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อสมการเชิงปริพันธ์อันดับเศษส่วนที่มีอิมพัลส์

Impulsive fractional integral inequalities

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บทคัดย่อ

งานวิจัยนี้ศึกษาอสมการเชิงปริพันธ์อันดับเศษส่วนที่มีอิมพัลส์ จากนั้นประยุกต์ผลลัพธ์ที่ได้กับปัญหาการควบคุมเหมาะสมที่สุดของสมการเชิงอนุพันธ์อันดับเศษส่วนที่มีอิมพัลส์ที่มีการหน่วงเชิงเวลา นอกจากนี้ในการวิจัยยังได้ยกตัวอย่างสมการความร้อนที่มีตัวควบคุมและใช้ผลจากทฤษฎียืนยันการมีผลเฉลยของปัญหาการควบคุมเหมาะสมที่สุดนี้อีกด้วย

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ABSTRACT

In this research, the impulsive fractional integral inequalities was studied. The results was applied to the optimize problem of the impulsive fractional differential equation with time delay. Furthermore, the optimize problem of the generalized heat equation with time delay was riased as an example that support our results.

Keywords : impulsive fractional integral inequalities, optimize problem, impulsive fractional differential equation, generalized heat equation

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Wichai Witayakiattilerd

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CHAPTER I

INTRODUCTION

The mathematical models of many real world problems can be described by impulsive differential equations. They have been studied quite extensively [4,8,9,14] because they have advantage over the traditional initial value problems. They can be used to model other phenomena that cannot be modeled by the traditional initial value, such as the dynamics of the systemic arterial pressure, the dynamics of populations subjected to abrupt changes (harvesting, diseases, etc.). Some phenomena in physics or in the other fields, some time they cannot be described or characterized by the differential equations of integer order but they can efficiently be described by the fractional order.

Among the previous research, little is concerned with integro-differential equations with fractional order and impulse.

In 2006, Chonwerayuth [2] proved the existence and uniqueness of a classical solution of an integro-differential equation;

$$\begin{cases} x^{\alpha}(t)+Ax(t)=f\left(t,x(t),x^{t}\right)+\int_{-r}^{t}h(t,s)g\left(s,Kx(s)\right)ds+Bu(t),\ t\in[0,T], \\ x(t)=\varphi(t)\ ,t\in[-r,0] \end{cases} \tag{1}$$

เมื่อ $x^{t}(\theta)=x(t+\theta),-r-t\le\theta\le 0$

In the same year 2006, Wei.W [14] has done a portion of work on the nonlinear impulsive integro-differential equation;

$$\begin{cases} x'(t)+Ax(t)=F(t,x(t),Gx(t),Sx(t)),\quad t\neq t_i\ ,\ t\in[0,T] \\ \triangle x(t_i)=J_i(x(t_i)\ \ ;\ i=1,2,\ldots,n \\ x(0)=x_0 \end{cases} \tag{2}$$

where G and S are nonlinear integral operators given by

$$Gx(t)=\int_0^t k(t,\tau)g(\tau,x(\tau))d\tau \ \text{ and } \ Sx(t)=\int_0^T h(t,\tau)s(\tau,x(\tau))d\tau\,.$$

In 2008, Gastao S.F.Frederico [3] has studied on the fractional optimal control in the sense of Caputo and the fractional Neether's theorem.

In 2009 , Gisele M.Mophou [4] proved existence and uniqueness of mild solution to impulsive fractional differential equations

$$\begin{cases} D_t^{\alpha} x(t) = Ax(t) + f(t, x(t)),\ t \neq t_i, t \in [0,T] \\ \Delta x(t_i) = J_i(x(t_i)),\ t = t_i, i = 1,2,\ldots,n; \\ x(0) = x_0 \end{cases} \tag{3}$$

where $0<\alpha<1$ and D_t^{α} denote the Caputo fractional derivative.

These researches motivate our research. In this research, we consider some implusive fractional integral inequalities on a Banach space X .

1.1 The purpose of the research project

1.1.1 To study the definitions and theories associated with the fractional integral inequalities.

1.1.2 To study and find the necessary and sufficient conditions for the inequality of fractional integrals.

1.1.3 To solve the integral inequality of fractions order.

1.1.4 Applied to the problem described by equations of fractional integro-differential with impulse.

1.2 The scope of the research project

1.2.1 Study definitions and theories associated with the fractional integral inequality.

1.2.2 Examined and the necessary and sufficient conditions for the inequality of fractional integrals.

1.2.3 Solve the fractional integral inequality and give somes examples to support the theory.

1.2.4 Apply the main results to the fractional differential equations.

CHAPTER II

PRELIMINARIES

In this chapter, we will state the definitions and theories that use in the research such as some definitions and theories on Banach space, fixed-point theory and fractional calculus theory etc.

2.1 Some definitions and theorems on Banach space

Functional analysis plays a central role in this reseach. In this section, Some of definitions and theories those are required in subsequent chapters, with appropriate references given wherever necessary.

Let X be a Banach space with norm $\|\cdot\|$. Let Y be another Banach space with norm $\|\cdot\|_Y$.A linear transformation from X into Y is bounded on a domain of T, $D(T)$ if there exist a constant c such that $\|Tx\|_Y \leq c\|x\|$ for all $x \in D(T)$.The linear space of all bounded linear operators from X into Y, denoted by $\mathcal{L}(X,Y)$ and denote $\mathcal{L}(X,X)$ by $\mathcal{L}(X)$.

Definition 2.1 A sequence x_n in X is said to be a strongly convergent to an element x in X if $\|x_n - x\| \to 0$ as $n \to \infty$. We denote by $x_n \to^s x$.

Theorem 2.2(Uniform Boundedness Principle). Let $\{T_\alpha \mid \alpha \in \Lambda\}$ be a family of operators from $\mathcal{L}(X,Y)$. If for each $x \in X$ there is a constant c_x such that $\sup\{\|T_\alpha \mid \alpha \in \Lambda\|\} \leq c_x$, then the operator $\{T_\alpha\}$ are uniformly bounded.

Let X be a Banach and X^* is its dual. Element of X^* can be used to generate a new topology for X called ***weak topology***. Note that the norm topology on X was called ***strong topology***. So the new topology is weaker than the strong (norm) topology. Particularly, the linear functional on X that are continues in the weak topology are precisely the functional in X^*. The concept of open (closed) sets, compactness, convergence, etc., are topological, hence they must be qualified by referring to the topology involved. In the case of norm linear spaces, when one speaks of open (closed) sets, compactness, convergence, etc., one refer to strong (norm) topology, while, with reference to its weak topology, they are called weakly open (weakly closed) sets, weak compactness, weak convergence, etc. Thus a sequence $\{x_n\}$ in X is said to converge weakly to an element x in X if, for every $x^* \in X^*$, $x^*(x_n) \to x^*(x)$, written by $x_n \to^w x$. Every weakly convergent sequence is bound. Every strongly convergent sequence is weakly convergent, but the converse is not true.

2.2 Extremal Set and Extremmal Points

Definition 2.3 A subset C of a real or complex vector space X is said to be a convex if, for $\alpha \in [0,1]$, $\alpha x_1 + (1-\alpha)x_2 \in C$ for every $x_1, x_2 \in X$.

Definition 2.4 Let K be a subset of a real or complex vector space X. A nonempty subset E of K is said to be an ***extremal subset*** of K if a proper convex combination $\alpha x_1 + (1-\alpha)x_2$, $0 < \alpha < 1$, of two point x_1, x_2 of K line in E only if both x_1 and x_2 in E.An extremal subset of K consisting of just one point is called an ***extremal point*** of K.

Definition 2.4 A real value function f defined on a topological vector space X is said to be ***convex*** (strictly) if for every $x_1, x_2 \in X$;

$$f(\alpha x_1 + (1-\alpha)x_2,) \le (<) \alpha f(x_1) + (1-\alpha) f(x_2), \text{ for all } 0 \le \alpha \le 1$$

This is a classical result from the theory of extremals.

Theorem 2.5 Let C be a weakly compact subset of a Banach space X and f a weakly lower semicontinuous function on C, that is, $f(x_0) \le \underline{\lim}_{n\to\infty} f(x_n)$ whenever $x_n \to x_0$. Then f attains its minimum on C. Furthermore, if C is also convex and f strictly convex, then it has a unique minimum in C.

2.3 Fixed-point Theorems

Fixed-point theorem on Banach spaces or contraction mapping is an advantage tool that is not difficult applying proving the existence and the uniqueness of equations.

Consider a function $\varphi : \Re \to \Re$ and suppose that we require solving the equation $\varphi(x) = 0$. This is equivalent to solving the equation $\psi(x) = x$ where $\psi(x) = \varphi(x) + x$ for all $x \in \Re$.

Thus x is a zero of φ if and only if x is a fixed point of φ, i.e. a point which is left unaltered after the application of ψ. More generally, many problems are equivalent to solving $Af = f$ where $A : D(A) \to R(A)$ is an operator (not necessarily linear), acting in some normed vector spaces , $D(A)$

and $R(A)$ are domain and range of A in X respectively i.e. we seek a fixed point $f \in D(A)$ of the operator A (for simplicity, we write Af rather than $A(f)$).

There are many fixed-point theorems, which guarantee existence and/or uniqueness of fixed points. We state here what is used in this thesis.

Definition 2.6 Let X be a normed vector space and let $A: D(A) \to R(A)$ be an operator (not necessarily linear). Then

1) A is a contraction if there exists a constant c with $0 \le c \le 1$ such that

$$\| Af_1 - Af_2 \| \le \| f_1 - f_2 \| \quad \text{for all} \quad f_1, f_2 \in D(A) \tag{1}$$

2) A is strictly contraction if there exists a constant c with $0 \le c < 1$ such that (1) holds.

Theorem 2.7 (The contraction mapping theorem; Banach fixed point theorem) Let X be a Banach space and let $A: X \to X$ be a strictly contraction. Then the equation $Af = f$ has a unique solution in X .i.e. A has a unique fixed point f.

The result of this theorem can easily generalize as follows:

Theorem 2.8 Let X_0 be a closed subset of the Banach space X and assume that the operator A map X_0 into itself and a strictly contraction on X_0. Then the equation $Af = f$ has a unique solution $f \in X_0$.

Theorem 2.9(Leray-Schauder theorem) Let T be a continuous and compact mapping of a Banach space X into itself such that the set $\{x \in X \mid x = \lambda Tx \text{ for some } 0 \le \lambda \le 1\}$ is bounded. Then T has a fixed point.

2.4 Semigroup of linear operators

Consider a dynamic system, the state of which is evolving with time according to some law. For example, we may be interested in the temperature distribution along a rod which is being heated at one end. Suppose the initial state of the system is x_0; in this case $x_0(z)$ would measure the initial temperature at the point z of the rod. At a subsequent time $t > 0$, the state of the system will be given by $x(z,t)$; this state would measure the temperature at the point x at time t. Since, for each $t > 0$, the state $x(z,t)$ is an element of a Banach space X .We shall use the symbol $u(t)$ to indicate such a state, i.e.

$x(t)(z) = x(z,t)$. The state $x(t)$ will be related to the original state x_0 by some transition operator $T(t)$ so that

$$x(t) = T(t)x_0, \quad \text{for all } t \geq 0$$

We shall thus obtain a family $\{T(t)\}_{t\geq 0}$ of such operators. It is natural to ask what properties this family should have.

Firstly, each operator $T(t)$ acts in a set of "state x_0", where the states can typically represented by functions. Hence the domain of $T(t)$ will be a subspaces of function.

Next, it is clear that $T(0)$ must be I, the identity operator on X since at $t = 0$ there is no transition. Further, for any $s,t \geq 0$ we should require that $T(s+t)x_0 = T(s)T(t)x_0$. Indeed, the left hand side describes the evolution over a time interval of length $s+t$. The right hand side effectively say that the system evolves from x_0 to $T(t)x_0$ in t units of time and then continues to evolve from $T(t)x_0$ to $T(s)[T(t)x_0]$ in a subsequent time interval of length s, from t to $s+t$. The net effect should be the same as going nonstop from 0 to $s+t$, without taking a snapshot at time t.

Thus, we led to the two conditions

$$T(0) = I, \; T(s)T(t) = T(s+t) \quad \text{for all } s,t \geq 0$$

Finally, it is natural to expect that if s is close to t, and then $T(s)x_0$ should be close to $T(t)x_0$ in some sense. This is concept to define a family of transition operator say "semigroup of operators". We are now ready to make the following formal definition. Throughout this section X be a Banach space.

Definition 2.10 A one-parameter family $\{T(t)\}_{t\geq 0}$ of bounded linear operators from X into X is a semigroup of bounded linear operators on X if

1) $T(0) = I$, (I is the identity operator on X);

2) $T(t+s) = T(t)T(s)$ for every $t,s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators $\{T(t)\}_{t_0}$ is uniformly continuous if

$$\lim_{t\to 0^+} \| T(t) - I \|_{L(X)} = 0.$$

The linear operator A defined by

$$D(A)=\{x\in X\mid \lim_{t\to 0^+}\frac{T(t)x-x}{t}\text{ exists}\}$$

and

$$Ax=\lim_{t\to 0^+}\frac{T(t)x-x}{t}=\frac{d^+T(t)x}{dt}\Big|_{t=0}\qquad\text{for all }\ x\in D(A)$$

is called the infinitesimal generator of semigroup $\{T(t)\}_{t\ge 0}$, $D(A)$ is the domain of A.

From definition 2.10, we have a semigroup $\{T(t)\}_{t\ge 0}$ with a unique infinitesimal generator. If $T(t)$ is uniformly continuous, its infinitesimal generator is a bounded operator. On the other hand, every bounded linear operator A is the infinitesimal generator of a uniformly continuous semigroup $\{T(t)\}_{t\ge 0}$ and this semigroup is unique.

Definition 2.11 A semigroup $\{T(t)\}_{t\ge 0}$ of a bounded linear operator on X is a strongly continuous semigroup of bounded linear operators if $\lim_{t\to 0^+}T(t)x=x$ for all $x\in X$.

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of a C_0- semigroup.

Theorem 2.12 Let $-A$ be an infinitesimal generator of the $C_0-semigroup$ $\{T(t)\}_{t\ge 0}$.Then

a) for all $x\in X$, $\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h}T(s)xds=T(t)x$;

b) for all $x\in X, \int_0^t T(s)xds\in D(A)$ and $A\int_0^t T(s)xds=x-T(t)x$;

c) for all $x\in D(A)$,$T(t)x\in D(A)$ and $\frac{d}{dt}T(t)x=-AT(t)x=-T(t)Ax$;

d) for all $x\in D(A)$, $T(s)x-T(t)x=\int_s^t T(\tau)Axd\tau=\int_s^t AT(\tau)xd\tau$.

Theorem 2.12 has some simple consequences that we now state.

Theorem 2.13 If $-A$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\ge 0}$ then $D(A)$ is dense in X and A is a closed linear operator.

2.5 Description of Fractional calculus

Definition 2.14 Let $f \in C(\Re,\Re)$ and $\alpha > 0$. Define the fractional derivative of function f by

$$D^{-\alpha} f(t) \equiv \frac{1}{\Gamma(\alpha)} \int_0^t f(s)(t-s)^{\alpha-1} ds$$

for positive, $D^{\alpha} f(t) = D^{n}(D^{\alpha-n} f(t))$ which $0 < n-\alpha < 1$, D^{n} denote the ordinary derivative of order integer n and $\Gamma(\cdot)$ denote the Gamma function.

Theorem 2.15 [Jumarie, G.] Given $0 < \alpha \leq 1$. Let f, g and u be the $\alpha - th$ differentiable functions. Then theses equalities hold;

1) $D^{\alpha}[f(t)g(t)] = f(t)D^{\alpha}g(t) + f(t)D^{\alpha}g(t)$,
2) $D^{\alpha} f(u(t)) = D_u^{\alpha} f(u)(\dfrac{du}{dt})^{\alpha}$.

CHAPTER III

IMPULSIVE FRACTIONAL INTEGRAL INEQUALITIES

In this chapter, we will discuss some impulsive fractional integral inequalities. Throughout this chapter, the family of all functions map from $\Re^+$ to $\Re$ such that their derivatives of order α exist on $\Re^+ - \{t_k\}$ and left continuous at t_k, $k \in \{1, 2, \ldots, n\}$ for $0 < t_k < t_{k+1}$ is represented by $PC^\alpha(\Re^+, \Re)$.

Theorem 3.1 Given $0 < \alpha \le 1$. Let $y \in PC^\alpha(\Re^+, \Re)$ which respect to,

$$y^\alpha(t) \le y(t)p(t) + q(t), t \ne t_k \tag{1}$$

$$y(t_k^+) \le a_k y(t_k), y(0) = a_0 \tag{2}$$

where a_k , $k = 1, 2, \ldots$ are non-negative constants and $p, q \in C(\Re^+, \Re)$. Then

$$y(t) \le \sum_{0 \le t_k \le t} \Big(\prod_{t_k \le t_j \le t} a_j e^{\int_{t_j}^{t_{j+1}} \phi_{\alpha-1}(t_{k+1}-s)p(s)ds} \Big) \cdot \Big(\int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1}-s)q(s)e^{-\int_{t_k}^{s} \phi_{\alpha-1}(s-r)p(r)dr} ds \Big) \tag{3}$$

where $\phi_\alpha(t) = \dfrac{t^\alpha}{\Gamma(\alpha+1)}$.

Proof. Let $t \in [0, t_1]$. Then, we get from inequality (1),

$$D_t^\alpha [y(t)e^{-\int_0^t \phi_{\alpha-1}(t-s)p(s)ds}] \le q(t)e^{-\int_0^t \phi_{\alpha-1}(t-s)p(s)ds}.$$

By integrating of order α from 0 to t,

\begin{align*}

$$\begin{aligned} y(t) &\le e^{\int_0^t \phi_{\alpha-1}(t-s)p(s)ds}[a_0 + \int_0^t \phi_{\alpha-1}(t-s)q(s)e^{-\int_0^s \phi_{\alpha-1}(s-r)p(r)dr} ds] \\ &= a_0 e^{\int_0^t \phi_{\alpha-1}(t-s)p(s)ds} + e^{\int_0^t \phi_{\alpha-1}(t-s)p(s)ds} \int_0^t \phi_{\alpha-1}(t-s)q(s)e^{-\int_0^s \phi_{\alpha-1}(s-r)p(r)dr} ds. \end{aligned}$$

On the interval $[t_1, t_2]$. By inequality (1), we get

$$D^\alpha [y(t)e^{-\int_{t_1}^t \phi_{\alpha-1}(t-s)p(s)ds}] \le q(t)e^{-\int_{t_1}^t \phi_{\alpha-1}(t-s)p(s)ds}$$

by integrating of order α from t_1 to t ,

$$y(t) \le y(t_1^+)e^{\int_{t_1}^t \phi_{\alpha-1}(t-s)p(s)ds} + e^{\int_{t_1}^t \phi_{\alpha-1}(t-s)p(s)ds} \int_{t_1}^t \phi_{\alpha-1}(t-s)q(s)e^{-\int_{t_1}^s \phi_{\alpha-1}(s-r)p(r)dr} ds$$

and from inequality (2), we get

$$y(t_1^+) \le a_1 y(t_1) \le a_0 a_1 e^{\int_0^{t_1} \phi_{\alpha-1}(t_1-s)p(s)ds} + a_1 e^{\int_0^{t_1} \phi_{\alpha-1}(t_1-s)p(s)ds} \int_0^{t_1} \phi_{\alpha-1}(t_1-s)q(s)e^{-\int_0^s \phi_{\alpha-1}(s-r)p(r)dr} ds.$$

Hence, we obtain for $t \in [t_1, t_2]$,

$$y(t)\le a_0a_1e^{\int_0^{t_1}\phi_{\alpha-1}(t_1-s)p(s)ds}e^{\int_{t_1}^{t}\phi_{\alpha-1}(t-s)p(s)ds}$$
$$+a_1e^{\int_0^{t_1}\phi_{\alpha-1}(t_1-s)p(s)ds}e^{\int_{t_1}^{t}\phi_{\alpha-1}(t-s)p(s)ds}\int_0^{t_1}\phi_{\alpha-1}(t_1-s)q(s)e^{-\int_0^{s}\phi_{\alpha-1}(s-r)p(r)dr}ds$$
$$+e^{\int_{t_1}^{t}\phi_{\alpha-1}(t-s)p(s)ds}\int_{t_1}^{t}\phi_{\alpha-1}(t-s)q(s)e^{-\int_{t_1}^{s}\phi_{\alpha-1}(s-r)p(r)dr}ds.$$

Assume that inequality (3) holds for $t\in[0,t_k]$ some integer $k>1$. Then for $t\in[t_k,t_{k+1}]$, the result of inequality (1) that

$$D^{\alpha}[y(t)e^{-\int_{t_k}^{t}\phi_{\alpha-1}(t-s)p(s)ds}]\le q(t)e^{-\int_{t_k}^{t}\phi_{\alpha-1}(t-s)p(s)ds}.$$

Therefore,

$$y(t)\le y(t_k^+)e^{\int_{t_k}^{t}\phi_{\alpha-1}(t-s)p(s)ds}+e^{\int_{t_k}^{t}\phi_{\alpha-1}(t-s)p(s)ds}\int_{t_k}^{t}\phi_{\alpha-1}(t-s)q(s)e^{-\int_{t_k}^{s}\phi_{\alpha-1}(s-r)p(r)dr}ds.$$

Using inequality (2), we obtain, for $t\in[t_k,t_{k+1}]$,

$$y(t)\le a_k y(t_k)e^{\int_{t_k}^{t}\phi_{\alpha-1}(t-s)p(s)ds}+e^{\int_{t_k}^{t}\phi_{\alpha-1}(t-s)p(s)ds}\int_{t_k}^{t}\phi_{\alpha-1}(t-s)q(s)e^{-\int_{t_k}^{s}\phi_{\alpha-1}(s-r)p(r)dr}ds.$$

By the induction hypothesis, this can reduced to

$$y(t)\le\sum_{0\le t_k\le t}\Big(\prod_{t_k\le t_j\le t}a_je^{\int_{t_j}^{t_{j+1}}\phi_{\alpha-1}(t_{k+1}-s)p(s)ds}\Big)\Big(\int_{t_k}^{t_{k+1}}\phi_{\alpha-1}(t_{k+1}-s)q(s)e^{-\int_{t_k}^{s}\phi_{\alpha-1}(s-r)p(r)dr}ds\Big)$$

which on simplification give the estimate (1), for $t\in[0,t_{k+1}]$. ■

Theorem 3.2 Let $y\in PC^{\alpha}(\Re^+,\Re)$, $0<\alpha<1$ which satisfies,

$$y(t)\le a+\sum_{0\le t_k\le t}\int_{t_k}^{t_{k+1}}\phi_{\alpha-1}(t_{k+1}-s)p(s)y(s)ds+\sum_{0<t_k<t}c_ky(t_k) \tag{4}$$

where a and $c_k\ge 0$ are constants and $\phi_{\alpha}(t)=\dfrac{t^{\alpha}}{\Gamma(\alpha+1)}$. Then,

$$y(t)\le\prod_{0\le t_k\le t}(1+c_k)e^{\int_{t_k}^{t_{k+1}}\phi_{\alpha-1}(t_{k+1}-s)p(s)m(s)ds}, \tag{5}$$

where $c_0=a-1$.

Proof. Setting the right hand side equal to $\mu(t)$. Then we have,

$$\begin{cases}\mu^{\alpha}(t)=p(t)y(t);\ \ t\ne t_k\\ \mu(t_k^+)=\mu(t_k)+c_ky(t_k),\ \mu(0)=a.\end{cases}$$

Since $y(t)\le\mu(t)$,

$$\begin{cases}\mu^{\alpha}(t)=p(t)\mu(t);\ t\ne t_k\\ \mu(t_k^+)=(1+c_k)\mu(t_k),\ \mu(0)=a=c_0+1.\end{cases}$$

Applying theorem 3.1, we obtain

$$y(t) \le \prod_{0\le t_k \le t} (1+c_k) e^{\int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1}-s)p(s)ds}. \qquad \blacksquare$$

Theorem 3.3 If $x \in PC^{\alpha}([-r,T],X)$, $0<\alpha<1$, such that

$$\begin{cases} \| x(t) \| \le a + \sum_{0\le t_k \le t} \int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1}-s) b(s) \| x(s) \| ds + \sum_{0\le t_k \le t} \int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1}-s) c(s) \| x_s \|_B \, ds \\ \qquad + \sum_{0<t_k<t} d_k \| x(x_k) \|, \ t \in \dot{I} \\ x(t) = \varphi(t), \ t \in [-r,0] \end{cases}.$$

Then $\| x(t) \| \le (a + \| \varphi \|_C) \prod_{0<t_k \le t} (1+d_k) e^{\frac{b(t_{k+1}-t_k)^{\alpha}}{\alpha\Gamma(\alpha)}}$, for all $t \in I$,

where $\phi_{\alpha}(t) = \dfrac{t^{\alpha}}{\Gamma(\alpha+1)}$ and $b = \sup_{s\in I}[b(s)+c(s)]$.

Proof. Note that $\| x(t) \| \le \| x_t \|_B$ for all $t \in I$. So

$$\| x(t) \| \le a + b \sum_{0\le t_k \le t} \int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1}-s) \| x_s \|_B \, ds.$$

Setting $g(t) = \sum_{0\le t_k \le t} \int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1}-s) \| x_s \|_B \, ds$, for all $t \in I$.

Then $g(t)$ is monotonous increasing. Indeed, for $t_k < \tau < t \le t_{k+1}$, $k = 1,2,\ldots$, we have

$$\begin{aligned} g(t) - g(\tau) &= \int_{t_k}^{t} \phi_{\alpha-1}(t-s) \| x_s \|_B \, ds - \int_{t_k}^{\tau} \phi_{\alpha-1}(\tau-s) \| x_s \|_B \, ds \\ &= \int_0^{t-t_k} u^{\alpha-1} \| x_{t-u} \|_B \, du - \int_0^{\tau-t_k} u^{\alpha-1} \| x_{\tau-u} \|_B \, du \\ &= \int_0^{\tau-t_k} u^{\alpha-1} [\| x_{t-u} \|_B - \| x_{\tau-u} \|_B] du + \int_0^{t-t_k} u^{\alpha-1} \| x_{t-u} \|_B \, du. \end{aligned}$$

Since $\| x_t \|_B$ is monotonous increasing, $\| x_{t-u} \|_B - \| x_{\tau-u} \|_B > 0$. Hence $g(t) > g(\tau)$. We know that

$$\| x_t \|_B \le \| \varphi \|_C + \sup_{t\in[0,T]} \| x(t) \| \le (a + \| \varphi \|_C) + b \sum_{0\le t_k \le t} \int_{t_k}^{t_{k+1}} \phi_{\alpha}(t_{k+1}-s) \| x_s \|_B \, ds.$$

Therefore by using theorem 3.2,

$$\| x(t) \| \le \| x_t \|_B \le (a + \| \varphi \|_C) \prod_{0<t_k \le t} (1+d_k) e^{\frac{b(t_{k+1}-t_k)^{\alpha}}{\alpha\Gamma(\alpha)}}, \text{ for all } t \in I. \qquad \blacksquare$$

CHAPTER IV

IMPULSIVE FRACTIONAL INTEGRAL INEQUALITIES

For this chapter, we discuss the nonlinear-retarded functional impulsive fractional differential equations

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), x(t-l), \dots, x(t-ml)) + \mathcal{B}(t)u(t), \ t \in I \setminus \mathcal{D} \\ \Delta x(t_i) = J_i(x(t_i)), \ t_i \in \mathcal{D} \\ x(0) = x_0, \quad x(t) = \varphi(t), \ t \in [-ml, 0) \end{cases} \quad (1)$$

for some positive integer m and positive real l where $I = [0,T]$, $\mathcal{D} = \{t_1, t_2, \dots, t_n\}$, A is a generator of a semigroup $\{S(t)\}_{t\geq 0}$ on Banach space (B-space) $\mathcal{X}$ with some conditions, $\Delta x(t_i) = x(t_i^+) - x(t_i)$ denote the incitement of state x at time t_i with the magnitude of incitement J_i, $i = 1, 2, \dots, n$. Then, we study a minimizing regulation problem of the system with the objective functional $\mathcal{P}$, that is, to search u_0 an element of a family of admissible regulators $\mathcal{A}_{ad}$ which

$$\mathcal{P}(u_0) \leq \mathcal{P}(u), \ \text{ for all } \ u \in \mathcal{A}_{ad}. \quad (2)$$

In this case, define the objective functional by $\mathcal{P}(u) = \int_0^T r(t, x(t), x_t, u(t))dt + g(x(T))$ for all $u \in \mathcal{A}_{ad}$ with a given running function r and a given terminal function g and x is a solution of the system consistent with a regulator $u \in \mathcal{A}_{ad}$.

The extent of this chapter starts with the generalized of the Ascoli-Arzela Theorem (AAT). The investigation of existence of solution for the system is then presented in Subsection 4.1. Furthermore, the optimization regulation problem is solved and written in section 4.2. In the last section (Section 4.3), we solve the minimization regulation problem of nonlinear fractional heat by using our main result.

4.1 Existence of solution

Theorem 4.1 (The generalized of the Ascoli-Arzela Theorem) Let $\varphi \in C([-r,0], \mathcal{X})$. Suppose that $\mathcal{G} \subseteq \{x \in PC([-r,T], \mathcal{X}) \mid x(t) = \varphi(t) \text{ for } t \in [-r,0]\}$. If these conditions hold;

1) $\mathcal{G}$ is a uniformly bounded subset of $PC(I, \mathcal{X})$

2) $\mathcal{G}$ is equicontinuous in $(0,t_1)$, (t_1,t_2) ,…, (t_n,T)

3) Its $t-$section, $\mathcal{G}(t) \equiv \{x(t) \mid x \in \mathcal{G}, t \in [-r,T] \setminus \{0, t_1, \dots, t_n, T\}\}$, $\mathcal{G}(t^+) \equiv \{x(t^+) \mid x \in \mathcal{G}\}$ and $\mathcal{G}(t^-) \equiv \{x(t^-) \mid x \in \mathcal{G}\}$ are relatively compact (RCP) subsets of $\mathcal{X}$.}

Then $\mathcal{G}$ is a RCP subset of $PC([-r,T],\mathcal{X})$.}

Proof. Let $\{x_m\}$ be any sequence of $\mathcal{G}$. We have $\{x_m|_{[0,t_1]}\} \subset C([0,t_1],\mathcal{X})$. Applying the AAT on interval $[0,t_1]$, there is a subsequence of $\{x_m\}$, labeled by $\{x_m\}$ again, s.t.

$$x_m|_{[0,t_1]} \to x^1 \quad \text{in } C([0,t_1],\mathcal{X}) \text{ as } m\to\infty.$$

See $\{x_m|_{[t_1,t_2]}\} \subset C([t_1,t_2],\mathcal{X})$ and let $x_m(t_1) = x_m(t_1^+)$.

By using the AAT on interval $[t_1,t_2]$, $\{x_m|_{[t_1,t_2]}\}$ is a RCP subset of $C([t_1,t_2],\mathcal{X})$.

So, there is a subsequence s.t.

$$x_m|_{[t_1,t_2]} \to x^2 \quad \text{in } C([t_1,t_2],\mathcal{X}) \text{ as } m\to\infty .$$

Continue this process until time interval $[t_m,T]$. Then there exists a subsequence $\{x_m\}$, s.t.

$$x_m|_{[t_n,T]} \to x^{n+1} \text{ in } C([t_n,T],\mathcal{X}) \text{ as } m\to\infty.$$

Define $x(t) = x^i(t)$, $t\in[t_{i-1},t_i]$ for each $i\in\{1,\ldots,n+1\}$. We obtain $x\in PC([-r,T],\mathcal{X})$ and

$$x_m|_{[t_n,T]} \to x \text{ in } PC([-r,T],\mathcal{X}) \text{ as } m\to\infty.$$

Therefore $\mathcal{G}$ is RCP. ■

Let us consider system (1) with assumptions $A1-A5$;

A1) $f: I\times\mathcal{X}^{m+1}\to\mathcal{X}$ is uniformly continuous in t and locally Lipchitz in $x_1,\ldots,x_{m+1}$, that is, for any $\delta>0$, there is some positive $a(\delta)$ s.t.

$$\| f(t,x_1,\ldots,x_{m+1}) - f(s,y_1,\ldots,y_{m+1})\| \le a(\delta)[\| x_1-y_1\| + \ldots + \| x_{m+1}-y_{m+1}\| + |t-s|]$$

provided $\| x_1-y_1\|,\ldots,\| x_{m+1}-y_{m+1}\|\le\delta$ and for all s, $t\in I$.

A2) there is some constant c s.t.

$$f(t,x_1,\ldots,x_{m+1}) \le c(1+\| x_1\| + \ldots + \| x_{m+1}\|) \tag{3}$$

for all $(x_1,\ldots,x_{m+1})\in\mathcal{X}^{m+1}$

A3) Y is another separable reflexive B-space that regulator u take the value, that is, $\mathcal{B}(s)\in\mathcal{L}(\mathcal{L}_q(I,Y),\mathcal{L}_p(I,\mathcal{X}))$ for all $s\in I$.

A4) $J_i:\mathcal{X}\to\mathcal{X}$ is an operator s.t. $J_i(\mathcal{X})$ is a bounded subset of $\mathcal{X}$ which there are $e_i>0$, $i=1,2,\ldots,n$ s.t.

$$\| J_i(x_1(t)) - J_i(x_2)(t)\| \le e_i \| x_1(t)-x_2(t)\|, \tag{4}$$

for all x_1, $x_2\in\mathcal{X}$ and $t\in I$.

A5) A is the infinitesimal generator of a compact semigroup $\{S(t)\}_{t\geq 0}$ such there exist $M \geq 1$, $\omega > 0$ that $\| S(t) \| \leq Me^{\omega t}$ for all $t \geq 0$.

Definition 4.2 For any $u \in \mathcal{A}_{ad}$, a function $x \in PC([-ml,T],\mathcal{X})$ is called a $PC-$ mild solution corresponding to a regulator u if it satisfies,

$$\begin{cases} x(t) = S(\phi(t))\varphi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{0\leq t_k \leq t} \int_{t_k}^{t_k+1} (t_{k+1}-s)^{\alpha-1} S(\phi(t)-\phi(s))[f(s,x(s),x(s-l),\ldots,x(s-ml)) \\ \qquad +\mathcal{B}(s)u(s)]ds + \displaystyle\sum_{0<t_k<t} S(\phi(t)-\phi(t_k))J_k x(t_k),\ t \in I \\ x(t) = \varphi(t),\ t \in [-ml,0] \end{cases} \tag{5}$$

With assumptions $A1-A5$, the existence of solution of system (1) is proved by using the LSFPT and the compactness of semigroup $\{S(t)\}_{t\geq 0}$. From the definition 6, define the operator F by

$$\begin{cases} Fx(t) = S(\phi(t))\varphi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{0\leq t_k \leq t} \int_{t_k}^{t_k+1} (t_{k+1}-s)^{\alpha-1} S(\phi(t)-\phi(s))[f(s,x(s),x(s-l),\ldots,x(s-ml)) \\ \qquad +\mathcal{B}(s)u(s)]ds + \displaystyle\sum_{0<t_k<t} S(\phi(t)-\phi(t_k))J_k x(t_k),\ t \in I \\ Fx(t) = \varphi(t),\ t \in [-ml,0] \end{cases} \tag{6}$$

for all $x \in PC([-ml,T],\mathcal{X})$ .Then F is well defined.

Let $x \in PC([-ml,T],\mathcal{X})$. By using the continuity of $\| x(t) \|$ and $\| x(t-ml) \|$ and assumption $A2$, there is some positive constant N s.t.

$$\| f(s,x(s),x(s-l),\ldots,x(s-kl)) \| \leq N \quad \text{for all } s \in I. \tag{7}$$

Therefore,

$$\begin{aligned} \| Fx(t) \| &\leq Me^{\omega\phi(t)} \| \varphi \|_C + \frac{Me^{\omega\phi(t)}}{\Gamma(\alpha)} \sum_{0\leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} [\| f(s,x(s),x(s-l),\ldots,x(s-ml)) \| + \| \mathcal{B}(s)u(s) \|]ds \\ &\quad + Me^{\omega\phi(t)} \sum_{0<t_k<t} \| J_k x(t_k) \| \\ &\leq Me^{\omega\phi(t)} \| \varphi \|_C + \frac{MNe^{\omega\phi(t)}}{\Gamma(\alpha)} \sum_{0\leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} ds \\ &\quad + \frac{Me^{\omega\phi(t)}}{\Gamma(\alpha)} \sum_{0\leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} \| \mathcal{B}(s)u(s) \| ds + Me^{\omega\phi(t)} \sum_{0<t_k<t} e_k \| x(t_k) \| \end{aligned}$$

$$\leq Me^{\omega\phi(t)}\|\varphi\|_C+\frac{MN(n+1)e^{\omega\phi(t)}T^\alpha}{\alpha\Gamma(\alpha)}+\frac{Me^{\omega\phi(t)}K}{\Gamma(\alpha)}\sum_{k=0}^{n}[\int_{t_k}^{t_{k+1}}(t_{k+1}-s)^{\frac{p(\alpha-1)}{p-1}}ds]^{\frac{p-1}{p}}[\int_{t_k}^{t_{k+1}}\|\mathcal{B}(s)u(s)\|^p ds]^{\frac{1}{p}}$$
$$+MNe^{\omega\phi(t)}\sum_{0<t_k<t}e_k$$

$$\leq Me^{\omega\phi(t)}\|\varphi\|_C+\frac{MNe^{\omega\phi(t)}T^\alpha}{\alpha\Gamma(\alpha)}+\frac{Me^{\omega\phi(t)}(p-1)(n+1)T^{\frac{p\alpha-1}{p-1}}\|\mathcal{B}(\cdot)u\|_{L_p(I,X)}}{(p\alpha-1)\Gamma(\alpha)}+MNe^{\omega\phi(t)}\sum_{k=1}^{n}e_k<\infty.$$

Therefore, the operator F is bounded. ■

Lemma 4.3 Assume that assumption $A1-A5$ holds. Then the bounded operator F is continuous.

Proof Suppose that x_n is a sequence in $PC([-ml,T],\mathcal{X})$ converging to x in $PC([-ml,T],\mathcal{X})$. Then there is $N_0>0$ and for every $n>N_0$, $\|x_n-x\|_{PC}\leq 1$. Then $\|x_n\|\leq 1+\|x\|\equiv\rho$. By using $A1$, for $s\in(0,T)$ there is $b(\rho)>0$ s.t.

$$\|f(s,x_n(s),x_n(s-l),\ldots,x_n(s-ml))-f(s,x(s),x(s-l),\ldots,x(s-ml))\|\leq b(\rho)\|x_n-x\|_{PC}$$

Therefore, we have

$$\begin{aligned}|Fx_n(t)-Fx(t)|\leq&\frac{1}{\Gamma(\alpha)}\sum_{0\leq t_k\leq t}\int_{t_k}^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}\|S(\phi(t)-\phi(s))\|\\&\cdot\|f(s,x_n(s),x_n(s-l),\ldots,x_n(s-ml))-f(s,x(s),x(s-l),\ldots,x(s-ml))\|ds\\&+\sum_{0<t_k<t}\|S(\phi(t)-\phi(t_k))\|\|J_kx_n(t_k)-J_kx(t_k)\|\\\leq&\frac{Me^{\omega T}}{\Gamma(\alpha)}b(\rho)\|x_n-x\|_{PC}\sum_{0\leq t_k\leq t}\int_{t_k}^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}ds+Me^{\omega T}\sum_{0<t_k<t}e_k\|x_n(t_k)-x(t_k)\|\\\leq&\frac{Me^{\omega T}}{\alpha\Gamma(\alpha)}b(\rho)\|x_n-x\|_{PC}(n+1)T^\alpha+\sum_{k=0}^{k=n}e_k\|x_n-x\|_{PC}.\end{aligned}$$

Since $\|x_n-x\|_{PC}\to 0$ as $n\to+\infty$, so $\|Fx_n-Fx\|\to 0$ as $n\to+\infty$. This implies that F is continuous on $PC([-ml,T],\mathcal{X})$. ■

Corollary 4.4 Assume that assumption $A1-A5$ holds. The operator F maps bounded set into itself.

Proof The proof is followed from the proof of Lemma 4.3. For each $r>0$, there is some $\delta>0$ s.t. for every $x\in B_\delta\equiv\{x\in PC([-r,T],\mathcal{X})\mid\|x\|_{PC}\leq\delta\}$, we have $\|Fx\|_{PC}\leq\delta$.

Lemma 4.5 Suppose assumptions $A1-A5$ holds. Then F is operator compact.

Proof Given a bounded subset B of $PC([-ml,T],\mathcal{X})$. By Corollary 8, $F(B)$ is bounded. Define

$$Q=F(B)\ \text{ and }\ Q(t)=\{Fx(t)\mid x\in B\}. \tag{8}$$

Clearly, for $t \in [-ml, 0]$, $Q(t) = \{\varphi(t)\}$ is compact.

We only necessary consider for $t > 0$. Given $\varepsilon > 0$. For $0 < \varepsilon \le t \le T$, for short we denote $f(s, x(s), x(s-l), \ldots, x(s-ml)) + \mathcal{B}(s)u(s))$ by $\widetilde{f}_u(s, \underline{x}(s))$ where $x = (x_1, \ldots, x_{m+1})$ which $x_1(s) = x(s-l)$,. . ., $x_m(s) = x(s-ml)$ and $x_{m+1}(s) = x(s)$. Define

$$Q_\varepsilon(t) \equiv F_\varepsilon(B)(t)$$

$$= S(\phi(\varepsilon))S(\phi(t)-\phi(\varepsilon))\varphi_0 + \frac{S(\phi(\varepsilon))}{\Gamma(\alpha)} \sum_{0 \le t_k < t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} S(\phi(t_{k+1}) - \phi(\varepsilon) - \phi(s)) \widetilde{f}_u(s, \underline{x}(s))ds$$

$$+ \frac{S(\phi(\varepsilon))}{\Gamma(\alpha)} \int_{t_k}^{t-\varepsilon} (t-s)^{\alpha-1} S(\phi(t) - \phi(\varepsilon) - \phi(s)) \widetilde{f}_u(s, \underline{x}(s))ds + S(\phi(\varepsilon)) \sum_{0 < t_k < t} S(\phi(t_k) - \phi(\varepsilon) - \phi(s)) J_k(x(t_k)) \tag{9}$$

By continuity of $\phi(t)$ and compactness of $S(t)$, set $\{Q_\varepsilon(t) \mid x \in B\}$ is RCP in $\mathcal{X}$ for every ε sufficiently small, $t \in [\varepsilon, T]$. For $t \in (0, t_1]$ the equation (9) reduce to

$$Q_\varepsilon(t) = F_\varepsilon(B)(t) = S(\phi(\varepsilon))S(\phi(t)-\phi(\varepsilon))\varphi_0 + \frac{S(\phi(\varepsilon))}{\Gamma(\alpha)} \int_0^{t-\varepsilon} (t-s)^{\alpha-1} S(\phi(t) - \phi(\varepsilon) - \phi(s)) \widetilde{f}_u(s, \underline{x}(s))ds$$

Furthermore, the continuity of $\| x(t) \|$ and $\| x(t-ml) \|$ on $(0, t_1)$ for all $m = 1, \ldots, k$ imply that there is a positive N s.t.

$$\| x(t) \|, \| x(t-ml) \| \le N. \tag{10}$$

By assumption $A2$ and assumption $A4$, there exist a positive constant c such that

$$\| \widetilde{f}_u(s, \underline{x}(s)) \| \le c(1 + (m+1) \| x \|_{PC}) + \| \mathcal{B}(\cdot)u \|_{L_p(I, \mathcal{X})} \equiv L_u. \tag{11}$$

Then for $t \in [\varepsilon, t_1]$,

$$\begin{aligned}
\sup_{x \in B} \| Fx(t) - F_\varepsilon x(t) \| &= \frac{1}{\Gamma(\alpha)} \sup_{x \in B} \| \int_0^t (t-s)^{\alpha-1} S(\phi(t) - \phi(s)) \widetilde{f}_u(s, \underline{x}(s))ds \\
&\quad - S(\phi(\varepsilon)) \int_0^{t-\varepsilon} (t-s)^{\alpha-1} S(\phi(t) - \phi(\varepsilon) - \phi(s)) \widetilde{f}_u(s, \underline{x}(s))ds \| \\
&= \frac{1}{\Gamma(\alpha)} \sup_{x \in B} \| \int_{t-\varepsilon}^t (t-s)^{\alpha-1} S(\phi(t) - \phi(s)) \widetilde{f}_u(s, \underline{x}(s))ds \| \\
&\le \frac{ML_u}{\Gamma(\alpha)} \int_{t-\varepsilon}^t (t-s)^{\alpha-1} ds = \frac{ML_u \varepsilon^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Hence, there exist RCP sets arbitrary close to the set $Q(t)$ for $t \in [0, t_1]$. Therefore, $Q(t)$ itself is RCP in $\mathcal{X}$ on the interval $[0, t_1]$. On the interval $(t_1, t_2]$, we define

$$Q(t_1^+) \equiv Q(t_1^-) + J_1(Q(t_1^-)) = Q(t_1) + J_1(Q(t_1)). \tag{12}$$

By the condition $A3$, we get $J_1(Q(t_1))$ is RCP and this implies $Q(t_1^+)$ is also RCP.

Let $x(t_1^+) = x_1$. Then for $t \in (t_1, t_2]$, the equation (9) reduce to

$$Q_\varepsilon(t) = F_\varepsilon(B)(t)$$
$$= S(\phi(\varepsilon))S(\phi(t)-\phi(t_1)-\phi(\varepsilon))x_1 + \frac{S(\phi(\varepsilon))}{\Gamma(\alpha)}\int_0^{t_{k+1}}(t_1-s)^{\alpha-1}S(\phi(t_1)-\phi(\varepsilon)-\phi(s))\widetilde{f}_u(s,\underline{x}(s))ds$$
$$+\frac{S(\phi(\varepsilon))}{\Gamma(\alpha)}\int_{t_1}^{t-\varepsilon}(t-s)^{\alpha-1}S(\phi(t)-\phi(\varepsilon)-\phi(s))\widetilde{f}_u(s,\underline{x}(s))ds + S(\phi(\varepsilon))S(\phi(t_1)-\phi(\varepsilon)-\phi(s))J_1(x(t_1))$$

Furthermore, for $t\in[t_1+\varepsilon, t_2]$,

$$\sup_{x\in B}\{\| Fx(t)-Fx(t)\|\} \le \frac{ML_u\varepsilon^\alpha}{\Gamma(\alpha+1)}. \tag{13}$$

Therefore $Q(t)$ is RCP on $[t_1,t_2]$.

Generally, given any $t_k\in\mathcal{D}=\{t_0=0,t_1,t_2,\ldots,t_n,t_{n+1}=T\}$, define $x(t_k^+)=x_k$ and

$$Q(t_k^+)\equiv Q(t_k^+)+J_k(Q(t_k^-)) = Q(t_k)+J_k(Q(t_k)) \text{ for } t_k\in\mathcal{D}. \tag{14}$$

Similarly, for $t\in(t_k,t_{k+1}]$ the equation (9) reduce to

$$Q_\varepsilon(t) = F_\varepsilon(B)(t)$$
$$= S(\phi(\varepsilon))S(\phi(t)-\phi(t_k)-\phi(\varepsilon))x_k + \frac{S(\phi(\varepsilon))}{\Gamma(\alpha)}\sum_{0\le t_k<t}\int_{t_k}^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}S(\phi(t_{k+1})-\phi(\varepsilon)-\phi(s))\widetilde{f}_u(s,\underline{x}(s))ds$$
$$+\frac{S(\phi(\varepsilon))}{\Gamma(\alpha)}\int_{t_k}^{t-\varepsilon}(t-s)^{\alpha-1}S(\phi(t)-\phi(\varepsilon)-\phi(s))\widetilde{f}_u(s,\underline{x}(s))ds + S(\phi(\varepsilon))\sum_{0<t_k<t}S(\phi(t_k)-\phi(\varepsilon)-\phi(s))J_k(x(t_k))$$

Furthermore, for $[t_k+\varepsilon, t_{k+1}]$,

$$\sup_{x\in B}\{\| Fx(t)-Fx(t)\|\} \le \frac{ML_u\varepsilon^\alpha}{\Gamma(\alpha+1)}. \tag{15}$$

By repeating these process till the time interval which expanded, $Q(t)$ is RCP for $t\in[-ml,T]\setminus\mathcal{D}$ and $Q(t_k^+)$ is RCP for $t_k\in\mathcal{D}$. Next, we will show that Q is equicontinuous on (t_k,t_{k+1}), $k=0,1,\ldots,n$. Since B is bounded and follow from the inequality (11), there is an $L_u>0$ such that

$$\|\widetilde{f}_u(s,\underline{x}(s))\|\le L_u. \tag{16}$$

Let $h>0$ and for $0<t<t+h<t_1$ and for $x\in B$. Then

$$\| Fx(t+h)-Fx(t)\|$$
$$\le\| S(\phi(t+h))\varphi(0)-S(\phi(t))\varphi(0)\| + \|\frac{1}{\Gamma(\alpha)}\int_0^{t+h}(t+h-s)^{\alpha-1}S(\phi(t+h)-\phi(s))\widetilde{f}_u(s,\underline{x}(s))ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}S(\phi(t)-\phi(s))\widetilde{f}_u(s,\underline{x}(s))ds\,\|$$
$$\le\| S(\phi(t))\|\|\| S(\phi(h))-I\|\|\|\varphi\|_C + \frac{1}{\Gamma(\alpha)}\int_t^{t+h}(t+h-s)^{\alpha-1}\| S(\phi(t+h)-\phi(s))\|\|\|\widetilde{f}_u(s,\underline{x}(s))\| ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t\| S(\phi(t)-\phi(s))\|\|\|(t-s+h)^{\alpha-1}S(\phi(h))-(t-s)^{\alpha-1}I\|\|\|\widetilde{f}_u(s,\underline{x}(s))\| ds$$

$$\leq Me^{\omega\phi(T)} \| \varphi \|_C \| S(\phi(h)) - I \| + \frac{Me^{\omega\phi(T)}}{\alpha\Gamma(\alpha)} L_u h^\alpha + \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} L_u \int_0^t \| (t-s+h)^{\alpha-1} S(h) - (t-s)^{\alpha-1} I \| ds$$

Since $\lim_{h\to 0} \| (t-s+h)^{\alpha-1} S(\phi(h)) - (t-s)^{\alpha-1} I \| = 0$ and $\lim_{h\to 0} \| S(\phi(h)) - I \| = 0$, so

$\| Fx(t+h) - Fx(t) \| \to 0$ as $h \to 0$. Hence F is equicontinuous on $(0, t_1)$.

In general, for (t_k, t_{k+1}), $k = 0,1,2,\dots,n$, for $t_k < t < t+h < t_{k+1}$

$$\| Fx(t+h) - Fx(t) \| \leq \| S(\phi(t)) \| \| S(\phi(h)) - I \| \| x_k \|$$

$$+ \frac{1}{\Gamma(\alpha)} \int_t^{t+h} (t+h-s)^{\alpha-1} \| S(\phi(t+h) - \phi(s)) \| \| \widetilde{f}_u(s, \underline{x}(s)) \| ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t \| S(\phi(t) - \phi(s)) \| \| (t-s+h)^{\alpha-1} S(\phi(h)) - (t-s)^{\alpha-1} I \| \| \widetilde{f}_u(s, \underline{x}(s)) \| ds$$

$$\leq Me^{\omega\phi(T)} \| x_k \| \| S(\phi(h)) - I \| + \frac{Me^{\omega\phi(T)}}{\alpha\Gamma(\alpha)} L_u h^\alpha$$

$$+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} L_u \int_0^t \| (t-s+h)^{\alpha-1} S(h) - (t-s)^{\alpha-1} I \| ds.$$

Using the same idea, we can show that Q is equicontinuous on (t_k, t_{k+1}) where $k = 0,1,2,\dots,n$.

Therefore, Theorem 4.1 implies that $F(B)$ is a RCP subset of $PC([-ml,T],\mathcal{X})$. Further, F is a compact operator. ∎

Lemma 4.6 The set $\Omega \equiv \{x \in PC([-ml,T],\mathcal{X}) \mid x = \sigma Fx, \sigma \in [0,1]\}$ is bounded on $PC([-ml,T],\mathcal{X})$

Proof Let $x \in \Omega$. Since φ is continuous, there exist $M_1 > 0$

$$\| x(t) \| = \| \sigma Fx(t) \| \leq \| Fx(t) \| = \| \varphi(t) \| \leq M_1 \quad \text{for all } t \in [-r, 0]. \tag{17}$$

By using assumptions $A2$, there are $N_1, N_2 > 0$ such that for $t \in I$

$$\| f(t, x(t), x(t-l), \dots, x(t-ml)) \| \leq N_1 + N_2 \| x(t) \| \tag{18}$$

Using assumption $A3$ and assumption $A4$, we get,

$$\| x(t) \| = \| \sigma Fx(t) \| \leq \| Fx(t) \|$$

$$\leq \| S(\phi(t)) \| \| \varphi \|_C + \frac{1}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} \| S(\phi(t)-\phi(s)) \| [\| f(s, x(s), x_1(s-l), \dots, x(s-ml)) \|$$

$$+ \| \mathcal{B}(s)u(s) \|] ds + \sum_{0 < t_k < t} \| S(\phi(t) - \phi(t_k)) \| \| J_k(x(t_k)) \|$$

$$\leq Me^{\omega\phi(T)} \| \varphi \|_C + \frac{MN_1 e^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} ds + \frac{MN_2 e^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} \| x(s) \| ds$$

$$+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} \| \mathcal{B}(s)u(s) \| ds + Me^{\omega\phi(T)} \sum_{0 < t_k < t} e_k \| x(t_k) \|$$

$$\leq Me^{\omega\phi(T)} \| \varphi \|_C + \frac{MN_1 e^{\omega\phi(T)}}{\alpha\Gamma(\alpha)} T^\alpha (n+1) + \frac{MN_2 e^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} \| x(s) \| ds$$

$$+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{\frac{p(\alpha-1)}{p-1}} ds \right]^{\frac{p-1}{p}} \left[\int_{t_k}^{t_{k+1}} \| \mathcal{B}(s) u(s) \|^p ds \right]^{\frac{1}{p}} + Me^{\omega\phi(T)} \sum_{0 < t_k < t} e_k \| x(t_k) \|$$

$$\leq Me^{\omega\phi(T)} \left[\| \varphi \|_C + \frac{N_1 T^\alpha (n+1)}{\alpha\Gamma(\alpha)} + \frac{T^{\frac{p\alpha-1}{p-1}} (p-1)(n+1 \| \mathcal{B}(\cdot) u \|_{L_p(I,X)}}{(p\alpha-1)\Gamma(\alpha)} \right]$$

$$+ \frac{MN_2 e^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} \| x(s) \| ds + Me^{\omega\phi(T)} \sum_{0 < t_k < t} e_k \| x(t_k) \|.$$

Using Theorem 3.3 , there is $M_2 > 0$ s.t. $\| x(t) \| \leq M_2$ for all $x \in \Omega$. Hence Ω is a bounded subset of $PC([-ml,T],\mathcal{X})$. ■

Theorem 4.7 Assume that assumptions $A1 - A5$ hold, then the system (1) has at least one $PC-$mild solution corresponding to a regulator $u \in \mathcal{A}_{ad}$ on $[-ml,T]$.

Proof Define the operator F as (6). Then by Lemma 4.3 and Lemma 4.5, we have F is continuous on $PC([-ml,T],\mathcal{X})$ and compact. Set $\Omega \equiv \{x \in PC([-ml,T],\mathcal{X}) \mid x = \sigma Fx, \sigma \in [0,1]\}$. The Lemma 4.6 implies Ω is a bounded subset of $PC([-ml,T],\mathcal{X})$. Then, the LSFPT to implies that F has a fixed point in $PC([-ml,T],\mathcal{X})$. Therefore, system (1) has at least $PC-$mild solution corresponding to the regulator $u \in \mathcal{A}_{ad}$ on $PC([-ml,T],\mathcal{X})$. ■

4.2 Existence of Optimal Regulation

Let $\mathcal{A}_{ad}$ be the admissible regulator set. Note that the result in Section 4.1 implies that for each regulator $u \in \mathcal{A}_{ad}$, there exists a PC-mild solution x corresponding to S the regulator u.

Let us consider the minimization problem (P) corresponding to system(1); Find a regulator $u^0 \in \mathcal{A}_{ad}$ s.t.

$$\mathcal{P}(u^0, x^0) \leq \mathcal{P}(u,x) \quad \text{for all } u \in \mathcal{A}_{ad} \tag{19}$$

which $\mathcal{P}(u,x) = \int_0^T r(t, x(t), x(t-l), \ldots, x(t-ml), u(t)) dt + g(x(T))$ and x is a mild solution of system (1) corresponding to a regulator $u \in \mathcal{A}_{ad}$, the order pair (u,x) is called the admissible pair. For convenience, $\mathcal{P}(u,x)$ is short written by $\mathcal{P}(u)$.

We solve the optimizing regulation problem in this paper under the following assumptions, label by $A6 - A10$;

A6) $r: I \times \mathcal{X}^{m+1} \times Y \to (-\infty, \infty]$ is Borel measurable

A7) $r(t,\cdot,\cdot,\cdot)$ is sequentially lower semicontinuous on $\mathcal{X}^{m+1} \times Y$ for a.e. on I.

A8) $r(t,\xi,\cdot)$ is convex on Y for all $\xi \in \mathcal{X}^{m+1}$ and for a.e. $t \in I$.

A9) There exist constants a_i, $b > 0$ for all $i = 1,\ldots,m+1$ and $\eta \in \mathcal{L}_1(I,\Re)$ s.t.

$$r(t,\xi,u) \geq \eta(t) + \sum_{i=1}^{m+1} a_i \| \xi_1 \| + b \| u \|_Y^q \tag{20}$$

where $\xi = (\xi_1,\ldots,\xi_{m+1}) \in \mathcal{X}^{m+1}$.

A10) $g: \mathcal{X} \to \Re$ is nonnegative continuous function.

Theorem 4.8 Suppose that assumption $A1 - A10$ hold. Then the problem (P) corresponding to system (1) has at least one solution, that is, there is an admissible pair (u^0, x^0) such that

$$\mathcal{P}(u^0,x^0) \leq \mathcal{P}(u,x) \text{ for all } u \in \mathcal{A}_{ad}.$$

Proof If $\inf\{\mathcal{P}(u) \mid u \in \mathcal{A}_{ad}\} = +\infty$, it is well done. Suppose that $\inf\{\mathcal{P}(u) \mid u \in \mathcal{A}_{ad}\} = w < +\infty$. By assumption $A9$, there are constants a_i, $b > 0$ for all $i = 0,\ldots,m$ and $\eta \in \mathcal{L}_1(I,\Re)$ such that

$$r(t,x(t),x(t-l),\ldots,x(t-ml),u) \geq \eta(t) + \sum_{i=0}^{m} a_i \| x(t-il) \| + b \| u \|_Y^q .$$

Since η is non-negative, we get

$$\mathcal{P}(u) \geq \int_0^T \eta(t)dt + \sum_{i=0}^{m} a_i \int_0^T || x(t-il) || \, dt + b\int_0^T || u(t) ||_Y^q \, dt \geq -\sigma > -\infty .$$

for some $\sigma > 0$, for all $u \in \mathcal{A}_{ad}$.

Hence $w \geq -\sigma > -\infty$. By definition of minimum, there exists a minimizing sequence $\{u_n\}$ of $\mathcal{P}$, that is $\lim_{n\to\infty} \mathcal{P}(u_n) = w$ and

$$\mathcal{P}(u_n) \geq \int_0^T \eta(t)dt + \sum_{i=0}^{m} a_i \int_0^T || x_n(t-il) || \, dt + b\int_0^T || u_n(t) ||_Y^q \, dt .$$

So, there exists $N_0 > 0$ such that

$$w + w_1 \geq \mathcal{P}(u_n) \geq c\int_0^T || u_n(t) ||_Y^q \, dt, \quad \text{for all} \quad n \geq N_0,$$

for some $w_1 > 0$ and hence $\| u_n \|_{L_q(I,Y)}^q \leq \dfrac{w_1 + w}{c}$.

Thus u_n is a bounded sequence containing in the reflexive B-space $\mathcal{L}_q(I,Y)$. Therefore, u_n has a convergence subsequence, relabeled as u_n and $u_n \to u_0$ for some $u_0 \in \mathcal{A}_{ad} = \mathcal{L}_q(I,Y)$. Let $x_n \in PC([-ml,T],\mathcal{X})$ be a sequence of PC-mild solutions of system (1) corresponding to the regulators sequence u_n ;

$$\begin{cases} x_n(t) = S(\phi(t))\varphi(0) + \dfrac{1}{\Gamma(\alpha)} \sum\limits_{0 \le t_k \le t} \int_{t_k}^{t_k+1} (t_{k+1}-s)^{\alpha-1} S(\phi(t)-\phi(s))[f(s, x_n(s), x_n(s-l), \dots, x_n(s-ml)) \\ \qquad \mathcal{B}(s)u_n(s)]ds + \sum\limits_{0<t_k<t} S(\phi(t)-\phi(t_k))J_k x_n(t_k),\ t \in I \\ x_n(t) = \varphi(t),\ t \in [-ml, 0] \end{cases} \tag{21}$$

The a priori estimate implies there is some positive constant ρ s.t.

$$\| x_n \|_{PC([-r,T],\mathcal{X})} \le \rho \text{ for all } n = 0,1,2,\dots$$

Let x^0 be a PC-mild solution of system (1) corresponding to regulator u^0, i.e.,

$$\begin{cases} x^0(t) = S(\phi(t))\varphi(0) + \dfrac{1}{\Gamma(\alpha)} \sum\limits_{0 \le t_k \le t} \int_{t_k}^{t_k+1} (t_{k+1}-s)^{\alpha-1} S(\phi(t)-\phi(s))[f(s, x^0(s), x^0(s-l), \dots, x^0(s-ml)) \\ \qquad +\mathcal{B}(s)u^0(s)]ds + \sum\limits_{0<t_k<t} S(\phi(t)-\phi(t_k))J_k x^0(t_k),\ t \in I \\ x^0(t) = \varphi(t),\ t \in [-ml, 0] \end{cases} \tag{22}$$

Given $\rho > 0$. By using assumption $A1$, there exist $b(\rho) > 0$ such that for $s \in (0,T)$,

$$\| f(s, x_n(s), x_n(s-l), \dots, x_n(s-ml)) - f(s, x^0(s), x^0(s-l), \dots, x^0(s-ml)) \| \le b(\rho) \| x_n - x^0 \|_{PC}. \tag{23}$$

We use the fact that $x_n(s) - x^0(s) = 0$ for $s \in [-r, 0]$, so we have

$$\begin{aligned} \| x_n(s) - x^0(s) \| &\le \frac{b(\rho) M e^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0 \le t_k \le t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} \| x_n(s) - x^0(s) \| ds \\ &+ \frac{M e^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{k=1}^{k=n} \left[\int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\frac{p\alpha-1}{p-1}} ds \right]^{\frac{p-1}{p}} \left[\int_{t_k}^{t_{k+1}} \| \mathcal{B}(s)u_n(s) - \mathcal{B}(s)u^0(s) \|^p ds \right]^{\frac{1}{p}} \\ &+ M e^{\omega\phi(T)} \sum_{0<t_k<t} e_k \| x_n(t_k) - x^0(t_k) \| \\ &\le \frac{M e^{\omega\phi(T)} (p-1)(n+1) T^{\frac{p\alpha-1}{p-1}} \| \mathcal{B}(\cdot)u_n - \mathcal{B}(\cdot)u^0 \|_p}{(p\alpha-1)\Gamma(\alpha)} \\ &+ \frac{b(\rho) M e^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0 \le t_k \le t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} \| x_n(s) - x^0(s) \| ds + M e^{\omega\phi(T)} \sum_{0<t_k<t} e_k \| x_n(t_k) - x^0(t_k) \|. \end{aligned}$$

Applying theorem 5, there exist $M_1 > 0$ independent on u, n and t such that

$$\| x_n(t) - x^0(t) \| \le M_1 \| \mathcal{B}(\cdot)u_n - \mathcal{B}(\cdot)u^0 \|_{\mathcal{L}_q(I,Y)} . \tag{24}$$

The strongly continuous of $\mathcal{B}(\cdot)$ implies $\| \mathcal{B}(\cdot)u_n - \mathcal{B}(\cdot)u^0 \|_{\mathcal{L}_q(I,Y)} \xrightarrow{s} 0$.

Consequently, $\| x_n - x^0 \| \xrightarrow{s} 0$ in $PC([-ml,T],\mathcal{X})$.

Let us set $r_n(t) = r(t, x_n(t), x_n(t-l), \ldots, x(t-ml), u_n(t))$ for all $t \in I$.

Then by assumption $A6$ and assumption $A9$, $\{r_n(t)\}$ is a sequence of non-negative measurable functions.

So, by applying Fatou 's Lemma,

$$\varliminf_{n\to\infty} \int_0^T r_n(t)dt \ge \int_0^T \varliminf_{n\to\infty} r_n(t)dt. \tag{25}$$

By assumption $A7$

$$m = \lim_{n\to\infty} \mathcal{P}(u_n) \ge \varliminf_{n\to\infty} [\int_0^T r_n(t)dt + \Phi(x_n(T))]$$

$$\ge \int_0^T \varliminf_{n\to\infty} r(t, x_n(t), (x_n)_t, u_n(t))dt + g(x^0(T)) \ge r(t, x^0(t), x_t^0, u^0(t))dt + g(x^0(T)) = \mathcal{P}(u^0)$$

Therefore, $\mathcal{P}(u^0) = m$. ■

4.3 Optimal regulation of Nonlinear Fractional Heat Equation

In this section, we apply the results of the reseach to the generalize heat eqution. Let us consider the boundary value problem with delay and regulation;

$$\frac{\partial^\alpha y(x,t)}{\partial t^\alpha} = \Delta y(x,t) + f(x,t,y(x,t),y(x,t-l),\ldots,y(x,t-ml)) + \int_\Omega \mathcal{B}(x,\xi)u(\xi,t)d\xi,\ (x,t) \in \Omega \times I \setminus \mathcal{D} \tag{26}$$

$$\Delta y(x,t_k) = J_k(y(x,t_k)),\ t_k \in \mathcal{D} \tag{27}$$

$$y(x,t) = \varphi(x,t),\ (x,t) \in \bar{\Omega} \times [-ml,0] \tag{28}$$

$$y(x,t) = 0,\ (x,t) \in \partial\Omega \times I \tag{29}$$

where $\mathcal{D} = \{t_1, t_2, \ldots, t_n\}$, Ω is boundary domain of $\Re^N$, $\varphi \in C(\bar{\Omega} \times [-ml,0])$, $u \in \mathcal{L}_q(\Omega \times I)$, $h \in C([-ml,T]^2, \Re)$ and $B : \bar{\Omega} \times \bar{\Omega} \to \Re$ is continuous. Suppose that these conditions hold;

HHf) $f : \bar{\Omega} \times I \times \Re \times \Re^N \to \Re$ and there are L_1, $L_2 \ge 0$ s.t.

$$| f(x,t,\xi,\eta) | \le L_1(1 + |\xi| + |\eta|), \tag{30}$$

and

$$| f(x,t,\xi,\eta) - f(x,s,\xi_1,\eta_1) | \le L_2(| t-s | + | \xi - \xi_1 | + | \eta - \eta_1 |). \tag{31}$$

HHJ) $J_k : \Re \to \Re$, $k = 1,2,\ldots,n$ and there exist $e_k > 0$, $k = 1,2,\ldots,n$ s.t.

$$| J_k(\xi) - J_k(\xi_1) | \le e_k | \xi - \xi_1 |. \tag{32}$$

In this system, $y(x,t)$ represents the temperature at the point $x \in \Omega$ at time t, the condition (28) represents the temperature at the histories time $t \in [-ml, 0]$. The condition (29) represents the temperature on the boundary $\partial\Omega$ that is equal to zero. The input function f means an external heat sources. In this case, the function f depend on the histories data $y(x,t-l),\ldots, y(x,t-ml)$, which is impacted from the initial delay function $\varphi(x,t)$ for $t \in [-ml, 0]$ in the condition (28). Furthermore, the system is regulated by the regulator u with the sensor mapping $\int_\Omega \mathcal{B}(x,\xi)u(\xi,t)d\xi$. Given an admissible regulator set $\mathcal{A}_{ad} = \mathcal{L}_q(\Omega \times I)$. We solve the optimization regulation problem (P_0) with the objective functional;

$$\mathcal{P}(u) = \int_0^T \int_\Omega |y(\xi,t)|^2 + \| y(\xi,t-l) \|^2 + \ldots + \| y(\xi,t-ml) \|^2 \, d\xi dt + \int_0^T \int_\Omega |u(\xi,t)|^2 \, d\xi dt + g(y(x,T)) \tag{33}$$

where $g \in C(\Re, \Re^+)$. That is, find $u_0 \in \mathcal{A}_{ad}$ that minimize the objective functional. Let $\mathcal{X} = \mathcal{L}_p(\Omega)$.

For $t \in [-ml, T]$, define $y(t) : \Omega \to \mathcal{X}$ by

$$y(t)(x) = y(x,t) \text{ for all } x \in \Omega, \tag{34}$$

and

$$D_t^\alpha y(t)x = \frac{\partial^\alpha y(x,t)}{\partial t^\alpha}, \text{ for all } y \in X,\ x \in \Omega. \tag{35}$$

We define

$$f(t, y(t), y(t-l),\ldots, y(t-ml))(x) = f(x,t,y(x,t), y(x,t-l),\ldots, y(x,t-ml)), \tag{36}$$

$$\mathcal{B}(t)u(t)(x) = \int_\Omega \mathcal{B}(x,\xi)u(\xi,t)d\xi \tag{37}$$

and

$$J_k(y(t))(x) = J_k(y(x,t)). \tag{38}$$

Define an operator $A : \mathcal{X} \to \mathcal{X}$ as

$$Ay = \Delta y \quad \text{for all } y \in D(A) \tag{39}$$

which $D(A)$ contains all $C^2(\overline{\Omega})$ function vanishing on $\partial\Omega$.

Now we introduce the eigenvalue problem for the negative Laplacian;

$$Ay = \lambda y \quad \text{for all } y \in D(A). \tag{40}$$

Using the standard definition of the inner product, we define that for any y_1, $y_2 \in D(A)$;

$$< Ay_1, y_2 >= \int_\Omega \overline{y_2} \Delta y_1 dy = \int_\Omega \overline{y_1} \Delta y_2 dy =< y_1, Ay_2 >. \tag{41}$$

Therefore, A is symmetric and its eigenvalues must be real. Furthermore, for any $y \in D(A)$, we have

$$< Ay, y >=< \Delta y, y >= \int_\Omega \overline{y} \Delta y dy = \int_\Omega |grad y|^2 \, dy \geq 0. \tag{42}$$

The right hand side vanishes only if y is constant but the only constant in $D(A)$ is the zero constant. Thus, we obtain

$$\lambda \| y \|^2 =< \lambda y, y >=< Ay, y >> 0, \text{for all } y \neq 0 \text{ in } D(A). \tag{43}$$

This is precisely the definition of a positive operator, A is actually strongly positive. Because of equation(43), the eigenvalues of A must be positive and we obtain a following lemma.

Lemma 4.9 The operator A is a infinitesimal generator of a compact C_0-semigroup on $\mathcal{X}$.

Applying Lemma 4.9 , so the system (26)-(29) can transform to the abstract form;

$$\begin{cases} D_t^\alpha y(t) = Ay(t) + f(t, y(t), y(t-l), \ldots, y(t-ml)) + \mathcal{B}(t)u(t), & t \in I \setminus \mathcal{D} \\ \Delta y(t_k) = J_k(y(t_k)), & t_k \in \mathcal{D} \\ y(t) = \varphi(t), & t \in [-ml, 0]. \end{cases} \tag{44}$$

Theorem 4.10 Suppose that the assumptions (HHf) and (HHJ) are satisfied. Then the problem (P_0) for the nonlinear fractional heat equation with delay in $\Re^N$, (26)-(29) has at least one solution.

Proof We solve the regulation problem (P_0) for system (26)-(29) pass the abstract form, (44). The definitions of f, J_k $(k = 1, 2, \ldots, n)$ and the objective functional $\mathcal{P}$ and the assumptions (HHf), (HHJ) imply the assumptions of Theorem 4.7 and Theorem 4.8.

Therefore, the problem (P_0) with system (26)-(29) has at least one solution.

CHAPTER V

CONCLUSION

In this reseach, we considered the fractional integral inequalities and the fractional nonlinear differential system (1) with time lag, when A is the infinitesimal generator of a compact semigroup $\{S(t)\}_{t\geq 0}$ that satisfies the exponential stability. We proved some properties ofthe fractional integral inequalities, the existence solution and solved the optimal regulation problem. We proposed a method for proving existence whose main component is the use of the Leray-Schauder Fixed Point Theorem (LSFPT). More precisely, we assume that the input function, f and operator J_k , $k \in \{1,2,...,n\}$ satisfy the condition $A1-A5$. We successfully applied this method and use these assumptions to prove the existence of PC-mild solution. For studying the optimization regulation problem, we win to prove that system (1) has at least one optimal regulator with conditions $A1-A10$. Beside the study of the solution and the optimization regulation problem, we give some examples (model of problem in the real world), we give example of f and J_k, $k \in \{1,2,...,n\}$ such that satisfying the conditions (HHf) and (HHJ). Then we transform them to the abstract form and use our main results to conclude that these systems have at least one optimal regulator.

Finally yet importantly, we should be interested in developing this method and use weakly assumptions to prove the existence and uniqueness of PC-mild solution a little further. Moreover, we should be interested in studying the others the solution behaviors for example; the stable property. Even though it seems likely that efforts in this direction can be successful, there no guarantee for that. Therefore, we can only hope for the best, but have to expect the worst.

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Place of birth: Bangkok, Thailand

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PRESENT POSITIONS Lecturer, Department of Mathematics
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EDUCATION

Undergraduate B.Sc. (Mathematics)
Silapakorn University, Nakhorn Pathom, Thailand, 2000-2003

Graduate: M.Sc. (Mathematics)
Chulalongkorn University, Bangkok, Thailand, 2004-2007

Doctorate: Ph.D.(Mathematics)
Chulalongkorn University, Bangkok, Thailand, 2008-2011

FIELD OF RESEARCH

- Mathematical control theory
- Fractional calculus

- Applied mathematics
- Financial mathematics

ABSTRACTS/PRESENTATIONS

1. Sirikul Siriteerakul& W.Witayakiatilerd "A Study on Heat Diffusion of Tile" The 5th Thailand-Japan International AcademicConference (5th TJIA 2012), on Oct 20, 2012, The University of Tokyo, Tokyo, Japan. (Oral)
2. W.Witayakiatilerd., "Optimal Regulation of Impulsive FractionalDifferential Equations with Delays" International conference on global optimization and its applications 2012, on Dec 13-15, 2012, Faculrty of Mathematics and Natural Sciences, University of Padjadjaran, Bundung, Indonesia. (Oral)

ORIGINAL ARTICLES

Local Publications

1. W.Witayakiatilerd "Fractional Calculus : Fractional Derivative and Application"Scientific journal Lad Krabang, Vol. 20 (2) pp.81-90,2012.

International Publications

1. W.Witayakiatilerd & A.Chonwerayuth ,Regularity of piecewise continuous almost periodic solutions for nonlinear impulsive systems, Pushpa Publishing House ,November 2009
2. W.Witayakiatilerd & A.Chonwerayuth , Fractional integro-differential equations of mixed type with solution operator and optimal controls , Journal of Mathematics Research(JMR), August 2011
3. W.Witayakiatilerd, Optimal Regulation of Impulsive Fractional Differential Equation with Delay and Application to Nonlinear Fractional Heat Equation , Journal of Mathematics Research(JMR), Vol 5, No 2 (2013)