รายงานการวิจัย
ซอร์ฟแวร์เชิงตัวเลขของแบบจำลองน้ำบาดาลเพื่อการพัฒนาแหล่งน้ำสำหรับพื้นที่การเกษตรที่ประสบภัยแล้ง
Numerical Software of a Ground Water Simulation for a Water Resource Development for Drought Agricultural Areas

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สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง

ได้รับทุนสนับสนุนงานวิจัยจากเงินรายได้ ประจำปีงบประมาณ 2557
คณะวิทยาศาสตร์
สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง

กิตติกรรมประกาศ

ขอขอบคุณคณะวิทยาศาสตร์ สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง ที่ให้การสนับสนุนทุนวิจัย ประเภทส่งเสริมนักวิจัยด้วยเงินรายได้คณะวิทยาศาสตร์ ประจำปีงบประมาณ 2557 จำนวน 350,000 บาท จนงานวิจัยนี้สำเร็จลุล่วงไปได้ด้วยดี

รายละเอียดเกี่ยวกับโครงการ

ชื่อโครงการวิจัย

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จำนวนเงิน 350,000 บาท

ระยะเวลาทำการวิจัย 2 ปี ตั้งแต่ 1 ตุลาคม 2557 ถึง 30 กันยายน 2559

หน่วยงานและผู้ดำเนินการวิจัยพร้อมหน่วยงานที่สังกัด

ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ สจล. โทร 02-326-4341 ต่อ 283

บทคัดย่อ

งานวิจัยนี้ได้ประยุกต์ตัวแบบเชิงคณิตศาสตร์เพื่อจำลองแบบการไหลของน้ำบาดาลในแต่ละฤดูกาล และพัฒนาระเบียบวิธีเชิงตัวเลขเพื่อการหาผลเฉลยเชิงตัวเลขของตัวแบบเชิงคณิตศาสตร์ของการไหลของน้ำบาดาลในแต่ละฤดูกาลโดยใช้วิธีเชิงตัวเลข สามารถนำตัวแบบเชิงคณิตศาสตร์และระเบียบวิธีเชิงตัวเลขมาบูรณาการเพื่อการแก้ปัญหาการไหลของน้ำบาดาลในแต่ละฤดูกาลในบริเวณพื้นที่การเกษตรที่ประสบภัยแล้ง โดยจำลองแบบการไหลของน้ำบาดาล โดยอาศัยข้อมูลแหล่งน้ำในอดีตและปัจจุบันว่าเป็นอย่างไร เพื่อจะคำนวณทิศทางการไหลและระดับของน้ำบาดาลในอนาคต เพื่อการแก้ปัญหาแก้ปัญหาน้ำแล้งในบริเวณเพาะปลูก

Abstract

The groundwater measurement is obliged to take care of the issue of need water assets in numerous dry season ranges for agricultural use. In this study, we propose a groundwater stream model demonstrate that gives the pumping rates and the infusion rates individually. The objective are to propose a simply and flexible groundwater simulation using the implicit and explicit traditional finite difference methods and alternating direction methods. The groundwater model is giving the water driven head that gives the groundwater level. The understood limited distinction technique is utilized to surmise the groundwater flow. The complex geometry in the model is considered by variable grid sizes aquifer parameters, sinks and source terms. The proposed alternating direction methods are shown that they are able to use in groundwater simulation for the real-world cases.

สารบัญ

CHAPTER I
Preliminary

Ground water simulation models are widely topics in the analysis and management of groundwater systems. The important of the utilization of groundwater resources continues to grow due to the increasing require of water for irrigation as well as drinking, agricultural, commercial and industrial purposes. Although, the amount.of groundwater resources have been decaying due to population growth, uncontrolled and unplanned urbanization, industrialization, and agricultural activities. Hence, the sustainable management planning must be developed for the groundwater systems. The management planning has to limit in the case of legal well drilling and limited-pumping. The groundwater model can be use to simulate the scenarios for the management planning.

The groundwater model is described by partial differential equations. The governing equation can be solved by analytical and numerical solution techniques. The simple and ideal cases with regularity shaped aquifers and homogeneous hydraulic properties can be solved by analytical methods. Most of groundwater modeling has the aquifer systems with the heterogeneous structure. On the other hand, numerical solutions have to be used if the aquifer system has complicated geometry or heterogeneous material properties.

In the case of steady-state groundwater model solutions can be obtained by many simply basic techniques. On the other hand, the case of transient groundwater model is solved by the advanced techniques due to the difficult in terms of time dimension in the governing equations. The finite difference [1,2,3] and finite elements methods [5,6] are most popular numerical solution techniques. The case of free surface flows can be considered into two groups: adaptive mesh methods and fixed mesh methods. The adaptive mesh method needs a large number of calculations and they also require some convergence conditions [4]. It follows that the fixed mesh techniques are more popular than adaptive mesh techniques. If the variation of geometry and material in the third dimension is constant, the two-dimensional modeling can be used. If the material properties and/or geometry vary along the third dimension, then the three-dimensional modeling may return better solutions than the two-dimensional modeling. A useful spreadsheet for one-, two- and three-dimensional steady-state and transient ground water numerical simulation is introduced in [10, 11].

It follows that the fixed mesh techniques are more popular than adaptive mesh techniques. If the variation of geometry and material in the third dimension is constant, then the two dimensional modeling can be used. Although, If the material properties and/or geometry vary any the third dimension, then the three-dimension modeling may return better solutions than the two-dimension modeling.

In this research, the objectives is to propose a simply and flexible groundwater numerical simulation by using implicit and explicit finite difference methods. The complex geometries in the model are considered by variable grid sizes aquifer parameters, sink and source terms. In the simulation process, the finite difference methods for models such as the alternating direction explicit method (ADEM), the alternating direction implicit method (ADIM), the forward time central space method (FTCS) and the backward time central space method (BTCS) are proposed. Finally, the numerical simulations results are compared with comprehensive solutions of MODFLOW.

CHAPTER II
Groundwater Measurement Model

2.1 The governing equation

The governing equation of vertically integrated Darcy's flow in a two-dimensional confined, compressible, isotropic, heterogeneous aquifer is [11]

$$S\frac{\partial H}{\partial t}=\frac{\partial}{\partial x}\left(K\frac{\partial H}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial H}{\partial y}\right)\pm W, \qquad (2.1)$$

where H hydraulic head (meter), K hydraulic conductivity (meter/day), W sinks and/or source (1/day) and S matrix of specific storage (1/meter). We assume the hydraulic conductivity is constant. It is obtain that

$$S\frac{\partial h}{\partial t}=K\left(\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}\right)+W, \qquad (2.2)$$

2.2The initial condition

The initial conditions at $t=0$, $0\le x\le L$ and $0\le y\le M$ where L,M are constant being specified,

$$H=H_0 \qquad (2.3)$$

2.3The boundary condition

The boundary condition for $t>0$ are specified,

$$\frac{\partial H}{\partial n}=B_N \qquad \text{at } 0\le x\le L \text{ and } y=M, \tag{2.4}$$

$$\frac{\partial H}{\partial n}=B_S \qquad \text{at } 0\le x\le L \text{ and } y=0, \tag{2.5}$$

$$\frac{\partial H}{\partial n}=B_W \qquad \text{at } x=0 \text{ and } 0\le y\le M, \tag{2.6}$$

$$\frac{\partial H}{\partial n}=B_E \qquad \text{at } x=0 \text{ and } 0\le y\le M. \tag{2.7}$$

and given pumping well point each,

$$Q(x,y)=Q_i \quad \text{for all } i=1,2,3,\ldots,q \tag{2.8}$$

in order to solve (2.2) in $\Omega\times[0,T]$ where $\Omega\in[0,\mathrm{L}]\times[0,\mathrm{M}]$.

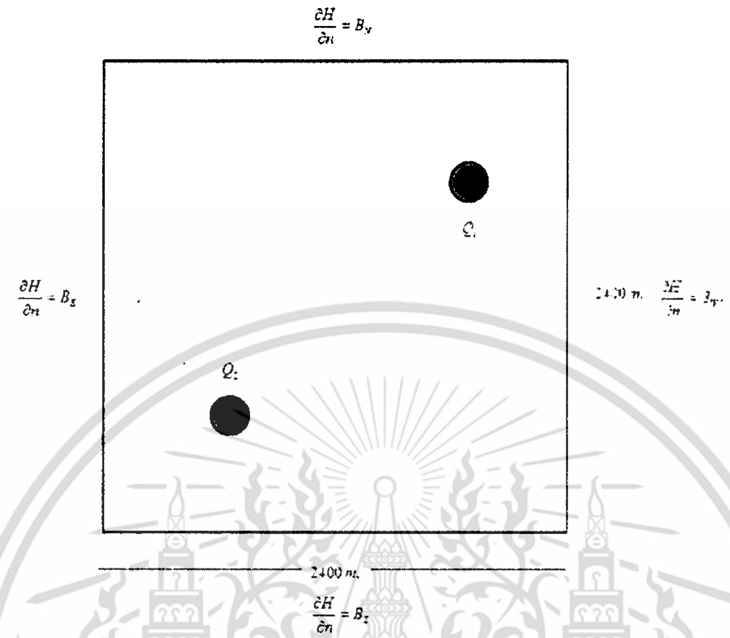

Figure 1: Boundary conditions of a transient groundwater flow model

From figure 1 the lengths of domain are $L \times M$, B_N, B_S, B_W and B_E are boundary conditions of model and Q_1, Q_2 are source and sinks term respectively.

CHAPTER III
Numerical Techniques

Numerical Techniques

We now discretize (2.2) by dividing the interval $[0,\mathrm{L}]$ in x-direction into I subintervals such that $I\Delta x = L$, the interval $[0,\mathrm{M}]$ in y-direction into J subintervals such that $J\Delta y = M$ and the interval $[0,T]$ in time into K subintervals such that $K\Delta t = T$. We can then approximate $H(x,y,t)$ by $H_{i,j}^{k}$, value of the difference approximation of $H(x,y,t)$ at point $x = i\Delta x$, $y = j\Delta y$ and $t = N\Delta t$, where $0 \le i \le I$, $0 \le j \le J$ and $0 \le n \le N$ which I,J and N are positive integers. In this paper, we used the finite difference methods for models such as the alternating direction explicit method (ADEM), the alternating direction implicit method (ADIM), the forward time central space method (FTCS) and the backward time central space method (BTCS).

3.1 Alternating direction explicit method (ADEM)

ADEM equation can be written as [11]

$$S_{i,j}\left(\frac{H_{i,j}^{n+1}-H_{i,j}^{n}}{\Delta t}\right)=K_{i+\frac{1}{2},j}\left(\frac{H_{i+1,j}^{n}-H_{i,j}^{n}}{(\Delta x)^{2}}\right)+K_{i-\frac{1}{2},j}\left(\frac{H_{i-1,j}^{n+1}-H_{i,j}^{n+1}}{(\Delta x)^{2}}\right)+K_{i,j+\frac{1}{2}}\left(\frac{H_{i,j+1}^{n}-H_{i,j}^{n}}{(\Delta y)^{2}}\right)+K_{i,j-\frac{1}{2}}\left(\frac{H_{i,j-1}^{n+1}-H_{i,j}^{n+1}}{(\Delta y)^{2}}\right)+W_{i,j}, \tag{3.1}$$

$$S_{i,j}\left(\frac{H_{i,j}^{n+1}-H_{i,j}^{n}}{\Delta t}\right)=K_{i-\frac{1}{2},j}\left(\frac{H_{i-1,j}^{n}-H_{i,j}^{n}}{(\Delta x)^{2}}\right)+K_{i+\frac{1}{2},j}\left(\frac{H_{i+1,j}^{n+1}-H_{i,j}^{n+1}}{(\Delta x)^{2}}\right)+K_{i,j-\frac{1}{2}}\left(\frac{H_{i,j-1}^{n}-H_{i,j}^{n}}{(\Delta y)^{2}}\right)+K_{i,j+\frac{1}{2}}\left(\frac{H_{i,j+1}^{n+1}-H_{i,j}^{n+1}}{(\Delta y)^{2}}\right)+W_{i,j} \tag{3.2}$$

We consider (3.1) and letting that

$$a_{i,j} = \frac{\Delta t}{S_{i,j}(\Delta x)^2} K_{i+\frac{1}{2},j},$$

$$b_{i,j} = \frac{\Delta t}{S_{i,j}(\Delta x)^2} K_{i-\frac{1}{2},j},$$

$$c_{i,j} = \frac{\Delta t}{S_{i,j}(\Delta y)^2} K_{i,j+\frac{1}{2}},$$

$$d_{i,j} = \frac{\Delta t}{S_{i,j}(\Delta y)^2} K_{i,j-\frac{1}{2}},$$

$$e_{i,j} = \frac{\Delta t}{S_{i,j}}, \tag{3.3}$$

for $i=1$ and $j=1$ at $t>0$, substituting the approximate unknown value of the bottom boundary and left boundary by

$H_{1,0}^{n+1} = H_{1,1}^{n+1} - (\Delta y) B_S$ and $H_{0,1}^{n+1} = H_{1,1}^{n+1} - (\Delta x) B_W$

respectively, and by rearranging, we obtain

$$H_{1,1}^{n+1} = a_{1,1}\left(H_{2,1}^{n} - H_{1,1}^{n}\right) - b_{1,1}(\Delta x) B_W + c_{1,1}\left(H_{1,2}^{n} - H_{1,1}^{n}\right) - d_{1,1}(\Delta y) B_S + H_{1,1}^{n} + e_{1,1} W_{1,1}, \tag{3.4}$$

for $i=1$ and $1<j<J-1$ at $t>0$, substituting the approximate unknown value of the left boundary by $H_{0,j}^{n+1} = H_{1,j}^{n+1} - (\Delta x) B_W$ and by rearranging, we obtain

$$H_{1,j}^{n+1} = \frac{a_{1,j}}{\left(1+d_{1,j}\right)}\left(H_{2,j}^{n} - H_{1,j}^{n}\right) - b_{1,j}(\Delta x) B_W + \frac{c_{1,j}}{\left(1+d_{1,j}\right)}\left(H_{1,j+1}^{n} - H_{1,j}^{n}\right) + \frac{d_{1,j}}{\left(1+d_{1,j}\right)} H_{1,j-1}^{n+1} + \frac{1}{\left(1+d_{1,j}\right)} H_{1,j}^{n} + \frac{e_{1,j}}{\left(1+d_{1,j}\right)} W_{1,j}, \tag{3.5}$$

for $1<i<I-1$ and $j=1$ at $t>0$, substituting the approximate unknown value of the bottom boundary by $H_{i,0}^{n+1} = H_{i,1}^{n+1} - (\Delta y) B_S$ and by rearranging, we obtain

$$H_{i,1}^{n+1}=\frac{a_{i,1}}{\left(1+b_{i,1}\right)}\left(H_{i+1,1}^{n}-H_{i,1}^{n}\right)+\frac{b_{i,1}}{\left(1+b_{i,1}\right)}H_{i-1,1}^{n+1}+\frac{c_{i,1}}{\left(1+b_{i,1}\right)}\left(H_{i,2}^{n}-H_{i,1}^{n}\right)$$
$$-\frac{d_{i,1}}{\left(1+b_{i,1}\right)}(\Delta y)B_S+\frac{1}{\left(1+b_{i,1}\right)}H_{i,1}^{n}+\frac{e_{i,1}}{\left(1+b_{i,1}\right)}W_{i,1}, \qquad (3.6)$$

for $1<i<I-1$ and $1<j<J-1$ at $t>0$, and by rearranging, we obtain

$$H_{i,j}^{n+1}=\frac{a_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(H_{i+1,j}^{n}-H_{i,j}^{n}\right)+\frac{b_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}H_{i-1,j}^{n+1}+\frac{c_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}\left(H_{i,j+1}^{n}-H_{i,j}^{n}\right)$$
$$+\frac{d_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}H_{i,j-1}^{n+1}+\frac{1}{\left(1+b_{i,j}+d_{i,j}\right)}H_{i,j}^{n}+\frac{e_{i,j}}{\left(1+b_{i,j}+d_{i,j}\right)}W_{i,j}.$$
$$(3.7)$$

In similary,we consider (3.2) and letting that,

$$aa_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i-\frac{1}{2},j},$$
$$bb_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i+\frac{1}{2},j},$$
$$cc_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i,j-\frac{1}{2}},$$
$$dd_{i,j}=\frac{\Delta t}{S_{i,j}\left(\Delta x\right)^2}K_{i,j+\frac{1}{2}},$$
$$ee_{i,j}=\frac{\Delta t}{S_{i,j}},$$
$$(3.8)$$

for $i=I-1$ and $j=J-1$ at $t>0$, substituting the approximate unknown values of the top boundary and right boundary by $H_{I-1,J}^{n+1}=H_{I-1,J-1}^{n+1}+(\Delta y)B_N$ and $H_{I,J-1}^{n+1}=H_{I-1,J-1}^{n+1}+(\Delta x)B_E$ respectively, and by rearranging, we obtain

$$H_{I-1,J-1}^{n+1}=aa_{I-1,J-1}\left(H_{I-2,J-1}^{n}-H_{I-1,J-1}^{n}\right)+bb_{I-1,J-1}\left(\Delta x\right)B_E+cc_{I-1,J-1}\left(H_{I-1,J-2}^{n}-H_{I-1,J-1}^{n}\right)$$
$$+dd_{I-1,J-1}\left(\Delta y\right)B_N+H_{I-1,J-1}^{n}+ee_{I-1,J-1}W_{I-1,J-1},$$
$$(3.9)$$

for $i=I-1$ and $1<j<J-1$ at $t>0$, substituting the approximate unknown value of the right boundary by $H_{I,j}^{n+1}=H_{I-1,j}^{n+1}+(\Delta x)B_E$ and by rearranging, we obtain

$$H_{I-1,j}^{n+1}=\frac{aa_{I-1,j}}{\left(1+dd_{I-1,j}\right)}\left(H_{I-2,j}^{n}-H_{I-1,j}^{n}\right)+bb_{I-1,j}\left(\Delta x\right)B_E+\frac{cc_{I-1,j}}{\left(1+dd_{I-1,j}\right)}\left(H_{I-1,j-1}^{n}-H_{I-1,j}^{n}\right)$$
$$+\frac{dd_{I-1,j}}{\left(1+dd_{I-1,j}\right)}H_{I-1,j+1}^{n+1}+\frac{1}{\left(1+dd_{I-1,j}\right)}H_{I-1,j}^{n}+\frac{ee_{I-1,j}}{\left(1+dd_{I-1,j}\right)}W_{I-1,j}, \tag{3.10}$$

for $1<i<I-1$ and $j=J-1$ at $t>0$, substituting the approximate unknown value of the top boundary by $H_{i,J}^{n+1}=H_{i,J-1}^{n+1}+\left(\Delta y\right)B_N$ and by rearranging, we obtain

$$H_{i,J-1}^{n+1}=\frac{aa_{i,J-1}}{\left(1+bb_{i,J-1}\right)}\left(H_{i-1,J-1}^{n}-H_{i,J-1}^{n}\right)+\frac{bb_{i,J-1}}{\left(1+bb_{i,J-1}\right)}H_{i+1,J-1}^{n+1}+\frac{cc_{i,J-1}}{\left(1+bb_{i,J-1}\right)}\left(H_{i,J}^{n}-H_{i,J-1}^{n}\right)$$
$$+dd_{i,J-1}\left(\Delta y\right)B_N+\frac{1}{\left(1+bb_{i,J-1}\right)}H_{i,J-1}^{n}+\frac{ee_{i,J-1}}{\left(1+bb_{i,J-1}\right)}W_{i,J-1}, \tag{3.11}$$

for $1<i<I-1$ and $1<j<J-1$ at $t>0$, and by rearranging, we obtain

$$H_{i,j}^{n+1}=\frac{aa_{i,j}}{\left(1+bb_{i,j}+dd_{i,j}\right)}\left(H_{i-1,j}^{n}-H_{i,j}^{n}\right)+\frac{cc_{i,j}}{\left(1+bb_{i,j}+dd_{i,j}\right)}\left(H_{i,j-1}^{n}-H_{i,j}^{n}\right)+\frac{bb_{i,j}}{\left(1+bb_{i,j}+dd_{i,j}\right)}H_{i+1,j}^{n+1}$$
$$+\frac{dd_{i,j}}{\left(1+bb_{i,j}+dd_{i,j}\right)}H_{i,j+1}^{n+1}+\frac{1}{\left(1+bb_{i,j}+dd_{i,j}\right)}H_{i,j}^{n}+\frac{ee_{i,j}}{\left(1+bb_{i,j}+dd_{i,j}\right)}W_{i,j}. \tag{3.12}$$

We will approximate (3.1) by calculation from left into right and from bottom into top, and approximate (3.2) by calculation from right into left and from top into bottom.

3.2Alternating direction implicit method (ADIM)

This method will divide into two parts. First, we using the central difference method with space and using forward difference method with time, we have

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H_{i-1,j}^{m+\frac{1}{2}} - 2H_{i,j}^{m+\frac{1}{2}} + H_{i+1,j}^{m+\frac{1}{2}}}{(\Delta x)^2},$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H_{i,j-1}^{m} - 2H_{i,j}^{m} + H_{i,j+1}^{m}}{(\Delta y)^2},$$

$$\frac{\partial h}{\partial t} \approx \frac{H_{i,j}^{m+\frac{1}{2}} - H_{i,j}^{m}}{\Delta t},$$

$$W_{i,j}^{m} = \pm \frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}},$$

(3.13)

Substituting (3.13) into (2.2), we get

$$S\left(\frac{H_{i,j}^{m+\frac{1}{2}} - H_{i,j}^{m}}{\Delta t}\right) = K\left[\left(\frac{H_{i-1,j}^{m+\frac{1}{2}} - 2H_{i,j}^{m+\frac{1}{2}} + H_{i+1,j}^{m+\frac{1}{2}}}{(\Delta x)^2}\right) + \left(\frac{H_{i,j-1}^{m} - 2H_{i,j}^{m} + H_{i,j+1}^{m}}{(\Delta y)^2}\right)\right] + \frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}},$$

(3.14)

Divide through by $\frac{S}{\Delta t}$, we will have

$$H_{i,j}^{m+\frac{1}{2}} - H_{i,j}^{m} = \frac{K(\Delta t)}{S(\Delta x)^2}\left(H_{i-1,j}^{m+\frac{1}{2}} - 2H_{i,j}^{m+\frac{1}{2}} + H_{i+1,j}^{m+\frac{1}{2}}\right) + \frac{K(\Delta t)}{S(\Delta y)^2}\left(H_{i,j-1}^{m} - 2H_{i,j}^{m} + H_{i,j+1}^{m}\right) + \frac{(\Delta t)}{S}\frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}},$$

(3.15)

Letting that

$$\alpha = \frac{(\Delta t)K}{(\Delta x)^2 S},$$
$$\beta = \frac{(\Delta t)K}{(\Delta y)^2 S},$$
$$\gamma = \frac{\Delta t}{S}, \tag{3.16}$$

for $i = 1$ and $1 < j < J-1$ at $t > 0$, substituting the approximate unknown value of the left boundary by $H_{0,j}^{m+\frac{1}{2}} = H_{1,j}^{m+\frac{1}{2}} - (\Delta x) B_W$ and by rearranging, we obtain

$$(1+\alpha) H_{1,j}^{m+\frac{1}{2}} - \alpha H_{2,j}^{m+\frac{1}{2}} = \beta H_{1,j-1}^{m} + (1-2\beta) H_{1,j}^{m} + \beta H_{1,j+1}^{m} + \gamma \frac{Q_{1,j}^{m}}{\Delta x \Delta y H_{1,j}^{m}} - \alpha (\Delta x) B_W, \tag{3.17}$$

for $1 < i < I-1$ and $1 < j < J-1$ at $t > 0$, and by rearranging, we obtain

$$-\alpha H_{i-1,j}^{m+\frac{1}{2}} + (1+2\alpha) H_{i,j}^{m+\frac{1}{2}} - \alpha H_{i+1,j}^{m+\frac{1}{2}} = \beta H_{i,j-1}^{m} + (1-2\beta) H_{i,j}^{m} + \beta H_{i,j+1}^{m} + \gamma \frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}}, \tag{3.18}$$

for $i = I-1$ and $1 < j < J-1$ at $t > 0$, substituting the approximate unknown value of the right boundary by $H_{I,j}^{m+\frac{1}{2}} = H_{I-1,j}^{m+\frac{1}{2}} + (\Delta x) B_E$ and by rearranging, we obtain

$$-\alpha H_{I-2,j}^{m+\frac{1}{2}} + (1+\alpha) H_{I-1,j}^{m+\frac{1}{2}} = \beta H_{I-1,j-1}^{m} + (1-2\beta) H_{I-1,j}^{m} + \beta H_{I-1,j+1}^{m} + \gamma \frac{Q_{I-1,j}^{m}}{\Delta x \Delta y H_{I-1,j}^{m}} + \alpha (\Delta x) B_E, \tag{3.19}$$

from (3.17) – (3.19), we write in matrix form, we will have

$$
\begin{bmatrix}
1+\alpha & -\alpha & & & \\
-\alpha & 1+2\alpha & -\alpha & & \\
 & \vdots & \vdots & \vdots & \\
 & & -\alpha & 1+2\alpha & -\alpha \\
 & & & -\alpha & 1+\alpha
\end{bmatrix}
\begin{bmatrix}
H_{1,j}^{m+\frac{1}{2}} \\
H_{1,j}^{m+\frac{1}{2}} \\
\vdots \\
H_{I-2,j}^{m+\frac{1}{2}} \\
H_{I-1,j}^{m+\frac{1}{2}}
\end{bmatrix}
=
\begin{bmatrix}
\beta H_{1,j-1}^{m} + (1-2\beta) H_{1,j}^{m} + \beta H_{1,j+1}^{m} + \gamma \dfrac{Q_{1,j}^{m}}{\Delta x \Delta y H_{1,j}^{m}} - \alpha (\Delta x) B_W \\
\beta H_{2,j-1}^{m} + (1-2\beta) H_{2,j}^{m} + \beta H_{2,j+1}^{m} + \gamma \dfrac{Q_{2,j}^{m}}{\Delta x \Delta y H_{2,j}^{m}} \\
\vdots \\
\beta H_{I-2,j-1}^{m} + (1-2\beta) H_{I-2,j}^{m} + \beta H_{I-2,j+1}^{m} + \gamma \dfrac{Q_{I-2,j}^{m}}{\Delta x \Delta y H_{I-2,j}^{m}} \\
\beta H_{I-1,j-1}^{m} + (1-2\beta) H_{I-1,j}^{m} + \beta H_{I-1,j+1}^{m} + \gamma \dfrac{Q_{I-1,j}^{m}}{\Delta x \Delta y H_{I-1,j}^{m}} + \alpha (\Delta x) B_E
\end{bmatrix},
\tag{3.20}
$$

for $j = 1,2,3,\cdots,J-1$

Next, we using the central difference method with space and using forward difference method with time, we have

$$
\begin{aligned}
\frac{\partial^2 H}{\partial x^2} &\approx \frac{H_{i-1,j}^{m+\frac{1}{2}} - 2H_{i,j}^{m+\frac{1}{2}} + H_{i+1,j}^{m+\frac{1}{2}}}{(\Delta x)^2}, \\
\frac{\partial^2 H}{\partial y^2} &\approx \frac{H_{i,j-1}^{m+1} - 2H_{i,j}^{m+1} + H_{i,j+1}^{m+1}}{(\Delta y)^2}, \\
\frac{\partial H}{\partial t} &\approx \frac{H_{i,j}^{m+1} - H_{i,j}^{m+\frac{1}{2}}}{\Delta t}, \\
W_{i,j}^{m+\frac{1}{2}} &= \pm \frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m+\frac{1}{2}}},
\end{aligned}
\tag{3.21}
$$

Substituting (3.21) into (2.2), we get

$$S\left(\frac{H_{i,j}^{m+1}-H_{i,j}^{m+\frac{1}{2}}}{\Delta t}\right)=K\left[\left(\frac{H_{i-1,j}^{m+\frac{1}{2}}-2H_{i,j}^{m+\frac{1}{2}}+H_{i+1,j}^{m+\frac{1}{2}}}{(\Delta x)^2}\right)+\left(\frac{H_{i,j-1}^{m+1}-2H_{i,j}^{m+1}+H_{i,j+1}^{m+1}}{(\Delta y)^2}\right)\right]+\frac{Q_{i,j}^{m}}{\Delta x\Delta yH_{i,j}^{m+\frac{1}{2}}}, \tag{3.22}$$

Divide through by $\frac{S}{\Delta t}$, we will have

$$H_{i,j}^{m+1}-H_{i,j}^{m+\frac{1}{2}}=\frac{K(\Delta t)}{S(\Delta x)^2}\left(H_{i-1,j}^{m+\frac{1}{2}}-2H_{i,j}^{m+\frac{1}{2}}+H_{i+1,j}^{m+\frac{1}{2}}\right)+\frac{K(\Delta t)}{S(\Delta y)^2}\left(H_{i,j-1}^{m+1}-2H_{i,j}^{m+1}+H_{i,j+1}^{m+1}\right)+\frac{(\Delta t)}{S}\frac{Q_{i,j}^{m}}{\Delta x\Delta yH_{i,j}^{m+\frac{1}{2}}}, \tag{3.23}$$

for $1<i<I-1$ and $j=1$ at $t>0$, substituting the approximate unknown value of the bottom boundary by $H_{i,0}^{m+1}=H_{i,1}^{m+1}-(\Delta y)B_S$ and by rearranging, we obtain

$$(1+\beta)H_{i,1}^{m+1}-\beta H_{i,2}^{m+1}=\alpha H_{i-1,1}^{m+\frac{1}{2}}+(1-2\alpha)H_{i,1}^{m+\frac{1}{2}}+\alpha H_{i+1,1}^{m+\frac{1}{2}}+\gamma\frac{Q_{i,1}^{m}}{\Delta x\Delta yH_{i,1}^{m+\frac{1}{2}}}-\beta(\Delta y)B_S, \tag{3.24}$$

for $1<i<I-1$ and $1<j<J-1$ at $t>0$, and by rearranging, we obtain

$$-\beta H_{i,j-1}^{m+1}+(1+2\beta)H_{i,j}^{m+1}-\beta H_{i,j+1}^{m+1}=\alpha H_{i-1,j}^{m+\frac{1}{2}}+(1-2\alpha)H_{i,j}^{m+\frac{1}{2}}+\alpha H_{i+1,j}^{m+\frac{1}{2}}+\gamma\frac{Q_{i,j}^{m}}{\Delta x\Delta yH_{i,j}^{m+\frac{1}{2}}}, \tag{3.25}$$

for $1<i<I-1$ and $j=J-1$ at $t>0$, substituting the approximate unknown value of the top boundary by $H_{i,J}^{m+1}=H_{i,J-1}^{m+1}+(\Delta y)B_N$ and by rearranging, we obtain

$$-\beta H_{i,J-2}^{m+1}+(1+\beta)H_{i,J-1}^{m+1}=\alpha H_{i-1,J-1}^{m+\frac{1}{2}}+(1-2\alpha)H_{i,J-1}^{m+\frac{1}{2}}+\alpha H_{i+1,J-1}^{m+\frac{1}{2}}+\gamma\frac{Q_{i,J-1}^{m}}{\Delta x\Delta yH_{i,J-1}^{m+\frac{1}{2}}}+\beta(\Delta y)B_N, \tag{3.26}$$

from (3.24) – (3.25), we write in matrix form, we will have

$$\begin{bmatrix} 1+\beta & -\beta & & & \\ -\beta & 1+2\beta & -\beta & & \\ & \vdots & \vdots & \vdots & \\ & & -\beta & 1+2\beta & -\beta \\ & & & -\beta & 1+\beta \end{bmatrix} \begin{bmatrix} H_{i,1}^{m+1} \\ H_{i,2}^{m+1} \\ \vdots \\ H_{i,J-2}^{m+1} \\ H_{i,J-1}^{m+1} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha H_{i-1,1}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,1}^{m+\frac{1}{2}} + \alpha H_{i+1,1}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,1}^{m}}{\Delta x \Delta y H_{i,1}^{m+\frac{1}{2}}} - \beta(\Delta y) B_S \\ \alpha H_{i-1,2}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,2}^{m+\frac{1}{2}} + \alpha H_{i+1,2}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,2}^{m}}{\Delta x \Delta y H_{i,2}^{m+\frac{1}{2}}} \\ \vdots \\ \alpha H_{i-1,J-2}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,J-2}^{m+\frac{1}{2}} + \alpha H_{i+1,J-2}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,J-2}^{m}}{\Delta x \Delta y H_{i,J-2}^{m+\frac{1}{2}}} \\ \alpha H_{i-1,J-1}^{m+\frac{1}{2}} + (1-2\alpha) H_{i,J-1}^{m+\frac{1}{2}} + \alpha H_{i+1,J-1}^{m+\frac{1}{2}} + \gamma \dfrac{Q_{i,J-1}^{m}}{\Delta x \Delta y H_{i,J-1}^{m+\frac{1}{2}}} + \beta(\Delta y) B_N \end{bmatrix}, \tag{3.27}$$

for $i = 1, 2, 3, \cdots, I-1$

3.3 Forward time central space explicit method (FTCS)

Using the central difference method with space and using forward difference method with time on equation (2.2), we have

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H_{i-1,j}^{m} - 2H_{i,j}^{m} + H_{i+1,j}^{m}}{(\Delta x)^2},$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H_{i,j-1}^{m} - 2H_{i,j}^{m} + H_{i,j+1}^{m}}{(\Delta y)^2},$$

$$\frac{\partial H}{\partial t} \approx \frac{H_{i,j}^{m+1} - H_{i,j}^{m}}{\Delta t},$$

$$W_{i,j}^{k} = \pm \frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}}, \tag{3.28}$$

Substituting (3.28) into (2.2), we get

$$S\left(\frac{H_{i,j}^{m+1} - H_{i,j}^{m}}{\Delta t}\right) = K\left[\left(\frac{H_{i-1,j}^{m} - 2H_{i,j}^{m} + H_{i+1,j}^{m}}{(\Delta x)^2}\right) + \left(\frac{H_{i,j-1}^{m} - 2H_{i,j}^{m} + H_{i,j+1}^{m}}{(\Delta y)^2}\right)\right] + \frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}}, \tag{3.29}$$

letting

$$\xi = \frac{(\Delta t)K}{(\Delta x)^2 S},$$

$$\eta = \frac{(\Delta t)K}{(\Delta y)^2 S},$$

$$\omega = \frac{\Delta t}{S}, \tag{3.30}$$

Divide through by $\frac{S}{\Delta t}$, we will have

$$H_{i,j}^{m+1} - H_{i,j}^{m} = \frac{K(\Delta t)}{S(\Delta x)^2}\left(H_{i-1,j}^{m} - 2H_{i,j}^{m} + H_{i+1,j}^{m}\right) + \frac{K(\Delta t)}{S(\Delta y)^2}\left(H_{i,j-1}^{m} - 2H_{i,j}^{m} + H_{i,j+1}^{m}\right) + \frac{(\Delta t)}{S}\frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}}, \tag{3.31}$$

for $1 < i < I-1$ and $1 < j < J-1$ at $t > 0$, and by rearranging, we obtain

$$H_{i,j}^{m+1} = H_{i,j}^{m} + \xi\left(H_{i+1,j}^{m} - 2H_{i,j}^{m} + H_{i-1,j}^{m}\right) + \eta\left(H_{i,j+1}^{m} - 2H_{i,j}^{m} + H_{i,j-1}^{m}\right) + \omega\left(\frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}}\right), \tag{3.32}$$

3.4 Backward time central space implicit method (BTCS)

Using the central difference method with space and using forward difference method with time on equation (2.2), we can obtain

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H_{i-1,j}^{m+1} - 2H_{i,j}^{m+1} + H_{i+1,j}^{m+1}}{(\Delta x)^2},$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H_{i,j-1}^{m+1} - 2H_{i,j}^{m+1} + H_{i,j+1}^{m+1}}{(\Delta y)^2},$$

$$\frac{\partial H}{\partial t} \approx \frac{H_{i,j}^{m+1} - H_{i,j}^{m}}{\Delta t},$$

$$W_{i,j}^{k} = \pm \frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}},$$

(3.33)

Substituting (3.33) into (2.2), we get

$$S\left(\frac{H_{i,j}^{m+1} - H_{i,j}^{m}}{\Delta t}\right) = K\left[\left(\frac{H_{i-1,j}^{m+1} - 2H_{i,j}^{m+1} + H_{i+1,j}^{m+1}}{(\Delta x)^2}\right) + \left(\frac{H_{i,j-1}^{m+1} - 2H_{i,j}^{m+1} + H_{i,j+1}^{m+1}}{(\Delta y)^2}\right)\right] + \frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}},$$

(3.34)

Divide through by $\frac{S}{\Delta t}$, we will have

$$H_{i,j}^{m+1} - H_{i,j}^{m} = \frac{K(\Delta t)}{S(\Delta x)^2}\left(H_{i-1,j}^{m+1} - 2H_{i,j}^{m+1} + H_{i+1,j}^{m+1}\right) + \frac{K(\Delta t)}{S(\Delta y)^2}\left(H_{i,j-1}^{m+1} - 2H_{i,j}^{m+1} + H_{i,j+1}^{m+1}\right) + \frac{(\Delta t)}{S}\frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}},$$

(3.35)

letting

$$\lambda = \frac{(\Delta t)K}{(\Delta x)^2 S},$$

$$\kappa = \frac{(\Delta t)K}{(\Delta y)^2 S},$$

$$\rho = \frac{\Delta t}{S},$$

(3.36)

for $1 < i < I-1$ and $1 < j < J-1$ at $t > 0$, and by rearranging, we obtain

$$-H_{i,j}^{m} - c\frac{Q_{i,j}^{m}}{\Delta x \Delta y H_{i,j}^{m}} = aH_{i-1,j}^{m+1} + aH_{i+1,j}^{m+1} + bH_{i,j-1}^{m+1} + bH_{i,j+1}^{m+1} - (1 + 2a + 2c)H_{i,j}^{m+1},$$

(3.37)

CHAPTER IV
Numerical Experiment

4.1Numerical Experiment

A transient groundwater flow model provide hydraulic head profile. The application of the numerical simulations of a transient groundwater flow model is tested using the hypothetical examples. The hypothetical example has homogeneous aquifer parameters as grid spacing, $\Delta x = \Delta y = 50\,m.$, number of grid spacing, $I = J = 49$, hydraulic conductivity, $K = 15\,m/day$, storage capacity, $S = 1\,m^{-1}$ and time step, $\Delta t = 1\,day$

Example1. We consider pump wells which pump water up from the groundwater only. Given boundary conditions and pumping well as;

Table 1: Boundary conditions and pumping well of Example 1

B_N	B_S	B_E	B_W	Q_1	Q_2	Q_3	Q_4
-0.005	0.005	-0.005	0.005	$-216\,m^3/day$	$-324\,m^3/day$	$-432\,m^3/day$	$-540\,m^3/day$

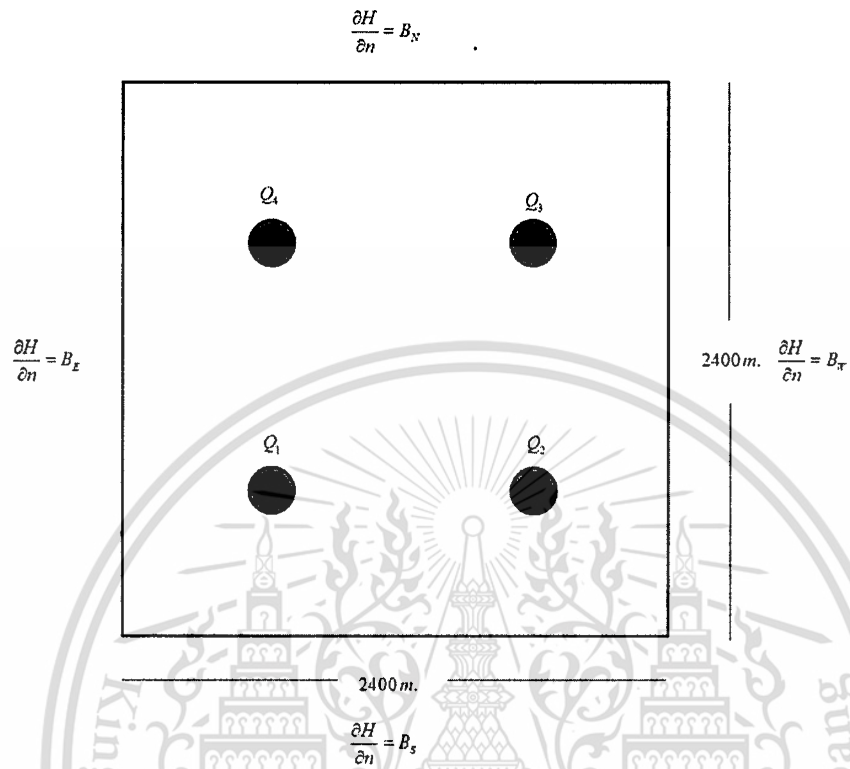

Figure2: Domain and boundary of Example 1

Table 2: Hydraulic Head (metre) $H(x, y)$ of FTCS technique, $\Delta x = \Delta y = 50\,m.$ and $\Delta t = 1\ day$

x\y	400	600	800	1000	1200	1400	1600	1800	2000	2200
400	14.7417	14.7728	14.8365	14.8790	14.8884	14.8692	14.8054	14.7187	14.7105	14.5694
600	14.7728	14.3846	14.8676	14.9452	14.9591	14.9314	14.8137	14.0848	14.7187	14.6313
800	14.8364	14.8674	14.9311	14.9737	14.9831	14.9639	14.9002	14.8135	14.8053	14.6642
1000	14.8780	14.9440	14.9728	14.9889	14.9930	14.9851	14.9631	14.9302	14.8682	14.6860
1200	14.8831	14.9523	14.9779	14.9908	14.9949	14.9908	14.9781	14.9525	14.8833	14.6920
1400	14.8475	14.9010	14.9424	14.9764	14.9886	14.9804	14.9529	14.9159	14.8581	14.6816
1600	14.7395	14.6989	14.8345	14.9416	14.9729	14.9520	14.8679	14.7574	14.7731	14.6529
1800	14.6037	13.4422	14.6988	14.8999	14.9457	14.9147	14.7573	13.7712	14.6624	14.6155
2000	14.6443	14.6037	14.7393	14.8466	14.8781	14.8571	14.7729	14.6624	14.6781	14.5581
2200	14.5463	14.5991	14.6412	14.6765	14.6894	14.6811	14.6529	14.6155	14.5581	14.3822

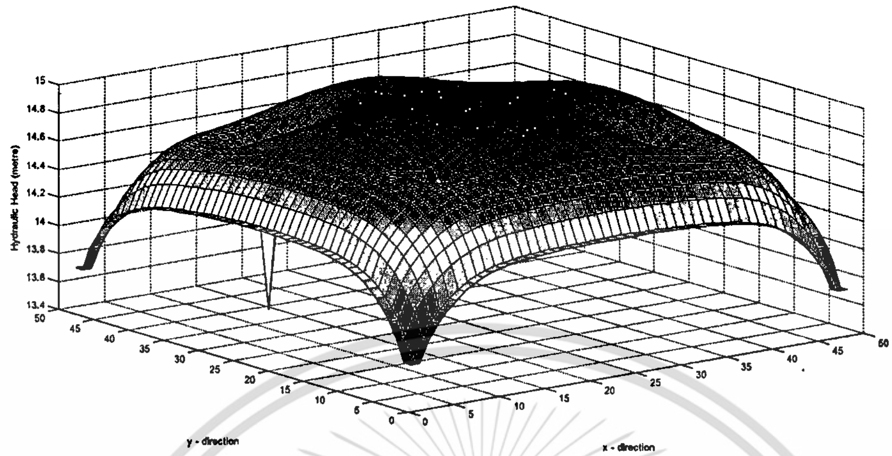

Figure3: The surface graph of hydraulic head (FTCS technique)

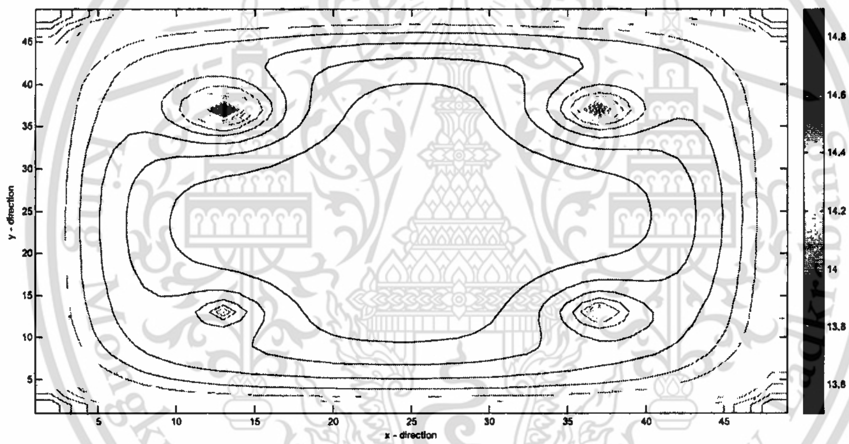

Figure4: The contour graph of Hydraulic Head (FTCS technique)

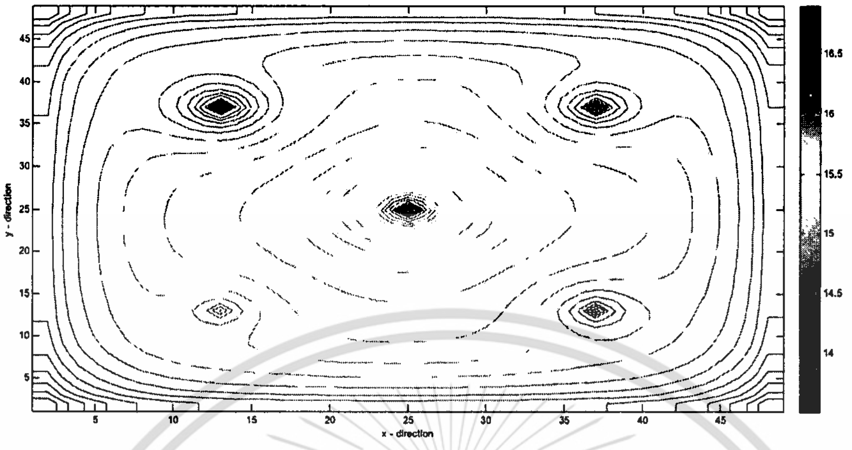

Figure7: The contour graph of Hydraulic Head (FTCS technique)

The approximation of the hydraulic head from the FTCS technique is shown in Tables 4. Three-dimensional representation of hydraulic head can be seen in Fig.6. This figure shows the level of hydraulic heads over the solution domain. Each contour of Fig.7 indicates the various levels of hydraulic heads. The contours of FTCS numerical model can be seen in Fig.6-7.

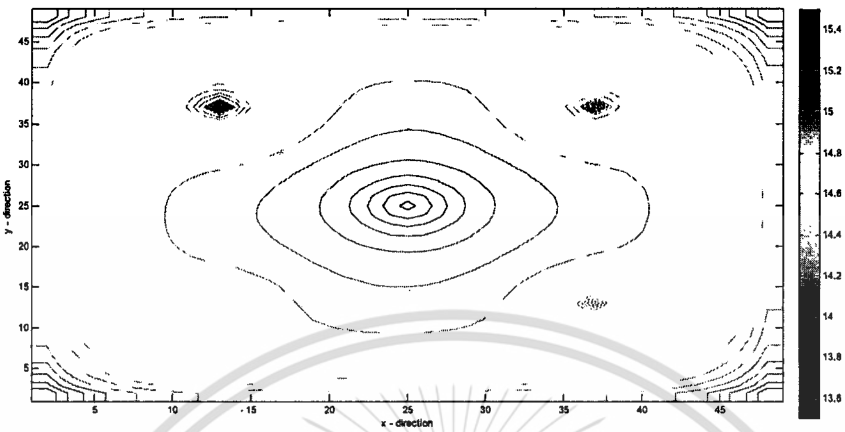

Figure10: The contour graph of Hydraulic Head (FTCS technique)

The approximation of the hydraulic head from the FTCS technique is shown in Tables 6. Three-dimensional representation of hydraulic head can be seen in Fig.9. This figure shows the level of hydraulic heads over the solution domain. Each contour of Fig.10 indicates the various levels of hydraulic heads. The contours of FTCS numerical model can be seen in Fig.9-10.

Table 7: Hydraulic Head (meter) $H(\mathrm{x},\mathrm{y})$ of BTCS technique, $\Delta x = \Delta y = 50\,m.$ and $\Delta t = 1\ day$

x\y	400	600	800	1000	1200	1400	1600	1800	2000	2200
400	14.7417	14.7730	14.8373	14.8807	14.8905	14.8709	14.8063	14.7189	14.7105	14.5693
600	14.7730	14.3860	14.8719	14.9544	14.9711	14.9406	14.8180	14.0861	14.7189	14.6312
800	14.8372	14.8717	14.9465	15.0105	15.0346	15.0008	14.9156	14.8179	14.8061	14.6642
1000	14.8797	14.9532	15.0097	15.0982	15.1820	15.0944	15.0000	14.9394	14.8699	14.6861
1200	14.8853	14.9643	15.0294	15.1797	15.5399	15.1798	15.0296	14.9645	14.8854	14.6923
1400	14.8492	14.9103	14.9793	15.0857	15.1776	15.0897	14.9898	14.9251	14.8598	14.6817
1600	14.7403	14.7033	14.8499	14.9785	15.0244	14.9889	14.8833	14.7618	14.7739	14.6530
1800	14.6039	13.4436	14.7031	14.9091	14.9577	14.9239	14.7616	13.7727	14.6627	14.6154
2000	14.6443	14.6039	14.7402	14.8483	14.8802	14.8588	14.7738	14.6626	14.6781	14.5579
2200	14.5462	14.5990	14.6412	14.6766	14.6896	14.6812	14.6529	14.6154	14.5579	14.3821

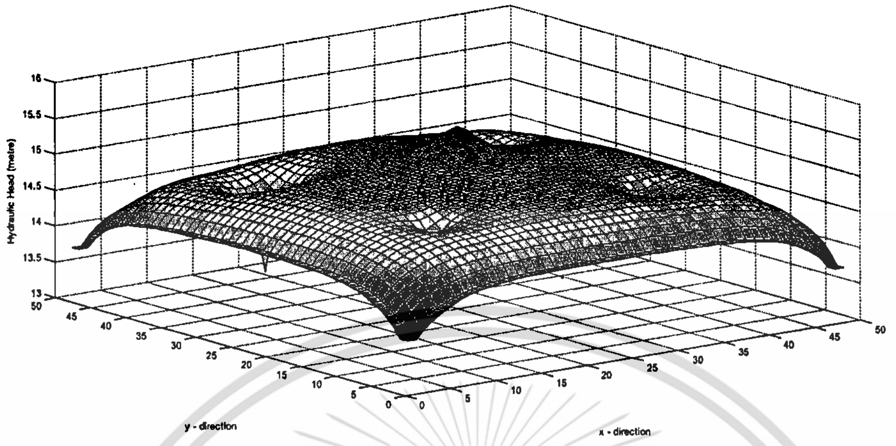

Figure11: The surface graph of hydraulic head (BTCS technique)

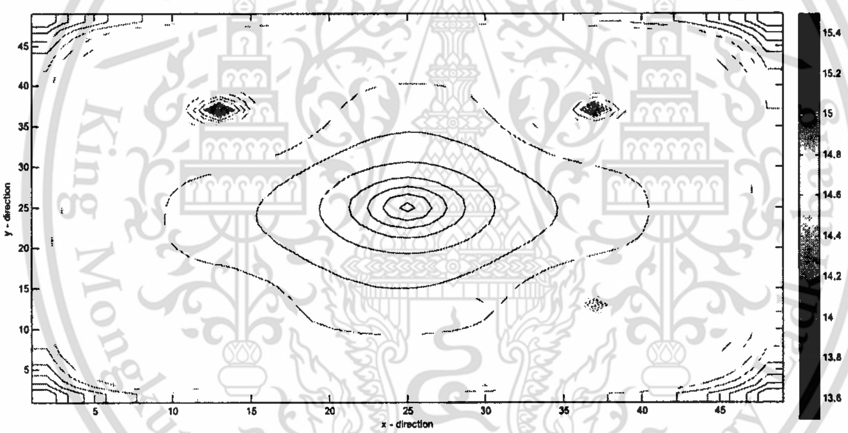

Figure12: The contour graph of Hydraulic Head (BTCS technique)

The approximation of the hydraulic head from the BTCS technique is shown in Tables 7. Three-dimensional representation of hydraulic head can be seen in Fig.11. This figure shows the level of hydraulic heads over the solution domain. Each contour of Fig.12 indicates the various levels of hydraulic heads. The contours of BTCS numerical model can be seen in Fig.11-12.

Table 8: Hydraulic Head (metre) $H(\mathrm{x},\mathrm{y})$ of ADEM technique, $\Delta x = \Delta y = 50\,m.$ and $\Delta t = 1\ day$

x\y	400	600	800	1000	1200	1400	1600	1800	2000	2200
400	14.3603	14.4655	14.5900	14.6701	14.6884	14.6476	14.5355	14.3827	14.3031	14.1040
600	14.4649	14.1472	14.7060	14.8319	14.8595	14.8042	14.6262	13.8101	14.3821	14.2378
800	14.5875	14.7038	14.8517	14.9656	15.0042	14.9444	14.7997	14.6241	14.5331	14.3295
1000	14.6617	14.8230	14.9593	15.1055	15.2085	15.0956	14.9384	14.7958	14.6396	14.3879
1200	14.6651	14.8333	14.9829	15.1973	15.4841	15.1978	14.9838	14.8344	14.6661	14.4032
1400	14.5908	14.7354	14.8902	15.0660	15.1866	15.0771	14.9137	14.7662	14.6158	14.3728
1600	14.4156	14.4508	14.6831	14.8842	14.9625	14.9075	14.7405	14.5391	14.4757	14.2988
1800	14.2035	13.0772	14.4486	14.7270	14.8082	14.7574	14.5369	13.4534	14.2950	14.1985
2000	14.1798	14.2030	14.4132	14.5828	14.6428	14.6075	14.4732	14.2945	14.2428	14.0709
2200	14.0361	14.1567	14.2644	14.3502	14.3858	14.3660	14.2966	14.1979	14.0708	13.8381

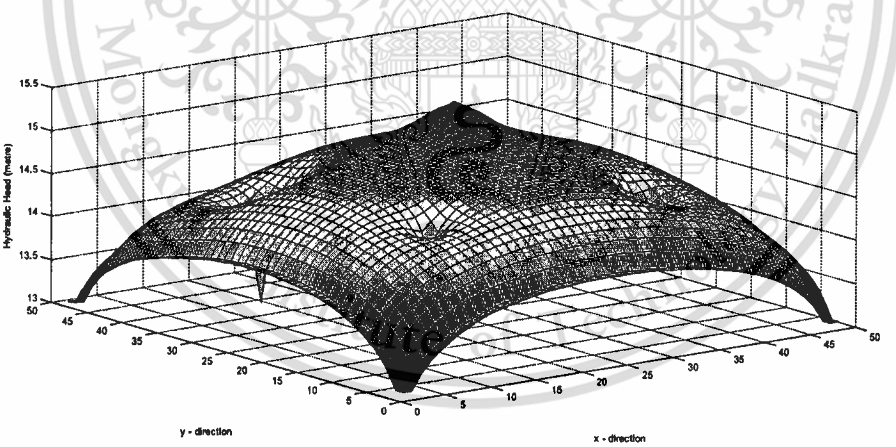

Figure13: The surface graph of hydraulic head (ADEM technique)

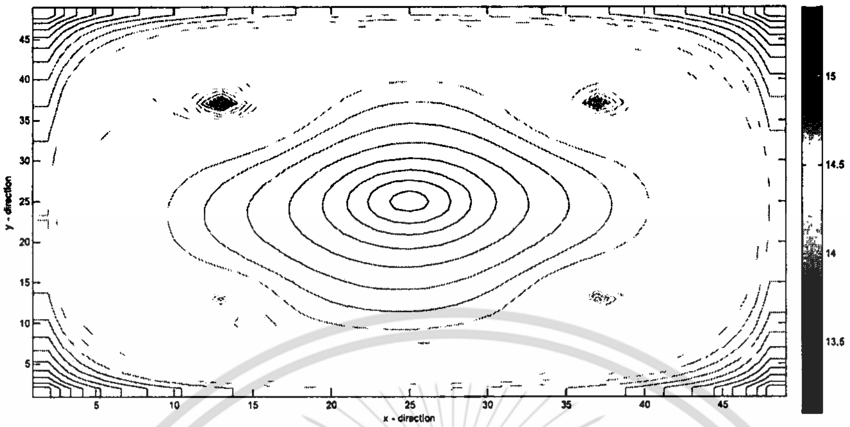

Figure14: The contour graph of Hydraulic Head (ADEM technique)

The approximation of the hydraulic head from the ADEM technique is shown in Tables 8. Three-dimensional representation of hydraulic head can be seen in Fig.13. This figure shows the level of hydraulic heads over the solution domain. Each contour of Fig.14 indicates the various levels of hydraulic heads. The contours of ADEM numerical model can be seen in Fig.13-14.

Table 9: Hydraulic Head (meter) $H(x, y)$ of ADIM technique, $\Delta x = \Delta y = 50\,m.$ and $\Delta t = 1\ day$

x\y	400	600	800	1000	1200	1400	1600	1800	2000	2200
400	14.7417	14.7735	14.8379	14.8812	14.8911	14.8715	14.8068	14.7194	14.7105	14.5682
600	14.7726	14.3860	14.8720	14.9545	14.9713	14.9408	14.8181	14.0861	14.7185	14.6298
800	14.8367	14.8716	14.9465	15.0106	15.0346	15.0009	14.9156	14.8178	14.8056	14.6626
1000	14.8792	14.9531	15.0096	15.0982	15.1820	15.0944	15.0000	14.9393	14.8694	14.6845
1200	14.8848	14.9642	15.0294	15.1797	15.5392	15.1798	15.0296	14.9644	14.8849	14.6906
1400	14.8487	14.9101	14.9793	15.0857	15.1776	15.0897	14.9898	14.9250	14.8593	14.6801
1600	14.7398	14.7031	14.8499	14.9785	15.0244	14.9889	14.8833	14.7617	14.7734	14.6514
1800	14.6035	13.4436	14.7032	14.9092	14.9578	14.9241	14.7617	13.7727	14.6623	14.6139
2000	14.6443	14.6043	14.7407	14.8489	14.8808	14.8594	14.7743	14.6631	14.6781	14.5569
2200	14.5473	14.6006	14.6429	14.6784	14.6914	14.6829	14.6546	14.6170	14.5591	14.3822

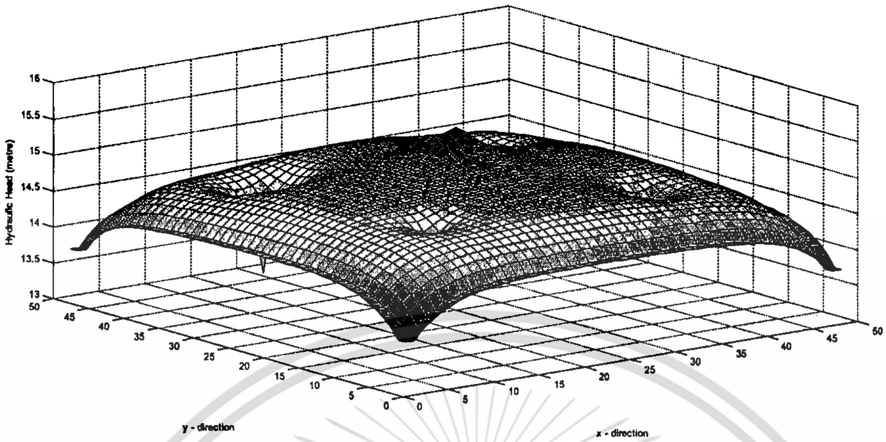

Figure15: The surface graph of hydraulic head (ADIM technique)

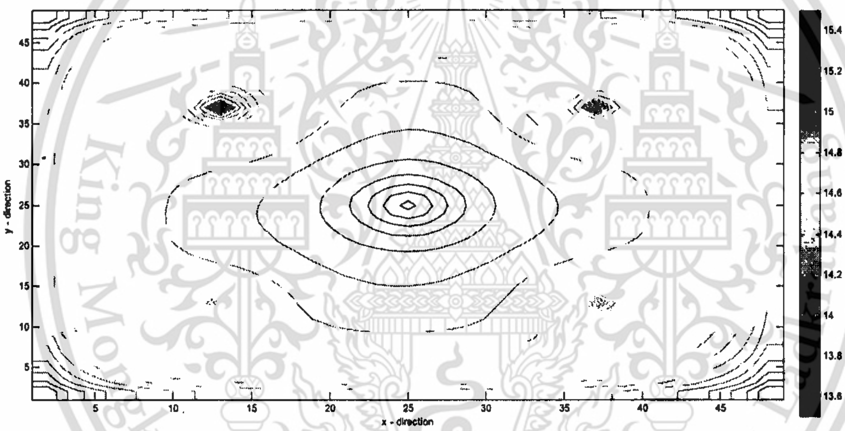

Figure16: The contour graph of Hydraulic Head (ADIM technique)

The approximation of the hydraulic head from the ADIM technique is shown in Tables 8. Three-dimensional representation of hydraulic head can be seen in Fig.15. This figure shows the level of hydraulic heads over the solution domain. Each contour of Fig.16 indicates the various levels of hydraulic heads. The contours of ADIM numerical model can be seen in Fig.15-16.

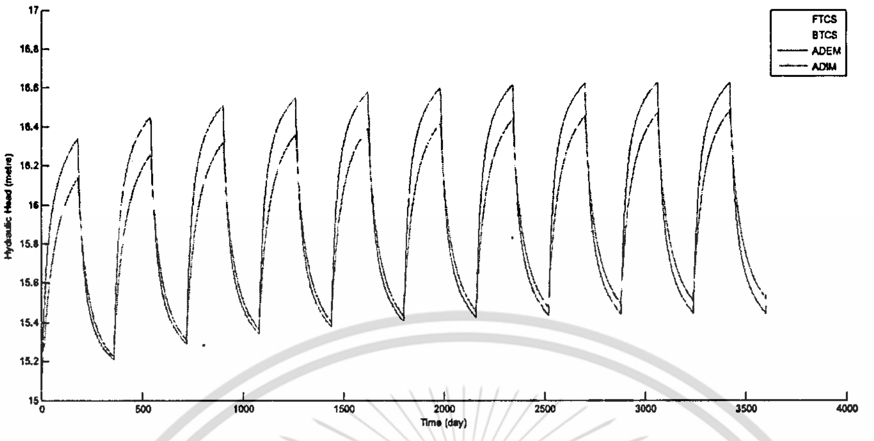

Figure 17: The variation of hydraulic head at $x = 1200\,m.$ and $y = 1200\,m.$ (FTCS technique)

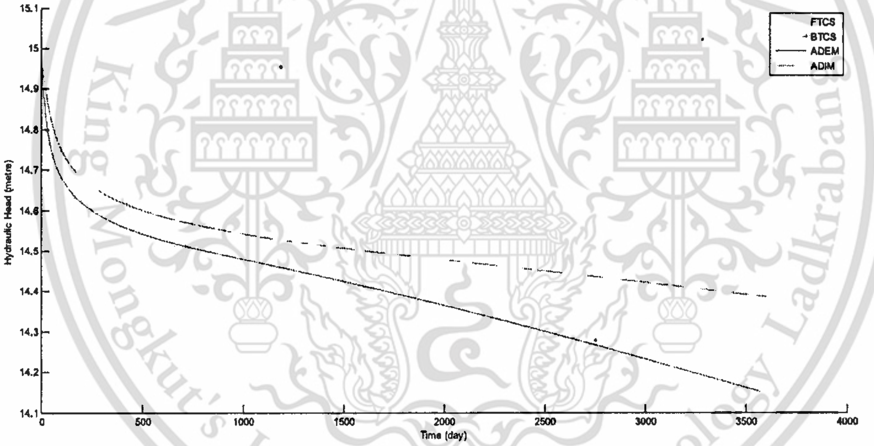

Figure 18: The variation of hydraulic head at $x = 600\,m.$ and $y = 600\,m.$ (FTCS technique)

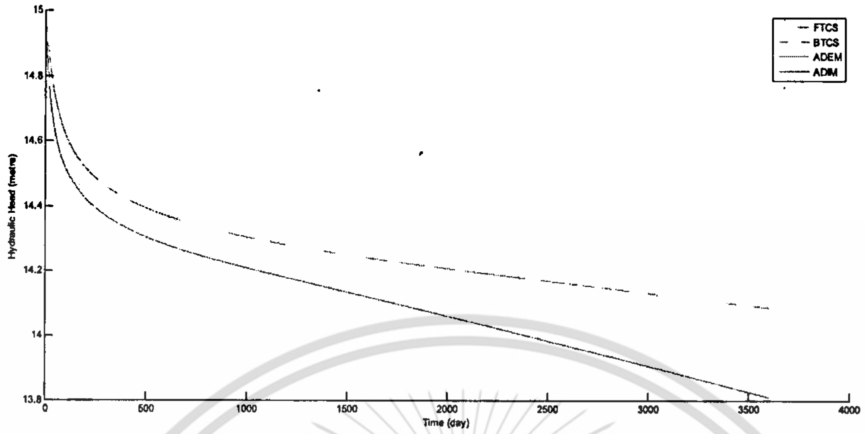

Figure 19: The variation of hydraulic head at $x = 1800\,m.$ and $y = 600\,m.$ (FTCS technique)

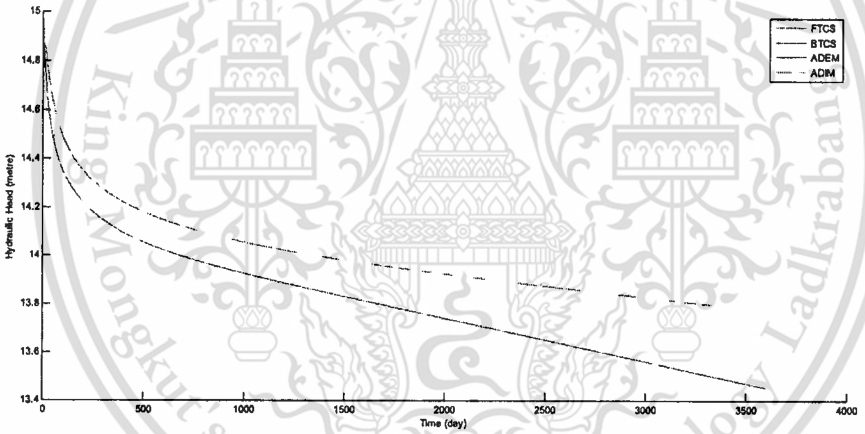

Figure 20: The variation of hydraulic head at $x = 1800\,m.$ and $y = 1800\,m.$ (FTCS technique)

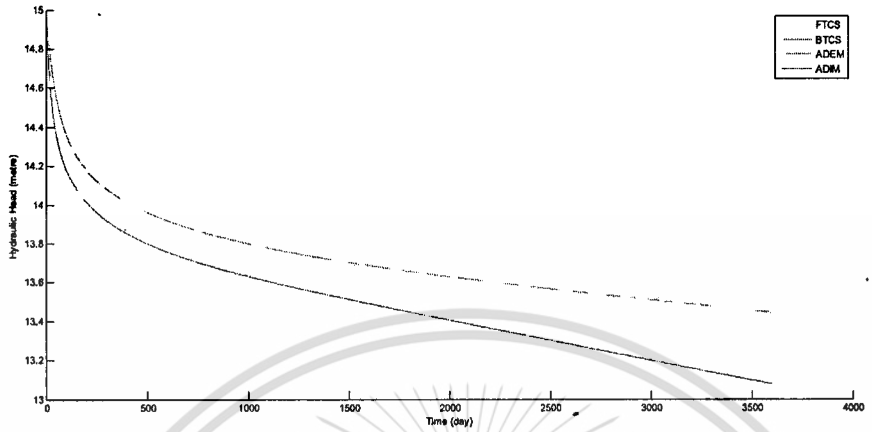

Figure 21: The variation of hydraulic head at $x = 600\,m.$ and $y = 1800\,m.$ (FTCS technique)

Variation of groundwater table at any node becomes steady-state as the time increases. Fig.17-21. Show the variation of groundwater table where the time indicator is changed from 1to3600.

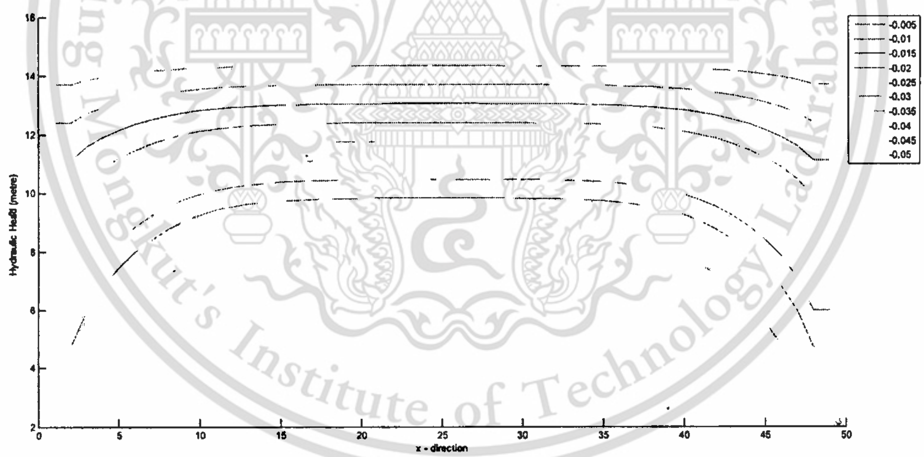

Figure 22: The cross-sectional line of hydraulic head at $y = 0\,m.$ if derivatives of boundary conditions are varied.

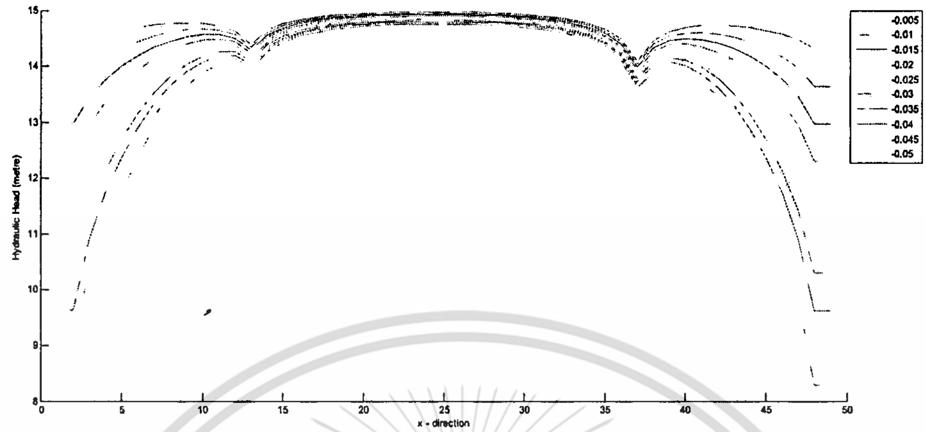

Figure 23: The cross-sectional line of hydraulic head at $y = 600\,m.$ if derivatives of boundary conditions are varied.

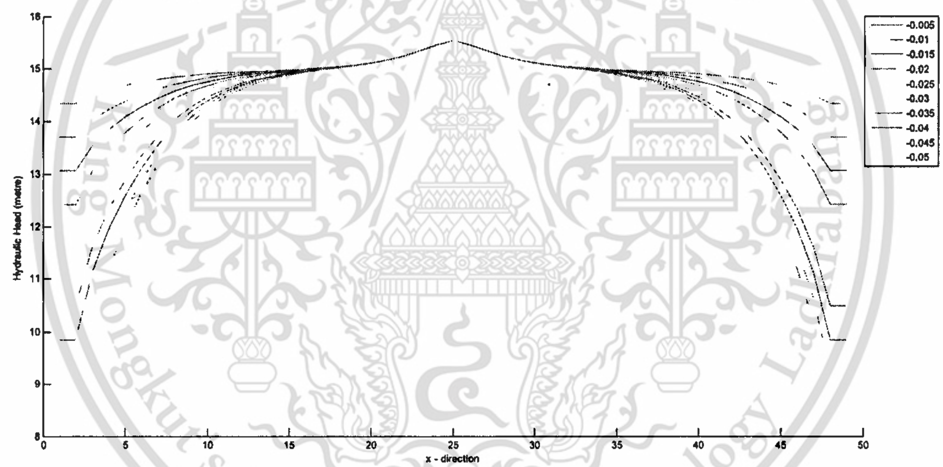

Figure 24: The cross-sectional line of hydraulic head at $y = 1200\,m.$ if derivatives of boundary conditions are varied

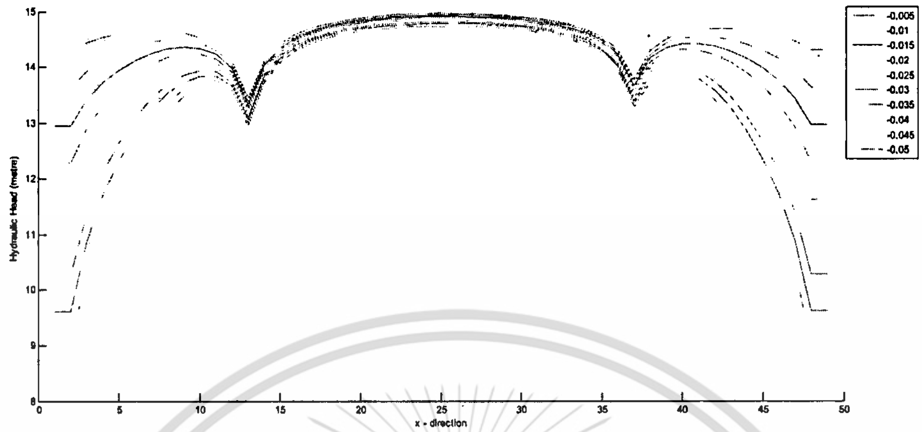

Figure 25: The cross-sectional line of hydraulic head at $y = 1800\,m.$ if derivatives of boundary conditions are varied.

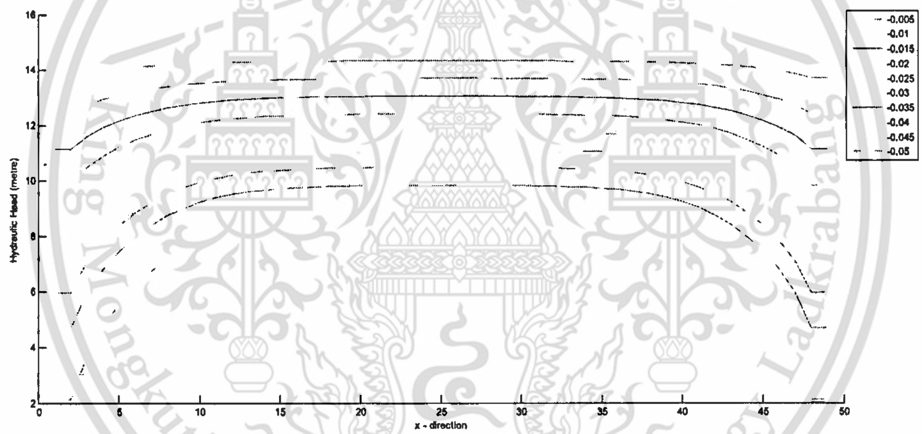

Figure 26: The cross-sectional line of hydraulic head at $y = 2400\,m.$ if derivatives of boundary conditions are varied.

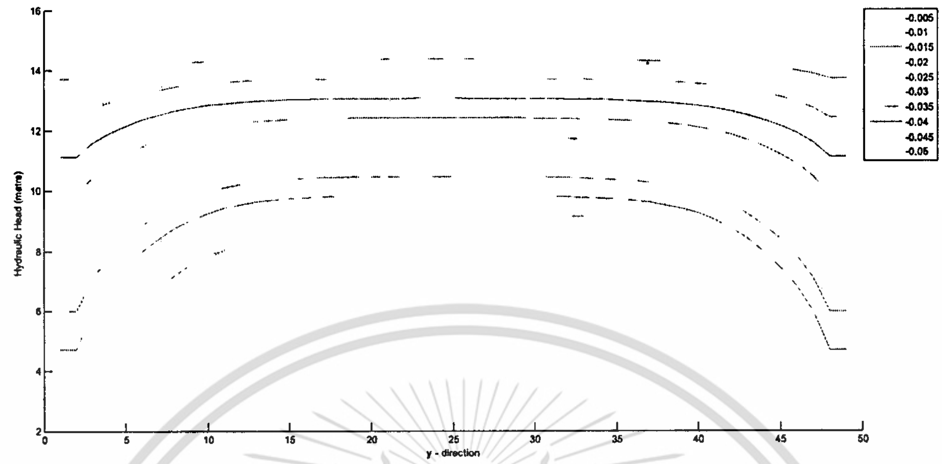

Figure 27: The cross-sectional line of hydraulichead at $x = 0\,m.$ if derivatives of boundary conditions are varied.

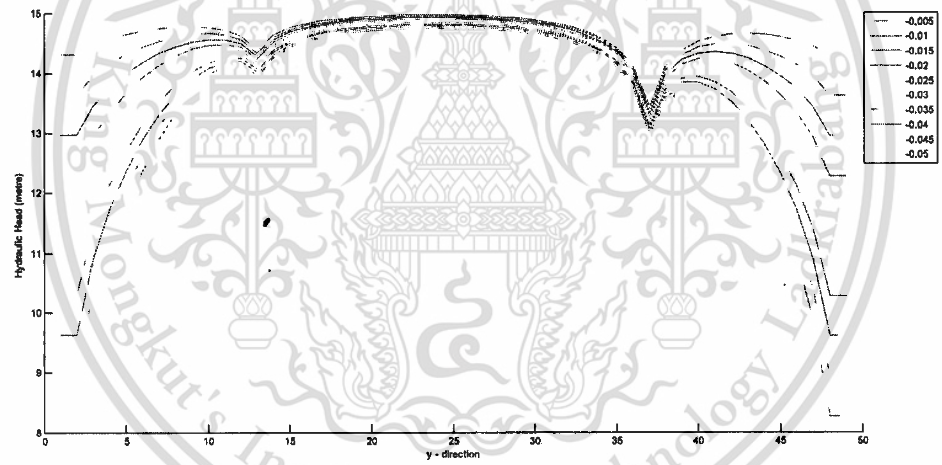

Figure 28: The cross-section Line of hydraulic head at $x = 600\,m.$ if derivatives of boundary conditions are varied.

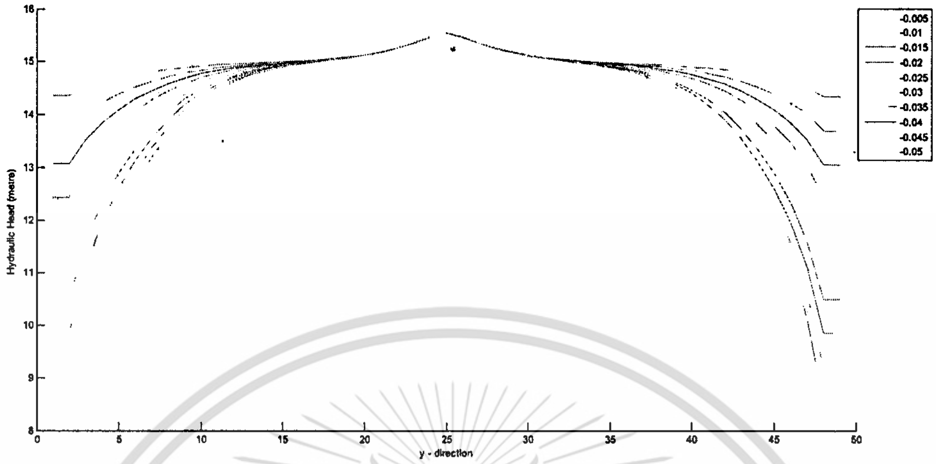

Figure 29: The cross-section line of hydraulic head at $x = 1200m.$ if derivatives of boundary conditions are varied

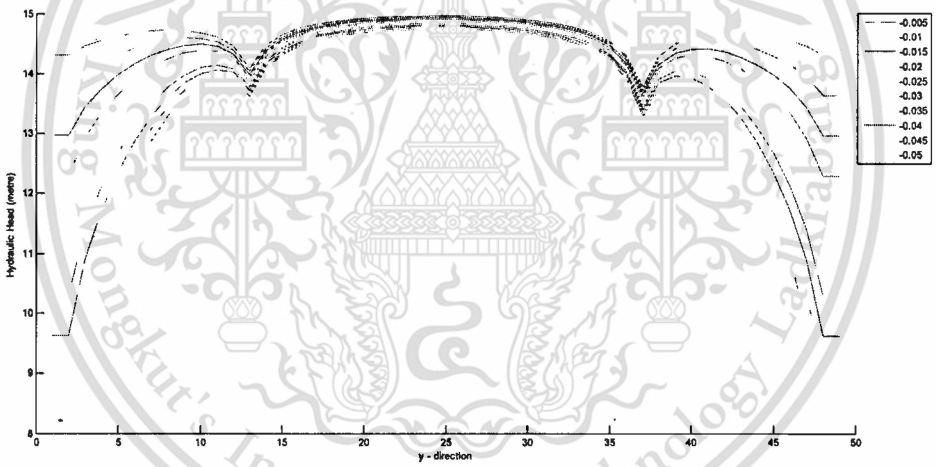

Figure 30: The cross-section line of hydraulic head at $x = 1800m.$ if derivatives of boundary conditions are varied.

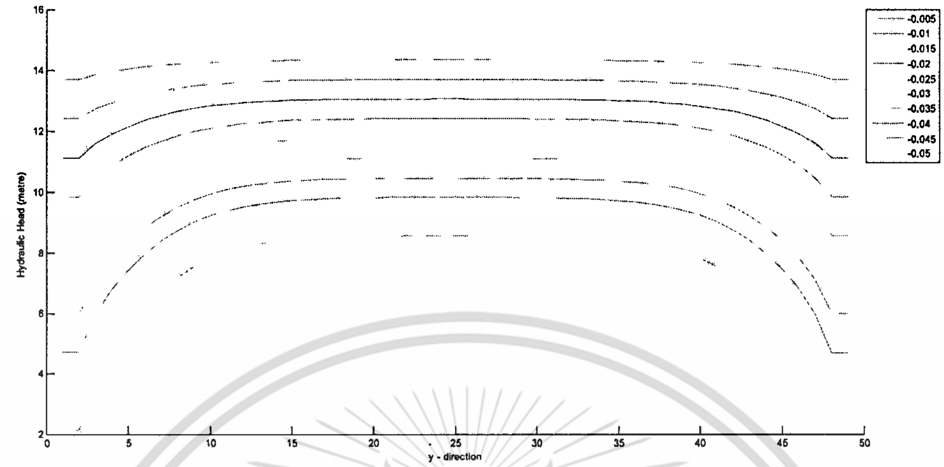

Figure 31: The cross-sectional line of hydraulic head at $x = 2400\,m.$ if derivatives of boundary conditions are varied.

Table 10: The stability of solution that obtained by FTCS, BTCS, ADEM and ADIM scheme.

Δx	Δy	Δt	FTCS	BTCS	ADEM	ADIM
		150	stable	stable	stable	stable
		155	stable	stable	stable	stable
		160	stable	stable	stable	stable
100	100	165	unstable	stable	stable	stable
		170	unstable	stable	stable	stable
		175	unstable	stable	stable	stable
		180	unstable	stable	stable	stable

CHAPTER V
Discussion and Conclusion

In this research, the applications are divided into three case. The case 1, we consider pump wells which pump water up from the groundwater only. The case 2, we consider pump wells which pump water up from the groundwater and pump water down the groundwater simultaneously. The case 3, we consider same the case 2 but pump water down the groundwater six month and stop pump down the groundwater six month every year. We can see in the case 3, it is corresponding with real-world problem.

The groundwater model is used in the problem. The FTCS, BTCS, ADEM and ADIM technique are used to approximate model. The calculated results give the hydraulic head profile. The easiest technique to calculate are FTCS and ADEM respectively. The most error result is result from ADEM technique.

The approximation of the hydraulic head from the FTCS, BTCS, ADEM and ADIM technique are shown in Table 3-6.respectively. It can be conclude that stability requirements is one of the disadvantage of the technique. The real-world problem require a small amount of time interval in obtaining accurate solutions. In Table 7, we can see that the FTCS scheme is not good agreement for real application. The BTCS, ADEM and ADIM have an advantage over compare FTCS. It is unconditionally stable.

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www.icmsa2015.kmitl.ac.th

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Mathematical Simulation of a Groundwater Management in a Drought Area Using an Implicit Finite Difference Method

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Abstract

The groundwater management is required to solve the problem of lack water resources in many drought areas for agricultural usage. In this study, we propose a groundwater flow model and a groundwater management model that provide the pumping rates and the injection rates respectively. The groundwater model is providing the hydraulic head that gives the groundwater level. The implicit finite difference method is used to approximate the groundwater flow directions. The objective of groundwater flow management model is the minimum cost of injection rates. These are then subjected to optimal management of the water injection stations to achieve minimum cost. The numerical experiments are also given.

Keywords: groundwater management; groundwater model; implicit method

1. Introduction

Groundwater modeling is a powerful tool for water resources management, groundwater protection and remediation. The models are decided by maker to predict the behavior of a groundwater system prior to implementation of a remediation plan. The significance of the utilization of water resources continues to grow due to the increasing require of water for irrigation as well as drinking, agriculture, commercial and industrial proposes. Although, the amount of groundwater resources have been decaying due population growth, uncontrolled and unplanned urbanization, industrialization, and agricultural activities. Hence, the sustainable management planning must be developed for the groundwater systems. The management planning have to limited in the case of legal well drilling and limited-pumpings. On the other hand, the partial differential equations governing the system is solved by model. The groundwater model are solved by analytical and numerical solution techniques. Analytical methods is not suitable application to require much data and their application is limited to simple problem. Numerical methods can solve more complex problem than analytical solutions. Now, rapid development of computer processors and increasing speed, numerical modeling has become tools more effective an easy to use. The finite difference method

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and the finite element are most tools used numerical modeling approaches. Each method has its advantages and limitations. Selecting numerical modeling approach depend on the problem of concern and the objectives of modeling. Most of groundwater modeling has the aquifer systems with the heterogeneous structure. In the case of the steady-state groundwater model solutions can be obtained by the simply basic techniques. On the other hand, the case of transient ground water model is solved by the advanced techniques due to the difficult in terms of time dimension in the governing equations. Theoretical solution of the governing equation of groundwater model need general assumptions such as ideal solution domains and homogeneous geometries.

Groundwater models can be simple, analytical solutions of one-dimensional is like solutions of spreadsheet models [1], for very complicated three-dimensional models. It is always introduced to start with a simple model, as long as the model concept satisfies modeling objectives, and then the model complex can be increased [2]. The finite difference [3-5] and finite elements [6-8] methods are the most popular numerical solution techniques. A simulation/optimization model is proposed for the identification of unknown groundwater well locations and pumping rates for two-dimensions and model is combined with genetic algorithm based optimization model [9].. A useful spreadsheet for two and three dimensional steady-state and transient groundwater numerical simulation is proposed in [10].

In this research, the objective of groundwater flow management model is the minimum cost of injection rates. These are then subjected to optimal management of the water injection stations to achieve minimum cost. The numerical experiments are also given.

2. The governing equation of groundwater steady-flow model

Mathematical model are all based on the water balance principle. Combine the mass balance equation and Darcy's law produce the governing equation for groundwater flow. The general equation that governs two-dimensional groundwater steady-flow in isotropic, homogeneous porous media is [11],

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}=0. \tag{1}$$

where H is hydraulic head (metre). We will consider source and sink term, from equation (1), we obtain,

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}+W=0, \tag{2}$$

where W is sinks and/or source (1/day). The boundary condition are specified, $0 \le x \le L$ and $0 \le y \le M$ where L, M are constant,

$$\frac{\partial H}{\partial n}=B_N \qquad at\ 0 \le x \le L \ \ and\ \ y=M, \tag{3}$$

$$\frac{\partial H}{\partial n}=B_S \qquad at\ 0 \le x \le L \ \ and\ \ y=0, \tag{4}$$

$$H=B_W \qquad at\ x=0 \ \ and\ \ 0 \le y \le M, \tag{5}$$

$$H=B_E \qquad at\ x=0 \ \ and\ \ 0 \le y \le M\ . \tag{6}$$

and given pumping well point each,

$$Q(x,y)=Q_i \qquad for\ \ i=1,2,3,\ldots. \tag{7}$$

in order to solve (2) in domain $[0,L]\times[0,M]$.

3. Numerical technique implicit finite difference method

The groundwater steady-flow not depend on time, then the implicit finite difference method is used to solve groundwater model, including it is appropriate with problem in form linear system and it's easy to combine with groundwater management model.

Taking the central difference scheme in space into each terms of equation (2), then

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H_{i-1,j} - 2H_{i,j} + H_{i+1,j}}{(\Delta x)^2} \quad (8)$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H_{i,j-1} - 2H_{i,j} + H_{i,j+1}}{(\Delta y)^2} \quad (9)$$

$$W_{i,j} = \pm \frac{Q_{i,j}}{\Delta x \Delta y H_{i,j}} \quad (10)$$

Substituting Eq.(8) – (10) into Eq.(2), for $1 < i < I-1$ and $1 < j < J-1$,

$$-(2a+2b)H_{i,j} + aH_{i-1,j} + aH_{i+1,j} + bH_{i,j-1} + bH_{i,j+1} = -W_{i,j}. \quad (11)$$

where $a = \frac{1}{(\Delta x)^2}$ and $a = \frac{1}{(\Delta y)^2}$. For $i=1$ and $j=1$, substituting the approximate unknown value of the bottom boundary by $H_{1,0} = H_{1,1}$,

$$-(2a+b)H_{1,1} + aH_{2,1} + bH_{1,2} = -W_{1,1} - aB_W. \quad (12)$$

For $1 < i < I-1$ and $j=1$, substituting the approximate unknown value of the bottom boundary by $H_{i,0} = H_{i,1}$,

$$-(2a+b)H_{i,1} + aH_{i-1,1} + aH_{i+1,1} + bH_{i,2} = -W_{i,1}. \quad (13)$$

For $i = I-1$ and $j=1$, substituting the approximate unknown value of the bottom boundary by $H_{I-1,0} = H_{I-1,1}$,

$$-(2a+b)H_{I-1,1} + aH_{I-2,1} + bH_{I-1,2} = -W_{I-1,1} - aB_E. \quad (14)$$

For $i=1$ and $1 < j < J-1$,

$$-(2a+2b)H_{1,j} + aH_{2,j} + bH_{1,j-1} + bH_{1,j+1} = -W_{1,j} - aB_W. \quad (15)$$

For $i = I-1$ and $1 < j < J-1$,

$$-(2a+2b)H_{I-1,j} + aH_{I-2,j} + bH_{I-1,j-1} + bH_{I-1,j+1} = -W_{I-1,j} - aB_E. \quad (16)$$

For $i=1$ and $j = J-1$, substituting the approximate unknown value of the top boundary by $H_{1,J} = H_{1,J-1}$,

$$-(2a+b)H_{1,J-1}+aH_{2,J-1}+bH_{1,J-2}=-W_{1,J-1}-aB_W. \quad (17)$$

For $1<i<I-1$ and $j=J-1$, substituting the approximate unknown value of the top boundary by $H_{i,J}=H_{i,J-1}$,

$$-(2a+b)H_{i,J-1}+aH_{i-1,J-1}+aH_{i+1,J-1}+bH_{i,J-2}=-W_{i,J-1}. \quad (18)$$

For $i=I-1$ and $j=J-1$, substituting the approximate unknown value of the top boundary by $H_{I-1,J}=H_{I-1,J-1}$,

$$-(2a+b)H_{I-1,J-1}+aH_{I-2,J-1}+bH_{I-1,J-2}=-W_{I-1,J-1}-aB_E. \quad (19)$$

The equations (11) – (19) can be written in matrix form as follow,

$$AH=B. \quad (20)$$

where

$$A=\begin{bmatrix} A_1 & A_3 & & & \\ A_3 & A_2 & A_3 & & \\ & \ddots & \ddots & \ddots & \\ & & A_3 & A_2 & A_3 \\ & & & A_3 & A_1 \end{bmatrix},\ A_1=\begin{bmatrix} -(2a+b) & a & & & \\ a & -(2a+b) & a & & \\ & \ddots & \ddots & \ddots & \\ & & a & -(2a+b) & a \\ & & & a & -(2a+b) \end{bmatrix},$$

$$A_2=\begin{bmatrix} -(2a+2b) & a & & & \\ a & -(2a+2b) & a & & \\ & \ddots & \ddots & \ddots & \\ & & a & -(2a+2b) & a \\ & & & a & -(2a+2b) \end{bmatrix},\ A_3=\begin{bmatrix} b & & & & \\ & b & & & \\ & & \ddots & & \\ & & & b & \\ & & & & b \end{bmatrix},$$

$$H=\begin{bmatrix} H_{1,1} \\ H_{2,1} \\ \vdots \\ H_{I-2,J-1} \\ H_{I-1,J-1} \end{bmatrix},\qquad B=\begin{bmatrix} -W_{1,1}-aB_W \\ -W_{2,1} \\ \vdots \\ -W_{I-2,1} \\ -W_{I-1,1}-aB_W \\ -W_{1,2}-aB_W \\ \vdots \\ -W_{i,j} \\ \vdots \\ -W_{I-1,J-2}-aB_E \\ -W_{1,J-1}-aB_W \\ -W_{2,J-1} \\ \vdots \\ -W_{I-2,J-1} \\ -W_{I-1,J-1}-aB_E \end{bmatrix}.$$

4. Groundwater management model

The objective function C is cost of pumping wells in the systems, so

$$C = \sum_{i=1}^{m} W_i Q_i. \tag{21}$$

where m is the number of pumping wells point, W_i is cost of pumping wells each point (Bath/ m^3) and Q_i is injection rate (m^3/day). The constraint are

$$H_l \le H_{ST}, \tag{22}$$

where H_l is monitoring point for measuring water requirement and H_{ST} is the standard water requirement each point. The upper bound of injection rate is

$$Q_i \le Q_{\max}, \tag{23}$$

the lower bound of injection rate is

$$Q_i \ge Q_{\min}, \tag{24}$$

and the monitoring point and injection rate are non-negative, that is

$$H_l, Q_i \ge 0. \tag{25}$$

where $Q_{\min}$ and $Q_{\max}$ are the lower and upper bounds respectively of the injection rate each point. The optimal control problem is solved by the simplex method.

5. Numerical experiment

We consider the land area width 2400 m and length 2400 m which is between two rivers. The area is meshed by 100 grids points with grid space is 240 m. The boundary conditions of the area is specified Eqs.(3)-(6) where $B_N = 0$, $B_S = 0$, $B_W = 20$ and $B_E = 19$. The four pumping wells is pumping the water from the ground. The pumping wells have the lower injection rates, upper injection rates and the cost of pumping wells as table 1. There are eight monitoring point for measuring water requirement. The monitoring point have the standard water requirement each point as table 2.

Table 1. The injection rates and the cost of each pumping wells.

	Q(1440 m,480 m)	Q(480 m,720 m)	Q(1680 m,1440 m)	Q(720 m,1680 m)
Lower	165 m^3/day	175 m^3/day	180 m^3/day	190 m^3/day
Upper	250 m^3/day	230 m^3/day	300 m^3/day	270 m^3/day
Cost	1.5 Bath/ m^3	1.9 Bath/ m^3	1.8 Bath/ m^3	1.6 Bath/ m^3

Table 2. The standard water requirement (SWR) each monitoring point.

Positions	(240,240)	(960,480)	(1920,720)	(720,1200)	(1200,1200)	(240,1440)	(1200,1920)	(1920,1920)
SWR	22 metre	23 metre	24 metre	24 metre	25 metre	23 metre	25 metre	24 metre

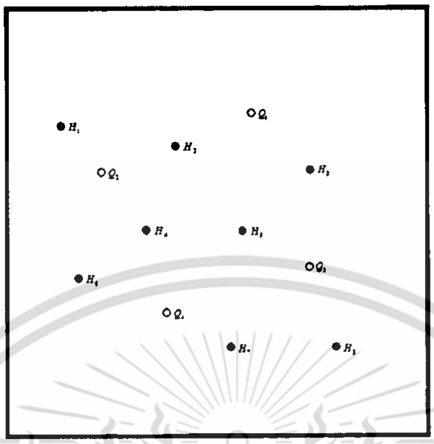

Fig. 1. Simulation of groundwater management .

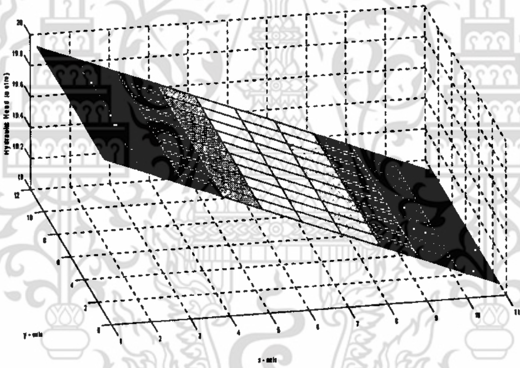

Fig. 2. The surface graph before optimal control of cost.

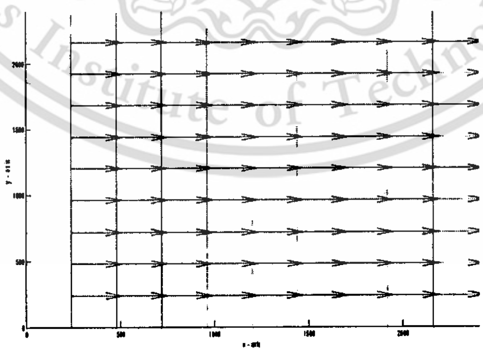

Fig. 3. The contour graph and direction flow before optimal control of cost.

Table 3. Table of the hydraulic head before optimal control of cost.

y\x	0	240	480	720	960	1200	1440	1680	1920	2160	2400
0	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
240	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
480	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
720	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
960	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1200	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1440	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1680	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
1920	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
2160	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000
2400	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000	19.4000	19.3000	19.2000	19.1000	19.0000

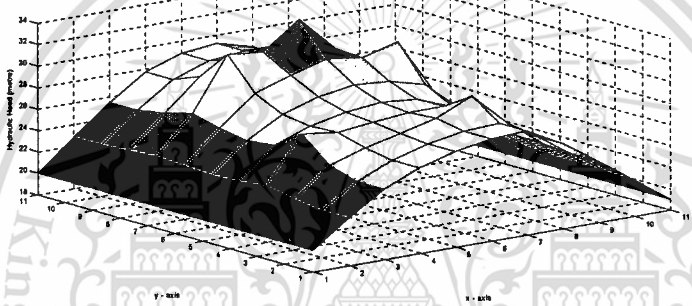

Fig. 4. The surface graph after optimal control of cost

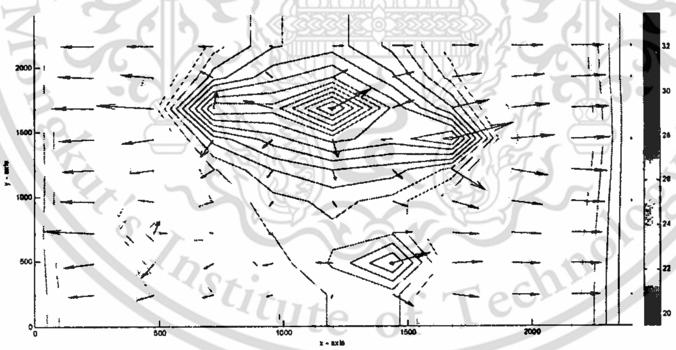

Fig. 5. The contour graph and direction flow after optimal control of cost.

Table 4. Table of the hydraulic head after optimal control of cost.

y\x	0	240	480	720	960	1200	1440	1680	1920	2160	2400
0	20.0000	22.4668	24.6064	25.9931	26.9362	27.5723	27.7052	25.9609	23.7603	21.4090	19.0000
240	20.0000	22.4668	24.6064	25.9931	26.9362	27.5723	27.7052	25.9609	23.7603	21.4090	19.0000
480	20.0000	22.7940	25.3593	26.4366	27.2432	28.0755	29.5824	26.4172	23.9110	21.4667	19.0000
720	20.0000	23.3498	27.6001	27.1510	27.5245	27.9041	27.8818	26.2143	24.0000	21.5467	19.0000
960	20.0000	23.0050	25.7905	27.0426	27.7999	28.1345	27.8265	26.5582	24.3280	21.7200	19.0000
1200	20.0000	22.8798	25.5142	27.4292	28.4979	29.0074	28.7315	27.8639	25.0340	22.0051	19.0000
1440	20.0000	23.0000	25.9574	28.6619	29.7552	30.6658	30.2281	31.1318	25.9390	22.2665	19.0000
1680	20.0000	23.1628	26.6534	31.5058	31.1952	33.6724	30.3834	28.5222	25.3236	22.1220	19.0000
1920	20.0000	22.9979	25.9875	28.7726	29.8473	30.4711	29.1111	27.2499	24.7114	21.8977	19.0000
2160	20.0000	22.8414	25.5262	27.7496	28.9502	29.2537	28.3398	26.6547	24.3745	21.7574	19.0000
2400	20.0000	22.8414	25.5262	27.7496	28.9502	29.2537	28.3398	26.6547	24.3745	21.7574	19.0000

Table 5. The optimal injection rates of minimum cost in the systems.

	Q(1440 m,480 m)	Q(480 m,720 m)	Q(1680 m,1440 m)	Q(720 m,1680 m)
Injection rate	165 m^3/day	175 m^3/day	239.48 m^3/day	214.81 m^3/day

6. Conclusion

We have established the groundwater management model, the first, we will measure hydraulic head from the groundwater steady-flow model by using implicit finite difference method. it can be obtain the system of linear equation. We bring the system of linear equation to construct groundwater management model for investigate the optimal least cost of the water injection in system which it is under condition. Although the water requirement and the injection cost of each monitoring point are unequal for difference struction, we will obtain the amount of groundwater follow water requirement of each monitoring point.

Acknowledgements

This paper is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand. The authors greatly appreciate valuable comments received from the anonymous reviewers.

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