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Tracy-Singh Products, Tracy-Singh Sums, Khatri-Rao Products,  
and Khatri-Rao Sums for Hilbert Space Operators

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### บทคัดย่อ

โครงการวิจัยนี้มุ่งพัฒนาองค์ความรู้ในทฤษฎีตัวดำเนินการเกี่ยวกับผลคูณและผลบวกแบบต่าง ๆ ของตัวดำเนินการเชิงเส้นที่มีขอบเขตบนปริภูมิฮิลเบิร์ต เราพิจารณาศึกษาสมบัติของผลคูณเทอร์ซี-ซิงท์ของตัวดำเนินการดังกล่าวที่เกี่ยวกับค่าเฉลี่ยเรขาคณิตแบบต่าง ๆ เราศึกษาสมบัติของผลคูณคาทรี-ราวของตัวดำเนินการดังกล่าวในแง่ของการวิเคราะห์และความสัมพันธ์กับตัวดำเนินการเลือก เราขยายแนวคิดของผลบวกเทอร์ซี-ซิงท์และผลบวกคาทรี-ราวสำหรับเมทริกซ์ไปสู่ผลบวกเทอร์ซี-ซิงท์และผลบวกคาทรี-ราวสำหรับตัวดำเนินการข้างต้น รวมทั้งศึกษาสมบัติเชิงพีชคณิต สมบัติเชิงอันดับ สมบัติเชิงการวิเคราะห์ และอสมการต่าง ๆ สำหรับผลบวกเทอร์ซี-ซิงท์และผลบวกคาทรี-ราว ผลลัพธ์ของโครงการวิจัยแสดงให้เห็นว่าผลคูณและผลบวกดังกล่าวข้างต้นมีความเข้ากันได้กับการดำเนินการเชิงพีชคณิต อันดับ และทอพอโลยีสำหรับตัวดำเนินการ

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<b>Research Title</b>	Tracy-Singh Products, Tracy-Singh Sums, Khatri-Rao Products, and Khatri-Rao Sums for Hilbert Space Operators
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## ABSTRACT

The research project aims to make a further development on operator theory concerning various kinds of products and sums of bounded linear operators on a Hilbert space. We investigate properties of the Tracy-Singh product of the operators involving various kinds of geometric means. We consider properties of the Khatri-Rao product of the operators in terms of analysis and relations to selection operators. We extend the notions of the Tracy-Singh sum and the Khatri-Rao sum for matrices to the Tracy-Singh sum and the Khatri-Rao sum for the operators. Moreover, we investigate algebraic, order, and analytic properties of the Tracy-Singh sum and the Khatri-Rao sum. The results in this project show that the products and the sums of operators are compatible with algebraic operations, order relations, and topologies for operators.

**Keywords :** Bounded linear operator, Hilbert space, Tracy-Singh product, Tracy-Singh sum, Khatri-Rao product, Khatri-Rao sum, operator inequality

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## King Mongkut's Institute of Technology Ladkrabang

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# CHAPTER 1

## Introduction

### 1.1 Research motivation

Motivated from computer science and statistics, a rectangular matrix is a two-dimensional array for collecting data. To produce a new data, we can use a variety of matrix products, e.g. ordinary product, Hadamard product, Kronecker product, Tracy-Singh product, and Khatri-Rao product. Each type of product is suitable for particular problems occurred in science, engineering, economics, etc.

One of the most important products is the Kronecker product, which has a rich and very pleasing algebra that supports a wide range of applications. These applications include signal processing, image processing, semidefinite programming, quantum computing, system identification, and other special fields. The Tracy-Singh product is a generalization of the Kronecker product in a block form. From mathematical point of view, the concept of the Kronecker product of matrices can be extended to the tensor product of bounded linear operators on a Hilbert space. Tensor products were investigated by many mathematicians from algebraic, order, structure, and analytical point of views.

Recently, the notions of Tracy-Singh product and Khatri-Rao product for matrices are extended to those for bounded linear operators on a complex Hilbert space. Algebraic, order, and analytic properties for Tracy-Singh products and Khatri-Rao products are also investigated.

This research is devoted to sustaining the theory of Tracy-Singh products and Khatri-Rao products. We investigate further properties of these products concerning inequalities, convexity, and geometric means. Then, we propose a natural definition of the Tracy-Singh sum and the Khatri-Rao sum for bounded linear operators acting on Hilbert spaces. We investigate algebraic, order, structure, and analytical properties of this kind of operator sum. Certain operator inequalities concerning the Tracy-Singh sum and the Khatri-Rao sum are also established.

## 1.2 Objectives

In this research, we develop further theory of Hilbert space operators as follows.

1. Investigate further properties of Tracy-Singh products and Khatri-Rao products concerning inequalities, convexity, and Moore-Penrose inverses.
2. Investigate relationship between Tracy-Singh products and Khatri-Rao products concerning various kinds of geometric means.
3. Provide a natural definition of the Tracy-Singh sum for Hilbert space operators and investigate its algebraic, order, structure, and analytic properties.
4. Provide a natural definition of the Khatri-Rao sum for Hilbert space operators and investigate its algebraic, order, structure, and analytic properties.
5. Establish operator inequalities concerning Tracy-Singh sums and Khatri-Rao sums.

## 1.3 Scopes

- All Hilbert spaces are over the complex numbers, assumed to be separable, and of finite or infinite dimensional.
- All operators are assumed to be linear and bounded.
- Geometric means considered here are the metric geometric mean, the spectral geometric mean, the Sagae-Tanabe metric geometric mean, and their weighted versions.
- Operator inequalities considered here are in the sense of positive-semidefinite ordering for Hermitian operators.

## 1.4 Research methodology

**Step 1.** Study advanced topics in operator theory, operator algebra, operator inequalities, and operator means from textbooks.

**Step 2.** Review literature results about Tracy-Singh product and Khatri-Rao product for matrices in the following details:

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- algebraic properties
- order properties
- operator inequalities
- relations between Tracy-Singh product and Khatri-Rao product.

**Step 3.** Review literature results about Tracy-Singh product and Khatri-Rao product for operators acting on a Hilbert space in the following details:

- algebraic properties
- order properties
- analytic properties
- operator inequalities
- relations between Tracy-Singh product and Khatri-Rao product.

**Step 4.** Investigate relations between the Tracy-Singh product of operators and the following geometric means:

- weighted metric geometric means
- weighted spectral geometric means
- Sagae-Tanabe metric geometric mean.

Then introduce Sagae-Tanabe spectral geometric mean and investigate its relations with the Tracy- Singh product.

**Step 5.** Establish operator inequalities involving Tracy-Singh products, Khatri-Rao products, and weighted Pythagorean means:

- weighted arithmetic means
- weighted harmonic means
- weighted (metric) geometric means.

**Step 6.** Introduce the Tracy-Singh sum for operators on a Hilbert space, and investigate the following properties:

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- relations with algebraic operations for operators, such as, the addition, the scalar multiplication, the usual multiplication, the adjoint, and the direct sum
- relations with operator orderings, including monotonicity, positivity, and strict positivity
- analytic properties, including norm bounds, continuity, and convergence.

**Step 7.** Introduce the Khatri-Rao sum for operators on a Hilbert space, and investigate the following properties:

- relations with algebraic operations for operators, such as, the addition, the scalar multiplication, the usual multiplication, the adjoint, and the direct sum
- relations with operator orderings, including monotonicity, positivity, and strict positivity
- analytic properties, including norm bounds, continuity, and convergence.

**Step 8.** Establish certain operator inequalities involving Khatri-Rao sums and Moore-Penrose inverses.

In Step 4- Step 8, we use the software Mathematica for computing Tracy-Singh products, Khatri-Rao products, Tracy-Singh sums, Khatri-Rao sums, norms, Moore-Penrose inverses, matrix decompositions, and functions of matrices. To write research papers, we use mathematical typesetting software called Latex.

# CHAPTER 2

## Tracy-Singh Products and Geometric Means for Operators

In this chapter, we investigate relationship between metric/spectral/Sagae-Tanabe geometric means for several positive operators and Tracy-Singh products in terms of identities and inequalities. In particular, we obtain various generalizations of arithmetic-geometric-harmonic means inequality and its reverse. Moreover, we introduce the weighted Sagae-Tanabe spectral geometric mean for several positive operators and deduce its properties related to Tracy-Singh products.

### 2.1 Introduction

It is well known that the tensor product (or Kronecker product) plays a fundamental role in linear algebra, functional analysis and related fields. Nowadays, theory of tensor products of operators is still developing, see [49] for instance. Recently, the authors of [31] introduced the Tracy-Singh product of operators which generalizes both tensor products of operators and Tracy-Singh products of complex matrices [40].

On the other hand, geometric means of positive definite matrices arise naturally in several areas of pure and applied mathematics. There are at least two types of geometric means. The first one is the metric geometric mean, introduced by Ando [3]. Recall that the set of  $n$ -by- $n$  positive definite matrices is a Riemannian manifold, in which the Riemannian metric between two matrices  $A$  and  $B$  is given by

$$\delta(A, B) = \|\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\|_2,$$

here,  $\|\cdot\|_2$  denotes the Frobenius norm. The metric geometric mean  $A\#B$  is indeed the unique metric midpoint of the geodesic line containing  $A$  and  $B$  (see, e.g., [21]).

The second one is the spectral geometric mean  $\natural$ , introduced by Fiedler and Pták [16]. In fact, the square of  $A\natural B$  is similar to the product  $AB$ , and in particular,

its eigenvalues coincide with the positive square roots of the eigenvalues of  $AB$ . See more information about metric/spectral geometric means in [4, 22] and [10, Chapters 4 and 6].

These two kinds of geometric means can be extended to multiple matrices by iterative processes, see e.g. [5]). Another such iterative geometric mean is the Sagae-Tanabe geometric mean, introduced in [37]. One of the most interesting properties of geometric means is the arithmetic-geometric-harmonic means (AM-GM-HM) inequalities. Indeed, the AM-GM-HM inequality concerning the metric geometric mean was established in [4]. Another version concerning the Sagae-Tanabe geometric mean were discussed in [37, 1].

The Kronecker product of matrices turns out to be compatible with the metric geometric mean in the sense that

$$(A\#B) \otimes (C\#D) = (A \otimes C)\#(B \otimes D) \quad (2.1)$$

for positive semidefinite matrices  $A, B, C, D$  of appropriate sizes (see [4]). Of course, the similar result for the spectral geometric mean is also true ([5]). Moreover, Kilicman and Al Zhour [5] discussed relations between Tracy-Singh products and metric geometric means, spectral geometric means, and Sagae-Tanabe geometric means of several positive definite matrices.

In this paper, we develop further theory for geometric means of Hilbert space operators. We investigate relationship between metric/spectral/Sagae-Tanabe geometric means and Tracy-Singh products in terms of operator identities and inequalities. In particular, we obtain various generalizations of the property (2.1) and the AM-GM-HM inequality and its reverse. Moreover, we introduce the weighted Sagae-Tanabe spectral geometric mean for several positive operators and deduce its properties related to Tracy-Singh products.

This chapter is organized as follows. Section 2.2 consists of prerequisites on Tracy-Singh and Khatri-Rao products for Hilbert space operators. In Section 2.3, we establish certain identities and inequalities between metric geometric means and Tracy-Singh products of several positive operators. Sections 2.4 and 2.5 deal with spectral geometric means and Sagae-Tanabe metric geometric means, respectively. In Section 2.6, we define the weighted Sagae-Tanabe spectral geometric mean for several positive operators and prove certain results related to Tracy-Singh products.

## 2.2 Preliminaries

Throughout this paper, let  $\mathcal{H}$  and  $\mathcal{K}$  be complex separable Hilbert spaces. When  $X$  and  $Y$  are Hilbert spaces, denote by  $\mathbb{B}(X, Y)$  the Banach space of bounded linear operators from  $X$  into  $Y$ , and abbreviate  $\mathbb{B}(X, X)$  to  $\mathbb{B}(X)$ . For Hermitian operators  $A, B \in \mathbb{B}(\mathcal{H})$ , the notation  $A \geq B$  means that  $A - B$  is a positive operator, while  $A > 0$  indicates that  $A$  is positive and invertible.

The projection theorem for Hilbert spaces allows us to decompose

$$\mathcal{H} = \bigoplus_{i=1}^m \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{l=1}^n \mathcal{K}_l$$

where all  $\mathcal{H}_i$  and  $\mathcal{K}_k$  are Hilbert spaces. Each operator  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  thus can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,m} \text{ and } B = [B_{kl}]_{k,l=1}^{n,n}$$

where  $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$  and  $B_{kl} \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k)$  for each  $i, j = 1, \dots, m$  and  $k, l = 1, \dots, n$ .

**Definition 2.1.** Let  $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$  be operator matrices defined as above. The *Tracy-Singh product* of  $A$  and  $B$  is defined to be the operator matrix

$$A \boxtimes B = [[A_{ij} \otimes B_{kl}]_{kl}]_{ij}, \quad (2.2)$$

which is a bounded linear operator from  $\bigoplus_{i,k=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_k$  into itself.

Note that if  $m = n = 1$ , the Tracy-Singh product  $A \boxtimes B$  reduces to the tensor product  $A \otimes B$ .

**Lemma 2.2** ([31, 32]). *Let  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$  be compatible operator matrices.*

(i) *The Tracy-Singh product is compatible with the usual product in the sense that*

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD.$$

(ii) *If  $A, B > 0$ , then  $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$  for any real number  $\alpha$ .*

(iii) *If  $A, B \geq 0$ , then  $A \boxtimes B \geq 0$ .*

(iv) If  $A, B > 0$ , then  $A \boxtimes B > 0$ .

(v) If  $A \geq C \geq 0$  and  $B \geq D \geq 0$ , then  $A \boxtimes B \geq C \boxtimes D$ .

The notion of Khatri-Rao product for operators was introduced in [33].

**Definition 2.3.** Let  $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{K})$  be operator matrices. The *Khatri-Rao product* of  $A$  and  $B$  is defined to be

$$A \boxtimes B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{m,m} \quad (2.3)$$

as a bounded linear operator from  $\bigoplus_{i=1}^m \mathcal{H}_i \otimes \mathcal{K}_i$  into itself.

Note that if  $m = 1$ , then  $A \boxtimes B = A \otimes B$ . We set  $\boxtimes_{i=1}^1 A_i = A_1 = \boxtimes_{i=1}^1 A_i$ . For  $r \in \mathbb{N} - \{1\}$  and a finite number of operator matrices  $A_i \in \mathbb{B}(\mathcal{H}_i)$  ( $i = 1, \dots, r$ ), denote

$$\boxtimes_{i=1}^r A_i = ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{r-1}) \boxtimes A_r,$$

$$\boxtimes_{i=1}^r A_i = ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{r-1}) \boxtimes A_r.$$

**Lemma 2.4** ([33]). *Let  $r \in \mathbb{N} - \{1\}$ . There exists an isometry  $Z$  such that*

$$\boxtimes_{i=1}^r A_i = Z^* \left( \boxtimes_{i=1}^r A_i \right) Z \quad (2.4)$$

for any  $A_i \in \mathbb{B}(\mathcal{H}_i)$ ,  $i = 1, \dots, r$ .

## 2.3 Metric geometric mean

In this section, we establish certain operator identities and inequalities involving metric geometric means and Tracy-Singh products. First of all, we recall some background about metric geometric means.

The metric geometric mean for matrices/operators was firstly defined by Ando [3]:

$$A \# B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}, \quad A, B > 0. \quad (2.5)$$

This formula comes from two natural requirements. First, it should coincide with the usual geometric mean for positive real numbers:  $A \# B = (AB)^{1/2}$  provided that  $AB = BA$ . The second condition is the congruent invariance

for any invertible  $T \in \mathbb{B}(\mathcal{H})$ . Now, consider positive invertible operators  $A$  and  $B$  in  $\mathbb{B}(\mathcal{H})$  and let  $w \in [0, 1]$ . The  $w$ -weighted geometric mean of  $A$  and  $B$  is defined by

$$A\#_w B = A^{1/2}(A^{-1/2}BA^{-1/2})^w A^{1/2}.$$

For arbitrary positive operators  $A$  and  $B$ , we define the  $w$ -weighted geometric mean of  $A$  and  $B$  to be

$$A\#_w B = \lim_{\varepsilon \rightarrow 0^+} (A + \varepsilon I)\#_w (B + \varepsilon I).$$

Here, the limit is taken in the strong-operator topology. For briefly, we write  $A\#B$  for  $A\#_{1/2}B$ .

**Theorem 2.1.** *Let  $A_1, A_2, B_1$  and  $B_2$  be positive operators in  $\mathbb{B}(\mathcal{H})$  and  $w \in [0, 1]$ . Then*

$$(A_1\#_w A_2) \boxtimes (B_1\#_w B_2) = (A_1 \boxtimes B_1)\#_w (A_2 \boxtimes B_2), \quad (2.6)$$

$$(A_1\#_w A_2) \square (B_1\#_w B_2) \leq (A_1 \square B_1)\#_w (A_2 \square B_2). \quad (2.7)$$

*Proof.* First, consider the case  $A_1, A_2, B_1, B_2 > 0$ . By using Lemma 2.2, we get

$$\begin{aligned} & (A_1 \boxtimes B_1)\#_w (A_2 \boxtimes B_2) \\ &= (A_1 \boxtimes B_1)^{1/2} [(A_1 \boxtimes B_1)^{-1/2} (A_2 \boxtimes B_2) (A_1 \boxtimes B_1)^{-1/2}]^w (A_1 \boxtimes B_1)^{1/2} \\ &= \left( A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[ \left( A_1^{-1/2} \boxtimes B_1^{-1/2} \right) (A_2 \boxtimes B_2) \left( A_1^{-1/2} \boxtimes B_1^{-1/2} \right) \right]^w \left( A_1^{1/2} \boxtimes B_1^{1/2} \right) \\ &= \left( A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[ \left( A_1^{-1/2} A_2 A_1^{-1/2} \right) \boxtimes \left( B_1^{-1/2} B_2 B_1^{-1/2} \right) \right]^w \left( A_1^{1/2} \boxtimes B_1^{1/2} \right) \\ &= \left( A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[ \left( A_1^{-1/2} A_2 A_1^{-1/2} \right)^w \boxtimes \left( B_1^{-1/2} B_2 B_1^{-1/2} \right)^w \right] \left( A_1^{1/2} \boxtimes B_1^{1/2} \right) \\ &= \left[ A_1^{1/2} \left( A_1^{-1/2} A_2 A_1^{-1/2} \right)^w A_1^{1/2} \right] \boxtimes \left[ B_1^{1/2} \left( B_1^{-1/2} B_2 B_1^{-1/2} \right)^w B_1^{1/2} \right] \\ &= (A_1\#_w A_2) \boxtimes (B_1\#_w B_2). \end{aligned}$$

Ando's result [4] states that if  $\Phi$  is a positive linear map, then for all  $A, B \geq 0$ ,

$$\Phi(A\#_w B) \leq \Phi(A)\#_w \Phi(B). \quad (2.8)$$

By applying Lemma 2.4 and inequality (2.8), we have

$$\begin{aligned} (A_1\#_w A_2) \square (B_1\#_w B_2) &= Z^* [(A_1\#_w A_2) \boxtimes (B_1\#_w B_2)] Z \\ &= Z^* [(A_1 \boxtimes A_2)\#_w (A_2 \boxtimes B_2)] Z \\ &\leq [Z^*(A_1 \boxtimes B_1)Z]\#_w [Z^*(A_2 \boxtimes B_2)Z] \\ &= (A_1 \square B_1)\#_w (A_2 \square B_2). \end{aligned}$$

For arbitrary  $A_1, A_2, B_1, B_2 \geq 0$ , perturb each of them with  $\varepsilon I$  and then take limit as  $\varepsilon \rightarrow 0^+$ .  $\square$

**Corollary 2.5.** *Let  $r \in \mathbb{N}$  and  $w \in [0, 1]$ . For each  $1 \leq i \leq r$ , let  $A_i, B_i \in \mathbb{B}(\mathcal{H})$  be positive operators. Then*

$$\boxtimes_{i=1}^r (A_i \#_w B_i) = \left( \boxtimes_{i=1}^r A_i \right) \#_w \left( \boxtimes_{i=1}^r B_i \right). \quad (2.9)$$

*Proof.* The proof is by induction on  $r$ .  $\square$

In [5], Kilicman and Al Zhou investigated weighted metric geometric means of any finite number of positive definite matrices. Now, we will extend this geometric mean to the case of finite number of positive operators.

**Definition 2.6.** Let  $r \in \mathbb{N} - \{1\}$ . For each  $1 \leq i \leq r$ , let  $A_i \in \mathbb{B}(\mathcal{H})$  be a positive operator. Let  $\alpha_i \in [0, 1]$  for each  $1 \leq i \leq r - 1$  and denote  $\alpha = (\alpha_1, \dots, \alpha_{r-1})$ . We define

$$\mathcal{G}_{\alpha_1}(A_1, A_2) = A_2 \#_{\alpha_1} A_1.$$

Now continue recurrently, setting

$$\mathcal{G}_{\alpha}(A_1, \dots, A_r) = \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1, \dots, A_{r-1}), A_r)$$

where  $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{r-2})$ . We call  $\mathcal{G}_{\alpha}(A_1, \dots, A_r)$  the *iterative  $\alpha$ -weighted metric geometric mean* of  $A_1, \dots, A_r$ .

The next two results asserts the compatibility between Tracy-Singh products and iterative weighted metric geometric means.

**Theorem 2.2.** *Let  $r \in \mathbb{N} - \{1\}$ . For each  $1 \leq i \leq r$ , let  $A_i, B_i \in \mathbb{B}(\mathcal{H})$  be positive operators. Let  $\alpha_i \in [0, 1]$  for each  $1 \leq i \leq r - 1$ . Then*

$$\mathcal{G}_{\alpha}(A_1 \boxtimes B_1, \dots, A_r \boxtimes B_r) = \mathcal{G}_{\alpha}(A_1, \dots, A_r) \boxtimes \mathcal{G}_{\alpha}(B_1, \dots, B_r) \quad (2.10)$$

*Proof.* We use induction on  $r$ . By continuity, we may assume that  $A_i, B_i > 0$  for all  $i = 1, \dots, r$ . When  $r = 2$ , we have by Proposition 2.1 that

$$\begin{aligned} \mathcal{G}_{\alpha}(A_1 \boxtimes B_1, A_2 \boxtimes B_2) &= (A_2 \boxtimes B_2) \#_{\alpha}(A_1 \boxtimes B_1) \\ &= (A_2 \#_{\alpha} A_1) \boxtimes (B_2 \#_{\alpha} B_1) \\ &= \mathcal{G}_{\alpha}(A_1, A_2) \boxtimes \mathcal{G}_{\alpha}(B_1, B_2) \end{aligned}$$

where  $\alpha \in [0, 1]$ . This gives the claim when  $r = 2$ . Suppose that the property (2.10) holds for  $r - 1$  ( $r \geq 3$ ). Let  $\alpha = (\alpha_1, \dots, \alpha_{r-1})$  and  $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{r-2})$  where  $\alpha_i \in [0, 1]$  for any  $1 \leq i \leq r - 1$ . Using Theorem 2.1, we have

$$\begin{aligned} \mathcal{G}_\alpha(A_1 \boxtimes B_1, \dots, A_r \boxtimes B_r) &= \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1 \boxtimes B_1, \dots, A_{r-1} \boxtimes B_{r-1}), A_r \boxtimes B_r) \\ &= \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1, \dots, A_{r-1}) \boxtimes \mathcal{G}_{\tilde{\alpha}}(B_1, \dots, B_{r-1}), A_r \boxtimes B_r) \\ &= \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1, \dots, A_{r-1}), A_r) \boxtimes \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(B_1, \dots, B_{r-1}), B_r) \\ &= \mathcal{G}_\alpha(A_1, \dots, A_r) \boxtimes \mathcal{G}_\alpha(B_1, \dots, B_r). \square \end{aligned}$$

**Corollary 2.7.** *Let  $r \in \mathbb{N} - \{1\}$  and  $s \in \mathbb{N}$ . For each  $1 \leq i \leq r$ ,  $1 \leq j \leq s$ , let  $A_{ij} \in \mathbb{B}(\mathcal{H})$  be a positive operator. Let  $\alpha_i \in [0, 1]$  for each  $1 \leq i \leq r - 1$ . Then*

$$\mathcal{G}_\alpha \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right) = \bigotimes_{j=1}^s \mathcal{G}_\alpha(A_{1j}, \dots, A_{rj}). \quad (2.11)$$

The *Thompson metric* [41] on the open convex cone of positive invertible operators is defined for each  $A, B > 0$  by

$$d(A, B) = \max\{\log M(A/B), \log M(B/A)\},$$

where  $M(A/B) = \inf\{\lambda > 0 : A \leq \lambda B\}$ . The *diameter* of  $\{A_1, \dots, A_r\}$  with respect to the Thompson metric  $d$  is defined by

$$\Delta(A_1, \dots, A_r) = \max\{d(A_i, A_j) : 1 \leq i, j \leq r\}.$$

**Lemma 2.8.** *Let  $r \in \mathbb{N} - \{1\}$ . Let  $A_i$  for each  $1 \leq i \leq r$  and  $B$  be positive invertible operators on  $\mathcal{H}$ . Let  $\alpha_i \in [0, 1]$  for each  $1 \leq i \leq r - 1$ . Then*

$$d(\mathcal{G}_\alpha(A_1, \dots, A_r), B) \leq \Delta(A_1, \dots, A_r, B). \quad (2.12)$$

*Proof.* See [1, Proposition 3.1]. □

The next result is a generalization of inequality (2.12).

**Proposition 2.9.** *Let  $r \in \mathbb{N} - \{1\}$  and  $s \in \mathbb{N}$ . For each  $1 \leq i \leq r$ ,  $1 \leq j \leq s$ , let  $A_{ij}, B_j \in \mathbb{B}(\mathcal{H})$  be positive invertible operators. Let  $\alpha_i \in [0, 1]$  for each  $1 \leq i \leq r - 1$ . Then*

$$d \left( \bigotimes_{j=1}^s \mathcal{G}_\alpha(A_{1j}, \dots, A_{rj}), \bigotimes_{j=1}^s B_j \right) \leq \Delta \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj}, \bigotimes_{j=1}^s B_j \right). \quad (2.13)$$

*Proof.* This proposition follows from Lemma 2.8 and Corollary 2.7. □

## 2.4 Spectral geometric mean

Recall that for positive definite matrices  $A$  and  $B$  of the same size, its spectral geometric mean [16] is defined by

$$A\sharp B = (A^{-1}\#B)^{\frac{1}{2}}A(A^{-1}\#B)^{\frac{1}{2}}.$$

Now, let  $A$  and  $B$  be positive invertible operators in  $\mathbb{B}(\mathcal{H})$  and  $w \in [0, 1]$ . The  $w$ -weighted spectral geometric mean of  $A$  and  $B$  is defined by

$$A\sharp_w B = (A^{-1}\#B)^w A(A^{-1}\#B)^w.$$

For arbitrary positive operators  $A$  and  $B$ , we define the  $w$ -weighted spectral geometric mean of  $A$  and  $B$  to be

$$A\sharp_w B = \lim_{\varepsilon \rightarrow 0^+} (A + \varepsilon I)\sharp_w (B + \varepsilon I).$$

**Theorem 2.3.** *Let  $A_1, A_2, B_1$  and  $B_2$  be positive operators in  $\mathbb{B}(\mathcal{H})$ . Then*

$$(A_1 \boxtimes B_1)\sharp_w (A_2 \boxtimes B_2) = (A_1\sharp_w A_2) \boxtimes (B_1\sharp_w B_2). \quad (2.14)$$

*Proof.* By continuity, we may assume that  $A_1, A_2, B_1, B_2 > 0$ . It follows from Lemma 2.2 and Proposition 2.1 that

$$\begin{aligned} & (A_1 \boxtimes B_1)\sharp_w (A_2 \boxtimes B_2) \\ &= [(A_1 \boxtimes B_1)^{-1}\#(A_2 \boxtimes B_2)]^w (A_1 \boxtimes B_1) [(A_1 \boxtimes B_1)^{-1}\#(A_2 \boxtimes B_2)]^w \\ &= [(A_1^{-1} \boxtimes B_1^{-1})\#(A_2 \boxtimes B_2)]^w (A_1 \boxtimes B_1) [(A_1^{-1} \boxtimes B_1^{-1})\#(A_2 \boxtimes B_2)]^w \\ &= [(A_1^{-1}\#A_2) \boxtimes (B_1^{-1}\#B_2)]^w (A_1 \boxtimes B_1) [(A_1^{-1}\#A_2) \boxtimes (B_1^{-1}\#B_2)]^w \\ &= [(A_1^{-1}\#A_2)^w \boxtimes (B_1^{-1}\#B_2)^w] (A_1 \boxtimes B_1) [(A_1^{-1}\#A_2)^w \boxtimes (B_1^{-1}\#B_2)^w] \\ &= [(A_1^{-1}\#A_2)^w A_1 (A_1^{-1}\#A_2)^w] \boxtimes [(B_1^{-1}\#B_2)^w B_1 (B_1^{-1}\#B_2)^w] \\ &= (A_1\sharp_w A_2) \boxtimes (B_1\sharp_w B_2). \square \end{aligned}$$

**Corollary 2.10.** *Let  $r \in \mathbb{N} - \{1\}$  and  $w \in [0, 1]$ . For each  $1 \leq i \leq r$ , let  $A_i, B_i \in \mathbb{B}(\mathcal{H})$  be positive operators. Then*

$$\left( \bigotimes_{i=1}^r A_i \right) \sharp_w \left( \bigotimes_{i=1}^r B_i \right) = \bigotimes_{i=1}^r (A_i \sharp_w B_i). \quad (2.15)$$

*Proof.* The proof is by induction on  $r$ . We have that the property (2.15) holds for  $r = 2$  by Lemma 2.3. Suppose that the property (2.15) holds for  $r - 1$  ( $r \geq 3$ ). By using Lemma 2.3, we get

$$\begin{aligned}
\left( \bigotimes_{i=1}^r A_i \right) \natural_w \left( \bigotimes_{i=1}^r B_i \right) &= \left[ \left( \bigotimes_{i=1}^{r-1} A_i \right) \otimes A_r \right] \natural_w \left[ \left( \bigotimes_{i=1}^{r-1} B_i \right) \otimes B_r \right] \\
&= \left[ \left( \bigotimes_{i=1}^{r-1} A_i \right) \natural_w \left( \bigotimes_{i=1}^{r-1} B_i \right) \right] \otimes (A_r \natural_w B_r) \\
&= \left( \bigotimes_{i=1}^{r-1} (A_i \natural_w B_i) \right) \otimes (A_r \natural_w B_r) \\
&= \bigotimes_{i=1}^r (A_i \natural_w B_i). \square
\end{aligned}$$

In [5], Kilicman and Al Zhou studied weighted spectral geometric means of any finite number of positive definite matrices and proved several properties related to Tracy-Singh products. Now, we will extend this geometric mean to the case of any finite number of positive operators.

**Definition 2.11.** Let  $r \in \mathbb{N} - \{1\}$ . For each  $1 \leq i \leq r$ , let  $A_i \in \mathbb{B}(\mathcal{H})$  be positive operators. Let  $\alpha_i \in [0, 1]$  for each  $1 \leq i \leq r - 1$ . We define

$$\mathcal{G}_{\alpha_1}^{sp}(A_1, A_2) = A_1 \natural_{\alpha_1} A_2.$$

Now continue recurrently, setting for each  $r \geq 3$ ,

$$\mathcal{G}_{\alpha}^{sp}(A_1, \dots, A_r) = \mathcal{G}_{\alpha_{r-1}}^{sp}(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1, \dots, A_{r-1}), A_r)$$

where  $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{r-2})$ . We call  $\mathcal{G}_{\alpha}^{sp}(A_1, \dots, A_r)$  the *iterated  $\alpha$ -weighted spectral geometric mean* of  $A_1, \dots, A_r$ .

From Definition 2.11, we can rewrite (2.15) in Corollary 2.10 to be

$$\mathcal{G}_{\alpha}^{sp} \left( \bigotimes_{i=1}^r A_i, \bigotimes_{i=1}^r B_i \right) = \bigotimes_{i=1}^r \mathcal{G}_{\alpha}^{sp}(A_i, B_i)$$

where  $\alpha = w$ .

**Corollary 2.12.** Let  $r \in \mathbb{N} - \{1\}$ . Let  $A_i$  and  $B_i$  be compatible positive operators in  $\mathbb{B}(\mathcal{H})$  for each  $i = 1, \dots, r$ . Then

$$\mathcal{G}_{\alpha}^{sp}(A_1 \otimes B_1, \dots, A_r \otimes B_r) = \mathcal{G}_{\alpha}^{sp}(A_1, \dots, A_r) \otimes \mathcal{G}_{\alpha}^{sp}(B_1, \dots, B_r). \quad (2.16)$$

*Proof.* The proof is by induction on  $r$ . By Theorem 2.3, we have that the property (2.16) is true for  $r = 2$ . Suppose that the property (2.16) is true for  $r - 1$ . By Theorem 2.3, we obtain

$$\begin{aligned}
& \mathcal{G}_\alpha^{sp}(A_1 \boxtimes B_1, \dots, A_r \boxtimes B_r) \\
&= \mathcal{G}_{\alpha_{r-1}}^{sp}(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1 \boxtimes B_1, \dots, A_{r-1} \boxtimes B_{r-1}), A_r \boxtimes B_r) \\
&= \mathcal{G}_{\alpha_{r-1}}^{sp}(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1, \dots, A_{r-1}) \boxtimes \mathcal{G}_{\tilde{\alpha}}^{sp}(B_1, \dots, B_{r-1}), A_r \boxtimes B_r) \\
&= \mathcal{G}_{\alpha_{r-1}}^{sp}(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1, \dots, A_{r-1}), A_r) \boxtimes \mathcal{G}_{\alpha_{r-1}}^{sp}(\mathcal{G}_{\tilde{\alpha}}^{sp}(B_1, \dots, B_{r-1}), B_r) \\
&= \mathcal{G}_\alpha^{sp}(A_1, \dots, A_r) \boxtimes \mathcal{G}_\alpha^{sp}(B_1, \dots, B_r). \square
\end{aligned}$$

**Corollary 2.13.** *Let  $r \in \mathbb{N} - \{1\}$  and  $s \in \mathbb{N}$ . Let  $A_{ij} \in \mathbb{B}(\mathcal{H})$  be a positive invertible operator for each  $i = 1, \dots, r$ ,  $j = 1, \dots, s$ . Then*

$$\mathcal{G}_\alpha^{sp} \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right) = \bigotimes_{j=1}^s \mathcal{G}_\alpha^{sp}(A_{1j}, \dots, A_{rj}). \quad (2.17)$$

*Proof.* The proof is by induction on  $s$ . □

## 2.5 Sagae-Tanabe metric geometric mean

Let  $r \in \mathbb{N} - \{1\}$ . For each  $1 \leq i \leq r$ , let  $A_i \in \mathbb{B}(\mathcal{H})$  be a positive invertible operator. Let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for each  $1 \leq i \leq r$  and  $\sum_{i=1}^r t_i = 1$ . The *weighted arithmetic* and *harmonic means* of  $A_1, \dots, A_r$  are defined by

$$\begin{aligned}
\mathcal{A}_t(A_1, \dots, A_r) &= \sum_{i=1}^r t_i A_i, \\
\mathcal{H}_t(A_1, \dots, A_r) &= \left( \sum_{i=1}^r t_i A_i^{-1} \right)^{-1}.
\end{aligned}$$

Sagae and Tanabe [37] proposed weighted geometric means of severable positive definite matrices as follows.

**Definition 2.14.** Let  $A$  and  $B$  be positive invertible operators in  $\mathbb{B}(\mathcal{H})$  and let  $v = (v_1, v_2)$  where  $v_1, v_2 \in [0, 1]$  and  $v_1 + v_2 = 1$ . We define

$$\mathcal{G}_v(A, B) = A \#_\alpha B$$

where  $\alpha = 1 - v_2$ . Let  $r \in \mathbb{N} - \{1\}$ . For each  $1 \leq i \leq r$ , let  $A_i \in \mathbb{B}(\mathcal{H})$  be a positive invertible operator. Let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for each  $1 \leq i \leq r$  and

$\sum_{i=1}^r t_i = 1$ . For each  $1 \leq i \leq r-1$ , let

$$\alpha_i = 1 - (t_{i+1} / \sum_{j=1}^{i+1} t_j).$$

The *Sagae-Tanabe weighted geometric mean* of  $A_1, \dots, A_r$  is defined by

$$\mathcal{G}_t(A_1, \dots, A_r) = \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{t}}(A_1, \dots, A_{r-1}), A_r)$$

where  $\mathcal{G}_{\tilde{t}}(A_1, \dots, A_{r-1})$  is the Sagae-Tanabe weighted geometric mean of  $A_1, \dots, A_{r-1}$  with weighted  $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{r-1})$  where  $\tilde{t}_i = t_i / \sum_{j=1}^{r-1} t_j$  for each  $1 \leq i \leq r-1$ .

Note that

$$\mathcal{G}_t(A_1, \dots, A_r) = \mathcal{G}_{\alpha}(A_1, \dots, A_r)$$

where  $\mathcal{G}_{\alpha}(A_1, \dots, A_r)$  is the weighted metric geometric mean of  $A_1, \dots, A_r$  in Definition 2.6 with weight  $\alpha = (\alpha_1, \dots, \alpha_{r-1})$ .

**Theorem 2.4.** *Let  $r \in \mathbb{N} - \{1\}$  and  $s \in \mathbb{N}$ . For each  $1 \leq i \leq r$ ,  $1 \leq j \leq s$ , let  $A_{ij} \in \mathbb{B}(\mathcal{H})$  be a positive invertible operator. Let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for each  $1 \leq i \leq r$  and  $\sum_{i=1}^r t_i = 1$ . Then*

$$\mathcal{G}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right) = \bigotimes_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}). \quad (2.18)$$

*Proof.* Let  $\alpha_i = 1 - (t_{i+1} / \sum_{j=1}^{i+1} t_j)$  for each  $1 \leq i \leq r-1$  and denote  $\alpha = (\alpha_1, \dots, \alpha_{r-1})$ . By Definition 2.14, we have

$$\mathcal{G}_t(A_{11} \otimes A_{12}, \dots, A_{r1} \otimes A_{r2}) = \mathcal{G}_{\alpha}(A_{11} \otimes A_{12}, \dots, A_{rj} \otimes B_{rj}).$$

Applying Theorem 2.2, we obtain

$$\mathcal{G}_{\alpha}(A_{11} \otimes A_{12}, \dots, A_{rj} \otimes B_{rj}) = \mathcal{G}_{\alpha}(A_{11}, \dots, A_{r1}) \otimes \mathcal{G}_{\alpha}(A_{12}, \dots, A_{r2}).$$

This implies that

$$\mathcal{G}_t(A_{11} \otimes A_{12}, \dots, A_{r1} \otimes A_{r2}) = \mathcal{G}_t(A_{11}, \dots, A_{r1}) \otimes \mathcal{G}_t(A_{12}, \dots, A_{r2}).$$

We get the result by using induction on  $s$ .  $\square$

**Lemma 2.15.** *Let  $r \in \mathbb{N} - \{1\}$ . For each  $1 \leq i \leq r$ , let  $A_i \in \mathbb{B}(\mathcal{H})$  be a positive invertible operator and let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for  $i = 1, \dots, r$  and  $\sum_{i=1}^r t_i = 1$ . Then*

$$\mathcal{H}_t(A_1, \dots, A_r) \leq \mathcal{G}_t(A_1, \dots, A_r) \leq \mathcal{A}_t(A_1, \dots, A_r). \quad (2.19)$$

*Proof.* See [1, Proposition 2.4].  $\square$

We extend [5, Theorem 4.6] to AM-GM-HM inequalities involving Tracy-Singh product of positive invertible operators as in the next two results.

**Corollary 2.16.** *Let  $r \in \mathbb{N} - \{1\}$  and  $s \in \mathbb{N}$ . For each  $1 \leq i \leq r, 1 \leq j \leq s$ , let  $A_{ij} \in \mathbb{B}(\mathcal{H})$  be a positive invertible operator and let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for  $i = 1, \dots, r$  and  $\sum_{i=1}^r t_i = 1$ . Then*

$$\bigotimes_{j=1}^s \mathcal{H}_t(A_{1j}, \dots, A_{rj}) \leq \mathcal{G}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right) \leq \bigotimes_{j=1}^s \mathcal{A}_t(A_{1j}, \dots, A_{rj}). \quad (2.20)$$

*Proof.* Lemma 2.15 tells us that

$$\mathcal{H}_t(A_{1j}, \dots, A_{rj}) \leq \mathcal{G}_t(A_{1j}, \dots, A_{rj}) \leq \mathcal{A}_t(A_{1j}, \dots, A_{rj})$$

for each  $1 \leq j \leq s$ . By using Lemma 2.2, we get

$$\bigotimes_{j=1}^s \mathcal{H}_t(A_{1j}, \dots, A_{rj}) \leq \bigotimes_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}) \leq \bigotimes_{j=1}^s \mathcal{A}_t(A_{1j}, \dots, A_{rj}).$$

Applying Theorem 2.4, we obtain

$$\bigotimes_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}) = \mathcal{G}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right)$$

and the inequality (2.20) follows.  $\square$

**Corollary 2.17.** *Let  $r \in \mathbb{N} - \{1\}$  and  $s \in \mathbb{N}$ . For each  $1 \leq i \leq r, 1 \leq j \leq s$ , let  $A_{ij} \in \mathbb{B}(\mathcal{H})$  be a positive invertible operator and let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for  $i = 1, \dots, r$  and  $\sum_{i=1}^r t_i = 1$ . Then*

$$\mathcal{H}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right) \leq \bigotimes_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}) \leq \mathcal{A}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right). \quad (2.21)$$

*Proof.* It follows directly from the AM-GM-HM inequality (2.19) and Theorem 2.4.  $\square$

We now turn to the AM-GM-HM inequality involving Khatri-Rao products.

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**Corollary 2.18.** Let  $A_{ij} \in \mathbb{B}(\mathcal{H})$  ( $1 \leq i \leq r$ ,  $1 \leq j \leq s$ ,  $r \geq 2$ ) be compatible positive invertible operators and let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for  $i = 1, \dots, r$  and  $\sum_{i=1}^r t_i = 1$ . Then

$$\boxed{\bullet}_{j=1}^s \mathcal{H}_t(A_{1j}, \dots, A_{rj}) \leq \boxed{\bullet}_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}) \leq \mathcal{A}_t \left( \boxed{\bullet}_{j=1}^s A_{1j}, \dots, \boxed{\bullet}_{j=1}^s A_{rj} \right). \quad (2.22)$$

*Proof.* We have by Lemmas 2.4 and 2.15 that

$$\begin{aligned} \boxed{\bullet}_{j=1}^s \mathcal{H}_t(A_{1j}, \dots, A_{rj}) &= Z^* \left( \boxed{\boxtimes}_{j=1}^s \mathcal{H}_t(A_{1j}, \dots, A_{rj}) \right) Z \\ &\leq Z^* \left( \boxed{\boxtimes}_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}) \right) Z \\ &= \boxed{\bullet}_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}). \end{aligned}$$

Using Lemma 2.4, we get

$$\begin{aligned} &Z^* \left[ \mathcal{A}_t \left( \boxed{\boxtimes}_{j=1}^s A_{1j}, \dots, \boxed{\boxtimes}_{j=1}^s A_{rj} \right) \right] Z \\ &= Z^* \left[ \sum_{i=1}^r t_i \left( \boxed{\boxtimes}_{j=1}^s A_{ij} \right) \right] Z = \sum_{i=1}^r t_i \left[ Z^* \left( \boxed{\boxtimes}_{j=1}^s A_{ij} \right) Z \right] \\ &= \sum_{i=1}^r t_i \left( \boxed{\bullet}_{j=1}^s A_{ij} \right) = \mathcal{A}_t \left( \boxed{\bullet}_{j=1}^s A_{1j}, \dots, \boxed{\bullet}_{j=1}^s A_{rj} \right). \end{aligned}$$

Applying Lemma 2.4 and Corollary 2.17, we obtain

$$\begin{aligned} \boxed{\bullet}_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}) &= Z^* \left( \boxed{\boxtimes}_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}) \right) Z \\ &\leq Z^* \left[ \mathcal{A}_t \left( \boxed{\boxtimes}_{j=1}^s A_{1j}, \dots, \boxed{\boxtimes}_{j=1}^s A_{rj} \right) \right] Z \\ &= \mathcal{A}_t \left( \boxed{\bullet}_{j=1}^s A_{1j}, \dots, \boxed{\bullet}_{j=1}^s A_{rj} \right). \square \end{aligned}$$

The next result is a generalization of Lemma 2.8.

**Proposition 2.19.** Let  $A_{ij}$  and  $B_j$  ( $1 \leq i \leq r$ ,  $1 \leq j \leq s$ ,  $r \geq 2$ ) be compatible positive invertible operators and let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for  $i = 1, \dots, r$

and  $\sum_{i=1}^r t_i = 1$ . Then

$$d\left(\bigotimes_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}), \bigotimes_{j=1}^s B_j\right) \leq \Delta\left(\bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj}, \bigotimes_{j=1}^s B_j\right). \quad (2.23)$$

*Proof.* The desired result follows from Lemma 2.8 and Corollary 2.7.  $\square$

For  $h, x \geq 1$ , the (generalized) Specht ratio is defined by

$$S_h(x) = \frac{(h^x - 1)h^{x(h^x - 1)^{-1}}}{e \log h^x} \text{ for } h \neq 1 \text{ and } S_1(x) = 1.$$

We denote  $S_h(1)$  by  $S_h$ . See [1, 17] for more information. The next result is a reverse version of AM-GM-HM inequality involving Tracy-Singh products via Specht ratio.

**Proposition 2.20.** *Let  $A_{ij}$  ( $1 \leq i \leq r$ ,  $1 \leq j \leq s$ ,  $r \geq 2$ ) be compatible positive invertible operators and let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for  $i = 1, \dots, r$  and  $\sum_{i=1}^r t_i = 1$ . Then*

$$\mathcal{A}_t\left(\bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj}\right) \leq S_h^{r-1} \cdot \left(\bigotimes_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj})\right) \quad (2.24)$$

where  $h = e^{\Delta(\bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj})}$ .

*Proof.* By using Lemma 2.8 and Corollary 2.7, we get the result.  $\square$

**Lemma 2.21.** *Let  $A_i \in \mathbb{B}(\mathcal{H})$  ( $1 \leq i \leq r$ ,  $r \geq 2$ ) be positive invertible operators and  $t_i$  ( $1 \leq i \leq r$ ) be real numbers such that  $t_1 > 0, t_i < 0$  ( $2 \leq i \leq r$ ) and  $\sum_{i=1}^r t_i = 1$ . Then*

$$\mathcal{A}_t(A_1, \dots, A_r) \leq \mathcal{G}_t(A_1, \dots, A_r). \quad (2.25)$$

If  $\sum_{i=1}^r t_i A_i^{-1} > 0$ , then

$$\mathcal{G}_t(A_1, \dots, A_r) \leq \mathcal{H}_t(A_1, \dots, A_r). \quad (2.26)$$

*Proof.* The proof is similar to the case of matrices, given in [2, Theorem 2.1].  $\square$

We now obtain reverse AM-GM-HM inequalities involving Tracy-Singh products as follows.

**Theorem 2.5.** Let  $A_{ij} \in \mathbb{B}(\mathcal{H})$  ( $1 \leq i \leq r$ ,  $1 \leq j \leq s$ ,  $r \geq 2$ ) be compatible positive invertible operators and  $t_i$  ( $1 \leq i \leq r$ ) be real numbers such that  $t_1 > 0, t_i < 0$  ( $2 \leq i \leq r$ ) and  $\sum_{i=1}^r t_i = 1$ . Then

$$\bigotimes_{j=1}^s \mathcal{A}_t(A_{1j}, \dots, A_{rj}) \leq \mathcal{G}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right). \quad (2.27)$$

If  $\mathcal{H}_t(A_{1j}, \dots, A_{rj}) > 0$  for all  $j = 1, \dots, s$ , then

$$\mathcal{G}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right) \leq \bigotimes_{j=1}^s \mathcal{H}_t(A_{1j}, \dots, A_{rj}). \quad (2.28)$$

*Proof.* It follows from Lemma 2.21 that

$$A_t(A_{1j}, \dots, A_{rj}) \leq \mathcal{G}_t(A_{1j}, \dots, A_{rj}) \leq \mathcal{H}_t(A_{1j}, \dots, A_{rj})$$

for each  $j = 1, \dots, s$ . Since  $A_t(A_{1j}, \dots, A_{rj}) \geq 0$  for all  $j = 1, \dots, s$ , we have by Lemmas 2.2 and 2.21 that

$$\mathcal{G}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right) = \bigotimes_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}) \geq \bigotimes_{j=1}^s \mathcal{A}_t(A_{1j}, \dots, A_{rj}).$$

Since  $\mathcal{H}_t(A_{1j}, \dots, A_{rj}) > 0$  for all  $j = 1, \dots, s$ , we obtain by Lemma 2.2 that

$$\bigotimes_{j=1}^s \mathcal{H}_t(A_{1j}, \dots, A_{rj}) > 0.$$

The proof is complete by applying Lemma 2.21 and Corollary 2.4.  $\square$

**Theorem 2.6.** Let  $A_{ij} \in \mathbb{B}(\mathcal{H})$  ( $1 \leq i \leq r$ ,  $1 \leq j \leq s$ ,  $r \geq 2$ ) be compatible positive invertible operators and  $t_i$  ( $1 \leq i \leq r$ ) be real numbers such that  $t_1 > 0, t_i < 0$  ( $2 \leq i \leq r$ ) and  $\sum_{i=1}^r t_i = 1$ . Then

$$\mathcal{A}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right) \leq \bigotimes_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rs}). \quad (2.29)$$

If  $\mathcal{H}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right) > 0$ , then

$$\bigotimes_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rs}) \leq \mathcal{H}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right). \quad (2.30)$$

*Proof.* By applying Lemma 2.21 and Corollary 2.4, we get the results.  $\square$

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**Corollary 2.22.** Let  $A_{ij} \in \mathbb{B}(\mathcal{H})$  ( $1 \leq i \leq r$ ,  $1 \leq j \leq s$ ,  $r \geq 2$ ) be compatible positive invertible operators and  $t_i$  ( $1 \leq i \leq r$ ) be real numbers such that  $t_1 > 0, t_i < 0$  ( $2 \leq i \leq r$ ) and  $\sum_{i=1}^r t_i = 1$ . Then

$$\mathcal{A}_t \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right) \leq \bigotimes_{j=1}^s \mathcal{G}_t(A_{1j}, \dots, A_{rj}) \leq \bigotimes_{j=1}^s \mathcal{H}_t(A_{1j}, \dots, A_{rj}). \quad (2.31)$$

*Proof.* This result is a direct consequence of Theorem 2.6 and Lemmas 2.4 and 2.15.  $\square$

## 2.6 Sagae-Tanabe spectral geometric mean

We introduce the following definition:

**Definition 2.23.** Let  $r \in \mathbb{N} - \{1\}$ . For each  $1 \leq i \leq r$ , let  $A_i \in \mathbb{B}(\mathcal{H})$  be a positive invertible operator. Let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for each  $1 \leq i \leq r$  and  $\sum_{i=1}^r t_i = 1$ . Let  $\alpha_i = 1 - (t_{i+1} / \sum_{j=1}^{i+1} t_j)$  for each  $1 \leq i \leq r-1$ . The *Sagae-Tanabe spectral geometric mean* of  $A_1, \dots, A_r$  is defined by

$$\mathcal{G}_t^{sp}(A_1, \dots, A_r) = \mathcal{G}_\alpha^{sp}(A_1, \dots, A_r)$$

where  $\alpha = (\alpha_1, \dots, \alpha_{r-1})$ .

**Proposition 2.24.** Let  $A_i$  and  $B_i$  ( $1 \leq i \leq r, r \geq 2$ ) be compatible positive operators in  $\mathbb{B}(\mathcal{H})$  and let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for  $i = 1, \dots, r-1$  and  $\sum_{i=1}^r t_i = 1$ . Then

$$\mathcal{G}_t^{sp}(A_1 \boxtimes B_1, \dots, A_r \boxtimes B_r) = \mathcal{G}_t^{sp}(A_1, \dots, A_r) \boxtimes \mathcal{G}_t^{sp}(B_1, \dots, B_r) \quad (2.32)$$

$$\mathcal{G}_t^{sp} \left( \bigotimes_{i=1}^r A_i, \bigotimes_{i=1}^r B_i \right) = \bigotimes_{i=1}^r \mathcal{G}_t^{sp}(A_i, B_i). \quad (2.33)$$

*Proof.* Let  $\alpha_i = 1 - (t_{i+1} / \sum_{j=1}^{i+1} t_j)$  for each  $1 \leq i \leq r-1$  and denote  $\alpha = (\alpha_1, \dots, \alpha_{r-1})$ . By Definition 2.23, we have

$$\begin{aligned} \mathcal{G}_t^{sp}(A_1 \boxtimes B_1, \dots, A_r \boxtimes B_r) &= \mathcal{G}_\alpha^{sp}(A_1 \boxtimes B_1, \dots, A_r \boxtimes B_r) \\ \mathcal{G}_t^{sp} \left( \bigotimes_{i=1}^r A_i, \bigotimes_{i=1}^r B_i \right) &= \mathcal{G}_\alpha^{sp} \left( \bigotimes_{i=1}^r A_i, \bigotimes_{i=1}^r B_i \right). \end{aligned}$$

By Corollary 2.12, we get (2.32). Applying Corollary 2.10, we obtain (2.33).  $\square$

**Corollary 2.25.** *Let  $A_{ij} \in \mathbb{B}(\mathcal{H})$  ( $1 \leq i \leq r$ ,  $1 \leq j \leq s$ ,  $r \geq 2$ ) be compatible positive invertible operators and let  $t = (t_1, \dots, t_r)$  where  $t_i \in [0, 1]$  for  $i = 1, \dots, r - 1$  and  $\sum_{i=1}^r t_i = 1$ . Then*

$$\mathcal{G}_t^{sp} \left( \bigotimes_{j=1}^s A_{1j}, \dots, \bigotimes_{j=1}^s A_{rj} \right) = \bigotimes_{j=1}^s \mathcal{G}_t^{sp}(A_{1j}, \dots, A_{rj}). \quad (2.34)$$

*Proof.* From (2.32), we have

$$\mathcal{G}_t^{sp}(A_{11} \boxtimes A_{12}, \dots, A_{r1} \boxtimes A_{r2}) = \mathcal{G}_t^{sp}(A_{11}, \dots, A_{r1}) \boxtimes \mathcal{G}_t^{sp}(A_{12}, \dots, A_{r2}).$$

We obtain (2.34) by induction on  $s$ . □

## 2.7 Conclusion

Several relations between metric/spectral/Sagae-Tanabe geometric means for several positive operators and Tracy-Singh products are established in terms of identities and inequalities. In particular, we obtain noncommutative arithmetic-geometric-harmonic means inequalities and their reverses. Moreover, we define the weighted Sagae-Tanabe spectral geometric mean for several positive operators and deduce its properties related to Tracy-Singh products.

# CHAPTER 3

## Khatri-Rao Products for Hilbert Space Operators

In this chapter, we introduce the notion of Khatri-Rao product for operator matrices acting on the direct sum of Hilbert spaces. This notion generalizes both tensor product for operators and Tracy-Singh product for complex matrices. We investigate algebraic and order properties of this kind of operator product. Moreover, Khatri-Rao product is related to Tracy-Singh product via an isometry.

### 3.1 Introduction

In matrix theory, there are many kinds of matrix products which are of interest in both theory and applications, such as Kronecker product, Hadamard product, and Khatri-Rao product. Denote by  $M_{m,n}(\mathbb{C})$  the set of  $m$ -by- $n$  complex matrices and abbreviate  $M_{n,n}(\mathbb{C})$  to  $M_n(\mathbb{C})$ . Recall that for  $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$  and  $B = [b_{ij}] \in M_{m,n}(\mathbb{C})$ , the Hadamard product of  $A$  and  $B$  is defined by the entrywise product

$$A \odot B = [a_{ij}b_{ij}] \in M_{m,n}(\mathbb{C}).$$

The Kronecker product of  $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$  and  $B \in M_{p,q}(\mathbb{C})$  is given by

$$A \otimes_K B = [a_{ij}B]_{ij} \in M_{mp,nq}(\mathbb{C}). \quad (3.1)$$

One of the most attractive properties of this product is the mixed product property:

$$(A \otimes B)(C \otimes D) = AC \otimes BD \quad (3.2)$$

for all  $A \in M_{m,n}(\mathbb{C})$ ,  $B \in M_{p,q}(\mathbb{C})$ ,  $C \in M_{n,r}(\mathbb{C})$  and  $D \in M_{q,s}(\mathbb{C})$ . Khatri and Rao [19] defined the Khatri-Rao product of partitioned matrices as follows. Let  $A$  and  $B$  be complex matrices partitioned so that their  $(i,j)$ th block are given by

$A_{ij}$  and  $B_{ij}$  for each  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$  (the sizes of  $A_{ij}$  and  $B_{ij}$  may be different). Then the Khatri-Rao product of  $A$  and  $B$  is defined by

$$A \boxtimes_K B = [A_{ij} \otimes_K B_{ij}]_{ij}. \quad (3.3)$$

Note that if  $A$  and  $B$  are non-partitioned (that is, each of only one block), then their Khatri-Rao product is just their Kronecker product. If  $A$  and  $B$  are entrywise partitioned (that is, each block is a  $1 \times 1$  matrix), then their Khatri-Rao product is just their Hadamard product. Interesting algebraic, order and topological properties of this product were studied in the literature; see, *e.g.*, [13, 15, 18, 24, 25, 45, 47]. Their applications go to statistics, computer science and related fields, *e.g.*, [23, 36].

The tensor product of Hilbert space operators can be viewed as a natural extension of the Kronecker product to infinite-dimensional setting. Let  $\mathcal{H}, \mathcal{H}', \mathcal{K}$  and  $\mathcal{K}'$  are complex Hilbert spaces. Recall that the tensor product of  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$  is the unique bounded linear operator from  $\mathcal{H} \otimes \mathcal{K}$  into  $\mathcal{H}' \otimes \mathcal{K}'$  such that

$$(A \otimes B)(x \otimes y) = Ax \otimes By \quad (3.4)$$

for all  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ .

In this chapter, we generalize the tensor product for operators to the Khatri-Rao product of operator matrices acting on the direct sum of Hilbert space in a natural way. It turns out that this operator product is compatible with algebraic operations and order relations for operators. Our results generalize the results known so far in the literature for both Khatri-Rao products of matrices and tensor products of operators.

This paper is organized as follows. In the next section, we provide some preliminaries results about Tracy-Singh products for operator matrices. Then, in Section 3.3, we introduce the Khatri-Rao product for operator matrices and deduce its algebraic. Section 3.4 discusses positivity and order properties of Khatri-Rao product. In the last section, we investigate relationship between Khatri-Rao product and Tracy-Singh product.

### 3.2 Preliminaries on Tracy-Singh products for operator matrices

Throughout, let  $\mathcal{H}$ ,  $\mathcal{H}'$ ,  $\mathcal{K}$  and  $\mathcal{K}'$  be complex separable Hilbert spaces. When  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert spaces, denote by  $\mathbb{B}(\mathcal{X}, \mathcal{Y})$  the Banach space of bounded linear operators from  $\mathcal{X}$  into  $\mathcal{Y}$ , and abbreviate  $\mathbb{B}(\mathcal{X}, \mathcal{X})$  to  $\mathbb{B}(\mathcal{X})$ . Let

$$\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^m \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{l=1}^q \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^p \mathcal{K}'_k$$

where each  $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$  are Hilbert spaces. For each  $j = 1, \dots, n$ , let  $M_j$  be the canonical embedding from  $\mathcal{H}_j$  into  $\mathcal{H}$ . Similarly, let  $N_l$  be the canonical embedding from  $\mathcal{K}_l$  into  $\mathcal{K}$  for each  $l = 1, \dots, q$ . For each  $i = 1, \dots, m$ , let  $P_i$  be the orthogonal projection from  $\mathcal{H}'$  onto  $\mathcal{H}'_i$ . Similarly, let  $Q_k$  be the orthogonal projection from  $\mathcal{K}'$  onto  $\mathcal{K}'_k$  for  $k = 1, \dots, p$ . Each operator  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  thus can be represented uniquely as an operator matrix

$$A = [A_{ij}]_{i,j=1}^{m,n}$$

where  $A_{ij} = P_i A M_j \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}'_i)$  for each  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . Similarly, an operator  $B$  in  $\mathbb{B}(\mathcal{K}, \mathcal{K}')$  can be written uniquely as

$$B = [B_{kl}]_{k,l=1}^{p,q}$$

where  $B_{kl} = Q_k B N_l \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}'_k)$  for each  $k = 1, \dots, p$  and  $l = 1, \dots, q$ .

**Definition 3.1.** Let  $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$  be operator matrices. We define the *Tracy-Singh product* of  $A$  and  $B$  to be the operator matrix

$$A \boxtimes B = [[A_{ij} \otimes B_{kl}]_{kl}]_{ij} \quad (3.5)$$

which is a bounded linear operator from  $\bigoplus_{j,l=1}^{n,q} \mathcal{H}_j \otimes \mathcal{K}_l$  to  $\bigoplus_{i,k=1}^{m,p} \mathcal{H}'_i \otimes \mathcal{K}'_k$ .

**Lemma 3.2** ([31]). *Let  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B, C \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$  be operator matrices. Then*

$$(A \boxtimes B)^* = A^* \boxtimes B^*, \quad (3.6)$$

$$A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C, \quad (3.7)$$

$$(B + C) \boxtimes A = B \boxtimes A + C \boxtimes A. \quad (3.8)$$

**Lemma 3.3** ([31]). Let  $\mathcal{H}, \mathcal{H}', \mathcal{H}'', \mathcal{K}, \mathcal{K}'$  and  $\mathcal{K}''$  be complex Hilbert spaces. Let  $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}', \mathcal{H}'')$ ,  $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K}', \mathcal{K}'')$ ,  $C = [C_{ij}]_{i,j=1}^{n,r} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $D = [D_{kl}]_{k,l=1}^{q,s} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ . Then

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD. \quad (3.9)$$

We call (3.9) the mixed product property.

**Lemma 3.4** ([31]). If  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  are invertible, then  $A \boxtimes B$  is invertible and

$$(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}. \quad (3.10)$$

**Lemma 3.5** ([31]). Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . If  $A, B \geq 0$ , then  $A \boxtimes B \geq 0$ .

### 3.3 Compatibility of Khatri-Rao products to operator operations

From now on, the following orthogonal decompositions of Hilbert spaces are fixed:

$$\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^m \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{j=1}^n \mathcal{K}_j, \quad \mathcal{K}' = \bigoplus_{i=1}^m \mathcal{K}'_i. \quad (3.11)$$

That is, we fix how to partition any operator matrix in  $\mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $\mathbb{B}(\mathcal{K}, \mathcal{K}')$ .

**Definition 3.6.** Let  $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B = [B_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$  be operators matrices partitioned according to the decomposition (4.3). The *Khatri-Rao product* of  $A$  and  $B$  is defined to be the operator matrix

$$A \boxtimes B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{m,n} \quad (3.12)$$

as a bounded linear operator from  $\bigoplus_{j=1}^n \mathcal{H}_j \otimes \mathcal{K}_j$  to  $\bigoplus_{i=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_i$ .

Note that

- (i) if both  $A$  and  $B$  are  $1 \times 1$  block operator matrices, their  $A \boxtimes B$  is  $A \otimes B$ .
- (ii) if  $\mathcal{H}_i = \mathcal{K}_i = \mathbb{C}$  and  $\mathcal{H}'_j = \mathcal{K}'_j = \mathbb{C}$  for all  $i, j$ , then their Khatri-Rao product is the Hadamard product of complex matrices.

For each  $A \in M_{m,n}(\mathbb{C})$  and  $B \in M_{p,q}(\mathbb{C})$ , the induced maps

$$L_A : \mathbb{C}^n \rightarrow \mathbb{C}^m, x \mapsto Ax \quad \text{and} \quad L_B : \mathbb{C}^q \rightarrow \mathbb{C}^p, x \mapsto Bx$$

are bounded linear operators. We identify  $\mathbb{C}^n \otimes \mathbb{C}^q$  with  $\mathbb{C}^{nq}$  together with the canonical bilinear map  $(x, y) \mapsto x \otimes_K y$  for each  $(x, y) \in \mathbb{C}^n \times \mathbb{C}^q$ .

**Lemma 3.7.** *For any  $A \in M_{m,n}(\mathbb{C})$  and  $B \in M_{p,q}(\mathbb{C})$ , we have*

$$L_A \otimes L_B = L_{A \otimes_K B}. \quad (3.13)$$

*Proof.* By using the mixed product property of the Kronecker product (3.2), we have

$$\begin{aligned} (L_A \otimes L_B)(x \otimes y) &= L_A(x) \otimes L_B(y) = L_A(x) \otimes_K L_B(y) \\ &= Ax \otimes_K By = (A \otimes_K B)(x \otimes_K y) \\ &= (A \otimes_K B)(x \otimes y) = L_{A \otimes_K B}(x \otimes y). \end{aligned}$$

for any  $x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^q$ . The uniqueness of tensor product implies that  $L_A \otimes L_B = L_{A \otimes_K B}$ .  $\square$

**Proposition 3.8.** *The Khatri-Rao product of two linear maps induced by matrices is just the linear map induced by the Khatri-Rao product of these matrices. More precisely, for any complex matrices  $A = [A_{ij}]$  and  $B = [B_{ij}]$  partitioned in block-matrix forms, we have*

$$L_A \boxtimes L_B = L_{A \boxtimes B}. \quad (3.14)$$

*Proof.* Recall that the  $(i, j)$ th block of the matrix representation of  $L_A$  is  $L_{A_{ij}}$ . By Lemma 3.7, we obtain  $L_A \boxtimes L_B = [L_{A_{ij}} \otimes L_{B_{ij}}]_{ij} = [L_{A_{ij} \otimes_K B_{ij}}]_{ij} = L_{A \boxtimes B}$ .  $\square$

**Proposition 3.9.** *Let  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B, C \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$  be operator matrices. Then*

$$(A \boxtimes B)^* = A^* \boxtimes B^* \quad (3.15)$$

$$A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C \quad (3.16)$$

$$(B + C) \boxtimes A = B \boxtimes A + C \boxtimes A \quad (3.17)$$

$$(\alpha A) \boxtimes B = \alpha(A \boxtimes B) = A \boxtimes (\alpha B) \quad (3.18)$$

for any  $\alpha \in \mathbb{C}$ .

*Proof.* Since  $A^* = [A_{ji}^*]_{ij}$  and  $B^* = [B_{ji}^*]_{ij}$  for all  $i, j$ , we obtain

$$(A \boxtimes B)^* = [(A_{ij} \otimes B_{ij})^*]_{ij} = [A_{ji}^* \otimes B_{ji}^*]_{ij} = A^* \boxtimes B^*.$$

By using the fact that  $(B + C)_{ij} = B_{ij} + C_{ij}$  for all  $i, j$  together with the left distributivity of the tensor product over the addition, we have

$$\begin{aligned} A \boxtimes (B + C) &= [A_{ij} \otimes (B_{ij} + C_{ij})]_{ij} \\ &= [(A_{ij} \otimes B_{ij}) + (A_{ij} \otimes C_{ij})]_{ij} \\ &= A \boxtimes B + A \boxtimes C. \end{aligned}$$

Similarly, we obtain the property (3.17). Since  $(\alpha A)_{ij} = \alpha A_{ij}$  for all  $i, j$ , we get

$$(\alpha A) \boxtimes B = [(\alpha A_{ij}) \otimes B_{ij}]_{ij} = [\alpha (A_{ij} \otimes B_{ij})]_{ij} = \alpha (A \boxtimes B).$$

Similarly,  $A \boxtimes (\alpha B) = \alpha (A \boxtimes B)$ .  $\square$

**Proposition 3.10.** *Let  $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B = [B_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$  and let  $r, s \in \mathbb{N}$  be such that  $r \leq m$  and  $s \leq n$ . We partition  $A$  to be  $A = [A_{kl}]_{k,l=1}^{r,s}$  where  $A_{kl}$  is  $m_k \times n_l$  block operator matrix such that  $(g, h)$ th of  $A_{kl}$  is  $(u, v)$ th of  $A$  and partition  $B$  to be  $B = [B_{kl}]_{k,l=1}^{r,s}$  where  $B_{kl}$  is  $m_k \times n_l$  block operator matrix such that  $(g, h)$ th of  $B_{kl}$  is  $(u, v)$ th of  $B$  where  $u = (k-1)m_k + g$ ,  $v = (l-1)n_l + h$ ,  $\sum_{k=1}^r m_k = m$  and  $\sum_{l=1}^s n_l = n$ . Then*

$$A \boxtimes B = [A_{kl} \boxtimes B_{kl}]_{kl} = \begin{bmatrix} A_{11} \boxtimes B_{11} & \cdots & A_{1s} \boxtimes B_{1s} \\ \vdots & \ddots & \vdots \\ A_{r1} \boxtimes B_{r1} & \cdots & A_{rs} \boxtimes B_{rs} \end{bmatrix}. \quad (3.19)$$

That is, the  $(k, l)$ th block of  $A \boxtimes B$  is just  $A_{kl} \boxtimes B_{kl}$ .

*Proof.* It follows from the definition of the Khatri-Rao product.  $\square$

Let  $A_i \in \mathbb{B}(\mathcal{H}_i, \mathcal{K}_i)$  for each  $i = 1, \dots, n$ , the operator  $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$  such that

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$$

is the *direct sum* of  $A_1, \dots, A_n$ , denoted by  $A_1 \oplus \dots \oplus A_n$ . The next result gives a relation between the direct sum and the Khatri-Rao product.

**Proposition 3.11.** *Let  $A, C \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B, D \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$  be compatible operator matrices. Then*

$$(A \oplus B) \boxtimes (C \oplus D) = (A \boxtimes C) \oplus (B \boxtimes D). \quad (3.20)$$

*Proof.* Since  $0 \boxtimes 0 = 0$ , we have by Proposition 3.10 that

$$(A \oplus B) \boxtimes (C \oplus D) = \begin{bmatrix} A \boxtimes C & 0 \\ 0 & B \boxtimes D \end{bmatrix} = (A \boxtimes C) \oplus (B \boxtimes D).$$

□

### 3.4 Khatri-Rao products and positivity

**Lemma 3.12** (Albert's Theorem). *Let*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{bmatrix} \quad (3.21)$$

*be operator in  $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$  where  $S_{11}$  and  $S_{22}$  are self-adjoint and  $S_{11}$  has a closed range. Then  $S \geq 0$  if and only if*

- (i)  $S_{11} \geq 0$ ,
- (ii)  $S_{12} = S_{11} S_{11}^\dagger S_{12}$ ,
- (iii)  $S_{22} \geq S_{12}^* S_{11}^\dagger S_{12}$ .

*Proof.* See [44].

□

**Lemma 3.13.** *Let  $S$  be operator in (3.21) where  $S_{11}$  and  $S_{22}$  are self-adjoint. Then  $S > 0$  if and only if*

- (i)  $S_{11} > 0$ ,
- (ii)  $S_{22} > 0$ ,
- (iii)  $S_{11} > S_{12} S_{22}^{-1} S_{12}^*$ ,
- (iv)  $S_{22} > S_{12}^* S_{11}^{-1} S_{12}$ .

*Proof. Sufficiency.* Since  $S > 0$ , for any  $\xi \in \mathcal{H}_1$  we have

$$\langle S_{11}\xi, \xi \rangle = \langle S(\xi \oplus 0), \xi \oplus 0 \rangle > 0,$$

which means  $S_{11} > 0$ . Similarly  $S_{22} > 0$ . We know, by [9], that

$$S^{-1} = \begin{bmatrix} (S_{11} - S_{12}S_{22}^{-1}S_{12}^*)^{-1} & * \\ * & (S_{22} - S_{12}^*S_{11}^{-1}S_{12})^{-1} \end{bmatrix}.$$

Since  $S^{-1} > 0$ , we get  $(S_{11} - S_{12}S_{22}^{-1}S_{12}^*)^{-1} > 0$  and  $(S_{22} - S_{12}^*S_{11}^{-1}S_{12})^{-1} > 0$ . This implies that  $S_{11} - S_{12}S_{22}^{-1}S_{12}^* > 0$  and  $S_{22} - S_{12}^*S_{11}^{-1}S_{12} > 0$ .

*Necessity.* By Lemma 6.17, we have  $S \geq 0$ . Let

$$\begin{aligned} A &= (S_{11} - S_{12}S_{22}^{-1}S_{12}^*)^{-1}, \\ B &= -S_{11}^{-1}S_{12}(S_{22} - S_{12}^*S_{11}^{-1}S_{12})^{-1}, \\ C &= (S_{22} - S_{12}^*S_{11}^{-1}S_{12})^{-1}. \end{aligned}$$

By [9], we obtain

$$S \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} S = I$$

i.e.  $S$  is invertible. Hence  $S > 0$ . □

Now, we will extend [20, Theorem 6] to the case of Khatri-Rao product of operator matrices.

**Theorem 3.1.** *Let  $A = [A_{ij}] \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$  and  $B = [B_{ij}] \in \mathbb{B}(\mathcal{K}_1 \oplus \mathcal{K}_2)$ . If  $A \geq 0$ ,  $A_{11} > 0$ ,  $A_{22} > 0$  and  $B > 0$ , then  $A \boxtimes B > 0$ .*

*Proof.* Consider

$$A \boxtimes B = \begin{bmatrix} A_{11} \otimes B_{11} & A_{12} \otimes B_{12} \\ A_{12}^* \otimes B_{12}^* & A_{22} \otimes B_{22} \end{bmatrix}.$$

We have  $A_{11} \otimes B_{11} > 0$  and  $A_{22} \otimes B_{22} > 0$  since  $A_{11}, A_{22}, B_{11}, B_{22} > 0$ . Applying Lemma 6.17 and Lemma 3.13, we get

$$\begin{aligned} & A_{11} \otimes B_{11} - (A_{12} \otimes B_{12})(A_{22} \otimes B_{22})^{-1}(A_{12} \otimes B_{12})^* \\ &= A_{11} \otimes B_{11} - A_{12}A_{22}^{-1}A_{12}^* \otimes B_{12}B_{22}^{-1}B_{12}^* \\ &> (A_{11} - A_{12}A_{22}^{-1}A_{12}^*) \otimes B_{12}B_{22}^{-1}B_{12}^* \\ &\geq 0. \end{aligned}$$

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Similarly, we obtain

$$A_{22} \otimes B_{22} - (A_{12} \otimes B_{12})^*(A_{11} \otimes B_{11})^{-1}(A_{12} \otimes B_{12}) > 0.$$

Then, we have from Lemma 3.13,  $A \boxtimes B > 0$ .  $\square$

**Corollary 3.14.** *Let  $A = [A_{ij}] \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$  and  $B = [B_{ij}] \in \mathbb{B}(\mathcal{K}_1 \oplus \mathcal{K}_2)$  be strictly positive operators. Then  $A \boxtimes B$  is also strictly positive.*

The next result provides the monotonicity of Khatri-Rao product.

**Theorem 3.2.** *Let  $A_1, A_2 \in \mathbb{B}(\mathcal{H})$  and  $B_1, B_2 \in \mathbb{B}(\mathcal{K})$ . If  $A_1 \geq A_2 \geq 0$  and  $B_1 \geq B_2$  with  $B_1 \geq 0$ , then*

$$A_1 \boxtimes B_1 \geq A_2 \boxtimes B_2 \geq 0. \quad (3.22)$$

*Proof.* By using the mixed product property and Lemma 3.5, we obtain

$$(A_1 \boxtimes B_1) - (A_2 \boxtimes B_2) = (A_1 - A_2) \boxtimes B_1 + A_2 \boxtimes (B_1 - B_2) \geq 0.$$

Applying Theorem 4.5, we get the result.  $\square$

### 3.5 Relationship between Khatri-Rao products and Tracy-Singh products

Liu [25] defined the two selection matrices  $Z_1$  and  $Z_2$  to show that the Khatri-Rao and Tracy-Singh products of matrices are related. Now, we extend the relationship between Khatri-Rao and Tracy-Singh products of matrices to relationship between Khatri-Rao and Tracy-Singh products of operators.

Consider the following operator matrix

$$Z_1 = \begin{bmatrix} E_1 \\ \vdots \\ E_m \end{bmatrix} \quad (3.23)$$

where  $E_r = \left[ E_{gh}^{(r)} \right]_{g,h=1}^{m,m}$  is a bounded linear operator from  $\bigoplus_{i=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_i$  to  $\bigoplus_{i=1}^m \mathcal{H}'_r \otimes \mathcal{K}'_i$  for  $r = 1, \dots, m$ , with  $E_{gh}^{(r)}$  is an identity operator if  $g = h = r$  and the others are zero operators. We also define

$$Z_2 = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix} \quad (3.24)$$

where  $F_s = \left[ F_{gh}^{(s)} \right]_{g,h=1}^n$  is a bounded linear operator from  $\bigoplus_{j=1}^n \mathcal{H}_j \otimes \mathcal{K}_j$  to  $\bigoplus_{j=1}^n \mathcal{H}_s \otimes \mathcal{K}_j$  for  $s = 1, \dots, n$ , with  $F_{gh}^{(s)}$  is an identity operator if  $g = h = s$  and the others are zero operators. In the case of  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have

$$Z_1 = Z_2 = Z. \quad (3.25)$$

For example, consider  $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}'_1 \oplus \mathcal{H}'_2)$  and  $\mathbb{B}(\mathcal{K}_1 \oplus \mathcal{K}_2, \mathcal{K}'_1 \oplus \mathcal{K}'_2)$ . We have

$$Z_1 = \begin{bmatrix} I_{11} & 0 \\ 0_{11} & 0 \\ 0_{21} & 0 \\ 0 & I_{21} \end{bmatrix}, \quad Z_2 = \begin{bmatrix} I_{12} & 0 \\ 0_{12} & 0 \\ 0_{22} & 0 \\ 0 & I_{22} \end{bmatrix}$$

where  $I_{11} \in \mathbb{B}(\mathcal{H}'_1 \otimes \mathcal{K}'_1)$ ,  $I_{21} \in \mathbb{B}(\mathcal{H}'_2 \otimes \mathcal{K}'_2)$ ,  $I_{12} \in \mathbb{B}(\mathcal{H}_1 \otimes \mathcal{K}_1)$ ,  $I_{22} \in \mathbb{B}(\mathcal{H}_2 \otimes \mathcal{K}_2)$ ,  $0_{11} \in \mathbb{B}(\mathcal{H}'_1 \otimes \mathcal{K}'_1, \mathcal{H}'_1 \otimes \mathcal{K}'_2)$ ,  $0_{21} \in \mathbb{B}(\mathcal{H}'_1 \otimes \mathcal{K}'_1, \mathcal{H}'_2 \otimes \mathcal{K}'_1)$ ,  $0_{12} \in \mathbb{B}(\mathcal{H}_1 \otimes \mathcal{K}_1, \mathcal{H}_1 \otimes \mathcal{K}_2)$  and  $0_{22} \in \mathbb{B}(\mathcal{H}_1 \otimes \mathcal{K}_1, \mathcal{H}_2 \otimes \mathcal{K}_1)$ . Next, we give the relationship between Khatri-Rao and Tracy-Singh products of operators which can be viewed as an operator extension of [20, Theorem 3].

**Lemma 3.15.** *Let  $Z_1$  and  $Z_2$  are the selection operator matrices in (3.23) and (3.24), respectively. Then for  $i = 1, 2$ ,*

1.  $Z_i^* Z_i = I$ ,

2.  $0 \leq Z_i Z_i^* \leq I$ .

*Proof.* A direct computation shows that  $Z_1^* Z_1 = I$  and  $Z_2^* Z_2 = I$ . We know that  $E_i E_i^*$  is  $m \times m$  block operator matrix such that consist only zero and identity operators which has an entry of identity operator in the  $(i, i)$ th block and  $E_i E_j^* = 0$  if  $i \neq j$  for  $i, j = 1, \dots, m$ . Then

$$Z_1 Z_1^* = \begin{bmatrix} E_1 E_1^* & E_1 E_2^* & \cdots & E_1 E_m^* \\ E_2 E_1^* & E_2 E_2^* & \cdots & E_2 E_m^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m E_1^* & E_m E_2^* & \cdots & E_m E_m^* \end{bmatrix} = \begin{bmatrix} E_1 E_1^* & 0 & \cdots & 0 \\ 0 & E_2 E_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_m E_m^* \end{bmatrix}.$$

Since  $E_i E_i^* \leq I$  for all  $i = 1, \dots, m$ , we have  $Z_1 Z_1^* \leq I$ . Similarly,  $Z_2 Z_2^* \leq I$ .  $\square$

**Theorem 3.3.** *For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , we have*

$$A \boxtimes B = Z_1^* (A \boxtimes B) Z_2, \quad (3.26)$$

where  $Z_1$  and  $Z_2$  are the selection operator matrices in (3.23) and (3.24), respectively. In the case of  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ , we have

$$A \boxdot B = Z^*(A \boxtimes B)Z, \quad (3.27)$$

where  $Z$  is the selection operator matrix in (3.25).

*Proof.* Let  $B(i)$  denote the  $i$ th column of  $B$  for  $i = 1, \dots, n$ . Then we have

$$\begin{aligned} Z_1^*(A \boxtimes B)Z_2 &= \begin{bmatrix} E_1^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1n} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{m1} \boxtimes B & \cdots & A_{mn} \boxtimes B \end{bmatrix} \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix} \\ &= \begin{bmatrix} E_1^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} (A_{11} \boxtimes B)F_1 + \cdots + (A_{1n} \boxtimes B)F_n \\ \vdots \\ (A_{m1} \boxtimes B)F_1 + \cdots + (A_{mn} \boxtimes B)F_n \end{bmatrix} \\ &= \begin{bmatrix} E_1^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} A_{11} \boxtimes B(1) & \cdots & A_{1n} \boxtimes B(n) \\ \vdots \\ A_{m1} \boxtimes B(1) & \cdots & A_{mn} \boxtimes B(n) \end{bmatrix} \\ &= \begin{bmatrix} A_{11} \boxtimes B_{11} & \cdots & A_{1n} \boxtimes B_{1n} \\ \vdots \\ A_{m1} \boxtimes B_{m1} & \cdots & A_{mn} \boxtimes B_{mn} \end{bmatrix} \\ &= A \boxdot B. \end{aligned}$$

□

**Remark 3.16.** If we partition  $A$  and  $B$  into row operator matrices, then  $A \boxdot B = (A \boxtimes B)Z_2$ . If both  $A$  and  $B$  are column operator matrices,  $A \boxdot B = Z_1^*(A \boxtimes B)$ .

Next, we extend the Khatri-Rao product inequality of matrices in [20, Theorem 4] to the Khatri-Rao product inequality of operators.

**Corollary 3.17.** Let  $A$  and  $B$  be operators in  $\mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $\mathbb{B}(\mathcal{K}, \mathcal{K}')$ , respectively. Then

$$A^*A \boxdot B^*B \geq (A^* \boxdot B^*)(A \boxdot B). \quad (3.28)$$

*Proof.* Since  $ZZ^* \leq I$ , we have

$$\begin{aligned} (A^* \boxdot B^*)(A \boxdot B) &= Z^*(A^* \boxtimes B^*)ZZ^*(A \boxtimes B)Z \\ &\leq Z^*(A^* \boxtimes B^*)I(A \boxtimes B)Z \\ &= Z^*(A^*A \boxtimes B^*B)Z \\ &= A^*A \boxdot B^*B. \end{aligned}$$

□

**Corollary 3.18.** Let  $A = A_1 \oplus \dots \oplus A_n$  and  $B = B_1 \oplus \dots \oplus B_n$  be operators in  $\mathbb{B}(\mathcal{H})$  and  $\mathbb{B}(\mathcal{K})$ , respectively. Then

$$Z^*(A \boxtimes B) = (A \boxdot B)Z^* \text{ and } (A \boxtimes B)Z = Z(A \boxdot B). \quad (3.29)$$

*Proof.* Using the fact that  $E_i E_i^* X = X_i = X E_i E_i^*$  and  $E_i E_j^* X = 0 = X E_i E_j^*$  if  $i \neq j$  where  $X = X_1 \oplus \dots \oplus X_n$ , we have

$$\begin{aligned} ZZ^*(A \boxtimes B) &= \begin{bmatrix} E_1 E_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_n E_n^* \end{bmatrix} \begin{bmatrix} A_1 \boxtimes B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \boxtimes B \end{bmatrix} \\ &= \begin{bmatrix} E_1 E_1^*(A_1 \boxtimes B) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_n E_n^*(A_n \boxtimes B) \end{bmatrix} \\ &= \begin{bmatrix} (A_1 \boxtimes B) E_1 E_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (A_n \boxtimes B) E_n E_n^* \end{bmatrix} \\ &= \begin{bmatrix} A_1 \boxtimes B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \boxtimes B \end{bmatrix} \begin{bmatrix} E_1 E_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_n E_n^* \end{bmatrix} \\ &= (A \boxtimes B) ZZ^*. \end{aligned}$$

By applying Theorem 4.5, we get

$$Z^*(A \boxtimes B) = Z^* ZZ^*(A \boxtimes B) = Z^*(A \boxtimes B) ZZ^* = (A \boxdot B) Z^*.$$

Similarly,  $(A \boxtimes B)Z = Z(A \boxdot B)$ . □

**Theorem 3.4.** Let  $A \in \mathbb{B}\left(\bigoplus_{i=1}^n \mathcal{H}, \bigoplus_{j=1}^m \mathcal{H}'\right)$  and  $B \in \mathbb{B}\left(\bigoplus_{i=1}^n \mathcal{K}, \bigoplus_{j=1}^m \mathcal{K}'\right)$ . Let  $Z_1$  and  $Z_2$  be selection operator matrices such that  $A \boxdot B = Z_1^*(A \boxtimes B)Z_2$ . Then there exist operators  $Q_1 \in \mathbb{B}\left(\bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^m \mathcal{H}' \otimes \mathcal{K}', \bigoplus_{i=1}^m \bigoplus_{j=1}^m \mathcal{H}' \otimes \mathcal{K}'\right)$  and  $Q_2 \in \mathbb{B}\left(\bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^n \mathcal{H} \otimes \mathcal{K}, \bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathcal{H} \otimes \mathcal{K}\right)$  such that

$$Z_i^* Q_i = 0, \quad Q_i^* Q_i = I, \quad Z_i Z_i^* + Q_i Q_i^* = I, \quad (3.30)$$

for  $i = 1, 2$ .

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*Proof.* Let

$$Q_1 = \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} F_2 & F_3 & \cdots & F_n \\ F_3 & F_4 & \cdots & F_1 \\ \vdots & \vdots & \ddots & \vdots \\ F_1 & F_2 & \cdots & F_{n-1} \end{bmatrix}. \quad (3.31)$$

Then calculations show that

$$\begin{aligned} Z_1^* Q_1 &= \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \\ Q_1^* Q_1 &= \begin{bmatrix} E_2^* & E_3^* & \cdots & E_1^* \\ E_3^* & E_4^* & \cdots & E_2^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m^* & E_1^* & \cdots & E_{m-1}^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} \\ &= \begin{bmatrix} \sum E_i^* E_i & 0 & \cdots & 0 \\ 0 & \sum E_i^* E_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum E_i^* E_i \end{bmatrix} \\ &= I. \end{aligned}$$

$$\begin{aligned}
& Q_1^* Q_1 + Z_1 Z_1^* \\
&= \begin{bmatrix} E_2^* & E_3^* & \cdots & E_1^* \\ E_3^* & E_4^* & \cdots & E_2^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m^* & E_1^* & \cdots & E_{m-1}^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix} \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i \neq 1} E_i E_j^* & 0 & \cdots & 0 \\ 0 & \sum_{i \neq 2} E_i E_j^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{i \neq m} E_i E_j^* \end{bmatrix} + \begin{bmatrix} E_1 E_1^* & 0 & \cdots & 0 \\ 0 & E_2 E_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_m E_m^* \end{bmatrix} \\
&= \begin{bmatrix} \sum E_i E_i^* & 0 & \cdots & 0 \\ 0 & \sum E_i E_i^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum E_i E_i^* \end{bmatrix} \\
&= I.
\end{aligned}$$

Similarly, we have  $Z_2^* Q_2 = 0$ ,  $Q_2^* Q_2 = I$ , and  $Z_2 Z_2^* + Q_2 Q_2^* = I$ .  $\square$

**Remark 3.19.** Operator matrices  $Q_1$  and  $Q_2$  are not unique. Consider

$$R_1 = \begin{bmatrix} E_m & E_{m-1} & \cdots & E_2 \\ E_1 & E_m & \cdots & E_3 \\ \vdots & \vdots & \ddots & \vdots \\ E_{m-1} & E_{m-2} & \cdots & E_1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} F_m & F_{m-1} & \cdots & F_2 \\ F_1 & F_m & \cdots & F_3 \\ \vdots & \vdots & \ddots & \vdots \\ F_{m-1} & F_{m-2} & \cdots & F_1 \end{bmatrix},$$

Then we have  $Z_i^* = 0$ ,  $R_i^* R_i = I$ , and  $Z_i Z_i^* + R_i R_i^* = I$  for  $i = 1, 2$ .

**Corollary 3.20.** Let  $A \in \mathbb{B} \left( \bigoplus_{i=1}^n \mathcal{H}, \bigoplus_{j=1}^m \mathcal{H}' \right)$  and  $B \in \mathbb{B} \left( \bigoplus_{i=1}^n \mathcal{K}, \bigoplus_{j=1}^m \mathcal{K}' \right)$ . Then

$$AA^* \boxplus BB^* = (A \boxplus B)(A \boxplus B)^* + Z_1^*(A \boxtimes B)Q_2Q_2^*(A \boxtimes B)^*Z_1. \quad (3.32)$$

*Proof.*

$$\begin{aligned}
AA^* \boxplus BB^* &= Z_1^*(AA^* \boxtimes BB^*)Z_1 \\
&= Z_1^*(A \boxtimes B)(A \boxtimes B)^*Z_1 \\
&= Z_1^*(A \boxtimes B)(Z_2Z_2^* + Q_2Q_2^*)(A \boxtimes B)^*Z_1 \\
&= Z_1^*(A \boxtimes B)Z_2Z_2^*(A \boxtimes B)^*Z_1 + Z_1^*(A \boxtimes B)Q_2Q_2^*(A \boxtimes B)^*Z_1 \\
&= (Z_1^*(A \boxtimes B)Z_2)(Z_1^*(A \boxtimes B)Z_2)^* + Z_1^*(A \boxtimes B)Q_2Q_2^*(A \boxtimes B)^*Z_1 \\
&= (A \boxplus B)(A \boxplus B)^* + Z_1^*(A \boxtimes B)Q_2Q_2^*(A \boxtimes B)^*Z_1.
\end{aligned}$$

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□

**Corollary 3.21.** Let  $A \in \mathbb{B} \left( \bigoplus_{i=1}^n \mathcal{H}, \bigoplus_{j=1}^m \mathcal{H}' \right)$  and  $B \in \mathbb{B} \left( \bigoplus_{i=1}^n \mathcal{K}, \bigoplus_{j=1}^m \mathcal{K}' \right)$ . Then

$$AA^* \square BB^* \geq (A \square B)(A \square B)^*. \quad (3.33)$$

**Theorem 3.5.** Let  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ . Let  $Z_1$  and  $Z_2$  be selection operator matrices such that  $A \square B = Z_1^*(A \boxtimes B)Z_2$ . Then there exist operator matrices

$$V \in \mathbb{B} \left( \bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_j, \bigoplus_{i=1}^m \bigoplus_{j=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_j \right)$$

and  $W \in \mathbb{B} \left( \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_j, \bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_j \right)$  such that

$$Z_1^*V = 0, \quad Z_2^*W = 0, \quad Z_1Z_1^* + VV^* = I, \quad Z_2Z_2^* + WW^* = I.$$

*Proof.* Let

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix} \quad (3.34)$$

where

$$V^{(r)} = \left[ V_{kl}^{(r)} \right]_{k,l=1}^{m, m^2-1} : \bigoplus_{i=1}^m \bigoplus_{j=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_i \rightarrow \bigoplus_{i=1}^m \mathcal{H}'_r \otimes \mathcal{K}'_i$$

for  $r = 1, \dots, m$ , with  $V_{kl}^{(r)}$  is an identity operator if  $k \neq r$  and  $l = m(r-1) + k$  and the other are zero operators. Consider

$$\begin{aligned} E_1^*V_1 &= \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= 0. \end{aligned}$$

For each  $r$ , we have

$$V_r V_r^* = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}.$$

Then we have

$$Z_1^*V = \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{bmatrix} = E_1^*V_1 + E_2^*V_2 + \cdots + E_m^*V_m = 0,$$

$$\begin{aligned} Z_1Z_1^* + VV^* &= \begin{bmatrix} E_1E_1^* & E_1E_2^* & \cdots & E_1E_m^* \\ E_2E_1^* & E_2E_2^* & \cdots & E_2E_m^* \\ \vdots & \vdots & \ddots & \vdots \\ E_mE_1^* & E_mE_2^* & \cdots & E_mE_m^* \end{bmatrix} + \begin{bmatrix} V_1V_1^* & V_1V_2^* & \cdots & V_1V_m^* \\ V_2V_1^* & V_2V_2^* & \cdots & V_2V_m^* \\ \vdots & \vdots & \ddots & \vdots \\ V_mEV_1^* & V_mV_2^* & \cdots & V_mV_m^* \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}. \end{aligned}$$

Let

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_m \end{bmatrix} \quad (3.35)$$

where

$$W^{(s)} = \left[ W_{kl}^{(s)} \right]_{k,l=1}^{n, n^2-1} ; \bigoplus_{i=1}^n \bigoplus_{\substack{j=1 \\ i+j < n^2}}^n \mathcal{H}_i \otimes \mathcal{K}_i \rightarrow \bigoplus_{i=1}^n \mathcal{H}_s \otimes \mathcal{K}_i$$

for  $s = 1, \dots, n$ , with  $W_{kl}^{(s)}$  is an identity operator if  $k \neq s$  and  $l = n(s-1) + k$  and the other are zero operators. A direct computation shows that  $Z_2^*W = 0$  and  $Z_2Z_2^* + WW^* = I$ .  $\square$

**Remark 3.22.** Operator matrices  $V$  and  $W$  are not unique. For example, consider  $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}'_1 \oplus \mathcal{H}'_2)$  and  $\mathbb{B}(\mathcal{K}_1 \oplus \mathcal{K}_2, \mathcal{K}'_1 \oplus \mathcal{K}'_2)$ . By Theorem 4.1, we have

$$V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ \text{---} & \text{---} & \text{---} \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}$$

Consider

$$\tilde{V} = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ \text{---} & \text{---} & \text{---} \\ 0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

Then  $Z_1^* \tilde{V} = 0$  and  $Z_1 Z_1^* + \tilde{V} (\tilde{V})^* = I$ .

**Corollary 3.23.** *Let  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ . Then*

$$AA^* \boxplus BB^* = (A \boxplus B)(A \boxplus B)^* + Z_1^*(A \boxtimes B)WW^*(A \boxtimes B)^*Z_1. \quad (3.36)$$

*Proof.* The proof is similar with Corollary 3.20.  $\square$

**Corollary 3.24.** *Let  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ . Then*

$$AA^* \boxplus BB^* \geq (A \boxplus B)(A \boxplus B)^*. \quad (3.37)$$

### 3.6 Norm bounds, continuity, and compactness of Khatri-Rao products in the operator-norm topology

In this section, we establish some attractive properties of the Khatri-Rao product from analytical point of view. These properties involve certain continuity, convergence and norm estimates.

Recall the following bounds for the operator norm of operator matrices.

**Lemma 3.25** ([11]). *Let  $A = [A_{ij}]_{i,j=1}^{n,n}$  be an operator matrix in  $\mathbb{B}(\mathcal{H})$ . Then*

$$n^{-2} \sum_{i,j=1}^n \|A_{ij}\|_\infty^2 \leq \|A\|_\infty^2 \leq \sum_{i,j=1}^n \|A_{ij}\|_\infty^2. \quad (3.38)$$

**Theorem 3.26.** *For any operator matrices  $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ , we have*

$$\|A \boxplus B\|_\infty \leq n^2 \|A\|_\infty \|B\|_\infty. \quad (3.39)$$

*Proof.* It follows from Lemma 3.25 that

$$\begin{aligned} \|A \boxplus B\|_\infty^2 &\leq \sum_{i,j} \|A_{ij} \otimes B_{ij}\|_\infty^2 = \sum_{i,j} \|A_{ij}\|_\infty^2 \|B_{ij}\|_\infty^2 \\ &\leq \left( \sum_{i,j} \|A_{ij}\|_\infty^2 \right) \left( \sum_{i,j} \|B_{ij}\|_\infty^2 \right) \\ &\leq n^4 \|A\|_\infty^2 \|B\|_\infty^2. \end{aligned}$$

Hence, we obtain the (5.12).  $\square$

**Theorem 3.27.** *Let  $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{ij}] \in \mathbb{B}(\mathcal{K})$  be operator matrices and let  $(A_r)_{r=1}^\infty$  and  $(B_r)_{r=1}^\infty$  be sequences in  $\mathbb{B}(\mathcal{H})$  and  $\mathbb{B}(\mathcal{K})$ , respectively. If  $A_r \rightarrow A$  and  $B_r \rightarrow B$ , then  $A_r \boxplus B_r \rightarrow A \boxplus B$ . That is, the map  $(A, B) \mapsto A \boxplus B$  is continuous with respect to the topology induced by the operator norm.*

*Proof.* Suppose that  $A_r \rightarrow A$  and  $B_r \rightarrow B$ . By using Lemma ?? and Theorem 3.26, we have

$$\begin{aligned} \|A_r \boxplus B_r - A \boxplus B\|_\infty &= \|A_r \boxplus B_r - A \boxplus B_r + A \boxplus B_r - A \boxplus B\|_\infty \\ &\leq \|(A_r - A) \boxplus B_r\|_\infty + \|A \boxplus (B_r - B)\|_\infty \\ &\leq n^2 \|A_r - A\|_\infty \|B_r\|_\infty + n^2 \|A\|_\infty \|B_r - B\|_\infty. \end{aligned}$$

It follows that  $A_r \boxplus B_r \rightarrow A \boxplus B$  in the operator-norm topology.  $\square$

**Lemma 3.28** ([26]). *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be nonzero operators. Then  $A \otimes B$  is compact if and only if both  $A$  and  $B$  are compact.*

**Theorem 3.29.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be nonzero operator matrices with  $n = q$ . Then  $A \boxplus B$  is compact if and only if both  $A$  and  $B$  are compact.*

*Proof.* Write  $A = [A_{ij}]$  and  $B = [B_{ij}]$ . For sufficiency, suppose that  $A$  and  $B$  are compact. By Lemma 3.30, we deduce that  $A_{ij}$  and  $B_{ij}$  are compact for all  $i, j$ . It follows from Lemma 3.28 that  $A_{ij} \otimes B_{ij}$  is compact for all  $i, j$ . Lemma 3.30 ensures the compactness of  $A \boxplus B$ . For necessity part, reverse the previous procedure.  $\square$

### 3.7 Khatri-Rao products on norm ideals of compact operators

In this section, we investigate the Khatri-Rao product on norm ideals of  $\mathbb{B}(\mathcal{H})$ . Recall that any proper ideal of  $\mathbb{B}(\mathcal{H})$  is contained in the ideal of compact operators.

For any compact operator  $A \in \mathbb{B}(\mathcal{H})$ , let  $(s_i(A))_{i=1}^{\infty}$  be the sequence of decreasingly ordered singular values of  $A$  (i.e. eigenvalues of  $|A|$ ). For each  $1 \leq p < \infty$ , the Schatten  $p$ -norm of  $A$  is defined by

$$\|A\|_p = \left( \sum_{i=1}^{\infty} s_i^p(A) \right)^{1/p}.$$

If  $\|A\|_p < \infty$ , then we say that  $A$  is a Schatten  $p$ -class operator. The Schatten  $\infty$ -norm is just the operator norm. For each  $1 \leq p \leq \infty$ , let  $\mathcal{S}_p$  be the Schatten  $p$ -class operators. In particular,  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are the trace class and the Hilbert-Schmidt class, respectively. Each Schatten  $p$ -norm induces a norm ideal of  $\mathbb{B}(\mathcal{H})$  and this ideal is closed under the topology generated by such norm.

**Lemma 3.30.** *Let  $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$  be an operator matrix. Then  $A$  is compact if and only if  $A_{ij}$  is compact for all  $i, j$ .*

*Proof.* If  $A$  is compact, then  $A_{ij} = P_i' A E_j$  is also compact for each  $i, j$  due to the fact that  $\mathcal{S}_{\infty}$  is an ideal of  $\mathbb{B}(\mathcal{H})$ . Conversely, suppose that  $A_{ij}$  is compact for all  $i, j$ . Recall that a bounded linear operator is compact if and only if it maps a bounded sequence into a sequence having a convergent subsequence. Consider a bounded sequence  $(x_r)_{r=1}^{\infty}$  in  $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$ . Write  $x_r = [x_r^{(1)} \ x_r^{(2)} \ \dots \ x_r^{(n)}]^T \in \mathcal{H}$  for each  $r \in \mathbb{N}$ . Consider

$$Ax_r = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_r^{(1)} \\ \vdots \\ x_r^{(n)} \end{bmatrix} = \begin{bmatrix} A_{11}x_r^{(1)} + \cdots + A_{1n}x_r^{(n)} \\ \vdots \\ A_{n1}x_r^{(1)} + \cdots + A_{nn}x_r^{(n)} \end{bmatrix}.$$

For each  $l = 1, 2, \dots, n$ , since  $(x_r^{(l)})_{r=1}^{\infty}$  is bounded, the sequence  $(A_{ij}x_r^{(l)})_{r=1}^{\infty}$  has a convergent subsequence, namely,  $(A_{ij}x_{r_k}^{(l)})_{k=1}^{\infty}$ . Hence,

$$\begin{bmatrix} A_{11}x_{r_k}^{(1)} + \cdots + A_{1n}x_{r_k}^{(n)} \\ \vdots \\ A_{n1}x_{r_k}^{(1)} + \cdots + A_{nn}x_{r_k}^{(n)} \end{bmatrix}$$

is a desired convergent subsequence of  $(Ax_r)_{r=1}^{\infty}$ .  $\square$

**Lemma 3.31** ([11]). *Let  $A = [A_{ij}]_{i,j=1}^{n,n}$  be an operator matrix in the Schatten  $p$ -class.*

(i) *For  $1 \leq p \leq 2$ , we have*

$$\sum_{i,j=1}^n \|A_{ij}\|_p^2 \leq \|A\|_p^2 \leq n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2. \quad (3.40)$$

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(ii) For  $2 \leq p < \infty$ , we have

$$n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2 \leq \|A\|_p^2 \leq \sum_{i,j=1}^n \|A_{ij}\|_p^2. \quad (3.41)$$

**Lemma 3.32.** *Let  $1 \leq p < \infty$ . An operator matrix  $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$  is a Schatten  $p$ -class operator if and only if  $A_{ij}$  is a Schatten  $p$ -class operator for all  $i, j$ .*

*Proof.* This is a direct consequence of the norm estimations in Lemma 3.31.  $\square$

**Lemma 3.33.** *Let  $1 \leq p \leq \infty$ . Let  $A = [A_{ij}]_{i,j=1}^{n,n}$  be an operator matrix in the class  $\mathcal{S}_p$  and let  $(A_r)_{r=1}^\infty$  be a sequence in  $\mathcal{S}_p$  where  $A_r = [A_{ij}^{(r)}]_{i,j=1}^{n,n}$  for each  $r \in \mathbb{N}$ . Then  $A_r \rightarrow A$  in  $\mathcal{S}_p$  if and only if  $A_{ij}^{(r)} \rightarrow A_{ij}$  in  $\mathcal{S}_p$  for all  $i, j = 1, \dots, n$ .*

*Proof.* Lemma 3.32 assures that  $A_{ij}$  and  $A_{ij}^{(r)}$  belong to  $\mathcal{S}_p$  for any  $i, j = 1, \dots, n$  and  $r \in \mathbb{N}$ . Consider the case  $1 \leq p \leq 2$ . Suppose that  $A_r \rightarrow A$  in  $\mathcal{S}_p$ . For any fixed  $i, j \in \{1, \dots, n\}$ , we have from the estimation (3.40) that

$$\|A_{ij}^{(r)} - A_{ij}\|_p^2 \leq \sum_{i,j=1}^n \|A_{ij}^{(r)} - A_{ij}\|_p^2 \leq \|A_r - A\|_p^2.$$

Hence,  $A_{ij}^{(r)} \rightarrow A_{ij}$  in  $\mathcal{S}_p$ .

Conversely, suppose that  $A_{ij}^{(r)} \rightarrow A_{ij}$  in  $\mathcal{S}_p$  for each  $i, j$ . Lemma 3.31 implies that

$$\|A_r - A\|_p^2 \leq n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}^{(r)} - A_{ij}\|_p^2.$$

Hence,  $A_r \rightarrow A$  in  $\mathcal{S}_p$ .

The case  $2 < p < \infty$  and the case  $p = \infty$  are done by using the norm estimations (3.41) and (3.38), respectively.  $\square$

**Theorem 3.34.** *The Khatri-Rao product is continuous with respect to the topology induced by the operator norm. More precisely, if a sequence  $(A_r)_{r=1}^\infty$  converges to  $A$  and a sequence  $(B_r)_{r=1}^\infty$  converges to  $B$  in the ideal of compact operator  $\mathcal{S}_\infty$ , then  $A_r \boxtimes B_r$  converges to  $A \boxtimes B$  in  $\mathcal{S}_\infty$ .*

*Proof.* Write  $A = [A_{ij}]$  and  $B = [B_{ij}]$ . Since  $A_r \rightarrow A$  and  $B_r \rightarrow B$  in  $\mathcal{S}_p$ , we have by Lemma 3.33 that  $A_{ij}^{(r)} \rightarrow A_{ij}$  and  $B_{ij}^{(r)} \rightarrow B_{ij}$  for all  $i, j$ . It follows that

$$\begin{aligned} & \|A_{ij}^{(r)} \otimes B_{ij}^{(r)} - A_{ij} \otimes B_{ij}\|_\infty \\ &= \|A_{ij}^{(r)} \otimes B_{ij}^{(r)} - A_{ij}^{(r)} \otimes B_{ij} + A_{ij}^{(r)} \otimes B_{ij} - A_{ij} \otimes B_{ij}\|_\infty \\ &\leq \|A_{ij}^{(r)} \otimes (B_{ij}^{(r)} - B_{ij})\|_\infty + \|(A_{ij}^{(r)} - A_{ij}) \otimes B_{ij}\|_\infty \\ &= \|A_{ij}^{(r)}\|_p \|B_{ij}^{(r)} - B_{ij}\|_\infty + \|A_{ij}^{(r)} - A_{ij}\|_\infty \|B_{ij}\|_\infty \\ &\rightarrow \|A_{ij}\|_\infty \cdot 0 + 0 \cdot \|B_{ij}\|_\infty = 0. \end{aligned}$$

Hence,  $A_{ij}^{(r)} \otimes B_{ij}^{(r)} \rightarrow A_{ij} \otimes B_{ij}$  in  $\mathcal{S}_\infty$  for all  $i, j$ . By using the fact that  $\mathcal{S}_\infty$  is closed set, we have  $A_r \boxdot B_r \rightarrow A \boxdot B$  in  $\mathcal{S}_\infty$ .  $\square$

The following theorem supplies bounds for Schatten 1-norm of the Khatri-Rao product of operators.

**Theorem 3.35.** *For any nonzero compact operator matrices  $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ , we have*

$$\|A \boxdot B\|_1 \leq n \|A\|_1 \|B\|_1. \quad (3.42)$$

*Hence,  $A \boxdot B$  is trace-class if and only if both  $A$  and  $B$  are trace-class.*

*Proof.* Suppose that both  $A$  and  $B$  are nonzero and compact. Then the operator  $A \boxdot B$  is compact by Theorem 3.29. It follows from the norm bound (3.40) that

$$\begin{aligned} \|A \boxdot B\|_1^2 &\leq n^2 \sum_{i,j} \|A_{ij} \otimes B_{ij}\|_1^2 = n^2 \sum_{i,j} \|A_{ij}\|_1^2 \|B_{ij}\|_1^2 \\ &\leq n^2 \left( \sum_{i,j} \|A_{ij}\|_1^2 \right) \left( \sum_{i,j} \|B_{ij}\|_1^2 \right) \leq n^2 \|A\|_1^2 \|B\|_1^2. \end{aligned}$$

Hence, we obtain the bound (3.42).  $\square$

**Theorem 3.36.** *For any nonzero compact operator matrices  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  with  $n = q$ , we have*

$$\|A \boxdot B\|_2 \leq \|A\|_2 \|B\|_2. \quad (3.43)$$

*Hence,  $A \boxdot B$  is a Hilbert-Schmidt operator if and only if both  $A$  and  $B$  are Hilbert-Schmidt operators.*

*Proof.* Since both  $A$  and  $B$  are nonzero and compact, the operator  $A \boxtimes B$  is compact by Theorem 3.29. Write  $A = [A_{ij}]_{i,j=1}^{n,n}$  and  $B = [B_{ij}]_{i,j=1}^{n,n}$ . Then by Lemma 3.31(ii), we have

$$\begin{aligned} \|A \boxtimes B\|_2^2 &= \sum_{i,j} \|A_{ij} \otimes B_{ij}\|_2^2 = \sum_{i,j} \|A_{ij}\|_2^2 \|B_{ij}\|_2^2 \\ &\leq \left( \sum_{i,j} \|A_{ij}\|_2^2 \right) \left( \sum_{i,j} \|B_{ij}\|_2^2 \right) = \|A\|_2^2 \|B\|_2^2. \end{aligned}$$

Hence, we obtain the bound (3.43).  $\square$

**Theorem 3.37.** *For any  $p \in \{1, 2\}$ , the Khatri-Rao product is continuous with respect to the topology induced by the Schatten  $p$ -norm. More precisely, if a sequence  $(A_r)_{r=1}^\infty$  converges to  $A$  and a sequence  $(B_r)_{r=1}^\infty$  converges to  $B$  in the norm ideal  $\mathcal{S}_p$ , then  $A_r \boxtimes B_r$  converges to  $A \boxtimes B$  in  $\mathcal{S}_p$ .*

*Proof.* Suppose that the sequences  $(A_r)_{r=1}^\infty$  and  $(B_r)_{r=1}^\infty$  converge to  $A$  and  $B$ , respectively, in  $\mathcal{S}_2$ . By applying Lemma ?? and the bound (3.43), we get

$$\begin{aligned} \|A_r \boxtimes B_r - A \boxtimes B\|_2 &= \|A_r \boxtimes B_r - A \boxtimes B_r + A \boxtimes B_r - A \boxtimes B\|_2 \\ &\leq \|(A_r - A) \boxtimes B_r\|_2 + \|A \boxtimes (B_r - B)\|_2 \\ &\leq \|A_r - A\|_2 \|B_r\|_2 + \|A\|_2 \|B_r - B\|_2 \\ &\rightarrow 0 \cdot \|B\|_2 + \|A\|_2 \cdot 0 = 0. \end{aligned}$$

Hence,  $A_r \boxtimes B_r \rightarrow A \boxtimes B$  in  $\mathcal{S}_2$ . Similarly, we obtain, by using the bound (3.42),  $A_r \boxtimes B_r \rightarrow A \boxtimes B$  in  $\mathcal{S}_1$ .  $\square$

# CHAPTER 4

## Khatri-Rao Products and Selection Operators

We develop further theory for Khatri-Rao products of Hilbert space operators in connections with selection operators. We provide two constructions related to selection operators. Then we establish certain identities and inequalities involving Khatri-Rao and Tracy-Singh products. As consequences, we obtain some characterizations for the mixed product property concerning the Khatri-Rao product of operators.

### 4.1 Introduction

This paper concerns operator extensions of certain matrix products, namely, the Kronecker (tensor) product, the Tracy-Singh product, and the Khatri-Rao product. Fundamental theory for these matrix products are collected, for instance, in [7, 8, 20, 30, 42, 43, 46] and references therein. Denote by  $M_{m,n}(\mathbb{C})$  the algebra of  $m$ -by- $n$  complex matrices. Recall that the Kronecker product of  $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$  and  $B \in M_{p,q}(\mathbb{C})$  is given by

$$A \hat{\otimes} B = [a_{ij}B]_{ij}.$$

Consider partitioned matrices  $A$  and  $B$  such that the  $(i, j)$ th block of  $A$  is  $A_{ij}$  and the  $(k, l)$ th block of  $B$  is  $B_{kl}$ . The Tracy-Singh product [40] of  $A$  and  $B$  is defined by

$$A \hat{\boxtimes} B = [[A_{ij} \hat{\otimes} B_{kl}]_{kl}]_{ij}. \quad (4.1)$$

The Khatri-Rao product [19] is defined for two partitioned matrices  $A = [A_{ij}]$  and  $B = [B_{ij}]$  as follows

$$A \hat{\boxdot} B = [A_{ij} \hat{\otimes} B_{ij}]_{ij}. \quad (4.2)$$

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The tensor product of Hilbert space operators can be viewed as an extension of the Kronecker product of complex matrices. Recall that the tensor product of  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$  is the unique bounded linear operator from  $\mathcal{H} \otimes \mathcal{K}$  into  $\mathcal{H}' \otimes \mathcal{K}'$  such that  $(A \otimes B)(x \otimes y) = Ax \otimes By$  for all  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ . Recently, the Tracy-Singh product and the Khatri-Rao product for matrices were generalized to those for operators acting on the direct sum of Hilbert spaces, see [31, 32, 33]. Fundamental algebraic and order properties of operator Khatri-Rao products are investigated in [33]. That paper also provides a construction of a unital positive linear map taking the Tracy-Singh product of two operators to their Khatri-Rao product. Such a linear map appears in the form  $X \mapsto Z^*AZ$  where  $Z$  is an isometry, called a selection operator. See details in Section 2.

The present chapter contains further development on operator Khatri-Rao products in relations with Tracy-Singh products and selection operators. First, we provide two constructions related to selection operators. Consequently, we establish some operator identities and inequalities involving Khatri-Rao and Tracy-Singh products. Finally, we obtain some characterizations for the mixed product property concerning the Khatri-Rao product of operators.

## 4.2 Tracy-Singh products and Khatri-Rao products for operators

Throughout this paper, let  $\mathcal{H}, \mathcal{H}', \mathcal{K}$  and  $\mathcal{K}'$  be complex separable Hilbert spaces. When  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert spaces, let us denote by  $\mathbb{B}(\mathcal{X}, \mathcal{Y})$  the space of all bounded linear operators from  $\mathcal{X}$  into  $\mathcal{Y}$  and abbreviate  $\mathbb{B}(\mathcal{X}, \mathcal{X})$  to  $\mathbb{B}(\mathcal{X})$ . Capital letters always denote a Hilbert space operator. In particular,  $I$  and  $O$  stand for the identity and the zero operator, respectively.

In order to define Tracy-Singh products of operators, we fix the following decompositions

$$\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^m \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{j=1}^q \mathcal{K}_j, \quad \mathcal{K}' = \bigoplus_{i=1}^p \mathcal{K}'_i. \quad (4.3)$$

where all of  $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$  are Hilbert spaces. For each  $j$  and  $l$ , let  $M_j : \mathcal{H}_j \rightarrow \mathcal{H}$  and  $N_l : \mathcal{K}_l \rightarrow \mathcal{K}$  be the canonical injections. For each  $i$  and  $k$ , let  $P_i : \mathcal{H}' \rightarrow \mathcal{H}'_i$  and  $Q_k : \mathcal{K}' \rightarrow \mathcal{K}'_k$  be the canonical projections. Given  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ , put  $A_{ij} = P_i A M_j \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}'_i)$  for each  $i, j$ . Thus we can write  $A$  in the operator-matrix form  $A = [A_{ij}]_{i,j=1}^{m,n}$ . Similarly, given  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , let  $B_{kl} = Q_k B N_l \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}'_k)$

for each  $k = 1, \dots, p$  and  $l = 1, \dots, q$ . We can identify  $B$  with the operator matrix  $B = [B_{kl}]_{k,l=1}^{p,q}$ .

**Definition 4.1.** The *Tracy-Singh product* of  $A$  and  $B$  is defined to be the bounded linear operator from  $\bigoplus_{j,l=1}^{n,q} \mathcal{H}_j \otimes \mathcal{K}_l$  to  $\bigoplus_{i,k=1}^{m,p} \mathcal{H}'_i \otimes \mathcal{K}'_k$  represented by

$$A \boxtimes B = [[A_{ij} \otimes B_{kl}]_{kl}]_{ij}. \quad (4.4)$$

If both factor  $A$  and  $B$  consist of only one block, then  $A \boxtimes B = A \otimes B$ .

**Lemma 4.2** ([31]). *The following properties of the Tracy-Singh product for operators hold (provided that each term is well-defined):*

1. *Compatibility with adjoints:*  $(A \boxtimes B)^* = A^* \boxtimes B^*$ .
2. *Mixed-product property:*  $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$ .
3. *Monotonicity:* if  $A \geq B \geq 0$  and  $C \geq D \geq 0$ , then  $A \boxtimes B \geq C \boxtimes D \geq 0$ .

From now on, we fix the decomposition (4.3), and assume  $n = q$  and  $m = p$ .

**Definition 4.3.** The *Khatri-Rao product* of  $A = [A_{ij}]_{i,j=1}^{m,n}$  and  $B = [B_{ij}]_{i,j=1}^{m,n}$  is defined to be a bounded linear operator from  $\bigoplus_{j=1}^n \mathcal{H}_j \otimes \mathcal{K}_j$  to  $\bigoplus_{i=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_i$  represented by the operator matrix

$$A \boxdot B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{m,n}. \quad (4.5)$$

**Lemma 4.4** ([33]). *For  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , we have  $(A \boxdot B)^* = A^* \boxdot B^*$ .*

Fix an ordered tuple  $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$  of Hilbert spaces. Define the ordered pair  $(Z_1, Z_2)$  of selection operators associated with  $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$  by [33]:

$$Z_1 = \begin{bmatrix} E_1 \\ \vdots \\ E_m \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}. \quad (4.6)$$

Here, for each  $r = 1, \dots, m$

$$E_r = \left[ E_{gh}^{(r)} \right]_{g,h=1}^{m,m} : \bigoplus_{k=1}^m \mathcal{H}'_k \otimes \mathcal{K}'_k \rightarrow \bigoplus_{l=1}^m \mathcal{H}'_r \otimes \mathcal{K}'_l$$

with  $E_{gh}^{(r)}$  is an identity operator if  $g = h = r$  and the others are zero operators. For each  $s = 1, \dots, n$ , the operator  $F_s$  is defined by

$$F_s = \left[ F_{gh}^{(s)} \right]_{g,h=1}^{n,n} : \bigoplus_{i=1}^n \mathcal{H}_i \otimes \mathcal{K}_i \rightarrow \bigoplus_{j=1}^n \mathcal{H}_s \otimes \mathcal{K}_j$$

with  $F_{gh}^{(s)}$  is an identity operator if  $g = h = s$  and the others are zero operators. From the construction, the operator  $Z_i$  is an isometry and  $Z_i Z_i^* \leq I$  for  $i = 1, 2$ . When  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $Z_1 = Z_2$ .

**Lemma 4.5** ([33]). *Let  $(Z_1, Z_2)$  be the ordered pair of selection operators associated with the ordered tuple  $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ . For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , we have*

$$A \boxdot B = Z_1^*(A \boxtimes B)Z_2. \quad (4.7)$$

For the case  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $Z_1 = Z_2 := Z$  and hence for any  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ ,

$$A \boxdot B = Z^*(A \boxtimes B)Z. \quad (4.8)$$

### 4.3 Two constructions related to selection operators

In this section, we construct certain operators related to selection operators.

**Theorem 4.1.** *Let  $(Z_1, Z_2)$  be the ordered pair of selection operators associated with an ordered tuple  $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ . Then there exist operators*

$$\begin{aligned} V &: \bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_j \rightarrow \bigoplus_{i=1}^m \bigoplus_{j=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_j, \\ W &: \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_j \rightarrow \bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_j \end{aligned}$$

such that  $Z_1^*V = 0$ ,  $Z_2^*W = 0$ ,  $Z_1 Z_1^* + VV^* = I$  and  $Z_2 Z_2^* + WW^* = I$ . If, in addition,  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $V = W$ .

*Proof.* Let

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix} \quad (4.9)$$

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where

$$V^{(r)} = \left[ V_{kl}^{(r)} \right]_{k,l=1}^{m,m^2-1} : \bigoplus_{i=1}^m \bigoplus_{\substack{j=1 \\ i+j < m^2}}^m \mathcal{H}'_i \otimes \mathcal{K}'_i \rightarrow \bigoplus_{i=1}^m \mathcal{H}'_r \otimes \mathcal{K}'_i$$

for  $r = 1, \dots, m$ , with  $V_{kl}^{(r)}$  is an identity operator if  $k \neq r$  and  $l = m(r-1) + k$  and the others are zero operators. Note that

$$\begin{aligned} & E_1^* V_1 \\ &= \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \\ &= 0. \end{aligned}$$

For each  $r$ , we have

$$V_r V_r^* = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}$$

Then we obtain

$$Z_1^* V = \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{bmatrix} = E_1^* V_1 + E_2^* V_2 + \cdots + E_m^* V_m = 0,$$

$$Z_1 Z_1^* + V V^*$$

$$\begin{aligned} &= \begin{bmatrix} E_1 E_1^* & E_1 E_2^* & \cdots & E_1 E_m^* \\ E_2 E_1^* & E_2 E_2^* & \cdots & E_2 E_m^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m E_1^* & E_m E_2^* & \cdots & E_m E_m^* \end{bmatrix} + \begin{bmatrix} V_1 V_1^* & V_1 V_2^* & \cdots & V_1 V_m^* \\ V_2 V_1^* & V_2 V_2^* & \cdots & V_2 V_m^* \\ \vdots & \vdots & \ddots & \vdots \\ V_m E V_1^* & V_m V_2^* & \cdots & V_m V_m^* \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}. \end{aligned}$$

Now, let

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_m \end{bmatrix} \quad (4.10)$$

where

$$W^{(s)} = \left[ W_{kl}^{(s)} \right]_{k,l=1}^{n,n^2-1} : \bigoplus_{\substack{i=1 \\ i+j < n^2}}^n \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_i \rightarrow \bigoplus_{i=1}^n \mathcal{H}_s \otimes \mathcal{K}_i$$

for  $s = 1, \dots, n$ , with  $W_{kl}^{(s)}$  is an identity operator if  $k \neq s$  and  $l = n(s-1) + k$  and others are zero operators. A direct computation shows that  $Z_2^*W = 0$  and  $Z_2Z_2^* + WW^* = I$ . When  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $V_i = W_i$  for all  $i = 1, \dots, m$ , i.e.  $V = W$ .  $\square$

**Theorem 4.2.** Fix the decomposition (4.3) with  $n = q$  and  $m = p$ . Suppose further that  $\mathcal{H}_i = \mathcal{X}$ ,  $\mathcal{K}_i = \mathcal{Y}$ ,  $\mathcal{H}'_j = \mathcal{X}'$  and  $\mathcal{K}'_j = \mathcal{Y}'$  for all  $i = 1, \dots, n$  and  $j = 1, \dots, m$ . Let  $(Z_1, Z_2)$  be the ordered pair of associated selection operators. Then there exist operators

$$Q_1 : \bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^m \mathcal{X}' \otimes \mathcal{Y}' \rightarrow \bigoplus_{i=1}^m \bigoplus_{j=1}^m \mathcal{X}' \otimes \mathcal{Y}',$$

$$Q_2 : \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^n \mathcal{X} \otimes \mathcal{Y} \rightarrow \bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathcal{X} \otimes \mathcal{Y}$$

such that  $Z_i^*Q_i = 0$ ,  $Q_i^*Q_i = I$  and  $Z_iZ_i^* + Q_iQ_i^* = I$  for  $i = 1, 2$ . If, in addition,  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $Q_1 = Q_2$ .

*Proof.* Consider

$$Q_1 = \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} F_2 & F_3 & \cdots & F_n \\ F_3 & F_4 & \cdots & F_1 \\ \vdots & \vdots & \ddots & \vdots \\ F_1 & F_2 & \cdots & F_{n-1} \end{bmatrix}. \quad (4.11)$$

Then calculations reveal that

$$Z_1^*Q_1 = \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix},$$

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$$\begin{aligned}
Q_1^*Q_1 &= \begin{bmatrix} E_2^* & E_3^* & \cdots & E_1^* \\ E_3^* & E_4^* & \cdots & E_2^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m^* & E_1^* & \cdots & E_{m-1}^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} \\
&= \begin{bmatrix} \sum E_i^* E_i & 0 & \cdots & 0 \\ 0 & \sum E_i^* E_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum E_i^* E_i \end{bmatrix} = I,
\end{aligned}$$

$$\begin{aligned}
Q_1^*Q_1 + Z_1Z_1^* &= \begin{bmatrix} E_2^* & E_3^* & \cdots & E_1^* \\ E_3^* & E_4^* & \cdots & E_2^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m^* & E_1^* & \cdots & E_{m-1}^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix} \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i \neq 1} E_i E_j^* & 0 & \cdots & 0 \\ 0 & \sum_{i \neq 2} E_i E_j^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{i \neq m} E_i E_j^* \end{bmatrix} + \begin{bmatrix} E_1 E_1^* & 0 & \cdots & 0 \\ 0 & E_2 E_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_m E_m^* \end{bmatrix} \\
&= \begin{bmatrix} \sum E_i E_i^* & 0 & \cdots & 0 \\ 0 & \sum E_i E_i^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum E_i E_i^* \end{bmatrix} = I.
\end{aligned}$$

Similarly, we have  $Z_2^*Q_2 = 0$ ,  $Q_2^*Q_2 = I$  and  $Z_2Z_2^* + Q_2Q_2^* = I$ . When  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $E_i = F_i$  for all  $i = 1, \dots, m$ , i.e.  $Q_1 = Q_2$ .  $\square$

#### 4.4 Operator identities and inequalities concerning Khatri-Rao products, Tracy-Singh products, and selection operators

In this section, we apply the construction in Section 3 to establish certain operator identities and inequalities concerning Khatri-Rao products, Tracy-Singh products, and selection operators.

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**Theorem 4.3.** Let  $(Z_1, Z_2)$  be the ordered pair of selection operators associated with an ordered tuple  $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ . Let  $V$  and  $W$  be operator matrices defined by (4.9) and (5.20). For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , we have

$$AA^* \square BB^* = (A \square B)(A^* \square B^*) + Z_1^*(A \boxtimes B)WW^*(A^* \boxtimes B^*)Z_1, \quad (4.12)$$

$$A^*A \square B^*B = (A^* \square B^*)(A \square B) + Z_2^*(A^* \boxtimes B^*)VV^*(A \boxtimes B)Z_2. \quad (4.13)$$

*Proof.* Since  $AA^* \in \mathbb{B}(\mathcal{H}')$  and  $BB^* \in \mathbb{B}(\mathcal{K}')$ , the ordered pair of selection operators associated with  $(\mathcal{H}', \mathcal{H}', \mathcal{K}', \mathcal{K}')$  is given by  $(Z_1, Z_1)$ . By using Lemmas 4.2 and 4.5, and Theorem 4.1, we get

$$\begin{aligned} AA^* \square BB^* &= Z_1^*(AA^* \boxtimes BB^*)Z_1 \\ &= Z_1^*(A \boxtimes B)(A \boxtimes B)^*Z_1 \\ &= Z_1^*(A \boxtimes B)(Z_2Z_2^* + WW^*)(A \boxtimes B)^*Z_1 \\ &= Z_1^*(A \boxtimes B)Z_2Z_2^*(A \boxtimes B)^*Z_1 + Z_1^*(A \boxtimes B)WW^*(A \boxtimes B)^*Z_1 \\ &= (A \square B)(A \square B)^* + Z_1^*(A \boxtimes B)WW^*(A \boxtimes B)^*Z_1. \end{aligned}$$

Now, for inequality (4.13), note that  $A^*A \in \mathbb{B}(\mathcal{H})$  and  $B^*B \in \mathbb{B}(\mathcal{K})$ . In this case, the pair of associated selection operators is  $(Z_2, Z_2)$ . It follows that

$$\begin{aligned} A^*A \square B^*B &= Z_2^*(A^*A \boxtimes B^*B)Z_2 \\ &= Z_2^*(A \boxtimes B)^*(A \boxtimes B)Z_2 \\ &= Z_2^*(A \boxtimes B)^*(Z_1Z_1^* + VV^*)(A \boxtimes B)Z_2 \\ &= Z_2^*(A \boxtimes B)^*Z_1Z_1^*(A \boxtimes B)Z_2 + Z_2^*(A \boxtimes B)^*VV^*(A \boxtimes B)Z_2 \\ &= (A^* \square B^*)(A \square B) + Z_2^*(A^* \boxtimes B^*)VV^*(A \boxtimes B)Z_2. \square \end{aligned}$$

**Corollary 4.6.** Let  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$  be operator matrices. Then

$$AA^* \square BB^* \geq (A \square B)(A^* \square B^*). \quad (4.14)$$

*Proof.* It follows immediately from Theorem 4.3.  $\square$

**Theorem 4.4.** Assume the hypothesis of Theorem 4.2. For any  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , we have

$$AA^* \square BB^* = (A \square B)(A^* \square B^*) + Z_1^*(A \boxtimes B)Q_2Q_2^*(A^* \boxtimes B^*)Z_1, \quad (4.15)$$

$$A^*A \square B^*B = (A^* \square B^*)(A \square B) + Z_2^*(A^* \boxtimes B^*)Q_1Q_1^*(A \boxtimes B)Z_2, \quad (4.16)$$

where  $Q_1$  and  $Q_2$  are operator matrices in (4.11).

*Proof.* The proof is similar to that of Theorem 4.3. Instead of Theorem 4.1, we apply Theorem 4.2.  $\square$

## 4.5 Characterizations of the mixed product property for Khatri-Rao products

In general, the mixed product property

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$$

does not hold for compatible operator matrices  $A, B, C, D$ . It is interesting to find necessary and sufficient conditions for which this property holds. Indeed, we have the following assertions.

**Theorem 4.5.** *Assume the notations in Theorem 4.3. For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , the following statements are equivalent:*

- (i)  $AC \boxtimes BD = (A \boxtimes B)(C \boxtimes D)$  for all  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ ,
- (ii)  $AA^* \boxtimes BB^* = (A \boxtimes B)(A^* \boxtimes B^*)$ ,
- (iii)  $Z_1^*(A \boxtimes B)W = 0$ .

*Proof.* It is clear that (i) $\Rightarrow$ (ii). To prove (ii) $\Rightarrow$ (iii), suppose (ii). By Theorem 4.3, (ii) holds only if

$$[Z_1^*(A \boxtimes B)W][W^*(A^* \boxtimes B^*)Z_1] = 0,$$

i.e.,  $Z_1^*(A \boxtimes B)W = 0$ .

(iii) $\Rightarrow$ (i): Assume the condition (iii) holds. Note that by Theorem 4.1 we have

$$Z_1^*(A \boxtimes B)(I - Z_2Z_2^*) = Z_1^*(A \boxtimes B)WW^* = 0,$$

and hence  $Z_1^*(A \boxtimes B) = Z_1^*(A \boxtimes B)Z_2Z_2^*$ . For any  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ , we have by Lemmas 4.2 and 4.5 that

$$\begin{aligned} AC \boxtimes BD &= Z_1^*(AC \boxtimes BD)Z_1 \\ &= Z_1^*(A \boxtimes B)(C \boxtimes D)Z_1 \\ &= Z_1^*(A \boxtimes B)Z_2Z_2^*(C \boxtimes D)Z_1 \\ &= (A \boxtimes B)(C \boxtimes D). \end{aligned}$$

Thus we arrive at (i).  $\square$

**Theorem 4.6.** *Assume the notations in Theorem 4.3. For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , the following statements are equivalent:*

- (i)  $CA \boxdot DB = (C \boxdot D)(A \boxdot B)$  for all  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ ,
- (ii)  $A^*A \boxdot B^*B = (A^* \boxdot B^*)(A \boxdot B)$ ,
- (iii)  $V^*(A \boxtimes B)Z_2 = 0$ .

*Proof.* Clearly, (i)  $\Rightarrow$  (ii). The assertion (ii)  $\Rightarrow$  (iii) follows from Theorem 4.3. Now, suppose that (iii) holds. Then  $VV^*(A \boxtimes B)Z_2 = 0$ . Using Theorem 4.1, we get

$$(I - Z_1Z_1^*)(A \boxtimes B)Z_1 = VV^*(A \boxtimes B)Z_1 = 0$$

which implies  $(A \boxtimes B)Z_1 = Z_1Z_1^*(A \boxtimes B)Z_1$ . For any  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ , we have by Lemmas 4.2 and 4.5 that

$$\begin{aligned} CA \boxdot DB &= Z_2^*(CA \boxtimes DB)Z_2 \\ &= Z_2^*(C \boxtimes D)(A \boxtimes B)Z_2 \\ &= Z_2^*(C \boxtimes D)Z_1Z_1^*(A \boxtimes B)Z_2 \\ &= (C \boxdot D)(A \boxdot B). \square \end{aligned}$$

**Theorem 4.7.** *Assume the hypothesis of Theorem 4.2. For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , the following conditions are equivalent:*

- (i)  $AC \boxdot BD = (A \boxdot B)(C \boxdot D)$  for all  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ ,
- (ii)  $AA^* \boxdot BB^* = (A \boxdot B)(A^* \boxdot B^*)$ ,
- (iii)  $Z_1^*(A \boxtimes B)Q_2 = 0$ .

*Proof.* The proof is similar to that of Theorem 4.5. □

**Theorem 4.8.** *Assume the hypothesis of Theorem 4.2. For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , the following statements are equivalent:*

- (i)  $CA \boxdot DB = (C \boxdot D)(A \boxdot B)$  for all  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ ,
- (ii)  $A^*A \boxdot B^*B = (A^* \boxdot B^*)(A \boxdot B)$ ,
- (iii)  $Q_1^*(A \boxtimes B)Z_2 = 0$ .

*Proof.* The proof is similar to that of Theorem 4.6. □

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# CHAPTER 5

## Tracy-Singh Sums for Hilbert space operators

In this paper, we define the Tracy-Singh sum for operator matrices and establish some properties involving this sum.

### 5.1 Introduction

In the areas of linear algebra, the Kronecker product is very important. Let  $M_n(\mathbb{C})$  denote the set of  $n$ -by- $n$  complex matrices. For any  $A \in M_m(\mathbb{C})$  and  $B \in M_n(\mathbb{C})$ , the Kronecker product of  $A$  and  $B$  is given by

$$A \hat{\otimes} B = [a_{ij}B]_{ij}. \quad (5.1)$$

This matrix product is generalized to the Tracy-Singh product of partitioned matrices by Tracy and Singh [40]. Let  $A = [A_{ij}]$  be partitioned with  $A_{ij}$  of order  $m_i \times m_j$  as the  $(i, j)$ th submatrix and let  $B = [B_{kl}]$  be a partitioned matrix with  $B_{kl}$  of order  $n_k \times n_l$  as the  $(k, l)$ th submatrix, where  $\sum_{i=1}^r m_i = m$  and  $\sum_{k=1}^s n_k = n$ . The Tracy-Singh product of  $A$  and  $B$  is defined by

$$A \hat{\boxtimes} B = [[A_{ij} \hat{\otimes} B_{kl}]_{kl}]_{ij}. \quad (5.2)$$

From Tracy-Singh products of matrices, Al Zhou and Kilicman [8] defined Tracy-Singh sum of matrices and gave some result of this sum. The Tracy-Singh sum is defined to be

$$A \hat{\boxplus} B = A \hat{\boxtimes} I_n + I_m \hat{\boxtimes} B. \quad (5.3)$$

where  $I_m$  and  $I_n$  are block identity matrices of order  $m \times m$  and  $n \times n$ , respectively.

In this paper, we define the Tracy-Singh sum for operator matrices and derive some results involving this sum. We establish equalities and inequalities by using basic properties of Tracy-Singh products [31, 32]. Some results in [8] are extended to the case of Tracy-Singh sum of operator matrices in this paper.

This chapter is organized as follows. In Section 5.2, we introduce the definition of the Tracy-Singh product and basic results. In Section 5.3, we investigate algebraic and order properties of Tracy-Singh sums for operators. In Section 5.4, we establish some results involving binomial theorem and Tracy-Singh sums. Analytic properties of Tracy-Singh sums is presented in Section 5.5.

## 5.2 Preliminaries

Throughout this chapter, let  $\mathcal{H}$  and  $\mathcal{K}$  be complex separable Hilbert spaces. When  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert spaces, denote by  $\mathbb{B}(\mathcal{X}, \mathcal{Y})$  the Banach space of bounded linear operators from  $\mathcal{X}$  into  $\mathcal{Y}$ , and abbreviate  $\mathbb{B}(\mathcal{X}, \mathcal{X})$  to  $\mathbb{B}(\mathcal{X})$ . Unless otherwise stated, capital letters mean operators on a Hilbert space.

The projection theorem for Hilbert spaces allows us to decompose

$$\mathcal{H} = \bigoplus_{i=1}^m \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{k=1}^n \mathcal{K}_k,$$

where  $\mathcal{H}_i$ 's and  $\mathcal{K}_k$ 's are Hilbert spaces. Thus any operator  $A \in \mathbb{B}(\mathcal{H})$  can be expressed uniquely as an operator matrix

$$A = [A_{ij}]_{i,j=1}^{m,m}$$

where  $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$  for each  $i, j = 1, \dots, m$ . Similarly, an operator  $B$  in  $\mathbb{B}(\mathcal{K})$  can be written uniquely as

$$B = [B_{kl}]_{k,l=1}^{n,n}$$

where  $B_{kl} \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k)$  for each  $k, l = 1, \dots, n$ .

Recall that the tensor product of  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  is the unique bounded linear operator from  $\mathcal{H} \otimes \mathcal{K}$  into itself such that

$$(A \otimes B)(x \otimes y) = Ax \otimes By \tag{5.4}$$

for all  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ .

**Definition 5.1** ([31]). Let  $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$  be operator matrices defined as above. The *Tracy-Singh product* of  $A$  and  $B$  We define the *Tracy-Singh product* of  $A$  and  $B$  to be the operator matrix

$$A \boxtimes B = [[A_{ij} \otimes B_{kl}]_{kl}]_{ij} \tag{5.5}$$

which is a bounded linear operator from  $\bigoplus_{i,k=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_k$  into itself.

Recall that for each  $A \in \mathbb{M}_m(\mathbb{C})$  and  $B \in \mathbb{M}_n(\mathbb{C})$ , the induced maps

$$L_A : \mathbb{C}^m \rightarrow \mathbb{C}^m, x \mapsto Ax \quad \text{and} \quad L_B : \mathbb{C}^n \rightarrow \mathbb{C}^n, x \mapsto Bx$$

are bounded linear operators.

**Lemma 5.2** ([31]). *For each complex matrices  $A \in \mathbb{M}_m(\mathbb{C})$  and  $B \in \mathbb{M}_n(\mathbb{C})$ , we have*

$$L_A \boxtimes L_B = L_{A \boxtimes B}. \quad (5.6)$$

**Proposition 5.3.** *Let  $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$  and let  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices. Then*

$$A \boxtimes B = [A_{ij} \boxtimes B]_{ij} = \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1m} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{m1} \boxtimes B & \cdots & A_{mm} \boxtimes B \end{bmatrix}.$$

Fundamental algebraic properties of Tracy-Singh products are listed below.

**Lemma 5.4** ([31]). *The following properties hold, provided that all operators are compatible.*

$$(A \boxtimes B)^* = A^* \boxtimes B^*, \quad (5.7)$$

$$(\alpha A) \boxtimes B = \alpha(A \boxtimes B) = A \boxtimes (\alpha B), \quad (5.8)$$

$$A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C, \quad (5.9)$$

$$(B + C) \boxtimes A = B \boxtimes A + C \boxtimes A. \quad (5.10)$$

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD. \quad (5.11)$$

**Lemma 5.5** ([31]). *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ .*

(i) *If  $A, B \geq 0$ , then  $A \boxtimes B \geq 0$ .*

(ii) *If  $A, B > 0$ , then  $A \boxtimes B > 0$ .*

**Lemma 5.6** ([32]). *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices, and let  $(A_r)_{r=1}^{\infty}$  and  $(B_r)_{r=1}^{\infty}$  be sequences in  $\mathbb{B}(\mathcal{H})$  and  $\mathbb{B}(\mathcal{K})$ , respectively. If  $A_r \rightarrow A$  and  $B_r \rightarrow B$ , then  $A_r \boxtimes B_r \rightarrow A \boxtimes B$ .*

**Lemma 5.7** ([32]). *For any operator matrices  $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ , we have*

$$\frac{1}{mn} \|A\| \|B\| \leq \|A \boxtimes B\| \leq mn \|A\| \|B\|. \quad (5.12)$$

**Lemma 5.8.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . Then  $A \boxtimes B = 0$  if and only if  $A = 0$  or  $B = 0$ .*

*Proof.* It follows directly from Lemma 5.7.  $\square$

**Lemma 5.9** ([32]). *Let  $A \in \mathbb{B}(\mathcal{H})$ .*

(i) *If  $f$  is an analytic function on a region containing the spectra of  $A$  and  $I \boxtimes A$ , then*

$$f(I \boxtimes A) = I \boxtimes f(A).$$

(ii) *If  $f$  is an analytic function on a region containing the spectra of  $A$  and  $A \boxtimes I$ , then*

$$f(A \boxtimes I) = f(A) \boxtimes I.$$

From now on, fix the following decompositions of Hilbert spaces:

$$\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{i=1}^n \mathcal{K}_i.$$

are fix.

**Definition 5.10.** Let  $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$  be operators matrices. We define the *Khatri-Rao product* of  $A$  and  $B$  to be the operator matrix

$$A \boxtimes B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{n,n} \tag{5.13}$$

which is a bounded linear operator from  $\bigoplus_{i=1}^n \mathcal{H}_i \otimes \mathcal{K}_i$  into itself.

Recall that for each  $A \in \mathbb{M}_n(\mathbb{C})$ , the induced map

$$L_A : \mathbb{C}^m \rightarrow \mathbb{C}^m, \quad x \mapsto Ax$$

is bounded linear operator.

**Lemma 5.11** ([33]). *For any complex matrices  $A = [A_{ij}]$  and  $B = [B_{ij}]$  partitioned in block-matrix forms, we have*

$$L_A \boxtimes L_B = L_{A \boxtimes B}. \tag{5.14}$$

**Lemma 5.12** ([33]). *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B, C \in \mathbb{B}(\mathcal{K})$  be operator matrices, and let  $\alpha \in \mathbb{C}$ . Then*

$$(A \boxplus B)^* = A^* \boxplus B^*, \quad (5.15)$$

$$(\alpha A) \boxplus B = \alpha(A \boxplus B) = A \boxplus (\alpha B), \quad (5.16)$$

$$A \boxplus (B + C) = A \boxplus B + A \boxplus C, \quad (5.17)$$

$$(B + C) \boxplus A = B \boxplus A + C \boxplus A. \quad (5.18)$$

**Lemma 5.13** ([33]). *There is an isometry  $Z$  such that  $Z^*Z = I$  and*

$$A \boxplus B = Z^*(A \boxtimes B)Z \quad (5.19)$$

for any  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . We call  $Z$  the selection operator associated with  $\{\mathcal{H}, \mathcal{K}\}$ .

**Lemma 5.14** ([34]). *Let  $Z$  be the selection operator associated with  $\{\mathcal{H}, \mathcal{K}\}$ . Then there exists an operator*

$$W : \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_j \rightarrow \bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_j$$

such that  $Z^*W = 0$  and  $ZZ^* + WW^* = I$ . We call  $W$  the operator associated with  $Z$ .

In [34], an operator  $W$  is defined as

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix} \quad (5.20)$$

where

$$W^{(s)} = \left[ W_{kl}^{(s)} \right]_{k,l=1}^{n, n^2-1} : \bigoplus_{i=1}^n \bigoplus_{\substack{j=1 \\ i+j < n^2}}^n \mathcal{H}_i \otimes \mathcal{K}_i \rightarrow \bigoplus_{i=1}^n \mathcal{H}_s \otimes \mathcal{K}_i$$

for  $s = 1, \dots, n$ , with  $W_{kl}^{(s)}$  is an identity operator if  $k \neq s$  and  $l = n(s-1) + k$  and others are zero operators

**Lemma 5.15** ([33]). *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices.*

This (i) If  $A \geq 0$  and  $B \geq 0$ , then  $A \boxplus B \geq 0$ .

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(ii) If  $A > 0$  and  $B > 0$ , then  $A \boxplus B > 0$ .

**Lemma 5.16** ([35]). Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices and let  $(A_r)_{r=1}^\infty$  and  $(B_r)_{r=1}^\infty$  be sequences in  $\mathbb{B}(\mathcal{H})$  and  $\mathbb{B}(\mathcal{K})$ , respectively. If  $A_r \rightarrow A$  and  $B_r \rightarrow B$ , then  $A_r \boxplus B_r \rightarrow A \boxplus B$ .

**Lemma 5.17** ([35]). For any operator matrices  $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ , we have

$$\|A \boxplus B\| \leq n^2 \|A\| \|B\|. \quad (5.21)$$

### 5.3 Algebraic and order properties of Tracy-Singh sums for operators

**Definition 5.18.** Let  $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ . We define the *Tracy-Singh sum* of  $A$  and  $B$  as follows:

$$A \boxplus B = A \boxtimes I_{\mathcal{K}} + I_{\mathcal{H}} \boxtimes B, \quad (5.22)$$

which belongs to  $\mathbb{B}\left(\bigoplus_{i,j=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_j\right)$ .

Note that if both  $A$  and  $B$  are  $1 \times 1$  block operator matrices i.e.  $m = n = 1$ , their Tracy-Singh sum  $A \boxplus B$  is known as the tensor sum [28]

$$A \oplus B = A \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes B.$$

We now consider the Tracy-Singh sum of two bounded linear operators induced by matrices.

**Proposition 5.19.** Let  $A = [A_{ij}]$  and  $B = [B_{kl}]$  be complex partitioned matrices. Then

$$L_A \boxplus L_B = L_{A \boxplus B}. \quad (5.23)$$

*Proof.* We know that the linear map induced by the identity matrix is the identity operator. By applying Lemma 5.2, we get

$$\begin{aligned} L_A \boxplus L_B &= L_A \boxtimes I + I \boxtimes L_B = L_A \boxtimes L_I + L_I \boxtimes L_B \\ &= L_{A \boxtimes I} + L_{I \boxtimes B} = L_{A \boxtimes I + I \boxtimes B} \\ &= L_{A \boxplus B}. \end{aligned}$$

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**Proposition 5.20.** *Let  $\alpha$  and  $\beta$  be scalars. Then for any  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$ ,*

$$(A \boxplus B)^* = A^* \boxplus B^*, \quad (5.24)$$

$$\alpha(A \boxplus B) = \alpha A \boxplus \alpha B, \quad (5.25)$$

$$(\alpha + \beta)(A \boxplus B) = \alpha A \boxplus \beta B + \beta A \boxplus \alpha B, \quad (5.26)$$

$$(A + C) \boxplus (B + D) = A \boxplus B + C \boxplus D. \quad (5.27)$$

We call (5.27) the mixed sum property.

*Proof.* Applying property (5.7) of Lemma 5.4, we obtain

$$\begin{aligned} (A \boxplus B)^* &= (A \boxtimes I + I \boxtimes B)^* \\ &= (A \boxtimes I)^* + (I \boxtimes B)^* \\ &= A^* \boxtimes I + I \boxtimes B^* \\ &= A^* \boxplus B^*. \end{aligned}$$

By using property 5.8 of Lemma 5.4, we get

$$\begin{aligned} \alpha(A \boxplus B) &= \alpha(A \boxtimes I + I \boxtimes B) \\ &= \alpha(A \boxtimes I) + \alpha(I \boxtimes B) \\ &= (\alpha A) \boxtimes I + I \boxtimes (\alpha B) \\ &= \alpha A \boxplus \alpha B. \end{aligned}$$

Similarly, we have property (5.26). By applying properties (5.9) and (5.10) Lemma 5.4, we get

$$\begin{aligned} (A + C) \boxplus (B + D) &= (A + C) \boxtimes I + I \boxtimes (B + D) \\ &= A \boxtimes I + C \boxtimes I + I \boxtimes B + I \boxtimes D \\ &= (A \boxtimes I + I \boxtimes B) + (C \boxtimes I + I \boxtimes D) \\ &= A \boxplus B + C \boxplus D. \square \end{aligned}$$

It follows from (5.24) that if  $A$  and  $B$  are Hermitian (skew-Hermitian), then so is  $A \boxplus B$ . The mixed sum property (5.27) implies that if  $A_1 = X_1 + iY_1$  and  $A_2 = X_2 + iY_2$  are the Cartesian decompositions of  $A_1$  and  $A_2$ , respectively, then  $A_1 \boxplus A_2 = X_1 \boxplus X_2 + i(Y_1 \boxplus Y_2)$  is the Cartesian decomposition of  $A_1 \boxplus A_2$ .

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**Lemma 5.21.** *Let  $A_1, A_2 \in \mathbb{B}(\mathcal{H})$  and  $B_1, B_2 \in \mathbb{B}(\mathcal{K})$  be nonzero operators. Then  $A_1 \otimes B_1 = A_2 \otimes B_2$  if and only if there exists  $\alpha \in \mathbb{C} \setminus \{0\}$  such that  $A_1 = \alpha A_2$  and  $B_1 = \alpha^{-1} B_2$ .*

*Proof.* See [39, Proposition 2.1]. □

**Theorem 5.1.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . Then  $A \boxplus B = 0$  if and only if there exists  $\alpha \in \mathbb{C} \setminus \{0\}$  such that  $A = \alpha I$  and  $B = -\alpha I$ .*

*Proof.* If  $A = \alpha I$  and  $B = -\alpha I$ , then  $A \boxplus B = 0$  by Lemma 5.4. Now, suppose that  $A \boxplus B = 0$ . Then, by Lemma 5.3,

$$0 = A \boxplus B = A \boxtimes I + I \boxtimes B = \begin{bmatrix} A_{11} \boxplus B & A_{12} \boxtimes I & \cdots & A_{1m} \boxtimes I \\ A_{21} \boxtimes I & A_{22} \boxplus B & \cdots & A_{2m} \boxtimes I \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} \boxtimes I & A_{m2} \boxtimes I & \cdots & A_{mm} \boxplus B \end{bmatrix}.$$

For  $i, j \in \{1, \dots, m\}$  such that  $i \neq j$ , since  $A_{ij} \boxtimes I = 0$ , by Lemma 5.8 we can deduce  $A_{ij} = 0$ . Let  $i \in \{1, \dots, m\}$  and consider

$$\begin{bmatrix} A_{ii} \oplus B_{11} & I \otimes B_{12} & \cdots & I \otimes B_{1n} \\ I \otimes B_{21} & A_{ii} \oplus B_{22} & \cdots & I \otimes B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I \otimes B_{n1} & I \otimes B_{n2} & \cdots & A_{ii} \oplus B_{nn} \end{bmatrix} = 0$$

Lemma 5.8 implies that  $B_{kl} = 0$  for any  $k \neq l$ . For  $k \in \{1, \dots, n\}$ , we have  $A_{ii} \oplus B_{kk} = 0$  if and only if  $A_{ii} \otimes (-I) = I \otimes B_{kk}$ . If  $A_{ii} = 0$ , then  $I \otimes B_{kk} = 0$  and hence  $B_{kk} = 0$  for all  $k = 1, \dots, n$  by Lemma 5.8. It follows that  $B = 0$  and  $A \boxtimes I = A \boxplus B = 0$ . Thus  $A = 0$  by Lemma 5.8. Assume that  $A_{ii} \neq 0$  for all  $i = 1, \dots, m$ . By Lemma 5.21, there exists  $\alpha_{ik} \in \mathbb{C} \setminus \{0\}$  such that  $A_{ii} = \alpha_{ik} I$  and  $-I = \alpha_{ik}^{-1} B_{kk}$ . For any fix  $i \in \{1, \dots, m\}$ , we have  $A_{ii} = \alpha_{ik} I$  for all  $k = 1, \dots, n$ . Hence,  $\alpha_{i1} = \alpha_{i2} = \dots = \alpha_{in} = \alpha_i$ . For any fix  $k \in \{1, \dots, n\}$ , we have  $B_{kk} = -\alpha_i I$  for all  $i = 1, \dots, m$ . This implies that  $\alpha_1 = \alpha_2 = \dots = \alpha_m = \alpha$ . Thus  $A_{ii} = \alpha I$  for all  $i = 1, \dots, m$  and  $B_{kk} = -\alpha I$  for all  $k = 1, \dots, n$ . Therefore  $A = \alpha I$  and  $B = -\alpha I$ . □

The next result is an operator extension of [?, Theorem 3.3].

**Proposition 5.22.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be invertible operators. If  $A \boxplus B$  and  $A^{-1} \boxplus B^{-1}$  are invertible, then*

$$(A \boxplus B)^{-1} = (A^{-1} \boxtimes I) (A^{-1} \boxplus B^{-1})^{-1} (I \boxtimes B^{-1}) \quad (5.28)$$

$$(A \boxplus B)^{-1} = (I \boxtimes B^{-1}) (A^{-1} \boxplus B^{-1})^{-1} (A^{-1} \boxtimes I) \quad (5.29)$$

$$(A \boxplus B)^{-1} = (A^{-1} \boxplus B^{-1})^{-1} (A^{-1} \boxtimes B^{-1}). \quad (5.30)$$

*Proof.* Applying Lemma 5.4, we have

$$\begin{aligned} (A \boxplus B)^{-1} &= (A \boxtimes I + I \boxtimes B)^{-1} \\ &= (A^{-1} \boxtimes I) [(I \boxtimes B + A \boxtimes I) (A^{-1} \boxtimes I)]^{-1} \\ &= (A^{-1} \boxtimes I) [A^{-1} \boxtimes B + I \boxtimes I]^{-1} \\ &= (A^{-1} \boxtimes I) [(I \boxtimes B^{-1}) (A^{-1} \boxtimes B + I \boxtimes I)]^{-1} (I \boxtimes B^{-1}) \\ &= (A^{-1} \boxtimes I) [A^{-1} \boxtimes I + I \boxtimes B^{-1}]^{-1} (I \boxtimes B^{-1}) \\ &= (A^{-1} \boxtimes I) (A^{-1} \boxplus B^{-1})^{-1} (I \boxtimes B^{-1}). \end{aligned}$$

Similarly, we obtain the properties (5.29) and (5.30). □

Recall that the *commutator* of  $A, B \in \mathbb{B}(\mathcal{H})$  is defined by

$$[A, B] = AB - BA.$$

The *anticommutator* of  $A$  and  $B$  is defined by

$$[A, B]_+ = AB + BA.$$

**Proposition 5.23.** *Let  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$ . Then*

$$[A \boxplus B, C \boxplus D] = [A, C] \boxplus [B, D]. \quad (5.31)$$

*Proof.* By using Lemma 5.4, we have

$$\begin{aligned} [A \boxplus B, C \boxplus D] &= (A \boxplus B)(C \boxplus D) - (C \boxplus D)(A \boxplus B) \\ &= AC \boxtimes I + I \boxtimes BD - CA \boxtimes I - I \boxtimes DB \\ &= (AC - CA) \boxtimes I + I \boxtimes (BD - DB) \\ &= (AC - CA) \boxplus (BD - DB) \\ &= [A, C] \boxplus [B, D]. \end{aligned}$$

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Recall that an operator  $A \in \mathbb{B}(\mathcal{H})$  is said to be *normal* if  $[A^*, A] = 0$ . It follows from Propositions 5.20 and 5.23 that if  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  are normal, then so is  $A \boxplus B$ .

**Proposition 5.24.** *Let  $A, A_1, A_2 \in \mathbb{B}(\mathcal{H})$  and  $B, B_1, B_2 \in \mathbb{B}(\mathcal{K})$ .*

(i) *If  $A \geq 0$  and  $B \geq 0$ , then  $A \boxplus B \geq 0$ .*

(ii) *If  $A > 0$  and  $B > 0$ , then  $A \boxplus B > 0$ .*

(iii) *If  $A_1 \geq A_2$  and  $B_1 \geq B_2$ , then  $A_1 \boxplus B_1 \geq A_2 \boxplus B_2$ .*

(iv) *If  $A_1 > A_2$  and  $B_1 > B_2$ , then  $A_1 \boxplus B_1 > A_2 \boxplus B_2$ .*

*Proof.* By applying Lemma 5.5 and Proposition 5.20, we get the results.  $\square$

**Proposition 5.25.** *Let  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$  be positive operators. Then*

$$(A \boxplus B)(C \boxplus D) \geq AC \boxplus BD. \quad (5.32)$$

*Proof.* Applying Lemma 5.4, we have

$$\begin{aligned} (A \boxplus B)(C \boxplus D) &= (A \boxtimes I + I \boxtimes B)(C \boxtimes I + I \boxtimes D) \\ &= AC \boxtimes I + A \boxtimes D + C \boxtimes B + I \boxtimes BD \\ &= AC \boxplus BD + A \boxtimes D + C \boxtimes B. \end{aligned}$$

Since  $A \boxtimes D + C \boxtimes B$  is positive, we get the result.  $\square$

**Corollary 5.26.** *Let  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$  be positive operator matrices. Then*

$$[A \boxplus B, C \boxplus D]_+ \geq [A, C]_+ \boxplus [B, D]_+. \quad (5.33)$$

*Proof.* By using Proposition 5.20 and 5.25, we obtain

$$\begin{aligned} [A \boxplus B, C \boxplus D]_+ &= (A \boxplus B)(C \boxplus D) + (C \boxplus D)(A \boxplus B) \\ &\geq AC \boxplus BD + CA \boxplus DB \\ &= (AC + CA) \boxplus (BD + DB) \\ &= [A, C]_+ \boxplus [B, D]_+. \end{aligned}$$

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## 5.4 Binomial theorem involving Tracy-Singh sums and its consequences

In this section, we prove an operator version of the binomial theorem in which the sum and the product are replaced by the Tracy-Singh sum and the Tracy-Singh product, respectively. Consequently, we obtain certain operator inequalities, including a Bernoulli-type inequality. Binomial theorem is also used to treat the nilpotency of the Tracy-Singh sum of two operators.

**Theorem 5.2.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be compatibly operator matrices. Then*

$$(A \boxplus B)^r = A^r \boxplus B^r + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k) \quad (5.34)$$

for any integer  $r \geq 2$ .

*Proof.* By using Lemma 5.4, we have

$$\begin{aligned} (A \boxplus B)^2 &= (A \boxtimes I + I \boxtimes B)(A \boxtimes I + I \boxtimes B) \\ &= A^2 \boxtimes I + 2A \boxtimes B + I \boxtimes B^2 \\ &= A^2 \boxplus B^2 + 2A \boxtimes B. \end{aligned}$$

This show that (5.34) is true for  $r = 2$ . Suppose that (5.34) holds for an integer

$r \geq 2$ . Then, by Lemma 5.4,

$$\begin{aligned}
(A \boxplus B)^{r+1} &= (A \boxplus B)^r (A \boxplus B) \\
&= \left( A^r \boxplus B^r + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k) \right) (A \boxplus B) \\
&= (A^r \boxplus B^r)(A \boxplus B) + \left( \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k) \right) (A \boxplus B) \\
&= (A^r \boxtimes I)(A \boxtimes I) + (A^r \boxtimes I)(I \boxtimes B) + (I \boxtimes B^r)(A \boxtimes I) \\
&\quad + (I \boxtimes B^r)(I \boxtimes B) + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k) (A \boxtimes I) \\
&\quad + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k) (I \boxtimes B) \\
&= A^{r+1} \boxtimes I + A^r \boxtimes B + A \boxtimes B^r + I \boxtimes B^{r+1} \\
&\quad + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k+1} \boxtimes B^k) + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^{k+1}) \\
&= A^{r+1} \boxplus B^{r+1} + \sum_{k=1}^r \binom{r}{k} (A^{(r+1)-k} \boxtimes B^k) \\
&\quad + \sum_{k=1}^r \binom{r}{k-1} (A^{(r+1)-k} \boxtimes B^k) \\
&= A^{r+1} \boxplus B^{r+1} + \sum_{k=1}^r \binom{r+1}{k} (A^{(r+1)-k} \boxtimes B^k).
\end{aligned}$$

□

**Corollary 5.27.** *Let  $A \geq 0$  and  $B \geq 0$  be compatibly operator matrices, then*

$$(A \boxplus B)^r \geq A^r \boxplus B^r \quad (5.35)$$

for any  $r \in \mathbb{N}$ .

*Proof.* It follows immediately from Theorem 5.2 and Lemma 5.5. □

Proposition 5.25, Theorem 5.2 and Corollary 5.27 are a generalization of a matrix version given in [?, Theorem 3.1]. The next result is a Bernoulli type inequality concerning Tracy-Singh sums.

**Corollary 5.28.** *Let  $A$  be a positive operator. Then*

$$(I \boxplus A)^r \geq I \boxplus (rA) \quad (5.36)$$

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*Proof.* Since  $A \geq 0$  we have  $I \boxplus A^r \geq I \boxtimes I$ . By Theorem 5.27, we obtain

$$\begin{aligned}
(I \boxplus A)^r &= I \boxplus A^r + \sum_{k=1}^{r-1} \binom{r}{k} (I \boxtimes A^k) \\
&= I \boxplus A^r + r(I \boxtimes A) + \sum_{k=2}^{r-1} \binom{r}{k} (I \boxtimes A^k) \\
&\geq I \boxplus A^r + r(I \boxtimes A) \\
&\geq I \boxtimes I + I \boxtimes (rA) \\
&= I \boxplus (rA).
\end{aligned}$$

□

**Corollary 5.29.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . If  $A$  and  $B$  are nilpotent, then  $A \boxplus B$  is also nilpotent.*

*Proof.* Suppose  $A^r = 0$  and  $B^s = 0$  for some  $r, s \in \mathbb{N}$ . From Theorem 5.2, we get

$$\begin{aligned}
(A \boxplus B)^{r+s} &= \sum_{k=0}^{r+s} \binom{r+s}{k} (A^{r+s-k} \boxtimes B^k) \\
&= \sum_{k=0}^{s-1} \binom{r+s}{k} (A^{r+s-k} \boxtimes B^k) + \sum_{k=s}^{r+s} \binom{r+s}{k} (A^{r+s-k} \boxtimes B^k).
\end{aligned}$$

If  $k \geq s$ , then  $B^k = 0$ . If  $k < s$ , then  $r+s-k > r$  and hence  $A^k = 0$ . Since  $A \boxtimes 0 = 0$  and  $0 \boxtimes B = 0$ , we have

$$\begin{aligned}
(A \boxplus B)^{r+s} &= \sum_{k=0}^{s-1} \binom{r+s}{k} (0 \boxtimes B^k) + \sum_{k=s}^{r+s} \binom{r+s}{k} (A^{r+s-k} \boxtimes 0) \\
&= \sum_{k=0}^{s-1} \binom{r+s}{k} (0 \boxtimes B)^k + \sum_{k=s}^{r+s} \binom{r+s}{k} (A \boxtimes 0)^{r+s-k} \\
&= 0.
\end{aligned}$$

This means that  $A \boxplus B$  is nilpotent. □

## 5.5 Analytic properties of Tracy-Singh sums

**Proposition 5.30.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices, and let  $(A_r)_{r=1}^\infty$  and  $(B_r)_{r=1}^\infty$  be sequences in  $\mathbb{B}(\mathcal{H})$  and  $\mathbb{B}(\mathcal{K})$ , respectively. If  $A_r \rightarrow A$  and  $B_r \rightarrow B$ , then  $A_r \boxplus B_r \rightarrow A \boxplus B$ .*

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*Proof.* By Lemma 5.6, we have  $A_r \boxtimes I \rightarrow A \boxtimes I$  and  $I \boxtimes B_r \rightarrow I \boxtimes B$ . Then  $A_r \boxtimes I + I \boxtimes B_r \rightarrow A \boxtimes I + I \boxtimes B$ , i.e.  $A_r \boxplus B_r \rightarrow A \boxplus B$ .  $\square$

We obtain a triangle-like inequality for Tracy-Singh sums as follows.

**Proposition 5.31.** *For any operators  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ , we have*

$$\frac{1}{mn} \|A \boxplus B\| \leq \|A\| + \|B\|. \quad (5.37)$$

*Proof.* It follows from the norm estimation in Lemma 5.7.  $\square$

For any an operator  $T \in \mathbb{B}(\mathcal{H})$ , we define

$$e^T = \sum_{k=0}^{\infty} \frac{1}{k!} T^k, \quad (5.38)$$

$$\sin(T) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} T^{2k+1}, \quad (5.39)$$

$$\cos(T) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} T^{2k}, \quad (5.40)$$

$$\sinh(T) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} T^{2k+1}, \quad (5.41)$$

$$\cosh(T) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} T^{2k}, \quad (5.42)$$

The series on the right hand side of (5.38)-(5.42) convergence in  $\mathbb{B}(\mathcal{H})$ . If  $T > 0$ , we define the logarithm of  $T$  to be an operator  $X$  such that  $e^X = T$  and write  $X = \log T$ . The following facts are well-known.

**Lemma 5.32.** *For any  $S, T \in \mathbb{B}(\mathcal{H})$  satisfying  $ST = TS$ , we have*

$$e^{S+T} = e^S e^T,$$

$$\sin(S+T) = \sin S \cos T + \cos S \sin T,$$

$$\cos(S+T) = \cos S \cos T - \sin S \sin T,$$

$$\sinh(S+T) = \sinh S \cosh T + \cosh S \sinh T,$$

$$\cosh(S+T) = \cosh S \cosh T - \sinh S \sinh T.$$

*If  $S, T > 0$  and  $ST = TS$ , we have*

$$\log ST = \log S + \log T.$$

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**Theorem 5.3.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . Then*

$$e^{A \boxplus B} = e^A \boxtimes e^B, \quad (5.43)$$

$$\sin(A \boxplus B) = \sin A \boxtimes \cos B + \cos A \boxtimes \sin B, \quad (5.44)$$

$$\cos(A \boxplus B) = \cos A \boxtimes \cos B - \sin A \boxtimes \sin B, \quad (5.45)$$

$$\sinh(A \boxplus B) = \sinh A \boxtimes \cosh B + \cosh A \boxtimes \sinh B, \quad (5.46)$$

$$\cosh(A \boxplus B) = \cosh A \boxtimes \cosh B - \sinh A \boxtimes \sinh B. \quad (5.47)$$

If  $A, B > 0$ , then

$$\log A \boxtimes B = \log A \boxplus \log B. \quad (5.48)$$

*Proof.* Since  $A \boxtimes I$  and  $I \boxtimes B$  commute, we have by Lemmas 5.9 and 5.32 that

$$\begin{aligned} e^{A \boxplus B} &= e^{A \boxtimes I + I \boxtimes B} \\ &= e^{A \boxtimes I} e^{I \boxtimes B} \\ &= (e^A \boxtimes I)(I \boxtimes e^B) \\ &= e^A \boxtimes e^B. \end{aligned}$$

Similarly, by using Lemma 5.9, we get the identities (5.44)-(5.48). □

# CHAPTER 6

## Khatri-Rao Sums for Hilbert Space Operators

In this chapter, we generalize the notions of Khatri-Rao sums for matrices and tensor sums for Hilbert space operators to Khatri-Rao sums for Hilbert space operators. This kind of operator sum is compatible with algebraic operations and order relations. We investigate its analytic properties, including continuity, convergence, and norm bounds. We also discuss the role of selection operator that relates Khatri-Rao sums to Tracy-Singh sums and Khatri-Rao products. Binomial theorem involving Khatri-Rao sums and its consequences are then established.

### 6.1 Algebraic and order properties of Khatri-Rao sums for operators

**Definition 6.1.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . We define the *Khatri-Rao sum* of  $A$  and  $B$  as follows:

$$A \boxtimes B = A \boxdot I_{\mathcal{K}} + I_{\mathcal{H}} \boxdot B \quad (6.1)$$

which belongs to  $\mathbb{B}(\bigoplus_{i=1}^n \mathcal{H}_i \otimes \mathcal{K}_i)$ .

Note that

- (i) If both  $A$  and  $B$  are  $1 \times 1$  block operator matrices, their Khatri-Rao sum  $A \boxtimes B$  is the tensor sum [28]  $A \oplus B := A \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes B$ .
- (ii) If  $\mathcal{H}_i = \mathcal{K}_i = \mathbb{C}$ , their the Khatri-Rao sum  $A \boxtimes B$  is known as the Hadamard sum of complex matrices (see e.g. [?]).

**Proposition 6.2.** Let  $A = [A_{ij}]$  and  $B = [B_{ij}]$  be compatible partitioned matrices of order  $m \times m$  and  $n \times n$ , respectively. Then

$$L_A \boxtimes L_B = L_{A \boxtimes B}. \quad (6.2)$$

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*Proof.* We know that the linear map induced by the identity matrix is the identity operator. By applying Lemma 5.11, we get

$$L_A \boxtimes L_B = L_A \boxdot I + L_I \boxdot L_B = L_{A \hat{\boxdot} I} + L_{I \hat{\boxdot} B} = L_{A \hat{\boxdot} B}.$$

□

**Proposition 6.3.** *Let  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$  be operator matrices, and let  $\alpha, \beta \in \mathbb{C}$ . Then*

$$(A \boxtimes B)^* = A^* \boxtimes B^*, \quad (6.3)$$

$$\alpha(A \boxtimes B) = \alpha A \boxtimes \alpha B, \quad (6.4)$$

$$(\alpha + \beta)(A \boxtimes B) = \alpha A \boxtimes \beta B + \beta A \boxtimes \alpha B, \quad (6.5)$$

$$(A + C) \boxtimes (B + D) = A \boxtimes B + C \boxtimes D. \quad (6.6)$$

We call property (6.6) mixed sum property.

*Proof.* Applying property (5.15) of Lemma 5.12, we obtain

$$\begin{aligned} (A \boxtimes B)^* &= (A \boxdot I + I \boxdot B)^* \\ &= (A \boxdot I)^* + (I \boxdot B)^* \\ &= A^* \boxdot I + I \boxdot B^* \\ &= A^* \boxtimes B^*. \end{aligned}$$

By using property 5.16 of Lemma 5.12, we get

$$\begin{aligned} \alpha(A \boxtimes B) &= \alpha(A \boxdot I + I \boxdot B) \\ &= \alpha(A \boxdot I) + \alpha(I \boxdot B) \\ &= (\alpha A) \boxdot I + I \boxdot (\alpha B) \\ &= \alpha A \boxtimes \alpha B. \end{aligned}$$

Similarly, we have property (6.5). By applying properties (5.17) and (5.18) Lemma 5.12, we get

$$\begin{aligned} (A + C) \boxtimes (B + D) &= (A + C) \boxdot I + I \boxdot (B + D) \\ &= A \boxdot I + C \boxdot I + I \boxdot B + I \boxdot D \\ &= (A \boxdot I + I \boxdot B) + (C \boxdot I + I \boxdot D) \\ &= A \boxtimes B + C \boxtimes D. \end{aligned}$$

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**Corollary 6.4.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . If  $A$  and  $B$  are Hermitian (skew-Hermitian), then  $A \boxtimes B$  is also Hermitian (skew-Hermitian).*

**Corollary 6.5.** *Let  $A_1 \in \mathbb{B}(\mathcal{H})$  and  $A_2 \in \mathbb{B}(\mathcal{K})$ . If  $A_1 = X_1 + iY_1$  and  $A_2 = X_2 + iY_2$  are the Cartesian decompositions of  $A_1$  and  $A_2$ , respectively, then  $A_1 \boxtimes A_2 = X_1 \boxtimes X_2 + i(Y_1 \boxtimes Y_2)$  is the Cartesian decomposition of  $A_1 \boxtimes A_2$ .*

**Proposition 6.6.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices.*

(i) *If  $A \geq 0$  and  $B \geq 0$ , then*

$$A \boxtimes B \geq 0. \quad (6.7)$$

(ii) *If  $A > 0$  and  $B > 0$ , then*

$$A \boxtimes B > 0. \quad (6.8)$$

*Proof.* It follows from Lemma 5.15. □

**Corollary 6.7.** *Let  $A_1, A_2 \in \mathbb{B}(\mathcal{H})$  and  $B_1, B_2 \in \mathbb{B}(\mathcal{K})$  be operator matrices.*

(i) *If  $A_1 \geq A_2$  and  $B_1 \geq B_2$ , then*

$$A_1 \boxtimes B_1 \geq A_2 \boxtimes B_2. \quad (6.9)$$

(ii) *If  $A_1 > A_2$  and  $B_1 > B_2$ , then*

$$A_1 \boxtimes B_1 > A_2 \boxtimes B_2. \quad (6.10)$$

*Proof.* (i) By Propositions 6.3 and 6.6, we obtain

$$A_1 \boxtimes B_1 - A_2 \boxtimes B_2 = (A_1 - A_2) \boxtimes (B_1 - B_2) \geq 0.$$

This implies that  $A_1 \boxtimes B_1 \geq A_2 \boxtimes B_2$

(ii) The proof is similarly with (i). □

## 6.2 Analytic properties of Khatri-Rao sums for operators

**Proposition 6.8.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operators matrices, and let  $(A_r)_{r=1}^{\infty}$  and  $(B_r)_{r=1}^{\infty}$  be sequences in  $\mathbb{B}(\mathcal{H})$  and  $\mathbb{B}(\mathcal{K})$ , respectively. If  $A_r \rightarrow A$  and  $B_r \rightarrow B$ , then  $A_r \boxtimes B_r \rightarrow A \boxtimes B$ .*

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*Proof.* By Lemma 5.16, we have  $A_r \boxdot I \rightarrow A \boxdot I$  and  $I \boxdot B_r \rightarrow I \boxdot B$ . Then  $A_r \boxdot I + I \boxdot B_r \rightarrow A \boxdot I + I \boxdot B$ , i.e.  $A_r \boxtimes B_r \rightarrow A \boxtimes B$ .  $\square$

**Proposition 6.9.** *For any operator matrices  $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ , we have*

$$\frac{1}{n^2} \|A \boxtimes B\| \leq \|A\| + \|B\|. \quad (6.11)$$

*Proof.* By using Lemma 5.17, we have

$$\begin{aligned} \|A \boxtimes B\| &= \|A \boxdot I + I \boxdot B\| \\ &\leq \|A \boxdot I\| + \|I \boxdot B\| \\ &\leq n^2 \|A\| \|I\| + n^2 \|I\| \|B\| \\ &= n^2 (\|A\| + \|B\|). \end{aligned}$$

$\square$

### 6.3 Khatri-Rao sums and selection operators

In this section, we investigate the role of selection operators that relates Khatri-Rao sums to Tracy-Singh sums and Khatri-Rao products.

**Proposition 6.10.** *There is an isometry  $Z$  such that  $Z^*Z = I$  and*

$$A \boxtimes B = Z^*(A \boxplus B)Z \quad (6.12)$$

for any  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ .

*Proof.* By Lemma 5.13, there is an isometry  $Z$  such that  $Z^*Z = I$  and

$$A \boxtimes B = A \boxdot I + I \boxdot B = Z^*(A \boxtimes I + I \boxtimes B)Z = Z^*(A \boxplus B)Z.$$

$\square$

**Proposition 6.11.** *Let  $A \in \mathbb{B}(\mathcal{H})$ ,  $B \in \mathbb{B}(\mathcal{K})$  and  $Z$  be the selection operator associated with  $\{\mathcal{H}, \mathcal{K}\}$ . Then*

$$A \boxdot B^* + A^* \boxdot B \leq (A \boxtimes B)(A \boxtimes B)^* + Z^*(A \boxplus B)W W^*(A \boxplus B)^*Z. \quad (6.13)$$

where  $W$  is the operator associated with  $Z$ .

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*Proof.* By Lemma 5.4, 5.14 and 5.13, and Theorem 6.10, we have

$$\begin{aligned}
(A \boxtimes B)(A \boxtimes B)^* &= Z^*(A \boxplus B)ZZ^*(A^* \boxplus B^*)Z \\
&= Z^*(A \boxplus B)(I - WW^*)(A^* \boxplus B^*)Z \\
&= Z^*(A \boxplus B)(A^* \boxplus B^*)Z - Z^*(A \boxplus B)WW^*(A^* \boxplus B^*)Z \\
&= Z^*(AA^* \boxplus BB^* + A \boxtimes B^* + A^* \boxtimes B)Z \\
&\quad - Z^*(A \boxplus B)WW^*(A^* \boxplus B^*)Z \\
&= AA^* \boxtimes BB^* + A \square B^* + A^* \square B \\
&\quad - Z^*(A \boxplus B)WW^*(A \boxplus B)^*Z.
\end{aligned}$$

By Proposition 6.6, we have  $AA^* \boxtimes BB^* \geq 0$  and hence

$$(A \boxtimes B)(A \boxtimes B)^* + Z^*(A \boxplus B)WW^*(A \boxplus B)^*Z - (A \square B^* + A^* \square B) \geq 0.$$

□

**Proposition 6.12.** *Let  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$  be positive operators. Then*

$$AC \boxtimes BD \leq (A \boxtimes B)(C \boxtimes D) + Z^*(A \boxplus B)WW^*(C \boxplus D)Z. \quad (6.14)$$

*Proof.* By replacing  $A^* = C$  and  $B^* = D$  in the proof of Proposition 6.11, we have

$$(A \boxtimes B)(C \boxtimes D) + Z^*(A \boxplus B)WW^*(C \boxplus D)Z - AC \boxtimes BD = A \square D + C \square B.$$

Since  $A \square D + C \square B$  is positive, we get the result. □

## 6.4 Binomial theorem involving Khatri-Rao sums and its consequences

In this section, we prove an operator version of binomial theorem concerning Khatri-Rao sums. As consequences, we obtain two operator inequalities, including Bernoulli type inequality.

**Lemma 6.13.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . We have  $W^*(A \boxplus B)Z = 0$  if and only if  $A = A_1 \oplus \dots \oplus A_n$  and  $B = B_1 \oplus \dots \oplus B_n$  where  $Z$  is the selection operator associated with  $\{\mathcal{H}, \mathcal{K}\}$  and  $W$  is an operator associated with  $Z$  in (5.20).*

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*Proof.* Consider  $A \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$  and  $B \in \mathbb{B}(\mathcal{K}_1 \oplus \mathcal{K}_2)$ . By a computation, we have

$$\begin{aligned}
& W^*(A \boxplus B)Z \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} A_{11} \oplus B_{11} & I \otimes B_{12} & A_{12} \otimes I & 0 \\ I \otimes B_{21} & A_{11} \oplus B_{21} & 0 & A_{12} \otimes I \\ A_{21} \otimes I & 0 & A_{22} \oplus B_{11} & I \otimes B_{12} \\ 0 & A_{21} \otimes I & I \otimes B_{21} & A_{22} \oplus B_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} A_{11} \oplus B_{11} & 0 \\ I \otimes B_{21} & A_{12} \otimes I \\ A_{21} \otimes I & I \otimes B_{21} \\ 0 & A_{22} \oplus B_{22} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ I \otimes B_{21} & A_{12} \otimes I \\ A_{21} \otimes I & I \otimes B_{12} \end{bmatrix}.
\end{aligned}$$

By Lemma 5.8, we obtain

$$\begin{aligned}
W^*(A \boxplus B)Z = 0 &\Leftrightarrow A_{12} \otimes I, A_{21} \otimes I, I \otimes B_{12} \text{ and } I \otimes B_{21} \text{ are zeros} \\
&\Leftrightarrow A_{12}, A_{21}, B_{12} \text{ and } B_{21} \text{ are zeros} \\
&\Leftrightarrow A = A_1 \oplus A_2 \text{ and } B = B_1 \oplus B_2.
\end{aligned}$$

By a direct computation, we can show that Lemma 6.13 holds for the case  $A = [A_{ij}]_{i,j=1}^{n,n}$  and  $B = [B_{ij}]_{i,j=1}^{n,n}$  for any integer  $n > 2$ .  $\square$

**Theorem 6.1.** *Let  $A = A_1 \oplus \dots \oplus A_n \in \mathbb{B}(\mathcal{H})$  and  $B = B_1 \oplus \dots \oplus B_n \in \mathbb{B}(\mathcal{K})$  be compatible operator matrices, then*

$$(A \boxtimes B)^r = A^r \boxtimes B^r + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k) \quad (6.15)$$

for any integer  $r \geq 2$ .

*Proof.* By the proof of Proposition 6.11, we have

$$(A \boxtimes B)^2 = A^2 \boxtimes B^2 + 2A \boxtimes B - Z^*(A \boxplus B)WZ.$$

Since  $W^*(A \boxplus B)Z = 0$ , by Lemma 6.13, we get

$$(A \boxtimes B)^2 = A^2 \boxtimes B^2 + 2A \boxtimes B.$$

This show that Theorem 6.1 is true for  $r = 2$ . Suppose that (6.15) hold for  $r \geq 2$ . Then, by Lemma 5.4,

$$\begin{aligned}
(A \boxtimes B)^{r+1} &= (A \boxtimes B)^r (A \boxtimes B) \\
&= \left( A^r \boxtimes B^r + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k) \right) (A \boxtimes B) \\
&= (A^r \boxtimes B^r)(A \boxtimes B) + \left( \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k) \right) (A \boxtimes B) \\
&= A^{r+1} \boxtimes B^{r+1} + A^r \boxtimes B + A \boxtimes B^r - Z^*(A^r \boxplus B^r)WW^*(A \boxplus B)Z \\
&\quad + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k+1} \boxtimes B^k - Z^*(A^{r-k} \boxtimes B^r)WW^*(A \boxtimes I)Z) \\
&\quad + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^{k+1} - Z^*(A^{r-k} \boxtimes B^r)WW^*(I \boxtimes B)Z) \\
&= A^{r+1} \boxtimes B^{r+1} + A^r \boxtimes B + A \boxtimes B^r - Z^*(A^r \boxplus B^r)WW^*(A \boxplus B)Z \\
&\quad + \sum_{k=1}^r \binom{r}{k} (A^{(r+1)-k} \boxtimes B^k) + \sum_{k=0}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^{k+1}) \\
&\quad - \sum_{k=1}^{r-1} \binom{r}{k} Z^*(A^{r-k} \boxtimes B^r)WW^*(A \boxplus B)Z \\
&= A^{r+1} \boxtimes B^{r+1} + A^r \boxtimes B + A \boxtimes B^r - Z^*(A^r \boxplus B^r)WW^*(A \boxplus B)Z \\
&\quad + \sum_{k=1}^r \binom{r}{k} (A^{(r+1)-k} \boxtimes B^k) - \sum_{k=1}^{r-1} \binom{r}{k} Z^*(A^{r-k} \boxtimes B^r)WW^*(A \boxplus B)Z.
\end{aligned}$$

By Lemma 6.13, we have  $W^*(A \boxplus B)Z = 0$ . Then

$$(A \boxtimes B)^{r+1} = A^{r+1} \boxtimes B^{r+1} + \sum_{k=1}^r \binom{r+1}{k} (A^{(r+1)-k} \boxtimes B^k).$$

□

**Corollary 6.14.** *Let  $A = A_1 \oplus \dots \oplus A_n \in \mathbb{B}(\mathcal{H})$  and  $B = B_1 \oplus \dots \oplus B_n \in \mathbb{B}(\mathcal{K})$  be positive operators, then*

$$(A \boxtimes B)^r \geq A^r \boxtimes B^r \tag{6.16}$$

for any  $r \in \mathbb{N}$ .

*Proof.* Inequality (6.16) is obviously true for  $r = 1$ . Let  $r \geq 2$ . Since  $A, B \geq 0$ , we have by Lemma 5.15 that  $\sum_{k=1}^r \binom{r}{k} (A^{r-k} \boxtimes B^k) \geq 0$ . By Theorem 6.1, we obtain  $(A \boxtimes B)^r \geq A^r \boxtimes B^r$ . □

**Corollary 6.15.** *Let  $A = A_1 \oplus \dots \oplus A_n \in \mathbb{B}(\mathcal{H})$  be a positive operator. Then*

$$(I \boxtimes A)^r \geq I \boxtimes (rA) \quad (6.17)$$

for  $r \in \mathbb{N}$ .

*Proof.* Since  $A \geq 0$  we have  $I \boxtimes A^r \geq I \boxtimes I$ . By Theorem 6.14, we obtain

$$\begin{aligned} (I \boxtimes A)^r &= I \boxtimes A^r + \sum_{k=1}^{r-1} \binom{r}{k} (I \boxtimes A^k) \\ &= I \boxtimes A^r + r(I \boxtimes A) + \sum_{k=2}^{r-1} \binom{r}{k} (I \boxtimes A^k) \\ &\geq I \boxtimes A^r + r(I \boxtimes A) \\ &\geq I \boxtimes I + I \boxtimes (rA) \\ &= I \boxtimes (rA). \end{aligned}$$

□

## 6.5 Inequalities for Khatri-Rao sums involving Moore-Penrose inverses

In this section, we establish certain inequalities for Khatri-Rao sums involving a kind of operator inverse, namely, the Moore-Penrose inverse. Recall that a *Moore-Penrose inverse* of  $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$  is an operator  $A^\dagger \in \mathbb{B}(\mathcal{K}, \mathcal{H})$  satisfying the following Penrose conditions

- (i)  $A^\dagger$  is an inner inverse of  $A$  ;
- (ii)  $A^\dagger$  is an outer inverse of  $A$  ;
- (iii)  $AA^\dagger$  is Hermitian ;
- (iv)  $A^\dagger A$  is Hermitian.

It is well known that the following statements are equivalent for  $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ :

- (1) a Moore-Penrose inverse of  $A$  exists ;

(2) a Moore-Penrose inverse of  $A$  is unique ;

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(3) the range of  $A$  is closed.

**Lemma 6.16.** *Let  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ . If  $A$  and  $B$  have closed ranges, then*

1. *the range of  $A \boxtimes B$  is closed ;*
2.  $(A \boxtimes B)^\dagger = A^\dagger \boxtimes B^\dagger$ .

*Proof.* Since the ranges of  $A$  and  $B$  are closed, the Moore-Penrose inverses  $A^\dagger$  and  $B^\dagger$  exist and are unique. It is straightforward to verify that  $A^\dagger \boxtimes B^\dagger$  satisfies the following Penrose equations:

- (i)  $(A \boxtimes B)(A^\dagger \boxtimes B^\dagger)(A \boxtimes B) = A \boxtimes B$
- (ii)  $(A^\dagger \boxtimes B^\dagger)(A \boxtimes B)(A^\dagger \boxtimes B^\dagger) = A^\dagger \boxtimes B^\dagger$
- (iii)  $((A \boxtimes B)(A^\dagger \boxtimes B^\dagger))^* = (A \boxtimes B)(A^\dagger \boxtimes B^\dagger)$
- (iv)  $((A^\dagger \boxtimes B^\dagger)(A \boxtimes B))^* = (A^\dagger \boxtimes B^\dagger)(A \boxtimes B)$ .

Hence, a Moore-Penrose inverse of  $A \boxtimes B$  exists and it is uniquely determined by  $A^\dagger \boxtimes B^\dagger$ . It follows that  $A \boxtimes B$  has a closed range.  $\square$

**Lemma 6.17** (see e.g. [44]). *Let*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{bmatrix}$$

*be an operator in  $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$  such that  $T_{11}$  has a closed range. Then  $T \geq 0$  if and only if*

- (i)  $T_{11} \geq 0$ ,
- (ii)  $T_{12} = T_{11} T_{11}^\dagger T_{12}$ ,
- (iii)  $T_{22}^* = T_{22} \geq T_{12}^* T_{11}^\dagger T_{12}$ .

**Theorem 6.2.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be positive invertible operators. Then*

$$(A \boxtimes B)(A \boxtimes B)^{-1}(A \boxtimes B) \leq A \boxtimes B^{-1} + A^{-1} \boxtimes B + 2I. \quad (6.18)$$

*Proof.* Denote

$$S = \begin{bmatrix} A^{1/2} \boxtimes B^{1/2} & A^{1/2} \boxtimes B^{-1/2} + A^{-1/2} \boxtimes B^{1/2} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$$

where  $Z$  is the selection operator associated with  $\{\mathcal{H}, \mathcal{K}\}$ . Using Lemma 5.4, we get

$$0 \leq S^*S = \begin{bmatrix} A \boxtimes B & A \boxplus B \\ A \boxplus B & A \boxtimes B^{-1} + A^{-1} \boxtimes B + 2I \boxtimes I \end{bmatrix}.$$

Pre- and post-multiplying  $S^*S$  by  $X$  and  $X^*$ , respectively, we obtain

$$0 \leq X^*S^*SX = \begin{bmatrix} A \boxdot B & A \boxtimes B \\ A \boxtimes B & A \boxtimes B^{-1} + A^{-1} \boxdot B + 2I \boxdot I \end{bmatrix}.$$

By Proposition 6.6, we have  $A \boxdot B > 0$  and then  $A \boxdot B$  is invertible. It follows that  $(A \boxdot B)^\dagger = (A \boxdot B)^{-1}$ . Applying Lemma 6.17, we get the result.  $\square$

The next Theorem extends the result of [?, Corollary 3.8] to the case of operator matrices.

**Theorem 6.3.** *Let  $A \in \mathbb{B}(\mathcal{H})$  be a positive operator matrix such that  $A$  and  $A \boxdot I$  have closed ranges, and  $A \boxtimes I = I \boxtimes A$ . Then*

$$(A \boxtimes AA^\dagger)(A \boxdot I)^\dagger(A \boxtimes AA^\dagger) \leq A \boxtimes A^\dagger + 2AA^\dagger \boxdot I. \quad (6.19)$$

*Proof.* Denote

$$S = \begin{bmatrix} A^{1/2} \boxtimes I & A^{1/2} \boxtimes I + (A^\dagger)^{1/2} \boxtimes I \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$$

where  $Z$  is the selection operator. Applying Lemmas 5.4 and 6.16, we get

$$0 \leq S^*S = \begin{bmatrix} A \boxtimes I & A \boxplus AA^\dagger \\ A \boxplus AA^\dagger & A \boxplus A^\dagger + 2AA^\dagger \boxtimes I \end{bmatrix}.$$

Pre- and post-multiplying by  $X^*$  and  $X$ , respectively, we obtain

$$0 \leq X^*S^*SX = \begin{bmatrix} A \boxdot I & A \boxtimes AA^\dagger \\ A \boxtimes AA^\dagger & A \boxtimes A^\dagger + 2AA^\dagger \boxdot I \end{bmatrix}.$$

The proof is complete by using Lemma 6.17.  $\square$

## 6.6 Kantorovich type inequalities for Khatri-Rao sums

**Lemma 6.18.** *Let  $S$  be positive invertible operator in  $\mathbb{B}(\mathcal{K})$  with  $0 < mI \leq S \leq MI$  and let  $X \in \mathbb{B}(\mathcal{H}, \mathcal{K})$  be such that  $X^*X = I$ , then*

$$(X^*SX)^{-1} \leq X^*S^{-1}X \leq \frac{(M+m)^2}{4Mm}(X^*SX)^{-1}, \quad (6.20)$$

$$(X^*SX)^2 \leq X^*S^2X \leq \frac{(M+m)^2}{4Mm}(X^*SX)^2. \quad (6.21)$$

*Proof.* The proof is similar to the case of matrices, given in [29].  $\square$

**Proposition 6.19.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be positive invertible operators. Then*

$$(A \boxtimes B)^{-1} \leq A^{-1} \boxtimes B^{-1} \leq \frac{(M+m)^2}{Mm}(A \boxtimes B)^{-1}, \quad (6.22)$$

$$(A \boxtimes B)^2 \leq A^2 \boxtimes B^2 \leq \frac{(M+m)^2}{Mm}(A \boxtimes B)^2. \quad (6.23)$$

where  $0 < mI \leq A \boxtimes I \oplus I \boxtimes B \leq MI$ .

*Proof.* Denote  $S = \begin{bmatrix} A \boxtimes I & 0 \\ 0 & I \boxtimes B \end{bmatrix}$  and  $X = \frac{1}{\sqrt{2}} \begin{bmatrix} Z \\ Z \end{bmatrix}$  where  $Z$  is the selection operator associated with  $\{\mathcal{H}, \mathcal{K}\}$ . Then  $X^*X = I$ . Using Lemmas ?? and 5.13, we have

$$\begin{aligned} X^*SX &= \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} \begin{bmatrix} A \boxtimes I & 0 \\ 0 & I \boxtimes B \end{bmatrix} \begin{bmatrix} Z \\ Z \end{bmatrix} = \frac{1}{2}(A \boxtimes B) \\ X^*S^{-1}X &= \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} \begin{bmatrix} A^{-1} \boxtimes I & 0 \\ 0 & I \boxtimes B^{-1} \end{bmatrix} \begin{bmatrix} Z \\ Z \end{bmatrix} = \frac{1}{2}(A^{-1} \boxtimes B^{-1}) \\ X^*S^2X &= \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} \begin{bmatrix} A^2 \boxtimes I & 0 \\ 0 & I \boxtimes B^2 \end{bmatrix} \begin{bmatrix} Z \\ Z \end{bmatrix} = \frac{1}{2}(A^2 \boxtimes B^2). \end{aligned}$$

Substitution in (6.20) and (6.21) of Lemma 6.18 lead to the results.  $\square$

**Lemma 6.20.** *Let  $S$  be positive invertible operator in  $\mathbb{B}(\mathcal{K})$  with  $0 < mI \leq S \leq MI$  and let  $X \in \mathbb{B}(\mathcal{H}, \mathcal{K})$  be such that  $X^*X = I$ , then*

$$X^*SX - (X^*S^{-1}X)^{-1} \leq (\sqrt{M} - \sqrt{m})^2 I, \quad (6.24)$$

$$X^*S^2X - (X^*SX)^2 \leq \frac{1}{4}(M - m)^2 I, \quad (6.25)$$

$$(X^*S^2X)^{1/2} - X^*SX \leq \frac{(M - m)^2}{4(M + m)} I. \quad (6.26)$$

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*Proof.* The proof is similar to the case of matrices, given in [29].  $\square$

**Proposition 6.21.** *Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be positive invertible operators. Then*

$$\frac{1}{2}(A \boxtimes B) - 2(A^{-1} \boxtimes B^{-1})^{-1} \leq (\sqrt{M} - \sqrt{m})^2 I, \quad (6.27)$$

$$2(A^2 \boxtimes B^2) - (A \boxtimes B)^2 \leq (M - m)^2 I, \quad (6.28)$$

$$(A^2 \boxtimes B^2)^{1/2} - \frac{1}{\sqrt{2}}(A \boxtimes B) \leq \frac{(M - m)^2}{2\sqrt{2}} I, \quad (6.29)$$

where  $0 < mI \leq A \boxtimes I \oplus I \boxtimes B \leq MI$ .

*Proof.* The proof is similar with Proposition 6.19.  $\square$

**Proposition 6.22.** *Let  $A \in \mathbb{B}(\mathcal{H})$  be a positive operator matrix such that  $A$  has a closed range and  $A^\dagger \boxtimes I = I \boxtimes A^\dagger$ . Then*

$$A \boxtimes A^\dagger \geq 2AA^\dagger \boxtimes I. \quad (6.30)$$

*Proof.* Denote  $S = A \boxtimes I \geq 0$ . By virtue of  $S + S^\dagger \geq 2SS^\dagger$ , we have, by using Lemmas 5.4 and 6.16,

$$\begin{aligned} A \boxplus A^\dagger &= A \boxtimes I + A^\dagger \boxtimes I = A \boxtimes I + (A \boxtimes I)^\dagger \\ &\geq 2(A \boxtimes I)(A \boxtimes I)^\dagger = 2AA^\dagger \boxtimes I. \end{aligned}$$

Pre- and post-multiplying by  $Z^*$  and  $Z$ , respectively, we get the result.  $\square$

We mention that Proposition 6.22 is an extension of [?, Corollary 3.6].

# CHAPTER 7

## Outputs of the Research

1. Ploymukda, A., Chansangiam, P., Khatri-Rao sums for Hilbert space operators, *Songklanakarın Journal of Science and Technology*, 40 (3), 595-601, May - Jun. 2018.

Database : Web of Science, SCOPUS

2. Ploymukda, A., Chansangiam, P., Norm estimations, continuity, and compactness for Khatri-Rao products of Hilbert Space operators, *Malaysian Journal of Fundamental and Applied Sciences*, Vol. 14, No. 4, 382-386, 2018.

Database : Web of Science

3. Ploymukda, A., Chansangiam, P., Operator inequalities involving Khatri-Rao sums and Moore-Penrose inverses, *Malaysian Journal of Fundamental and Applied Sciences*, Vol. 13, No. 4 (2017) 742-746.

Database : Web of Science

4. Ploymukda, A., Chansangiam, P., Algebraic, order, and analytic properties of Tracy-Singh sums for Hilbert space operators, *Songklanakarın Journal of Science and Technology*, 2019, in press.

Database : Web of Science, SCOPUS

5. Ploymukda, A., Chansangiam, P., Khatri-Rao Products and Selection Operators, *Journal of Computational Analysis and Applications*, 27(2), 316-329, 2019.

Database : Web of Science, SCOPUS

## References

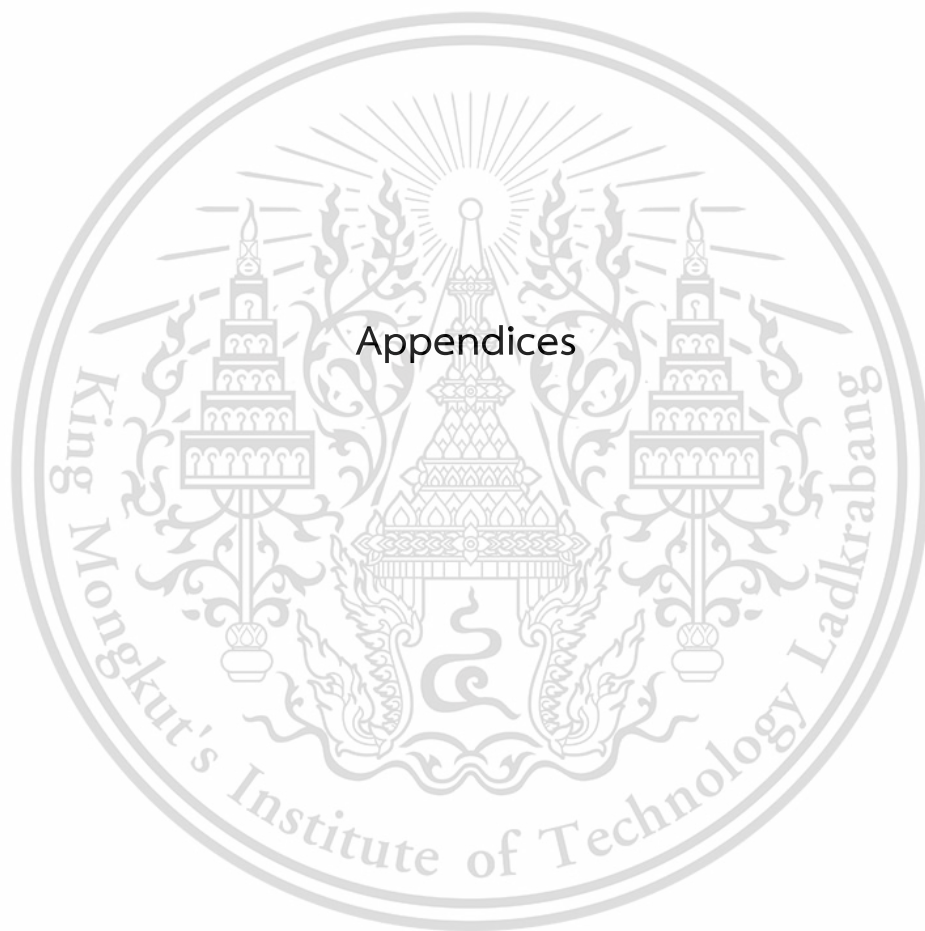
- [1] E. Ahn, S. Kim, H. Lee and Y. Lim, Sagae-Tanabe weighted mean and reverse inequalities, *Kyungpook Math. J.* 47, 595–600, (2007).
- [2] M. Alic, B. Mond, J. Pecaric and V. Volenec, The arithmetic-geometric-harmonic-mean and related matrix inequalities, *Linear Algebra and its Applications.* 40264, 55–62, (1997).
- [3] T. Ando, Topics on Operator Inequalities, Lecture Notes Hokkaido Univ., Sapporo, (1978).
- [4] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Linear Algebra Appl.* 26, 203–241, (1979).
- [5] A. Kiliçman and Z. Al Zhour, Improvements on geometric means related to the Tracy-Singh products of positive matrices, *Mathematica.* 21, 49–65, (2005).
- [6] Albert, A.: Conditions for positive and nonnegative definiteness in terms of pseudoinverses, *SIAM L. Appl. Math.* 17, 434–440 (1969).
- [7] Al Zhour, Z.A., Kiliçman, A.: Extension and generalization inequalities involving the Khatri-Rao product of several positive matrices. *J. Inequal. Appl.*, Article ID 80878 (2006)
- [8] Z. A. Al Zhour, A. Kiliçman, Matrix equalities and inequalities involving Khatri-Rao and Tracy-Singh sums, *J. Inequal. Appl.*, 7, 11–17 (2006).
- [9] H. Bart, I. Gohberg, M.A. Kaashoek, A.C.M. Ran.: *Factorization of Matrix and Operator Functions: The State Space Method*, Birkhauser Verlag, Basel (2008).
- [10] R. Bhatia, Positive Definite Matrices, Princeton University Press, New Jersey, (2007).
- [11] Bhatia, R., Kittaneh, F.: Norm inequalities for partitioned operators and an application. *Math. Ann.* 287, 719-726 (1990)
- [12] Cao, C.G., Zhang, X. and Yang, Z.P., Some inequalities for the Khatri-Rao product of matrices, *Electro. J. Linear Algebra*, 9 (2002) 276-281 (electronic).
- [13] Civciv, H. and Taurkmen, R., On the bounds for  $l^p$  norms of Khatri-Rao and Tracy-Singh products of Cauchy-Toeplitz matrices, *Seluk J. Appl. Math.* 6 (2005) no. 2, 43-52.
- [14] Civciv, H., Taurkmen, R.: On the bounds for  $l^p$  norms of Khatri-Rao and Tracy-Singh products of Cauchy-Toeplitz matrices. *Seluk J. Appl. Math.* 6, 43-52 (2005)

- [15] Feng, X.X. and Yang, Z.P., Lowner partial ordering inequalities on the Khatri-Rao product of matrices, *Gongcheng Shuxue Xuebao*, 19 (2002) no. 3, 106-110.
- [16] M. Fiedler and V. Pták, A new positive definite geometric mean of two positive definite matrices, *Linear Algebra Appl.* 251, 1–20, (1997).
- [17] J.I. Fujii, Y. Seo and M. Tominaga, Kantorovich type operator inequalities via the Specht ratio, *Linear Algebra and its Applications.* 377, 69–81, (2004).
- [18] Horn, R.A. and Mathias, R., Block-matrix generalizations of Schur's basic theorems on Hadamard products, *Linear Algebra Appl.*, 172 (1992) 337-346.
- [19] C. G. Khatri, C. R. Rao, Solutions to some functional equations and their applications to characterization of probability distributions, *Sankhyā*, 26, 167–180 (1968).
- [20] S. Liu, Matrix results on the Khatri-Rao and Tracy-Singh products, *Linear Algebra Appl.*, 289, 267–277 (1999).
- [21] S. Lang, *Fundamentals of Differential Geometry*, Graduate Texts in Mathematics, Springer, Heidelberg, (1999).
- [22] J. Lawson and Y. Lim, The geometric mean, matrices, metrics, and more, *Am. Math. Monthly.* 108, 797–812, (2001).
- [23] Liu, S., *Contributions to Matrix Calculus and Applications in Econometrics*, Tinbergen Institute Research Series, no. 106, Thesis Publishers, Amsterdam, The Netherlands, 1995.
- [24] Liu, S.: Matrix results on the Khatri-Rao and Tracy-Singh products, *Linear Algebra and its Applications* **289**, 267-277 (1999).
- [25] Liu, S.: Several inequalities involving KhatriRao products of positive semidefinite matrices, *Linear Algebra and its Applications* **354**, 175-186 (2002).
- [26] Kubrusly, C.S. and Vieira P.C.M.: Convergence and Decomposition for Tensor Products of Hilbert Space Operators. *Oper. Matrices.* 2(3), 407- 416 (2008)
- [27] Liu, S.: *Contributions to Matrix Calculus and Applications in Econometrics*, Tinbergen Institute Research Series, no. 106, Thesis Publishers, Amsterdam, The Netherlands, 1995.
- [28] C.S. Kubrusly and N. Levan: Preservation of tensor sum and tensor product, *Acta Math. Univ. Comenianae* 80(1), 133–142 (2011).
- [29] S. Liu and H. Neudecker: A survey of Cauchy-Schwarz and Kantorovich-type matrix inequalities, *Statistical Papers* **40**, 55–73 (1999).

- [30] S. Liu, G. Trenkler, Hadamard, Khatri-Rao, Kronecker and other matrix products, *International Journal of Information and Systems Sciences*, 4, 160–177 (2008).
- [31] A. Ploymukda, P. Chansangaim, W. Lewkeeratiyutkul, Algebraic and order properties of Tracy-Singh products for operator matrices, *J. Comput. Anal. Appl.*, 24, 656–664 (2018).
- [32] A. Ploymukda, P. Chansangaim, W. Lewkeeratiyutkul, Analytic properties of Tracy-Singh products for operator matrices, *J. Comput. Anal. Appl.*, 24, 665–674 (2018).
- [33] A. Ploymukda, P. Chansangaim, Khatri-Rao products for operator matrices acting on the direct sum of Hilbert spaces, *Journal of Mathematics*, 2016, Article ID 8301709, 7 pages, <http://dx.doi.org/10.1155/2016/8301709>, (2016).
- [34] A. Ploymukda and P. Chansangaim, Khatri-Rao products and selection operators, *J. Comput. Anal. Appl.*, 27(2), 316–325, (2019).
- [35] A. Ploymukda and P. Chansangaim, Norm estimations, continuity, and compactness for Khatri-Rao products of Hilbert Space operators. *Malaysian Journal of Fundamental and Applied Sciences*, 14(4), 382–386 (2018).
- [36] C.R. Rao, M.B. Rao, *Matrix Algebra and Its Applications to Statistics and Econometrics*, World Scientific, Singapore, 1998
- [37] M. Sagae and K. Tanabe, Upper and lower bounds for the arithmetic-geometric-harmonic means of positive definite matrices, *Linear and Multilinear Algebra*. 37, 279–282, (1994).
- [38] Schechter, M.: On the Spectra of Operators on Tensor Products. *J. Funct. Anal.* 4(1), 95-99 (1969)
- [39] J. Stochel, *Seminormality of Operators from Their Tensor Product*, Proceedings of the American Mathematical Society 124 (1996), 135–140.
- [40] Tracy, D.S. and Singh, R.P. 1972. A New Matrix Product and Its Applications in Partitioned Matrix Differentiation. *Stat. Neerl.* 26(4) : 143–157.
- [41] A.C. Thompson, On certain contraction mappings in a partially ordered vector space, *Proc. Amer. Math. Soc.* 14, 438–443, (1963).
- [42] C. F. Van Loan, The ubiquitous Kronecker product, *Int. J. Comput. Appl. Math.*, 123, 85–100 (2000).
- [43] G. Visick, A quantitative version of the observation that the Hadamard product is a principal submatrix of the Kronecker product, *Linear Algebra Appl.*, 304, 45–68 (2000).
- [44] Q. Xu, L. Sheng, Positive semi-definite matrices of adjointable operators on Hilbert  $C^*$ -modules, *Linear Algebra Appl.*, **428** (2008), 992–1000.

- [45] Yang, Z.P., Zhang, X. and Cao, C.G., Inequalities involving Khatri-Rao products of Hermitian matrices, *Korean J. Comput. Appl. Math.*, 9 (2002) no. 1, 125-133.
- [46] H. Zhang, F. Ding, On the Kronecker products and their applications, *J. Appl. Math.*, 2013, 8 pages (2013).
- [47] Zhang, X., Yang, Z.P. and Cao, C.G., Matrix inequalities involving the Khatri-Rao product, *Arch. Math. (Brno)*, 38 (2002) no. 4, 265-272.
- [48] Zhou, J.H. and Wang, G.R., Generalized Schur complement and Khatri-Rao product, *J. Shanghai Normal University (Natural Sciences)*, 35 (2006) no. 1, 22-26.
- [49] J. Zanni and C.S. Kubrusly, A note on compactness of tensor products, *Acta Math. Univ. Comenianae.* 84, 59–62, (2015).





## Appendix A

### Research Papers

1. Ploymukda, A., **Chansangiam, P.**, Khatri-Rao sums for Hilbert space operators, *Songklanakarin Journal of Science and Technology*, 40 (3), 595-601, May - Jun. 2018.  
Database : Web of Science, SCOPUS
2. Ploymukda, A., **Chansangiam, P.**, Norm estimations, continuity, and compactness for Khatri-Rao products of Hilbert Space operators, *Malaysian Journal of Fundamental and Applied Sciences*, Vol. 14, No. 4, 382-386, 2018.  
Database : Web of Science
3. Ploymukda, A., **Chansangiam, P.**, Operator inequalities involving Khatri-Rao sums and Moore-Penrose inverses, *Malaysian Journal of Fundamental and Applied Sciences*, Vol. 13, No. 4 (2017) 742-746.  
Database : Web of Science
4. Ploymukda, A., **Chansangiam, P.**, Algebraic, order, and analytic properties of Tracy-Singh sums for Hilbert space operators, *Songklanakarin Journal of Science and Technology*, 2019, in press.  
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5. Ploymukda, A., **Chansangiam, P.**, Khatri-Rao Products and Selection Operators, *Journal of Computational Analysis and Applications*, 27(2), 316-329, 2019.  
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Original Article

# Khatri-Rao sums for Hilbert space operators

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## Abstract

We generalize the notions of Khatri-Rao sums for matrices and tensor sums for Hilbert space operators to Khatri-Rao sums for Hilbert space operators. This kind of operator sum is compatible with algebraic operations and order relations. We investigate its analytic properties, including continuity, convergence, and norm bounds. We also discuss the role of selection operator that relates Khatri-Rao sums to Tracy-Singh sums and Khatri-Rao products. Binomial theorem involving Khatri-Rao sums and its consequences are then established.

**Keywords:** tensor product, Khatri-Rao product (sum), Tracy-Singh product (sum)

## 1. Introduction

In matrix and operator theory, there are many kinds of products which are of interest from both theory and application points of views. These products include the Kronecker (tensor) product, the Tracy-Singh product and the Khatri-Rao product. Let  $M_n(\mathbb{C})$  denote the set of  $n$ -by- $n$  complex matrices. Recall that for any  $A \in M_m(\mathbb{C})$  and  $B \in M_n(\mathbb{C})$ , the Kronecker product of  $A$  and  $B$  is given by

$$A \hat{\otimes} B = [a_{ij} B]_{ij}. \tag{1}$$

The Tracy-Singh product of partitioned matrices, as a generalized Kronecker product, was introduced by Tracy and Singh (1972). Let  $A = [A_{ij}]$  be partitioned with  $A_{ij}$  of order  $m_i \times m_j$  as the  $(i, j)$  th submatrix and let  $B = [B_{kl}]$  be a partitioned matrix with  $B_{kl}$  of order  $n_k \times n_l$  as the  $(k, l)$  th submatrix where  $\sum_{i=1}^r m_i = m$ ,  $\sum_{k=1}^s n_k = n$ . The Tracy-Singh product of  $A$  and  $B$  is defined by

$$A \hat{\boxtimes} B = \left[ [A_{ij} \hat{\otimes} B_{kl}]_{kl} \right]_{ij}. \tag{2}$$

If  $r$  and  $s$  are equal, the Khatri-Rao product of  $A$  and  $B$  can be defined to be (Khatri and Rao, 1968)

$$A \hat{\square} B = [A_{ij} \hat{\otimes} B_{ij}]_{ij}. \tag{3}$$

See more information about the development of this theory in Al Zhou (2012), Al Zhou and Kilicman (2006b), Cao, Zhang and Yang (2002), Liu and Trenkler (2008), Shuangzhe (1999), Van Loan (2000).

From Tracy-Singh and Khatri-Rao products of matrices, Al Zhou and Kilicman (2006a) defined Tracy-Singh sums and Khatri-Rao sums of matrices and established certain properties for these sums. Indeed, the Tracy-Singh and Khatri-Rao sums are respectively defined to be

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$$A \hat{\boxplus} B = A \hat{\boxtimes} I_n + I_m \hat{\boxtimes} B, \quad A \hat{\boxcirc} B = A \hat{\boxminus} I_n + I_m \hat{\boxminus} B \tag{4}$$

where  $I_m$  and  $I_n$  are block identity matrices of order  $m \times m$  and  $n \times n$ , respectively.

The theory of Kronecker products of matrices was extended to tensor products of bounded linear operators on a Hilbert space. The notion of tensor sum for operators was investigated in Kubrusly and Levan (2011). Indeed, the tensor sum of  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  is defined to be  $A \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes B$ , here  $\otimes$  denotes the tensor product. Recently, the Tracy-Singh product and the Khatri-Rao product for operators were investigated by the authors (Ploymukda & Chansangiam, 2016).

From the discussion above, it is natural to generalize the Khatri-Rao sum for matrices and the tensor sum of operators to the Khatri-Rao sum for operators. We will show in this paper that the Khatri-Rao sum is compatible with algebraic operations and order relations. We discuss analytic properties of this sum, including continuity, convergence, norm bounds. We investigate the role of selection operator that relates Khatri-Rao sums to Tracy-Singh sums and Khatri-Rao products. Binomial theorem involving Khatri-Rao sums and its consequences are then established.

This paper is organized as follows. Section 2 supplies preliminary results on Tracy-Singh products and Khatri-Rao products for Hilbert space operators. In Section 3, we introduce Khatri-Rao sums for operators and investigate its algebraic, order, and analytic properties. Section 4 discusses relationship between Khatri-Rao sums and selection operators. In Section 5, we prove an operator version of binomial theorem and deduce some consequences inequalities.

## 2. Preliminaries

In what follows,  $\mathcal{H}$  and  $\mathcal{K}$  are complex separable Hilbert spaces. When  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert spaces, denote by  $\mathbb{B}(\mathcal{X}, \mathcal{Y})$  the Banach space of bounded linear operators from  $\mathcal{X}$  into  $\mathcal{Y}$ , and abbreviate  $\mathbb{B}(\mathcal{X}, \mathcal{X})$  to  $\mathbb{B}(\mathcal{X})$ . For Hermitian operators  $A, B \in \mathbb{B}(\mathcal{H})$ , the partial order  $A \geq B$  indicates that  $A - B$  is a positive operator, while  $A > 0$  means that  $\langle Ax, x \rangle > 0$  for all nonzero vectors  $x$  in  $\mathcal{H}$ .

### 2.1 Tracy-Singh products for operators

Recall that the tensor product of  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  is the unique bounded linear operator from  $\mathcal{H} \otimes \mathcal{K}$  into itself such that

$$(A \otimes B)(x \otimes y) = Ax \otimes By \tag{5}$$

for all  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ . To define the Tracy-Singh product of operators, we first use projection theorem for Hilbert spaces to decompose

$$\mathcal{H} = \bigoplus_{i=1}^m \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{k=1}^n \mathcal{K}_k$$

where each  $\mathcal{H}_i$  and  $\mathcal{K}_k$  are Hilbert spaces. Thus any operator  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  can be uniquely represented by operator matrices  $A = [A_{ij}]_{i,j=1}^{m,m}$  and  $B = [B_{kl}]_{k,l=1}^{n,n}$  where  $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$  and  $B_{kl} \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k)$  for each  $i, j, k, l$ .

**Definition 1.** According to the previous discussion, the *Tracy-Singh product* of  $A$  and  $B$  is defined to be the bounded linear operator from  $\bigoplus_{i,k=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_k$  into itself represented by the operator matrix

$$A \boxtimes B = \left[ [A_{ij} \otimes B_{kl}]_{kl} \right]_{ij} \tag{6}$$

**Lemma 2** (Ploymukda & Chansangiam, 2016). The following properties hold, provided that all operators are compatible.

$$(A \boxtimes B)^* = A^* \boxtimes B^* \tag{7}$$

$$(\alpha A) \boxtimes B = \alpha(A \boxtimes B) = A \boxtimes (\alpha B) \tag{8}$$

$$A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C \tag{9}$$

$$(B + C) \boxtimes A = B \boxtimes A + C \boxtimes A \tag{10}$$

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD \tag{11}$$

**Lemma 3** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . Then  $A \boxtimes B = 0$  if and only if  $A = 0$  or  $B = 0$ .

**Proof.** Write  $A = [A_{ij}]_{i,j=1}^{m,m}$  and  $B = [B_{kl}]_{k,l=1}^{n,n}$ . We have the following norm bounds (Ploymukda & Chansangiam, 2016)

$$\frac{1}{mn} \|A\| \|B\| \leq \|A \boxtimes B\| \leq mn \|A\| \|B\|.$$

Now, the claim follows.

**Definition 4.** Let  $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ . We define the *Tracy-Singh sum* of  $A$  and  $B$  as follows:

$$A \boxplus B = A \boxtimes I_{\mathcal{K}} + I_{\mathcal{H}} \boxtimes B \tag{12}$$

which belongs to  $\mathbb{B}\left(\bigoplus_{i,j=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_j\right)$ .

**2.2 Khatri-Rao products for operators**

From now on, fix the following decompositions of Hilbert spaces:

$$\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{i=1}^n \mathcal{K}_i.$$

**Definition 5.** Let  $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$  be operator matrices. We define the *Khatri-Rao product* of  $A$  and  $B$  to be the operator matrix

$$A \square B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{n,n} \tag{13}$$

which is a bounded linear operator from  $\bigoplus_{i=1}^n \mathcal{H}_i \otimes \mathcal{K}_i$  into itself.

Recall that for each  $A \in \mathbb{M}_n(\mathbb{C})$ , the induced map  $L_A : \mathbb{C}^m \rightarrow \mathbb{C}^m$ ,  $x \mapsto Ax$  is a bounded linear operator.

**Lemma 6** (Ploymukda & Chansangiam, 2016). For any complex matrices  $A = [A_{ij}]$  and  $B = [B_{ij}]$  partitioned in block-matrix forms, we have

$$L_A \square L_B = L_{A \square B}. \tag{14}$$

**Lemma 7** (Ploymukda & Chansangiam, 2016). Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B, C \in \mathbb{B}(\mathcal{K})$  be operator matrices, and let  $\alpha \in \mathbb{C}$ . Then

$$(A \square B)^* = A^* \square B^*, \tag{15}$$

$$(\alpha A) \square B = \alpha(A \square B) = A \square (\alpha B), \tag{16}$$

$$A \square (B + C) = A \square B + A \square C, \tag{17}$$

$$(B + C) \square A = B \square A + C \square A. \tag{18}$$

**Lemma 8** (Ploymukda & Chansangiam, 2016). There is an isometry  $Z$  such that  $ZZ^* \leq I$  and

$$A \square B = Z^* (A \boxtimes B) Z \tag{19}$$

for any  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . We call  $Z$  the selection operator associated with the ordered tuple  $(\mathcal{H}, \mathcal{K})$ .

**Lemma 9** (Ploymukda & Chansangiam, 2016). Let  $Z$  be the selection operator associated with  $(\mathcal{H}, \mathcal{K})$ . Then there exists an operator

$$W : \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_j \rightarrow \bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_j$$

such that  $Z^*W = 0$  and  $ZZ^* + WW^* = I$ . Indeed,  $W$  is defined as

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix} \tag{20}$$

where  $W^{(s)} = [W_{kl}^{(s)}]_{k,l=1}^{n,n^2-1} : \bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_i \rightarrow \bigoplus_{i=1}^n \mathcal{H}_s \otimes \mathcal{K}_i$  for  $s = 1, \dots, n$ , with  $W_{kl}^{(s)}$  is an identity operator if  $k \neq s$  and

$l = n(s - 1) + k$  and others are zero operators.

**Lemma 10** (Ploymukda & Chansangiam, 2016). Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices.

(i) If  $A \geq 0$  and  $B \geq 0$ , then  $A \square B \geq 0$ .

(ii) If  $A > 0$  and  $B > 0$ , then  $A \square B > 0$ .

**Lemma 11** (Ploymukda & Chansangiam, 2016). Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices and let  $(A_r)_{r=1}^\infty$  and  $(B_r)_{r=1}^\infty$  be sequences in  $\mathbb{B}(\mathcal{H})$  and  $\mathbb{B}(\mathcal{K})$ , respectively. If  $A_r \rightarrow A$  and  $B_r \rightarrow B$ , then  $A_r \boxdot B_r \rightarrow A \boxdot B$ .

**Lemma 12** (Ploymukda & Chansangiam, 2016). For any operator matrices  $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ , we have

$$\|A \boxdot B\| \leq n^2 \|A\| \|B\|. \tag{21}$$

### 3. Algebraic, Order, and Analytic Properties of Khatri-Rao Sums for Operators

In this section, we generalize the Khatri-Rao sum for matrices and the tensor sum for operators to the Khatri-Rao sum for operators. This kind of operator sum turns out to be compatible with algebraic operations and order relations for operators. Binomial theorem involving Tracy-Singh sums and its consequences are also established. We investigate continuity, convergence, norm bounds for Tracy-Singh sums of operators.

**Definition 13.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . We define the *Khatri-Rao sum* of  $A$  and  $B$  as follows:

$$A \otimes B = A \boxdot I_{\mathcal{K}} + I_{\mathcal{H}} \boxdot B \tag{22}$$

which belongs to  $\mathbb{B}(\bigoplus_{i=1}^n \mathcal{H}_i \otimes \mathcal{K}_i)$ .

Note that if both  $A$  and  $B$  are  $1 \times 1$  block operator matrices, their Khatri-Rao sum  $A \otimes B$  becomes the tensor sum (Kubrusly & Levan, 2011)

$$A \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes B.$$

If  $\mathcal{H}_i = \mathcal{K}_i = \mathbb{C}$ , the Khatri-Rao sum  $A \otimes B$  reduces to the Hadamard sum of complex matrices (Al Zhou & Kilicman, 2006a).

**Proposition 14.** Let  $A = [A_{ij}]$  and  $B = [B_{ij}]$  be compatible partitioned matrices of order  $m \times m$  and  $n \times n$ , respectively. Then

$$L_A \otimes L_B = L_{A \otimes B}. \tag{23}$$

**Proof.** We know that the linear map induced by the identity matrix is the identity operator. By applying Lemma 6, we get

$$L_A \otimes L_B = L_A \boxdot L_I + L_I \boxdot L_B = L_{A \boxdot I} + L_{I \boxdot B} = L_{A \otimes B}.$$

**Proposition 15.** Let  $\alpha$  and  $\beta$  be scalars. The following properties hold, provided that all operators are compatible.

$$(A \otimes B)^* = A^* \otimes B^*, \tag{24}$$

$$\alpha (A \otimes B) = \alpha A \otimes \alpha B, \tag{25}$$

$$(\alpha + \beta)(A \otimes B) = \alpha A \otimes \beta B + \beta A \otimes \alpha B, \tag{26}$$

$$(A_1 + A_2) \otimes (B_1 + B_2) = A_1 \otimes B_1 + A_2 \otimes B_2, \tag{27}$$

$$(A_1 \oplus A_2) \otimes (B_1 \oplus B_2) = (A_1 \otimes B_1) \oplus (A_2 \otimes B_2). \tag{28}$$

Here the symbol  $\oplus$  denotes the direct sum for operators. We call property (27) mixed sum property.

**Proof.** Using Lemma 7, we get (24)-(26). By applying properties (17) and (18) Lemma 7, we get

$$\begin{aligned} (A+C) \otimes (B+D) &= (A+C) \boxdot I + I \boxdot (B+D) \\ &= A \boxdot I + C \boxdot I + I \boxdot B + I \boxdot D \\ &= (A \boxdot I + I \boxdot B) + (C \boxdot I + I \boxdot D) \\ &= A \otimes B + C \otimes D. \end{aligned}$$

By the definition of Khatri-Rao sum, we have

$$\begin{aligned} (A_1 \otimes B_1) \oplus (A_2 \otimes B_2) &= \begin{bmatrix} A_1 \boxdot I + I \boxdot B_1 & 0 \\ 0 & A_2 \boxdot I + I \boxdot B_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 \boxdot I & 0 \\ 0 & A_2 \boxdot I \end{bmatrix} + \begin{bmatrix} I \boxdot B_1 & 0 \\ 0 & I \boxdot B_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \boxdot \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \boxdot \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \end{aligned}$$

$$= (A_1 \oplus A_2) \otimes (B_1 \oplus B_2).$$

**Corollary 16.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . If  $A$  and  $B$  are Hermitian (skew-Hermitian), then  $A \otimes B$  is also Hermitian (skew-Hermitian).

**Corollary 17.** Let  $A_1 \in \mathbb{B}(\mathcal{H})$  and  $A_2 \in \mathbb{B}(\mathcal{K})$ . If  $A_1 = X_1 + iY_1$  and  $A_2 = X_2 + iY_2$  are the Cartesian decompositions of  $A_1$  and  $A_2$ , respectively, then  $A_1 \otimes A_2 = X_1 \otimes X_2 + i(Y_1 \otimes Y_2)$  is the Cartesian decomposition of  $A_1 \otimes A_2$ .

**Proposition 18.** Let  $A, A_1, A_2 \in \mathbb{B}(\mathcal{H})$  and  $B, B_1, B_2 \in \mathbb{B}(\mathcal{K})$  be operator matrices.

- (i) If  $A \geq 0$  and  $B \geq 0$ , then  $A \otimes B \geq 0$ .
- (ii) If  $A > 0$  and  $B > 0$ , then  $A \otimes B > 0$ .
- (iii) If  $A_1 \geq A_2$  and  $B_1 \geq B_2$ , then  $A_1 \otimes B_1 \geq A_2 \otimes B_2$ .
- (iv) If  $A_1 > A_2$  and  $B_1 > B_2$ , then  $A_1 \otimes B_1 > A_2 \otimes B_2$ .

**Proof.** It follows from Lemma 10 and Proposition 15.

The next proposition asserts that the Khatri-Rao sum is (jointly) continuous with respect to the operator-norm topology.

**Proposition 19.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operators matrices, and let  $(A_r)_{r=1}^\infty$  and  $(B_r)_{r=1}^\infty$  be sequences in  $\mathbb{B}(\mathcal{H})$  and  $\mathbb{B}(\mathcal{K})$ , respectively. If  $A_r \rightarrow A$  and  $B_r \rightarrow B$ , then  $A_r \otimes B_r \rightarrow A \otimes B$ .

**Proof.** By Lemma 11, we have  $A_r \square I \rightarrow A \square I$  and  $I \square B_r \rightarrow I \square B$ . Then  $A_r \square I + I \square B_r \rightarrow A \square I + I \square B$ , i.e.  $A_r \otimes B_r \rightarrow A \otimes B$ .

The next result is a triangle-like inequality involving Khatri-Rao sums.

**Proposition 20.** For any operator matrices  $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ , we have

$$\frac{1}{n^2} \|A \otimes B\| \leq \|A\| + \|B\|. \tag{29}$$

**Proof.** By using Lemma 12, we have

$$\begin{aligned} \|A \otimes B\| &= \|A \square I + I \square B\| \leq \|A \square I\| + \|I \square B\| \\ &\leq n^2 \|A\| \|I\| + n^2 \|I\| \|B\| = n^2 (\|A\| + \|B\|). \end{aligned}$$

#### 4. Khatri-Rao Sums and Selection Operators

In this section, we investigate the role of selection operators that relates Khatri-Rao sums to Tracy-Singh sums and Khatri-Rao products.

**Proposition 21.** There is an isometry  $Z$  such that  $ZZ^* \leq I$  and

$$A \otimes B = Z^*(A \boxplus B)Z \tag{30}$$

for any  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ .

**Proof.** By Lemma 8, there is an isometry  $Z$  such that  $ZZ^* \leq I$  and

$$A \otimes B = A \square I + I \square B = Z^*(A \boxtimes I + I \boxtimes B)Z = Z^*(A \boxplus B)Z.$$

**Proposition 22.** Let  $A \in \mathbb{B}(\mathcal{H})$ ,  $B \in \mathbb{B}(\mathcal{K})$ , and let  $Z$  be the selection operator associated with  $(\mathcal{H}, \mathcal{K})$ . Then

$$A \square B^* + A^* \square B \leq (A \otimes B)(A \otimes B)^* + Z^*(A \boxplus B)WW^*(A \boxplus B)^*Z. \tag{31}$$

where  $W$  is the operator defined in Lemma 9.

**Proof.** By Lemmas 2, 8 and 9, and Theorem 21, we have

$$\begin{aligned} (A \otimes B)(A \otimes B)^* &= Z^*(A \boxplus B)ZZ^*(A^* \boxplus B^*)Z \\ &= Z^*(A \boxplus B)(I - WW^*)(A^* \boxplus B^*)Z \\ &= Z^*(A \boxplus B)(A^* \boxplus B^*)Z - Z^*(A \boxplus B)WW^*(A^* \boxplus B^*)Z \\ &= Z^*(AA^* \boxplus BB^* + A \boxtimes B^* + A^* \boxtimes B)Z - Z^*(A \boxplus B)WW^*(A^* \boxplus B^*)Z \\ &= AA^* \otimes BB^* + A \square B^* + A^* \square B - Z^*(A \boxplus B)WW^*(A \boxplus B)^*Z. \end{aligned}$$

By Proposition 18, we have  $AA^* \otimes BB^* \geq 0$  and hence

$$(A \otimes B)(A \otimes B)^* + Z^*(A \boxplus B)WW^*(A \boxplus B)^*Z - (A \square B^* + A^* \square B) \geq 0.$$

**Proposition 23.** Let  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$  be positive operators. Then

$$AC \otimes BD \leq (A \otimes B)(C \otimes D) + Z^*(A \boxplus B)WW^*(C \boxplus D)Z. \tag{32}$$

**Proof.** By replacing  $A^* = C$  and  $B^* = D$  in the proof of Proposition 22, we have

$$(A \otimes B)(C \otimes D) + Z^*(A \boxplus B)WW^*(C \boxplus D)Z - AC \otimes BD = A \square D + C \square B.$$

Since  $A \square D + C \square B$  is positive, we get the result.

### 5. Binomial Theorem Involving Khatri-Rao Sums and its Consequences

In this section, we prove an operator version of binomial theorem concerning Khatri-Rao sums. As consequences, we obtain two operator inequalities, including Bernoulli type inequality.

**Lemma 24** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . We have  $W^*(A \boxplus B)Z = 0$  if and only if  $A = A_1 \oplus \dots \oplus A_n$  and  $B = B_1 \oplus \dots \oplus B_n$  where  $Z$  is the selection operator associated with  $(\mathcal{H}, \mathcal{K})$  and  $W$  is the operator defined by (20).

**Proof.** Consider  $A \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$  and  $B \in \mathbb{B}(\mathcal{K}_1 \oplus \mathcal{K}_2)$ . By a computation, we have

$$\begin{aligned} W^*(A \boxplus B)Z &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} A_{11} \oplus B_{11} & I \otimes B_{12} & A_{12} \otimes I & 0 \\ I \otimes B_{21} & A_{11} \oplus B_{21} & 0 & A_{12} \otimes I \\ A_{21} \otimes I & 0 & A_{22} \oplus B_{11} & I \otimes B_{12} \\ 0 & A_{21} \otimes I & I \otimes B_{21} & A_{22} \oplus B_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} \oplus B_{11} & 0 \\ I \otimes B_{21} & A_{12} \otimes I \\ A_{21} \otimes I & I \otimes B_{21} \\ 0 & A_{22} \oplus B_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I \otimes B_{21} & A_{12} \otimes I \\ A_{21} \otimes I & I \otimes B_{12} \end{bmatrix}. \end{aligned}$$

By Lemma 3, we obtain

$$\begin{aligned} W^*(A \boxplus B)Z = 0 &\Leftrightarrow A_{12} \otimes I, A_{21} \otimes I, I \otimes B_{12} \text{ and } I \otimes B_{21} \text{ are zeros.} \\ &\Leftrightarrow A_{12}, A_{21}, B_{12} \text{ and } B_{21} \text{ are zeros.} \\ &\Leftrightarrow A = A_1 \oplus A_2 \text{ and } B = B_1 \oplus B_2. \end{aligned}$$

A direct computation shows that Lemma 24 holds for the case  $A = [A_{ij}]_{i,j=1}^{n,n}$  and  $B = [B_{ij}]_{i,j=1}^{n,n}$  for any integer  $n > 2$ .

**Theorem 25.** Let  $A = A_1 \oplus \dots \oplus A_n \in \mathbb{B}(\mathcal{H})$  and  $B = B_1 \oplus \dots \oplus B_n \in \mathbb{B}(\mathcal{K})$  be compatible operator matrices, then for any integer  $r \geq 2$ ,

$$(A \otimes B)^r = A^r B^r + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \square B^k). \tag{33}$$

**Proof.** By the proof of Proposition 22, we have

$$(A \otimes B)^2 = A^2 \otimes B^2 + 2A \square B - Z^*(A \boxplus B)WW^*(A \boxplus B)Z.$$

Since  $W^*(A \boxplus B)Z = 0$ , by Lemma 24, we get

$$(A \otimes B)^2 = A^2 \otimes B^2 + 2A \square B.$$

This show that Theorem 25 is true for  $r = 2$ . Suppose that (33) hold for  $r \geq 2$ . Then, by Lemma 2,

$$\begin{aligned} (A \otimes B)^{r+1} &= (AB)^r (AB) \\ &= \left( A^r B^r + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \square B^k) \right) (AB) \\ &= (A^r B^r)(AB) + \left( \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \square B^k) \right) (AB) \\ &= A^{r+1} B^{r+1} + A^r \square B + A \square B^r - Z^*(A^r \boxplus B^r)WW^*(A \boxplus B)Z \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k+1} \square B^k - Z^*(A^{r-k} \boxtimes B^r)WW^*(A \boxtimes I)Z) \\
 & + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \square B^{k+1} - Z^*(A^{r-k} \boxtimes B^r)WW^*(I \boxtimes B)Z) \\
 = & A^{r+1}B^{r+1} + A^r \square B + A \square B^r - Z^*(A^r \boxplus B^r)WW^*(A \boxplus B)Z \\
 & + \sum_{k=1}^r \binom{r}{k} (A^{(r+1)-k} \square B^k) + \sum_{k=0}^{r-1} \binom{r}{k} (A^{r-k} \square B^{k+1}) \\
 & - \sum_{k=1}^{r-1} \binom{r}{k} Z^*(A^{r-k} \boxtimes B^r)WW^*(A \boxplus B)Z \\
 = & A^{r+1}B^{r+1} + A^r \square B + A \square B^r - Z^*(A^r \boxplus B^r)WW^*(A \boxplus B)Z \\
 & + \sum_{k=1}^r \binom{r}{k} (A^{(r+1)-k} \square B^k) - \sum_{k=1}^{r-1} \binom{r}{k} Z^*(A^{r-k} \boxtimes B^r)WW^*(A \boxplus B)Z.
 \end{aligned}$$

Since  $W^*(A \boxplus B)Z = 0$  (by Lemma 24), we arrive at (33).

**Corollary 26.** Let  $A = A_1 \oplus \dots \oplus A_n \in \mathbb{B}(\mathcal{H})$  and  $B = B_1 \oplus \dots \oplus B_n \in \mathbb{B}(\mathcal{K})$  be positive operators, then for any  $r \in \mathbb{N}$ ,

$$(A \otimes B)^r \geq A^r \otimes B^r. \tag{34}$$

**Proof.** Inequality (34) is obviously true for  $r = 1$ . Let  $r \geq 2$ . Since  $A, B \geq 0$ , we have by Lemma 10 that

$$\sum_{k=1}^r \binom{r}{k} (A^{r-k} \square B^k) \geq 0. \text{ By Theorem 25, we obtain } (A \otimes B)^r \geq A^r \otimes B^r.$$

**Corollary 27.** Let  $A = A_1 \oplus \dots \oplus A_n \in \mathbb{B}(\mathcal{H})$  be a positive operator. Then for any  $r \in \mathbb{N}$ ,

$$(I \otimes A)^r \geq I \otimes (rA). \tag{35}$$

**Proof.** Since  $A \geq 0$  we have  $I \otimes A^r \geq I \square I$ . By Theorem 26, we obtain

$$\begin{aligned}
 (I \otimes A)^r & = I \otimes A^r + \sum_{k=1}^{r-1} \binom{r}{k} (I \square A^k) = I \otimes A^r + r(I \square A) + \sum_{k=2}^{r-1} \binom{r}{k} (I \square A^k) \\
 & \geq I \otimes A^r + r(I \square A) \geq I \square I + I \square (rA) = I \otimes (rA).
 \end{aligned}$$

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**References**

Al Zhou, Z. A. (2012). New Holder-type inequalities for the Tracy-Singh and Khatri-Rao products of positive matrices. *International Journal of Computer Engineering Research*, 3(3), 50-54.

Al Zhou, Z. A., & Kilicman, A. (2006a). Matrix equalities and inequalities involving Khatri-Rao and Tracy-Singh sums. *Journal of Inequalities in Pure and Applied Mathematics*, 7(1), 11-17

Al Zhou, Z. A., & Kilicman, A. (2006b). Extension and generalization inequalities involving the Khatri-Rao product of several positive matrices, *Journal of Inequalities and Applications*, 1-21.

Cao, C. G., Zhang, X., & Yang, Z. P. (2002). Some inequalities for the Khatri-Rao product of matrices. *Electronic Journal of Linear Algebra*, 9, 276-281.

Khatri, C. G., & Rao, C. R. (1968). Solutions to some functional equations and their applications to characterization of probability distributions. *Sankhya*, 26(4), 167-180.

Kubrusly, C. S., & Levan, N. (2011). Preservation of tensor sum and tensor product. *Acta Mathematica Universitatis Comenianae*, 80(1), 133-142.

Liu, S., & Trenkler, G. (2008). Hadamard, Khatri-Rao, Kronecker and other matrix products. *International Journal of Information and Systems Sciences*, 4(1), 160-177.

Ploymukda, A., & Chansangiam, P. (2016). Khatri-Rao products for operator matrices acting on the direct sum of Hilbert spaces. *Journal of Mathematics*, 2016, Article ID 8301709, 7 pages.

Shuangzhe, L. (1999). Matrix results on the Khatri-Rao and Tracy-Singh products. *Linear Algebra and its Applications*, 289(1-3), 267-277.

Tracy, D. S., & Singh, R. P. (1972). A new matrix product and its applications in partitioned matrix differentiation. *Statistica Neerlandica*, 26(4), 143-157.

Van Loan, C. F. (2000). The ubiquitous Kronecker product. *Journal of Computational and Applied Mathematics*, 123, 85-100.

# Norm estimations, continuity, and compactness for Khatri-Rao products of Hilbert Space operators

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## Abstract

We provide estimations for the operator norm, the trace norm, and the Hilbert-Schmidt norm for Khatri-Rao products of Hilbert space operators. It follows that the Khatri-Rao product is continuous on norm ideals of compact operators equipped with the topologies induced by such norms. Moreover, if two operators are represented by block matrices in which each block is nonzero, then their Khatri-Rao product is compact if and only if both operators are compact. The Khatri-Rao product of two operators are trace-class (Hilbert-Schmidt class) if and only if each operator is trace-class (Hilbert-Schmidt class, respectively).

**Keywords:** Khatri-Rao product, compact operator, Schatten  $p$ -class operator, trace-class operator, Hilbert-Schmidt class operator.

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## INTRODUCTION

Matrices and operators are fundamental tools in mathematics and related fields from viewpoints of theory, computations, and applications. A variety of ways to multiply matrices has been investigated in the literature. For instance, the Kronecker product and the Khatri-Rao product. Denote by  $M_{m,n}(\mathbb{C})$  the set of  $m$ -by- $n$  complex matrices and abbreviate  $M_{n,n}(\mathbb{C})$  to  $M_n(\mathbb{C})$ . Recall that the Kronecker product of  $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$  and  $B \in M_{p,q}(\mathbb{C})$  is given by

$$A \hat{\otimes} B = [a_{ij} B]_{ij}.$$

The notion of Kronecker product was generalized to the Khatri-Rao product as follows. Consider two complex matrices  $A$  and  $B$  partitioned so that their  $(i, j)$  th block are given by  $A_{ij}$  and  $B_{ij}$  for each  $i, j$  (the sizes of  $A_{ij}$  and  $B_{ij}$  may be different). Then the Khatri-Rao product [1] of  $A$  and  $B$  is defined by

$$A \hat{*} B = [A_{ij} \hat{\otimes} B_{ij}]_{ij}. \tag{1}$$

When  $A$  and  $B$  are of only one block, their Khatri-Rao product is just their Kronecker product. See more information in [2-6] and references therein.

The notion of Kronecker product of matrices is extended to the tensor product of operators on a Hilbert space. Certain algebraic, order, and analytic properties of tensor product of operators have been established, see [7-9]. In [10-11], the authors study the notion of tensor product for operators to the Tracy-Singh product of operators. Recently in [12-13], the authors introduced the Khatri-Rao product and the

Khatri-Rao sum for operators acting on the direct sum of Hilbert spaces. This construction provides a natural extension for both the Khatri-Rao product/sum for matrices, and the tensor product/sum of operators (see Section 2 for details). Fundamental algebraic, order, and structure properties of the Khatri-Rao product were investigated in [12].

In this paper, we continue developing this theory by discussing analytic properties of Khatri-Rao products for bounded linear operators acting on a Hilbert space. Under the assumption that two operators are represented by block matrices whose each block is nonzero, we will show that their Khatri-Rao product is compact if and only if both factors are compact. We provide norm bounds for the operator-norm, and the Schatten  $p$ -norms for  $p = 1, 2, \infty$ . Then we show that the Khatri-Rao product is (jointly) continuous with respect to the topologies induced by such norms. The norm bounds imply that the Khatri-Rao product of two operators are trace-class (Hilbert-Schmidt class) if and only if each operator is trace-class (Hilbert-Schmidt class, respectively).

This paper is organized as follows. In Section 2, we explain the notion of Khatri-Rao product for operators. In Section 3, we establish analytic properties involving norm bounds, continuity, convergence, and compactness of the Khatri-Rao product of operators in the operator-norm topology. The last section discusses the same properties of the Khatri-Rao product on norm ideals of compact operators.

## PRELIMINARIES ON KHATRI-RAO PRODUCTS FOR OPERATORS

Throughout this paper, let  $H, H', K$  and  $K'$  be complex Hilbert spaces. When  $X$  and  $Y$  are Hilbert spaces, let  $B(X, Y)$  stand for the Banach space of all bounded linear operators from  $X$  into  $Y$ , equipped with the operator norm  $\|\cdot\|_{\infty}$ . We abbreviate  $B(X, X)$  to  $B(X)$ .

Recall that the tensor product of  $A \in B(H, H')$  and  $B \in B(K, K')$  is a unique bounded linear operator  $A \otimes B$  in  $B(H \otimes K, H' \otimes K')$  such that

$$(A \otimes B)(x \otimes y) = Ax \otimes By$$

for all  $x \in H$  and  $y \in K$ . The tensor product is bilinear and continuous with respect to the topology induced by the operator norm.

From now on, fix the following orthogonal decompositions:

$$H = \bigoplus_{j=1}^n H_j, \quad H' = \bigoplus_{i=1}^m H'_i, \quad K = \bigoplus_{j=1}^n K_j, \quad K' = \bigoplus_{i=1}^m K'_i,$$

where all  $H_j, H'_i, K_j, K'_i$  are Hilbert spaces. For each  $j = 1, \dots, n$ , let

$E_j : H_j \rightarrow H$  and  $F_j : K_j \rightarrow K$  be the canonical embeddings defined by

$$E_j : x \mapsto (0, \dots, 0, x, 0, \dots, 0) \text{ (} x \text{ is in the } j\text{-th position),}$$

$$F_j : y \mapsto (0, \dots, 0, y, 0, \dots, 0) \text{ (} y \text{ is in the } j\text{-th position).}$$

For each  $i = 1, \dots, m$ , let  $P_i : H' \rightarrow H'_i$  and  $Q_i : K' \rightarrow K'_i$  be the orthogonal projections. Hence any operator  $A \in B(H, H')$  and  $B \in B(K, K')$  can be represented uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,n} \text{ and } B = [B_{ij}]_{i,j=1}^{m,n} \quad (2)$$

where  $A_{ij} = P_i A E_j \in B(H_j, H'_i)$  and  $B_{ij} = Q_i B F_j \in B(K_j, K'_i)$  for each  $i, j$ . We can perform the addition, the scalar multiplication, the adjointation, and the usual multiplication of operator matrices in a similar way to those of matrices.

We define the Khatri-Rao product of  $A$  and  $B$  to be a bounded linear operator from  $\bigoplus_{j=1}^n H_j \otimes K_j$  to  $\bigoplus_{i=1}^m H'_i \otimes K'_i$  represented by an operator matrix

$$A * B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{m,n}. \quad (3)$$

When  $m = n = 1$ , the Khatri-Rao product  $A * B$  is reduced to the tensor product  $A \otimes B$ .

**Lemma 1** ([12]). The Khatri-Rao product of operators is bilinear. More precisely,

$$A * (B + C) = A * B + A * C,$$

$$(B + C) * A = B * A + C * A,$$

$$A * (\alpha B) = \alpha(A * B) = (\alpha A) * B$$

for any compatible operators  $A, B, C$  and for any scalar  $\alpha$ .

### NORM BOUNDS, CONTINUITY, CONVERGENCE AND COMPACTNESS OF KHATRI-RAO PRODUCTS IN THE OPERATOR-NORM TOPOLOGY

In this section, we discuss norm estimation, continuity, convergence, and compactness of the Khatri-Rao product of operators with respect to the topology induced by the operator norm.

Recall the following bounds for the operator norm of operator matrices.

**Lemma 2** ([14]). Let  $A = [A_{ij}]_{i,j=1}^{n,n}$  be an operator matrix in  $B(H)$ . Then

$$n^{-2} \sum_{i,j=1}^n \|A_{ij}\|_\infty^2 \leq \|A\|_\infty^2 \leq \sum_{i,j=1}^n \|A_{ij}\|_\infty^2. \quad (4)$$

The next theorem provides an upper estimate for the operator norm of Khatri-Rao product. Such bound depends on the number of blocks in the representation (2).

**Theorem 3.** For any operator matrices  $A = [A_{ij}]_{i,j=1}^{n,n} \in B(H)$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in B(K)$ , we have

$$\|A * B\|_\infty \leq n^2 \|A\|_\infty \|B\|_\infty. \quad (5)$$

**Proof.** Since each  $(i, j)$ -th block of  $A * B$  is given by  $A_{ij} \otimes B_{ij}$ , it follows from Lemma 2 and the Cauchy-Schwarz inequality that

$$\begin{aligned} \|A * B\|_\infty^2 &\leq \sum_{i,j} \|A_{ij} \otimes B_{ij}\|_\infty^2 \\ &= \sum_{i,j} \|A_{ij}\|_\infty^2 \|B_{ij}\|_\infty^2 \\ &\leq \left( \sum_{i,j} \|A_{ij}\|_\infty^2 \right) \left( \sum_{i,j} \|B_{ij}\|_\infty^2 \right) \\ &\leq n^4 \|A\|_\infty^2 \|B\|_\infty^2. \end{aligned}$$

Hence, we obtain the bound (5).

The next result asserts that the map  $(A, B) \mapsto A * B$  is (sequentially) continuous with respect to the operator-norm topology.

**Theorem 4.** Let  $A = [A_{ij}]_{i,j=1}^{n,n} \in B(H)$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in B(K)$  be operator matrices and let  $(A_r)_{r=1}^\infty$  and  $(B_r)_{r=1}^\infty$  be sequences in  $B(H)$  and  $B(K)$ , respectively. If  $A_r \rightarrow A$  and  $B_r \rightarrow B$  (in operator norm), then  $A_r * B_r \rightarrow A * B$ .

**Proof.** Suppose that  $A_r \rightarrow A$  and  $B_r \rightarrow B$  in operator norm. By the continuity of the operator norm, we have  $\|B_r\|_\infty \rightarrow \|B\|_\infty$  as  $r \rightarrow \infty$ . By using Lemma 1 and Theorem 3, we have

$$\begin{aligned} &\|A_r * B_r - A * B\|_\infty \\ &\leq \| (A_r - A) * B_r + A * (B_r - B) \|_\infty \\ &\leq n^2 \|A_r - A\|_\infty \|B_r\|_\infty + n^2 \|A\|_\infty \|B_r - B\|_\infty \\ &\rightarrow n^2 \cdot 0 \cdot \|B\|_\infty + n^2 \cdot \|A\|_\infty \cdot 0 = 0. \end{aligned}$$

It follows that  $A_r * B_r \rightarrow A * B$  in the operator-norm topology.

Next, we consider the compactness of the Khatri-Rao product of two operators. Recall that a linear operator  $T : H \rightarrow H$  is said to be compact if and only if it can be written in the form

$$T = \sum_{n=1}^\infty \lambda_n \langle x_n, \cdot \rangle y_n$$

where  $\{x_n\}_{n \in \mathbb{N}}$  and  $\{y_n\}_{n \in \mathbb{N}}$  are orthonormal sets in  $H$ , and  $(\lambda_n)_{n \in \mathbb{N}}$  is a sequence of positive real numbers with limit zero, called the singular values of  $T$ . The zero operator is an example of a compact operator. Every finite rank operator (between infinite/finite dimensional spaces) is compact. Every compact operator is always bounded and continuous with respect to the operator-norm topology.

**Lemma 5** ([7]). Let  $A = [A_{ij}] \in B(H)$  and  $B = [B_{ij}] \in B(K)$  be operators such that  $A_{ij}$  and  $B_{ij}$  are nonzero for all  $i, j$ . Then the tensor product  $A \otimes B$  is compact if and only if both  $A$  and  $B$  are compact.

**Lemma 6** ([11]). Let  $A = [A_{ij}] \in B(H)$  be an operator matrix. Then  $A$  is compact if and only if  $A_{ij}$  is compact for all  $i, j$ .

The next result provides a necessary and sufficient condition for  $A^* B$  to be compact.

**Theorem 7.** Let  $A = [A_{ij}] \in B(H)$  and  $B = [B_{ij}] \in B(K)$  be operators such that  $A_{ij}$  and  $B_{ij}$  are nonzero for all  $i, j$ . Then the Khatri-Rao product  $A^* B$  is compact if and only if both  $A$  and  $B$  are compact.

**Proof.** First, suppose that  $A$  and  $B$  are compact. By Lemma 6, we deduce that  $A_{ij}$  and  $B_{ij}$  are compact for all  $i, j$ . It follows from Lemma 5 that  $A_{ij} \otimes B_{ij}$  is compact for all  $i, j$ . Lemma 6 ensures the compactness of  $A^* B$ . For necessity part, reverse the previous procedure.

### KHATRI-RAO PRODUCTS ON NORM IDEALS OF COMPACT OPERATORS

In this section, we investigate the Khatri-Rao product on several norm ideals of  $B(H)$ . Recall that any proper ideal of the algebra  $B(H)$  is contained in the ideal of compact operators. For any compact operator  $A \in B(H)$ , let  $(s_i(A))_{i=1}^\infty$  be the sequence of decreasingly ordered singular values of  $A$ . For each  $1 \leq p < \infty$ , the Schatten  $p$ -norm of  $A$  is defined by

$$\|A\|_p = \left( \sum_{i=1}^\infty s_i^p(A) \right)^{1/p}$$

or equivalently,

$$\|A\|_p = \left( \text{tr}|A|^p \right)^{1/p}$$

where  $|A|^p$  is defined by the functional calculus. If  $\|A\|_p$  is a nonnegative real number, then  $A$  is called a Schatten  $p$ -class operator. The Schatten  $\infty$ -norm is just the operator norm. For each  $1 \leq p \leq \infty$ , denote by  $S_p$  the Schatten  $p$ -class operators. In particular,  $S_1$  and  $S_2$  are known as the trace class and the Hilbert-Schmidt class, respectively. Each Schatten  $p$ -norm induces a norm ideal of  $B(H)$  and this ideal is closed under the topology generated by such norm.

In order to proceed, some auxiliary results are needed.

**Lemma 8.** ([14]). Let  $A = [A_{ij}]_{i,j=1}^{n,n}$  be an operator matrix in  $S_p$ .

(i) For  $1 \leq p \leq 2$ , we have

$$\sum_{i,j=1}^n \|A_{ij}\|_p^2 \gg \|A\|_p^2 \gg n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2. \tag{6}$$

(ii) For  $2 \leq p \leq \infty$ , we have

$$n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}\|_p^2 \gg \|A\|_p^2 \gg \sum_{i,j=1}^n \|A_{ij}\|_p^2. \tag{7}$$

**Lemma 9.** Let  $1 \leq p \leq \infty$ . An operator matrix  $A = [A_{ij}] \in B(H)$  is a Schatten  $p$ -class operator if and only if  $A_{ij}$  is a Schatten  $p$ -class operator for all  $i, j$ .

**Proof.** This is a direct consequence of the norm estimations in Lemma 8.

**Lemma 10.** Let  $1 \leq p \leq \infty$ . Let  $A = [A_{ij}]$  be an operator matrix in the class  $S_p$  and let  $(A_r)_{r=1}^\infty$  be a sequence in  $S_p$  where  $A_r = [A_{ij}^{(r)}]_{i,j=1}^{n,n}$  for each  $r \in \mathbb{N}$ . Then  $A_r \rightarrow A$  in  $S_p$  if and only if  $A_{ij}^{(r)} \rightarrow A_{ij}$  in  $S_p$  for all  $i, j = 1, \dots, n$ .

**Proof.** Lemma 9 assures that  $A_{ij}$  and  $A_{ij}^{(r)}$  belong to  $S_p$  for any  $i, j = 1, \dots, n$  and  $r \in \mathbb{N}$ . Consider the case  $1 \leq p \leq 2$  and suppose that  $A_r \rightarrow A$  in  $S_p$ . For any fixed  $i, j \in \{1, \dots, n\}$ , we have from the estimation (6) that

$$\|A_{ij}^{(r)} - A_{ij}\|_p^2 \gg \sum_{i,j=1}^n \|A_{ij}^{(r)} - A_{ij}\|_p^2 \gg \|A_r - A\|_p^2.$$

Hence,  $A_{ij}^{(r)} \rightarrow A_{ij}$  in  $S_p$ . Conversely, suppose that  $A_{ij}^{(r)} \rightarrow A_{ij}$  in  $S_p$  for each  $i, j$ . Lemma 8 implies that

$$\|A_r - A\|_p^2 \gg n^{4/p-2} \sum_{i,j=1}^n \|A_{ij}^{(r)} - A_{ij}\|_p^2.$$

Hence,  $A_r \rightarrow A$  in  $S_p$ .

Consider the case  $2 < p < \infty$  and suppose that  $A_r \rightarrow A$  in  $S_p$ . Using the norm estimations (7), we obtain

$$\|A_{ij}^{(r)} - A_{ij}\|_p^2 \gg \sum_{i,j=1}^n \|A_{ij}^{(r)} - A_{ij}\|_p^2 \gg n^{2-4/p} \|A_r - A\|_p^2.$$

in for any  $i, j \in \{1, \dots, n\}$ . Since  $A_r \rightarrow A$  in  $S_p$ , we have  $A_{ij}^{(r)} \rightarrow A_{ij}$  in  $S_p$ . Conversely, suppose that  $A_{ij}^{(r)} \rightarrow A_{ij}$  in  $S_p$  for each  $i, j$ . Applying Lemma 8, we get

$$\|A_r - A\|_p^2 \gg \sum_{i,j=1}^n \|A_{ij}^{(r)} - A_{ij}\|_p^2.$$

Hence,  $A_r \rightarrow A$  in  $S_p$ .

Consider the case  $p = \infty$ . Let  $A_r \rightarrow A$  in  $S_\infty$ . For any fixed  $i, j \in \{1, \dots, n\}$ , we have from Lemma 2 that

$$\|A_{ij}^{(r)} - A_{ij}\|_\infty^2 \gg \sum_{i,j=1}^n \|A_{ij}^{(r)} - A_{ij}\|_\infty^2 \gg n^2 \|A_r - A\|_\infty^2.$$

Hence,  $A_{ij}^{(r)} \rightarrow A_{ij}$  in  $S_\infty$ . Conversely, suppose that  $A_{ij}^{(r)} \rightarrow A_{ij}$  in  $S_\infty$  for each  $i, j$ . The estimation (4) implies that

$$\|A_r - A\|_\infty^2 \leq \sum_{i,j=1}^n \|A_{ij}^{(r)} - A_{ij}\|_\infty^2.$$

Thus  $A_r \rightarrow A$  in  $S_\infty$ .

The following result asserts that the Khatri-Rao product is continuous on the ideal of compact operators  $S_\infty$ .

**Theorem 11.** If a sequence  $(A_r)_{r=1}^\infty$  converges to  $A$  and a sequence  $(B_r)_{r=1}^\infty$  converges to  $B$  in  $S_\infty$ , then  $A_r * B_r$  converges to  $A * B$  in  $S_\infty$ .

**Proof.** It follows from Theorem 4, Theorem 7, and the fact that  $S_\infty$  is a closed set.

The following theorem supplies an upper bound for the Schatten 1-norm of the Khatri-Rao product of operators.

**Theorem 12.** Let  $A \in B(H)$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in B(K)$ . If  $A$  and  $B$  are compact, then we have the following bound:

$$\|A * B\|_1 \leq n \|A\|_1 \|B\|_1. \tag{8}$$

Hence,  $A * B$  is trace-class if and only if both  $A$  and  $B$  are trace-class.

**Proof.** Suppose that  $A_{ij}$  and  $B_{ij}$  are nonzero operators for all  $i, j$ . Then the Khatri-Rao product  $A * B$  is also compact by Theorem 7. It follows from the norm bound (6) and the Cauchy-Schwarz inequality that

$$\begin{aligned} \|A * B\|_1^2 &\leq \sum_{i,j} \|A_{ij} \otimes B_{ij}\|_1^2 \\ &= \sum_{i,j} \|A_{ij}\|_1^2 \|B_{ij}\|_1^2 \\ &\leq \left( \sum_{i,j} \|A_{ij}\|_1^2 \right) \left( \sum_{i,j} \|B_{ij}\|_1^2 \right) \\ &= n^2 \|A\|_1^2 \|B\|_1^2. \end{aligned}$$

Hence, we obtain the bound (8). If there is a zero block of  $A_{ij}$  or  $B_{ij}$ , then  $A_{ij} \otimes B_{ij} = 0$  is compact, and we can apply the above procedure.

**Theorem 13.** Let  $A \in B(H)$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in B(K)$ . If  $A$  and  $B$  are compact, then

$$\|A * B\|_2 \leq \|A\|_2 \|B\|_2. \tag{9}$$

Hence,  $A * B$  is a Hilbert-Schmidt operator if and only if both  $A$  and  $B$  are Hilbert-Schmidt operators.

**Proof.** We may suppose that  $A_{ij}$  and  $B_{ij}$  are nonzero for all  $i, j$ . Then the operator  $A * B$  is compact by Theorem 7. Then by Lemma 8(ii) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|A * B\|_2^2 &= \sum_{i,j} \|A_{ij} \otimes B_{ij}\|_2^2 \\ &= \sum_{i,j} \|A_{ij}\|_2^2 \|B_{ij}\|_2^2 \end{aligned}$$

$$\begin{aligned} &\leq \left( \sum_{i,j} \|A_{ij}\|_2^2 \right) \left( \sum_{i,j} \|B_{ij}\|_2^2 \right) \\ &= \|A\|_2^2 \|B\|_2^2. \end{aligned}$$

Hence, we obtain the bound (9).

The final result states that the Khatri-Rao product is sequentially continuous on the norm ideal of trace-class operators and the norm ideal of Hilbert-Schmidt class operators.

**Theorem 14.** Let  $p \in \{1, 2\}$ . If a sequence  $(A_r)_{r=1}^\infty$  converges to  $A$  and a sequence  $(B_r)_{r=1}^\infty$  converges to  $B$  in the norm ideal  $S_p$ , then  $A_r * B_r$  converges to  $A * B$  in  $S_p$ .

**Proof.** Suppose that the sequences  $(A_r)_{r=1}^\infty$  and  $(B_r)_{r=1}^\infty$  converge to  $A$  and  $B$ , respectively, in  $S_1$ . By applying Lemma 1 and the bound (8), we get

$$\begin{aligned} \|A_r * B_r - A * B\|_1 &= \|A_r * B_r - A * B_r + A * B_r - A * B\|_1 \\ &\leq \|(A_r - A) * B_r\|_1 + \|A * (B_r - B)\|_1 \\ &\leq n \|(A_r - A)\|_1 \|B_r\|_1 + n \|A\|_1 \|(B_r - B)\|_1 \\ &\rightarrow 0 \cdot n \|B\|_1 + n \|A\|_1 \cdot 0 = 0. \end{aligned}$$

Hence,  $A_r * B_r \rightarrow A * B$  in  $S_1$ .

Let the sequences  $(A_r)_{r=1}^\infty$  and  $(B_r)_{r=1}^\infty$  converge to  $A$  and  $B$ , respectively, in  $S_2$ . By using Lemma 1 and the bound (9), we have

$$\begin{aligned} \|A_r * B_r - A * B\|_2 &= \|A_r * B_r - A * B_r + A * B_r - A * B\|_2 \\ &\leq \|(A_r - A) * B_r\|_2 + \|A * (B_r - B)\|_2 \\ &\leq \|(A_r - A)\|_2 \|B_r\|_2 + \|A\|_2 \|(B_r - B)\|_2 \\ &\rightarrow 0 \cdot \|B\|_2 + \|A\|_2 \cdot 0 = 0. \end{aligned}$$

Hence,  $A_r * B_r \rightarrow A * B$  in  $S_2$ .

## CONCLUSION

We have provided a necessary and sufficient condition for the Khatri-Rao product of operators to be compact. Indeed, for two operators in which every block is nonzero, their Khatri-Rao product is compact if and only if both factors are compact. We establish estimations for the operator norm, the trace norm, and the Hilbert-Schmidt norm for Khatri-Rao products of Hilbert space operators. The Khatri-Rao product is continuous with respect to the topologies induced by such norms. It follows that the Khatri-Rao product of two operators are trace-class (Hilbert-Schmidt class) if and only if each operator is trace-class (Hilbert-Schmidt class, respectively).

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## REFERENCES

- [1] Khatri, C.G. and Rao, C.R., Solutions to some functional equations and their applications to characterization of probability distributions, *Sankhya*, 30(2), 167-180, 1968.
- [2] Al, Zhou, Z.A. and Kilicman, A., Extension and generalization inequalities involving the Khatri-Rao product of several positive matrices, *J. Inequal. Appl.*, Article ID 80878, 21 pages, 2006.

- [3] Civciv, H. and Taurkmen, R., On the bounds for norms of Khatri-Rao and Tracy-Singh products of Cauchy-Toeplitz matrices, *Seluk J. Appl. Math.*, 6(2), 43-52, 2005.
- [4] Liu, S., *Contributions to Matrix Calculus and Applications in Econometrics*, PhD dissertation, Institute of Actuarial Science and Econometrics, University of Amsterdam, Amsterdam, 1995.
- [5] Rao, C.R. and Rao, M.B., *Matrix Algebra and Its Applications to Statistics and Econometrics*, World Scientific, 1998.
- [6] Van Loan, C.F., The ubiquitous Kronecker product, *J. Comp. Appl. Math.*, 123, 85-100, 2000.
- [7] Kubrusly, C.S. and Vieira P.C.M., Convergence and decomposition for tensor products of Hilbert space operators, *Oper. Matrices*, 2(3), 407-416, 2008.
- [8] Schechter, M., On the spectra of operators on tensor products, *J. Funct. Anal.*, 4(1), 95-99, 1969.
- [9] Zanni, J. and Kubrsly, C.S., A note on compactness of tensor product, *Acta Math. Univ. Comenian.*, 84(1), 59-62, 2015.
- [10] Ploymukda, A. Chansangiam, P. and W. Lewkeeratiyutkul, Algebraic and order properties of Tracy-Singh products for operator matrices, *J. Comput. Anal. Appl.*, 24(4), 656-664, 2018.
- [11] Ploymukda, A. Chansangiam, P. and W. Lewkeeratiyutkul, Analytic properties of Tracy-Singh products for operator matrices, *Journal of Computational Analysis and Applications*, 24(4), 665-674, 2018.
- [12] Ploymukda, A. and Chansangiam, P., Khatri-Rao products for operator matrices acting on the direct sum of Hilbert spaces, *J. Math.*, 2016 (Article ID 8301709), 7 pages, 2016.
- [13] Ploymukda, A. and Chansangiam, P., Operator Inequalities Involving Khatri-Rao Sums and Moore-Penrose Inverses, *Mal. J. Fund. Appl. Sci.*, 13(4), 742-746, 2017.
- [14] Bhatia, R. and Kittaneh, F., Norm inequalities for partitioned operators and an application, *Math. Ann.*, 287(1), 719-726, 1990.



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# Operator inequalities involving Khatri-Rao sums and Moore-Penrose inverses

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## Abstract

We establish relations between the Khatri-Rao sum of Hilbert space operators and ordinary products, powers, ordinary inverses, and Moore-Penrose inverses in terms of inequalities, including arithmetic-geometric mean inequality and Kantorovich type inequalities. In particular, such relations hold for the tensor sum of operators and the Khatri-Rao sum of complex matrices.

**Keywords:** Tensor product, Khatri-Rao sum (product), Tracy-Singh sum (product), Moore-Penrose inverse, operator inequality

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## INTRODUCTION

In mathematics, there are many kinds of matrix products/sums which have rich theory and numerous applications. Such matrix products include the Kronecker (tensor) product, the Tracy-Singh product, and the Khatri-Rao product. Recall that the Kronecker product of two complex matrices  $A$  and  $B$  is defined by

$$A \hat{\otimes} B = [a_{ij}B]_{ij}, \tag{1}$$

that is, the  $(i,j)$ th block of  $A \hat{\otimes} B$  is given by  $a_{ij}B$ . The Kronecker sum of an  $n \times n$  matrix  $A$  and an  $m \times m$  matrix  $B$  is defined as

$$A \hat{\oplus} B = A \otimes I_n + I_m \otimes B. \tag{2}$$

Here,  $I_k$  denotes the identity matrix of size  $k \times k$  for any natural number  $k$ . The Tracy-Singh product, introduced in [1], is a generalization of the Kronecker product. Indeed, partition  $A = [A_{ij}]$  and  $B = [B_{kl}]$ , where the submatrices  $A_{ij}$  and  $B_{kl}$  can be of arbitrary sizes. Then the Tracy-Singh product of  $A$  and  $B$  is defined as

$$A \hat{\boxtimes} B = \left[ [A_{ij} \hat{\otimes} B_{kl}]_{kl} \right]_{ij}.$$

If  $A$  and  $B$  have the same form of partitioning, we can define their Khatri-Rao product by [2]

$$A \hat{\square} B = [A_{ij} \hat{\otimes} B_{ij}]_{ij}.$$

See more information about theory of matrix products in [3-6].

The notions of Tracy-Singh sum and Khatri-Rao sum for matrices are respectively defined by (see [7])

$$A \hat{\boxplus} B = A \hat{\boxtimes} I_n + I_m \hat{\boxtimes} B, \tag{3}$$

$$A \hat{\squareplus} B = A \hat{\square} I_n + I_m \hat{\square} B. \tag{4}$$

Here, we partition  $I_m$  and  $I_n$  so that their diagonal blocks are identity matrices. If  $A$  and  $B$  are of only one block, then the Tracy-Singh sum (3) and the Khatri-Rao sum (4) are reduced to the Kronecker sum (2).

A significant development in operator theory is to introduce the tensor product of Hilbert space operators, generalizing the Kronecker product of matrices. From now on, let  $\mathcal{H}$  and  $\mathcal{K}$  be complex separable Hilbert spaces. When  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert spaces, denote by  $\mathbb{B}(\mathcal{X}, \mathcal{Y})$  the algebra of bounded linear operators from  $\mathcal{X}$  into  $\mathcal{Y}$ , and abbreviate  $\mathbb{B}(\mathcal{X}, \mathcal{X})$  to  $\mathbb{B}(\mathcal{X})$ . The identity operator on the space  $\mathcal{X}$  is denoted by  $I_{\mathcal{X}}$  or  $I$  if the underlying space is clear from the context. Using the universal mapping property, the tensor product of  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  is the unique bounded linear operator from the tensor space  $\mathcal{H} \otimes \mathcal{K}$  into itself satisfying

$$(A \otimes B)(x \otimes y) = Ax \otimes By$$

for all  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ . The tensor sum of  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  is defined to be (see [8])

$$A \oplus B = A \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes B. \tag{5}$$

Recently, the Tracy-Singh product, the Khatri-Rao product, and the Khatri-Rao sum for Hilbert space operators were investigated by the authors [9-13]. The notion of Khatri-Rao sum of operators includes the tensor sum of operators and the Khatri-Rao sum of complex matrices as special cases.

In this paper, we develop further theory of operator products/sums by establishing certain inequalities for Khatri-Rao sums of operators. These inequalities involve ordinary products and powers, ordinary and

Moore-Penrose inverses. We also deduce Kantorovich type inequalities concerning Khatri-Rao sums. Our results generalize some matrix inequalities in [7]. In operator case, we require some mild assumptions such as the closeness of their ranges. Moreover, new operator inequalities are established by means of block partitioning technique.

The paper is organized as follows. The next section supplies preliminaries on Tracy-Singh products/sums and Khatri-Rao products/sums for Hilbert space operators. The third section deals with certain operator inequalities concerning Khatri-Rao sums and Moore-Penrose inverses. In the final section, we establish Kantorovich type operator inequalities involving Khatri-Rao sums.

**PRELIMINARIES**

For Hermitian operators  $A, B \in \mathbb{B}(\mathcal{H})$ , the partial order  $A \geq B$  means that  $A - B$  is a positive operator.

**Tracy-Singh products/sums for operators**

To define the Tracy-Singh product of operators, we first fix the decompositions of Hilbert spaces (this can be done by using the projection theorem):

$$\mathcal{H} = \bigoplus_{i=1}^m \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{k=1}^n \mathcal{K}_k$$

where  $\mathcal{H}_i$  and  $\mathcal{K}_k$  are Hilbert spaces for all  $i, k$ . It follows that any operator  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  can be uniquely represented as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,m} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{n,n}$$

where  $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$  and  $B_{kl} \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k)$  for each  $i, j, k, l$ . Basic algebraic operations for operator matrices, such as, the addition, the scalar multiplication, the usual multiplication, and the adjoints can be performed in the same way as those of block matrices.

**Definition 1.** According to the previous setup, the Tracy-Singh product of  $A$  and  $B$  is defined to be the bounded linear operator from  $\bigoplus_{i,k=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_k$  into itself, represented by the operator matrix

$$A \boxtimes B = \left[ [A_{ij} \otimes B_{kl}]_{kl} \right]_{ij} \tag{6}$$

Recall that a Moore-Penrose inverse of  $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$  is an operator  $A^\dagger \in \mathbb{B}(\mathcal{K}, \mathcal{H})$  satisfying the following conditions (see [14]):

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

The existence of  $A^\dagger$  is equivalent to the closeness of the range of  $A$ , and in this case the operator  $A^\dagger$  is unique (see, e.g., [15]).

**Lemma 2** ([9]). The Tracy-Singh product fulfills the following properties, assuming that all operators are compatible:

$$(A \boxtimes B)^* = A^* \boxtimes B^* \tag{7}$$

$$(\alpha A) \boxtimes B = \alpha(A \boxtimes B) = A \boxtimes (\alpha B), \tag{8}$$

$$A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C, \tag{9}$$

$$(B + C) \boxtimes A = B \boxtimes A + C \boxtimes A, \tag{10}$$

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD, \tag{11}$$

$$(A \boxtimes B)^\dagger = A^\dagger \boxtimes B^\dagger. \tag{12}$$

**Lemma 3** ([9]). Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices. If  $A \geq 0$  and  $B \geq 0$ , then  $A \boxtimes B \geq 0$ .

**Definition 4.** Let  $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ . We define the Tracy-Singh sum of  $A$  and  $B$  as follows:

$$A \boxplus B = A \boxtimes I_{\mathcal{K}} + I_{\mathcal{H}} \boxtimes B \tag{13}$$

which belongs to  $\mathbb{B}\left(\bigoplus_{i,j=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_j\right)$ .

**Khatri-Rao Products/Summs for Operators**

From now on, fix the following Hilbert space direct sums:

$$\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{i=1}^n \mathcal{K}_i.$$

**Definition 5.** Let  $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$  be operator matrices. We define the Khatri-Rao product of  $A$  and  $B$  to be the bounded linear operator from  $\bigoplus_{i=1}^n \mathcal{H}_i \otimes \mathcal{K}_i$  into itself, represented as follows:

$$A \boxdot B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{n,n} \tag{14}$$

**Lemma 6** ([11]). There is an isometry  $Z$  such that  $ZZ^* \leq I$  and

$$A \boxdot B = Z^* (A \boxtimes B) Z \tag{15}$$

for any  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . We call  $Z$  the selection operator associated with the ordered tuple  $(\mathcal{H}, \mathcal{K})$ .

**Definition 7.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . We define the Khatri-Rao sum of  $A$  and  $B$  as follows:

$$A \otimes B = A \boxdot I_{\mathcal{K}} + I_{\mathcal{H}} \boxdot B \tag{16}$$

which belongs to  $\mathbb{B}\left(\bigoplus_{i=1}^n \mathcal{H}_i \otimes \mathcal{K}_i\right)$ .

Note that if both  $A$  and  $B$  are  $1 \times 1$  block operator matrices, their Khatri-Rao sum  $A \otimes B$  becomes the tensor sum. If  $\mathcal{H}_i = \mathcal{K}_i = \mathbb{C}$ , the Khatri-Rao sum  $A \otimes B$  reduces to the Hadamard sum of complex matrices (see e.g. [7]).

**Lemma 8** ([13]). There is an isometry  $Z$  such that  $ZZ^* \leq I$  and

$$A \otimes B = Z^* (A \boxplus B) Z \tag{17}$$

for any  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ .

**OPERATOR INEQUALITIES INVOLVING KHATRI-RAO SUMS AND MOORE-PENROSE INVERSES**

In this section, we derive certain operator inequalities involving Khatri-Rao sums and Moore-Penrose inverses. Roughly speaking, we may consider the Khatri-Rao sum and the Khatri-Rao product as the “sum” and the “product”, respectively. The Moore-Penrose inverse plays a role like the “inverse” for operators. To ensure the existence of

the Moore-Penrose inverse of an operator, we must impose the closeness of its range. The results in this section include those for the tensor sum of operators and the Khatri-Rao sum of complex matrices as special cases.

To derive operator inequalities in this section, we apply a block-partitioning technique, which is explained in the next lemma.

**Lemma 9** (see e.g. [16]) Let  $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{bmatrix}$  be an operator in  $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$  such that  $T_{11}$  has a closed range. Then  $T \geq 0$  if and only if

- (i)  $T_{11} \geq 0$ ,
- (ii)  $T_{12} = T_{11}T_{11}^\dagger T_{12}$ ,
- (iii)  $T_{22} = T_{22} \geq T_{12}^*T_{11}^\dagger T_{12}$ .

Recall that for any positive real numbers  $a$  and  $b$ , we have

$$\frac{(a+b)^2}{ab} \leq \frac{a}{b} + \frac{b}{a} + 2$$

(indeed, both sides are equal). Let us generalize this fact to operators in which we consider the Khatri-Rao sum as the “sum” and the Moore-Penrose inverse as the “inverse”.

**Theorem 10.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be positive invertible operators such that  $A \square B$  has a closed range. Then

$$(A \otimes B)(A \square B)^\dagger (A \otimes B) \leq A \square B^{-1} + A^{-1} \square B + 2I. \tag{18}$$

**Proof.** Denote

$$S = \begin{bmatrix} A^{1/2} \otimes B^{1/2} & A^{1/2} \otimes B^{-1/2} + A^{-1/2} \otimes B^{1/2} \end{bmatrix}$$

and  $X = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$  where  $Z$  is the selection operator associated with  $(\mathcal{H}, \mathcal{K})$ . Using Lemma 2, we get

$$\begin{aligned} 0 &\leq S^*S \\ &= \begin{bmatrix} A \otimes B & A \otimes I + I \otimes B \\ A \otimes I + I \otimes B & A \otimes B^{-1} + A^{-1} \otimes B + 2I \otimes I \end{bmatrix} \\ &= \begin{bmatrix} A \otimes B & A \boxplus B \\ A \boxplus B & A \otimes B^{-1} + A^{-1} \otimes B + 2I \otimes I \end{bmatrix}. \end{aligned}$$

Pre- and post-multiplying  $S^*S$  by  $X^*$  and  $X$ , respectively, we obtain

$$\begin{aligned} 0 &\leq X^*S^*SX \\ &= \begin{bmatrix} Z^*(A \otimes B)Z & Z^*(A \boxplus B)Z \\ Z^*(A \boxplus B)Z & Z^*(A \otimes B^{-1} + A^{-1} \otimes B + 2I \otimes I)Z \end{bmatrix} \\ &= \begin{bmatrix} A \square B & A \otimes B \\ A \otimes B & A \otimes B^{-1} + A^{-1} \square B + 2I \square I \end{bmatrix}. \end{aligned}$$

Applying Lemma 9, we get the inequality (18).

Recall that for any positive real numbers  $a$  and  $b$ , we have

$$\frac{(a+1)^2}{a} \leq a + 2 + \frac{1}{a}$$

(indeed, both sides are equal). The next theorem generalize this fact to operators; it is also an extension of [7, Corollary 3.8].

**Theorem 11.** Let  $A \in \mathbb{B}(\mathcal{H})$  be a positive operator such that  $A$  and  $A \square I$  have closed ranges, and  $A \boxtimes I = I \boxtimes A$ . Then

$$(A \otimes AA^\dagger)(A \square I)^\dagger (A \otimes AA^\dagger) \leq A \otimes A^\dagger + 2AA^\dagger \square I. \tag{19}$$

**Proof.** Note first that  $A^\dagger$  and  $(A \square I)^\dagger$  exist due to the assumption that  $A$  and  $A \square I$  have closed ranges. Now, denote

$$S = \begin{bmatrix} A^{1/2} \boxtimes I & A^{1/2} \boxtimes I + (A^\dagger)^{1/2} \boxtimes I \end{bmatrix} \text{ and } X = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$$

where  $Z$  is the selection operator. Since  $A \boxtimes I = I \boxtimes A$ , we have, by Lemma 2,  $A^\dagger \boxtimes I = I \boxtimes A^\dagger$  and  $AA^\dagger \boxtimes I = I \boxtimes AA^\dagger$ . Then

$$\begin{aligned} 0 &\leq S^*S \\ &= \begin{bmatrix} A \boxtimes I & A \boxtimes I + AA^\dagger \boxtimes I \\ A \boxtimes I + AA^\dagger \boxtimes I & A \boxtimes I + A^\dagger \boxtimes I + 2AA^\dagger \boxtimes I \end{bmatrix} \\ &= \begin{bmatrix} A \boxtimes I & A \boxtimes I + I \boxtimes AA^\dagger \\ A \boxtimes I + I \boxtimes AA^\dagger & A \boxtimes I + I \boxtimes A^\dagger + 2AA^\dagger \boxtimes I \end{bmatrix} \\ &= \begin{bmatrix} A \boxtimes I & A \boxplus AA^\dagger \\ A \boxplus AA^\dagger & A \boxplus A^\dagger + 2AA^\dagger \boxtimes I \end{bmatrix}. \end{aligned}$$

Pre- and post-multiplying by  $X^*$  and  $X$ , respectively, we obtain

$$\begin{aligned} 0 &\leq X^*S^*SX \\ &= \begin{bmatrix} Z^*(A \boxtimes I)Z & Z^*(A \boxplus AA^\dagger)Z \\ Z^*(A \boxplus AA^\dagger)Z & Z^*(A \boxplus A^\dagger + 2AA^\dagger \boxtimes I)Z \end{bmatrix} \\ &= \begin{bmatrix} A \square I & A \otimes AA^\dagger \\ A \otimes AA^\dagger & A \otimes A^\dagger + 2AA^\dagger \square I \end{bmatrix}. \end{aligned}$$

The proof is complete by using Lemma 9.

An equivalent form the arithmetic-geometric mean inequality is that for any real number  $a > 0$  we have

$$a + \frac{1}{a} \geq 2.$$

The next theorem is a generalization of this fact; it is also an operator extension of [7, Corollary 3.6].

**Theorem 12.** Let  $A \in \mathbb{B}(\mathcal{H})$  be a positive operator such that  $A$  has a closed range and  $A^\dagger \square I = I \square A^\dagger$ . Then

$$A \otimes A^\dagger \geq 2AA^\dagger \square I.$$

**Proof.** The Moore-Penrose inverse of  $A$  exists since its range is closed. Now, denote  $S = A \boxtimes I \geq 0$ . The spectral theorem implies that  $S + S^\dagger \geq 2SS^\dagger$ . We have by using Lemma 2 that

$$\begin{aligned} A \boxplus A^\dagger &= A \boxtimes I + A^\dagger \boxtimes I \\ &= A \boxtimes I + (A \boxtimes I)^\dagger \\ &\geq 2(A \boxtimes I)(A \boxtimes I)^\dagger \\ &= 2AA^\dagger \boxtimes I. \end{aligned}$$

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We get the desired result by pre- and post-multiplying by  $Z^*$  and  $Z$ , respectively.

**KANTOROVICH TYPE INEQUALITIES INVOLVING KHATRI-RAO SUMS**

In this section, Kantorovich type inequalities involving Khatri-Rao sums are established. We begin with an auxiliary lemma.

**Lemma 13.** Let  $S$  be a positive invertible operator in  $\mathbb{B}(\mathcal{K})$  with  $mI \leq S \leq MI$  where  $m, M$  are positive constants. The following inequalities hold:

$$S \leq (m + M)I - mMS^{-1}, \tag{20}$$

$$S^2 \leq (m + M)S - mM I. \tag{21}$$

**Proof.** Since  $mI \leq S \leq MI$ , we have  $(MI - S)(mI - S) \leq 0$  and hence

$$(MI - S)(mI - S)S^{-1} = S^{-1/2}(MI - S)(mI - S)S^{-1/2} \leq 0$$

Now, the desired inequalities follow easily.

The next lemma provides certain operator inequalities, generalizing matrix results in [17].

**Lemma 14.** Let  $S$  be a positive invertible operator in  $\mathbb{B}(\mathcal{K})$  with  $mI \leq S \leq MI$  where  $m, M$  are positive constants. For any  $X \in \mathbb{B}(\mathcal{H}, \mathcal{K})$  such that  $X^*X = I$ , we have

$$(X^*SX)^2 \leq X^*S^2X \leq \frac{(M + m)^2}{4Mm}(X^*SX)^2, \tag{22}$$

$$(X^*SX)^{-1} \leq X^*S^{-1}X \leq \frac{(M + m)^2}{4Mm}(X^*SX)^{-1}. \tag{23}$$

**Proof.** Since  $X^*X = I$ , we have  $X^\dagger = X^*$ . Since  $XX^\dagger$  is Hermitian and idempotent, it is a projection and thus  $XX^\dagger \leq I$ . Then

$$(X^*SX)^2 = (SX)^*XX^*(SX) \leq (SX)^*I(SX) = X^*S^2X.$$

It follows from (21) in Lemma 13 that

$$\begin{aligned} X^*S^2X &\leq (m + M)X^*SX - mMX^* \\ &= \frac{(m + M)^2}{4mM}(X^*SX)^2 - \left(\frac{m + M}{2\sqrt{mM}}X^*SX - \sqrt{mMI}\right)^2 \\ &\leq \frac{(m + M)^2}{4mM}(X^*SX)^2. \end{aligned}$$

Since  $S^{1/2}X(X^*SX)^{-1}X^*S$  is Hermitian and idempotent, it is a projection and thus  $S^{1/2}X(X^*SX)^{-1}X^*S \leq I$ . It follows from (20) that

$$\begin{aligned} X^*S^{-1}X &\leq \frac{(m + M)}{mM}XX^* - mMXS^* \\ &= \frac{(m + M)^2}{4mM}(X^*SX)^{-1} \end{aligned}$$

$$\begin{aligned} &-\left[\frac{m + M}{2\sqrt{mM}}(X^*SX)^{-1/2} - \frac{1}{\sqrt{mM}}(X^*SX)^{1/2}\right]^2 \\ &\leq \frac{(m + M)^2}{4mM}(X^*SX)^{-1}. \end{aligned}$$

The next theorem establishes Kantorovich type inequalities concerning ordinary powers and inverses of operators.

**Theorem 15.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be positive invertible operators and  $mI \leq A \boxtimes I \oplus I \boxtimes B \leq MI$  where  $m, M$  are positive constants. Then

$$(A \otimes B)^2 \leq A^2 \otimes B^2 \leq \frac{(M + m)^2}{Mm}(A \otimes B)^2. \tag{24}$$

If  $A \otimes B$  is invertible, then

$$(A \otimes B)^{-1} \leq A^{-1} \otimes B^{-1} \leq \frac{(M + m)^2}{Mm}(A \otimes B)^{-1}. \tag{25}$$

**Proof.** Denote

$$S = \begin{bmatrix} A \boxtimes I & 0 \\ 0 & I \boxtimes B \end{bmatrix} \text{ and } X = \frac{1}{\sqrt{2}} \begin{bmatrix} Z \\ Z \end{bmatrix}$$

where  $Z$  is the selection operator associated with  $(\mathcal{H}, \mathcal{K}), S \geq 0$  and  $X^*X = I$ . By Lemma 3,  $A \boxtimes I \geq 0$  and  $I \boxtimes B \geq 0$ . We have  $S \geq 0$  by Lemma 14. Since  $A$  and  $B$  are invertible, we conclude that  $S$  is invertible. Using Lemma 8, we have

$$X^*SX = \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} \begin{bmatrix} A \boxtimes I & 0 \\ 0 & I \boxtimes B \end{bmatrix} \begin{bmatrix} Z \\ Z \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} (A \boxtimes I + I \boxtimes B) \begin{bmatrix} Z \\ Z \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} (A \boxplus B) \begin{bmatrix} Z \\ Z \end{bmatrix}$$

$$= \frac{1}{2} (A \otimes B),$$

$$X^*S^{-1}X = \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} \begin{bmatrix} A^{-1} \boxtimes I & 0 \\ 0 & I \boxtimes B^{-1} \end{bmatrix} \begin{bmatrix} Z \\ Z \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} (A^{-1} \boxtimes I + I \boxtimes B^{-1}) \begin{bmatrix} Z \\ Z \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} (A^{-1} \boxplus B^{-1}) \begin{bmatrix} Z \\ Z \end{bmatrix}$$

$$= \frac{1}{2} (A^{-1} \otimes B^{-1}),$$

$$X^*S^2X = \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} \begin{bmatrix} A^2 \boxtimes I & 0 \\ 0 & I \boxtimes B^2 \end{bmatrix} \begin{bmatrix} Z \\ Z \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} (A^2 \boxtimes I + I \boxtimes B^2) \begin{bmatrix} Z \\ Z \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} Z^* & Z^* \end{bmatrix} (A^2 \boxplus B^2) \begin{bmatrix} Z \\ Z \end{bmatrix}$$

$$= \frac{1}{2} (A^2 \otimes B^2).$$

Substitution in (22) and (23) of Lemma 14 leads to the results.

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**Lemma 16.** Let  $S$  be a positive invertible operator in  $\mathbb{B}(\mathcal{K})$  with  $mI \leq S \leq MI$  where  $m, M$  are positive constants. For any  $X \in \mathbb{B}(\mathcal{H}, \mathcal{K})$  such that  $X^*X = I$ , we have

$$X^*SX - (X^*S^{-1}X)^{-1} \leq (\sqrt{M} - \sqrt{m})^2 \tag{26}$$

$$X^*S^2X - (X^*SX)^2 \leq \frac{1}{4} (M - m)^2 \tag{27}$$

$$(X^*S^2X)^{1/2} - X^*SX \leq \frac{(M - m)^2}{4(M + m)} I. \tag{28}$$

**Proof.** Using (20), we obtain

$$\begin{aligned} X^*SX - (X^*S^{-1}X)^{-1} &\leq (m + M)I - mMX^*S^{-1}X - (X^*S^{-1}X)^{-1} \\ &= (\sqrt{M} - \sqrt{m})^2 I - \left[ \sqrt{mM} (X^*S^{-1}X)^{1/2} - (X^*S^{-1}X)^{1/2} \right]^2 \\ &\leq (\sqrt{M} - \sqrt{m})^2 I \end{aligned}$$

It follows from (21) that

$$\begin{aligned} X^*S^2X - (X^*SX)^2 &\leq (m + M)X^*SX - mMI - (X^*SX)^2 \\ &= \frac{1}{4}(M - m)^2 I - \left[ X^*SX - \frac{1}{2}(m + M)I \right]^2 \\ &\leq \frac{1}{4}(M - m)^2 I \end{aligned}$$

On the other hand, we have from (21) that

$$\begin{aligned} (X^*S^2X)^{1/2} - X^*SX &\leq (X^*S^2X)^{1/2} - \frac{1}{m + M}S^2 - \frac{mM}{m + M}I \\ &= \frac{(M - m)^2}{4(m + M)} I - \left[ \frac{1}{\sqrt{mM}}(X^*S^2X)^{1/2} - \frac{\sqrt{m + M}}{2} I \right]^2 \\ &\leq \frac{(M - m)^2}{4(m + M)} I. \end{aligned}$$

Our last theorem provides another forms of Kantorovich inequality concerning ordinary powers and inverses.

**Theorem 17.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be positive invertible operators and  $mI \leq (A \otimes B) \oplus (B \otimes A) \leq MI$  where  $m, M$  are positive constants. Then

$$2(A^2 \otimes B^2) - (A \otimes B)^2 \leq (M - m)^2 I, \tag{29}$$

$$(A^2 \otimes B^2)^{1/2} - \frac{1}{\sqrt{2}}(A \otimes B) \leq \frac{(M - m)^2}{2\sqrt{2}} I. \tag{30}$$

In addition, if  $A^{-1} \otimes B^{-1}$  is invertible, then

$$\frac{1}{2}(A \otimes B) - 2(A^{-1} \otimes B^{-1})^{-1} \leq (\sqrt{M} - \sqrt{m})^2 I. \tag{31}$$

**Proof.** The proof can be omitted since it is similar to that of Theorem 15. Instead of Lemma 14, we apply Lemma 16.

### CONCLUSION

We provide relations between the Khatri-Rao sum of operators and various kinds of operator operations, namely, ordinary products and powers, ordinary and Moore-Penrose inverses. These relations appear in terms of inequalities, including arithmetic-geometric mean inequality and Kantorovich type inequalities. The results involving Moore-Penrose inverses are valid under the assumption of closedness of certain operators. Our results show that the Khatri-Rao sum and the Khatri-Rao product can be regarded as the “sum” and the “product” of operators, respectively.

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### REFERENCES

- [1] Tracy, D. S. and Singh, R. P. 1972. A new matrix product and its applications in partitioned matrix differentiation. *Statistica Neerlandica*. 26(4), 143-157.
- [2] Khatri, C. G. and Rao, C. R. 1968. Solutions to some functional equations and their applications to characterization of probability distributions. *Sankhya*. 26(4), 167-180.
- [3] Al Zhou, Z. A. and Kilicman, A. 2006. Extension and generalization inequalities involving the Khatri-Rao product of several positive matrices, *Journal of Inequalities and Applications*, Article ID 80878.
- [4] Liu, S. and Trenkler, G. 2008. Hadamard, Khatri-Rao, Kronecker and other matrix products, *International Journal of Information and Systems Sciences*, 4(1), 160-177.
- [5] Shuangzhe, L. 1999. Matrix results on the Khatri-Rao and Tracy-Singh products, *Linear Algebra and its Applications*. 289(1-3), 267-277.
- [6] Van Loan, C. F. 2000. The ubiquitous Kronecker product, *Journal of Computational and Applied Mathematics*. 123(1-2), 85-100.
- [7] Al Zhou, Z. A. and Kilicman, A. 2006. Matrix equalities and inequalities involving Khatri-Rao and Tracy-Singh sums, *Journal of Inequalities in Pure and Applied Mathematics*. 7(1), 11-17.
- [8] Kubrusly, C. S. and Levan, N. 2011. Preservation of tensor sum and tensor product, *Acta Mathematica Universitatis Comenianae*. 80(1), 133-142.
- [9] Ploymukda, A. Chansangiam, P. and Lewkeeratiyutkul, W. 2017. Algebraic and order properties of Tracy-Singh products for operator matrices, *Journal of Computational Analysis and Applications*, 24(4), 656-664.
- [10] Ploymukda, A. Chansangiam, P. and Lewkeeratiyutkul, W. 2017. Analytic properties of Tracy-Singh products for operator matrices, *Journal of Computational Analysis and Applications*, 24(4), 665-674
- [11] Ploymukda, A. and Chansangiam, P. 2016. Khatri-Rao products for operator matrices acting on the direct sum of Hilbert spaces, *Journal of Mathematics*, Article ID 8301709.
- [12] Ploymukda, A. and Chansangiam, P. 2017. Several inequalities for Khatri-Rao products of Hilbert space operators, *Communications in Mathematics and Applications*. 8(1), 45-60.
- [13] Ploymukda, A. and Chansangiam, P. 2018. Khatri-Rao sums for Hilbert space operators, *Songklanakarinn Journal of Science and Technology*. 40(5), (in press).
- [14] Penrose, R. 1955. A generalized inverse for matrices, *Mathematical Proceedings of the Cambridge Philosophical Society*. 51(3), 406-413.
- [15] Caradus, S. R. 1978. *Generalized inverses and operator theory*, Queen's Papers in Pure and Applied Mathematics no. 50, Queen's University, Kingston, Ontario.
- [16] Xu, Q. and Sheng, L. 2008. Positive semi-definite matrices of adjointable operators on Hilbert-modules, *Linear Algebra and its Applications*. 428(4), 992-1000.
- [17] Liu, S. and Neudecker, H. 1999. A survey of Cauchy-Schwarz and Kantorovich-type matrix inequalities, *Statistical Papers*. 40(1), 55-73.

## Original Article

# Algebraic, Order, and Analytic Properties of Tracy-Singh Sums for Hilbert Space Operators

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### Abstract

We introduce the Tracy-Singh sum for operators on a Hilbert space, generalizing both the Tracy-Singh sum for matrices and the tensor sum for operators. The Tracy-Singh sum is shown to be compatible with algebraic operations and order relations. Then we establish binomial theorem involving Tracy-Singh sums and its consequences. We also investigate continuity, convergence, and norm bounds for Tracy-Singh sums. Moreover, we derive operator identities involving Tracy-Singh sums and certain operator functions defined by power series.

**Keywords:** operator matrix, tensor product, Tracy-Singh product, Tracy-Singh sum, analytic functions of operators.

### 1. Introduction

It is well known that the Kronecker product and the Tracy-Singh product for matrices possess pleasing properties in algebraic, order, and analytic viewpoints. Such matrix products have applications in various fields, including quantum physics,

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thermodynamics, control and system theory, signal processing, image compression, computer science, statistics. See, e.g., Steeb and Hardy (2011) for an excellent source on this subject.

Let  $M_n$  denote the set of  $n$ -by- $n$  complex matrices for each natural number  $n$ .

For any  $A \in M_m$  and  $B \in M_n$ , the Kronecker product of  $A$  and  $B$  is given by

$$A \hat{\otimes} B = [a_{ij}B]_{ij}. \quad (1)$$

This matrix product was generalized to the Tracy-Singh product of partitioned matrices in Tracy and Singh (1972). Let  $A = [A_{ij}]$  be partitioned with  $A_{ij}$  of order  $m_i \times m_j$  as the  $(i, j)$ th submatrix and let  $B = [B_{kl}]$  be a partitioned matrix with  $B_{kl}$  of order  $n_k \times n_l$  as the  $(k, l)$ th submatrix, where  $\sum_{i=1}^r m_i = m$  and  $\sum_{k=1}^s n_k = n$ . The Tracy-Singh product of  $A$  and  $B$  is defined by

$$A \hat{\boxtimes} B = \left[ [A_{ij} \hat{\otimes} B_{kl}]_{kl} \right]_{ij}. \quad (2)$$

Tracy-Singh products for matrices have been developed by many authors in both theory and applications, see e.g., Tracy and Singh (1972), Tracy and Jinadasa (1989), Shuangzhe (1999). The Tracy-Singh sum, introduced by Al Zhour and Kilicman (2006), is defined for each  $A \in M_n$  and  $B \in M_m$  by

$$A \hat{\boxplus} B = A \hat{\boxtimes} I_n + I_m \hat{\boxtimes} B, \quad (3)$$

where  $I_m$  and  $I_n$  are block identity matrices of order  $m \times m$  and  $n \times n$ , respectively. If both  $A$  and  $B$  consist of only one block, then (3) reduces to the Kronecker sum.

The notion of Kronecker product (sum) for matrices was then generalized to tensor product (sum) of bounded linear operators on a Hilbert space. Let  $\mathcal{H}$  and  $\mathcal{K}$  be

Hilbert spaces. The tensor product of two bounded linear operators  $A: \mathcal{H} \rightarrow \mathcal{H}$  and  $B: \mathcal{K} \rightarrow \mathcal{K}$  is the unique bounded linear operator from  $\mathcal{H} \otimes \mathcal{K}$  into itself such that

$$(A \otimes B)(x \otimes y) = Ax \otimes By \quad (4)$$

for all  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ . The theory of tensor product for operators was focused by many authors from the past until nowadays, see e.g. Zanni and Kubrusly (2015), Kubrusly and Levan (2011). The tensor sum for operators was investigated in Kubrusly and Levan (2011).

Recently, the notion of Tracy-Singh product for matrices was generalized to Tracy-Singh product for Hilbert space operators; see Ploymukda, Chansangiam, and Lewkeeratiyutkul (2018a). To define the Tracy-Singh product, we apply the projection theorem for Hilbert spaces to decompose

$$\mathcal{H} = \bigoplus_{i=1}^m \mathcal{H}_i \quad \text{and} \quad \mathcal{K} = \bigoplus_{k=1}^n \mathcal{K}_k$$

where all  $\mathcal{H}_i$ 's and  $\mathcal{K}_k$ 's are Hilbert spaces. Thus every bounded linear operator  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,m} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{n,n}$$

where  $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$  and  $B_{kl} \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k)$  for each  $i, j, k, l$ .

**Definition 1.** According to the above notations, the Tracy-Singh product of  $A$  and  $B$  is defined to be the operator matrix

$$A \boxtimes B = \left[ [A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}, \quad (5)$$

which is a bounded linear operator from  $\bigoplus_{i,k=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_k$  into itself.

Note that when  $m = n = 1$ , the Tracy-Singh product  $A \boxtimes B$  reduces to the tensor product  $A \otimes B$ . Algebraic, order, and analytic properties of the Tracy-Singh product of operators were investigated in Ploymukda, Chansangiam, and Lewkeeratiyutkul (2018a) and (2018b).

In this paper, we generalize the notions of Tracy-Singh sum for matrices and the tensor sum of operators to the Tracy-Singh sum for operators. This kind of operator sum turns out to be compatible with certain algebraic operations and operator orderings. Binomial theorem involving Tracy-Singh sums and its consequences are then established. Moreover, we investigate continuity, convergence, norm bounds for Tracy-Singh sums of operators. Finally, we derive operator identities concerning Tracy-Singh sums and certain operator functions defined by power series.

This paper is organized as follows. In Section 2, we provide preliminaries on the Tracy-Singh product for operators. We introduce the Tracy-Singh sum for operators in Section 3 and then investigate its algebraic and order properties. In Section 4, we prove the binomial theorem involving Tracy-Singh sums and deduce some consequences. Analytic properties of Tracy-Singh sums are presented in Section 5.

## 2. Preliminaries on Tracy-Singh products for operators

Throughout this paper, let  $\mathcal{H}$  and  $\mathcal{K}$  be complex Hilbert spaces. When  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert spaces, denote by  $\mathbb{B}(\mathcal{X}, \mathcal{Y})$  the Banach space of bounded linear operators from  $\mathcal{X}$  into  $\mathcal{Y}$ , and abbreviate  $\mathbb{B}(\mathcal{X}, \mathcal{X})$  to  $\mathbb{B}(\mathcal{X})$ . Unless otherwise stated, capital letters mean operators on a Hilbert space. In particular,  $I$  stands for the identity operator. The positive-semidefinite ordering between two Hermitian operators  $A$  and  $B$  in  $\mathbb{B}(\mathcal{H})$  is defined as follows:  $A \leq B$  means that  $B - A$  is a positive operator.

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Fundamental properties of the Tracy-Singh product for operators are provided here; see Ployemukda, Chansangiam, and Lewkeeratiyutkul (2018a) and (2018b). Recall that each  $A \in M_m$  corresponds to a bounded linear operator  $L_A : \mathbb{C}^m \rightarrow \mathbb{C}^m$ ,  $x \mapsto Ax$ .

**Lemma 2.** For complex matrices  $A \in M_m$  and  $B \in M_n$ , we have

$$L_A \boxtimes L_B = L_{A \boxtimes B}. \quad (6)$$

**Proposition 3.** Let  $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices. Then

$$A \boxtimes B = [A_{ij} \boxtimes B]_{ij} = \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1n} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{n1} \boxtimes B & \cdots & A_{nn} \boxtimes B \end{bmatrix}.$$

**Lemma 4.** The following properties hold, provided that all operators are compatible.

$$(A \boxtimes B)^* = A^* \boxtimes B^*, \quad (7)$$

$$(\alpha A) \boxtimes B = \alpha(A \boxtimes B) = A \boxtimes (\alpha B), \quad (8)$$

$$A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C, \quad (9)$$

$$(B + C) \boxtimes A = B \boxtimes A + C \boxtimes A, \quad (10)$$

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD. \quad (11)$$

**Lemma 5.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . If  $A, B \geq 0$ , then  $A \boxtimes B \geq 0$ .

The next lemma asserts the continuity of the map  $(A, B) \mapsto A \boxtimes B$ .

**Lemma 6.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices, and let  $(A_r)_{r=1}^\infty$  and  $(B_r)_{r=1}^\infty$  be sequences in  $\mathbb{B}(\mathcal{H})$  and  $\mathbb{B}(\mathcal{K})$ , respectively. If  $A_r \rightarrow A$  and  $B_r \rightarrow B$ , then  $A_r \boxtimes B_r \rightarrow A \boxtimes B$ .

**Lemma 7.** For any operator  $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ , we have

$$\frac{1}{mn} \|A\| \|B\| \leq \|A \boxtimes B\| \leq mn \|A\| \|B\|. \quad (12)$$

The next lemma is a direct consequence of Lemma 7.

**Lemma 8.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . Then  $A \boxtimes B = 0$  if and only if  $A = 0$  or  $B = 0$ .

**Lemma 9.** Let  $A \in \mathbb{B}(\mathcal{H})$ .

- (i) If  $f$  is an analytic function on a region containing the spectra of  $A$  and  $I \boxtimes A$ , then  $f(I \boxtimes A) = I \boxtimes f(A)$ .
- (ii) If  $f$  is an analytic function on a region containing the spectra of  $A$  and  $A \boxtimes I$ , then  $f(A \boxtimes I) = f(A) \boxtimes I$ .

### 3. Algebraic and order properties of Tracy-Singh sums for operators

In this section, we define the Tracy-Singh sum for Hilbert space operators and investigate its algebraic and order properties. It turns out that the Tracy-Singh sum is compatible with the adjoint operation, the scalar multiplication, the ordinary sum and commutators. Moreover, positivity and positive-semidefinite ordering of operators are preserved by Tracy-Singh sums.

**Definition 10.** Let  $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ . We define the

**Tracy-Singh sum** of  $A$  and  $B$  as follows:

$$A \boxplus B = A \boxtimes I_{\mathcal{K}} + I_{\mathcal{H}} \boxtimes B, \quad (13)$$

which belongs to  $\mathbb{B}\left(\bigoplus_{i,j=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_j\right)$ .

Note that if both  $A$  and  $B$  are  $1 \times 1$  block operator matrices i.e.  $m = n = 1$ , their Tracy-Singh sum  $A \boxplus B$  is known as the tensor sum (Kubrusly and Levan, 2011)

$$A \oplus B = A \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes B.$$

The next result asserts that the Tracy-Singh sum of two linear maps induced by matrices is just the linear map induced by the Tracy-Singh sum of those matrices.

**Proposition 11.** Let  $A = [A_{ij}] \in M_n(\mathbb{C})$  and let  $B = [B_{kl}] \in M_m(\mathbb{C})$  be complex partitioned matrices. Then

$$L_A \boxplus L_B = L_{A \boxplus B}. \quad (14)$$

It follows that if  $\mathcal{E}$  is a standard ordered basis of  $M_{mn}$ , then

$$[L_A \boxplus L_B]_{\mathcal{E}} = A \hat{\boxplus} B.$$

**Proof.** We know that the linear map induced by the identity matrix is the identity operator. By applying Lemma 2, we get

$$\begin{aligned} L_A \boxplus L_B &= L_A \boxtimes I + I \boxtimes L_B = L_A \boxtimes I_I + L_I \boxtimes L_B \\ &= L_{A \hat{\boxplus} I} + L_{I \hat{\boxplus} B} = L_{A \hat{\boxplus} I + I \hat{\boxplus} B} = L_{A \hat{\boxplus} B}. \end{aligned}$$

Proposition 11 says that the matrix representation of the Tracy-Singh sum of linear maps induced by two matrices with respect to the standard ordered basis is just the Tracy-Singh sum of these matrices.

**Proposition 12.** Let  $\alpha$  and  $\beta$  be complex scalars. Then for any  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$ , we have

$$(A \boxplus B)^* = A^* \boxplus B^*, \quad (15)$$

$$\alpha(A \boxplus B) = (\alpha A) \boxplus (\alpha B), \quad (16)$$

$$(\alpha + \beta)(A \boxplus B) = \alpha A \boxplus \beta B + \beta A \boxplus \alpha B, \quad (17)$$

$$(A+C) \boxplus (B+D) = A \boxplus B + C \boxplus D. \quad (18)$$

We call (18) the mixed sum property.

**Proof.** Using Lemma 4, we obtain (15)-(17). By applying properties (9) and (10) of Lemma 4, we get

$$\begin{aligned} (A+C) \boxplus (B+D) &= (A+C) \boxtimes I + I \boxtimes (B+D) \\ &= A \boxtimes I + C \boxtimes I + I \boxtimes B + I \boxtimes D \\ &= (A \boxtimes I + I \boxtimes B) + (C \boxtimes I + I \boxtimes D) \\ &= A \boxplus B + C \boxplus D. \end{aligned}$$

It follows from (15) that if  $A$  and  $B$  are Hermitian (skew-Hermitian), then so is  $A \boxplus B$ . The mixed sum property (18) implies that if  $A_1 = X_1 + iY_1$  and  $A_2 = X_2 + iY_2$  are the Cartesian decompositions of  $A_1$  and  $A_2$ , respectively, then  $A_1 \boxplus A_2 = X_1 \boxplus X_2 + i(Y_1 \boxplus Y_2)$  is the Cartesian decomposition of  $A_1 \boxplus A_2$ .

**Lemma 13.** Let  $A_1, A_2 \in \mathbb{B}(\mathcal{H})$  and  $B_1, B_2 \in \mathbb{B}(\mathcal{K})$  be nonzero operators. Then  $A_1 \otimes B_1 = A_2 \otimes B_2$  if and only if there exists  $\alpha \in \mathbb{C} \setminus \{0\}$  such that  $A_1 = \alpha A_2$  and  $B_1 = \alpha^{-1} B_2$ .

**Proof.** See Proposition 2.1 of Stochel (1996).

**Theorem 14.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . Then  $A \boxplus B = 0$  if and only if there exists  $\alpha \in \mathbb{C} \setminus \{0\}$  such that  $A = \alpha I$  and  $B = -\alpha I$ .

**Proof.** If  $A = \alpha I$  and  $B = -\alpha I$ , then  $A \boxplus B = 0$  by Lemma 4. Now, suppose that  $A \boxplus B = 0$ . Then by Lemma 3,

$$0 = A \boxplus B = \begin{bmatrix} A_{11} \boxplus B & A_{12} \boxtimes I & \cdots & A_{1m} \boxtimes I \\ A_{21} \boxtimes I & A_{22} \boxplus B & \cdots & A_{2m} \boxtimes I \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} \boxtimes I & A_{m2} \boxtimes I & \cdots & A_{mm} \boxplus B \end{bmatrix}.$$

For any  $i, j \in \{1, \dots, m\}$  such that  $i \neq j$ , since  $A_{ij} \boxtimes I = 0$ , by Lemma 8 we can deduce  $A_{ij} = 0$ . Let  $i \in \{1, \dots, m\}$  and consider

$$\begin{bmatrix} A_{ii} \oplus B_{11} & I \otimes B_{12} & \cdots & I \otimes B_{1n} \\ I \otimes B_{21} & A_{ii} \oplus B_{22} & \cdots & I \otimes B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I \otimes B_{n1} & I \otimes B_{n2} & \cdots & A_{ii} \oplus B_{nn} \end{bmatrix} = 0.$$

Lemma 8 implies that  $B_{kl} = 0$  for any  $k \neq l$ . For  $k \in \{1, \dots, n\}$ , we have  $A_{ii} \oplus B_{kk} = 0$  if and only if  $A_{ii} \otimes (-I) = I \otimes B_{kk}$ . If  $A_{ii} \otimes (-I) = I \otimes B_{kk}$ , then  $I \otimes B_{kk} = 0$  and hence  $B_{kk} = 0$  for all  $k = 1, \dots, n$  by Lemma 8. It follows that  $B = 0$  and  $A \boxtimes I = A \boxplus B = 0$ .

Thus  $A = 0$  by Lemma 8. Assume that  $A_{ii} \neq 0$  for all  $i = 1, \dots, m$ . By Lemma 13, there exists  $\alpha_{ik} \in \mathbb{C} \setminus \{0\}$  such that  $A_{ii} = \alpha_{ik} I$  and  $-I = \alpha_{ik}^{-1} B_{kk}$ . For any fixed  $i$ , we have  $A_{ii} = \alpha_{ik} I$  for all  $k = 1, \dots, n$ . Hence,  $\alpha_{i1} = \alpha_{i2} = \dots = \alpha_{in} = \alpha_i$ . For any fixed  $k \in \{1, \dots, n\}$ , we have  $B_{kk} = -\alpha_i I$  for all  $i = 1, \dots, m$ . This implies that  $\alpha_1 = \alpha_2 = \dots = \alpha_m = \alpha$ . Thus  $A_{ii} = \alpha I$  for all  $i = 1, \dots, m$  and  $B_{kk} = -\alpha I$  for all  $k = 1, \dots, n$ . Therefore  $A = \alpha I$  and  $B = -\alpha I$ .

The next result provides formulas for the inverses of  $A \boxplus B$  in terms of Tracy-Singh products.

**Proposition 15.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be invertible operators. If  $A \boxplus B$  and  $A^{-1} \boxplus B^{-1}$  are invertible, then

$$(A \boxplus B)^{-1} = (A^{-1} \boxtimes I)(A^{-1} \boxplus B^{-1})^{-1}(I \boxtimes B^{-1}), \quad (19)$$

$$(A \boxplus B)^{-1} = (I \boxtimes B^{-1})(A^{-1} \boxplus B^{-1})^{-1}(A^{-1} \boxtimes I), \quad (20)$$

$$(A \boxplus B)^{-1} = (A^{-1} \boxplus B^{-1})^{-1}(A^{-1} \boxtimes B^{-1}). \quad (21)$$

**Proof.** Applying Lemma 4, we have

$$\begin{aligned} (A \boxplus B)^{-1} &= (A \boxtimes I + I \boxtimes B)^{-1} \\ &= (A^{-1} \boxtimes I) \left[ (I \boxtimes B + A \boxtimes I)(A^{-1} \boxtimes I) \right]^{-1} \\ &= (A^{-1} \boxtimes I) \left[ A^{-1} \boxtimes B + I \boxtimes I \right]^{-1} \\ &= (A^{-1} \boxtimes I) \left[ (I \boxtimes B^{-1})(A^{-1} \boxtimes B + I \boxtimes I) \right]^{-1} (I \boxtimes B^{-1}) \\ &= (A^{-1} \boxtimes I) \left[ A^{-1} \boxtimes I + I \boxtimes B^{-1} \right]^{-1} (I \boxtimes B^{-1}) \\ &= (A^{-1} \boxtimes I)(A^{-1} \boxplus B^{-1})^{-1}(I \boxtimes B^{-1}). \end{aligned}$$

Similarly, we obtain the properties (20) and (21).

Recall that the **commutator** and the **anticommutator** of  $A, B \in \mathbb{B}(\mathcal{H})$  are defined respectively by

$$[A, B] = AB - BA,$$

$$[A, B]_+ = AB + BA.$$

**Proposition 16.** Let  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$ . Then

$$[A \boxplus B, C \boxplus D] = [A, C] \boxplus [B, D]. \quad (22)$$

**Proof.** It follows directly from Lemma 4.

Recall that an operator  $A \in \mathbb{B}(\mathcal{H})$  is said to be normal if  $[A^*, A] = 0$ . It follows from Propositions 12 and 16 that if  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  are normal, then so is  $A \boxplus B$ . The next result shows that the Tracy-Singh sum preserves positivity and order.

**Proposition 17.** Let  $A, A_1, A_2 \in \mathbb{B}(\mathcal{H})$  and  $B, B_1, B_2 \in \mathbb{B}(\mathcal{K})$ .

- (i) If  $A \geq 0$  and  $B \geq 0$ , then  $A \boxplus B \geq 0$ .
- (ii) If  $A_1 \geq A_2$  and  $B_1 \geq B_2$ , then  $A_1 \boxplus B_1 \geq A_2 \boxplus B_2$ .

**Proof.** It follows directly from Lemma 5 and Proposition 12.

**Proposition 18.** Let  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$  be positive operators. Then

$$(A \boxplus B)(C \boxplus D) \geq AC \boxplus BD. \quad (23)$$

**Proof.** Applying Lemma 4, we have

$$(A \boxplus B)(C \boxplus D) = AC \boxplus BD + A \boxtimes D + C \boxtimes B.$$

Since  $A \boxtimes D + C \boxtimes B$  is positive, we get the result.

**Corollary 19.** Let  $A, C \in \mathbb{B}(\mathcal{H})$  and  $B, D \in \mathbb{B}(\mathcal{K})$  be positive operators. Then

$$[A \boxplus B, C \boxplus D]_+ \geq [A, C]_+ \boxplus [B, D]_+. \quad (24)$$

**Proof.** It follows directly from Propositions 12 and 18.

#### 4. Binomial theorem involving Tracy-Singh sums and its consequences

In this section, we prove an operator version of the binomial theorem in which the sum and the product are replaced by the Tracy-Singh sum and the Tracy-Singh product, respectively. Consequently, we obtain certain operator inequalities, including a Bernoulli-type inequality. Binomial theorem is also used to treat the nilpotency of the Tracy-Singh sum of two operators.

**Theorem 20.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be compatible operator matrices. Then for any integer  $r \geq 2$ ,

$$(A \boxplus B)^r = A^r \boxplus B^r + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k). \quad (25)$$

**Proof.** By using Lemma 4, we have

$$\begin{aligned} (A \boxplus B)^2 &= (A \boxtimes I + I \boxtimes B)(A \boxtimes I + I \boxtimes B) \\ &= A^2 \boxtimes I + 2A \boxtimes B + I \boxtimes B^2 = A^2 \boxplus B^2 + 2A \boxtimes B. \end{aligned}$$

This show that (25) is true for  $r = 2$ . Suppose that (25) holds for an integer  $r \geq 2$ . Then, by Lemma 4,

$$\begin{aligned} (A \boxplus B)^{r+1} &= (A \boxplus B)^r (A \boxplus B) \\ &= \left( A^r \boxplus B^r + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k) \right) (A \boxplus B) \\ &= (A^r \boxplus B^r)(A \boxplus B) + \left( \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k) \right) (A \boxplus B) \\ &= (A^r \boxtimes I)(A \boxtimes I) + (A^r \boxtimes I)(I \boxtimes B) + (I \boxtimes B^r)(A \boxtimes I) \\ &\quad + (I \boxtimes B^r)(I \boxtimes B) + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k)(A \boxtimes I) + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^k)(I \boxtimes B) \\ &= A^{r+1} \boxtimes I + A^r \boxtimes B + A \boxtimes B^r + I \boxtimes B^{r+1} \end{aligned}$$

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$$\begin{aligned}
& + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k+1} \boxtimes B^k) + \sum_{k=1}^{r-1} \binom{r}{k} (A^{r-k} \boxtimes B^{k+1}) \\
& = A^{r+1} \boxplus B^{r+1} + \sum_{k=1}^r \binom{r}{k} (A^{(r+1)-k} \boxtimes B^k) + \sum_{k=1}^r \binom{r}{k-1} (A^{(r+1)-k} \boxtimes B^k) \\
& = A^{r+1} \boxplus B^{r+1} + \sum_{k=1}^r \binom{r+1}{k} (A^{(r+1)-k} \boxtimes B^k).
\end{aligned}$$

**Corollary 21.** Let  $A \geq 0$  and  $B \geq 0$  be compatibly operator matrices. Then

$$(A \boxplus B)^r \geq A^r \boxplus B^r \quad (26)$$

for any  $r \in \mathbb{N}$ .

**Proof.** It follows immediately from Theorem 20 and Lemma 5.

The next result is a Bernoulli type inequality concerning Tracy-Singh sums.

**Corollary 22.** Let  $A$  be a positive operator. Then for any  $r \in \mathbb{N}$ ,

$$(I \boxplus A)^r \geq I \boxplus (rA). \quad (27)$$

**Proof.** Since  $A \geq 0$ , we have  $I \boxplus A^r \geq I \boxtimes I$ . By Theorem 21, we obtain

$$\begin{aligned}
(I \boxplus A)^r & = I \boxplus A^r + \sum_{k=1}^{r-1} \binom{r}{k} (I \boxtimes A^k) = I \boxplus A^r + r(I \boxtimes A) + \sum_{k=2}^{r-1} \binom{r}{k} (I \boxtimes A^k) \\
& \geq I \boxplus A^r + r(I \boxtimes A) \geq I \boxtimes I + I \boxtimes (rA) = I \boxplus (rA).
\end{aligned}$$

**Corollary 23.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . If  $A$  and  $B$  are nilpotent, then  $A \boxplus B$  is also nilpotent.

**Proof.** Suppose  $A^r = 0$  and  $B^s = 0$  for some  $r, s \in \mathbb{N}$ . From Theorem 20, we get

$$\begin{aligned}
(A \boxplus B)^{r+s} &= \sum_{k=0}^{r+s} \binom{r+s}{k} (A^{r+s-k} \boxtimes B^k) \\
&= \sum_{k=0}^{s-1} \binom{r+s}{k} (A^{r+s-k} \boxtimes B^k) + \sum_{k=s}^{r+s} \binom{r+s}{k} (A^{r+s-k} \boxtimes B^k).
\end{aligned}$$

If  $k \geq s$ , then  $B^k = 0$ . If  $k < s$ , then  $r+s-k > r$  and hence  $A^k = 0$ . Now,

$$\begin{aligned}
(A \boxplus B)^{r+s} &= \sum_{k=0}^{s-1} \binom{r+s}{k} (0 \boxtimes B^k) + \sum_{k=s}^{r+s} \binom{r+s}{k} (A^{r+s-k} \boxtimes 0) \\
&= \sum_{k=0}^{s-1} \binom{r+s}{k} (0 \boxtimes B)^k + \sum_{k=s}^{r+s} \binom{r+s}{k} (A \boxtimes 0)^{r+s-k} \\
&= 0.
\end{aligned}$$

This means that  $A \boxplus B$  is nilpotent.

## 5. Analytic properties of Tracy-Singh sums

In this section, we investigate continuity, convergence, norm bounds for Tracy-Singh sums of operators. We also discuss the relationship between Tracy-Singh sums and certain functions of operators defined by power series.

**Proposition 24.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$  be operator matrices, and let  $(A_r)_{r=1}^\infty$  and  $(B_r)_{r=1}^\infty$  be sequences in  $\mathbb{B}(\mathcal{H})$  and  $\mathbb{B}(\mathcal{K})$ , respectively. If  $A_r \rightarrow A$  and  $B_r \rightarrow B$ , then  $A_r \boxplus B_r \rightarrow A \boxplus B$ .

**Proof.** By Lemma 6, we have  $A_r \boxtimes I \rightarrow A \boxtimes I$  and  $I \boxtimes B_r \rightarrow I \boxtimes B$ . Thus  $A_r \boxtimes I + I \boxtimes B_r \rightarrow A \boxtimes I + I \boxtimes B$ , i.e.  $A_r \boxplus B_r \rightarrow A \boxplus B$ .

Proposition 24 asserts that the Tracy-Singh sum is (jointly) continuous with respect to the operator-norm topology. The next result is a triangle-like inequality for Tracy-Singh sums.

**Proposition 25.** For any  $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$  and  $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ , we have

$$\frac{1}{mn} \|A \boxplus B\| \leq \|A\| + \|B\|. \quad (28)$$

**Proof.** It follows from the norm estimation in Lemma 7.

In the rest of paper, we shall establish operator identities involving Tracy-Singh sums and functions of operators defined by power series. Recall that for any  $T \in \mathbb{B}(\mathcal{H})$ , we define

$$e^T = \sum_{k=0}^{\infty} \frac{1}{k!} T^k, \quad (29)$$

$$\sin(T) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} T^{2k+1}, \quad (30)$$

$$\cos(T) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} T^{2k}, \quad (31)$$

$$\sinh(T) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} T^{2k+1}, \quad (32)$$

$$\cosh(T) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} T^{2k}. \quad (33)$$

The series on the right hand side of (29)-(33) converges in the norm topology of  $\mathbb{B}(\mathcal{H})$ .

If  $T$  is positive and invertible, we define  $\log T$  to be the operator  $X$  such that  $e^X = T$ .

The following facts are well-known.

**Lemma 26.** For any  $S, T \in \mathbb{B}(\mathcal{H})$  satisfying  $ST = TS$ , we have

$$e^{S+T} = e^S e^T,$$

$$\sin(S+T) = \sin S \cos T + \cos S \sin T,$$

$$\cos(S+T) = \cos S \cos T - \sin S \sin T,$$

$$\sinh(S+T) = \sinh S \cosh T + \cosh S \sinh T,$$

$$\cosh(S+T) = \cosh S \cosh T - \sinh S \sinh T.$$

If  $S$  and  $T$  are invertible positive operators such that  $ST = TS$ , then

$$\log ST = \log S + \log T.$$

**Theorem 27.** Let  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ . Then

$$e^{A \boxplus B} = e^A \boxtimes e^B, \quad (34)$$

$$\sin(A \boxplus B) = \sin A \boxtimes \cos B + \cos A \boxtimes \sin B, \quad (35)$$

$$\cos(A \boxplus B) = \cos A \boxtimes \cos B - \sin A \boxtimes \sin B, \quad (36)$$

$$\sinh(A \boxplus B) = \sinh A \boxtimes \cosh B + \cosh A \boxtimes \sinh B, \quad (37)$$

$$\cosh(A \boxplus B) = \cosh A \boxtimes \cosh B - \sinh A \boxtimes \sinh B. \quad (38)$$

If  $A$  and  $B$  are invertible positive operators, then

$$\log(A \boxtimes B) = \log A \boxplus \log B. \quad (39)$$

**Proof.** By property (11) in Lemma 4, we have  $(A \boxtimes I)(I \boxtimes B) = (I \boxtimes B)(A \boxtimes I)$ . It

follows that

$$\begin{aligned} e^{A \boxplus B} &= e^{A \boxtimes I + I \boxtimes B} = e^{A \boxtimes I} e^{I \boxtimes B} \quad (\text{by Lemma 26}) \\ &= (e^A \boxtimes I)(I \boxtimes e^B) \quad (\text{by Lemma 9}) \end{aligned}$$

$$= e^A \boxtimes e^B \quad (\text{by Lemma 4, property (11)}).$$

Similarly, we get the identities (35)-(39).

Our results in this paper suggest that the Tracy-Singh sum is a “sum”, while the Tracy-Singh product is a “product” for operators.

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### References

- Al Hour, Z. & Kilicman, A. (2006). Matrix equalities and inequalities involving Khatri-Rao and Tracy-Singh sums. *Journal of Inequalities in Pure and Applied Mathematics*, 7(1), 11-17. Retrieved from <http://eudml.org/doc/128511>
- Kubrusly, C.S. & Levan, N. (2011). Preservation of tensor sum and tensor product. *Acta Mathematica Universitatis Comenianae*, 80(1), 133-142. Retrieved from <https://www.researchgate.net/publication/264702227>
- Ploymukda, A., Chansangiam, P., & Lewkeeratiyutkul, W. (2018a). Algebraic and order properties of Tracy-Singh product for operator matrices. *Journal of Computational Analysis and Applications*, 24(4), 656-664.
- Ploymukda, A., Chansangiam, P., & Lewkeeratiyutkul, W. (2018b). Analytic properties of Tracy-Singh products for operator matrices. *Journal of Computational Analysis and Applications*, 24(4), 665-674.
- Shuangzhe, L. (1999). Matrix results on the Khatri-Rao and Tracy-Singh products. *Linear Algebra and Its Applications* 289(1-3), 267-277. doi.org/10.1016/S0024-3795(98)10209-4

- Steeb, W. & Hardy, Y. (2011). *Matrix calculus and Kronecker product: A practical approach to linear and multilinear algebra* (2nd ed). Singapore: World Scientific Publishing.
- Stochel, J. (1996). Seminormality of operators from their tensor product. *Proceedings of the American Mathematical Society*, 124(1), 135-140. doi: 10.1090/S0002-9939-96-03017-1
- Tracy, D.S. & Jinadasa, K.G. (1989). Partitioned Kronecker products of matrices and applications. *The Canadian Journal of Statistics*, 17(1), 107-120. doi: 10.2307/3314768
- Tracy, D.S. & Singh, R.P. (1972). A new matrix product and its applications in partitioned matrix differentiation. *Statistica Neerlandica*, 26(4), 143-157. doi: 10.1111/j.1467-9574.1972.tb00199.x
- Zanni, J. & Kubrusly, C.S. (2015). A note on compactness of tensor products. *Acta Mathematica Universitatis Comenianae*, 89(1), 59-62. Retrieved from <http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/47>

# Khatri-Rao Products and Selection Operators

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## Abstract

We develop further theory for Khatri-Rao products of Hilbert space operators in connections with selection operators. We provide two constructions related to selection operators. Then we establish certain identities and inequalities involving Khatri-Rao and Tracy-Singh products. As consequences, we obtain some characterizations for the mixed product property concerning the Khatri-Rao product of operators.

**Keywords:** tensor product, Khatri-Rao product, Tracy-Singh product, operator matrix

**Mathematics Subject Classifications 2010:** 47A80, 15A69, 47A05.

## 1 Introduction

This paper concerns operator extensions of certain matrix products, namely, the Kronecker (tensor) product, the Tracy-Singh product, and the Khatri-Rao product. Fundamental theory for these matrix products are collected, for instance, in [1, 2, 4, 5, 10, 11, 12] and references therein. Denote by  $M_{m,n}(\mathbb{C})$  the algebra of  $m$ -by- $n$  complex matrices. Recall that the Kronecker product of  $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$  and  $B \in M_{p,q}(\mathbb{C})$  is given by

$$A \hat{\otimes} B = [a_{ij}B]_{ij}.$$

Consider partitioned matrices  $A$  and  $B$  such that the  $(i, j)$ th block of  $A$  is  $A_{ij}$  and the  $(k, l)$ th block of  $B$  is  $B_{kl}$ . The Tracy-Singh product [9] of  $A$  and  $B$  is defined by

$$A \hat{\boxtimes} B = [[A_{ij} \hat{\otimes} B_{kl}]_{kl}]_{ij}. \quad (1)$$

The Khatri-Rao product [3] is defined for two partitioned matrices  $A = [A_{ij}]$  and  $B = [B_{ij}]$  as follows

$$A \hat{\square} B = [A_{ij} \hat{\otimes} B_{ij}]_{ij}. \quad (2)$$

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The tensor product of Hilbert space operators can be viewed as an extension of the Kronecker product of complex matrices. Recall that the tensor product of  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$  is the unique bounded linear operator from  $\mathcal{H} \otimes \mathcal{K}$  into  $\mathcal{H}' \otimes \mathcal{K}'$  such that  $(A \otimes B)(x \otimes y) = Ax \otimes By$  for all  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ . Recently, the Tracy-Singh product and the Khatri-Rao product for matrices were generalized to those for operators acting on the direct sum of Hilbert spaces, see [6, 7, 8]. Fundamental algebraic and order properties of operator Khatri-Rao products are investigated in [8]. That paper also provides a construction of a unital positive linear map taking the Tracy-Singh product of two operators to their Khatri-Rao product. Such a linear map appears in the form  $X \mapsto Z^*AZ$  where  $Z$  is an isometry, called a selection operator. See details in Section 2.

The present paper contains further development on operator Khatri-Rao products in relations with Tracy-Singh products and selection operators. First, we provide two constructions related to selection operators (see Section 3). Consequently, we establish some operator identities and inequalities involving Khatri-Rao and Tracy-Singh products (see Section 4). Finally, we obtain some characterizations for the mixed product property concerning the Khatri-Rao product of operators (see Section 5).

## 2 Tracy-Singh products and Khatri-Rao products for operators

Throughout this paper, let  $\mathcal{H}, \mathcal{H}', \mathcal{K}$  and  $\mathcal{K}'$  be complex separable Hilbert spaces. When  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert spaces, let us denote by  $\mathbb{B}(\mathcal{X}, \mathcal{Y})$  the space of all bounded linear operators from  $\mathcal{X}$  into  $\mathcal{Y}$  and abbreviate  $\mathbb{B}(\mathcal{X}, \mathcal{X})$  to  $\mathbb{B}(\mathcal{X})$ . Capital letters always denote a Hilbert space operator. In particular,  $I$  and  $O$  stand for the identity and the zero operator, respectively.

In order to define Tracy-Singh products of operators, we fix the following decompositions

$$\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^m \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{j=1}^q \mathcal{K}_j, \quad \mathcal{K}' = \bigoplus_{i=1}^p \mathcal{K}'_i. \quad (3)$$

where all of  $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$  are Hilbert spaces. For each  $j$  and  $l$ , let  $M_j : \mathcal{H}_j \rightarrow \mathcal{H}$  and  $N_l : \mathcal{K}_l \rightarrow \mathcal{K}$  be the canonical injections. For each  $i$  and  $k$ , let  $P_i : \mathcal{H}' \rightarrow \mathcal{H}'_i$  and  $Q_k : \mathcal{K}' \rightarrow \mathcal{K}'_k$  be the canonical projections. Given  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ , put  $A_{ij} = P_i A M_j \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}'_i)$  for each  $i, j$ . Thus we can write  $A$  in the operator-matrix form  $A = [A_{ij}]_{i,j=1}^{m,n}$ . Similarly, given  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , let  $B_{kl} = Q_k B N_l \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}'_k)$  for each  $k = 1, \dots, p$  and  $l = 1, \dots, q$ . We can identify  $B$  with the operator matrix  $B = [B_{kl}]_{k,l=1}^{p,q}$ .

**Definition 1.** *The Tracy-Singh product of  $A$  and  $B$  is defined to be the bounded linear operator from  $\bigoplus_{j,l=1}^{n,q} \mathcal{H}_j \otimes \mathcal{K}_l$  to  $\bigoplus_{i,k=1}^{m,p} \mathcal{H}'_i \otimes \mathcal{K}'_k$  represented by*

$$A \boxtimes B = [[A_{ij} \otimes B_{kl}]_{kl}]_{ij}. \quad (4)$$

If both factor  $A$  and  $B$  consist of only one block, then  $A \boxtimes B = A \otimes B$ .

**Lemma 2** ([6]). *The following properties of the Tracy-Singh product for operators hold (provided that each term is well-defined):*

1. *Compatibility with adjoints:*  $(A \boxtimes B)^* = A^* \boxtimes B^*$ .
2. *Mixed-product property:*  $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$ .
3. *Monotonicity:* if  $A \geq B \geq 0$  and  $C \geq D \geq 0$ , then  $A \boxtimes B \geq C \boxtimes D \geq 0$ .

From now on, we fix the decomposition (3), and assume  $n = q$  and  $m = p$ .

**Definition 3.** *The Khatri-Rao product of  $A = [A_{ij}]_{i,j=1}^{m,n}$  and  $B = [B_{ij}]_{i,j=1}^{m,n}$  is defined to be a bounded linear operator from  $\bigoplus_{j=1}^n \mathcal{H}_j \otimes \mathcal{K}_j$  to  $\bigoplus_{i=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_i$  represented by the operator matrix*

$$A \boxtimes B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{m,n}. \tag{5}$$

**Lemma 4** ([8]). *For  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , we have  $(A \boxtimes B)^* = A^* \boxtimes B^*$ .*

Fix an ordered tuple  $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$  of Hilbert spaces. Define the ordered pair  $(Z_1, Z_2)$  of selection operators associated with  $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$  by [8]:

$$Z_1 = \begin{bmatrix} E_1 \\ \vdots \\ E_m \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}. \tag{6}$$

Here, for each  $r = 1, \dots, m$

$$E_r = [E_{gh}^{(r)}]_{g,h=1}^{m,m} : \bigoplus_{k=1}^m \mathcal{H}'_k \otimes \mathcal{K}'_k \rightarrow \bigoplus_{l=1}^m \mathcal{H}'_l \otimes \mathcal{K}'_l$$

with  $E_{gh}^{(r)}$  is an identity operator if  $g = h = r$  and the others are zero operators. For each  $s = 1, \dots, n$ , the operator  $F_s$  is defined by

$$F_s = [F_{gh}^{(s)}]_{g,h=1}^{n,n} : \bigoplus_{i=1}^n \mathcal{H}_i \otimes \mathcal{K}_i \rightarrow \bigoplus_{j=1}^n \mathcal{H}_s \otimes \mathcal{K}_j$$

with  $F_{gh}^{(s)}$  is an identity operator if  $g = h = s$  and the others are zero operators. From the construction, the operator  $Z_i$  is an isometry and  $Z_i Z_i^* \leq I$  for  $i = 1, 2$ . When  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $Z_1 = Z_2$ .

**Lemma 5** ([8]). *Let  $(Z_1, Z_2)$  be the ordered pair of selection operators associated with the ordered tuple  $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ . For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , we have*

$$A \boxtimes B = Z_1^*(A \boxtimes B)Z_2. \tag{7}$$

For the case  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $Z_1 = Z_2 := Z$  and hence for any  $A \in \mathbb{B}(\mathcal{H})$  and  $B \in \mathbb{B}(\mathcal{K})$ ,

$$A \boxtimes B = Z^*(A \boxtimes B)Z. \tag{8}$$

### 3 Two constructions related to selection operators

In this section, we construct certain operators related to selection operators.

**Theorem 6.** *Let  $(Z_1, Z_2)$  be the ordered pair of selection operators associated with an ordered tuple  $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ . Then there exist operators*

$$V : \bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_j \rightarrow \bigoplus_{i=1}^m \bigoplus_{j=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_j,$$

$$W : \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_j \rightarrow \bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathcal{H}_i \otimes \mathcal{K}_j$$

such that  $Z_1^*V = 0, Z_2^*W = 0, Z_1Z_1^* + VV^* = I$  and  $Z_2Z_2^* + WW^* = I$ . If, in addition,  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $V = W$ .

*Proof.* Let

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix} \tag{9}$$

where

$$V^{(r)} = \left[ V_{kl}^{(r)} \right]_{k,l=1}^{m, m^2-1} : \bigoplus_{i=1}^m \bigoplus_{j=1}^m \mathcal{H}'_i \otimes \mathcal{K}'_i \rightarrow \bigoplus_{i=1}^m \mathcal{H}'_r \otimes \mathcal{K}'_i$$

$i+j < m^2$

for  $r = 1, \dots, m$ , with  $V_{kl}^{(r)}$  is an identity operator if  $k \neq r$  and  $l = m(r-1) + k$  and the others are zero operators. Note that

$$E_1^* V_1 = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$= 0.$

For each  $r$ , we have

$$V_r V_r^* = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}.$$

Then we obtain

$$Z_1^*V = [E_1^* \ E_2^* \ \dots \ E_m^*] \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{bmatrix} = E_1^*V_1 + E_2^*V_2 + \dots + E_m^*V_m = 0,$$

$$Z_1Z_1^* + VV^*$$

$$= \begin{bmatrix} E_1E_1^* & E_1E_2^* & \dots & E_1E_m^* \\ E_2E_1^* & E_2E_2^* & \dots & E_2E_m^* \\ \vdots & \vdots & \ddots & \vdots \\ E_mE_1^* & E_mE_2^* & \dots & E_mE_m^* \end{bmatrix} + \begin{bmatrix} V_1V_1^* & V_1V_2^* & \dots & V_1V_m^* \\ V_2V_1^* & V_2V_2^* & \dots & V_2V_m^* \\ \vdots & \vdots & \ddots & \vdots \\ V_mV_1^* & V_mV_2^* & \dots & V_mV_m^* \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{bmatrix} = \begin{bmatrix} I & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{bmatrix}$$

Now, let

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_m \end{bmatrix} \tag{10}$$

where

$$W^{(s)} = [W_{kl}^{(s)}]_{k,l=1}^{n,n^2-1} : \bigoplus_{i=1}^n \bigoplus_{\substack{j=1 \\ i+j < n^2}}^n \mathcal{H}_i \otimes \mathcal{K}_i \rightarrow \bigoplus_{i=1}^n \mathcal{H}_s \otimes \mathcal{K}_i$$

for  $s = 1, \dots, n$ , with  $W_{kl}^{(s)}$  is an identity operator if  $k \neq s$  and  $l = n(s-1) + k$  and others are zero operators. A direct computation shows that  $Z_2^*W = 0$  and  $Z_2Z_2^* + WW^* = I$ . When  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $V_i = W_i$  for all  $i = 1, \dots, m$ , i.e.  $V = W$ .  $\square$

**Theorem 7.** Fix the decomposition (3) with  $n = q$  and  $m = p$ . Suppose further that  $\mathcal{H}_i = \mathcal{X}$ ,  $\mathcal{K}_i = \mathcal{Y}$ ,  $\mathcal{H}'_j = \mathcal{X}'$  and  $\mathcal{K}'_j = \mathcal{Y}'$  for all  $i = 1, \dots, n$  and  $j = 1, \dots, m$ . Let  $(Z_1, Z_2)$  be the ordered pair of associated selection operators. Then there exist operators

$$Q_1 : \bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^m \mathcal{X}' \otimes \mathcal{Y}' \rightarrow \bigoplus_{i=1}^m \bigoplus_{j=1}^m \mathcal{X}' \otimes \mathcal{Y}',$$

$$Q_2 : \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^n \mathcal{X} \otimes \mathcal{Y} \rightarrow \bigoplus_{i=1}^n \bigoplus_{j=1}^n \mathcal{X} \otimes \mathcal{Y}$$

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such that  $Z_i^*Q_i = 0$ ,  $Q_i^*Q_i = I$  and  $Z_iZ_i^* + Q_iQ_i^* = I$  for  $i = 1, 2$ . If, in addition,  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $Q_1 = Q_2$ .

*Proof.* Consider

$$Q_1 = \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} F_2 & F_3 & \cdots & F_n \\ F_3 & F_4 & \cdots & F_1 \\ \vdots & \vdots & \ddots & \vdots \\ F_1 & F_2 & \cdots & F_{n-1} \end{bmatrix}. \quad (11)$$

Then calculations reveal that

$$Z_1^*Q_1 = [E_1^* \ E_2^* \ \cdots \ E_m^*] \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} = [0 \ 0 \ \cdots \ 0],$$

$$Q_1^*Q_1 = \begin{bmatrix} E_2^* & E_3^* & \cdots & E_m^* \\ E_3^* & E_4^* & \cdots & E_1^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m^* & E_1^* & \cdots & E_{m-1}^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} \\ = \begin{bmatrix} \sum E_i^*E_i & 0 & \cdots & 0 \\ 0 & \sum E_i^*E_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum E_i^*E_i \end{bmatrix} = I,$$

$$Q_1^*Q_1 + Z_1Z_1^* \\ = \begin{bmatrix} E_2^* & E_3^* & \cdots & E_m^* \\ E_3^* & E_4^* & \cdots & E_1^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m^* & E_1^* & \cdots & E_{m-1}^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix} [E_1^* \ E_2^* \ \cdots \ E_m^*] \\ = \begin{bmatrix} \sum_{i \neq 1} E_i E_j^* & 0 & \cdots & 0 \\ 0 & \sum_{i \neq 2} E_i E_j^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{i \neq m} E_i E_j^* \end{bmatrix} + \begin{bmatrix} E_1 E_1^* & 0 & \cdots & 0 \\ 0 & E_2 E_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_m E_m^* \end{bmatrix} \\ = \begin{bmatrix} \sum E_i E_i^* & 0 & \cdots & 0 \\ 0 & \sum E_i E_i^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum E_i E_i^* \end{bmatrix} = I.$$

Similarly, we have  $Z_2^*Q_2 = 0$ ,  $Q_2^*Q_2 = I$  and  $Z_2Z_2^* + Q_2Q_2^* = I$ . When  $\mathcal{H} = \mathcal{H}'$  and  $\mathcal{K} = \mathcal{K}'$ , we have  $E_i = F_i$  for all  $i = 1, \dots, m$ , i.e.  $Q_1 = Q_2$ .  $\square$

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## 4 Operator identities and inequalities concerning Khatri-Rao products, Tracy-Singh products, and selection operators

In this section, we apply the construction in Section 3 to establish certain operator identities and inequalities concerning Khatri-Rao products, Tracy-Singh products, and selection operators.

**Theorem 8.** *Let  $(Z_1, Z_2)$  be the ordered pair of selection operators associated with an ordered tuple  $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ . Let  $V$  and  $W$  be operator matrices defined by (9) and (10). For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , we have*

$$AA^* \square BB^* = (A \square B)(A^* \square B^*) + Z_1^*(A \boxtimes B)WW^*(A^* \boxtimes B^*)Z_1, \quad (12)$$

$$A^*A \square B^*B = (A^* \square B^*)(A \square B) + Z_2^*(A^* \boxtimes B^*)VV^*(A \boxtimes B)Z_2. \quad (13)$$

*Proof.* Since  $AA^* \in \mathbb{B}(\mathcal{H}')$  and  $BB^* \in \mathbb{B}(\mathcal{K}')$ , the ordered pair of selection operators associated with  $(\mathcal{H}', \mathcal{H}', \mathcal{K}', \mathcal{K}')$  is given by  $(Z_1, Z_1)$ . By using Lemmas 2 and 5, and Theorem 6, we get

$$\begin{aligned} AA^* \square BB^* &= Z_1^*(AA^* \boxtimes BB^*)Z_1 \\ &= Z_1^*(A \boxtimes B)(A \boxtimes B)^*Z_1 \\ &= Z_1^*(A \boxtimes B)(Z_2Z_2^* + WW^*)(A \boxtimes B)^*Z_1 \\ &= Z_1^*(A \boxtimes B)Z_2Z_2^*(A \boxtimes B)^*Z_1 + Z_1^*(A \boxtimes B)WW^*(A \boxtimes B)^*Z_1 \\ &= (A \square B)(A \square B)^* + Z_1^*(A \boxtimes B)WW^*(A \boxtimes B)^*Z_1. \end{aligned}$$

Now, for inequality (13), note that  $A^*A \in \mathbb{B}(\mathcal{H})$  and  $B^*B \in \mathbb{B}(\mathcal{K})$ . In this case, the pair of associated selection operators is  $(Z_2, Z_2)$ . It follows that

$$\begin{aligned} A^*A \square B^*B &= Z_2^*(A^*A \boxtimes B^*B)Z_2 \\ &= Z_2^*(A \boxtimes B)^*(A \boxtimes B)Z_2 \\ &= Z_2^*(A \boxtimes B)^*(Z_1Z_1^* + VV^*)(A \boxtimes B)Z_2 \\ &= Z_2^*(A \boxtimes B)^*Z_1Z_1^*(A \boxtimes B)Z_2 + Z_2^*(A \boxtimes B)^*VV^*(A \boxtimes B)Z_2 \\ &= (A^* \square B^*)(A \square B) + Z_2^*(A^* \boxtimes B^*)VV^*(A \boxtimes B)Z_2. \quad \square \end{aligned}$$

**Corollary 9.** *Let  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$  be operator matrices. Then*

$$AA^* \square BB^* \geq (A \square B)(A^* \square B^*). \quad (14)$$

*Proof.* It follows immediately from Theorem 8.  $\square$

**Theorem 10.** *Assume the hypothesis of Theorem 7. For any  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , we have*

$$AA^* \square BB^* = (A \square B)(A^* \square B^*) + Z_1^*(A \boxtimes B)Q_2Q_2^*(A^* \boxtimes B^*)Z_1, \quad (15)$$

$$A^*A \square B^*B = (A^* \square B^*)(A \square B) + Z_2^*(A^* \boxtimes B^*)Q_1Q_1^*(A \boxtimes B)Z_2, \quad (16)$$

where  $Q_1$  and  $Q_2$  are operator matrices in (11).

*Proof.* The proof is similar to that of Theorem 8. Instead of Theorem 6, we apply Theorem 7. □

## 5 Characterizations of the mixed product property for Khatri-Rao products

In general, the mixed product property

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$$

does not hold for compatible operator matrices  $A, B, C, D$ . It is interesting to find necessary and sufficient conditions for which this property holds. Indeed, we have the following assertions.

**Theorem 11.** *Assume the notations in Theorem 8. For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , the following statements are equivalent:*

- (i)  $AC \boxtimes BD = (A \boxtimes B)(C \boxtimes D)$  for all  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ ,
- (ii)  $AA^* \boxtimes BB^* = (A \boxtimes B)(A^* \boxtimes B^*)$ ,
- (iii)  $Z_1^*(A \boxtimes B)W = 0$ .

*Proof.* It is clear that (i)  $\Rightarrow$  (ii). To prove (ii)  $\Rightarrow$  (iii), suppose (ii). By Theorem 8, (ii) holds only if

$$[Z_1^*(A \boxtimes B)W][W^*(A^* \boxtimes B^*)Z_1] = 0,$$

i.e.,  $Z_1^*(A \boxtimes B)W = 0$ .

(iii)  $\Rightarrow$  (i): Assume the condition (iii) holds. Note that by Theorem 6 we have

$$Z_1^*(A \boxtimes B)(I - Z_2Z_2^*) = Z_1^*(A \boxtimes B)WW^* = 0,$$

and hence  $Z_1^*(A \boxtimes B) = Z_1^*(A \boxtimes B)Z_2Z_2^*$ . For any  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ , we have by Lemmas 2 and 5 that

$$\begin{aligned} AC \boxtimes BD &= Z_1^*(AC \boxtimes BD)Z_1 \\ &= Z_1^*(A \boxtimes B)(C \boxtimes D)Z_1 \\ &= Z_1^*(A \boxtimes B)Z_2Z_2^*(C \boxtimes D)Z_1 \\ &= (A \boxtimes B)(C \boxtimes D). \end{aligned}$$

Thus we arrive at (i). □

**Theorem 12.** *Assume the notations in Theorem 8. For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , the following statements are equivalent:*

- (i)  $CA \boxtimes DB = (C \boxtimes D)(A \boxtimes B)$  for all  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ ,
- (ii)  $A^*A \boxtimes B^*B = (A^* \boxtimes B^*)(A \boxtimes B)$ ,

$$(iii) V^*(A \boxtimes B)Z_2 = 0.$$

*Proof.* Clearly, (i)  $\Rightarrow$  (ii). The assertion (ii)  $\Rightarrow$  (iii) follows from Theorem 8. Now, suppose that (iii) holds. Then  $VV^*(A \boxtimes B)Z_2 = 0$ . Using Theorem 6, we get

$$(I - Z_1Z_1^*)(A \boxtimes B)Z_1 = VV^*(A \boxtimes B)Z_1 = 0$$

which implies  $(A \boxtimes B)Z_1 = Z_1Z_1^*(A \boxtimes B)Z_1$ . For any  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ , we have by Lemmas 2 and 5 that

$$\begin{aligned} CA \boxplus DB &= Z_2^*(CA \boxtimes DB)Z_2 \\ &= Z_2^*(C \boxtimes D)(A \boxtimes B)Z_2 \\ &= Z_2^*(C \boxtimes D)Z_1Z_1^*(A \boxtimes B)Z_2 \\ &= (C \boxplus D)(A \boxplus B). \end{aligned} \quad \square$$

**Theorem 13.** Assume the hypothesis of Theorem 7. For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , the following conditions are equivalent:

- (i)  $AC \boxplus BD = (A \boxplus B)(C \boxplus D)$  for all  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ ,
- (ii)  $AA^* \boxplus BB^* = (A \boxplus B)(A^* \boxplus B^*)$ ,
- (iii)  $Z_1^*(A \boxtimes B)Q_2 = 0$ .

*Proof.* The proof is similar to that of Theorem 11. □

**Theorem 14.** Assume the hypothesis of Theorem 7. For any operator matrices  $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$  and  $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ , the following statements are equivalent:

- (i)  $CA \boxplus DB = (C \boxplus D)(A \boxplus B)$  for all  $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$  and  $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$ ,
- (ii)  $A^*A \boxplus B^*B = (A^* \boxplus B^*)(A \boxplus B)$ ,
- (iii)  $Q_1^*(A \boxtimes B)Z_2 = 0$ .

*Proof.* The proof is similar to that of Theorem 12. □

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## References

- [1] Z. A. Al Zhour, A. Kilicman, Extension and generalization in- equalities involving the Khatri-Rao product of several positive matrices, *J. Inequal. Appl.*, 2006, 21 pages (2006).
- [2] Z. A. Al Zhour, A. Kilicman, Matrix equalities and inequalities involving Khatri-Rao and Tracy-Singh sums, *J. Inequal. Appl.*, 7, 11-17 (2006).

## Khatri-Rao Products and Selection Operators

- [3] C. G. Khatri, C. R. Rao, Solutions to some functional equations and their applications to characterization of probability distributions, *Sankhyā*, 26, 167–180 (1968).
- [4] S. Liu, Matrix results on the Khatri-Rao and Tracy-Singh products, *Linear Algebra Appl.*, 289, 267–277 (1999).
- [5] S. Liu, G. Trenkler, Hadamard, Khatri-Rao, Kronecker and other matrix products, *International Journal of Information and Systems Sciences*, 4, 160–177 (2008).
- [6] A. Ploymukda, P. Chansangaim, W. Lewkeeratiyutkul, Algebraic and order properties of Tracy-Singh products for operator matrices, *J. Comput. Anal. Appl.*, 24, 656–664 (2018).
- [7] A. Ploymukda, P. Chansangaim, W. Lewkeeratiyutkul, Analytic properties of Tracy-Singh products for operator matrices, *J. Comput. Anal. Appl.*, 24, 665–674 (2018).
- [8] A. Ploymukda, P. Chansangaim, Khatri-Rao products for operator matrices acting on the direct sum of Hilbert spaces, *Journal of Mathematics*, 2016, Article ID 8301709, 7 pages, <http://dx.doi.org/10.1155/2016/8301709>, (2016).
- [9] D. S. Tracy, R. P. Singh, A new matrix product and its applications in partitioned matrix differentiation, *Stat. Neerl.*, 26, 143–157 (1972).
- [10] C. F. Van Loan, The ubiquitous Kronecker product, *Int. J. Comput. Appl. Math.*, 123, 85–100 (2000).
- [11] G. Visick, A quantitative version of the observation that the Hadamard product is a principal submatrix of the Kronecker product, *Linear Algebra Appl.*, 304, 45–68 (2000).
- [12] H. Zhang, F. Ding, On the Kronecker products and their applications, *J. Appl. Math.*, 2013, 8 pages (2013).

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แบบรายงานการใช้จ่ายเงินโครงการวิจัย

สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง

รายงานความก้าวหน้า ครั้งที่ 1 รอบ 12 เดือน ประจำปีงบประมาณ 2561

แหล่งงบประมาณแผ่นดิน (แบบปกติ)  แหล่งเงินรายได้

ชื่อโครงการ (ภาษาไทย) ผลคูณเทรซี-ซิงห์ ผลบวกเทรซี-ซิงห์ ผลคูณคาทรี-ราวและผลบวกคาทรี-ราวสำหรับตัวดำเนินการบนปริภูมิฮิลเบิร์ต

(ภาษาอังกฤษ) Tracy-Singh Products, Tracy-Singh Sums, Khatri-Rao Products, and Khatri-Rao Sums for Hilbert Space Operators

ชื่อ-สกุลหัวหน้าโครงการวิจัยผู้รับทุน/ผู้วิจัย ผศ.ดร. ภัทราวุธ จันท์เสียม

รายงานในช่วงตั้งแต่วันที่ 22 มกราคม 2561 ถึง 21 มกราคม 2562

ระยะเวลาดำเนินการ 1 ปี - เดือน 22 มกราคม 2561 ถึง 21 มกราคม 2562

ข้อมูลการรายงานค่าใช้จ่ายงบประมาณโครงการวิจัย

1. การเบิกจ่ายงบประมาณ (กรณีการจ่ายเงินถ้าจ่ายงวดเดียวให้ลบข้อที่ไม่เกี่ยวข้องออก)

งวดที่ 1 300,400 บาท % วันที่ได้รับอนุมัติให้เบิกจ่ายเงิน (ว/ด/ป) 2561/03/07

งวดที่ 2 56,600 บาท % วันที่ได้รับอนุมัติให้เบิกจ่ายเงิน (ว/ด/ป)

2. สรุปงบประมาณค่าใช้จ่ายที่ใช้บังคับตั้งแต่เริ่มทำการวิจัยถึงปัจจุบัน (จำแนกตามหมวดค่าใช้จ่าย)

รายการ	จำนวนเงินที่ใช้			
	ปีที่ 1	รวม	งบประมาณ ที่ตั้งไว้	คงเหลือ (หรือเกิน)
<b>1. ค่าตอบแทน</b>	<b>50,400</b>	<b>50,400</b>	<b>107,000</b>	<b>เหลือ 56,600</b>
1.1 ค่าตอบแทนนักวิจัย	-	-	56,600	เหลือ 56,600
1.2 ค่าจ้างผู้ช่วยวิจัย	50,400	50,400	50,400	-
<b>2. ค่าใช้สอย</b>	<b>40,000</b>	<b>40,000</b>	<b>40,000</b>	<b>-</b>
2.1 ค่าจ้างพิมพ์รายงานต่าง ๆ	5,500	5,500	5,000	เกิน 500
2.2 ค่าจ้างพิมพ์บทความวิจัยด้านคณิตศาสตร์ 5 บทความ โดยใช้ซอฟต์แวร์ Latex	10,000	10,000	10,000	-
2.3 ค่าเดินทาง / ค่าที่พัก / ค่าพาหนะ / ค่าน้ำมัน เชื้อเพลิง (สำหรับหัวหน้าโครงการและผู้ช่วยวิจัย)	18,000	18,000	18,000	-
2.4 ค่าบริการพิมพ์งานจากเครื่องพิมพ์และค่าสแกน เอกสาร	6,000	6,000	6,000	-
2.5 ค่าเช่าเล่มรายงานโครงการวิจัย	500	500	1,000	เหลือ 500
<b>3. ค่าวัสดุ</b>	<b>190,000</b>	<b>190,000</b>	<b>190,000</b>	<b>-</b>
3.1 ค่าวัสดุสำนักงาน (รวมค่ากระดาษ)	42,000	42,000	40,000	เกิน 2,000
3.2 ค่าวัสดุคอมพิวเตอร์ (หมึกพิมพ์, CD/DVD, Flash drive, อุปกรณ์อะไหล่ของคอมพิวเตอร์, แบตเตอรี่, adapter, mouse, external hard disk)	76,000	76,000	70,000	เกิน 6,000
3.3 ค่าเอกสารงานวิจัยทางด้านคณิตศาสตร์ที่เกี่ยวข้อง (ไม่มีในสำนักหอสมุดกลาง)	72,000	72,000	80,000	เหลือ 8,000
<b>4. ค่าสาธารณูปโภค</b>	<b>20,000</b>	<b>20,000</b>	<b>20,000</b>	<b>-</b>
4.1 ค่าไปรษณีย์ ทั้งภายในประเทศและต่างประเทศ	5,900	5,900	6,000	เหลือ 100
4.2 ค่าธรรมเนียมในการโอนเงิน (ในประเทศ/ ต่างประเทศ)	5,500	5,500	6,000	เหลือ 500
4.3 ค่าบริการโทรศัพท์และอินเทอร์เน็ต	8,600	8,600	8,000	เกิน 600

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ลงนามหัวหน้าโครงการวิจัยผู้รับทุน

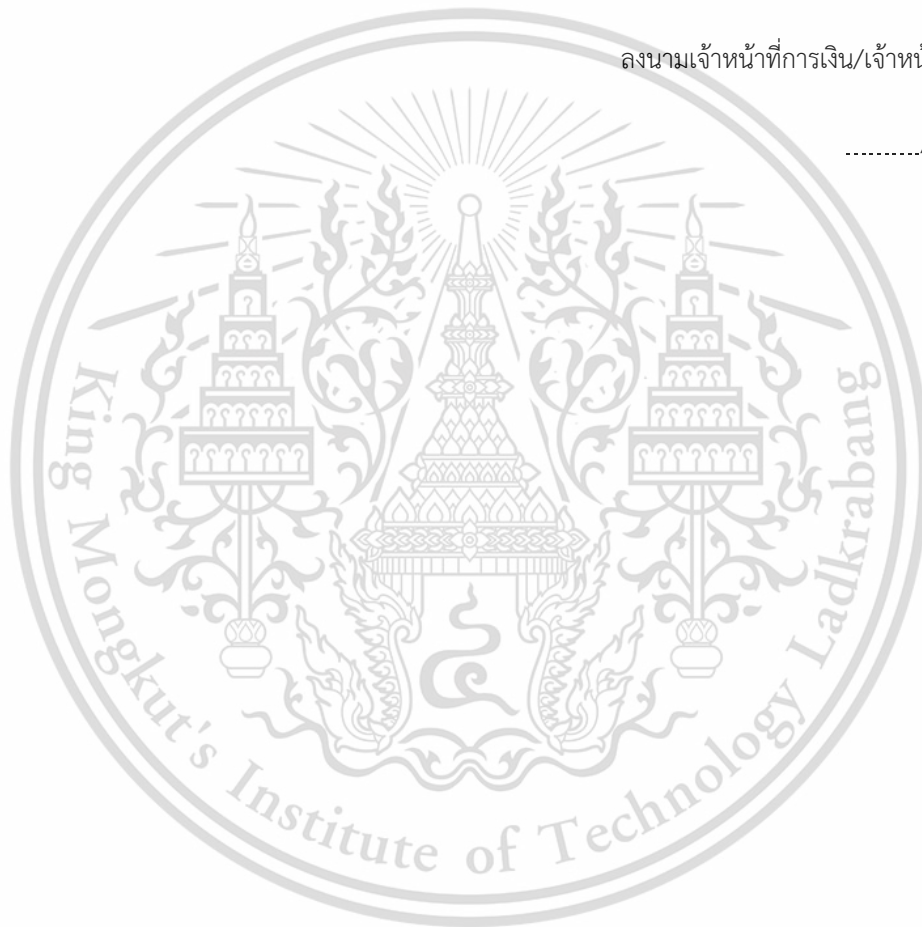
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ลงนามเจ้าหน้าที่การเงิน/เจ้าหน้าที่ที่เกี่ยวข้อง

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## Author Biography

### Personal biography

Name : Patrawut Chansangiam

Current Position :

- Assistant Professor, Department of Mathematics, Faculty of Science, KMITL
- Head of M.S. Curriculum in Applied Mathematics, Department of Mathematics, KMITL
- Committee of Ph.D. Curriculum in Applied Mathematics, Department of Mathematics, KMITL

### Educational biography

Academic Degree	Major	Institute	Graduated Year
<b>D.Sc.</b>	<b>Mathematics</b>	Chulalongkorn University	<b>2013</b>
<b>M.S.</b>	<b>Applied Mathematics</b>	<b>KMITL</b>	<b>2007</b>
<b>B.E.</b>	<b>Electronics Engineering</b>	<b>KMITL</b>	<b>2005</b>

### Research Areas:

- Functional Analysis
- Matrix Theory
- Inequalities
- Mathematical Aspects for Engineering

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**Awards**

Year	Award	Institute
1998	1st rank (team) in Secondary School Math Test	Mathematical Association of Thailand
2001	1st rank (individual) in High School Math Test	Sermpanya Co. Ltd.
2007	Excellent grades in Master of Science (Mathematics)	The Professor Dr. Tab Nilanidhi Foundation

**Academic and Research Scholarship**

Year	Academic and Research Scholarship	Institute
2014	<b>Axiomatic Means for Positive Operators and Applications</b>	<b>KMITL Research Fund</b>
2014	โครงการวิจัยจากงบประมาณเงินรายได้ประจำปี 2558	คณะวิทยาศาสตร์ สจล.
2009	The 72nd anniversary of His Majesty King Bhumibol Adulyadej	Graduate School at Chulalongkorn University
2012	The 90th anniversary of Chulalongkorn University Fund (Ratchadaphiseksomphot Endowment Fund)	Graduate School at Chulalongkorn University
2012	ทุนนำเสนอผลงานทางวิชาการในต่างประเทศ	Graduate School at Chulalongkorn University
2012	ทุนนำเสนอผลงานทางวิชาการในต่างประเทศ	Dept. of Mathematics and Computer Science at Chulalongkorn University

## Publications

1. Arnon Ploymukda and Patrawut Chansangiam, Khatri-Rao Products for Operator Matrices Acting on the Direct Sum of Hilbert Spaces, Journal of Mathematics, Volume 2016, Article ID 8301709, 7 pages
2. Patrawut Chansangiam, Adjointations of Operator Inequalities and Characterizations of Operator Monotonicity via Operator Means, Communications in Mathematics and Applications, Vol. 7, No. 2, pp. 93–103, 2016
3. Patrawut Chansangiam. Mathematical aspects for electrical network connections. KKU Engineering Journal. 2016 ;1: 21-28.
4. Patrawut Chansangiam. Certain integral inequalities involving tensor products, positive linear maps, and operator means. Journal of Inequalities and Applications. 2016 ;Article 121: 1-13.[ IF : 0.630 ]
5. Patrawut Chansangiam. Integral Inequalities of Kantorovich and Fiedler Types for Hadamard Products of Operators. Mathematical Inequalities and Applications. 2016 ;: .[ IF : 0.636 ]
6. Patrawut Chansangiam. Positivity, Betweenness, and Strictness of Operator Means. Abstract and Applied Analysis. 2015 ;ArticleID 5: 1-5.[ IF : 1.274 ]
7. Patrawut Chansangiam; Wicharn Lewkeeratiyutkul. Operator connections and Borel measures on the unit interval. ScienceAsia. 2015 ;4: 273-279.[ IF : 0.347 ]
8. Patrawut Chansangiam; Arnon Ploymukda. Significant Properties and Examples of Unitary Matrices. KKU Science Journal. 2015 ;1: 28-38.
9. Patrawut Chansangiam. Properties and Examples of Means for Positive Operators. SWU Science Journal. 2015 ;1: 205-218.
10. Patrawut Chansangiam; Arnon Ploymukda. Applications of Householder Transformations to Matrix Decompositions. Science and Technology Journal. 2015 ;4: 539-547.
11. Patrawut Chansangiam. Cancellability and regularity of operator connections with applications to nonlinear operator equations involving means. Journal of Inequalities and Applications. 2015 ;Article No. 411: 1-13.[ IF : 0.770 ]
12. Patrawut Chansangiam; Arnon Ploymukda. Significant Properties of Householder Transformation. วารสารวิจัยและพัฒนา มจร. 2015 ;3: 201-209.
13. Patrawut Chansangiam. A Survey on Operator Monotonicity, Operator Convexity, and Operator Means. International Journal of Analysis. 2015 ;: 1-8.

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14. Patrawut Chansangiam. Geometric Means for Positive Operators on a Hilbert Space. *KMITL Science Journal*. 2014 ;1: 100-109.
15. Patrawut Chansangiam; Wicharn Lewkeeratiyutkul. Characterizations of connections for positive operators. *Southeast Asean Bulletin of Mathematics*. 2013 ;; 645-657.
16. Patrawut Chansangiam. An Order-Preserving Linear Map from Matrices to Banach \*-Algebras and Applications. *Journal of Mathematics*. 2013 ;; 1-5
17. Patrawut Chansangiam; Wicharn Lewkeeratiyutkul. The normed ordered cone of operator connections. *Chamchuri Journal of Mathematics*. 2013 ;5: 1-11.
18. Patrawut Chansangiam. Bohr's inequality and its extensions in Banach -algebras. *International Mathematical Forum*. 2012 ;7: 369-383.
19. Patrawut Chansangiam. Bohr inequalities in  $C^*$ -algebras. *ScienceAsia*. 2010 ;36: 326-332. [ IF : 0.347 ]
20. Sowaros Toim and Patrawut Chansangiam. Semi-tensor Product of Matrices over a Commutative Semiring. *Proceedings of THE 5th KMITL-TKU International Joint Symposium on Mathematics and Applied Mathematics 2016*. 73-88.
21. Arnon Ploymukda and Patrawut Chansangiam, Tracy-Singh Products for Bounded Linear Operator on a Hilbert Space, *Proceedings of THE 5th KMITL-TKU International Joint Symposium on Mathematics and Applied Mathematics 2016*. 39-60.
22. P. Chansangiam, P. Hemchote and P. Pantaraphong, Generalizations of Bohr inequality for Hilbert space operators, *Journal of Mathematical Analysis and Applications*, 356, 525–536, 2009.
23. P. Chansangiam, P. Hemchote and P. Pantaraphong, Inequalities for Kronecker products and Hadamard products of positive definite matrices, *ScienceAsia*, 35, 106–110, 2009. [ IF : 0.347 ]
24. P. Pranayanuntana and P. Chansangiam, Inequalities of elementary symmetric polynomial for positive definite symmetric matrix, *Laos Journal on Applied Science*, 1 (2006): 471–480.
25. P. Pranayanuntana and P. Chansangiam, A proof of  $\sum_{j_1 \cdots j_{n-i}} a_{j_1} \cdots a_{j_{n-i}} \geq \sum_{j_1 \cdots j_{n-i}} \lambda_{j_1} \cdots \lambda_{j_{n-i}}$ ,  $0 \leq i \leq n-1$  where  $a_{jk} \in \{a_{11}, \dots, a_{nn}\}$  and  $a_{jk} \in \sigma(A)$  for  $A = [a_{ij}]_{n \times n} > 0$ , using mixed determinants, *Proceedings of the 10th WSEAS International Conference on Applied Mathematics*, (2006): 159–163.

### Oral Presentation

1. Inequalities of elementary symmetric polynomial for positive definite symmetric matrix, International Conference On Applied Science, Vientiane, Laos, 5-7 November 2006.
2. Inequalities involving squares of linear combination of Hermitian matrices, The 14th Conference of the International Linear Algebra Society, Shanghai, China, 16-20 July 2007.
3. New matrix inequalities for the Kronecker product of positive definite matrices, The 14th Conference of the International Linear Algebra Society, Shanghai, China, 16-20 July 2007.
4. Extensions and variations of Bohr inequality for operator algebras, Annual Pure and Applied Mathematics Conference 2009, Bangkok, Thailand, 25-26 May 2009.
5. Improvement and characterizations of operator connections and operator means, Annual Pure and Applied Mathematics Conference 2012, Bangkok, Thailand, 28-29 May 2012.
6. Characterizations of connections and means for positive operators, International Workshop on Operator Theory and Applications 2012, Sydney, Australia, 16-20 July 2012.
7. Axiomatic Characterizations of Operator Connections, The 8th Mathematics and Physical Science Graduate Congress 2012, Bangkok, Thailand, 8-10 December 2012.
8. P. Chansangiam and W. Lewkeeratiyutkul, Operator connections and Borel measures on the unit interval, Workshop on Operator Theory and Operator Algebras 2012, Lisbon, Portugal, 11-14 September 2012.
9. Integral representations and decompositions of operator connections, Annual Pure and Applied Mathematics Conference 2013, Bangkok, Thailand, 9-10 May 2013.
10. Characterizations of operator monotone functions via Borel measures, The 19th Annual Meeting in Mathematics (AMM 2014), Pattaya, Thailand, 20-22 March 2014.
11. Cancellability of operator connections and applications to certain operator equations, Annual Pure and Applied Mathematics Conference 2014 (APAM 2014), Bangkok, Thailand, 1-2 May 2014.
12. Characterizations of Strictness for Operator Connections, Annual Pure and Applied Mathematics Conference 2015 (APAM 2015), Nakorn Prathom, Thailand, 28-29 May 2015.

**Parts of Books**

1. P. Chansangiam, Operator Means and Applications, Linear Algebra–Theorems and Applications, 2012, ISBN: 978-953-51-0669-2, doi: 10.5772/3107, pp. 163–188.
2. P. Chansangiam, Inequalities in the Löwner Partial Order, 2011, LAP LAMBERT Academic Publishing, Saarbrücken, ISBN: 978-3-8443-0145-8

