APPROXIMATION METHODS FOR SOLVING NEW PROBLEMS AND SPLIT GENERAL SYSTEM OF VARIATIONAL INEQUALITIES

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN APPLIED MATHEMATICS

DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

2019

KMITL-2019-SC-D-001-033

หัวข้อวิทยานิพนธ์	วิธีการประมาณค่าสำหรับการแก้ปัญหาใหม่ของอสมการแปรผัน และระบบทั่วไปของอสมการแปรผันแบบแยก
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บทคัดย่อ

วิทยานิพนธ์ฉบับนี้มีจุดประสงค์เพื่อ แนะนำปัญหาแบบใหม่ของระบบอสมการแปรผันทั่วไป ซึ่งได้แรงบันดาลใจมาจากปัญหาระบบอสมการแปรผันทั่วไป ของ Ceng และคณะ [6] และแนวคิดของปัญหาความเป็นไปได้แบบแยกในปริภูมิฮิลเบิร์ต จากนั้นได้สร้างกระบวนการทำซ้ำ เพื่อหาสมาชิกร่วมระหว่างผลเฉลยของปัญหาใหม่ที่ได้กล่าวไปข้างต้น และจุดตรึงของการส่งแบบไม่ขยาย และได้พิสูจน์ทฤษฎีการลู่เข้าแบบเข้มของลำดับที่เกิดจากกระบวนการทำซ้ำที่สร้างขึ้น มากไปกว่านั้นปัญหาใหม่สามารถลดรูปไปเป็นปัญหาความเป็นไปได้แบบแยก ปัญหาอสมการแปรผันแบบแยก และปัญหาค่าต่ำสุดจำกัดได้ ซึ่งในวิทยานิพนธ์ฉบับนี้ได้แนะนำวิธีการแก้ปัญหาใหม่นั้น และนั่นหมายความว่ามันสามารถแก้ไขปัญหาทั้ง 3 นั้นได้เช่นเดียวกัน โดยผลลัพธ์ที่ได้จากวิทยานิพนธ์ฉบับนี้นั้นยังเป็นการปรับปรุง และขยายงานวิจัยอื่นๆ ที่เกี่ยวข้องอีกด้วย ท้ายที่สุดมีการยกตัวอย่างเชิงตัวเลขเพื่อสนับสนุนวิทยานิพนธ์ฉบับนี้ในเซตของจำนวนจริง

คำสำคัญ : การส่งแบบไม่ขยาย จุดตรึง ปัญหาระบบอสมการแปรผันทั่วไปแบบแยก

Thesis Title	Approximation Methods for Solving New Problems and Split General System of Variational inequalities
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Degree	Doctor of Philosophy (Applied Mathematics)
Department	Mathematics
Year	2019
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Abstract

The purpose of this thesis is to propose new problems of the general system of variational inequalities problem, inspired and motivated by Ceng et al. [6] and the concept of the split feasibility problem in Hilbert spaces. Then, we establish iterative methods for finding a common element of the solution set of the proposed problems and the set of fixed point of nonexpansive mapping and prove strong convergence theorems of sequences generated by these iterative methods. Moreover, our new problems can be reduced to the split feasibility (SFP), split variational inequality (SVIP) and constrain minimization problem. In this thesis, we introduced the method for solving our new problems and that means we can solve SFP, SVIP and minimization problem as well. Our main result improve and extend some corresponding results in the literature. Finally, numerical examples are also presented to support our main results in the space of real numbers.

Keywords : Fixed point, Nonexpansive mapping, Split feasibility problem, Split general system of variational inequalities

Acknowledgements

This thesis would not have been possible without all the advice, help and support from the following persons who in one way or another contributed to the completion of this thesis.

First of all, I would like to express my sincere gratitude to my thesis advisor, Associate Professor Dr. Atid Kangtunyakarn. Without his advice, assistance and dedicated involvement in all part of the process, this thesis would have never been accomplished. I would like to thank you very much for your support and understanding over these past four years. Next, I would also like to thank all my thesis committees, Professor Dr. Suthep Suantai, Assistant Professor Dr. Nopparat Pochai, Assistant Professor Dr. Kanchana Kumnungkit, and Assistant Professor Dr. Pattrawut Chansangiam, for their valuable comments, advice and encouragement. In addition, I would like to thank every lecturers and professors of the Department of Mathematics, King Mongkut's Institute of Technology Ladkrabang, for all knowledge and their kind support.

Furthermore, my sincere thank also goes to Ms. Kanyarat Cheawchan for your warm encouragement and all your suggestions. In October 2018, I went to Gyeongsang National University for several weeks to study with Distinguished Professor Dr. Yeol Je Cho. My time at Gyeongsang National University has been highly productive and working with him was a very good experience. So, I am very appreciative to Professor Yeol Je Cho and his wife for everything they have done for me and Ms. Kanyarat Cheawchan.

Finally, I am very grateful to my beloved family, my parent Mr. Chulert Siriyan, Mrs. Sirikron Chalearnchai and my sibling Ms. Kanyapak Siriyan, who have supported me along the way. I really appreciate it. I do love you always and forever.

Keerati Siriyan

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Chapter 1

Introduction

1.1 Research Motivation

Currently, there are many studies and inventions about the method for solving problems, creation or development of new technologies to apply and solve many problems efficiently in various disciplines such as optimization, economy, engineering, computer science and etc. The basis of invention of technologies are science and mathematical. Sometime, if we solve problems by the direct method, we will find that it is harder and more complicated to solve than converted these problems into a mathematical model. Moreover, many problems in science are usually converted into a mathematical model. The one of an effective mathematical tool for solving these problems is fixed point theory, which it can be applied to solve problems in various disciplines as mention previously. Over the past decade, many mathematicians have been widely studied and developed the fixed point theory for nonlinear mapping such as nonexpansive, quasi-nonexpansive, nonspreading and etc., see [28, 45].

In this section, we suggest basic knowledge about the definition of mappings, operator, iterative method and important theorems to give motivation for our main result.

Throughout in this section, let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let C be a nonempty closed and convex subset of H. The notation of weak and strong convergence are denoted by " $\rightharpoonup$ " and " $\to$ ", respectively.

Let T be a mapping of C into itself. The fixed point of a mapping T is to find $x \in C$ such that $x = Tx$. The set of fixed point of T is denoted by $F(T)$, i.e., $F(T) = \{x \in C : x = Tx\}$.

A mapping $T : C \to C$ is called contraction on C if there exists $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \tag{1.1}$$

for all $x, y \in C$. If the inequality (1.1) holds and $\alpha = 1$, then T is called nonexpansive mapping.

A mapping $A : H \to H$ is called strongly positive operator if there exists a constant $\lambda > 0$ such that

$$\langle Ax, x \rangle \geq \lambda \|x\|^2, \tag{1.2}$$

for all $x \in H$.

A mapping $A : C \to H$ is called γ- inverse strongly monotone if there exists a positive real number $\gamma > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2, \tag{1.3}$$

for all $x, y \in C$.

After all this time, many author have introduced several types of iterative method find the fixed point of nonlinear mappings.

In 1953, Mann [29] introduced the Mann iteration to find a fixed point of nonexpansive mapping $T : C \to C$ as follow:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \tag{1.4}$$

for each $n \geq 0$ and $x_0 \in C$ where $\{\alpha_n\} \subset (0, 1)$. The Mann iteration obtain only weak convergence. Later, many authors have studied and modified Mann iteration to obtain new iterative methods and strong convergence theorems.

In 1967, Halpern [16] introduced the Halpern iteration to find a fixed point of nonexpansive mapping $T : C \to C$ as follow:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T x_n, \tag{1.5}$$

for each $n \geq 0$ and $x_0 = u \in C$ where $\{\alpha_n\} \subset (0, 1)$. He proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T. In 1992, Wittmann [49] proved that sequence $\{x_n\}$ converges strongly to $P_{F(T)}u$, where $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, where $P_{F(T)}$ is a metric projection onto $F(T)$.

In 1974, Ishikawa [18] introduced the Ishikawa iteration, by modification of Mann iteration, to find a fixed point of nonexpansive mapping $T : C \to C$ as follow:

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n)T x_n, \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)T y_n, \end{aligned} \tag{1.6}$$

for each $n \geq 0$ and $x_0 \in C$ where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. The Ishikawa iteration obtain strong convergence theorem. It easy to see that if we put $\beta_n = 1$, then the Ishikawa iteration reduces to Mann iteration.

In 2000, Moudafi [33] introduced the viscosity approximation method for nonexpansive mapping. The viscosity approximation method has been used and studied in several references, which formally generates the sequence $\{x_n\}$ by the following iteration formula:

$$x_{n+1} = (1 - \alpha_n)T x_n + \alpha_n f(x_n), \tag{1.7}$$

for each $n \geq 0$, where $\{\alpha_n\} \subset (0, 1)$ and $f : C \to C$ is a contraction mapping. He proved that the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$, where $x^* = P_{F(T)} f(x^*)$.

In 2006, Marino and Xu [30] introduced the general iterative method, by using the concept of the viscosity approximation method, to find a fixed point of nonexpansive mapping T as follow:

$$x_{n+1} = (I - \alpha_n A)T x_n + \alpha_n \gamma f(x_n), \tag{1.8}$$

for each $n \geq 0$, where $\{\alpha_n\} \subset (0,1)$, $\gamma \in (0, \overline{\gamma}/\alpha)$, $f : C \to C$ is a α-contraction mapping and $A : H \to H$ is a strongly positive bounded linear operator with $\overline{\gamma} > 0$. They proved that the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$ which solves the following variational inequality:

$$\langle (A - \gamma f)x^*, x^* - z\rangle \leq 0, z \in F(T). \tag{1.9}$$

1.1.1 Variational inequality problems

In 1964, Lions and Stampacchia [26] first introduced the following variational inequality problem: find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \tag{1.10}$$

for all $v \in C$. The set of the solutions of (1.10) is denoted by $VI(C, A)$.

The variational inequality problem had been extended and widely studied in several literature; see [15, 20, 53, 54]. It is well known that variational inequalities cover as various disciplines such as optimization, finance, mechanic and etc., see [7, 15, 53].

In 2008, Ceng et al. [6] introduced the *general system of variational inequalities* to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \ \ \forall x \in C, \\ &\langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, \ \ \forall x \in C, \end{aligned} \tag{1.11}$$

where $\lambda, \mu > 0$ are $A, B : C \to H$ two constants and be two different mappings. In particular, if $A = B$, then problem (1.11) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \ \ \forall x \in C, \\ &\langle \mu Ax^* + y^* - x^*, x - y^*\rangle \geq 0, \ \ \forall x \in C, \end{aligned} \tag{1.12}$$

which is called the new system of variational inequalities introduced by Verma [47], in 1999. Moreover, if we put $x^* = y^*$, then problem (1.12) reduces to the variational inequality.

In order to find the common element of the solutions of the general system of variational inequalities problem and the fixed point of a nonexpansive mapping, Ceng et.al [6] proved the following strong convergence theorem by a relaxed extragradient method:

Theorem 1.1. Let the mappings $A, B : C \to H$ be α, β inverse strongly monotone mappings, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap F(G) \neq \emptyset$, where a mapping $G : C \to C$ is defined by $G(x) = P_C\,[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)]$, $\forall x \in C$. Suppose that $x_1 = u \in C$ and $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n), \end{cases} \tag{1.13}$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Then $\{x_n\}$ converges strongly to some point $x^* \in C$ and (x^*, y^*) is a solution of the general system of variational inequalities (1.13), where $y^* = P_C(I - \mu B)x^*$.

1.1.2 Split feasibility and split variational inequality problem

Let H_1, H_2 be two real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $D : H_1 \to H_2$ be bounded linear operator with adjoint D^* .

In 1994, Censor and Elfving [8] introduced the split feasibility problem (in short SFP) in the finite dimensional Hilbert spaces, which is to find a point x such that

$$x \in C \text{ and } Dx \in Q. \tag{1.14}$$

The set of all solutions of (1.14) is denoted by Γ. The split feasibility problem has been widely studied by many authors, see [3, 4, 9, 51]. Moreover, this problem is useful in various disciplines such as signal processing, image reconstruction and computer tomograph, see [4, 9, 11].

Assume that the split feasibility problem (SFP) is consistent, i.e. (1.14) has a solution. It easy to see that $x \in C$ solves (1.14) if and only if it solves the following fixed point equation

$$x = P_C(I - \gamma D^*(I - P_Q)D)x, \forall x \in C, \tag{1.15}$$

where γ is any positive constant; see [52].

A popular algorithm that solves (1.15) is the CQ algorithm of Byrne [3], in 2002, which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C(I - \gamma D^*(I - P_Q)D)x_n, \forall n \in \mathbb{N}, \tag{1.16}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator D^*D.

In 2012, Censor et al. [10] introduced the split variational inequality problem (in short SVIP) which is to find $\overline{x} \in C$ such that

$$\langle f\overline{x}, x - \overline{x}\rangle \geq 0, \forall x \in C, \tag{1.17}$$

and find $\overline{y} = D\overline{x} \in Q$ such that

$$\langle g\overline{y}, y - \overline{y}\rangle \geq 0, \forall y \in Q, \tag{1.18}$$

where $f : C \to H_1$ and $g : C \to H_2$ are nonlinear mappings. The set of solution of (1.18) is denoted by $\Omega = \{\overline{x} \in VI(C, f) : \overline{y} \in VI(C, g)\}$. The split variational inequality problem is reduced to the split feasibility problem if we put $f \equiv g \equiv 0$. Many researchers have been studied, used and extended the SVIP in various literature; see [46, 55, 56].

In this thesis, we propose new problems of the general system of variational inequalities and prove strong convergence theorems to solve proposed problems and the fixed point problem of nonexpansive mapping under the suitable conditions in Hilbert space. By applying our main results, strong convergence theorems of the constrained minimization, the split feasibility problem and the split variational inequality are obtained. Moreover, we give some numerical examples to support our main theorems in $\mathbb{R}$ and $\mathbb{R}^3$ spaces.

1.2 Objectives of the study

1) To propose new problems of the general system of variational inequalities that can solve complicated problems more than the original problem.
2) To establish new iterative methods for finding a common element of the solution set of proposed problems and the set of fixed point problem of nonexpansive mapping and prove strong convergence theorems for fixed point problem, system of variational inequality problem and proposed problems in Hilbert space.
3) To apply our main theorems to solve other problems such as the constrained minimization, the split feasibility problem and the split variational inequality problem.
4) To give numerical examples to support our main results.

1.3 Scopes of the study

1) Study and give basic knowledge about the fixed point problem, variational inequality problem, the strong convergence theorems of nonlinear mappings and other, which we can use in our research.
2) Create new problems of the general system of variational inequalities in Hilbert space.
3) Our main theorems are proved in framework of Hilbert space for solving proposed problems and fixed point problems of nonexpansive mapping.
4) Give numerical examples to support our main results in $\mathbb{R}$ and $\mathbb{R}^3$ spaces.

1.4 Research methodology

1) Study theorems, definitions, lemmas and other knowledge about fixed point theory, nonlinear mappings and techniques from textbooks and related researches.
2) Study suitable conditions and properties for our research.

3) Study and establish new problems and iterative method in Hilbert space.
4) Establish and prove strong convergence theorems of the proposed problems and the fixed point problem of nonexpansive mapping.
5) Apply our main theorems to prove strong convergence theorems of the constrained minimization, the split feasibility problem and the split variational inequality problem.
6) Create and give numerical examples to support our main results in $\mathbb{R}$ and $\mathbb{R}^3$ spaces.
7) Write the thesis.

1.5 Benefits of the study

1) Obtain new problems of the general system of variational inequalities.
2) Obtain strong convergence theorems for finding a common element of the solution of proposed problems and the fixed point set of nonexpansive mapping.
3) Obtain strong convergence theorems of the constrained minimization, the split feasibility problem and the split variational inequality problem, by applying our main theorems.
4) Our strong convergence theorems can be applied to solve various problems in the real world such as the optimization problem, computer and economic problem.
5) Obtain new mathematical tools for solving the solution of fixed point problem of nonexpansive mapping in Hilbert space.

This thesis is divided into 5 chapters

In chapter 1, we review the background of this thesis whether it be definitions of mappings, operators, iterative methods and related theorems for inspiration in our research.

In chapter 2, we give definitions, lemmas, properties and basic knowledge of fixed point theory, variational inequality problem and other to prove our main theorem in chapters 3.

In chapter 3, we introduce new problems of the general system of variational inequalities. Some Lemmas for these problems are obtained. Then, we prove strong convergence theorems for finding the common element of a solution of the proposed problem and the fixed point of nonexpansive mapping in Hilbert space.

In chapter 4, we apply our main results to obtain strong convergence theorems of the constrained minimization, the split feasibility problem and the split variational inequality problem and give numerical examples to support our main results.

In chapter 5, we give the conclusion of this thesis.

In chapter 6, we describe the conclusion of the thesis.

Chapter 2

Preliminaries

In this chapter, we will look into fundamental knowledge involving Hilbert spaces. Furthermore, we give several useful definitions, lemmas, remarks and properties that will be needed for our thesis in the next sections.

2.1 Fundamental knowledge

Before we can talk about the important mathematical tools and properties for solving fixed point problems in Hilbert spaces in our thesis, we need to know and study the basic knowledge such as the definitions, lemmas and remarks of linear, Banach and Hilbert spaces. After that, we will study about the fixed point theory, fundamental properties and some useful lemmas and definitions in Hilbert spaces.

2.1.1 Linear spaces

In this section, we will first deal with linear spaces. We present the definitions and basic facts of linear spaces.

Definition 2.1. [1] A *linear space* or vector space X over the field $\mathbb{F}$ (the real field $\mathbb{R}$ or the complex field $\mathbb{C}$) is a set X together with an internal binary operation "$+$" called *addition* and a *scalar multiplication* carrying (α, x) in $\mathbb{F} \times X$ to αx in X satisfying the following for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$:

(L1) $x + y = y + x$,

(L2) $(x + y) + z = x + (y + z)$,

(L3) there exists an element $0 \in E$ called the *zero element* such that $x + 0 = x$ for all $x \in E$,

(L4) for every element $x \in E$ there exists an element $-x \in E$ and called the *additive inverse* or the *negative* of x, such that $x + (-x) = 0$.

(L5) $\alpha(x + y) = \alpha x + \alpha y$,

(L6) $(\alpha + \beta)x = \alpha x + \beta x$,

(L7) $\alpha(\beta x) = (\alpha\beta)x$,

(L8) $1 \cdot x = x$.

The element of a vector space X are called *vector*, and the element of $\mathbb{K}$ are called *scalars*. In the sequel, unless otherwise stated, X denotes a linear space over field $\mathbb{R}$.

Example 2.1. [1]

(1) $X = \{x = (a_1, a_2, ...) : a_i \in \mathbb{R}\}$ is a linear space.

(2) With the usual addition and multiplication, $\mathbb{R}$ and $\mathbb{C}$ are linear space over $\mathbb{R}$

Remark 2.2. [44] Since we admit the real numbers as scalars, a linear space is also called a real linear space.

Definition 2.2. [1] Let C be a subset of a linear space X. Then C is said to be *convex* if $(1-\alpha)x + \alpha y \in C$ for all $x, y \in C$ and all scalar $\alpha \in [0, 1]$.

2.1.2 Banach and Hilbert spaces

In this section, we give the definition of Banach and Hilbert spaces. Moreover, we also give some fundamental properties that are useful for these two spaces. It is well known that a Hilbert space is always a Banach space, the converse need not hold.

Definition 2.3. [12] Let X be a linear space. A norm is a function $\|\cdot\| : X \to [0, +\infty)$ which satisfies the following properties: For every x, y in X

(N1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;

(N2) $\|cx\| = |c| \, \|x\|$ for every scalar c;

(N3) $\|x + y\| \leq \|x\| + \|y\|$.

The ordered pair $(X, \|\cdot\|)$ is called a *normed space*. In this case, we write X.

Definition 2.4. [12] Let X be a normed space and let $\{x_n\}$ be a sequence of element of X.

(i) A sequence $\{x_n\}$ *converges strongly* to $z \in X$ if $\lim_{n\to\infty} \|x_n - z\| = 0$.
In this case , we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(ii) A sequence $\{x_n\}$ is *Cauchy* if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$\|x_m - x_n\| < \varepsilon.$$

Remark 2.3. [57] Every convergent sequence in a normed linear space is Cauchy sequence.

Definition 2.5. [57] A normed space X is *complete* if every Cauchy sequence in X converges in X.

Definition 2.6. [48] A complete normed linear space is called a *Banach space*.

Example 2.4. [48, 44]

(1) $\mathbb{R}$ is a Banach space with the norm $\|x\| = |x|$.

(2) $\mathbb{C}$ is also a Banach space.

(3) The space l^∞ of all bounded sequences $x = (x_1, x_2, x_3, ..., x_n, ...)$ of real numbers is a Banach space with the norm defined by $\|x\| = \sup_n |x_n|$.

Now, we give the definition of inner product and Hilbert spaces.

Definition 2.7. [1] Let X be a linear space over field $\mathbb{C}$. An *inner product* on X is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ with the following properties:

(I1) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(I2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ where the bar denotes complex conjugation;

(I3) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{C}$.

The ordered pair $(X, \langle \cdot, \cdot \rangle)$ is called an *innner product space* and $\langle x, y \rangle$ is called inner product of two elements $x, y \in X$.

Theorem 2.5. [36] For an inner product space X, $x, y, z \in K$ and $\alpha \in \mathbb{C}$. Then, the following properties hold:

(i) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$;

(ii) $\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$;

(iii) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$;

(iv) $\langle x, x \rangle = 0$ if and only if $x = 0$;

(v) if $\langle x, y \rangle = \langle x, z \rangle$, then $y = z$.

Remark 2.6. [44] An inner product space is called a *real inner product space* for the case when the scalars are the real numbers and $\langle x, y \rangle$ is a real number. For the case, (I2) means

$$\langle x, y \rangle = \langle y, x \rangle \tag{2.1}$$

Remark 2.7. [44] Let X be an inner product space. For each $x \in X$, we define its norm $\|x\|$ by

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Theorem 2.8 (The Cauchy-Schwarz inequality [44]). Let X be an inner product space, then

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| \text{ for all } x, y \in X.$$

Definition 2.8 (Strong convergence [13]). A sequence $\{x_n\}$ of vectors in an inner product space X is called *strongly convergent* to a vector x in X if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.9 (Weak convergence [13]). A sequence $\{x_n\}$ of vectors in an inner product space X is called *weakly convergent* to a vector x in X if

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ as } n \to \infty,$$

for every $y \in X$.

Remark 2.9. Throughout this thesis, strong and weak convergence are denoted by " $\to$ "," $\rightharpoonup$ ", respectively.

Theorem 2.10. [13] A strongly convergence sequence is weakly convergence (to the same limit), that is, $x_n \to x$ implies $x_n \rightharpoonup x$.

Remark 2.11. [44] If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Lemma 2.12. [44] Let $\{x_n\}$ be a Cauchy sequence of an inner product space X such that $x_n \rightharpoonup x$. Then $x_n \to x$.

Definition 2.10. [44] A complete inner product space is called a *Hilbert space*.

Example 2.13. [57] $\mathbb{R}, \mathbb{R}^n, l^2, L^2(a,b)$ are Hilbert spaces.

Remark 2.14. [44] Let H be an inner product space. Then we know that the following (i) and (ii) are equivalent:

i) H is complete,

ii) each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

2.2 Fundamental properties in Hilbert spaces

After we studied the fundamental knowledge about various spaces especially Hilbert space. Next, we give some important properties in Hilbert space that be used in many of the proof in our thesis.

Lemma 2.15. [44, 35] Let H be a real Hilbert space. Then the following properties hold for all $x, y \in H$ and $\alpha \in [0,1]$:

(i) $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2\langle x, y\rangle$.

(ii) $\langle x+y, x-y\rangle = \|x\|^2 - \|y\|^2$.

(iii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$.

(iv) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$

(v) $|\|x\| - \|y\|| \le \|x+y\| \le \|x\| + \|y\|$.

Theorem 2.16. [44] Let $\{a_n\}$ be a bounded sequence of real numbers. Then, there exists subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Theorem 2.17. [44] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Suppose that $\{x_n\} \subset C$ and $x_n \rightharpoonup x$. Then $x \in C$.

Definition 2.11. [44] Let C be a nonempty closed convex subset of a Hilbert space H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called

i) *lower semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\}$$

is closed.

ii) *upper semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \geq a\}$$

is closed.

iii) *convex* if for any $x, y \in C$ and $t \in (0,1)$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

iv) *concave* if for any $x, y \in C$ and $t \in (0,1)$,

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y).$$

Theorem 2.18. [44] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightharpoonup x_0$. Then

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

Lemma 2.19. [34] Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{u_n\} \subset H$ with $u_n \rightharpoonup u$, the inequality

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

holds for every $v \in H$ with $v \neq u$.

Theorem 2.20. [13] Weakly convergent sequence in a Hilbert space H are *bounded*, that is, if $\{x_n\}$ is a weakly convergent sequence, then there exists a number $M > 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Theorem 2.21. (**Double extract subsequence principle** [32]) Let $\{x_n\}$ be a sequence in a Hilbert space H and $x \in H$. If every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a further subsequence $\left\{x_{n_{k_l}}\right\}$ such that $\lim_{l\to\infty} x_{n_{k_l}} = x$, then $\lim_{n\to\infty} x_n = x$.

Next, we recall the definition of the metric projection operator.

Definition 2.12 (Metric projection [44]). Let C be a nonempty closed convex subset of a real Hilbert space H. For every $x \in H$, there exists a unique nearest point $P_C x \in C$ such that

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

P_C is called *metric projection* of H onto C.

Lemma 2.22 ([43]). Let C be a nonempty closed convex subset of a real Hilbert space H. Let $P_C : H \to C$ be the metric projection. For a given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if the following inequality holds:

$$\langle x - y, y - z\rangle \geq 0, \forall v \in C.$$

It is well-known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \forall x, y \in H.$$

Obviously, it implies that

$$\|(x - y) - (P_C x - P_C y)\|^2 \leq \|x - y\|^2 - \|P_C x - P_C y\|^2, \forall x, y \in H.$$

Lemma 2.23. [43] Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

2.3 Fixed point of nonexpansive mapping with some properties

In this section, we study about the existence and properties of fixed points of nonexpansive mapping. First, we recall the definition of the fixed points set.

Let K be a nonempty set and T be a mapping from K into itself. We denote $F(T)$ by the set of all fixed points of a mapping T, that is,

$$F(T) = \{x \in K : x = Tx\}.$$

Example 2.24. Let $K = \mathbb{R}$.

i) If $Tx = \frac{x+1}{2}$, then $F(T) = \{1\}$.

ii) If $Tx = \frac{x^3-2x}{2}$, then $F(T) = \{0, 2\}$.

iii) If $Tx = x$, then $F(T) = \mathbb{R}$.

iv) If $Tx = x + 1$, then $F(T) = \emptyset$.

Definition 2.13. Let C be a nonempty subset of H. A mapping $T : C \to C$ is called

1) a *nonexpansive mapping* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$, which is equivalent to

$$\langle Ty - Tx, (I - T)\,x - (I - T)y\rangle \leq \frac{1}{2}\|(I - T)\,x - (I - T)y\|^2,$$

for all $x, y \in C$.

2) a α-*contractive mapping* if there exists $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha\,\|x - y\|$$

for all $x, y \in C$.

It is obvious that if T is contractive, then T is nonexpansive. That is, the class of nonexpansive mappings includes the class of contractive mappings.

Example 2.25. Let $\mathbb{R}$ be the set of real numbers and let T, f be mappings from $\mathbb{R}$ to $\mathbb{R}$ defined by $Tx = \frac{2x-1}{3}$ and $fx = \frac{3x}{5}$, for all $x \in \mathbb{R}$, respectively. Then T is nonexpansive mapping and f is $\frac{3}{5}$-contractive mapping.

Solution. Let $x, y \in \mathbb{R}$.

Consider,

$$\begin{aligned} |Tx - Ty| &= \left|\frac{2x-1}{3} - \frac{2y-1}{3}\right| \\ &= \frac{2}{3}\,|x - y| \\ &\leq |x - y| \end{aligned}$$

Then, we have T is nonexpansive mapping.

Next, we consider

$$\begin{aligned} |fx - fy| &= \left|\frac{3x}{5} - \frac{3y}{5}\right| \\ &= \frac{3}{5}\,|x - y| \end{aligned}$$

Then, we have f is $\frac{3}{5}$-contractive mapping.

Theorem 2.26. [44] Let C be a nonempty bounded closed convex subset of a Hilbert space H. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Theorem 2.27. [44] Let C be a nonempty bounded closed convex subset of a Hilbert space H. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.

Lemma 2.28 (Demiclosedness principle [23]). Let T be a nonexpansive self-mapping of closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y it follows that $(I - T)x = y$, where I is the identity mapping of H.

2.4 Properties of strongly positive bounded linear operators in Hilbert spaces

Next, we present some basic definitions and properties of linear operators. An adjoint operator, self adjoint operator, strongly positive, inverse strongly monotone, spectral radius and etc. are discussed in this section.

Definition 2.14. [1] Let X and Y be linear spaces with the same scalars, and let T be a mapping of X into Y. Then T is called *linear* if

(i) T is additive: $T(x + y) = T(x) + T(y)$ for all $x, y \in X$;

(ii) T is homogeneous: $T(\alpha x) = \alpha T(x)$ for all $x \in X$ and scalar $\alpha \in \mathbb{R}$.

The linear operator T is called a *linear functional* if $Y = \mathbb{R}$.

Definition 2.15. [44] Let X and Y be two normed linear spaces and let T be a linear mapping of X into Y. Then T is said to be *bounded* if there exists a $M > 0$ such that

$$\|T(x)\| \leq M\|x\|,$$

for all $x \in X$.

Let T be a bounded linear mapping of X into Y. Then there exists $M > 0$ such that

$$\|T(x)\| \leq M\|x\|,$$

for all $x \in X$. So, we have that for $x \in X$ with $\|x\| \leq 1$,

$$\|T(x)\| \leq M,$$

where $T(x)$ is denoted by Tx.

For a bounded linear mapping $T : X \to Y$, we define its norm by

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|.$$

For such $\|T\|$, we have the following results

Theorem 2.29. [44] Let X and Y be normed linear spaces and let T be a bounded linear mapping of X into Y. Then the following hold:

i) $\|Tx\| \leq \|T\|\|x\|$ for all $x \in X$,

ii) $\|T\| = \sup_{\|x\|=1} \|Tx\|$.

Definition 2.16. [14] Let A be a bounded operator on a Hilbert space H. The operator $A^* : H \to H$ defined by

$$\langle Ax, y\rangle = \langle x, A^*y\rangle\,, \tag{2.2}$$

for all $x, y \in H$, is called the *adjoint operator* of A.

Definition 2.17. [14] Let A be a bounded operator on a Hilbert space H. If $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x \in H$, then A is called the *self-adjoint* operator.

Theorem 2.30. [13] The adjoint operator A^* of a bounded operator A is bounded. Moreover, we have $\|A\| = \|A^*\|$ and $\|A^*A\| = \|A\|^2$.

Remark 2.31. [13] If A is a bounded operator on a Hilbert space H, then A^*A is self-adjoint operator.

Theorem 2.32. [14] Let T be a bounded linear self-adjoint operator on a Hilbert space H. Then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|\,.$$

Definition 2.18. (Normal Operator [13]) A bounded T is called a *normal operator* if $TT^* = T^*T$.

Theorem 2.33. [13] A bounded operator T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.

Definition 2.19. [19] The *spectrum* of the operator T is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : (T - I\lambda)\,(x) = 0, \text{ for some } 0 \neq x \in X\}$$

Definition 2.20. [19] Let H be a Hilbert space. For every bounded operator T on H. The *spectral radius* of T, denoted by $r_\sigma(T)$, is the number defined by

$$r_\sigma(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}\,,$$

Theorem 2.34. [19] Let T be a normal bounded linear operator on a Hilbert space H. Then T is is self-adjoint operator if and only if $\sigma(T) \subset \mathbb{R}$.

Definition 2.21. [14] An operator A is called *positive* if it is self-adjoint and $\langle Ax, x\rangle \geq 0$ for all $x \in H$.

Definition 2.22. [31] A self-adjoint operator A is a strongly positive operator on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\,\|x\|^2,$$

for all $x \in H$.

Definition 2.23. [17] Let A be an operator of C into H. Then, A is called

i) an *α-strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\,\|x - y\|^2,$$

for all $x, y \in C$.

ii) an *α-inverse strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\,\|Ax - Ay\|^2,$$

for all $x, y \in C$.

Lemma 2.35. [30] Let A be a strongly positive linear bounded self-adjoint operator on a Hilbert space H with coefficient $\alpha > 0$ and $0 < \delta < \|A\|^{-1}$. Then $\|I - \delta A\| \leq 1 - \delta\alpha$.

Definition 2.24. [44] Let H be a Hilbert space and let $f : H \to (-\infty, \infty]$ be a proper convex function. Then, we define the *subdifferential* ∂f of f by

$$\partial f(x) = \{z \in H : f(y) \geq \langle y - x, z\rangle + f(x)\},$$

foa all $x, y \in H$. If $f(x) = \infty$, then $\partial f(x) = \emptyset$.

Remark 2.36. [2] Let H be a Hilbert space and let $f : H \to \mathbb{R}$ be a function. If f is a convex and differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$, i.e., its gradient is its only subgradient.

Remark 2.37. Let C and Q be nonempty subset of real Hilbert spaces H_1 and H_2, respectively and let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Let $f : H_1 \to \mathbb{R}$ be a convex, continuous differentiable function defined by $f(x) = \frac{1}{2}\|Ax - P_Q Ax\|^2$, for all $x \in H_1$. Then, $A^*(I - P_Q)Ax = \nabla f(x)$.

Proof. Let $x, y \in H_1$.
Consider,

$$\begin{aligned}
\frac{1}{2}\|Ay - P_Q Ay\|^2 - \frac{1}{2}\|Ax - P_Q Ax\|^2 - \langle A^*Ax - P_Q Ax, y - x\rangle \\
&= \frac{1}{2}\Big(\|Ay - P_Q Ay\|^2 - \|Ax - P_Q Ax\|^2 - 2\langle A^*Ax - P_Q Ax, y - x\rangle\Big) \\
&= \frac{1}{2}\Big(\|Ay - P_Q Ay\|^2 - \|Ax - P_Q Ax\|^2 \\
&\quad -2\langle (I - P_Q)Ax, Ay - P_Q Ax + P_Q Ax - Ax\rangle\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\|Ay - P_QAy\|^2 - \|Ax - P_QAx\|^2 - 2\langle (I-P_Q)Ax, P_QAx - Ax\rangle \\
&\qquad -2\langle (I-P_Q)Ax, Ay - P_QAx\rangle\Big) \\
&= \frac{1}{2}\Big(\|Ay - P_QAy\|^2 + \|Ax - P_QAx\|^2 - 2\langle (I-P_Q)Ax, Ay - P_QAx\rangle\Big) \\
&= \frac{1}{2}\Big(\|Ay - P_QAy\|^2 + \|Ax - P_QAx\|^2 \\
&\qquad -2\langle (I-P_Q)Ax, Ay - P_QAy + P_QAy - P_QAx\rangle\Big) \\
&= \frac{1}{2}\Big(\|Ay - P_QAy\|^2 + \|Ax - P_QAx\|^2 - 2\langle (I-P_Q)Ax, Ay - P_QAy\rangle \\
&\qquad -2\langle (I-P_Q)Ax, P_QAy - P_QAx\rangle\Big) \\
&= \frac{1}{2}\Big(\|Ay - P_QAy - (Ax - P_QAx)\|^2 + 2\langle (I-P_Q)Ax, P_QAx - P_QAy\rangle\Big) \\
&\geq 0
\end{aligned}
$$

From Definition 2.24 and Remark 2.36, we have $A^*(I-P_Q)Ax \in \partial f(x) = \{\nabla f(x)\}$. Hence $A^*(I-P_Q)Ax = \nabla f(x)$. □

2.5 Some useful lemmas

In this final section of this chapter, we give some lemmas that are useful to prove our main theorems.

Lemma 2.38. [50] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n, \forall n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the conditions :

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty}\beta_n \leq 0$ or $\sum\limits_{n=1}^{\infty} |\alpha_n\beta_n| = \infty$

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.39. [41] Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n \leq 1$. Suppose that

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) z_n$$

for all integers $n \geq 1$ and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.40. [40] Let C be a nonempty closed and convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let A_i be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\gamma_i > 0$ and $\overline{\gamma} = \min_{i=1,2,..,N}\gamma_i$. Let $\{a_i\}_{i=1}^N \subseteq (0,1)$, with $\sum_{i=1}^N a_i = 1$. Then the following properties hold:

(i) $\left\|I - \rho \sum_{i=1}^{N} a_i A_i\right\| \leq 1 - \rho\overline{\gamma}$ and $I - \rho \sum_{i=1}^{N} a_i A_i$ is a nonexpansive mapping for every $0 < \rho < \|A_i\|^{-1} (i = 1, 2, ..., N)$.

(ii) $VI(C, \sum_{i=1}^{N} a_i A_i) = \bigcap_{i=1}^{N} VI(C, A_i)$.

Lemma 2.41. [6] For given $x^*, y^* \in C, (x^*, y^*)$ is a solution of problem (1.11) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)], \ \ \forall x \in C,$$

where $y^* = P_C(x^* - \mu Bx^*)$.

Lemma 2.42. [50] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions

(i) $\{\alpha_n\} \subset [0, 1], \ \sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \leq 0 \ or \ \sum\limits_{n=1}^{\infty} |\beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Chapter 3

Main results

In this chapter, we introduce new problems of the general system of variational inequalities problem and prove strong convergence theorems for finding the common element of the set of fixed point of nonexpansive mapping and the solution set of the proposed problems. We divided all content in this chapter into 3 parts as follow:

3.1 The split general system of variational inequalities problem

In this section, inspired and motivated by Ceng et al.[6] and the concept of the split feasibility problem, we introduce *the split general system of variational inequalities problem* (SGSV) which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ &\langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, \ \forall x \in C, \end{aligned} \tag{3.1}$$

and find $(\overline{x^*} = Dx^*, \overline{y^*} = Dy^*) \in Q \times Q$ such that

$$\begin{aligned} &\langle \alpha \overline{A}\overline{y^*} + \overline{x^*} - \overline{y^*}, \overline{x} - \overline{x^*} \rangle \geq 0, \ \forall \overline{x} \in Q, \\ &\langle \gamma \overline{B}\overline{x^*} + \overline{y^*} - \overline{x^*}, \overline{x} - \overline{y^*} \rangle \geq 0, \ \forall \overline{x} \in Q, \end{aligned} \tag{3.2}$$

where $A, B : C \to H_1$ and $\overline{A}, \overline{B} : Q \to H_2$ are four different mappings, $\lambda, \mu, \alpha, \gamma > 0$ and $D : H_1 \to H_2$ is a bounded linear operator. The set of all solutions of (3.1) and (3.2) are denoted by $\Omega_{A,B}$ and $\Omega_{\overline{A},\overline{B}}$, respectively. The set of all solutions of the split general system of variational inequalities problem (SGSV) is denoted by $\Omega^{A,B}_{\overline{A},\overline{B}}$, that is, $\Omega^{A,B}_{\overline{A},\overline{B}} = \left\{(x^*, y^*) \in \Omega_{A,B} \ : \ (\overline{x^*}, \overline{y^*}) \in \Omega_{\overline{A},\overline{B}}\right\}$, where $\overline{x^*} = Dx^*$ and $\overline{y^*} = Dy^*$.

Now, we give the following example to support the SGSV.

Example 3.1. Let $\mathbb{R}$ be the set of real numbers and A, B be mappings from $[-20, 20]$ to $\mathbb{R}$ defined by $Ax = \frac{2x}{3}$ and $Bx = x - 3$, respectively. Let $\overline{A}, \overline{B}$ be mappings from $[-10, 10]$ to $\mathbb{R}$ defined by $\overline{A}x = x - 1$ and $\overline{B}x = 2x - 3$, respectively. Let D be a mapping from $\mathbb{R}$ to $\mathbb{R}$ defined by $Dx = \frac{x}{2}$. We choose $\lambda = \frac{1}{2}, \mu = 1, \alpha = 1, \gamma = \frac{1}{2}$. Then, we have $(2, 3) \in \Omega^{A,B}_{\overline{A},\overline{B}}$.

Lemma 3.2. Let C, Q be nonempty subsets of H_1, H_2, respectively. Let $A, B : C \to H_1$ be a, b-inverse strongly monotone with $\lambda, \mu \in (0, 2\overline{d})$ where $\overline{d} = \min\{a, b\}$. Let $\overline{A}, \overline{B} : Q \to H_2$ be $\overline{a}, \overline{b}$-inverse strongly monotone with $\alpha, \gamma \in (0, 2\widehat{d})$ where $\widehat{d} = \min\{\overline{a}, \overline{b}\}$. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator D^*D. Define $G_C : C \to C$ by $G_C(x) = P_C(I - \lambda A)P_C(I - \mu B)x$, for all $x \in C$ and define $G_Q : Q \to Q$ by $G_Q(\hat{x}) = P_Q(I - \alpha\bar{A})P_Q(I - \gamma\bar{B})\hat{x}$, for all $\widehat{x} \in Q$. Assume $\Omega_{\overline{A},\overline{B}}^{A,B} = \left\{(x^*, y^*) \in \Omega_{A,B} \quad (\overline{x^*}, \overline{y^*}) \in \Omega_{\overline{A},\overline{B}}\right\} \neq \emptyset$. Then the following are equivalent:

(i) $(x^*, y^*) \in \Omega_{\overline{A},\overline{B}}^{A,B}$,

(i) $x^* = G_C(x^* - \eta D^*(I - G_Q)Dx^*)$,

where $y^* = P_C(I - \mu B)x^*$ and $\overline{y^*} = P_Q(I - \gamma\overline{B})\overline{x^*}$ with $\overline{x^*} = Dx^*$ and $\overline{y^*} = Dy^*$.

Proof. Let conditions hold.

$(i) \Rightarrow (ii)$ Let $(x^*, y^*) \in \Omega_{\overline{A},\overline{B}}^{A,B}$, we have $(x^*, y^*) \in \Omega_{A,B}$ and $(\overline{x^*}, \overline{y^*}) \in \Omega_{\overline{A},\overline{B}}$, with $\overline{x^*} = Dx^*$ and $\overline{y^*} = Dy^*$.

Since $(x^*, y^*) \in \Omega_{A,B}$, we obtain

$$\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \ \forall x \in C,$$
$$\langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \ \ \forall x \in C,$$

Then, we have $x^* = P_C(I - \lambda A)y^*$ and $y^* = P_C(I - \mu B)x^*$,
that is,

$$x^* = P_C(I - \lambda A)P_C(I - \mu B)x^* = G_C x^*. \tag{3.3}$$

Since $(\overline{x^*}, \overline{y^*}) \in \Omega_{\overline{A},\overline{B}}$, we obtain

$$\langle \alpha\overline{A}\overline{y^*} + \overline{x^*} - \overline{y^*}, \overline{x} - \overline{x^*} \rangle \geq 0, \ \ \forall \overline{x} \in Q,$$
$$\langle \gamma\overline{B}\overline{x^*} + \overline{y^*} - \overline{x^*}, \overline{x} - \overline{y^*} \rangle \geq 0, \ \ \forall \overline{x} \in Q,$$

Then, we have $\overline{x^*} = P_Q\left(I - \alpha\overline{A}\right)\overline{y^*}$ and $\overline{y^*} = P_Q\left(I - \gamma\overline{B}\right)\overline{x^*}$,
that is,

$$\overline{x^*} = P_Q\left(I - \alpha\overline{A}\right)P_Q\left(I - \gamma\overline{B}\right)\overline{x^*} = G_Q\overline{x^*} = G_Q Dx^*. \tag{3.4}$$

It implies that $x^* = G_C(x^* - \eta D^*(I - G_Q)Dx^*)$.

$(ii) \Rightarrow (i)$ Let $x^* = G_C(x^* - \eta D^*(I - G_Q)Dx^*)$ and $(w, w^*) \in \Omega_{\overline{A},\overline{B}}^{A,B}$, that is, $(w, w^*) \in \Omega_{A,B}$ and $(\overline{w}, \overline{w^*}) \in \Omega_{\overline{A},\overline{B}}$, where $w^* = P_C(I - \mu B)w$, $\overline{w} = Dw$ and $\overline{w^*} = Dw^* = P_Q\left(I - \gamma\overline{B}\right)\overline{w}$.
Since A is a-inverse strongly monotone mapping, we have

$$\begin{aligned}
\|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\lambda a\|Ax - Ay\|^2 + \lambda^2\|Ax - Ay\|^2 \\
&= \|x - y\|^2 - \lambda(2a - \lambda)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - \lambda(2\overline{d} - \lambda)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

It implie that

$$\|(I - \lambda A)x - (I - \lambda A)y\| \leq \|x - y\|. \tag{3.5}$$

Hence $(I-\lambda A)$ is a nonexpansive mapping.
By using the same method as (3.5), we have $(I-\mu B),(I-\alpha\overline{A})$ and $(I-\gamma\overline{B})$ are non-expansive mappings. Then, we obtain that $P_C(I-\lambda A), P_C(I-\mu B), P_Q(I-\alpha\overline{A})$ and $P_Q(I-\gamma\overline{B})$ are nonexpansive mappings.
Since $P_C(I-\lambda A)$ and $P_C(I-\mu B)$ are nonexpansive mappings, we obtain G_C is a non-expansive mapping. Since $P_Q(I-\alpha\overline{A})$ and $P_Q(I-\gamma\overline{B})$ are nonexpansive mappings, we obtain G_Q is nonexpansive mapping.
From $(w,w^*)\in\Omega_{\overline{A},\overline{B}}^{A,B}$ and $(i)\Rightarrow(ii)$, we have $w=G_C(w-\eta D^*(I-G_Q)Dw)$.
By using the same method as (3.4), we have $Dw=G_QDw$.
Then, we have

$$\begin{aligned}
\|x^*-w\|^2 &= \|G_C(x^*-\eta D^*(I-G_Q)Dx^*)-G_C(w-\eta D^*(I-G_Q)Dw)\|^2\\
&\le \|(x^*-\eta D^*(I-G_Q)Dx^*)-(w-\eta D^*(I-G_Q)Dw)\|^2\\
&= \|x^*-w-\eta(D^*(I-G_Q)Dx^*-D^*(I-G_Q)Dw)\|^2\\
&= \|x^*-w-\eta(D^*(I-G_Q)Dx^*)\|^2\\
&= \|x^*-w\|^2-2\eta\langle x^*-w, D^*(I-G_Q)Dx^*\rangle+\eta^2\|D^*(I-G_Q)Dx^*\|^2\\
&= \|x^*-w\|^2-2\eta\langle D(x^*-w),(I-G_Q)Dx^*\rangle+\eta^2\|D^*(I-G_Q)Dx^*\|^2\\
&= \|x^*-w\|^2+2\eta\langle Dw-G_QDx^*+G_QDx^*-Dx^*,(I-G_Q)Dx^*\rangle\\
&\quad+\eta^2\|D^*(I-G_Q)Dx^*\|^2\\
&= \|x^*-w\|^2+2\eta(\langle Dw-G_QDx^*,(I-G_Q)Dx^*\rangle\\
&\quad+\langle G_QDx^*-Dx^*,(I-G_Q)Dx^*\rangle)+\eta^2\|D^*(I-G_Q)Dx^*\|^2\\
&= \|x^*-w\|^2+2\eta\left(\langle Dw-G_QDx^*,(I-G_Q)Dx^*\rangle-\|(I-G_Q)Dx^*\|^2\right)\\
&\quad+\eta^2\langle D^*(I-G_Q)Dx^*, D^*(I-G_Q)Dx^*\rangle\\
&\le \|x^*-w\|^2+2\eta\left(\frac{1}{2}\|(I-G_Q)Dx^*\|^2-\|(I-G_Q)Dx^*\|^2\right)\\
&\quad+\eta^2L\|(I-G_Q)Dx^*\|^2\\
&= \|x^*-w\|^2-\eta\|(I-G_Q)Dx^*\|^2+\eta^2L\|(I-G_Q)Dx^*\|^2\\
&= \|x^*-w\|^2-\eta(1-\eta L)\|(I-G_Q)Dx^*\|^2. \qquad (3.6)
\end{aligned}$$

From (3.6), we have

$$Dx^*\in F(G_Q), \qquad (3.7)$$

that is, $\overline{x^*}=G_Q\overline{x^*}=P_Q\left(I-\alpha\bar{A}\right)P_Q\left(I-\gamma\bar{B}\right)\overline{x^*}$.
It implies that $\overline{x^*}=P_Q\left(I-\alpha\overline{A}\right)\overline{y^*}$, where $\overline{y^*}=P_Q\left(I-\gamma\overline{B}\right)\overline{x^*}$.
By the property of P_C, we obtain

$$\begin{aligned}
\left\langle\alpha\overline{A}\overline{y^*}+\overline{x^*}-\overline{y^*},\overline{x}-\overline{x^*}\right\rangle &\ge 0,\ \ \forall\overline{x}\in Q,\\
\left\langle\gamma\overline{B}\overline{x^*}+\overline{y^*}-\overline{x^*},\overline{x}-\overline{y^*}\right\rangle &\ge 0,\ \ \forall\overline{x}\in Q,
\end{aligned}$$

that is,

$$(\overline{x^*},\overline{y^*})\in\Omega_{\overline{A},\overline{B}}. \qquad (3.8)$$

From the definition of x^* and (3.7), we have

$$x^* = G_C(x^* - \eta D^*(I - G_Q)Dx^*) = G_C(x^*).$$

That is, $x^* \in F(G_C)$.
Then, we have $x^* = G_C x^* = P_C(I - \lambda A) P_C(I - \mu B) x^*$.
It implies that $x^* = P_C(I - \lambda A) y^*$, where $y^* = P_C(I - \mu B) x^*$.
By property of P_C, we obtain

$$\langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, \ \ \forall x \in C,$$
$$\langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, \ \ \forall x \in C,$$

that is,

$$(x^*, y^*) \in \Omega_{A,B}. \tag{3.9}$$

From (3.8) and (3.9), we have $(x^*, y^*) \in \Omega_{A,B}^{\bar{A},\bar{B}}$. □

Now, we give the following example to support Lemma 3.2

Example 3.3. Let $\mathbb{R}$ be the set of real number and A, B be mappings from $[-20, 20]$ to $\mathbb{R}$ defined by $Ax = \frac{x-3}{5}$ and $Bx = \frac{x-3}{9}$ for all $x \in [-20, 20]$, respectively. Let $\overline{A}, \overline{B}$ be mappings from $[-10, 10]$ to $\mathbb{R}$ defined by $\overline{A}x = \frac{x-1}{3}$ and $\overline{B}x = \frac{x-1}{7}$ for all $x \in [-10, 10]$, respectively. Let D be a mapping from $\mathbb{R}$ to $\mathbb{R}$ defined by $Dx = \frac{x}{3}$, for all $x \in \mathbb{R}$. It is easy to see that $A, B, \overline{A}, \overline{B}$ are 1-inverse strongly monotone with $\lambda, \mu, \alpha, \gamma \in (0, 2)$. Then, we choose $\lambda = 1, \mu = 1.5, \alpha = 0.5, \gamma = 1$. Since $Dx = \frac{x}{3}$, we have $D^*x = \frac{x}{3}$ is an adjoint of D. From D and D^*, we obtain $L = \frac{1}{9}$, L is the spectral radius of the operator D^*D. Then, we can choose $\eta = 2$. Define $G_{[-20,20]} : [-20, 20] \to [-20, 20]$ by

$$G_{[-20,20]}(x) = P_{[-20,20]}(I - A)\left(P_{[-20,20]}(x - 1.5Bx)\right), \ \forall x \in [-20, 20],$$

and define $G_{[-10,10]} : [-10, 10] \to [-10, 10]$ by

$$G_{[-10,10]}(\hat{x}) = P_{[-10,10]}(I - 0.5\bar{A})\left(P_{[-10,10]}(\hat{x} - \bar{B}\hat{x})\right), \ \forall \hat{x} \in [-10, 10].$$

It is easy to see that $(3, 3) \in \Omega_{A,B}^{\bar{A},\bar{B}}$. By Lemma 3.2, we have $3 = G_{[-20,20]}(3 - 2D^*(I - G_{[-10,10]})D(3))$, where $y^* = P_{[-20,20]}(3 - 1.5B(3))$ and $\overline{y^*} = P_{[-10,10]}(\overline{x^*} - \bar{B}(\overline{x^*}))$ with $\overline{x^*} = D(3)$ and $\overline{y^*} = D(3)$.

Next, we prove a strong convergence theorem for finding a common element of the set of fixed points of mapping G define as in 3.2, which is the solution of the split general system of variational inequalities problem (SGSV), and the set of fixed points of nonexpansive mapping.

Theorem 3.4. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A, B : C \to H_1$ be a, b-inverse strongly monotone mappings with $d = \min\{a, b\}$, respectively. Let $\overline{A}, \overline{B} : Q \to H_2$ be $\bar{a}, \bar{b}$-inverse strongly monotone mappings with $\overline{d} = \min\{\overline{a}, \overline{b}\}$, respectively. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Define the mapping $G_C : C \to C$ by $G_C(x) = P_C(I - \lambda A)P_C(I - \mu B)x$, for all $x \in C$ and define the mapping $G_Q : Q \to Q$ by $G_Q(\widehat{x}) = P_Q(I - \alpha\bar{A})P_Q(I - \gamma\bar{B})\widehat{x}$, for all $\widehat{x} \in Q$. Define $G : C \to C$ by $G(x) = G_C(x - \eta D^*(I - G_Q)Dx)$ for all $x \in C$. Let $T : C \to C$ be a nonexpansive mapping. Assume $\Im = F(G) \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} y_n &= G_C(x_n - \eta D^*(I - G_Q)Dx_n), \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)Ty_n, \end{aligned} \tag{3.10}$$

where $\{\alpha_n\} \subset [0, 1]$, $\lambda, \mu \in (0, 2d)$, $\alpha, \gamma \in (0, 2\overline{d})$, $\eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_\Im u$ which $(x_0, y_0) \in \Omega^{A,B}_{\overline{A},\overline{B}}$, where $y_0 = P_C(I - \mu B)x_0$ and $\overline{y_0} = P_Q(I - \gamma\overline{B})\,\overline{x_0}$ with $\overline{x_0} = Dx_0$ and $\overline{y_0} = Dy_0$.

Proof. Let $z \in \Im$. Then, we have $z = G(z) = G_C(z - \eta D^*(I - G_Q)Dz)$.
By Lemma 3.2, we have $(z, z_0) \in \Omega^{A,B}_{\overline{A},\overline{B}}$, where $z_0 = P_C(I - \mu B)z$ and $\overline{z_0} = P_C(I - \gamma\overline{B})\overline{z}$ with $\overline{z} = Dz$ and $\overline{z_0} = Dz_0$.
Since $(z, z_0) \in \Omega^{A,B}_{\overline{A},\overline{B}}$, we have $(z, z_0) \in \Omega_{A,B}$ and $(Dz, Dz_0) \in \Omega_{\overline{A},\overline{B}}$.
Since $(Dz, Dz_0) \in \Omega_{\overline{A},\overline{B}}$, we obtain

$$\begin{aligned} &\langle \alpha\overline{A}Dz_0 + Dz - Dz_0, \overline{x} - Dz\rangle \geq 0, \quad \forall \overline{x} \in Q, \\ &\langle \gamma\overline{B}Dz + Dz_0 - Dz, \overline{x} - Dz_0\rangle \geq 0, \quad \forall \overline{x} \in Q, \end{aligned}$$

Then, we have $Dz = P_Q(I - \alpha\overline{A})Dz_0$ and $Dz_0 = P_Q(I - \gamma\overline{B})Dz$.
It implies that

$$Dz = P_Q(I - \alpha\overline{A})P_Q(I - \gamma\overline{B})Dz = G_Q Dz. \tag{3.11}$$

Step 1. Show that $\{x_n\}$ is bounded.
Applying (3.6) and the definition of y_n, we have

$$\|y_n - z\| \leq \|x_n - z\|. \tag{3.12}$$

From definition of x_n and (3.12), we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n u + (1 - \alpha_n)Ty_n - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|Ty_n - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|y_n - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|x_n - z\| \\ &\leq \max\{\|u - z\|, \|x_1 - z\|\}. \end{aligned}$$

By induction, we get that $\|x_n - z\| \leq \max\{\|u - z\|, \|x_1 - z\|\}$, for all $n \in \mathbb{N}$.
It implies that the sequence $\{x_n\}$ is bounded and so is $\{y_n\}$.
Step 2. Show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
From the definition of y_n, we have

$$\begin{aligned}
\|y_n - y_{n-1}\|^2 &= \|G_C(x_n - \eta D^*(I - G_Q)Dx_n) - G_C(x_{n-1} - \eta D^*(I - G_Q Dx_{n-1}))\|^2 \\
&\leq \|(x_n - \eta D^*(I - G_Q)Dx_n) - (x_{n-1} - \eta D^*(I - G_Q)Dx_{n-1})\|^2 \\
&= \|(x_n - x_{n-1}) - \eta D^*(I - G_Q)Dx_n + \eta D^*(I - G_Q)Dx_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\eta\langle x_n - x_{n-1}, D^*((I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\rangle \\
&\quad + \eta^2\|D^*((I - G_Q)Dx_n - (I - G_Q)Dx_{n-1})\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 - 2\eta\langle Dx_n - G_Q Dx_n + G_Q Dx_n - G_Q Dx_{n-1} \\
&\quad + G_Q Dx_{n-1} - Dx_{n-1}, (I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\rangle \\
&\quad + \eta^2 L\|(I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\eta[\langle (I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}, (I - G_Q) \\
&\quad \times Dx_n - (I - G_Q)Dx_{n-1}\rangle - \langle G_Q Dx_{n-1} - G_Q Dx_n, (I - G_Q)Dx_n \\
&\quad - (I - G_Q)Dx_{n-1}\rangle] + \eta^2 L\|(I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 + 2\eta\Big[-\|(I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\|^2 \\
&\quad + \langle G_Q Dx_{n-1} - G_Q Dx_n, (I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\rangle\Big] \\
&\quad + \eta^2 L\|(I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 + 2\eta\Big[-\|(I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\|^2 \\
&\quad + \frac{1}{2}\|(I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\|^2\Big] \\
&\quad + \eta^2 L\|(I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 - \eta\|(I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\|^2 \\
&\quad + \eta^2 L\|(I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 - \eta(1 - \eta L)\|(I - G_Q)Dx_n - (I - G_Q)Dx_{n-1}\|^2 \\
&\leq \|x_n - x_{n-1}\|^2.
\end{aligned}$$

This implies that

$$\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\|. \tag{3.13}$$

From the definition of x_n and (3.13), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n u + (1 - \alpha_n)Ty_n - (\alpha_{n-1}u + (1 - \alpha_{n-1})Ty_{n-1})\| \\
&\leq |\alpha_n - \alpha_{n-1}|\,\|u\| + (1 - \alpha_n)\|Ty_n - Ty_{n-1}\| + |\alpha_n - \alpha_{n-1}|\,\|Ty_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}|\,\|u\| + (1 - \alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\,\|Ty_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}|\,\|u\| + (1 - \alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\,\|Ty_{n-1}\|. \qquad (3.14)
\end{aligned}$$

From Lemma 2.42, conditions (i), (ii) and (3.14), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.15}$$

Applying (3.6) and the definition of y_n, we have

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 - \eta(1-\eta L)\|(I-G_Q)Dx_n\|^2.$$

From the definition of x_n,

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n u + (1-\alpha_n)Ty_n - z\|^2 \\
&\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\|Ty_n - z\|^2 \\
&\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\|y_n - z\|^2 \\
&\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - (1-\alpha_n)\eta(1-\eta L)\|(I-G_Q)Dx_n\|^2.
\end{aligned}$$

It implies that

$$\begin{aligned}
(1-\alpha_n)\eta(1-\eta L)\|(I-G_Q)Dx_n\|^2 &\leq \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\leq \alpha_n\|u-z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\,\|x_{n+1} - x_n\|.
\end{aligned}$$

By (3.15) and condition (i), we obtain

$$\lim_{n\to\infty} \|(I-G_Q)Dx_n\| = 0. \tag{3.16}$$

Step 3. Show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $\lim_{n\to\infty}\|Ty_n - y_n\| = 0$.
From the definition of x_n, we have

$$\begin{aligned}
x_{n+1} - x_n &= \alpha_n u + (1-\alpha_n)Ty_n - x_n \\
&= \alpha_n(u - x_n) + (1-\alpha_n)(Ty_n - x_n)
\end{aligned} \tag{3.17}$$

Then, by (3.15), (3.17) and condition (i), we obtain

$$\lim_{n\to\infty} \|Ty_n - x_n\| = 0 \tag{3.18}$$

Put $h_n = P_C(I-\mu B)(x_n - \eta D^*(I-G_Q)Dx_n)$ and $h^* = P_C(I-\mu B)(z - \eta D^*(I-G_Q)Dz)$, we can rewrite y_n by

$$y_n = P_C(I-\lambda A)h_n \;\; , \forall n \geq 1,$$

and $z = P_C(I-\lambda A)h^*$.
By the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n u + (1-\alpha_n)Ty_n - z\|^2 \\
&\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\|Ty_n - z\|^2 \\
&\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\|y_n - z\|^2 \\
&= \alpha_n\|u-z\|^2 + (1-\alpha_n)\|P_C(I-\lambda A)h_n - P_C(I-\lambda A)h^*\|^2 \\
&\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\|(I-\lambda A)h_n - (I-\lambda A)h^*\|^2
\end{aligned}$$

$$= \alpha_n\|u-z\|^2 + (1-\alpha_n)\|(h_n-h^*) - \lambda(Ah_n - Ah^*)\|^2$$
$$= \alpha_n\|u-z\|^2 + (1-\alpha_n)\Big[\|h_n-h^*\|^2 - 2\lambda\langle h_n-h^*, Ah_n - Ah^*\rangle$$
$$+\lambda^2\|Ah_n-Ah^*\|^2\Big]$$
$$\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\Big[\|h_n-h^*\|^2 - 2\lambda a\|Ah_n-Ah^*\|^2 + \lambda^2\|Ah_n-Ah^*\|^2\Big]$$
$$= \alpha_n\|u-z\|^2 + (1-\alpha_n)\Big[\|h_n-h^*\|^2 - \lambda(2d-\lambda)\|Ah_n-Ah^*\|^2\Big]$$
$$\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\Big[\|x_n-z\|^2 - \lambda(2d-\lambda)\|Ah_n-Ah^*\|^2\Big]$$

It implies that

$$\lambda(1-\alpha_n)(2d-\lambda)\|Ah_n-Ah^*\|^2 \leq \alpha_n\|u-z\|^2 + \|x_n-z\|^2 - \|x_{n+1}-z\|^2$$
$$\leq \alpha_n\|u-z\|^2 + (\|x_n-z\| + \|x_{n+1}-z\|)\,\|x_{n+1}-x_n\|.$$

From (3.15) and condition (i), we have

$$\lim_{n\to\infty}\|Ah_n - Ah^*\| = 0. \tag{3.19}$$

Put $k_n = x_n - \eta D^*(I-G_Q)Dx_n$ and $k^* = z - \eta D^*(I-G_Q)Dz$, we can rewrite y_n by

$$y_n = P_C(I-\lambda A)P_C(I-\mu B)k_n, \ \ \forall n \geq 1,$$

and $z = P_C(I-\lambda A)P_C(I-\mu B)k^*$.
From the definition of x_n, we have

$$\|x_{n+1}-z\|^2 = \|\alpha_n u + (1-\alpha_n)Ty_n - z\|^2$$
$$\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\|Ty_n - z\|^2$$
$$\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\|y_n - z\|^2$$
$$= \alpha_n\|u-z\|^2 + (1-\alpha_n)\,\|P_C(I-\lambda A)P_C(I-\mu B)k_n$$
$$- P_C(I-\lambda A)P_C(I-\mu B)k^*\|^2$$
$$\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\,\|P_C(I-\mu B)k_n - P_C(I-\mu B)k^*\|^2$$
$$\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\,\|(I-\mu B)k_n - (I-\mu B)k^*\|^2$$
$$= \alpha_n\|u-z\|^2 + (1-\alpha_n)\,\|(k_n-k^*) - \mu(Bk_n - Bk^*)\|^2$$
$$= \alpha_n\|u-z\|^2 + (1-\alpha_n)\Big[\|k_n-k^*\|^2 - 2\mu\langle k_n-k^*, Bk_n-Bk^*\rangle$$
$$+\mu^2\|Bk_n-Bk^*\|^2\Big]$$
$$\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\Big[\|k_n-k^*\|^2 - 2\mu b\|Bk_n-Bk^*\|^2$$
$$+\mu^2\|Bk_n-Bk^*\|^2\Big]$$
$$= \alpha_n\|u-z\|^2 + (1-\alpha_n)\Big[\|k_n-k^*\|^2 - \mu(2d-\mu)\|Bk_n-Bk^*\|^2\Big]$$
$$\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\Big[\|x_n-z\|^2 - \mu(2d-\mu)\|Bk_n-Bk^*\|^2\Big].$$

It implies that

$$\mu(1-\alpha_n)(2d-\mu)\|Bk_n - Bk^*\|^2 \leq \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2$$
$$\leq \alpha_n\|u-z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\|.$$

From (3.15) and condition (i), we have

$$\lim_{n\to\infty} \|Bk_n - Bk^*\| = 0. \tag{3.20}$$

By the definition of y_n, we have

$$\begin{aligned}
\|y_n - z\|^2 &= \|P_C(I-\lambda A)h_n - P_C(I-\lambda A)h^*\|^2 \\
&\leq \langle (h_n - \lambda Ah_n) - (h^* - \lambda Ah^*), y_n - z\rangle \\
&= \frac{1}{2}\Big[\|(h_n - \lambda Ah_n) - (h^* - \lambda Ah^*)\|^2 + \|y_n - z\|^2 \\
&\quad - \|(h_n - \lambda Ah_n) - (h^* - \lambda Ah^*) - (y_n - z)\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|h_n - h^*\|^2 + \|y_n - z\|^2 - \|(h_n - \lambda Ah_n) - (h^* - \lambda Ah^*) - (y_n - z)\|^2\Big] \\
&= \frac{1}{2}\Big[\|h_n - h^*\|^2 + \|y_n - z\|^2 - \|(h_n - y_n) - (h^* - z) - \lambda(Ah_n - Ah^*)\|^2\Big] \\
&= \frac{1}{2}\Big[\|h_n - h^*\|^2 + \|y_n - z\|^2 - \|(h_n - y_n) - (h^* - z)\|^2 \\
&\quad + 2\lambda\langle (h_n - y_n) - (h^* - z), Ah_n - Ah^*\rangle - \lambda^2\|Ah_n - Ah^*\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|x_n - z\|^2 + \|y_n - z\|^2 - \|(h_n - y_n) - (h^* - z)\|^2 \\
&\quad + 2\lambda\|(h_n - y_n) - (h^* - z)\|\,\|Ah_n - Ah^*\| - \lambda^2\|Ah_n - Ah^*\|^2\Big].
\end{aligned}$$

This implies that

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 - \|(h_n - y_n) - (h^* - z)\|^2 + 2\lambda\|(h_n - y_n) - (h^* - z)\|\,\|Ah_n - Ah^*\|. \tag{3.21}$$

From (3.21) and the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n u + (1-\alpha_n)Ty_n - z\|^2 \\
&\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\|Ty_n - z\|^2 \\
&\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\|y_n - z\|^2 \\
&\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\Big[\|x_n - z\|^2 - \|(h_n - y_n) - (h^* - z)\|^2 \\
&\quad + 2\lambda\|(h_n - y_n) - (h^* - z)\|\,\|Ah_n - Ah^*\|\,\Big] \\
&\leq \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - (1-\alpha_n)\|(h_n - y_n) - (h^* - z)\|^2 \\
&\quad + 2\lambda(1-\alpha_n)\|(h_n - y_n) - (h^* - z)\|\,\|Ah_n - Ah^*\|]
\end{aligned}$$

It implies that

$$(1-\alpha_n)\|(h_n - y_n) - (h^* - z)\|^2 \leq \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2$$
$$+ 2\lambda(1-\alpha_n)\|(h_n - y_n) - (h^* - z)\|\,\|Ah_n - Ah^*\|$$

$$\leq \alpha_n\|u-z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|)\|x_n - x_{n+1}\| + 2\lambda(1-\alpha_n)\|(h_n - y_n) - (h^* - z)\|\|Ah_n - Ah^*\|$$

From (3.15), (3.19) and condition (i), we have

$$\lim_{n\to\infty}\|(h_n - y_n) - (h^* - z)\| = 0. \tag{3.22}$$

From (3.11), we have

$$\begin{aligned}\|(z-k^*) - (x_n - k_n)\|^2 &= \|z - (z - \eta D^*(I - G_Q)Dz) \\ &\quad -(x_n - (x_n - \eta D^*(I-G_Q)Dx_n))\|^2 \\ &= \eta^2\|D^*(I-G_Q)Dx_n\|^2 \\ &\leq \eta^2 L\|(I-G_Q)Dx_n\|^2.\end{aligned}$$

From (3.16), we have

$$\lim_{n\to\infty}\|(z-k^*) - (x_n - k_n)\| = 0. \tag{3.23}$$

Consider,

$$\begin{aligned}\|(x_n - h_n) + (h^* - z)\|^2 &= \|(x_n - P_C(I-\mu B)k_n) + (P_C(I-\mu B)k^* - z)\|^2 \\ &= \|x_n - \mu Bx_n - (I-\mu B)\eta D^*(I-G_Q)Dx_n \\ &\quad + (I-\mu B)\eta D^*(I-G_Q)Dx_n + \mu Bx_n - P_C(I-\mu B)k_n \\ &\quad + P_C(I-\mu B)k^* - \mu Bz + (I-\mu B)\eta D^*(I-G_Q)Dz \\ &\quad -(I-\mu B)\eta D^*(I-G_Q)Dz + \mu Bz - z\|^2 \\ &= \|(I-\mu B)k_n - (I-\mu B)k^* - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] \\ &\quad + \mu(Bk_n - Bk^*) + \eta D^*(I-G_Q)Dx_n - \eta D^*(I-G_Q)Dz\|^2 \\ &= \|(I-\mu B)k_n - (I-\mu B)k^* - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] \\ &\quad + \mu(Bk_n - Bk^*) + \eta D^*(I-G_Q)Dx_n\|^2 \\ &\leq \|(I-\mu B)k_n - (I-\mu B)k^* - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] \\ &\quad + \mu(Bk_n - Bk^*)\|^2 + 2\eta\langle (I-G_Q)Dx_n, D[(x_n - h_n) + (h^* - z)]\rangle \\ &\leq \|(I-\mu B)k_n - (I-\mu B)k^* - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] \\ &\quad + \mu(Bk_n - Bk^*)\|^2 + 2\eta\|(I-G_Q)Dx_n\|\|D[(x_n - h_n) + (h^* - z)]\| \\ &\leq \|(I-\mu B)k_n - (I-\mu B)k^* - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*]\|^2 \\ &\quad + 2\mu\langle Bk_n - Bk^*, (I-\mu B)k_n - (I-\mu B)k^* \\ &\quad - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] + \mu(Bk_n - Bk^*)\rangle \\ &\quad + 2\eta\|(I-G_Q)Dx_n\|\|D[(x_n - h_n) + (h^* - z)]\| \\ &\leq \|(I-\mu B)k_n - (I-\mu B)k^*\|^2 - \|P_C(I-\mu B)k_n - P_C(I-\mu B)k^*\|^2 \\ &\quad + 2\mu\|Bk_n - Bk^*\|\|(I-\mu B)k_n - (I-\mu B)k^* \\ &\quad - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] + \mu(Bk_n - Bk^*)\| \\ &\quad + 2\eta\|(I-G_Q)Dx_n\|\|D[(x_n - h_n) + (h^* - z)]\|\end{aligned}$$

$$
\begin{aligned}
&= \|(I-\mu B)k_n-(I-\mu B)k^*\|^2 - \|h_n-h^*\|^2 + 2\mu\,\|Bk_n-Bk^*\| \\
&\quad \times \|(I-\mu B)k_n-(I-\mu B)k^* - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] \\
&\quad +\mu(Bk_n-Bk^*)\| + 2\eta\,\|(I-G_Q)Dx_n\|\,\|D\,[(x_n-h_n)+(h^*-z)]\| \\
&\le \|(I-\mu B)k_n-(I-\mu B)k^*\|^2 - \|P_C(I-\lambda A)h_n - P_C(I-\lambda A)h^*\|^2 \\
&\quad + 2\mu\,\|Bk_n-Bk^*\|\,\|(I-\mu B)k_n-(I-\mu B)k^* - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] \\
&\quad +\mu(Bk_n-Bk^*)\| + 2\eta\,\|(I-G_Q)Dx_n\|\,\|D\,[(x_n-h_n)+(h^*-z)]\| \\
&= \|(I-\mu B)k_n-(I-\mu B)k^*\|^2 - \|y_n-z\|^2 + 2\mu\,\|Bk_n-Bk^*\| \\
&\quad \times \|(I-\mu B)k_n-(I-\mu B)k^* - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] \\
&\quad +\mu(Bk_n-Bk^*)\| + 2\eta\,\|(I-G_Q)Dx_n\|\,\|D\,[(x_n-h_n)+(h^*-z)]\| \\
&\le \|(I-\mu B)k_n-(I-\mu B)k^*\|^2 - \|Ty_n-Tz\|^2 + 2\mu\,\|Bk_n-Bk^*\| \\
&\quad \times \|(I-\mu B)k_n-(I-\mu B)k^* - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] \\
&\quad +\mu(Bk_n-Bk^*)\| + 2\eta\,\|(I-G_Q)Dx_n\|\,\|D\,[(x_n-h_n)+(h^*-z)]\| \\
&\le \|(I-\mu B)k_n-(I-\mu B)k^* - (Ty_n-z)\|\,(\|(I-\mu B)k_n-(I-\mu B)k^*\| + \|Ty_n-z\|) \\
&\quad + 2\mu\,\|Bk_n-Bk^*\|\,\|(I-\mu B)k_n-(I-\mu B)k^* - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] \\
&\quad +\mu(Bk_n-Bk^*)\| + 2\eta\,\|(I-G_Q)Dx_n\|\,\|D\,[(x_n-h_n)+(h^*-z)]\| \\
&= \|k_n-\mu Bk_n-k^*+\mu Bk^*-x_n+x_n-Ty_n+z\|\,(\|(I-\mu B)k_n-(I-\mu B)k^*\| \\
&\quad + \|Ty_n-z\|) + 2\mu\,\|Bk_n-Bk^*\|\,\|(I-\mu B)k_n-(I-\mu B)k^* \\
&\quad - [P_C(I-\mu B)k_n - P_C(I-\mu B)k^*] +\mu(Bk_n-Bk^*)\| + 2\eta\,\|(I-G_Q)Dx_n\| \\
&\quad \times \|D\,[(x_n-h_n)+(h^*-z)]\| \\
&= \|(x_n-Ty_n)+(z-k^*)-(x_n-k_n)-\mu(Bk_n-Bk^*)\|\,(\|(I-\mu B)k_n-(I-\mu B)k^*\| \\
&\quad + \|Ty_n-z\|) + 2\mu\,\|Bk_n-Bk^*\|\,\|(I-\mu B)k_n-(I-\mu B)k^* - [P_C(I-\mu B)k_n \\
&\quad - P_C(I-\mu B)k^*] +\mu(Bk_n-Bk^*)\| + 2\eta\,\|(I-G_Q)Dx_n\|\,\|D\,[(x_n-h_n)+(h^*-z)]\| \\
&\le (\|x_n-Ty_n\| + \|(z-k^*)-(x_n-k_n)\| + \mu\,\|Bk_n-Bk^*\|)\,(\|(I-\mu B)k_n-(I-\mu B)k^*\| \\
&\quad + \|Ty_n-z\|) + 2\mu\,\|Bk_n-Bk^*\|\,\|(I-\mu B)k_n-(I-\mu B)k^* - [P_C(I-\mu B)k_n \\
&\quad - P_C(I-\mu B)k^*] +\mu(Bk_n-Bk^*)\| + 2\eta\,\|(I-G_Q)Dx_n\|\,\|D\,[(x_n-h_n)+(h^*-z)]\|
\end{aligned}
$$

From (3.16), (3.18), (3.20) and (3.23), we obtain

$$\lim_{n\to\infty} \|(x_n-h_n)+(h^*-z)\| = 0 \tag{3.24}$$

From (3.18), (3.22), (3.24) and

$$
\begin{aligned}
\|Ty_n-y_n\| &= \|Ty_n-x_n+x_n-h_n+h_n-h^*+h^*-z+z-y_n\| \\
&\le \|Ty_n-x_n\| + \|(x_n-h_n)+(h^*-z)\| + \|(h_n-y_n)-(h^*-z)\|,
\end{aligned}
$$

we get

$$\lim_{n\to\infty} \|Ty_n-y_n\| = 0. \tag{3.25}$$

From (3.18), (3.25) and

$$\|x_n - y_n\| \leq \|x_n - Ty_n\| + \|Ty_n - y_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.26}$$

Step 4. Show that $\limsup_{n\to\infty} \langle u - x_0, x_n - x_0\rangle \leq 0$ where $x_0 = P_{\Im}u$.
To show this inequality, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - x_0, x_n - x_0\rangle = \lim_{j\to\infty} \langle u - x_0, x_{n_j} - x_0\rangle. \tag{3.27}$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_{n_j} \rightharpoonup q$ as $j \to \infty$, where $q \in C$.
From (3.27) and $x_{n_j} \rightharpoonup q$ as $j \to \infty$, we have

$$\limsup_{n\to\infty} \langle u - x_0, x_n - x_0\rangle = \langle u - x_0, q - x_0\rangle. \tag{3.28}$$

From (3.26), $x_{n_j} \rightharpoonup q$ as $j \to \infty$, we have $y_{n_j} \rightharpoonup q$ as $j \to \infty$. From (3.25) and Lemma 2.28, we obtain

$$q \in F(T). \tag{3.29}$$

Assume that $q \neq F(G)$. By Opial's condition and (3.26), we have

$$\begin{aligned}
\liminf_{j\to\infty} \|x_{n_j} - q\| &< \liminf_{j\to\infty} \|x_{n_j} - G(q)\| \\
&= \liminf_{j\to\infty} \|x_{n_j} - y_{n_j} + y_{n_j} - G_C(q - \eta D^*(I - G_Q)Dq)\| \\
&= \liminf_{j\to\infty} \|x_{n_j} - y_{n_j} + G_C(x_{n_j} - \eta D^*(I - G_Q)Dx_{n_j}) \\
&\qquad - G_C(q - \eta D^*(I - G_Q)Dq)\| \\
&\leq \liminf_{j\to\infty} (\|x_{n_j} - y_{n_j}\| + \|x_{n_j} - q\|) \\
&= \liminf_{j\to\infty} \|x_{n_j} - q\|.
\end{aligned}$$

This is a contradiction. Then, we have

$$q \in F(G). \tag{3.30}$$

From (3.29) and (3.30), we have

$$q \in \Im. \tag{3.31}$$

From (3.28) and (3.31), we obtain

$$\limsup_{n\to\infty} \langle u - x_0, x_n - x_0\rangle \leq 0. \tag{3.32}$$

Step 5. Finally, we show that $\lim_{n\to\infty} \|x_n - x_0\| = 0$, where $x_0 = P_{\Im}u$.
From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - x_0\|^2 &= \|\alpha_n u + (1 - \alpha_n)Ty_n - x_0\|^2 \\
&= \|\alpha_n(u - x_0) + (1 - \alpha_n)(Ty_n - x_0)\|^2 \\
&\leq (1 - \alpha_n)\|Ty_n - x_0\|^2 + 2\alpha_n \langle u - x_0, x_{n+1} - x_0\rangle
\end{aligned}$$

$$\leq (1-\alpha_n)\|y_n - x_0\|^2 + 2\alpha_n \langle u - x_0, x_{n+1} - x_0\rangle$$
$$\leq (1-\alpha_n)\|x_n - x_0\|^2 + 2\alpha_n \langle u - x_0, x_{n+1} - x_0\rangle .$$

From (3.32), condition (i) and Lemma 2.38, we can conclude that the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Im}u$. From Lemma 3.2, we get that $(x_0, y_0) \in \Omega^{A,B}_{\overline{A},\overline{B}}$, where $y_0 = P_C(I - \lambda B)x_0$ and $\overline{y_0} = P_Q(I - \gamma\overline{B})\overline{x_0}$ with $\overline{x_0} = D\overline{x_0}$ and $\overline{y_0} = D\overline{y_0}$. This completes the proof. □

The following corollary is consequence which is applied by Theorem 3.4.

Corollary 3.5. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A, B : C \to H_1$ be a, b-inverse strongly monotone mappings with $d = \min\{a, b\}$, respectively. Let $\overline{A}, \overline{B} : Q \to H_2$ be $\overline{a}, \overline{b}$-inverse strongly monotone mappings with $\overline{d} = \min\{\overline{a}, \overline{b}\}$, respectively. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Define the mapping $G_C : C \to C$ by $G_C(x) = P_C(I - \lambda A)P_C(I - \mu B)x$, for all $x \in C$ and define the mapping $G_Q : Q \to Q$ by $G_Q(\widehat{x}) = P_Q(I - \alpha\bar{A})P_Q(I - \gamma\bar{B})\widehat{x}$, for all $\widehat{x} \in Q$. Define $G : C \to C$ by $G(x) = G_C(x - \eta D^*(I - G_Q)Dx)$ for all $x \in C$. Assume $\Im = F(G) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} y_n &= G_C(x_n - \eta D^*(I - G_Q)Dx_n), \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n, \end{aligned} \tag{3.33}$$

where $\{\alpha_n\} \subset [0,1]$, $\lambda, \mu \in (0, 2d)$, $\alpha, \gamma \in (0, 2\bar{d})$, $\eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Im}u$ which $(x_0, y_0) \in \Omega^{A,B}_{\overline{A},\overline{B}}$, where $y_0 = P_C(I - \mu B)x_0$ and $\overline{y_0} = P_Q\left(I - \gamma\bar{B}\right)\overline{x_0}$ with $\overline{x_0} = Dx_0$ and $\overline{y_0} = Dy_0$.

Proof. Put $T = I$. Then, by Theorem 3.4, we obtain the desired conclusion. □

3.2 The general system of variational inequalities improvement

In this section, we present about the improvement of the general system of variational inequalities, Ceng et al. [6]. In the first part, we prove a strong convergence theorem for the general system of variational inequalities problem and nonexpansive mapping. The other part, we introduce a new problem which motivated by Ceng et al. [6] and prove a strong convergence for the proposed problem.

3.2.1 Convergence analysis for relaxed extragradient method and variational inequality problem with numerical example

Now, we introduce the iterative scheme with the inverse strongly monotone mapping and prove a strong convergence for finding the common element of the solution of the general system of variational inequalities and the set of the solution of variational inequality problem in a real Hilbert space.

Theorem 3.6. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, D_1, D_2 : C \to H$ be d, d_1, d_2-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \lambda_1 D_1) P_C(I - \lambda_2 D_2)x$, for all $x \in C$ and $a \in [0, 1)$. Let f be an α-contraction mapping on H. For $k = 1, 2, ..., N$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{N} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,N} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Suppose that $\Im = F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= (1 - \beta_n)x_n + \beta_n P_C(I - \lambda D)y_n, \\ y_n &= \alpha_n \gamma f(x_n) + (I - \alpha_n \overline{A}) G x_n, \end{aligned} \tag{3.34}$$

where $\{\alpha_n\} \subset [0, 1]$, $\lambda \in (0, 2d)$, $\lambda_1, \lambda_2 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2\}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n \leq c < 1$;

(iii) $0 < c_k < 1$ and $\sum_{k=1}^{N} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0) is a solution of (1.11) where $y_0 = P_C(x_0 - \lambda_2 D_2 x_0)$.

Proof. Since $\alpha_n \to 0$ as $n \to \infty$, without loss of generality, we may assume that $\alpha_n < \frac{1}{\|A_i\|}$, $\forall n \in \mathbb{N}$ and $i = 1, 2, ..., N$. Let $x, y \in C$. Since D is d-inverse strongly monotone mapping with $\lambda \in (0, 2d)$, we obtain

$$\begin{aligned} \|(I - \lambda D)x - (I - \lambda D)y\|^2 &= \|x - y - \lambda(Dx - Dy)\|^2 \\ &= \|x - y\|^2 - 2\lambda \langle x - y, Dx - Dy \rangle - \lambda^2 \|Dx - Dy\|^2 \\ &\leq \|x - y\|^2 - 2\lambda d \|Dx - Dy\|^2 - \lambda^2 \|Dx - Dy\|^2 \\ &\leq \|x - y\|^2 - \lambda(2d - \lambda)\|Dx - Dy\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

This implie that

$$\|(I - \lambda D)x - (I - \lambda D)y\| \leq \|x - y\|, \tag{3.35}$$

that is, $(I-\lambda D)$ is a nonexpansive mapping. Then, we have $P_C(I-\lambda D)$ is a nonexpansive mapping. By using the same method as (3.35), we have $P_C(I-\lambda_1 D_1)$ and $P_C(I-\lambda_2 D_2)$ are nonexpansive mappings. Then G is a nonexpansive mapping.

The proof will be divided into five steps.

Step 1. We will show that $\{x_n\}$ is bounded.

Let $x^* \in \Im$. From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|(1-\beta_n)x_n+\beta_n P_C(I-\lambda D)y_n - x^*\| \\
&\leq (1-\beta_n)\|x_n-x^*\| + \beta_n \|P_C(I-\lambda D)y_n - x^*\| \\
&\leq (1-\beta_n)\|x_n-x^*\| + \beta_n \|y_n - x^*\| \\
&= (1-\beta_n)\|x_n-x^*\| + \beta_n \left\|\alpha_n\gamma f(x_n) + (I-\alpha_n\bar{A})Gx_n - x^*\right\| \\
&\leq (1-\beta_n)\|x_n-x^*\| + \beta_n\alpha_n \|\gamma f(x_n) - \bar{A}x^*\| + \beta_n \|I-\alpha_n\bar{A}\|\|Gx_n-x^*\| \\
&\leq (1-\beta_n)\|x_n-x^*\| + \beta_n\alpha_n\gamma \|f(x_n)-f(x^*)\| + \beta_n\alpha_n\|\gamma f(x^*)-\bar{A}x^*\| \\
&\quad + \beta_n(1-\alpha_n\bar{\gamma})\|Gx_n - x^*\| \\
&\leq (1-\beta_n)\|x_n-x^*\| + \beta_n\alpha_n\gamma\alpha\|x_n-x^*\| + \beta_n\alpha_n\|\gamma f(x^*)-\bar{A}x^*\| \\
&\quad + \beta_n(1-\alpha_n\bar{\gamma})\|x_n-x^*\| \\
&= (1-\beta_n+\beta_n(\alpha_n\gamma\alpha+1-\alpha_n\bar{\gamma}))\|x_n-x^*\| + \beta_n\alpha_n\|\gamma f(x^*)-\bar{A}x^*\| \\
&= (1-\beta_n+\beta_n(1-\alpha_n(\bar{\gamma}-\gamma\alpha)))\|x_n-x^*\| + \beta_n\alpha_n\|\gamma f(x^*)-\bar{A}x^*\| \\
&= (1-\beta_n\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n-x^*\| + \beta_n\alpha_n\|\gamma f(x^*)-\bar{A}x^*\| \\
&\leq \max\left\{\|x_1-x^*\|, \frac{\|\gamma f(x^*)-\bar{A}x^*\|}{\bar{\gamma}-\gamma\alpha}\right\}.
\end{aligned}$$

By induction, we have $\|x_n-x^*\| \leq \max\left\{\|x_1-x^*\|, \frac{\|\gamma f(x^*)-\bar{A}x^*\|}{\bar{\gamma}-\gamma\alpha}\right\}, \forall n\in\mathbb{N}$.

Hence $\{x_n\}$ is bounded and so is $\{y_n\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

From the definition of y_n, we have

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \left\|\alpha_{n+1}\gamma f(x_{n+1}) + (I-\alpha_{n+1}\overline{A})Gx_{n+1} - \alpha_n\gamma f(x_n) - (I-\alpha_n\overline{A})Gx_n\right\| \\
&\leq \alpha_{n+1}\gamma\|f(x_{n+1})-f(x_n)\| + \gamma|\alpha_{n+1}-\alpha_n|\|f(x_n)\| \\
&\quad + \|I-\alpha_{n+1}\overline{A}\|\|Gx_{n+1}-Gx_n\| + \left\|(I-\alpha_{n+1}\overline{A})Gx_n - (I-\alpha_n\overline{A})Gx_n\right\| \\
&\leq \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\| + \gamma|\alpha_{n+1}-\alpha_n|\|f(x_n)\| + (1-\alpha_{n+1}\overline{\gamma})\|x_{n+1}-x_n\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\|\overline{A}Gx_n\| \\
&= (1-\alpha_{n+1}(\overline{\gamma}-\gamma\alpha))\|x_{n+1}-x_n\| + \gamma|\alpha_{n+1}-\alpha_n|\|f(x_n)\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\|\overline{A}Gx_n\|.
\end{aligned} \tag{3.36}$$

From the definition of x_n and (3.36), we have

$$\|x_{n+1}-x_n\| = \|(1-\beta_n)x_n + \beta_n P_C(I-\lambda D)y_n - (1-\beta_{n-1})x_{n-1} - \beta_{n-1}P_C(I-\lambda D)y_{n-1}\|$$

$$
\begin{aligned}
&= \|(1-\beta_n)x_n + \beta_n P_C(I-\lambda D)y_n - (1-\beta_n)x_{n-1} + (1-\beta_n)x_{n-1} \\
&\quad - (1-\beta_{n-1})x_{n-1} - \beta_n P_C(I-\lambda D)y_{n-1} + \beta_n P_C(I-\lambda D)y_{n-1} \\
&\quad -\beta_{n-1} P_C(I-\lambda D)y_{n-1}\| \\
&\le (1-\beta_n)\|x_n - x_{n-1}\| + \beta_n \|P_C(I-\lambda D)y_n - P_C(I-\lambda D)y_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\,\|x_{n-1}\| + |\beta_n - \beta_{n-1}|\,\|P_C(I-\lambda D)y_{n-1}\| \\
&\le (1-\beta_n)\|x_n - x_{n-1}\| + \beta_n \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\,\|x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\,\|P_C(I-\lambda D)y_{n-1}\| \\
&\le (1-\beta_n)\|x_n - x_{n-1}\| + \beta_n \left((1-\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n - x_{n-1}\| \right. \\
&\quad \left. +\gamma|\alpha_n - \alpha_{n-1}|\,\|f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\,\|\bar{A}Gx_{n-1}\|\right) \\
&\quad + |\beta_n - \beta_{n-1}|\,\|x_{n-1}\| + |\beta_n - \beta_{n-1}|\,\|P_C(I-\lambda D)y_{n-1}\| \\
&= (1-\beta_n)\|x_n - x_{n-1}\| + \beta_n(1-\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n - x_{n-1}\| \\
&\quad + \beta_n\gamma|\alpha_n - \alpha_{n-1}|\,\|f(x_{n-1})\| + \beta_n|\alpha_n - \alpha_{n-1}|\,\|\bar{A}Gx_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\,\|x_{n-1}\| + |\beta_n - \beta_{n-1}|\,\|P_C(I-\lambda D)y_{n-1}\| \\
&= (1-\beta_n + \beta_n(1-\alpha_n(\bar{\gamma}-\gamma\alpha)))\|x_n - x_{n-1}\| + \beta_n\gamma|\alpha_n - \alpha_{n-1}| \\
&\quad \times \|f(x_{n-1})\| + \beta_n|\alpha_n - \alpha_{n-1}|\,\|\bar{A}Gx_{n-1}\| + |\beta_n - \beta_{n-1}|\,\|x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\,\|P_C(I-\lambda D)y_{n-1}\| \\
&\le (1-\beta_n\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\,\|f(x_{n-1})\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\,\|\bar{A}Gx_{n-1}\| + |\beta_n - \beta_{n-1}|\,\|x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\,\|P_C(I-\lambda D)y_{n-1}\|
\end{aligned}
$$

This together with conditions (i),(iii) and Lemma 2.42 , we get that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.37}$$

From conditions (i),(iii), (3.36), and (3.37), we obtain $\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$.
From the definition of y_n, we have

$$
\begin{aligned}
\|y_n - x^*\|^2 &= \left\|\alpha_n\gamma f(x_n) + \left(I - \alpha_n\bar{A}\right)Gx_n - x^*\right\|^2 \\
&= \left\|(Gx_n - x^*) + (\alpha_n\gamma f(x_n) - \alpha_n\bar{A}Gx_n)\right\|^2 \\
&\le \|Gx_n - x^*\|^2 + 2\alpha_n\left\langle \gamma f(x_n) - \bar{A}Gx_n, y_n - x^*\right\rangle \\
&\le \|x_n - x^*\|^2 + 2\alpha_n\left\|\gamma f(x_n) - \bar{A}Gx_n\right\|\|y_n - x^*\|.
\end{aligned} \tag{3.38}
$$

From nonexpansiveness of P_C and (3.38), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|P_C(I-\lambda D)y_n - x^*\|^2 \\
&\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|(I-\lambda D)y_n - (I-\lambda D)x^*\|^2 \\
&= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|y_n - x^* - \lambda(Dy_n - Dx^*)\|^2 \\
&= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|y_n - x^*\|^2 \\
&\quad - 2\lambda\beta_n\langle y_n - x^*, Dy_n - Dx^*\rangle + \beta_n\lambda^2\|Dy_n - Dx^*\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + 2\beta_n\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \\
&\quad \times \|y_n - x^*\| - 2\lambda d\beta_n\|Dy_n - Dx^*\|^2 + \beta_n\lambda^2\|Dy_n - Dx^*\|^2 \\
&= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + 2\beta_n\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \\
&\quad \times \|y_n - x^*\| - \lambda\beta_n(2d-\lambda)\|Dy_n - Dx^*\|^2 \\
&= \|x_n - x^*\|^2 + 2\beta_n\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\| \\
&\quad - \lambda\beta_n(2d-\lambda)\|Dy_n - Dx^*\|^2.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\lambda\beta_n(2d-\lambda)\|Dy_n - Dx^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2\beta_n\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\| \\
&\leq \left(\|x_n - x^*\| + \|x_{n+1} - x^*\|\right) \|x_{n+1} - x_n\| \\
&\quad + 2\beta_n\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\|. \qquad (3.39)
\end{aligned}
$$

Form conditions (i), (ii), (3.37) and (3.39), we have

$$\lim_{n\to\infty} \|Dy_n - Dx^*\| = 0. \qquad (3.40)$$

Step 3. Show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, $\lim_{n\to\infty} \|x_n - Gx_n\| = 0$, $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$, $\lim_{n\to\infty} \|y_n - Gy_n\| = 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$.
From the definition of x_n and (3.38), we obtain

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\beta_n)x_n + \beta_n P_C(I-\lambda D)y_n - x^*\|^2 \\
&= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|P_C(I-\lambda D)y_n - x^*\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|y_n - x^*\|^2 - \beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + 2\beta_n\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\| \\
&\quad - \beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2 \\
&= \|x_n - x^*\|^2 + 2\beta_n\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\| \\
&\quad - \beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2\beta_n\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\| \\
&\leq \left(\|x_n - x^*\| + \|x_{n+1} - x^*\|\right) \|x_{n+1} - x_n\| \\
&\quad + 2\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\|. \qquad (3.41)
\end{aligned}
$$

Form conditions (i), (ii), (3.37) and (3.41), we have

$$\lim_{n\to\infty} \|x_n - P_C(I-\lambda D)y_n\| = 0. \qquad (3.42)$$

By Lemma 2.22 and (3.38), we obtain

$$\begin{aligned}
\|P_C(I-\lambda D)y_n - x^*\|^2 &\leq \langle (I-\lambda D)y_n - (I-\lambda D)x^*, P_C(I-\lambda D)y_n - x^*\rangle \\
&= \frac{1}{2}\Big(\|(I-\lambda D)y_n - (I-\lambda D)x^*\|^2 + \|P_C(I-\lambda D)y_n - x^*\|^2 \\
&\quad -\|(I-\lambda D)y_n - (I-\lambda D)x^* - (P_C(I-\lambda D)y_n - x^*)\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|y_n - x^*\|^2 + \|P_C(I-\lambda D)y_n - x^*\|^2 \\
&\quad -\|y_n - P_C(I-\lambda D)y_n - \lambda(Dy_n - Dx^*)\|^2\Big) \\
&\leq \frac{1}{2}\Big(\Big(\|x_n - x^*\|^2 + 2\alpha_n\left\|\gamma f(x_n) - \bar{A}Gx_n\right\|\|y_n - x^*\| \\
&\quad + \|P_C(I-\lambda D)y_n - x^*\|^2 - \|y_n - P_C(I-\lambda D)y_n\|^2 \\
&\quad +2\lambda\langle y_n - P_C(I-\lambda D)y_n, Dy_n - Dx^*\rangle - \lambda^2\|Dy_n - Dx^*\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|x_n - x^*\|^2 + 2\alpha_n\left\|\gamma f(x_n) - \bar{A}Gx_n\right\|\|y_n - x^*\| \\
&\quad + \|P_C(I-\lambda D)y_n - x^*\|^2 - \|y_n - P_C(I-\lambda D)y_n\|^2 \\
&\quad +2\lambda\|y_n - P_C(I-\lambda D)y_n\|\|Dy_n - Dx^*\|\Big).
\end{aligned}$$

It follow that

$$\begin{aligned}
\|P_C(I-\lambda D)y_n - x^*\|^2 \leq\ & \|x_n - x^*\|^2 + 2\alpha_n\left\|\gamma f(x_n) - \bar{A}Gx_n\right\|\|y_n - x^*\| \\
& - \|y_n - P_C(I-\lambda D)y_n\|^2 + 2\lambda\|y_n - P_C(I-\lambda D)y_n\| \\
& \times \|Dy_n - Dx^*\|.
\end{aligned} \tag{3.43}$$

From (3.43), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|P_C(I-\lambda D)y_n - x^*\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\Big(\|x_n - x^*\|^2 + 2\alpha_n\left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \\
&\quad \times \|y_n - x^*\| - \|y_n - P_C(I-\lambda D)y_n\|^2 + 2\lambda\|y_n - P_C(I-\lambda D)y_n\| \\
&\quad \times \|Dy_n - Dx^*\|\Big) \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + 2\alpha_n\left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \\
&\quad \times \|y_n - x^*\| - \beta_n\|y_n - P_C(I-\lambda D)y_n\|^2 + 2\lambda\|y_n - P_C(I-\lambda D)y_n\| \\
&\quad \times \|Dy_n - Dx^*\| \\
&= \|x_n - x^*\|^2 + 2\alpha_n\left\|\gamma f(x_n) - \bar{A}Gx_n\right\|\|y_n - x^*\| \\
&\quad - \beta_n\|y_n - P_C(I-\lambda D)y_n\|^2 + 2\lambda\|y_n - P_C(I-\lambda D)y_n\|\|Dy_n - Dx^*\|.
\end{aligned}$$

It implies that

$$\begin{aligned}
\beta_n\|y_n - P_C(I-\lambda D)y_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n\left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \\
&\quad \times \|y_n - x^*\| + 2\lambda\|y_n - P_C(I-\lambda D)y_n\|\|Dy_n - Dx^*\| \\
&\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|
\end{aligned}$$

$$+ 2\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\| + 2\lambda \|y_n - P_C(I - \lambda D)y_n\| \|Dy_n - Dx^*\|. \quad (3.44)$$

From conditions (i), (ii), (3.37), (3.40) and (3.44), we get that

$$\lim_{n\to\infty} \|y_n - P_C(I - \lambda D)y_n\| = 0. \quad (3.45)$$

Consider,

$$\|x_n - y_n\| \leq \|x_n - P_C(I - \lambda D)y_n\| + \|P_C(I - \lambda D)y_n - y_n\|.$$

From (3.42) and (3.45), we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \quad (3.46)$$

From definition of y_n and condition (i), we have

$$\begin{aligned}\|y_n - Gx_n\| &= \left\|\alpha_n \gamma f(x_n) + (I - \alpha_n \overline{A})Gx_n - Gx_n\right\| \\ &= \alpha_n \left\|\gamma f(x_n) - \overline{A}Gx_n\right\| \to 0 \ \ as \ n \to \infty. \end{aligned} \quad (3.47)$$

Consider,

$$\|x_n - Gx_n\| \leq \|x_n - y_n\| + \|y_n - Gx_n\|.$$

By (3.46) and (3.47), we have

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \quad (3.48)$$

From (3.46), (3.48) and

$$\begin{aligned}\|y_n - Gy_n\| &\leq \|y_n - x_n\| + \|x_n - Gx_n\| + \|Gx_n - Gy_n\| \\ &\leq \|y_n - x_n\| + \|x_n - Gx_n\| + \|x_n - y_n\|,\end{aligned}$$

we get that

$$\lim_{n\to\infty} \|y_n - Gy_n\| = 0. \quad (3.49)$$

Step 4. We will show that $\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0 \rangle \leq 0$, where $x_0 = P_{\Im}(I - \overline{A} + \gamma f)x_0$.

To show this, choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0 \rangle = \lim_{k\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_{n_k} - x_0 \rangle. \quad (3.50)$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup q$ as $k \to \infty$, where $q \in C$. Then, from (3.46) and $x_{n_k} \rightharpoonup q$, we obtain $y_{n_k} \rightharpoonup q$ as $k \to \infty$.
From (3.50) and $y_{n_k} \rightharpoonup q$ as $k \to \infty$, we have

$$\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0 \rangle = \langle \gamma f(x_0) - \overline{A}x_0, q - x_0 \rangle. \quad (3.51)$$

In order to show $\langle \gamma f(x_0) - \overline{A}x_0, q - x_0 \rangle \leq 0$, we need to show that $q \in \Im = F(G) \cap VI(C, D)$. Assume that $q \notin F(G)$. It implies that $q \neq Gq$. From Lemma 2.19 and (3.49), we have

$$\begin{aligned}\liminf_{n\to\infty}\|y_{n_k}-q\| &< \liminf_{n\to\infty}\|y_{n_k}-Gq\| \\ &\le \liminf_{n\to\infty}(\|y_{n_k}-Gy_{n_k}\|+\|Gy_{n_k}-Gq\|) \\ &\le \liminf_{n\to\infty}(\|y_{n_k}-Gy_{n_k}\|+\|y_{n_k}-q\|) \\ &\le \liminf_{n\to\infty}\|y_{n_k}-q\|.\end{aligned}$$

This is a contraction, that is,

$$q \in F(G). \tag{3.52}$$

Next, we will show that $q \in VI(C,D)$.

Assume that $q \notin VI(C,D)$. Since $VI(C,D) = F(P_C(I-\lambda D))$, we have $q \neq P_C(I-\lambda D)q$. From Lemma 2.19 and (3.45), we obtain

$$\begin{aligned}\liminf_{n\to\infty}\|y_{n_k}-q\| &< \liminf_{n\to\infty}\|y_{n_k}-P_C(I-\lambda D)q\| \\ &\le \liminf_{n\to\infty}(\|y_{n_k}-P_C(I-\lambda D)y_{n_k}\|+\|P_C(I-\lambda D)y_{n_k}-P_C(I-\lambda D)q\|) \\ &\le \liminf_{n\to\infty}(\|y_{n_k}-P_C(I-\lambda D)y_{n_k}\|+\|y_{n_k}-q\|) \\ &\le \liminf_{n\to\infty}\|y_{n_k}-q\|.\end{aligned}$$

This is a contraction, that is,

$$q \in VI(C,D). \tag{3.53}$$

From (3.52) and (3.53), we have $q \in \Im = F(G)\cap VI(C,D)$.

By (3.51) and Lemma 2.22, we have

$$\limsup_{n\to\infty}\langle \gamma f(x_0)-\overline{A}x_0, y_n-x_0\rangle = \langle \gamma f(x_0)-\overline{A}x_0, q-x_0\rangle \le 0.$$

Step 5. Finally, We will show that $\{x_n\}$ converges strongly to x_0, where $x_0 = P_{\Im}(I-\overline{A}+\gamma f)x_0$.

From the definition of x_n and $x_0 = P_{\Im}(I-\overline{A}+\gamma f)x_0$, we have

$$\begin{aligned}\|x_{n+1}-x_0\|^2 &= \|(1-\beta_n)x_n+\beta_nP_C(I-\lambda D)y_n-x_0\|^2 \\ &\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n\|P_C(I-\lambda D)y_n-x_0\|^2 \\ &\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n\|y_n-x_0\|^2 \\ &= (1-\beta_n)\|x_n-x_0\|^2+\beta_n\left\|\alpha_n\gamma f(x_n)+\left(I-\alpha_n\bar{A}\right)Gx_n-x_0\right\|^2 \\ &\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n\left(\left\|\left(I-\alpha_n\bar{A}\right)(Gx_n-x_0)\right\|^2\right. \\ &\quad \left.+2\alpha_n\left\langle\gamma f(x_n)-\bar{A}x_0, y_n-x_0\right\rangle\right) \\ &\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n\left((1-\alpha_n\bar{\gamma})^2\|x_n-x_0\|^2\right. \\ &\quad \left.+2\alpha_n\gamma\langle f(x_n)-f(x_0), y_n-x_0\rangle+2\alpha_n\left\langle\gamma f(x_0)-\bar{A}x_0, y_n-x_0\right\rangle\right) \\ &\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n\left((1-\alpha_n\bar{\gamma})^2\|x_n-x_0\|^2\right. \\ &\quad \left.+2\alpha_n\gamma\|f(x_n)-f(x_0)\|\,\|y_n-x_0\|+2\alpha_n\left\langle\gamma f(x_0)-\bar{A}x_0, y_n-x_0\right\rangle\right)\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n(1-\alpha_n\bar{\gamma})^2\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\beta_n \|x_n - x_0\| \|y_n - x_0\| + 2\alpha_n\beta_n \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n(1-\alpha_n\bar{\gamma})^2\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\beta_n \|x_n - x_0\| \left(\alpha_n \|\gamma f(x_n) - \bar{A}x_0\| + (1-\alpha_n\bar{\gamma}) \|Gx_n - x_0\|\right) \\
&\quad + 2\alpha_n\beta_n \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n(1-\alpha_n\bar{\gamma})^2\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\beta_n \|x_n - x_0\| \left(\alpha_n\gamma\alpha \|x_n - x_0\| + \alpha_n \|\gamma f(x_0) - \bar{A}x_0\|\right. \\
&\quad \left. + (1-\alpha_n\bar{\gamma}) \|x_n - x_0\|\right) + 2\alpha_n\beta_n \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle \\
&= (1-\beta_n)\|x_n - x_0\|^2 + \beta_n(1-\alpha_n\bar{\gamma})^2\|x_n - x_0\|^2 + 2\alpha_n^2\gamma^2\alpha^2\beta_n\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2\gamma\alpha\beta_n \|\gamma f(x_0) - \bar{A}x_0\| \|x_n - x_0\| + 2\alpha_n\gamma\alpha\beta_n (1-\alpha_n\bar{\gamma}) \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\beta_n \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle \\
&\leq (1-\beta_n)\|x_n - x_0\|^2 + \beta_n (1-\alpha_n\bar{\gamma}) \|x_n - x_0\|^2 + 2\alpha_n^2\bar{\gamma}^2\beta_n\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2\bar{\gamma}\beta_n \|\gamma f(x_0) - \bar{A}x_0\| \|x_n - x_0\| + 2\alpha_n\gamma\alpha\beta_n\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\beta_n \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle \\
&= (1-\beta_n + \beta_n - \beta_n\alpha_n\bar{\gamma} + 2\alpha_n\gamma\alpha\beta_n) \|x_n - x_0\|^2 + 2\alpha_n^2\bar{\gamma}^2\beta_n\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2\bar{\gamma}\beta_n \|\gamma f(x_0) - \bar{A}x_0\| \|x_n - x_0\| + 2\alpha_n\beta_n \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle \\
&= (1 - \alpha_n\beta_n(\bar{\gamma} - 2\gamma\alpha)) \|x_n - x_0\|^2 + \alpha_n\beta_n \left(2\alpha_n\bar{\gamma}^2\|x_n - x_0\|^2\right. \\
&\quad \left. + 2\alpha_n\bar{\gamma} \|\gamma f(x_0) - \bar{A}x_0\| \|x_n - x_0\| + 2\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle\right) \\
&= (1 - \alpha_n\beta_n(\bar{\gamma} - 2\gamma\alpha)) \|x_n - x_0\|^2 + \alpha_n\beta_n(\bar{\gamma} - 2\gamma\alpha) \left(\frac{2\alpha_n\bar{\gamma}^2\|x_n - x_0\|^2}{(\bar{\gamma} - 2\gamma\alpha)}\right. \\
&\quad \left. + \frac{2\alpha_n\bar{\gamma} \|\gamma f(x_0) - \bar{A}x_0\| \|x_n - x_0\|}{(\bar{\gamma} - 2\gamma\alpha)} + \frac{2\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle}{(\bar{\gamma} - 2\gamma\alpha)}\right).
\end{aligned}
$$

By step 4, condition (i) and Lemma 2.38, we can conclude that $\{x_n\}$ converges strongly to $x_0 = P_{\Im}(I - \overline{A} + \gamma f)x_0$. Then, from Lemma 2.41, we have (x_0, y_0) is a solution of the problem (1.11) where $y_0 = P_C(x_0 - \lambda_2 D_2 x_0)$. This completes the proof. □

The following corollary is directed result from Theorem 3.6

Corollary 3.7. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, D_1 : C \to H$ be d, d_1-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \lambda_1 D_1) P_C(I - \lambda_2 D_1)x$, for all $x \in C$ and $a \in [0,1)$. Let f be an α-contraction mapping on H. For $k = 1, 2, ..., N$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{N} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,N}\gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Suppose that $\Im = F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$
\begin{aligned}
x_{n+1} &= (1-\beta_n)x_n + \beta_n P_C(I - \lambda D)y_n, \\
y_n &= \alpha_n\gamma f(x_n) + \left(I - \alpha_n\overline{A}\right)Gx_n,
\end{aligned}
\tag{3.54}
$$

where $\{\alpha_n\} \subset [0,1]$, $\lambda \in (0, 2d)$, $\lambda_1, \lambda_2 \in (0, 2d_1)$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n \leq c < 1$;

(iii) $0 < c_k < 1$ and $\sum_{k=1}^{N} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0) is a solution of (1.12), where $y_0 = P_C(x_0 - \lambda_2 D_1 x_0)$.

Proof. If we put $D_1 = D_2$, in Theorem 3.6, we have the desired conclusion. □

3.2.2 The modified generalized system of variational inequalities

Motivated by the general system of variational inequalities problem, Ceng et al. [6], we introduce a new problem involving the general system of variational inequalities in a real Hilbert space as follow:

Let $D_1, D_2, D_3 : C \to H$ be three mappings. We consider the problem for finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, \ \forall x \in C, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, \ \forall x \in C, \\ \langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, \ \forall x \in C. \end{cases} \tag{3.55}$$

where $\lambda_1, \lambda_2, \lambda_3 > 0$ and $a \in [0,1]$, which is called the *modified generalized system of variational inequalities*.

The following example is given for supporting the proposed problem.

Example 3.8. Let $\mathbb{R}$ be the set of real numbers and D_1, D_2, D_3 be mappings from $[-20, 20]$ to $\mathbb{R}$ defined by $D_1 x = \frac{4x-17}{8}$, $D_2 x = \frac{4x-7}{4}$ and $D_3 x = x - 2$, respectively. We choose $a = \frac{3}{4}, \lambda_1 = \frac{1}{3}, \lambda_2 = \frac{2}{3}$ and $\lambda_3 = 2$. Then, we have $(3, 2, 1)$ is the solution of the modified generalized system of variational inequalities provlem.

Now, we prove lemma and remark that will be useful for proving the next theorem.

Lemma 3.9. Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2, D_3 : C \to H$ are three mappings. For every $\lambda_1, \lambda_2, \lambda_3 > 0$ and $a \in [0,1]$. The following statement are equivalent

(i) $(x^*, y^*, z^*) \in C \times C \times C$ is a solution of problem (3.55)

(ii) x^* is a fixed point of the mapping G, i.e. $x^* \in F(G)$, define the mapping $G : C \to C$ by

$$G(x) = P_C(I - \lambda_1 D_1)\left(ax + (1-a)P_C(I - \lambda_2 D_2)\left(ax + (1-a)P_C(I - \lambda_3 D_3)x\right)\right), \forall x \in C,$$

where $y^* = P_C(I - \lambda_2 D_2)(ax^* + (1-a)z^*)$ and $z^* = P_C(I - \lambda_3 D_3)x^*$.

Proof. Let conditions hold.

$(i) \Rightarrow (ii)$ Suppose that $(x^*, y^*, z^*) \in C \times C \times C$ is a solution of (3.55). For every $x \in C$, we have

$$\langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0,$$
$$\langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0,$$
$$\langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0.$$

From properties of P_C, we have

$$x^* = P_C(I - \lambda_1 D_1)(ax^* + (1-a)y^*),$$
$$y^* = P_C(I - \lambda_2 D_2)(ax^* + (1-a)z^*),$$
$$z^* = P_C(I - \lambda_3 D_3)x^*.$$

It implies that

$$\begin{aligned} x^* &= P_C(I - \lambda_1 D_1)(ax^* + (1-a)P_C(I - \lambda_2 D_2)(ax^* + (1-a)P_C(I - \lambda_3 D_3)x^*)) \\ &= G(x^*). \end{aligned}$$

It follow that $x^* \in F(G)$, where $y^* = P_C(I - \lambda_2 D_2)(ax^* + (1-a)z^*)$ and $z^* = P_C(I - \lambda_3 D_3)x^*$.

$(ii) \Rightarrow (i)$ Let $x^* \in F(G)$, $y^* = P_C(I - \lambda_2 D_2)(ax^* + (1-a)z^*)$ and $z^* = P_C(I - \lambda_3 D_3)x^*$. Since $x^* \in F(G)$, we have

$$\begin{aligned} x^* &= P_C(I - \lambda_1 D_1)(ax^* + (1-a)P_C(I - \lambda_2 D_2)(ax^* + (1-a)P_C(I - \lambda_3 D_3)x^*)) \\ &= P_C(I - \lambda_1 D_1)(ax^* + (1-a)y^*). \end{aligned} \quad (3.56)$$

From (3.56), $y^* = P_C(I - \lambda_2 D_2)(ax^* + (1-a)z^*)$ and $z^* = P_C(I - \lambda_3 D_3)x^*$, we have

$$\langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, \ \ \forall x \in C,$$
$$\langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, \ \ \forall x \in C,$$
$$\langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, \ \ \forall x \in C.$$

It follow that $(x^*, y^*, z^*) \in C \times C \times C$ is a solution of (3.55). □

The following example is given for supporting Lemma 3.9.

Example 3.10. Let $\mathbb{R}$ be the set of real numbers and D_1, D_2, D_3 be mappings from $[0, 20]$ to $\mathbb{R}$ defined by $D_1 x = x - 1, D_2 x = x + 2$ and $D_3 x = x - 22$, for all $x \in [0, 20]$, respectively. Let mapping $G : [0, 20] \to [0, 20]$ be defined by

$$G(x) = P_{[0,20]}(I - \frac{1}{2}D_1)\left(0.5x + 0.5P_{[0,20]}(I - \frac{1}{3}D_2)\left(0.5x + 0.5P_{[0,20]}(I - \frac{1}{2}D_3)x\right)\right),$$

where $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{3}, \lambda_3 = \frac{1}{2}$ and $a = 0.5$. Then, we have $2 \in F(G)$, where $z^* = P_{[0,20]}\left(I - \frac{1}{2}D_3\right)x^*$. and $y^* = P_{[0,20]}\left(I - \frac{1}{3}D_2\right)(0.5(x^*) + 0.5(z^*))$. Hence $(x^*, y^*, z^*) = (2, 4, 12)$ is solution of (3.55), by Lemma 3.9.

Remark 3.11. If D_1, D_2, D_3, in Lemma 3.9, are d_1, d_2, d_3-inverse strongly monotone, respectively, then G is nonexpansive mapping, where $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2, d_3\}$.

Proof. Since D_1 is d_1-inverse strongly monotone mapping, we have

$$\begin{aligned}
\|(I-\lambda_1 D_1)x - (I-\lambda_1 D_1)y\|^2 &= \|x-y-\lambda_1(D_1x - D_1y)\|^2 \\
&= \|x-y\|^2 - 2\lambda_1 \langle x-y, D_1x - D_1y\rangle + {\lambda_1}^2\|D_1x - D_1y\|^2 \\
&\le \|x-y\|^2 - 2\lambda_1 d_1\|D_1x - D_1y\|^2 + {\lambda_1}^2\|D_1x - D_1y\|^2 \\
&= \|x-y\|^2 - \lambda_1(2d_1 - \lambda_1)\|D_1x - D_1y\|^2 \\
&\le \|x-y\|^2 - \lambda_1(2\overline{d} - \lambda_1)\|D_1x - D_1y\|^2 \\
&\le \|x-y\|^2.
\end{aligned}$$

It implie that

$$\|(I-\lambda_1 D_1)x - (I-\lambda_1 D_1)y\| \le \|x-y\|. \tag{3.57}$$

Hence $(I-\lambda_1 D_1)$ is nonexpansive.
By using the same method as (3.57), we have $(I-\lambda_2 D_2)$ and $(I-\lambda_3 D_3)$ are nonexpansive. Then, we obtain that $P_C(I-\lambda_1 D_1), P_C(I-\lambda_2 D_2)$ and $P_C(I-\lambda_3 D_3)$ are nonexpansive. From $P_C(I-\lambda_1 D_1), P_C(I-\lambda_2 D_2)$ and $P_C(I-\lambda_3 D_3)$ are nonexpansive, we have

$$\begin{aligned}
\|Gx - Gy\| &= \|P_C(I-\lambda_1 D_1)\,(ax + (1-a)P_C(I-\lambda_2 D_2)\,(ax + (1-a)P_C(I-\lambda_3 D_3)x)) \\
&\quad - P_C(I-\lambda_1 D_1)\,(ay + (1-a)P_C(I-\lambda_2 D_2)\,(ay + (1-a)P_C(I-\lambda_3 D_3)y))\| \\
&\le \|(ax + (1-a)P_C(I-\lambda_2 D_2)\,(ax + (1-a)P_C(I-\lambda_3 D_3)x)) \\
&\quad - (ay + (1-a)P_C(I-\lambda_2 D_2)\,(ay + (1-a)P_C(I-\lambda_3 D_3)y))\| \\
&\le a\|x-y\| + (1-a)\,\|P_C(I-\lambda_2 D_2)\,(ax + (1-a)P_C(I-\lambda_3 D_3)x) \\
&\quad - P_C(I-\lambda_2 D_2)\,(ay + (1-a)P_C(I-\lambda_3 D_3)y)\| \\
&\le a\|x-y\| + (1-a)\,\|(ax + (1-a)P_C(I-\lambda_3 D_3)x) \\
&\quad - (ay + (1-a)P_C(I-\lambda_3 D_3)y)\| \\
&\le a\|x-y\| + (1-a)\,(a\|x-y\| + (1-a)\,\|P_C(I-\lambda_3 D_3)x - P_C(I-\lambda_3 D_3)y\|) \\
&\le a\|x-y\| + (1-a)\,(a\|x-y\| + (1-a)\,\|x-y\|) \\
&= \|x-y\|.
\end{aligned}$$

Therefore G is a nonexpansive mapping. □

Next, we prove a strong convergence theorem for finding a common element of the set of fixed points of nonexpansive mapping, the set of fixed points of mapping G define as in Lemma 3.9, which is the solution of the modified generalized system of variational inequalities, and the solution set of the variational inequality.

Theorem 3.12. Let C be a nonempty closed convex subset of a real Hilbert space H and let $D, D_1, D_2, D_3 : C \to H$ be d, d_1, d_2, d_3-inverse strongly monotone mappings, respectively. Define the mapping G as in Lemma 3.9 and $a \in [0,1)$. For $k = 1,2,...,\overline{N}$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,\overline{N}} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Let $T : C \to C$ be a nonexpansive mapping with $\Omega = F(T) \cap F(G) \cap VI(C,D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n, \\ y_n &= \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n, \end{aligned} \tag{3.58}$$

where f is α-contraction mapping on C, $\{\alpha_n\} \subset [0,1], \lambda \in (0,2d),\ \lambda_1, \lambda_2, \lambda_3 \in \left(0, 2\overline{d}\right)$ with $\overline{d} = \min\{d_1, d_2, d_3\}$ and $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n^i \leq c < 1$ for all $i = 1,2,3$;

(iii) $0 < c_k < 1$ and $\sum\limits_{k=1}^{\overline{N}} c_k = 1$;

(iv) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum\limits_{n=1}^{\infty} |\beta_{n+1}^i - \beta_n^i| < \infty$, for all $i = 1,2,3$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (3.55) where $y_0 = P_C(I - \lambda_2 D_2)(ax_0 + (1-a)z_0)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$.

Proof. Since $\alpha_n \to 0$ as $n \to \infty$, without loss of generality, we may assume that $\alpha_n < \frac{1}{\|A_i\|}$, $\forall n \in \mathbb{N}$ and $i = 1,2,...,\overline{N}$. By Lemma 2.40, we have $\|I - \alpha_n \overline{A}\| \leq 1 - \alpha_n \overline{\gamma}$.
Since D is d-inverse strongly monotone mapping with $\lambda \in (0,2d)$ by using the same method as (3.57), we can conclude that $(I - \lambda D)$ is a nonexpansive mapping. Then, we obtain $P_C(I - \lambda D)$ is a nonexpansive mapping.
From Remark 3.11, we get that G is a nonexpansive mapping.
Step 1. We show that $\{x_n\}$ is bounded.
Let $x^* \in \Omega$.
From the definition of x_n, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \left\|\beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n - x^*\right\| \\ &\leq \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|T x_n - x^*\| + \beta_n^3 \|P_C(I - \lambda D) y_n - P_C(I - \lambda D) x^*\| \\ &\leq \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^3 \|y_n - x^*\| \\ &= \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^3 \left\|\alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n - x^*\right\| \\ &\leq \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^3 \alpha_n \left\|\gamma f(x_n) - \overline{A} x^*\right\| \\ &\quad + \beta_n^3 \left\|I - \alpha_n \overline{A}\right\| \|G x_n - x^*\| \\ &\leq \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^3 \alpha_n \gamma \|f(x_n) - f(x^*)\| \\ &\quad + \beta_n^3 \alpha_n \left\|\gamma f(x^*) - \overline{A} x^*\right\| + \beta_n^3 (1 - \alpha_n \overline{\gamma}) \|G x_n - x^*\| \\ &\leq \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^3 \alpha_n \gamma \alpha \|x_n - x^*\| \\ &\quad + \beta_n^3 (1 - \alpha_n \overline{\gamma}) \|x_n - x^*\| + \beta_n^3 \alpha_n \left\|\gamma f(x^*) - \overline{A} x^*\right\| \end{aligned}$$

$$= \left(\beta_n^1 + \beta_n^2 + \beta_n^3 \alpha_n \gamma\alpha + \beta_n^3 \left(1 - \alpha_n \overline{\gamma}\right)\right) \|x_n - x^*\| + \beta_n^3 \alpha_n \left\|\gamma f(x^*) - \overline{A}x^*\right\|$$
$$= \left(1 - \beta_n^3 + \beta_n^3 \left(\alpha_n \gamma\alpha + 1 - \alpha_n \overline{\gamma}\right)\right) \|x_n - x^*\| + \beta_n^3 \alpha_n \left\|\gamma f(x^*) - \overline{A}x^*\right\|$$
$$= \left(1 - \beta_n^3 + \beta_n^3 \left(1 - \alpha_n \left(\overline{\gamma} - \gamma\alpha\right)\right)\right) \|x_n - x^*\| + \beta_n^3 \alpha_n \left\|\gamma f(x^*) - \overline{A}x^*\right\|$$
$$= \left(1 - \beta_n^3 \alpha_n \left(\overline{\gamma} - \gamma\alpha\right)\right) \|x_n - x^*\| + \beta_n^3 \alpha_n \left\|\gamma f(x^*) - \overline{A}x^*\right\|$$
$$\leq \max\left\{\|x_1 - x^*\|, \frac{\left\|\gamma f(x^*) - \overline{A}x^*\right\|}{\overline{\gamma} - \gamma\alpha}\right\}.$$

By induction, we get that $\|x_n - x^*\| \leq \max\left\{\|x_1 - x^*\|, \frac{\|\gamma f(x^*) - \overline{A}x^*\|}{\overline{\gamma} - \gamma\alpha}\right\}, \forall n \in \mathbb{N}$.
Hence $\{x_n\}$ is bounded and so is $\{y_n\}$.

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
From the definition of y_n, we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \left\|\alpha_{n+1}\gamma f(x_{n+1}) + \left(I - \alpha_{n+1}\overline{A}\right) Gx_{n+1} - \alpha_n \gamma f(x_n) - \left(I - \alpha_n \overline{A}\right) Gx_n\right\| \\
&= \|\alpha_{n+1}\gamma f(x_{n+1}) - \alpha_{n+1}\gamma f(x_n) + \alpha_{n+1}\gamma f(x_n) - \alpha_n \gamma f(x_n) \\
&\quad + \left(I - \alpha_{n+1}\overline{A}\right) Gx_{n+1} - \left(I - \alpha_{n+1}\overline{A}\right) Gx_n + \left(I - \alpha_{n+1}\overline{A}\right) Gx_n \\
&\quad - \left(I - \alpha_n \overline{A}\right) Gx_n\| \\
&\leq \alpha_{n+1}\gamma \|f(x_{n+1}) - f(x_n)\| + \gamma |\alpha_{n+1} - \alpha_n| \|f(x_n)\| \\
&\quad + \left\|I - \alpha_{n+1}\overline{A}\right\| \|Gx_{n+1} - Gx_n\| + \left\|\left(I - \alpha_{n+1}\overline{A}\right) Gx_n - \left(I - \alpha_n \overline{A}\right) Gx_n\right\| \\
&\leq \alpha_{n+1}\gamma\alpha \|x_{n+1} - x_n\| + \gamma |\alpha_{n+1} - \alpha_n| \|f(x_n)\| \\
&\quad + \left(1 - \alpha_{n+1}\overline{\gamma}\right) \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n| \left\|\overline{A}Gx_n\right\| \\
&= \left(1 - \alpha_{n+1}\left(\overline{\gamma} - \gamma\alpha\right)\right) \|x_{n+1} - x_n\| + \gamma |\alpha_{n+1} - \alpha_n| \|f(x_n)\| \\
&\quad + |\alpha_{n+1} - \alpha_n| \left\|\overline{A}Gx_n\right\| \\
&\leq \|x_{n+1} - x_n\| + \gamma |\alpha_{n+1} - \alpha_n| \|f(x_n)\| + |\alpha_{n+1} - \alpha_n| \left\|\overline{A}Gx_n\right\|. \qquad (3.59)
\end{aligned}$$

Let

$$x_{n+1} = \left(1 - \beta_n^1\right) z_n + \beta_n^1 x_n, \qquad (3.60)$$

where $z_n = \frac{x_{n+1} - \beta_n^1 x_n}{1 - \beta_n^1}$.
Since $x_{n+1} - \beta_n^1 x_n = \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n$ and (3.60), we have

$$\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}^1 x_{n+1}}{1 - \beta_{n+1}^1} - \frac{x_{n+1} - \beta_n^1 x_n}{1 - \beta_n^1} \\
&= \frac{\beta_{n+1}^2 T x_{n+1} + \beta_{n+1}^3 P_C(I - \lambda D) y_{n+1}}{1 - \beta_{n+1}^1} - \frac{\beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n}{1 - \beta_n^1} \\
&= \frac{\beta_{n+1}^2}{1 - \beta_{n+1}^1} T x_{n+1} - \frac{\beta_n^2}{1 - \beta_n^1} T x_n + \frac{\beta_{n+1}^3}{1 - \beta_{n+1}^1} P_C(I - \lambda D) y_{n+1} \\
&\quad - \frac{\beta_n^3}{1 - \beta_n^1} P_C(I - \lambda D) y_n \\
&= \frac{\beta_{n+1}^2}{1 - \beta_{n+1}^1} T x_{n+1} - \frac{\beta_{n+1}^2}{1 - \beta_{n+1}^1} T x_n + \frac{\beta_{n+1}^2}{1 - \beta_{n+1}^1} T x_n - \frac{\beta_n^2}{1 - \beta_n^1} T x_n \\
&\quad + \frac{\beta_{n+1}^3}{1 - \beta_{n+1}^1} P_C(I - \lambda D) y_{n+1} - \frac{\beta_{n+1}^3}{1 - \beta_{n+1}^1} P_C(I - \lambda D) y_n
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\beta^3_{n+1}}{1-\beta^1_{n+1}} P_C(I-\lambda D)y_n - \frac{\beta^3_n}{1-\beta^1_n} P_C(I-\lambda D)y_n \\
&= \frac{\beta^2_{n+1}}{1-\beta^1_{n+1}} (Tx_{n+1} - Tx_n) + \left(\frac{\beta^2_{n+1}}{1-\beta^1_{n+1}} - \frac{\beta^2_n}{1-\beta^1_n} \right) Tx_n \\
&+ \frac{\beta^3_{n+1}}{1-\beta^1_n} (P_C(I-\lambda D)y_{n+1} - P_C(I-\lambda D)y_n) \\
&+ \left(\frac{\beta^3_{n+1}}{1-\beta^1_{n+1}} - \frac{\beta^3_n}{1-\beta^1_n} \right) P_C(I-\lambda D)y_n.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{\beta^2_{n+1}}{1-\beta^1_{n+1}} \|Tx_{n+1} - Tx_n\| + \left| \frac{\beta^2_{n+1}}{1-\beta^1_{n+1}} - \frac{\beta^2_n}{1-\beta^1_n} \right| \|Tx_n\| + \frac{\beta^3_{n+1}}{1-\beta^1_n} \\
&\quad \times \|P_C(I-\lambda D)y_{n+1} - P_C(I-\lambda D)y_n\| + \left| \frac{\beta^3_{n+1}}{1-\beta^1_{n+1}} - \frac{\beta^3_n}{1-\beta^1_n} \right| \|P_C(I-\lambda D)y_n\| \\
&\leq \frac{\beta^2_{n+1}}{1-\beta^1_{n+1}} \|x_{n+1} - x_n\| + \frac{\beta^3_{n+1}}{1-\beta^1_n} \|y_{n+1} - y_n\| \\
&\quad + \left| \frac{\beta^2_{n+1}}{1-\beta^1_{n+1}} - \frac{\beta^2_n}{1-\beta^1_{n+1}} + \frac{\beta^2_n}{1-\beta^1_{n+1}} - \frac{\beta^2_n}{1-\beta^1_n} \right| \|Tx_n\| \\
&\quad + \left| \frac{\beta^3_{n+1}}{1-\beta^1_{n+1}} - \frac{\beta^3_n}{1-\beta^1_{n+1}} + \frac{\beta^3_n}{1-\beta^1_{n+1}} - \frac{\beta^3_n}{1-\beta^1_n} \right| \|P_C(I-\lambda D)y_n\| \\
&\leq \frac{\beta^2_{n+1}}{1-\beta^1_{n+1}} \|x_{n+1} - x_n\| + \frac{\beta^3_{n+1}}{1-\beta^1_n} \|y_{n+1} - y_n\| + \left| \frac{\beta^2_{n+1} - \beta^2_n}{1-\beta^1_{n+1}} \right| \|Tx_n\| \\
&\quad + \beta^2_n \left| \frac{\beta^1_n - \beta^1_{n+1}}{(1-\beta^1_{n+1})(1-\beta^1_{n+1})} \right| \|Tx_n\| + \left| \frac{\beta^3_{n+1} - \beta^3_n}{1-\beta^1_{n+1}} \right| \|P_C(I-\lambda D)y_n\| \\
&\quad + \beta^3_n \left| \frac{\beta^1_n - \beta^1_{n+1}}{(1-\beta^1_{n+1})(1-\beta^1_{n+1})} \right| \|P_C(I-\lambda D)y_n\| \\
&\leq \frac{\beta^2_{n+1}}{1-\beta^1_{n+1}} \|x_{n+1} - x_n\| + \frac{\beta^3_{n+1}}{1-\beta^1_n} (\|x_{n+1} - x_n\| + \gamma |\alpha_{n+1} - \alpha_n| \|f(x_n)\| \\
&\quad + |\alpha_{n+1} - \alpha_n| \|\bar{A}Gx_n\|) + \left| \frac{\beta^2_{n+1} - \beta^2_n}{1-\beta^1_{n+1}} \right| \|Tx_n\| + \left| \frac{\beta^1_n - \beta^1_{n+1}}{(1-\beta^1_{n+1})(1-\beta^1_{n+1})} \right| \|Tx_n\| \\
&\quad + \left| \frac{\beta^3_{n+1} - \beta^3_n}{1-\beta^1_{n+1}} \right| \|P_C(I-\lambda D)y_n\| + \left| \frac{\beta^1_n - \beta^1_{n+1}}{(1-\beta^1_{n+1})(1-\beta^1_{n+1})} \right| \|P_C(I-\lambda D)y_n\| \\
&= \frac{\beta^2_{n+1}}{1-\beta^1_{n+1}} \|x_{n+1} - x_n\| + \frac{\beta^3_{n+1}}{1-\beta^1_{n+1}} \|x_{n+1} - x_n\| + \frac{\beta^3_{n+1}}{1-\beta^1_n} \gamma |\alpha_{n+1} - \alpha_n| \|f(x_n)\| \\
&\quad + \frac{\beta^3_{n+1}}{1-\beta^1_n} |\alpha_{n+1} - \alpha_n| \|\bar{A}Gx_n\| + \left| \frac{\beta^2_{n+1} - \beta^2_n}{1-\beta^1_{n+1}} \right| \|Tx_n\| + \left| \frac{\beta^1_n - \beta^1_{n+1}}{(1-\beta^1_{n+1})(1-\beta^1_{n+1})} \right| \\
&\quad \times \|Tx_n\| + \left| \frac{\beta^3_{n+1} - \beta^3_n}{1-\beta^1_{n+1}} \right| \|P_C(I-\lambda D)y_n\| + \left| \frac{\beta^1_n - \beta^1_{n+1}}{(1-\beta^1_{n+1})(1-\beta^1_{n+1})} \right| \|P_C(I-\lambda D)y_n\| \\
&= \|x_{n+1} - x_n\| + \frac{\beta^3_{n+1}}{1-\beta^1_n} \gamma |\alpha_{n+1} - \alpha_n| \|f(x_n)\| + \frac{\beta^3_{n+1}}{1-\beta^1_n} |\alpha_{n+1} - \alpha_n| \|\bar{A}Gx_n\| \\
&\quad + \left| \frac{\beta^2_{n+1} - \beta^2_n}{1-\beta^1_{n+1}} \right| \|Tx_n\| + \left| \frac{\beta^1_n - \beta^1_{n+1}}{(1-\beta^1_{n+1})(1-\beta^1_{n+1})} \right| \|Tx_n\| \\
&\quad + \left| \frac{\beta^3_{n+1} - \beta^3_n}{1-\beta^1_{n+1}} \right| \|P_C(I-\lambda D)y_n\| + \left| \frac{\beta^1_n - \beta^1_{n+1}}{(1-\beta^1_{n+1})(1-\beta^1_{n+1})} \right| \|P_C(I-\lambda D)y_n\|.
\end{aligned}
$$

From conditions (ii), (iv), we have $\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$.

By Lemma 2.39 and (3.60), we get that $\lim_{n\to\infty} \|z_n - x_n\| = 0.$
Since $x_{n+1} - x_n = (1-\beta_n^1)(z_n - x_n)$, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.61}$$

From (3.59), (3.61) and condition (iv), we obtain $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0.$
By the definition of y_n, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \left\|\alpha_n \gamma f(x_n) + (I - \alpha_n \bar{A})\, Gx_n - x^*\right\|^2 \\
&= \left\|(Gx_n - x^*) + (\alpha_n \gamma f(x_n) - \alpha_n \bar{A} Gx_n)\right\|^2 \\
&\le \|Gx_n - x^*\|^2 + 2\alpha_n \langle \gamma f(x_n) - \bar{A} Gx_n, y_n - x^* \rangle \\
&\le \|x_n - x^*\|^2 + 2\alpha_n \left\|\gamma f(x_n) - \bar{A} Gx_n\right\| \|y_n - x^*\|.
\end{aligned} \tag{3.62}$$

From nonexpansiveness of P_C and (3.62), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \beta_n^1 \|x_n - x^*\|^2 + \beta_n^2 \|Tx_n - x^*\|^2 + \beta_n^3 \|P_C(I - \lambda D)y_n - x^*\|^2 \\
&\le \beta_n^1 \|x_n - x^*\|^2 + \beta_n^2 \|x_n - x^*\|^2 + \beta_n^3 \|(I - \lambda D)y_n - (I - \lambda D)x^*\|^2 \\
&= \beta_n^1 \|x_n - x^*\|^2 + \beta_n^2 \|x_n - x^*\|^2 + \beta_n^3 \|y_n - x^* - \lambda(Dy_n - Dx^*)\|^2 \\
&= \beta_n^1 \|x_n - x^*\|^2 + \beta_n^2 \|x_n - x^*\|^2 + \beta_n^3 \|y_n - x^*\|^2 \\
&\quad - 2\lambda \beta_n^3 \langle y_n - x^*, Dy_n - Dx^* \rangle + \beta_n^3 \lambda^2 \|Dy_n - Dx^*\|^2 \\
&\le \beta_n^1 \|x_n - x^*\|^2 + \beta_n^2 \|x_n - x^*\|^2 + \beta_n^3 \|y_n - x^*\|^2 - 2\lambda d \beta_n^3 \|Dy_n - Dx^*\|^2 \\
&\quad + \beta_n^3 \lambda^2 \|Dy_n - Dx^*\|^2 \\
&\le \beta_n^1 \|x_n - x^*\|^2 + \beta_n^2 \|x_n - x^*\|^2 + \beta_n^3 \|x_n - x^*\|^2 \\
&\quad + 2\beta_n^3 \alpha_n \left\|\gamma f(x_n) - \overline{A} Gx_n\right\| \|y_n - x^*\| - \lambda \beta_n^3 (2d - \lambda) \|Dy_n - Dx^*\|^2 \\
&= \|x_n - x^*\|^2 + 2\beta_n^3 \alpha_n \left\|\gamma f(x_n) - \overline{A} Gx_n\right\| \|y_n - x^*\| \\
&\quad - \lambda \beta_n^3 (2d - \lambda) \|Dy_n - Dx^*\|^2.
\end{aligned}$$

It implies that

$$\begin{aligned}
\lambda \beta_n^3 (2d - \lambda) \|Dy_n - Dx^*\|^2 &\le 2\beta_n^3 \alpha_n \left\|\gamma f(x_n) - \overline{A} Gx_n\right\| \|y_n - x^*\| + \|x_n - x^*\|^2 \\
&\quad - \|x_{n+1} - x^*\|^2 \\
&\le 2\beta_n^3 \alpha_n \left\|\gamma f(x_n) - \overline{A} Gx_n\right\| \|y_n - x^*\| \\
&\quad + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.63}$$

Form conditions (i), (ii), (3.61) and (3.63), we have

$$\lim_{n\to\infty} \|Dy_n - Dx^*\| = 0. \tag{3.64}$$

Step 3. We show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, $\lim_{n\to\infty} \|x_n - Gx_n\| = 0$, $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$, $\lim_{n\to\infty} \|y_n - Gy_n\| = 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0.$

From the definition of x_n and (3.62), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \left\|\beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I-\lambda D)y_n - x^*\right\|^2 \\
&\le \beta_n^1\|x_n-x^*\|^2 + \beta_n^2\|Tx_n-x^*\|^2 + \beta_n^3\|P_C(I-\lambda D)y_n-x^*\|^2 \\
&\quad - \beta_n^1\beta_n^2\|x_n-Tx_n\|^2 - \beta_n^1\beta_n^3\|x_n-P_C(I-\lambda D)y_n\|^2 \\
&\le \beta_n^1\|x_n-x^*\|^2 + \beta_n^2\|x_n-x^*\|^2 + \beta_n^3\|y_n-x^*\|^2 - \beta_n^1\beta_n^2\|x_n-Tx_n\|^2 \\
&\quad - \beta_n^1\beta_n^3\|x_n-P_C(I-\lambda D)y_n\|^2 \\
&\le \beta_n^1\|x_n-x^*\|^2 + \beta_n^2\|x_n-x^*\|^2 + \beta_n^3\|x_n-x^*\|^2 + 2\beta_n^3\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\| \\
&\quad \times \|y_n-x^*\| - \beta_n^1\beta_n^2\|x_n-Tx_n\|^2 - \beta_n^1\beta_n^3\|x_n-P_C(I-\lambda D)y_n\|^2 \\
&= \|x_n-x^*\|^2 + 2\beta_n^3\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\|y_n-x^*\| - \beta_n^1\beta_n^2\|x_n-Tx_n\|^2 \\
&\quad - \beta_n^1\beta_n^3\|x_n-P_C(I-\lambda D)y_n\|^2.
\end{aligned}$$

It follow that

$$\begin{aligned}
\beta_n^1\beta_n^2\|x_n-Tx_n\|^2 &\le 2\beta_n^3\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\|y_n-x^*\| + \|x_n-x^*\|^2 \\
&\quad - \|x_{n+1}-x^*\|^2 - \beta_n^1\beta_n^3\|x_n-P_C(I-\lambda D)y_n\|^2 \\
&\le 2\beta_n^3\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\|y_n-x^*\| \\
&\quad + (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|. \qquad (3.65)
\end{aligned}$$

Form conditions (i), (ii), (3.61) and (3.65), we have

$$\lim_{n\to\infty}\|x_n-Tx_n\| = 0. \qquad (3.66)$$

By using the same method as (3.66), we get that

$$\lim_{n\to\infty}\|x_n-P_C(I-\lambda D)y_n\| = 0. \qquad (3.67)$$

From property of P_C and (3.62), we have

$$\begin{aligned}
\|P_C(I-\lambda D)y_n-x^*\|^2 &\le \langle (I-\lambda D)y_n-(I-\lambda D)x^*, P_C(I-\lambda D)y_n-x^*\rangle \\
&= \frac{1}{2}\Big(\|(I-\lambda D)y_n-(I-\lambda D)x^*\|^2 + \|P_C(I-\lambda D)y_n-x^*\|^2 \\
&\quad -\|(I-\lambda D)y_n-(I-\lambda D)x^*-(P_C(I-\lambda D)y_n-x^*)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|y_n-x^*\|^2 + \|P_C(I-\lambda D)y_n-x^*\|^2 \\
&\quad -\|y_n-P_C(I-\lambda D)y_n-\lambda(Dy_n-Dx^*)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|y_n-x^*\|^2 + \|P_C(I-\lambda D)y_n-x^*\|^2 - \|y_n-P_C(I-\lambda D)y_n\|^2 \\
&\quad +2\lambda\langle y_n-P_C(I-\lambda D)y_n, Dy_n-Dx^*\rangle - \lambda^2\|Dy_n-Dx^*\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n-x^*\|^2 + 2\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\|y_n-x^*\| \\
&\quad + \|P_C(I-\lambda D)y_n-x^*\|^2 - \|y_n-P_C(I-\lambda D)y_n\|^2 \\
&\quad +2\lambda\|y_n-P_C(I-\lambda D)y_n\|\|Dy_n-Dx^*\|\Big).
\end{aligned}$$

It implies that

$$\|P_C(I-\lambda D)y_n - x^*\|^2 \leq \|x_n - x^*\|^2 + 2\alpha_n \left\|\gamma f(x_n) - \overline{A}Gx_n\right\| \|y_n - x^*\|$$
$$- \|y_n - P_C(I-\lambda D)y_n\|^2 + 2\lambda \|y_n - P_C(I-\lambda D)y_n\|$$
$$\times \|Dy_n - Dx^*\|. \quad (3.68)$$

From definition of x_n and (3.68), we get that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n^1\|x_n - x^*\|^2 + \beta_n^2\|Tx_n - x^*\|^2 + \beta_n^3\|P_C(I-\lambda D)y_n - x^*\|^2 \\
&\leq \beta_n^1\|x_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 + \beta_n^3 \Big(\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \left\|\gamma f(x_n) - \bar{A}Gx_n\right\| \|y_n - x^*\| - \|y_n - P_C(I-\lambda D)y_n\|^2 \\
&\quad + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \|Dy_n - Dx^*\|\Big) \\
&\leq \beta_n^1\|x_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 + \beta_n^3\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \left\|\gamma f(x_n) - \overline{A}Gx_n\right\| \|y_n - x^*\| - \beta_n^3\|y_n - P_C(I-\lambda D)y_n\|^2 \\
&\quad + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \|Dy_n - Dx^*\| \\
&= \|x_n - x^*\|^2 + 2\alpha_n \left\|\gamma f(x_n) - \overline{A}Gx_n\right\| \|y_n - x^*\| \\
&\quad - \beta_n^3\|y_n - P_C(I-\lambda D)y_n\|^2 + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \\
&\quad \times \|Dy_n - Dx^*\|.
\end{aligned}$$

It implies that

$$\begin{aligned}
\beta_n^3\|y_n - P_C(I-\lambda D)y_n\|^2 &\leq 2\alpha_n \left\|\gamma f(x_n) - \overline{A}Gx_n\right\| \|y_n - x^*\| + \|x_n - x^*\|^2 \\
&\quad - \|x_{n+1} - x^*\|^2 + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \|Dy_n - Dx^*\| \\
&\leq 2\alpha_n \left\|\gamma f(x_n) - \overline{A}Gx_n\right\| \|y_n - x^*\| \\
&\quad + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\| \\
&\quad + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \|Dy_n - Dx^*\|.
\end{aligned} \quad (3.69)$$

Form condition (i), (ii), (3.61), (3.64) and (3.69), we have

$$\lim_{n\to\infty} \|y_n - P_C(I-\lambda D)y_n\| = 0. \quad (3.70)$$

Consider,

$$\|x_n - y_n\| \leq \|x_n - P_C(I-\lambda D)y_n\| + \|P_C(I-\lambda D)y_n - y_n\|.$$

By (3.67) and (3.70), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \quad (3.71)$$

From definition of y_n and condition (i), we obtain

$$\begin{aligned}
\|y_n - Gx_n\| &= \left\|\alpha_n\gamma f(x_n) + (I - \alpha_n\overline{A})Gx_n - Gx_n\right\| \\
&= \alpha_n \left\|\gamma f(x_n) - \overline{A}Gx_n\right\| \to 0 \;\; as\; n \to \infty.
\end{aligned} \quad (3.72)$$

Since

$$\|x_n - Gx_n\| \le \|x_n - y_n\| + \|y_n - Gx_n\|,$$

(3.71) and (3.72), we get that

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{3.73}$$

Moreover, from (3.66), (3.71) and

$$\begin{aligned}\|y_n - Ty_n\| &\le \|y_n - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Ty_n\| \\ &\le \|y_n - x_n\| + \|x_n - Tx_n\| + \|x_n - y_n\|,\end{aligned}$$

we have

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0. \tag{3.74}$$

Again, from (3.71), (3.73) and

$$\begin{aligned}\|y_n - Gy_n\| &\le \|y_n - x_n\| + \|x_n - Gx_n\| + \|Gx_n - Gy_n\| \\ &\le \|y_n - x_n\| + \|x_n - Gx_n\| + \|x_n - y_n\|,\end{aligned}$$

we have

$$\lim_{n\to\infty} \|y_n - Gy_n\| = 0. \tag{3.75}$$

Step 4. We show that $\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0\rangle \le 0$, where $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$. To show this inequality, take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0\rangle = \lim_{i\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_{n_i} - x_0\rangle. \tag{3.76}$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_{n_i} \rightharpoonup q$ as $i \to \infty$, where $q \in C$. From (3.71) and $x_{n_i} \rightharpoonup q$ as $i \to \infty$, we get that $y_{n_i} \rightharpoonup q$ as $i \to \infty$. From (3.76) and $y_{n_i} \rightharpoonup q$ as $i \to \infty$, we have

$$\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0\rangle = \langle \gamma f(x_0) - \overline{A}x_0, q - x_0\rangle. \tag{3.77}$$

In order to show $\langle \gamma f(x_0) - \overline{A}x_0, q - x_0\rangle \le 0$, we need to show that $q \in \Omega = F(T) \cap F(G) \cap VI(C, D)$.

First, we show that $q \in F(T)$.

Assume that $q \notin F(T)$. Then, we have $q \ne Tq$. From (3.74) and Opial's condition, we obtain

$$\begin{aligned}\liminf_{n\to\infty} \|y_{n_i} - q\| &< \liminf_{n\to\infty} \|y_{n_i} - Tq\| \\ &\le \liminf_{n\to\infty} (\|y_{n_i} - Ty_{n_i}\| + \|Ty_{n_i} - Tq\|) \\ &\le \liminf_{n\to\infty} (\|y_{n_i} - Ty_{n_i}\| + \|y_{n_i} - q\|) \\ &\le \liminf_{n\to\infty} \|y_{n_i} - q\|.\end{aligned}$$

This is a contraction, that is ,

$$q \in F(T). \tag{3.78}$$

Show that $q \in F(G)$.

Assume that $q \notin F(G)$. Then, we have $q \neq Gq$. From (3.75) and Opial's condition, we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \|y_{n_i} - q\| &< \liminf_{n\to\infty} \|y_{n_i} - Gq\| \\
&\leq \liminf_{n\to\infty} (\|y_{n_i} - Gy_{n_i}\| + \|Gy_{n_i} - Gq\|) \\
&\leq \liminf_{n\to\infty} (\|y_{n_i} - Gy_{n_i}\| + \|y_{n_i} - q\|) \\
&\leq \liminf_{n\to\infty} \|y_{n_i} - q\|.
\end{aligned}$$

This is a contraction, that is ,

$$q \in F(G). \tag{3.79}$$

Show that $q \in VI(C, D)$.

Assume that $q \notin VI(C, D)$. Since $VI(C, D) = F(P_C(I - \lambda D))$, we have $q \neq P_C(I - \lambda D)q$. By nonexpansiveness of $P_C(I - \lambda D)$, (3.70) and Opial's condition, we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \|y_{n_i} - q\| &< \liminf_{n\to\infty} \|y_{n_i} - P_C(I - \lambda D)q\| \\
&\leq \liminf_{n\to\infty} (\|y_{n_i} - P_C(I - \lambda D)y_{n_i}\| + \|P_C(I - \lambda D)y_{n_i} - P_C(I - \lambda D)q\|) \\
&\leq \liminf_{n\to\infty} (\|y_{n_i} - P_C(I - \lambda D)y_{n_i}\| + \|y_{n_i} - q\|) \\
&\leq \liminf_{n\to\infty} \|y_{n_i} - q\|.
\end{aligned}$$

This is a contraction, that is ,

$$q \in VI(C, D). \tag{3.80}$$

From (3.78), (3.79) and (3.80), we obtain $q \in \Omega = F(T) \cap F(G) \cap VI(C, D)$.

From (3.77) and property of P_C, we have

$$\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0\rangle = \langle \gamma f(x_0) - \overline{A}x_0, q - x_0\rangle \leq 0.$$

Step 5. We show that $\{x_n\}$ convergence strongly to x_0, where $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$. From the definition of x_n and $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$, we have

$$\begin{aligned}
\|x_{n+1} - x_0\|^2 &= \left\|\beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_c(I - \lambda D)y_n - x_0\right\|^2 \\
&\leq \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|Tx_n - x_0\|^2 + \beta_n^3 \|P_c(I - \lambda D)y_n - x_0\|^2 \\
&\leq \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 \|y_n - x_0\|^2 \\
&= \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 \left\|\alpha_n \gamma f(x_n) + \left(I - \alpha_n \bar{A}\right) G x_n - x_0\right\|^2 \\
&\leq \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 \left(\left\|\left(I - \alpha_n \bar{A}\right)(Gx_n - x_0)\right\|^2\right. \\
&\quad \left. + 2\alpha_n \left\langle \gamma f(x_n) - \bar{A}x_0, y_n - x_0\right\rangle\right) \\
&\leq \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 \left((1 - \alpha_n \overline{\gamma})^2 \|x_n - x_0\|^2\right. \\
&\quad \left. + 2\alpha_n \gamma \left\langle f(x_n) - f(x_0), y_n - x_0\right\rangle + 2\alpha_n \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n^1\|x_n - x_0\|^2 + \beta_n^2\|x_n - x_0\|^2 + \beta_n^3 \Big((1-\alpha_n\overline{\gamma})^2\|x_n - x_0\|^2 \\
&\quad +2\alpha_n\gamma\, \|f(x_n) - f(x_0)\|\, \|y_n - x_0\| + 2\alpha_n \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle\Big) \\
&\leq \beta_n^1\|x_n - x_0\|^2 + \beta_n^2\|x_n - x_0\|^2 + \beta_n^3(1-\alpha_n\overline{\gamma})^2\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\beta_n^3\, \|x_n - x_0\|\, \|y_n - x_0\| + 2\alpha_n\beta_n^3 \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle \\
&\leq \beta_n^1\|x_n - x_0\|^2 + \beta_n^2\|x_n - x_0\|^2 + \beta_n^3(1-\alpha_n\overline{\gamma})^2\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\beta_n^3\, \|x_n - x_0\| \left(\alpha_n \left\|\gamma f(x_n) - \bar{A}x_0\right\| + (1-\alpha_n\overline{\gamma})\, \|Gx_n - x_0\|\right) \\
&\quad + 2\alpha_n\beta_n^3 \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle \\
&\leq \beta_n^1\|x_n - x_0\|^2 + \beta_n^2\|x_n - x_0\|^2 + \beta_n^3(1-\alpha_n\overline{\gamma})^2\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\beta_n^3\, \|x_n - x_0\| \left(\alpha_n\gamma\alpha\, \|x_n - x_0\| + \alpha_n \left\|\gamma f(x_0) - \bar{A}x_0\right\|\right. \\
&\quad \left. + (1-\alpha_n\overline{\gamma})\, \|x_n - x_0\|\right) + 2\alpha_n\beta_n^3 \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle \\
&= (1-\beta_n^3)\|x_n - x_0\|^2 + \beta_n^3(1-\alpha_n\overline{\gamma})^2\|x_n - x_0\|^2 + 2\alpha_n^2\gamma^2\alpha^2\beta_n^3\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2\gamma\alpha\beta_n^3 \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\| + 2\alpha_n\gamma\alpha\beta_n^3\,(1-\alpha_n\overline{\gamma})\, \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\beta_n^3 \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle \\
&\leq (1-\beta_n^3)\|x_n - x_0\|^2 + \beta_n^3\,(1-\alpha_n\bar{\gamma})\, \|x_n - x_0\|^2 + 2\alpha_n^2\bar{\gamma}^2\beta_n^3\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2\bar{\gamma}\beta_n^3 \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\| + 2\alpha_n\gamma\alpha\beta_n^3\|x_n - x_0\|^2 \\
&\quad + 2\alpha_n\beta_n^3 \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle \\
&= \left(1 - \beta_n^3 + \beta_n^3 - \beta_n^3\alpha_n\bar{\gamma} + 2\alpha_n\gamma\alpha\beta_n^3\right) \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2\bar{\gamma}^2\beta_n^3\|x_n - x_0\|^2 + 2\alpha_n^2\bar{\gamma}\beta_n^3 \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\| \\
&\quad + 2\alpha_n\beta_n^3 \left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle \\
&= \left(1 - \alpha_n\beta_n^3\,(\bar{\gamma} - 2\gamma\alpha)\right) \|x_n - x_0\|^2 + \alpha_n\beta_n^3 \Big(2\alpha_n\bar{\gamma}^2\|x_n - x_0\|^2 \\
&\quad +2\alpha_n\bar{\gamma} \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\| + 2\left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle\Big) \\
&= \left(1 - \alpha_n\beta_n^3\,(\bar{\gamma} - 2\gamma\alpha)\right) \|x_n - x_0\|^2 + \alpha_n\beta_n^3\,(\bar{\gamma} - 2\gamma\alpha) \left(\frac{2\alpha_n\bar{\gamma}^2\|x_n - x_0\|^2}{(\bar{\gamma} - 2\gamma\alpha)}\right. \\
&\quad \left. + \frac{2\alpha_n\bar{\gamma} \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\|}{(\bar{\gamma} - 2\gamma\alpha)} + \frac{2\left\langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\right\rangle}{(\bar{\gamma} - 2\gamma\alpha)}\right).
\end{aligned}
$$

From step 4, condition(i) and Lemma 2.38, we can conclude that $\{x_n\}$ converges strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and by Lemma 3.9, we have (x_0, y_0, z_0) is a solution of (3.55) where $y_0 = P_C(I - \lambda_2 D_2)\,(ax_0 + (1-a)z_0)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$. This completes the proof. □

The following corollary is consequence which is applied by Theorem 3.12.

Corollary 3.13. Let C be a nonempty closed convex subset of a real Hilbert space H and let $D, D_1, D_2, : C \to H$ be d, d_1, d_2-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \lambda_1 D_1)\,(P_C(I - \lambda_2 D_2)x)$, for all $x \in C, \lambda_1, \lambda_2 > 0$. For $k = 1, 2, ..., \overline{N}$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient

$\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,\ldots,\overline{N}} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Let T be a nonexpansive mapping with $\Omega = F(T) \cap F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n, \\ y_n &= \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n, \end{aligned} \tag{3.81}$$

where f is α-contraction mapping on H, $\{\alpha_n\} \subset [0,1], \lambda \in (0, 2d)$, $\lambda_1, \lambda_2 \in \left(0, 2\overline{d}\right)$ with $\overline{d} = \min\{d_1, d_2\}$ and $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n^i \leq c < 1$ for all $i = 1, 2, 3$;

(iii) $0 < c_k < 1$ and $\sum\limits_{k=1}^{\overline{N}} c_k = 1$;

(iv) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum\limits_{n=1}^{\infty} \left|\beta_{n+1}^i - \beta_n^i\right| < \infty$, for all $i = 1, 2, 3$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (1.11) where $y_0 = P_C(I - \lambda_2 D_2)(z_0)$.

Proof. From Theorem 3.12, If we put $a = 0$ and $D_3 \equiv 0$, we have the desired conclusion. □

Chapter 4

Applications

In this chapter, we apply the results of previous chapter for solving the constrained minimization, the split feasibility and the split variational inequality problems in a real Hilbert space.

4.1 The split feasibility problem

Throughout this section, assume that the split feasibility problem (SFP) is consistent, that is, Γ is nonempty. It is easy to see that $x^* \in C$ solves SFP if and only if it solves the fixed point equation

$$x^* = P_C(x^* - \eta D^*(I - P_Q)Dx^*), \tag{4.1}$$

where $\eta > 0$, P_C and P_Q are the orthogonal projection onto C and Q, respectively, and A^* is the adjoint of A. The most popular method for solving (4.1) is Byrne's CQ algorithm [3], which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C\left(x_{n+1} - \eta A^*(I - P_Q)Ax_{n+1}\right),\ \forall n \in \mathbb{N}, \tag{4.2}$$

where $\eta \in (0, \frac{2}{\gamma})$ with γ being the spectral radius of the operator A^*A.

4.1.1 Application of the split general system of variational inequalities problem

In this section, we utilize Theorem 3.4 to prove Theorem 4.1 for finding the common element of the solution set of the split feasibility problem and the fixed point of nonexpansive mapping.

Theorem 4.1. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Let $T : C \to C$ be a nonexpansive mapping. Assume $\Im = \Gamma \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} y_n &= P_C(x_n - \eta D^*(I - P_Q)Dx_n), \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)Ty_n, \end{aligned} \tag{4.3}$$

where $\{\alpha_n\} \subset [0, 1]$, $\eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Im}u$ which (x_0, Dx_0) is the solution of the split feasibility problem (SFP).

Proof. If we put $A \equiv B \equiv \overline{A} \equiv \overline{B} \equiv 0,\ x^* = y^*$ and $\overline{x^*} = \overline{y^*}$, in Theorem 3.4, we obtain the desired conclusion. □

Remark 4.2. If we take $T \equiv I$, in Theorem 4.1, then we obtain

$$\begin{aligned} y_n &= P_C(x_n - \eta D^*(I - P_Q)Dx_n), \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)y_n. \end{aligned} \tag{4.4}$$

From Theorem 4.1, we have $\{x_n\}$ generated by (4.4) converges strongly to $x_0 = P_{\Im}u$ which is a solution of the split feasibility problem (SFP). Moreover, we have Corollary 3.7 of Xu [51] is a special case of our main theorem, Theorem 3.4.

4.1.2 Application of the modified generalized system of variational inequalities

In this section, we utilize Theorem 3.12 to prove Theorem 4.4 for finding the common element of the set of fixed points of nonexpansive mapping and the solution set of the split feasibility, the modified generalized system of variational inequalities problem.

To prove Theorem 4.4, we need the following proposition which is introduced by Ceng et al. [5]

Proposition 4.3. [5] Given $x^* \in H_1$, the following statements are equivalent.

(i) $x^* \in \Gamma$;

(ii) x^* solves the equation (4.1).

(iii) x^* solves the variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle A^*(I - P_Q)Ax^*, x - x^* \rangle \geq 0,\ \forall x \in C,$$

where A^* is the adjoint of A.

Theorem 4.4. Let C and Q be nonempty closed convex subsets of real Hilbert space H_1, H_2, respectively and let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3- inverse strongly monotone mapping, respectively. Define the mapping G as in Lemma 3.9 and $a \in [0, 1)$. For $k = 1, 2, ..., \overline{N}$, define $\overline{A} : H_1 \to H_1$ by $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H_1 with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,\overline{N}} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Let $T : C \to C$ be a nonexpansive mapping with $\Omega = F(T) \cap F(G) \cap \Gamma \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda A^*(I - P_Q)A)y_n, \\ y_n &= \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) Gx_n, \end{aligned} \tag{4.5}$$

where f is α-contraction mapping on H_1, $\{\alpha_n\} \subset [0,1], \lambda \in (0, \frac{2}{L})$ with L being the spectral radius of the operator A^*A, $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2, d_3\}$ and $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n^i \leq c < 1$ for all $i = 1, 2, 3$;

(iii) $0 < c_k < 1$ and $\sum_{k=1}^{\overline{N}} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} \left|\beta_{n+1}^i - \beta_n^i\right| < \infty$, for all $i = 1, 2, 3$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (3.55) where $y_0 = P_C(I - \lambda_2 D_2)(ax_0 + (1-a)z_0)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$.

Proof. Let $x, y \in H_1$.
We will show that $A^*(I - P_Q)A$ is $\frac{1}{L}$-inverse strongly monotone.
By the definition of the spectral radius L, we have

$$\begin{aligned}\|A^*(I-P_Q)Ax - A^*(I-P_Q)Ay\|^2 &= \langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, A^*(I-P_Q)\\ &\quad \times Ax - A^*(I-P_Q)Ay\rangle\\ &= \langle (I-P_Q)Ax - (I-P_Q)Ay, AA^*(I-P_Q)Ax\\ &\quad - AA^*(I-P_Q)Ay\rangle\\ &\leq L\|(I-P_Q)Ax - (I-P_Q)Ay\|^2.\end{aligned}$$

Consider,

$$\begin{aligned}\|(I-P_Q)Ax - (I-P_Q)Ay\|^2 &= \langle (I-P_Q)Ax - (I-P_Q)Ay, (I-P_Q)Ax - (I-P_Q)Ay\rangle\\ &= \langle (I-P_Q)Ax - (I-P_Q)Ay, Ax - Ay\rangle\\ &\quad - \langle (I-P_Q)Ax - (I-P_Q)Ay, P_QAx - P_QAy\rangle\\ &= \langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x - y\rangle\\ &\quad - \langle (I-P_Q)Ax - (I-P_Q)Ay, P_QAx - P_QAy\rangle\\ &= \langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x - y\rangle\\ &\quad - \langle (I-P_Q)Ax, P_QAx - P_QAy\rangle\\ &\quad + \langle (I-P_Q)Ay, P_QAx - P_QAy\rangle\\ &\leq \langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x - y\rangle.\end{aligned}$$

It implies that

$$\langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x - y\rangle \geq \frac{1}{L}\|A^*(I-P_Q)Ax - A^*(I-P_Q)Ay\|^2.$$

Hence, $A^*(I - P_Q)A$ is $\frac{1}{L}$- inverse strongly monotone.
From Theorem 3.12 and Proposition 4.3, we obtain the desired conclusion. □

4.2 The minimization problem

Let C and Q be a closed convex subset of a real Hilbert space H_1 and H_2, respectively. Let $D : H_1 \to H_2$ be a bounded linear operator. The *constrained minimization problem* is to find $x^* \in C$ such that

$$f(x^*) = \min_{x \in C} f(x), \tag{4.6}$$

where $f : H_1 \to \mathbb{R}$ is a convex, continuous differentiable function. The set of all solutions of (4.6) is denoted by Γ_f.

It is obvious that $x^* \in \Gamma$ if and only if $x^* \in C$ and $Dx^* - P_Q Dx^* = 0$. Define the proximity function f by

$$f(x) = \frac{1}{2}\|Dx - P_Q Dx\|^2. \tag{4.7}$$

for all $x \in H_1$. Note that f is continuously differentiable function.
Consider the constrained minimization problem, we obtain

$$f(x^*) = \min_{x \in C} f(x) = \min_{x \in C} \frac{1}{2}\|Dx - P_Q Dx\|^2. \tag{4.8}$$

Then, we have $x^* \in \Gamma$ if and only if x^* solves the minimization problem (4.8) with the minimization equal to 0. Furthermore, by the definition of the proximity function f, we have

$$\nabla f(x) = D^*(I - P_Q)Dx,$$

where ∇f is a gradient of f.

4.2.1 Application of the split general system of variational inequalities problem

In this section, we prove Theorem 4.7 for approximation the solutions of the minimization problem and the fixed points of nonexpansive mapping, by applying Theorem 4.1.

To prove Theorem 4.7, we need the following proposition.

Proposition 4.5. [5] Given $x^* \in H_1$, the following statements are equivalent.

(i) $x^* \in \Gamma$;

(ii) x^* solves the equation (4.1);

(iii) x^* solves the variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \ \forall x \in C,$$

where $\nabla f(x) = D^*(I - P_Q)Dx$. and D^* is the adjoint of D.

Remark 4.6. Moreover, $x^* \in \Gamma$ if and only if $x^* \in \Gamma_f$.

Proof. "Necessity". Suppose $z \in \Gamma$, that is, $z \in C$ and $Dz \in Q$.
Consider,

$$\begin{aligned} f(z) &= \frac{1}{2}\|Dz - P_Q Dz\|^2 \\ &= 0 \\ &\leq f(x), \end{aligned}$$

for all $x \in C$. Hence $z \in \Gamma_f$.
"Sufficiency". Suppose $z \in \Gamma_f$, we have

$$f(z) \leq f(x), \tag{4.9}$$

for all $x \in C$.
Since $\Gamma \neq \emptyset$, there exists $\overline{x} \in C$ and $D\overline{x} \in Q$.
From (4.9), we obtain

$$\frac{1}{2}\|Dz - P_Q Dz\|^2 \leq \frac{1}{2}\|Dx - P_Q Dx\|^2, \ \forall x \in C. \tag{4.10}$$

Since $\overline{x} \in C$, we have

$$\frac{1}{2}\|Dz - P_Q Dz\|^2 \leq \frac{1}{2}\|D\overline{x} - P_Q D\overline{x}\|^2. \tag{4.11}$$

Since $D\overline{x} \in Q$, we get that $D\overline{x} = P_Q D\overline{x}$.
From (4.11), we have

$$\frac{1}{2}\|Dz - P_Q Dz\|^2 = 0.$$

This implies that $Dz = P_Q Dz$, that is $Dz \in Q$.
Hence $z \in \Gamma$. □

Theorem 4.7. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $f : H_1 \to \mathbb{R}$ be a continuous differentiable function defined by (4.7), with the gradient ∇f. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Let $T : C \to C$ be a nonexpansive mapping. Assume $\Im = \Gamma \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} y_n &= P_C(x_n - \eta \nabla f x_n), \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n) T y_n, \end{aligned} \tag{4.12}$$

where $\{\alpha_n\} \subset [0,1]$, $\eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Im} u$ which x_0 is the solution of the minimization problem (4.8), that is, $x_0 \in \Gamma_f$.

Proof. Let $x, y \in H_1$.
First, we show that ∇f is $\frac{1}{L}$-inverse strongly monotone.
Consider,

$$\begin{aligned}\|\nabla f(x) - \nabla f(y)\|^2 &= \|D^*(I - P_Q)Dx - D^*(I - P_Q)Dy\|^2 \\ &= \langle D^*(I - P_Q)Dx - D^*(I - P_Q)Dy, D^*(I - P_Q)Dx - D^*(I - P_Q)Dy\rangle \\ &= \langle (I - P_Q)Dx - (I - P_Q)Dy, DD^*(I - P_Q)Dx - DD^*(I - P_Q)Dy\rangle \\ &\leq L\|(I - P_Q)Dx - (I - P_Q)Dy\|^2.\end{aligned}$$

From the property of P_C, we have

$$\begin{aligned}\|(I - P_Q)Dx - (I - P_Q)Dy\|^2 &= \langle (I - P_Q)Dx - (I - P_Q)Dy, (I - P_Q)Dx - (I - P_Q)Dy\rangle \\ &= \langle (I - P_Q)Dx - (I - P_Q)Dy, Dx - Dy\rangle \\ &\quad - \langle (I - P_Q)Dx - (I - P_Q)Dy, P_QDx - P_QDy\rangle \\ &= \langle D^*(I - P_Q)Dx - D^*(I - P_Q)Dy, x - y\rangle \\ &\quad - \langle (I - P_Q)Dx - (I - P_Q)Dy, P_QDx - P_QDy\rangle \\ &= \langle D^*(I - P_Q)Dx - D^*(I - P_Q)Dy, x - y\rangle \\ &\quad - \langle (I - P_Q)Dx, P_QDx - P_QDy\rangle + \langle (I - P_Q)Dy, P_QDx - P_QDy\rangle \\ &\leq \langle D^*(I - P_Q)Dx - D^*(I - P_Q)Dy, x - y\rangle .\end{aligned}$$

Since $\nabla f(x) = D^*(I - P_Q)Dx$, we obtain

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2.$$

Thus, ∇f is $\frac{1}{L}$- inverse strongly monotone.
From Proposition 4.5, Theorem 4.1 and Remark 4.6, we have the desired conclusion. □

Remark 4.8. If we take $T \equiv I$, in Theorem 4.7, then we obtain

$$\begin{aligned} y_n &= P_C(x_n - \eta\nabla f x_n), \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)y_n. \end{aligned} \tag{4.13}$$

From Theorem 4.7, we have the sequence $\{x_n\}$ generated by (4.13) converges strongly to $x_0 = P_{\mathfrak{I}}u$ which is a solution of the minimization problem (4.8). Moreover, we have Theorem 5.2 of Lopez et al. [27] is a special case of our main theorem, Theorem 3.4.

4.2.2 Application of the modified generalized system of variational inequalities

In this section, we prove Theorem 4.10 for approximation the solution of the standard constrained convex optimization problem and the split feasibility problem, by applying our main result Theorem 3.12.

To prove Theorem 4.10, we need the following lemma.

Lemma 4.9. [39] (Optimality condition) A necessary condition of Optimality for a point $x^* \in C$ to be a solution of the minimization problem (4.8) is that x^* solves the variational inequality

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \tag{4.14}$$

for all $x \in C$. Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C(I - \lambda \nabla f)x^*,$$

for every $\lambda > 0$. If, in addition, f is convex, then the optimality condition (4.14) is also sufficient.

Theorem 4.10. Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2, D_3 : C \to H$ be d_1, d_2, d_3- inverse strongly monotone mapping, respectively. Let $\overline{f} : C \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla \overline{f}$ is $\frac{1}{L_{\overline{f}}}$-inverse strongly monotone and continuous with $L_{\overline{f}} > 0$. Define the mapping G as in Lemma 3.9 and $a \in [0, 1)$. For $k = 1, 2, ..., \overline{N}$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,\overline{N}} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Let $T : C \to C$ be a nonexpansive mapping with $\Omega = F(T) \cap F(G) \cap \Gamma_f \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda \nabla \overline{f}) y_n, \\ y_n &= \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n, \end{aligned} \tag{4.15}$$

where f is α-contraction mapping on H, $\{\alpha_n\} \subset [0, 1]$, $\lambda \in (0, \frac{2}{L_{\overline{f}}})$, $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2, d_3\}$ and $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n^i \leq c < 1$ for all $i = 1, 2, 3$;

(iii) $0 < c_k < 1$ and $\sum_{k=1}^{\overline{N}} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1}^i - \beta_n^i| < \infty$, for all $i = 1, 2, 3$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (3.55) where $y_0 = P_C(I - \lambda_2 D_2)(ax_0 + (1-a)z_0)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$.

Proof. By using Lemma 4.9 and Theorem 3.12, we obtain the desired conclusion. □

4.3 The split variational inequality

The split variational inequality problem (SVIP) is to find $\widehat{x} \in C$ such that

$$\langle f_1 \widehat{x}, x - \widehat{x} \rangle \geq 0, \ \forall x \in C, \tag{4.16}$$

and find $\widehat{y} = D\widehat{x} \in Q$ such that

$$\langle f_2\widehat{y}, x - \widehat{y}\rangle \geq 0,\ \forall y \in Q, \tag{4.17}$$

where $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings and $D : H_1 \to H_2$ is a bounded linear operator. The set of all solution of the split variational inequality problem (SVIP) is denoted by $\Phi = \{\widehat{x} \in VI(C, f_1) : \widehat{y} \in VI(C, f_1)\}$. The split variational inequality problem (SVIP) is reduced to the split feasibility problem (SFP) if $f_1 \equiv f_2 \equiv 0$.

4.3.1 Application of the split general system of variational inequalities problem

In this section, we utilize Theorem 3.4 to prove Theorem 4.11 for finding the common element of the solution set of the split variational inequality problem and the fixed point of nonexpansive mapping.

Theorem 4.11. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : C \to H_1$ be a-inverse strongly monotone mapping. Let $\overline{A} : Q \to H_2$ be $\overline{a}$-inverse strongly monotone mapping. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Define the mapping $G_C : C \to C$ by $G_C(x) = P_C(I - \lambda A)x$, for all $x \in C$ and define the mapping $G_Q : Q \to Q$ by $G_Q(\widehat{x}) = P_Q(I - \alpha\bar{A})\widehat{x}$, for all $\widehat{x} \in Q$. Define $G : C \to C$ by $G(x) = G_C(x - \eta D^*(I - G_Q)Dx)$ for all $x \in C$. Let $T : C \to C$ be a nonexpansive mapping. Assume $\Im = \Phi \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} y_n &= G_C(x_n - \eta D^*(I - G_Q)Dx_n),\\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)Ty_n, \end{aligned} \tag{4.18}$$

where $\{\alpha_n\} \subset [0,1]$, $\eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum\limits_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_\Im u$.

Proof. Putting $B \equiv 0$, $\overline{B} \equiv 0$, $x^* = y^*$ and $\overline{x^*} = \overline{y^*}$, in Theorem 3.4, then we obtain the desired conclusion. □

Chapter 5

Numerical example

In this chapter, we give numerical examples for supporting our main theorems and some applications.

5.1 Numerical examples of the split general system of variational inequalities problem and applications

The following four examples are given for supporting Theorem 3.4 in Chapter 3 and Theorem 4.1, Theorem 4.7, Theorem 4.11 in Chapter 4, respectively.

Example 5.1. Let $\mathbb{R}$ be the set of real numbers and let $<.,.>: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be an inner product defined by $\langle \mathsf{x}, \mathsf{y} \rangle = \mathsf{x} \cdot \mathsf{y} = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$, for all $\mathsf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\mathsf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. Let $H_1 = H_2 = \mathbb{R}^3$, $C = [-50, 50] \times [-50, 50] \times [-50, 50]$ and $Q = [-60, 60] \times [-60, 60] \times [-60, 60]$. Let A, B be mappings from C to $\mathbb{R}^3$ defined by

$$A\mathsf{x} = \left(\frac{x_1 - 2}{3}, \frac{2x_2 - 4}{5}, \frac{x_3 - 2}{4} \right), \forall \mathsf{x} \in C,$$

and

$$B\mathsf{x} = \left(x_1 - 2, \frac{x_2 - 2}{3}, \frac{x_3 - 2}{5} \right), \forall \mathsf{x} \in C,$$

and let $\overline{A}$, $\overline{B}$ be mappings from Q to $\mathbb{R}^3$ defined by

$$\overline{A}\mathsf{x} = \left(\frac{x_1}{5}, \frac{x_2}{2}, \frac{2x_3}{3} \right), \forall \mathsf{x} \in Q \ \ and \ \ \overline{B}\mathsf{x} = \left(\frac{x_1}{3}, \frac{x_2}{5}, \frac{x_3}{7} \right), \forall \mathsf{x} \in Q.$$

It is easy to see that $A, B, \overline{A}, \overline{B}$ are 1-inverse strongly monotone with $\lambda, \mu, \alpha, \gamma \in (0, 2)$. Then, we can choose $\lambda = 0.5, \mu = 1, \alpha = 1, \gamma = 0.5$. Let the mapping $D : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$D\mathsf{x} = (2x_1 - x_2 - x_3, x_1 - 2x_2 + x_3, x_1 + x_2 - 2x_3), \forall \mathsf{x} \in \mathbb{R}^3,$$

and $D^* : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$D^*\mathsf{x} = (2x_1 + x_2 + x_3, -x_1 - 2x_2 + x_3, -x_1 + x_2 - 2x_3), \forall \mathsf{x} \in \mathbb{R}^3.$$

Then, the spectral radius of the operator D^*D is 9 and also we can choose $\eta = 0.1$. Define $G_C : C \to C$ by

$$G_C(\mathsf{x}) = P_C(I - 0.5A)P_C(I - B)\mathsf{x}, \ \forall \mathsf{x} \in C,$$

and define $G_Q : Q \to Q$ by

$$G_Q(\mathsf{x}) = P_Q(I - \bar{A})P_Q(I - 0.5\bar{B})\mathsf{x}, \ \forall \mathsf{x} \in Q.$$

Let the mapping $G : C \to C$ be defined by

$$G(x) = G_C(x_n - 0.11D^*(I - G_Q)Dx_n), \forall \mathsf{x} \in C.$$

Let T be a mapping from C into itself defined by $T\mathsf{x} = (\frac{x_1+2}{2}, \frac{x_2+4}{3}, \frac{3x_3+2}{4})$, $\forall \mathsf{x} \in C$. Let $\mathsf{x}_1 = (x_1^1, x_1^2, x_1^3) \in C$ and $\mathsf{x}_n = (x_n^1, x_n^2, x_n^3)$ be generated by (3.10), where $\alpha_n = \frac{1}{8n}$, for every $n \in \mathbb{N}$. Put u = **5**, where **5** = (5,5,5). By the definition of $A, B, \overline{A}, \overline{B}, D$ and T, we have $\mathbf{2} \in F(G) \cap F(T)$, where $\mathbf{2} = (2, 2, 2)$. From Theorem 3.4, we can conclude that the sequence $\mathsf{x}_n = (x_n^1, x_n^2, x_n^3)$ converges strongly to **2**. For every $n \in \mathbb{N}$, we can rewrite (3.10) as follow:

$$\begin{aligned} \mathsf{y}_n &= G_C\left(\mathsf{x}_n - 0.1D^*\left(I - G_Q\right)D\mathsf{x}_n\right), \\ \mathsf{x}_{n+1} &= \tfrac{1}{8n}\mathsf{u} + \left(\tfrac{8n-1}{8n}\right)T\mathsf{y}_n. \end{aligned} \tag{5.1}$$

By using the algorithm (5.1), the following table and figure show the values of sequences x_n and y_n, where $\mathsf{x}_1 = \mathbf{10} = (10, 10, 10)$ and $n = N = 20$.

Table 5.1: The values of x_n and y_n with $\mathsf{x}_1 = \mathbf{10}$ and $N = 20$ of the iterative (5.1).

n	$\mathsf{x}_n = (x_n^1, x_n^2, x_n^3)$	$\mathsf{y}_n = (y_n^1, y_n^2, y_n^3)$
1	(10.000000, 10.000000, 10.000000)	(2.000000, 6.266667, 7.600000)
2	(2.375000, 3.619444, 2.375000)	(2.000000, 3.429975, 3.141172)
3	(2.187500, 2.634367, 2.187500)	(2.000000, 2.406361, 2.416339)
4	(2.125000, 2.254810, 2.125000)	(2.000000, 2.172967, 2.174208)
⋮	⋮	⋮
10	(2.041667, 2.054015, 2.041667)	(2.000000, 2.033119, 2.037592)
⋮	⋮	⋮
17	(2.023438, 2.029874, 2.023438)	(2.000000, 2.018215, 2.020806)
18	(2.022059, 2.028086, 2.022059)	(2.000000, 2.017118, 2.019561)
19	(2.020833, 2.026500, 2.020833)	(2.000000, 2.016145, 2.018457)
20	(2.019737, 2.025083, 2.019737)	(2.000000, 2.015278, 2.017471)

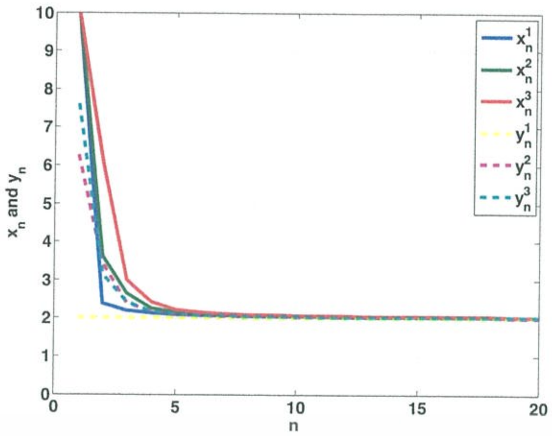


Figure 5.1: The convergence of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 20$.

Example 5.2. In this example, we use the same mappings and parameters as in Example 5.1 except the following mapping G_C and G_Q. Define G_C by $G_C(x) = P_C(x),\ \forall x \in C$ and define G_Q by $G_Q(\widehat{x}) = P_Q(\widehat{x}),\ \forall \widehat{x} \in Q$. By the definition of D and T, we have $\mathbf{2} \in \Gamma \cap F(T)$, where $\mathbf{2} = (2, 2, 2)$. From Theorem 4.1, we can conclude that the sequence $\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$ converges strongly to $\mathbf{2}$. For every $n \in \mathbb{N}$, we can rewrite (4.3) as follow:

$$\begin{aligned} \mathbf{y}_n &= P_C\left(\mathbf{x}_n - 0.1D^*\left(I - P_Q\right)D\mathbf{x}_n\right), \\ \mathbf{x}_{n+1} &= \tfrac{1}{8n}\mathbf{u} + \left(\tfrac{8n-1}{8n}\right)T\mathbf{y}_n. \end{aligned} \tag{5.2}$$

By using the algorithm (5.2), the following table and figure show the values of sequences $\mathbf{x}_n$ and $\mathbf{y}_n$, where $\mathbf{x}_1 = \mathbf{10} = (10, 10, 10)$ and $N = 40$.

Table 5.2: The values of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 40$ of the iterative (5.2).

n	$\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$	$\mathbf{y}_n = (y_n^1, y_n^2, y_n^3)$
1	(10.000000, 10.000000, 10.000000)	(10.000000, 10.000000, 10.000000)
2	(5.875000, 4.708333, 5.875000)	(5.875000, 4.708333, 7.625000)
3	(4.003906, 3.033854, 4.003906)	(4.003906, 3.033854, 6.142578)
4	(3.085205, 2.455259, 3.085205)	(3.085205, 2.455259, 5.102478)
⋮	⋮	⋮
20	(2.041668, 2.030369, 2.041668)	(2.041668, 2.030369, 2.119598)
⋮	⋮	⋮
37	(2.021386, 2.015826, 2.021386)	(2.021386, 2.015826, 2.045701)
38	(2.020792, 2.015393, 2.020792)	(2.020792, 2.015393, 2.044295)
39	(2.020230, 2.014982, 2.020230)	(2.020230, 2.014982, 2.042981)
40	(2.019698, 2.014594, 2.019698)	(2.019698, 2.014594, 2.041747)

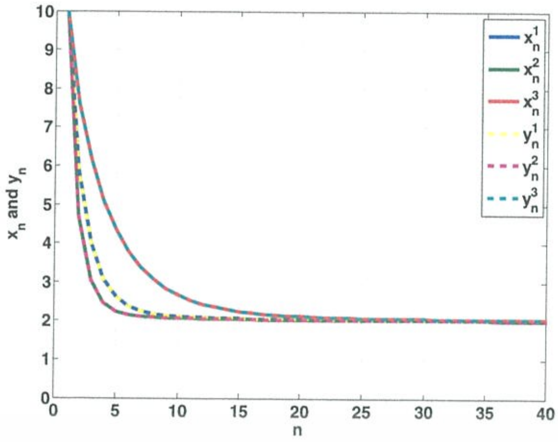


Figure 5.2: The convergence of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 40$.

Example 5.3. In this example, we use the same mappings and parameters as in Example 5.2 except the following mapping f. Let f be a mapping from $\mathbb{R}^3$ into $\mathbb{R}$ defined by $f(x) = \frac{1}{2}\|Dx - P_Q Dx\|^2, \forall x \in \mathbb{R}^3$, with $\nabla f(x) = D^*(I - P_Q)Dx, \forall x \in \mathbb{R}^3$. By the definition of $f, \nabla f, D$ and T, we have $\mathbf{2} \in \Gamma \cap F(T)$, where $\mathbf{2} = (2, 2, 2)$. From Theorem 4.7, we can conclude that the sequence $\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$ converges strongly to $\mathbf{2}$. For every $n \in \mathbb{N}$, we can rewrite (4.12) as follow:

$$\begin{aligned} \mathbf{y}_n &= P_C(\mathbf{x}_n - 0.1\nabla f\mathbf{x}_n), \\ \mathbf{x}_{n+1} &= \tfrac{1}{8n}\mathbf{u} + \left(\tfrac{8n-1}{8n}\right) T\mathbf{y}_n. \end{aligned} \tag{5.3}$$

By using the algorithm (5.3), the following table and figure show the values of sequences $\mathbf{x}_n$ and $\mathbf{y}_n$, where $\mathbf{x}_1 = \mathbf{10} = (10, 10, 10)$ and $N = 40$.

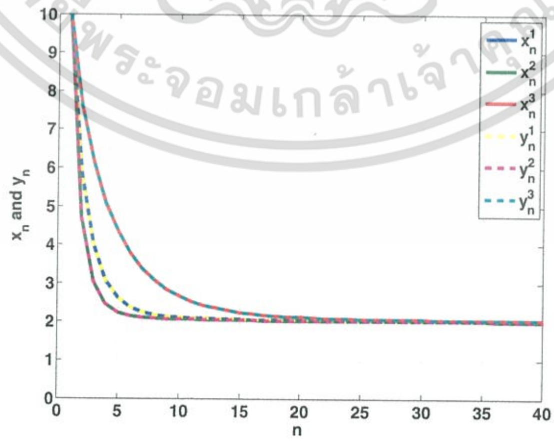


Figure 5.3: The convergence of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 40$.

Table 5.3: The values of x_n and y_n with $\mathsf{x}_1 = \mathbf{10}$ and $N = 40$ of the iterative (5.3).

n	$\mathsf{x}_n = (x_n^1, x_n^2, x_n^3)$	$\mathsf{y}_n = (y_n^1, y_n^2, y_n^3)$
1	(10.000000, 10.000000, 10.000000)	(10.000000, 10.000000, 10.000000)
2	(5.875000, 4.708333, 5.875000)	(5.875000, 4.708333, 7.625000)
3	(4.003906, 3.033854, 4.003906)	(4.003906, 3.033854, 6.142578)
4	(3.085205, 2.455259, 3.085205)	(3.085205, 2.455259, 5.102478)
⋮	⋮	⋮
20	(2.041668, 2.030369, 2.041668)	(2.041668, 2.030369, 2.119598)
⋮	⋮	⋮
37	(2.021386, 2.015826, 2.021386)	(2.021386, 2.015826, 2.045701)
38	(2.020792, 2.015393, 2.020792)	(2.020792, 2.015393, 2.044295)
39	(2.020230, 2.014982, 2.020230)	(2.020230, 2.014982, 2.042981)
40	(2.019698, 2.014594, 2.019698)	(2.019698, 2.014594, 2.041747)

Example 5.4. In this example, we use the same mappings and parameters as in Example 5.1 except the following mapping G_C and G_Q. Define G_C by $G_C(x) = P_C(I - 0.5A)x$, $\forall x \in C$ and define G_Q by $G_Q(\widehat{x}) = P_Q(I - \overline{A})\widehat{x}$, $\forall \widehat{x} \in Q$. By the definition of D and T, we have $\mathbf{2} \in \Phi \cap F(T)$, where $\mathbf{2} = (2, 2, 2)$. From Theorem 4.11, we can conclude that the sequence $\mathsf{x}_n = (x_n^1, x_n^2, x_n^3)$ converges strongly to $\mathbf{2}$. For every $n \in \mathbb{N}$, we can rewrite (4.18) as follow:

$$\begin{aligned} \mathsf{y}_n &= G_C\left(\mathsf{x}_n - 0.1D^*\left(I - G_Q\right)D\mathsf{x}_n\right), \\ \mathsf{x}_{n+1} &= \tfrac{1}{8n}\mathsf{u} + \left(\tfrac{8n-1}{8n}\right)T\mathsf{y}_n. \end{aligned} \tag{5.4}$$

By using the algorithm (5.4), the following table and figure show the values of sequences x_n and y_n, where $\mathsf{x}_1 = \mathbf{10} = (10, 10, 10)$ and $N = 40$.

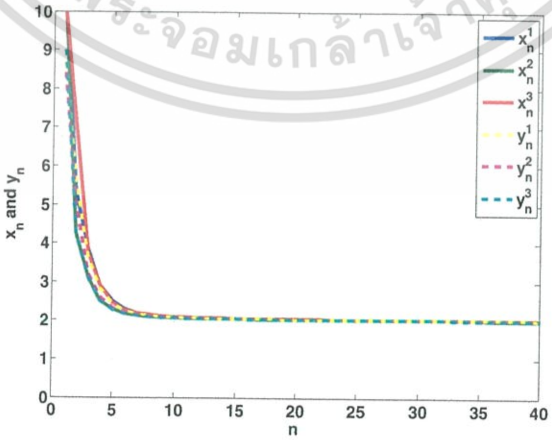


Figure 5.4: The convergence of x_n and y_n with $\mathsf{x}_1 = \mathbf{10}$ and $N = 40$.

Table 5.4: The values of x_n and y_n with $\mathsf{x}_1 = 10$ and $N = 40$ of the iterative (5.4).

n	$\mathsf{x}_n = (x_n^1, x_n^2, x_n^3)$	$\mathsf{y}_n = (y_n^1, y_n^2, y_n^3)$
1	(10.000000, 10.000000, 10.000000)	(8.666667, 8.400000, 9.000000)
2	(5.291667, 4.241667, 5.291667)	(5.533027, 4.974856, 4.225897)
3	(3.843606, 3.117142, 3.843606)	(3.641449, 3.197288, 3.091111)
4	(2.911528, 2.507467, 2.911528)	(2.835120, 2.594672, 2.509903)
⋮	⋮	⋮
20	(2.037771, 2.029968, 2.037771)	(2.034291, 2.029198, 2.027707)
⋮	⋮	⋮
37	(2.019482, 2.015585, 2.019482)	(2.017676, 2.015117, 2.014384)
38	(2.018943, 2.015157, 2.018943)	(2.017187, 2.014701, 2.013989)
39	(2.018434, 2.014753, 2.018434)	(2.016724, 2.014307, 2.013615)
40	(2.017951, 2.014369, 2.017951)	(2.016286, 2.013933, 2.013260)

5.2 Numerical example of Convergence analysis for relaxed extragradient method and variational inequality problem

The following example is given for supporting Theorem 3.6 in Chapter 3.

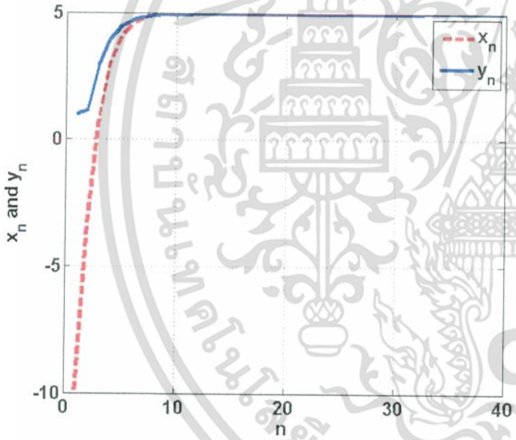
Example 5.5. Let $\mathbb{R}$ be the set of real numbers. Let D, D_1, D_2 be a mapping from $[-50, 50]$ to $\mathbb{R}$ defined by $Dx = \frac{2x-10}{3}, D_1x = \frac{x-5}{2}$ and $D_2x = \frac{3x-15}{4}$, for all $x \in [-50, 50]$. Let mapping $f : \mathbb{R} \to \mathbb{R}$ be defined by $fx = \frac{5x}{9}$, for every $x \in \mathbb{R}$. For $k = 1, 2, ..., N$, let $c_k = \frac{2}{3^k} + \frac{1}{N3^N}$ and let the mapping $A : \mathbb{R} \to \mathbb{R}$ be defined by $A_kx = \frac{kx}{5}$, for every $x \in \mathbb{R}$. Let $x_1 \in \mathbb{R}$ and $\{x_n\}$ be generated by (3.34) where $\lambda = 1.5, \lambda_1 = 0.5, \lambda_2 = 0.5, \alpha = 1, \gamma = 0.05$, $\alpha_n = \frac{2}{5n}$ and $\beta_n = \frac{5n-1}{9n}$. By the definition of D, D_1, D_2, A and f, we have $5 \in F(G) \cap VI(C, D)$. Then, from Theorem 3.6, the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to 5. We can rewritten (3.34) as follow:

$$\begin{aligned} x_{n+1} &= \left(\tfrac{3n+1}{8n}\right) x_n + \left(\tfrac{5n-1}{8n}\right) P_{[-50,50]}(I - (1.5)D)y_n, \\ y_n &= \tfrac{0.1}{5n} f(x_n) + \left(I - \left(\tfrac{2}{5n}\right)\bar{A}\right) Gx_n. \end{aligned} \tag{5.5}$$

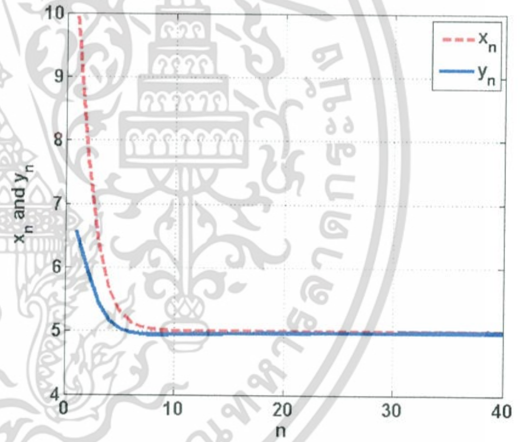
The following table and figure shows the values of the sequence $\{x_n\}$ and $\{y_n\}$ of iterative (5.5), where $x_1 = -10$, $x_1 = 10$ and $n = N = 40$.

Table 5.5: The values of $\{x_n\}$ and $\{y_n\}$ with different initial value x_1

	$x_1 = -10$		$x_1 = 10$	
n	x_n	y_n	x_n	y_n
1	−10.000000	0.988889	10.000000	6.573611
2	−3.333333	1.156481	7.777778	5.967168
3	0.833333	2.928086	6.388889	5.448663
4	2.993827	3.946134	5.668724	5.169807
⋮	⋮	⋮	⋮	⋮
20	4.999993	4.972774	5.000002	4.972779
⋮	⋮	⋮	⋮	⋮
37	5.000000	4.985285	5.000000	4.985285
38	5.000000	4.985673	5.000000	4.985673
39	5.000000	4.986040	5.000000	4.986040
40	5.000000	4.986389	5.000000	4.986389



(a) $x_1 = -10$



(b) $x_1 = 10$

Figure 5.5: The convergence of the sequence $\{x_n\}$ and $\{y_n\}$ with different initial value x_1 and $n = N = 40$.

5.3 Numerical examples of the modified generalized system of variational inequalities and application

The following two examples are given for supporting Theorem 3.12 in Chapter 3 and Theorem 4.10 in Chapter 4, respectively.

Example 5.6. Let $\mathbb{R}$ be the set of real numbers and D, D_1, D_2, D_3 be a mapping from $[0, 20]$ to $\mathbb{R}$ defined by $D = \frac{x-1}{3}, D_1 = \frac{x-1}{4}, D_2 = \frac{x-1}{5}$, and $D_3 = \frac{x-1}{6}$, respectively. Let T be a mapping from $[0, 20]$ into itself defined by $Tx = \frac{x+2}{3}, \forall x \in [0, 20]$. For $k = 1, 2, ..., \overline{N}$, let $c_k = \frac{5}{6^k} + \frac{1}{\overline{N}6^{\overline{N}}}$ and let the mapping $A : \mathbb{R} \to \mathbb{R}$ be defined by $A_k = \frac{kx}{4}$, for every $x \in \mathbb{R}$. Let mapping $f : \mathbb{R} \to \mathbb{R}$ be defined by $fx = \frac{3x}{5}$, for every $x \in \mathbb{R}$. Let $x_1 \in \mathbb{R}$ and $\{x_n\}$ generated by (3.58) where $a = 0.5, \lambda = 1, \lambda_1 = 0.5, \lambda_2 = 1, \lambda_3 = 1.5, \alpha = 1, \gamma = 0.095$, $\alpha_n = \frac{1}{4n}, \beta_n^1 = \frac{3n-1}{12n}, \beta_n^2 = \frac{5n-1}{12n}$, and $\beta_n^3 = \frac{4n+2}{12n}$. By the definition of D, D_1, D_2, D_3, A, f, and T, we have $1 \in F(T) \cap F(G) \cap VI(C, D)$. From Theorem 3.12, we can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to 1. We can rewritten (3.58) as follow:

$$\begin{aligned} x_{n+1} &= \left(\tfrac{3n-1}{12n}\right) x_n + \left(\tfrac{5n-1}{12n}\right) Tx_n + \left(\tfrac{4n+2}{12n}\right) P_C(I-D)y_n, \\ y_n &= \tfrac{0.095}{4n} f(x_n) + \left(I - \left(\tfrac{1}{4n}\right) \overline{A}\right) Gx_n. \end{aligned} \tag{5.6}$$

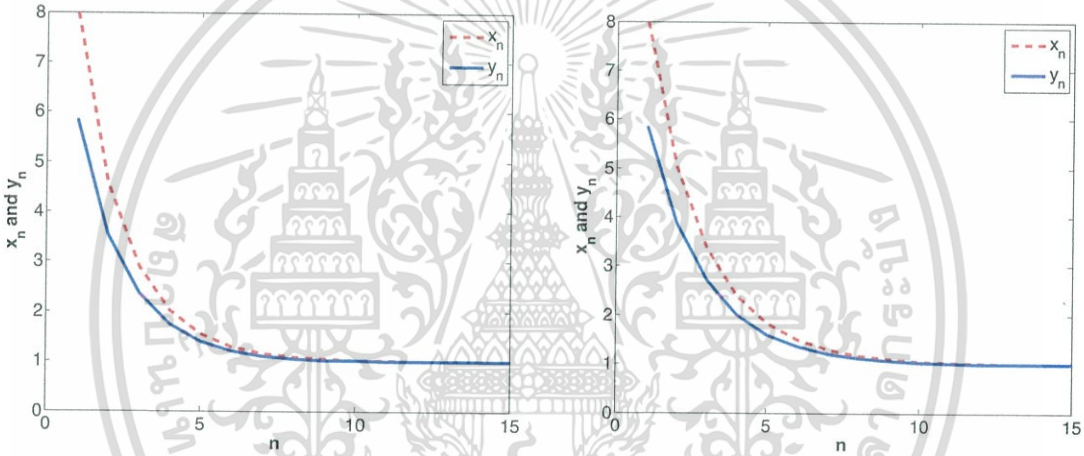
Example 5.7. In this example, we use the same mappings and parameters as in Example 5.6 except a following mapping $\overline{f}$. Let the mapping $\overline{f} : [0, 20] \to \mathbb{R}$ be defined by $\overline{f}(x) = \frac{x^2-2x}{14}$, with $\nabla\overline{f}x = \frac{x-1}{7}$, for all $x \in [0, 20]$. By the definition of $\nabla\overline{f}, D_1, D_2, D_3, A, f$, and T, we have $1 \in F(T) \cap F(G) \cap \Gamma_{\overline{f}}$. From Theorem 4.10, we can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to 1. We can rewritten (4.15) as follow:

$$\begin{aligned} x_{n+1} &= \left(\tfrac{3n-1}{12n}\right) x_n + \left(\tfrac{5n-1}{12n}\right) Tx_n + \left(\tfrac{4n+2}{12n}\right) P_C(I-\nabla\overline{f})y_n, \\ y_n &= \tfrac{0.095}{4n} f(x_n) + \left(I - \left(\tfrac{1}{4n}\right) \overline{A}\right) Gx_n. \end{aligned} \tag{5.7}$$

The following table and figure shows the values of the sequence $\{x_n\}$ and $\{y_n\}$ of iterative (5.6) and (5.7), where $x_1 = 8$ and $n = \overline{N} = 15$.

Table 5.6: The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = 8$ and $n = N = \overline{N} = 15$ of the iterative (5.6) and (5.7).

	Iterative (5.6)		Iterative (5.7)	
n	x_n	y_n	x_n	y_n
1	8.000000	5.854781	8.000000	5.854781
2	4.562705	3.545405	5.025065	3.879684
3	2.894625	2.362649	3.370147	2.709735
4	2.019907	1.732779	2.403852	2.014352
⋮	⋮	⋮	⋮	⋮
8	1.086276	1.056126	1.172564	1.119855
⋮	⋮	⋮	⋮	⋮
12	1.005057	0.998681	1.018323	1.008502
13	1.001615	0.996523	1.009486	1.002352
14	0.999812	0.995521	1.004306	0.998850
15	0.998897	0.995133	1.001300	0.996913



(a) Iterative (5.6).

(b) Iterative (5.7).

Figure 5.6: The convergence of $\{x_n\}$ and $\{y_n\}$ with initial value $x_1 = 8$ and $n = N = \overline{N} = 15$.

Chapter 6

Conclusions

In this final chapter, we summarize all main theorems, corollaries and applications obtained in this thesis.

6.1 Convergence theorems for the split general system of variational inequalities problem and a nonexpansive mapping

(1) Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A, B : C \to H_1$ be a, b-inverse strongly monotone mappings with $d = \min\{a, b\}$, respectively. Let $\overline{A}, \overline{B} : Q \to H_2$ be $\overline{a}, \overline{b}$-inverse strongly monotone mappings with $\overline{d} = \min\{\overline{a}, \overline{b}\}$, respectively. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Define the mapping $G_C : C \to C$ by $G_C(x) = P_C(I - \lambda A)P_C(I - \mu B)x$, for all $x \in C$ and define the mapping $G_Q : Q \to Q$ by $G_Q(\widehat{x}) = P_Q(I - \alpha\bar{A})P_Q(I - \gamma\bar{B})\widehat{x}$, for all $\widehat{x} \in Q$. Define $G : C \to C$ by $G(x) = G_C(x - \eta D^*(I - G_Q)Dx)$ for all $x \in C$. Let $T : C \to C$ be a nonexpansive mapping. Assume $\Im = F(G) \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by

$$y_n = G_C(x_n - \eta D^*(I - G_Q)Dx_n),$$
$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Ty_n,$$

where $\{\alpha_n\} \subset [0, 1]$, $\lambda, \mu \in (0, 2d)$, $\alpha, \gamma \in (0, 2\overline{d})$, $\eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum\limits_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_\Im u$ which $(x_0, y_0) \in \Omega^{A,B}_{\overline{A},\overline{B}}$, where $y_0 = P_C(I - \mu B)x_0$ and $\overline{y_0} = P_Q\left(I - \gamma\bar{B}\right)\overline{x_0}$ with $\overline{x_0} = Dx_0$ and $\overline{y_0} = Dy_0$.

(2) Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A, B : C \to H_1$ be a, b-inverse strongly monotone mappings with $d = \min\{a, b\}$, respectively. Let $\overline{A}, \overline{B} : Q \to H_2$ be $\overline{a}, \overline{b}$-inverse strongly monotone mappings with $\overline{d} = \min\{\overline{a}, \overline{b}\}$, respectively. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Define the mapping $G_C : C \to C$ by $G_C(x) = P_C(I - \lambda A)P_C(I - \mu B)x$, for all $x \in C$ and define the mapping $G_Q : Q \to Q$ by $G_Q(\widehat{x}) = P_Q(I - \alpha\bar{A})P_Q(I - \gamma\bar{B})\widehat{x}$, for all $\widehat{x} \in Q$. Define $G : C \to C$ by $G(x) = G_C(x - \eta D^*(I - G_Q)Dx)$ for all $x \in C$. Assume $\Im = F(G) \neq \emptyset$. For given

$u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by

$$y_n = G_C(x_n - \eta D^*(I - G_Q)Dx_n),$$
$$x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n,$$

where $\{\alpha_n\} \subset [0,1],\ \lambda, \mu \in (0, 2d),\ \alpha, \gamma \in (0, 2\bar{d}),\ \eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Im}u$ which $(x_0, y_0) \in \Omega^{A,B}_{\bar{A},\bar{B}}$, where $y_0 = P_C(I - \mu B)x_0$ and $\overline{y_0} = P_Q\left(I - \gamma \bar{B}\right)\overline{x_0}$ with $\overline{x_0} = Dx_0$ and $\overline{y_0} = Dy_0$.

(3) Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Let $T : C \to C$ be a nonexpansive mapping. Assume $\Im = \Gamma \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by

$$y_n = P_C(x_n - \eta D^*(I - P_Q)Dx_n),$$
$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Ty_n,$$

where $\{\alpha_n\} \subset [0,1],\ \eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Im}u$ which (x_0, Dx_0) is the solution of the split feasibility problem (SFP).

(4) Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $f : H_1 \to \mathbb{R}$ be a continuous differentiable function defined by (4.7), with the gradient ∇f. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Let $T : C \to C$ be a nonexpansive mapping. Assume $\Im = \Gamma \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by

$$y_n = P_C(x_n - \eta \nabla f x_n),$$
$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Ty_n,$$

where $\{\alpha_n\} \subset [0,1],\ \eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Im}u$ which x_0 is the solution of the minimization problem (4.8), that is, $x_0 \in \Gamma_f$.

(5) Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : C \to H_1$ be a-inverse strongly monotone mapping. Let $\overline{A} : Q \to H_2$ be $\overline{a}$-inverse strongly monotone mapping. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Define the mapping $G_C : C \to C$ by $G_C(x) = P_C(I - \lambda A)x$, for all $x \in C$ and define the mapping $G_Q : Q \to Q$ by $G_Q(\widehat{x}) = P_Q(I - \alpha\bar{A})\widehat{x}$, for all $\widehat{x} \in Q$. Define $G : C \to C$ by $G(x) = G_C(x - \eta D^*(I - G_Q)Dx)$ for all $x \in C$. Let $T : C \to C$ be a nonexpansive mapping. Assume $\Im = \Phi \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by

$$y_n = G_C(x_n - \eta D^*(I - G_Q)Dx_n),$$
$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Ty_n,$$

where $\{\alpha_n\} \subset [0, 1]$, $\eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum\limits_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Im}u$.

6.2 Convergence theorems for the general system of variational inequalities problem

(1) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, D_1, D_2 : C \to H$ be d, d_1, d_2-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \lambda_1 D_1) P_C(I - \lambda_2 D_2)x$, for all $x \in C$ and $a \in [0, 1)$. Let f be an α-contraction mapping on H. For $k = 1, 2, ..., N$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{N} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,N}\gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Suppose that $\Im = F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C(I - \lambda D)y_n,$$
$$y_n = \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) Gx_n,$$

where $\{\alpha_n\} \subset [0, 1]$, $\lambda \in (0, 2d)$, $\lambda_1, \lambda_2 \in \left(0, 2\overline{d}\right)$ with $\overline{d} = \min\{d_1, d_2\}$. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n \leq c < 1$;

(iii) $0 < c_k < 1$ and $\sum_{k=1}^{N} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0) is a solution of (1.11) where $y_0 = P_C(x_0 - \lambda_2 D_2 x_0)$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, D_1 : C \to H$ be d, d_1-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \lambda_1 D_1) P_C(I - \lambda_2 D_1) x$, for all $x \in C$ and $a \in [0, 1)$. Let f be an α-contraction mapping on H. For $k = 1, 2, ..., N$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{N} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,N} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Suppose that $\Im = F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C(I - \lambda D)y_n,$$
$$y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n \overline{A})\, Gx_n,$$

where $\{\alpha_n\} \subset [0, 1]$, $\lambda \in (0, 2d)$, $\lambda_1, \lambda_2 \in (0, 2d_1)$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n \leq c < 1$;

(iii) $0 < c_k < 1$ and $\sum_{k=1}^{N} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0) is a solution of (1.12), where $y_0 = P_C(x_0 - \lambda_2 D_1 x_0)$.

6.3 Convergence theorems for the modified generalized system of variational inequalities and a nonexpansive mapping

(1) Let C be a nonempty closed convex subset of a real Hilbert space H and let $D, D_1, D_2, D_3 : C \to H$ be d, d_1, d_2, d_3-inverse strongly monotone mappings, respectively. Define the mapping G as in Lemma 3.9 and $a \in [0, 1)$. For $k = 1, 2, ..., \overline{N}$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,\overline{N}} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Let $T : C \to C$ be a nonexpansive mapping with $\Omega = F(T) \cap F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$x_{n+1} = \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D)y_n,$$
$$y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n \overline{A})\, Gx_n,$$

where f is α-contraction mapping on C, $\{\alpha_n\} \subset [0,1], \lambda \in (0,2d),\ \lambda_1, \lambda_2, \lambda_3 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2, d_3\}$ and $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n^i \leq c < 1$ for all $i = 1,2,3$;

(iii) $0 < c_k < 1$ and $\sum\limits_{k=1}^{\overline{N}} c_k = 1$;

(iv) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum\limits_{n=1}^{\infty} |\beta_{n+1}^i - \beta_n^i| < \infty$, for all $i = 1,2,3$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (3.55) where $y_0 = P_C(I - \lambda_2 D_2)(ax_0 + (1-a)z_0)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H and let $D, D_1, D_2, : C \to H$ be d, d_1, d_2-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \lambda_1 D_1)(P_C(I - \lambda_2 D_2)x)$, for all $x \in C, \lambda_1, \lambda_2 > 0$. For $k = 1, 2, ..., \overline{N}$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,\overline{N}} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Let T be a nonexpansive mapping with $\Omega = F(T) \cap F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$x_{n+1} = \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n,$$
$$y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n \overline{A})\, G x_n,$$

where f is α-contraction mapping on H, $\{\alpha_n\} \subset [0,1], \lambda \in (0,2d),\ \lambda_1, \lambda_2 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2\}$ and $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n^i \leq c < 1$ for all $i = 1,2,3$;

(iii) $0 < c_k < 1$ and $\sum\limits_{k=1}^{\overline{N}} c_k = 1$;

(iv) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum\limits_{n=1}^{\infty} |\beta_{n+1}^i - \beta_n^i| < \infty$, for all $i = 1,2,3$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (1.11) where $y_0 = P_C(I - \lambda_2 D_2)(z_0)$.

(3) Let C and Q be nonempty closed convex subsets of real Hilbert space H_1, H_2, respectively and let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3- inverse strongly monotone mapping, respectively. Define the mapping G as in Lemma 3.9 and $a \in [0,1)$. For $k = 1, 2, ..., \overline{N}$, define $\overline{A} : H_1 \to H_1$ by $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H_1 with coefficient $\gamma_k > 0, \overline{\gamma} =$

$\min_{k=1,2,\ldots,\overline{N}}\gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Let $T : C \to C$ be a nonexpansive mapping with $\Omega = F(T) \cap F(G) \cap \Gamma \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$x_{n+1} = \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda A^*(I - P_Q)A)y_n,$$
$$y_n = \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n,$$

where f is α-contraction mapping on H_1, $\{\alpha_n\} \subset [0,1], \lambda \in (0, \frac{2}{L})$ with L being the spectral radius of the operator A^*A, $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2, d_3\}$ and $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n^i \leq c < 1$ for all $i = 1, 2, 3$;

(iii) $0 < c_k < 1$ and $\sum\limits_{k=1}^{\overline{N}} c_k = 1$;

(iv) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum\limits_{n=1}^{\infty} \left|\beta_{n+1}^i - \beta_n^i\right| < \infty$, for all $i = 1, 2, 3$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (3.55) where $y_0 = P_C(I - \lambda_2 D_2)\left(a x_0 + (1-a) z_0\right)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$.

(4) Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2, D_3 : C \to H$ be d_1, d_2, d_3- inverse strongly monotone mapping, respectively. Let $\overline{f} : C \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla\overline{f}$ is $\frac{1}{L_{\overline{f}}}$-inverse strongly monotone and continuous with $L_{\overline{f}} > 0$. Define the mapping G as in Lemma 3.9 and $a \in [0,1)$. For $k = 1, 2, \ldots, \overline{N}$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,\ldots,\overline{N}}\gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Let $T : C \to C$ be a nonexpansive mapping with $\Omega = F(T) \cap F(G) \cap \Gamma_f \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$x_{n+1} = \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda\nabla\overline{f})y_n,$$
$$y_n = \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n,$$

where f is α-contraction mapping on H, $\{\alpha_n\} \subset [0,1], \lambda \in (0, \frac{2}{L_{\overline{f}}})$, $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2, d_3\}$ and $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n^i \leq c < 1$ for all $i = 1, 2, 3$;

(iii) $0 < c_k < 1$ and $\sum\limits_{k=1}^{\overline{N}} c_k = 1$;

(iv) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum\limits_{n=1}^{\infty} \left|\beta_{n+1}^i - \beta_n^i\right| < \infty$, for all $i = 1, 2, 3$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (3.55) where $y_0 = P_C(I - \lambda_2 D_2)\left(a x_0 + (1-a) z_0\right)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$.

6.4 Examples and numerical results

6.4.1 Conclusions for Example 5.1, 5.2, 5.3 and 5.4

(1) Table 5.1 and Figure 5.1 show that the sequence $\{\mathsf{x}_n = (x_n^1, x_n^2, x_n^3)\}$ and $\{\mathsf{y}_n = (y_n^1, y_n^2, y_n^3)\}$ converges strongly to $\mathbf{2}$, where $\mathsf{x}_1 = (10, 10, 10)$, $N = 20$ and $\mathbf{2} = (2, 2, 2) \in F(G) \cap F(T)$.

(2) Table 5.2 and Figure 5.2 show that the sequence $\{\mathsf{x}_n = (x_n^1, x_n^2, x_n^3)\}$ and $\{\mathsf{y}_n = (y_n^1, y_n^2, y_n^3)\}$ converges strongly to $\mathbf{2}$, where $\mathsf{x}_1 = (10, 10, 10)$, $N = 40$ and $\mathbf{2} = (2, 2, 2) \in \Gamma \cap F(T)$.

(3) Table 5.3 and Figure 5.3 show that the sequence $\{\mathsf{x}_n = (x_n^1, x_n^2, x_n^3)\}$ and $\{\mathsf{y}_n = (y_n^1, y_n^2, y_n^3)\}$ converges strongly to $\mathbf{2}$, where $\mathsf{x}_1 = (10, 10, 10)$, $N = 40$ and $\mathbf{2} = (2, 2, 2) \in \Phi \cap F(T)$.

(4) Table 5.4 and Figure 5.4 show that the sequence $\{\mathsf{x}_n = (x_n^1, x_n^2, x_n^3)\}$ and $\{\mathsf{y}_n = (y_n^1, y_n^2, y_n^3)\}$ converges strongly to $\mathbf{2}$, where $\mathsf{x}_1 = (10, 10, 10)$, $N = 40$ and $\mathbf{2} = (2, 2, 2) \in \Gamma \cap F(T)$.

(5) Theorem 3.4, 4.1, 4.7 and 4.11 guarantee the convergence of sequence $\{\mathsf{x}_n = (x_n^1, x_n^2, x_n^3)\}$ and $\{\mathsf{y}_n = (y_n^1, y_n^2, y_n^3)\}$ in Example 5.1, 5.2, 5.3 and 5.4, respectively.

(6) The iteration (5.1), in Example 5.1, converges faster than the iteration (5.2), (5.3), and (5.4) in Example 5.2, 5.3 and 5.4, respectively.

6.4.2 Conclusions for Example 5.5

(1) Table 5.5 and Figure 5.5 show that the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to 5, where $x_1 = 10$ and -10, $N = 40$ and $5 \in F(G) \cap VI(C, D)$.

(2) Theorem 3.6 guarantees the convergence of sequence $\{x_n\}$ and $\{y_n\}$ in Example 5.5.

6.4.3 Conclusions for Example 5.6 and 5.7

(1) Table 5.6 and Figure 5.6, (a), show that the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to 1, where $x_1 = 8$, $N = 15$ and $1 \in F(T) \cap F(G) \cap VI(C, D)$.

(2) Table 5.6 and Figure 5.6, (b), show that the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to 1, where $x_1 = 8$, $N = 15$ and $1 \in F(T) \cap F(G) \cap \Phi_f$.

(3) Theorem 3.12 and 4.10 guarantee the convergence of sequence $\{x_n\}$ and $\{y_n\}$ in Example 5.6 and 5.7, respectively.

(4) The iteration (5.6), in Example 5.6, is converging slightly faster than the iteration (5.7) in Example 5.7.

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Appendix

Appendix A

The research papers

Received: 4 June 2018
DOI: 10.1002/mma.5240

RESEARCH ARTICLE

WILEY

Algorithm method for solving the split general system of variational inequalities problem and fixed point problem of nonexpansive mapping with application

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MSC Classification: 47H09; 47H10

The purpose of this research is to introduce the split general system of variational inequalities problem, by using the concept of the general system of variational inequalities problem in Ceng et al. Then, we establish and prove a strong convergence theorem for finding a common element of the set of solutions of this problem and the set of fixed points of a nonexpansive mapping for the first time in this research. By using our main theorem, strong convergence theorems of the split feasibility problem, the split variational inequality problem, and the minimization problem are obtained, respectively. In order to show the potential of our main result, we give three numerical examples of these classical problems introduced by previous studies, with the numerical example of our main theorem to compare the rate of convergence.

KEYWORDS
fixed point, nonexpansive mappings, the split general system of variational inequalities problem

1 | INTRODUCTION

Throughout this article, let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\| \, . \, \|$. Let C be a nonempty closed convex subset of H. Let $A : H \to H$ be a strongly positive linear bounded operator if there exists a constant $\alpha > 0$ with the property

$$\langle Ax, x \rangle \geq \alpha \|x\|^2,$$

for all $x \in H$.

A mapping $A : C \to H$ is called γ-inverse strongly monotone if there exists a positive real number $\gamma > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \gamma \|Ax - Ay\|^2,$$

for all $x, y \in C$.

Let $T : C \to C$ be a nonlinear mapping. We denote the set of fixed points of a mapping T by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$. The fixed point problem can be applied and used in many disciplines such as the neural networks, engineering, and the vibration of messes attached to strings or nets, see Haykin[1] and Cheng.[2]

A mapping $T : C \to C$ is called nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. Moreover, it is also known that if T is a nonexpansive mapping of H into itself, we have

$$\langle Ty - Tx, (I-T)x - (I-T)y \rangle \le \frac{1}{2}\|(I-T)x - (I-T)y\|^2,$$

for all $x, y \in H$.

In 1967, Halpern[3] introduced the following iteration scheme for a nonexpansive mapping T, which is referred as the *Halpern iteration*,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \tag{1}$$

for each $n \ge 1$ and $x_0 = u \in C$ where $\{\alpha_n\} \subset (0,1)$. Wittmann,[4] in 1992, proved that sequence $\{x_n\}$ converges strongly to $P_{F(T)}u$, where $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, where $P_{F(T)}$ is the metric projection onto $F(T)$.

The *equilibrium problem* is to find a point $x \in C$

$$\phi(x, y) \ge 0, \ \forall y \in C, \tag{2}$$

where $\phi : C \times C \to \mathbb{R}$ is bifunction. The set of all solutions of the equilibrium problem is denoted by $EP(\phi)$.

The *variational inequality problem* (VIP) is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \ge 0, \tag{3}$$

for all $v \in C$. The set of all solutions of the variational inequality is denoted by $VI(C, A)$. In 1964, the variational inequality was introduced by Lions and Stampacchia.[5] It is well known that the variational inequality covers various disciplines such as finance, physic, optimization, and mechanics, see literature.[6-8]

In 2008, Ceng et al[9] considered *the general system of variational inequalities problem*, which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda A y^* + x^* - y^*, x - x^* \rangle \ge 0, \ \forall x \in C, \\ &\langle \mu B x^* + y^* - x^*, x - y^* \rangle \ge 0, \ \forall x \in C, \end{aligned} \tag{4}$$

where $A, B : C \to H$ are two mappings and $\lambda, \mu > 0$ are two constants. In particular, if we put $A = B$, then the problem (4) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda A y^* + x^* - y^*, x - x^* \rangle \ge 0, \ \forall x \in C, \\ &\langle \mu A x^* + y^* - x^*, x - y^* \rangle \ge 0, \ \forall x \in C, \end{aligned} \tag{5}$$

which is introduced by Verma,[10] in 1999, and is called *the new system of variational inequalities*. Further, if we add up the requirement that $x^* = y^*$, then the problem (5) reduces to the variational inequality $VI(C, A)$.

In 2009, Kumam and Kumam[11] proved strong convergence theorem for finding the common element of the solution of the general system of variational inequalities problem (4), the fixed point of nonexpansive mapping S, and the solution of the equilibrium problem. They defined $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ in the following way:

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \ge 0, \ \forall y \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \ \forall n \in \mathbb{N}, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in [0, 1], and proved that $\{x_n\}$ converges strongly to $z = P_{F(S)\cap\Omega\cap EP(\phi)} f(z)$ and (z, y) is a solution of problem (4), where $y = P_C(z - \mu B z)$, under some suitable conditions on $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{r_n\}$, and bifunction ϕ. Furthermore, the general system of variational inequalities problem has been studied and developed in many literatures, see previous studies.[12-21]

Let H_1, H_2 be two real Hilbert spaces. Let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively.

In 1994, Censor and Elfving[22] introduced the *split feasibility problem* (SFP) in the finite-dimensional Hilbert spaces, which is to find a point x such that

$$x \in C \ \text{and} \ Dx \in Q, \tag{6}$$

where $D : H_1 \to H_2$ is a bounded linear operator. The set of all solutions of (6) is denoted by Γ. The SFP is useful in various disciplines such as signal processing, image reconstruction, and computer tomograph, see other works.[23-25] The SFP has been studied by many researchers, see previous works.[23,24,26] Later, Byrne[23,27] proposed the CQ algorithm for solving the SFP. Recently, in 2009, Censor and Segal[28] introduced the *split common fixed point problem* (SCFP), which is to find $x \in Fix(S)$ such that $Dx \in Fix(T)$, where $S : C \to C$ and $T : Q \to Q$ are two operators and $D : H_1 \to H_2$ is a bounded linear operator. The SCFP is a generalization of SFP, that is, if we put $S \equiv T \equiv I$, then the SCFP reduces to the SFP. Recently, in 2012, Censor et al[29] proposed the *split variational inequality problem* (SVIP). Obviously, the SFP is a special case of the SVIP.

Inspired and motivated by Ceng et al[9] and the concept of the SFP, we introduced *the split general system of variational inequalities problem* (SGSV), which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{7}$$

and find $(\overline{x^*} = Dx^*, \overline{y^*} = Dy^*) \in Q \times Q$ such that

$$\begin{aligned} &\left\langle \alpha \overline{A}\overline{y^*} + \overline{x^*} - \overline{y^*}, \bar{x} - \overline{x^*} \right\rangle \geq 0, \quad \forall \bar{x} \in Q, \\ &\left\langle \gamma \overline{B}\overline{x^*} + \overline{y^*} - \overline{x^*}, \bar{x} - \overline{y^*} \right\rangle \geq 0, \quad \forall \bar{x} \in Q, \end{aligned} \tag{8}$$

where $A, B : C \to H_1$ and $\overline{A}, \overline{B} : Q \to H_2$ are four different mappings, $\lambda, \mu, \alpha, \gamma > 0$, and $D : H_1 \to H_2$ is a bounded linear operator. The set of all solutions of (7) and (8) are denoted by $\Omega_{A,B}$ and $\Omega_{\overline{A},\overline{B}}$, respectively. The set of all solutions of the SGSV is denoted by $\Omega^{A,B}_{\overline{A},\overline{B}}$, that is, $\Omega^{A,B}_{\overline{A},\overline{B}} = \{(x^*, y^*) \in \Omega_{A,B} \; : \; (\overline{x^*}, \overline{y^*}) \in \Omega_{\overline{A},\overline{B}}\}$, where $\overline{x^*} = Dx^*$ and $\overline{y^*} = Dy^*$. Moreover, if we put $A \equiv B \equiv \overline{A} \equiv \overline{B} \equiv 0$, $x^* = y^*$ and $\overline{x^*} = \overline{y^*}$ in (7) and (8), we obtain the SGSV is reduced to the SFP.

Now, we give the following example to support the SGSV.

Example 1. Let $\mathbb{R}$ be the set of real numbers and A, B be mappings from $[-20, 20]$ to $\mathbb{R}$ defined by $Ax = \frac{2x}{3}$ and $Bx = x - 3$, respectively. Let $\overline{A}, \overline{B}$ be mappings from $[-10, 10]$ to $\mathbb{R}$ defined by $\overline{A}x = x - 1$ and $\overline{B}x = 2x - 3$, respectively. Let D be a mapping from $\mathbb{R}$ to $\mathbb{R}$ defined by $Dx = \frac{x}{2}$. We choose $\lambda = \frac{1}{2}, \mu = 1, \alpha = 1, \gamma = \frac{1}{2}$. Then, we have $(2, 3) \in \Omega^{A,B}_{\overline{A},\overline{B}}$.

In this paper, motivated by Ceng et al[9] and the modified Halpern iteration, we introduced a method for solving the SGSV. Then, we prove a strong convergence theorem for finding a common element of the set of fixed points of a nonexpansive mapping and the set of the solutions of the SGSV under some suitable conditions. Moreover, in application, we apply our main theorem to prove strong convergence theorems for finding solutions of the SFP, the SVIP, and the minimization problem. In the last section, we give numerical examples of our main theorem and applications to support our research, and we get that a numerical example of our main theorem converges faster than another examples.

2 | PRELIMINARIES

Let C be a nonempty closed and convex subset of a real Hilbert space H. We will use the notations '$\to$' and '$\rightharpoonup$' to denote the strong and weak convergence, respectively. It is well known that each Hilbert space H satisfies Opial's condition,[30] that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H$ with $x \neq y$.

Recall that the nearest point projection of H onto C is denoted by $P_C x$ for all $x \in H$, that is,

$$\|x - P_C y\| \leq \|x - y\|,$$

for all $y \in C$. Such an operator P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, that is,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle,$$

for all $x, y \in H$. Furthermore, this immediately implies that

$$\|(x-y)-(P_C x - P_C y)\|^2 \le \|x-y\|^2 - \|P_C x - P_C y\|^2,$$

for all $x, y \in H$.

We shall need to use the following lemmas to prove our main theorem.

Lemma 1. (Takahashi[31]) *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality*

$$\langle x-y, y-z \rangle \ge 0,$$

for all $z \in C$.

Lemma 2. (Xu[32]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1-\alpha_n)s_n + \beta_n, \qquad \forall n \ge 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions

$$(i)\quad \{\alpha_n\} \subset [0,1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$(ii)\quad \limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \le 0 \text{ or } \sum_{n=1}^{\infty} |\beta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 3. (Osilike and Isiogugu[33]) *In a real Hilbert spaces H, the following results hold:*

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y \rangle$ for all $x, y \in H$.
(ii) For all $x, y \in H$ and $\alpha \in [0,1]$,

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Lemma 4. (Browder[34]) *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E, and $S : C \to C$ be a nonexpansive mapping with a fixed point. Then $I - S$ is demiclosed at zero.*

Lemma 5. *Let C, Q be nonempty subsets of H_1, H_2, respectively. Let $A, B : C \to H_1$ be a, b-inverse strongly monotone with $\lambda, \mu \in (0, 2\overline{d})$ where $\overline{d} = \min\{a, b\}$. Let $\overline{A}, \overline{B} : Q \to H_2$ be $\overline{a}, \overline{b}$-inverse strongly monotone with $\alpha, \gamma \in (0, 2\widehat{d})$ where $\widehat{d} = \min\left\{\overline{a}, \overline{b}\right\}$. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator D^*D. Define $G_C : C \to C$ by $G_C(x) = P_C(I - \lambda A)P_C(I - \mu B)x$, for all $x \in C$, and define $G_Q : Q \to Q$ by $G_Q(\hat{x}) = P_Q(I - \alpha\bar{A})P_Q(I - \gamma\bar{B})\hat{x}$, for all $\widehat{x} \in Q$. Assume $\Omega^{A,B}_{\overline{A},\overline{B}} = \{(x^*, y^*) \in \Omega_{A,B} \ \ (\overline{x^*}, \overline{y^*}) \in \Omega_{\overline{A},\overline{B}}\} \ne \emptyset$. Then the following are equivalent:*

$$(i)\quad (x^*, y^*) \in \Omega^{A,B}_{\overline{A},\overline{B}},$$

$$(ii)\quad x^* = G_C(x^* - \eta D^*(I - G_Q)Dx^*).$$

where $y^* = P_C(I - \mu B)x^*$ and $\overline{y^*} = P_Q(I - \gamma\overline{B})\overline{x^*}$ with $\overline{x^*} = Dx^*$ and $\overline{y^*} = Dy^*$.

Proof. Let the following conditions hold. (*i*) ⇒ (*ii*) Let $(x^*, y^*) \in \Omega^{A,B}_{\overline{A},\overline{B}}$. We have $(x^*, y^*) \in \Omega_{A,B}$ and $(\overline{x^*}, \overline{y^*}) \in \Omega_{\overline{A},\overline{B}}$, with $\overline{x^*} = Dx^*$ and $\overline{y^*} = Dy^*$. Since $(x^*, y^*) \in \Omega_{A,B}$, we obtain

$$\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \ge 0, \quad \forall x \in C,$$
$$\langle \mu Bx^* + y^* - x^*, x - y^* \rangle \ge 0, \quad \forall x \in C.$$

Then, we have $x^* = P_C(I - \lambda A)y^*$ and $y^* = P_C(I - \mu B)x^*$, that is,

$$x^* = P_C(I - \lambda A)P_C(I - \mu B)x^* = G_C x^*. \tag{9}$$

Since $(\overline{x^*}, \overline{y^*}) \in \Omega_{\overline{A},\overline{B}}$, we obtain

$$\left\langle \alpha\overline{A}\overline{y^*} + \overline{x^*} - \overline{y^*}, \overline{x} - \overline{x^*} \right\rangle \geq 0, \quad \forall \overline{x} \in Q,$$
$$\left\langle \gamma\overline{B}\overline{x^*} + \overline{y^*} - \overline{x^*}, \overline{x} - \overline{y^*} \right\rangle \geq 0, \quad \forall \overline{x} \in Q.$$

Then, we have $\overline{x^*} = P_Q\left(I - \alpha\overline{A}\right)\overline{y^*}$ and $\overline{y^*} = P_Q\left(I - \gamma\overline{B}\right)\overline{x^*}$, that is,

$$\overline{x^*} = P_Q\left(I - \alpha\overline{A}\right)P_Q\left(I - \gamma\overline{B}\right)\overline{x^*} = G_Q\overline{x^*} = G_Q D x^*. \tag{10}$$

It implies that $x^* = G_C(x^* - \eta D^*(I - G_Q)Dx^*)$.

$(ii) \Rightarrow (i)$ Let $x^* = G_C(x^* - \eta D^*(I - G_Q)Dx^*)$ and $(w, w^*) \in \Omega^{A,B}_{\overline{A},\overline{B}}$, that is, $(w, w^*) \in \Omega_{A,B}$ and $\left(\overline{w}, \overline{w^*}\right) \in \Omega_{\overline{A},\overline{B}}$, where $w^* = P_C(I - \mu B)w$, $\overline{w} = Dw$, and $\overline{w^*} = Dw^* = P_Q\left(I - \gamma\overline{B}\right)\overline{w}$. Since A is a-inverse strongly monotone mapping, we have

$$\begin{aligned}
\|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle - \lambda^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\lambda a\|Ax - Ay\|^2 - \lambda^2\|Ax - Ay\|^2 \\
&= \|x - y\|^2 - \lambda(2a - \lambda)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - \lambda(2d - \lambda)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

It implies that

$$\|(I - \lambda A)x - (I - \lambda A)y\| \leq \|x - y\|. \tag{11}$$

Hence, $(I - \lambda A)$ is a nonexpansive mapping. By using the same method as (11), we have that $(I - \mu B)$, $(I - \alpha\overline{A})$, and $(I - \gamma\overline{B})$ are nonexpansive mappings. Then, we obtain that $P_C(I - \lambda A)$, $P_C(I - \mu B)$, $P_Q(I - \alpha\overline{A})$, and $P_Q(I - \gamma\overline{B})$ are nonexpansive mappings. Since $P_C(I - \lambda A)$ and $P_C(I - \mu B)$ are nonexpansive mappings, we obtain that G_C is a nonexpansive mapping. Since $P_Q(I - \alpha\overline{A})$ and $P_Q(I - \gamma\overline{B})$ are nonexpansive mappings, we obtain that G_Q is nonexpansive mapping. From $(w, w^*) \in \Omega^{A,B}_{\overline{A},\overline{B}}$ and $(i) \Rightarrow (ii)$, we have $w = G_C(w - \eta D^*(I - G_Q)Dw)$. By using the same method as (10), we have $Dw = G_Q Dw$. Then, we have

$$\begin{aligned}
\|x^* - w\|^2 &= \left\|G_C(x^* - \eta D^*(I - G_Q)Dx^*) - G_C(w - \eta D^*(I - G_Q)Dw)\right\|^2 \\
&\leq \left\|(x^* - \eta D^*(I - G_Q)Dx^*) - (w - \eta D^*(I - G_Q)Dw)\right\|^2 \\
&= \left\|x^* - w - \eta(D^*(I - G_Q)Dx^* - D^*(I - G_Q)Dw)\right\|^2 \\
&= \left\|x^* - w - \eta(D^*(I - G_Q)Dx^*)\right\|^2 \\
&= \|x^* - w\|^2 - 2\eta\left\langle x^* - w, D^*(I - G_Q)Dx^*\right\rangle + \eta^2\left\|D^*(I - G_Q)Dx^*\right\|^2 \\
&= \|x^* - w\|^2 - 2\eta\left\langle D(x^* - w), (I - G_Q)Dx^*\right\rangle + \eta^2\left\|D^*(I - G_Q)Dx^*\right\|^2 \\
&= \|x^* - w\|^2 + 2\eta\left\langle Dw - G_QDx^* + G_QDx^* - Dx^*, (I - G_Q)Dx^*\right\rangle + \eta^2\left\|D^*(I - G_Q)Dx^*\right\|^2 \\
&= \|x^* - w\|^2 + 2\eta\left(\left\langle Dw - G_QDx^*, (I - G_Q)Dx^*\right\rangle\right. \\
&\quad \left. + \left\langle G_QDx^* - Dx^*, (I - G_Q)Dx^*\right\rangle\right) + \eta^2\left\|D^*(I - G_Q)Dx^*\right\|^2 \\
&= \|x^* - w\|^2 + 2\eta\left(\left\langle Dw - G_QDx^*, (I - G_Q)Dx^*\right\rangle - \left\|(I - G_Q)Dx^*\right\|^2\right) \\
&\quad + \eta^2\left\langle D^*(I - G_Q)Dx^*, D^*(I - G_Q)Dx^*\right\rangle \\
&\leq \|x^* - w\|^2 + 2\eta\left(\frac{1}{2}\left\|(I - G_Q)Dx^*\right\|^2 - \left\|(I - G_Q)Dx^*\right\|^2\right) + \eta^2 L\left\|(I - G_Q)Dx^*\right\|^2 \\
&= \|x^* - w\|^2 - \eta\left\|(I - G_Q)Dx^*\right\|^2 + \eta^2 L\left\|(I - G_Q)Dx^*\right\|^2 \\
&= \|x^* - w\|^2 - \eta(1 - \eta L)\left\|(I - G_Q)Dx^*\right\|^2.
\end{aligned} \tag{12}$$

From (12), we have

$$Dx^* \in F(G_Q), \tag{13}$$

that is, $\overline{x^*} = G_Q\overline{x^*} = P_Q\left(I - \alpha\bar{A}\right)P_Q\left(I - \gamma\bar{B}\right)\overline{x^*}$. It implies that $\overline{x^*} = P_Q\left(I - \alpha\overline{A}\right)\overline{y^*}$, where $\overline{y^*} = P_Q\left(I - \gamma\overline{B}\right)\overline{x^*}$. By the property of P_C, we obtain

$$\left\langle \alpha\overline{A}\overline{y^*} + \overline{x^*} - \overline{y^*}, \bar{x} - \overline{x^*}\right\rangle \geq 0, \quad \forall \bar{x} \in Q,$$
$$\left\langle \gamma\overline{B}\overline{x^*} + \overline{y^*} - \overline{x^*}, \bar{x} - \overline{y^*}\right\rangle \geq 0, \quad \forall \bar{x} \in Q,$$

that is,

$$(\overline{x^*}, \overline{y^*}) \in \Omega_{\overline{A},\overline{B}}. \tag{14}$$

From the definition of x^* and (13), we have

$$x^* = G_C(x^* - \eta D^*(I - G_Q)Dx^*) = G_C(x^*).$$

That is, $x^* \in F(G_C)$. Then, we have $x^* = G_Cx^* = P_C(I - \lambda A)P_C(I - \mu B)x^*$. It implies that $x^* = P_C(I - \lambda A)y^*$, where $y^* = P_C(I - \mu B)x^*$. By property of P_C, we obtain

$$\langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C,$$
$$\langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, \quad \forall x \in C,$$

that is,

$$(x^*, y^*) \in \Omega_{A,B}. \tag{15}$$

From (14) and (15), we have $(x^*, y^*) \in \Omega^{A,B}_{\overline{A},\overline{B}}$. □

We give the following example to support Lemma 5.

Example 2. Let $\mathbb{R}$ be the set of real number and A, B be mappings from $[-20, 20]$ to $\mathbb{R}$ defined by $Ax = \frac{x-3}{5}$ and $Bx = \frac{x-3}{9}$ for all $x \in [-20, 20]$, respectively. Let $\overline{A}, \overline{B}$ be mappings from $[-10, 10]$ to $\mathbb{R}$ defined by $\overline{A}x = \frac{x-1}{3}$ and $\overline{B}x = \frac{x-1}{7}$ for all $x \in [-10, 10]$, respectively. Let D be a mapping from $\mathbb{R}$ to $\mathbb{R}$ defined by $Dx = \frac{x}{3}$, for all $x \in \mathbb{R}$. It is easy to see that $A, B, \overline{A}, \overline{B}$ are 1-inverse strongly monotone with $\lambda, \mu, \alpha, \gamma \in (0, 2)$. Then, we choose $\lambda = 1, \mu = 1.5, \alpha = 0.5, \gamma = 1$. Since $Dx = \frac{x}{3}$, we have $D^*x = \frac{x}{3}$ is an adjoint of D. From D and D^*, we obtain $L = \frac{1}{9}$ and L is the spectral radius of the operator D^*D. Then, we can choose $\eta = 2$. Define $G_{[-20,20]} : [-20, 20] \to [-20, 20]$ by

$$G_{[-20,20]}(x) = P_{[-20,20]}(I - A)\left(P_{[-20,20]}(x - 1.5Bx)\right), \ \forall x \in [-20, 20],$$

and define $G_{[-10,10]} : [-10, 10] \to [-10, 10]$ by

$$G_{[-10,10]}(\hat{x}) = P_{[-10,10]}(I - 0.5\bar{A})\left(P_{[-10,10]}(\hat{x} - \bar{B}\hat{x})\right), \ \forall \hat{x} \in [-10, 10].$$

It is easy to see that $(3, 3) \in \Omega^{A,B}_{\overline{A},\overline{B}}$. By Lemma 5, we have $3 = G_{[-20,20]}(3 - 2D^*(I - G_{[-10,10]})D(3))$, where $y^* = P_{[-20,20]}(3 - 1.5B(3))$ and $\overline{y^*} = P_{[-10,10]}(\overline{x^*} - \bar{B}(\overline{x^*}))$ with $\overline{x^*} = D(3)$ and $\overline{y^*} = D(3)$.

3 | MAIN RESULT

In this section, we prove the strong convergence theorem, by using Lemma 5 as an important tool for finding the solution of the SGSV and the fixed point of nonexpansive mapping.

Theorem 1. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A, B : C \to H_1$ be a, b-inverse strongly monotone mappings with $d = \min\{a, b\}$, respectively. Let $\overline{A}, \overline{B} : Q \to H_2$ be $\overline{a}, \overline{b}$-inverse strongly monotone mappings with $\overline{d} = \min\{\overline{a}, \overline{b}\}$, respectively. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Define the mapping $G_C : C \to C$ by $G_C(x) = P_C(I - \lambda A)P_C(I - \mu B)x$, for all $x \in C$, and define the mapping $G_Q : Q \to Q$ by $G_Q(\hat{x}) = P_Q(I - \alpha\bar{A})P_Q(I - \gamma\bar{B})\hat{x}$, for all $\hat{x} \in Q$. Define $G : C \to C$ by $G(x) = G_C(x - \eta D^*(I - G_Q)Dx)$ for all $x \in C$. Let $T : C \to C$ be a nonexpansive mapping. Assume $\mathfrak{F} = F(G) \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by*

$$y_n = G_C(x_n - \eta D^*(I - G_Q)Dx_n),$$
$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Ty_n, \tag{16}$$

where $\{\alpha_n\} \subset [0,1]$, $\lambda, \mu \in (0, 2d)$, $\alpha, \gamma \in (0, 2\overline{d})$, $\eta \in (0, \frac{1}{L})$ *with L is the spectral radius of the operator* D^*D. *Suppose the following conditions hold:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ *converges strongly to* $x_0 = P_{\mathfrak{F}} u$, *which* $(x_0, y_0) \in \Omega^{A,B}_{\overline{A},\overline{B}}$, *where* $y_0 = P_C(I - \mu B)x_0$ *and* $\overline{y_0} = P_Q\left(I - \gamma \bar{B}\right)\overline{x_0}$ *with* $\overline{x_0} = Dx_0$ *and* $\overline{y_0} = Dy_0$.

Proof. Let $z \in \mathfrak{F}$. Then, we have $z = G(z) = G_C(z - \eta D^*(I - G_Q)Dz)$. By Lemma 5, we have $(z, z_0) \in \Omega^{A,B}_{\overline{A},\overline{B}}$, where $z_0 = P_C(I - \mu B)z$ and $\overline{z_0} = P_C(I - \gamma\overline{B})\overline{z}$ with $\overline{z} = Dz$ and $\overline{z_0} = Dz_0$. Since $(z, z_0) \in \Omega^{A,B}_{\overline{A},\overline{B}}$, we have $(z, z_0) \in \Omega_{A,B}$ and $(Dz, Dz_0) \in \Omega_{\overline{A},\overline{B}}$. Since $(Dz, Dz_0) \in \Omega_{\overline{A},\overline{B}}$, we obtain

$$\left\langle \alpha\overline{A}Dz_0 + Dz - Dz_0, \bar{x} - Dz \right\rangle \geq 0, \quad \forall \bar{x} \in Q,$$
$$\left\langle \gamma\overline{B}Dz + Dz_0 - Dz, \bar{x} - Dz_0 \right\rangle \geq 0, \quad \forall \bar{x} \in Q.$$

Then, we have $Dz = P_Q\left(I - \alpha\overline{A}\right)Dz_0$ and $Dz_0 = P_Q\left(I - \gamma\overline{B}\right)Dz$. It implies that

$$Dz = P_Q\left(I - \alpha\overline{A}\right)P_Q\left(I - \gamma\overline{B}\right)Dz = G_Q Dz. \tag{17}$$

Step 1. Show that $\{x_n\}$ is bounded. Applying (12) and the definition of $\{y_n\}$, we have

$$\|y_n - z\| \leq \|x_n - z\|. \tag{18}$$

From definition of $\{x_n\}$ and (18), we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n u + (1 - \alpha_n)Ty_n - z\| \\
&\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|Ty_n - z\| \\
&\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|y_n - z\| \\
&\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|x_n - z\| \\
&\leq \max\{\|u - z\|, \|x_1 - z\|\}.
\end{aligned}$$

By induction, we get that $\|x_n - z\| \leq \max\{\|u - z\|, \|x_1 - z\|\}$, for all $n \in \mathbb{N}$. It implies that the sequence $\{x_n\}$ is bounded and so is $\{y_n\}$.

Step 2. Show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. From the definition of $\{y_n\}$, we have

$$\begin{aligned}
\|y_n - y_{n-1}\|^2 &= \left\|G_C\left(x_n - \eta D^*\left(I - G_Q\right)Dx_n\right) - G_C\left(x_{n-1} - \eta D^*\left(I - G_Q Dx_{n-1}\right)\right)\right\|^2 \\
&\leq \left\|\left(x_n - \eta D^*\left(I - G_Q\right)Dx_n\right) - \left(x_{n-1} - \eta D^*\left(I - G_Q\right)Dx_{n-1}\right)\right\|^2 \\
&\leq \left\|(x_n - x_{n-1}) - \eta D^*\left(I - G_Q\right)Dx_n - \eta D^*\left(I - G_Q\right)Dx_{n-1}\right\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\eta\left\langle x_n - x_{n-1}, D^*\left(\left(I - G_Q\right)Dx_n - \left(I - G_Q\right)Dx_{n-1}\right\rangle\right. \\
&\quad + \eta^2\left\|D^*\left(\left(I - G_Q\right)Dx_n - \left(I - G_Q\right)Dx_{n-1}\right)\right\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 - 2\eta\left\langle Dx_n - G_Q Dx_n + G_Q Dx_n - G_Q Dx_{n-1}\right. \\
&\quad \left. + G_Q Dx_{n-1} - Dx_{n-1}, \left(I - G_Q\right)Dx_n - \left(I - G_Q\right)Dx_{n-1}\right\rangle \\
&\quad + \eta^2 L\left\|\left(I - G_Q\right)Dx_n - \left(I - G_Q\right)Dx_{n-1}\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - x_{n-1}\|^2 - 2\eta \left[\left\langle (I - G_Q) Dx_n - (I - G_Q) Dx_{n-1}, (I - G_Q) \right.\right. \\
&\quad \times Dx_n - (I - G_Q) Dx_{n-1} \rangle - \langle G_Q Dx_{n-1} - G_Q Dx_n, (I - G_Q) Dx_n \\
&\quad \left. - (I - G_Q) Dx_{n-1} \rangle \right] + \eta^2 L \left\| (I - G_Q) Dx_n - (I - G_Q) Dx_{n-1} \right\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 + 2\eta \left[- \left\| (I - G_Q) Dx_n - (I - G_Q) Dx_{n-1} \right\|^2 \right. \\
&\quad \left. + \langle G_Q Dx_{n-1} - G_Q Dx_n, (I - G_Q) Dx_n - (I - G_Q) Dx_{n-1} \rangle \right] \\
&\quad + \eta^2 L \left\| (I - G_Q) Dx_n - (I - G_Q) Dx_{n-1} \right\|^2 \\
&= \|x_n - x_{n-1}\|^2 + 2\eta \left[- \left\| (I - G_Q) Dx_n - (I - G_Q) Dx_{n-1} \right\|^2 \right. \\
&\quad \left. + \frac{1}{2} \left\| (I - G_Q) Dx_n - (I - G_Q) Dx_{n-1} \right\|^2 \right] \\
&\quad + \eta^2 L \left\| (I - G_Q) Dx_n - (I - G_Q) Dx_{n-1} \right\|^2 \\
&= \|x_n - x_{n-1}\|^2 - \eta \left\| (I - G_Q) Dx_n - (I - G_Q) Dx_{n-1} \right\|^2 \\
&\quad + \eta^2 L \left\| (I - G_Q) Dx_n - (I - G_Q) Dx_{n-1} \right\|^2 \\
&= \|x_n - x_{n-1}\|^2 - \eta(1 - \eta L) \left\| (I - G_Q) Dx_n - (I - G_Q) Dx_{n-1} \right\|^2 \\
&= \|x_n - x_{n-1}\|^2.
\end{aligned}
$$

This implies that

$$
\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\|. \tag{19}
$$

From the definition of x_n and (19), we obtain

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n u + (1 - \alpha_n) T y_n - (\alpha_{n-1} u + (1 - \alpha_{n-1}) T y_{n-1})\| \\
&\leq |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|T y_n - T y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|T y_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|T y_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|T y_{n-1}\|.
\end{aligned} \tag{20}
$$

From Lemma 2, conditions (i) and (ii), and (20), we have

$$
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{21}
$$

Applying (12) and the definition of $\{y_n\}$, we have

$$
\|y_n - z\|^2 \leq \|x_n - z\|^2 - \eta(1 - \eta L) \|(I - G_Q) Dx_n\|^2.
$$

From the definition of $\{x_n\}$,

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n u + (1 - \alpha_n) T y_n - z\|^2 \\
&\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|T y_n - z\|^2 \\
&\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|y_n - z\|^2 \\
&\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|x_n - z\|^2 - (1 - \alpha_n)\eta(1 - \eta L) \|(I - G_Q) Dx_n\|^2.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
(1 - \alpha_n)\eta(1 - \eta L) \|(I - G_Q) Dx_n\|^2 &\leq \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\leq \alpha_n \|u - z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|) \|x_{n+1} - x_n\|.
\end{aligned}
$$

By (21) and condition (i), we obtain

$$
\lim_{n \to \infty} \|(I - G_Q) Dx_n\| = 0. \tag{22}
$$

Step 3. Show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $\lim_{n\to\infty}\|Ty_n - y_n\| = 0$.

From the definition of x_n, we have

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n u + (1-\alpha_n)Ty_n - x_n \\ &= \alpha_n(u - x_n) + (1-\alpha_n)(Ty_n - x_n). \end{aligned} \tag{23}$$

Then, by (21), (23), and condition (i), we obtain

$$\lim_{n\to\infty}\|Ty_n - x_n\| = 0. \tag{24}$$

Putting $h_n = P_C(I - \mu B)(x_n - \eta D^*(I - G_Q)Dx_n)$ and $h^* = P_C(I - \mu B)(z - \eta D^*(I - G_Q)Dz)$, we can rewrite y_n by

$$y_n = P_C(I - \lambda A)h_n \quad ,\forall n \geq 1,$$

and $z = P_C(I - \lambda A)h^*$. By the definition of x_n, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n u + (1-\alpha_n)Ty_n - z\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\|Ty_n - z\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\|y_n - z\|^2 \\ &= \alpha_n\|u - z\|^2 + (1-\alpha_n)\|P_C(I - \lambda A)h_n - P_C(I - \lambda A)h^*\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\|(I - \lambda A)h_n - (I - \lambda A)h^*\|^2 \\ &= \alpha_n\|u - z\|^2 + (1-\alpha_n)\|(h_n - h^*) - \lambda(Ah_n - Ah^*)\|^2 \\ &= \alpha_n\|u - z\|^2 + (1-\alpha_n)\left[\|h_n - h^*\|^2 - 2\lambda\langle h_n - h^*, Ah_n - Ah^*\rangle + \lambda^2\|Ah_n - Ah^*\|^2\right] \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\left[\|h_n - h^*\|^2 - 2\lambda a\|Ah_n - Ah^*\|^2 + \lambda^2\|Ah_n - Ah^*\|^2\right] \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\left[\|h_n - h^*\|^2 - \lambda(2d - \lambda)\|Ah_n - Ah^*\|^2\right] \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\left[\|x_n - z\|^2 - \lambda(2d - \lambda)\|Ah_n - Ah^*\|^2\right]. \end{aligned}$$

It implies that

$$\begin{aligned} \lambda(1-\alpha_n)(2d - \lambda)\|Ah_n - Ah^*\|^2 &\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|)\|x_{n+1} - x_n\|. \end{aligned}$$

From (21) and condition (i), we have

$$\lim_{n\to\infty}\|Ah_n - Ah^*\| = 0. \tag{25}$$

Putting $k_n = x_n - \eta D^*(I - G_Q)Dx_n$ and $k^* = z - \eta D^*(I - G_Q)Dz$, we can rewrite y_n by

$$y_n = P_C(I - \lambda A)P_C(I - \mu B)k_n, \quad \forall n \geq 1,$$

and $z = P_C(I - \lambda A)P_C(I - \mu B)k^*$. From the definition of x_n, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n u + (1-\alpha_n)Ty_n - z\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\|Ty_n - z\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\|y_n - z\|^2 \\ &= \alpha_n\|u - z\|^2 + (1-\alpha_n)\|P_C(I - \lambda A)P_C(I - \mu B)k_n - P_C(I - \lambda A)P_C(I - \mu B)k^*\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\|P_C(I - \mu B)k_n - P_C(I - \mu B)k^*\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\|(I - \mu B)k_n - (I - \mu B)k^*\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\|(k_n - k^*) - \mu(Bk_n - Bk^*)\|^2 \\ &= \alpha_n\|u - z\|^2 + (1-\alpha_n)\left[\|k_n - k^*\|^2 - 2\mu\langle k_n - k^*, Bk_n - Bk^*\rangle + \mu^2\|Bk_n - Bk^*\|^2\right] \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\left[\|k_n - k^*\|^2 - 2\mu b\|Bk_n - Bk^*\|^2 + \mu^2\|Bk_n - Bk^*\|^2\right] \\ &= \alpha_n\|u - z\|^2 + (1-\alpha_n)\left[\|k_n - k^*\|^2 - \mu(2d - \mu)\|Bk_n - Bk^*\|^2\right] \\ &\leq \alpha_n\|u - z\|^2 + (1-\alpha_n)\left[\|x_n - z\|^2 - \mu(2d - \mu)\|Bk_n - Bk^*\|^2\right]. \end{aligned}$$

It implies that

$$\mu(1-\alpha_n)(2d-\mu)\|Bk_n - Bk^*\|^2 \le \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1}-z\|^2$$
$$\le \alpha_n \|u-z\|^2 + (\|x_n - z\| - \|x_{n+1}-z\|)\|x_{n+1}-x_n\|.$$

From (21) and condition (i), we have

$$\lim_{n\to\infty} \|Bk_n - Bk^*\| = 0. \tag{26}$$

By the definition of y_n, we have

$$\begin{aligned}
\|y_n - z\|^2 &= \|P_C(I-\lambda A)h_n - P_C(I-\lambda A)h^*\|^2 \\
&\le \langle (h_n - \lambda Ah_n) - (h^* - \lambda Ah^*), y_n - z\rangle \\
&= \frac{1}{2}\left[\|(h_n - \lambda Ah_n) - (h^* - \lambda Ah^*)\|^2 + \|y_n - z\|^2 - \|(h_n - \lambda Ah_n) - (h^* - \lambda Ah^*) - (y_n - z)\|^2\right] \\
&\le \frac{1}{2}\left[\|h_n - h^*\|^2 + \|y_n - z\|^2 - \|(h_n - \lambda Ah_n) - (h^* - \lambda Ah^*) - (y_n - z)\|^2\right] \\
&= \frac{1}{2}\left[\|h_n - h^*\|^2 + \|y_n - z\|^2 - \|(h_n - y_n) - (h^* - z) - \lambda(Ah_n - Ah^*)\|^2\right] \\
&= \frac{1}{2}\left[\|h_n - h^*\|^2 + \|y_n - z\|^2 - \|(h_n - y_n) - (h^* - z)\|^2\right. \\
&\quad \left. + 2\lambda\langle (h_n - y_n) - (h^* - z), Ah_n - Ah^*\rangle - \lambda^2\|Ah_n - Ah^*\|^2\right] \\
&\le \frac{1}{2}\left[\|x_n - z\|^2 + \|y_n - z\|^2 - \|(h_n - y_n) - (h^* - z)\|^2\right. \\
&\quad \left. + 2\lambda\|(h_n - y_n) - (h^* - z)\|\,\|Ah_n - Ah^*\| - \lambda^2\|Ah_n - Ah^*\|^2\right].
\end{aligned}$$

This implies that

$$\|y_n - z\|^2 \le \|x_n - z\|^2 - \|(h_n - y_n) - (h^* - z)\|^2 + 2\lambda\|(h_n - y_n) - (h^* - z)\|\,\|Ah_n - Ah^*\|. \tag{27}$$

From (27) and the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n u + (1-\alpha_n)Ty_n - z\|^2 \\
&\le \alpha_n \|u - z\|^2 + (1-\alpha_n)\|Ty_n - z\|^2 \\
&\le \alpha_n \|u - z\|^2 + (1-\alpha_n)\|y_n - z\|^2 \\
&\le \alpha_n \|u - z\|^2 + (1-\alpha_n)\left[\|x_n - z\|^2 - \|(h_n - y_n) - (h^* - z)\|^2\right. \\
&\quad \left. + 2\lambda\|(h_n - y_n) - (h^* - z)\|\,\|Ah_n - Ah^*\|\right] \\
&\le \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - (1-\alpha_n)\|(h_n - y_n) - (h^* - z)\|^2 \\
&\quad + 2\lambda(1-\alpha_n)\|(h_n - y_n) - (h^* - z)\|\,\|Ah_n - Ah^*\|\big].
\end{aligned}$$

It implies that

$$\begin{aligned}
(1-\alpha_n)\|(h_n - y_n) - (h^* - z)\|^2 &\le \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\quad + 2\lambda(1-\alpha_n)\|(h_n - y_n) - (h^* - z)\|\,\|Ah_n - Ah^*\| \\
&\le \alpha_n \|u - z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|)\|x_n - x_{n+1}\| \\
&\quad + 2\lambda(1-\alpha_n)\|(h_n - y_n) - (h^* - z)\|\,\|Ah_n - Ah^*\|.
\end{aligned}$$

From (21), (25), and condition (i), we have

$$\lim_{n\to\infty} \|(h_n - y_n) - (h^* - z)\| = 0. \tag{28}$$

From (17), we have

$$\begin{aligned}\|(z-k^*)-(x_n-k_n)\|^2 &= \|z-(z-\eta D^*(I-G_Q)Dz)\\ &\quad -(x_n-(x_n-\eta D^*(I-G_Q)Dx_n))\|^2\\ &= \eta^2\|D^*(I-G_Q)Dx_n)\|^2\\ &\le \eta^2 L\|(I-G_Q)Dx_n)\|^2.\end{aligned}$$

From (22), we have

$$\lim_{n\to\infty}\|(z-k^*)-(x_n-k_n)\|=0. \tag{29}$$

Consider

$$\begin{aligned}\|(x_n-h_n)+(h^*-z)\|^2 &= \|(x_n-P_C(I-\mu B)k_n)+(P_C(I-\mu B)k^*-z)\|^2\\
&= \|x_n-\mu Bx_n-(I-\mu B)\eta D^*(I-G_Q)Dx_n+(I-\mu B)\eta D^*(I-G_Q)Dx_n+\mu Bx_n\\
&\quad -P_C(I-\mu B)k_n+P_C(I-\mu B)k^*-\mu Bz+(I-\mu B)\eta D^*(I-G_Q)Dz\\
&\quad -(I-\mu B)\eta D^*(I-G_Q)Dz+\mu Bz-z\|^2\\
&= \|(I-\mu B)k_n-(I-\mu B)k^*-\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]+\mu(Bk_n-Bk^*)\\
&\quad +\eta D^*(I-G_Q)Dx_n-\eta D^*(I-G_Q)Dz\|^2\\
&= \|(I-\mu B)k_n-(I-\mu B)k^*-\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]\\
&\quad +\mu(Bk_n-Bk^*)+\eta D^*(I-G_Q)Dx_n\|^2\\
&\le \|(I-\mu B)k_n-(I-\mu B)k^*-\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]+\mu(Bk_n-Bk^*)\|^2\\
&\quad +2\eta\left\langle (I-G_Q)Dx_n, D\left[(x_n-h_n)+(h^*-z)\right]\right\rangle\\
&\le \|(I-\mu B)k_n-(I-\mu B)k^*-\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]+\mu(Bk_n-Bk^*)\|^2\\
&\quad +2\eta\|(I-G_Q)Dx_n\|\left\|D\left[(x_n-h_n)+(h^*-z)\right]\right\|\\
&\le \|(I-\mu B)k_n-(I-\mu B)k^*-\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]\|^2\\
&\quad +2\mu\langle Bk_n-Bk^*,(I-\mu B)k_n-(I-\mu B)k^*\\
&\quad -\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]+\mu(Bk_n-Bk^*)\rangle\\
&\quad +2\eta\|(I-G_Q)Dx_n\|\left\|D\left[(x_n-h_n)+(h^*-z)\right]\right\|\\
&\le \|(I-\mu B)k_n-(I-\mu B)k^*\|^2-\|P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\|^2\\
&\quad +2\mu\|Bk_n-Bk^*\|\,\|(I-\mu B)k_n-(I-\mu B)k^*\\
&\quad -\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]+\mu(Bk_n-Bk^*)\|\\
&\quad +2\eta\|(I-G_Q)Dx_n\|\left\|D\left[(x_n-h_n)+(h^*-z)\right]\right\|\\
&= \|(I-\mu B)k_n-(I-\mu B)k^*\|^2-\|h_n-h^*\|^2+2\mu\|Bk_n-Bk^*\|\\
&\quad \times\|(I-\mu B)k_n-(I-\mu B)k^*-\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]\\
&\quad +\mu(Bk_n-Bk^*)\|+2\eta\|(I-G_Q)Dx_n\|\left\|D\left[(x_n-h_n)+(h^*-z)\right]\right\|\\
&\le \|(I-\mu B)k_n-(I-\mu B)k^*\|^2-\|P_C(I-\lambda A)h_n-P_C(I-\lambda A)h^*\|^2\\
&\quad +2\mu\|Bk_n-Bk^*\|\left\|(I-\mu B)k_n-(I-\mu B)k^*-\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]\right.\\
&\quad \left.+\mu(Bk_n-Bk^*)\right\|+2\eta\|(I-G_Q)Dx_n\|\left\|D\left[(x_n-h_n)+(h^*-z)\right]\right\|\\
&= \|(I-\mu B)k_n-(I-\mu B)k^*\|^2-\|y_n-z\|^2+2\mu\|Bk_n-Bk^*\|\,\|(I-\mu B)k_n-(I-\mu B)k^*\\
&\quad -\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]+\mu(Bk_n-Bk^*)\|+2\eta\|(I-G_Q)Dx_n\|\\
&\quad \times\left\|D\left[(x_n-h_n)+(h^*-z)\right]\right\|\end{aligned}$$

$$\leq \|(I-\mu B)k_n-(I-\mu B)k^*\|^2-\|Ty_n-Tz\|^2+2\mu\|Bk_n-Bk^*\|\|(I-\mu B)k_n-(I-\mu B)k^* - \left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]+\mu(Bk_n-Bk^*)\Big\|+2\eta\|(I-G_Q)Dx_n\|\left\|D\left[(x_n-h_n)+(h^*-z)\right]\right\|$$
$$\leq \|(I-\mu B)k_n-(I-\mu B)k^*-(Ty_n-z)\|(\|(I-\mu B)k_n-(I-\mu B)k^*\|+\|Ty_n-z\|) + 2\mu\|Bk_n-Bk^*\|\left\|(I-\mu B)k_n-(I-\mu B)k^*-\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right] +\mu(Bk_n-Bk^*)\right\|+2\eta\|(I-G_Q)Dx_n\|\left\|D\left[(x_n-h_n)+(h^*-z)\right]\right\|$$
$$= \|k_n-\mu Bk_n-k^*+\mu Bk^*-x_n+x_n-Ty_n+z\|(\|(I-\mu B)k_n-(I-\mu B)k^*\|+\|Ty_n-z\|) + 2\mu\|Bk_n-Bk^*\|\left\|(I-\mu B)k_n-(I-\mu B)k^*-\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right] +\mu(Bk_n-Bk^*)\right\|+2\eta\|(I-G_Q)Dx_n\|\left\|D\left[(x_n-h_n)+(h^*-z)\right]\right\|$$
$$= \|(x_n-Ty_n)+(z-k^*)-(x_n-k_n)-\mu(Bk_n-Bk^*)\|(\|(I-\mu B)k_n-(I-\mu B)k^*\|+\|Ty_n-z\|) + 2\mu\|Bk_n-Bk^*\|\|(I-\mu B)k_n-(I-\mu B)k^*-\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]+\mu(Bk_n-Bk^*)\| + 2\eta\|(I-G_Q)Dx_n\|\left\|D\left[(x_n-h_n)+(h^*-z)\right]\right\|$$
$$\leq (\|x_n-Ty_n\|+\|(z-k^*)-(x_n-k_n)\|+\mu\|Bk_n-Bk^*\|)(\|(I-\mu B)k_n-(I-\mu B)k^*\|+\|Ty_n-z\|) + 2\mu\|Bk_n-Bk^*\|\|(I-\mu B)k_n-(I-\mu B)k^*-\left[P_C(I-\mu B)k_n-P_C(I-\mu B)k^*\right]+\mu(Bk_n-Bk^*)\| + 2\eta\|(I-G_Q)Dx_n\|\left\|D\left[(x_n-h_n)+(h^*-z)\right]\right\|.$$

From (22), (24), (26), and (29), we obtain

$$\lim_{n\to\infty}\|(x_n-h_n)+(h^*-z)\|=0. \tag{30}$$

From (24), (28), (30), and

$$\|Ty_n-y_n\|=\|Ty_n-x_n+x_n-h_n+h_n-h^*+h^*-z+z-y_n\| \leq \|Ty_n-x_n\|+\|(x_n-h_n)+(h^*-z)\|+\|(h_n-y_n)-(h^*-z)\|,$$

we get

$$\lim_{n\to\infty}\|Ty_n-y_n\|=0. \tag{31}$$

From (24), (31), and

$$\|x_n-y_n\|\leq\|x_n-Ty_n\|+\|Ty_n-y_n\|,$$

we have

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{32}$$

Step 4. Show that $\limsup_{n\to\infty}\langle u-x_0,x_n-x_0\rangle\leq 0$ where $x_0=P_{\mathfrak{F}}u$.

To show this inequality, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-x_0,x_n-x_0\rangle=\lim_{j\to\infty}\left\langle u-x_0,x_{n_j}-x_0\right\rangle. \tag{33}$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_{n_j}\rightharpoonup q$ as $j\to\infty$, where $q\in C$. From (33) and $x_{n_j}\rightharpoonup q$ as $j\to\infty$, we have

$$\limsup_{n\to\infty}\langle u-x_0,x_n-x_0\rangle=\langle u-x_0,q-x_0\rangle. \tag{34}$$

From (32) and $x_{n_j}\rightharpoonup q$ as $j\to\infty$, we have $y_{n_j}\rightharpoonup q$ as $j\to\infty$. From (31) and Lemma 4, we obtain

$$q\in F(T). \tag{35}$$

Assume that $q \neq F(G)$. By Opial's condition and (32), we have

$$\begin{aligned}\liminf_{j\to\infty}\left\|x_{n_j}-q\right\| &< \liminf_{j\to\infty}\left\|x_{n_j}-G(q)\right\|\\ &= \liminf_{j\to\infty}\left\|x_{n_j}-y_{n_j}+y_{n_j}-G_C(q-\eta D^*(I-G_Q)Dq)\right\|\\ &= \liminf_{j\to\infty}\left\|x_{n_j}-y_{n_j}+G_C(x_{n_j}-\eta D^*(I-G_Q)Dx_{n_j})-G_C(q-\eta D^*(I-G_Q)Dq)\right\|\\ &\le \liminf_{j\to\infty}\left(\left\|x_{n_j}-y_{n_j}\right\|+\left\|x_{n_j}-q\right\|\right)\\ &= \liminf_{j\to\infty}\left\|x_{n_j}-q\right\|.\end{aligned}$$

This is a contradiction. Then, we have

$$q \in F(G). \tag{36}$$

From (35) and (36), we have

$$q \in \mathfrak{I}. \tag{37}$$

From (34) and (37), we obtain

$$\limsup_{n\to\infty}\langle u-x_0, x_n-x_0\rangle \le 0. \tag{38}$$

Step 5. Finally, we show that $\lim_{i\to\infty}\|x_n-x_0\|=0$, where $x_0 \in P_{\mathfrak{I}}u$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-x_0\|^2 &= \|\alpha_n u+(1-\alpha_n)Ty_n-x_0\|^2\\ &= \|\alpha_n(u-x_0)+(1-\alpha_n)(Ty_n-x_0)\|^2\\ &\le (1-\alpha_n)\|Ty_n-x_0\|^2+2\alpha_n\langle u-x_0,x_{n+1}-x_0\rangle\\ &\le (1-\alpha_n)\|y_n-x_0\|^2+2\alpha_n\langle u-x_0,x_{n+1}-x_0\rangle\\ &\le (1-\alpha_n)\|x_n-x_0\|^2+2\alpha_n\langle u-x_0,x_{n+1}-x_0\rangle.\end{aligned}$$

From (38), condition (i), and Lemma 2, we can conclude that the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\mathfrak{I}}u$. From Lemma 5, we get that $(x_0, y_0) \in \Omega_{\bar{A},\bar{B}}^{A,B}$, where $y_0 = P_C(I-\lambda B)x_0$ and $\overline{y_0} = P_Q(I-\gamma\bar{B})\overline{x_0}$ with $\overline{x_0} = D\overline{x_0}$ and $\overline{y_0} = D\overline{y_0}$. This completes the proof. □

Corollary 1. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A, B : C \to H_1$ be a, b-inverse strongly monotone mappings with $d = \min\{a,b\}$, respectively. Let $\bar{A}, \bar{B} : Q \to H_2$ be $\bar{a}, \bar{b}$-inverse strongly monotone mappings with $\bar{d} = \min\{\bar{a},\bar{b}\}$, respectively. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Define the mapping $G_C : C \to C$ by $G_C(x) = P_C(I-\lambda A)P_C(I-\mu B)x$, for all $x \in C$, and define the mapping $G_Q : Q \to Q$ by $G_Q(\hat{x}) = P_Q(I-\alpha\bar{A})P_Q(I-\gamma\bar{B})\hat{x}$, for all $\hat{x} \in Q$. Define $G : C \to C$ by $G(x) = G_C(x-\eta D^*(I-G_Q)Dx)$ for all $x \in C$. Assume $\mathfrak{I} = F(G) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by*

$$\begin{aligned}y_n &= G_C(x_n-\eta D^*(I-G_Q)Dx_n),\\ x_{n+1} &= \alpha_n u+(1-\alpha_n)y_n,\end{aligned} \tag{39}$$

where $\{\alpha_n\} \subset [0,1]$, $\lambda, \mu \in (0,2d)$, $\alpha, \gamma \in (0,2\bar{d})$, $\eta \in (0,\frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$.

(ii) $\sum_{i=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\mathfrak{I}}u$, which $(x_0, y_0) \in \Omega_{\bar{A},\bar{B}}^{A,B}$, where $y_0 = P_C(I-\mu B)x_0$ and $\overline{y_0} = P_Q\left(I-\gamma\bar{B}\right)\overline{x_0}$ with $\overline{x_0} = Dx_0$ and $\overline{y_0} = Dy_0$.

Proof. Put $T = I$. Then, by Theorem 1, we obtain the desired conclusion. □

Remark 1. As shown in this section, we introduced a modified problem of the general system of variational inequalities problem and prove the strong convergence to solve the proposed problem and the fixed point problem of nonexpansive mapping. In recent years, many authors have studied, modified, and extended the general system of variational inequalities problem in many literatures, which are interesting researches and useful in this field, see literature.[11,35-38]

4 | APPLICATION

In this section, we utilize our main theorem to obtain Theorems 2, 3, and 4, which solve the SFP, the split variational inequality, and the minimization problem, respectively.

Let H_1 and H_2 be real Hilbert spaces, and let C, Q be nonempty closed convex subsets of H_1, H_2, respectively. Let $D : H_1 \to H_2$ be a bounded linear operator with the adjoint D^*.

4.1 | The split feasibility problem

Throughout this section, assume that the SFP is consistent, that is, Γ is nonempty. It is easy to see that $x \in C$ solves SFP if and only if it solves the fixed point equation

$$x = P_C(x - \eta D^*(I - P_Q)Dx), \tag{40}$$

for all $x \in C$, where P_C and P_Q are the orthogonal projections onto C and Q, respectively, and $\eta > 0$; see Xu[39] for more details. In fact, various results about the SFP have been widely studied, see other studies.[40-44] To solve (40), in 2002, Byrne[23] introduced the CQ algorithm for solving the SFP as follows: Let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = P_C(x_n - \eta D^*(I - P_Q)Dx_n), \ \forall n \in \mathbb{N}, \tag{41}$$

where $\eta \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator D^*D.

Theorem 2. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Let $T : C \to C$ be a nonexpansive mapping. Assume $\mathfrak{I} = \Gamma \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by*

$$\begin{aligned} y_n &= P_C(x_n - \eta D^*(I - P_Q)Dx_n), \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)Ty_n, \end{aligned} \tag{42}$$

where $\{\alpha_n\} \subset [0, 1]$, $\eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\mathfrak{I}}u$, which (x_0, Dx_0) is the solution of the SFP.

Proof. If we put $A \equiv B \equiv \overline{A} \equiv \overline{B} \equiv 0$, $x^* = y^*$, and $\overline{x^*} = \overline{y^*}$, in Theorem 1, we obtain the desired conclusion. □

Remark 2. If we take $T \equiv I$, in Theorem 2, then we obtain

$$\begin{aligned} y_n &= P_C(x_n - \eta D^*(I - P_Q)Dx_n), \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)y_n. \end{aligned} \tag{43}$$

From Theorem 2, we have that $\{x_n\}$ generated by (43) converges strongly to $x_0 = P_{\mathfrak{I}}u$, which is a solution of the SFP. Moreover, we have that corollary 3.7 of Xu[26] is a special case of our main theorem.

4.2 | The split variational inequality

In 2012, Censor et al[29] introduced SVIP, which is to find $\hat{x} \in C$ such that

$$\langle f_1\hat{x}, x - \hat{x} \rangle \geq 0, \ \forall x \in C, \tag{44}$$

and find $\hat{y} = D\hat{x} \in Q$ such that

$$\langle f_2\hat{y}, x - \hat{y}\rangle \geq 0,\ \forall y \in Q, \tag{45}$$

where $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings and $D : H_1 \to H_2$ is a bounded linear operator. The set of all solution of the SVIP is denoted by $\Phi = \{\hat{x} \in VI(C, f_1) : \hat{y} \in VI(C, f_1)\}$. The SVIP is reduced to the SFP if $f_1 \equiv f_2 \equiv 0$.

Theorem 3. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : C \to H_1$ be a-inverse strongly monotone mapping. Let $\overline{A} : Q \to H_2$ be $\overline{a}$-inverse strongly monotone mapping. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Define the mapping $G_C : C \to C$ by $G_C(x) = P_C(I - \lambda A)x$, for all $x \in C$, and define the mapping $G_Q : Q \to Q$ by $G_Q(\hat{x}) = P_Q(I - \alpha\bar{A})\hat{x}$, for all $\hat{x} \in Q$. Define $G : C \to C$ by $G(x) = G_C(x - \eta D^*(I - G_Q)Dx)$ for all $x \in C$. Let $T : C \to C$ be a nonexpansive mapping. Assume $\Im = \Phi \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by*

$$\begin{aligned} y_n &= G_C(x_n - \eta D^*(I - G_Q)Dx_n), \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)Ty_n, \end{aligned} \tag{46}$$

where $\{\alpha_n\} \subset [0, 1]$, $\eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Im}u$.

Proof. Putting $B \equiv 0$, $\overline{B} \equiv 0$, $x^* = y^*$ and $\overline{x^*} = \overline{y^*}$, in Theorem 1, then we obtain the desired conclusion. □

4.3 | The minimization problem

Let C be a closed convex subset of H_1. The *constrained minimization problem* is to find $x^* \in C$ such that

$$f(x^*) = \min_{x \in C} f(x), \tag{47}$$

where $f : H_1 \to \mathbb{R}$ is a continuous differentiable function. The set of all solutions of (47) is denoted by Γ_f. It is obvious that $x^* \in \Gamma$ if and only if $x^* \in C$ and $Dx^* - P_Q Dx^* = 0$. Define the proximity function f by

$$f(x) = \frac{1}{2}\|Dx - P_Q Dx\|^2. \tag{48}$$

Considering the constrained minimization problem, we obtain

$$f(x^*) = \min_{x \in C} f(x) = \min_{x \in C} \frac{1}{2}\|Dx - P_Q Dx\|^2. \tag{49}$$

Then, we have $x^* \in \Gamma$ if and only if x^* solves the minimization problem (49) with the minimization equal to 0. Furthermore, by the definition of the proximity function f, we have

$$\nabla f(x) = D^*(I - P_Q)Dx,$$

where ∇f is a gradient of f.

From (40), we get that $x^* \in C$ solves SFP (6) if and only if it solves the fixed point equation

$$x^* = P_C(I - \eta D^*(I - P_Q)D)x^* = P_C(I - \eta\nabla f)x^*. \tag{50}$$

Proposition 1. (Ceng et al[42]) *Given $x^* \in H_1$, the following statements are equivalent.*

1. $x^* \in \Gamma$.

2. *x^* solves Equation 50.*
3. *x^* solves the VIP of finding $x^* \in C$ such that*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0,\ \forall x \in C,$$

where $\nabla f(x) = D^*(I - P_Q)Dx$ and D^* is the adjoint of D.

Remark 3. Moreover, $x^* \in \gamma$ if and only if $x^* \in \gamma_f$.

Proof. "Necessity." Suppose $z \in \gamma$, that is, $z \in C$ and $Dz \in Q$. Consider

$$\begin{aligned} f(z) &= \frac{1}{2}\|Dz - P_Q Dz\|^2 \\ &= 0 \\ &\leq f(x), \end{aligned}$$

for all $x \in C$. Hence, $z \in \Gamma_f$.

"Sufficiency." Suppose $z \in \Gamma_f$; we have

$$f(z) \leq f(x), \tag{51}$$

for all $x \in C$.

Since $\Gamma \neq \emptyset$, there exists $\bar{x} \in C$ and $D\bar{x} \in Q$.

From (51), we obtain

$$\frac{1}{2}\|Dz - P_Q Dz\|^2 \leq \frac{1}{2}\|Dx - P_Q Dx\|^2,\ \forall x \in C. \tag{52}$$

Since $\bar{x} \in C$, we have

$$\frac{1}{2}\|Dz - P_Q Dz\|^2 \leq \frac{1}{2}\|D\bar{x} - P_Q D\bar{x}\|^2. \tag{53}$$

Since $D\bar{x} \in Q$, we get that $D\bar{x} = P_Q D\bar{x}$. From (53), we have

$$\frac{1}{2}\|Dz - P_Q Dz\|^2 = 0.$$

This implies that $Dz = P_Q Dz$, that is, $Dz \in Q$. Hence, $z \in \Gamma$. □

Theorem 4. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $f : H_1 \to \mathbb{R}$ be a continuous differentiable function defined by (48), with the gradient ∇f. Let $D : H_1 \to H_2$ be a bounded linear operator with adjoint D^*. Let $T : C \to C$ be a nonexpansive mapping. Assume $\mathfrak{I} = \Gamma \cap F(T) \neq \emptyset$. For given $u, x_1 \in C$ and let the sequence $\{x_n\}$ be generated by*

$$\begin{aligned} y_n &= P_C(x_n - \eta \nabla f x_n), \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n) T y_n, \end{aligned} \tag{54}$$

where $\{\alpha_n\} \subset [0, 1]$, $\eta \in (0, \frac{1}{L})$ with L is the spectral radius of the operator D^*D. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\mathfrak{I}} u$, which x_0 is the solution of the minimization problem (49), that is, $x_0 \in \Gamma_f$.

Proof. Let $x, y \in H_1$. First, we show that ∇f is $\frac{1}{L}$-inverse strongly monotone. Consider

$$\begin{aligned}\|\nabla f(x) - \nabla f(y)\|^2 &= \|D^*(I-P_Q)Dx - D^*(I-P_Q)Dy\|^2\\ &= \left\langle D^*(I-P_Q)Dx - D^*(I-P_Q)Dy, D^*(I-P_Q)Dx - D^*(I-P_Q)Dy\right\rangle\\ &= \left\langle (I-P_Q)Dx - (I-P_Q)Dy, DD^*(I-P_Q)Dx - DD^*(I-P_Q)Dy\right\rangle\\ &\le L\|(I-P_Q)Dx - (I-P_Q)Dy\|^2.\end{aligned}$$

From the property of P_C, we have

$$\begin{aligned}\|(I-P_Q)Dx - (I-P_Q)Dy\|^2 &= \left\langle (I-P_Q)Dx - (I-P_Q)Dy, (I-P_Q)Dx - (I-P_Q)Dy\right\rangle\\ &= \left\langle (I-P_Q)Dx - (I-P_Q)Dy,\ Dx - Dy\right\rangle\\ &\quad - \left\langle (I-P_Q)Dx - (I-P_Q)Dy,\ P_QDx - P_QDy\right\rangle\\ &= \left\langle D^*(I-P_Q)Dx - D^*(I-P_Q)Dy, x - y\right\rangle\\ &\quad - \left\langle (I-P_Q)Dx - (I-P_Q)Dy,\ P_QDx - P_QDy\right\rangle\\ &= \left\langle D^*(I-P_Q)Dx - D^*(I-P_Q)Dy, x - y\right\rangle\\ &\quad - \left\langle (I-P_Q)Dx,\ P_QDx - P_QDy\right\rangle + \left\langle (I-P_Q)Dy, P_QDx - P_QDy\right\rangle\\ &\le \left\langle D^*(I-P_Q)Dx - D^*(I-P_Q)Dy, x - y\right\rangle.\end{aligned}$$

Since $\nabla f(x) = D^*(I - P_Q)Dx$, we obtain

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \ge \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2.$$

Thus, ∇f is $\frac{1}{L}$-inverse strongly monotone. From Proposition 1, Theorem 2, and Remark 3, we have the desired conclusion. □

Remark 4. If we take $T \equiv I$, in Theorem 4, then we obtain

$$\begin{aligned} y_n &= P_C(x_n - \eta\nabla f x_n),\\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n. \end{aligned} \tag{55}$$

From Theorem 4, we have that the sequence $\{x_n\}$ generated by (55) converges strongly to $x_0 = P_{\mho}u$, which is a solution of the minimization problem (49). Moreover, we have that theorem 5.2 of Lopez et al[45] is a special case of our main theorem.

5 | EXAMPLE AND NUMERICAL RESULT

Example 3. Let $\mathbb{R}$ be the set of real numbers, and let $<.,.>: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be an inner product defined by $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}\cdot\mathbf{y} = x_1\cdot y_1 + x_2\cdot y_2 + x_3\cdot y_3$, for all $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. Let $H_1 = H_2 = \mathbb{R}^3$, $C = [-50, 50]\times[-50, 50]\times[-50, 50]$, and $Q = [-60, 60]\times[-60, 60]\times[-60, 60]$. Let A, B be mappings from C to $\mathbb{R}^3$ defined by

$$A\mathbf{x} = \left(\frac{x_1-2}{3}, \frac{2x_2-4}{5}, \frac{x_3-2}{4}\right), \forall \mathbf{x} \in C,$$

and

$$B\mathbf{x} = \left(x_1 - 2, \frac{x_2-2}{3}, \frac{x_3-2}{5}\right), \forall \mathbf{x} \in C,$$

and let $\overline{A}$, $\overline{B}$ be mappings from Q to $\mathbb{R}^3$ defined by

$$\overline{A}\mathbf{x} = \left(\frac{x_1}{5}, \frac{x_2}{2}, \frac{2x_3}{3}\right), \forall \mathbf{x} \in Q \ \ and \ \ \overline{B}\mathbf{x} = \left(\frac{x_1}{3}, \frac{x_2}{5}, \frac{x_3}{7}\right), \forall \mathbf{x} \in Q.$$

It is easy to see that $A, B, \overline{A}, \overline{B}$ are 1-inverse strongly monotone with $\lambda, \mu, \alpha, \gamma \in (0, 2)$. Then, we can choose $\lambda = 0.5, \mu = 1, \alpha = 1, \gamma = 0.5$. Let the mapping $D : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$D\mathbf{x} = (2x_1 - x_2 - x_3, x_1 - 2x_2 + x_3, x_1 + x_2 - 2x_3), \forall \mathbf{x} \in \mathbb{R}^3,$$

and $D^* : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$D^*\mathbf{x} = (2x_1 + x_2 + x_3, -x_1 - 2x_2 + x_3, -x_1 + x_2 - 2x_3), \forall \mathbf{x} \in \mathbb{R}^3.$$

Then, the spectral radius of the operator D^*D is 9, and also, we can choose $\eta = 0.1$. Define $G_C : C \to C$ by

$$G_C(\mathbf{x}) = P_C(I - 0.5A)P_C(I - B)\mathbf{x}, \ \forall \mathbf{x} \in C,$$

and define $G_Q : Q \to Q$ by

$$G_Q(\mathbf{x}) = P_Q(I - \bar{A})P_Q(I - 0.5\bar{B})\mathbf{x}, \ \forall \mathbf{x} \in Q.$$

Let the mapping $G : C \to C$ be defined by

$$G(x) = G_C(x_n - 0.11D^*(I - G_Q)Dx_n), \forall \mathbf{x} \in C.$$

Let T be a mapping from C into itself defined by $T\mathbf{x} = \left(\frac{x_1+2}{2}, \frac{x_2+4}{3}, \frac{3x_1+2}{4}\right)$, $\forall \mathbf{x} \in C$. Let $\mathbf{x}_1 = (x_1^1, x_1^2, x_1^3) \in C$ and $\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$ be generated by (16), where $\alpha_n = \frac{1}{8n}$, for every $n \in \mathbb{N}$. Put $\mathbf{u} = \mathbf{5}$, where $\mathbf{5} = (5, 5, 5)$. By the definition of $A, B, \overline{A}, \overline{B}, D$ and T, we have $\{2\} \in F(G) \cap F(T)$, where $\mathbf{2} = (2, 2, 2)$. From Theorem 1, we can conclude that the sequence $\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$ converges strongly to $\mathbf{2}$. For every $n \in \mathbb{N}$, we can rewrite (16) as follows:

$$\begin{aligned} \mathbf{y}_n &= G_C\left(\mathbf{x}_n - 0.1D^*\left(I - G_Q\right)D\mathbf{x}_n\right), \\ \mathbf{x}_{n+1} &= \frac{1}{8n}\mathbf{u} + \left(\frac{8n-1}{8n}\right)T\mathbf{y}_n. \end{aligned} \tag{56}$$

By using the algorithm (56), the following table and figure show the values of sequences $\mathbf{x}_n$ and $\mathbf{y}_n$, where $\mathbf{x}_1 = \mathbf{10} = (10, 10, 10)$ and $n = N = 20$.

Example 4. In this example, we use the same mappings and parameters as in Example 3 except the following mappings G_C and G_Q. Define G_C by $G_C(x) = P_C(x), \forall x \in C$, and define G_Q by $G_Q(\hat{x}) = P_Q(\hat{x}), \ \forall \hat{x} \in Q$. By the definition of D and T, we have $\mathbf{2} \in \Gamma \cap F(T)$, where $\mathbf{2} = (2, 2, 2)$. From Theorem 2, we can conclude that the sequence $\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$ converges strongly to $\mathbf{2}$. For every $n \in \mathbb{N}$, we can rewrite (42) as follows:

$$\begin{aligned} \mathbf{y}_n &= P_C\left(\mathbf{x}_n - 0.1D^*\left(I - P_Q\right)D\mathbf{x}_n\right), \\ \mathbf{x}_{n+1} &= \frac{1}{8n}\mathbf{u} + \left(\frac{8n-1}{8n}\right)T\mathbf{y}_n. \end{aligned} \tag{57}$$

By using the algorithm (57), the following table and figure show the values of sequences $\mathbf{x}_n$ and $\mathbf{y}_n$, where $\mathbf{x}_1 = \mathbf{10} = (10, 10, 10)$ and $N = 40$.

Example 5. In this example, we use the same mappings and parameters as in Example 3 except the following mappings G_C and G_Q. Define G_C by $G_C(x) = P_C(I - 0.5A)x, \forall x \in C$, and define G_Q by $G_Q(\hat{x}) = P_Q(I - \overline{A})\hat{x}, \ \forall \hat{x} \in Q$. By the definition of D and T, we have $\mathbf{2} \in \Phi \cap F(T)$, where $\mathbf{2} = (2, 2, 2)$. From Theorem 3, we can conclude that the sequence $\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$ converges strongly to $\mathbf{2}$. For every $n \in \mathbb{N}$, we can rewrite (46) as follows:

$$\begin{aligned} \mathbf{y}_n &= G_C\left(\mathbf{x}_n - 0.1D^*\left(I - G_Q\right)D\mathbf{x}_n\right), \\ \mathbf{x}_{n+1} &= \frac{1}{8n}\mathbf{u} + \left(\frac{8n-1}{8n}\right)T\mathbf{y}_n. \end{aligned} \tag{58}$$

By using the algorithm (58), the following table and figure show the values of sequences $\mathbf{x}_n$ and $\mathbf{y}_n$, where $\mathbf{x}_1 = \mathbf{10} = (10, 10, 10)$ and $N = 40$.

Example 6. In this example, we use the same mappings and parameters as in Example 4 except the following mapping f. Let f be a mapping from $\mathbb{R}^3$ into $\mathbb{R}$ defined by $f(x) = \frac{1}{2}\|Dx - P_Q Dx\|^2, \forall x \in \mathbb{R}^3$, with $\nabla f(x) = D^*(I - P_Q)Dx, \ \forall x \in \mathbb{R}^3$. By the definition of f, ∇f, D and T, we have $\mathbf{2} \in \Gamma \cap F(T)$, where $\mathbf{2} = (2,2,2)$. From Theorem 4, we can conclude that the sequence $\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$ converges strongly to $\mathbf{2}$. For every $n \in \mathbb{N}$, we can rewrite (54) as follows:

$$\mathbf{y}_n = P_C(\mathbf{x}_n - 0.1\nabla f \mathbf{x}_n),$$
$$\mathbf{x}_{n+1} = \frac{1}{8n}\mathbf{u} + \left(\frac{8n-1}{8n}\right) T\mathbf{y}_n. \tag{59}$$

By using the algorithm (59), the following table and figure show the values of sequences $\mathbf{x}_n$ and $\mathbf{y}_n$, where $\mathbf{x}_1 = \mathbf{10} = (10, 10, 10)$ and $N = 40$.

6 | CONCLUSION

1. Table 1 and Figure 1 show that sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ converge to $\mathbf{2} = (2,2,2)$, where $\{\mathbf{2}\} \in F(G) \cap F(T)$. The convergence of the sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ of Example 3 can be guaranteed by Theorem 1.
2. Table 2 and Figure 2 show that sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ converge to $\mathbf{2} = (2,2,2)$, where $\{\mathbf{2}\} \in \Gamma \cap F(T)$. The convergence of the sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ of Example 4 can be guaranteed by Theorem 2.
3. Table 3 and Figure 3 show that sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ converge to $\mathbf{2} = (2,2,2)$, where $\{\mathbf{2}\} \in \Phi \cap F(T)$. The convergence of the sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ of Example 5 can be guaranteed by Theorem 3.
4. Table 4 and Figure 4 show that sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ converge to $\mathbf{2} = (2,2,2)$, where $\{\mathbf{2}\} \in \Gamma \cap F(T)$. The convergence of the sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ of Example 6 can be guaranteed by Theorem 4.

TABLE 1 The values of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 20$ of the iterative (56)

n	$\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$	$\mathbf{y}_n = (y_n^1, y_n^2, y_n^3)$
1	(10.000000, 10.000000, 10.000000)	(2.000000, 6.266667, 7.600000)
2	(2.375000, 3.619444, 2.375000)	(2.000000, 3.429975, 3.141172)
3	(2.187500, 2.634367, 2.187500)	(2.000000, 2.406361, 2.416339)
4	(2.125000, 2.254810, 2.125000)	(2.000000, 2.172967, 2.174208)
⋮	⋮	⋮
10	(2.041667, 2.054015, 2.041667)	(2.000000, 2.033119, 2.037592)
⋮	⋮	⋮
17	(2.023438, 2.029874, 2.023438)	(2.000000, 2.018215, 2.020806)
18	(2.022059, 2.028086, 2.022059)	(2.000000, 2.017118, 2.019561)
19	(2.020833, 2.026500, 2.020833)	(2.000000, 2.016145, 2.018457)
20	(2.019737, 2.025083, 2.019737)	(2.000000, 2.015278, 2.017471)

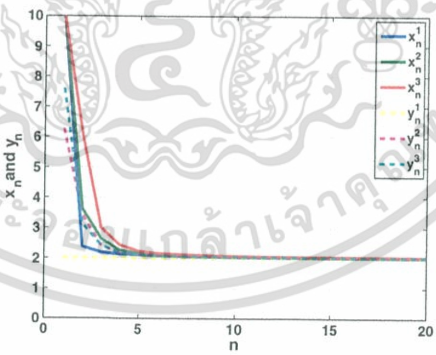

FIGURE 1 The convergence of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 20$ [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 2 The values of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 40$ of the iterative (57)

n	$\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$	$\mathbf{y}_n = (y_n^1, y_n^2, y_n^3)$
1	(10.000000, 10.000000, 10.000000)	(10.000000, 10.000000, 10.000000)
2	(5.875000, 4.708333, 5.875000)	(5.875000, 4.708333, 7.625000)
3	(4.003906, 3.033854, 4.003906)	(4.003906, 3.033854, 6.142578)
4	(3.085205, 2.455259, 3.085205)	(3.085205, 2.455259, 5.102478)
⋮	⋮	⋮
20	(2.041668, 2.030369, 2.041668)	(2.041668, 2.030369, 2.119598)
⋮	⋮	⋮
37	(2.021386, 2.015826, 2.021386)	(2.021386, 2.015826, 2.045701)
38	(2.020792, 2.015393, 2.020792)	(2.020792, 2.015393, 2.044295)
39	(2.020230, 2.014982, 2.020230)	(2.020230, 2.014982, 2.042981)
40	(2.019698, 2.014594, 2.019698)	(2.019698, 2.014594, 2.041747)

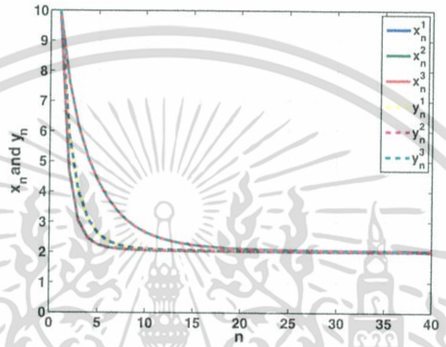

FIGURE 2 The convergence of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 40$ [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 3 The values of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 40$ of the iterative (58)

n	$\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$	$\mathbf{y}_n = (y_n^1, y_n^2, y_n^3)$
1	(10.000000, 10.000000, 10.000000)	(8.666667, 8.400000, 9.000000)
2	(5.291667, 4.241667, 5.291667)	(5.533027, 4.974856, 4.225897)
3	(3.843606, 3.117142, 3.843606)	(3.641449, 3.197288, 3.091111)
4	(2.911528, 2.507467, 2.911528)	(2.835120, 2.594672, 2.509903)
⋮	⋮	⋮
20	(2.037771, 2.029968, 2.037771)	(2.034291, 2.029198, 2.027707)
⋮	⋮	⋮
37	(2.019482, 2.015585, 2.019482)	(2.017676, 2.015117, 2.014384)
38	(2.018943, 2.015157, 2.018943)	(2.017187, 2.014701, 2.013989)
39	(2.018434, 2.014753, 2.018434)	(2.016724, 2.014307, 2.013615)
40	(2.017951, 2.014369, 2.017951)	(2.016286, 2.013933, 2.013260)

5. From these examples in the last section, we obtain that the sequences $\{\mathbf{x_n}\}$ and $\{\mathbf{y_n}\}$ in Example 3 converge faster than the sequences $\{\mathbf{x_n}\}$ and $\{\mathbf{y_n}\}$ in other example.
6. The SGSV can be reduced to the SFP, the SVIP, and the minimization problem in Section 4. It is well known that the methods for solving the SFP, SVIP, and minimization problem have been widely studied by many researchers. In this paper, we introduced the method for solving the SGSV for the first time and that means we can also solve the SFP, SVIP, and the minimization problem.

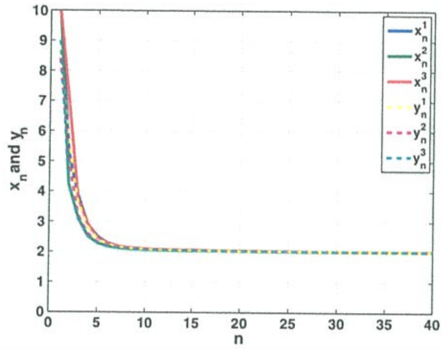

FIGURE 3 The convergence of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 40$ [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 4 The values of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 40$ of the iterative (59)

n	$\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$	$\mathbf{y}_n = (y_n^1, y_n^2, y_n^3)$
1	(10.000000, 10.000000, 10.000000)	(10.000000, 10.000000, 10.000000)
2	(5.875000, 4.708333, 5.875000)	(5.875000, 4.708333, 7.625000)
3	(4.003906, 3.033854, 4.003906)	(4.003906, 3.033854, 6.142578)
4	(3.085205, 2.455259, 3.085205)	(3.085205, 2.455259, 5.102478)
⋮	⋮	⋮
20	(2.041668, 2.030369, 2.041668)	(2.041668, 2.030369, 2.119598)
⋮	⋮	⋮
37	(2.021386, 2.015826, 2.021386)	(2.021386, 2.015826, 2.045701)
38	(2.020792, 2.015393, 2.020792)	(2.020792, 2.015393, 2.044295)
39	(2.020230, 2.014982, 2.020230)	(2.020230, 2.014982, 2.042981)
40	(2.019698, 2.014594, 2.019698)	(2.019698, 2.014594, 2.041747)

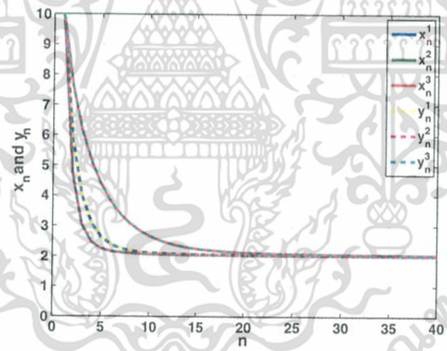

FIGURE 4 The convergence of $\mathbf{x}_n$ and $\mathbf{y}_n$ with $\mathbf{x}_1 = \mathbf{10}$ and $N = 40$ [Colour figure can be viewed at wileyonlinelibrary.com]

ACKNOWLEDGMENT

This research was supported by Research and Innovation Services of King Mongkut's Institute of Tecnology Ladkrabang.

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How to cite this article: Siriyan K, Kangtunyakarn A. Algorithm method for solving the split general system of variational inequalities problem and fixed point problem of nonexpansive mapping with application. *Math Meth Appl Sci.* 2018;1–23. https://doi.org/10.1002/mma.5240

Thai Journal of Mathematics
Volume 16 (2018) Number 1 : 203–218
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Convergence Analysis for Relaxed Extragradient Method and Variational Inequality Problem with Numerical Example

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Abstract : The purpose of this paper is to introduce an iterative method for finding a common element of fixed point of nonexpansive mapping which is generated by the general system of variational inequalities with inverse strongly monotone mappings and the set of the solution of variational inequality. By using our main result, we obtain the strong convergence theorem of the proposed iterative method and another corollary in a real Hilbert space.

Keywords : fixed point; nonexpansive mappings; variational inequalities.
2010 Mathematics Subject Classification : 47H09; 47H10; 49J40.

1 Introduction

Throughout this paper, let H be a real Hilbert space with the inner product $\langle .,. \rangle$ and the norm $\| . \|$ and let C be a nonempty closed and convex subset of H. We call $A : H \to H$ a *strongly positive bounded linear operator* if there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2,$$

for all $x, y \in C$.

A mapping $A : C \to H$ is called α-*inverse strongly monotone* if there exists a

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positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$.

Let $T : C \to C$ be a mapping. A point x is called fixed point of T if and only if $Tx = x$. We denote the set of solutions of fixed point of T by $Fix(T)$. It is well known that $F(T)$ is always closed convex and also nonempty provided T has a bounded trajectory, by Goebel and Kirk [1]. Recall the following mappings:

A mapping $f : H \to H$ is called *contraction* if there exists $\alpha \in (0, 1)$ such that

$$\|fx - fy\| \leq \alpha \|x - y\|,$$

for all $x, y \in H$.

A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$.

Let $A : C \to H$. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \tag{1.1}$$

for all $v \in C$. The set of the solutions of (1.1) is denoted by $VI(C, A)$.

The variational inequality problem, which were introduced by Lions and Stampacchia [2] in 1964. It has been widely studied in the literature, see [3–6].

In 1953, Mann [7] introduced the following iteration to find a fixed point of nonexpansive mapping T, which referred as the Mann iteration,

$$x_{n+1} = \beta_n T x_n + (1 - \beta_n) x_n, \tag{1.2}$$

for each $n \geq 1$ and $x_1 \in C$ where $\{\beta_n\}$ in $[0, 1]$.

In 2006, Marino and Xu [8] introduced the general iterative method and proved the following theorem:

Theorem 1.1. *Let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $A : H \to H$ be a strongly positive bounded linear operator and $f : H \to H$ be a contraction mapping and let $\{x_n\}$ be generated by*

$$\begin{cases} x_0 \in H \\ x_{n+1} = (I - \alpha_n A) T x_n + \alpha_n \gamma f(x_n), \ n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in (0,1) satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of T.

Let $A, B : C \to H$ be two different mappings. In 2008, Ceng et al. [9] introduced the *general system of variational inequalities* to find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \ \forall x \in C, \\ &\langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \ \ \forall x \in C, \end{aligned} \tag{1.4}$$

where $\lambda, \mu > 0$ are two constants. In particular, if $A = B$, then problem (1.4) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \ \forall x \in C, \\ &\langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, \ \ \forall x \in C, \end{aligned} \tag{1.5}$$

which is called *the new system of variational inequalities* introduced by Verma [10], in 1999. Moreover, if we put $x^* = y^*$, then problem (1.5) reduces to the variational inequality problem.

In order to find the common element of the solutions of the general system of variational inequalities problem (1.4) and the set of fixed point of a nonexpansive mapping, Ceng et.al [9] proved the strong convergence theorem by a relaxed extragradient method as follow:

Theorem 1.2. *Let the mappings $A, B : C \to H$ be α, β inverse strongly monotone mappings, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap F(G) \neq \emptyset$, where a mapping $G : C \to C$ is defined by $G(x) = P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)]$, $\forall x \in C$. Suppose that $x_1 = u \in C$ and $\{x_n\}$ is generated by*

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n), \end{cases} \tag{1.6}$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to some point $x^* \in C$ and (x^*, y^*) is a solution of the general system of variational inequalities (1.4), where $y^* = P_C(x^* - \mu Bx^*)$.

In this paper, motivated and inspired by the iterative scheme in Mann [7], Marino and Xu [8] and Ceng et.al [9], we introduce an iterative scheme for finding the solution of the problem (1.4). Then, we prove a strong convergence theorem, that the iterative sequence $\{x_n\}$ converges strongly to some point $x^* \in C$ and (x^*, y^*) is the solution of (1.4) under some proper conditions in a real Hilbert space.

2 Preliminaries

In this section, we collect some lemmas which will be needed to prove our main theorem in the next section.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let P_C be the metric projection of H onto C, i.e., for $x \in H$, P_C satisfies the property

$$\|x - P_C x\| \leq \|x - y\|,$$

for all $y \in C$.

The following lemmas characterizes the projection P_C.

Lemma 2.1. [11] *For a given $x \in H$ and $z \in C$,*

$$x = P_C y \Leftrightarrow \langle x - y, z - x \rangle \geq 0, \quad \forall z \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \ \forall x, y \in H.$$

Lemma 2.2. [12] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions:

(i) $\{\alpha_n\} \subset (0,1), \ \sum_{n=1}^{\infty} \alpha_n = \infty;$

(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < \infty.$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. [11] *Let H be a Hilbert space, let C be a nonempty closed convex of H, and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.4. [13] *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $\{x_n\} \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

Lemma 2.5. [14] *Let C be a nonempty closed and convex subset of a real Hilbert space H. For every $i = 1,2,...,N$, let A_i be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\gamma_i > 0$ and $\overline{\gamma} = \min_{i=1,2,..,N}\gamma_i$. Let $\{a_i\}_{i=1}^N \subseteq (0,1)$, with $\sum_{i=1}^N a_i = 1$. Then the following properties hold:*

(i) $\left\|I - \rho\sum_{i=1}^N a_iA_i\right\| \leq 1 - \rho\overline{\gamma}$ *and* $I - \rho\sum_{i=1}^N a_iA_i$ *is a nonexpansive mapping for every* $0 < \rho < \|A_i\|^{-1}(i = 1,2,...,N)$.

(ii) $VI(C, \sum_{i=1}^N a_iA_i) = \bigcap_{i=1}^N VI(C, A_i)$.

Lemma 2.6. [9] *For given $x^*, y^* \in C, (x^*, y^*)$ is a solution of problem (1.4) if and only if x^* is a point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)], \ \forall x \in C,$$

where $y^* = P_C(x - \mu Bx)$.

Lemma 2.7. [15] *In a real Hilbert spaces H, the following inequalities hold: for all $x, y \in H$ and $\alpha \in [0,1]$,*

(i) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$,

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ *for all* $x, y \in H$.

3 Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, D_1, D_2 : C \to H$ be d, d_1, d_2-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \lambda_1 D_1) P_C(I - \lambda_2 D_2)x$, for all $x \in C$ and $a \in [0,1)$. Let f be an α-contraction mapping on H. For $k = 1,2,...,N$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^N c_kA_kx$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,N}\gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Suppose that $\Im = F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{aligned} x_{n+1} &= (1-\beta_n)x_n + \beta_nP_C(I - \lambda D)y_n, \\ y_n &= \alpha_n\gamma f(x_n) + (I - \alpha_n\overline{A})\, Gx_n, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\} \subset [0,1]$, $\lambda \in (0, 2d)$, $\lambda_1, \lambda_2 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2\}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n \leq c < 1$;

(iii) $0 < c_k < 1$ *and* $\sum_{k=1}^{N} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ convergence strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0) is a solution of (1.4) where $y_0 = P_C(x_0 - \lambda_2 D_2 x_0)$.

Proof. Since $\alpha_n \to 0$ as $n \to \infty$, without loss of generality, we may assume that $\alpha_n < \frac{1}{\|A_i\|}$, $\forall n \in \mathbb{N}$ and $i = 1, 2, ..., N$. Let $x, y \in C$. Since D is d-inverse strongly monotone mapping with $\lambda \in (0, 2d)$, we obtain

$$\begin{aligned}\|(I - \lambda D)x - (I - \lambda D)y\|^2 &= \|x - y - \lambda(Dx - Dy)\|^2 \\ &= \|x - y\|^2 - 2\lambda \langle x - y, Dx - Dy\rangle - \lambda^2\|Dx - Dy\|^2 \\ &\leq \|x - y\|^2 - 2\lambda d\|Dx - Dy\|^2 - \lambda^2\|Dx - Dy\|^2 \\ &\leq \|x - y\|^2 - \lambda(2d - \lambda)\|Dx - Dy\|^2 \\ &\leq \|x - y\|^2.\end{aligned}$$

This implie that

$$\|(I - \lambda D)x - (I - \lambda D)y\| \leq \|x - y\|, \tag{3.2}$$

that is, $(I - \lambda D)$ is a nonexpansive mapping. Then, we have $P_C(I - \lambda D)$ is a nonexpansive mapping. By using the same method as (3.2), we have $P_C(I - \lambda_1 D_1)$ and $P_C(I - \lambda_2 D_2)$ are nonexpansive mappings. Then G is a nonexpansive mapping.

The proof will be divided into five steps.

Step 1. We will show that $\{x_n\}$ is bounded.

Let $x^* \in \Im$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1 - \beta_n)x_n + \beta_n P_C(I - \lambda D)y_n - x^*\| \\ &\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n \|P_C(I - \lambda D)y_n - x^*\| \\ &\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n \|y_n - x^*\| \\ &= (1 - \beta_n)\|x_n - x^*\| + \beta_n \|\alpha_n \gamma f(x_n) + (I - \alpha_n \bar{A})Gx_n - x^*\| \\ &\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n \alpha_n \|\gamma f(x_n) - \bar{A}x^*\| \\ &\quad + \beta_n \|I - \alpha_n \bar{A}\| \|Gx_n - x^*\| \\ &\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n \alpha_n \gamma \|f(x_n) - f(x^*)\| \\ &\quad + \beta_n \alpha_n \|\gamma f(x^*) - \bar{A}x^*\| + \beta_n (1 - \alpha_n \bar{\gamma})\|Gx_n - x^*\| \\ &\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n \alpha_n \gamma \alpha \|x_n - x^*\| + \beta_n \alpha_n \|\gamma f(x^*) - \bar{A}x^*\| \\ &\quad + \beta_n (1 - \alpha_n \bar{\gamma})\|x_n - x^*\| \\ &= (1 - \beta_n + \beta_n (\alpha_n \gamma \alpha + 1 - \alpha_n \bar{\gamma}))\|x_n - x^*\| + \beta_n \alpha_n \|\gamma f(x^*) - \bar{A}x^*\| \\ &= (1 - \beta_n + \beta_n (1 - \alpha_n (\bar{\gamma} - \gamma\alpha)))\|x_n - x^*\| + \beta_n \alpha_n \|\gamma f(x^*) - \bar{A}x^*\| \\ &= (1 - \beta_n \alpha_n (\bar{\gamma} - \gamma\alpha))\|x_n - x^*\| + \beta_n \alpha_n \|\gamma f(x^*) - \bar{A}x^*\| \\ &\leq \max\left\{\|x_1 - x^*\|, \frac{\|\gamma f(x^*) + \bar{A}x^*\|}{\bar{\gamma} - \gamma\alpha}\right\}.\end{aligned}$$

By induction, we have $\|x_n - x^*\| \leq \max\left\{\|x_1 - x^*\|, \frac{\|\gamma f(x^*)+\overline{A}x^*\|}{\overline{\gamma}-\gamma\alpha}\right\}, \forall n \in \mathbb{N}$.
Hence $\{x_n\}$ is bounded and so is $\{y_n\}$.
Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
From the definition of $\{y_n\}$, we have

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \left\|\alpha_{n+1}\gamma f(x_{n+1}) + \left(I-\alpha_{n+1}\overline{A}\right)Gx_{n+1} - \alpha_n\gamma f(x_n) - \left(I-\alpha_n\overline{A}\right)Gx_n\right\| \\
&\leq \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + \gamma|\alpha_{n+1}-\alpha_n|\|f(x_n)\| \\
&\quad + \left\|I-\alpha_{n+1}\overline{A}\right\|\|Gx_{n+1}-Gx_n\| + \left\|\left(I-\alpha_{n+1}\overline{A}\right)Gx_n - \left(I-\alpha_n\overline{A}\right)Gx_n\right\| \\
&\leq \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\| + \gamma|\alpha_{n+1}-\alpha_n|\|f(x_n)\| \\
&\quad + (1-\alpha_{n+1}\overline{\gamma})\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\left\|\overline{A}Gx_n\right\| \\
&= (1-\alpha_{n+1}(\overline{\gamma}-\gamma\alpha))\|x_{n+1}-x_n\| + \gamma|\alpha_{n+1}-\alpha_n|\|f(x_n)\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\left\|\overline{A}Gx_n\right\|. \qquad (3.3)
\end{aligned}$$

From the definition of $\{x_n\}$ and (3.3), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|(1-\beta_n)x_n + \beta_n P_C(I-\lambda D)y_n - (1-\beta_{n-1})x_{n-1} \\
&\quad - \beta_{n-1}P_C(I-\lambda D)y_{n-1}\| \\
&= \|(1-\beta_n)x_n + \beta_n P_C(I-\lambda D)y_n - (1-\beta_n)x_{n-1} + (1-\beta_n)x_{n-1} \\
&\quad - (1-\beta_{n-1})x_{n-1} - \beta_n P_C(I-\lambda D)y_{n-1} + \beta_n P_C(I-\lambda D)y_{n-1} \\
&\quad - \beta_{n-1}P_C(I-\lambda D)y_{n-1}\| \\
&\leq (1-\beta_n)\|x_n-x_{n-1}\| + \beta_n\|P_C(I-\lambda D)y_n - P_C(I-\lambda D)y_{n-1}\| \\
&\quad + |\beta_n-\beta_{n-1}|\|x_{n-1}\| + |\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\| \\
&\leq (1-\beta_n)\|x_n-x_{n-1}\| + \beta_n\|y_n-y_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad + |\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\| \\
&\leq (1-\beta_n)\|x_n-x_{n-1}\| + \beta_n((1-\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n-x_{n-1}\| \\
&\quad + \gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| + |\alpha_n-\alpha_{n-1}|\|\bar{A}Gx_{n-1}\|) \\
&\quad + |\beta_n-\beta_{n-1}|\|x_{n-1}\| + |\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\| \\
&= (1-\beta_n)\|x_n-x_{n-1}\| + \beta_n(1-\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n-x_{n-1}\| \\
&\quad + \beta_n\gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| + \beta_n|\alpha_n-\alpha_{n-1}|\|\bar{A}Gx_{n-1}\| \\
&\quad + |\beta_n-\beta_{n-1}|\|x_{n-1}\| + |\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\| \\
&= (1-\beta_n+\beta_n(1-\alpha_n(\bar{\gamma}-\gamma\alpha)))\|x_n-x_{n-1}\| \\
&\quad + \beta_n\gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| + \beta_n|\alpha_n-\alpha_{n-1}|\|\bar{A}Gx_{n-1}\| \\
&\quad + |\beta_n-\beta_{n-1}|\|x_{n-1}\| + |\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\| \\
&\leq (1-\beta_n\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n-x_{n-1}\| + \gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad + |\alpha_n-\alpha_{n-1}|\|\bar{A}Gx_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad + |\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)y_{n-1}\|.
\end{aligned}$$

This together with conditions $(i),(iii)$ and Lemma 2.2 , we get that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.4}$$

From conditions $(i),(iii)$, (3.3), and (3.4), we obtain $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$.
From the definition of y_n, we have

$$\begin{aligned}\|y_n-x^*\|^2&=\left\|\alpha_n\gamma f(x_n)+\left(I-\alpha_n\bar{A}\right)Gx_n-x^*\right\|^2\\&=\left\|(Gx_n-x^*)+\left(\alpha_n\gamma f(x_n)-\alpha_n\bar{A}Gx_n\right)\right\|^2\\&\le\|Gx_n-x^*\|^2+2\alpha_n\left\langle\gamma f(x_n)-\bar{A}Gx_n,y_n-x^*\right\rangle\\&\le\|x_n-x^*\|^2+2\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\|y_n-x^*\|.\end{aligned}\tag{3.5}$$

From nonexpansiveness of P_C and (3.5), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2&\le(1-\beta_n)\|x_n-x^*\|^2+\beta_n\|P_C(I-\lambda D)y_n-x^*\|^2\\&\le(1-\beta_n)\|x_n-x^*\|^2+\beta_n\|(I-\lambda D)y_n-(I-\lambda D)x^*\|^2\\&=(1-\beta_n)\|x_n-x^*\|^2+\beta_n\|y_n-x^*-\lambda(Dy_n-Dx^*)\|^2\\&=(1-\beta_n)\|x_n-x^*\|^2+\beta_n\|y_n-x^*\|^2\\&\quad-2\lambda\beta_n\langle y_n-x^*,Dy_n-Dx^*\rangle+\beta_n\lambda^2\|Dy_n-Dx^*\|^2\\&\le(1-\beta_n)\|x_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+2\beta_n\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\\&\quad\times\|y_n-x^*\|-2\lambda d\beta_n\|Dy_n-Dx^*\|^2+\beta_n\lambda^2\|Dy_n-Dx^*\|^2\\&=(1-\beta_n)\|x_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+2\beta_n\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\\&\quad\times\|y_n-x^*\|-\lambda\beta_n(2d-\lambda)\|Dy_n-Dx^*\|^2\\&=\|x_n-x^*\|^2+2\beta_n\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\|y_n-x^*\|\\&\quad-\lambda\beta_n(2d-\lambda)\|Dy_n-Dx^*\|^2.\end{aligned}$$

It implies that

$$\begin{aligned}\lambda\beta_n(2d-\lambda)\|Dy_n-Dx^*\|^2&\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\&\quad+2\beta_n\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\|y_n-x^*\|\\&\le(\|x_n-x^*\|-\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|\\&\quad+2\beta_n\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\|y_n-x^*\|.\end{aligned}\tag{3.6}$$

Form conditions $(i),(ii)$, (3.4) and (3.6), we have

$$\lim_{n\to\infty}\|Dy_n-Dx^*\|=0. \tag{3.7}$$

Step 3. Show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, $\lim_{n\to\infty}\|x_n-Gx_n\|=0$, $\lim_{n\to\infty}\|y_n-Ty_n\|$

$= 0$, $\lim_{n\to\infty} \|y_n - Gy_n\| = 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$.
From the definition of x_n and (3.5), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(1-\beta_n)x_n + \beta_n P_C(I-\lambda D)y_n - x^*\|^2\\ &\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|P_C(I-\lambda D)y_n - x^*\|^2\\ &\quad - \beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2\\ &\le (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|y_n-x^*\|^2-\beta_n(1-\beta_n)\|x_n-P_C(I-\lambda D)y_n\|^2\\ &\le (1-\beta_n)\|x_n-x^*\|^2 + \beta_n\|x_n-x^*\|^2 + 2\beta_n\alpha_n\|\gamma f(x_n) - \bar{A}Gx_n\|\\ &\quad \times \|y_n - x^*\| - \beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2\\ &\le \|x_n - x^*\|^2 + 2\beta_n\alpha_n\|\gamma f(x_n) - \bar{A}Gx_n\|\,\|y_n - x^*\|\\ &\quad - \beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2.\end{aligned}$$

It implies that

$$\begin{aligned}\beta_n(1-\beta_n)\|x_n - P_C(I-\lambda D)y_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1}-x^*\|^2\\ &\quad + 2\beta_n\alpha_n\|\gamma f(x_n) - \bar{A}Gx_n\|\,\|y_n - x^*\|\\ &\le (\|x_n - x^*\| - \|x_{n+1}-x^*\|)\,\|x_{n+1}-x_n\|\\ &\quad + 2\alpha_n\|\gamma f(x_n) - \bar{A}Gx_n\|\,\|y_n - x^*\|. \quad (3.8)\end{aligned}$$

Form conditions $(i), (ii)$, (3.4) and (3.8), we have

$$\lim_{n\to\infty}\|x_n - P_C(I-\lambda D)y_n\| = 0. \quad (3.9)$$

By Lemma 2.1 and (3.5), we obtain

$$\begin{aligned}\|P_C(I-\lambda D)y_n - x^*\|^2 &\le \langle (I-\lambda D)y_n - (I-\lambda D)x^*, P_C(I-\lambda D)y_n - x^*\rangle\\ &= \frac{1}{2}\Big(\|(I-\lambda D)y_n - (I-\lambda D)x^*\|^2 + \|P_C(I-\lambda D)y_n - x^*\|^2\\ &\quad -\|(I-\lambda D)y_n - (I-\lambda D)x^* - (P_C(I-\lambda D)y_n - x^*)\|^2\Big)\\ &\le \frac{1}{2}\Big(\|y_n - x^*\|^2 + \|P_C(I-\lambda D)y_n - x^*\|^2\\ &\quad -\|y_n - P_C(I-\lambda D)y_n - \lambda(Dy_n - Dx^*)\|^2\Big)\\ &\le \frac{1}{2}\Big(\|x_n - x^*\|^2 + 2\alpha_n\|\gamma f(x_n) - \bar{A}Gx_n\|\,\|y_n - x^*\|\\ &\quad + \|P_C(I-\lambda D)y_n - x^*\|^2 - \|y_n - P_C(I-\lambda D)y_n\|^2\\ &\quad + 2\lambda\langle y_n - P_C(I-\lambda D)y_n, Dy_n - Dx^*\rangle - \lambda^2\|Dy_n - Dx^*\|^2\Big)\\ &\le \frac{1}{2}\Big(\|x_n - x^*\|^2 + 2\alpha_n\|\gamma f(x_n) - \bar{A}Gx_n\|\,\|y_n - x^*\|\\ &\quad + \|P_C(I-\lambda D)y_n - x^*\|^2 - \|y_n - P_C(I-\lambda D)y_n\|^2\\ &\quad + 2\lambda\|y_n - P_C(I-\lambda D)y_n\|\,\|Dy_n - Dx^*\|\Big).\end{aligned}$$

It follow that

$$\|P_C(I-\lambda D)y_n - x^*\|^2 \leq \|x_n - x^*\|^2 + 2\alpha_n \|\gamma f(x_n) - \overline{A}Gx_n\| \|y_n - x^*\| - \|y_n - P_C(I-\lambda D)y_n\|^2 + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \times \|Dy_n - Dx^*\|. \quad (3.10)$$

From (3.10), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|P_C(I-\lambda D)y_n - x^*\|^2 \\ &\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \Big(\|x_n - x^*\|^2 + 2\alpha_n \|\gamma f(x_n) - \bar{A}Gx_n\| \\ &\quad \times \|y_n - x^*\| - \|y_n - P_C(I-\lambda D)y_n\|^2 + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \\ &\quad \times \|Dy_n - Dx^*\|\Big) \\ &\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + 2\alpha_n \|\gamma f(x_n) - \bar{A}Gx_n\| \\ &\quad \times \|y_n - x^*\| - \beta_n\|y_n - P_C(I-\lambda D)y_n\|^2 + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \\ &\quad \times \|Dy_n - Dx^*\| \\ &= \|x_n - x^*\|^2 + 2\alpha_n \|\gamma f(x_n) - \bar{A}Gx_n\| \|y_n - x^*\| \\ &\quad - \beta_n\|y_n - P_C(I-\lambda D)y_n\|^2 + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \|Dy_n - Dx^*\|.\end{aligned}$$

It implies that

$$\begin{aligned}\beta_n\|y_n - P_C(I-\lambda D)y_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n \|\gamma f(x_n) - \bar{A}Gx_n\| \\ &\quad \times \|y_n - x^*\| + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \|Dy_n - Dx^*\| \\ &\leq (\|x_n - x^*\| - \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\| \\ &\quad + 2\alpha_n \|\gamma f(x_n) - \bar{A}Gx_n\| \|y_n - x^*\| \\ &\quad + 2\lambda \|y_n - P_C(I-\lambda D)y_n\| \|Dy_n - Dx^*\|. \quad (3.11)\end{aligned}$$

From conditions (i), (ii), (3.4), (3.7) and (3.11), we get

$$\lim_{n\to\infty} \|y_n - P_C(I-\lambda D)y_n\| = 0. \quad (3.12)$$

Consider,

$$\|x_n - y_n\| \leq \|x_n - P_C(I-\lambda D)y_n\| + \|P_C(I-\lambda D)y_n - y_n\|.$$

From (3.9) and (3.12), we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \quad (3.13)$$

From definition of y_n and condition (i), we have

$$\begin{aligned}\|y_n - Gx_n\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n\overline{A})Gx_n - Gx_n\| \\ &= \alpha_n \|\gamma f(x_n) + \overline{A}Gx_n\| \to 0 \ \ as\ n \to \infty. \quad (3.14)\end{aligned}$$

Consider,

$$\|x_n - Gx_n\| \leq \|x_n - y_n\| + \|y_n - Gx_n\|.$$

By (3.13) and (3.14), we have

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{3.15}$$

From (3.13), (3.15) and

$$\begin{aligned}\|y_n - Gy_n\| &\leq \|y_n - x_n\| + \|x_n - Gx_n\| + \|Gx_n - Gy_n\| \\ &\leq \|y_n - x_n\| + \|x_n - Gx_n\| + \|x_n - y_n\|,\end{aligned}$$

we get that

$$\lim_{n\to\infty} \|y_n - Gy_n\| = 0. \tag{3.16}$$

Step 4. We will show that $\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0\rangle \leq 0$, where $x_0 = P_{\Im}(I - \overline{A} + \gamma f)x_0$.

To show this, choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0\rangle = \lim_{k\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_{n_k} - x_0\rangle. \tag{3.17}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup q$ as $k \to \infty$, where $q \in C$. Then, from (3.13) and $x_{n_k} \rightharpoonup q$, we obtain $y_{n_k} \rightharpoonup q$ as $k \to \infty$. From (3.17) and $y_{n_k} \rightharpoonup q$ as $k \to \infty$, we have

$$\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0\rangle = \langle \gamma f(x_0) - \overline{A}x_0, q - x_0\rangle. \tag{3.18}$$

In order to show $\langle \gamma f(x_0) - \overline{A}x_0, q - x_0\rangle \leq 0$, we need to show that $q \in \Im = F(G) \cap VI(C, D)$. Assume that $q \notin F(G)$. It implies that $q \neq Gq$. From Lemma 2.4 and (3.16), we have

$$\begin{aligned}\liminf_{n\to\infty} \|y_{n_k} - q\| &< \liminf_{n\to\infty} \|y_{n_k} - Gq\| \\ &\leq \liminf_{n\to\infty} (\|y_{n_k} - Gy_{n_k}\| + \|Gy_{n_k} - Gq\|) \\ &\leq \liminf_{n\to\infty} (\|y_{n_k} - Gy_{n_k}\| + \|y_{n_k} - q\|) \\ &\leq \liminf_{n\to\infty} \|y_{n_k} - q\|.\end{aligned}$$

This is a contraction, that is,

$$q \in F(G). \tag{3.19}$$

Next, we will show that $q \in VI(C, D)$.
Assume that $q \notin VI(C, D)$. Since $VI(C, D) = F(P_C(I - \lambda D))$, we have $q \neq$

$P_C(I-\lambda D)q$. From Lemma 2.4 and (3.12), we obtain

$$
\begin{aligned}
\liminf_{n\to\infty}\|y_{n_k}-q\| &< \liminf_{n\to\infty}\|y_{n_k}-P_C(I-\lambda D)q\| \\
&\le \liminf_{n\to\infty}\left(\|y_{n_k}-P_C(I-\lambda D)y_{n_k}\|+\|P_C(I-\lambda D)y_{n_k}-P_C(I-\lambda D)q\|\right) \\
&\le \liminf_{n\to\infty}\left(\|y_{n_k}-P_C(I-\lambda D)y_{n_k}\|+\|y_{n_k}-q\|\right) \\
&\le \liminf_{n\to\infty}\|y_{n_k}-q\|.
\end{aligned}
$$

This is a contraction, that is,

$$q\in VI(C,D). \tag{3.20}$$

From (3.19) and (3.20), we have $q\in\Im=F(G)\cap VI(C,D)$.By (3.18) and Lemma 2.1, we have

$$\limsup_{n\to\infty}\langle\gamma f(x_0)-\overline{A}x_0, y_n-x_0\rangle=\langle\gamma f(x_0)-\overline{A}x_0, q-x_0\rangle\le 0.$$

Step 5. Finally, We will show that $\{x_n\}$ converges strongly to x_0, where $x_0=P_\Im(I-\overline{A}+\gamma f)x_0$.
From the definition of x_n and $x_0=P_\Im(I-\overline{A}+\gamma f)x_0$, we have

$$
\begin{aligned}
\|x_{n+1}-x_0\|^2 &= \|(1-\beta_n)x_n+\beta_nP_C(I-\lambda D)y_n-x_0\|^2 \\
&\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n\|P_C(I-\lambda D)y_n-x_0\|^2 \\
&\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n\|y_n-x_0\|^2 \\
&\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n\left\|\alpha_n\gamma f(x_n)+(I-\alpha_n\bar{A})\,Gx_n-x_0\right\|^2 \\
&\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n\Big(\left\|(I-\alpha_n\bar{A})\,(Gx_n-x_0)\right\|^2 \\
&\quad +2\alpha_n\langle\gamma f(x_n)-\bar{A}x_0, y_n-x_0\rangle\Big) \\
&\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n\Big((1-\alpha_n\bar{\gamma})^2\|x_n-x_0\|^2 \\
&\quad +2\alpha_n\gamma\langle f(x_n)-f(x_0), y_n-x_0\rangle+2\alpha_n\langle\gamma f(x_0)-\bar{A}x_0, y_n-x_0\rangle\Big) \\
&\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n\Big((1-\alpha_n\bar{\gamma})^2\|x_n-x_0\|^2 \\
&\quad +2\alpha_n\gamma\|f(x_n)-f(x_0)\|\,\|y_n-x_0\|+2\alpha_n\langle\gamma f(x_0)-\bar{A}x_0, y_n-x_0\rangle\Big) \\
&\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n(1-\alpha_n\bar{\gamma})^2\|x_n-x_0\|^2 \\
&\quad +2\alpha_n\gamma\alpha\beta_n\|x_n-x_0\|\,\|y_n-x_0\|+2\alpha_n\beta_n\langle\gamma f(x_0)-\bar{A}x_0, y_n-x_0\rangle \\
&\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n(1-\alpha_n\bar{\gamma})^2\|x_n-x_0\|^2 \\
&\quad +2\alpha_n\gamma\alpha\beta_n\|x_n-x_0\|\left(\alpha_n\|\gamma f(x_n)-\bar{A}x_0\|+(1-\alpha_n\bar{\gamma})\|Gx_n-x_0\|\right) \\
&\quad +2\alpha_n\beta_n\langle\gamma f(x_0)-\bar{A}x_0, y_n-x_0\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n(1-\alpha_n\bar{\gamma})^2\|x_n-x_0\|^2 \\
&\quad +2\alpha_n\gamma\alpha\beta_n\|x_n-x_0\|\left(\alpha_n\gamma\alpha\|x_n-x_0\|+\alpha_n\|\gamma f(x_0)-\bar{A}x_0\|\right.\\
&\quad \left.+(1-\alpha_n\bar{\gamma})\|x_n-x_0\|\right)+2\alpha_n\beta_n\langle\gamma f(x_0)-\bar{A}x_0,y_n-x_0\rangle \\
&=(1-\beta_n)\|x_n-x_0\|^2+\beta_n(1-\alpha_n\bar{\gamma})^2\|x_n-x_0\|^2+2\alpha_n^2\gamma^2\alpha^2\beta_n\|x_n-x_0\|^2 \\
&\quad +2\alpha_n^2\gamma\alpha\beta_n\|\gamma f(x_0)-\bar{A}x_0\|\|x_n-x_0\|+2\alpha_n\gamma\alpha\beta_n(1-\alpha_n\bar{\gamma})\|x_n-x_0\|^2 \\
&\quad +2\alpha_n\beta_n\langle\gamma f(x_0)-\bar{A}x_0,y_n-x_0\rangle \\
&\le (1-\beta_n)\|x_n-x_0\|^2+\beta_n(1-\alpha_n\bar{\gamma})\|x_n-x_0\|^2+2\alpha_n^2\bar{\gamma}^2\beta_n\|x_n-x_0\|^2 \\
&\quad +2\alpha_n^2\bar{\gamma}\beta_n\|\gamma f(x_0)-\bar{A}x_0\|\|x_n-x_0\|+2\alpha_n\gamma\alpha\beta_n\|x_n-x_0\|^2 \\
&\quad +2\alpha_n\beta_n\langle\gamma f(x_0)-\bar{A}x_0,y_n-x_0\rangle \\
&=(1-\beta_n+\beta_n-\beta_n\alpha_n\bar{\gamma}+2\alpha_n\gamma\alpha\beta_n)\|x_n-x_0\|^2+2\alpha_n^2\bar{\gamma}^2\beta_n\|x_n-x_0\|^2 \\
&\quad +2\alpha_n^2\bar{\gamma}\beta_n\|\gamma f(x_0)-\bar{A}x_0\|\|x_n-x_0\|+2\alpha_n\beta_n\langle\gamma f(x_0)-\bar{A}x_0,y_n-x_0\rangle \\
&=(1-\alpha_n\beta_n(\bar{\gamma}-2\gamma\alpha))\|x_n-x_0\|^2+\alpha_n\beta_n\Big(2\alpha_n\bar{\gamma}^2\|x_n-x_0\|^2 \\
&\quad +2\alpha_n\bar{\gamma}\|\gamma f(x_0)-\bar{A}x_0\|\|x_n-x_0\|+2\langle\gamma f(x_0)-\bar{A}x_0,y_n-x_0\rangle\Big) \\
&=(1-\alpha_n\beta_n(\bar{\gamma}-2\gamma\alpha))\|x_n-x_0\|^2+\alpha_n\beta_n(\bar{\gamma}-2\gamma\alpha)\left(\frac{2\alpha_n\bar{\gamma}^2\|x_n-x_0\|^2}{(\bar{\gamma}-2\gamma\alpha)}\right. \\
&\quad \left.+\frac{2\alpha_n\bar{\gamma}\|\gamma f(x_0)-\bar{A}x_0\|\|x_n-x_0\|}{(\bar{\gamma}-2\gamma\alpha)}+\frac{2\langle\gamma f(x_0)-\bar{A}x_0,y_n-x_0\rangle}{(\bar{\gamma}-2\gamma\alpha)}\right).
\end{aligned}
$$

By step 4, condition (i) and Lemma 2.2, we can conclude that $\{x_n\}$ converges strongly to $x_0 = P_{\Im}(I - \overline{A} + \gamma f)x_0$. Then, from Lemma 2.6, we have (x_0, y_0) is a solution of the problem (1.4) where $y_0 = P_C(x_0 - \lambda_2 D_2 x_0)$. This completes the proof. □

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, D_1 : C \to H$ be d, d_1-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $G(x) = P_C(I - \lambda_1 D_1) P_C(I - \lambda_2 D_2)x$, for all $x \in C$ and $a \in [0,1)$. Let f be an α-contraction mapping on H. For $k = 1, 2, ..., N$, define $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{N} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,N}\gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Suppose that $\Im = F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$
\begin{aligned}
x_{n+1} &= (1-\beta_n)x_n + \beta_n P_C(I-\lambda D)y_n, \\
y_n &= \alpha_n\gamma f(x_n) + \left(I - \alpha_n\overline{A}\right)Gx_n,
\end{aligned} \tag{3.21}
$$

where $\{\alpha_n\} \subset [0,1]$, $\lambda \in (0, 2d)$, $\lambda_1, \lambda_2 \in (0, 2d_1)$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n \leq c < 1$;

(iii) $0 < c_k < 1$ *and* $\sum_{k=1}^{N} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ *convergence strongly to* $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ *and* (x_0, y_0) *is a solution of* (1.5) *where* $y_0 = P_C(x_0 - \lambda_2 D_1 x_0)$.

Proof. If we put $D_1 = D_2$ in Theorem 3.1, we have the desired conclusion. □

4 Example and Numerical Results

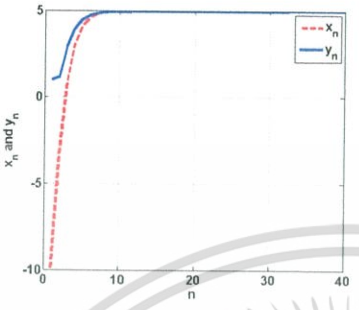
Example 4.1. Let $\mathbb{R}$ be the set of real numbers. Let D, D_1, D_2 be a mapping from $[-50, 50]$ to $\mathbb{R}$ defined by $Dx = \frac{2x-10}{3}$, $D_1 x = \frac{x-5}{2}$ and $D_2 x = \frac{3x-15}{4}$, for all $x \in [-50, 50]$. Let mapping $f : \mathbb{R} \to \mathbb{R}$ be defined by $fx = \frac{5x}{9}$, for every $x \in \mathbb{R}$. For $k = 1, 2, ..., N$, let $c_k = \frac{2}{3^k} + \frac{1}{N3^N}$ and let the mapping $A : \mathbb{R} \to \mathbb{R}$ be defined by $A_k x = \frac{kx}{5}$, for every $x \in \mathbb{R}$. Let $x_1 \in \mathbb{R}$ and $\{x_n\}$ be generated by (3.1) where $\lambda = 1.5, \lambda_1 = 0.5, \lambda_2 = 0.5, \alpha = 1, \gamma = 0.05$, $\alpha_n = \frac{2}{5n}$ and $\beta_n = \frac{5n-1}{9n}$. By the definition of D, D_1, D_2, A and f, we have $5 \in F(G) \cap VI(C, D)$. Then, from Theorem 3.1, the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to 5. We can rewritten (3.1) as follow:

$$\begin{aligned} x_{n+1} &= \left(\tfrac{3n+1}{8n}\right) x_n + \left(\tfrac{5n-1}{8n}\right) P_{[-50,50]}(I - (1.5)D)y_n, \\ y_n &= \tfrac{0.1}{5n} f(x_n) + \left(I - \left(\tfrac{2}{5n}\right) \bar{A}\right) Gx_n. \end{aligned} \tag{4.1}$$

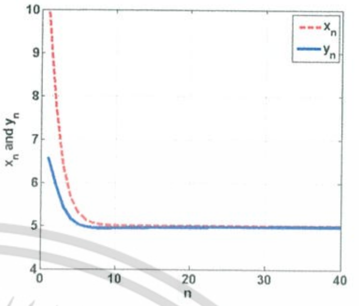
The following table and figure shows the values of the sequence $\{x_n\}$ and $\{y_n\}$ of iterative (4.1), where $x_1 = -10$, $x_1 = 10$ and $n = N = 40$.

	$x_1 = -10$		$x_1 = 10$	
n	x_n	y_n	x_n	y_n
1	−10.000000	0.988889	10.000000	6.573611
2	−3.333333	1.156481	7.777778	5.967168
3	0.833333	2.928086	6.388889	5.448663
⋮	⋮	⋮	⋮	⋮
20	4.999993	4.972774	5.000002	4.972779
⋮	⋮	⋮	⋮	⋮
38	5.000000	4.985673	5.000000	4.985673
39	5.000000	4.986040	5.000000	4.986040
40	5.000000	4.986389	5.000000	4.986389

Table 1: The values of $\{x_n\}$ and $\{y_n\}$ with different initial value x_1



(a) $x_1 = -10$



(b) $x_1 = 10$

Figure 1: The convergence of the sequence $\{x_n\}$ and $\{y_n\}$ with different initial value x_1 and $n = N = 40$.

From Table 1 and Figure 1 (a) and (b), we can observe that $\{x_n\}$ and $\{y_n\}$ converge to 5, where $5 \in F(G) \cap VI(C, D)$. The convergence of $\{x_n\}$ and $\{y_n\}$ of Example 4.1 can be guaranteed by Theorem 3.1.

Acknowledgement(s) : This research was supported by Research and Innovation Services of King Mongkut's Institute of Tecnology Ladkrabang.

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(Received 27 September 2017)
(Accepted 12 December 2017)

Numer Algor
https://doi.org/10.1007/s11075-018-0537-8

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ORIGINAL PAPER

A new general system of variational inequalities for convergence theorem and application

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Received: 26 December 2017 / Accepted: 27 April 2018

Abstract In this present paper, we propose a modified form of generalized system of variational inequalities and introduce an iterative scheme for finding a common element of the set of fixed points of nonexpansive mapping and the solution set of the proposed problem in the framework of real Hilbert spaces. We prove a strong convergence theorem of the proposed iterative scheme. Applying our main result, we prove strong convergence theorems of the standard constrained convex optimization problem and the split feasibility problem. In support of our main result, a numerical example is also presented.

Keywords Fixed point · System of variational inequalities · Nonexpansive mappings

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle .,. \rangle$ and the norm $\| . \|$. Let $T : C \to C$ be a mapping. Then, T is called *contraction* if there exists $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \forall x, y \in C.$$

If the last inequality holds for $\alpha = 1$, then T is called *nonexpansive*. The set of fixed points of a mapping $T : C \to C$ is denoted by $F(T)$, that is $F(T) = \{x \in C : Tx = x\}$.

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Published online: 04 May 2018

A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A bounded linear operator A on H is called *strongly positive* with coefficient $\overline{\gamma}$ if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \ \forall x \in H.$$

The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \ \forall v \in C. \tag{1.1}$$

The set of the solutions of the variational inequality is denoted by $VI(C, A)$. It is well known that numerous problems in optimization, physic, finance, and minimax problem reduce to find element of (1.1). The variational inequality has been widely studied in the literature (see [4, 13, 16, 17, 27, 29]).

In 1976, Korpelevich [18] introduced the *extragradient method*, for solving the variational inequality problem in the Euclidean space $\mathbb{R}^n$, as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases} \tag{1.2}$$

for all $n \geq 0$, where $\lambda \in \left(0, \frac{1}{\kappa}\right)$ and A is a monotone and κ- Lipschitz continuous mapping of C into $\mathbb{R}^n$. It is obvious that if $VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.2) converges weakly to an element in $VI(C, A)$. Later, motivated by Korpelevich [18], Nadezhkina and Takahashi [19] and Zeng and Yao [30] proposed some iterative schemes for finding the common elements in $F(T) \cap VI(C, A)$. In 2007, Yao and Yao [28] introduced a new iterative scheme for finding an element in $F(T) \cap VI(C, A)$ under some suitable conditions and they obtained the strong convergence theorem in a real Hilbert space.

Recently, in 2008, Ceng et al.[11] introduced the following *general system of variational inequalities*, which involves finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.3}$$

where $A, B : C \to H$ are two different mappings and $\lambda, \mu > 0$ are two constants. The general system of variational inequalities has been studied and developed in literatures (see [2, 6–10, 12, 14]).

In order to find the common element of the solutions of the general system of variational inequality problem (1.3), Ceng et al. [11] introduced the following iterative scheme:

Let the mappings $A, B : C \to H$ be α, β inverse strongly monotone mappings, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap F(G) \neq \emptyset$, where a mapping $G : C \to C$ is defined by $G(x) =$

$P_C[P_C(x-\mu Bx)-\lambda AP_C(x-\mu Bx)], \forall x\in C$. Suppose that $x_1=u\in C$ and $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n), \end{cases} \tag{1.4}$$

where $\lambda\in(0,2\alpha)$, $\mu\in(0,2\beta)$, and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are three sequences in $[0,1]$ such that

(i) $\alpha_n+\beta_n+\gamma_n=1$;
(ii) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(iii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$.

Then, they proved that the sequence $\{x_n\}$, defined by the iterative (1.4), converges strongly to some point $x^*\in C$ and (x^*,y^*) is a solution of the general system of variational inequalities (1.3), where $y^*=P_C(I-\mu B)x^*$.

Motivated by the problem (1.3) of Ceng et al. [11], we introduce a new problem of the system of variational inequalities in a real Hilbert space as follows:

Let $D_1,D_2,D_3:C\to H$ be three mappings. We consider the problem for finding $(x^*,y^*,z^*)\in C\times C\times C$ such that

$$\begin{cases} \langle x^*-(I-\lambda_1D_1)(ax^*+(1-a)y^*), x-x^*\rangle\ge 0, & \forall x\in C, \\ \langle y^*-(I-\lambda_2D_2)(ax^*+(1-a)z^*), x-y^*\rangle\ge 0, & \forall x\in C, \\ \langle z^*-(I-\lambda_3D_3)x^*, x-z^*\rangle\ge 0, & \forall x\in C. \end{cases} \tag{1.5}$$

where $\lambda_1,\lambda_2,\lambda_3>0$ and $a\in[0,1]$, which is called *modified generalized system of variational inequalities*.

If putting $a=0$, in (1.5), we have

$$\begin{array}{ll} \langle x^*-(I-\lambda_1D_1)y^*, x-x^*\rangle\ge 0, & \forall x\in C, \\ \langle y^*-(I-\lambda_2D_2)z^*, x-y^*\rangle\ge 0, & \forall x\in C, \\ \langle z^*-(I-\lambda_3D_3)x^*, x-z^*\rangle\ge 0, & \forall x\in C. \end{array} \tag{1.6}$$

which is a generalized system of variational inequalities modified by Ceng et al. [11], in the sense that if we put $D_3\equiv 0$ and $x^*=z^*$, then the problem (1.6) reduces to (1.3).

In this paper, motivated by the above related literature, we introduce a new problem (1.5) and the iterative scheme for finding a common element of the set of fixed point of a nonexpansive mapping and the set of solution of the problem (1.5) in a real Hilbert space. Then, we establish and prove the strong convergence theorem under some proper conditions. Furthermore, we also give a numerical example to support our main theorem.

2 Preliminaries

In this section, we give some useful lemmas and remarks that will be needed to prove our main result.

Let C be a nonempty closed convex subset of a real Hilbert space H. We denote weak convergence and strong convergence by notations $\rightarrow$ and $\rightharpoonup$, respectively. For every $x \in H$, there exists a unique nearest point $P_C x$ in C such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called *metric projection* of H onto C.

It is well known that metric projection P_C has the following properties:

(i) P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \forall x, y \in H.$$

Obviously, it implies that

$$\|(x - y) - (P_C x - P_C y)\|^2 \leq \|x - y\|^2 - \|P_C x - P_C y\|^2, \forall x, y \in H.$$

(ii) For each $x \in H$,

$$y = P_C x \Leftrightarrow \langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Lemma 2.1 [26] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions

(i) $\{\alpha_n\} \subset [0, 1], \ \sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty.$

Then, $\lim_{n\to\infty} s_n = 0.$

Lemma 2.2 [20] *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $\{x_n\} \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

Lemma 2.3 [25] *Let H be a Hilbert space, let C be a nonempty closed convex of H, and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \ \Leftrightarrow \ u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.4 [24] *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequence in a Banach space* X. *Let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. *Suppose that*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n, \quad \forall n \ge 0$$

and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then, $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.5 [21] *In real Hilbert spaces* H, *the following well-known results hold:*

(i) *For all* $x, y \in H$ *and* $\alpha \in [0, 1]$,

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2,$$

(ii) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$ *for all* $x, y \in H$.

Lemma 2.6 [23] *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *For every* $i = 1, 2, ..., N$, *let* A_i *be a strongly positive linear bounded operator on a Hilbert space* H *with coefficient* $\gamma_i > 0$ *and* $\overline{\gamma} = \min_{i=1,2,...,N}\gamma_i$. *Let* $\{a_i\}_{i=1}^N \subseteq (0, 1)$, *with* $\sum_{i=1}^N a_i = 1$. *Then, the following properties hold:*

(i) $\left\|I - \rho\sum_{i=1}^N a_i A_i\right\| \le 1 - \rho\overline{\gamma}$ *and* $I - \rho\sum_{i=1}^N a_i A_i$ *is a nonexpansive mapping for every* $0 < \rho < \|A_i\|^{-1} (i = 1, 2, ..., N)$.
(ii) $VI(C, \sum_{i=1}^N a_i A_i) = \bigcap_{i=1}^N VI(C, A_i)$.

Lemma 2.7 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $D_1, D_2, D_3 : C \to H$ *be three mappings. For every* $\lambda_1, \lambda_2, \lambda_3 > 0$ *and* $a \in [0, 1]$. *The following statements are equivalent*

(a) $(x^*, y^*, z^*) \in C \times C \times C$ *is a solution of problem* (1.5)
(b) x^* *is a fixed point of the mapping* G, *i.e.* $x^* \in F(G)$, *defined the mapping* $G : C \to C$ *by*

$$G(x) = P_C(I-\lambda_1 D_1)\,(ax+(1-a)P_C(I-\lambda_2 D_2)\,(ax+(1-a)P_C(I-\lambda_3 D_3)x)),$$

$\forall x \in C$, *where* $y^* = P_C(I - \lambda_2 D_2)\left(ax^* + (1 - a)z^*\right)$ *and* $z^* = P_C(I - \lambda_3 D_3)x^*$.

Proof Let conditions hold.

(a) ⇒ (b) Suppose that $(x^*, y^*, z^*) \in C \times C \times C$ is a solution of (1.5). For every $x \in C$, we have

$$\begin{aligned}\langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), x - x^* \rangle &\geq 0,\\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), x - y^* \rangle &\geq 0,\\ \langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle &\geq 0.\end{aligned}$$

From properties of P_C, we have

$$\begin{aligned}x^* &= P_C(I - \lambda_1 D_1)\left(ax^* + (1-a)y^*\right),\\ y^* &= P_C(I - \lambda_2 D_2)\left(ax^* + (1-a)z^*\right),\\ z^* &= P_C(I - \lambda_3 D_3)x^*.\end{aligned}$$

It implies that

$$\begin{aligned}x^* &= P_C(I - \lambda_1 D_1)\left(ax^* + (1-a)P_C(I - \lambda_2 D_2)\left(ax^* + (1-a)P_C(I - \lambda_3 D_3)x^*\right)\right)\\ &= G(x^*).\end{aligned}$$

It follows that $x^* \in F(G)$, where $y^* = P_C(I - \lambda_2 D_2)(ax^* + (1-a)z^*)$ and $z^* = P_C(I - \lambda_3 D_3)x^*$.

(b) ⇒ (a) Let $x^* \in F(G)$, $y^* = P_C(I - \lambda_2 D_2)(ax^* + (1-a)z^*)$ and $z^* = P_C(I - \lambda_3 D_3)x^*$.

Since $x^* \in F(G)$, we have

$$\begin{aligned}x^* &= P_C(I - \lambda_1 D_1)\left(ax^* + (1-a)P_C(I - \lambda_2 D_2)\left(ax^* + (1-a)P_C(I - \lambda_3 D_3)x^*\right)\right)\\ &= P_C(I - \lambda_1 D_1)(ax^* + (1-a)y^*).\end{aligned} \tag{2.1}$$

From (2.1), $y^* = P_C(I - \lambda_2 D_2)(ax^* + (1-a)z^*)$ and $z^* = P_C(I - \lambda_3 D_3)x^*$, we have

$$\begin{aligned}\langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), x - x^* \rangle &\geq 0, \quad \forall x \in C,\\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), x - y^* \rangle &\geq 0, \quad \forall x \in C,\\ \langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle &\geq 0, \quad \forall x \in C.\end{aligned}$$

It follows that $(x^*, y^*, z^*) \in C \times C \times C$ is a solution of (1.5). □

Example 2.8. *Let* $\mathbb{R}$ *be the set of real numbers and* D_1, D_2, D_3 *be mappings from* $[0, 20]$ *to* $\mathbb{R}$ *defined by* $D_1 x = x - 1$, $D_2 x = x + 2$ *and* $D_3 x = x - 22$, *for all* $x \in [0, 20]$, *respectively. Let mapping* $G : [0, 20] \to [0, 20]$ *be defined by*

$$\begin{aligned}G(x) = P_{[0,20]}\left(I - \frac{1}{2}D_1\right)&\left(0.5x + 0.5P_{[0,20]}\left(I - \frac{1}{3}D_2\right)\right.\\ &\left.\times\left(0.5x + 0.5P_{[0,20]}\left(I - \frac{1}{2}D_3\right)x\right)\right),\end{aligned}$$

where $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{3}, \lambda_3 = \frac{1}{2}$ *and* $a = 0.5$. *Then, we have* $2 \in F(G)$, *where* $z^* = P_{[0,20]}\left(I - \frac{1}{2}D_3\right)x^*$. *and* $y^* = P_{[0,20]}\left(I - \frac{1}{3}D_2\right)(0.5(x^*) + 0.5(z^*))$. *Hence,* $(x^*, y^*, z^*) = (2, 4, 12)$ *is a solution of* (1.5), *by Lemma 2.6.*

Remark 2.9 If D_1, D_2, D_3, in Lemma 2.7, are d_1, d_2, d_3-inverse strongly monotone, respectively, then G is nonexpansive mapping, where $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2, d_3\}$.

Proof Since D_1 is d_1-inverse strongly monotone mapping, we have

$$\begin{aligned}
\|(I - \lambda_1 D_1)x - (I - \lambda_1 D_1)y\|^2 &= \|x - y - \lambda_1(D_1x - D_1y)\|^2 \\
&= \|x - y\|^2 - 2\lambda_1 \langle x - y, D_1x - D_1y\rangle \\
&\quad -\lambda_1{}^2\|D_1x - D_1y\|^2 \\
&\le \|x - y\|^2 - 2\lambda_1 d_1\|D_1x - D_1y\|^2 \\
&\quad -\lambda_1{}^2\|D_1x - D_1y\|^2 \\
&= \|x - y\|^2 - \lambda_1(2d_1 - \lambda_1)\|D_1x - D_1y\|^2 \\
&\le \|x - y\|^2 - \lambda_1(2\overline{d} - \lambda_1)\|D_1x - D_1y\|^2 \\
&\le \|x - y\|^2.
\end{aligned}$$

It implies that

$$\|(I - \lambda_1 D_1)x - (I - \lambda_1 D_1)y\| \le \|x - y\|. \tag{2.2}$$

Hence, $(I - \lambda_1 D_1)$ is nonexpansive.

By using the same method as (2.2), we have $(I - \lambda_2 D_2)$ and $(I - \lambda_3 D_3)$ is nonexpansive. Then, we obtain that $P_C(I - \lambda_1 D_1)$, $P_C(I - \lambda_2 D_2)$, and $P_C(I - \lambda_3 D_3)$ are nonexpansive.

From $P_C(I - \lambda_1 D_1)$, $P_C(I - \lambda_2 D_2)$, and $P_C(I - \lambda_3 D_3)$ are nonexpansive, we have

$$\begin{aligned}
\|Gx - Gy\| &= \|P_C(I - \lambda_1 D_1)(ax + (1 - a)P_C(I - \lambda_2 D_2)(ax + (1 - a) \\
&\qquad \times P_C(I - \lambda_3 D_3)x)) \\
&\qquad - P_C(I - \lambda_1 D_1)(ay + (1 - a)P_C(I - \lambda_2 D_2)(ay + (1 - a) \\
&\qquad \times P_C(I - \lambda_3 D_3)y))\| \\
&\le \|(ax + (1 - a)P_C(I - \lambda_2 D_2)(ax + (1 - a)P_C(I - \lambda_3 D_3)x)) \\
&\qquad - (ay + (1 - a)P_C(I - \lambda_2 D_2)(ay + (1 - a)P_C(I - \lambda_3 D_3)y))\| \\
&\le a\|x - y\| + (1 - a)\|P_C(I - \lambda_2 D_2)(ax + (1 - a)P_C(I - \lambda_3 D_3)x) \\
&\qquad - P_C(I - \lambda_2 D_2)(ay + (1 - a)P_C(I - \lambda_3 D_3)y)\| \\
&\le a\|x - y\| + (1 - a)\|(ax + (1 - a)P_C(I - \lambda_3 D_3)x) \\
&\qquad - (ay + (1 - a)P_C(I - \lambda_3 D_3)y)\| \\
&\le a\|x - y\| + (1 - a)(a\|x - y\| \\
&\qquad + (1 - a)\|P_C(I - \lambda_3 D_3)x - P_C(I - \lambda_3 D_3)y\|) \\
&\le a\|x - y\| + (1 - a)(a\|x - y\| + (1 - a)\|x - y\|) \\
&= \|x - y\|.
\end{aligned}$$

Therefore, G is a nonexpansive mapping. □

Remark 2.10 If $a = 0$, in Lemma 2.7, then the following statements are equivalent

(a) $(x^*, y^*, z^*) \in C \times C \times C$ is a solution of problem (1.6)
(b) x^* is a fixed point of the mapping G, i.e. $x^* \in F(G)$, defined the mapping $G : C \to C$ by $G(x) = P_C(I-\lambda_1 D_1)P_C(I-\lambda_2 D_2)P_C(I-\lambda_3 D_3)x,\ \forall x \in C$, where $y^* = P_C(I - \lambda_2 D_2)z^*$ and $z^* = P_C(I - \lambda_3 D_3)x^*$.

Remark 2.11 If $x^* = y^* = z^*$, in Remark 2.10, then the following statements are equivalent

(a) $x^* \in VI(C, D_1) \cap VI(C, D_2) \cap VI(C, D_3)$
(b) x^* is a fixed point of the mapping G, i.e. $x^* \in F(G)$, defined the mapping $G : C \to C$ by $G(x) = P_C(I-\lambda_1 D_1)P_C(I-\lambda_2 D_2)P_C(I-\lambda_3 D_3)x,\ \forall x \in C$, where $x^* = P_C(I - \lambda_2 D_2)x^* = P_C(I - \lambda_3 D_3)x^*$.

3 Main result

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D, D_1, D_2, D_3 : C \to H$ be d, d_1, d_2, d_3-inverse strongly monotone mappings, respectively. Defined the mapping G as in Lemma (2.7) and $a \in [0, 1)$. For $k = 1, 2, ..., \overline{N}$, defined $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,\overline{N}} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Let $T : C \to C$ be a nonexpansive mapping with $\Omega = F(T) \cap F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{aligned} x_{n+1} &= \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n, \\ y_n &= \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n, \end{aligned} \tag{3.1}$$

where f is α-contraction mapping on C, $\{\alpha_n\} \subset [0, 1]$, $\lambda \in (0, 2d)$, $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2, d_3\}$ and $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < b \le \beta_n^i \le c < 1$ *for all* $i = 1, 2, 3$;
(iii) $0 < c_k < 1$ *and* $\sum_{k=1}^{\overline{N}} c_k = 1$;
(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} \left|\beta_{n+1}^i - \beta_n^i\right| < \infty$, *for all* $i = 1, 2, 3$.

Then, $\{x_n\}$ converges strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (1.5) where $y_0 = P_C(I - \lambda_2 D_2)(ax_0 + (1 - a)z_0)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$.

Proof Since $\alpha_n \to 0$ as $n \to \infty$, without loss of generality, we may assume that $\alpha_n < \frac{1}{\|A_i\|}$, $\forall n \in \mathbb{N}$ and $i = 1, 2, ..., \overline{N}$. By Lemma 2.6, we have $\|I - \alpha_n \overline{A}\| \leq 1 - \alpha_n \overline{\gamma}$.

Since D is d-inverse strongly monotone mapping with $\lambda \in (0, 2d)$ by using the same method as (2.2), we can conclude that $(I - \lambda D)$ is a nonexpansive mapping. Then, we obtain $P_C(I - \lambda D)$ is a nonexpansive mapping.

From Remark 2.9, we get that G is a nonexpansive mapping.

Step 1. We show that $\{x_n\}$ is bounded.

Let $x^* \in \Omega$.

From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|\beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n - x^*\right\| \\
&\leq \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|T x_n - x^*\| + \beta_n^3 \|P_C(I - \lambda D) y_n \\
&\quad - P_C(I - \lambda D) x^*\| \\
&\leq \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^3 \|y_n - x^*\| \\
&= \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^3 \|\alpha_n \gamma f(x_n) \\
&\quad + \left(I - \alpha_n \overline{A}\right) G x_n - x^*\| \\
&\leq \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^3 \alpha_n \|\gamma f(x_n) + \overline{A} x^*\| \\
&\quad + \beta_n^3 \|I - \alpha_n \overline{A}\| \|G x_n - x^*\| \\
&\leq \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^3 \alpha_n \gamma \|f(x_n) + f(x^*)\| \\
&\quad + \beta_n^3 \alpha_n \|\gamma f(x^*) + \overline{A} x^*\| + \beta_n^3 (1 - \alpha_n \overline{\gamma}) \|G x_n - x^*\| \\
&\leq \beta_n^1 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^3 \alpha_n \gamma \alpha \|x_n + x^*\| \\
&\quad + \beta_n^3 (1 - \alpha_n \overline{\gamma}) \|x_n - x^*\| + \beta_n^3 \alpha_n \|\gamma f(x^*) + \overline{A} x^*\| \\
&= \left(\beta_n^1 + \beta_n^2 + \beta_n^3 \alpha_n \gamma \alpha + \beta_n^3 (1 - \alpha_n \overline{\gamma})\right) \|x_n - x^*\| \\
&\quad + \beta_n^3 \alpha_n \|\gamma f(x^*) + \overline{A} x^*\| \\
&= \left(1 - \beta_n^3 + \beta_n^3 (\alpha_n \gamma \alpha + 1 - \alpha_n \overline{\gamma})\right) \|x_n - x^*\| \\
&\quad + \beta_n^3 \alpha_n \|\gamma f(x^*) + \overline{A} x^*\| \\
&= \left(1 - \beta_n^3 + \beta_n^3 (1 - \alpha_n (\overline{\gamma} - \gamma \alpha))\right) \|x_n - x^*\| \\
&\quad + \beta_n^3 \alpha_n \|\gamma f(x^*) + \overline{A} x^*\| \\
&= \left(1 - \beta_n^3 \alpha_n (\overline{\gamma} - \gamma \alpha)\right) \|x_n - x^*\| + \beta_n^3 \alpha_n \|\gamma f(x^*) + \overline{A} x^*\| \\
&\leq \max \left\{ \|x_1 - x^*\|, \frac{\|\gamma f(x^*) + \overline{A} x^*\|}{\overline{\gamma} - \gamma \alpha} \right\}.
\end{aligned}$$

By induction, we get that $\|x_n - x^*\| \leq \max\left\{\|x_1 - x^*\|, \frac{\|\gamma f(x^*)+\overline{A}x^*\|}{\overline{\gamma}-\gamma\alpha}\right\}$, $\forall n \in \mathbb{N}$.

Hence, $\{x_n\}$ is bounded and so is $\{y_n\}$.

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

From the definition of $\{y_n\}$, we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \left\|\alpha_{n+1}\gamma f(x_{n+1}) + \left(I - \alpha_{n+1}\overline{A}\right) Gx_{n+1} - \alpha_n \gamma f(x_n) - \left(I - \alpha_n \overline{A}\right) Gx_n\right\| \\
&= \|\alpha_{n+1}\gamma f(x_{n+1}) - \alpha_{n+1}\gamma f(x_n) + \alpha_{n+1}\gamma f(x_n) - \alpha_n \gamma f(x_n) \\
&\quad + \left(I - \alpha_{n+1}\overline{A}\right) Gx_{n+1} - \left(I - \alpha_{n+1}\overline{A}\right) Gx_n + \left(I - \alpha_{n+1}\overline{A}\right) Gx_n \\
&\quad - \left(I - \alpha_n \overline{A}\right) Gx_n\| \\
&\leq \alpha_{n+1}\gamma \|f(x_{n+1}) - f(x_n)\| + \gamma |\alpha_{n+1} - \alpha_n| \|f(x_n)\| \\
&\quad + \left\|I - \alpha_{n+1}\overline{A}\right\| \|Gx_{n+1} - Gx_n\| \\
&\quad + \left\|\left(I - \alpha_{n+1}\overline{A}\right) Gx_n - \left(I - \alpha_n \overline{A}\right) Gx_n\right\| \\
&\leq \alpha_{n+1}\gamma\alpha \|x_{n+1} - x_n\| + \gamma |\alpha_{n+1} - \alpha_n| \|f(x_n)\| \\
&\quad + (1 - \alpha_{n+1}\overline{\gamma}) \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n| \left\|\overline{A}Gx_n\right\| \\
&= (1 - \alpha_{n+1}(\overline{\gamma} - \gamma\alpha)) \|x_{n+1} - x_n\| + \gamma |\alpha_{n+1} - \alpha_n| \|f(x_n)\| \\
&\quad + |\alpha_{n+1} - \alpha_n| \left\|\overline{A}Gx_n\right\| \\
&\leq \|x_{n+1} - x_n\| + \gamma |\alpha_{n+1} - \alpha_n| \|f(x_n)\| + |\alpha_{n+1} - \alpha_n| \left\|\overline{A}Gx_n\right\|. \qquad (3.2)
\end{aligned}$$

Let

$$x_{n+1} = \left(1 - \beta_n^1\right) z_n + \beta_n^1 x_n, \qquad (3.3)$$

where $z_n = \frac{x_{n+1} - \beta_n^1 x_n}{1 - \beta_n^1}$.

Since $x_{n+1} - \beta_n^1 x_n = \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n$ and (3.2), we have

$$\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}^1 x_{n+1}}{1 - \beta_{n+1}^1} - \frac{x_{n+1} - \beta_n^1 x_n}{1 - \beta_n^1} \\
&= \frac{\beta_{n+1}^2 T x_{n+1} + \beta_{n+1}^3 P_C(I - \lambda D) y_{n+1}}{1 - \beta_{n+1}^1} - \frac{\beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n}{1 - \beta_n^1} \\
&= \frac{\beta_{n+1}^2}{1 - \beta_{n+1}^1} T x_{n+1} - \frac{\beta_n^2}{1 - \beta_n^1} T x_n + \frac{\beta_{n+1}^3}{1 - \beta_{n+1}^1} P_C(I - \lambda D) y_{n+1} \\
&\quad - \frac{\beta_n^3}{1 - \beta_n^1} P_C(I - \lambda D) y_n \\
&= \frac{\beta_{n+1}^2}{1 - \beta_{n+1}^1} T x_{n+1} - \frac{\beta_{n+1}^2}{1 - \beta_{n+1}^1} T x_n + \frac{\beta_{n+1}^2}{1 - \beta_{n+1}^1} T x_n - \frac{\beta_n^2}{1 - \beta_n^1} T x_n
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta_{n+1}^{3}}{1-\beta_{n+1}^{1}} P_C(I-\lambda D)y_{n+1} - \frac{\beta_{n+1}^{3}}{1-\beta_{n+1}^{1}} P_C(I-\lambda D)y_n \\
&+\frac{\beta_{n+1}^{3}}{1-\beta_{n+1}^{1}} P_C(I-\lambda D)y_n - \frac{\beta_{n}^{3}}{1-\beta_{n}^{1}} P_C(I-\lambda D)y_n \\
=\ & \frac{\beta_{n+1}^{2}}{1-\beta_{n+1}^{1}} (Tx_{n+1}-Tx_n) + \left(\frac{\beta_{n+1}^{2}}{1-\beta_{n+1}^{1}} - \frac{\beta_{n}^{2}}{1-\beta_{n}^{1}}\right) Tx_n \\
&+\frac{\beta_{n+1}^{3}}{1-\beta_{n}^{1}} (P_C(I-\lambda D)y_{n+1} - P_C(I-\lambda D)y_n) \\
&+\left(\frac{\beta_{n+1}^{3}}{1-\beta_{n+1}^{1}} - \frac{\beta_{n}^{3}}{1-\beta_{n}^{1}}\right) P_C(I-\lambda D)y_n.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
\|z_{n+1}-z_n\| \le\ & \frac{\beta_{n+1}^{2}}{1-\beta_{n+1}^{1}} \|Tx_{n+1}-Tx_n\| + \left|\frac{\beta_{n+1}^{2}}{1-\beta_{n+1}^{1}} - \frac{\beta_{n}^{2}}{1-\beta_{n}^{1}}\right| \|Tx_n\| + \frac{\beta_{n+1}^{3}}{1-\beta_{n}^{1}} \\
&\times \|P_C(I-\lambda D)y_{n+1} - P_C(I-\lambda D)y_n\| \\
&+\left|\frac{\beta_{n+1}^{3}}{1-\beta_{n+1}^{1}} - \frac{\beta_{n}^{3}}{1-\beta_{n}^{1}}\right| \|P_C(I-\lambda D)y_n\| \\
\le\ & \frac{\beta_{n+1}^{2}}{1-\beta_{n+1}^{1}} \|x_{n+1}-x_n\| + \frac{\beta_{n+1}^{3}}{1-\beta_{n}^{1}} \|y_{n+1}-y_n\| \\
&+\left|\frac{\beta_{n+1}^{2}}{1-\beta_{n+1}^{1}} - \frac{\beta_{n}^{2}}{1-\beta_{n+1}^{1}} + \frac{\beta_{n}^{2}}{1-\beta_{n+1}^{1}} - \frac{\beta_{n}^{2}}{1-\beta_{n}^{1}}\right| \|Tx_n\| \\
&+\left|\frac{\beta_{n+1}^{3}}{1-\beta_{n+1}^{1}} - \frac{\beta_{n}^{3}}{1-\beta_{n+1}^{1}} + \frac{\beta_{n}^{3}}{1-\beta_{n+1}^{1}} - \frac{\beta_{n}^{3}}{1-\beta_{n}^{1}}\right| \|P_C(I-\lambda D)y_n\| \\
\le\ & \frac{\beta_{n+1}^{2}}{1-\beta_{n+1}^{1}} \|x_{n+1}-x_n\| + \frac{\beta_{n+1}^{3}}{1-\beta_{n}^{1}} \|y_{n+1}-y_n\| + \left|\frac{\beta_{n+1}^{2}-\beta_{n}^{2}}{1-\beta_{n+1}^{1}}\right| \|Tx_n\| \\
&+\beta_n^2 \left|\frac{\beta_{n}^{1}-\beta_{n+1}^{1}}{(1-\beta_{n+1}^{1})(1-\beta_{n+1}^{1})}\right| \|Tx_n\| + \left|\frac{\beta_{n+1}^{3}-\beta_{n}^{3}}{1-\beta_{n+1}^{1}}\right| \|P_C(I-\lambda D)y_n\| \\
&+\beta_n^3 \left|\frac{\beta_{n}^{1}-\beta_{n+1}^{1}}{(1-\beta_{n+1}^{1})(1-\beta_{n+1}^{1})}\right| \|P_C(I-\lambda D)y_n\| \\
\le\ & \frac{\beta_{n+1}^{2}}{1-\beta_{n+1}^{1}} \|x_{n+1}-x_n\| + \frac{\beta_{n+1}^{3}}{1-\beta_{n}^{1}} (\|x_{n+1}-x_n\| + \gamma\,|\alpha_{n+1}-\alpha_n|\,\|f(x_n)\| \\
&+|\alpha_{n+1}-\alpha_n|\,\|\bar{A}Gx_n\|) + \left|\frac{\beta_{n+1}^{2}-\beta_{n}^{2}}{1-\beta_{n+1}^{1}}\right| \|Tx_n\| + \left|\frac{\beta_{n}^{1}-\beta_{n+1}^{1}}{(1-\beta_{n+1}^{1})(1-\beta_{n+1}^{1})}\right| \|Tx_n\| \\
&+\left|\frac{\beta_{n+1}^{3}-\beta_{n}^{3}}{1-\beta_{n+1}^{1}}\right| \|P_C(I-\lambda D)y_n\| + \left|\frac{\beta_{n}^{1}-\beta_{n+1}^{1}}{(1-\beta_{n+1}^{1})(1-\beta_{n+1}^{1})}\right| \|P_C(I-\lambda D)y_n\| \\
=\ & \frac{\beta_{n+1}^{2}}{1-\beta_{n+1}^{1}} \|x_{n+1}-x_n\| + \frac{\beta_{n+1}^{3}}{1-\beta_{n+1}^{1}} \|x_{n+1}-x_n\| + \frac{\beta_{n+1}^{3}}{1-\beta_{n}^{1}} \gamma\,|\alpha_{n+1}-\alpha_n|\,\|f(x_n)\| \\
&+\frac{\beta_{n+1}^{3}}{1-\beta_{n}^{1}} |\alpha_{n+1}-\alpha_n|\,\|\bar{A}Gx_n\| + \left|\frac{\beta_{n+1}^{2}-\beta_{n}^{2}}{1-\beta_{n+1}^{1}}\right| \|Tx_n\| + \left|\frac{\beta_{n}^{1}-\beta_{n+1}^{1}}{(1-\beta_{n+1}^{1})(1-\beta_{n+1}^{1})}\right|
\end{aligned}
$$

$$\times \|Tx_n\| + \left|\frac{\beta_{n+1}^3 - \beta_n^3}{1-\beta_{n+1}^1}\right| \|P_C(I-\lambda D)y_n\| + \left|\frac{\beta_n^1 - \beta_{n+1}^1}{\left(1-\beta_{n+1}^1\right)\left(1-\beta_{n+1}^1\right)}\right| \|P_C(I-\lambda D)y_n\|$$

$$= \|x_{n+1}-x_n\| + \frac{\beta_{n+1}^3}{1-\beta_n^1}\gamma\,|\alpha_{n+1}-\alpha_n|\,\|f(x_n)\| + \frac{\beta_{n+1}^3}{1-\beta_n^1}\,|\alpha_{n+1}-\alpha_n|\,\|\bar{A}Gx_n\|$$

$$+ \left|\frac{\beta_{n+1}^2 - \beta_n^2}{1-\beta_{n+1}^1}\right| \|Tx_n\| + \left|\frac{\beta_n^1 - \beta_{n+1}^1}{\left(1-\beta_{n+1}^1\right)\left(1-\beta_{n+1}^1\right)}\right| \|Tx_n\|$$

$$+ \left|\frac{\beta_{n+1}^3 - \beta_n^3}{1-\beta_{n+1}^1}\right| \|P_C(I-\lambda D)y_n\| + \left|\frac{\beta_n^1 - \beta_{n+1}^1}{\left(1-\beta_{n+1}^1\right)\left(1-\beta_{n+1}^1\right)}\right| \|P_C(I-\lambda D)y_n\|.$$

From conditions (ii) and (iii), we have $\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0$.
By Lemma 2.4 and (3.3), we get that $\lim_{n\to\infty}\|z_n - x_n\| = 0$.
Since $x_{n+1}-x_n = \left(1-\beta_n^1\right)(z_n - x_n)$, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.4}$$

From (3.2) and (3.4) and condition (iii), we obtain $\lim_{n\to\infty}\|y_{n+1}-y_n\| = 0$.
By the definition of y_n, we have

$$\begin{aligned}\|y_n - x^*\|^2 &= \|\alpha_n\gamma f(x_n) + \left(I-\alpha_n\bar{A}\right)Gx_n - x^*\|^2\\ &= \left\|\left(Gx_n - x^*\right) + \left(\alpha_n\gamma f(x_n) - \alpha_n\bar{A}Gx_n\right)\right\|^2\\ &\le \|Gx_n - x^*\|^2 + 2\alpha_n\left\langle\gamma f(x_n) - \bar{A}Gx_n, y_n - x^*\right\rangle\\ &\le \|x_n - x^*\|^2 + 2\alpha_n\|\gamma f(x_n) - \bar{A}Gx_n\|\,\|y_n - x^*\|. \end{aligned} \tag{3.5}$$

From nonexpansiveness of P_C and (3.5), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \beta_n^1\|x_n - x^*\|^2 + \beta_n^2\|Tx_n - x^*\|^2 + \beta_n^3\|P_C(I-\lambda D)y_n - x^*\|^2\\ &\le \beta_n^1\|x_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 + \beta_n^3\|(I-\lambda D)y_n - (I-\lambda D)x^*\|^2\\ &= \beta_n^1\|x_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 + \beta_n^3\|y_n - x^* - \lambda(Dy_n - Dx^*)\|^2\\ &= \beta_n^1\|x_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 + \beta_n^3\|y_n - x^*\|^2\\ &\quad -2\lambda\beta_n^3\langle y_n - x^*, Dy_n - Dx^*\rangle + \beta_n^3\lambda^2\|Dy_n - Dx^*\|^2\\ &\le \beta_n^1\|x_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 + \beta_n^3\|y_n - x^*\|^2 - 2\lambda d\beta_n^3\|Dy_n - Dx^*\|^2\\ &\quad +\beta_n^3\lambda^2\|Dy_n - Dx^*\|^2\\ &\le \beta_n^1\|x_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 + \beta_n^3\|x_n - x^*\|^2\\ &\quad +2\beta_n^3\alpha_n\|\gamma f(x_n) - \overline{A}Gx_n\|\,\|y_n - x^*\| - \lambda\beta_n^3(2d-\lambda)\|Dy_n - Dx^*\|^2\\ &= \|x_n - x^*\|^2 + 2\beta_n^3\alpha_n\|\gamma f(x_n) - \overline{A}Gx_n\|\,\|y_n - x^*\|\\ &\quad -\lambda\beta_n^3(2d-\lambda)\|Dy_n - Dx^*\|^2. \end{aligned}$$

It implies that

$$\begin{aligned}\lambda\beta_n^3(2d-\lambda)\left\|Dy_n-Dx^*\right\|^2 &\le 2\beta_n^3\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\left\|y_n-x^*\right\|+\left\|x_n-x^*\right\|^2\\&\quad-\left\|x_{n+1}-x^*\right\|^2\\&\le 2\beta_n^3\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\left\|y_n-x^*\right\|\\&\quad+\left(\left\|x_n-x^*\right\|+\left\|x_{n+1}-x^*\right\|\right)\|x_{n+1}-x_n\|. \qquad (3.6)\end{aligned}$$

From conditions (i) and (ii) and (3.4) and (3.6), we have

$$\lim_{n\to\infty}\left\|Dy_n-Dx^*\right\|=0. \qquad (3.7)$$

Step 3. We show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, $\lim_{n\to\infty}\|x_n-Gx_n\|=0$, $\lim_{n\to\infty}\|y_n-Ty_n\|=0$, $\lim_{n\to\infty}\|y_n-Gy_n\|=0$ and $\lim_{n\to\infty}\|x_n-y_n\|=0$.
From the definition of x_n and (3.5), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \left\|\beta_n^1x_n+\beta_n^2Tx_n+\beta_n^3P_C(I-\lambda D)y_n-x^*\right\|^2\\&\le \beta_n^1\left\|x_n-x^*\right\|^2+\beta_n^2\left\|Tx_n-x^*\right\|^2+\beta_n^3\left\|P_C(I-\lambda D)y_n-x^*\right\|^2\\&\quad-\beta_n^1\beta_n^2\|x_n-Tx_n\|^2-\beta_n^1\beta_n^3\|x_n-P_C(I-\lambda D)y_n\|^2\\&\le \beta_n^1\left\|x_n-x^*\right\|^2+\beta_n^2\left\|x_n-x^*\right\|^2+\beta_n^3\left\|y_n-x^*\right\|^2-\beta_n^1\beta_n^2\|x_n-Tx_n\|^2\\&\quad-\beta_n^1\beta_n^3\|x_n-P_C(I-\lambda D)y_n\|^2\\&\le \beta_n^1\left\|x_n-x^*\right\|^2+\beta_n^2\left\|x_n-x^*\right\|^2+\beta_n^3\left\|x_n-x^*\right\|^2+2\beta_n^3\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\\&\quad\times\left\|y_n-x^*\right\|-\beta_n^1\beta_n^2\|x_n-Tx_n\|^2-\beta_n^1\beta_n^3\|x_n-P_C(I-\lambda D)y_n\|^2\\&= \left\|x_n-x^*\right\|^2+2\beta_n^3\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\left\|y_n-x^*\right\|-\beta_n^1\beta_n^2\|x_n-Tx_n\|^2\\&\quad-\beta_n^1\beta_n^3\|x_n-P_C(I-\lambda D)y_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\beta_n^1\beta_n^2\|x_n-Tx_n\|^2 &\le 2\beta_n^3\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\left\|y_n-x^*\right\|+\left\|x_n-x^*\right\|^2\\&\quad-\left\|x_{n+1}-x^*\right\|^2-\beta_n^1\beta_n^3\|x_n-P_C(I-\lambda D)y_n\|^2\\&\le 2\beta_n^3\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\left\|y_n-x^*\right\|\\&\quad+\left(\left\|x_n-x^*\right\|+\left\|x_{n+1}-x^*\right\|\right)\|x_{n+1}-x_n\|. \qquad (3.8)\end{aligned}$$

From conditions (i) and (ii) and (3.4) and (3.8), we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \qquad (3.9)$$

By using the same method as (3.9), we get that

$$\lim_{n\to\infty}\|x_n-P_C(I-\lambda D)y_n\|=0. \qquad (3.10)$$

From property of P_C and (3.5), we have

$$\begin{aligned}\left\|P_C(I-\lambda D)y_n-x^*\right\|^2 &\le \left\langle (I-\lambda D)y_n-(I-\lambda D)x^*, P_C(I-\lambda D)y_n-x^*\right\rangle\\ &= \frac{1}{2}\Big(\left\|(I-\lambda D)y_n-(I-\lambda D)x^*\right\|^2+\left\|P_C(I-\lambda D)y_n-x^*\right\|^2\\ &\quad -\left\|(I-\lambda D)y_n-(I-\lambda D)x^*-\left(P_C(I-\lambda D)y_n-x^*\right)\right\|^2\Big)\\ &\le \frac{1}{2}\Big(\left\|y_n-x^*\right\|^2+\left\|P_C(I-\lambda D)y_n-x^*\right\|^2\\ &\quad -\left\|y_n-P_C(I-\lambda D)y_n-\lambda\left(Dy_n-Dx^*\right)\right\|^2\Big)\end{aligned}$$

$$\begin{aligned}&\le \frac{1}{2}\Big(\left\|y_n-x^*\right\|^2+\left\|P_C(I-\lambda D)y_n-x^*\right\|^2-\|y_n-P_C(I-\lambda D)y_n\|^2\\ &\quad +2\lambda\left\langle y_n-P_C(I-\lambda D)y_n, Dy_n-Dx^*\right\rangle-\lambda^2\left\|Dy_n-Dx^*\right\|^2\Big)\\ &\le \frac{1}{2}\Big(\left\|x_n-x^*\right\|^2+2\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\left\|y_n-x^*\right\|\\ &\quad +\left\|P_C(I-\lambda D)y_n-x^*\right\|^2-\|y_n-P_C(I-\lambda D)y_n\|^2\\ &\quad +2\lambda\|y_n-P_C(I-\lambda D)y_n\|\left\|Dy_n-Dx^*\right\|\Big).\end{aligned}$$

It implies that

$$\begin{aligned}\left\|P_C(I-\lambda D)y_n-x^*\right\|^2 &\le \left\|x_n-x^*\right\|^2+2\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\left\|y_n-x^*\right\|\\ &\quad -\|y_n-P_C(I-\lambda D)y_n\|^2+2\lambda\|y_n-P_C(I-\lambda D)y_n\|\\ &\quad \times\left\|Dy_n-Dx^*\right\|. \end{aligned}\tag{3.11}$$

From definition of x_n and (3.11), we get that

$$\begin{aligned}\left\|x_{n+1}-x^*\right\|^2 &\le \beta_n^1\left\|x_n-x^*\right\|^2+\beta_n^2\left\|Tx_n-x^*\right\|^2+\beta_n^3\left\|P_C(I-\lambda D)y_n-x^*\right\|^2\\ &\le \beta_n^1\left\|x_n-x^*\right\|^2+\beta_n^2\left\|x_n-x^*\right\|^2+\beta_n^3\Big(\left\|x_n-x^*\right\|^2\\ &\quad +2\alpha_n\left\|\gamma f(x_n)-\bar{A}Gx_n\right\|\left\|y_n-x^*\right\|-\|y_n-P_C(I-\lambda D)y_n\|^2\\ &\quad +2\lambda\|y_n-P_C(I-\lambda D)y_n\|\left\|Dy_n-Dx^*\right\|\Big)\\ &\le \beta_n^1\left\|x_n-x^*\right\|^2+\beta_n^2\left\|x_n-x^*\right\|^2+\beta_n^3\left\|x_n-x^*\right\|^2\\ &\quad +2\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\left\|y_n-x^*\right\|-\beta_n^3\|y_n-P_C(I-\lambda D)y_n\|^2\\ &\quad +2\lambda\|y_n-P_C(I-\lambda D)y_n\|\left\|Dy_n-Dx^*\right\|\\ &= \left\|x_n-x^*\right\|^2+2\alpha_n\left\|\gamma f(x_n)-\overline{A}Gx_n\right\|\left\|y_n-x^*\right\|\\ &\quad -\beta_n^3\|y_n-P_C(I-\lambda D)y_n\|^2+2\lambda\|y_n-P_C(I-\lambda D)y_n\|\\ &\quad \times\left\|Dy_n-Dx^*\right\|.\end{aligned}$$

It implies that

$$\begin{aligned}\beta_n^3\|y_n - P_C(I-\lambda D)y_n\|^2 &\le 2\alpha_n\|\gamma f(x_n) - \overline{A}Gx_n\|\|y_n - x^*\| + \|x_n - x^*\|^2 \\ &\quad -\|x_{n+1} - x^*\|^2 + 2\lambda\|y_n - P_C(I-\lambda D)y_n\|\|Dy_n - Dx^*\| \\ &\le 2\alpha_n\|\gamma f(x_n) - \overline{A}Gx_n\|\|y_n - x^*\| \\ &\quad + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\ &\quad + 2\lambda\|y_n - P_C(I-\lambda D)y_n\|\|Dy_n - Dx^*\|. \end{aligned} \tag{3.12}$$

From conditions (i) and (ii) and (3.4), (3.7), and (3.12), we have

$$\lim_{n\to\infty}\|y_n - P_C(I-\lambda D)y_n\| = 0. \tag{3.13}$$

Consider,

$$\|x_n - y_n\| \le \|x_n - P_C(I-\lambda D)y_n\| + \|P_C(I-\lambda D)y_n - y_n\|.$$

By (3.10) and (3.13), we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.14}$$

From definition of y_n and condition (i), we obtain

$$\begin{aligned}\|y_n - Gx_n\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n\overline{A})Gx_n - Gx_n\| \\ &= \alpha_n\|\gamma f(x_n) + \overline{A}Gx_n\| \to 0 \ \ as\ n \to \infty. \end{aligned} \tag{3.15}$$

Since,

$$\|x_n - Gx_n\| \le \|x_n - y_n\| + \|y_n - Gx_n\|,$$

(3.14) and (3.15), we get that

$$\lim_{n\to\infty}\|x_n - Gx_n\| = 0. \tag{3.16}$$

Moreover, from (3.9), (3.14), and

$$\begin{aligned}\|y_n - Ty_n\| &\le \|y_n - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Ty_n\| \\ &\le \|y_n - x_n\| + \|x_n - Tx_n\| + \|x_n - y_n\|,\end{aligned}$$

we have

$$\lim_{n\to\infty}\|y_n - Ty_n\| = 0. \tag{3.17}$$

Again, from (3.14), (3.16), and

$$\begin{aligned}\|y_n - Gy_n\| &\le \|y_n - x_n\| + \|x_n - Gx_n\| + \|Gx_n - Gy_n\| \\ &\le \|y_n - x_n\| + \|x_n - Gx_n\| + \|x_n - y_n\|,\end{aligned}$$

we have

$$\lim_{n\to\infty}\|y_n - Gy_n\| = 0. \tag{3.18}$$

Step 4. We show that $\limsup_{n\to\infty}\langle\gamma f(x_0) - \overline{A}x_0, y_n - x_0\rangle \le 0$, where $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$.

To show this inequality, take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle\gamma f(x_0)-\overline{A}x_0,\, y_n-x_0\rangle=\lim_{i\to\infty}\langle\gamma f(x_0)-\overline{A}x_0,\, y_{n_i}-x_0\rangle. \tag{3.19}$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_{n_i}\rightharpoonup q$ as $i\to\infty$, where $q\in C$. From (3.14) and $x_{n_i}\rightharpoonup q$ as $i\to\infty$, we get that $y_{n_i}\rightharpoonup q$ as $i\to\infty$.

From (3.19) and $y_{n_i}\rightharpoonup q$ as $i\to\infty$, we have

$$\limsup_{n\to\infty}\langle\gamma f(x_0)-\overline{A}x_0,\, y_n-x_0\rangle=\langle\gamma f(x_0)-\overline{A}x_0, q-x_0\rangle. \tag{3.20}$$

In order to show $\langle\gamma f(x_0)-\overline{A}x_0, q-x_0\rangle\le 0$, we need to show that $q\in\Omega=F(T)\cap F(G)\cap VI(C,D)$.

First, we show that $q\in F(T)$.

Assume that $q\notin F(T)$. Then, we have $q\neq Tq$. From (3.17) and Opial's condition, we obtain

$$\begin{aligned}\liminf_{n\to\infty}\|y_{n_i}-q\| &< \liminf_{n\to\infty}\|y_{n_i}-Tq\|\\ &\le \liminf_{n\to\infty}\left(\|y_{n_i}-Ty_{n_i}\|+\|Ty_{n_i}-Tq\|\right)\\ &\le \liminf_{n\to\infty}\left(\|y_{n_i}-Ty_{n_i}\|+\|y_{n_i}-q\|\right)\\ &\le \liminf_{n\to\infty}\|y_{n_i}-q\|.\end{aligned}$$

This is a contradiction, that is,

$$q\in F(T). \tag{3.21}$$

Show that $q\in F(G)$.

Assume that $q\notin F(G)$. Then, we have $q\neq Gq$. From (3.18) and Opial's condition , we obtain

$$\begin{aligned}\liminf_{n\to\infty}\|y_{n_i}-q\| &< \liminf_{n\to\infty}\|y_{n_i}-Gq\|\\ &\le \liminf_{n\to\infty}\left(\|y_{n_i}-Gy_{n_i}\|+\|Gy_{n_i}-Gq\|\right)\\ &\le \liminf_{n\to\infty}\left(\|y_{n_i}-Gy_{n_i}\|+\|y_{n_i}-q\|\right)\\ &\le \liminf_{n\to\infty}\|y_{n_i}-q\|.\end{aligned}$$

This is a contradiction, that is,

$$q\in F(G). \tag{3.22}$$

Show that $q\in VI(C,D)$.

Assume that $q\notin VI(C,D)$. Since $VI(C,D)=F(P_C(I-\lambda D))$, we have $q\neq P_C(I-\lambda D)q$. By nonexpansiveness of $P_C(I-\lambda D)$, (3.13) and Opial's condition, we obtain

$$\begin{aligned}\liminf_{n\to\infty}\|y_{n_i}-q\| &< \liminf_{n\to\infty}\|y_{n_i}-P_C(I-\lambda D)q\|\\ &\le \liminf_{n\to\infty}\left(\|y_{n_i}-P_C(I-\lambda D)y_{n_i}\|+\|P_C(I-\lambda D)y_{n_i}-P_C(I-\lambda D)q\|\right)\\ &\le \liminf_{n\to\infty}\left(\|y_{n_i}-P_C(I-\lambda D)y_{n_i}\|+\|y_{n_i}-q\|\right)\\ &\le \liminf_{n\to\infty}\|y_{n_i}-q\|.\end{aligned}$$

This is a contradiction, that is,

$$q \in VI(C, D). \tag{3.23}$$

From (3.21), (3.22), and (3.23), we obtain $q \in \Omega = F(T) \cap F(G) \cap VI(C, D)$. From (3.20) and property of P_C, we have

$$\limsup_{n\to\infty} \langle \gamma f(x_0) - \overline{A}x_0, y_n - x_0 \rangle = \langle \gamma f(x_0) - \overline{A}x_0, q - x_0 \rangle \le 0.$$

Step 5. We show that $\{x_n\}$ converges strongly to x_0, where $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$. From the definition of x_n and $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$, we have

$$\begin{aligned}
\|x_{n+1} - x_0\|^2 &= \left\|\beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_c(I - \lambda D) y_n - x_0\right\|^2 \\
&\le \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|T x_n - x_0\|^2 + \beta_n^3 \|P_c(I - \lambda D) y_n - x_0\|^2 \\
&\le \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 \|y_n - x_0\|^2 \\
&\le \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 \left\|\alpha_n \gamma f(x_n) + \left(I - \alpha_n \bar{A}\right) G x_n - x_0\right\|^2 \\
&\le \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 \left(\left\|\left(I - \alpha_n \bar{A}\right)(G x_n - x_0)\right\|^2\right. \\
&\quad \left. + 2\alpha_n \langle \gamma f(x_n) - \bar{A}x_0, y_n - x_0\rangle\right) \\
&\le \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 \left((1 - \alpha_n \overline{\gamma})^2 \|x_n - x_0\|^2\right. \\
&\quad \left. + 2\alpha_n \gamma \langle f(x_n) - f(x_0), y_n - x_0\rangle + 2\alpha_n \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle\right) \\
&\le \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 \left((1 - \alpha_n \overline{\gamma})^2 \|x_n - x_0\|^2\right. \\
&\quad \left. + 2\alpha_n \gamma \|f(x_n) - f(x_0)\| \|y_n - x_0\| + 2\alpha_n \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle\right) \\
&\le \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 (1 - \alpha_n \overline{\gamma})^2 \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n \gamma \alpha \beta_n^3 \|x_n - x_0\| \|y_n - x_0\| + 2\alpha_n \beta_n^3 \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle \\
&\le \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 (1 - \alpha_n \overline{\gamma})^2 \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n \gamma \alpha \beta_n^3 \|x_n - x_0\| \left(\alpha_n \left\|\gamma f(x_n) - \bar{A}x_0\right\| + (1 - \alpha_n \overline{\gamma}) \|G x_n - x_0\|\right) \\
&\quad + 2\alpha_n \beta_n^3 \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle \\
&\le \beta_n^1 \|x_n - x_0\|^2 + \beta_n^2 \|x_n - x_0\|^2 + \beta_n^3 (1 - \alpha_n \overline{\gamma})^2 \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n \gamma \alpha \beta_n^3 \|x_n - x_0\| \left(\alpha_n \gamma \alpha \|x_n - x_0\| + \alpha_n \left\|\gamma f(x_0) - \bar{A}x_0\right\|\right. \\
&\quad \left. + (1 - \alpha_n \overline{\gamma}) \|x_n - x_0\|\right) + 2\alpha_n \beta_n^3 \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle \\
&= (1 - \beta_n^3) \|x_n - x_0\|^2 + \beta_n^3 (1 - \alpha_n \overline{\gamma})^2 \|x_n - x_0\|^2 + 2\alpha_n^2 \gamma^2 \alpha^2 \beta_n^3 \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2 \gamma \alpha \beta_n^3 \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\| + 2\alpha_n \gamma \alpha \beta_n^3 (1 - \alpha_n \overline{\gamma}) \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n \beta_n^3 \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle \\
&\le (1 - \beta_n^3) \|x_n - x_0\|^2 + \beta_n^3 (1 - \alpha_n \bar{\gamma}) \|x_n - x_0\|^2 + 2\alpha_n^2 \bar{\gamma}^2 \beta_n^3 \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2 \bar{\gamma} \beta_n^3 \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\| + 2\alpha_n \gamma \alpha \beta_n^3 \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n \beta_n^3 \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle \\
&= \left(1 - \beta_n^3 + \beta_n^3 - \beta_n^3 \alpha_n \bar{\gamma} + 2\alpha_n \gamma \alpha \beta_n^3\right) \|x_n - x_0\|^2 \\
&\quad + 2\alpha_n^2 \bar{\gamma}^2 \beta_n^3 \|x_n - x_0\|^2 + 2\alpha_n^2 \bar{\gamma} \beta_n^3 \left\|\gamma f(x_0) - \bar{A}x_0\right\| \|x_n - x_0\| \\
&\quad + 2\alpha_n \beta_n^3 \langle \gamma f(x_0) - \bar{A}x_0, y_n - x_0\rangle
\end{aligned}$$

$$= \Big(1 - \alpha_n \beta_n^3 (\bar{\gamma} - 2\gamma\alpha)\Big) \|x_n - x_0\|^2 + \alpha_n \beta_n^3 \Big(2\alpha_n \bar{\gamma}^2 \|x_n - x_0\|^2$$
$$+ 2\alpha_n \bar{\gamma} \left\|\gamma f(x_0) - \bar{A} x_0\right\| \|x_n - x_0\| + 2\langle \gamma f(x_0) - \bar{A} x_0, y_n - x_0\rangle\Big)$$
$$= \Big(1 - \alpha_n \beta_n^3 (\bar{\gamma} - 2\gamma\alpha)\Big) \|x_n - x_0\|^2 + \alpha_n \beta_n^3 (\bar{\gamma} - 2\gamma\alpha) \left(\frac{2\alpha_n \bar{\gamma}^2 \|x_n - x_0\|^2}{(\bar{\gamma} - 2\gamma\alpha)}\right.$$
$$\left. + \frac{2\alpha_n \bar{\gamma} \left\|\gamma f(x_0) - \bar{A} x_0\right\| \|x_n - x_0\|}{(\bar{\gamma} - 2\gamma\alpha)} + \frac{2\langle \gamma f(x_0) - \bar{A} x_0, y_n - x_0\rangle}{(\bar{\gamma} - 2\gamma\alpha)}\right).$$

From step 4, condition (i) and Lemma 2.1, we can conclude that $\{x_n\}$ converges strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and by Lemma 2.7, we have (x_0, y_0, z_0) is a solution of (1.5) where $y_0 = P_C(I - \lambda_2 D_2)(ax_0 + (1-a)z_0)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$. This completes the proof. □

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D, D_1, D_2, D_3 : C \to H$ be d, d_1, d_2, d_3-inverse strongly monotone mappings, respectively. Defined the mapping $G : C \to C$ by $G(x) = P_C(I - \lambda_1 D_1)(P_C(I - \lambda_2 D_2)(P_C(I - \lambda_3 D_3)x))$, for all $x \in C$, $\lambda_1, \lambda_2, \lambda_3 > 0$. For $k = 1, 2, ..., \overline{N}$, defined $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,\overline{N}} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Let T be a nonexpansive mapping with $\Omega = F(T) \cap F(G) \cap VI(C, D) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{aligned} x_{n+1} &= \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n, \\ y_n &= \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n, \end{aligned} \quad (3.24)$$

where f is α-contraction mapping on H, $\{\alpha_n\} \subset [0, 1]$, $\lambda \in (0, 2d)$, $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2, d_3\}$ and $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \le \beta_n^i \le c < 1$ *for all* $i = 1, 2, 3$;

(iii) $0 < c_k < 1$ *and* $\sum_{k=1}^{\overline{N}} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} \left|\beta_{n+1}^i - \beta_n^i\right| < \infty$, *for all* $i = 1, 2, 3$.

Then, $\{x_n\}$ converges strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (1.6) where $y_0 = P_C(I - \lambda_2 D_2)(z_0)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$.

Proof From Theorem 3.1, if we put $a = 0$, we have the desired conclusion. □

4 Application

In this section, applying our main result Theorem 3.1, we can prove strong convergence theorems for approximating the solution of the standard constrained convex optimization problem and the split feasibility problem.

4.1 Minimization problem

Let C be closed convex subset of H. The *standard constrained convex optimization problem* is to find $x^* \in C$ such that

$$f(x^*) = \min_{x \in C} f(x), \tag{4.1}$$

where $f : C \to \mathbb{R}$ is a convex, Fr′echet differentiable function. The set of all solutions of (4.1) is denoted by Φ_f.

Lemma 4.1 [22] *(Optimality condition) A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem* (4.1) *is that x^* solves the variational inequality*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \tag{4.2}$$

for all $x \in C$. Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C(I - \lambda \nabla f)x^*,$$

for every $\lambda > 0$. If, in addition, f is convex, then the optimality condition (4.2) *is also sufficient.*

Theorem 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2, D_3 : C \to H$ be d_1, d_2, d_3- inverse strongly monotone mapping, respectively. Let $\overline{f} : C \to \mathbb{R}$ be a real-valued convex function with the gradient $\nabla \overline{f}$ is $\frac{1}{L_{\overline{f}}}$- inverse strongly monotone and continuous with $L_{\overline{f}} > 0$. Defined the mapping G as in Lemma 2.7 and $a \in [0, 1)$. For $k = 1, 2, ..., \overline{N}$, defined $\overline{A} : H \to H$ by $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, for all $x \in H$, where A_k is a strongly positive bounded linear operator on H with coefficient $\gamma_k > 0, \overline{\gamma} = \min_{k=1,2,...,\overline{N}} \gamma_k$ and $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. Let $T : C \to C$ be a nonexpansive mapping with $\Omega = F(T) \cap F(G) \cap \Phi_f \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{aligned} x_{n+1} &= \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda \nabla \overline{f}) y_n, \\ y_n &= \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n, \end{aligned} \tag{4.3}$$

where f is α-contraction mapping on H, $\{\alpha_n\} \subset [0, 1], \lambda \in (0, \frac{2}{L_{\overline{f}}})$, $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\overline{d})$ with $\overline{d} = \min\{d_1, d_2, d_3\}$ and $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n^i \leq c < 1$ *for all* $i = 1, 2, 3$;

(iii) $0 < c_k < 1$ *and* $\sum_{k=1}^{\overline{N}} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} \left|\beta_{n+1}^i - \beta_n^i\right| < \infty$, *for all* $i = 1, 2, 3$.

Then, $\{x_n\}$ *converges strongly to* $x_0 = P_\Omega(I-\overline{A}+\gamma f)x_0$ *and* (x_0, y_0, z_0) *is a solution of* (1.5) *where* $y_0 = P_C(I-\lambda_2 D_2)(ax_0+(1-a)z_0)$ *and* $z_0 = P_C(I-\lambda_3 D_3)x_0$.

Proof By using Lemma 4.1 and Theorem 3.1, we obtain the desired conclusion. □

4.2 The split feasibility problem

Let H_1, H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1, H_2, respectively.

The *split feasibility problem* (SFP) is to find a point x such that

$$x \in C \ \ and \ \ Ax \in Q, \tag{4.4}$$

where $A : H_1 \to H_2$ is a bounded linear operator and which is introduced by Censor and Elfving [15]. The set of solution of the split feasibility problem (SFP) is denoted by Γ. Many authors have studied the SFP (see for more details [1, 5]).

Note that if SFP (4.4) is consistent (that is, Γ is nonempty), then $x^* \in \Gamma$ can solve the following fixed point equation

$$x^* = P_C\left(x - \eta A^*(I-P_Q)Ax\right),\ \forall x \in C, \tag{4.5}$$

where P_C and P_Q are the orthogonal projection onto C and Q, respectively, $\eta > 0$ and A^* is the adjoint of A. The most popular method for solving (4.5) is Byrne's CQ algorithm [3], which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C\left(x_{n+1} - \eta A^*(I-P_Q)Ax_{n+1}\right),\ \forall n \in \mathbb{N}, \tag{4.6}$$

where $\eta \in (0, \frac{2}{\gamma})$ with γ being the spectral radius of the operator A^*A.

To prove Theorem 4.4, we need the following proposition which is introduced by Ceng et al. [5].

Proposition 4.3 [5] *Given* $x^* \in H_1$, *the following statements are equivalent.*

(i) $x^* \in \Gamma$;
(ii) x^* *solves the* (4.5).
(iii) x^* *solves the variational inequality problem (VIP) of finding* $x^* \in C$ *such that*

$$\langle A^*(I-P_Q)Ax^*, x - x^*\rangle \geq 0,\ \forall x \in C,$$

where A^* *is the adjoint of* A.

Theorem 4.4 *Let C and Q be nonempty closed convex subsets of real Hilbert spaces* H_1, H_2, *respectively and let* $A : H_1 \to H_2$ *be a bounded linear operator with adjoint* A^*. *Let* $D_1, D_2, D_3 : C \to H_1$ *be* d_1, d_2, d_3*- inverse strongly monotone mapping, respectively. Defined the mapping G as in Lemma 2.7 and* $a \in [0, 1)$. *For* $k = 1, 2, ..., \overline{N}$, *defined* $\overline{A} : H_1 \to H_1$ *by* $\overline{A}x = \sum_{k=1}^{\overline{N}} c_k A_k x$, *for all* $x \in H$, *where* A_k *is a strongly positive bounded linear operator on* H_1 *with coefficient* $\gamma_k > 0, \overline{\gamma} =$

$\min_{k=1,2,\ldots,\overline{N}} \gamma_k$ *and* $0 < \gamma < \frac{\overline{\gamma}}{2\alpha}$. *Let* $T : C \to C$ *be a nonexpansive mapping with* $\Omega = F(T) \cap F(G) \cap \Gamma \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{aligned} x_{n+1} &= \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda A^*(I - P_Q)A) y_n, \\ y_n &= \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n, \end{aligned} \tag{4.7}$$

where f *is* α*-contraction mapping on* H_1, $\{\alpha_n\} \subset [0, 1]$, $\lambda \in (0, \frac{2}{L})$ *with* L *being the spectral radius of the operator* A^*A, $\lambda_1, \lambda_2, \lambda_3 \in \left(0, 2\overline{d}\right)$ *with* $\overline{d} = \min\{d_1, d_2, d_3\}$ *and* $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$. *Suppose the following conditions hold:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n^i \leq c < 1$ *for all* $i = 1, 2, 3$;

(iii) $0 < c_k < 1$ *and* $\sum_{k=1}^{\overline{N}} c_k = 1$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} \left|\beta_{n+1}^i - \beta_n^i\right| < \infty$, *for all* $i = 1, 2, 3$.

Then, $\{x_n\}$ *converges strongly to* $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ *and* (x_0, y_0, z_0) *is a solution of* (1.5) *where* $y_0 = P_C(I - \lambda_2 D_2)(ax_0 + (1 - a)z_0)$ *and* $z_0 = P_C(I - \lambda_3 D_3)x_0$.

Proof Let $x, y \in H_1$.

We show that $A^*(I - P_Q)A$ is $\frac{1}{L}$-inverse strongly monotone.

By the definition of the spectral radius L, we have

$$\begin{aligned} \left\|A^*(I - P_Q)Ax - A^*(I - P_Q)Ay\right\|^2 &= \left\|A^*(I - P_Q)Ax - A^*(I - P_Q)Ay\right\|^2 \\ &= \langle A^*(I - P_Q)Ax - A^*(I - P_Q)Ay, A^*(I - P_Q) \\ &\quad \times Ax - A^*(I - P_Q)Ay \rangle \\ &= \langle (I - P_Q)Ax - (I - P_Q)Ay, AA^*(I - P_Q)Ax \\ &\quad - AA^*(I - P_Q)Ay \rangle \\ &\leq L\left\|(I - P_Q)Ax - (I - P_Q)Ay\right\|^2. \end{aligned} \tag{4.8}$$

Consider,

$$\begin{aligned} \left\|(I - P_Q)Ax - (I - P_Q)Ay\right\|^2 &= \langle (I - P_Q)Ax - (I - P_Q)Ay, (I - P_Q)Ax - (I - P_Q)Ay \rangle \\ &= \langle (I - P_Q)Ax - (I - P_Q)Ay, Ax - Ay \rangle \\ &\quad - \langle (I - P_Q)Ax - (I - P_Q)Ay, P_QAx - P_QAy \rangle \\ &= \langle A^*(I - P_Q)Ax - A^*(I - P_Q)Ay, x - y \rangle \\ &\quad - \langle (I - P_Q)Ax - (I - P_Q)Ay, P_QAx - P_QAy \rangle \\ &= \langle A^*(I - P_Q)Ax - A^*(I - P_Q)Ay, x - y \rangle \\ &\quad - \langle (I - P_Q)Ax, P_QAx - P_QAy \rangle \\ &\quad + \langle (I - P_Q)Ay, P_QAx - P_QAy \rangle \\ &\leq \langle A^*(I - P_Q)Ax - A^*(I - P_Q)Ay, x - y \rangle. \end{aligned}$$

It implies that

$$\left\langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x-y\right\rangle \geq \frac{1}{L}\left\|A^*(I-P_Q)Ax - A^*(I-P_Q)Ay\right\|^2. \tag{4.9}$$

Hence, $A^*(I-P_Q)A$ is $\frac{1}{L}$- inverse strongly monotone.

From Theorem 3.1 and Proposition 4.3, we obtain the desired conclusion. □

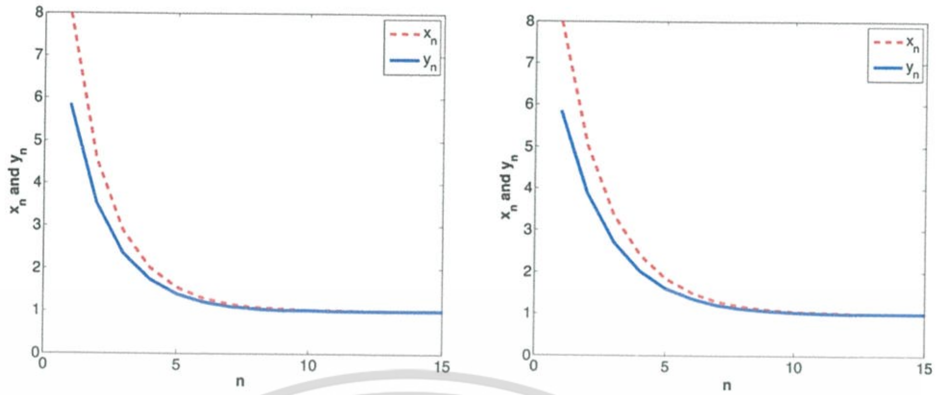
5 Example and numerical results

Example 5.1. *Let $\mathbb{R}$ be the set of real numbers and D, D_1, D_2, D_3 be a mapping from* $[0, 20]$ *to $\mathbb{R}$ defined by $D = \frac{x-1}{3}, D_1 = \frac{x-1}{4}, D_2 = \frac{x-1}{5}$, and $D_3 = \frac{x-1}{6}$, respectively. Let T be a mapping from* $[0, 20]$ *into itself defined by $Tx = \frac{x+2}{3}, \forall x \in$* $[0, 20]$. *For $k = 1, 2, ..., \overline{N}$, let $c_k = \frac{5}{6^k} + \frac{1}{N6^N}$ and let the mapping $A : \mathbb{R} \to \mathbb{R}$ be defined by $A_k = \frac{kx}{4}$, for every $x \in \mathbb{R}$. Let mapping $f : \mathbb{R} \to \mathbb{R}$ be defined by $fx = \frac{3x}{5}$, for every $x \in \mathbb{R}$. Let $x_1 \in \mathbb{R}$ and $\{x_n\}$ generated by* (3.1) *where $a = 0.5, \lambda = 1, \lambda_1 = 0.5, \lambda_2 = 1, \lambda_3 = 1.5, \alpha = 1, \gamma = 0.095, \alpha_n = \frac{1}{4n}, \beta_n^1 = \frac{3n-1}{12n}, \beta_n^2 = \frac{5n-1}{12n}$, and $\beta_n^3 = \frac{4n+2}{12n}$. By the definition of D, D_1, D_2, D_3, A, f, and T, we have $\{1\} \in F(T) \cap F(G) \cap VI(C, D)$. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 1. We can rewrite* (3.1) *as follows:*

$$\begin{aligned} x_{n+1} &= \left(\tfrac{3n-1}{12n}\right)x_n + \left(\tfrac{5n-1}{12n}\right)Tx_n + \left(\tfrac{4n+2}{12n}\right)P_C(I-D)y_n, \\ y_n &= \tfrac{0.095}{4n}f(x_n) + \left(I - \left(\tfrac{1}{4n}\right)\overline{A}\right)Gx_n. \end{aligned} \tag{5.1}$$

Table 1 The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = 8$ and $n = N = \overline{N} = 15$ of the iteratives (5.1) and (5.2)

	Iterative (5.1)		Iterative (5.2)	
n	x_n	y_n	x_n	y_n
1	8.000000	5.854781	8.000000	5.854781
2	4.562705	3.545405	5.025065	3.879684
3	2.894625	2.362649	3.370147	2.709735
4	2.019907	1.732779	2.403852	2.014352
⋮	⋮	⋮	⋮	⋮
8	1.086276	1.056126	1.172564	1.119855
⋮	⋮	⋮	⋮	⋮
12	1.005057	0.998681	1.018323	1.008502
13	1.001615	0.996523	1.009486	1.002352
14	0.999812	0.995521	1.004306	0.998850
15	0.998897	0.995133	1.001300	0.996913



(a) Iterative (5.1). (b) Iterative (5.2).

Fig. 1 The convergence of $\{x_n\}$ and $\{y_n\}$ with initial value $x_1 = 8$ and $n = N = \overline{N} = 15$

Example 5.2. *In this example, we use the same mappings and parameters as in Example 5.1 except the following mapping* $\overline{f}$. *Let the mapping* $\overline{f} : [0, 20] \to \mathbb{R}$ *be defined by* $f(x) = \frac{x^2-2x}{14}$, *with* $\nabla \overline{f} x = \frac{x-1}{7}$, *for all* $x \in [0, 20]$. *By the definition of* $\nabla \overline{f}, D_1, D_2, D_3, A, f,$ *and* T, *we have* $\{1\} \in F(T) \cap F(G) \cap \Phi_{\overline{f}}$. *From Theorem 4.2, we can conclude that the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to 1. We can rewrite* (4.3) *as follows:*

$$\begin{aligned} x_{n+1} &= \left(\frac{3n-1}{12n}\right) x_n + \left(\frac{5n-1}{12n}\right) T x_n + \left(\frac{4n+2}{12n}\right) P_C(I - \nabla \overline{f}) y_n, \\ y_n &= \frac{0.095}{4n} f(x_n) + \left(I - \left(\frac{1}{4n}\right) \overline{A}\right) G x_n. \end{aligned} \tag{5.2}$$

The following table and figure show the values of the sequences $\{x_n\}$ and $\{y_n\}$ of iteratives (5.1) and (5.2), where $x_1 = 8$ and $n = \overline{N} = 15$.

6 Conclusion

1. Table 1 and Fig. 1a show that $\{x_n\}$ and $\{y_n\}$ converge to 1, where $\{1\} \in F(T) \cap F(G) \cap VI(C, D)$. The convergence of $\{x_n\}$ and $\{y_n\}$ of Example 5.1 can be guaranteed by Theorem 3.1.
2. Table 1 and Fig. 1b show that $\{x_n\}$ and $\{y_n\}$ converge to 1, where $\{1\} \in F(T) \cap F(G) \cap VI(C, VI(C, \nabla \overline{f}))$. The convergence of $\{x_n\}$ and $\{y_n\}$ of Example 5.2 can be guaranteed by Theorem 4.2.
3. From these Examples, we obtain that the sequences $\{x_n\}$ and $\{y_n\}$ in Example 5.1 converge faster than the sequences $\{x_n\}$ and $\{y_n\}$ in Example 5.2.

Funding information This research was supported by Research and Innovation Services of King Mongkut's Institute of Tecnology Ladkrabang.

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