THREE ASYMPTOTIC ESTIMATES OF r-FREE INTEGERS UNDER CERTAIN RESTRICTIONS

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Abstract

Let $r \geq 2$ be a fixed integer. A positive integer n is called r-free if in the canonical representation of n into prime powers each exponent is less than r. The integer 1 is considered to be r-free. As usual, 2-free and 3-free integers are called square-free and cube-free, respectively. In this thesis, we investigate the asymptotic estimates of r-free integers under certain restrictions. First we use an elementary method to give an asymtotical ratio of odd to even r-free integers. Next, for the finite set of prime number P, we prove the proportion of all r-free numbers which are divisible by at least one element in P and coprime to all of primes in P.

Furthermore, we let $\alpha > 1$ be an irrational number and with bounded partial quotients, $\beta \in [0, \alpha)$. The Beatty sequence of parameters α and β is defined by

$$\{\lfloor \alpha n + \beta \rfloor\}_{n \in \mathbb{N}},$$

where $\lfloor z \rfloor$ is the integer part of $z \in \mathbb{R}$. We generalize the previous result on the distribution of consecutive r-free integers in Beatty sequences.

Keywords : Square-free integer, Cube-free integer, r-free integer, Asymtotical ratio, Elementary method, Beatty sequences.

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Sunanta Srisopha

Table of Contents

List of Tables

Notations

Throughout this thesis, the following symbols is adopted.

Symbol	Meaning
$\mathbb{R}$	the set of real numbers
$\mathbb{N}$	the set of positive integers
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{C}$	the set of complex numbers
P	the finite set of prime numbers
N_r	the set of r-free integers
N_o	the set of odd r-free integers
N_e	the set of even r-free integers
C_{odd}	the set of odd square-free integers
C_{even}	the set of even square-free integers
C_P	the set of r-free integers not divisible by any $p \in P$
C'_P	the set of r-free integers divisible by some $p \in P$
$f(x) = O(g(x))$	$f(x)$ is big oh of $g(x)$.
$f(x) \ll g(x)$	$f(x)$ is big oh of $g(x)$.
$f(x) \sim g(x)$	$f(x)$ is asymptotic to $g(x)$ as $x \to \infty$.
$Q_{k,r}(x;\alpha,\beta)$	the number of (k,r)-integers of Beatty sequence $\lfloor \alpha n + \beta \rfloor$, $1 \le n \le x$
$T_{k,r}(x)$	the number of positive integers $n \le x$ such that $\lfloor \alpha n+\beta \rfloor$ and $\lfloor \alpha n+\beta \rfloor+1$ are (k,r)-integers

Chapter 1

Introduction

This chapter consists of five sections: Research motivation, objective of the study, scopes of the study, benefit of the study and research methodology.

1.1 Research Motivation

In this thesis, we divide works into three parts. The first part is about the odd/even dichotomy for the set of r-free integers. The second is about the proportion of all r-free integers which are divisible by at least one element in P and coprime to all of primes in P, where P denote the finite set of prime numbers. The third is about consecutive generalized r-free integers in Beatty sequences.

Let $r > 1$ be a fixed integer. A positive integer n is said to be an r-free integer if each of its prime factors appears to the power at most $r-1$. The integer 1 is considered to be r-free. For $r = 2, 3$, these numbers are called square-free and cube-free respectively. On the other hands, a positive integer n is said to be an r-full integer if each of its prime factors appears to the power at least r. As usual, 2-full and 3-full numbers are called square-full and cube-full, respectively.

In the first part, let $N_r(x)$ be the number of r-free integers not greater than x. It well know that for r fixed

$$N_r(x) = \frac{1}{\zeta(r)}x + O(x^{1/r}), \tag{1.1}$$

where ζ is the Riemann zeta function. The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{if } s > 1,$$

and by

$$\zeta(s) = \lim_{x\to\infty} \left(\sum_{n\le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right) \qquad \text{if } 0 < s < 1.$$

For a study of these asymptotic formulae, we refer to [1, Equation 14.24].

The motivation follows from work of Scott [2] and Jameson [3], where it was shown that the ratio of odd to even square-free numbers is asymptotically $2:1$. In 2020, Srichan [4] used an elementary method to prove the odd/even dichotomy for the set of square-full. In 2021, Puttasontiphot and Srichan [5] extended the method in [4] to the case of cube-full numbers. Later, Jameson [6] used this to gave a new proof for his paper in [3]. Thus, it would be interesting to generalize these results to the odd/even dichotomy for the set of r-free integers by using the method in [4].

In the second part, we consider the natural density of the set of square-free integer which is studied first by Gegenbauer. In 1979, Hardy and Wright [7] proved that the natural density of the set of square-free integer is $6/\pi^2$. Later, in [2] Scott gave a conjecture on the natural density of the set of odd square-free integers is $4/\pi^2$ and it was proven by Jameson in [3]. In 2021, authors [8] generalized this problem to the case of r-free integers by using an elementary method and showed that the asymptotical ratio of odd to even r-free numbers is asymptotically $2^r : 2^r - 2$. In the same year, Brown [9] reproved Jameson's result and generalized it. Brown proved that the proportion of all numbers which are square-free and divisible by all of the primes in T and by none of the primes in P is

$$\frac{6}{\pi^2} \prod_{p \in T} \frac{1}{1+p} \prod_{p \in P} \frac{p}{1+p}, \tag{1.2}$$

where P and T are disjoint sets of prime numbers with T finite. Now we use the elementary method in [4] to prove a similar result as in [9] and generalize it to r-free integers.

In the third part, let $\alpha > 1$ be irrational and with bounded partial quotients, $\beta \in [0, \alpha)$. The Beatty sequence of parameter α and β is defined by

$$\{\lfloor \alpha n + \beta \rfloor\}_{n \in \mathbb{N}}$$

where $\lfloor z \rfloor$ is the integer part of $z \in \mathbb{R}$. The problem for the existence of square -free numbers in the Beatty sequences arose in 2008. Gülolu and Nevans [10] proved that

$$\sum_{\substack{n \le x \\ \lfloor \alpha n \rfloor \text{ is square-free}}} 1 = \frac{x}{\zeta(2)} + O\Big(\frac{x \log\log x}{\log x}\Big),$$

where $\alpha > 1$ is an irrational number of finite type. In 2009, Abercrombie and Banks [11] showed that

$$\sum_{\substack{n \le x \\ \lfloor \alpha n \rfloor \text{ is square-free}}} 1 = \frac{x}{\zeta(2)} + O\Big(x^{2/3} \log x\Big),$$

for almost all $\alpha > 1$. In 2013, Victorovich [20] showed that

$$\sum_{\substack{n \le x \\ \lfloor \alpha n \rfloor \text{ is square-free}}} 1 = \frac{x}{\zeta(2)} + O\Big(A x^{5/6} \log^5 x\Big), \tag{1.3}$$

where $\alpha > 1$ is an irrational number with bounded partial quotient or an irrational algebraic number. Here $A = \max\{\tau(m), 1 \le m \le x^2\}$. In 2021, Veasna et al. [12] showed that,

$$\sum_{\substack{n \le x \\ \lfloor \alpha n + \beta \rfloor \text{ is } r\text{-free}}} 1 = \frac{x}{\zeta(r)} + O(x^{(1/2r+1/2)} \log^3 x), \tag{1.4}$$

where $\alpha > 1$ be an irrational number and with bounded partial quotients, $\beta \in [0, \alpha)$, as $x \to \infty$. In the case of $r = 2$, their result in (1.4) improves Victorovich's result in (1.3). Moreover, they used the technique of Tangsupphathawat et al. [13] to improve the previous result on the distribution of consecutive square-free integers in Beatty sequences and showed that, for $\alpha > 1$ irrational and with bounded partial quotients, $\beta \in [0, \alpha)$ and sufficiently small $\varepsilon > 0$, as $x \to \infty$ we have

$$\sum_{\substack{n \le x \\ \lfloor \alpha n+\beta \rfloor,\ \lfloor \alpha n+\beta \rfloor+1 \text{ are square-free}}} 1 = \prod_{p} \Big(1 - \frac{2}{p^2}\Big) x + O\Big(\alpha x^{\frac{3}{4}+\varepsilon} \log^3 x\Big). \quad (1.5)$$

It would be interesting to generalize (1.5) to a larger class of integers.

In this part, we shall use the technique of Tangsupphathawat et al. [13] to generalize the previous result on the distribution of consecutive square-free integers in Beatty sequences.

1.2 Objectives of the study

1) To show an asymptotical ratio of odd to even r-free integers by using an elementary method.
2) To derive asymptotic formulae for the number of all r-free integers which are divisible by at least one element in P and coprime to all of primes in P and generalize Brown's result to r-free integers.
3) To generalize the result on the distribution of consecutive r-free integers in Beatty sequences.

1.3 Scopes of the study

Our main objects are r-free integers, i.e., the integers whose the canonical representation of them into prime powers each exponent is less than r. Aspects of these integers to be investigated are

1) the odd/even dichotomy for the set of r-free integers,
2) the proportion of all r free integers which are divisible by at least one element in P and coprime to all of primes in P,
3) the elementary method and generalization for r-free and (k, r)-integers.

1.4 Benefits of the study

1) An asymptotical ratio of odd to even r-free integers are obtained.
2) Asymtotic formulae from the problems of counting r-free integers which are divisible by at least one element in P and coprime to all of primes in P are obtained.
3) Generalization for the result on the distribution of consecutive r-free integers in Beatty sequences are shown.

1.5 Research Methodology

1) Study the fundamental theorem of arithmetic, arithmetic functions, asymtotic equality of function and the unique factorization theorem.

2) Study works of Scott [2], Jameson [3, 6], Srichan [4], Brown [9], Veasna et al. [12] and Tangsupphathawat et al. [13].

3) Apply the methods of Srichan [4] to show an asymptotical ratio of odd to even r-free integers.

4) Apply the elementary method to prove a similar result as in [9] and generalize it to r-free integers.

5) Apply the technique of Tangsupphathawat et al. [13] to generalize the previous result on the distribution of consecutive squree-free integers in Beatty sequences.

6) Summarize all the results so obtained and write a thesis.

Table 1.1: The research schedule

Activity	Time frame					
	2019	2020		2021		2022
	Agu.-Dec.	Jan.-Jun.	Jul.-Dec.	Jan.-Jun.	Jul.-Dec.	Jan.-Jun.
Step 1	———	——→				
Step 2		———	——→			
Step 3			———	——→		
Step 4				———	——→	
Step 5					———	——→
Step 6						——→

Chapter 2

Preliminaries

In this chapter, we will recall some definitions, properties, theorems and examples that we will be used throughout our study.

2.1 Some basic of arithmetic

In this section, we will review some definitions, theorems and some examples of arithmetic.

Definition 2.1. ([14]) A **divisor** of n is an integer d that divides n and write $d|n$ whenever $d = nc$ for some integer c. If d does not divide n, write $d \nmid n$.

Theorem 2.1. ([14], Theorem 1.3 on p.15) Given integers a and b, there is one and only one number d with the following properites:

1) $d \geqslant 0$,

2) $d|a$ and $d|b$,

3) if there is a number c such that $c|a$ and $c|b$ then $c|d$.

Definition 2.2. ([14]) The number d of Theorem 2.1 is called the **greatest common divisor** of a and b which is denoted by gcd(a, b).

Definition 2.3. ([14]) The integers a and b are said to be **relatively prime** if gcd$(a, b) = 1$.

Theorem 2.2. ([14], Theorem 1.5 on p.16) If $a|bc$ and gcd$(a, b) = 1$, then $a|c$.

Definition 2.4. ([14]). An integer $n(> 1)$ is called a **prime** if the only positive divisors of n are 1 and n. If $n(> 1)$ is not prime, then n is called a **composite**.

Example 2.1. The prime numbers less than 100 are $2, 3, 5, 7, 11, 13, 17, 19.23, 29, 31, 37, 41,$ $43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89$ and 97.

2.2 The unique factorization theorem

Theorem 2.3. ([14], Theorem 1.6 on p.16) Every integer $n > 1$ is either a prime number or product of prime numbers.

Theorem 2.4. ([14], Theorem 1.7 on p.16) There are infinitely many prime numbers.

Theorem 2.5. ([14], Theorem 1.8 on p.17) If a prime p does not divide a, then gcd$(p, a) = 1$.

Theorem 2.6. ([14], Theorem 1.9 on p.17) If a prime p divides ab, then $p|a$ or $p|b$. More generally, if a prime p divides a product $a_1 \dots a_n$, then p divides at least one of the factors.

Theorem 2.7. ([14], Theorem 1.10 on p.17, The unique factorization theorem) Every integer $n > 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors.

Theorem 2.8. ([14], Theorem 1.12 on p.18) If two positive integers a and b have the factorizations

$$a = \prod_{i=1}^{\infty} p_i^{a_i}, \qquad b = \prod_{i=1}^{\infty} p_i^{b_i},$$

then their gcd has the factorization

$$\text{gcd}(a, b) = \prod_{i=1}^{\infty} p_i^{c_i}$$

where each $c_i = \min\{a_i, b_i\}$, the smaller of a_i and b_i.

Definition 2.5. ([14]) A positive integer $n > 1$ with **unique prime factorization**

$$n = p_1^{a_1} p_2^{a_2} \dots p_s^{a_s}$$

where $p_1, \dots, p_s$ are distinct primes and $a_1, \dots, a_s \in \mathbb{N}$, is r**-free** whenever $a_i < r$ for all $i = 1, 2, ..., s$.
In case $r = 2$, we call n, a **squarefree**.

Example 2.2. Consider

1) If $r = 2$, then $n_1 = 110 = 2 \cdot 5 \cdot 11,\ n_2 = 42 = 2 \cdot 3 \cdot 7$ are squarefree numbers,

2) If $r = 3$, then $n_1 = 98 = 2 \cdot 7^2,\ n_2 = 100 = 2^2 \cdot 5^2$ are 3-free (or cube-free) numbers,

3) If $r = 5$, then $n_1 = 1224 = 2^3 \cdot 3^2 \cdot 17,\ n_2 = 30800 = 2^4 \cdot 5^2 \cdot 7 \cdot 11$ are 5-free numbers.

2.3 Some basic properties of congruences

Definition 2.6. ([14]) Given $a, b, m \in \mathbb{Z}$ with $m > 0$. We say that a is **congruent to** b **modulo** m, if $m|(a - b)$ and write

$$a \equiv b \pmod{m}.$$

Theorem 2.9. ([14], Theorem 5.3 on p. 108, Theorem 5.4 on p. 109) Let $a, b, c \in \mathbb{Z}$.

1) If $c > 0$ then $a \equiv b \pmod{m}$ if and only if $ac \equiv bc \pmod{mc}$.

2) If $ac \equiv bc \pmod{m}$ and $d = \text{gcd}(m, c)$, then $a \equiv b \pmod{m/d}$.

Definition 2.7. ([14]) A congruence of the form

$$ax \equiv b \pmod{m}$$

where x is an unknown integer is called a **linear congruence** in one variable.

Theorem 2.10. ([14], Theorem 5.12, 5.13, 5.14 on pp.111-112) Given $a, b, m \in \mathbb{Z}$ with $m > 0$.

1) If $\gcd(a, m) = 1$, then the linear congruence $ax \equiv b \pmod{m}$ has exactly one solution modulo m.

2) If $\gcd(a, m) = d$, then the linear congruence $ax \equiv b \pmod{m}$ has solutions if and only if $d|b$.

3) Assume that $\gcd(a, m) = d$ and suppose that $d|b$. Then the linear congruence $ax \equiv b \pmod{m}$ has exactly d solutions modulo m. These are given by

$$t,\, t + \frac{m}{d},\, t + 2\frac{m}{d}, \ldots, t + (d-1)\frac{m}{d},$$

where t is the solution, unique modulo m/d, of the linear congruence

$$\frac{a}{d}x \equiv \frac{b}{d} \left(\bmod \frac{m}{d}\right).$$

2.4 Properties of arithmetic function

Definition 2.8. ([14, 15]) A complex - valued function defined on the positive integers is called an **arithmetic function**. Denote by $\mathcal{A}$ the **set of all arithmetic functions**.

Examples of arithmetic functions are:

1) The **Möbius function**, $\mu(n)$, is defined by, [19],

$$\mu(n) := \begin{cases} (-1)^s & \text{if } n = p_1 p_2 \cdots p_s \text{ for distinct primes } p_1, \ldots, p_s, \\ 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

2) For $\alpha \in \mathbb{R}$, the **divisor function**, $\sigma_\alpha(n)$, is defined to be the sum of the αth power of divisors of n, [14],

$$\sigma_\alpha(n) := \sum_{d|n} d^\alpha;$$

when $\alpha = 0$, σ_0 is the number of divisors of n denoted by $d(n)$;

when $\alpha = 1$, σ_1 is the sum of divisors of n denoted by $\sigma(n)$.

3) The **Euler phi function**, $\varphi(n)$, is defined to be the number of positive integers not exceeding n that are relatively prime to n, i.e.,

$$\varphi(n) := \sum_{\substack{x \le n \\ \gcd(x,n)=1}} 1.$$

The Euler phi function satisfies, [19],

$$\varphi(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right),$$

where the sum is extended over divisors d of n.

4) The **Liouville's function**, $\lambda(n)$, is defined by, [14], $\lambda(1) = 1$ and if the prime factorization of n is $p_1^{a_1} \cdots p_s^{a_s}$, then

$$\lambda(n) := (-1)^{a_1+\cdots+a_s}.$$

5) The **unit function**, $U(n)$, is defined by, [14],

$$U(n) := 1.$$

Definition 2.9. ([14]) For every integer $n \geq 1$ we define **Mangoldt's function** Λ by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.10. ([14]) For $x > 0$ we define **Chebychev's Ψ-function** by the formula

$$\Psi(x) = \sum_{n \leq x} \Lambda(n).$$

Definition 2.11. ([14]) Let $k \in \mathbb{N}$. An arithmetic function f is said to be **periodic** with period k if

$$f(n+k) = f(n) \qquad (n \in \mathbb{N}).$$

Definition 2.12. ([14]) An arithmetic function f is called **multiplicative** if f is not identically zero, $f(1) = 1$, and

$$f(mn) = f(m)f(n) \qquad (\gcd(m,n) = 1, m, n \in, \mathbb{N}).$$

Denote $\mathcal{M}$ be the **set of multiplicative functions**. Clearly, if $f \in \mathcal{A}$ with $f(1) = 1$ then $f \in \mathcal{M}$ if and only if

$$f(p_1^{a_1} \ldots p_s^{a_s}) = f(p_1^{a_1}) \cdots f(p_s^{a_s})$$

for all prime p_i's and positive integer a_i's. Note that $\mu(n)$, $\sigma_\alpha(n)$, $\varphi(n)$, $U(n)$ and $\lambda(n)$ are multiplicative on n (see also [14, 19]).

Definition 2.13. ([14]) If $f, g \in \mathcal{A}$, the **Dirichlet convolution** of f and g is defined by

$$(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

It is well-known that $(\mathcal{A}, +, *)$ is an integral domain. Note that $\zeta_1(n) = n$.

Example 2.3. Let $n = 20$. Then we get

$$\begin{aligned}(\zeta_1 * U)(20) &= \sum_{d|20} \zeta_1(d) U(20/d) \\ &= \zeta_1(1)U(20) + \zeta_1(2)U(10) + \zeta_1(4)U(5) + \zeta_1(5)U(4) + \zeta_1(10)U(2) + \zeta_1(20)U(1) \\ &= 1 + 2 + 4 + 5 + 10 + 20 = 42.\end{aligned}$$

Theorem 2.11. ([14], Theorem 2.14 on p.35) For $f, g \in \mathcal{A}$, the Dirichlet convolution of multiplicative functions f, g is also multiplicative.

Theorem 2.12. ([14], Theorem 2.9 on p.30) The **identity function** with respect to Dirichlet convolution is the arithmetic function

$$I(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Theorem 2.13. ([14], Theorem 2.8 on p.30) For $f \in \mathcal{A}$ with $f(1) \neq 0$, its Dirichlet inverse f^{-1*} exists and is given by

$$f^{-1*}(1) = \frac{1}{f(1)}, \qquad f^{-1*}(n) = \sum_{\substack{d|n \\ d>1}} f\left(\frac{n}{d}\right) f^{-1*}(d) \qquad (n > 1).$$

2.5 The big oh notation and asymtotic equality

Definition 2.14. [14] If $g(x) > 0$ for all $x \geq a$, we write

$$f(x) = O(g(x)) \qquad \text{(read: "}f(x)\text{ is big oh of }g(x)\text{")}$$

to mean that the quotient $f(x)/g(x)$ is bounded for $x \geq a$; that is there exists a constant $M > 0$ such that

$$| f(x) | \leq Mg(x) \qquad \text{for all } x \geq a.$$

For the convenient, we write $f(x) \ll g(x)$ instead of $f(x) = O(g(x))$. An equation of the form

$$f(x) = h(x) + O(g(x))$$

means that $f(x) - h(x) = O(g(x))$. We note that $f(t) = O(g(t))$ for $t \geq a$ implies $\int_a^x f(t)dt = O(\int_a^x g(t)dt)$ for $x \geq a$.

Example 2.4. 1) Let $f(x) = x^4 - 8x^3 + 5x^2 - 5x + 6$. Then we can write

$$f(x) = x^4 + O(x^3).$$

The symbol $O(x^3)$ represents an unspecified function of x which grows no faster than some constant time x^3.

2) For all $x \geq 1$, (see [14], Theorem 3.2)

$$\sum_{n \leq x} d(n) = x \log x + (2C - 1)x + O(\sqrt{x}),$$

where C is Euler's constant ($C = \lim_{n\to\infty} (\sum_{k=1}^{n} \frac{1}{k} - \log n)$).

Definition 2.15. [14] If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

we say that $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$, and we write

$$f(x) \sim g(x) \text{ as } x \to \infty.$$

Example 2.5. 1) Let $f(x) = 2x+1$ and $g(x) = 2x+5$, we have $\lim_{x\to\infty} \frac{2x+1}{2x+5} = 1.$ Thus $f(x) \sim g(x)$ as $x \to \infty$.

2) $\sum_{n\le x} d(n) \sim x\log x$ as $x \to \infty$. (see [14], Theorem 3.3)

Since $\sum_{n\le x} d(n) = x\log x + (2C-1)x + O(\sqrt{x})$, we have

$$\lim_{x\to\infty} \left(\frac{\sum_{n\le x} d(n)}{x\log x} \right) = 1 + \lim_{x\to\infty} \left(\frac{(2C-1)x + O(\sqrt{x})}{x\log x} \right) = 1.$$

Definition 2.16. [14] The **Riemann zeta function** $\zeta(s)$ defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{if } s > 1,$$

and by

$$\zeta(s) = \lim_{x\to\infty} \left(\sum_{n\le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right) \qquad \text{if } 0 < s < 1.$$

Theorem 2.14. ([14], Theorem 3.2 on p.55) If $x \ge 1$ we have:

1. $\sum_{n\le x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right).$
2. $\sum_{n\le x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$ if $s > 0, s \ne 1.$
3. $\sum_{n> x} \frac{1}{n^s} = O(x^{1-s})$ if $s > 1.$
4. $\sum_{n\le x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha})$ if $\alpha \ge 0.$

2.6 Average of arithmetic functions

Theorem 2.15. ([14], Theorem 3.3 on p.57) For all $x \ge 1$ we have

$$\sum_{n\le x} d(n) = x\log x + (2C-1)x + O(\sqrt{x}),$$

where C is Euler's constant.

Theorem 2.16. ([14], Theorem 3.4 on p.60) For all $x \ge 1$ we have

$$\sum_{n\le x} \sigma_1(n) = \frac{1}{2}\zeta(2)x^2 + O(x\log x).$$

Theorem 2.17. ([14], Theorem 3.5 on p.60) If $x \geq 1$ and $\alpha > 0, \alpha \neq 1$, we have

$$\sum_{n \leq x} \sigma_\alpha(n) = \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + O(x^\beta)$$

where $\beta = \max\{1, \alpha\}$.

Theorem 2.18. ([14], Theorem 3.6 on p.61) If $\beta > 0$ let $\delta = \max\{0, 1-\beta\}$. Then if $x > 1$ we have

$$\sum_{n \leq x} \sigma_{-\beta}(n) = \zeta(\beta+1)x + O(x^\delta) \qquad \text{if } \beta \neq 1,$$

$$= \zeta(2)x + O(\log x) \qquad \text{if } \beta = 1.$$

Theorem 2.19. ([14], Theorem 3.7 on p.62) For $x > 1$ we have

$$\sum_{n \leq x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x),$$

so the average order of $\phi(n)$ is $3n/\pi^2$.

2.7 Beatty sequences

Definition 2.17. ([16]) Let $\alpha > 1$ be an irrational number and with bounded partial quotients, $\beta \in [0, \alpha)$. **The Beatty Sequences** of parameter α and β is defined by

$$\{\lfloor \alpha n + \beta \rfloor\}_{n \in \mathbb{N}},$$

where $\lfloor z \rfloor$ is the integer part of $z \in \mathbb{R}$.

Theorem 2.20 (Beatty Theorem, [16], Theorem 12.2 on p.94)**.** Let X be any positive irrational number and Y its reciprocal. Then the two sequences

$$1+X, 2(1+X), 3(1+X), \ldots,$$

$$1+Y, 2(1+Y), 3(1+Y), \ldots$$

together contain exactly one number from each of the intervals $(n, n+1)$ between consecutive positive integers $(n = 1, 2, 3, \ldots)$.

Corollary 2.21. ([16], Corollary 12.5 on p.94) The sequences $\lfloor n(1+X) \rfloor, \lfloor n(1+Y) \rfloor$, called Beatty Sequences corresponding to the irrational number X, together contain each natural number exactly once.

Example 2.6. Let $X = \sqrt{2} \approx 1.4142$ is irrational number. Then $Y = X/(X-1) = 3.4143$ and for $n = 1, 2, 3, \ldots,$ we get the sequences

$$n(1+X) := \{2.4142, 4.8284, 7.2426, 9.6568, \ldots\};$$

$$n(1+Y) := \{4.4143, 8.8286, 13.2429, 17.6572, \ldots\}.$$

Thus we have the Beatty sequences

$$\lfloor n(1+X) \rfloor_{n \geq 1} := \{2, 4, 7, 9, \ldots\};$$

$$\lfloor n(1+Y) \rfloor_{n \geq 1} := \{4, 8, 13, 17, \ldots\}.$$

2.8 Chinese remainder theorem

Theorem 2.22 (Chinese remainder theorem, [14], Theorem 5.26 on p.117)**.** Assume $m_1, \dots, m_r$ are positive integers, relatively prime in pairs:

$$(m_i, m_k) = 1 \qquad \text{if } i \neq k.$$

Let $b_1, \dots, b_r$ be arbitray integers. Then the system of congruences

$$\begin{aligned} x &\equiv b_1 \pmod{m_1} \\ &\vdots \\ x &\equiv b_r \pmod{m_r} \end{aligned}$$

has exactly one solution modulo the product $m_1 \cdots m_r$.

Theorem 2.23. ([14], Theorem 5.27 on p.118) Assume $m_1, \dots, m_r$ are relatively prime in pairs. Let $b_1, \dots, b_r$ be arbitray integers and let $a_1, \dots, a_r$ satisfy

$$(a_k, m_k) = 1 \qquad \text{if } k = 1, 2, \dots, r.$$

Then the linear system of system of congruences

$$\begin{aligned} a_1 x &\equiv b_1 \pmod{m_1} \\ &\vdots \\ a_r x &\equiv b_r \pmod{m_r} \end{aligned}$$

has exactly one solution modulo the product $m_1 \cdots m_r$.

Theorem 2.24. ([14], Theorem 5.28 on p.118) Let f be a polynomial with integer coefficients, let $m_1, m_2, \dots, m_r$ be positive integers relative prime in pairs, and let $m = m_1 m_2 \cdots m_r$. Then the congruence

$$f(x) \equiv 0 \pmod{m}$$

has a solution if and only if each of congruences

$$f(x) \equiv 0 \pmod{m_i} \qquad (i = 1, 2, \dots, r)$$

has a solution. Moreover, if $v(m)$ and $v(m_i)$ denote the number of solutions of two equations above respectively, then

$$v(m) = v(m_1)v(m_2) \cdots v(m_r).$$

2.9 Literature review

2.9.1 The work of Scott [2]

In 2008, Scott [2] consider the 'odd/even dichotomy' for the set S of square-free positive integers. First, he calculated the asymptotic density of the square-free integers.

Asymptotic density for S

For any prime p, the fraction of positive integers divisible by p^2 is $1/p^2$, so the asymptotic density for S is given by the following infinite product over the primes:

$$P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\cdots.$$

He next introduced the well-known infinite series

$$Q = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{6}.$$

Multiplying Q by $1 - 1/2^2$ clearly removes all even terms from the series. Indeed, He got that multiplication of the series Q by each subsequent term of P systematically removes the remaining odd fractional terms from the series for Q, whence $PQ = 1$ or $P = 6/\pi^2$.

Parity within S

By the formulae

$$\sum_{n\in S} \frac{1}{n^2} = \frac{15}{\pi^2} \quad \text{and} \quad \sum_{n\in S} \frac{(-1)^{n+1}}{n^2} = \frac{9}{\pi^2}.$$

He got $O + E = 15/\pi^2$ and $O - E = 9/\pi^2$, where O and E denote the number of odd and even square-free positive integers, respectively, leading to the estimate $O/E = 4$ for his postulated asymptotic 'odd/even' ratio ρ. Note in passing that this estimate is clearly weighted in favour of the early terms where there is marked irregularity.

Next consider the following asymptotic sums:

$$\sum_{n\in S}^{n\le N} \frac{1}{n} \sim \frac{6}{\pi^2}\log N \quad \text{and} \quad \sum_{n\in S}^{n\le N} \frac{(-1)^{n+1}}{n} \sim \frac{2}{\pi^2}\log N.$$

In this case, he got $O + E \sim (6/\pi^2)\log N$ and $O - E \sim (2/\pi^2)\log N$ and the more trustworthy estimate $\rho = O/E \approx 2$.

2.9.2 The work of Jameson [3, 6]

In 2010, Jameson [3] proved that the ratio of odd to even square-free numbers is asymptotically 2:1. His proof is most efficiently presented in the language of Dirichlet series and convolutions.

For any arithmetic function $a(n)$, there is a corresponding Dirichlet series $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, defining a function $F_a(s)$ where it converges. He multiplied two Dirichlet series and collect the terms in the obvious way (which is valid provided that both series converge absolutely),

$$F_a(s)F_b(s) = F_{a*b}(s), \tag{2.1}$$

where the Dirichlet convolution $a * b$ is defined by

$$(a * b)(n) = \sum_{jk=n} a(j)b(k) = \sum_{j|n} a(j)b(n/j).$$

Let e_1 be the sequence having 1 in place 1 and 0 elsewhere. Then $a*e_1 = a$ for any arithmetic function a, so e_1 is the identity for convolution. Recall that the **Möbius function** μ takes the value 1 at 1 and $(-1)^k$ at a square-free integer with k prime factors. The unit function u is defined by $u(n) = 1$ for all n.

Lemma 2.25. We have $u * \mu = e_1$. Hence for all $n > 1$, $\sum_{j|n} \mu(j) = 0$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$ for $s > 1$, it follows from (2.1) that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$. (Alternatively, this identity, together with the definition of $\mu(n)$ itself, can be derived directly from the Euler product.) In particular,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

It will help to introduce the following notation:

$$\nu(n) = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

Clearly,

$$\sum_{n=1}^{\infty} \frac{\nu(n)}{n^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Let $A(x)$ be the number of odd square-free numbers not greater than x.
Clearly,

$$A(x) = \sum_{n \le x} |\mu(n)|\nu(n).$$

Lemma 2.26. We have $\mu\nu * \nu = e_1$, hence $\sum_{n=1}^{\infty} \frac{\mu(n)\nu(n)}{n^2} = \frac{8}{\pi^2}$.

Lemma 2.27. For all n, we have $|\mu(n)|\nu(n) = \sum_{m^2|n} \mu(m)\nu(m)\nu(n/m)$.

Theorem 2.28. $A(x) = \frac{4}{\pi^2}x + q(x)$, where $|q(x)| \le 3x^{1/2}$.

In 2021, Jameson [6] used an elementary method to give a new proof in [3] and showed that, asymptotically, two thirds of the square free numbers are odd and one third even.

2.9.3 The work of Srichan [4]

In the work of Srichan [4], he considered the asymptotical ratio of odd to even square-full numbers.

A positive integer n is called square-full if $p^2|n$ for every prime factor p of n. Let $N(x)$, $N_o(x)$ and $N_e(x)$ be the number of members of set of all square-full numbers, odd square-full numbers and even square-full in the interval $[1, x]$, respectively.

Theorem 2.29. As $x \to \infty$, we have

$$\frac{N_o(x)}{N_e(x)} \sim 2 - \sqrt{2}.$$

Remark: The result in this theorem indicates that the ratio of odd to even square-full numbers is asymptotically $1 : 1 + \frac{\sqrt{2}}{2}$.

2.9.4 The work of Brown [9]

In 2021, Brown [9] use the classical result for all square-free numbers to reprove Jameson's result [3] and indeed to generalize it.

Theorem 2.30. Let P and T be disjoint sets of prime numbers with T finite. Then the proportion of all numbers which are square-free and divisible by all the primes in T and by none of the primes in P is

$$\frac{6}{\pi^2} \prod_{p \in T} \frac{1}{1+p} \prod_{p \in P} \frac{p}{1+p}.$$

2.9.5 The work of Tangsupphathawat, Srichan and Laohakosol [13]

In the work of Tangsupphathawat et al. [13], they used the similar method due to Rieger to prove that the Piateske-Shapiro sequence defined by

$$\mathbb{N}^c = \{\lfloor n^c \rfloor : n \in \mathbb{N}, c \in \mathbb{R}, c > 1\}$$

contains infinitely many consecutive square-free integers whenever $1 < c < 3/2$. To do this, they let ε be arbitrary small positive number, not necessarily the same in different occurrences and then they got some lemmas as following:

Lemma 2.31. For $1 < c < 2$, let x be a positive real number and let q and a be two integers such that $0 \le a < q \le x^c$. Then

$$\sum_{\substack{n \le x \\ \lfloor n^c \rfloor \equiv a \pmod{q}}} 1 = \frac{x}{q} + \begin{cases} O\left(\frac{x^{(c+4)/7}}{q^{1/7}}\right) & \text{for } q < x^{c-5/4}, \\ O\left(\frac{x^{(c+1)/3}}{q^{1/3}}\right) & \text{for } x^{c-5/4} \le q < x^{c-1/2}, \\ O\left(\frac{x^c}{q}\right) & \text{for } x^{c-1/2} \le q < x^c. \end{cases}$$

Lemma 2.32. For each $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for all $n \ge 1$,

$$d(n) \le C_\varepsilon n^\varepsilon.$$

Lemma 2.33. For a fixed real $y > 1$,

$$\sum_{\substack{d,t \\ \gcd(d,t)=1 \\ dt \le y}} \frac{\mu(d)\mu(t)}{d^2t^2} = \prod_{p \text{ prime}} \left(1 - \frac{2}{p^2}\right) + O\left(y^{-1+\varepsilon}\right).$$

Lemma 2.34. Let $1 < c < 2$ and let x be a positive real number.

(I) If $A_c^2(x)$ denotes the number of quadruples d, t, u, v of positive integers satifying the conditions

$$t^2v - d^2u = 1, \qquad d^2u \le x^c, \qquad x^{c/2} < dt \le x^{2c/3},$$

then

$$A_c^2(x) \ll x^{2c/3+\varepsilon}.$$

(II) If $B_c^2(x)$ denotes the number of quadruples d, t, u, v of positive integers satifying the conditions

$$t^2v - d^2u = 1, \qquad d^2u \le x^c, \qquad dt > x^{2c/3},$$

then

$$B_c^2(x) \ll x^{2c/3+\varepsilon}.$$

Theorem 2.35. For $1 < c < 3/2$ and sufficiently small $\varepsilon > 0$,

$$\sum_{\substack{n \le x \\ \lfloor n^c \rfloor, \lfloor n^c \rfloor + 1 \text{ are square-free}}} 1 = \prod_p \left(1 - \frac{2}{p^2}\right) x + O\left(x^{(2c+1)/4+\varepsilon}\right) \quad \text{as } x \to \infty.$$

2.9.6 The work of Veasna, Srichan and Mavecha [12]

In the work of Veasna et al. [12], they gave asymptotic formula for the problem about the existence of square -free numbers in the Beatty sequences arose in 2008 by using the result on the number of values of Beatty sequence $\lfloor \alpha n + \beta \rfloor$, in an arithmetic progression in [17].

Theorem 2.36. Let $\alpha > 1$ be an irrational number and with bounded partial quotients, $\beta \in [0, \alpha)$. As $x \to \infty$, then

$$Q_r(x; \alpha, \beta) = \frac{x}{\zeta(r)} + O(x^{(r+1)/2r} \log^3 x),$$

where $Q_r(x; \alpha, \beta)$ is the number of r-free integers of Beatty sequence $\lfloor \alpha n + \beta \rfloor$, $1 \le n \le x$.

In the case of $r = 2$, they gave the improvement of (1.3) in following corollary.

Corollary 2.37. Let $\alpha > 1$ be an irrational number and with bounded partial quotients, $\beta \in [0, \alpha)$. As $x \to \infty$, then

$$Q_2(x; \alpha, \beta) = \frac{x}{\zeta(2)} + O(x^{3/4} \log^3 x).$$

Moreover, they also proved there exist infinitely many consecutive square-free numbers of the forms $\lfloor \alpha n+\beta \rfloor$, $\lfloor \alpha n+\beta \rfloor+1$, which improves Dimitrov's result in 2019.

Theorem 2.38. For $\alpha > 1$ irrational and with bounded partial quotients, $\beta \in [0,\alpha)$ and sufficiently small $\varepsilon > 0$, as $x \to \infty$, then

$$\sum_{\substack{n\le x \\ \lfloor \alpha n+\beta \rfloor,\ \lfloor \alpha n+\beta \rfloor+1 \text{ are square-free}}} 1 = \prod_{p}\Big(1-\frac{2}{p^2}\Big)x + O\Big(\alpha x^{\frac{3}{4}+\varepsilon}\log^3 x\Big).$$

Chapter 3

Odd/even r-free numbers

Let $r > 1$ be a fixed integer. A positive integer n is *r-free* if each of its prime factors appears to the power at most $r-1$. The integer 1 is considered to be r-free. As usual, 2-free and 3-free integers are called square-free and cube-free, respectively. In this chapter, we study the odd/even dichotomy for the set of r-free numbers. Moreover, we use an elementary method to give an asymtotical ratio of odd to even r-free integers.

3.1 Main Results

Let $A \subset \mathbb{N}$. Let $A(x)$ denote the number of elements in A not greater than x.

Theorem 3.1. As $x \to \infty$, we have

$$\frac{N_o(x)}{N_e(x)} \sim \frac{2^r}{2^r - 2}, \tag{3.1}$$

where $N_o(x)$ and $N_e(x)$ denote the number of odd and even r-free positive integers not greater than x, respectively.

Proof of Theorem 3.1. Let N_e and N_o be the set of all even and odd r-free integers, respectively. We assume that,

$$N_o(x) \sim ax \qquad \text{and} \qquad N_e(x) \sim bx, \qquad \text{for some } a, b \in \mathbb{R}^+. \tag{3.2}$$

Denote

$$A_1 = \{n \in N_e : 4 \nmid n\}, \qquad A_1^{'} = \{n \in N_e : 4 \mid n\},$$

for $2 \le k \le r-2$,

$$A_k = \{n \in A_{k-1}^{'} : 2^{k+1} \nmid n\}, \qquad A_k^{'} = \{n \in A_{k-1}^{'} : 2^{k+1} \mid n\}.$$

We note that $A_k^{'} = A_{k+1} \cup A_{k+1}^{'}$. Thus,

$$N_e = \Big(\bigcup_{k=1}^{r-2} A_k\Big) \cup A_{r-2}^{'}. \tag{3.3}$$

For $1 \le k \le r-2$, A_k's are disjoint set. Thus, from (3.3), we have

$$N_e(x) = \Big(\sum_{k=1}^{r-2} A_k(x)\Big) + A_{r-2}^{'}(x). \tag{3.4}$$

Now, we note that the element $n \in A_k$ is the form $n = 2^{k+1}m + 2^k$, for some $m \in \mathbb{N} \cup \{0\}$. Thus, $\frac{n}{2^k}$ is odd and r-free. This implies that, for $1 \le k \le r-2$,

$$A_k(x) = N_o\Big(\frac{x}{2^k}\Big). \tag{3.5}$$

Similary, the element $h \in A'_{r-2}$ is the form $h = 2^r m_1 + 2^{r-1}$, for some $m_1 \in \mathbb{N} \cup \{0\}$. Thus, $\frac{h}{2^{r-1}}$ is odd and r-free. This implies that,

$$A'_{r-2}(x) = N_o\Big(\frac{x}{2^{r-1}}\Big). \tag{3.6}$$

Inserting (3.5) and (3.6) in (3.4), we have

$$N_e(x) = \sum_{k=1}^{r-2} N_o\Big(\frac{x}{2^k}\Big) + N_o\Big(\frac{x}{2^{r-1}}\Big) = \sum_{k=1}^{r-1} N_o\Big(\frac{x}{2^k}\Big). \tag{3.7}$$

In view of (3.2) and (3.7), we have

$$bx = \sum_{k=1}^{r-1} a\frac{x}{2^k} = ax\Big(1 - 2^{1-r}\Big).$$

This proves (3.1).

Now it remains to prove the existence of a and b.

In view of (3.7), we write

$$N_r(x) = N_o(x) + N_e(x) = \sum_{k=0}^{r-1} N_o\Big(\frac{x}{2^k}\Big). \tag{3.8}$$

We replace x in (3.8) by $x/2$ and subtract this with (3.8). We have

$$N_r(x) - N_r\Big(\frac{x}{2}\Big) = N_o(x) - N_o\Big(\frac{x}{2^r}\Big). \tag{3.9}$$

Replace x in (3.9) by $x/2^r$, we have

$$N_r\Big(\frac{x}{2^r}\Big) - N_r\Big(\frac{x}{2^{r+1}}\Big) = N_o\Big(\frac{x}{2^r}\Big) - N_o\Big(\frac{x}{2^{2r}}\Big). \tag{3.10}$$

In view of (3.9) and (3.10), we have

$$N_r(x) - N_r\Big(\frac{x}{2}\Big) + N_r\Big(\frac{x}{2^r}\Big) - N_r\Big(\frac{x}{2^{r+1}}\Big) = N_o(x) - N_o\Big(\frac{x}{2^{2r}}\Big).$$

Repeating this, we have

$$N_o(x) - N_o\Big(\frac{x}{2^{r(k+1)}}\Big) = \sum_{i=0}^{k} N_r\Big(\frac{x}{2^{ri}}\Big) - \sum_{i=0}^{k} N_r\Big(\frac{x}{2^{ri+1}}\Big). \tag{3.11}$$

The asymptotic formula (1.1) implies $N_r(x) \sim cx$, where $c = 1/\zeta(r)$. Then, for $\epsilon > 0$, we take x_0 such that

$$(c-\epsilon)x \le N_r(x) \le (c+\epsilon)x, \text{ for } x \ge x_0. \tag{3.12}$$

To apply inequality (3.12) with (3.11), we take k such that $\frac{x}{2^{rk+r+1}} < x_0 \le \frac{x}{2^{rk+1}}$. Then, we have

$$(c-\epsilon)\frac{x}{2^{ri+1}} \le N_r(\frac{x}{2^{ri+1}}) \le (c+\epsilon)\frac{x}{2^{ri+1}}, \tag{3.13}$$

and

$$(c-\epsilon)\frac{x}{2^{ri}} \le N_r(\frac{x}{2^{ri}}) \le (c+\epsilon)\frac{x}{2^{ri}}, \tag{3.14}$$

for $0 \le i \le k$. In view of (3.11), (3.13) and (3.14), we have

$$N_o(x) - N_o\Big(\frac{x}{2^{r(k+1)}}\Big) \le \sum_{i=0}^{k}(c+\epsilon)\frac{x}{2^{ri}} - \sum_{i=0}^{k}(c-\epsilon)\frac{x}{2^{ri+1}}$$
$$= x\Big(\frac{c}{2}+\frac{3\epsilon}{2}\Big)\sum_{i=0}^{k}\frac{1}{2^{ri}}$$
$$\le x\Big(\frac{c}{2}+\frac{3\epsilon}{2}\Big)\sum_{i=0}^{\infty}\frac{1}{2^{ri}}$$
$$= x\Big(\frac{c}{2}+\frac{3\epsilon}{2}\Big)\frac{2^r}{2^r-1}.$$

From the choosing k such that $\frac{x}{2^{rk+r+1}} < x_0 \le \frac{x}{2^{r(k+1)}}$,
we have $N_o\Big(\frac{x}{2^{r(k+1)}}\Big) \le N_o(2x_0) < 2x_0$. Then, we have

$$N_o(x) \le x\Big(c+3\epsilon\Big)\frac{2^{r-1}}{2^r-1} + 2x_0 \le x\Big(c+3\epsilon\Big)\frac{2^{r-1}}{2^r-1} + \frac{2^{r+1}}{2^r-1}x_0.$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$N_o(x) \le x\Big(c+3\epsilon\Big)\frac{2^{r-1}}{2^r-1} + \frac{2^{r-1}}{2^r-1}4x\epsilon = x\Big(c+7\epsilon\Big)\frac{2^{r-1}}{2^r-1}.$$

By the similar proof we deal with the lower bound. In view of (3.11), (3.13) and (3.14), we have

$$N_o(x) - N_o\Big(\frac{x}{2^{r(k+1)}}\Big) \ge \sum_{i=0}^{k}(c-\epsilon)\frac{x}{2^{ri}} - \sum_{i=0}^{k}(c+\epsilon)\frac{x}{2^{ri+1}}$$
$$= x\Big(\frac{c}{2}-\frac{3\epsilon}{2}\Big)\sum_{i=0}^{k}\frac{1}{2^{ri}}$$
$$= x\Big(\frac{c}{2}-\frac{3\epsilon}{2}\Big)\frac{2^r}{2^r-1} - x\Big(\frac{c}{2}-\frac{3\epsilon}{2}\Big)\frac{2^{-rk}}{2^r-1}$$
$$\ge x\Big(c-3\epsilon\Big)\frac{2^{r-1}}{2^r-1} - \frac{cx}{2^{rk+1}(2^r-1)}.$$

We note that $2^r x_0 \ge \frac{x}{2^{rk+1}}$. Then, we have

$$N_o(x) \ge N_o(x) - N_o\Big(\frac{x}{2^{r(k+1)}}\Big) \ge x\Big(c-3\epsilon\Big)\frac{2^{r-1}}{2^r-1} - cx_0\frac{2^r}{2^r-1}.$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$N_o(x) \ge N_o(x) - O\Big(\frac{x}{2^{r(k+1)}}\Big) \ge x\Big(c-3\epsilon-2c\epsilon\Big)\frac{2^{r-1}}{2^r-1}.$$

This proves the existence of a and consequently b also exists, in fact $b = c - a$. □

Corollary 3.2. As $x \to \infty$, we have

$$\frac{C_{odd}(x)}{C_{even}(x)} \sim 2, \tag{3.15}$$

where $C_{odd}(x)$ and $C_{even}(x)$ denote the number of odd and even square-free positive integers not greater than x, respectively.

Example 3.1. Let $N_r(x)$ be the set of r-free integers not greater than x, we have

$$N_2(60) = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 37, 38, 39, 41, 42, 43, 46, 47, 51, 53, 55, 57, 58, 59\},$$

$$N_3(60) = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, 31, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 57, 58, 59, 60\}.$$

Thus we get the following:

x	5	10	20	30	40	60
$C_{odd}(x)/C_{even}(x)$	$3/1$	$4/3$	$9/4$	$13/7$	$17/9$	$25/12$
$N_o(x)/N_e(x)$	$3/2$	$5/4$	$10/8$	$14/12$	$19/15$	$29/22$

As $x \to \infty$, we can see that $\frac{C_{odd}(x)}{C_{even}(x)} \sim 2$ and $\frac{N_o(x)}{N_e(x)} \sim \frac{2^3}{2^3-2} = \frac{4}{3}$.

Chapter 4

On the natural density of r-free numbers

The density of a finite set of distinct positive integers, $\delta[A]$, is the ratio of the number of its elements to its largest element. The natural density of an infinite increasing sequence of positive integers, a_n, is

$$\delta[A] = \lim_{n\to\infty} \frac{n}{a_n}.$$

In this chapter, we use the elementary method in [4] to prove a similar result as in [9] and generalize it to r-free integers.

4.1 Auxilary Lemma

Let A be a given set, for $x > 1$, we denote $A(x)$ be the number of elements in $A \cap [1, x]$. Let $P = \{p_1, p_2, ..., p_k\}$ be a finite set of prime numbers. For any integer $r \geq 2$, let N_r be the set of all r-free numbers. Let $C_P = \{n \in N_r | (n, P) = 1\}$ and $C'_P = \{n \in N_r | (n, P) \neq 1\}$. Here (n, P) denotes the greatest common divisor of n and all prime in P. For $0 \leq \alpha_i \leq r-1$, $1 \leq i \leq k$, we define by

$$\mathcal{A}_{\alpha_1,\cdots,\alpha_k} := \left\{ n \in C'_P \middle| n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} m,\ m \in C_P \right\}$$

and $\mathcal{A}_{0,\cdots,0} = C_P$.

The following Lemma will be used in our proof.

Lemma 4.1. Let p be a given prime number and $P = \{p\}$. As $x \to \infty$, we have $\frac{C_P(x)}{C'_P(x)} \sim \frac{p^r - p^{r-1}}{p^{r-1}-1}$.

Proof. The proof is similar to the proof of Theorem 3.1.

First we assume that,

$$C_P(x) \sim ax \qquad \text{and} \qquad C'_P(x) \sim bx, \qquad \text{for some } a, b \in \mathbb{R}^+. \tag{4.1}$$

First, we wish to show that,

$$\frac{a}{b} = \frac{p^r - p^{r-1}}{p^{r-1} - 1}. \tag{4.2}$$

For $1 \leq i \leq r-1$, we denote by

$$A_i = \{n \in C'_P : n = p^i m,\ m \in C_P\}.$$

We note that, for $1 \leq i \leq r-1$, A_i are disjoint sets. Thus,

$$C'_P = \bigcup_{i=1}^{r-1} A_i. \tag{4.3}$$

For $1 \leq i \leq r-1$, we note that, each element $n \in A_i$ is the form $n = p^i m$ such that $(p,m) = 1$ for some $m \in C_P$. Thus, $\frac{n}{p^i}$ is an element in C_P. This implies that, for $1 \leq i \leq r-1$,

$$A_i(x) = C_P\Big(\frac{x}{p^i}\Big). \tag{4.4}$$

In view of (4.3) and (4.4), we have

$$C'_P(x) = \sum_{i=1}^{r-1} C_P\Big(\frac{x}{p^i}\Big). \tag{4.5}$$

From (4.1) and (4.5), we get

$$bx = \sum_{i=1}^{r-1} a\frac{x}{p^i} = ax\Big(\frac{p^{r-1}-1}{p^r - p^{r-1}}\Big).$$

This proves (4.2). Now it remains to prove the existence of a and b.
In view of (4.5), we write

$$N_r(x) = C_P(x) + C'_P(x) = \sum_{i=0}^{r-1} C_P\Big(\frac{x}{p^k}\Big). \tag{4.6}$$

We replace x in (4.6) by x/p and substract this with (4.6). We have

$$N_r(x) - N_r\Big(\frac{x}{p}\Big) = C_P(x) - C_P\Big(\frac{x}{p^r}\Big). \tag{4.7}$$

Replace x in (4.7) by x/p^r, we have

$$N_r\Big(\frac{x}{p^r}\Big) - N_r\Big(\frac{x}{p^{r+1}}\Big) = C_P\Big(\frac{x}{p^r}\Big) - C_P\Big(\frac{x}{p^{2r}}\Big). \tag{4.8}$$

In view of (4.7) and (4.8), we have

$$N_r(x) - N_r\Big(\frac{x}{p}\Big) + N_r\Big(\frac{x}{p^r}\Big) - N_r\Big(\frac{x}{p^{r+1}}\Big) = C_P(x) - C_P\Big(\frac{x}{p^{2r}}\Big).$$

Repeating this, we have

$$C_P(x) - C_P\Big(\frac{x}{p^{r(k+1)}}\Big) = \sum_{i=0}^{k} N_r\Big(\frac{x}{p^{ri}}\Big) - \sum_{i=0}^{k} N_r\Big(\frac{x}{p^{ri+1}}\Big), \text{ for some } k \in \mathbb{N}. \tag{4.9}$$

It is well known that, $N_r(x) \sim cx$, where $c = 1/\zeta(r)$. Then, for $\epsilon > 0$, we take x_0 such that

$$(c-\epsilon)x \leq N_r(x) \leq (c+\epsilon)x, \text{ for } x \geq x_0. \tag{4.10}$$

To apply inequality (4.10) with (4.9), we take k such that $\frac{x}{p^{rk+r+1}} < x_0 \leq \frac{x}{p^{rk+1}}$ and from (4.9) and (4.10), we have

$$\begin{aligned} C_P(x) - C_P\Big(\frac{x}{p^{r(k+1)}}\Big) &\leq \sum_{i=0}^{k}(c+\epsilon)\frac{x}{p^{ri}} - \sum_{i=0}^{k}(c-\epsilon)\frac{x}{p^{ri+1}} \\ &= x\Big(\frac{p(c+\epsilon)-(c-\epsilon)}{p}\Big)\sum_{i=0}^{k}\frac{1}{p^{ri}} \\ &\leq x\Big(\frac{p(c+\epsilon)-(c-\epsilon)}{p}\Big)\frac{p^r}{p^r-1}. \end{aligned}$$

From the choosing k such that $\frac{x}{p^{rk+r+1}} < x_0 \leq \frac{x}{p^{r(k+1)}}$, we have $C_p\left(\frac{x}{p^{r(k+1)}}\right) \leq N_r(px_0) < px_0$. Then, we have

$$\begin{aligned} C_P(x) &\leq x\Big(p(c+\epsilon)-(c-\epsilon)\Big)\frac{p^{r-1}}{p^r-1}+px_0 \\ &\leq x\Big(p(c+\epsilon)-(c-\epsilon)\Big)\frac{p^{r-1}}{p^r-1}+\frac{p^{r+1}}{p^r-1}x_0. \end{aligned}$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$\begin{aligned} C_P(x) &\leq x\Big(p(c+\epsilon)-(c-\epsilon)\Big)\frac{p^{r-1}}{p^r-1}-(p^2\epsilon x)\frac{p^{r-1}}{p^r-1} \\ &= x\Big(p(c+\epsilon)-p^2\epsilon-c+\epsilon\Big)\frac{p^{r-1}}{p^r-1}. \end{aligned}$$

By a similar proof we deal with the lower bound. In view of (4.9) and (4.10), for the integer k such that $\frac{x}{p^{rk+r+1}} < x_0 \leq \frac{x}{p^{rk+1}}$, we have

$$\begin{aligned} C_P(x)-C_P\Big(\frac{x}{p^{r(k+1)}}\Big) &\geq \sum_{i=0}^{k}(c-\epsilon)\frac{x}{p^{ri}}-\sum_{i=0}^{k}(c+\epsilon)\frac{x}{p^{ri+1}} \\ &= x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\sum_{i=0}^{k}\frac{1}{p^{ri}} \\ &= x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\Big(\frac{p^r-p^{r-rk}}{p^r-1}\Big) \\ &= x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\Big(\frac{p^r}{p^r-1}\Big)-x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\Big(\frac{p^{r-rk}}{p^r-1}\Big) \\ &\geq \frac{x\Big(p(c-\epsilon)-(c+\epsilon)\Big)(p^{r-1})}{p^r-1}-cx\Big(\frac{p^{r-rk}}{p^r-1}\Big). \end{aligned}$$

We note that $p^{r+1}x_0 > \frac{x}{p^{rk}}$. Then, we have

$$\begin{aligned} C_P(x) &\geq C_P(x)-C_P\Big(\frac{x}{2^{r(k+1)}}\Big) \\ &\geq \frac{x\big(p(c-\epsilon)-(c+\epsilon)\big)(p^{r-1})}{p^r-1}-\frac{cx_0p^{2r+1}}{p^r-1}. \end{aligned}$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$C_P(x) \geq \frac{x\big(p(c-\epsilon)-(c+\epsilon)\big)(p^{r-1})}{p^r-1}-\frac{cx\epsilon p^{2r+1}}{p^r-1}=\frac{x\big(p(c-\epsilon)-(c+\epsilon)-c\epsilon p^{r+2}\big)(p^{r-1})}{p^r-1}.$$

This proves the existence of a. The existence of b follows from the existence of a, since $N_r(x) = C_P(x) + C'_P(x)$. A second proof of the same sort is not needed. □

4.2 Main Results

Theorem 4.2. Let $P = \{p_1, p_2, ..., p_k\}$ be a finite set of prime numbers. For any $r \geq 2$ be integers, let N_r be the set of all r-free numbers. As $x \to \infty$, we have

$$\frac{C'_P(x)}{C_P(x)} \sim \prod_{p\in P}\frac{p^r-1}{p^r-p^{r-1}}-1,$$

where $C_P = \{n \in N_r | (n, P) = 1\}$ and $C'_P = \{n \in N_r | (n, P) \neq 1\}$. Here (n, P) denotes the greatest common divisor of n and all prime in P.

Proof of Theorem 4.2. The proof is similar to the proof of Lemma 4.1 but much more complex. First, we assume that

$$C_P(x) \sim \delta x \qquad \text{and} \qquad C'_P(x) \sim \beta x, \qquad \text{for some } \delta, \beta \in \mathbb{R}^+. \tag{4.11}$$

Now, we note that each element $n \in \mathcal{A}_{\alpha_1,\cdots,\alpha_k}$ is the form $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}m,\ m \in C_P$, $0 \le \alpha_i \le r-1$ and for some $\alpha_i \ne 0$, $1 \le i \le k$. Thus $\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}$ is an element in C_P. This implies that, for $0 \le \alpha_i \le r-1$, $1 \le i \le k$,

$$\mathcal{A}_{\alpha_1,\cdots,\alpha_k}(x) = C_P\Big(\frac{x}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}\Big). \tag{4.12}$$

From (4.12), we sum $\mathcal{A}_{\alpha_1,\cdots,\alpha_k}(x)$ for all $0 \le \alpha_i \le r-1$ but not all zero, $1 \le i \le k$, and get

$$\begin{aligned} C'_P(x) = & \sum_{\substack{0\le\alpha_i\le r-1 \\ \alpha_i \text{ are not all zero}}} C_P\Big(\frac{x}{\prod_{i=1}^k p_i^{\alpha_i}}\Big) \\ = & \sum_{1\le i\le k}\sum_{1\le\alpha_i\le r-1} C_P\left(\frac{x}{p_i^{\alpha_i}}\right) + \sum_{1\le i_1<i_2\le k}\sum_{1\le\alpha_{i_1},\,\alpha_{i_2}\le r-1} C_P\left(\frac{x}{p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}}\right) \\ & + \cdots + \sum_{1\le\alpha_1,\cdots,\alpha_k\le r-1} C_P\left(\frac{x}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}\right). \end{aligned} \tag{4.13}$$

In view of (4.11), we have

$$\begin{aligned} \beta x = & \delta x \sum_{1\le i\le k}\sum_{1\le\alpha_i\le r-1}\left(\frac{1}{p_i^{\alpha_i}}\right) + \delta x \sum_{1\le i_1<i_2\le k}\sum_{1\le\alpha_{i_1},\,\alpha_{i_2}\le r-1}\left(\frac{1}{p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}}\right) \\ & + \cdots + \delta x \sum_{1\le\alpha_1,\cdots,\alpha_k\le r-1}\left(\frac{1}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}\right) \\ = & \delta x\Big(\prod_{p\in P}\frac{p^r-1}{p^r-p^{r-1}}-1\Big). \end{aligned} \tag{4.14}$$

Now, it remains to show the existence of δ and β. To do this, we use the mathematical induction on k, where k is the number of primes in P. For the convenient, we let $P_{k-1} = \{p_1,\cdots,p_{k-1}\}$ and $C_{P_k} = \{n \in N_r | (n, P_k) = 1\}$.

We assume that, for $1 \le j < k$, the exists δ_j such that $C_{P_j}(x) \sim \delta_j x$. From Lemma 4.1, δ_1 exists. Let q be a prime number with $q \notin P_{k-1}$. Note that

$$C_{P_{k-1}}(x) = C_{P_{k-1}\cup\{q\}}(x) + C^*_{P_{k-1}\cup\{q\}}(x), \tag{4.15}$$

where $C^*_{P_{k-1}\cup\{q\}} = \{n \in N_r | (n, P_{k-1}) = 1 \text{ and } q \mid n\}$. From the same reason in (4.12), we have

$$C^*_{P_{k-1}\cup\{q\}}(x) = \sum_{1\le\alpha\le r-1} C_{P_{k-1}}\left(\frac{x}{q^\alpha}\right). \tag{4.16}$$

In view of (4.15) and (4.16), we have

$$C_{P_{k-1}}(x) = C_{P_{k-1}\cup\{q\}}(x) + \sum_{1\le\alpha\le r-1} C_{P_{k-1}}\left(\frac{x}{q^\alpha}\right). \tag{4.17}$$

Since $C_{P_{k-1}}(x) \sim \delta_{k-1}x$, we have

$$C_{P_{k-1}\cup\{q\}}(x) \sim \Big(1 - \sum_{1\le\alpha\le r-1}\Big(\frac{1}{q^{\alpha}}\Big)\Big)\delta_{k-1}x.$$

This shows that δ_k exists. By the mathematical induction, the proof is completed. $\square$

Remark 4.3. From

$$\frac{C'_P(x)}{C_P(x)} = \frac{N_r(x) - C_P(x)}{C_P(x)} = \frac{N_r(x)}{C_P(x)} - 1,$$

we have, as $x \to \infty$,

$$\frac{C_P(x)}{N_r(x)} \sim \prod_{p\in P}\frac{p^r - p^{r-1}}{p^r - 1}. \tag{4.18}$$

In the case $T = \emptyset$, we can see that, the equation (4.18) covers Brown's result in (B).

Example 4.1. 1) Setting $P = \{2\}$ and $r = 3$ we have, as $x \to \infty$,

$$\frac{C'_P(x)}{C_P(x)} \sim \frac{2^3-1}{2^3-2^2} - 1 = \frac{7}{4} - 1 = \frac{3}{4}.$$

We can see that the asymptotic ratio $C'_P(x)/C_P(x)$ is similar to $N_o(x)/N_e(x)$ in Example 3.1.

2) Setting $P = \{2, 3, 5\}$ and $r = 3$ we have, as $x \to \infty$,

$$\frac{C'_P(x)}{C_P(x)} \sim \Big(\frac{2^3-1}{2^3-2^2} - 1\Big)\Big(\frac{3^3-1}{3^3-3^2} - 1\Big)\Big(\frac{5^3-1}{5^3-5^2} - 1\Big) = (\frac{3}{4})(\frac{4}{9})(\frac{6}{25}) = \frac{2}{25}.$$

3) Setting $P = \{2, 3, 11, 17\}$ and $r = 4$ we have, as $x \to \infty$,

$$\begin{aligned}\frac{C'_P(x)}{C_P(x)} &\sim \Big(\frac{2^4-1}{2^4-2^3} - 1\Big)\Big(\frac{3^4-1}{3^4-3^3} - 1\Big)\Big(\frac{11^4-1}{11^4-11^3} - 1\Big)\Big(\frac{17^4-1}{17^4-17^3} - 1\Big)\\ &= (\frac{7}{8})(\frac{13}{27})(\frac{133}{1331})(\frac{4912}{78608})\\ &= \frac{3715621}{1412467848}.\end{aligned}$$

Chapter 5

Consecutive generalized r-free integers in Beatty sequences

Let $\alpha > 1$ be an irrational number and with bounded partial quotients, $\beta \in [0, \alpha)$. The Beatty sequence of parameter α and β is defined by

$$\{\lfloor \alpha n + \beta \rfloor\}_{n \in \mathbb{N}},$$

where $\lfloor z \rfloor$ is the integer part of $z \in \mathbb{R}$.

Let k and r be fixed positive integers with $1 < r < k$. A positive integer n is called a (k, r)-*integer* if n is of the form $n = a^k b$, where $a, b \in \mathbb{N}$ and b is r-free. In 1966, Subbarao and Harris [21] remarked that in the limiting case when $k \to \infty$, a (k, r)-integer becomes an r-free integer. Thus, the (k, r)-integer is a generalized r-free integer. A positive integer n is called semi r-free if in the canonical factorization of n no exponent is equal to r. The (k, r)-integers also include the semi r-free integers when $k = r + 1$.

Example 5.1. Let $k = 5,\ r = 3$, then

$$n_1 = 288 = 2^5 \cdot 3^2,$$

$$n_2 = 38880 = 6^5 \cdot 5^2 \cdot 2$$

are $(5, 3)$- integers.

Let $\lambda_{k,r}(n)$ be the multiplicative function defined by

$$\lambda_{k,r}(p^a) = \begin{cases} 1, & \text{if } a \equiv 0 \text{ (mod } k\text{)}; \\ -1, & \text{if } a \equiv r \text{ (mod } k\text{)}; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{m=1}^{\infty} \frac{\lambda_{k,r}(m)}{m^s} = \frac{\zeta(ks)}{\zeta(rs)}, \qquad s > \frac{1}{r}. \tag{5.1}$$

In this chapter, we shall use the technique of Tangsupphathawat et al. [13] to generalize the previous result on the distribution of consecutive square-free integers in Beatty sequences.

5.1 Auxilary Lemmas

We now denote $f(x) \ll g(x)$ means $f(x) = O(g(x))$ and collect some lemmas needed later.

Lemma 5.1. [22, Lemma 2.6] Let $q_{k,r}(n)$ denote the characteristic function of the set of (k,r)-integers. Then

$$q_{k,r}(n) = \sum_{c|n} \lambda_{k,r}(c) = \sum_{a^k b^r c = n} \mu(b).$$

Let $d(r,k,n)$ denote the number of ways of writing an integer n in the form $n = n_1^r n_2^k$, and put

$$D(r,k,x) = \sum_{n \le x} d(r,k,n).$$

In the proof of our main result, we need the following estimate for the function $D(r,k,x)$ whose proof can be found in [1, Section 14.3].

Lemma 5.2. ([1]) For a sufficiently large $x \in \mathbb{R}$, we have

$$D(r,k,x) \ll x^{1/r}.$$

Lemma 5.3. For $x \ge 1$, we have

$$\sum_{m \le x} \lambda_{k,r}(m) \ll x^{1/r}.$$

Proof. In view of Lemmas 5.1 and 5.2, we have

$$\sum_{m \le x} \lambda_{k,r}(m) = \sum_{a^k b^r \le x} \mu(b) \ll \sum_{a^k b^r \le x} 1 \ll x^{1/r}.$$

□

The following lemma is a technique of Tangsupphathawat et al.[13] that is used to study the consecutive square-free in Piatetski-Shapiro sequences.

Lemma 5.4. Let $A_{\alpha,\beta}(x;k,r)$ and $B_{\alpha,\beta}(x;k,r)$ denote the number of 6-tuples (d_1,t_1,d_2,t_2,u,v) satisfying the conditions

$$d_2^k t_2^r v - d_1^k t_1^r u = 1, \;\; d_1^k t_1^r u \le \alpha x + \beta. \tag{5.2}$$

I) If $x^{1/2} < d_1^k t_1^r d_2^k t_2^r \le x^{3/2-1/2r}$, then

$$A_{\alpha,\beta}(x;k,r) \ll \alpha x^{1/2r+1/2} \log x.$$

II) If $x^{3/2-1/2r} < d_1^k t_1^r d_2^k t_2^r$, then

$$B_{\alpha,\beta}(x;k,r) \ll \alpha x^{1/2r+1/2} \log x.$$

Proof. I) For a fixed choice of d_1, t_1, d_2 and t_2 satisfying (5.2), we have $d_1^k t_1^r u \equiv -1 \pmod{d_2^k t_2^r}$, which fixes the value of u modulo $d_2^k t_2^r$. From (5.2), the total number of possibilities for u is $O(1 + \frac{(\alpha x+\beta)}{d_1^k t_1^r d_2^k t_2^r})$. By (5.2), the value of v is fixed for a given choice of

d_1, t_1, d_2, t_2, u. Then by Lemma 5.1, we have

$$
\begin{aligned}
A_{\alpha,\beta}(x;k,r) &\ll \sum_{x^{1/2} < d_1^k d_2^k t_1^r t_2^r \le x^{3/2-1/2r}} \Big(1 + \frac{\alpha x + \beta}{d_1^k t_1^r d_2^k t_2^r}\Big) \\
&\ll \sum_{x^{1/2} < m \le x^{3/2-1/2r}} \tau(m) d(k,r,m) \Big(1 + \frac{\alpha x + \beta}{m}\Big) \\
&\ll \sum_{x^{1/2} < m \le x^{3/2-1/2r}} d(k,r,m) \Big(\log m + \frac{\alpha x + \beta}{m} \log m\Big) \\
&\ll \big(x^{3/2-1/2r}\big)^{1/r} \log x + (\alpha x + \beta)\big(x^{1/2}\big)^{1/r-1} \log x \\
&\ll x^{3/2r-1/2r^2} \log x + (\alpha x + \beta)(x^{1/2r-1/2}) \log x \\
&\ll x^{3/2r-1/2r^2} \log x + \alpha x^{1/2r+1/2} \log x \\
&\ll \alpha x^{1/2r+1/2} \log x.
\end{aligned}
$$

II) From (5.2), we have $uvd_1^k t_1^r d_2^k t_2^r \ll (\alpha x + \beta)(\alpha x + \beta + 1)$, whence $uv \ll (\alpha x + \beta)(\alpha x + \beta + 1) x^{-3/2+1/2r}$ for every 6-tuple counted by $B_{\alpha,\beta}(x)$. From a divisor argument, the total number of choices for u, v is therefore $O(\alpha^2 x^{1/2r+1/2} \log x)$. For every such choice u, v, the number of solutions in d_1, t_1, d_2, t_2 of the equation $d_2^k t_2^r v - d_1^k t_1^r u = 1$ is $O(\log x)$. □

Lemma 5.5. ([17]) For $\alpha > 1$ irrational and with bounded partial quotients, $\beta \in [0, \alpha)$, and positive integer $d \ge 2, 0 \le a < d$, we have

$$
\sum_{\substack{n \le x \\ \lfloor \alpha n + \beta \rfloor \equiv a (\bmod d)}} 1 = \frac{x}{d} + O(d \log^3 x) \quad \text{as } x \to \infty.
$$

For growing difference d the result is non-trivial provided $d \ll \sqrt{x} \log^{-3/2-\varepsilon} x$, for $\varepsilon > 0$.

5.2 Main Results

Theorem 5.6. Let $\alpha > 1$ be an irrational number and with bounded partial quotients, $\beta \in [0, \alpha)$. As $x \to \infty$, we have

$$
Q_{k,r}(x;\alpha,\beta) = x\frac{\zeta(k)}{\zeta(r)} + O(x^{(1/2r+1/2)} \log^3 x),
$$

where $Q_{k,r}(x;\alpha,\beta)$ is the number of (k,r)-integers of Beatty sequence $\lfloor \alpha n + \beta \rfloor$, $1 \le n \le x$.

Proof of Theorem 5.6. Let $x > 1$, we write

$$\begin{aligned}
Q_{k,r}(x;\alpha,\beta) &= \sum_{\substack{n\le x\\ \lfloor \alpha n+\beta\rfloor \text{ is } (k,\, r)\text{-integer}}} 1\\
&= \sum_{n\le x} q_{k,r}(\lfloor \alpha n+\beta\rfloor)\\
&= \sum_{n\le x}\Big(\sum_{d\mid\lfloor \alpha n+\beta\rfloor}\lambda_{k,r}(d)\Big)\\
&= \sum_{d\le \alpha x+\beta}\lambda_{k,r}(d)\sum_{\substack{n\le x\\ \lfloor \alpha n+\beta\rfloor\equiv 0(\bmod d)}} 1\\
&= \sum_{d\le \sqrt{x}}\lambda_{k,r}(d)\sum_{\substack{n\le x\\ \lfloor \alpha n+\beta\rfloor\equiv 0(\bmod d)}} 1 + \sum_{\sqrt{x}<d\le \alpha x+\beta}\lambda_{k,r}(d)\sum_{\substack{n\le x\\ \lfloor \alpha n+\beta\rfloor\equiv 0(\bmod d)}} 1.
\end{aligned}$$

In view of Lemma 5.5, we get

$$\begin{aligned}
\sum_{d\le \sqrt{x}}\lambda_{k,r}(d)\sum_{\substack{n\le x\\ \lfloor \alpha n+\beta\rfloor\equiv 0(\bmod d)}} 1 &= \sum_{d\le\sqrt{x}}\lambda_{k,r}(d)\Big(\frac{x}{d}+O(d\log^3 x)\Big)\\
&= x\sum_{d\le\sqrt{x}}\frac{\lambda_{k,r}(d)}{d}+O\Big(\log^3 x\Big|\sum_{d\le\sqrt{x}}\lambda_{k,r}(d)d\Big|\Big)\\
&= x\sum_{d\le\sqrt{x}}\frac{\lambda_{k,r}(d)}{d}+O\big(\log^3 x\cdot x^{1/2r+1/2}\big).
\end{aligned}$$

We note that

$$\begin{aligned}
\sum_{\sqrt{x}<d\le \alpha x+\beta}\lambda_{k,r}(d)\sum_{\substack{n\le x\\ \lfloor \alpha n+\beta\rfloor\equiv 0(\bmod d)}} 1 &\ll \sum_{\sqrt{x}<d\le \alpha x+\beta}\lambda_{k,r}(d)\Big(\frac{x}{d}\Big)\\
&\ll x\sum_{\sqrt{x}<d\le \alpha x+\beta}\frac{\lambda_{k,r}(d)}{d}\\
&\ll x(x^{1/2})^{1/r-1}\\
&\ll x^{1/2r+1/2}.
\end{aligned}$$

This proves Theorem 5.6. □

In the case of $k = r+1$, we obtain the following corollary.

Corollary 5.7. Let $\alpha > 1$ be an irrational number and with bounded partial quotients, $\beta \in [0,\alpha)$. As $x \to \infty$, we have

$$Q_{r+1,r}(x;\alpha,\beta) = x\frac{\zeta(r+1)}{\zeta(r)} + O(x^{(1/2r+1/2)}\log^3 x),$$

where $Q_{r+1,r}(x;\alpha,\beta)$ is the number of semi r-free integers of Beatty sequence $\lfloor \alpha n+\beta\rfloor$, $1 \le n \le x$.

Theorem 5.8. For a large $x \in \mathbb{R}$, let $T_{k,r}(x)$ denote the number of positive integers $n \le x$ such that $\lfloor \alpha n+\beta\rfloor$ and $\lfloor \alpha n+\beta\rfloor + 1$ are (k,r)-integers. We have, as $x \to \infty$,

$$T_{k,r}(x) = x\sum_{m=1}^{\infty}\frac{\tau(m)\lambda_{k,r}(m)}{m} + O(\alpha x^{(1/2r+1/2)+\epsilon}\log^3 x),$$

where $\tau(n)$ denote the number of divisors of n.

Proof of Theorem 5.8. Let k, r be integers with $1 < r < k$. For a large $x \in \mathbb{R}$, let

$$\begin{aligned} T_{\alpha,\beta}(x;k,r) &= \sum_{n\le x} q_{k,r}(\lfloor \alpha n+\beta\rfloor) q_{k,r}(\lfloor \alpha n+\beta\rfloor+1) \\ &= \sum_{n\le x}\Bigg(\sum_{d\mid\lfloor \alpha n+\beta\rfloor}\lambda_{k,r}(d)\Bigg)\Bigg(\sum_{t\mid\lfloor \alpha n+\beta\rfloor+1}\lambda_{k,r}(t)\Bigg) \\ &= \Bigg(\sum_{\substack{d,t\\ \gcd(d,t)=1\\ dt\le\sqrt{\alpha x+\beta}}} + \sum_{\substack{d,t\\ \gcd(d,t)=1\\ dt>\sqrt{\alpha x+\beta}}}\Bigg)\lambda_{k,r}(d)\lambda_{k,r}(t) \sum_{\substack{n\le x\\ \lfloor \alpha n+\beta\rfloor\equiv 0 \,(\mathrm{mod}\, d)\\ \lfloor \alpha n+\beta\rfloor+1\equiv 0\,(\mathrm{mod}\, t)}} 1. \end{aligned}$$

In view of Lemma 5.4, we have

$$T_{\alpha,\beta}(x;k,r) = \sum_{\substack{d,t\\ \gcd(d,t)=1\\ dt\le\sqrt{\alpha x+\beta}}}\lambda_{k,r}(d)\lambda_{k,r}(t) \sum_{\substack{n\le x\\ \lfloor \alpha n+\beta\rfloor\equiv 0 \,(\mathrm{mod}\, d)\\ \lfloor \alpha n+\beta\rfloor+1\equiv 0\,(\mathrm{mod}\, t)}} 1 + O(\alpha x^{1/2r+1/2}\log x).$$

By the Chinese remainder theorem, there is a positive integer ω, unique modulo dt, satisfying the congruence system $\omega \equiv 0 \pmod d$ and $\omega + 1 \equiv 0 \pmod t$. Thus,

$$\sum_{\substack{d,t\\ \gcd(d,t)=1\\ dt\le\sqrt{\alpha x+\beta}}}\lambda_{k,r}(d)\lambda_{k,r}(t) \sum_{\substack{n\le x\\ \lfloor \alpha n+\beta\rfloor\equiv 0 \,(\mathrm{mod}\, d)\\ \lfloor \alpha n+\beta\rfloor+1\equiv 0\,(\mathrm{mod}\, t)}} 1 = \sum_{\substack{d,t\\ \gcd(d,t)=1\\ dt\le\sqrt{\alpha x+\beta}}}\lambda_{k,r}(d)\lambda_{k,r}(t) \sum_{\substack{n\le N\\ \lfloor \alpha n+\beta\rfloor\equiv \omega \,(\mathrm{mod}\, dt)}} 1.$$

In view of lemma 5.5, we have

$$\sum_{\substack{d,t\\ \gcd(d,t)=1\\ dt\le\sqrt{\alpha x+\beta}}}\lambda_{k,r}(d)\lambda_{k,r}(t) \sum_{\substack{n\le x\\ \lfloor \alpha n+\beta\rfloor\equiv 0 \,(\mathrm{mod}\, d)\\ \lfloor \alpha n+\beta\rfloor+1\equiv 0\,(\mathrm{mod}\, t)}} 1 = x\sum_{\substack{d,t\\ \gcd(d,t)=1\\ dt\le\sqrt{\alpha x+\beta}}}\frac{\lambda_{k,r}(d)\lambda_{k,r}(t)}{dt} + O\Bigg(\log^3 x \sum_{\substack{d,t\\ \gcd(d,t)=1\\ dt\le\sqrt{\alpha x+\beta}}} dt\lambda_{k,r}(d)\lambda_{k,r}(t)\Bigg).$$

In view of Lemma 5.3, we have

$$\begin{aligned} \sum_{\substack{d,t\\ \gcd(d,t)=1\\ dt\le\sqrt{\alpha x+\beta}}} dt\lambda_{k,r}(d)\lambda_{k,r}(t) &= \sum_{m\le\sqrt{\alpha x+\beta}} m\tau(m)\lambda_{k,r}(m) \\ &\ll \sum_{m\le\sqrt{\alpha x+\beta}} m^{1+\epsilon}\lambda_{k,r}(m) \\ &\ll (\alpha x+\beta)^{1+1/2r+\epsilon}, \end{aligned}$$

and

$$\begin{aligned} \sum_{\substack{d,t\\ \gcd(d,t)=1\\ dt\le\sqrt{\alpha x+\beta}}}\frac{\lambda_{k,r}(d)\lambda_{k,r}(t)}{dt} &= \sum_{\substack{d,t\\ \gcd(d,t)=1}}\frac{\lambda_{k,r}(d)\lambda_{k,r}(t)}{dt} + \sum_{\substack{d,t\\ \gcd(d,t)=1\\ dt>\sqrt{\alpha x+\beta}}}\frac{\lambda_{k,r}(d)\lambda_{k,r}(t)}{dt} \\ &= \sum_{\substack{d,t\\ \gcd(d,t)=1}}\frac{\lambda_{k,r}(d)\lambda_{k,r}(t)}{dt} + O\Bigg(\sum_{m>\sqrt{\alpha x+\beta}}\frac{\tau(m)\lambda_{k,r}(m)}{m}\Bigg) \\ &= \sum_{\substack{d,t\\ \gcd(d,t)=1}}\frac{\lambda_{k,r}(d)\lambda_{k,r}(t)}{dt} + O\Big((\alpha x+\beta)^{(1/2)(1/r-1)+\epsilon}\Big). \qquad (5.3) \end{aligned}$$

Since $\frac{1}{r}-1<0$, the infinite series on the right side of (5.3) converges when $r \ge 2$. Then the proof of Theorem 5.8 completes. □

Corollary 5.9. For a large $x \in \mathbb{R}$, let $T_{r+1,r}(x)$ denote the number of positive integers $n \le x$ such that $\lfloor \alpha n + \beta \rfloor$ and $\lfloor \alpha n + \beta \rfloor + 1$ are semi r-free integers. We have, as $x \to \infty$,

$$T_{r+1,r}(x) = x \sum_{m=1}^{\infty} \frac{\tau(m)\lambda_{r+1,r}(m)}{m} + O(\alpha x^{(1/2r+1/2)+\epsilon} \log^3 x).$$

Remark 5.10. For a fixed large x, every positive integer a, and a positive r-free integer b, the inequality $a^{\lfloor x \rfloor} b \le x$ holds only when $a = 1$. Thus, in the interval $[1, x]$, every $(\lfloor x \rfloor, r)$-integer is an r-free integer. Thus, as $x \to \infty$, the function $T_{\lfloor x \rfloor, r}(x)$ counts the number of positive integers $n \le x$ such that $\lfloor \alpha n + \beta \rfloor$ and $\lfloor \alpha n + \beta \rfloor + 1$ are r-free integers. In view of (5.1), we have

$$\sum_{m=1}^{\infty} \frac{\lambda_{\lfloor x \rfloor, r}(m)}{m^s} = \frac{\zeta(\lfloor x \rfloor s)}{\zeta(rs)}, \qquad s > \frac{1}{r}.$$

When $x \to \infty$, since $\zeta(\lfloor x \rfloor s) = 1$, we have,

$$\sum_{m=1}^{\infty} \frac{\lambda_{\lfloor x \rfloor, r}(m)}{m^s} = \frac{1}{\zeta(rs)} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{rs}}.$$

Remark 5.10 indicates that Theorem 5.6 is a generalized result of (1.4). Moreover, in the case $r = 2$, Theorem 5.8 covers the result (1.5) of Veasna al et. as the following corollary.

Corollary 5.11. For a large $x \in \mathbb{R}$, we have, as $x \to \infty$,

$$\sum_{\substack{n \le x \\ \lfloor \alpha n + \beta \rfloor,\ \lfloor \alpha n + \beta \rfloor + 1 \text{ are } r\text{-free}}} 1 = x \prod_{p} \left(1 - \frac{2}{p^r}\right) + O(\alpha x^{(1/2r+1/2)+\epsilon} \log^3 x).$$

Example 5.2. For irrational $\alpha = \sqrt{2} \approx 1.4142136 > 1$ and $\beta = 0.4142 \in [0, \alpha)$, we have the Beatty sequence of the form $\lfloor \alpha n + \beta \rfloor$ and $\lfloor \alpha n + \beta \rfloor + 1$ as following:

n	1	2	3	4	5	6	7	8	9	10	...
$\lfloor \alpha n + \beta \rfloor$	1	3	4	6	7	8	10	11	13	14	...
$\lfloor \alpha n + \beta \rfloor + 1$	2	4	5	7	8	9	11	12	14	15	...

1. The elements $1, 3, 4, 6, 7, 10, 11, 13$ and 14 are 3-free integers. Thus $Q_3(10; \alpha, \beta) = 9$.
2. The elements $1, 3, 6, 7, 8, 10, 11, 13$ and 14 are $(3, 2)$-integeres. Thus $Q_{3,2}(10; \alpha, \beta) = 9$.
   On the other side, from Theorem 5.6 we have
   $Q_{3,2}(10; \alpha, \beta) = (10)\frac{\zeta(3)}{\zeta(2)} + O(10^{(1/4+1/2)} \log^3 10) = (10)\left(\frac{1.20206}{1.64493}\right) + O(10^{0.75}) = 7.30767.$
3. From the table we can see that $T_{3,2}(10) = 6$.
   On the other side, from Theorem 5.8 we have
   $T_{3,2}(10) = (10) \sum_{m=1}^{\infty} \frac{\tau(m)\lambda_{3,2}(m)}{m} + O(\sqrt{2} x^{(1/4+1/2)+\epsilon} \log^3 10) = 5.12015.$

Chapter 6

Conclusion

The thesis consists of three main parts. In the first part, we establish an asymtotical ratio of odd to even r-free integers. In the second part, for the finite set of prime number P, we prove the proportion of all r free numbers which are divisible by at least one element in P and coprime to all of primes in P. In the third part, we generalize a result on the distribution of consecutive square-free integers in Beatty sequences.

Main results of the first part are now described.

1. For any $r \geq 2$ be integers. As $x \to \infty$, we have

$$\frac{N_o(x)}{N_e(x)} \sim \frac{2^r}{2^r - 2}, \tag{6.1}$$

where $N_o(x)$ and $N_e(x)$ denote the number of odd and even r-free positive integers not greater than x, respectively.

2. As $x \to \infty$, we have

$$\frac{C_{odd}(x)}{C_{even}(x)} \sim 2, \tag{6.2}$$

where $C_{odd}(x)$ and $C_{even}(x)$ denote the number of odd and even square-free positive integers not greater than x, respectively. Note that the asymptotical ratio (6.1) covers the result (6.2) of Scott [2] and Jameson [3].

Main results of the second part are now described.

For any $r \geq 2$ be integers, let N_r be the set of all r-free numbers.

1. Let p be a given prime number and $P = \{p\}$. As $x \to \infty$, we have

$$\frac{C_P(x)}{C'_P(x)} \sim \frac{p^r - p^{r-1}}{p^{r-1} - 1}.$$

2. Let $P = \{p_1, p_2, ..., p_k\}$ be a finite set of prime numbers. As $x \to \infty$, we have

$$\frac{C'_P(x)}{C_P(x)} \sim \prod_{p \in P} \frac{p^r - 1}{p^r - p^{r-1}} - 1,$$

where $C_P = \{n \in N_r | (n, P) = 1\}$ and $C'_P = \{n \in N_r | (n, P) \neq 1\}$. Here (n, P) denotes the greatest common divisor of n and all prime in P.

Main results of the third part are now described.

Let k and r be fixed positive integers with $1 < r < k$.

1. Let $\alpha > 1$ be an irrational number and with bounded partial quotients, $\beta \in [0, \alpha)$. As $x \to \infty$, we have

$$Q_{k,r}(x; \alpha, \beta) = x\frac{\zeta(k)}{\zeta(r)} + O(x^{(1/2r+1/2)} \log^3 x),$$

where $Q_{k,r}(x;\alpha,\beta)$ is the number of (k,r)-integers of Beatty sequence $\lfloor \alpha n+\beta \rfloor$, $1 \le n \le x$.

2. For a large $x \in \mathbb{R}$, let $T_{k,r}(x)$ denote the number of positive integers $n \le x$ such that $\lfloor \alpha n+\beta \rfloor$ and $\lfloor \alpha n+\beta \rfloor + 1$ are (k,r)-integers. We have, as $x \to \infty$,

$$T_{k,r}(x) = x \sum_{m=1}^{\infty} \frac{\tau(m)\lambda_{k,r}(m)}{m} + O(\alpha x^{(1/2r+1/2)+\epsilon} \log^3 x),$$

where $\tau(n)$ denote the number of divisors of n.

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Appendix

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Odd/even r-free numbers *

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Received xxxx

Abstract

In this paper we use an elementary method to give an asymptotical ratio of odd to even r-free numbers and show that it is asymptotically $2^r : 2^r - 2$.

1 Introduction and results

Let $r > 1$ be a fixed integer. A positive integer n is *r-free* if each of its prime factors appears to the power at most $r - 1$. A positive integer n is *r-full* if each of its prime factors appears to the power at least r. As usual, 2-full and 3-full numbers are called square-full and cube-full, respectively.

Let $N_r(x)$ be the number of r-free integers $\le x$. It well know that for r fixed

$$N_r(x) = \frac{1}{\zeta(r)}x + O(x^{1/r}). \tag{1}$$

For a study of these asymptotic formulae, we refer to [2, Equation 14.24].

In this paper, we study the odd/even dichotomy for the set of r-free numbers. The motivation follows from work of Scott [5] and Jameson [3], where it was shown that the ratio of odd to even square-free numbers is asymptotically 2 : 1. In 2020, the second author [6] used an elementary method to prove the odd/even dichotomy for the set of square-full numbers. In 2021, Tippawan Puttasontiphot and Teerapat Srichan [7] extended the method in [6] to the case of cube-full numbers. Very recently, Jameson [4] used this to give a new proof in [3]. Thus, it would be interesting to generalize these results to the odd/even dichotomy for the set of r-free numbers by using the method in [6].

Here we prove the following results.

Theorem 1 *As $x \to \infty$, we have*

$$\frac{O(x)}{E(x)} \sim \frac{2^r}{2^r - 2}, \tag{2}$$

where $O(x)$ and $E(x)$ denote the number of odd and even r-free positive integers not greater than x, respectively.

Corollary 2 *As $x \to \infty$, we have*

$$\frac{C_{odd}(x)}{C_{even}(x)} \sim 2, \tag{3}$$

where $C_{odd}(x)$ and $C_{even}(x)$ denote the number of odd and even square-free positive integers not greater than x, respectively.

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Notations

Let A be a given set, for $x > 1$, we denote $A(x)$ be the number of elements in A. $f(x) \sim g(x)$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and we say that $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$.

2 Proofs

Proof of Theorem 1. Let E and O be the set of all even and odd r-free integers, respectively. We assume that,

$$O(x) \sim ax \qquad \text{and} \qquad E(x) \sim bx, \qquad \text{for some } a, b \in \mathbb{R}^+. \tag{4}$$

Denote

$$A_1 = \{n \in E : 4 \nmid n\}, \quad A_1^{'} = \{n \in E : 4 \mid n\},$$

for $2 \le k \le r-2$,

$$A_k = \{n \in A_{k-1}^{'} : 2^{k+1} \nmid n\}, \quad A_k^{'} = \{n \in A_{k-1}^{'} : 2^{k+1} \mid n\}.$$

We note that $A_k^{'} = A_{k+1} \cup A_{k+1}^{'}$. Thus,

$$E = \Big(\bigcup_{k=1}^{r-2} A_k\Big) \cup A_{r-2}^{'}. \tag{5}$$

For $1 \le k \le r-2$, set A_k are disjoint set. Thus, from (5), we have

$$E(x) = \Big(\sum_{k=1}^{r-2} A_k(x)\Big) + A_{r-2}^{'}(x). \tag{6}$$

Now, we note that the element $n \in A_k$ is the form $n = 2^{k+1}m + 2^k$, for some $m \in \mathbb{N} \cup \{0\}$. Thus, $\frac{n}{2^k}$ is odd and r-free. This implies that, for $1 \le k \le r-2$,

$$A_k(x) = O\Big(\frac{x}{2^k}\Big). \tag{7}$$

Similary, the element $h \in A_{r-2}^{'}$ is the form $h = 2^r m_1 + 2^{r-1}$, for some $m_1 \in \mathbb{N} \cup \{0\}$. Thus, $\frac{h}{2^{r-1}}$ is odd and r-free. This implies that,

$$A_{r-2}^{'}(x) = O\Big(\frac{x}{2^{r-1}}\Big). \tag{8}$$

Inserting (7) and (8) in (6), we have

$$E(x) = \sum_{k=1}^{r-2} O\Big(\frac{x}{2^k}\Big) + O\Big(\frac{x}{2^{r-1}}\Big) = \sum_{k=1}^{r-1} O\Big(\frac{x}{2^k}\Big). \tag{9}$$

In view of (4) and (9), we have

$$bx = \sum_{k=1}^{r-1} a\frac{x}{2^k} = ax\Big(1 - 2^{1-r}\Big).$$

This proves (2).

Now it remains to prove the existence of a and b.

In view of (9), we write

$$N_r(x) = O(x) + E(x) = \sum_{k=0}^{r-1} O\Big(\frac{x}{2^k}\Big). \tag{10}$$

We replace x in (10) by $x/2$ and subtract this with (10). We have

$$N_r(x) - N_r\Big(\frac{x}{2}\Big) = O(x) - O\Big(\frac{x}{2^r}\Big). \tag{11}$$

Replace x in (11) by $x/2^r$, we have

$$N_r\Big(\frac{x}{2^r}\Big) - N_r\Big(\frac{x}{2^{r+1}}\Big) = O\Big(\frac{x}{2^r}\Big) - O\Big(\frac{x}{2^{2r}}\Big). \tag{12}$$

In view of (11) and (12), we have

$$N_r(x) - N_r\Big(\frac{x}{2}\Big) + N_r\Big(\frac{x}{2^r}\Big) - N_r\Big(\frac{x}{2^{r+1}}\Big) = O(x) - O\Big(\frac{x}{2^{2r}}\Big).$$

Repeating this, we have

$$O(x) - O\Big(\frac{x}{2^{r(k+1)}}\Big) = \sum_{i=0}^{k} N_r\Big(\frac{x}{2^{ri}}\Big) - \sum_{i=0}^{k} N_r\Big(\frac{x}{2^{ri+1}}\Big). \tag{13}$$

The asymptotic formula (1) implies $N_r(x) \sim cx$, where $c = 1/\zeta(r)$. Then, for $\epsilon > 0$, we take x_0 such that

$$(c-\epsilon)x \le N_r(x) \le (c+\epsilon)x, \quad \text{for } x \ge x_0. \tag{14}$$

To apply inequality (14) with (13), we take k such that $\frac{x}{2^{rk+r+1}} < x_0 \le \frac{x}{2^{rk+1}}$. Then, we have

$$(c-\epsilon)\frac{x}{2^{ri+1}} \le N_r(\frac{x}{2^{ri+1}}) \le (c+\epsilon)\frac{x}{2^{ri+1}}, \tag{15}$$

and

$$(c-\epsilon)\frac{x}{2^{ri}} \le N_r(\frac{x}{2^{ri}}) \le (c+\epsilon)\frac{x}{2^{ri}}, \tag{16}$$

for $0 \le i \le k$. In view of (13), (15) and (16), we have

$$\begin{aligned} O(x) - O\Big(\frac{x}{2^{r(k+1)}}\Big) &\le \sum_{i=0}^{k}(c+\epsilon)\frac{x}{2^{ri}} - \sum_{i=0}^{k}(c-\epsilon)\frac{x}{2^{ri+1}} \\ &= x\Big(\frac{c}{2} + \frac{3\epsilon}{2}\Big)\sum_{i=0}^{k}\frac{1}{2^{ri}} \\ &\le x\Big(\frac{c}{2} + \frac{3\epsilon}{2}\Big)\sum_{i=0}^{\infty}\frac{1}{2^{ri}} \\ &= x\Big(\frac{c}{2} + \frac{3\epsilon}{2}\Big)\frac{2^r}{2^r - 1}. \end{aligned}$$

From the choosing k such that $\frac{x}{2^{rk+r+1}} < x_0 \le \frac{x}{2^{r(k+1)}}$, we have $O\Big(\frac{x}{2^{r(k+1)}}\Big) \le O(2x_0) < 2x_0$. Then, we have

$$O(x) \le x\Big(c + 3\epsilon\Big)\frac{2^{r-1}}{2^r - 1} + 2x_0 \le x\Big(c + 3\epsilon\Big)\frac{2^{r-1}}{2^r - 1} + \frac{2^{r+1}}{2^r - 1}x_0.$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$O(x) \le x\Big(c + 3\epsilon\Big)\frac{2^{r-1}}{2^r - 1} + \frac{2^{r-1}}{2^r - 1}4x\epsilon = x\Big(c + 7\epsilon\Big)\frac{2^{r-1}}{2^r - 1}.$$

By the similar proof we deal with the lower bound. In view of (13), (15) and (16), we have

$$\begin{aligned}
O(x) - O\Big(\frac{x}{2^{r(k+1)}}\Big) &\geq \sum_{i=0}^{k}(c-\epsilon)\frac{x}{2^{ri}} - \sum_{i=0}^{k}(c+\epsilon)\frac{x}{2^{ri+1}} \\
&= x\Big(\frac{c}{2} - \frac{3\epsilon}{2}\Big)\sum_{i=0}^{k}\frac{1}{2^{ri}} \\
&= x\Big(\frac{c}{2} - \frac{3\epsilon}{2}\Big)\frac{2^r}{2^r-1} - x\Big(\frac{c}{2} - \frac{3\epsilon}{2}\Big)\frac{2^{-rk}}{2^r-1} \\
&\geq x\Big(c - 3\epsilon\Big)\frac{2^{r-1}}{2^r-1} - \frac{cx}{2^{rk+1}(2^r-1)}.
\end{aligned}$$

We note that $2^r x_0 \geq \frac{x}{2^{rk+1}}$. Then, we have

$$O(x) \geq O(x) - O\Big(\frac{x}{2^{r(k+1)}}\Big) \geq x\Big(c - 3\epsilon\Big)\frac{2^{r-1}}{2^r-1} - cx_0\frac{2^r}{2^r-1}.$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$O(x) \geq O(x) - O\Big(\frac{x}{2^{r(k+1)}}\Big) \geq x\Big(c - 3\epsilon - 2c\epsilon\Big)\frac{2^{r-1}}{2^r-1}.$$

This proves the existence of a and consequently b also exists, in fact $b = c - a$. ■

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Note on the natural density of r-free numbers

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Abstract: Let P be a finite set of prime numbers. By using an elementary method the proportion of all r-free numbers which are divisible by at least one element in P and co-prime to all of primes in P is studied.
Keywords: Natural density, r-free numbers.
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1 Introduction and results

Let $r > 1$ be a fixed integer. A positive integer n is r-free if each of its prime factors appears to the power at most $r - 1$. As usual, 2-free and 3-free numbers are called square-free and cube-free, respectively. The density of a finite set of distinct positive integers, $\delta[A]$, is the ratio of the number of its elements to its largest element. The natural density of an infinite increasing sequence of positive integers, a_n, is

$$\delta[A] = \lim_{n\to\infty} \frac{n}{a_n}.$$

The natural density of the set of square-free integer is studied first by Gegenbauer. He proved that the natural density of the set of square-free integer is $6/\pi^2$ [2]. Later, in [5] Scott gave a conjecture on the natural density of the set of odd square-free integers is $4/\pi^2$ and it was proven by Jameson in [3]. Very recently, authors [7] generalized this problem to the case of r-free integers by using an elementary method and showed that the asymptotical ratio of odd to even r-free numbers is asymptotically $2^r : 2^r - 2$. In the same year, Brown [1] reproved Jameson's result and generalized it. Brown proved that the proportion of all numbers which are square-free and divisible by all of the primes in T and by none of the primes in P is

$$\frac{6}{\pi^2}\prod_{p\in T}\frac{1}{1+p}\prod_{p\in P}\frac{p}{1+p}, \tag{1}$$

where P and T are disjoint sets of prime numbers with T finite. In this paper, we use the elementary method in [6] to prove a similar result as in [1] and generalize it to r-free integers. We prove the following Theorem.

Theorem 1.1. Let $P = \{p_1, p_2, ..., p_k\}$ be a finite set of prime numbers. For any integer $r \geq 2$, let Q be the set of all r-free numbers. As $x \to \infty$, we have

$$\frac{C'_P(x)}{C_P(x)} \to \prod_{p\in P}\frac{p^r - 1}{p^r - p^{r-1}} - 1,$$

where $C_P = \{n \in Q | (n, P) = 1\}$ and $C'_P = \{n \in Q | (n, P) \neq 1\}$. Here (n, P) denotes the greatest common divisor of $p_1p_2 \cdots p_k$ and n.

Remark 1.2. From

$$\frac{C'_P(x)}{C_P(x)} = \frac{Q(x) - C_P(x)}{C_P(x)} = \frac{Q(x)}{C_P(x)} - 1,$$

we have, as $x \to \infty$,

$$\frac{C_P(x)}{Q(x)} \to \prod_{p\in P}\frac{p^r - p^{r-1}}{p^r - 1}. \tag{2}$$

In the case $T = \emptyset$, we can see that, the equation (2) covers Brown's result in (1).

2 Notations and Lemmas

Let A be a given set, for $x > 1$, we denote $A(x)$ be the number of elements in $A \cap [1, x]$. $f(x) \sim g(x)$ means $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = 1$ and we say that $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$. Let $P = \{p_1, p_2, ..., p_k\}$ be a finite set of prime numbers. For $r \geq 2$ be integers, let Q be the set of all r-free numbers. Let $C_P = \{n \in Q | (n, P) = 1\}$ and $C'_P = \{n \in Q | (n, P) \neq 1\}$. Here (n, P) denotes the greatest common divisor of $p_1p_2 \cdots p_k$ and n. For $0 \leq \alpha_i \leq r - 1$, $1 \leq i \leq k$, we denote $\mathcal{A}_{\alpha_1,\cdots,\alpha_k} := \left\{ n \in C'_P \middle| n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}m,\ m \in C_P \right\}$ and $\mathcal{A}_{0,\cdots,0} = C_P$.

The following Lemma will be used in our proof.

Lemma 2.1. Let p be a given prime number and $P = \{p\}$. As $x \to \infty$, we have $\frac{C_P(x)}{C'_P(x)} \to \frac{p^r - p^{r-1}}{p^{r-1}-1}$.

Proof. The proof is very similar of the proof of Theorem 1 in [7].

First we assume that,

$$C_P(x) \sim ax \qquad \text{and} \qquad C'_P(x) \sim bx, \qquad \text{for some } a, b \in \mathbb{R}^+. \tag{3}$$

First, we wish to show that,

$$\frac{a}{b} = \frac{p^r - p^{r-1}}{p^{r-1} - 1}. \tag{4}$$

For $1 \le i \le r-1$, we denote by

$$A_i = \{n \in C'_P : n = p^i m,\ m \in C_P\}.$$

We note that, for $1 \le i \le r-1$, A_i are disjoint sets. Thus,

$$C'_P = \bigcup_{i=1}^{r-1} A_i. \tag{5}$$

For $1 \le i \le r-1$, we note that, each element $n \in A_i$ is the form $n = p^i m$ such that $(p, m) = 1$ for some $m \in C_P$. Thus, $\frac{n}{p^i}$ is an element in C_P. This implies that, for $1 \le i \le r-1$,

$$A_i(x) = C_P\Big(\frac{x}{p^i}\Big). \tag{6}$$

In view of (5) and (6), we have

$$C'_P(x) = \sum_{i=1}^{r-1} C_P\Big(\frac{x}{p^i}\Big). \tag{7}$$

From (3) and (7), we get

$$bx = \sum_{i=1}^{r-1} a\frac{x}{p^i} = ax\frac{p^{r-1}-1}{p^r - p^{r-1}}.$$

This proves (4). Now it remains to prove the existence of a and b.
In view of (7), we write

$$Q(x) = C_P(x) + C'_P(x) = \sum_{i=0}^{r-1} C_P\Big(\frac{x}{p^i}\Big). \tag{8}$$

We replace x in (8) by x/p and subtract this with (8). We have

$$Q(x) - Q\Big(\frac{x}{p}\Big) = C_P(x) - C_P\Big(\frac{x}{p^r}\Big). \tag{9}$$

Replace x in (9) by x/p^r, we have

$$Q\Big(\frac{x}{p^r}\Big) - Q\Big(\frac{x}{p^{r+1}}\Big) = C_P\Big(\frac{x}{p^r}\Big) - C_P\Big(\frac{x}{p^{2r}}\Big). \tag{10}$$

In view of (9) and (10), we have

$$Q(x) - Q\Big(\frac{x}{p}\Big) + Q\Big(\frac{x}{p^r}\Big) - Q\Big(\frac{x}{p^{r+1}}\Big) = C_P(x) - C_P\Big(\frac{x}{p^{2r}}\Big).$$

Repeating this, we have

$$C_P(x) - C_P\Big(\frac{x}{p^{r(k+1)}}\Big) = \sum_{i=0}^{k} Q\Big(\frac{x}{p^{ri}}\Big) - \sum_{i=0}^{k} Q\Big(\frac{x}{p^{ri+1}}\Big), \text{ for some } k \in \mathbb{N}. \tag{11}$$

It is well known that, $Q(x) \sim cx$, where $c = 1/\zeta(r)$. Then, for $\epsilon > 0$, we take x_0 such that

$$(c-\epsilon)x \le Q(x) \le (c+\epsilon)x, \text{ for } x \ge x_0. \tag{12}$$

To apply inequality (12) with (11), we take k such that $\frac{x}{p^{rk+r+1}} < x_0 \le \frac{x}{p^{rk+1}}$ and from (11) and (12), we have

$$\begin{aligned} C_P(x) - C_P\Big(\frac{x}{p^{r(k+1)}}\Big) &\le \sum_{i=0}^{k}(c+\epsilon)\frac{x}{p^{ri}} - \sum_{i=0}^{k}(c-\epsilon)\frac{x}{p^{ri+1}} \\ &= x\Big(\frac{p(c+\epsilon)-(c-\epsilon)}{p}\Big)\sum_{i=0}^{k}\frac{1}{p^{ri}} \\ &\le x\Big(\frac{p(c+\epsilon)-(c-\epsilon)}{p}\Big)\frac{p^r}{p^r-1}. \end{aligned}$$

From the choosing k such that $\frac{x}{p^{rk+r+1}} < x_0 \le \frac{x}{p^{r(k+1)}}$, we have $N_p\Big(\frac{x}{p^{r(k+1)}}\Big) \le Q_p(px_0) < px_0$. Then, we have

$$\begin{aligned} C_P(x) &\le x\Big(p(c+\epsilon)-(c-\epsilon)\Big)\frac{p^{r-1}}{p^r-1} + px_0 \\ &\le x\Big(p(c+\epsilon)-(c-\epsilon)\Big)\frac{p^{r-1}}{p^r-1} + \frac{p^{r+1}}{p^r-1}x_0. \end{aligned}$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$\begin{aligned} C_P(x) &\le x\Big(p(c+\epsilon)-(c-\epsilon)\Big)\frac{p^{r-1}}{p^r-1} - (p^2\epsilon x)\frac{p^{r-1}}{p^r-1} \\ &= x\Big(p(c+\epsilon) - p^2\epsilon - c + \epsilon\Big)\frac{p^{r-1}}{p^r-1}. \end{aligned}$$

By a similar proof we deal with the lower bound. In view of (11) and (12), for the integer

k such that $\frac{x}{p^{rk+r+1}} < x_0 \le \frac{x}{p^{rk+1}}$, we have

$$\begin{aligned}
C_P(x) - C_P\Big(\frac{x}{p^{r(k+1)}}\Big) &\ge \sum_{i=0}^{k}(c-\epsilon)\frac{x}{p^{ri}} - \sum_{i=0}^{k}(c+\epsilon)\frac{x}{p^{ri+1}} \\
&= x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\sum_{i=0}^{k}\frac{1}{p^{ri}} \\
&= x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\big(\frac{p^r - p^{r-rk}}{p^r-1}\big) \\
&= x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\big(\frac{p^r}{p^r-1}\big) - x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\big(\frac{p^{r-rk}}{p^r-1}\big) \\
&\ge \frac{x\Big(p(c-\epsilon)-(c+\epsilon)\Big)(p^{r-1})}{p^r-1} - cx\big(\frac{p^{r-rk}}{p^r-1}\big).
\end{aligned}$$

We note that $p^{r+1}x_0 > \frac{x}{p^{rk}}$. Then, we have

$$\begin{aligned}
C_P(x) &\ge C_P(x) - C_P\Big(\frac{x}{2^{r(k+1)}}\Big) \\
&\ge \frac{x\big(p(c-\epsilon)-(c+\epsilon)\big)(p^{r-1})}{p^r-1} - \frac{cx_0p^{2r+1}}{p^r-1}.
\end{aligned}$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$C_P(x) \ge \frac{x\big(p(c-\epsilon)-(c+\epsilon)\big)(p^{r-1})}{p^r-1} - \frac{cx\epsilon p^{2r+1}}{p^r-1} = \frac{x\big(p(c-\epsilon)-(c+\epsilon)-c\epsilon p^{r+2}\big)(p^{r-1})}{p^r-1}.$$

This proves the existence of a. The existence of b follows from the existence of a, since $Q(x) = C_P(x) + C'_P(x)$. A second proof of the same sort is not needed. □

3 Proof of Theorem 1.1

Proof of Theorem 1.1. The proof is similar to the proof of Lemma 2.1 but much more complex. First, we assume that

$$C_P(x) \sim \delta x \quad \text{and} \quad C'_P(x) \sim \beta x, \quad \text{for some } \delta, \beta \in \mathbb{R}^+. \tag{13}$$

Now, we note that each element $n \in \mathcal{A}_{\alpha_1,\cdots,\alpha_k}$ is the form $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}m,\ m \in C_P$, $0 \le \alpha_i \le r-1$ and for some $\alpha_i \ne 0$, $1 \le i \le k$. Thus $\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}$ is an element in C_P. This implies that, for $0 \le \alpha_i \le r-1$, $1 \le i \le k$,

$$\mathcal{A}_{\alpha_1,\cdots,\alpha_k}(x) = C_P\Big(\frac{x}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}\Big). \tag{14}$$

From (14), we sum $\mathcal{A}_{\alpha_1,\cdots,\alpha_k}(x)$ for all $0 \le \alpha_i \le r-1$ but not all zero, $1 \le i \le k$, and get

$$\begin{aligned} C'_P(x) &= \sum_{\substack{0\le\alpha_i\le r-1 \\ \alpha_i \text{ are not all zero}}} C_P\Big(\frac{x}{\prod_{i=1}^k p_i^{\alpha_i}}\Big) \\ &= \sum_{1\le i\le k}\sum_{1\le\alpha_i\le r-1} C_P\left(\frac{x}{p_i^{\alpha_i}}\right) + \sum_{1\le i_1<i_2\le k}\sum_{1\le\alpha_{i_1},\alpha_{i_2}\le r-1} C_P\left(\frac{x}{p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}}\right) \\ &\quad + \cdots + \sum_{1\le\alpha_1,\cdots,\alpha_k\le r-1} C_P\left(\frac{x}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}\right). \end{aligned} \tag{15}$$

In view of (13), we have

$$\begin{aligned} \beta x =& \delta x \sum_{1\le i\le k}\sum_{1\le\alpha_i\le r-1}\left(\frac{1}{p_i^{\alpha_i}}\right) + \delta x \sum_{1\le i_1<i_2\le k}\sum_{1\le\alpha_{i_1},\alpha_{i_2}\le r-1}\left(\frac{1}{p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}}\right) \\ &+ \cdots + \delta x \sum_{1\le\alpha_1,\cdots,\alpha_k\le r-1}\left(\frac{1}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}\right) \\ &= \delta x\Big(\prod_{p\in P}\frac{p^r-1}{p^r-p^{r-1}} - 1\Big). \end{aligned} \tag{16}$$

Now, it remains to show the existence of δ and β. To do this, we use the mathematical induction on k, where k is the number of primes in P. For the convenient, we let $P_{k-1} = \{p_1, \cdots, p_{k-1}\}$ and $C_{P_k} = \{n \in Q | (n, P_k) = 1\}$.

We assume that, for $1 \le j < k$, the exists δ_j such that $C_{P_j}(x) \sim \delta_j x$. From Lemma 2.1, δ_1 exists. Let q be a prime number with $q \notin P_{k-1}$. Note that

$$C_{P_{k-1}}(x) = C_{P_{k-1}\cup\{q\}}(x) + C^*_{P_{k-1}\cup\{q\}}(x), \tag{17}$$

where $C^*_{P_{k-1}\cup\{q\}} = \{n \in Q | (n, P_{k-1}) = 1 \text{ and } q \mid n\}$. From the same reason in (14), we have

$$C^*_{P_{k-1}\cup\{q\}}(x) = \sum_{1\le\alpha\le r-1} C_{P_{k-1}}\left(\frac{x}{q^\alpha}\right). \tag{18}$$

In view of (17) and (18), we have

$$C_{P_{k-1}}(x) = C_{P_{k-1}\cup\{q\}}(x) + \sum_{1\le\alpha\le r-1} C_{P_{k-1}}\left(\frac{x}{q^\alpha}\right). \tag{19}$$

Since $C_{P_{k-1}}(x) \sim \delta_{k-1}x$, we have

$$C_{P_{k-1}\cup\{q\}}(x) \sim \Big(1 - \sum_{1\le\alpha\le r-1}\left(\frac{1}{q^\alpha}\right)\Big)\delta_{k-1}x.$$

This shows that δ_k exists. By the mathematical induction, the proof is completed. □

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