x_n\| + \|x_n - T_n^{\widetilde{f}_1}x_n\| + \|T_n^{\widetilde{f}_1}x_n - T_n^{\widetilde{f}_1}W_n\| \\ &\le \|W_n - x_n\| + \|x_n - T_n^{\widetilde{f}_1}x_n\| + \|x_n - W_n\| \\ &= 2\|x_n - W_n\| + \|T_n^{\widetilde{f}_1}x_n - x_n\|.\end{aligned}$$

From (3.14) and (3.16), we get

$$\|T_n^{\widetilde{f}_1}W_n - W_n\| \to 0 \text{ as } n \to \infty. \tag{3.17}$$

Observe that

$$\begin{aligned}\|P_C(I - \lambda_n^1\nabla\widetilde{f}_1)W_n - W_n\| &= \|s_n^1 W_n + (1 - s_n^1)T_n^{\widetilde{f}_1}W_n - W_n\| \\ &= (1 - s_n^1)\|T_n^{\widetilde{f}_1}W_n - W_n\| \\ &\le \|T_n^{\widetilde{f}_1}W_n - W_n\|,\end{aligned} \tag{3.18}$$

where $s_n^1 = \frac{2-\lambda_n^1 L_1}{4} \in (0, \frac{1}{2})$.
From (3.18), we have

$$\begin{aligned}&\|P_C(I - \frac{2}{L_1}\nabla\widetilde{f}_1)W_n - W_n\| \\ &\le \|P_C(I - \frac{2}{L_1}\nabla\widetilde{f}_1)W_n - P_C(I - \lambda_n^1\nabla\widetilde{f}_1)W_n\| + \|P_C(I - \lambda_n^1\nabla\widetilde{f}_1)W_n - W_n\| \\ &\le \|(I - \frac{2}{L_1}\nabla\widetilde{f}_1)W_n - (I - \lambda_n^1\nabla\widetilde{f}_1)W_n\| + \|P_C(I - \lambda_n^1\nabla\widetilde{f}_1)W_n - W_n\| \\ &\le (\frac{2}{L_1} - \lambda_n^1)\|\nabla\widetilde{f}_1(W_n)\| + \|T_n^{\widetilde{f}_1}W_n - W_n\|.\end{aligned}$$

From the boundedness of $\{W_n\}$, (3.17), and the condition iv), we conclude that

$$\lim_{n\to\infty}\|W_n - P_C(I - \frac{2}{L_1}\nabla\widetilde{f}_1)W_n\| = 0. \tag{3.19}$$

Applying the same method as (3.19), we also have

$$\lim_{n\to\infty}\|V_n - P_C(I - \frac{2}{L_2}\nabla\widetilde{f}_2)V_n\| = 0. \tag{3.20}$$

Next, we show that $\limsup_{n\to\infty}\langle f(y^*) - x^*, W_n - x^*\rangle \le 0$, where $x^* = P_{U_{\tilde{f}_1}}f(y^*)$ and $\limsup_{n\to\infty}\langle g(x^*) - y^*, V_n - y^*\rangle \le 0$, where $y^* = P_{U_{\tilde{f}_2}}g(x^*)$.

Indeed, take a subsequence $\{W_{n_k}\}$ of $\{W_n\}$ such that

$$\limsup_{n\to\infty}\langle f(y^*) - x^*, W_n - x^*\rangle = \limsup_{k\to\infty}\langle f(y^*) - x^*, W_{n_k} - x^*\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_k} \rightharpoonup \widehat{x}$ as $k \to \infty$. From (3.14), we obtain $W_{n_k} \rightharpoonup \widehat{x}$ as $k \to \infty$.

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Assume that $\widehat{x} \neq P_C(I - \frac{2}{L_1}\nabla \widetilde{f}_1)\widehat{x}$.
By nonexpansiveness of $P_C(I - \frac{2}{L_1}\nabla \widetilde{f}_1)$, (3.19) and Opial's property, we have

$$\begin{aligned}\liminf_{k\to\infty} \|W_{n_k} - \widehat{x}\| &< \liminf_{k\to\infty} \|W_{n_k} - P_C(I - \frac{2}{L_1}\nabla \widetilde{f}_1)\widehat{x}\| \\ &\leq \liminf_{k\to\infty} (\|W_{n_k} - P_C(I - \frac{2}{L_1}\nabla \widetilde{f}_1)W_{n_k}\| \\ &\quad + \|P_C(I - \frac{2}{L_1}\nabla \widetilde{f}_1)W_{n_k} - P_C(I - \frac{2}{L_1}\nabla \widetilde{f}_1)\widehat{x}\|) \\ &\leq \liminf_{k\to\infty} \|W_{n_k} - \widehat{x}\|.\end{aligned}$$

This is a contradiction, then we have

$$\widehat{x} \in F\Big(P_C(I - \frac{2}{L_1}\nabla \widetilde{f}_1)\Big) = U_{\widetilde{f}_1}. \tag{3.21}$$

Since $W_{n_k} \rightharpoonup \widehat{x}$ as $k \to \infty$, (3.21) and Lemma 2.11 (a), we can derive that

$$\begin{aligned}\limsup_{n\to\infty} \langle f(y^*) - x^*, W_n - x^*\rangle &= \limsup_{k\to\infty} \langle f(y^*) - x^*, W_{n_k} - x^*\rangle \\ &= \langle f(y^*) - x^*, \widehat{x} - x^*\rangle \\ &\leq 0.\end{aligned} \tag{3.22}$$

Similarly, indeed, take a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that

$$\limsup_{n\to\infty} \langle g(x^*) - y^*, V_n - y^*\rangle = \limsup_{k\to\infty} \langle g(x^*) - y^*, V_{n_k} - y^*\rangle.$$

Since $\{y_n\}$ is bounded, without loss of generality, we may assume that $y_{n_k} \rightharpoonup \widehat{y}$ as $k \to \infty$. From (3.15), we obtain $V_{n_k} \rightharpoonup \widehat{y}$ as $k \to \infty$.
Following the same method as (3.22), we easily obtain that

$$\limsup_{n\to\infty} \langle g(x^*) - y^*, V_n - y^*\rangle \leq 0. \tag{3.23}$$

Finally, we show that $\{x_n\}$ converses strongly to x^*, where $x^* = P_{U_{\widetilde{f}_1}} f(y^*)$, and $\{y_n\}$ converses strongly to y^*, where $y^* = P_{U_{\widetilde{f}_2}} g(x^*)$.

Let $W_n = \alpha_n f(y_n) + (1 - \alpha_n)T_n^{\widetilde{f}_1} x_n$ and $V_n = \alpha_n g(x_n) + (1 - \alpha_n)T_n^{\widetilde{f}_2} y_n$.
From the definition of x_n, we get

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ =&\|(1 - \mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1 - \alpha_n)T_n^{\widetilde{f}_1} x_n) - x^*\|^2 \\ =&\|(1 - \mu_n)(x_n - x^*) + \mu_n(P_C(\alpha_n f(y_n) + (1 - \alpha_n)T_n^{\widetilde{f}_1} x_n) - x^*)\|^2 \\ =&(1 - \mu_n)\|x_n - x^*\|^2 + \mu_n\|P_C(\alpha_n f(y_n) + (1 - \alpha_n)T_n^{\widetilde{f}_1} x_n) - x^*\|^2 \\ \leq&(1 - \mu_n)\|x_n - x^*\|^2 + \mu_n\|\alpha_n f(y_n) + (1 - \alpha_n)T_n^{\widetilde{f}_1} x_n - x^*\|^2 \\ =&(1 - \mu_n)\|x_n - x^*\|^2 + \mu_n\|\alpha_n(f(y_n) - x^*) + (1 - \alpha_n)(T_n^{\widetilde{f}_1} x_n - x^*)\|^2 \\ \leq&(1 - \mu_n)\|x_n - x^*\|^2 + \mu_n((1 - \alpha_n)\|T_n^{\widetilde{f}_1} x_n - x^*\|^2 + 2\alpha_n\langle f(y_n) - x^*, W_n - x^*\rangle) \\ \leq&(1 - \mu_n)\|x_n - x^*\|^2 + \mu_n(1 - \alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\mu_n\langle f(y_n) - x^*, W_n - x^*\rangle\end{aligned}$$

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$$
\begin{aligned}
&=(1-\alpha_n\mu_n)\|x_n-x^*\|^2+2\alpha_n\mu_n\langle f(y_n)-x^*,W_n-x^*\rangle\\
&=(1-\alpha_n\mu_n)\|x_n-x^*\|^2+2\alpha_n\mu_n(\langle f(y_n)-f(y^*),W_n-x^*\rangle+\langle f(y^*)-x^*,W_n-x^*\rangle)\\
&\leq(1-\alpha_n\mu_n)\|x_n-x^*\|^2+2\alpha_n\mu_n(\|f(y_n)-f(y^*)\|\|W_n-x^*\|+\langle f(y^*)-x^*,W_n-x^*\rangle)\\
&\leq(1-\alpha_n\mu_n)\|x_n-x^*\|^2+2\alpha_n\mu_n\|f(y_n)-f(y^*)\|(\|W_n-x_{n+1}\|+\|x_{n+1}-x^*\|)\\
&\quad+2\alpha_n\mu_n\langle f(y^*)-x^*,W_n-x^*\rangle\\
&\leq(1-\alpha_n\mu_n)\|x_n-x^*\|^2+2\alpha_n\mu_n a\|y_n-y^*\|\|W_n-x_{n+1}\|+2\alpha_n\mu_n a\|y_n-y^*\|\|x_{n+1}-x^*\|\\
&\quad+2\alpha_n\mu_n\langle f(y^*)-x^*,W_n-x^*\rangle\\
&\leq(1-\alpha_n\mu_n)\|x_n-x^*\|^2+2\alpha_n\mu_n a\|y_n-y^*\|\|W_n-x_{n+1}\|+\alpha_n\mu_n a[\|y_n-y^*\|^2+\|x_{n+1}-x^*\|^2\\
&\quad+2\alpha_n\mu_n\langle f(y^*)-x^*,W_n-x^*\rangle,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2\leq&\frac{1-\alpha_n\mu_n}{1-\alpha_n\mu_n a}\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|\|W_n-x_{n+1}\|\\
&+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle f(y^*)-x^*,W_n-x^*\rangle\\
=&\Big(1-\frac{\alpha_n\mu_n-\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\Big)\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|\|W_n-x_{n+1}\|\\
&+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle f(y^*)-x^*,W_n-x^*\rangle\\
=&\Big(1-\frac{\alpha_n\mu_n(1-a)}{1-\alpha_n\mu_n a}\Big)\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|\|W_n-x_{n+1}\|\\
&+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle f(y^*)-x^*,W_n-x^*\rangle. \qquad (3.24)
\end{aligned}
$$

Similarly, as derived above, we also have

$$
\begin{aligned}
\|y_{n+1}-y^*\|^2\leq&\Big(1-\frac{\alpha_n\mu_n(1-a)}{1-\alpha_n\mu_n a}\Big)\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|x_n-x^*\|\|V_n-y_{n+1}\|\\
&+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle g(x^*)-y^*,V_n-y^*\rangle. \qquad (3.25)
\end{aligned}
$$

From (3.24) and (3.25), we deduce that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\leq&\Big(1-\frac{\alpha_n\mu_n(1-a)}{1-\alpha_n\mu_n a}\Big)(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\
&+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}(\|y_n-y^*\|\|W_n-x_{n+1}\|+\|x_n-x^*\|\|V_n-y_{n+1}\|)\\
&+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\
&+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}(\langle f(y^*)-x^*,W_n-x^*\rangle+\langle g(x^*)-y^*,V_n-y^*\rangle)\\
=&\Big(1-\frac{\alpha_n\mu_n(1-2a)}{1-\alpha_n\mu_n a}\Big)(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\
&+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}(\|y_n-y^*\|\|W_n-x_{n+1}\|+\|x_n-x^*\|\|V_n-y_{n+1}\|)\\
&+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}(\langle f(y^*)-x^*,W_n-x^*\rangle+\langle g(x^*)-y^*,V_n-y^*\rangle).
\end{aligned}
$$

(3.26)

By (3.7), (3.14), (3.15), (3.22), (3.23), the condition (i) and Lemma 2.12, we have $\lim_{n\to\infty}(\|x_n-x^*\|+\|y_n-y^*\|)=0$. It implies that the sequences $\{x_n\}$ and $\{y_n\}$ converge to

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$x^* = P_{U_{\widetilde{f}_1}} f(y^*)$ and $y^* = P_{U_{\widetilde{f}_2}} g(x^*)$, respectively.
This completes the proof. $\square$

As a direct result of Theorem 3.1, we get the following result.

Corollary 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\widetilde{f} : C \to \mathbb{R}$ be a real-valued convex function and assume that $\nabla \widetilde{f}$ is an $\frac{1}{L}$-inverse strongly monotone with $L \geq 0$ and $U_{\widetilde{f}} = VI(C, \nabla \widetilde{f}) \neq \emptyset$. Let $f : H \to H$ be a-contraction mapping with $a \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and

$$x_{n+1} = (1-\mu_n)x_n + \mu_n P_C(\alpha_n f(x_n) + (1-\alpha_n)T_n^{\widetilde{f}} x_n), \tag{3.27}$$

for all $n \in \mathbb{N}$, where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n^i \nabla \widetilde{f}) = s_n I + (1-s_n)T_n^{\widetilde{f}}$, $T_n^{\widetilde{f}}$ is nonexpansive, $s_n = \frac{2-\lambda_n L}{4}$, and $\{\lambda_n\} \subset (0, \frac{2}{L})$. Assume that the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \overline{\theta} \leq \mu_n \leq \theta$, for all $n \in \mathbb{N}$ and for some $\overline{\theta}, \theta > 0$,

(iii) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum\limits_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$,

(iv) $\lim\limits_{n\to\infty} \lambda_n = \dfrac{2}{L}$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* = P_{U_{\widetilde{f}}} f(x^*)$.

Proof. If we put $f \equiv g$, $x_n = y_n$, in Theorem 3.1, we obtain the desired conclusion. $\square$

3.2 The new subgradient for the modification of equilibrium problems and variational inequality problems

We present the new subgradient extragradient algorithm and the new iterative method for proving weak and strong convergence theorem of $\{x_n\}$ generated by the following algorithm:

Algorithm 3.3. Let $A, B, \overline{A}, \overline{B} : C \to H$ be $a, b, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $F, Q : C \times C \to \mathbb{R}$ be bifunctions satisfying A1)-A4) and define the mapping $\varphi : C \to C$ by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$, for all $x \in C$, $r > 0$, $\eta = \min\{a, b\}$, $\lambda, \beta \in (0, 2\eta)$ where T_r^F, T_r^Q define by $T_r^F(\bar{x}) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - \bar{x}\rangle \geq 0, \forall y \in C\}$ and $T_r^Q(\bar{x}) = \{z \in C : Q(z,y) + \frac{1}{r}\langle y - z, z - \bar{x}\rangle \geq 0, \forall y \in C\}$, for all $\bar{x} \in H$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$, and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1 \in C$ and

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) + \beta_n \varphi(x_n), \end{cases} \tag{3.28}$$

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for all $n \in \mathbb{N}$, where $\bar{a} \in (0,1)$ and $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$.

The following lemma is crucial for proving our main theorem.

Lemma 3.4. Let C be a nonempty closed convex subset of a real Hilbert space H and let $\overline{A}, \overline{B} : C \to H$ be $\bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $x^* \in VI(C, \overline{A}) \cap VI(C, \overline{B})$, $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$ and $\bar{a} \in (0,1)$, we have

$$
\begin{aligned}
&\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n) - x^*\|^2 \\
&\qquad\qquad \leq \|x_n - x^*\|^2 - (1 - \frac{\gamma}{\bar{\eta}})\|x_n - y_n\|^2 \\
&\qquad\qquad\quad - (1 - \frac{\gamma}{\bar{\eta}})\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n\|^2,
\end{aligned}
$$

where sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.28).

Proof. Let $x^* \in VI(C, \overline{A}) \cap VI(C, \overline{B})$.
By the property of P_C, we have

$$
\begin{aligned}
&\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*\|^2 \\
&\quad \leq \|x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n - x^*\|^2 \\
&\qquad - \|x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\quad = \|x_n - x^*\|^2 - 2\gamma\langle x_n - x^*, (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\rangle + \gamma^2\|(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\|^2 \\
&\qquad - \Big(\|x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 + \gamma^2\|(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\|^2 \\
&\qquad - 2\gamma\langle x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n), (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\rangle\Big) \\
&\quad = \|x_n - x^*\|^2 - \|x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\qquad - 2\gamma\langle P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*, (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\rangle. \qquad (3.29)
\end{aligned}
$$

From monotonicity of $(\bar{a}\overline{A} + (1-\bar{a})\overline{B})$ and Lemma 2.26, we have

$$
\begin{aligned}
0 &\leq \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n - (\bar{a}\overline{A} + (1-\bar{a})\overline{B})x^*, y_n - x^*\rangle \\
&= \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - x^*\rangle - \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})x^*, y_n - x^*\rangle \\
&\leq \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - x^*\rangle \\
&= \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\rangle \\
&\quad + \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*\rangle.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
&\langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, x^* - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\rangle \\
&\qquad \leq \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\rangle. \qquad (3.30)
\end{aligned}
$$

From (3.29) and (3.30), we get

$$
\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*\|^2
$$

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$$\begin{aligned}
&\leq\|x_n-x^*\|^2-\|x_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\|^2\\
&\quad-2\gamma\langle P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*,(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n\rangle\\
&\leq\|x_n-x^*\|^2-\|x_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\|^2\\
&\quad+2\gamma\langle(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n,y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\rangle\\
&=\|x_n-x^*\|^2-\|x_n-y_n\|^2-\|y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\|^2\\
&\quad-2\langle x_n-y_n,y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\rangle\\
&\quad+2\gamma\langle(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n,y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\rangle\\
&=\|x_n-x^*\|^2-\|x_n-y_n\|^2-\|y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\|^2\\
&\quad+2\langle y_n-x_n+\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n,y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\rangle\\
&=\|x_n-x^*\|^2-\|x_n-y_n\|^2-\|y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\|^2\\
&\quad+2\langle x_n-y_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n,P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-y_n\rangle\\
&=\|x_n-x^*\|^2-\|x_n-y_n\|^2-\|y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\|^2\\
&\quad+2\langle(I-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B}))x_n-y_n,P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-y_n\rangle\\
&\quad+2\langle\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n,P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-y_n\rangle\\
&\leq\|x_n-x^*\|^2-\|x_n-y_n\|^2-\|y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\|^2\\
&\quad+2\langle\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n,P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-y_n\rangle\\
&\leq\|x_n-x^*\|^2-\|x_n-y_n\|^2-\|y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\|^2\\
&\quad+2\gamma\|(\bar{a}\overline{A}+(1-\bar{a})\overline{B})x_n-(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n\|\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-y_n\|\\
&\leq\|x_n-x^*\|^2-\|x_n-y_n\|^2-\|y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\|^2\\
&\quad+2(\frac{\gamma}{\bar{\eta}})\|x_n-y_n\|\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-y_n\|\\
&\leq\|x_n-x^*\|^2-\|x_n-y_n\|^2-\|y_n-P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)\|^2\\
&\quad+\frac{\gamma}{\bar{\eta}}(\|x_n-y_n\|^2+\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-y_n\|^2)\\
&=\|x_n-x^*\|^2-(1-\frac{\gamma}{\bar{\eta}})\|x_n-y_n\|^2-(1-\frac{\gamma}{\bar{\eta}})\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-y_n\|^2.
\end{aligned}$$

□

Now, we present the following Lemma 3.5 for proving a strong convergence theorem.

Lemma 3.5. Let H be a real Hilbert space and let S be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in S$,

$$\|x_{n+1}-u\| \leq \|x_n-u\|+b_n,$$

for every $n = 0, 1, 2, \ldots$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then $\{P_S x_n\}$ converges strongly to some $z \in S$.

Proof. Let $\epsilon > 0$. From Lemma 2.7, then $\lim_{n\to\infty}\|x_n-u\|$ exists and we have that $\{\|x_n-u\|\}$ is bounded, for all $u \in S$.

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Put $u_n = P_S x_n$. From parallelogram law, we get

$$\begin{aligned}\|u_{n+1}-u_n\|^2 =& 2\|x_{n+1}-u_{n+1}\|^2 + 2\|x_{n+1}-u_n\|^2 - 4\|x_{n+1} - \frac{1}{2}(u_{n+1}+u_n)\|^2 \\ \leq& 2\|x_{n+1}-u_{n+1}\|^2 + 2\|x_{n+1}-u_n\|^2 - 4\|x_{n+1}-u_{n+1}\|^2 \\ =& 2\|x_{n+1}-u_n\|^2 - 2\|x_{n+1}-u_{n+1}\|^2 \\ \leq& 2\|x_n-u_n\|^2 - 2\|x_{n+1}-u_{n+1}\|^2 + 2\bar{b}_n, \end{aligned} \tag{3.31}$$

where $\bar{b}_n = b_n(b_n + 2\|x_n - u_n\|)$.
It implies that

$$\|x_{n+1}-u_{n+1}\|^2 \leq \|x_n - u_n\|^2 + \bar{b}_n.$$

Since $\{x_n\}$ and $\{u_n\}$ are bounded and $\sum_{n=1}^{\infty} b_n < \infty$, we obtain that $\sum_{n=1}^{\infty} \bar{b}_n < \infty$.
From Lemma 2.7, then $\lim_{n\to\infty} \|x_n - u_n\|$ there exists.
By (3.31), we have $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$, then there exist $N \in \mathbb{N}$, such that

$$\|u_{n+1} - u_n\| \leq \frac{\epsilon}{2^n}, \qquad \text{for all } n \geq N.$$

Thus, for every $p \in \mathbb{N}$, we have

$$\|u_{n+p} - u_n\| \leq \sum_{k=n}^{n+p-1} \|u_{k+1} - u_k\| \leq \epsilon \sum_{k=n}^{n+p-1} \frac{1}{2^k} \leq \epsilon(\frac{1}{2^{n-1}}) < \epsilon. \tag{3.32}$$

From (3.32), we have that $\{u_n\}$ is a cauchy sequence. Hence, $\{u_n\}$ converges strongly to some $z \in S$. □

Next, we prove the weak and strong convergence theorem of the sequence $\{x_n\}$ generated by the new subgradient extragradient algorithm for finding a common element of the set of solutions of the MEP and two sets of the variational inequality problems.

Theorem 3.6. Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B, \overline{A}, \overline{B} : C \to H$ be $a, b, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $F, Q : C \times C \to \mathbb{R}$ be bifunctions satisfying A1)-A4). Define the mapping $\varphi : C \to C$ by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$, for all $r > 0$, $x \in C$, the mappings T_r^F, T_r^Q define as the same in Algorithm 3.3, where $\bar{a} \in (0,1)$ and $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by (3.28), satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_\xi(x_n).$$

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Proof. First, we show that φ is a nonexpansive mapping for every $\lambda, \beta \in (0, 2\eta)$, where $\eta = \min\{a, b\}$.

Let $x, y \in C$. Since A is a-inverse strongly monotone, we have

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 =&\|x-y-\lambda(Ax-Ay)\|^2\\ =&\|x-y\|^2-2\lambda\langle x-y, Ax-Ay\rangle+\lambda^2\|Ax-Ay\|^2\\ \leq&\|x-y\|^2-2\lambda a\|Ax-Ay\|^2+\lambda^2\|Ax-Ay\|^2\\ =&\|x-y\|^2-\lambda(2a-\lambda)\|Ax-Ay\|^2\\ \leq&\|x-y\|^2-\lambda(2\eta-\lambda)\|Ax-Ay\|^2\\ \leq&\|x-y\|^2. \end{aligned} \tag{3.33}$$

Thus $(I-\lambda A)$ is a nonexpansive mapping. By using the same method as (3.33), we have $(I-\beta B)$ is a nonexpansive mapping. Hence, $T_r^F(I-\lambda A)$ and $T_r^Q(I-\beta B)$ are nonexpansive mappings. It is easy to see that the mapping φ is a nonexpansive mapping.

Let $x^* \in VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi)$, and $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$. We divide the proof of this result into 3 steps.

Step 1. Show that $\{x_n\}_{n=0}^{\infty}$ is bounded.

From the definition of x_n, Lemma 2.8 and Lemma 3.4, we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\ =&\|\alpha_n P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)+\beta_n\varphi(x_n)-x^*\|^2\\ =&\|\alpha_n(P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*)+\beta_n(\varphi(x_n)-x^*)-(1-\alpha_n-\beta_n)x^*\|^2\\ \leq&\|\alpha_n(P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*)+\beta_n(\varphi(x_n)-x^*)\|^2\\ &-2(1-\alpha_n-\beta_n)\langle x^*, x_{n+1}-x^*\rangle\\ \leq&\|\alpha_n(P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*)+\beta_n(\varphi(x_n)-x^*)\|^2\\ &+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ =&\alpha_n(\alpha_n+\beta_n)\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*\|^2\\ &+\beta_n(\alpha_n+\beta_n)\|\varphi(x_n)-x^*\|^2-\alpha_n\beta_n\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-\varphi(x_n)\|^2\\ &+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ \leq&\alpha_n(\alpha_n+\beta_n)\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*\|^2\\ &+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ \leq&\alpha_n(\alpha_n+\beta_n)\Big(\|x_n-x^*\|^2-(1-\frac{\gamma}{\bar{\eta}})\|x_n-y_n\|^2\\ &-(1-\frac{\gamma}{\bar{\eta}})\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-y_n\|^2\Big)\\ &+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ \leq&\alpha_n(\alpha_n+\beta_n)\|x_n-x^*\|^2+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ =&(\alpha_n+\beta_n)^2\|x_n-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\ \leq&(\alpha_n+\beta_n)^2\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|x^*\|^2+(1-\alpha_n-\beta_n)\|x_{n+1}-x^*\|^2, \end{aligned}$$

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which implies that

$$\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2 + \frac{1-\alpha_n-\beta_n}{\alpha_n+\beta_n}\|x^*\|^2, \tag{3.34}$$

there exists $M>0$, such that

$$\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2 + (1-\alpha_n-\beta_n)M\|x^*\|^2. \tag{3.35}$$

By (3.35) and Lemma 2.7, then $\lim_{n\to\infty}\|x_n-x^*\|$, $\forall x^*\in\xi$ exists. So, we have the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded.

Step 2. Show that $\lim_{n\to\infty}\|\varphi(x_n)-x_n\|=0$. Let $W_n = x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n$. From the definition of x_n, Lemma 2.8 and Lemma 3.4, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
=&\|\alpha_n(P_{Q_n}W_n-x^*)+\beta_n(\varphi(x_n)-x^*)-(1-\alpha_n-\beta_n)x^*\|^2\\
\le&\|\alpha_n(P_{Q_n}W_n-x^*)+\beta_n(\varphi(x_n)-x^*)\|^2-2(1-\alpha_n-\beta_n)\langle x^*,x_{n+1}-x^*\rangle\\
\le&\|\alpha_n(P_{Q_n}W_n-x^*)+\beta_n(\varphi(x_n)-x^*)\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
\le&\alpha_n(\alpha_n+\beta_n)\|P_{Q_n}W_n-x^*\|^2+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2\\
&-\alpha_n\beta_n\|P_{Q_n}W_n-\varphi(x_n)\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
\le&\alpha_n(\alpha_n+\beta_n)\Big(\|x_n-x^*\|^2-(1-\frac{\gamma}{\bar{\eta}})\|x_n-y_n\|^2-(1-\frac{\gamma}{\bar{\eta}})\|P_{Q_n}W_n-y_n\|^2\Big)\\
&+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2-\alpha_n\beta_n\|P_{Q_n}W_n-\varphi(x_n)\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
=&(\alpha_n+\beta_n)^2\|x_n-x^*\|^2\\
&-\alpha_n(\alpha_n+\beta_n)(1-\frac{\gamma}{\bar{\eta}})\Big(\|x_n-y_n\|^2+\|P_{Q_n}W_n-y_n\|^2\Big)\\
&-\alpha_n\beta_n\|P_{Q_n}W_n-\varphi(x_n)\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|,
\end{aligned}$$

which yields that

$$\begin{aligned}
\alpha_n(\alpha_n+\beta_n)(1-\frac{\gamma}{\bar{\eta}})\Big(\|x_n-y_n\|^2+\|P_{Q_n}W_n-y_n\|^2\Big)+\alpha_n\beta_n\|P_{Q_n}W_n-\varphi(x_n)\|^2&\\
\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|.&
\end{aligned} \tag{3.36}$$

From (3.36), $\lim_{n\to\infty}\|x_n-x^*\|^2$ exists and condition (ii), we have

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|P_{Q_n}W_n-y_n\|=\lim_{n\to\infty}\|P_{Q_n}W_n-\varphi(x_n)\|=0. \tag{3.37}$$

Since,

$$\|x_n-\varphi(x_n)\|\le\|x_n-y_n\|+\|y_n-P_{Q_n}W_n\|+\|P_{Q_n}W_n-\varphi(x_n)\|,$$

and (3.37), we get

$$\lim_{n\to\infty}\|\varphi(x_n)-x_n\|=0. \tag{3.38}$$

Step 3. Show that $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z\in\xi$ and $z=\lim_{n\to\infty}P_{\xi}(x_n)$. Therefore it has at least one weak accumulation point. If $\bar{x}$ is a weak limit point of some subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$, then $x_{n_k}\rightharpoonup\bar{x}$ as $k\to\infty$.

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Assume that $\bar{x} \neq \varphi(\bar{x})$. By nonexpansiveness of φ, (3.38), and Opial's property, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| &< \liminf_{k\to\infty} \|x_{n_k} - \varphi(\bar{x})\| \\
&\leq \liminf_{k\to\infty} (\|x_{n_k} - \varphi(x_{n_k})\| + \|\varphi(x_{n_k}) - \varphi(\bar{x})\|) \\
&\leq \liminf_{k\to\infty} (\|x_{n_k} - \varphi(x_{n_k})\| + \|x_{n_k} - \bar{x}\|) \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\|.
\end{aligned}$$

This is a contradiction, then we have

$$\bar{x} \in F(\varphi). \tag{3.39}$$

Assume that $\bar{x} \neq P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))\bar{x}$. By nonexpansiveness of $P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))$, (3.37), and Opial's property, we have

$$\begin{aligned}
&\liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| \\
&\quad < \liminf_{k\to\infty} \|x_{n_k} - P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))\bar{x}\| \\
&\quad \leq \liminf_{k\to\infty} (\|x_{n_k} - P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_{n_k}\| \\
&\qquad + \|P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_{n_k} - P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))\bar{x}\|) \\
&\quad \leq \liminf_{k\to\infty} (\|x_{n_k} - P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_{n_k}\| + \|x_{n_k} - \bar{x}\|) \\
&\quad \leq \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\|.
\end{aligned}$$

This is a contradiction, then we have

$$\bar{x} \in F(P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))). \tag{3.40}$$

By (3.40), Lemma 2.26, and Lemma 2.25, we get

$$\bar{x} \in VI(C, \overline{A}) \cap VI(C, \overline{B}). \tag{3.41}$$

From (3.39) and (3.41), we have

$$\bar{x} \in \xi.$$

In order to show that the entire sequence $\{x_n\}$ weakly converges to $\bar{x}$, assume $x_{n_j} \rightharpoonup \bar{x}'$ as $j \to \infty$, with $\bar{x}' \neq \bar{x}$ and $\bar{x}' \in VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi)$.
By the Opial condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \bar{x}\| &= \liminf_{j\to\infty} \|x_{n_j} - \bar{x}\| \\
&< \liminf_{j\to\infty} \|x_{n_j} - \bar{x}'\| \\
&< \liminf_{j\to\infty} \|x_{n_j} - \bar{x}\| \\
&= \lim_{n\to\infty} \|x_n - \bar{x}\|,
\end{aligned}$$

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and this is a contradiction, thus $\bar{x}' = \bar{x}$. This implies that the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to the same point $\bar{x} \in \xi$. Finally, if we take

$$u_n = P_\xi x_n, \tag{3.42}$$

then by (3.35) and Lemma 3.5, we see that $\{P_\xi x_n\}_{n=0}^{\infty}$ converges strongly to some $z \in \xi$.

From (3.42) and Lemma 2.11 (a), we get

$$\langle x_n - u_n, u_n - \bar{x}\rangle \geq 0, \qquad \forall \bar{x} \in \xi.$$

Take $n \to \infty$, we also have

$$\langle \bar{x} - z, z - \bar{x}\rangle \geq 0,$$

and hence $z = \bar{x}$. Therefore u_n converges strongly to $\bar{x} \in \xi$.
This completes the proof. ☐

The following Corollary 3.7 is a special case of Theorem 3.6 if we put $\overline{A} \equiv \overline{B}$ in Theorem 3.6.

Corollary 3.7. Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B, \overline{A} : C \to H$ be $a, b, \bar{\alpha}$-inverse strongly monotone, respectively. Let $F, Q : C \times C \to \mathbb{R}$ be bifunctions satisfying A1)-A4). Define the mapping $\varphi : C \to C$ by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$, for all $r > 0$, $x \in C$, the mappings T_r^F, T_r^Q define as the same in Algorithm 3.3. Assume that $\xi = VI(C, \overline{A}) \cap F(\varphi) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(x_n - \gamma \overline{A}(x_n)), \\ Q_n = \{z \in H : \langle x_n - \gamma \overline{A}(x_n) - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma \overline{A}(y_n)) + \beta_n \varphi(x_n), \end{cases} \tag{3.43}$$

for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$, $\eta = \min\{a, b\}$, $\lambda, \beta \in (0, 2\eta)$, and $\gamma \in (0, 2\bar{\alpha})$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_\xi(x_n).$$

Proof. Putting $\overline{A} \equiv \overline{B}$ in Theorem 3.6, then we obtain the desired conclusion. ☐

3.3 Applications

In this section, we show the application of the theorems in Sections 3.1 and 3.2.

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3.3.1 Strong convergence theorem of the split feasibility problem

Let H_1, H_2 be two real Hilbert spaces. Let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively.

In 1994, Censor and Elfving [43] introduced the *split feasibility problem* (SFP), which is to find a point x such that

$$x \in C \textit{ and } Dx \in Q,$$

where $D : H_1 \to H_2$ is a bounded linear operator.

Throughout this section, we assume that the SFP is consistent, that is, the solution set Γ of the SFP is nonempty. Let $f : \mathcal{H}_1 \to \mathbb{R}$ be a continuous differentiable function. The minimization problem

$$\min_{x \in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 \tag{3.44}$$

is ill-posed.

Before proving Theorem 3.9, we need the following proposition 3.8.

Proposition 3.8. (See [44]) Given $x^* \in \mathcal{H}_1$, the following statements are equivalent.

(i) x^* solves the SFP;

(ii) $P_C(I - \lambda\nabla f)x^* = P_C(I - \lambda A^*(I - P_Q)A)x^* = x^*$;

(iii) x^* solves the variational inequality problem of finding $x^* \in C$ such that

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in C, \tag{3.45}$$

where $\nabla f = A^*(I - P_Q)A$, and A^* is the adjoint of A.

By using Proposition 3.8 and Theorem 3.1, we prove a strong convergence theorem for the two-step intermixed iteration to solve the split feasibility problems as follows:

Theorem 3.9. Let C be a nonempty closed convex subset of a real Hilbert space H. Let C and Q are the nonempty closed convex subsets from H_1 to H_2 and let $A_i : H_1 \to H_2$ are bounded linear operators, for all $i = 1, 2$, with L_i is a spectral radius of $A_i^* A_i$, for all $i = 1, 2$, with $\Gamma_i \neq 0$. Let $f, g : H \to H$ be a_f and a_g-contraction mapping, respectively, with $a_f, a_g \in (0, 1)$ and $a = \max\{a_f, a_g\}$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1, y_1 \in C$ and

$$\begin{cases} x_{n+1} = (1 - \mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1 - \alpha_n)T_n^{a_1} x_n), \\ y_{n+1} = (1 - \mu_n)y_n + \mu_n P_C(\alpha_n g(x_n) + (1 - \alpha_n)T_n^{a_2} y_n), \end{cases} \tag{3.46}$$

for all $n \in \mathbb{N}$, where $\{\mu_n\}, \{\alpha_n\} \subseteq [0, 1]$, $P_C(I - \lambda_n^i(A_i^*(I - P_Q)A_i)) = s_n^i I + (1 - s_n^i)T_n^{a_i}$, for all $i = 1, 2$, $T_n^{a_i}$ are nonexpansive, $s_n^i = \frac{2 - \lambda_n^i L_i}{4}$, and $\{\lambda_n^i\} \subset (0, \frac{2}{L_i})$. Assume that the following conditions hold:

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(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \overline{\theta} \leq \mu_n \leq \theta$, for all $n \in \mathbb{N}$ and for some $\overline{\theta}, \theta > 0$,

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$,

(iv) $\lim_{n\to\infty} \lambda_n^i = \frac{2}{L_i}$, for all $i = 1, 2$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_{\Gamma_1} f(y^*)$ with $\Gamma_1 = \{x \in C; A_1 x \in Q\}$, and $y^* = P_{\Gamma_2} g(x^*)$ with $\Gamma_2 = \{\bar{x} \in C; A_2 \bar{x} \in Q\}$, respectively.

Proof. Let $x, y \in C$ and $\nabla f_i = A_i^*(I - P_Q)A_i$, for all $i = 1, 2$, we have

$$\begin{aligned} \|\nabla f_i(x) - \nabla f_i(y)\|^2 =& \|A_i^*(I - P_Q)A_i x - A_i^*(I - P_Q)A_i y\|^2 \\ \leq& L_i \|(I - P_Q)A_i x - (I - P_Q)A_i y\|^2. \end{aligned} \tag{3.47}$$

From property of P_C, we have

$$\begin{aligned} &\|(I - P_Q)A_i x - (I - P_Q)A_i y\|^2 \\ &\quad= \langle (I - P_Q)A_i x - (I - P_Q)A_i y, (I - P_Q)A_i x - (I - P_Q)A_i y \rangle \\ &\quad= \langle (I - P_Q)A_i x - (I - P_Q)A_i y, A_i x - A_i y \rangle \\ &\qquad - \langle (I - P_Q)A_i x - (I - P_Q)A_i y, P_Q A_i x - P_Q A_i y \rangle \\ &\quad= \langle A_i^*(I - P_Q)A_i x - A_i^*(I - P_Q)A_i y, x - y \rangle \\ &\qquad - \langle (I - P_Q)A_i x - (I - P_Q)A_i y, P_Q A_i x - P_Q A_i y \rangle \\ &\quad= \langle A_i^*(I - P_Q)A_i x - A_i^*(I - P_Q)A_i y, x - y \rangle \\ &\qquad - \langle (I - P_Q)A_i x, P_Q A_i x - P_Q A_i y \rangle \\ &\qquad + \langle (I - P_Q)A_i y, P_Q A_i x - P_Q A_i y \rangle \\ &\quad\leq \langle A_i^*(I - P_Q)A_i x - A_i^*(I - P_Q)A_i y, x - y \rangle. \end{aligned} \tag{3.48}$$

Substitute (3.48) into (3.47), we have

$$\begin{aligned} \|\nabla f_i(x) - \nabla f_i(y)\|^2 \leq& L_i \langle A_i^*(I - P_Q)A_i x - A_i^*(I - P_Q)A_i y, x - y \rangle \\ =& L_i \langle \nabla f_i(x) - \nabla f_i(y), x - y \rangle. \end{aligned}$$

It follows that

$$\langle \nabla f_i(x) - \nabla f_i(y), x - y \rangle \geq \frac{1}{L_i} \|\nabla f_i(x) - \nabla f_i(y)\|^2.$$

Then ∇f_i are $\frac{1}{L_i}$-inverse strongly monotone, for all $i = 1, 2$.
From Proposition 3.8 and Theorem 3.1, we can conclude of Theorem 3.9. ☐

As a direct result of Theorem 3.9, we get Corollary 3.10 as follows:

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Corollary 3.10. Let C be a nonempty closed convex subset of a real Hilbert space H. Let C and Q are the nonempty closed convex subsets from H_1 to H_2 and let $A : H_1 \to H_2$ be bounded linear operator with L is a spectral radius of A^*A with $\Gamma \neq 0$. Let $f : H \to H$ be a-contraction mapping with $a \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and

$$x_{n+1} = (1-\mu_n)x_n + \mu_n P_C(\alpha_n f(x_n) + (1-\alpha_n)T_n^{a_1}x_n), \tag{3.49}$$

for all $n \in \mathbb{N}$, where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n(A^*(I-P_Q)A)) = s_nI + (1-s_n)T_n^{a_1}$, $T_n^{a_1}$ is nonexpansive, $s_n = \frac{2-\lambda_n L}{4}$, and $\{\lambda_n\} \subset (0, \frac{2}{L})$. Assume that the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \overline{\theta} \leq \mu_n \leq \theta$, for all $n \in \mathbb{N}$ and for some $\overline{\theta}, \theta > 0$,

(iii) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum\limits_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$,

(iv) $\lim\limits_{n\to\infty} \lambda_n = \dfrac{2}{L}$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* = P_\Gamma f(x^*)$ with $\Gamma = \{x \in C; Ax \in Q\}$.

Proof. If we put $f \equiv g$, $x_n = y_n$, in Theorem 3.9. The conclusion of Corollary 3.10 can be obtained from Theorem 3.9. ☐

3.3.2 Weak and strong convergence theorems to solve the generalized equilibrium problems and the system of equilibrium problems

In this section, we obtain the following weak and strong convergence theorems to solve the generalized equilibrium and the system of equilibrium problems.

Put $A \equiv B \equiv 0$, in (1.11), the MEP is reduced to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} F(x^*,y) + \frac{1}{r}\langle y - x^*, x^* - y^*\rangle \geq 0, & \forall y \in C, \\ Q(y^*,x) + \frac{1}{r}\langle x - y^*, y^* - x^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{3.50}$$

(3.50) is called *the system of equilibrium problems*.

If $A \equiv B$, $F \equiv Q$, $r = 1$, $x^* = y^*$ and $\lambda = \beta = 1$, in (1.11), then the MEP reduced to find $x^* \in C$ such that

$$F(x^*,y) + \langle Ax^*, y - x^*\rangle \geq 0, \qquad \forall y \in C, \tag{3.51}$$

where $A : C \to H$ is a mapping, the problem (3.51) is called *the generalized equilibrium problem*. The set of solutions of (3.51) is denoted by $EP(F,A)$. The problem (3.50) and (3.51) covers various disciplines such as optimization problems, variational inequalities and the Nash equilibrium problem in noncooperative games, see literature in [40, 41].

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Theorem 3.11. Let C be a nonempty closed convex subset of a real Hilbert space H and let $\overline{A}, \overline{B} : C \to H$ be $\bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $F, Q : C \times C \to \mathbb{R}$ be bifunctions satisfying A1)-A4). Define the mapping $\bar{\varphi} : C \to C$ by $\bar{\varphi}(x) = T_r^F(T_r^Q x)$, for all $r > 0$, $x \in C$, the mappings T_r^F, T_r^Q define as the same in Algorithm 3.3. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\bar{\varphi}) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n - y_n, y_n - z \rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) + \beta_n \bar{\varphi}(x_n), \end{cases} \tag{3.52}$$

for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$, $\bar{a} \in (0,1)$, and $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

Proof. If we put $A \equiv B \equiv 0$, in Theorem 3.6. The conclusion of Theorem 3.11 can be obtained from Theorem 3.6. □

Theorem 3.12. Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, \overline{A}, \overline{B} : C \to H$ be $a, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying A1)-A4). Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap EP(F, A) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} F(u_n, y) + \frac{1}{r}\langle Ax_n, y - u_n \rangle + \frac{1}{r}\langle y - u_n, u_n - x_n \rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n - y_n, y_n - z \rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) + \beta_n u_n, \end{cases} \tag{3.53}$$

for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\alpha_n + \beta_n \leq 1$, $r > 0$, $\bar{a} \in (0,1)$, and $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

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Proof. If we put $A \equiv B$, $F \equiv Q$ and $\lambda = \beta = 1$, in Theorem 3.6, then we obtain the desired conclusion. □

3.3.3 Weak and strong convergence theorems to solve the variational inequality problem and the general system of variational inequalities problem

In this section, we obtain the following weak and strong convergence theorems to solve the variational inequality and the general system of variational inequalities problem.

In 2008, Ceng et al. [42] introduced *the general system of variational inequalities problem (GSVIP)*, which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, \qquad \forall x \in C,\\ &\langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, \qquad \forall x \in C, \end{aligned} \tag{3.54}$$

where $A, B : C \to H$ are two mappings and $\lambda, \mu > 0$ are two constants. Further, if we put $A \equiv B$ and $x^* = y^*$, then the problem (3.54) reduces to the variational inequality $VI(C, A)$.

Remark 3.13. Putting $F \equiv Q \equiv 0$ in (1.11), we have that (1.11) is reduced to GSVIP. So, (3.54) is a spacial case of MEP.

Lemma 3.14. (See [42]) For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (3.54) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by

$$G(x) = P_C\big(P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)\big), \qquad \forall x \in C,$$

where $y^* = P_C(x^* - \mu Bx^*)$.

By using Theorem 3.6, we give a theorem involving finding the solution of the GSVIP as follows.

Theorem 3.15. Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B, \overline{A}, \overline{B} : C \to H$ be $a, b, \bar{a}, \bar{\beta}$-inverse strongly monotone, respectively. Define the mapping $\psi : C \to C$ by $\psi(x) = P_C(I - \lambda A)P_C(I - \beta B)x$, for all $x \in C$. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\psi) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n,\\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n - y_n, y_n - z \rangle \geq 0\},\\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B})y_n) + \beta_n \psi(x_n), \end{cases} \tag{3.55}$$

for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n \leq 1$, $\bar{a} \in (0, 1)$, $\eta = \min\{a, b\}$, $\gamma \leq \bar{\eta} = \min\{\bar{a}, \bar{\beta}\}$, and $\lambda, \beta \in (0, 2\eta)$, satisfying the following conditions hold:

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(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

Proof. From Lemma 3.14 and putting $F \equiv Q \equiv 0$, in Theorem 3.6, then we obtain the desired conclusion. ☐

Remark 3.16. If $T : C \to C$ be nonexpansive mapping with $F(T) \neq \emptyset$, then $F(T) = VI(C, I-T)$.

By using Theorem 3.15 and Remark 3.16, we give the corollary involving finding the solution of the fixed point problems as follows.

Corollary 3.17. Let C be a nonempty closed convex subset of a real Hilbert space H and let $T_i : C \to C$ be nonexpansive mappings, for all $i = 1, 2, 3, 4$. Define the mapping $\varphi^* : C \to C$ by $\varphi^*(x) = P_C(I - \lambda(I-T_1))P_C(I-\beta(I-T_2))x$, for all $x \in C$. Assume that $\xi = \bigcap_{i=1}^{4} F(T_i) \neq \emptyset$. For given $x_1 \in C$, and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}(I-T_3) + (1-\bar{a})(I-T_4)))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}(I-T_3) + (1-\bar{a})(I-T_4)))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}(I-T_3) + (1-\bar{a})(I-T_4))y_n) + \beta_n \varphi^*(x_n), \end{cases} \tag{3.56}$$

for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\alpha_n + \beta_n \leq 1$, $\bar{a} \in (0,1)$, and $\lambda, \beta, \gamma \in (0, \frac{1}{2})$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

Proof. The conclusion of Corollary 3.17 can be obtained from Theorem 3.15 and Remark 3.16. ☐

Corollary 3.18. Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, \overline{A}, \overline{B} : C \to H$ be $a, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap VI(C, A) \neq \emptyset$. For given $x_1 \in C$, and let the sequences

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$\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) + \beta_n P_C(I - \lambda A)x_n, \end{cases} \tag{3.57}$$

for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\alpha_n + \beta_n \leq 1$, $\lambda \in (0, 2a)$, $\bar{a} \in (0,1)$, and $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\displaystyle\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

Proof. If we put $A \equiv B$ and $\lambda = \beta$, in Theorem 3.15, then we obtain the desired conclusion. □

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Chapter 4

Examples and Numerical Results

In this chapter, we give the following examples to support our main theorem. In Example 4.1, we give computer programming to support Theorem 3.1.

Example 4.1. Let $C = [-10, 10] \times [-10, 10]$ and let $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an inner product defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2$, for all $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. For every $i = 1, 2$, let $\widetilde{f}_i : C \to \mathbb{R}$ be defined by $\widetilde{f}_1(x_1, x_2) = 2x_1^2 + x_2$ and $\widetilde{f}_2(x_1, x_2) = (x_1 - 1) + x_2^2$, $\forall x_1, x_2 \in \mathbb{R}$. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x_1, x_2) = (\frac{x_1}{3}, \frac{x_2}{3})$ and $g(x_1, x_2) = (\frac{x_1}{4}, \frac{x_2}{4})$, which f, g are $\frac{1}{2}$ and $\frac{1}{3}$-contraction mapping, respectively, and $a = \max\{\frac{1}{2}, \frac{1}{3}\} = \frac{1}{2}$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1, y_1 \in C$. Put $\alpha_n = \frac{1}{4n}$ and $\mu_n = \frac{3n+1}{7n}$, we can rewrite (3.1) as follows:

$$\begin{cases} x_{n+1} = (\frac{4n-1}{7n})x_n + (\frac{3n+1}{7n})P_C(\frac{1}{4n}f(y_n) + (\frac{4n-1}{4n})T_n^{\widetilde{f}_1}x_n), \\ y_{n+1} = (\frac{4n-1}{7n})y_n + (\frac{3n+1}{7n})P_C(\frac{1}{4}g(x_n) + (\frac{4n-1}{4n})T_n^{\widetilde{f}_2}y_n), \end{cases} \tag{4.1}$$

for all $n \in \mathbb{N}$, where $P_C(x_1, x_2) = (\max\{\min\{x_1, 10\}, -10\}, \max\{\min\{x_2, 10\}, -10\})$, $P_C(I - \lambda_n^i \nabla \widetilde{f}_i) = s_n^i I + (1 - s_n^i)T_n^{\widetilde{f}_i}$, and $s_n^i = \frac{2 - \lambda_n^i(16)}{4}$, where $\lambda_n^i = \frac{n^2}{8n^2+1}$ for all $i = 1, 2$.

Solution Since $\widetilde{f}_1(x_1, x_2) = 2x_1^2 + x_2$ and $\widetilde{f}_2(x_1, x_2) = (x_1 - 1) + x_2^2$, we have

$$\nabla \widetilde{f}_1(x_1, x_2) = (4x_1, 1) \text{ and } \nabla \widetilde{f}_2(x_1, x_2) = (1, 2x_2).$$

It obvious to check that $\nabla \widetilde{f}_i$ is an $\frac{1}{16}$-inverse strongly monotone, for all $i = 1, 2$.
Let $(0, -10), (-10, 0) \in [-10, 10] \times [-10, 10]$.
Consider,

$$\begin{aligned} P_C(I - \lambda_n^1 \nabla \widetilde{f}_1)(0, -10) =& P_{[-10,10]\times[-10,10]}(I - \frac{1}{16}\nabla \widetilde{f}_1)(0, -10) \\ =& P_{[-10,10]\times[-10,10]}(0, \frac{-161}{16}) \\ =& (P_{[-10,10]}(0), P_{[-10,10]}(\frac{-161}{16})) \\ =& (\max\{\min\{0, 10\}, -10\}, \max\{\min\{\frac{-161}{16}, 10\}, -10\}) \\ =& (0, -10), \end{aligned}$$

then $(0, -10) \in U_{\widetilde{f}_1}$.
Similarly,

$$\begin{aligned} P_C(I - \lambda_n^2 \nabla \widetilde{f}_2)(-10, 0) =& P_{[-10,10]\times[-10,10]}(I - \frac{1}{16}\nabla \widetilde{f}_2)(-10, 0) \\ =& P_{[-10,10]\times[-10,10]}(\frac{-161}{16}, 0) \\ =& (P_{[-10,10]}(\frac{-161}{16}), P_{[-10,10]}(0)) \\ =& (\max\{\min\{\frac{-161}{16}, 10\}, -10\}, \max\{\min\{0, 10\}, -10\}) \\ =& (-10, 0), \end{aligned}$$

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then $(-10, 0) \in U_{\widetilde{f}_2}$.

It is clear that the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfy all the conditions of Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0, -10)$ and $(-10, 0)$, respectively. The Table 4.1 and Figure 4.1 show the values of $\{x_n\}$ and $\{y_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, $y_n^1 = 10$, $y_n^2 = -10$ and $n = N = 400$.

Table 4.1: The values of $\{x_n\}$ and $\{y_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, $y_n^1 = 10$, $y_n^2 = -10$ and $n = N = 400$

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(-10.0000,10.0000)	(10.0000,-10.0000)
2	(-6.0784,8.0448)	(8.1639,-7.2059)
3	(-4.2192,7.3380)	(7.5048,-5.8538)
4	(-3.0380,6.9152)	(7.1109,-4.9143)
⋮	⋮	⋮
250	(-0.0050,-7.7785)	(-7.5696,-0.0076)
⋮	⋮	⋮
396	(-0.0043,-9.9937)	(-9.9937,-0.0065)
397	(-0.0042,-9.9937)	(-9.9937,-0.0064)
398	(-0.0042,-9.9937)	(-9.9937,-0.0064)
399	(-0.0042,-9.9937)	(-9.9937,-0.0064)
400	(-0.0042,-9.9937)	(-9.9937,-0.0064)

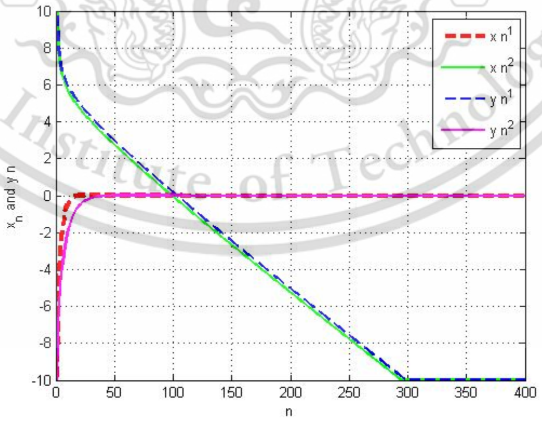

Figure 4.1: The convergences of $\{x_n\}$ and $\{y_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, $y_n^1 = 10$, $y_n^2 = -10$ and $n = N = 400$

In Example 4.2, we give computer programming to support Corollary 3.2.

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Example 4.2. In this example, we use the same mappings and parameters as in Example 4.1. If we choose $f \equiv g$ and $x_n = y_n$ in Example 4.1, we can rewrite (3.27) as follows:

$$x_{n+1} = (\frac{4n-1}{7n})x_n + (\frac{3n+1}{7n})P_C(\frac{1}{4n}f(x_n) + (\frac{4n-1}{4n})T_n^{\widetilde{f}}x_n),$$

for all $n \in \mathbb{N}$, where $P_C(x_1, x_2) = (\max\{\min\{x_1, 10\}, -10\}, \max\{\min\{x_2, 10\}, -10\})$, $P_C(I - \lambda_n \nabla \widetilde{f}) = s_n I + (1 - s_n)T_n^{\widetilde{f}}$, and $s_n = \frac{2-\lambda_n(16)}{4}$, where $\lambda_n = \frac{n^2}{8n^2+1}$. From Corollary 3.2, we can conclude that the sequence $\{x_n\}$ converges strongly to $(0, -10)$. The Table 4.2 and Figure 4.2 show the value of $\{x_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$ and $n = N = 400$.

Table 4.2*:* The value of $\{x_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$ and $n = N = 400$

n	$x_n = (x_n^1, x_n^2)$
1	(-10.0000,10.0000)
2	(-7.0308,8.9972)
3	(-5.2235,8.5685)
4	(-3.9531,8.2876)
⋮	⋮
250	(0.0000,-6.2974)
⋮	⋮
396	(0.0000,-9.9958)
397	(0.0000,-9.9958)
398	(0.0000,-9.9958)
399	(0.0000,-9.9958)
400	(0.0000,-9.9958)

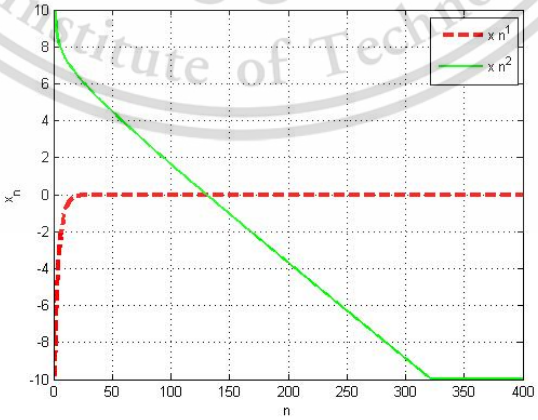

Figure 4.2*:* The convergence of $\{x_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$ and $n = N = 400$

In Example 4.3, we give computer programming to support Theorem 3.6 with

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the sum of the parameters α_n and β_n less than 1.

Example 4.3. Let $\mathbb{R}$ be the set of real numbers, and let $\langle\cdot,\cdot\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an inner product defined by $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}\cdot\mathbf{y} = x_1 \cdot y_1 + x_2 \cdot y_2$, for all $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ and a usual norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Let $H = \mathbb{R}^2$, $C = [-100, 100] \times [-100, 100]$. Let $A, B, \overline{A}, \overline{B} : C \to \mathbb{R}^2$ be defined by $A\mathbf{x} = (\frac{x_1}{2}, \frac{2x_2}{3})$, $B\mathbf{x} = (\frac{x_1}{3}, \frac{x_2}{4})$, $\overline{A}\mathbf{x} = (\frac{x_1}{3}, \frac{x_2}{3})$, and $\overline{B}\mathbf{x} = (\frac{x_1}{4}, \frac{x_2}{4})$, for all $\mathbf{x} \in C$. Let the mappings $F, Q : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(\mathbf{x}, \mathbf{y}) = \frac{-(x_1)^2 - (x_2)^2 + (y_1)^2 + (y_2)^2}{4}, \quad \forall \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \quad \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2,$$

and

$$Q(\mathbf{x}, \mathbf{y}) = \frac{-(x_1)^2 - (x_2)^2 + (y_1)^2 + (y_2)^2}{5}, \quad \forall \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \quad \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2.$$

Let $r, \beta, \lambda = 1$, the sequences $x^* = (x_1^*, x_2^*)$, $y^* = (y_1^*, y_2^*)$ and $y = (y_1, y_2)$

$$\begin{aligned}
0 \le & F(x^*, y) + \langle y - x^*, x^* - (I - A)y^* \rangle \\
= & \frac{-(x_1^*)^2 - (x_2^*)^2 + (y_1)^2 + (y_2)^2}{4} + \langle (y_1 - x_1^*, y_2 - x_2^*), (\frac{2x_1^* - y_1^*}{2}, \frac{3x_2^* - y_2^*}{3}) \rangle \\
= & \frac{-(x_1^*)^2 - (x_2^*)^2 + (y_1)^2 + (y_2)^2}{4} + (y_1 - x_1^*)(\frac{2x_1^* - y_1^*}{2}) + (y_2 - x_2^*)(\frac{3x_2^* - y_2^*}{3}) \\
= & \left(\frac{3(y_1)^2 + (12x_1^* - 6y_1^*)y_1 - 15(x_1^*)^2 + 6x_1^* y_1^*}{12} \right) \\
& + \left(\frac{3(y_2)^2 + (12x_2^* - 4y_2^*)y_2 - 15(x_2^*)^2 + 4x_2^* y_2^*}{12} \right) \\
= & G_1(y_1) + G_2(y_2),
\end{aligned}$$

where $G_1(y_1) = \left(\frac{3(y_1)^2 + (12x_1^* - 6y_1^*)y_1 - 15(x_1^*)^2 + 6x_1^* y_1^*}{12} \right)$ and $G_2(y_2) = \left(\frac{3(y_2)^2 + (12x_2^* - 4y_2^*)y_2 - 15(x_2^*)^2 + 4x_2^* y_2^*}{12} \right)$. $G_1(y_1)$ and $G_2(y_2)$ are quadratic functions with coefficients $a_1 = \frac{1}{4}$, $b_1 = x_1^* - \frac{y_1^*}{2}$, and $c_1 = \frac{-5(x_1^*)^2}{4} + \frac{x_1^* y_1^*}{2}$ of $G_1(y_1)$ and coefficients $a_2 = \frac{1}{4}, b_2 = x_2^* - \frac{y_2^*}{3}$, and $c_2 = \frac{-5(x_2^*)^2}{4} + \frac{x_2^* y_2^*}{3}$ of $G_2(y_2)$, respectively. Determine the discriminant Δ_1 of G_1 as follows:

$$\begin{aligned}
\Delta_1 = & b_1^2 - 4a_1 c_1 \\
= & (x_1^* - \frac{y_1^*}{2})^2 - 4\Big(\frac{1}{4}\Big)\Big(\frac{-5(x_1^*)^2}{4} + \frac{x_1^* y_1^*}{2}\Big) \\
= & (\frac{3x_1^* - y_1^*}{2})^2.
\end{aligned}$$

We know that $G_1(y_1) \ge 0$, for all $z \in \mathbb{R}$. If it has most one solution in $\mathbb{R}$, then $\Delta_1 \le 0$. So, we obtain $x_1^* = \frac{y_1^*}{3}$. Next, we determine the discriminant Δ_2 of G_2 by using the same method as above, we obtain $x_2^* = \frac{2y_2^*}{9}$. That is $T_r^F(I - \lambda A)y^* = (\frac{y_1^*}{3}, \frac{2y_2^*}{9})$. After that, we find the solution of $y^* = (y_1^*, y_2^*)$ in this inequality $0 \le Q(y^*, z) + \langle z - y^*, y^* - (I - B)x^* \rangle$. By using the same method as $T_r^F(I - \lambda A)y^*$, we obtain $T_r^Q(I - \beta B)x^* = (\frac{10x_1^*}{21}, \frac{15x_2^*}{28})$. That

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is $\varphi(x) = T_r^F(I-\lambda A)T_r^Q(I-\beta B)x = T_r^F(I-\lambda A)(\frac{10x_1}{21}, \frac{15x_2}{28}) = (\frac{10x_1}{63}, \frac{5x_2}{42})$.

Let $x_1 = (x_1^1, x_1^2)$ and $y_1 = (y_1^1, y_1^2) \in \mathbb{R}^2$. The sequences $\{x_n\}$ and $\{y_n\}$ are generated by (3.28), it is easy to see that $A, B, \overline{A}, \overline{B}$ are 1- inverse strongly monotone with $\eta, \bar{\eta}, \gamma = 1$ and we can choose $\bar{a} = \frac{1}{2}$, $\alpha_n = \frac{1}{n+1} - \frac{1}{(n+1)^{20}}$ and $\beta_n = 1 - \frac{1}{n+1}$, for all $n \in \mathbb{N}$. From the definitions of $A, B, \overline{A}, \overline{B}$ and φ, we have

$$VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi) = (0, 0).$$

From Theorem 3.6, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0, 0)$. For each $n \in \mathbb{N}$, we can rewrite (3.28) as follows:

$$\begin{cases} y_n &= P_C(I - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B})))x_n, \\ Q_n &= \{z \in H : \langle (I - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B})))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} &= (\frac{1}{n+1} - \frac{1}{(n+1)^{20}})P_{Q_n}(x_n - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B}))y_n) + (1 - \frac{1}{n+1})\varphi(x_n), \end{cases}$$

where $P_C(x_1, x_2) =$ (max{min{$x_1, 100$}, -100}, max{min{$x_2, 100$}, -100}). The Table 4.3 and Figure 4.3 show the values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-100, 100)$ and $n = 1, 2, ..., 13$.

Table 4.3: The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-100, 100)$ and $n = 1, 2, ..., 13$

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(-100.0000,100.0000)	(-70.8333,70.8333)
2	(-47.6066,45.6224)	(-33.7213,32.3159)
3	(-17.6281,15.6865)	(-12.4866,11.1113)
⋮	⋮	⋮
7	(-0.1054,0.0571)	(-0.0747,0.0405)
⋮	⋮	⋮
11	(-0.0003,0.0001)	(-0.0002,0.0001)
12	(-0.0001,0.0000)	(0.0000,0.0000)
13	(0.0000,0.0000)	(0.0000,0.0000)

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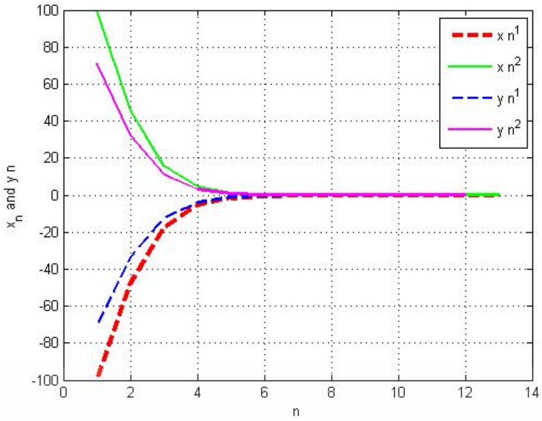

Figure 4.3*:* The convergences of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-100, 100)$ and $n = 1, 2, ..., 13$

In Example 4.4, we give computer programming to support Corollary 3.7.

Example 4.4. In this example, we use the same mappings and parameters as in Example 4.3. If we choose $\overline{A} \equiv \overline{B}$ in Example 4.3, we can rewrite (3.43) as follows:

$$\begin{cases} y_n = P_C(x_n - \overline{A}(x_n)), \\ Q_n = \{z \in H : \langle x_n - \overline{A}(x_n) - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = (\frac{1}{n+1} - \frac{1}{(n+1)^{20}})P_{Q_n}(x_n - \overline{A}(y_n)) + (1 - \frac{1}{n+1})\varphi(x_n), \end{cases}$$

for all $n \in \mathbb{N}$, where $P_C(x_1, x_2) = (\max\{\min\{x_1, 100\}, -100\}, \max\{\min\{x_2, 100\}, -100\})$. From Corollary 3.7, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0, 0)$. The Table 4.4 and Figure 4.4 show the values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-100, 100)$ and $n = 1, 2, ..., 13$.

Table 4.4*:* The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-100, 100)$ and $n = 1, 2, ..., 13$

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(-100.0000,100.0000)	(-66.6667,66.6667)
2	(-46.8253,44.8412)	(-31.2169,29.8941)
3	(-17.0950,15.1843)	(-11.3966,10.1229)
⋮	⋮	⋮
7	(-0.0980,0.0527)	(-0.0653,0.0351)
⋮	⋮	⋮
11	(-0.0002,0.0001)	(-0.0002,0.0000)
12	(-0.0001,0.0000)	(0.0000,0.0000)
13	(0.0000,0.0000)	(0.0000,0.0000)

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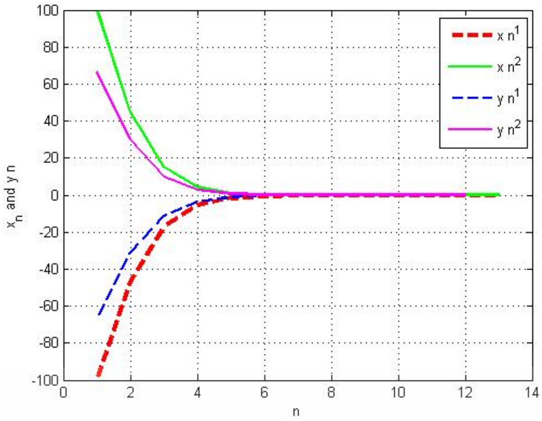

Figure 4.4*:* The convergences of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-100, 100)$ and $n = 1, 2, ..., 13$

Remark 4.5. If $y_n = x_n$ and $x_{n+1} = x_n$ with $\alpha_n + \beta_n = 1$, then **STOP**. ($x_n \in VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi)$ in Example 4.3 and $x_n \in VI(C, \overline{A}) \cap F(\varphi)$ in Remark 4.4).

In Example 4.6, we give computer programming to support Theorem 3.6 with the sum of the parameters α_n and β_n being 1.

Example 4.6. In this example, we use the same mappings in Example 4.3. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by (3.28), where $\alpha_n = \frac{1}{n^2}$ and $\beta_n = 1 - \frac{1}{n^2}$ for all $n \in \mathbb{N}$. From the definitions of $A, B, \overline{A}, \overline{B}$ and φ, we have

$$VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi) = (0, 0).$$

We can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0, 0)$. For each $n \in \mathbb{N}$, we can rewrite (3.28) as follows:

$$\begin{cases} y_n = P_C(I - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B})))x_n, \\ Q_n = \{z \in H : \langle (I - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B})))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = (\frac{1}{n^2})P_{Q_n}(x_n - (\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B}))y_n) + (1 - \frac{1}{n^2})\varphi(x_n), \end{cases}$$

where $P_C(x_1, x_2) = (\max\{\min\{x_1, 100\}, -100\}, \max\{\min\{x_2, 100\}, -100\})$. The Table 4.5 and Figure 4.5 show the values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = 1, 2, ..., 20$.

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Table 4.5*:* The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = 1, 2, ..., 20$

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(-10.0000,10.0000)	(-7.0833,7.0833)
2	(-7.9340,7.9340)	(-5.6199,5.6199)
3	(-2.5182,2.2821)	(-1.7838,1.6165)
⋮	⋮	⋮
10	(0.0000,0.0000)	(0.0000,0.0000)
⋮	⋮	⋮
18	(0.0000,0.0000)	(0.0000,0.0000)
19	(0.0000,0.0000)	(0.0000,0.0000)
20	(0.0000,0.0000)	(0.0000,0.0000)

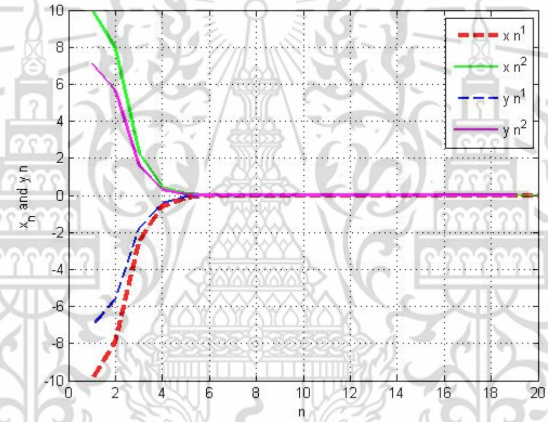

Figure 4.5*:* The convergences of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = 1, 2, ..., 20$

Remark 4.7. In case $\alpha_n = \frac{1}{n^2}$ and $\beta_n = 1 - \frac{1}{n^2}$, which $\alpha_n + \beta_n = 1$, we can see that Example 4.6 is a special case of Example 4.3. However, the both examples we have $\{x_n\}$ and $\{y_n\}$ are converge to $(0, 0)$.

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Chapter 5

Conclusions and Suggestions

In this chapter, we summarize all theorems and corollaries given in this thesis.

5.1 The modification of intermixed iteration for a constrained convex minimization problem

(1) Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2$, $\widetilde{f}_i : C \to \mathbb{R}$ be real-valued convex functions and assume that $\nabla\widetilde{f}_i$ are $\frac{1}{L_i}$-inverse strongly monotones with $L_i \geq 0$ and $U_{\widetilde{f}_i} = VI(C, \nabla\widetilde{f}_i) \neq \emptyset$. Let $f, g : H \to H$ be a_f and a_g-contraction mapping, respectively, with $a_f, a_g \in (0,1)$ and $a = \max\{a_f, a_g\}$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1, y_1 \in C$ and

$$\begin{cases} x_{n+1} = (1-\mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n), \\ y_{n+1} = (1-\mu_n)y_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)T_n^{\widetilde{f}_2}y_n), \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n^i \nabla\widetilde{f}_i) = s_n^i I + (1-s_n^i)T_n^{\widetilde{f}_i}$, $T_n^{\widetilde{f}_i}$ are nonexpansive, $s_n^i = \frac{2-\lambda_n^i L_i}{4}$, and $\{\lambda_n^i\} \subset (0, \frac{2}{L_i})$, for all $i = 1, 2$. Assume that the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \overline{\theta} \leq \mu_n \leq \theta$, for all $n \in \mathbb{N}$ and for some $\overline{\theta}, \theta > 0$,

(iii) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum\limits_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$,

(iv) $\lim\limits_{n\to\infty} \lambda_n^i = \dfrac{2}{L_i}$, for all $i = 1, 2$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_{U_{\widetilde{f}_1}} f(y^*)$ and $y^* = P_{U_{\widetilde{f}_2}} g(x^*)$, respectively.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\widetilde{f} : C \to \mathbb{R}$ be a real-valued convex function and assume that $\nabla\widetilde{f}$ is an $\frac{1}{L}$-inverse strongly monotone with $L \geq 0$ and $U_{\widetilde{f}} = VI(C, \nabla\widetilde{f}) \neq \emptyset$. Let $f : H \to H$ be a-contraction mapping with $a \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and

$$x_{n+1} = (1-\mu_n)x_n + \mu_n P_C(\alpha_n f(x_n) + (1-\alpha_n)T_n^{\widetilde{f}}x_n),$$

for all $n \in \mathbb{N}$, where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n^i \nabla\widetilde{f}) = s_n I + (1-s_n)T_n^{\widetilde{f}}$, $T_n^{\widetilde{f}}$ is nonexpansive, $s_n = \frac{2-\lambda_n L}{4}$, and $\{\lambda_n\} \subset (0, \frac{2}{L})$. Assume that the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

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(ii) $0 < \overline{\theta} \le \mu_n \le \theta$, for all $n \in \mathbb{N}$ and for some $\overline{\theta}, \theta > 0$,

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$,

(iv) $\lim_{n\to\infty} \lambda_n = \frac{2}{L}$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* = P_{U_{\tilde{f}}} f(x^*)$.

5.2 The new subgradient for the modification of equilibrium problems and variational inequality problems

(1) Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B, \overline{A}, \overline{B} : C \to H$ be $a, b, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $F, Q : C \times C \to \mathbb{R}$ be bifunctions satisfying A1)-A4). Define the mapping $\varphi : C \to C$ by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$, for all $r > 0$, $x \in C$, the mappings T_r^F, T_r^Q define as the same in Algorithm 3.3, where $\bar{a} \in (0, 1)$ and $\gamma \le \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi) \ne \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by (3.28), satisfying the following conditions hold:

(i) $0 < c \le \beta_n \le d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

(2) Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B, \overline{A} : C \to H$ be $a, b, \bar{\alpha}$-inverse strongly monotone, respectively. Let $F, Q : C \times C \to \mathbb{R}$ be bifunctions satisfying A1)-A4). Define the mapping $\varphi : C \to C$ by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$, for all $r > 0$, $x \in C$, the mappings T_r^F, T_r^Q define as the same in Algorithm 3.3. Assume that $\xi = VI(C, \overline{A}) \cap F(\varphi) \ne \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(x_n - \gamma\overline{A}(x_n)), \\ Q_n = \{z \in H : \langle x_n - \gamma\overline{A}(x_n) - y_n, y_n - z\rangle \ge 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma\overline{A}(y_n)) + \beta_n \varphi(x_n), \end{cases}$$

for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n \le 1$, for all $n \in \mathbb{N}$, $\eta = \min\{a, b\}$, $\lambda, \beta \in (0, 2\eta)$, and $\gamma \in (0, 2\bar{\alpha})$, satisfying the following conditions hold:

(i) $0 < c \le \beta_n \le d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$.

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Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

5.3 Strong convergence theorem of the split feasibility problem

(1) Let C be a nonempty closed convex subset of a real Hilbert space H. Let C and Q are the nonempty closed convex subsets from H_1 to H_2 and let $A_i : H_1 \to H_2$ are bounded linear operators, for all $i = 1, 2$, with L_i is a spectral radius of $A_i^* A_i$, for all $i = 1, 2$, with $\Gamma_i \neq 0$. Let $f, g : H \to H$ be a_f and a_g-contraction mapping, respectively, with $a_f, a_g \in (0,1)$ and $a = \max\{a_f, a_g\}$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1, y_1 \in C$ and

$$\begin{cases} x_{n+1} = (1-\mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{a_1} x_n), \\ y_{n+1} = (1-\mu_n)y_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)T_n^{a_2} y_n), \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n^i(A_i^*(I-P_Q)A_i)) = s_n^i I + (1-s_n^i)T_n^{a_i}$, for all $i = 1, 2$, $T_n^{a_i}$ are nonexpansive, $s_n^i = \frac{2-\lambda_n^i L_i}{4}$, and $\{\lambda_n^i\} \subset (0, \frac{2}{L_i})$. Assume that the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \overline{\theta} \leq \mu_n \leq \theta$, for all $n \in \mathbb{N}$ and for some $\overline{\theta}, \theta > 0$,

(iii) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum\limits_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$,

(iv) $\lim\limits_{n\to\infty} \lambda_n^i = \dfrac{2}{L_i}$, for all $i = 1, 2$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_{\Gamma_1} f(y^*)$ with $\Gamma_1 = \{x \in C; A_1 x \in Q\}$, and $y^* = P_{\Gamma_2} g(x^*)$ with $\Gamma_2 = \{\bar{x} \in C; A_2 \bar{x} \in Q\}$, respectively.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H. Let C and Q are the nonempty closed convex subsets from H_1 to H_2 and let $A : H_1 \to H_2$ be bounded linear operator with L is a spectral radius of $A^* A$ with $\Gamma \neq 0$. Let $f : H \to H$ be a-contraction mapping with $a \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and

$$x_{n+1} = (1-\mu_n)x_n + \mu_n P_C(\alpha_n f(x_n) + (1-\alpha_n)T_n^{a_1} x_n),$$

for all $n \in \mathbb{N}$, where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n(A^*(I-P_Q)A)) = s_n I + (1-s_n)T_n^{a_1}$, $T_n^{a_1}$ is nonexpansive, $s_n = \frac{2-\lambda_n L}{4}$, and $\{\lambda_n\} \subset (0, \frac{2}{L})$. Assume that the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \overline{\theta} \leq \mu_n \leq \theta$, for all $n \in \mathbb{N}$ and for some $\overline{\theta}, \theta > 0$,

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(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty,$

(iv) $\lim_{n\to\infty} \lambda_n = \frac{2}{L}.$

Then, the sequence $\{x_n\}$ converges strongly to $x^* = P_\Gamma f(x^*)$ with $\Gamma = \{x \in C; Ax \in Q\}$.

5.4 Weak and strong convergence theorems to solve the generalized equilibrium problems and the system of equilibrium problems

(1) Let C be a nonempty closed convex subset of a real Hilbert space H and let $\overline{A}, \overline{B} : C \to H$ be $\bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $F, Q : C \times C \to \mathbb{R}$ be bifunctions satisfying A1)-A4). Define the mapping $\bar{\varphi} : C \to C$ by $\bar{\varphi}(x) = T_r^F(T_r^Q x)$, for all $r > 0$, $x \in C$, the mappings T_r^F, T_r^Q define as the same in Algorithm 3.3. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\bar{\varphi}) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) + \beta_n \bar{\varphi}(x_n), \end{cases}$$

for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$, $\bar{a} \in (0,1)$, and $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_\xi(x_n).$$

(2) Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, \overline{A}, \overline{B} : C \to H$ be $a, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying A1)-A4). Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap EP(F, A) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} F(u_n, y) + \frac{1}{r}\langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) + \beta_n u_n, \end{cases}$$

for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\alpha_n + \beta_n \leq 1$, $r > 0$, $\bar{a} \in (0,1)$, and $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$, satisfying the following conditions hold:

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(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

5.5 Weak and strong convergence theorems to solve the variational inequality problem and the general system of variational inequalities problem

(1) Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B, \overline{A}, \overline{B} : C \to H$ be $a, b, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Define the mapping $\psi : C \to C$ by $\psi(x) = P_C(I-\lambda A)P_C(I-\beta B)x$, for all $x \in C$. Assume that $\xi = VI(C,\overline{A}) \cap VI(C,\overline{B}) \cap F(\psi) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n) + \beta_n \psi(x_n), \end{cases}$$

for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\alpha_n + \beta_n \leq 1$, $\bar{a} \in (0,1)$, $\eta = \min\{a,b\}$, $\gamma \leq \bar{\eta} = \min\{\bar{\alpha},\bar{\beta}\}$, and $\lambda, \beta \in (0, 2\eta)$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

(2) Let C be a nonempty closed convex subset of a real Hilbert space H and let $T_i : C \to C$ be nonexpansive mappings, for all $i = 1,2,3,4$. Define the mapping $\varphi^* : C \to C$ by $\varphi^*(x) = P_C(I-\lambda(I-T_1))P_C(I-\beta(I-T_2))x$, for all $x \in C$. Assume that $\xi = \bigcap_{i=1}^{4} F(T_i) \neq \emptyset$. For given $x_1 \in C$, and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I-\gamma(\bar{a}(I-T_3)+(1-\bar{a})(I-T_4)))x_n, \\ Q_n = \{z \in H : \langle (I-\gamma(\bar{a}(I-T_3)+(1-\bar{a})(I-T_4)))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}(I-T_3)+(1-\bar{a})(I-T_4))y_n) + \beta_n \varphi^*(x_n), \end{cases}$$

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for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\alpha_n + \beta_n \leq 1$, $\bar{a} \in (0,1)$, and $\lambda, \beta, \gamma \in (0, \frac{1}{2})$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

(3) Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, \overline{A}, \overline{B} : C \to H$ be $a, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap VI(C, A) \neq \emptyset$. For given $x_1 \in C$, and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) + \beta_n P_C(I - \lambda A)x_n, \end{cases}$$

for all $n \in \mathbb{N}$, where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\alpha_n + \beta_n \leq 1$, $\lambda \in (0, 2a)$, $\bar{a} \in (0,1)$, and $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

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Appendix

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Appendix A

The research papers

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An intermixed iteration for constrained convex minimization problem and split feasibility problem

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Abstract

In this paper, we first introduce the two-step intermixed iteration for finding the common solution of a constrained convex minimization problem, and also we prove a strong convergence theorem for the intermixed algorithm. By using our main theorem, we prove a strong convergence theorem for the split feasibility problem. Finally, we apply our main theorem for the numerical example.

MSC: 46N10; 47H09; 74G60

Keywords: Constrained convex minimization problem; Split feasibility problem; Variational inequality

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of a real Hilbert space H.

We denote the fixed point set of a mapping T by $F(T)$. Fixed point theory can be applied to variational inequality problems, equilibrium problems, split feasibility problems, optimization problems, etc. These problems are encountered in various fields such as engineering, physics, game theory, and economics.

A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In mathematics, conventional optimization problems arise in the process of making a trading system more effective and are usually stated in terms of minimization problems. In this paper, we give a new iteration for solving two constrained convex minimization problems.

Convex constrained minimization problem is popular and very important to various branches in physics, engineering and economics, e.g., to find the minimum travel distance or to find the lowest cost. Consider the constrained convex minimization problem as follows:

$$\text{minimize } \{f(x) : x \in C\}, \tag{1}$$

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where $f : C \to \mathbb{R}$ is a real-valued convex function. If f is (Fréchet) differentiable, then the gradient-projection algorithm (GPA) generates a sequence $\{x_n\}$ using the following recursive formula:

$$x_{n+1} = P_C\bigl(x_n - \lambda \nabla f(x_n)\bigr), \quad \forall n \geq 0, \tag{2}$$

or more generally,

$$x_{n+1} = P_C\bigl(x_n - \lambda_n \nabla f(x_n)\bigr), \quad \forall n \geq 0, \tag{3}$$

where both in (2) and (3) the initial guess x_0 is taken from C arbitrarily, and the parameters, λ or λ_n, are positive real numbers satisfying certain conditions. The convergence of the algorithms (2) and (3) depends on the behavior of the gradient ∇f. In fact, it is known that if ∇f is α-strongly monotone and L-Lipschitz with constants $\alpha, L \geq 0$, then the operator

$$T := P_C(I - \lambda \nabla f) \tag{4}$$

is a contraction; hence, the sequence $\{x_n\}$ defined by the algorithm (2) converges in norm to the unique minimizer of (1). However, if the gradient ∇f fails to be strongly monotone, the operator T defined by (4) could fail to be contractive; consequently, the sequence $\{x_n\}$ generated by the algorithm (2) may fail to converge strongly [1]. If ∇f is Lipschitz, then the algorithms (2) and (3) can still converge in the weak topology under certain conditions [2–4].

The variational inequality problem is to find a point $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C. \tag{5}$$

We denote the set of solutions of the variational inequality by $\mathrm{VI}(C, A)$. Many models of variational inequalities are used in practice, including a mathematical theory, some interesting connections to numerous disciplines and a wide range of important applications in engineering, physics, optimization, minimax problems, game theory, and economics; for more details, see [5, 6].

Su and Xu [3] introduced the relation of a solution to the minimization problem (1) and solutions of the variational inequality (5) as stated in the following Lemma 1, and this lemma helps to prove the theorem about the minimization problem more effectively; for more details, see [7–9].

Lemma 1 (Optimality condition, [3]) *A necessary condition for a point $x^* \in C$ to be a solution of the minimization problem (1) is that x^* solves the variational inequality*

$$\bigl\langle \nabla f\bigl(x^*\bigr), x - x^* \bigr\rangle \geq 0, \quad \forall x \in C. \tag{6}$$

Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C\bigl(x^* - \lambda \nabla f\bigl(x^*\bigr)\bigr),$$

for every constant $\lambda > 0$. If, in addition, f is convex, then the optimality condition (6) is also sufficient.

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By U_f we denote the set of solutions of (1).

In 2011, Ceng et al. [10] introduced the following iterative scheme that generates a sequence $\{x_n\}$ in an explicit way:

$$x_{n+1} = P_C\big[s_n\gamma Vx_n + (I - s_n\mu F)T_nx_n\big], \quad \forall n \geq 0,$$

where $s_n = \frac{2-\lambda_n L}{4}$ and $P_C(I - \lambda_n\nabla f) = s_nI + (1 - s_n)T_n$ for each $n \geq 0$. He proved that the sequence $\{x_n\}$ converges strongly to a minimizer $x^* \in S$ of (1).

In 2014, Ming and Lei [11] introduced an explicit composite iterative method for finding the common element of the set of solutions to an equilibrium problem and the solution set to a constrained convex minimization problem, as well as proved a strong convergence theorem, as follows:

Algorithm 1 Given $x_1 \in C$, let the sequences $\{u_n\}$ and $\{x_n\}$ be generated iteratively by

$$\begin{cases} \phi(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n\gamma Vu_n + (I - \alpha_nA)T_nu_n, & \forall n \in \mathbb{N}, \end{cases}$$

where T_n is a nonexpansive mapping from $P_C(I - \lambda_n\nabla f) = s_nI + (1 - s_n)T_n$, which is $\frac{2+\lambda_nL}{4}$-averaged with $s_n = \frac{2-\lambda_nL}{4}$, and ∇f is an L-Lipschitz mapping, for all $L \geq 0$, $V : C \to C$ is an l-Lipschitz mapping with constant $l \geq 0$, $A : C \to C$ is a strongly positive bounded linear operator with coefficient $\bar{\gamma} \geq 0$ and $0 < \gamma < \frac{\bar{\gamma}}{l}$, $u_n = Q_{\beta_n}x_n$, $\{\lambda_n\} \subset (0, \frac{2}{L})$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, \infty)$ and $\{s_n\} \subset (0, \frac{1}{2})$.

In 2015, Yao et al. [12] introduced the intermixed algorithm for two strict pseudocontractions S and T as follows:

Algorithm 2 For arbitrarily given $x_0 \in C$, $y_0 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_nP_C[\alpha_nf(y_n) + (1 - k - \alpha_n)x_n + kTx_n], & n \geq 0, \\ y_{n+1} = (1 - \beta_n)y_n + \beta_nP_C[\alpha_ng(x_n) + (1 - k - \alpha_n)y_n + kSy_n], & n \geq 0, \end{cases} \tag{7}$$

where $S, T : C \to C$ are λ-strictly pseudocontractions, $f : C \to H$ is a ρ_1-contraction, and $g : C \to H$ is a ρ_2-contraction, $k \in (0, 1 - \lambda)$ is a constant, and $\{\alpha_n\}$, $\{\beta_n\}$ are two real number sequences in $(0, 1)$.

Furthermore, under some control conditions, they proved that the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (7) converge independently to $P_{F(T)}f(y^*)$ and $P_{F(S)}g(x^*)$, respectively, where $x^* \in F(T) = \{z \in C : Tz = z\}$ and $y^* \in F(S) = \{z^* \in C : Tz^* = z^*\}$.

Motivated by Yao et al. [12] and Ming et al. [11], we introduce the new iterative method as follows:

Algorithm 3 Given $x_1, y_1 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be defined by

$$\begin{cases} x_{n+1} = (1 - \mu_n)x_n + \mu_nP_C(\alpha_nf(y_n) + (1 - \alpha_n)T_n^{\widetilde{f}_1}x_n), \\ y_{n+1} = (1 - \mu_n)y_n + \mu_nP_C(\alpha_ng(x_n) + (1 - \alpha_n)T_n^{\widetilde{f}_2}y_n), \end{cases} \tag{8}$$

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where $f,g : H \to H$ are a_f- and a_g-contraction mappings with $a_f, a_g \in (0,1)$ and $a = \max\{a_f, a_g\}$, $\nabla\widetilde{f_i}$ is an $\frac{1}{L_i}$-inverse strongly monotone with $L_i \geq 0$, for all $i = 1,2$, $\{\mu_n\}$, $\{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n^i \nabla\widetilde{f_i}) = s_n^i I + (1 - s_n^i)T_n^{\widetilde{f_i}}$, $\forall i = 1,2$ and $s_n^i = \frac{2-\lambda_n^i L_i}{4}$, $\{\lambda_n^i\} \subset (0, \frac{2}{L_i})$ and $0 < \overline{\theta} \leq \mu_n \leq \theta$ for all $n \in \mathbb{N}$ and for some $\overline{\theta}, \theta > 0$.

The purpose of this article is to combine the GPA and averaged mapping approach to design a two-step intermixed iteration for finding the common solution of a constrained convex minimization problem, and also prove a strong convergence theorem for the intermixed algorithm generated by (8). Applying our main result, we prove a strong convergence theorem for the split feasibility problem. Moreover, we utilize our main theorem in the numerical example.

2 Preliminaries

Throughout this article, we always assume that C is a nonempty, closed, and convex subset of a real Hilbert space H. We use "$\rightharpoonup$" for weak convergence and "$\rightarrow$" for strong convergence. For every $x \in H$, there is a unique nearest point $P_C x$ in C such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Such an operator P_C is called the metric projection of H onto C.

Assume that C is a nonempty closed and convex subset of H. A mapping $V : C \to C$ is said to be an l-Lipschitz if there exists a constant $l \geq 0$ such that

$$\|Vx - Vy\| \leq l\|x - y\|, \quad \forall x, y \in C.$$

If $l \in [0,1)$, then V is called a contraction. Obviously, if $l = 1$, V is a nonexpansive mapping.

Definition 1 A mapping $T : H \to H$ is said to be firmly nonexpansive if and only if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad x, y \in H.$$

Alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive.

Definition 2 (Positive operator) An operator A is called *positive* if it is self-adjoint and $\langle Ax, x\rangle \geq 0$ for all $x \in H$.

An operator A on H is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

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Lemma 2 ([13]) *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \quad \Longleftrightarrow \quad \langle u - z, v - u\rangle \geq 0, \quad \forall v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C.

Lemma 3 ([14]) *Let H be a real Hilbert space. Then the following results hold*:

(i) *For all $x, y \in H$ and $\alpha \in [0,1]$,*

$$\left\|\alpha x + (1-\alpha)y\right\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,$$

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, *for each $x, y \in H$.*

Lemma 4 ([4]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Definition 3 A mapping $T: H \to H$ is said to be an *averaged mapping* if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T = (1-\alpha)I + \alpha S, \tag{9}$$

where α is a number in $(0,1)$ and $S: H \to H$ is nonexpansive. More precisely, when (9) holds, we say that T is α-averaged.

Clearly, a firmly nonexpansive mapping is a $\frac{1}{2}$-averaged mapping.

Proposition 1 *For given operators $S, T, V: H \to H$:*

(i) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$, S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composition of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composition $T_1 \circ T_2 \circ \cdots \circ T_N$. In particular, if T_1 is α_1-averaged, and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composition $T_1 \circ T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

Lemma 5 ([11]) *For given $x \in H$ and let $P_C: H \to C$ be a metric projection. Then*

(a) *$z = P_C x$ if and only if $\langle x - z, y - z\rangle \leq 0$, $\forall y \in C$.*

(b) *$z = P_C x$ if and only if $\|x-z\|^2 \leq \|x-y\|^2 - \|y-z\|^2$, $\forall y \in C$.*

(c) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$, $\forall x, y \in H$.

Consequently, P_C is nonexpansive and monotone.

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Lemma 6 ([15]) *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{u_n\} \subset H$ with $u_n \rightharpoonup u$, the inequality*

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

holds for every $v \in H$ with $v \neq u$.

Definition 4 A nonlinear operator T whose domain $D(T) \subseteq H$ and range $R(T) \subseteq H$ is said to be:

(a) monotone if

$$\langle x - y, Tx - Ty\rangle \geq 0, \quad \forall x, y \in D(T);$$

(b) β-strongly monotone if there exists $\beta > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in D(T);$$

(c) ν-inverse strongly monotone (for short, ν-ism) if there exists $\nu > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \nu\|Tx - Ty\|^2, \quad \forall x, y \in D(T).$$

Proposition 2 *Let T be an operator from H to itself. Then*
(a) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism;*
(b) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism;*
(c) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Lemma 7 ([16]) *Assume $A : H \to H$ is a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < t \leq \|A\|^{-1}$. Then $\|I - tA\| \leq 1 - t\overline{\gamma}$.*

3 Main results

Let $V : C \to C$ be l-Lipschitz with coefficient $l \geq 0$, and $A : C \to C$ a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ and $0 < \gamma < \frac{\overline{\gamma}}{l}$. Let $f : C \to \mathbb{R}$ be a real-valued convex function and assume that ∇f is an L-Lipschitz mapping with $L \geq 0$. From Xu [1], we have that $P_C(I - \lambda\nabla f)$ is $\frac{2+\lambda L}{4}$-averaged for $0 < \lambda < \frac{2}{L}$ and for each $n \in \mathbb{N}$, that is, we can write

$$P_C(I - \lambda_n\nabla f) = (1 - s_n)I + s_n T_n^f,$$

where T_n^f is nonexpansive and $s_n = \frac{2+\lambda_n L}{4}$.

Theorem 1 *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1,2, \widetilde{f_i} : C \to \mathbb{R}$ be a real-valued convex function and assume that $\nabla\widetilde{f_i}$ is an $\frac{1}{L_i}$-inverse strongly monotone with $L_i > 0$ and $U_{\widetilde{f_i}} \neq \emptyset$. Let $f, g : H \to H$ be a_f- and a_g-contraction mappings, respectively, with $a_f, a_g \in (0,1)$ and $a = \max\{a_f, a_g\}$. Let the sequences $\{x_n\}$, $\{y_n\}$*

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be generated by $x_1, y_1 \in C$ *and*

$$\begin{cases} x_{n+1} = (1-\mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n), \\ y_{n+1} = (1-\mu_n)y_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)T_n^{\widetilde{f}_2} y_n), \end{cases} \tag{10}$$

where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n^i \nabla\widetilde{f}_i) = s_n^i I + (1-s_n^i)T_n^{\widetilde{f}_i}$, $s_n^i = \frac{2-\lambda_n^i L_i}{4}$ *and* $\{\lambda_n^i\} \subset (0, \frac{2}{L_i})$ *for all* $i = 1,2$. *Assume that the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \overline{\theta} \le \mu_n \le \theta$ *for all* $n \in \mathbb{N}$ *and for some* $\overline{\theta}, \theta > 0$,

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$.

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly as* $s_n^i \to 0$ ($\Longleftrightarrow \lambda_n^i \to \frac{2}{L_i}$) $\forall i = 1,2$, *to* $x^* = P_{U_{\widetilde{f}_1}} f(y^*)$ *and* $y^* = P_{U_{\widetilde{f}_2}} g(x^*)$, *respectively.*

Proof First, we show that $\{x_n\}$ and $\{y_n\}$ are bounded. Assume that $\widetilde{x} \in U_{\widetilde{f}_1}$ and $\widetilde{y} \in U_{\widetilde{f}_2}$. Then we have

$$\begin{aligned} \|x_{n+1} - \widetilde{x}\| &= \left\|(1-\mu_n)x_n + \mu_n P_C\bigl(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n\bigr) - \widetilde{x}\right\| \\ &= \left\|(1-\mu_n)(x_n - \widetilde{x}) + \mu_n\bigl(P_C\bigl(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n\bigr) - \widetilde{x}\bigr)\right\| \\ &\le (1-\mu_n)\|x_n - \widetilde{x}\| + \mu_n\left\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n - \widetilde{x}\right\| \\ &\le (1-\mu_n)\|x_n - \widetilde{x}\| + \mu_n\bigl(\alpha_n\left\|f(y_n) - \widetilde{x}\right\| + (1-\alpha_n)\left\|T_n^{\widetilde{f}_1} x_n - \widetilde{x}\right\|\bigr) \\ &\le (1-\mu_n)\|x_n - \widetilde{x}\| + \mu_n\bigl(\alpha_n\left\|f(y_n) - \widetilde{x}\right\| + (1-\alpha_n)\|x_n - \widetilde{x}\|\bigr) \\ &= (1-\alpha_n\mu_n)\|x_n - \widetilde{x}\| + \alpha_n\mu_n\left\|f(y_n) - \widetilde{x}\right\| \\ &\le (1-\alpha_n\mu_n)\|x_n - \widetilde{x}\| + \alpha_n\mu_n\bigl(\left\|f(y_n) - f(\widetilde{y})\right\| + \left\|f(\widetilde{y}) - \widetilde{x}\right\|\bigr) \\ &\le (1-\alpha_n\mu_n)\|x_n - \widetilde{x}\| + \alpha_n\mu_n a\|y_n - \widetilde{y}\| + \alpha_n\mu_n\left\|f(\widetilde{y}) - \widetilde{x}\right\|. \end{aligned} \tag{11}$$

Similarly, we get

$$\|y_{n+1} - \widetilde{y}\| \le (1-\alpha_n\mu_n)\|y_n - \widetilde{y}\| + \alpha_n\mu_n a\|x_n - \widetilde{x}\| + \alpha_n\mu_n\left\|g(\widetilde{x}) - \widetilde{y}\right\|. \tag{12}$$

Combining (11) and (12), we have

$$\begin{aligned} \|x_{n+1} - \widetilde{x}\| + \|y_{n+1} - \widetilde{y}\| &\le \bigl(1 - \alpha_n\mu_n(1-a)\bigr)\bigl(\|x_n - \widetilde{x}\| + \|y_n - \widetilde{y}\|\bigr) \\ &\quad + \alpha_n\mu_n\bigl(\left\|f(\widetilde{y}) - \widetilde{x}\right\| + \left\|g(\widetilde{x}) - \widetilde{y}\right\|\bigr). \end{aligned}$$

By induction, we can derive that

$$\|x_n - \widetilde{x}\| + \|y_n - \widetilde{y}\| \le \max\bigl\{\|x_1 - \widetilde{x}\| + \|y_1 - \widetilde{y}\|, \left\|f(\widetilde{y}) - \widetilde{x}\right\| + \left\|g(\widetilde{x}) - \widetilde{y}\right\|\bigr\},$$

for every $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{y_n\}$ are bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$. Observe that

$$\begin{aligned} &\left\|T_n^{\widetilde{f}_1} x_n - T_{n-1}^{\widetilde{f}_1} x_{n-1}\right\| \\ &\quad \le \left\|T_n^{\widetilde{f}_1} x_n - T_n^{\widetilde{f}_1} x_{n-1}\right\| + \left\|T_n^{\widetilde{f}_1} x_{n-1} - T_{n-1}^{\widetilde{f}_1} x_{n-1}\right\| \end{aligned}$$

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$$
\begin{aligned}
&\le \|x_n - x_{n-1}\| + \left\| \left(\frac{4P_C(I - \lambda_n^1 \nabla \widetilde{f}_1) - (2 - \lambda_n^1 L_1)}{2 + \lambda_n^1 L_1} \right) x_{n-1} \right. \\
&\qquad \left. - \left(\frac{4P_C(I - \lambda_{n-1}^1 \nabla \widetilde{f}_1) - (2 - \lambda_{n-1}^1 L_1)}{2 + \lambda_{n-1}^1 L_1} \right) x_{n-1} \right\| \\
&\le \|x_n - x_{n-1}\| + \left\| \left(\frac{4P_C(I - \lambda_n^1 \nabla \widetilde{f}_1)}{2 + \lambda_n^1 L_1} \right) x_{n-1} - \left(\frac{4P_C(I - \lambda_{n-1}^1 \nabla \widetilde{f}_1)}{2 + \lambda_{n-1}^1 L_1} \right) x_{n-1} \right\| \\
&\qquad + \left\| \left(\frac{2 - \lambda_{n-1}^1 L_1}{2 + \lambda_{n-1}^1 L_1} \right) x_{n-1} - \left(\frac{2 - \lambda_n^1 L_1}{2 + \lambda_n^1 L_1} \right) x_{n-1} \right\| \\
&= \|x_n - x_{n-1}\| \\
&\qquad + \left\| \frac{4(2 + \lambda_{n-1}^1 L_1) P_C(I - \lambda_n^1 \nabla \widetilde{f}_1) x_{n-1} - 4(2 + \lambda_n^1 L_1) P_C(I - \lambda_{n-1}^1 \nabla \widetilde{f}_1) x_{n-1}}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right\| \\
&\qquad + \left\| \frac{(2 - \lambda_{n-1}^1 L_1)(2 + \lambda_n^1 L_1) x_{n-1} - (2 - \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1) x_{n-1}}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right\| \\
&= \|x_n - x_{n-1}\| \\
&\qquad + \left\| \frac{4(2 + \lambda_{n-1}^1 L_1) P_C(I - \lambda_n^1 \nabla \widetilde{f}_1) x_{n-1} - 4(2 + \lambda_n^1 L_1) P_C(I - \lambda_{n-1}^1 \nabla \widetilde{f}_1) x_{n-1}}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right\| \\
&\qquad + \left(\frac{4L_1 |\lambda_n^1 - \lambda_{n-1}^1|}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right) \|x_{n-1}\| \\
&= \|x_n - x_{n-1}\| \\
&\qquad + \left\| \frac{4L_1(\lambda_{n-1}^1 - \lambda_n^1) P_C(I - \lambda_n^1 \nabla \widetilde{f}_1) x_{n-1}}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right. \\
&\qquad \left. + \frac{4(2 + \lambda_n^1 L_1)(P_C(I - \lambda_n^1 \nabla \widetilde{f}_1) x_{n-1} - P_C(I - \lambda_{n-1}^1 \nabla \widetilde{f}_1) x_{n-1})}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right\| \\
&\qquad + \left(\frac{4L_1 |\lambda_n^1 - \lambda_{n-1}^1|}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right) \|x_{n-1}\| \\
&\le \|x_n - x_{n-1}\| \\
&\qquad + \frac{4L_1 |\lambda_{n-1}^1 - \lambda_n^1| \|P_C(I - \lambda_n^1 \nabla \widetilde{f}_1) x_{n-1}\|}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} + \frac{4\|\lambda_{n-1}^1 \nabla \widetilde{f}_1 x_{n-1} - \lambda_n^1 \nabla \widetilde{f}_1 x_{n-1}\|}{2 + \lambda_{n-1}^1 L_1} \\
&\qquad + \left(\frac{4L_1 |\lambda_n^1 - \lambda_{n-1}^1|}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right) \|x_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + L_1 \left| \lambda_{n-1}^1 - \lambda_n^1 \right| \left\| P_C \left(I - \lambda_n^1 \nabla \widetilde{f}_1 \right) x_{n-1} \right\| \\
&\qquad + 4 \left| \lambda_{n-1}^1 - \lambda_n^1 \right| \|\nabla \widetilde{f}_1 x_{n-1}\| + L_1 \left| \lambda_n^1 - \lambda_{n-1}^1 \right| \|x_{n-1}\| \\
&\le \|x_n - x_{n-1}\| \\
&\qquad + \left| \lambda_{n-1}^1 - \lambda_n^1 \right| \left(L_1 \left\| P_C \left(I - \lambda_n^1 \nabla \widetilde{f}_1 \right) x_{n-1} \right\| + 4\|\nabla \widetilde{f}_1 x_{n-1}\| + L_1 \|x_{n-1}\| \right) \\
&\le \|x_n - x_{n-1}\| + M_1 \left| \lambda_{n-1}^1 - \lambda_n^1 \right|, \qquad (13)
\end{aligned}
$$

for some $M_1 > 0$ such that $M_1 \ge L_1 \|P_C(I - \lambda_n^1 \nabla \widetilde{f}_1) x_{n-1}\| + 4\|\nabla \widetilde{f}_1 x_{n-1}\| + L_1 \|x_{n-1}\|$, $\forall n \ge 1$.

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From the definition of x_n and (13), we have

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \Big\|(1-\mu_n)x_n + \mu_n P_C\big(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n\big) \\
&\qquad - \big((1-\mu_{n-1})x_{n-1} + \mu_{n-1}P_C\big(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big)\big)\Big\| \\
&\quad \leq (1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1} - \mu_n|\|x_{n-1}\| \\
&\qquad + \mu_n\big\|P_C\big(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n\big) \\
&\qquad - P_C\big(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big)\big\| \\
&\qquad + |\mu_n - \mu_{n-1}|\big\|P_C\big(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big)\big\| \\
&\quad \leq (1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1} - \mu_n|\|x_{n-1}\| \\
&\qquad + \mu_n\big\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \big(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big)\big\| \\
&\qquad + |\mu_n - \mu_{n-1}|\big\|P_C\big(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big)\big\| \\
&\quad \leq (1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1} - \mu_n|\|x_{n-1}\| \\
&\qquad + \mu_n\big(\big\|\alpha_n f(y_n) - \alpha_{n-1}f(y_{n-1})\big\| + \big\|(1-\alpha_n)T_n^{\widetilde{f}_1}x_n - (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big\|\big) \\
&\qquad + |\mu_n - \mu_{n-1}|\big\|P_C\big(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big)\big\| \\
&\quad \leq (1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1} - \mu_n|\|x_{n-1}\| \\
&\qquad + \mu_n\big(\alpha_n\big\|f(y_n) - f(y_{n-1})\big\| + |\alpha_n - \alpha_{n-1}|\big\|f(y_{n-1})\big\| \\
&\qquad + (1-\alpha_n)\big\|T_n^{\widetilde{f}_1}x_n - T_{n-1}^{\widetilde{f}_1}x_{n-1}\big\| + |\alpha_{n-1} - \alpha_n|\big\|T_{n-1}^{\widetilde{f}_1}x_{n-1}\big\|\big) \\
&\qquad + |\mu_n - \mu_{n-1}|\big\|P_C\big(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big)\big\| \\
&\quad \leq (1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1} - \mu_n|\|x_{n-1}\| \\
&\qquad + \mu_n\big(\alpha_n\big\|f(y_n) - f(y_{n-1})\big\| + |\alpha_n - \alpha_{n-1}|\big\|f(y_{n-1})\big\| \\
&\qquad + (1-\alpha_n)\big(\|x_n - x_{n-1}\| + M_1\big|\lambda_{n-1}^1 - \lambda_n^1\big|\big) + |\alpha_{n-1} - \alpha_n|\big\|T_{n-1}^{\widetilde{f}_1}x_{n-1}\big\|\big) \\
&\qquad + |\mu_n - \mu_{n-1}|\big\|P_C\big(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big)\big\| \\
&\quad \leq (1-\mu_n)\|x_n - x_{n-1}\| + |\mu_{n-1} - \mu_n|\|x_{n-1}\| \\
&\qquad + \mu_n\Bigg(\alpha_n\big\|f(y_n) - f(y_{n-1})\big\| + |\alpha_n - \alpha_{n-1}|\big\|f(y_{n-1})\big\| \\
&\qquad + (1-\alpha_n)\|x_n - x_{n-1}\| + (1-\alpha_n)\frac{4M_1}{L_1}\big|s_n^1 - s_{n-1}^1\big| \\
&\qquad + |\alpha_{n-1} - \alpha_n|\big\|T_{n-1}^{\widetilde{f}_1}x_{n-1}\big\|\Bigg) \\
&\qquad + |\mu_n - \mu_{n-1}|\big\|P_C\big(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big)\big\| \\
&\quad \leq (1-\mu_n\alpha_n)\|x_n - x_{n-1}\| + |\mu_{n-1} - \mu_n|\|x_{n-1}\| \\
&\qquad + |\mu_n - \mu_{n-1}|\big\|P_C\big(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big)\big\|
\end{aligned}
$$

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$$
\begin{aligned}
&+\mu_n\Big(\alpha_n a\|y_n-y_{n-1}\|+|\alpha_n-\alpha_{n-1}|\big\|f(y_{n-1})\big\| \\
&+(1-\alpha_n)\frac{4M_1}{L_1}\big|s_n^1-s_{n-1}^1\big|+|\alpha_{n-1}-\alpha_n|\big\|T_{n-1}^{\widetilde{f_1}}x_{n-1}\big\|\Big).
\end{aligned}
\tag{14}
$$

Using the same method as derived in (14), we have

$$
\begin{aligned}
&\|y_{n+1}-y_n\| \\
&\quad\le(1-\mu_n\alpha_n)\|y_n-y_{n-1}\|+|\mu_{n-1}-\mu_n|\|y_{n-1}\| \\
&\qquad+|\mu_n-\mu_{n-1}|\big\|P_C\big(\alpha_{n-1}g(x_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_2}}y_{n-1}\big)\big\| \\
&\qquad+\mu_n\Big(\alpha_n a\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\big\|g(x_{n-1})\big\| \\
&\qquad+(1-\alpha_n)\frac{4M_2}{L_2}\big|s_n^2-s_{n-1}^2\big|+|\alpha_{n-1}-\alpha_n|\big\|T_{n-1}^{\widetilde{f_2}}y_{n-1}\big\|\Big),
\end{aligned}
\tag{15}
$$

for some $M_2>0$ such that $M_2\ge L_2\|P_C(I-\lambda_n^2\nabla\widetilde{f_2})y_{n-1}\|+4\|\nabla\widetilde{f_2}y_{n-1}\|+L_2\|y_{n-1}\|$, $\forall n\ge 1$.

From (14) and (15), we have

$$
\begin{aligned}
&\|x_{n+1}-x_n\|+\|y_{n+1}-y_n\| \\
&\quad\le\big(1-(1-a)\mu_n\alpha_n\big)\big(\|x_n-x_{n-1}\|+\|y_n-y_{n-1}\|\big) \\
&\qquad+|\mu_{n-1}-\mu_n|\big(\|x_{n-1}\|+\|y_{n-1}\| \\
&\qquad+\big\|P_C\big(\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big)\big\| \\
&\qquad+\big\|P_C\big(\alpha_{n-1}g(x_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_2}}y_{n-1}\big)\big\|\big) \\
&\qquad+|\alpha_n-\alpha_{n-1}|\big(\big\|f(y_{n-1})\big\|+\big\|g(x_{n-1})\big\|+\big\|T_{n-1}^{\widetilde{f_1}}x_{n-1}\big\|+\big\|T_{n-1}^{\widetilde{f_2}}y_{n-1}\big\|\big) \\
&\qquad+(1-\alpha_n)\bigg(\frac{4M_1}{L_1}\big|s_n^1-s_{n-1}^1\big|+\frac{4M_2}{L_2}\big|s_n^2-s_{n-1}^2\big|\bigg).
\end{aligned}
$$

Applying Lemma 4 and condition (iii), we can conclude that

$$
\|x_{n+1}-x_n\|\to 0\quad\text{and}\quad\|y_{n+1}-y_n\|\to 0\quad\text{as } n\to\infty.
\tag{16}
$$

Next, we show that $\|x_n-W_n\|\to 0$ where $W_n=\alpha_n f(y_n)+(1-\alpha_n)T_n^{\widetilde{f_1}}x_n$ and $\|y_n-V_n\|\to 0$ where $V_n=\alpha_n g(x_n)+(1-\alpha_n)T_n^{\widetilde{f_2}}y_n$. Let $\widetilde{x}\in\mathcal{U}_{\widetilde{f_1}}$ and $\widetilde{y}\in\mathcal{U}_{\widetilde{f_2}}$. Then we derive that

$$
\begin{aligned}
\|x_{n+1}-\widetilde{x}\|^2&=\big\|(1-\mu_n)x_n+\mu_nP_CW_n-\widetilde{x}\big\|^2 \\
&=\big\|(1-\mu_n)(x_n-\widetilde{x})+\mu_n(P_CW_n-\widetilde{x})\big\|^2 \\
&=(1-\mu_n)\|x_n-\widetilde{x}\|^2+\mu_n\|P_CW_n-\widetilde{x}\|^2 \\
&\quad-(1-\mu_n)\mu_n\|x_n-P_CW_n\|^2 \\
&\le(1-\mu_n)\|x_n-\widetilde{x}\|^2+\mu_n\big\|\alpha_nf(y_n)+(1-\alpha_n)T_n^{\widetilde{f_1}}x_n-\widetilde{x}\big\|^2 \\
&\quad-(1-\mu_n)\mu_n\|x_n-P_CW_n\|^2
\end{aligned}
$$

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$$
\begin{aligned}
&= (1-\mu_n)\|x_n-\widetilde{x}\|^2 + \mu_n\big\|\alpha_n\big(f(y_n) - T_n^{\widetilde{f}_1}x_n\big) + T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\|^2 \\
&\quad - (1-\mu_n)\mu_n\|x_n - P_C W_n\|^2 \\
&\le (1-\mu_n)\|x_n-\widetilde{x}\|^2 + \mu_n\big(\big\|T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\|^2 \\
&\quad + 2\alpha_n\big\langle f(y_n) - T_n^{\widetilde{f}_1}x_n, \alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\rangle\big) \\
&\quad - (1-\mu_n)\mu_n\|x_n - P_C W_n\|^2 \\
&\le (1-\mu_n)\|x_n-\widetilde{x}\|^2 + \mu_n\big(\big\|T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\|^2 \\
&\quad + 2\alpha_n\big\|f(y_n) - T_n^{\widetilde{f}_1}x_n\big\|\big\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\|\big) \\
&\quad - (1-\mu_n)\mu_n\|x_n - P_C W_n\|^2 \\
&\le (1-\mu_n)\|x_n-\widetilde{x}\|^2 + \mu_n\|x_n - \widetilde{x}\|^2 \\
&\quad + 2\alpha_n\mu_n\big\|f(y_n) - T_n^{\widetilde{f}_1}x_n\big\|\big\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\| \\
&\quad - (1-\mu_n)\mu_n\|x_n - P_C W_n\|^2 \\
&= \|x_n - \widetilde{x}\|^2 \\
&\quad + 2\alpha_n\mu_n\big\|f(y_n) - T_n^{\widetilde{f}_1}x_n\big\|\big\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\| \\
&\quad - (1-\mu_n)\mu_n\|x_n - P_C W_n\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(1-\mu_n)\mu_n\|x_n - P_C W_n\|^2 \\
&\quad \le \|x_n-\widetilde{x}\|^2 - \|x_{n+1}-\widetilde{x}\|^2 \\
&\qquad + 2\alpha_n\mu_n\big\|f(y_n) - T_n^{\widetilde{f}_1}x_n\big\|\big\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\| \\
&\quad \le \|x_n - x_{n+1}\|\big(\|x_n-\widetilde{x}\| + \|x_{n+1}-\widetilde{x}\|\big) \\
&\qquad + 2\alpha_n\mu_n\big\|f(y_n) - T_n^{\widetilde{f}_1}x_n\big\|\big\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\|.
\end{aligned}
$$

By (16), as well as conditions (i) and (ii), we get

$$\|P_C W_n - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{17}$$

From definition of x_n and applying the same method as (17), we have

$$\|P_C V_n - y_n\| \to 0 \quad \text{as } n \to \infty. \tag{18}$$

Considering

$$
\begin{aligned}
\|P_C W_n - \widetilde{x}\|^2 &= \|P_C W_n - P_C\widetilde{x}\|^2 \\
&\le \langle W_n - \widetilde{x}, P_C W_n - \widetilde{x}\rangle \\
&= \frac{1}{2}\big(\|W_n-\widetilde{x}\|^2 + \|P_C W_n - \widetilde{x}\|^2 - \|W_n - P_C W_n\|^2\big)
\end{aligned}
$$

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implies that

$$\|P_C W_n - \widetilde{x}\| \le \|W_n - \widetilde{x}\|^2 - \|W_n - P_C W_n\|^2. \tag{19}$$

Observe that

$$\begin{aligned}
\|W_n - \widetilde{x}\|^2 &= \big\|\alpha_n\big(f(y_n) - \widetilde{x}\big) + (1-\alpha_n)\big(T_n^{\widetilde{f}_1} x_n - \widetilde{x}\big)\big\|^2 \\
&\le \alpha_n \big\|f(y_n) - \widetilde{x}\big\|^2 + (1-\alpha_n)\big\|T_n^{\widetilde{f}_1} x_n - \widetilde{x}\big\|^2 \\
&\le \alpha_n \big\|f(y_n) - \widetilde{x}\big\|^2 + (1-\alpha_n)\|x_n - \widetilde{x}\|^2.
\end{aligned} \tag{20}$$

From (19) and (20), we obtain

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}\|^2 &= \big\|(1-\mu_n)x_n + \mu_n P_C\big(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n\big) - \widetilde{x}\big\|^2 \\
&= \big\|(1-\mu_n)(x_n - \widetilde{x}) + \mu_n(P_C W_n - \widetilde{x})\big\|^2 \\
&\le (1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n\|P_C W_n - \widetilde{x}\|^2 \\
&\le (1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n\big(\|W_n - \widetilde{x}\|^2 - \|W_n - P_C W_n\|^2\big) \\
&\le (1-\mu_n)\|x_n - \widetilde{x}\|^2 \\
&\quad + \mu_n\big(\alpha_n\big\|f(y_n) - \widetilde{x}\big\|^2 + (1-\alpha_n)\|x_n - \widetilde{x}\|^2 - \|W_n - P_C W_n\|^2\big),
\end{aligned}$$

implying that

$$\begin{aligned}
\mu_n\|W_n - P_C W_n\|^2 &\le (1-\alpha_n\mu_n)\|x_n - \widetilde{x}\|^2 - \|x_{n+1} - \widetilde{x}\|^2 + \alpha_n\mu_n\big\|f(y_n) - \widetilde{x}\big\|^2 \\
&\le \|x_n - \widetilde{x}\|^2 - \|x_{n+1} - \widetilde{x}\|^2 + \alpha_n\mu_n\big\|f(y_n) - \widetilde{x}\big\|^2 \\
&\le \|x_n - x_{n+1}\|\big(\|x_n - \widetilde{x}\| + \|x_{n+1} - \widetilde{x}\|\big) + \alpha_n\mu_n\big\|f(y_n) - \widetilde{x}\big\|^2.
\end{aligned}$$

From $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and condition (i), we have

$$\|W_n - P_C W_n\| \to 0 \quad \text{as } n \to \infty. \tag{21}$$

From definition of V_n and applying the same argument as (21), we also obtain

$$\|V_n - P_C V_n\| \to 0 \quad \text{as } n \to \infty. \tag{22}$$

Since

$$\begin{aligned}
\|x_n - W_n\| &= \|x_n - P_C W_n + P_C W_n - W_n\| \\
&\le \|x_n - P_C W_n\| + \|P_C W_n - W_n\|.
\end{aligned}$$

From (17) and (21), we have

$$\|x_n - W_n\| \to 0 \quad \text{as } n \to \infty. \tag{23}$$

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From definition of y_n and applying the same method as in (23), we also have

$$\|y_n - V_n\| \to 0 \quad \text{as } n \to \infty. \tag{24}$$

Next, we show that $\|W_n - P_C(I - \frac{2}{L_1}\nabla\widetilde{f}_1)W_n\| \to 0$ as $n \to \infty$ and $\|V_n - P_C(I - \frac{2}{L_2}\nabla\widetilde{f}_2)V_n\| \to 0$ as $n \to \infty$. Observe that

$$W_n - x_n = \alpha_n\big(f(y_n) - x_n\big) + (1 - \alpha_n)\big(T_n^{\widetilde{f}_1}x_n - x_n\big),$$

which yields

$$(1 - \alpha_n)\big\|T_n^{\widetilde{f}_1}x_n - x_n\big\| \le \|W_n - x_n\| + \alpha_n\big\|f(y_n) - x_n\big\|.$$

From (23) and condition (i), we have

$$\big\|T_n^{\widetilde{f}_1}x_n - x_n\big\| \to 0 \quad \text{as } n \to \infty. \tag{25}$$

Since

$$\begin{aligned}
\big\|W_n - T_n^{\widetilde{f}_1}W_n\big\| &= \big\|W_n - x_n + x_n - T_n^{\widetilde{f}_1}x_n + T_n^{\widetilde{f}_1}x_n - T_n^{\widetilde{f}_1}W_n\big\| \\
&\le \|W_n - x_n\| + \big\|x_n - T_n^{\widetilde{f}_1}x_n\big\| + \big\|T_n^{\widetilde{f}_1}x_n - T_n^{\widetilde{f}_1}W_n\big\| \\
&\le \|W_n - x_n\| + \big\|x_n - T_n^{\widetilde{f}_1}x_n\big\| + \|x_n - W_n\| \\
&= 2\|x_n - W_n\| + \big\|T_n^{\widetilde{f}_1}x_n - x_n\big\|.
\end{aligned}$$

From (23) and (25), we get

$$\big\|T_n^{\widetilde{f}_1}W_n - W_n\big\| \to 0 \quad \text{as } n \to \infty. \tag{26}$$

Observe that

$$\begin{aligned}
\big\|P_C\big(I - \lambda_n^1\nabla\widetilde{f}_1\big)W_n - W_n\big\| &= \big\|s_n^1 W_n + \big(1 - s_n^1\big)T_n^{\widetilde{f}_1}W_n - W_n\big\| \\
&= \big(1 - s_n^1\big)\big\|T_n^{\widetilde{f}_1}W_n - W_n\big\| \\
&\le \big\|T_n^{\widetilde{f}_1}W_n - W_n\big\|,
\end{aligned} \tag{27}$$

where $s_n^1 = \frac{2-\lambda_n^1 L_1}{4} \in (0, \frac{1}{2})$.

From (27), we have

$$\begin{aligned}
&\left\|P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)W_n - W_n\right\| \\
&\quad\le \left\|P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)W_n - P_C\big(I - \lambda_n^1\nabla\widetilde{f}_1\big)W_n\right\| + \big\|P_C\big(I - \lambda_n^1\nabla\widetilde{f}_1\big)W_n - W_n\big\| \\
&\quad\le \left\|\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)W_n - \big(I - \lambda_n^1\nabla\widetilde{f}_1\big)W_n\right\| + \big\|P_C\big(I - \lambda_n^1\nabla\widetilde{f}_1\big)W_n - W_n\big\| \\
&\quad\le \left(\frac{2}{L_1} - \lambda_n^1\right)\big\|\nabla\widetilde{f}_1(W_n)\big\| + \big\|T_n^{\widetilde{f}_1}W_n - W_n\big\|.
\end{aligned}$$

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From the boundedness of $\{W_n\}$, $s_n^1 \to 0$ ($\Longleftrightarrow \lambda_n^1 \to \frac{2}{L_1}$) and (26), we conclude that

$$\lim_{n\to\infty} \left\| W_n - P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)W_n \right\| = 0. \tag{28}$$

Applying the same method as for (28), we also have

$$\lim_{n\to\infty} \left\| V_n - P_C\left(I - \frac{2}{L_2}\nabla\widetilde{f}_2\right)V_n \right\| = 0. \tag{29}$$

Next, we show that $\limsup_{n\to\infty}\langle f(y^*) - x^*, W_n - x^*\rangle \leq 0$, where $x^* = P_{U_{\widetilde{f}_1}} f(y^*)$ and $\limsup_{n\to\infty}\langle g(x^*) - y^*, V_n - y^*\rangle \leq 0$, where $y^* = P_{U_{\widetilde{f}_2}} g(x^*)$.

Indeed, take a subsequence $\{W_{n_k}\}$ of $\{W_n\}$ such that

$$\limsup_{n\to\infty}\bigl\langle f\bigl(y^*\bigr) - x^*, W_n - x^*\bigr\rangle = \limsup_{k\to\infty}\bigl\langle f\bigl(y^*\bigr) - x^*, W_{n_k} - x^*\bigr\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_k} \rightharpoonup \widehat{x}$ as $k \to \infty$. From (23), we obtain $W_{n_k} \rightharpoonup \widehat{x}$ as $k \to \infty$. Assume that $\widehat{x} \neq P_C(I - \frac{2}{L_1}\nabla\widetilde{f}_1)\widehat{x}$. By nonexpansiveness of $P_C(I - \frac{2}{L_1}\nabla\widetilde{f}_1)$, (28) and Opial's property, we have

$$\begin{aligned}
\liminf_{k\to\infty}\|W_{n_k} - \widehat{x}\| &< \liminf_{k\to\infty}\left\| W_{n_k} - P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)\widehat{x}\right\| \\
&\leq \liminf_{k\to\infty}\left(\left\| W_{n_k} - P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)W_{n_k}\right\| \right. \\
&\quad \left. + \left\| P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)W_{n_k} - P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)\widehat{x}\right\|\right) \\
&\leq \liminf_{k\to\infty}\|W_{n_k} - \widehat{x}\|.
\end{aligned}$$

This is a contradiction, thus we have

$$\widehat{x} \in F\left(P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)\right) = U_{\widetilde{f}_1}. \tag{30}$$

Since $W_{n_k} \rightharpoonup \widehat{x}$ as $k \to \infty$, due to (30) and Lemma 2, we can derive that

$$\begin{aligned}
\limsup_{n\to\infty}\bigl\langle f\bigl(y^*\bigr) - x^*, W_n - x^*\bigr\rangle &= \limsup_{k\to\infty}\bigl\langle f\bigl(y^*\bigr) - x^*, W_{n_k} - x^*\bigr\rangle \\
&= \bigl\langle f\bigl(y^*\bigr) - x^*, \widehat{x} - x^*\bigr\rangle \\
&\leq 0.
\end{aligned} \tag{31}$$

Similarly, take a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that

$$\limsup_{n\to\infty}\bigl\langle g\bigl(x^*\bigr) - y^*, V_n - y^*\bigr\rangle = \limsup_{k\to\infty}\bigl\langle g\bigl(x^*\bigr) - y^*, V_{n_k} - y^*\bigr\rangle.$$

Since $\{y_n\}$ is bounded, without loss of generality, we may assume that $y_{n_k} \rightharpoonup \widehat{y}$ as $k \to \infty$. From (24), we obtain $V_{n_k} \rightharpoonup \widehat{y}$ as $k \to \infty$. Following the same method as for (31), we easily

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obtain that

$$\limsup_{n\to\infty}\langle g(x^*)-y^*, V_n-y^*\rangle \le 0. \tag{32}$$

Finally, we show that $\{x_n\}$ converges strongly to x^*, where $x^* = P_{U_{\widetilde{f_1}}} f(y^*)$ and $\{y_n\}$ converges strongly to y^*, where $y^* = P_{U_{\widetilde{f_2}}} g(x^*)$.

Let $W_n = \alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f_1}} x_n$ and $V_n = \alpha_n g(x_n) + (1-\alpha_n)T_n^{\widetilde{f_2}} y_n$. From the definition of x_n, we get

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\quad= \|(1-\mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f_1}} x_n) - x^*\|^2\\
&\quad= \|(1-\mu_n)(x_n-x^*) + \mu_n(P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f_1}} x_n) - x^*)\|^2\\
&\quad= (1-\mu_n)\|x_n-x^*\|^2 + \mu_n\|P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f_1}} x_n) - x^*\|^2\\
&\quad\le (1-\mu_n)\|x_n-x^*\|^2 + \mu_n\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f_1}} x_n - x^*\|^2\\
&\quad= (1-\mu_n)\|x_n-x^*\|^2\\
&\qquad + \mu_n\|\alpha_n(f(y_n)-x^*) + (1-\alpha_n)(T_n^{\widetilde{f_1}} x_n - x^*)\|^2\\
&\quad\le (1-\mu_n)\|x_n-x^*\|^2\\
&\qquad + \mu_n((1-\alpha_n)\|T_n^{\widetilde{f_1}} x_n - x^*\|^2 + 2\alpha_n\langle f(y_n)-x^*, W_n-x^*\rangle)\\
&\quad\le (1-\mu_n)\|x_n-x^*\|^2\\
&\qquad + \mu_n(1-\alpha_n)\|x_n-x^*\|^2 + 2\alpha_n\mu_n\langle f(y_n)-x^*, W_n-x^*\rangle\\
&\quad= (1-\alpha_n\mu_n)\|x_n-x^*\|^2 + 2\alpha_n\mu_n\langle f(y_n)-x^*, W_n-x^*\rangle\\
&\quad= (1-\alpha_n\mu_n)\|x_n-x^*\|^2\\
&\qquad + 2\alpha_n\mu_n(\langle f(y_n)-f(y^*), W_n-x^*\rangle + \langle f(y^*)-x^*, W_n-x^*\rangle)\\
&\quad\le (1-\alpha_n\mu_n)\|x_n-x^*\|^2\\
&\qquad + 2\alpha_n\mu_n(\|f(y_n)-f(y^*)\|\|W_n-x^*\| + \langle f(y^*)-x^*, W_n-x^*\rangle)\\
&\quad\le (1-\alpha_n\mu_n)\|x_n-x^*\|^2\\
&\qquad + 2\alpha_n\mu_n\|f(y_n)-f(y^*)\|(\|W_n-x_{n+1}\| + \|x_{n+1}-x^*\|)\\
&\qquad + 2\alpha_n\mu_n\langle f(y^*)-x^*, W_n-x^*\rangle\\
&\quad\le (1-\alpha_n\mu_n)\|x_n-x^*\|^2\\
&\qquad + 2\alpha_n\mu_n a\|y_n-y^*\|\|W_n-x_{n+1}\| + 2\alpha_n\mu_n a\|y_n-y^*\|\|x_{n+1}-x^*\|\\
&\qquad + 2\alpha_n\mu_n\langle f(y^*)-x^*, W_n-x^*\rangle\\
&\quad\le (1-\alpha_n\mu_n)\|x_n-x^*\|^2\\
&\qquad + 2\alpha_n\mu_n a\|y_n-y^*\|\|W_n-x_{n+1}\| + \alpha_n\mu_n a(\|y_n-y^*\|^2 + \|x_{n+1}-x^*\|^2)\\
&\qquad + 2\alpha_n\mu_n\langle f(y^*)-x^*, W_n-x^*\rangle,
\end{aligned}$$

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which yields

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad\le \frac{1-\alpha_n\mu_n}{1-\alpha_n\mu_n a}\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|\|W_n-x_{n+1}\| \\
&\qquad+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle f(y^*)-x^*,W_n-x^*\rangle \\
&\quad=\left(1-\frac{\alpha_n\mu_n-\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\right)\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|\|W_n-x_{n+1}\| \\
&\qquad+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle f(y^*)-x^*,W_n-x^*\rangle \\
&\quad=\left(1-\frac{\alpha_n\mu_n(1-a)}{1-\alpha_n\mu_n a}\right)\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|\|W_n-x_{n+1}\| \\
&\qquad+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle f(y^*)-x^*,W_n-x^*\rangle.
\end{aligned} \tag{33}$$

Similarly, as derived above, we also have

$$\begin{aligned}
&\|y_{n+1}-y^*\|^2 \\
&\quad\le\left(1-\frac{\alpha_n\mu_n(1-a)}{1-\alpha_n\mu_n a}\right)\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|x_n-x^*\|\|V_n-y_{n+1}\| \\
&\qquad+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle g(x^*)-y^*,V_n-y^*\rangle.
\end{aligned} \tag{34}$$

From (33) and (34), we deduce that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 \\
&\quad\le\left(1-\frac{\alpha_n\mu_n(1-a)}{1-\alpha_n\mu_n a}\right)\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right) \\
&\qquad+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\left(\|y_n-y^*\|\|W_n-x_{n+1}\|+\|x_n-x^*\|\|V_n-y_{n+1}\|\right) \\
&\qquad+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right) \\
&\qquad+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\left(\langle f(y^*)-x^*,W_n-x^*\rangle+\langle g(x^*)-y^*,V_n-y^*\rangle\right) \\
&\quad=\left(1-\frac{\alpha_n\mu_n(1-2a)}{1-\alpha_n\mu_n a}\right)\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right) \\
&\qquad+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\left(\|y_n-y^*\|\|W_n-x_{n+1}\|+\|x_n-x^*\|\|V_n-y_{n+1}\|\right) \\
&\qquad+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\left(\langle f(y^*)-x^*,W_n-x^*\rangle+\langle g(x^*)-y^*,V_n-y^*\rangle\right).
\end{aligned} \tag{35}$$

By (16), (23), (24), (31), (32), condition (i) and Lemma 4, we have $\lim_{n\to\infty}(\|x_n-x^*\|+\|y_n-y^*\|)=0$. It implies that the sequences $\{x_n\}$, $\{y_n\}$ converge to $x^*=P_{U_{\tilde{f}_1}}f(y^*)$, $y^*=P_{U_{\tilde{f}_2}}g(x^*)$, respectively. This completes the proof. □

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Corollary 1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\widetilde{f}: C \to \mathbb{R}$ be a real-valued convex function and assume that $\nabla\widetilde{f}$ is $\frac{1}{L}$-inverse strongly monotone with $L > 0$ and $U_{\widetilde{f}} \neq \emptyset$. Let $f: H \to H$ be an a-contraction mapping with $a \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and*

$$x_{n+1} = (1-\mu_n)x_n + \mu_n P_C\big(\alpha_n f(x_n) + (1-\alpha_n)T_n^{\widetilde{f}}x_n\big), \tag{36}$$

where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I-\lambda_n^i\nabla\widetilde{f}) = s_nI + (1-s_n)T_n^{\widetilde{f}}$ and $s_n = \frac{2-\lambda_nL}{4}$, $\{\lambda_n\} \subset (0, \frac{2}{L})$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $0 < \overline{\theta} \leq \mu_n \leq \theta$ *for all* $n \in \mathbb{N}$ *and for some* $\overline{\theta}, \theta > 0$,

(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\mu_{n+1} - \mu_n| < \infty$.

Then $\{x_n\}$ converges strongly, as $s_n \to 0$ ($\Longleftrightarrow \lambda_n \to \frac{2}{L}$), to $x^* = P_{U_{\widetilde{f}}}f(x^*)$.

Proof If we put $f \equiv g$, $x_n = y_n$, in Theorem 1, we obtain the desired conclusion. □

4 Application

Let H_1, H_2 be two real Hilbert spaces. Let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively.

In 1994, Censor and Elfving [17] introduced the *split feasibility problem* (SFP), which is to find a point x such that

$$x \in C \quad \text{and} \quad Dx \in Q,$$

where $D: H_1 \to H_2$ is a bounded linear operator.

Throughout this paper, we assume that the SFP is consistent, that is, the solution set Γ of the SFP is nonempty. Let $f: \mathcal{H}_1 \to \mathbb{R}$ be a continuous differentiable function. The minimization problem

$$\min_{x\in C} f(x) := \min_{x\in C} \frac{1}{2}\|Ax - P_QAx\|^2 \tag{37}$$

is ill-posed.

Before proving Theorem 2, we need the following:

Proposition 3 ([18]) *Given $x^* \in \mathcal{H}_1$, the following statements are equivalent:*

(i) *x^* solves the SFP;*

(ii) $P_C(I-\lambda\nabla f)x^* = P_C(I-\lambda A^*(I-P_Q)A)x^* = x^*$;

(iii) *x^* solves the variational inequality problem of finding $x^* \in C$ such that*

$$\big\langle \nabla f(x^*), x - x^* \big\rangle \geq 0, \quad \forall x \in C, \tag{38}$$

where $\nabla f = A^*(I-P_Q)A$ and A^* is the adjoint of A.

Theorem 2 *Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively, and let $A_i: H_1 \to H_2$ be bounded linear operators for all $i = 1,2$ with L_i being the spectral radius of $A_i^*A_i$ for all $i = 1,2$ with $\Gamma_i \neq \emptyset$. Let $f, g: H \to H$ be a_f- and a_g-contraction*

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mappings with $a_f, a_g \in (0,1)$ *and* $a = \max\{a_f, a_g\}$. *Let the sequences* $\{x_n\}$, $\{y_n\}$ *be generated by* $x_1, y_1 \in C$ *and*

$$\begin{cases} x_{n+1} = (1-\mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{a_1}x_n), \\ y_{n+1} = (1-\mu_n)y_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)T_n^{a_2}y_n), \end{cases} \tag{39}$$

where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n^i(A_i^*(I-P_Q)A_i)) = s_n^i I + (1-s_n^i)T_n^{a_i}$, $\forall i = 1,2$ *and* $s_n^i = \frac{2-\lambda_n^i L_i}{4}$, $\{\lambda_n^i\} \subset (0, \frac{2}{L_i})$. *Assume that the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \overline{\theta} \le \mu_n \le \theta$ *for all* $n \in \mathbb{N}$ *and for some* $\overline{\theta}, \theta > 0$,

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$.

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly, as* $s_n^i \to 0$ $(\Longleftrightarrow \lambda_n^i \to \frac{2}{L_i})$ $\forall i = 1,2$, *to* $x^* = P_{\Gamma_1} f(y^*)$ *with* $\Gamma_1 = \{x \in C; A_1 x \in Q\}$ *and* $y^* = P_{\Gamma_2} g(x^*)$ *with* $\Gamma_2 = \{\bar{x} \in C; A_2\bar{x} \in Q\}$, *respectively.*

Proof Letting $x, y \in C$ and $\nabla f_i = A_i^*(I-P_Q)A_i$ for all $i = 1,2$, we have

$$\begin{aligned} \|\nabla f_i(x) - \nabla f_i(y)\|^2 &= \|A_i^*(I-P_Q)A_i x - A_i^*(I-P_Q)A_i y\|^2 \\ &\le L_i \|(I-P_Q)A_i x - (I-P_Q)A_i y\|^2. \end{aligned} \tag{40}$$

From the property of P_C, we have

$$\begin{aligned} &\|(I-P_Q)A_i x - (I-P_Q)A_i y\|^2 \\ &\quad = \langle (I-P_Q)A_i x - (I-P_Q)A_i y, (I-P_Q)A_i x - (I-P_Q)A_i y\rangle \\ &\quad = \langle (I-P_Q)A_i x - (I-P_Q)A_i y, A_i x - A_i y\rangle \\ &\qquad - \langle (I-P_Q)A_i x - (I-P_Q)A_i y, P_Q A_i x - P_Q A_i y\rangle \\ &\quad = \langle A_i^*(I-P_Q)A_i x - A_i^*(I-P_Q)A_i y, x - y\rangle \\ &\qquad - \langle (I-P_Q)A_i x - (I-P_Q)A_i y, P_Q A_i x - P_Q A_i y\rangle \\ &\quad = \langle A_i^*(I-P_Q)A_i x - A_i^*(I-P_Q)A_i y, x - y\rangle \\ &\qquad - \langle (I-P_Q)A_i x, P_Q A_i x - P_Q A_i y\rangle \\ &\qquad + \langle (I-P_Q)A_i y, P_Q A_i x - P_Q A_i y\rangle \\ &\quad \le \langle A_i^*(I-P_Q)A_i x - A_i^*(I-P_Q)A_i y, x - y\rangle. \end{aligned} \tag{41}$$

Substituting (41) into (40), we have

$$\begin{aligned} \|\nabla f_i(x) - \nabla f_i(y)\|^2 &\le L_i \langle A_i^*(I-P_Q)A_i x - A_i^*(I-P_Q)A_i y, x - y\rangle \\ &= L_i \langle \nabla f_i(x) - \nabla f_i(y), x - y\rangle. \end{aligned}$$

It follows that

$$\langle \nabla f_i(x) - \nabla f_i(y), x - y\rangle \ge \frac{1}{L_i}\|\nabla f_i(x) - \nabla f_i(y)\|^2.$$

Then ∇f_i is $\frac{1}{L_i}$-inverse strongly monotone, for all $i = 1,2$.

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From Proposition 3 and Theorem 1, we can conclude that Theorem 2 is true. ☐

Corollary 2 *Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be bounded linear operator with L being the spectral radius of A^*A with $\Gamma \neq \emptyset$. Let $f : H \to H$ be an a-contraction mapping with $a \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and*

$$x_{n+1} = (1-\mu_n)x_n + \mu_n P_C\big(\alpha_n f(x_n) + (1-\alpha_n)T_n^{a_1}x_n\big), \tag{42}$$

where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n(A^*(I-P_Q)A)) = s_nI + (1-s_n)T_n^{a_1}$ and $s_n = \frac{2-\lambda_nL}{4}$, $\{\lambda_n\} \subset (0, \frac{2}{L})$. Assume that the following conditions hold:

- (i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $0 < \overline{\theta} \leq \mu_n \leq \theta$ *for all* $n \in \mathbb{N}$ *and for some* $\overline{\theta}, \theta > 0$,
- (iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$.

Then $\{x_n\}$ converges strongly, as $s_n \to 0$ ($\Longleftrightarrow \lambda_n \to \frac{2}{L}$), to $x^* = P_\Gamma f(x^*)$ with $\Gamma = \{x \in C; Ax \in Q\}$.

Proof If we put $f \equiv g$, $x_n = y_n$ in Theorem 2, then the conclusion follows. ☐

5 Numerical examples

Example 1 Let $C = [-10,10] \times [-10,10]$ and let $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an inner product defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2$, for all $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$ and $\mathbf{y} = (y_1,y_2) \in \mathbb{R}^2$. For every $i = 1,2$, let $\widetilde{f}_i : C \to \mathbb{R}$ be defined by $\widetilde{f}_1(x_1,x_2) = 2x_1^2 + x_2$ and $\widetilde{f}_2(x_1,x_2) = (x_1-1) + x_2^2$, $\forall x_1,x_2 \in \mathbb{R}$. Let $f,g : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $f(x_1,x_2) = (\frac{x_1}{3}, \frac{x_2}{3})$ and $g(x_1,x_2) = (\frac{x_1}{4}, \frac{x_2}{4})$, be $\frac{1}{2}$- and $\frac{1}{3}$-contraction mappings and $a = \max\{\frac{1}{2}, \frac{1}{3}\} = \frac{1}{2}$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by $x_1, y_1 \in C$. Putting $\alpha_n = \frac{1}{4n}$ and $\mu_n = \frac{3n+1}{7n}$, we can rewrite (10) as follows:

$$\begin{cases} x_{n+1} = (\frac{4n-1}{7n})x_n + (\frac{3n+1}{7n})P_C(\frac{1}{4n}f(y_n) + (\frac{4n-1}{4n})T_n^{\widetilde{f}_1}x_n), \\ y_{n+1} = (\frac{4n-1}{7n})y_n + (\frac{3n+1}{7n})P_C(\frac{1}{4}g(x_n) + (\frac{4n-1}{4n})T_n^{\widetilde{f}_2}y_n), \end{cases} \tag{43}$$

where $P_C(x_1,x_2) = (\max\{\min\{x_1,10\},-10\}, \max\{\min\{x_2,10\},-10\})$ and also $P_C(I - \lambda_n^i \nabla\widetilde{f}_i) = s_n^iI + (1-s_n^i)T_n^{\widetilde{f}_i}$ and $s_n^i = \frac{2-\lambda_n^i(16)}{4}$, where $\lambda_n^i = \frac{n^2}{8n^2+1}$ $\forall i = 1,2$.

Then, since $\widetilde{f}_1(x_1,x_2) = 2x_1^2 + x_2$ and $\widetilde{f}_2(x_1,x_2) = (x_1-1) + x_2^2$, we have

$$\nabla\widetilde{f}_1(x_1,x_2) = (4x_1, 1) \quad \text{and} \quad \nabla\widetilde{f}_2(x_1,x_2) = (1, 2x_2).$$

It is obvious that $\nabla\widetilde{f}_i$ is a $\frac{1}{16}$-inverse strongly monotone, $\forall i = 1,2$.

Consider $(0,-10), (-10,0) \in [-10,10] \times [-10,10]$ for which

$$\begin{aligned} P_C\big(I - \lambda_n^1 \nabla\widetilde{f}_1\big)(0,-10) &= P_{[-10,10]\times[-10,10]}\left(I - \frac{1}{16}\nabla\widetilde{f}_1\right)(0,-10) \\ &= P_{[-10,10]\times[-10,10]}\left(0, \frac{-161}{16}\right) \\ &= \left(P_{[-10,10]}(0), P_{[-10,10]}\left(\frac{-161}{16}\right)\right) \\ &= \left(\max\big\{\min\{0,10\},-10\big\}, \max\left\{\min\left\{\frac{-161}{16}, 10\right\}, -10\right\}\right) \end{aligned}$$

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$$= (0, -10),$$

thus $(0, -10) \in U_{\widetilde{f_1}}$.

Similarly,

$$\begin{aligned} P_C\big(I - \lambda_n^2 \nabla\widetilde{f_2}\big)(-10, 0) &= P_{[-10,10]\times[-10,10]}\left(I - \frac{1}{16}\nabla\widetilde{f_2}\right)(-10, 0) \\ &= P_{[-10,10]\times[-10,10]}\left(\frac{-161}{16}, 0\right) \\ &= \left(P_{[-10,10]}\left(\frac{-161}{16}\right), P_{[-10,10]}(0)\right) \\ &= \left(\max\left\{\min\left\{\frac{-161}{16}, 10\right\}, -10\right\}, \max\big\{\min\{0, 10\}, -10\big\}\right) \\ &= (-10, 0), \end{aligned}$$

thus $(-10, 0) \in U_{\widetilde{f_2}}$.

It is clear that the sequences $\{\alpha_n\}$, $\{\mu_n\}$ satisfy all the conditions of Theorem 1, so we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0, -10)$ and $(-10, 0)$, respectively. Table 1 shows the values of $\{x_n\}$ and $\{y_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, $y_n^1 = 10$, $y_n^2 = -10$, and $n = N = 400$.

Table 1 The values of $\{x_n\}$ and $\{y_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, $y_n^1 = 10$, $y_n^2 = -10$, and $n = N = 400$

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(−10.0000, 10.0000)	(10.0000, −10.0000)
2	(−6.0784, 8.0448)	(8.1639, −7.2059)
3	(−4.2192, 7.3380)	(7.5048, −5.8538)
4	(−3.0380, 6.9152)	(7.1109, −4.9143)
⋮	⋮	⋮
250	(−0.0050, −7.7785)	(−7.5696, −0.0076)
⋮	⋮	⋮
396	(−0.0043, −9.9937)	(−9.9937, −0.0065)
397	(−0.0042, −9.9937)	(−9.9937, −0.0064)
398	(−0.0042, −9.9937)	(−9.9937, −0.0064)
399	(−0.0042, −9.9937)	(−9.9937, −0.0064)
400	(−0.0042, −9.9937)	(−9.9937, −0.0064)

Figure 1 The convergence of $\{x_n\}$ and $\{y_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, $y_n^1 = 10$, $y_n^2 = -10$, and $n = N = 400$

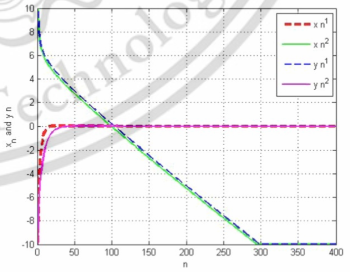

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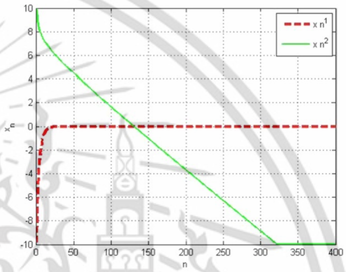
Table 2 The values of $\{x_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, and $n = N = 400$

n	$x_n = (x_n^1, x_n^2)$
1	(−10.0000, 10.0000)
2	(−7.0308, 8.9972)
3	(−5.2235, 8.5685)
4	(−3.9531, 8.2876)
⋮	⋮
250	(0.0000,-6.2974)
⋮	⋮
396	(0.0000, −9.9958)
397	(0.0000, −9.9958)
398	(0.0000, −9.9958)
399	(0.0000, −9.9958)
400	(0.0000, −9.9958)

Figure 2 The convergence of $\{x_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, and $n = N = 400$



Remark 1 If we choose $f \equiv g$ and $x_n = y_n$ in Example 1, we can rewrite (36) as follows:

$$x_{n+1} = \left(\frac{4n-1}{7n}\right)x_n + \left(\frac{3n+1}{7n}\right)P_C\left(\frac{1}{4n}f(x_n) + \left(\frac{4n-1}{4n}\right)T_n^{\tilde{f}}x_n\right),$$

where $P_C(x_1, x_2) = (\max\{\min\{x_1, 10\}, -10\}, \max\{\min\{x_2, 10\}, -10\})$ and also $P_C(I - \lambda_n \nabla\widetilde{f}) = s_n I + (1 - s_n)T_n^{\tilde{f}}$ and $s_n = \frac{2-\lambda_n(16)}{4}$, where $\lambda_n = \frac{n^2}{8n^2+1}$. From Corollary 1, we can conclude that the sequence $\{x_n\}$ converges strongly to $(0, -10)$. Table 2 shows the values of $\{x_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, and $n = N = 400$.

Conclusion

1. Theorem 1 guarantees the convergence of $\{x_n\}$ and $\{y_n\}$ in Example 1.
2. Corollary 1 guarantees the convergence of $\{x_n\}$ in Remark 1.
3. By using the concepts of an intermixed algorithm and gradient-projection algorithm (GPA), we give a new iteration for solving two constrained convex minimization problems.

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The subgradient extragradient method for approximation of fixed-point problem and modification of equilibrium problem

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ABSTRACT

In this paper, we consider the modification of equilibrium problem (MEP) and new subgradient extragradient algorithm by using the concept of the set of solutions of the modified variational inequality problem introduced by [Kangtunyakarn A. A new iterative scheme for fixed-point problems of infinite family of κ_i pseudo contractive mappings, equilibrium problem, variational inequality problems. J Optim Theory Appl. 2013;56:1543–1562.]. Then, we establish and prove weak and strong convergence theorem of the new subgradient extragradient algorithm for finding a common element of the set of solutions of the MEP and two sets of the variational inequality problems under some suitable conditions on α_n and β_n with $\alpha_n + \beta_n \leq 1$. Moreover, we apply our main theorem to prove weak and strong convergence theorems to solve the generalized equilibrium problem, the system of equilibrium problem, the variational inequality problem and the general system of variational inequality problems. Finally, we give two numerical examples to support our main result.

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Equilibrium problem; subgradient extragradient algorithm; variational inequality problem; fixed-point problem

1. Introduction

Throughout this article, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H.

The equilibrium problem (EP) is famous in many fields of pure and applied sciences. The EP can apply to physics, finance, network, optimization problems, variational inequality problems, and Nash equilibrium problems as special cases. Let $F : C \times C \to \mathbb{R}$ be a bifunction. *The equilibrium problem* of F is to find a point $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C, \tag{1}$$

which is introduced by Blum and Oettli [1], in 1994.

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Recently, many researchers have constructed their theorem to solve the equilibrium problem as follows.

Kim et al. [2] introduced a new extended extragradient iteration algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for a monotone and Lipschitz-type continuous mapping, and they proved a strong convergence theorem. Furthermore, Shang et al. [3] introduced a general iterative scheme using the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space; see more detail in [4,5].

If $F(x, y) := \langle A(x), y - x\rangle$ for all $x, y \in C$, where $A : C \to H$, then problem (1) is equivalent to finding x^* such that

$$\langle A(x^*), y - x^*\rangle \geq 0, \quad \forall\, y \in C. \tag{2}$$

The set of all solutions of (2) is denoted by $VI(C, A)$, which is known as *the variational inequality problem (VIP)*, introduced by Lions and Stampacchia [6] in 1964. Numerous problem in physic, game theory, finance, optimization, and mechanics reduce to find an element of (2); see more detail in [6–9].

Many mathematicians have modified the VIP as follows;

In 1999, Verma [10] introduced *the new system of variational inequalities problem*, which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall\, x \in C, \\ \langle \beta Ax^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall\, x \in C. \end{cases} \tag{3}$$

Further, if we put $x^* = y^*$, then the problem (3) reduces to the problem (2).

In 2013, Kangtunyakarn [5] modified the set of variational inequality problem as follows :

$$\begin{aligned} &VI(C, aA + (1 - a)B) \\ &\quad = \{x \in C : \langle y - x, (aA + (1 - a)B)x\rangle \geq 0,\ \forall\, y \in C,\ a \in (0, 1)\}, \end{aligned} \tag{4}$$

where $A, B : C \to H$ are two mappings. In particular, if we put $A \equiv B$, then we have $VI(C, aA + (1 - a)B) = VI(C, A)$, and he proved a strong convergence theorem to an element of $VI(C, aA + (1 - a)B)$ under suitable condition; see more detail in [5].

Many authors introduced their algorithms for solving the VIP, see, for instance [11,12].

In 1976, Korpelevich [11] proposed an algorithm for solving the VIP in Euclidean space and introduced the sequences $\{x_n\}$ and $\{y_n\}$ generated by $x_0 \in C$

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and

$$\begin{cases} y_n = P_C(x_n - \tau f(x_n)), \\ x_{n+1} = P_C(x_n - \tau f(y_n)), \end{cases} \tag{5}$$

where $f : H \to H$ is Lipschitz continuous on C with constant $L > 0$, $\tau \in (0, \frac{1}{L})$ and P_C denotes by the metric projection onto C, iteration (5) is called *the Extragradient Method*. If the solution set $VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (5) converges weakly to an element in $VI(C, A)$.

After that, using the concept of half-space, Censor et al. [12] modified Korpelevich [11] by replacing the second projection onto the closed and convex subset C of Hilbert space with the one onto the subgradient half-space T_n. The sequences $\{x_n\}$ and $\{y_n\}$ generated by $x_0 \in H$ and

$$\begin{cases} y_n = P_C(x_n - \tau f(x_n)), \\ T_n := \{w \in H | \langle (x_n - \tau f(x_n)) - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \tau f(y_n)), \end{cases} \tag{6}$$

where $f : H \to H$ is Lipschitz continuous on C with constant $L > 0$ and $\tau \in (0, \frac{1}{L})$, iteration (6) is called *the subgradient extragradient method*. Censor et al. [12] proved that the sequence $\{x_n\}$ generated by (6) converges weakly to a solution of the variational inequality and used Lemma 3.2 [13] for proved the strong convergence theorem of this sequence. A few years later, many mathematicians introduced the new problem and new iteration, which was developed and modified the subgradient extragradient method; see more detail in [14–18].

Many researchers have studied the iterative scheme to approximate the fixed-point problem of nonlinear mapping as follows.

In 1953, Mann [19] introduced the sequence $\{x_n\}$ generated by $x_0 \in H$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \forall\, n \geq 0, \tag{7}$$

where C is a nonempty closed convex subset of a normed space, $T : C \to C$ is a mapping and the sequence $\{\alpha_n\}$ is in the interval $(0, 1)$, iteration (7) is called *Mann iteration*. If T is a nonexpansive mapping under some suitable condition α_n satisfying $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by (7) converge weakly to element of the set of fixed points of T (i.e. $F(T) = \{x \in H : Tx = x\}$). Many authors have been trying to modify Mann's iteration to solve various problems such as the fixed point problem, split feasibility problem, equilibrium problem, monotone inclusion and image restoration problem; see more detail in [2,20,21].

In 2000, Moudafi [20] proposed *the viscosity approximation method* for nonexpansive mapping S and the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$x_{n+1} = \frac{1}{1 + \epsilon_n} S x_n + \frac{\epsilon_n}{1 + \epsilon_n} f(x_n), \quad \forall\, n \in \mathbb{N}, \tag{8}$$

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where $\{\epsilon_n\} \subset (0,1)$ satisfies certain conditions, $S : C \to C$ is a nonexpansive mapping and $f : C \to C$ is a contraction. Then he proved the sequence $\{x_n\}$ converges strongly to $z = P_{F(S)}f(z)$. Moreover, the viscosity approximation method has been studied and developed in many pieces of research to approximate the convex feasibility problem, hierarchical fixed-point problem, variational inequality problem, split common null point problem, see previous studies in [22,23]. Notice that the sum of coefficients $\frac{1}{1+\epsilon_n}$ and $\frac{\epsilon_n}{1+\epsilon_n}$ in (8) is equal to 1.

Recently, Kanzow and Shehu [24] modified inexact Krasnoselskii-Mann iteration and introduced the sequence $\{x_n\}$ generated by $x_1 \in H$, $u \in C$ and

$$x_{n+1} = \delta_n u + \alpha_n x_n + \beta_n T x_n + r_n, \quad \forall\, n \geq 1, \tag{9}$$

where r_n represents the residual, the nonnegative real numbers $\alpha_n, \beta_n, \delta_n$ are chosen such that $\alpha_n + \beta_n + \delta_n \leq 1$, for all $n \geq 1$, and $T : H \to C$ is a nonexpansive mapping. The sequence $\{x_n\}$ generated by (9) converges strongly to a point in $F(T)$, which is the nearest point projection of u onto $F(T)$. Observe that iterative scheme (9) is a modified Halpern's iterative scheme. Notice that the coefficient of α_n, β_n, and δ_n do not necessarily sum up to one.

Based on the result mentioned above, inspired and a motivated by the concept of the equilibrium problem and the new system of variational inequalities problem, we introduce *modification of equilibrium problem (MEP)*, which is to find $(x^*, y^*) \in C \times C$, such that

$$\begin{cases} F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - y^* + \lambda A y^*\rangle \geq 0, & \forall\, y \in C, \\ Q(y^*, x) + \frac{1}{r}\langle x - y^*, y^* - x^* + \beta B x^*\rangle \geq 0, & \forall\, x \in C, \end{cases} \tag{10}$$

where $F, Q : C \times C \to \mathbb{R}$ are bifunctions and $A, B : C \to H$ are mappings, $\lambda, \beta, r > 0$ are three constants. If we put $F \equiv Q \equiv 0$ and $A \equiv B$ in (10), then the problem (3) is a special case of the problem (10). The MEP can apply across many disciplines in mathematics and sciences such as economics, finance, network analysis, transportation, elasticity, and optimization.

In this paper, we prove weak convergence theorem of the sequence $\{x_n\}$ generated by the new subgradient extragradient for finding a common element of the set of solutions of the MEP and two sets of the variational inequality problems under some suitable conditions of α_n and β_n with $\alpha_n + \beta_n \leq 1$ in our main theorem. Moreover, we give a more comprehensive lemma than Lemma 3.2 [13] for an essential tool to prove the strong convergence theorem of sequence $\{x_n\}$.

The paper is therefore organized as follows: We first recall some basic definitions, and we give a lemma, which is an essential tool for proof weak and strong convergence of our main theorem in Section 2. We prove weak and strong convergence theorem of the sequence $\{x_n\}$ generated by the new subgradient extragradient for finding a common element of the set of solutions of the MEP and two sets of the variational inequality problems under some suitable conditions of our parameters α_n and β_n with $\alpha_n + \beta_n \leq 1$ in Section 3. We apply our

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main theorem to prove weak and strong convergence theorem to solve the generalized equilibrium problem, the system of equilibrium problem, the variational inequality problem, and the general system of variational inequality problem in Section 4. We give two numerical examples to support our main result in the last section.

2. Preliminaries

This section provides definitions and lemmas that will use for our main result in the next section.

We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x.

The following definition is important to our main theorem and its applications.

Definition 2.1: Let $A : C \to H$ and $T : C \to C$ be a mapping. Then

(i) a mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all } x, y \in C,$$

(ii) a mapping T is called *contraction* if there exists $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad \text{for all } x, y \in C,$$

(iii) a mapping T is called *quasi-nonexpansive* if

$$\|Tx - y\| \leq \|x - y\|, \quad \text{for all } x \in C \quad \text{and} \quad y \in F(T),$$

(iv) a mapping T is called *lipschitz continuous on C* if there exists $L > 0$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\|, \quad \text{for all } x, y \in C,$$

(v) a mapping A is called α*-inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \text{for all } x, y \in C,$$

(vi) a mapping A is called *firmly nonexpansive* if

$$\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle, \quad \text{for all } x, y \in C.$$

For each point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$. That is,

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall\, y \in C.$$

The mapping $P_C : H \to C$ is called *the metric projection of H onto C*.

The following Lemma 2.1 show that characterizes the projection P_C (see, for instance, [25,26]).

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Lemma 2.1: *Let $P_C : H \to C$ be the metric projection from H onto C. Then*

(i) *P_C is a nonexpansive mapping, i.e.*

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|, \quad \forall\, x, y \in H.$$

(ii) *P_C is firmly nonexpansive mapping, i.e.*

$$\langle P_C(x) - P_C(y), x - y\rangle \geq \|P_C(x) - P_C(y)\|^2, \quad \forall\, x, y \in H.$$

(iii) *For each $x \in H$, $y \in C$,*

$$\|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \leq \|x - y\|^2.$$

(iv) *$y = P_C(x)$ if and only if*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall\, z \in C.$$

Lemma 2.2 (See [25]): *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \;\Leftrightarrow\; u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Remark 2.1: From T_n in (6), we have $y_n = P_{T_n}(I - \tau f)x_n$ and if $x_n = y_n$, then $x_n \in VI(C, f)$ (Lemma 5.1 in [12]).

For solving the equilibrium problems for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0, \quad \forall x \in C,$
(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0, \quad \forall x, y \in C,$
(A3) $\forall\, x, y, z \in C,$
$\lim_{t \to 0^+} F(tz + (1 - t)x, y) \leq F(x, y),$
(A4) $\forall\, x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.3 (See [1]): *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall\, y \in C.$$

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Lemma 2.4 (See [27]): *Assume that* $F : C \times C \to \mathbb{R}$ *satisfies (A1)–(A4). For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall\, y \in C\},$$

for all $z \in H$. *Then, the following hold:*

1. T_r *is single-valued,*
2. T_r *is firmly nonexpansive i.e.*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle, \quad \forall\, x, y \in H,$$

3. $F(T_r) = EP(F)$,
4. $EP(F)$ *is closed and convex.*

Lemma 2.5 (See [24]): *Let X be a real inner product space. Then:*

(a) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$, $\forall\, x, y \in X$.
(b) $\|tx + sy\|^2 = t(t + s)\|x\|^2 + s(t + s)\|y\|^2 - st\|x - y\|^2$, $\forall\, x, y \in X$, $\forall\, s, t \in \mathbb{R}$.

Lemma 2.6: *Let C be a nonempty closed convex subset of a real Hilbert spaces and let* $F, Q : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1)–(A4). Let* $r > 0$, *then the following equivalent.*

(i) (x^*, y^*) *is a solution of (10),*
(ii) x^* *is a fixed point of a mapping* $\varphi : C \to C$ *defined by* $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$ *for all* $\beta, \lambda > 0$ *and* $x \in C$, *where* $y^* = T_r^Q(I - \beta B)x^*$.

Proof: From Lemma 2.4 and applied methods in [15], then we obtain the desired conclusion. ■

Lemma 2.7: *Let* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers satisfying*

$$a_{n+1} \leq a_n + b_n, \quad \text{for all } n \geq 0.$$

(i) *If* $\sum_{n=0}^{\infty} b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists.*
(ii) *If* $\sum_{n=0}^{\infty} b_n < \infty$ *and* $\{a_n\}_{n=0}^{\infty}$ *has a subsequence converging to zero, then*

$$\lim_{n\to\infty} a_n = 0.$$

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Lemma 2.8 (See [5]): *Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α and β-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C,A) \bigcap VI(C,B) \neq \emptyset$. Then*

$$VI(C, aA + (1-a)B) = VI(C,A) \bigcap VI(C,B), \quad \forall\, a \in (0,1).$$

Furthermore if $0 < \gamma < \min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1-a)B)$ is a nonexpansive mapping.

Remark 2.2: It is well known that $(aA + (1-a)B)$ is η-inverse strongly monotone, where $\eta = \min\{\alpha, \beta\}$.

Next, we present the new subgradient extragradient algorithm and the new iterative method for proving weak and strong convergence theorem of $\{x_n\}$ generated by the following algorithm:

Algorithm 2.1: Let $A, B, \overline{A}, \overline{B} : C \to H$ be $a, b, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $F, Q : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and define the mapping $\varphi : C \to C$ by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$, for all $x \in C$, $r > 0$, $\eta = \min\{a, b\}$, $\lambda, \beta \in (0, 2\eta)$ where T_r^F, T_r^Q define by $T_r^F(\bar{x}) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - \bar{x}\rangle \geq 0, \forall y \in C\}$ and $T_r^Q(\bar{x}) = \{z \in C : Q(z,y) + \frac{1}{r}\langle y - z, z - \bar{x}\rangle \geq 0, \forall y \in C\}$, for all $\bar{x} \in H$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$, and let the sequences $\{x_n\}$ and $\{y_n\}$ generated by $x_1 \in C$ and

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) + \beta_n \varphi(x_n), \end{cases} \tag{11}$$

for all $n \in \mathbb{N}$, where $\bar{a} \in (0,1)$ and $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$.

The following lemma is crucial for proving our main theorem.

Lemma 2.9: *Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $\overline{A}, \overline{B} : C \to H$ be $\bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $x^* \in VI(C, \overline{A}) \cap VI(C, \overline{B})$, $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$ and $\bar{a} \in (0,1)$, we have*

$$\begin{aligned} &\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*\|^2 \\ &\quad \leq \|x_n - x^*\|^2 - \left(1 - \frac{\gamma}{\bar{\eta}}\right)\|x_n - y_n\|^2 \\ &\qquad - \left(1 - \frac{\gamma}{\bar{\eta}}\right)\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n\|^2, \end{aligned}$$

where sequences $\{x_n\}$ and $\{y_n\}$ generated by (11).

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Proof: Let $x^* \in VI(C, \overline{A}) \cap VI(C, \overline{B})$.

By the property of P_C, we have

$$
\begin{aligned}
&\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*\|^2 \\
&\quad \leq \|x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n - x^*\|^2 \\
&\qquad - \|x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\quad = \|x_n - x^*\|^2 - 2\gamma\langle x_n - x^*, (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\rangle \\
&\qquad + \gamma^2\|(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\|^2 \\
&\qquad - \Big(\|x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 + \gamma^2\|(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\|^2 \\
&\qquad - 2\gamma\langle x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n), (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\rangle\Big) \\
&\quad = \|x_n - x^*\|^2 - \|x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\qquad - 2\gamma\langle P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*, (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\rangle. \qquad (12)
\end{aligned}
$$

From monotonicity of $(\bar{a}\overline{A} + (1-\bar{a})\overline{B})$, we have

$$
\begin{aligned}
0 &\leq \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n - (\bar{a}\overline{A} + (1-\bar{a})\overline{B})x^*, y_n - x^*\rangle \\
&= \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - x^*\rangle - \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})x^*, y_n - x^*\rangle \\
&\leq \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - x^*\rangle \\
&= \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\rangle \\
&\quad + \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*\rangle.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
&\langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, x^* - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\rangle \\
&\quad \leq \langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\rangle. \qquad (13)
\end{aligned}
$$

From (12) and (13), we get

$$
\begin{aligned}
&\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*\|^2 \\
&\quad \leq \|x_n - x^*\|^2 - \|x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\qquad - 2\gamma\langle P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - x^*, (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\rangle \\
&\quad \leq \|x_n - x^*\|^2 - \|x_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\qquad + 2\gamma\langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\rangle \\
&\quad = \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\qquad - 2\langle x_n - y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\rangle \\
&\qquad + 2\gamma\langle(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\rangle
\end{aligned}
$$

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$$\begin{aligned}
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\quad + 2\langle y_n - x_n + \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\quad + 2\langle x_n - y_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\quad + 2\langle (I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_n - y_n, P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n\rangle \\
&\quad + 2\langle \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, \\
&\quad P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n\rangle \\
&\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\quad + 2\langle \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n, \\
&\quad P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n\rangle \\
&\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\quad + 2\gamma\|(\bar{a}\overline{A} + (1-\bar{a})\overline{B})x_n - (\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n\| \\
&\quad \|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n\| \\
&\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\quad + 2\left(\frac{\gamma}{\bar{\eta}}\right)\|x_n - y_n\|\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n\| \\
&\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n)\|^2 \\
&\quad + \frac{\gamma}{\bar{\eta}}(\|x_n - y_n\|^2 + \|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n\|^2) \\
&= \|x_n - x^*\|^2 - (1 - \frac{\gamma}{\bar{\eta}})\|x_n - y_n\|^2 \\
&\quad - \left(1 - \frac{\gamma}{\bar{\eta}}\right)\|P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B})y_n) - y_n\|^2. \qquad \blacksquare
\end{aligned}$$

Now, we present the following Lemma 2.10 for proving strong convergence theorem.

Lemma 2.10: *Let H be a real Hilbert space and let S be a nonempty closed convex subset of H. Let* $\{x_n\}$ *be a sequence in H. Suppose that, for all* $u \in S$,

$$\|x_{n+1} - u\| \leq \|x_n - u\| + b_n,$$

for every $n = 0, 1, 2, \ldots$ *and* $\sum_{n=1}^{\infty} b_n < \infty$. *Then* $\{P_S x_n\}$ *converges strongly to some* $z \in S$.

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Proof: Let $\epsilon > 0$. From Lemma 2.7, then $\lim_{n\to\infty} \|x_n - u\|$ exists and we have that $\{\|x_n - u\|\}$ is bounded, for all $u \in S$.

Put $u_n = P_S x_n$. We get

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &= 2\|x_{n+1} - u_{n+1}\|^2 + 2\|x_{n+1} \\ &\quad - u_n\Big\|^2 - 4\Big\|x_{n+1} - \frac{1}{2}(u_{n+1} + u_n)\Big\|^2 \\ &\leq 2\|x_{n+1} - u_{n+1}\|^2 + 2\|x_{n+1} - u_n\|^2 - 4\|x_{n+1} - u_{n+1}\|^2 \\ &= 2\|x_{n+1} - u_n\|^2 - 2\|x_{n+1} - u_{n+1}\|^2 \\ &\leq 2\|x_n - u_n\|^2 - 2\|x_{n+1} - u_{n+1}\|^2 + 2\bar{b}_n, \end{aligned} \tag{14}$$

where $\bar{b}_n = b_n(b_n + 2\|x_n - u_n\|)$.

It implies that

$$\|x_{n+1} - u_{n+1}\|^2 \leq \|x_n - u_n\|^2 + \bar{b}_n.$$

Since $\{x_n\}$ and $\{u_n\}$ are bounded and $\sum_{n=1}^{\infty} b_n < \infty$, we obtain that $\sum_{n=1}^{\infty} \bar{b}_n < \infty$.

From Lemma 2.7, then $\{\|x_n - u_n\|\}$ there exists.

By (14), we have $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$, then there exist $N \in \mathbb{N}$, such that $\|u_{n+1} - u_n\| \leq \frac{\epsilon}{2^n}$, for all $n \geq N$.

Thus, for every $p \in \mathbb{N}$, we have

$$\|u_{n+p} - u_n\| \leq \sum_{k=n}^{n+p-1} \|u_{k+1} - u_k\| \leq \epsilon \sum_{k=n}^{n+p-1} \frac{1}{2^k} \leq \epsilon\left(\frac{1}{2^{n-1}}\right) < \epsilon. \tag{15}$$

From (15), we have that $\{u_n\}$ is a cauchy sequence. Hence, $\{u_n\}$ converges strongly to some $z \in S$. ■

3. Main results

In this section, we prove weak and strong convergence theorem of sequence $\{x_n\}$ generated by the new subgradient extragradient algorithm for finding a common element of the set of solutions of the MEP and two sets of the variational inequality problems.

Theorem 3.1: *Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $A, B, \overline{A}, \overline{B} : C \to H$ be $a, b, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $F, Q : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Define the mapping $\varphi : C \to C$ by $\varphi(x) = T_r^F(I - \lambda A)T_r^Q(I - \beta B)x$, for all $r > 0$, $x \in C$, the mappings T_r^F, T_r^Q define as the same in Algorithm 2.1. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap$*

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$F(\varphi) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by (11), satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, *for all* $n \in \mathbb{N}$,

(ii) $\displaystyle\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

Proof: First, we show that φ is a nonexpansive mapping for every $\lambda, \beta \in (0, 2\eta)$, where $\eta = \min\{a, b\}$. Let $x, y \in C$. Since A is a-inverse strongly monotone, we have

$$\begin{aligned}
\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|x-y-\lambda(Ax-Ay)\|^2 \\
&= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2 - 2\lambda a\|Ax-Ay\|^2 + \lambda^2\|Ax-Ay\|^2 \\
&= \|x-y\|^2 - \lambda(2a-\lambda)\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2 - \lambda(2\eta-\lambda)\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2. \qquad (16)
\end{aligned}$$

Thus $(I-\lambda A)$ is a nonexpansive mapping. By using the same method as (16), we have $(I-\beta B)$ is a nonexpansive mapping. Hence, $T_r^F(I-\lambda A)$ and $T_r^Q(I-\beta B)$ are nonexpansive mappings. It is easy to see that the mapping φ is a nonexpansive mapping.

Let $x^* \in VI(C,\overline{A}) \cap VI(C,\overline{B}) \cap F(\varphi)$ and $\gamma \leq \bar{\eta} = min\{\bar{\alpha}, \bar{\beta}\}$. We divide the proof of this result into three steps.

Step 1. Show that $\{x_n\}_{n=0}^{\infty}$ is bounded.

From the definition of x_n, Lemma 2.5 and Lemma 2.9, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad= \|\alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n) + \beta_n\varphi(x_n) - x^*\|^2 \\
&\quad= \|\alpha_n(P_{Q_n}(x_n - \gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n) - x^*) + \beta_n(\varphi(x_n)-x^*) \\
&\qquad - (1-\alpha_n-\beta_n)x^*\|^2 \\
&\quad\leq \|\alpha_n(P_{Q_n}(x_n - \gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n) - x^*) + \beta_n(\varphi(x_n)-x^*)\|^2 \\
&\qquad - 2(1-\alpha_n-\beta_n)\langle x^*, x_{n+1}-x^*\rangle \\
&\quad\leq \|\alpha_n(P_{Q_n}(x_n - \gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n) - x^*) + \beta_n(\varphi(x_n)-x^*)\|^2 \\
&\qquad + 2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|
\end{aligned}$$

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$$
\begin{aligned}
&= \alpha_n(\alpha_n+\beta_n)\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*\|^2\\
&\quad+\beta_n(\alpha_n+\beta_n)\|\varphi(x_n)-x^*\|^2\\
&\quad-\alpha_n\beta_n\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-\varphi(x_n)\|^2\\
&\quad+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
&\le \alpha_n(\alpha_n+\beta_n)\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-x^*\|^2\\
&\quad+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
&\le \alpha_n(\alpha_n+\beta_n)\Big(\|x_n-x^*\|^2-(1-\frac{\gamma}{\bar{\eta}})\|x_n-y_n\|^2\\
&\quad-(1-\frac{\gamma}{\bar{\eta}})\|P_{Q_n}(x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n)-y_n\|^2\Big)\\
&\quad+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
&\le \alpha_n(\alpha_n+\beta_n)\|x_n-x^*\|^2+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2\\
&\quad+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
&= (\alpha_n+\beta_n)^2\|x_n-x^*\|^2+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
&\le (\alpha_n+\beta_n)^2\|x_n-x^*\|^2+(1-\alpha_n-\beta_n)\|x^*\|^2\\
&\quad+(1-\alpha_n-\beta_n)\|x_{n+1}-x^*\|^2,
\end{aligned}
$$

which implies that

$$\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2+\frac{1-\alpha_n-\beta_n}{\alpha_n+\beta_n}\|x^*\|^2, \tag{17}$$

there exists $M>0$, such that

$$\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2+(1-\alpha_n-\beta_n)M\|x^*\|^2. \tag{18}$$

By (18) and Lemma 2.7, then $\lim_{n\to\infty}\|x_n-x^*\|$, $\forall\, x^*\in\xi$ exists. So, we have the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded.

Step 2. Show that $\lim_{n\to\infty}\|\varphi(x_n)-x_n\|=0$.

Let $W_n = x_n-\gamma(\bar{a}\overline{A}+(1-\bar{a})\overline{B})y_n$.

From the definition of x_n, Lemma 2.5 and Lemma 2.9, we have

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\quad= \|\alpha_n(P_{Q_n}W_n-x^*)+\beta_n(\varphi(x_n)-x^*)-(1-\alpha_n-\beta_n)x^*\|^2\\
&\quad\le \|\alpha_n(P_{Q_n}W_n-x^*)+\beta_n(\varphi(x_n)-x^*)\|^2-2(1-\alpha_n-\beta_n)\langle x^*,x_{n+1}-x^*\rangle\\
&\quad\le \|\alpha_n(P_{Q_n}W_n-x^*)+\beta_n(\varphi(x_n)-x^*)\|^2\\
&\qquad+2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
&\quad\le \alpha_n(\alpha_n+\beta_n)\|P_{Q_n}W_n-x^*\|^2+\beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2
\end{aligned}
$$

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$$
\begin{aligned}
&\quad - \alpha_n\beta_n\|P_{Q_n}W_n - \varphi(x_n)\|^2 + 2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\| \\
&\le \alpha_n(\alpha_n+\beta_n)\left(\|x_n-x^*\|^2 - \left(1-\frac{\gamma}{\bar{\eta}}\right)\|x_n-y_n\|^2\right. \\
&\quad \left. - \left(1-\frac{\gamma}{\bar{\eta}}\right)\|P_{Q_n}W_n - y_n\|^2\right) \\
&\quad + \beta_n(\alpha_n+\beta_n)\|x_n-x^*\|^2 \\
&\quad - \alpha_n\beta_n\|P_{Q_n}W_n - \varphi(x_n)\|^2 + 2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\| \\
&= (\alpha_n+\beta_n)^2\|x_n-x^*\|^2 \\
&\quad - \alpha_n(\alpha_n+\beta_n)\left(1-\frac{\gamma}{\bar{\eta}}\right)\left(\|x_n-y_n\|^2 + \|P_{Q_n}W_n-y_n\|^2\right) \\
&\quad - \alpha_n\beta_n\|P_{Q_n}W_n - \varphi(x_n)\|^2 + 2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
&\alpha_n(\alpha_n+\beta_n)\left(1-\frac{\gamma}{\bar{\eta}}\right)\left(\|x_n-y_n\|^2 + \|P_{Q_n}W_n-y_n\|^2\right) \\
&\quad + \alpha_n\beta_n\|P_{Q_n}W_n - \varphi(x_n)\|^2 \le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 \\
&\quad + 2(1-\alpha_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|.
\end{aligned} \tag{19}
$$

From (19), $\lim_{n\to\infty}(\|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2) = 0$ and condition (ii), we have

$$\lim_{n\to\infty}\|x_n-y_n\| = \lim_{n\to\infty}\|P_{Q_n}W_n - y_n\| = \lim_{n\to\infty}\|P_{Q_n}W_n - \varphi(x_n)\| = 0. \tag{20}$$

Since,

$$\|x_n - \varphi(x_n)\| \le \|x_n - y_n\| + \|y_n - P_{Q_n}W_n\| + \|P_{Q_n}W_n - \varphi(x_n)\|,$$

and (20), we get

$$\lim_{n\to\infty}\|\varphi(x_n) - x_n\| = 0. \tag{21}$$

Step 3. Show that $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and $z = \lim_{n\to\infty} P_{\xi}(x_n)$.

Therefore, it has at least one weak accumulation point. If $\bar{x}$ is a weak limit point of some subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$, then $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$.

Assume that $\bar{x} \ne \varphi(\bar{x})$. By nonexpansiveness of φ, (21) and Opial's property, we have

$$
\begin{aligned}
\liminf_{k\to\infty}\|x_{n_k} - \bar{x}\| &< \liminf_{k\to\infty}\|x_{n_k} - \varphi(\bar{x})\| \\
&\le \liminf_{k\to\infty}(\|x_{n_k} - \varphi(x_{n_k})\| + \|\varphi(x_{n_k}) - \varphi(\bar{x})\|) \\
&\le \liminf_{k\to\infty}(\|x_{n_k} - \varphi(x_{n_k})\| + \|x_{n_k} - \bar{x}\|)
\end{aligned}
$$

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$$\leq \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\|.$$

This is a contradiction, then we have

$$\bar{x} \in F(\varphi). \tag{22}$$

Assume that $\bar{x} \neq P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))\bar{x}$. By nonexpansiveness of $P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))$, (20) and Opial's property, we have

$$\begin{aligned}
&\liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| \\
&\quad < \liminf_{k\to\infty} \|x_{n_k} - P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))\bar{x}\| \\
&\quad \leq \liminf_{k\to\infty} (\|x_{n_k} - P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_{n_k}\| \\
&\qquad + \|P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_{n_k} - P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))\bar{x}\|) \\
&\quad \leq \liminf_{k\to\infty} (\|x_{n_k} - P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))x_{n_k}\| + \|x_{n_k} - \bar{x}\|) \\
&\quad \leq \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\|.
\end{aligned}$$

This is a contradiction, then we have

$$\bar{x} \in F(P_C(I - \gamma(\bar{a}\overline{A} + (1-\bar{a})\overline{B}))). \tag{23}$$

By (23), Lemma 2.8 and Lemma 2.2, we get

$$\bar{x} \in VI(C, \overline{A}) \cap VI(C, \overline{B}). \tag{24}$$

From (22) and (24), we have

$$\bar{x} \in \xi.$$

In order to show that the entire sequence $\{x_n\}$ weakly converges to $\bar{x}$, assume $x_{n_k} \rightharpoonup \bar{x}'$ as $k \to \infty$, with $\bar{x}' \neq \bar{x}$ and $\bar{x}' \in VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi)$.

By the Opial condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \bar{x}\| &= \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| \\
&< \liminf_{k\to\infty} \|x_{n_k} - \bar{x}'\| \\
&= \lim_{n\to\infty} \|x_n - \bar{x}'\| \\
&= \liminf_{k\to\infty} \|x_{n_k} - \bar{x}'\| \\
&< \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| \\
&= \lim_{n\to\infty} \|x_n - \bar{x}\|,
\end{aligned}$$

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and this is a contradiction, thus $\bar{x}' = \bar{x}$. This implies that the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to the same point $\bar{x} \in \xi$. Finally, if we take

$$u_n = P_{\xi} x_n, \tag{25}$$

then by (18) and Lemma 2.10, we see that $\{P_{\xi} x_n\}_{n=0}^{\infty}$ converges strongly to some $z \in \xi$. From (25) and Lemma 2.1(iv), we get

$$\langle x_n - u_n, u_n - \bar{x} \rangle \geq 0, \quad \forall\, \bar{x} \in \xi.$$

Take $n \to \infty$, we also have

$$\langle \bar{x} - z, z - \bar{x} \rangle \geq 0,$$

and hence $z = \bar{x}$. Therefore u_n converges strongly to $\bar{x} \in \xi$, this completes the proof. ■

The following Corollary 3.1 is a special case of Theorem 3.1 if we put $\overline{A} \equiv \overline{B}$ in Theorem 3.1.

Corollary 3.1: *Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $A, B, \overline{A} : C \to H$ be $a, b, \bar{\alpha}$-inverse strongly monotone, respectively. Let $F, Q : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Define the mapping $\varphi : C \to C$ by $\varphi(x) = T_r^F (I - \lambda A) T_r^Q (I - \beta B)x$, for all $r > 0$, $x \in C$, the mappings T_r^F, T_r^Q define as the same in Algorithm 2.1. Assume that $\xi = VI(C, \overline{A}) \cap F(\varphi) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} y_n = P_C(x_n - \gamma \overline{A}(x_n)), \\ Q_n = \{z \in H : \langle x_n - \gamma \overline{A}(x_n) - y_n, y_n - z \rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma \overline{A}(y_n)) + \beta_n \varphi(x_n), \end{cases} \tag{26}$$

where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$ and $\eta = \min\{a, b\}$, $\lambda, \beta \in (0, 2\eta)$, $\gamma \in (0, 2\bar{\alpha})$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, *for all* $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n \to \infty} P_{\xi}(x_n).$$

Proof: Putting $\overline{A} \equiv \overline{B}$ in Theorem 3.1, then we obtain the desired conclusion. ■

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4. Application

4.1. The generalized equilibrium and the system of equilibrium problems.

In this section, we obtain the following weak and strong convergence theorems to solve the generalized equilibrium and the system of equilibrium problems.

Put $A \equiv B \equiv 0$, in (10), the MEP is reduced to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - y^* \rangle \geq 0, & \forall\, y \in C, \\ Q(y^*, x) + \frac{1}{r}\langle x - y^*, y^* - x^* \rangle \geq 0, & \forall\, x \in C, \end{cases} \tag{27}$$

(27) is called *the system of equilibrium problem.*

If $A \equiv B$, $F \equiv Q$, $r = 1$, $x^* = y^*$ and $\lambda = \beta = 1$, in (10), then the MEP reduced to find $x^* \in C$ such that

$$F(x^*, y) + \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall\, y \in C, \tag{28}$$

where $A : C \to H$ is mapping, the problem (28) is called *the generalized equilibrium problem.* The set of solutions of (28) is denoted by $EP(F, A)$. The problem (27) and (28) covers various disciplines such as optimization problems, variational inequalities and the Nash equilibrium problem in noncooperative games, see literature in [27,28].

Theorem 4.1: *Let C be a nonempty closed convex subset of a real Hilbert spaces H and let* $\overline{A}, \overline{B} : C \to H$ *be* $\bar{\alpha}, \bar{\beta}$*-inverse strongly monotone, respectively. Let* $F, Q : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1)–(A4). Define the mapping* $\bar{\varphi} : C \to C$ *by* $\bar{\varphi}(x) = T_r^F(T_r^Q x)$*, for all* $r > 0$, $x \in C$*, the mappings* T_r^F, T_r^Q *define as the same in Algorithm 2.1. Assume that* $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\bar{\varphi}) \neq \emptyset$*. For given* $x_1 \in C$ *and let the sequences* $\{x_n\}$ *and* $\{y_n\}$ *be generated by*

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n - y_n, y_n - z \rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B})y_n) + \beta_n \bar{\varphi}(x_n), \end{cases} \tag{29}$$

where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ *with* $\alpha_n + \beta_n \leq 1$*, for all* $n \in \mathbb{N}$, $\bar{a} \in (0, 1)$ *and* $\gamma \leq \bar{\eta} = min\{\bar{\alpha}, \bar{\beta}\}$*, satisfying the following conditions hold:*

(i) $0 < c \leq \beta_n \leq d < 1$*, for all* $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ *converges weakly to* $z \in \xi$ *and furthermore,*

$$z = \lim_{n \to \infty} P_{\xi}(x_n).$$

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Proof: If we put $A \equiv B \equiv 0$, in Theorem 3.1. The conclusion of Theorem 4.1 can be obtained from Theorem 3.1. ■

Theorem 4.2: *Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $A, \overline{A}, \overline{B} : C \to H$ be $a, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying A1)-A4). Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap EP(F, A) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} F(u_n, y) + \frac{1}{r}\langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B})y_n) + \beta_n u_n, \end{cases} \tag{30}$$

where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$, $r > 0$, $\bar{a} \in (0, 1)$ and $\gamma \leq \bar{\eta} = min\{\bar{\alpha}, \bar{\beta}\}$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, *for all* $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n\to\infty} P_{\xi}(x_n).$$

Proof: If we put $A \equiv B$, $F \equiv Q$ and $\lambda = \beta = 1$, in Theorem 3.1, then we obtain the desired conclusion. ■

4.2. *The variational inequality and the general system of variational inequality problems*

In this section, we obtain the following weak and strong convergence theorems to solve the variational inequality and the general system of variational inequality problems.

In 2008, Ceng et al. [29] introduced *the general system of variational inequalities problem (GSVIP)*, which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall\, x \in C, \\ &\langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, \quad \forall\, x \in C, \end{aligned} \tag{31}$$

where $A, B : C \to H$ are two mappings and $\lambda, \mu > 0$ are two constants. Further, if we put $A \equiv B$ and $x^* = y^*$, then the problem (31) reduces to the variational inequality $VI(C, A)$.

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Remark 4.1: Put $F \equiv Q \equiv 0$ in (10), we have (10) is reduced to GSVIP. So, (31) is a spacial case of MEP.

Lemma 4.1 (See [29]): *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (31) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C\big(P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)\big), \quad \forall\, x \in C,$$

where $y^* = P_C(x^* - \mu Bx^*)$.

By use Theorem 3.1, we give a theorem involving to finding the solution of the GSVIP as follows.

Theorem 4.3: *Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $A, B, \overline{A}, \overline{B} : C \to H$ be $a, b, \bar{\alpha}, \bar{\beta}$-inverse strongly monotone, respectively. Define the mapping $\psi : C \to C$ by $\psi(x) = P_C(I - \lambda A)P_C(I - \beta B)x$, for all $x \in C$. Assume that $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\psi) \neq \emptyset$. For given $x_1 \in C$ and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B})y_n) + \beta_n \psi(x_n), \end{cases} \tag{32}$$

where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\alpha_n + \beta_n \leq 1$, for all $n \in \mathbb{N}$, $\bar{a} \in (0, 1)$, $\eta = \min\{a, b\}$, $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$ and $\lambda, \beta \in (0, 2\eta)$, satisfying the following conditions hold:

(i) $0 < c \leq \beta_n \leq d < 1$, *for all* $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to $z \in \xi$ and furthermore,

$$z = \lim_{n \to \infty} P_\xi(x_n).$$

Proof: From Lemma 4.1 and putting $F \equiv Q \equiv 0$, in Theorem 3.1, then we obtain the desired conclusion. ■

Next, we prove the fixed-point problem, which uses our main theorem.

Remark 4.2: Let $T : C \to C$ be nonexpansive mapping with $F(T) \neq \emptyset$. Then $F(T) = VI(C, I - T)$.

Corollary 4.1: *Let C be a nonempty closed convex subset of a real Hilbert spaces H and let $T_i : C \to C$ be a nonexpansive mappings, for all i = 1, 2, 3, 4. Define*

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the mapping $\varphi^* : C \to C$ *by* $\varphi^*(x) = P_C(I - \lambda(I - T_1))P_C(I - \beta(I - T_2))x$, *for all* $x \in C$. *Assume that* $\xi = \bigcap_{i=1}^{4} F(T_i) \neq \emptyset$. *For given* $x_1 \in C$ *and let the sequences* $\{x_n\}$ *and* $\{y_n\}$ *be generated by*

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}(I - T_3) + (1 - \bar{a})(I - T_4)))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}(I - T_3) + (1 - \bar{a})(I - T_4)))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}(I - T_3) + (1 - \bar{a})(I - T_4))y_n) + \beta_n \varphi^*(x_n), \end{cases} \tag{33}$$

where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ *with* $\alpha_n + \beta_n \leq 1$, *for all* $n \in \mathbb{N}$, $\bar{a} \in (0, 1)$ *and* $\lambda, \beta, \gamma \in (0, \frac{1}{2})$, *satisfying the following conditions hold:*

(i) $0 < c \leq \beta_n \leq d < 1$, *for all* $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ *converges weakly to* $z \in \xi$ *and furthermore,*

$$z = \lim_{n\to\infty} P_\xi(x_n).$$

Proof: The conclusion of Corollary 4.1 can be obtained from Theorem 4.3 and Remark 4.2. ■

Corollary 4.2: *Let* C *be a nonempty closed convex subset of a real Hilbert spaces* H *and let* $A, \overline{A}, \overline{B} : C \to H$ *be* $a, \bar{\alpha}, \bar{\beta}$*-inverse strongly monotone, respectively. Assume that* $\xi = VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap VI(C, A) \neq \emptyset$. *For given* $x_1 \in C$ *and let the sequences* $\{x_n\}$ *and* $\{y_n\}$ *be generated by*

$$\begin{cases} y_n = P_C(I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n, \\ Q_n = \{z \in H : \langle (I - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B}))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n P_{Q_n}(x_n - \gamma(\bar{a}\overline{A} + (1 - \bar{a})\overline{B})y_n) + \beta_n P_C(I - \lambda A)x_n, \end{cases} \tag{34}$$

where sequences $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ *with* $\alpha_n + \beta_n \leq 1$, *for all* $n \in \mathbb{N}$, $\lambda \in (0, 2a)$, $\bar{a} \in (0, 1)$ *and* $\gamma \leq \bar{\eta} = \min\{\bar{\alpha}, \bar{\beta}\}$, *satisfying the following conditions hold:*

(i) $0 < c \leq \beta_n \leq d < 1$, *for all* $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ *converges weakly to* $z \in \xi$ *and furthermore,*

$$z = \lim_{n\to\infty} P_\xi(x_n).$$

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Proof: If we put $A \equiv B$ and $\lambda = \beta$, in Theorem 4.3, then we obtain the desired conclusion. ■

5. Numerical

In this section, we give the following examples to support our main theorem.

Example 5.1: Let $\mathbb{R}$ be the set of real numbers, and let $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an inner product defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 \cdot y_1 + x_2 \cdot y_2$, for all $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ and a usual norm $\| \cdot \| : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Let $H = \mathbb{R}^2$, $C = [-100, 100] \times [-100, 100]$. Let $A, B, \overline{A}, \overline{B} : C \to \mathbb{R}^2$ defined by $A\mathbf{x} = (\frac{x_1}{2}, \frac{2x_2}{3})$, $B\mathbf{x} = (\frac{x_1}{3}, \frac{x_2}{4})$, $\overline{A}\mathbf{x} = (\frac{x_1}{3}, \frac{x_2}{3})$ and $\overline{B}\mathbf{x} = (\frac{x_1}{4}, \frac{x_2}{4})$, $\forall \mathbf{x} \in C$. Let the mapping $F, Q : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(\mathbf{x}, \mathbf{y}) = \frac{-(x_1)^2 - (x_2)^2 + (y_1)^2 + (y_2)^2}{4}, \quad \forall\, \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \\ \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2,$$

and

$$Q(\mathbf{x}, \mathbf{y}) = \frac{-(x_1)^2 - (x_2)^2 + (y_1)^2 + (y_2)^2}{5}, \quad \forall\, \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \\ \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2.$$

Let $r, \beta, \lambda = 1$, the sequence $x^* = (x_1^*, x_2^*)$, $y^* = (y_1^*, y_2^*)$ and $y = (y_1, y_2)$

$$\begin{aligned}
0 &\leq F(x^*, y) + \langle y - x^*, x^* - (I - A)y^* \rangle \\
&= \frac{-(x_1^*)^2 - (x_2^*)^2 + (y_1)^2 + (y_2)^2}{4} \\
&\quad + \langle (y_1 - x_1^*, y_2 - x_2^*), \left(\frac{2x_1^* - y_1^*}{2}, \frac{3x_2^* - y_2^*}{3} \right) \rangle \\
&= \frac{-(x_1^*)^2 - (x_2^*)^2 + (y_1)^2 + (y_2)^2}{4} + (y_1 - x_1^*) \left(\frac{2x_1^* - y_1^*}{2} \right) \\
&\quad + (y_2 - x_2^*) \left(\frac{3x_2^* - y_2^*}{3} \right) \\
&= \left(\frac{3(y_1)^2 + (12x_1^* - 6y_1^*)y_1 - 15(x_1^*)^2 + 6x_1^* y_1^*}{12} \right) \\
&\quad + \left(\frac{3(y_2)^2 + (12x_2^* - 4y_2^*)y_2 - 15(x_2^*)^2 + 4x_2^* y_2^*}{12} \right) \\
&= G_1(y_1) + G_2(y_2),
\end{aligned}$$

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where $G_1(y_1)=\left(\frac{3(y_1)^2+(12x_1^*-6y_1^*)y_1-15(x_1^*)^2+6x_1^*y_1^*}{12}\right)$ and $G_2(y_2)=\left(\frac{3(y_2)^2+(12x_2^*-4y_2^*)y_2-15(x_2^*)^2+4x_2^*y_2^*}{12}\right)$. $G_1(y_1)$ and $G_2(y_2)$ are quadratic functions with coefficients $a_1=\frac{1}{4}$, $b_1=x_1^*-\frac{y_1^*}{2}$, and $c_1=\frac{-5(x_1^*)^2}{4}+\frac{x_1^*y_1^*}{2}$ of $G_1(y_1)$ and coefficients $a_2=\frac{1}{4}, b_2=x_2^*-\frac{y_2^*}{3}$, and $c_2=\frac{-5(x_2^*)^2}{4}+\frac{x_2^*y_2^*}{3}$ of $G_2(y_2)$, respectively. Determine the discriminant Δ_1 of G_1 as follows:

$$
\begin{aligned}
\Delta_1 &= b_1^2-4a_1c_1\\
&=(x_1^*-\frac{y_1^*}{2})^2-4\left(\frac{1}{4}\right)\left(\frac{-5(x_1^*)^2}{4}+\frac{x_1^*y_1^*}{2}\right)\\
&=\left(\frac{3x_1^*-y_1^*}{2}\right)^2.
\end{aligned}
$$

We know that $G_1(y_1)\geq 0$, for all $z\in\mathbb{R}$. If it has most one solution in $\mathbb{R}$, then $\Delta_1\leq 0$. So, we obtain $x_1^*=\frac{y_1^*}{3}$. Next, we determine the discriminant Δ_2 of G_2 by using the same method as above, we obtain $x_2^*=\frac{2y_2^*}{9}$. That is $T_r^F(I-\lambda A)y^*=(\frac{y_1^*}{3},\frac{2y_2^*}{9})$. After that, we find the solution of $y^*=(y_1^*,y_2^*)$ in this inequality $0\leq Q(y^*,z)+\langle z-y^*,y^*-(I-B)x^*\rangle$. By using the same method as $T_r^F(I-\lambda A)y^*$, we obtain $T_r^Q(I-\beta B)x^*=(\frac{10x_1^*}{21},\frac{15x_2^*}{28})$. That is $\varphi(x)=T_r^F(I-\lambda A)T_r^Q(I-\beta B)x=T_r^F(I-\lambda A)(\frac{10x_1}{21},\frac{15x_2}{28})=(\frac{10x_1}{63},\frac{5x_2}{42})$.

Let $x_1=(x_1^1,x_1^2)$ and $y_1=(y_1^1,y_1^2)\in\mathbb{R}^2$. The sequences $\{x_n\}$ and $\{y_n\}$ are generated by (11), it is easy to see that $A,B,\overline{A},\overline{B}$ are 1- inverse strongly monotone with $\eta,\bar{\eta},\gamma=1$ and we can choose $\bar{a}=\frac{1}{2},\alpha_n=\frac{1}{n+1}-\frac{1}{(n+1)^{20}}$ and $\beta_n=1-\frac{1}{n+1}$, for all $n\in\mathbb{N}$. From the definition of $A,B,\overline{A},\overline{B}$ and φ, we have $VI(C,\overline{A})\cap VI(C,\overline{B})\cap F(\varphi)=(0,0)$. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to $(0,0)$. For each $n\in\mathbb{N}$, we can rewrite (11) as follows:

$$
\begin{aligned}
y_n &= P_C\left(I-\left(\frac{1}{2}(\overline{A})+\frac{1}{2}(\overline{B})\right)\right)x_n,\\
Q_n &= \left\{z\in H:\left\langle\left(I-\left(\frac{1}{2}(\overline{A})+\frac{1}{2}(\overline{B})\right)\right)x_n-y_n,y_n-z\right\rangle\geq 0\right\},\\
x_{n+1} &= \left(\frac{1}{n+1}-\frac{1}{(n+1)^{20}}\right)P_{Q_n}\left(x_n-\left(\frac{1}{2}(\overline{A})+\frac{1}{2}(\overline{B})\right)y_n\right)\\
&\quad+\left(1-\frac{1}{n+1}\right)\varphi(x_n),
\end{aligned}
$$

where $P_C(x_1,x_2)=(\max\{\min\{x_1,100\},-100\},\max\{\min\{x_2,100\},-100\})$.

Table 1 and Figure 1 shows the values of $\{x_n\}$ and $\{y_n\}$ with $x_1=(-100,100)$ and $n=1,2,\dots,13$.

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Table 1. The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-100, 100)$ and $n = 1, 2, \ldots, 13$.

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(−100.0000,100.0000)	(−70.8333,70.8333)
2	(−47.6066,45.6224)	(−33.7213,32.3159)
3	(−17.6281,15.6865)	(−12.4866,11.1113)
⋮	⋮	⋮
7	(-0.1054,0.0571)	(-0.0747,0.0405)
⋮	⋮	⋮
11	(−0.0003,0.0001)	(−0.0002,0.0001)
12	(−0.0001,0.0000)	(0.0000,0.0000)
13	(0.0000,0.0000)	(0.0000,0.0000)

Table 2. The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-100, 100)$ and $n = 1, 2, \ldots, 13$.

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(−100.0000,100.0000)	(−66.6667,66.6667)
2	(−46.8253,44.8412)	(−31.2169,29.8941)
3	(−17.0950,15.1843)	(−11.3966,10.1229)
⋮	⋮	⋮
7	(−0.0980,0.0527)	(−0.0653,0.0351)
⋮	⋮	⋮
11	(−0.0002,0.0001)	(−0.0002,0.0000)
12	(−0.0001,0.0000)	(0.0000,0.0000)
13	(0.0000,0.0000)	(0.0000,0.0000)

Remark 5.1: If we choose $\overline{A} \equiv \overline{B}$ in Example 5.1, we can rewrite (26) as follows:

$$
\begin{aligned}
y_n =& P_C(x_n - \overline{A}(x_n)), \\
Q_n =& \{z \in H : \langle x_n - \overline{A}(x_n) - y_n, y_n - z\rangle \geq 0\}, \\
x_{n+1} =& \left(\frac{1}{n+1} - \frac{1}{(n+1)^{20}}\right) P_{Q_n}(x_n - \overline{A}(y_n)) + \left(1 - \frac{1}{n+1}\right)\varphi(x_n),
\end{aligned}
$$

where $P_C(x_1, x_2) = (\max\{\min\{x_1, 100\}, -100\}, \max\{\min\{x_2, 100\}, -100\})$. From Corollary 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to $(0, 0)$.

Table 2 and Figure 2 shows the values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-100, 100)$ and $n = 1, 2, \ldots, 13$.

Remark 5.2: If $y_n = x_n$ and $x_{n+1} = x_n$ with $\alpha_n + \beta_n = 1$, then **STOP**. ($x_n \in VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi)$ in Example 5.1 and $x_n \in VI(C, \overline{A}) \cap F(\varphi)$ in Remark 5.1).

Example 5.2: In this example, we use the same mappings in Example 5.1. Let the sequence $\{x_n\}$ and $\{y_n\}$ be generated by (11), where $\alpha_n = \frac{1}{n^2}$ and $\beta_n = 1 - \frac{1}{n^2}$ for all $n \in \mathbb{N}$. From the definition of $A, B, \overline{A}, \overline{B}$ and φ, we have $VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap$

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Table 3. The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = 1, 2, \dots, 20$.

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(−10.0000,10.0000)	(−7.0833,7.0833)
2	(−7.9340,7.9340)	(−5.6199,5.6199)
3	(−2.5182,2.2821)	(−1.7838,1.6165)
⋮	⋮	⋮
10	(0.0000,0.0000)	(0.0000,0.0000)
⋮	⋮	⋮
18	(0.0000,0.0000)	(0.0000,0.0000)
19	(0.0000,0.0000)	(0.0000,0.0000)
20	(0.0000,0.0000)	(0.0000,0.0000)

$F(\varphi) = (0, 0)$. We can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to $(0, 0)$. For each $n \in \mathbb{N}$, we can rewrite (11) as follows:

$$\begin{aligned} y_n =& P_C\left(I - \left(\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B})\right)\right)x_n, \\ Q_n =& \left\{z \in H : \left\langle \left(I - \left(\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B})\right)\right)x_n - y_n, y_n - z\right\rangle \geq 0\right\}, \\ x_{n+1} =& \left(\frac{1}{n^2}\right)P_{Q_n}\left(x_n - \left(\frac{1}{2}(\overline{A}) + \frac{1}{2}(\overline{B})\right)y_n\right) + \left(1 - \frac{1}{n^2}\right)\varphi(x_n), \end{aligned}$$

where $P_C(x_1, x_2) = (\max\{\min\{x_1, 100\}, -100\}, \max\{\min\{x_2, 100\}, -100\})$.

Table 3 and Figure 3 shows the values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = 1, 2, \dots, 20$.

Remark 5.3: In case $\alpha_n = \frac{1}{n^2}$ and $\beta_n = 1 - \frac{1}{n^2}$ which $\alpha_n + \beta_n = 1$, we can see that Example 5.2 is a special case of Example 5.1. However, both examples we have $\{x_n\}$ and $\{y_n\}$ are converges to $(0, 0)$.

6. Conclusion

(1) We first introduce a modification of equilibrium problem (MEP).

(2) The MEP can be reduced to the system of equilibrium problem, the generalized equilibrium problem, and the GSVSP. It is well known that the methods for solving the system of equilibrium problem, the generalized equilibrium problem, and the GSVSP have been studying a broad class of problems arising in finance, economics, elasticity, transportation, and optimization, etc. In this paper, we can also solve the system of equilibrium problem, the generalized equilibrium problem, and the GSVSP in Section 4.

(3) By using the concept of Korpelevich's extragradient method, Censor et al. [12] defined the subgradient extragradient method as form (6), then the sequence $\{x_n\}$ converges weakly and strongly to a solution of the VIP. In Theorem 3.1, we use the concept of the subgradient extragradient method

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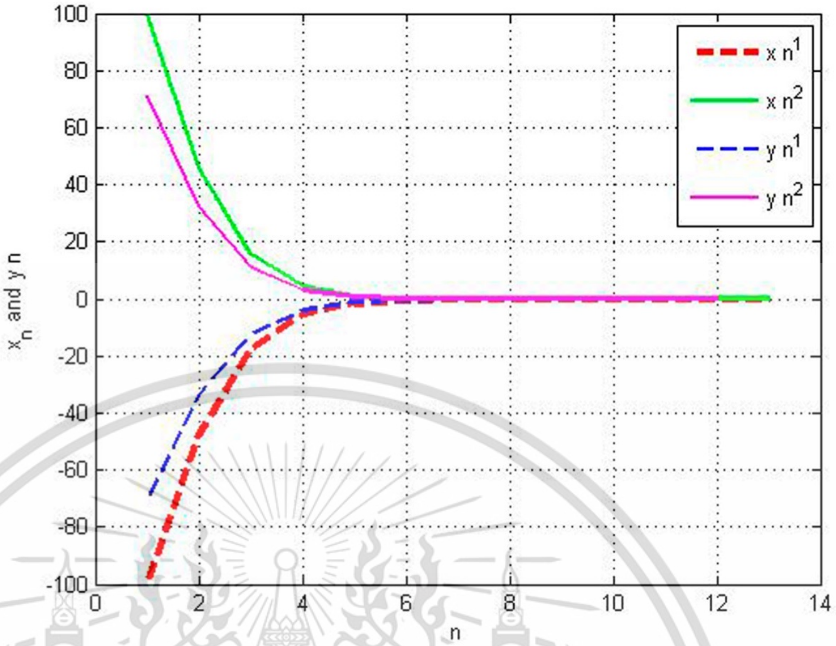

Figure 1. The convergences of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-100, 100)$ and $n = 1, 2, \ldots, 13$.

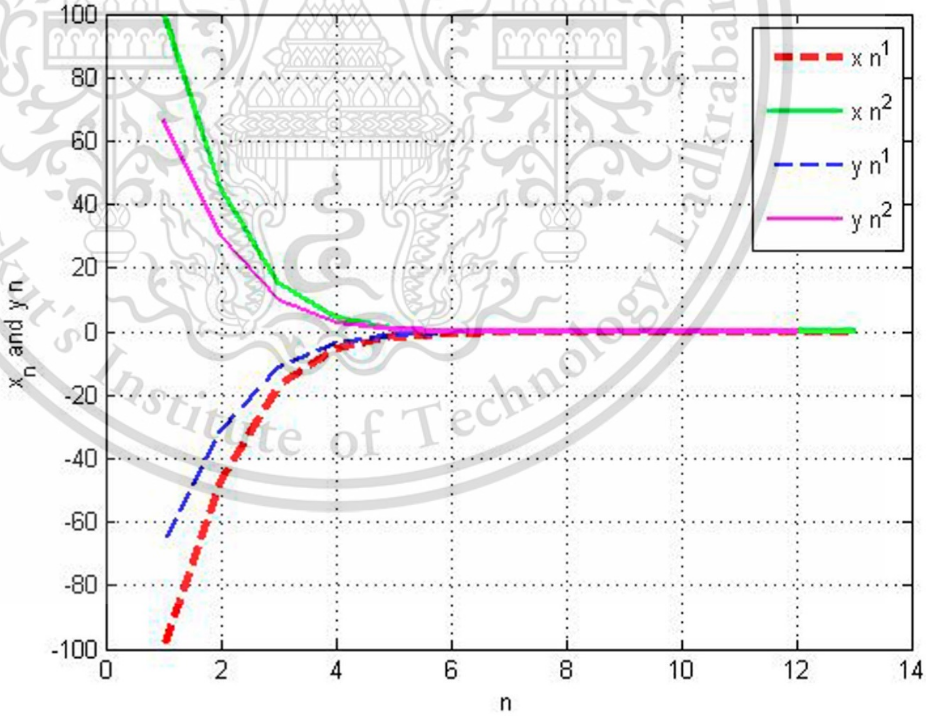

Figure 2. The convergences of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-100, 100)$ and $n = 1, 2, \ldots, 13$.

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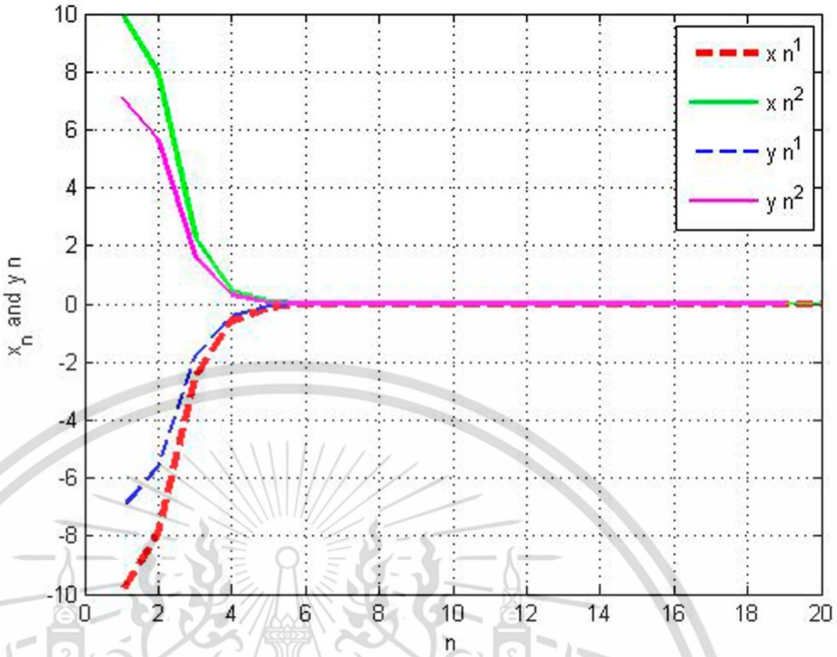

Figure 3. The convergences of $\{x_n\}$ and $\{y_n\}$ with $x_1 = (-10, 10)$ and $n = 1, 2, \ldots, 20$.

and suitable conditions of the parameters $\eta, \bar{\eta}, \gamma, \lambda, \beta, r, \bar{a}, \{\alpha_n\}$ and $\{\beta_n\}$, the sequences $\{x_n\}$ and $\{y_n\}$ defined by (11) converges weakly and strongly to $z \in VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi)$.

(4) Table 1 and Figure 1 show that the sequences $\{x_n\}$ and $\{y_n\}$ converge to $(0, 0)$, where $(0, 0) \in VI(C, \overline{A}) \cap VI(C, \overline{B}) \cap F(\varphi)$. The convergence of the sequences $\{x_n\}$ and $\{y_n\}$ of Example 5.1 can be guaranteed by Theorem 3.1.

(5) Table 2 and Figure 2 show that the sequences $\{x_n\}$ and $\{y_n\}$ converge to $(0, 0)$, where $(0, 0) \in VI(C, \overline{A}) \cap F(\varphi)$. The convergence of the sequences $\{x_n\}$ and $\{y_n\}$ of Remark 5.1 can be guaranteed by Corollary 3.1.

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Scholarship	KMITL Doctoral Scholarships

Academic Publications

1. Saechou, K. and Kangtunyakarn, A. 2019. ”An intermixed iteration for constrained convex minimization problem and split feasibility problem.” *Journal of Inequalities and Applications.* 2019 (1).
2. Saechou, K. and Kangtunyakarn, A. 2021. ”The subgradient extragradient method for approximation of fixed-point problem and modification of equilibrium problem.” *Optimization.* doi.org/10.1080/02331934.2021.1970751.
3. Saechou, K. and Kangtunyakarn, A. 2021. ”Modified intermixed iteration for solving the split general system of variational inequality problems and applications.” *Computational and Applied Mathematics.* Vol. 40(8), 264.
4. Saechou, K. and Kangtunyakarn, A. 2022. ”Application of The Combination of Variational Inequalities for Fixed Point Problems and Optimization Problems.” *Thai Journal of Mathematics.* Vol. 20(1), 79–98.
5. Saechou, K. and Kangtunyakarn, A. 2022. ”The method for solving the extension of general of the split feasibility problem and fixed point problem of the cutter.” *Journal of Inequalities and Applications.* Vol. 144, doi.org/10.1186/s13660-022-02881-2.
6. Saechou, K. and Kangtunyakarn, A. 2023. ”Convergence theorem for an intermixed iteration in p-uniformly convex metric space.”

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