ITERATIVE METHODS FOR SOLVING SPLIT MODIFIED GENERALIZED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF NONLINEAR MAPPINGS

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หัวข้อวิทยานิพนธ์	วิธีทำซ้ำสำหรับการแก้ปัญหาดุลยภาพที่วางนัยทั่วไปซึ่งถูกดัดแปรแล้วแบบแยก และปัญหาจุดตรึงของการส่งไม่เชิงเส้น
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บทคัดย่อ

วัตถุประสงค์ของวิทยานิพนธ์นี้คือแนะนำปัญหาใหม่ ปัญหาดุลยภาพทั่วไปแบบแยกที่ถูกปรับปรุง ซึ่งขยายปัญหาดุลยภาพทั่วไป ปัญหาเชิงดุลยภาพแบบแยก และปัญหาอสมการการแปรผันแบบแยก เราพิสูจน์ทฤษฎีบทการลู่เข้าแบบอ่อนสำหรับการหาสมาชิกร่วมของกลุ่มของผลเฉลยของปัญหาเชิงดุลยภาพ และกลุ่มของผลเฉลยของปัญหาอสมการการแปรผัน และกลุ่มของจุดตรึงของการส่งไม่กระจายในปริภูมิฮิลเบิร์ต และทฤษฎีบทการลู่เข้าแบบเข้มสำหรับการหาสมาชิกร่วมของกลุ่มของผลเฉลยของปัญหาอสมการการแปรผัน และกลุ่มของผลเฉลยของปัญหาดุลยภาพทั่วไปแบบแยกที่ถูกปรับปรุง และกลุ่มของจุดตรึงร่วมของการส่งกึ่งไม่ขยายแบบวงศ์จำกัดในปริภูมิฮิลเบิร์ต นอกจากนี้เราได้ยกตัวอย่างการคำนวณเพื่อสนับสนุนทฤษฎีบทหลักของวิทยานิพนธ์ฉบับนี้ จากตาราง 4.1 และรูป 4.1 การลู่เข้าของ $x_1 = -8$ จะเร็วกว่า $x_1 = 10$ เนื่องจาก C เป็นเซตของจำนวนจริงในช่วง $[0, 100]$ จากตาราง 4.2 และรูป 4.2 จะได้ว่าค่าของ N ไม่มีผลต่ออัตราการลู่เข้าของตัวอย่างที่ 4.1 โดยทฤษฎีบท 3.1 จะช่วยยืนยันการลู่เข้าของ $\{x_n\}$ และ $\{u_n\}$ ในตัวอย่างที่ 4.1 และทฤษฎีบท 3.5 จะช่วยยืนยันการลู่เข้าของ $\{x_n\}$, $\{y_n\}$ และ $\{u_n\}$ ในตัวอย่างที่ 4.2 และตัวอย่างที่ 4.3 อีกด้วย

คำสำคัญ : การส่งกึ่งไม่ขยาย การส่งแบบไม่ขยาย ปัญหาจุดตรึง ปัญหาดุลยภาพทั่วไปแบบแยกที่ถูกปรับปรุง ปัญหาอสมการการแปรผัน

Thesis Title	Iterative Methods for Solving Split Modified Generalized Equilibrium Problems and Fixed Point Problems of Nonlinear Mappings
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Abstract

The objective of this thesis is to introduce a new problem, the modified split generalized equilibrium problem, which extends the generalized equilibrium problem, the split equilibrium problem and the split variational inequality problem. We prove weak convergence theorems for finding a common element of the set of solutions of the equilibrium problem and the set of solutions of the variational inequality problem and the set of fixed points of a nonspreading mapping in Hilbert space and strong convergence theorems for finding a common element of the set of solutions of variational inequality problems and the set of solutions of the modified split generalized equilibrium problem and the set of common fixed points of a finite family of quasi-nonexpansive mappings in Hilbert spaces. In addition, we give numerical examples to support our main results. From Table 4.1 and Figure 4.1, so that the convergence of $x_1 = -8$ is faster than $x_1 = 10$ because C is the set of real numbers in the interval $[0, 100]$. From Table 4.2 and Figure 4.2, so that the value of N does not affect the rate of convergence in the Example 4.1. Theorem 3.1 assure the convergence of $\{x_n\}$ and $\{u_n\}$ in the Example 4.1 and Theorem 3.5 guarantees the convergence of $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ in Example 4.2 and Example 4.3.

Keywords : Quasi-nonexpansive mapping, Nonexpansive mapping, Fixed point problem, Modified split generalized equilibrium problem, Variational inequality problem

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Kanyarat Cheawchan

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Chapter 1

Introduction

1.1 Background and signification of the research

For the past decade, new technologies have been developed rapidly. Mathematics is one of the important tools for developing new technologies. The original of new technologies was happened from observing, suspicion and questioning. Normally, the most problems are nonlinear problems. We transformed the problem into many mathematical models and used different methods to solve the problem. The fixed point theory is one of the important mathematical tools applied to solve problems in many branches of science and technology. In the past few years, many mathematicians have been developed and widely studied fixed point theory. Finding the answer of the equation by using the fixed point theory may have the answer or no answer. Therefore, fixed point theory is involved with finding conditions on the set X and the mapping $T : X \to X$ to guarantee the existence and uniqueness of fixed points. Moreover, researchers have been studying about the structure of fixed point set and the approximation of fixed points. Iterative schemes for finding the solution set of nonlinear mappings such as nonexpansive mappings, quasi-nonexpansive mappings, nonspreading mappings have been increasingly studied by many mathematicians. They have introduced various types of iterative methods to approximate fixed points.

Throughout this paper, let H_1, H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator.

For a mapping T of C into itself, we denote $F(T)$ by the set of all *fixed points* of T i.e.,

$$F(T) = \{x \in C : Tx = x\}.$$

Example 1.1.

1. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = \frac{x+1}{2}$, then $F(T) = \{1\}$.
2. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = x^2$, then $F(T) = \{0, 1\}$.
3. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = x + 5$, then $F(T) = \emptyset$.
4. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = x$, then $F(T) = \mathbb{R}$.

In 1999, the W-mapping was introduced by Atsushiba and Takahashi [1]. They defined a mapping $W : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= \beta_1 T_1 U_0 + (1-\beta_1) I,\\
U_2 &= \beta_2 T_2 U_1 + (1-\beta_2) I,\\
U_3 &= \beta_3 T_3 U_2 + (1-\beta_3) I,\\
&\vdots\\
U_{N-1} &= \beta_{N-1} T_{N-1} U_{N-2} + (1-\beta_{N-1}) I,\\
W = U_N &= \beta_N T_N U_{N-1} + (1-\beta_N) I,
\end{aligned}$$

where $\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings of C into itself, I is the identity mapping and $\{\beta_i\}_{i=1}^N$ is a sequence in $[0,1]$.

In 2009, Kangtunyakarn and Suantai [2] introduced the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. They defined a mapping $K : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= \lambda_1 T_1 U_0 + (1-\lambda_1) U_0,\\
U_2 &= \lambda_2 T_2 U_1 + (1-\lambda_2) U_1,\\
U_3 &= \lambda_3 T_3 U_2 + (1-\lambda_3) U_2,\\
&\vdots\\
U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1}) U_{N-2},\\
K = U_N &= \lambda_N T_N U_{N-1} + (1-\lambda_N) U_{N-1},
\end{aligned}$$

where $\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings of C into itself and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ are real numbers such that $0 \le \lambda_i \le 1$ for every $i = 1, 2, ..., N$.

In 2009, Kangtunyakarn and Suantai [3] introduced the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. They defined a mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\vdots\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I,
\end{aligned}$$

where $\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings of C into itself and $\alpha_j = \left(\alpha_1^j, \alpha_2^j, \alpha_3^j\right) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, for every $j = 1, 2, \ldots, N$.

Remark 1.2. For the special cases of S-mapping, we have

1. If we put $\alpha_1^j = \beta_j$ and $\alpha_2^j = 0$, for all $j = 1, 2, \ldots, N$, then the S-mapping is reduced to the W-mapping.
2. If we put $\alpha_1^j = \lambda_j$ and $\alpha_3^j = 0$, for all $j = 1, 2, \ldots, N$, then the S-mapping is reduced to the K-mapping.

Let T be a mapping from C into itself. A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. Goebel and Kirk [4] showed that $F(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory. The problem for finding a common fixed point of a family of nonexpansive mappings has been studied by authors. Many well known problems are reduced for finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings; see [5, 6].

A mapping $T : C \to C$ is said to be *nonspreading* [7] if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|Tx - y\|^2,$$

for all $x, y \in C$. Iemoto and Takahashi [8] proved that a mapping T is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle,$$

for all $x, y \in C$.

A mapping $T : C \to C$ is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|,$$

for all $x \in C$ and $p \in F(T)$. Every nonexpansive mapping and nonspreading mapping with a nonempty fixed point set is clearly quasi-nonexpansive mapping.

A mapping $T : H_1 \to H_1$ is said to be *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle,$$

for all $x, y \in H_1$. The mapping T is equivalent to

$$\langle Tx - Ty, (I - T)x - (I - T)y\rangle \geq 0,$$

for all $x, y \in H_1$; see [9].

A mapping $A : C \to H_1$ is called *α-inverse-strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$. They have been investigated in the following literature; see [10, 11].

Let $A : C \to H_1$ be a nonlinear mapping. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \tag{1.1}$$

for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C, A)$.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \ \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(F)$. Equilibrium problems were introduced by Blum and Oettli [12] in 1994 and included many well-known problems such as variational inequality problem, optimization problem and fixed point problem; see [13, 14, 15, 16]. Several iterative methods have been proposed to solve the equilibrium problem; for instance, see [12, 17, 18]. In 2005, Combettes and Hirstoaga [17] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and also proved a strong convergence theorem.

Numerous problems in physics, optimization, economics, etc are reduced to find element of (1.1) and (1.2).

The problem for finding a common element of $EP(F)$ and the set of all common fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and importance. Many iterative methods are purposed for finding a common element of the solutions of the equilibrium problem and fixed point problem of nonexpansive mappings; see [2, 19, 20].

In 2007, Takahashi and Takahashi [20] introduced a general iterative method for finding a common element of $EP(F)$ and $F(S)$. They defined $\{x_n\}$ as follows:

$$\begin{cases} x_1 \in C, \\ F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) S z_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where S is a nonexpansive mapping of C into itself, $\{\beta_n\} \subset [0, 1]$, and proved strong convergence of the method (1.3) to $z = P_{F(S) \bigcap EP(F)} f(z)$ in the framework of a Hilbert space, under suitable conditions on $\{\beta_n\}$, $\{\lambda_n\}$ and bifunction F.

Let F be a function of $C \times C$ into $\mathbb{R}$ and let $f : H_1 \to H_1$ be a mapping. The *generalized equilibrium problem* is to find $x \in C$ such that

$$F(x, y) + \langle f(x), y - x \rangle \geq 0, \tag{1.4}$$

for all $y \in C$. The set of solutions of (1.4) is denoted by $EP(F, f)$. When $f \equiv 0$, $EP(F, f)$ is denoted by $EP(F)$ and $F \equiv 0$, $EP(F, f)$ is denoted by $VI(C, f)$.

In 1994, Censor and Elfving [21] introduced the *split feasibility problem* (in short, SFP) which is to find a point $x \in C$ such that $Ax \in Q$. The set of all solutions of split feasibility problem is denoted by $\varphi = \{x \in C : Ax \in Q\}$.

To solve the SFP, Byrne [22] introduced CQ algorithm whose sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_C\left(x_n - \gamma A^*\left(I - P_Q\right)Ax_n\right),$$

where initial $x_0 \in H_1$ and $\gamma \in (0, 2/L), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. Then CQ algorithm converges to a solution of the SFP, whenever solutions exist. If there are no solutions of the SFP, the CQ algorithm converges to a minimizer of the function

$$\frac{1}{2}\|(I - P_Q)Ax\|^2,$$

whenever such minimizers exist.

Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two nonlinear operators. The *split common fixed points problem* (SCFPP) [23, 24] is to find a point x^* such that

$$x^* \in F(U) \text{ and } Ax^* \in F(T).$$

The solution set of SCFPP is denoted by $\Phi = \{p^* \in F(U) : Ap^* \in F(T)\}$. The split common fixed point problem is a generalization of the split feasibility problem.

In 2017, Wang [25] introduced a new method for solving SCFPP as follows:

$$x_{n+1} = x_n - \rho_n\left((I - U)x_n + A^*(I - T)Ax_n\right),$$

where $\rho_n \subset (0, \infty)$ is chosen such that

$$\rho_n = \frac{\|(I - U)x_n\|^2 + \|(I - T)Ax_n\|^2}{\|(I - U)x_n + A^*(I - T)Ax_n\|^2} \tag{1.5}$$

and U and T are firmly quasi-nonexpansive mappings. Then the sequence $\{x_n\}$ converges weakly to z, where $z = \lim_{n\to\infty} P_\Phi x_n$.

Censor et. al. [23, 26] introduced the prototypical *split inverse problem* (SIP) which is a generalization of the split common fixed points problem. In this, there are given two vector spaces X and Y and a linear operator $A : X \to Y$. In addition, two inverse problems are involved. The first one, denoted by IP_1, is formulated in the space X and the second one, denoted by IP_2, is formulated in the space Y. Given these data, the split inverse problem is formulated as follows:

$$\text{find a point } x^* \in X \text{ that solves IP}_1, \tag{1.6}$$

and such that

$$\text{find a point } y^* \in Y \text{ that solves IP}_2. \tag{1.7}$$

This problem is used in many modeling arising in sensor networks, radiation therapy treatment planning, color imaging, etc.

The *split equilibrium problem* (SEP) [24] is to find $\widehat{x} \in C$ such that

$$F_1(\widehat{x}, x) \geq 0, \forall x \in C, \tag{1.8}$$

and such that

$$\widehat{y} = A\widehat{x} \in Q \text{ solves } F_2(\widehat{y}, y) \geq 0, \forall y \in Q, \tag{1.9}$$

where $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions. If we consider only problem (1.8), it is the equilibrium problem and we denoted its solution set by $EP(F_1)$. The solution set of SEP is denoted by $\Gamma = \{\widehat{p} \in EP(F_1) : A\widehat{p} \in EP(F_2)\}$. SEP is reduced to $EP(F)$, where $H_1 \equiv H_2, F_1 \equiv F_2$ and $A \equiv I$. $EP(F)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc.

The *split variational inequality problems* (in short, SVIP) were introduced and studied by Cencor et al. [23]: find $\overline{x} \in C$ such that

$$\langle f_1(\overline{x}), x - \overline{x} \rangle \geq 0, \forall x \in C, \tag{1.10}$$

and such that

$$\overline{y} = A\overline{x} \in Q \text{ solves } \langle f_2(\overline{y}), y - \overline{y} \rangle \geq 0, \forall y \in Q, \tag{1.11}$$

where $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings. The solution set of SVIP is denoted by $\Psi = \{\overline{p} \in VI(C, f_1) : A\overline{p} \in VI(Q, f_2)\}$. The split variational inequality problems have already been studied and used in practice as a model in intensity-modulated radiation therapy (IMRT) treatment planning; see [27] and the modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see [22, 28].

By invitigating SEP and SVIP, we introduce the *modified split generalized equilibrium problem* (MSGEP) which is to find $x^* \in C$ such that

$$F_1(x^*, x) + \langle f_1(x^*), x - x^* \rangle \geq 0, \forall x \in C, \tag{1.12}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) + \langle f_2(y^*), y - y^* \rangle \geq 0, \forall y \in Q, \tag{1.13}$$

where $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ be nonlinear mappings. The solution set of MSGEP is denoted by $\Omega = \{p^* \in EP(F_1, f_1) : Ap^* \in EP(F_2, f_2)\}$.

Remark 1.3. For the special cases of MSGEP, we have

1. If we put $f_1 \equiv f_2 \equiv 0$ in MSGEP then the MSGEP is reduced to SEP.

2. If we put $F_1 \equiv F_2 \equiv 0$ in MSGEP then the MSGEP is reduced to SVIP.

3. In the case of bifunctions F_1 and F_2 are according to (A1)-(A4). From (1.12) and (1.13), we have $x^* \in F(T_r^{F_1}(I - rf_1))$ and $Ax^* \in F(T_s^{F_2}(I - sf_2))$, for all $r, s > 0$. So, MSGEP can be viewed as SCFPP.

MSGEP is a generalization of the generalized equilibrium problem, the split equilibrium problem and the split variational inequality problem. So, this problem can be used in sensor networks, data compression, practice as a model in intensity-modulated radiation therapy (IMRT) treatment planning, robustness to marginal changes and equilibrium stability etc.

Example 1.4. Let $H_1 = [0,6]$, $H_2 = [0,18]$, $C = [2,5]$ and $Q = [6,10]$. Let $A : H_1 \to H_2$ defined by $Ax = 3x$ for all $x \in H_1$. Let the mapping $F_1 : C \times C \to \mathbb{R}$ defined by

$$F_1(x^*, x) = -(x^* - 2)^2 + (x - 2)^2, \forall x, y \in C,$$

and $F_2 : Q \times Q \to \mathbb{R}$ defined by

$$F_2(y^*, y) = -(y^* - 6)^2 + (y - 6)^2, \forall x, y \in Q.$$

Let the mapping $f_1 : C \to H_1$ defined by $f_1 x = \frac{x-2}{9}, \forall x \in C$ and the mapping $f_2 : Q \to H_2$ defined by $f_2 x = \frac{x-6}{7}, \forall x \in Q$.

Then $2 \in \Omega$. Therefore 2 is a solution of MSGEP.

In 2012, Tain and Jin [29] introduceed iterative algorithms involving quasi-nonexpansive mapping. They generated the iterative as follows;

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T_\omega x_n,$$

where A is a bounded linear operator on H_1, T is a quasi-nonexpansive mapping on H_1, f is a contraction with coefficient a under suitable conditions of the parameters α_n, γ and ω. By assuming $\omega \in (0, \frac{1}{2})$, $T_\omega := (1-\omega)I + \omega T$ and T is demiclosed on H_1 that is the difficult proof in a framework of Hilbert space.

1.2 Objectives of the research

1) To propose new iterative schemes for finding the solutions of variational inequality problems, equilibrium problems and fixed point problems in a framework of Hilbert space.

2) To propose new iterative schemes for finding the solutions of modified split generalized equilibrium problems and fixed point problems of quasi-nonexpansive mappings and nonspreading mappings in a framework of Hilbert spaces.

3) To propose new mathematical tool for common fixed points problems.

4) To give numerical examples for supporting our main results.

1.3 Scope of the research

1) Study variational inequality problems and equilibrium problems and introduce modified split generalized equilibrium problems in a Hilbert space.

2) Study the fixed point problems of nonlinear mappings including quasi-nonexpansive mappings and nonspreading mappings in a Hilbert space.

3) All strong and weak convergence theorems are considered and proved in a Hilbert spaces.

4) Give numerical examples for supporting our main results in $\mathbb{R}$, $\mathbb{R}^2$ and ℓ_2 spaces.

1.4 Expected benefits

1) Obtain weak convergence theorems for finding a common element of the set of solutions of the equilibrium problem and the set of solutions of the variational inequality problem and the set of fixed points of a nonspreading mapping in a framework of Hilbert space.

2) Obtain strong convergence theorems for finding a common element of the set of solutions of variational inequality problems and the set of solutions of the modified split generalized equilibrium problem and the set of common fixed points of a finite family of quasi-nonexpansive mappings and in a framework of Hilbert spaces.

3) Obtain mathematical tools for common fixed points problems.

This thesis consists of five chapters as follows:

In chapter 1, we show the background of this thesis, that is, iterative methods for fixed point theorems and the definitions, properties and the relation of a finite family of nonlinear mappings.

In chapter 2, we give the fundamental concepts, definitions, lemmas, remarks and useful results that are necessary to prove our main theorems in the next chapter.

In chapter 3, we prove the following strong convergence theorems:

1) Weak convergence theorems for finding common elements of equilibrium problems, variational inequality problems and fixed point problems.

2) Strong convergence theorems for finding a common element of the set of solutions of variational inequality problems and the set of common fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of the modified split generalized equilibrium problem.

Moreover, we apply our main results by using the relation between quasi-nonexpansive mappings and nonspreading mappings.

In chapter 4, we give numerical examples to support our main results in the previous chapter.

In chapter 5, we describe the conclusion of the thesis.

Chapter 2

Preliminaries

The purpose of this chapter is to explain fundamental concepts and definitions used throughout this thesis. Moreover, we give some lemmas, remarks and useful results used in the later chapters. Throughout this chapter, we use the letter $\mathbb{R}$ for the set of all real numbers, $\mathbb{C}$ for the set of all complex numbers and $\mathbb{F}$ for the set of all real or complex numbers.

2.1 Linear spaces

Definition 2.1. [30] Let E be a nonempty set, and assume that each pair of elements x and y in E can be combined by a process called addition to yield an element z in E denoted by $z = x + y$. Assume also that this operation of addition satisfies the following conditions (v1) $\sim$ (v4):

(v1) $(x + y) + z = x + (y + z)$,

(v2) $x + y = y + x$,

(v3) there exists a unique element in E denoted by 0 and called the zero element, or the origin, such that $x + 0 = x$ for all $x \in E$,

(v4) to each $x \in E$ there corresponds a unique element in E denoted by $-x$ and called the negative of x such that $x + (-x) = 0$.

We also assume that each scalar $\alpha \in \mathbb{R}$ and each element x in E can be combined by a process called scalar multiplication to yield an element y in E denoted by $y = \alpha x$ satisfying ($\tilde{\text{v}}$1) $\sim$ ($\tilde{\text{v}}$4):

($\tilde{\text{v}}$1) $\alpha(\beta x) = (\alpha\beta)x$,

($\tilde{\text{v}}$2) $1 \cdot x = x$,

($\tilde{\text{v}}$3) $(\alpha + \beta)x = \alpha x + \beta x$,

($\tilde{\text{v}}$4) $\alpha(x + y) = \alpha x + \alpha y$.

The algebraic system E defined by these operations and axioms is called a *linear space*. A linear space is often called a vector space.

Remark 2.1. [30] Since we admit the real numbers as scalars, a linear space is also called a real linear space.

Definition 2.2. [31] A set E in a vector space is called *convex* if for any $x, y \in E$ and $\alpha \in [0, 1]$, we have $\alpha x + (1 - \alpha)y \in E$.

2.2 Banach spaces and Hilbert spaces

Definition 2.3. [32] Let X be a linear space (or vector space) over the field $\mathbb{F}$. A *norm* on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(L1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

(L2) $\|\alpha x\| = |\alpha| \, \|x\|$,

(L3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

The ordered pair $(X, \|\cdot\|)$ is called a *normed space or normed vector space or normed linear space*.

Definition 2.4. (**Cauchy sequence** [31]) A sequence of vectors $\{x_n\}$ in a normed space X is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n > N$.

Definition 2.5. [31] A normed space X is called *complete* if every Cauchy sequence in X converges to an element of X.

Definition 2.6. [30] A complete normed linear space is called a *Banach space*.

Example 2.2. [30] Let $X = \ell_p^n, n > 1$ and $1 \leq p < \infty$. The sequence space defined by

$$\ell_p^n = \{x : x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n\}.$$

Then ℓ_p^n is a Banach space with the norm defined by $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$.

Example 2.3. Let $X = c_{00}$, the sequence space defined by

$$c_{00} = \left\{x = \{x_i\}_{i=1}^{\infty} \in \ell_\infty : \{x_i\}_{i=1}^{\infty} \text{ has only a finite number of nonzero terms}\right\}.$$

Then c_{00} space is a normed space with norm $\|\cdot\|_\infty$ but not a Banach space.

Theorem 2.4. [31] A subset S of a normed space X is *closed* if and only if every sequence of elements of S convergent in X has its limit in S, i.e.,

$$\{x_n\} \subseteq S \text{ and } x_n \to x \text{ implies } x \in S.$$

Next, we give the definition and important tools of Hilbert space. First of all, we introduce the definition of inner product space.

Definition 2.7. [33] An inner product on a vector space K over the field $\mathbb{F}$ is a function $\langle \cdot, \cdot \rangle : K \times K \to \mathbb{F}$, that assigns a scalar $\langle x, y \rangle$ for every $x, y \in K$, such that for all $x, y, z \in K$ and $\alpha \in \mathbb{F}$:

(I1) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$, (I2) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$,

(I3) $\overline{\langle x, y\rangle} = \langle y, x\rangle$,

(I4) $\langle x, x\rangle > 0 \Leftrightarrow x \neq 0$,

A vector space K over $\mathbb{F}$ with a specific inner product is called an inner product space. If $\mathbb{F} = \mathbb{C}$ is a complex inner product space, and if $\mathbb{F} = \mathbb{R}$, K is a real inner product space.

Theorem 2.5. [33] For an inner product space K, $x, y, z \in K$ and $\alpha \in \mathbb{F}$:

(J1) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$,

(J2) $\langle x, \alpha y\rangle = \bar{\alpha} \langle x, y\rangle$,

(J3) $\langle x, 0\rangle = \langle 0, x\rangle = 0$,

(J4) $\langle x, x\rangle = 0 \Leftrightarrow x = 0$,

(J5) If $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in K$ then $y = z$.

Remark 2.6. [30] An inner product space is called a real inner product space for the case when the scalars are the real numbers and $\langle x, y\rangle$ is a real number. For the case, (I3) means

$$\langle x, y\rangle = \langle y, x\rangle.$$

Remark 2.7. [30] Using (J1) and (J2), we obtain that for $x, y \in K$ and $\alpha, \beta \in \mathbb{C}$

$$\langle x, \alpha y + \beta z\rangle = \bar{\alpha} \langle x, y\rangle + \bar{\beta} \langle x, z\rangle .$$

Definition 2.8. [30] A complete inner product space is called a *Hilbert space*.

Example 2.8. Let $X = \ell_2$, the set of all sequences of complex numbers $(x_1, x_2, ..., x_i, ...)$ with $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. Then the function $\langle \cdot, \cdot\rangle : \ell_2 \times \ell_2 \to \mathbb{C}$ defined by

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \bar{y_i} \text{ for all } x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in X.$$

Then ℓ_2 is a Hilbert space.

Example 2.9. Let $X = C[a, b]$, the linear space of all scalar-valued continuous functions on $[a, b]$. Then the function $\langle \cdot, \cdot\rangle : C[a, b] \times C[a, b] \to \mathbb{C}$ defined by

$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}dt \text{ for all } f, g \in C[a, b].$$

Then $X = C[a, b]$ is a inner product space but not a Hilbert space.

Example 2.10. ℓ_p^n is a finite-dimensional Banach space that is not a Hilbert space for $p \neq 2$.

Theorem 2.11. [30] The inner product in an inner product space K is jointly continuous:

$$x_n \to x \text{ and } y_n \to y \Rightarrow \langle x_n, y_n\rangle \to \langle x, y\rangle.$$

Remark 2.12. [30] We of course obtain from Theorem 2.11 that if $x_n \to x$, then for a fixed $y \in K$,

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ and } \langle y, x_n\rangle \to \langle y, x\rangle$$

Remark 2.13. [30] Let K be an inner product space. For each x in K, we define its *norm* $\|x\|$ by

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}}.$$

Theorem 2.14. **(Schwarz inequality** [30]) Let K be an inner product space and let x and y be elements in K. Then the following holds:

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Definition 2.9. **(Strong convergence** [31]) A sequence $\{x_n\}$ of vectors in an inner product space K is called *strongly convergent* to x in K if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.10. **(Weak convergence** [31]) A sequence $\{x_n\}$ of vectors in an inner product space K is called *weakly convergent* to x in K if

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ as } n \to \infty \text{ for all } y \in X.$$

Remark 2.15. We represent weak and strong convergence by "$\rightharpoonup$" and "$\to$", respectively.

Theorem 2.16. [31] A strongly convergence sequence is weakly convergence (to the same limit), i.e., $x_n \to x$ implies $x_n \rightharpoonup x$.

Remark 2.17. [30] If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Lemma 2.18. [30] Let $\{x_n\}$ be a Cauchy sequence of an inner product space K such that $x_n \rightharpoonup x$. Then $x_n \to x$.

Theorem 2.19. [30] Let H be a Hilbert space and let C be a nonempty closed convex subset of H with $\{x_n\} \subset C$ and $x_n \rightharpoonup x$, then $x \in C$.

Theorem 2.20. [30] Let $\{a_n\}$ be a bounded of real numbers. Then, there exists subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Remark 2.21. [30] Let H be an inner product space. Then we know that the following (1) and (2) are equivalent:

(1) H is complete,

(2) each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Definition 2.11. **(Metric projection** [30]) The (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.22. [34] Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Given $x \in H$ and $y \in C$, then following holds

$$P_C x = y \Leftrightarrow \langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

Lemma 2.23. [35] Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Then the following hods:

(1) $\|P_C x - P_C y\| \leq \|x - y\|$, for all $x, y \in H$,

(2) $\langle y - x, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$, for all $x, y \in H$.

Theorem 2.24. **(Opial's theorem** [30]) Let H be a Hilbert space and suppose $x_n \rightharpoonup x$. Then

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

for any $y \in H$ with $x \neq y$.

Definition 2.12. **(Lower semicontinuous** [30]) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\} \text{ is closed.}$$

Moreover, f is called convex if for any $x_1, x_2 \in C$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Theorem 2.25. [30] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightharpoonup x_0$. Then

$$f(x_0) \leq \liminf_{n \to \infty} f(x_n).$$

Theorem 2.26. [31] Weakly convergent sequences in a Hilbert space H are bounded, i.e., if $\{x_n\}$ is a weakly convergent sequence, then there exists a number M such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Theorem 2.27. **(Double extract subsequence principle** [32]) Let $\{x_n\}$ be a sequence in a Hilbert space H and $x \in H$. If every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a further subsequence $\left\{x_{n_{k_l}}\right\}$ such that $\lim_{l \to \infty} x_{n_{k_l}} = x$, then $\lim_{n \to \infty} x_n = x$.

2.3 Bounded linear operators

Definition 2.13. [31] Let C be a nonempty closed convex subset of a real Hilbert space H. Then

(1) A is called a *linear operator* if

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y),$$

for all $x, y \in H$ and all scalars α, β.

(2) A is called *bounded* if there is a number K such that

$$\|Ax\| \leq K\|x\|,$$

for all $x \in H$.

Definition 2.14. (**Adjoint Operator** [31]) Let A be a bounded operator on a Hilbert space H. The operator $A^* : H \to H$ defined by

$$\langle Ax, y\rangle = \langle x, A^*y\rangle, \quad \text{for all} \quad x, y \in H$$

is called the *adjoint operator* of A.

Theorem 2.28. [31] The adjoint operator A^* of a bounded operator A is bounded. Moreover, we have $\|A\| = \|A^*\|$ and $\|A^*A\| = \|A\|^2$.

Definition 2.15. (**Self-Adjoint Operator** [31]) Let A be a bounded operator on a Hilbert space H. If $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x \in H$, then A is called the *self-adjoint operator*.

Remark 2.29. If A is a bounded operator on a Hilbert space H, then A^*A is self-adjoint operator.

Theorem 2.30. [31] Let T be a bounded linear self-adjoint operator on a Hilbert space H. Then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Definition 2.16. (**Normal Operator** [31]) A bounded operator T on a Hilbert space H is called a *normal operator* if $TT^* = T^*T$.

Theorem 2.31. [31] A bounded operator T on a Hilbert space H is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.

Definition 2.17. [31] An operator A is called *positive* if it is self-adjoint and

$$\langle Ax, x\rangle \geq 0, \quad \text{for all} \quad x \in H.$$

Definition 2.18. [36] Let T be a bounded linear operator on a Hilbert space H. The spectral radius of T, denoted by $r_\sigma(T)$, is the number defined by

$$r_\sigma(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

where $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)(x) = 0, \ \text{for some } 0 \neq x \in H\}$.

Theorem 2.32. [36] Let T be a normal bounded linear operator on a Hilbert space H. Then T is is self-adjoint operator if and only if $\sigma(T) \subset \mathbb{R}$.

Example 2.33. Let $\mathbb{R}^2$ be the two dimensional space of real numbers with an inner product $\langle \cdot,\cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\langle u,v \rangle = u \cdot v = u_1v_1 + u_2v_2$ and a usual norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ given by $\|u\| = \sqrt{u_1^2 + u_2^2}$, for all $u = (u_1,u_2), v = (v_1,v_2) \in \mathbb{R}^2$. Let an operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Ax = (2x_1 - x_2, x_1 + 2x_2)$ for all $x = (x_1,x_2) \in \mathbb{R}^2$. Then A is a bounded linear on $\mathbb{R}^2$.

Solution. Let $\alpha, \beta \in \mathbb{R}$ and $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$. Thus we derive

$$\begin{aligned}
A(\alpha x + \beta y) &= (2(\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2), (\alpha x_1 + \beta y_1) + 2(\alpha x_2 + \beta y_2)) \\
&= (2\alpha x_1 - \alpha x_2 + 2\beta y_1 - \beta y_2, \alpha x_1 + 2\alpha x_2 + \beta y_1 + 2\beta y_2) \\
&= (2\alpha x_1 - \alpha x_2, \alpha x_1 + 2\alpha x_2) + (2\beta y_1 - \beta y_2, \beta y_1 + 2\beta y_2) \\
&= \alpha(2x_1 - x_2, x_1 + 2x_2) + \beta(2y_1 - y_2, y_1 + 2y_2) \\
&= \alpha Ax + \beta Ay.
\end{aligned}$$

This implies that A is linear. Observe that

$$\begin{aligned}
\|Ax\| &= \|(2x_1 - x_2, x_1 + 2x_2)\| \\
&= \sqrt{(2x_1 - x_2)^2 + (x_1 + 2x_2)^2} \\
&= \sqrt{4x_1^2 - 4x_1x_2 + x_2^2 + x_1^2 + 4x_1x_2 + 4x_2^2} \\
&= \sqrt{5}\,\|x\| \\
&= M\,\|x\|.
\end{aligned}$$

Then A is bounded.

2.4 Variational inequality problems and Equilibrium problems

The variational inequalities were initially studied and introduced by Stampacchia [37, 38]. This problem is widely used in economics, social sciences and other fields; see [39, 40]. This is the property of variational inequality problems.

Theorem 2.34. [30] Let H be a real Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $\alpha > 0$ and let $A : C \to H$ be α-inverse strongly monotone. Then $VI(C,A) \neq \emptyset$.

Lemma 2.35. [35] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Equilibrium problems are well-known problems, introduced by Blum and Oettli [12] in 1994. If we give a mapping $T : C \to H_1$, let $F(x, y) = \langle Ax, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Ax, y - x\rangle \geq 0$ for all $y \in C$ that is z is a solution of the variational inequality problem.

For solving the equilibrium problem, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfy the following conditions:
(A1) $F(x, x) = 0$ for all $x \in C$,
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$,
(A3) for each $x, y, z \in C, \lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$,
(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.36. [12] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.37. [17] Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:

(1) T_r is single-valued,

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle,$$

(3) $F(T_r) = EP(F)$,

(4) $EP(F)$ is closed and convex.

Example 2.38. Let $\mathbb{R}^2$ be the two dimensional space of real numbers with an inner product $\langle \cdot, \cdot\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\langle u, v\rangle = u \cdot v = u_1v_1 + u_2v_2$ and a usual norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ given by $\|u\| = \sqrt{u_1^2 + u_2^2}$, for all $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$. Let $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x, y) = -x^2 + y^2, \forall x, y \in \mathbb{R}^2.$$

Then a bifunction F satisfies the condition (A1)-(A4) and $(0,0) \in EP(F)$.

Solution. From the definition of F, we obtain

$$\begin{aligned} F(x,y) &= -x^2 + y^2 \\ &= -(x_1,x_2)\cdot(x_1,x_2) + (y_1,y_2)\cdot(y_1,y_2) \\ &= -\left(x_1^2 + x_2^2\right) + \left(y_1^2 + y_2^2\right). \end{aligned}$$

Let $x,y,z \in \mathbb{R}^2$. That is

$$F(x,x) = -\left(x_1^2 + x_2^2\right) + \left(x_1^2 + x_2^2\right) = 0.$$

Thus F satisfies (A1). Next, observe that

$$\begin{aligned} F(x,y) + F(y,x) &= -\left(x_1^2 + x_2^2\right) + \left(y_1^2 + y_2^2\right) - \left(y_1^2 + y_2^2\right) + \left(x_1^2 + x_2^2\right) \\ &= 0. \end{aligned}$$

Then (A2) is true. Let $t \in [0,1]$, derive that

$$\begin{aligned} \lim_{t\to 0^+} F(tz + (1-t)x, y) &= \lim_{t\to 0^+} \left(-(tz_1 + (1-t)x_1, tz_2 + (1-t)x_2)^2 + (y_1,y_2)^2\right) \\ &= \lim_{t\to 0^+} \left(-((1-t)x_1 + tz_1)^2 - ((1-t)x_2 + tz_2)^2 + y_1^2 + y_2^2\right) \\ &= -x_1^2 - x_2^2 + y_1^2 + y_2^2 \\ &= F(x,y). \end{aligned}$$

Therefore (A3) holds. Let $t \in (0,1)$, we have

$$\begin{aligned} F(x, tz + (1-t)y) &= -(x_1,x_2)^2 + (tz_1 + (1-t)y_1, tz_2 + (1-t)y_2)^2 \\ &= -\left(x_1^2 + x_2^2\right) + \left((tz_1 + (1-t)y_1)^2 + (tz_2 + (1-t)y_2)^2\right) \\ &= -(t + (1-t))\left(x_1^2 + x_2^2\right) + \left(t^2 z_1^2 + 2t(1-t)z_1 y_1 + (1-t)^2 y_1^2\right) \\ &\quad + \left(t^2 z_2^2 + 2t(1-t)z_2 y_2 + (1-t)^2 y_2^2\right) \\ &\leq -(t + (1-t))\left(x_1^2 + x_2^2\right) + \left(t^2 z_1^2 + t(1-t)(z_1^2 + y_1^2) + (1-t)^2 y_1^2\right) \\ &\quad + \left(t^2 z_2^2 + t(1-t)(z_2^2 + y_2^2) + (1-t)^2 y_2^2\right) \\ &= t\left(-\left(x_1^2 + x_2^2\right) + \left(z_1^2 + z_2^2\right)\right) + (1-t)\left(-\left(x_1^2 + x_2^2\right) + \left(y_1^2 + y_2^2\right)\right) \\ &= tF(x,z) + (1-t)F(x,y). \end{aligned}$$

This yields that F is convex. Suppose that $\{y_n\} \subset \mathbb{R}^2$ with $y_n = (y_n^1, y_n^2) \to (y_1,y_2)$ as $n \to \infty$. Thus we get

$$\begin{aligned} \lim_{n\to\infty} F(x,y_n) &= \lim_{n\to\infty} \left(-\left(x_1^2 + x_2^2\right) + \left(y_n^1\right)^2 + \left(y_n^2\right)^2\right) \\ &= -\left(x_1^2 + x_2^2\right) + \left(y_1^2 + y_2^2\right) \\ &= F(x,y). \end{aligned}$$

Hence F is lower semicontinuous. So the condition (A4) holds.

Because $F((0,0),y) = y_1^2 + y_2^2 \geq 0$, then $(0,0) \in EP(F)$.

2.5 Fixed points of nonexpansive mappings, nonspreading mappings and quasi-nonexpansive mappings

The fundamental concepts of fixed point theory is used in many fields such as in economics, a Nash equilibrium of a game is a fixed point of the game's best response correspondence. John Nash finished his seminal paper by using fixed point theorem and won the Nobel prize in economics. In this section, we give the property for fixed point theory.

Theorem 2.39. [30] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Theorem 2.40. [30] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.

Lemma 2.41. (**Demiclosedness principle** [41]) Assume that T is a nonexpansive self-mapping of closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y it follows that $(I-T)x = y$. Here, I is the identity mapping of H.

Example 2.42. Let $T : [1, 100] \to [1, 100]$ be defined by $Tx = \frac{3x+5}{8}$. Then T is a nonexpansive mapping.

Solution. Let $x, y \in [1, 100]$. Thus we get

$$|Tx - Ty| = \left|\frac{3x+5}{8} - \frac{3y+5}{8}\right| = \left|\frac{3}{8}(x-y)\right| \leq |x-y|.$$

Hence T is a nonexpansive mapping.

By the concept of quasi-nonexpansive mapping and nonexpansive mapping, we know that a nonexpansive mapping with at least one fixed point in C is quasi-nonexpansive mapping but the inverse may be not true. It implies that the class of quasi-nonexpansive mapping generalizes the class of nonexpansive mapping. Next is the property of quasi-nonexpansive mapping.

Theorem 2.43. [42] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a quasi-nonexpansive mapping of C into itself. Then $F(T)$ is a nonempty closed convex set on which T is continuous.

Example 2.44. Let $T : [1, 100] \to [1, 100]$ be defined by $Tx = \frac{3x+5}{8}$. Then T is a quasi-nonexpansive mapping.

Solution. Let $x, y \in [1, 100]$. We observe that $Fix(T) = \{\mathbf{1}\}$. For every $u \in [1, 100]$ and $\mathbf{1} \in Fix(T)$, we have

$$|Tu - T\mathbf{1}| = \left|\frac{3u+5}{8} - \mathbf{1}\right| = \frac{3}{8}|u - \mathbf{1}| \leq |u - \mathbf{1}|.$$

Therefore T is a quasi-nonexpansive mapping.

The fixed point of the nonexpansive mapping T in Example 2.42 is $\mathbf{1}$ and Example 2.44 guarantees that T is quasi-nonexpansive mapping. This ensures that a nonexpansive mapping with at least one fixed point in C is quasi-nonexpansive mapping. The next example shows that the inverse may be not true.

Example 2.45. Let $T : [0, 2] \to [0, 2]$ be defined by

$$Tu = \begin{cases} \frac{u+2}{2} & \text{if } u \in (1, 2], \\ \frac{u}{2} & \text{if } u \in [0, 1]. \end{cases}$$

Solution. First, show that T is quasi-nonexpansive for all $u \in [0, 2]$.
Observe that $Fix(T) = \{\mathbf{2}\}$ if $x \in (1, 2]$ and $Fix(T) = \{\mathbf{0}\}$ if $u \in [0, 1]$.
For any $u \in (1, 2]$, we have

$$|Tu - T\mathbf{2}| = \left|\frac{u+2}{2} - \mathbf{2}\right| = \left|\frac{u-2}{2}\right| = \frac{1}{2}|u - \mathbf{2}| < |u - \mathbf{2}|.$$

For every $u \in [0, 1]$, we obtain

$$|Tu - T\mathbf{0}| = \left|\frac{u}{2} - \mathbf{0}\right| = \frac{1}{2}|u - \mathbf{0}| < |u - \mathbf{0}|.$$

Therefore T is a quasi-nonexpansive for all $u \in [0, 2]$.
Next, we show that T is not a nonexpansive mapping.
Choose $u = \frac{3}{2}$ and $v = \frac{1}{2}$, we have

$$\begin{aligned} |Tu - Tv| &= \left|T\left(\frac{3}{2}\right) - T\left(\frac{1}{2}\right)\right| \\ &= \left|\frac{7}{4} - \frac{1}{4}\right| \\ &= \left|\frac{3}{2}\right|. \end{aligned}$$

Thus we get

$$\begin{aligned} |u - v| &= \left|\frac{3}{2} - \frac{1}{2}\right| \\ &= 1. \end{aligned}$$

Hence we have

$$|Tu - Tv| > |u - v|.$$

That is, T is not a nonexpansive mapping.

A mapping T on a closed convex subset C of a Hilbert space H is said to be nonspreading mapping if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|Tx - y\|^2, \tag{2.1}$$

for all $x, y \in C$. In 2009, Iemoto and Takahashi [8] proved that (2.1) is equivalent to (2.2).

Lemma 2.46. Let C be a nonempty closed convex subset of a real Hilbert space H. Then the mapping $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \tag{2.2}$$

for all $x, y \in C$.

Lemma 2.47. [7] Let H be a Hilbert space, let C be a nonempty closed convex subset of H , and let T be a nonspreading mapping of C into itself. Then $F(T)$ is closed and convex.

Remark 2.48. A nonspreading mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive mapping. But the converse is not true.

Example 2.49. Let $T : [1, 100] \to [1, 100]$ be defined by $Tx = \frac{2x+7}{9}$. Then T is a nonspreading mapping.
Solution. Let $x, y \in [1, 100]$. Thus we get

$$|Tx - Ty|^2 = \left|\frac{2x+7}{9} - \frac{2y+7}{9}\right|^2 = \left|\frac{2}{9}(x-y)\right|^2 = \frac{4}{81}|x-y|^2$$

and

$$\begin{aligned} 2\langle x - Tx, y - Ty\rangle &= 2\left(x - \frac{2x+7}{9}\right)\left(y - \frac{2y+7}{9}\right) \\ &= 2\left(\frac{7x-7}{9}\right)\left(\frac{7y-7}{9}\right) \\ &= \frac{98}{81}(x-1)(y-1) \geq 0, (\text{ Since } x, y \geq 1). \end{aligned}$$

Therefore

$$\begin{aligned} |x-y|^2 + 2\langle x - Tx, y - Ty\rangle &\geq |x-y|^2 \\ &\geq \frac{4}{81}|x-y|^2 \\ &= |Tx - Ty|^2. \end{aligned}$$

Hence T is a nonspreading mapping.

The following example support Remark 2.48.

Example 2.50. Let $T : [1, 100] \to [1, 100]$ be defined by $Tx = \frac{2x+7}{9}$. Then T is a quasi-nonexpansive mapping.

Solution. Let $x, y \in [1, 100]$. By Example 2.49, we have T is a nonspreading mapping. We observe that $Fix(T) = \{1\}$. For every $u \in [1, 100]$ and $1 \in Fix(T)$, we have

$$|Tu - T1| = \left|\frac{2u+7}{9} - 1\right| = \frac{2}{9}|u-1| \leq |u-1|.$$

Therefore T is a quasi-nonexpansive mapping.

The following example shows that the converse of Remark 2.48 does not hold.

Example 2.51. Let $T : [0, 2] \to [0, 2]$ be defined by

$$Tu = \begin{cases} \frac{u+2}{2} & \text{if } u \in (1, 2], \\ \frac{u}{2} & \text{if } u \in [0, 1]. \end{cases}$$

Solution. First, show that T is quasi-nonexpansive for all $u \in [0, 2]$.
Observe that $Fix(T) = \{2\}$ if $x \in (1, 2]$ and $Fix(T) = \{0\}$ if $u \in [0, 1]$.
For any $u \in (1, 2]$, we have

$$|Tu - T2| = \left|\frac{u+2}{2} - 2\right| = \left|\frac{u-2}{2}\right| = \frac{1}{2}|u-2| < |u-2|.$$

For every $u \in [0, 1]$, we obtain

$$|Tu - T0| = \left|\frac{u}{2} - 0\right| = \frac{1}{2}|u-0| < |u-0|.$$

Therefore T is a quasi-nonexpansive for all $u \in [0, 2]$.
Next, we show that T is not a nonspreading mapping.
Choose $u = \frac{3}{2}$ and $v = \frac{1}{2}$, we have

$$\begin{aligned} |Tu - Tv|^2 &= \left|T\left(\frac{3}{2}\right) - T\left(\frac{1}{2}\right)\right|^2 \\ &= \left|\frac{7}{4} - \frac{1}{4}\right|^2 \\ &= \left|\frac{6}{4}\right|^2 \\ &= \frac{9}{4}. \end{aligned}$$

Thus we get

$$\begin{aligned} |u - v|^2 + 2\langle u - Tu, v - Tv\rangle &= \left|\frac{3}{2} - \frac{1}{2}\right|^2 + 2\left\langle \frac{3}{2} - T\left(\frac{3}{2}\right), \frac{1}{2} - T\left(\frac{1}{2}\right)\right\rangle \\ &= |1|^2 + 2\left\langle \frac{3}{2} - \frac{7}{4}, \frac{1}{2} - \frac{1}{4}\right\rangle \\ &= 1 + 2\left(\frac{3}{2} - \frac{7}{4}\right) \cdot \left(\frac{1}{2} - \frac{1}{4}\right) \\ &= 1 + 2\left(-\frac{1}{4}\right) \cdot \left(\frac{1}{4}\right) \\ &= 1 - \frac{1}{8} \\ &= \frac{7}{8}. \end{aligned}$$

Hence we have

$$|Tu - Tv|^2 > |u - v|^2 + 2\langle u - Tu, v - Tv\rangle.$$

That is, T is not a nonspreading mapping.

2.6 Some useful Lemmas and Theorems

This section is an important section to prove our main results. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. In this thesis, we represent weak and strong convergence by " $\rightharpoonup$" and " $\to$", respectively.

Lemma 2.52. Let H be a real Hilbert space. Then there holds the following well-known results:

(1) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$,

(2) $\|x + y\|^2 \leq \|x\|^2 + \langle y, x + y\rangle$,

(3) $\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2$,

for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.53. [43] Let H be a real Hibert space. If $\{x_n\}$ is a sequence in H weakly convergent to z, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2, \tag{2.3}$$

for all $y \in H$.

Lemma 2.54. [44] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

The next important result is used for proving our main result.

Lemma 2.55. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into H with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\sum_{i=1}^N a_i = 1$, where $a_i > 0$ for all $i = 1, 2, ..., N$. Then

$$\bigcap_{i=1}^N F(T_i) = VI\left(C, \sum_{i=1}^N a_i (I - T_i)\right).$$

Proof. In this Lemma, we show that $\bigcap_{i=1}^{N} F(T_i) = \bigcap_{i=1}^{N} VI(C, I-T_i)$ and $\bigcap_{i=1}^{N} VI(C, I-T_i) = VI\left(C, \sum_{i=1}^{N} a_i (I-T_i)\right)$. Lastly, we have

$$\bigcap_{i=1}^{N} F(T_i) = VI\left(C, \sum_{i=1}^{N} a_i (I-T_i)\right).$$

In the beginning, it is easy to see that $\bigcap_{i=1}^{N} F(T_i) \subseteq \bigcap_{i=1}^{N} VI(C, I-T_i)$. Next, we show that $\bigcap_{i=1}^{N} VI(C, I-T_i) \subseteq \bigcap_{i=1}^{N} F(T_i)$. Let $u \in \bigcap_{i=1}^{N} VI(C, I-T_i)$ and $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. So, we get $u \in VI(C, I-T_i)$, $\forall i = 1, 2, ..., N$. We may write

$$\langle u - v, (I - T_i)u \rangle \leq 0, \ \forall v \in C. \tag{2.4}$$

There exists $v^* \in C$ such that $v^* = T_i v^*$, $\forall i = 1, 2, ..., N$. Since T_i is quasi-nonexpansive mappings, $\forall i = 1, 2, ..., N$, it follows that

$$\begin{aligned} \|T_i u - v^*\|^2 &= \|(u - v^*) - (I - T_i) u\|^2 \\ &= \|u - v^*\|^2 - 2\langle u - v^*, (I - T_i) u \rangle + \|(I - T_i) u\|^2 \\ &\leq \|u - v^*\|^2 . \end{aligned} \tag{2.5}$$

By using (2.4) and (2.5), we conclude that

$$\|(I - T_i) u\|^2 \leq 2\langle u - v^*, (I - T_i) u \rangle \leq 0.$$

It implies that $u \in \bigcap_{i=1}^{N} F(T_i)$. Therefore $\bigcap_{i=1}^{N} VI(C, I-T_i) \subseteq \bigcap_{i=1}^{N} F(T_i)$. Hence

$$\bigcap_{i=1}^{N} F(T_i) = \bigcap_{i=1}^{N} VI(C, I-T_i).$$

After that, we show $\bigcap_{i=1}^{N} VI(C, I-T_i) = VI\left(C, \sum_{i=1}^{N} a_i (I-T_i)\right)$ where $a_i > 0$ and $\sum_{i=1}^{N} a_i = 1$. Observe that

$$u \in \bigcap_{i=1}^{N} VI(C, I-T_i)$$

$\Leftrightarrow$

$$u \in VI(C, I-T_i), \forall i = 1, 2, ..., N$$

$\Leftrightarrow$

$$\langle (I - T_i) u, v - u \rangle \geq 0, \forall v \in C \text{ and } \forall i = 1, 2, ..., N$$

$\Leftrightarrow$

$$\sum_{i=1}^{N} a_i \langle (I - T_i) u, v - u \rangle \geq 0, \forall v \in C$$

$\Leftrightarrow$

$$\left\langle \sum_{i=1}^{N} a_i \left(I - T_i\right) u, v - u \right\rangle \geq 0, \forall v \in C$$

$\Leftrightarrow$

$$u \in VI\left(C, \sum_{i=1}^{N} a_i \left(I - T_i\right)\right).$$

Therefore $\bigcap_{i=1}^{N} VI\left(C, I - T_i\right) = VI\left(C, \sum_{i=1}^{N} a_i \left(I - T_i\right)\right)$.

Hence $\bigcap_{i=1}^{N} F\left(T_i\right) = VI\left(C, \sum_{i=1}^{N} a_i \left(I - T_i\right)\right)$. □

Remark 2.56. From Lemma 2.35 and Lemma 2.55, we have

$$\bigcap_{i=1}^{N} F\left(T_i\right) = VI\left(C, \sum_{i=1}^{N} a_i \left(I - T_i\right)\right) = F\left(P_C\left(I - \lambda\left(\sum_{i=1}^{N} a_i \left(I - T_i\right)\right)\right)\right),$$

where $a_i > 0$ for all $i = 1, 2, ..., N$ with $\sum_{i=1}^{N} a_i = 1$ and $\lambda > 0$.

Chapter 3

Convergence theorems in Hilbert space and its application

3.1 Weak convergence theorems for finding common elements of equilibrium problems, variational inequality problems and fixed point problems

In this section, we introduce weak convergence theorems, motivated by Takahashi and Takahashi [20], for finding a common element of the set of solutions of the equilibrium problem, the set of solutions of the variational inequality problem and the set of fixed point problem in Hilbert spaces. In addition, our results improve existing results in [20].

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$, let $A : C \to H$ be a α-inverse strongly monotone mapping and let $T : C \to C$ be a nonspreading mapping with $\mathfrak{F} = F(T) \bigcap EP(F) \bigcap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n P_C(I - \lambda A)u_n + \gamma_n T P_C(I - \lambda A)x_n, \qquad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \subset [a, b] \subset (0, 1)$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$, $\{\gamma_n\} \subset [e, f] \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, and $r_n \in [g,\ h] \subset (0,\ 2\alpha)$, $\lambda \in (0,\ 2\alpha)$. Then the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $\mathfrak{F}$.

Proof. First, we show that $P_C(I - \lambda A)$ is nonexpansive. Let $x, y \in C$. Since A is α-inverse strongly monotone and $\lambda < 2\alpha$, we have

$$\begin{aligned} \|P_C(I - \lambda A)x - P_C(I - \lambda A)y\|^2 &\leq \|(I - \lambda A)x - (I - \lambda A)y\|^2 \\ &= \|x - y - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\alpha\lambda\|Ax - Ay\|^2 + \lambda^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Thus $P_C(I - \lambda A)$ is nonexpansive. We shall show that the sequence $\{x_n\}$ is bounded. Since

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

By Lemma 2.37, we have $u_n = T_{r_n}x_n$ and $EP(F) = F(T_{r_n})$. Let $z \in \mathfrak{F}$. By nonexpansiveness of $P_C(I-\lambda A)$ and T_{r_n}, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 =&\|\alpha_n(x_n-z)+\beta_n(P_C(I-\lambda A)u_n-z)\\
&+\gamma_n(TP_C(I-\lambda A)x_n-z)\|^2\\
=&\alpha_n\|x_n-z\|^2+\beta_n\|P_C(I-\lambda A)u_n-z\|^2\\
&+\gamma_n\|TP_C(I-\lambda A)x_n-z\|^2-\alpha_n\beta_n\|x_n-P_C(I-\lambda A)u_n\|^2\\
&-\alpha_n\gamma_n\|x_n-TP_C(I-\lambda A)x_n\|^2\\
&-\beta_n\gamma_n\|P_C(I-\lambda A)u_n-TP_C(I-\lambda A)x_n\|^2\\
\leq&(1-\beta_n)\|x_n-z\|^2+\beta_n\|T_{r_n}x_n-z\|^2\\
&-\alpha_n\beta_n\|x_n-P_C(I-\lambda A)u_n\|^2-\alpha_n\gamma_n\|x_n-TP_C(I-\lambda A)x_n\|^2\\
&-\beta_n\gamma_n\|P_C(I-\lambda A)u_n-TP_C(I-\lambda A)x_n\|^2 \qquad (3.2)\\
=&\|x_n-z\|^2-\alpha_n\beta_n\|x_n-P_C(I-\lambda A)u_n\|^2\\
&-\alpha_n\gamma_n\|x_n-TP_C(I-\lambda A)x_n\|^2\\
&-\beta_n\gamma_n\|P_C(I-\lambda A)u_n-TP_C(I-\lambda A)x_n\|^2 \qquad (3.3)\\
\leq&\|x_n-z\|^2.
\end{aligned}$$

This implies that $\lim_{n\to\infty}\|x_n-z\|$ exists for every $z\in\mathfrak{F}$. Then we have $\{x_n\}$ is bounded, so is $\{u_n\}$. By (3.3), we have

$$\begin{aligned}
\alpha_n\beta_n\|x_n-P_C(I-\lambda A)u_n\|^2 \leq&\|x_n-z\|^2-\|x_{n+1}-z\|^2\\
&-\alpha_n\gamma_n\|x_n-TP_C(I-\lambda A)x_n\|^2\\
&-\beta_n\gamma_n\|P_C(I-\lambda A)u_n-TP_C(I-\lambda A)x_n\|\\
\leq&\|x_n-z\|^2-\|x_{n+1}-z\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n-z\|$ exists for every $z\in\mathfrak{F}$, we have

$$\lim_{n\to\infty}\|x_n-P_C(I-\lambda A)u_n\|=0. \qquad (3.4)$$

By using the same method as (3.4), we have

$$\lim_{n\to\infty}\|x_n-TP_C(I-\lambda A)x_n\|=0. \qquad (3.5)$$

Next, we will show that $\lim_{n\to\infty}\|x_n-u_n\|=0$. Let $z\in\mathfrak{F}$. Since $u_n=T_{r_n}x_n$ and T_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}
\|z-T_{r_n}x_n\|^2 =&\|T_{r_n}z-T_{r_n}x_n\|^2\\
\leq&\langle T_{r_n}z-T_{r_n}x_n, z-x_n\rangle\\
=&\frac{1}{2}(\|T_{r_n}x_n-z\|^2+\|x_n-z\|^2-\|T_{r_n}x_n-x_n\|^2).
\end{aligned}$$

Hence

$$\|u_n-z\|^2\leq\|x_n-z\|^2-\|u_n-x_n\|^2. \qquad (3.6)$$

By (3.2) and (3.6), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq&(1-\beta_n)\|x_n-z\|^2+\beta_n\|u_n-z\|^2\\&-\alpha_n\beta_n\|x_n-P_C(I-\lambda A)u_n\|^2-\alpha_n\gamma_n\|x_n-TP_C(I-\lambda A)x_n\|^2\\&-\beta_n\gamma_n\|P_C(I-\lambda A)u_n-TP_C(I-\lambda A)x_n\|\\\leq&(1-\beta_n)\|x_n-z\|^2+\beta_n(\|x_n-z\|^2-\|u_n-x_n\|^2)\\=&\|x_n-z\|^2-\beta_n\|u_n-x_n\|^2.\end{aligned}$$

It implies that

$$\beta_n\|u_n-x_n\|^2 \leq \|x_n-z\|^2-\|x_{n+1}-z\|^2.$$

Since $\lim_{n\to\infty}\|x_n-z\|$ exists for every $z \in \mathfrak{F}$, we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.7}$$

Since

$$\begin{aligned}\|x_n-P_C(I-\lambda A)x_n\| \leq&\|x_n-P_C(I-\lambda A)u_n\|+\|P_C(I-\lambda A)u_n-P_C(I-\lambda A)x_n\|\\\leq&\|x_n-P_C(I-\lambda A)u_n\|+\|u_n-x_n\|,\end{aligned}$$

from (3.4) and (3.7), we have

$$\lim_{n\to\infty}\|x_n-P_C(I-\lambda A)x_n\|=0. \tag{3.8}$$

By (3.5) and (3.8), we have

$$\lim_{n\to\infty}\|P_C(I-\lambda A)x_n-TP_C(I-\lambda A)x_n\|=0. \tag{3.9}$$

Next, we will show that $\omega(x_n) \subset \mathfrak{F}$ where $\omega(x_n)=\{x : x_{n_m} \rightharpoonup x$ for some subsequence $\{n_m\}$ of $\{n\}\}$. Since $\{x_n\}$ is bounded in H, we have $\omega(x_n) \neq \emptyset$. Let $\omega \in \omega(x_n)$. Thus, there is a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ which converges weakly to ω. By (3.8), nonexpansiveness of $P_C(I-\lambda A)$ and Lemma 2.41, we obtain that $\omega \in F(P_C(I-\lambda A)$. By Lemma 2.35, we have $\omega \in VI(C,A)$. Since $\|u_{n_m}-x_{n_m}\| \to 0$ as $m\to\infty$, we have $u_{n_m} \rightharpoonup \omega$ as $m \to \infty$. Since

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,\ u_n-x_n\rangle \geq 0, \quad \forall y \in C,$$

it follows by $(A2)$ that

$$\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq F(y,u_n), \quad \forall y \in C.$$

In particular,

$$\left\langle y-u_{n_m},\frac{1}{r_{n_m}}(u_{n_m}-x_{n_m})\right\rangle \geq F(y,u_{n_m}). \tag{3.10}$$

By condition $(A4)$, $F(y,\cdot)$ is convex and lower semicontinuous, and thus weakly semicontinuous. By (3.7) imply that $\frac{1}{r_{n_m}}(u_{n_m}-x_{n_m}) \to 0$ in norm. Therefore, letting $m\to\infty$ in (3.10), we have

$$F(y,\omega) \leq \lim_{m\to\infty} F(y,u_{n_m}) \leq 0, \quad \forall y \in C. \tag{3.11}$$

Replacing y with $y_t := ty + (1-t)\omega$, $t \in (0,1]$, we have $y_t \in C$ and using $(A1)$, $(A4)$, and (3.11), we obtain

$$0 = F(y_t, y_t) \le tF(y_t, y) + (1-t)F(y_t, \omega) \le tF(y_t, y).$$

Hence $F(ty + (1-t)\omega, y) \ge 0$, $\forall t \in (0,1]$ and $\forall y \in C$. Letting $t \to 0^+$ and using assumption $(A3)$, we can conclude that

$$F(\omega, y) \ge 0, \quad y \in C.$$

Therefore, $\omega \in EP(F)$.

Next, we will show that $\omega \in F(T)$. Assume $\omega \ne T\omega$, using Opial's property and (3.5), we have

$$\begin{aligned}\liminf_{m\to\infty} \|x_{n_m} - \omega\| &< \liminf_{m\to\infty} \|x_{n_m} - T\omega\| \\ &\le \liminf_{m\to\infty} (\|x_{n_m} - TP_C(I-\lambda A)x_{n_m}\| \\ &\quad + \|TP_C(I-\lambda A)x_{n_m} - T\omega\|) \\ &= \liminf_{m\to\infty} \|TP_C(I-\lambda A)x_{n_m} - T\omega\|. \end{aligned} \tag{3.12}$$

By nonexpansiveness of $P_C(I-\lambda A)$ and $\omega \in F(P_C(I-\lambda A))$, we have

$$\begin{aligned}\|TP_C(I-\lambda A)x_{n_m} - T\omega\|^2 =& \|TP_C(I-\lambda A)x_{n_m} - TP_C(I-\lambda A)\omega\|^2 \\ \le& \|P_C(I-\lambda A)x_{n_m} - P_C(I-\lambda A)\omega\|^2 \\ &+ 2\langle P_C(I-\lambda A)x_{n_m} - TP_C(I-\lambda A)x_{n_m}, \omega - T\omega\rangle \\ \le& \|x_{n_m} - \omega\|^2 \\ &+ 2\|P_C(I-\lambda A)x_{n_m} - TP_C(I-\lambda A)x_{n_m}\|\|\omega - T\omega\|.\end{aligned}$$

It implies from (3.9) that

$$\liminf_{m\to\infty} \|TP_C(I-\lambda A)x_{n_m} - T\omega\| \le \liminf_{m\to\infty} \|x_{n_m} - \omega\|. \tag{3.13}$$

By (3.12) and (3.13), we have

$$\liminf_{m\to\infty} \|x_{n_m} - \omega\| < \liminf_{m\to\infty} \|x_{n_m} - \omega\|.$$

This is a contradiction. Then we have $\omega \in F(T)$. Hence $\omega(x_n) \subset \mathfrak{F}$.

Finally, we show that $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $\mathfrak{F}$. Claim that $\omega(x_n)$ is a singleton set. Let $\omega_1, \omega_2 \in \omega(x_n)$ and let $\{x_{k_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{k_i} \rightharpoonup \omega_1$ as $i \to \infty$ and $x_{m_j} \rightharpoonup \omega_2$ as $j \to \infty$. Since $\lim_{n\to\infty} \|x_n - q\|$ exists for each $q \in \mathfrak{F}$ and $\omega_1, \omega_2 \in \mathfrak{F}$, it follows by Lemma 2.53 that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \omega_1\|^2 &= \lim_{j\to\infty} \|x_{m_j} - \omega_1\|^2 \\ &= \lim_{j\to\infty} \|x_{m_j} - \omega_2\|^2 + \|\omega_2 - \omega_1\|^2 \\ &= \lim_{i\to\infty} \|x_{k_i} - \omega_2\|^2 + \|\omega_2 - \omega_1\|^2\end{aligned}$$

$$= \lim_{i\to\infty} \|x_{k_i} - \omega_1\|^2 + \|\omega_1 - \omega_2\|^2 + \|\omega_2 - \omega_1\|^2$$
$$= \lim_{n\to\infty} \|x_n - \omega_1\|^2 + 2\|\omega_1 - \omega_2\|^2.$$

Hence, $\omega_1 = \omega_2$. This shows that $\omega(x_n)$ is a singleton set. Therefore, we can conclude that $x_n \rightharpoonup \omega \in \mathfrak{F}$ as $n \to \infty$. It follows from (3.7) that $u_n \rightharpoonup \omega \in \mathfrak{F}$ as $n \to \infty$. So the proof is complete. □

Using Theorem 3.1, we obtain the following weakly convergence theorem for finding a common element of the set of solutions of the variational inequality problem and the set of fixed points of a nonspreading mapping in a real Hilbert space.

Corollary 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a α−inverse strongly monotone mapping and let $T : C \to C$ be a nonspreading mapping with $\mathfrak{F} = F(T) \bigcap VI(C,\ A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + \beta_n P_C(I - \lambda A)x_n + \gamma_n T P_C(I - \lambda A)x_n, \qquad \forall n \geq 1,$$

where $\{\alpha_n\} \subset [a, b] \subset (0, 1)$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$, $\{\gamma_n\} \subset [e, f] \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, and $\lambda \in (0,\ 2\alpha)$. Then the sequence $\{x_n\}$ converges weakly to an element of $\mathfrak{F}$.

Proof. From Theorem 3.1, putting $F \equiv 0$, we have $u_n = P_C x_n$. Since $x_n \in C$ for all $n \geq 1$, we have $x_n = P_C x_n$. Then $u_n = x_n$ and the desired result is directly obtained by Theorem 3.1. □

The following result we obtain weakly convergence theorem for finding a common element of the set of solutions of the equilibrium problem, the set of solutions of the variational inequality problem and the set of fixed points of a quasi-nonexpansive mapping in Hilbert spaces. Therefore, we omit the proof.

Corollary 3.3. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunctions from $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$, let $A : C \to H$ be a α-inverse strongly monotone mapping and let $T : C \to C$ be a quasi-nonexpansive mapping T such that $I - T$ is demiclosed at zero with $\mathfrak{F} = F(T) \bigcap EP(F) \bigcap VI(C,\ A) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n P_C(I - \lambda A)u_n + \gamma_n T P_C(I - \lambda A)x_n, \qquad \forall n \geq 1, \end{cases} \tag{3.14}$$

where $\{\alpha_n\} \subset [a, b] \subset (0, 1)$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$, $\{\gamma_n\} \subset [e, f] \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, and $r_n \in [g,\ h] \subset (0,\ 2\alpha)$, $\lambda \in (0,\ 2\alpha)$. Then the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $\mathfrak{F}$.

3.2 Strong convergence theorems for finding a common element of the set of solutions of variational inequality problems and the set of common fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of the modified split generalized equilibrium problem

Motivated by SEP, SVIP and [29], we introduce strong convergence theorems for finding a common element of the set of solutions of variational inequality problems and the set of common fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of the modified split generalized equilibrium problem without assuming demiclose condition and $T_\omega := (1-\omega)I+\omega T$, where T is a quasi-noneapansive mapping and $\omega \in (0, \frac{1}{2})$ that is the difficult proof in a framework of Hilbert space. First, we introduce Lemma 3.4 that is the important tool used to prove Theorem 3.5.

Lemma 3.4. Let C and Q be nonempty closed convex subsets of a real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)-(A4). Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Then

1. $T_r^{F_1}(I-rf_1)$ and $T_s^{F_2}(I-sf_2)$ are nonexpansive mapping.
2. $$\begin{aligned}&\|T_r^{F_1}(I-rf_1)\left(p+\gamma A^*\left(T_s^{F_2}(I-sf_2)-I\right)Ap\right)\\ &\quad - T_r^{F_1}(I-rf_1)\left(q+\gamma A^*\left(T_s^{F_2}(I-sf_2)-I\right)Aq\right)\|^2\\ &\leq \|p-q\|^2+\gamma(\gamma L-1)\left\|\left(T_s^{F_2}(I-sf_2)-I\right)Ap-\left(T_s^{F_2}(I-sf_2)-I\right)Aq\right\|^2,\end{aligned}$$

for all $p,\ q \in C$, where $r \in (0, 2\rho),\ s \in (0,1),\ \gamma \in (0, 1/L),\ L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A, $T_r^{F_1} : H_1 \to C$ defined by

$$T_r^{F_1}(x) = \{z \in C : F_1(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C\},$$

for all $x \in H_1$ and $T_s^{F_2} : H_2 \to Q$ defined by

$$T_s^{F_2}(\overline{x}) = \{\overline{z} \in Q : F_2(\overline{z},y) + \frac{1}{s}\langle y-\overline{z}, \overline{z}-\overline{x}\rangle \geq 0, \forall y \in Q\},$$

for all $\overline{x} \in H_2$.

Proof. Let $p, q \in C$. First, we show 1 is true. Since f_1 is a ρ-inverse strongly monotone mapping and $r \in (0, 2\rho)$, we obtain

$$\begin{aligned}\left\|T_r^{F_1}(I-rf_1)p - T_r^{F_1}(I-rf_1)q\right\|^2 &\leq \|p-q\|^2 - 2r\langle p-q, f_1p-f_1q\rangle + r^2\|f_1p-f_1q\|^2\\ &\leq \|p-q\|^2 + r(r-2\rho)\|f_1p-f_1q\|^2\\ &\leq \|p-q\|^2.\end{aligned}$$

Thus $T_r^{F_1}(I-rf_1)$ is a nonexpansive mapping. Since f_2 is a firmly nonexpansive mapping and $s \in (0,1)$, we get

$$\begin{aligned}\left\|T_s^{F_2}(I-sf_2)\overline{p}-T_s^{F_2}(I-sf_2)\overline{q}\right\|^2 &\le \|\overline{p}-\overline{q}\|^2-2s\langle \overline{p}-\overline{q}, f_2\overline{p}-f_2\overline{q}\rangle+s^2\|f_2\overline{p}-f_2\overline{q}\|^2\\ &\le \|\overline{p}-\overline{q}\|^2-s(2-s)\|f_2\overline{p}-f_2\overline{q}\|^2\\ &\le \|\overline{p}-\overline{q}\|^2,\end{aligned}$$

for all $\overline{p},\overline{q} \in Q$. Therefore $T_s^{F_2}(I-sf_2)$ is a nonexpansive mapping.

Next, we show 2 is true. From Lemma 3.4 (1), we have

$$\begin{aligned}&\left\|T_r^{F_1}(I-rf_1)\left(p+\gamma A^*\left(T_s^{F_2}(I-sf_2)-I\right)Ap\right)\right.\\ &\quad\left.-T_r^{F_1}(I-rf_1)\left(q+\gamma A^*\left(T_s^{F_2}(I-sf_2)-I\right)Aq\right)\right\|^2\\ &\le \left\|(p-q)+\gamma\left(A^*\left(T_s^{F_2}(I-sf_2)-I\right)Ap-A^*\left(T_s^{F_2}(I-sf_2)-I\right)Aq\right)\right\|^2\\ &\le \|p-q\|^2+2\gamma\left\langle Ap-Aq,\left(T_s^{F_2}(I-sf_2)-I\right)Ap-\left(T_s^{F_2}(I-sf_2)-I\right)Aq\right\rangle\\ &\quad+\gamma^2L\left\|\left(T_s^{F_2}(I-sf_2)-I\right)Ap-\left(T_s^{F_2}(I-sf_2)-I\right)Aq\right\|^2.\end{aligned} \tag{3.15}$$

From the property of $T_s^{F_2}$, we get

$$\begin{aligned}&\|(I-sf_2)Ap-(I-sf_2)Aq\|^2\\ &\ge \left\|T_s^{F_2}(I-sf_2)Ap-T_s^{F_2}(I-sf_2)Aq-(Ap-Aq)+(Ap-Aq)\right\|^2\\ &=\left\|\left(T_s^{F_2}(I-sf_2)-I\right)Ap-\left(T_s^{F_2}(I-sf_2)-I\right)Aq\right\|^2\\ &\quad+2\left\langle\left(T_s^{F_2}(I-sf_2)-I\right)Ap-\left(T_s^{F_2}(I-sf_2)-I\right)Aq, Ap-Aq\right\rangle\\ &\quad+\|Ap-Aq\|^2.\end{aligned} \tag{3.16}$$

Since

$$\begin{aligned}\|(I-sf_2)Ap-(I-sf_2)Aq\|^2 &= \|Ap-Aq\|^2-2s\langle Ap-Aq, f_2Ap-f_2Aq\rangle\\ &\quad+s^2\|f_2Ap-f_2Aq\|^2.\end{aligned} \tag{3.17}$$

From (3.16), (3.17) and the property of firmly nonexpansive mapping, we get

$$\begin{aligned}&2\left\langle\left(T_s^{F_2}(I-sf_2)-I\right)Ap-\left(T_s^{F_2}(I-sf_2)-I\right)Aq, Ap-Aq\right\rangle\\ &\le -\left\|\left(T_s^{F_2}(I-sf_2)-I\right)Ap-\left(T_s^{F_2}(I-sf_2)-I\right)Aq\right\|^2\\ &\quad-2s\langle Ap-Aq, f_2Ap-f_2Aq\rangle+s^2\|f_2Ap-f_2Aq\|^2\\ &\le -\left\|\left(T_s^{F_2}(I-sf_2)-I\right)Ap-\left(T_s^{F_2}(I-sf_2)-I\right)Aq\right\|^2.\end{aligned}$$

That is

$$\begin{aligned}&2\gamma\left\langle\left(T_s^{F_2}(I-sf_2)-I\right)Ap-\left(T_s^{F_2}(I-sf_2)-I\right)Aq, Ap-Aq\right\rangle\\ &\le -\gamma\left\|\left(T_s^{F_2}(I-sf_2)-I\right)Ap-\left(T_s^{F_2}(I-sf_2)-I\right)Aq\right\|^2.\end{aligned} \tag{3.18}$$

Substituting (3.18) in (3.15), we obtain

$$\begin{aligned}
&\left\|T_r^{F_1}(I - rf_1)\left(p + \gamma A^*\left(T_s^{F_2}(I - sf_2) - I\right)Ap\right)\right.\\
&\left.\quad - T_r^{F_1}(I - rf_1)\left(q + \gamma A^*\left(T_s^{F_2}(I - sf_2) - I\right)Aq\right)\right\|^2\\
&\leq \|p - q\|^2 - \gamma\left\|\left(T_s^{F_2}(I - sf_2) - I\right)Ap - \left(T_s^{F_2}(I - sf_2) - I\right)Aq\right\|^2\\
&\quad + \gamma^2 L\left\|\left(T_s^{F_2}(I - sf_2) - I\right)Ap - \left(T_s^{F_2}(I - sf_2) - I\right)Aq\right\|^2\\
&= \|p - q\|^2 + \gamma(\gamma L - 1)\left\|\left(T_s^{F_2}(I - sf_2) - I\right)Ap - \left(T_s^{F_2}(I - sf_2) - I\right)Aq\right\|^2.
\end{aligned}$$

□

Next, we prove a strong convergence theorem for solving the modified split generalized equilibrium problem (MSGEP).

Theorem 3.5. Let C and Q be nonempty closed convex subsets of a real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)-(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases}
u_n = T_r^{F_1}(I - rf_1)\left(x_n + \gamma A^*\left(T_s^{F_2}(I - sf_2) - I\right)Ax_n\right),\\
y_n = P_C(I - d_1 D_1)\left(au_n + (1 - a)P_C(I - d_2 D_2)u_n\right),\\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_n,\ \forall n \in \mathbb{N},
\end{cases} \tag{3.19}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), r \in (0, 2\rho), s \in (0, 1), a \in [0, 1], 0 < k_i < 1$ with $\sum_{i=1}^N k_i = 1, \gamma \in (0, 1/L), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for some $c, d > 0$ for all $n \geq 1$,

(iii) $\sum_{n=1}^\infty \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(iv) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}} u$.

Proof. Let $x, y \in C$ and $z \in \mathfrak{F}$. First, we show that $(I - d_1D_1)$ is a nonexpansive mapping. Since D_1 is an α-inverse strongly monotone mapping, we obtain

$$\begin{aligned}
&\|(I - d_1D_1)x - (I - d_1D_1)y\|^2 \\
&= \|x - y\|^2 - 2d_1\langle x - y, D_1x - D_1y\rangle + d_1^2 \|D_1x - D_1y\|^2 \\
&\leq \|x - y\|^2 + d_1(d_1 - 2\alpha) \|D_1x - D_1y\|^2 \\
&\leq \|x - y\|^2 .
\end{aligned}$$

Thus $(I - d_1D_1)$ is a nonexpansive mapping. By using the same method as above, we see that $(I - d_2D_2)$ is a nonexpansive mapping. Since f_1 is a ρ-inverse strongly monotone mapping and f_2 is a firmly nonexpansive mapping. From Lemma 3.4 (1), we have $(T_r^{F_1}(I - rf_1))$ and $(T_s^{F_2}(I - sf_2))$ are nonexpansive mappings.

Since T_i is a finite family of quasi-nonexpansive mappings, $\forall i = 1, 2, ..., N$, we get

$$\begin{aligned}
\|T_iy_n - z\|^2 &= \|(y_n - z) - (I - T_i)y_n\|^2 \\
&= \|y_n - z\|^2 - 2\langle y_n - z, (I - T_i)y_n\rangle + \|(I - T_i)y_n\|^2 \\
&\leq \|y_n - z\|^2 .
\end{aligned}$$

We can conclude that

$$\|(I - T_i)y_n\|^2 \leq 2\langle y_n - z, (I - T_i)y_n\rangle . \tag{3.20}$$

From Remark 2.56 and $z \in \bigcap_{i=1}^N F(T_i)$, we have $z \in F\left(P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)\right)$ and $z = T_iz, \forall i = 1, 2, ..., N$. Since P_C is nonexpansive mapping and (3.20), we have

$$\begin{aligned}
&\left\|P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_n - z\right\|^2 \\
&= \left\|P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_n - P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)z\right\|^2 \\
&\leq \left\|(y_n - z) - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)y_n\right\|^2 \\
&= \|y_n - z\|^2 - 2\lambda_n\left\langle y_n - z, \sum_{i=1}^N k_i(I - T_i)y_n\right\rangle + \lambda_n^2\left\|\sum_{i=1}^N k_i(I - T_i)y_n\right\|^2 \\
&\leq \|y_n - z\|^2 - 2\lambda_n\sum_{i=1}^N k_i\langle y_n - z, (I - T_i)y_n\rangle + \lambda_n^2\sum_{i=1}^N k_i\|(I - T_i)y_n\|^2 \\
&\leq \|y_n - z\|^2 - \lambda_n\sum_{i=1}^N k_i\|(I - T_i)y_n\|^2 + \lambda_n^2\sum_{i=1}^N k_i\|(I - T_i)y_n\|^2 \\
&= \|y_n - z\|^2 + \lambda_n(\lambda_n - 1)\sum_{i=1}^N k_i\|(I - T_i)y_n\|^2 \\
&\leq \|y_n - z\|^2 .
\end{aligned} \tag{3.21}$$

Since $z \in VI(C, D_1)$ and $z \in VI(C, D_2)$ and using the property of $(I - d_1D_1)$ and $(I - d_2D_2)$, we get

$$\|y_n - z\|^2 = \|P_C (I - d_1 D_1)(a u_n + (1-a) P_C (I - d_2 D_2) u_n) - P_C (I - d_1 D_1) z\|^2$$
$$\leq a \|u_n - z\|^2 + (1-a) \|P_C (I - d_2 D_2) u_n - z\|^2 \quad (3.22)$$
$$\leq \|u_n - z\|^2 . \quad (3.23)$$

Since $z \in \Omega$, we have $z = T_r^{F_1}(I - rf_1) z$ and $Az = T_s^{F_2}(I - sf_2) Az$. From Lemma 3.4 (2) and $\gamma \in (0, 1/L)$, we obtain

$$\|u_n - z\|^2 = \left\|T_r^{F_1}(I - rf_1)\left(x_n + \gamma A^* \left(T_s^{F_2}(I - sf_2) - I\right) Ax_n\right) - T_r^{F_1}(I - rf_1) z\right\|^2$$
$$\leq \|x_n - z\|^2 + \gamma (L\gamma - 1) \left\|\left(T_s^{F_2}(I - sf_2) - I\right) Ax_n\right\|^2 \quad (3.24)$$
$$\leq \|x_n - z\|^2. \quad (3.25)$$

Using the definition of x_n, (3.21), (3.23) and (3.25), we get

$$\begin{aligned}\|x_{n+1} - z\| = &\|\alpha_n(u - z) + \beta_n(x_n - z) \\ &+ \gamma_n \left(P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i)\right)\right) y_n - z\right)\Bigg\| \\ \leq &\alpha_n\|u - z\| + \beta_n\|x_n - z\| + \gamma_n\|y_n - z\| \\ \leq &\alpha_n\|u - z\| + \beta_n\|x_n - z\| + \gamma_n\|u_n - z\| \\ \leq &\alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|.\end{aligned}$$

Using induction, we can conclude that

$$\|x_n - z\| \leq max\{\|u - z\|, \|x_1 - z\|\}$$

for all $n \geq 1$. This implies that the sequence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{u_n\}$. From Lemma 3.4 (2) and $\gamma \in (0, 1/L)$, we obtain

$$\begin{aligned}&\|u_n - u_{n-1}\|^2 \\ &= \|T_r^{F_1}(I - rf_1)\left(x_n + \gamma A^* \left(T_s^{F_2}(I - sf_2) - I\right) Ax_n\right) \\ &\quad - T_r^{F_1}(I - rf_1)\left(x_{n-1} + \gamma A^* \left(T_s^{F_2}(I - sf_2) - I\right) Ax_{n-1}\right)\|^2 \\ &\leq \|x_n - x_{n-1}\|^2 + \gamma(\gamma L - 1) \left\|\left(T_s^{F_2}(I - sf_2) - I\right) Ax_n - \left(T_s^{F_2}(I - sf_2) - I\right) Ax_{n-1}\right\|^2 \\ &\leq \|x_n - x_{n-1}\|^2 . \end{aligned} \quad (3.26)$$

Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. According to equation (3.26), we have

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &= \Bigg\|\left(\alpha_n u + \beta_n x_n + \gamma_n P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i)\right)\right) y_n\right) \\ &\quad - \left(\alpha_{n-1} u + \beta_{n-1} x_{n-1} + \gamma_{n-1} P_C \left(I - \lambda_{n-1} \left(\sum_{i=1}^{N} k_i (I - T_i)\right)\right) y_{n-1}\right)\Bigg\|\end{aligned}$$

$$
\begin{aligned}
&\leq |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \gamma_n \|y_n - y_{n-1}\| \\
&\quad + \lambda_n \left\| \sum_{i=1}^{N} k_i (I - T_i) y_n - \sum_{i=1}^{N} k_i (I - T_i) y_{n-1} \right\| \\
&\quad + |\lambda_n - \lambda_{n-1}| \left\| \sum_{i=1}^{N} k_i (I - T_i) y_{n-1} \right\| \\
&\quad + |\gamma_n - \gamma_{n-1}| \left\| P_C \left(I - \lambda_{n-1} \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_{n-1} \right\| \\
&\leq (1 - \alpha_n) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|u\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| \\
&\quad + \lambda_n \left\| \sum_{i=1}^{N} k_i (I - T_i) y_n - \sum_{i=1}^{N} k_i (I - T_i) y_{n-1} \right\| \\
&\quad + |\lambda_n - \lambda_{n-1}| \left\| \sum_{i=1}^{N} k_i (I - T_i) y_{n-1} \right\| \\
&\quad + |\gamma_n - \gamma_{n-1}| \left\| P_C \left(I - \lambda_{n-1} \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_{n-1} \right\| \\
&\leq (1 - \alpha_n) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M + |\beta_n - \beta_{n-1}| M + \lambda_n M \\
&\quad + |\lambda_n - \lambda_{n-1}| M + |\gamma_n - \gamma_{n-1}| M,
\end{aligned}
$$

where $M := \max_{n \in \mathbb{N}} \left\{ \|u\|, \|x_n\|, \left\| \sum_{i=1}^{N} k_i (I - T_i) y_{n+1} - \sum_{i=1}^{N} k_i (I - T_i) y_n \right\|, \left\| \sum_{i=1}^{N} k_i (I - T_i) y_n \right\|, \left\| P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n \right\| \right\}.$

From condition (i), (iii), (iv) and Lemma 2.54, we have

$$
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.27}
$$

According to equation (3.21), (3.23) and (3.24), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \alpha_n \|u - z\|^2 + \gamma_n \left\| P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n - z \right\|^2 \\
&\quad + \beta_n \|x_n - z\|^2 - \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n \right\|^2 \\
&\leq \alpha_n \|u - z\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \|y_n - z\|^2 \\
&\quad - \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n \right\|^2 && (3.28) \\
&\leq \alpha_n \|u - z\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \|u_n - z\|^2 \\
&\quad - \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n \right\|^2 && (3.29) \\
&\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|x_n - z\|^2 + \gamma_n \gamma (L\gamma - 1) \left\| \left(T_s^{F_2} (I - s f_2) - I \right) A x_n \right\|^2 \\
&\quad - \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n \right\|^2.
\end{aligned}
$$

Which implies that

$$\gamma_n\gamma(1-L\gamma)\left\|\left(T_s^{F_2}(I-sf_2)-I\right)Ax_n\right\|^2 \\ \leq \alpha_n\|u-z\|^2+\|x_n-x_{n+1}\|\left(\|x_n-z\|+\|x_{n+1}-z\|\right).$$

By using condition (i) and (3.27), we have

$$\lim_{n\to\infty}\left\|\left(T_s^{F_2}(I-sf_2)-I\right)Ax_n\right\|=0. \tag{3.30}$$

By using the same method as (3.30), we have

$$\lim_{n\to\infty}\left\|x_n-P_C\left(I-\lambda_n\left(\sum_{i=1}^{N}k_i(I-T_i)\right)\right)y_n\right\|=0. \tag{3.31}$$

Let $M_n=x_n+\gamma A^*\left(T_s^{F_2}(I-sf_2)-I\right)Ax_n$. Applying the inequality (3.25), we have

$$\|M_n-z\|\leq\|x_n-z\|. \tag{3.32}$$

Using the property of inverse strongly monotone operators and (3.32), we have

$$\begin{aligned}\|u_n-z\|^2&=\left\|T_r^{F_1}(I-rf_1)M_n-T_r^{F_1}(I-rf_1)z\right\|^2\\&\leq\|(I-rf_1)M_n-(I-rf_1)z\|^2\\&=\|M_n-z\|^2-2r\langle M_n-z,f_1M_n-f_1z\rangle+r^2\|f_1M_n-f_1z\|^2\\&\leq\|x_n-z\|^2+r(r-2\rho)\|f_1M_n-f_1z\|^2.\end{aligned} \tag{3.33}$$

Substituting (3.33) in (3.29), we have

$$\begin{aligned}\|x_{n+1}-z\|^2\leq&\alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2\\&+\gamma_n\left(\|x_n-z\|^2+r(r-2\rho)\|f_1M_n-f_1z\|^2\right)\\\leq&\alpha_n\|u-z\|^2+(1-\alpha_n)\|x_n-z\|^2+\gamma_nr(r-2\rho)\|f_1M_n-f_1z\|^2.\end{aligned}$$

That is

$$\gamma_nr(2\rho-r)\|f_1M_n-f_1z\|^2\leq\alpha_n\|u-z\|^2+\|x_n-x_{n+1}\|\left(\|x_n-z\|+\|x_{n+1}-z\|\right).$$

According to condition (i) and (3.27), we get

$$\lim_{n\to\infty}\|f_1M_n-f_1z\|=0. \tag{3.34}$$

By the property of firmly nonexpansive mappings, we have

$$\begin{aligned}\|u_n-z\|^2&=\left\|T_r^{F_1}(I-rf_1)M_n-T_r^{F_1}(I-rf_1)z\right\|^2\\&\leq\langle u_n-z,(I-rf_1)M_n-(I-rf_1)z\rangle\\&=\frac{1}{2}\left(\|u_n-z\|^2+\|(I-rf_1)M_n-(I-rf_1)z\|^2\right.\\&\quad\left.-\|(u_n-z)-((I-rf_1)M_n-(I-rf_1)z)\|^2\right).\end{aligned} \tag{3.35}$$

That is

$$\begin{aligned}\|u_n - z\|^2 \leq& \|(I - rf_1)M_n - (I - rf_1)z\|^2 - \|(u_n - M_n) + r\,(f_1M_n - f_1z)\,\|^2\\ \leq& \|M_n - z\|^2 - \big(\|u_n - M_n\|^2 + 2r\langle u_n - M_n, f_1M_n - f_1z\rangle\\ &+ r^2\|f_1M_n - f_1z\|^2\big)\\ \leq& \|M_n - z\|^2 - \|u_n - M_n\|^2 + 2r\|u_n - M_n\|\|f_1M_n - f_1z\|\\ &- r^2\|f_1M_n - f_1z\|^2. \end{aligned} \tag{3.36}$$

Substituting (3.36) in (3.29), we get

$$\begin{aligned}\|x_{n+1} - z\|^2 \leq& \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left(\|M_n - z\|^2 - \|u_n - M_n\|^2\right.\\ &\left.+ 2r\|u_n - M_n\|\|f_1M_n - f_1z\| - r^2\|f_1M_n - f_1z\|^2\right)\\ \leq& \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - \gamma_n\|u_n - M_n\|^2\\ &+ 2r\gamma_n\|u_n - M_n\|\|f_1M_n - f_1z\|.\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n\|u_n - M_n\|^2 \leq& \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - z\| + \|x_{n+1} - z\|\right)\\ &+ 2r\gamma_n\|u_n - M_n\|\|f_1M_n - f_1z\|.\end{aligned}$$

From condition (i), (3.27) and (3.34), we ensure that

$$\lim_{n\to\infty}\|u_n - M_n\| = 0. \tag{3.37}$$

From (3.30) and (3.37), we also have

$$\begin{aligned}\|u_n - x_n\| \leq& \|u_n - M_n\| + \|M_n - x_n\|\\ =& \|u_n - M_n\| + \|x_n + \gamma A^*\left(T_s^{F_2}\left(I - sf_2\right) - I\right)Ax_n - x_n\|\\ \leq& \|u_n - M_n\| + \gamma\|A\|\|\left(T_s^{F_2}\left(I - sf_2\right) - I\right)Ax_n\|.\end{aligned}$$

Then, we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.38}$$

By using the same method as (3.33), we have

$$\|P_C\left(I - d_2D_2\right)u_n - z\|^2 \leq \|x_n - z\|^2 + d_2(d_2 - 2\beta)\|D_2u_n - D_2z\|^2. \tag{3.39}$$

Substituting (3.22) and (3.39) in (3.28), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 \leq& \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\Big(a\,\|u_n - z\|^2\\ &+ (1 - a)\,\|P_C\left(I - d_2D_2\right)u_n - z\|^2\Big)\\ \leq& \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2\\ &+ \gamma_n\left(1 - a\right)d_2(d_2 - 2\beta)\|D_2u_n - D_2z\|^2.\end{aligned}$$

We can conclude that

$$\begin{aligned}&\gamma_n (1-a)d_2(2\beta - d_2)\|D_2u_n - D_2z\|^2 \\ &\qquad \leq \alpha_n\|u-z\|^2 + \|x_n - x_{n+1}\| \left(\|x_n - z\| + \|x_{n+1} - z\|\right).\end{aligned}$$

According to condition (i) and (3.27), we get

$$\lim_{n\to\infty} \|D_2u_n - D_2z\| = 0. \tag{3.40}$$

Since P_C is a firmly nonexpansive mapping and using the same method as (3.35), we get

$$\begin{aligned}&\|P_C(I-d_2D_2)u_n - z\|^2 \\ &\leq \frac{1}{2}\left(\|P_C(I-d_2D_2)u_n - z\|^2 + \|(I-d_2D_2)u_n - (I-d_2D_2)z\|^2\right. \\ &\quad \left. - \|P_C(I-d_2D_2)u_n - z - (I-d_2D_2)u_n + (I-d_2D_2)z\|^2\right).\end{aligned}$$

That is

$$\begin{aligned}\|P_C(I-d_2D_2)u_n - z\|^2 &\leq \|u_n - z\|^2 - \|(P_C(I-d_2D_2)u_n - u_n) + d_2(D_2u_n - D_2z)\|^2 \\ &\leq \|x_n - z\|^2 - \|P_C(I-d_2D_2)u_n - u_n\|^2 \\ &\quad + 2d_2\|P_C(I-d_2D_2)u_n - u_n\|\|D_2u_n - D_2z\| \\ &\quad - d_2^2\|D_2u_n - D_2z\|^2. \end{aligned} \tag{3.41}$$

Substituting (3.22) and (3.41) in (3.28), we have

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &\leq \alpha_n\|u-z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left(a\|u_n - z\|^2 + (1-a)\|P_C(I-d_2D_2)u_n - z\|^2\right) \\ &\leq \alpha_n\|u-z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left(a\|x_n - z\|^2 + (1-a)(\|x_n - z\|^2\right. \\ &\quad - \|P_C(I-d_2D_2)u_n - u_n\|^2 + 2d_2\|P_C(I-d_2D_2)u_n - u_n\|\|D_2u_n - D_2z\| \\ &\quad \left. - d_2^2\|D_2u_n - D_2z\|^2)\right) \\ &\leq \alpha_n\|u-z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \gamma_n(1-a)\|P_C(I-d_2D_2)u_n - u_n\|^2 \\ &\quad + 2d_2\gamma_n(1-a)\|P_C(I-d_2D_2)u_n - u_n\|\|D_2u_n - D_2z\|.\end{aligned}$$

Therefore

$$\begin{aligned}&\gamma_n(1-a)\|P_C(I-d_2D_2)u_n - u_n\|^2 \\ &\qquad \leq \alpha_n\|u-z\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - z\| + \|x_{n+1} - z\|\right) \\ &\qquad\quad + 2d_2\gamma_n(1-a)\|P_C(I-d_2D_2)u_n - u_n\|\|D_2u_n - D_2z\|.\end{aligned}$$

From condition (i), (3.27) and (3.40), we get

$$\lim_{n\to\infty} \|P_C(I-d_2D_2)u_n - u_n\| = 0. \tag{3.42}$$

Let $k_n = au_n + (1-a)\, P_C\,(I - d_2D_2)\,u_n$. By using the same method as (3.33), we have

$$\|y_n - z\|^2 \le \|x_n - z\|^2 + d_1(d_1 - 2\alpha)\|D_1k_n - D_1z\|^2. \tag{3.43}$$

Substituting (3.43) in (3.28), we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
\le&\alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left(\|x_n - z\|^2 + d_1(d_1 - 2\alpha)\|D_1k_n - D_1z\|^2\right) \\
\le&\alpha_n\|u - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 + d_1(d_1 - 2\alpha)\gamma_n\|D_1k_n - D_1z\|^2.
\end{aligned}$$

Which implies that

$$d_1(2\alpha - d_1)\gamma_n\|D_1k_n - D_1z\|^2 \le \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - z\| + \|x_{n+1} - z\|\right).$$

According to condition (i) and (3.27), we have

$$\lim_{n\to\infty}\|D_1k_n - D_1z\| = 0. \tag{3.44}$$

By using the same method as (3.35), we have

$$\begin{aligned}
\|y_n - z\|^2 =&\|P_C\,(I - d_1D_1)\,k_n - P_C\,(I - d_1D_1)\,z\|^2 \\
\le&\langle y_n - z, (I - d_1D_1)\,k_n - (I - d_1D_1)\,z\rangle \\
=&\frac{1}{2}\left(\|y_n - z\|^2 + \|(I - d_1D_1)\,k_n - (I - d_1D_1)\,z\|^2\right. \\
&\left. - \|(y_n - k_n) + d_1(D_1k_n - D_1z)\|^2\right).
\end{aligned}$$

That is

$$\begin{aligned}
\|y_n - z\|^2 \le&\|k_n - z\|^2 - \left(\|y_n - k_n\|^2 + 2d_1\langle y_n - k_n, D_1k_n - D_1z\rangle\right. \\
&\left. + d_1^2\|D_1k_n - D_1z\|^2\right) \\
\le&\|x_n - z\|^2 - \|y_n - k_n\|^2 + 2d_1\|y_n - k_n\|\|D_1k_n - D_1z\| \\
&- d_1^2\|D_1k_n - D_1z\|^2.
\end{aligned} \tag{3.45}$$

Substituting (3.45) in (3.28),we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 \le&\alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left(\|x_n - z\|^2 - \|y_n - k_n\|^2\right. \\
&\left. + 2d_1\|y_n - k_n\|\|D_1k_n - d_1z\| - d_1^2\|D_1k_n - D_1z\|^2\right) \\
\le&\alpha_n\|u - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \gamma_n\|y_n - k_n\|^2 \\
&+ 2\gamma_nd_1\|y_n - k_n\|\|D_1k_n - D_1z\|.
\end{aligned} \tag{3.46}$$

Which implies that

$$\begin{aligned}
\gamma_n\|y_n - k_n\|^2 \le&\alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - z\| + \|x_{n+1} - z\|\right) \\
&+ 2\gamma_nd_1\|y_n - k_n\|\|D_1k_n - D_1z\|.
\end{aligned}$$

According to condition (i), (3.27) and (3.44), we get

$$\lim_{n\to\infty} \|y_n - k_n\| = 0. \tag{3.47}$$

From (3.42) and (3.47)

$$\begin{aligned}\|y_n - u_n\| &\leq \|y_n - k_n\| + \|k_n - u_n\| \\ &\leq \|y_n - k_n\| + (1-a)\|P_C\left(I - d_2 D_2\right) u_n - u_n\|,\end{aligned}$$

we conclude that

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.48}$$

By (3.38) and (3.48), we also conclude that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.49}$$

Afterward, we show that $\limsup_{n\to\infty} \langle u - z, x_n - z\rangle \leq 0$, where $z = P_{\mathfrak{F}} u$. To show this, choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - z, x_n - z\rangle = \lim_{j\to\infty} \langle u - z, x_{n_j} - z\rangle . \tag{3.50}$$

Without loss of generality, we may assume that $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$. From (3.49), we obtain $y_{n_j} \rightharpoonup \omega$ as $j \to \infty$. From Lemma 2.35, we have $VI(C, D_1) = F(P_C(I - d_1 D_1))$. Assume that $\omega \notin VI(C, D_1)$, we have $\omega \neq P_C(I - d_1 D_1)\omega$. Using Opial's condition, (3.47), we obtain

$$\begin{aligned}\liminf_{j\to\infty} \|y_{n_j} - \omega\| &< \liminf_{j\to\infty} \|y_{n_j} - P_C(I - d_1 D_1)\omega\| \\ &\leq \liminf_{j\to\infty} \left(\|P_C(I - d_1 D_1)k_{n_j} - P_C(I - d_1 D_1)y_{n_j}\|\right. \\ &\quad \left. + \|P_C(I - d_1 D_1)y_{n_j} - P_C(I - d_1 D_1)\omega\|\right) \\ &\leq \liminf_{j\to\infty} \left(\|k_{n_j} - y_{n_j}\| + \|y_{n_j} - \omega\|\right) \\ &\leq \liminf_{j\to\infty} \|y_{n_j} - \omega\| .\end{aligned}$$

This is a contradiction, so we have

$$\omega \in VI\left(C, D_1\right). \tag{3.51}$$

From (3.38), we have $u_{n_j} \rightharpoonup \omega$ as $j \to \infty$. By (3.42) and using the same method as (3.51), we obtain

$$\omega \in VI\left(C, D_2\right). \tag{3.52}$$

Next, we show that $\omega \in \bigcap_{i=1}^{N} F\left(T_i\right)$. From Lemma 2.56, we have

$$\bigcap_{i=1}^{N} F\left(T_i\right) = F\left(P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^{N} k_i\left(I - T_i\right)\right)\right)\right).$$

Assume that $\omega \notin \bigcap_{i=1}^{N} F(T_i)$, that in $\omega \neq P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^{N} k_i (I - T_i)\right)\right)\omega$. Using Opial's condition, (3.31) and (3.49), we obtain

$$\begin{aligned}
&\liminf_{j\to\infty} \|x_{n_j} - \omega\| \\
&< \liminf_{j\to\infty} \left\| x_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^{N} k_i (I - T_i)\right)\right)\omega \right\| \\
&\leq \liminf_{j\to\infty} \left(\left\| x_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^{N} k_i (I - T_i)\right)\right) y_{n_j} \right\| \right. \\
&\quad + \left\| P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^{N} k_i (I - T_i)\right)\right) y_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^{N} k_i (I - T_i)\right)\right) x_{n_j} \right\| \\
&\quad \left. + \left\| P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^{N} k_i (I - T_i)\right)\right) x_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^{N} k_i (I - T_i)\right)\right) \omega \right\| \right) \\
&\leq \liminf_{j\to\infty} \left(\|y_{n_j} - x_{n_j}\| + \lambda_{n_j} \left\| \sum_{i=1}^{N} k_i (I - T_i) y_{n_j} - \sum_{i=1}^{N} k_i (I - T_i) x_{n_j} \right\| \right. \\
&\quad \left. + \|x_{n_j} - \omega\| + \lambda_{n_j} \left\| \sum_{i=1}^{N} k_i (I - T_i) x_{n_j} - \sum_{i=1}^{N} k_i (I - T_i) \omega \right\| \right) \\
&\leq \liminf_{j\to\infty} \|x_{n_j} - \omega\|.
\end{aligned}$$

This is a contradiction, so we have

$$\omega \in \bigcap_{i=1}^{N} F(T_i). \tag{3.53}$$

After that, we show that $\omega \in \Omega$. Assume $\omega \notin EP(F_1, f_1)$. Since $EP(F_1, f_1) = F(T_r^{F_1}(I - rf_1))$, we obtain $\omega \neq T_r^{F_1}(I - rf_1)\omega$. Using Opial's condition and (3.37), we get

$$\begin{aligned}
\liminf_{j\to\infty} \|u_{n_j} - \omega\| &< \liminf_{j\to\infty} \|u_{n_j} - T_r^{F_1}(I - rf_1)\omega\| \\
&\leq \liminf_{j\to\infty} \left(\|T_r^{F_1}(I - rf_1) M_{n_j} - T_r^{F_1}(I - rf_1) u_{n_j}\| \right. \\
&\quad \left. + \|T_r^{F_1}(I - rf_1) u_{n_j} - T_r^{F_1}(I - rf_1)\omega\| \right) \\
&\leq \liminf_{j\to\infty} \left(\|M_{n_j} - u_{n_j}\| + \|u_{n_j} - \omega\| \right) \\
&\leq \liminf_{j\to\infty} \|u_{n_j} - \omega\|.
\end{aligned}$$

This is a contradiction, so we have

$$\omega \in EP(F_1, f_1). \tag{3.54}$$

Next, we show that $A\omega \in EP(F_2, f_2)$. Since A is bounded linear operator so that $Ax_{n_j} \rightharpoonup A\omega$ as $j \to \infty$. Assume $A\omega \notin EP(F_2, f_2)$. Since $EP(F_2, f_2) = F(T_s^{F_2}(I - sf_2))$, we obtain $A\omega \neq T_s^{F_2}(I - sf_s) A\omega$. Using Opial's condition and (3.30), so we have

$$A\omega \in EP(F_2, f_2). \tag{3.55}$$

We can conclude that $\omega \in \Omega$. Therefore $\omega \in \mathfrak{F}$. Since $\langle u - z, x_n - z\rangle$ is bounded and

$x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, we have

$$\begin{aligned} \limsup_{n\to\infty} \langle u - z, x_n - z\rangle &= \lim_{j\to\infty} \langle u - z, x_{n_j} - z\rangle \\ &= \langle u - z, \omega - z\rangle \le 0. \end{aligned} \tag{3.56}$$

Finally, we show that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathfrak{F}}u$. By (3.21), (3.23) and (3.25), we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \left\| \alpha_n(u - z) + \beta_n(x_n - z) + \gamma_n \left(P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n - z \right) \right\|^2 \\ &\le \left\| \beta_n(x_n - z) + \gamma_n \left(P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n - z \right) \right\|^2 \\ &\quad + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\le (\beta_n \|x_n - z\| + \gamma_n \|u_n - z\|)^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\le (1 - \alpha_n) \|x_n - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle. \end{aligned}$$

According to condition (i), (3.56) and Lemma 2.54, we can conclude that $\{x_n\}$ converges strongly to $z = P_{\mathfrak{F}}u$. By (3.38) and (3.49), we have $\{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}}u$. So the proof is complete. □

From Theorem 3.5, if we take $N = 1$, we have the following corollary:

Corollary 3.6. Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)-(A4). Let T be a quasi-nonexpansive mapping of C into itself. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap F(T) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1} (I - rf_1) \left(x_n + \gamma A^* \left(T_s^{F_2} (I - sf_2) - I\right) Ax_n\right), \\ y_n = P_C (I - d_1 D_1) (au_n + (1 - a) P_C (I - d_2 D_2) u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C (I - \lambda_n (I - T)) y_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), r \in (0, 2\rho), s \in (0, 1), a \in [0, 1], \gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions (i) – (iv) of Theorem 3.5 holds. Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}}u$.

From Theorem 3.5, if we take $H_1 \equiv H_2, F_1 \equiv F_2, f_1 \equiv f_2$ and $A \equiv I$, we have the following corollary:

Corollary 3.7. Let C be nonempty closed convex subset of a real Hilbert space H_1. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ be the bifunction satisfying (A1)-(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap EP(F_1, f_1) \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)x_n, \\ y_n = P_C(I - d_1D_1)(au_n + (1-a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), r \in (0, 2\rho), a \in [0,1], 0 < k_i < 1$ with $\sum_{i=1}^{N} k_i = 1$. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions $(i) - (iv)$ of Theorem 3.5 holds. Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}}u$.

From Theorem 3.5, if we take $f_1 \equiv f_2 \equiv 0$, we have the following corollary:

Corollary 3.8. Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)-(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Gamma \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1}(x_n + \gamma A^*(T_s^{F_2} - I)Ax_n), \\ y_n = P_C(I - d_1D_1)(au_n + (1-a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), a \in [0,1], 0 < k_i < 1$ with $\sum_{i=1}^{N} k_i = 1, \gamma \in (0, 1/L), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions $(i) - (iv)$ of Theorem 3.5 holds. Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}}u$.

Remark 3.9. By using the concept of Picard iteration, Wang [25] defined the iterative scheme $\{x_n\}$ for solving SCFPP as follows:

$$\begin{aligned} x_{n+1} &= x_n - \rho_n((I - U)x_n + A^*(I - T)Ax_n) \\ &= (I - \rho_n((I - U) + A^*(I - T)A))x_n, \end{aligned} \tag{3.57}$$

where ρ_n is according to (1.5) and U and T are firmly quasi-nonexpansive mappings. Then the sequence $\{x_n\}$ converges weakly to z, where $z = \lim_{n\to\infty} P_{\Phi}x_n$. In Theorem 3.5, we use the concept of Halpern iteration and suitable conditions of the parameters $d_1, d_2, r, s, a, \gamma, L, \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$, the sequence $\{x_n\}$ defined by (3.19) converges strongly to $z = P_{\mathfrak{F}}u$ which is a different method from (3.57).

3.3 Application

In this section, to show the application of section 3.2, the Corollary 3.6 is applied by using the relation between quasi-nonexpansive mappings and nonspreading mappings.

3.3.1 Strong convergence theorems for finding a common element of the set of solutions of variational inequality problems and the set of common fixed points of a finite family of nonspreading mappings and the set of solutions of the modified split generalized equilibrium problem

In 2009, Kangtunyakarn and Suantai [3] introduced the S-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$ as follows:

Definition 3.1. Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$
\begin{aligned}
U_0 &= I, \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\
&\ \ \vdots \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}
$$

This mapping is called an *S-mapping* generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 3.10. [45] Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (0, 1]$, $\alpha_3^N \in [0, 1)$, $\alpha_2^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let

S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^{N} F(T_i)$ and S is a quasi-nonexpansive mapping.

By using these results, we obtain the following Theorems.

Theorem 3.11. Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)-(A4). Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0,1], \alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (0,1], \alpha_3^N \in [0,1), \alpha_2^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^{N} F(T_i) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)\left(x_n + \gamma A^*\left(T_s^{F_2}(I - sf_2) - I\right)Ax_n\right), \\ y_n = P_C(I - d_1D_1)\left(au_n + (1-a)P_C(I - d_2D_2)u_n\right), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(I - S))y_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), r \in (0, 2\rho), s \in (0,1), a \in [0,1], \gamma \in (0, 1/L), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions $(i) - (iv)$ of Theorem 3.5 holds. Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}}u$.

Proof. By using Corollary 3.6 and Lemma 3.10, we obtain the conclusion. □

From Theorem 3.11, if we take $N = 1$, we have the following corollary:

Corollary 3.12. Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)-(A4). Let T be a nonspreading mapping of C into C with $F(T) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap F(T) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)\left(x_n + \gamma A^*\left(T_s^{F_2}(I - sf_2) - I\right)Ax_n\right), \\ y_n = P_C(I - d_1D_1)\left(au_n + (1-a)P_C(I - d_2D_2)u_n\right), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(I - T))y_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), r \in (0, 2\rho), s \in (0, 1), a \in [0, 1], \gamma \in (0, 1/L), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions $(i) - (iv)$ of Theorem 3.5 holds. Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}}u$.

Chapter 4

Examples and Numerical Results

In this section, Example 4.1 is given for supporting Theorem 3.1. Example 4.2 and Example 4.3 are given for supporting Theorem 3.5.

Example 4.1. Let $\mathbb{R}$ be the set of real numbers and let A be a mapping from $[0,100]$ to $\mathbb{R}$ defined by $Ax = \frac{x-1}{5}$, $\forall x \in [0,100]$ and T be a mapping from $[0,100]$ into itself defined by $Tx = \frac{x+1}{2}$, $\forall x \in [0,100]$. Let $F : [0,100] \times [0,100] \to \mathbb{R}$ defined by

$$F(x,y) = -(x-1)^2 + (y-1)^2, \forall x, y \in \mathbb{R}.$$

By the definition of F, we have

$$\begin{aligned}
0 &\leq F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \\
&= -(u_n - 1)^2 + (y-1)^2 + \frac{1}{r_n}(y - u_n)(u_n - x_n) \\
&= -(u_n - 1)^2 + (y-1)^2 + \frac{1}{r_n}(yu_n - yx_n - u_n^2 + u_n x_n) \\
&\Leftrightarrow \\
0 &\leq r_n\left(-(u_n-1)^2 + (y-1)^2\right) + (yu_n - yx_n - u_n^2 + u_n x_n) \\
&= 2r_n u_n - u_n^2 - r_n u_n^2 + u_n x_n + (-2r_n + u_n - x_n)y + r_n y^2.
\end{aligned}$$

Let $G(y) = r_n y^2 + (-2r_n + u_n - x_n)y + 2r_n u_n - u_n^2 - r_n u_n^2 + u_n x_n$ which is a quadratic function of y with coefficient $a = r_n$, $b = -2r_n + u_n - x_n$, and $c = 2r_n u_n - u_n^2 - r_n u_n^2 + u_n x_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned}
\Delta =& b^2 - 4ac \\
=& (-2r_n + u_n - x_n)^2 - 4r_n(2r_n u_n - u_n^2 - r_n u_n^2 + u_n x_n) \\
=& 4r_n^2 - 4r_n u_n - 8r_n^2 u_n + u_n^2 + 4r_n u_n^2 + 4r_n^2 u_n^2 + 4r_n x_n - 2u_n x_n - 4r_n u_n x_n + x_n^2 \\
=& (-2r_n + u_n + 2r_n u_n - x_n)^2.
\end{aligned}$$

We know that $G(y) \geq 0, \forall y \in [0,100]$. If it has most one solution in $\mathbb{R}$, then $\Delta \leq 0$. So we obtain

$$u_n = \frac{2r_n + x_n}{1 + 2r_n}. \tag{4.1}$$

Let $x_1 \in \mathbb{R}$ and $\{x_n\}$ generated by (3.14), where $\lambda = 1, r_n = \frac{n+1}{7n}, \alpha_n = \frac{n+1}{3n}, \beta_n = \frac{7n-3}{15n}$ and $\gamma_n = \frac{3n-2}{15n}$ for all $n \in \mathbb{N}$. By the definition of F, A and T, we have $1 \in F(T) \cap EP(F) \cap VI(C,A)$. Therefore the sequences $\{x_n\}$ and $\{u_n\}$ converge to 1. They are rewritten as follows:

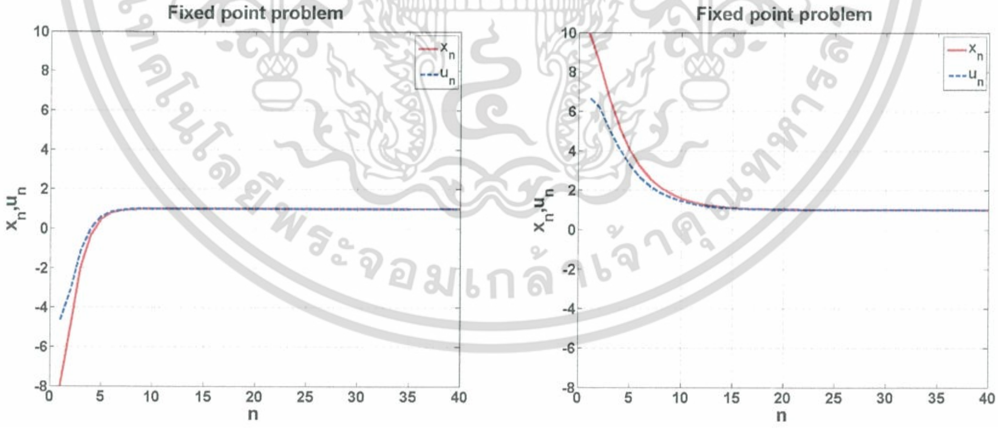
$$\begin{cases}
u_n = \frac{2r_n + x_n}{1+2r_n}, \\
x_{n+1} = \left(\frac{n+1}{3n}\right)x_n + \left(\frac{7n-3}{15n}\right)P_{[0,100]}(I-A)u_n + \left(\frac{3n-2}{15n}\right)TP_{[0,100]}(I-A)x_n, \ \forall n \geq 1,
\end{cases}$$

The following table shows the values of sequences $\{x_n\}$ and $\{u_n\}$ when $x_1 = -8$ and $x_1 = 10$ and $n = 40$.

Table 4.1: The values of $\{x_n\}$ and $\{u_n\}$ when $n = 40$.

	$x_1 = -8$		$x_1 = 10$	
n	x_n	u_n	x_n	u_n
1	-8.0000	-4.7273	10.0000	6.7273
2	-5.0000	-3.2000	8.4618	6.2233
3	-2.0000	-1.1724	6.6610	5.0994
4	-0.3333	0.0175	5.1801	4.0800
⋮	⋮	⋮	⋮	⋮
20	1.0000	1.0000	1.0191	1.0147
⋮	⋮	⋮	⋮	⋮
37	1.0000	1.0000	1.0001	1.0000
38	1.0000	1.0000	1.0000	1.0000
39	1.0000	1.0000	1.0000	1.0000
40	1.0000	1.0000	1.0000	1.0000

The figure 4.1 is the values of sequences $\{x_n\}$ and $\{u_n\}$ that correspond to the Table 4.1.



(a) $x_1 = -8$ (b) $x_1 = 10$

Figure 4.1: The convergence comparison of the sequences $\{x_n\}$ and $\{u_n\}$ with different initial values x_1.

The sequences $\{x_n\}$ and $\{u_n\}$ in Table 4.1 and Figure 4.1 converge to 1, where $1 \in F(T) \cap EP(F) \cap VI(C, A)$. From Table 4.1 and Figure 4.1, so that the convergence of $x_1 = -8$ is faster than $x_1 = 10$ because C is the set of real numbers in the interval $[0, 100]$. Therefore Theorem 3.1 assure the convergence of $\{x_n\}$ and $\{u_n\}$ in the Example 4.1.

In Example 4.2, we only instance an example in infinite dimensional Hilbert space for supporting Theorem 3.5. We omit the computer programming.

Example 4.2. Let $H_1 = H_2 = C = Q = \ell_2$ be the linear space whose elements consist of all 2-summable sequences $(x_1, x_2, ..., x_j, ...)$ of scalars, i.e.,

$$\ell_2 = \left\{ x : x = (x_1, x_2, ..., x_j, ...) \text{ and } \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\},$$

with an inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j$ where $x = \{x_j\}_{j=1}^{\infty}, y = \{y_j\}_{j=1}^{\infty} \in \ell_2$ and a norm $\|\cdot\| : \ell_2 \to \mathbb{R}$ defined by $\|x\|_2 = \left(\sum_{j=1}^{\infty} |x_j|^2\right)^{\frac{1}{2}}$ where $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. Let the mapping $A : \ell_2 \to \ell_2$ defined by $Ax = (\frac{x_1}{3}, \frac{x_2}{3}, ..., \frac{x_j}{3}, ...)$ for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$ and $A^* : \ell_2 \to \ell_2$ defined by $A^*z = (\frac{z_1}{3}, \frac{z_2}{3}, ..., \frac{z_j}{3}, ...)$ for all $z = \{z_j\}_{j=1}^{\infty} \in \ell_2$. Let $D_1, D_2 : \ell_2 \to \ell_2$ defined by $D_1 x = (\frac{x_1}{6}, \frac{x_2}{6}, ..., \frac{x_j}{6}, ...)$ and $D_2 x = (\frac{x_1}{5}, \frac{x_2}{5}, ..., \frac{x_j}{5}, ...), \forall x = \{x_j\}_{j=1}^{\infty} \in \ell_2$, respectively. Let the mapping $T_i : \ell_2 \to \ell_2$ defined by $T_i x = \left(\frac{3ix_1}{5i+1}, \frac{3ix_2}{5i+1}, ..., \frac{3ix_j}{5i+1}, ...\right), \forall x = \{x_j\}_{j=1}^{\infty} \in \ell_2$ and $k_i = \frac{6}{7^i} + \frac{1}{N7^N}$ for every $i = 1, 2, ..., N$. Let the mapping $F_1, F_2 : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by

$$F_1(x, y) = -x^2 + y^2, \quad \forall x = \{x_j\}_{j=1}^{\infty}, y = \{y_j\}_{j=1}^{\infty} \in \ell_2,$$

and

$$F_2(x, y) = -2x^2 + xy + y^2, \quad \forall x = \{x_j\}_{j=1}^{\infty}, y = \{y_j\}_{j=1}^{\infty} \in \ell_2.$$

Let the mapping $f_1 : \ell_2 \to \ell_2$ defined by $f_1 x = (\frac{x_1}{5}, \frac{x_2}{5}, ..., \frac{x_j}{5}, ...), \forall x = \{x_j\}_{j=1}^{\infty} \in \ell_2$ and the mapping $f_2 : \ell_2 \to \ell_2$ defined by $f_2 x = (\frac{x_1}{7}, \frac{x_2}{7}, ..., \frac{x_j}{7}, ...), \forall x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. Let $r = 1$ and $s = 0.5$. Since $L = \frac{1}{9}$, we choose $\gamma = 0.5$. Let $x_1 = (x_1^1, x_1^2, ..., x_1^j, ...)$ and $u = (u_1, u_2, ..., u_j, ...) \in \ell_2$ and let the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are generated by (3.19) as follows:

$$\begin{cases} u_n = T_1^{F_1}(I - f_1)\left(x_n + 0.5A^*\left(T_{0.5}^{F_2}(I - 0.5f_2) - I\right)Ax_n\right), \\ y_n = (I - D_1)(0.5u_n + 0.5(I - D_2)u_n), \\ x_{n+1} = \frac{1}{2n}u + \frac{7n-4}{12n}x_n + \frac{5n-2}{12n}\left(y_n - \left((\frac{1}{2n^2})\left(\sum_{i=1}^{N}(\frac{6}{7^i} + \frac{1}{N7^N})(y_n - T_i y_n)\right)\right)\right), \end{cases}$$

for all $n \geq 1$, where $x_n = (x_n^1, x_n^2, ..., x_n^j, ...), y_n = (y_n^1, y_n^2, ..., y_n^j, ...)$ and $u_n = (u_n^1, u_n^2, ..., u_n^j, ...)$. It easy to see that $D_1, D_2, T_i, F_1, F_2, f_1$ and f_2 satisfies Theorem 3.5. Moreover, we have $VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^{N} F(T_i) \cap \Omega = \{0\}$, where $\rho = d_1 = d_2 = 1$. From Theorem 3.5, we can conclude that the sequence $\{x_n\}, \{y_n\}$ and $\{u_n\}$ converge strongly to 0.

In Example 4.3, we give computer programming to support Theorem 3.5.

Example 4.3. Let $H_1 = H_2 = C = Q = \mathbb{R}^2$ be the two-dimensional Euclidean space of the real number with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\langle x, y \rangle = x \cdot y = x_1 y_1 + x_2 y_2$ where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and a usual norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$ where $x = (x_1, x_2) \in \mathbb{R}^2$. Let the mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$

defined by $Ax = (2x_1 - x_2, x_1 + 2x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $A^* : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A^*z = (2z_1 + z_2, 2z_2 - z_1)$ for all $z = (z_1, z_2) \in \mathbb{R}^2$. Let $D_1, D_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $D_1x = \left(\frac{x_1}{6}, \frac{x_2}{6}\right)$ and $D_2x = \left(\frac{x_1}{2}, \frac{x_2}{3}\right), \forall x = (x_1, x_2) \in \mathbb{R}^2$, respectively. Let the mapping $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_ix = \left(\frac{3ix_1}{3i+1}, \frac{3ix_2}{3i+2}\right), \forall x = (x_1, x_2) \in \mathbb{R}^2$ and $k_i = \frac{6}{7^i} + \frac{1}{N7^N}$ for every $i = 1, 2, ..., N$. Let the mapping $F_1, F_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F_1(x, y) = -x^2 + y^2, \forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$$

and

$$F_2(x, y) = -2x^2 + xy + y^2, \forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

Let the mapping $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f_1x = \left(\frac{x_1}{5}, \frac{x_2}{5}\right), \forall x = (x_1, x_2) \in \mathbb{R}^2$ and the mapping $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f_2x = \left(\frac{x_1}{7}, \frac{x_2}{7}\right), \forall x = (x_1, x_2) \in \mathbb{R}^2$. Let $r = 1$ and $s = 0.5$, the sequence $z_n = (z_n^1, z_n^2), x_n = (x_n^1, x_n^2), u_n = (u_n^1, u_n^2), y = (y_1, y_2) \in \mathbb{R}^2$. By the definition of f_1 and f_2, we get

$$\begin{aligned}
0 \leq& F_1(z_n, y) + \langle f_1(x_n), y - z_n\rangle + \frac{1}{r}\langle y - z_n, z_n - x_n\rangle \\
=& (y_1)^2 + (y_2)^2 + \frac{1}{5}x_n^1(y_1 - z_n^1) + \frac{1}{5}x_n^2(y_2 - z_n^2) + (y_1 - z_n^1)(z_n^1 - x_n^1) \\
&+ (y_2 - z_n^2)(z_n^2 - x_n^2) \\
=& \left((y_1)^2 + \left(-\frac{4}{5}x_n^1 + z_n^1\right)y_1 + \frac{4}{5}x_n^1z_n^1 - 2(z_n^1)^2\right) \\
&+ \left((y_2)^2 + \left(-\frac{4}{5}x_n^2 + z_n^2\right)y_2 + \frac{4}{5}x_n^2z_n^2 - 2(z_n^2)^2\right) \\
=& G_1(y_1) + G_2(y_2).
\end{aligned}$$

Let $G_1(y_1) = (y_1)^2 + \left(-\frac{4}{5}x_n^1 + z_n^1\right)y_1 + \frac{4}{5}x_n^1z_n^1 - 2(z_n^1)^2$ and $G_2(y_2) = (y_2)^2 + \left(-\frac{4}{5}x_n^2 + z_n^2\right)y_2 + \frac{4}{5}x_n^2z_n^2 - 2(z_n^2)^2$. $G_1(y_1)$ and $G_2(y_2)$ are quadratic function with coefficient $a_1 = 1$, $b_1 = -\frac{4}{5}x_n^1 + z_n^1$, and $c_1 = \frac{4}{5}x_n^1z_n^1 - 2(z_n^1)^2$ of $G_1(y_1)$ and coefficient $a_2 = 1$, $b_2 = -\frac{4}{5}x_n^2 + z_n^2$, and $c_2 = \frac{4}{5}x_n^2z_n^2 - 2(z_n^2)^2$ of $G_2(y_2)$, respectively. Determine the discriminant Δ_1 of G_1 as follows:

$$\begin{aligned}
\Delta_1 =& b_1^2 - 4a_1c_1 \\
=& \left(-\frac{4}{5}x_n^1 + z_n^1\right)^2 - 4(1)\left(\frac{4}{5}x_n^1z_n^1 - 2(z_n^1)^2\right) \\
=& \frac{1}{25}(4x_n^1 - 15z_n^1)^2.
\end{aligned}$$

We know that $G_1(y_1) \geq 0, \forall y \in \mathbb{R}$. If it has most one solution in $\mathbb{R}$, then $\Delta_1 \leq 0$, so we obtain $z_n^1 = \dfrac{4x_n^1}{15}$. Next, we determine the discriminant Δ_2 of G_2 by using the same method as above, we obtain $z_n^2 = \dfrac{4x_n^2}{15}$. That is $T_r^{F_1}(I - rf_1)x_n = \left(\frac{4x_n^1}{15}, \frac{4x_n^2}{15}\right)$. After that, we find the solution of $v_n = (v_n^1, v_n^2)$ in this inequality $0 \leq F_2(v_n, y) + \langle f_2(x_n), y - v_n\rangle + \dfrac{1}{s}\langle y - v_n, v_n - x_n\rangle$. By using the same method as $z_n = (z_n^1, z_n^2)$, we obtain

$$v_n = (v_n^1, v_n^2) = \left(\frac{5x_n^1}{49}, \frac{5x_n^2}{49}\right). \tag{4.2}$$

That is $T_s^{F_2}(I - sf_2)x_n = \left(\frac{5x_n^1}{49}, \frac{5x_n^2}{49}\right)$.

Let $x_1 = (x_1^1, x_1^2)$ and $u = (u_1, u_2) \in \mathbb{R}^2$. The sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are generated by (3.19), where $k_i = \frac{6}{7^i} + \frac{1}{N7^N}, d_1 = 1, d_2 = 1, a = 0.5, \alpha_n = \frac{1}{2n}, \beta_n = \frac{7n-4}{12n}, \gamma_n = \frac{5n-2}{12n}$ and $\lambda_n = \frac{1}{2n^2}$ for all $n \in \mathbb{N}$. Since $L = 5$, we choose $\gamma = 0.1$. From the definition of $D_1, D_2, T_i, F_1, F_2, f_1$ and f_2, we have $VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega = \{0\}$. From Theorem 3.5, we can conclude that the sequence $\{x_n\}, \{y_n\}$ and $\{u_n\}$ converge strongly to 0. We can rewrite (3.19) as follows:

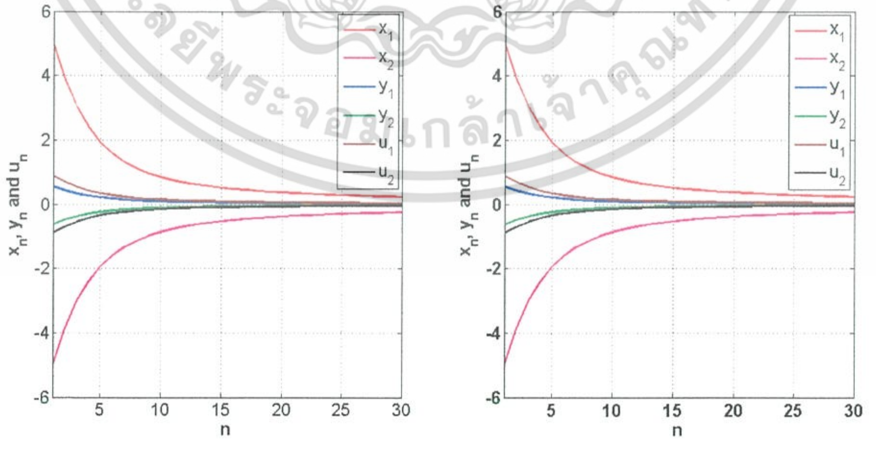
$$\begin{cases} u_n = T_1^{F_1}(I - f_1)\left(x_n + 0.1A^*\left(T_{0.5}^{F_2}(I - 0.5f_2) - I\right)Ax_n\right), \\ y_n = (I - D_1)(0.5u_n + 0.5(I - D_2)u_n), \\ x_{n+1} = \frac{1}{2n}u + \frac{7n-4}{12n}x_n + \frac{5n-2}{12n}\left(y_n - \left(\left(\frac{1}{2n^2}\right)\left(\sum_{i=1}^N\left(\frac{6}{7^i} + \frac{1}{N7^N}\right)(y_n - T_iy_n)\right)\right)\right), \end{cases}$$

for all $n \geq 1$, where $x_n = (x_n^1, x_n^2), y_n = (y_n^1, y_n^2)$ and $u_n = (u_n^1, u_n^2)$.

The following table shows the values of sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ when $u = (5, -5), x_1 = (5, -5)$ and $n = 30$.

Table 4.2: The values of $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ when $u = (5, -5), x_1 = (5, -5)$ and $n = 30$.

	$N = 1$			$N = 20$		
n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$	$u_n = (u_n^1, u_n^2)$	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$	$u_n = (u_n^1, u_n^2)$
1	(5.0000, -5.0000)	(0.4592, -0.5102)	(0.7347, -0.7347)	(5.0000, -5.0000)	(0.4592, -0.5102)	(0.7347, -0.7347)
2	(3.8504, -3.8616)	(0.3536, -0.3940)	(0.5658, -0.5674)	(3.8514, -3.8626)	(0.3537, -0.3941)	(0.5659, -0.5676)
⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	(0.5062, -0.5129)	(0.0465, -0.0523)	(0.0744, -0.0754)	(0.5062, -0.5129)	(0.0465, -0.0523)	(0.0744, -0.0754)
⋮	⋮	⋮	⋮	⋮	⋮	⋮
29	(0.2430, -0.2459)	(0.0223, -0.0251)	(0.0357, -0.0361)	(0.2430, -0.2459)	(0.0223, -0.0251)	(0.0357, -0.0361)
30	(0.2343, -0.2371)	(0.0215, -0.0242)	(0.0344, -0.0348)	(0.2343, -0.2371)	(0.0215, -0.0242)	(0.0344, -0.0348)



(a) $N = 1$ (b) $N = 20$

Figure 4.2: The convergence comparison with different values N.

Table 4.2 and Figure 4.2 in Example 4.3 show that the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$

converge to 0, where $\{0\} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^{N} F(T_i) \cap \Omega$. From Table 4.2 and Figure 4.2, so that the value of N does not affect the convergence of the iterative scheme (3.19). Therefore Theorem 3.5 guarantees the convergence of $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ in Example 4.2 and Example 4.3.

Chapter 5

Conclusions and Suggestions

In this chapter, we conclude all theorems and corollaries obtained in this thesis.

5.1 Weak convergence theorems for nonspreading mapping in Hilbert space

(1) Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunctions from $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$, let $A : C \to H$ be a α-inverse strongly monotone mapping, and let $T : C \to C$ be a nonspreading mapping with $\mathfrak{F} = F(T) \bigcap EP(F) \bigcap VI(C,\ A) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n P_C(I - \lambda A)u_n + \gamma_n T P_C(I - \lambda A)x_n, & \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [a, b] \subset (0, 1)$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$, $\{\gamma_n\} \subset [e, f] \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, and $r_n \in [g,\ h] \subset (0,\ 2\alpha)$, $\lambda \in (0,\ 2\alpha)$. Then the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $\mathfrak{F}$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a $\alpha-$inverse strongly monotone mapping, and let $T : C \to C$ be a nonspreading mapping with $\mathfrak{F} = F(T) \bigcap VI(C,\ A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + \beta_n P_C(I - \lambda A)x_n + \gamma_n T P_C(I - \lambda A)x_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset [a, b] \subset (0, 1)$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$, $\{\gamma_n\} \subset [e, f] \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, and $\lambda \in (0,\ 2\alpha)$. Then the sequence $\{x_n\}$ converges weakly to an element of $\mathfrak{F}$.

5.2 Weak convergence theorems for quasi-nonexpansive mapping in Hilbert space

(1) Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunctions from $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$, let $A : C \to H$ be a α-inverse strongly monotone mapping, and let $T : C \to C$ be a quasi-nonexpansive mapping with $\mathfrak{F} = F(T) \bigcap EP(F) \bigcap VI(C,\ A) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated

by $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n P_C(I - \lambda A)u_n + \gamma_n T P_C(I - \lambda A)x_n, \qquad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [a,b] \subset (0,1)$, $\{\beta_n\} \subset [c,d] \subset (0,1)$, $\{\gamma_n\} \subset [e,f] \subset (0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, and $r_n \in [g,\ h] \subset (0,\ 2\alpha)$, $\lambda \in (0,\ 2\alpha)$. Then the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $\mathfrak{F}$.

5.3 Strong convergence theorems for quasi-nonexpansive mappings in Hilbert space

(1) Let C and Q be nonempty closed convex subsets of a real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)-(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)\left(x_n + \gamma A^*\left(T_s^{F_2}(I - sf_2) - I\right)Ax_n\right), \\ y_n = P_C(I - d_1 D_1)\left(au_n + (1-a)P_C(I - d_2 D_2)u_n\right), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), r \in (0, 2\rho), s \in (0,1), a \in [0,1], 0 < k_i < 1$ with $\sum_{i=1}^N k_i = 1, \gamma \in (0, 1/L), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for some $c, d > 0$ for all $n \geq 1$,

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}} u$.

(2) Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$

be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)-(A4). Let T be a quasi-nonexpansive mapping of C into itself. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap F(T) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1} (I - rf_1) \left(x_n + \gamma A^* \left(T_s^{F_2} (I - sf_2) - I\right) Ax_n\right), \\ y_n = P_C (I - d_1 D_1) (au_n + (1-a) P_C (I - d_2 D_2) u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C (I - \lambda_n (I - T)) y_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), r \in (0, 2\rho), s \in (0,1), a \in [0,1], \gamma \in (0, 1/L), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for some $c, d > 0$ for all $n \geq 1$,

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}} u$.

(3) Let C be nonempty closed convex subset of a real Hilbert space H_1. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ be the bifunction satisfying (A1)-(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap EP(F_1, f_1) \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1} (I - rf_1) x_n, \\ y_n = P_C (I - d_1 D_1) (au_n + (1-a) P_C (I - d_2 D_2) u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C \left(I - \lambda_n \left(\sum_{i=1}^N k_i (I - T_i)\right)\right) y_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), r \in (0, 2\rho), a \in [0,1], 0 < k_i < 1$ with $\sum_{i=1}^N k_i = 1$. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for some $c, d > 0$ for all $n \geq 1$,

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}} u$.

(4) Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)-(A4). Let $\{T_i\}_{i=1}^{N}$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^{N} F(T_i) \cap \Gamma \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1}\left(x_n + \gamma A^*\left(T_s^{F_2} - I\right) A x_n\right), \\ y_n = P_C\left(I - d_1 D_1\right)\left(a u_n + (1-a) P_C\left(I - d_2 D_2\right) u_n\right), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C\left(I - \lambda_n\left(\sum_{i=1}^{N} k_i\left(I - T_i\right)\right)\right) y_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), a \in [0, 1], 0 < k_i < 1$ with $\sum_{i=1}^{N} k_i = 1, \gamma \in (0, 1/L), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for some $c, d > 0$ for all $n \geq 1$,

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}} u$.

5.4 Strong convergence theorems for nonspreading mappings in Hilbert space

(1) Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)-(A4). Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and let $\alpha_j =$

$(\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, ..., N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)\left(x_n + \gamma A^*\left(T_s^{F_2}(I - sf_2) - I\right)Ax_n\right), \\ y_n = P_C(I - d_1D_1)(au_n + (1-a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(I - S))y_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), r \in (0, 2\rho), s \in (0,1), a \in [0,1], \gamma \in (0, 1/L), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for some $c, d > 0$ for all $n \geq 1$,

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}}u$.

(2) Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)-(A4). Let T be a nonspreading mapping of C into C with $F(T) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathfrak{F} = VI(C, D_1) \cap VI(C, D_2) \cap F(T) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)\left(x_n + \gamma A^*\left(T_s^{F_2}(I - sf_2) - I\right)Ax_n\right), \\ y_n = P_C(I - d_1D_1)(au_n + (1-a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(I - T))y_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha), d_2 \in (0, 2\beta), r \in (0, 2\rho), s \in (0,1), a \in [0,1], \gamma \in (0, 1/L), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$.Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for some $c, d > 0$ for all $n \geq 1$,

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathfrak{F}}u$.

5.5 Suggestions

In this thesis, we obtain some results of variational inequality problem, the modified split generalized equilibrium problem and fixed point problem in a Hilbert space. For those who would like to extend these results, we suggest that proving our results in a Banach space and investigate the fixed point problem for demicontractive mappings which is more general than quasi-nonexpansive mappings.

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Appendix

Appendix A

The research papers

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ORIGINAL PAPER

A new approximation method for finding common elements of equilibrium problems, variational inequality problems and fixed point problems of nonspreading mappings

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Abstract In this paper, we introduce an iterative scheme for finding a common element of the set of solutions of the equilibrium problem, the set of solutions of the variational inequality problem and the set of fixed points of a nonspreading mapping in Hilbert spaces. Under suitable assumptions, weak convergence theorems have been proved in the framework of a Hilbert space. Our results improve and extend the corresponding results existing in the current literature. In addition, a numerical result indicate that the proposed method is quite effective.

Keywords Nonspreading mapping · Inverse strongly monotone mapping · Equilibrium problem · Variational inequality problem

Mathematics Subject Classification 47H09 · 47H10 · 90C33

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We

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denote $F(T)$ by the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. Goebel and Kirk [1] showed that $F(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory.

A mapping $T : C \to C$ is said to be *nonspreading* [2] if

$$2\|Tx - Ty\|^2 \le \|x - Ty\|^2 + \|Tx - y\|^2$$

for all $x, y \in C$. Iemoto and Takahashi [3] proved that T is nonspreading if and only if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$. A mapping $T : C \to C$ is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - z\| \le \|x - z\|$ for all $x \in C$ and $z \in F(T)$. Clearly, every nonspreading mapping with a nonempty fixed point set is quasi-nonexpansive.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* for F is to find $x \in C$ such that

$$F(x, y) \ge 0, \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Many problems in physics, optimization, and economics are seeking some elements of $EP(F)$; see [4,5]. Several iterative methods have been proposed to solve the equilibrium problem; for instance, see [4–6]. In 2005, Combettes and Hirstoaga [5] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and also proved a strong convergence theorem.

Let $A : C \to H$ be a nonlinear mapping. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle v - u, Au\rangle \ge 0, \quad \text{for all } v \in C. \tag{1.2}$$

The set of solutions of the variational inequality is denoted by $VI(C, A)$. Numerous problems in physics, optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games reduce to find element of (1.1) and (1.2).

A mapping $A : C \to H$ is called *inverse-strongly monotone* [7], if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \ge \alpha\|Ax - Ay\|^2, \quad \text{for all } x, y \in C.$$

The problem for finding a common fixed point of a family of nonexpansive mappings has been studied by many authors. The well known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings; see [8,9].

The problem for finding a common element of $EP(F)$ and the set of all common fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and importance. Many iterative methods are purposed for finding a common element of the solutions of the equilibrium problem and fixed point problem of nonexpansive mappings; see [10–12].

In 2007, Takahashi and Takahashi [12] introduced a general iterative method for finding a common element of $EP(F)$ and $F(S)$. They defined $\{x_n\}$ in the following way:

$$\begin{cases} x_1 \in C, \\ F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \quad \forall\, y \in C, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)Sz_n, \forall\, n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where S is a nonexpansive mapping of C into itself, $\{\beta_n\} \subset [0, 1]$, and proved strong convergence of the method (1.3) to $z = P_{F(S)\bigcap EP(F)} f(z)$ in the framework of a Hilbert space, under some suitable conditions on $\{\beta_n\}$, $\{\lambda_n\}$ and bifunction F.

Motivated by the above result and related literature, we propose an iterative scheme for finding a common element of the set of solutions of the equilibrium problem, the set of solutions of the variational inequality problem and the set of fixed points of a nonspreading mapping in a real Hilbert space. Then we show that the sequence converges weakly to a common element of three sets.

2 Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Let C be a nonempty closed convex subset of a real Hilbert space H. It is also known that a Hilbert space H satisfies *Opial's property* [13], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Let P_C be the metric projection of H onto C, *i.e.*, for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

The following characterizes the projection P_C.

Lemma 2.1 ([14]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $P_C : H \to C$ be the metric projection. Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if the following holds:*

$$\langle x - y, y - z\rangle \geq 0, \quad \textit{for all } z \in C.$$

Lemma 2.2 ([15]) *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E, and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

$(A1)$ $F(x, x) = 0, \quad \forall\, x \in C$;
$(A2)$ F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0, \quad \forall\, x, y \in C$;
$(A3)$ $\forall\, x, y, z \in C$,

$$\lim_{t\to 0^+} F(tz + (1-t)x, y) \leq F(x, y);$$

$(A4)$ $\forall\, x \in C, \; y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemmas appear implicitly in [4].

Lemma 2.3 ([4]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \tag{2.1}$$

for all $x \in C$.

Lemma 2.4 ([5]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall\, y \in C \right\}. \tag{2.2}$$

for all $z \in H$. Then the following hold:

(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive,* i.e., $\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle, \quad \forall\, x, y \in H$;
(3) *$F(T_r) = EP(F)$;*
(4) *$EP(F)$ is closed and convex.*

Lemma 2.5 ([16]) *Let H be a Hibert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u \in P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.6 ([17]) *Let H be a real Hibert space. If $\{x_n\}$ is a sequence in H weakly convergent to z, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2 \tag{2.3}$$

for all $y \in H$.

3 Main result

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunctions from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4), let $A : C \to H$ be a α-inverse strongly monotone mapping, and let $T : C \to C$ be a nonspreading mapping with $\mathfrak{F} = F(T) \bigcap EP(F) \bigcap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n P_C(I - \lambda A)u_n + \gamma_n T P_C(I - \lambda A)x_n, \quad \forall\, n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \subset [a, b] \subset (0, 1)$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$, $\{\gamma_n\} \subset [e, f] \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, and $r_n \in [g, h] \subset (0, 2\alpha)$, $\lambda \in (0, 2\alpha)$. Then the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $\mathfrak{F}$.

Proof First, we show that $P_C(I - \lambda A)$ is nonexpansive. Let $x, y \in C$. Since A is α-inverse strongly monotone and $\lambda < 2\alpha$, we have

$$\begin{aligned} \|P_C(I - \lambda A)x - P_C(I - \lambda A)y\|^2 &\leq \|(I - \lambda A)x - (I - \lambda A)y\|^2 \\ &= \|x - y - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\alpha\lambda\|Ax - Ay\|^2 + \lambda^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Thus $P_C(I-\lambda A)$ is nonexpansive. We shall show that the sequence $\{x_n\}$ is bounded.
Since

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C.$$

By Lemma 2.4, we have $u_n = T_{r_n}x_n$ and $EP(F) = F(T_{r_n})$. Let $z \in \mathfrak{F}$. By nonexpansiveness of $P_C(I-\lambda A)$ and T_{r_n}, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n(x_n - z) + \beta_n(P_C(I-\lambda A)u_n - z) \\
&\quad + \gamma_n(TP_C(I-\lambda A)x_n - z)\|^2 \\
&= \alpha_n\|x_n - z\|^2 + \beta_n\|P_C(I-\lambda A)u_n - z\|^2 \\
&\quad + \gamma_n\|TP_C(I-\lambda A)x_n - z\|^2 - \alpha_n\beta_n\|x_n - P_C(I-\lambda A)u_n\|^2 \\
&\quad - \alpha_n\gamma_n\|x_n - TP_C(I-\lambda A)x_n\|^2 \\
&\quad - \beta_n\gamma_n\|P_C(I-\lambda A)u_n - TP_C(I-\lambda A)x_n\|^2 \\
&\leq (1-\beta_n)\|x_n - z\|^2 + \beta_n\|T_{r_n}x_n - z\|^2 \\
&\quad - \alpha_n\beta_n\|x_n - P_C(I-\lambda A)u_n\|^2 - \alpha_n\gamma_n\|x_n - TP_C(I-\lambda A)x_n\|^2 \\
&\quad - \beta_n\gamma_n\|P_C(I-\lambda A)u_n - TP_C(I-\lambda A)x_n\|^2 \qquad (3.2) \\
&= \|x_n - z\|^2 - \alpha_n\beta_n\|x_n - P_C(I-\lambda A)u_n\|^2 \\
&\quad - \alpha_n\gamma_n\|x_n - TP_C(I-\lambda A)x_n\|^2 \\
&\quad - \beta_n\gamma_n\|P_C(I-\lambda A)u_n - TP_C(I-\lambda A)x_n\|^2 \\
&\leq \|x_n - z\|^2. \qquad (3.3)
\end{aligned}$$

This implies that $\lim_{n\to\infty}\|x_n - z\|$ exists for every $z \in \mathfrak{F}$. Then we have $\{x_n\}$ is bounded, so is $\{u_n\}$. By (3.3), we have

$$\begin{aligned}
\alpha_n\beta_n\|x_n - P_C(I-\lambda A)u_n\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\quad - \alpha_n\gamma_n\|x_n - TP_C(I-\lambda A)x_n\|^2 \\
&\quad - \beta_n\gamma_n\|P_C(I-\lambda A)u_n - TP_C(I-\lambda A)x_n\| \\
&\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - z\|$ exists for every $z \in \mathfrak{F}$, we have

$$\lim_{n\to\infty}\|x_n - P_C(I-\lambda A)u_n\| = 0. \qquad (3.4)$$

By using the same method as (3.4), we have

$$\lim_{n\to\infty}\|x_n - TP_C(I-\lambda A)x_n\| = 0. \qquad (3.5)$$

Next, we will show that $\lim_{n\to\infty}\|x_n - u_n\| = 0$.
Let $z \in \mathfrak{F}$. Since $u_n = T_{r_n}x_n$ and T_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}
\|z - T_{r_n}x_n\|^2 &= \|T_{r_n}z - T_{r_n}x_n\|^2 \\
&\leq \langle T_{r_n}z - T_{r_n}x_n, z - x_n\rangle \\
&= \frac{1}{2}\left(\|T_{r_n}x_n - z\|^2 + \|x_n - z\|^2 - \|T_{r_n}x_n - x_n\|^2\right).
\end{aligned}$$

Hence

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \qquad (3.6)$$

By (3.2) and (3.6), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-\beta_n)\|x_n-z\|^2+\beta_n\|u_n-z\|^2\\ &\quad -\alpha_n\beta_n\|x_n-P_C(I-\lambda A)u_n\|^2-\alpha_n\gamma_n\|x_n-TP_C(I-\lambda A)x_n\|^2\\ &\quad -\beta_n\gamma_n\|P_C(I-\lambda A)u_n-TP_C(I-\lambda A)x_n\|\\ &\le (1-\beta_n)\|x_n-z\|^2+\beta_n\left(\|x_n-z\|^2-\|u_n-x_n\|^2\right)\\ &= \|x_n-z\|^2-\beta_n\|u_n-x_n\|^2.\end{aligned}$$

It implies that

$$\beta_n\|u_n-x_n\|^2 \le \|x_n-z\|^2-\|x_{n+1}-z\|^2.$$

Since $\lim_{n\to\infty}\|x_n-z\|$ exists for every $z\in\mathfrak{F}$, we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.7}$$

Since

$$\begin{aligned}\|x_n-P_C(I-\lambda A)x_n\| &\le \|x_n-P_C(I-\lambda A)u_n\|+\|P_C(I-\lambda A)u_n-P_C(I-\lambda A)x_n\|\\ &\le \|x_n-P_C(I-\lambda A)u_n\|+\|u_n-x_n\|,\end{aligned}$$

from (3.4) and (3.7), we have

$$\lim_{n\to\infty}\|x_n-P_C(I-\lambda A)x_n\|=0. \tag{3.8}$$

By (3.5) and (3.8), we have

$$\lim_{n\to\infty}\|P_C(I-\lambda A)x_n-TP_C(I-\lambda A)x_n\|=0. \tag{3.9}$$

Next, we will show that $\omega(x_n)\subset\mathfrak{F}$ where $\omega(x_n)=\{x : x_{n_m}\rightharpoonup x$ for some subsequence $\{n_m\}$ of $\{n\}\}$. Since $\{x_n\}$ is bounded in H, we have $\omega(x_n)\neq\emptyset$. Let $\omega\in\omega(x_n)$. Thus, there is a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ which converges weakly to ω. By (3.8), nonexpansiveness of $P_C(I-\lambda A)$ and Lemma 2.2, we obtain that $\omega\in F(P_C(I-\lambda A))$. By Lemma 2.5, we have $\omega\in VI(C,A)$. Since $\|u_{n_m}-x_{n_m}\|\to 0$ as $m\to\infty$, we have $u_{n_m}\rightharpoonup\omega$ as $m\to\infty$. Since

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad \forall\, y\in C,$$

it follows by $(A2)$ that

$$\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge F(y,u_n),\quad \forall\, y\in C.$$

In particular,

$$\left\langle y-u_{n_m},\frac{1}{r_{n_m}}(u_{n_m}-x_{n_m})\right\rangle\ge F(y,u_{n_m}). \tag{3.10}$$

By condition $(A4)$, $F(y,\cdot)$ is convex and lower semicontinuous, and thus weakly semicontinuous. By (3.7) imply that $\frac{1}{r_{n_m}}(u_{n_m}-x_{n_m})\to 0$ in norm. Therefore, letting $m\to\infty$ in (3.10), we have

$$F(y,\omega)\le\lim_{m\to\infty}F(y,u_{n_m})\le 0,\quad \forall\, y\in C. \tag{3.11}$$

Replacing y with $y_t := ty+(1-t)\omega$, $t\in(0,1]$, we have $y_t\in C$ and using $(A1)$, $(A4)$, and (3.11), we obtain

$$0=F(y_t,y_t)\le tF(y_t,y)+(1-t)F(y_t,\omega)\le tF(y_t,y).$$

Hence $F(ty + (1-t)\omega, y) \geq 0$, $\forall t \in (0,1]$ and $\forall y \in C$. Letting $t \to 0^+$ and using assumption $(A3)$, we can conclude that

$$F(\omega, y) \geq 0, \quad y \in C.$$

Therefore, $\omega \in EP(F)$.

Next, we will show that $\omega \in F(T)$. To show this, we suppose that $\omega \neq T\omega$. Using Opial's property and (3.5), we have

$$\begin{aligned}\liminf_{m\to\infty} \|x_{n_m} - \omega\| &< \liminf_{m\to\infty} \|x_{n_m} - T\omega\| \\ &\leq \liminf_{m\to\infty} (\|x_{n_m} - TP_C(I-\lambda A)x_{n_m}\| \\ &\quad + \|TP_C(I-\lambda A)x_{n_m} - T\omega\|) \\ &= \liminf_{m\to\infty} \|TP_C(I-\lambda A)x_{n_m} - T\omega\|. \end{aligned} \tag{3.12}$$

By nonexpansiveness of $P_C(I-\lambda A)$ and $\omega \in F(P_C(I-\lambda A))$, we have

$$\begin{aligned}\|TP_C(I-\lambda A)x_{n_m} - T\omega\|^2 &= \|TP_C(I-\lambda A)x_{n_m} - TP_C(I-\lambda A)\omega\|^2 \\ &\leq \|P_C(I-\lambda A)x_{n_m} - P_C(I-\lambda A)\omega\|^2 \\ &\quad + 2\langle P_C(I-\lambda A)x_{n_m} - TP_C(I-\lambda A)x_{n_m}, \omega - T\omega\rangle \\ &\leq \|x_{n_m} - \omega\|^2 \\ &\quad + 2\|P_C(I-\lambda A)x_{n_m} - TP_C(I-\lambda A)x_{n_m}\|\|\omega - T\omega\|.\end{aligned}$$

It implies from (3.9) that

$$\liminf_{m\to\infty} \|TP_C(I-\lambda A)x_{n_m} - T\omega\| \leq \liminf_{m\to\infty} \|x_{n_m} - \omega\|. \tag{3.13}$$

By (3.12) and (3.13), we have

$$\liminf_{m\to\infty} \|x_{n_m} - \omega\| < \liminf_{m\to\infty} \|x_{n_m} - \omega\|.$$

This is a contradiction. Then we have $\omega \in F(T)$. Hence $\omega(x_n) \subset \mathfrak{F}$.

Finally, we show that $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $\mathfrak{F}$. Claim that $\omega(x_n)$ is a singleton set. Let $\omega_1, \omega_2 \in \omega(x_n)$ and let $\{x_{k_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{k_i} \rightharpoonup \omega_1$ as $i \to \infty$ and $x_{m_j} \rightharpoonup \omega_2$ as $j \to \infty$. Since $\lim_{n\to\infty} \|x_n - q\|$ exists for each $q \in \mathfrak{F}$ and $\omega_1, \omega_2 \in \mathfrak{F}$, it follows by Lemma 2.6 that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \omega_1\|^2 &= \lim_{j\to\infty} \|x_{m_j} - \omega_1\|^2 \\ &= \lim_{j\to\infty} \|x_{m_j} - \omega_2\|^2 + \|\omega_2 - \omega_1\|^2 \\ &= \lim_{i\to\infty} \|x_{k_i} - \omega_2\|^2 + \|\omega_2 - \omega_1\|^2 \\ &= \lim_{i\to\infty} \|x_{k_i} - \omega_1\|^2 + \|\omega_1 - \omega_2\|^2 + \|\omega_2 - \omega_1\|^2 \\ &= \lim_{n\to\infty} \|x_n - \omega_1\|^2 + 2\|\omega_1 - \omega_2\|^2.\end{aligned}$$

Hence, $\omega_1 = \omega_2$. This shows that $\omega(x_n)$ is a singleton set. Therefore, we can conclude that $x_n \rightharpoonup \omega \in \mathfrak{F}$ as $n \to \infty$. It follows from (3.7) that $u_n \rightharpoonup \omega \in \mathfrak{F}$ as $n \to \infty$. This completes the proof. □

Using Theorem 3.1, we obtain the following weakly convergence theorem for finding a common element of the set of solutions of the variational inequality problem and the set of fixed points of a nonspreading mapping in a real Hilbert space.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a α−inverse strongly monotone mapping, and let $T : C \to C$ be a nonspreading mapping with $\mathfrak{F} = F(T) \bigcap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n x_n + \beta_n P_C(I - \lambda A)x_n + \gamma_n T P_C(I - \lambda A)x_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset [a, b] \subset (0, 1)$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$, $\{\gamma_n\} \subset [e, f] \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, and $\lambda \in (0, 2\alpha)$. Then the sequence $\{x_n\}$ converges weakly to an element of $\mathfrak{F}$.

Proof From Theorem 3.1, putting $F \equiv 0$, we have $u_n = P_C x_n$. Since $x_n \in C$ for all $n \geq 1$, we have $x_n = P_C x_n$. Then $u_n = x_n$ and the desired result is directly obtained by Theorem 3.1. □

We know that every nonspreading mapping with a nonempty fixed point set is quasi-nonexpansive. The following result is using the same method as Theorem 3.1, we obtain weakly convergence theorem of quasi-nonexpansive mapping without demiclosed condition of T with different from the results of Kim [18]. Therefore, we omit the proof.

Theorem 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunctions from $C \times C$ into $\mathbb{R}$ satisfying (A1) − (A4), let $A : C \to H$ be a α-inverse strongly monotone mapping, and let $T : C \to C$ be a quasi-nonexpansive mapping with $\mathfrak{F} = F(T) \bigcap EP(F) \bigcap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n P_C(I - \lambda A)u_n + \gamma_n T P_C(I - \lambda A)x_n, & \forall n \geq 1, \end{cases} \tag{3.14}$$

where $\{\alpha_n\} \subset [a, b] \subset (0, 1)$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$, $\{\gamma_n\} \subset [e, f] \subset (0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, and $r_n \in [g, h] \subset (0, 2\alpha)$, $\lambda \in (0, 2\alpha)$. Then the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $\mathfrak{F}$.

4 Example and numerical results

In this section, we give an example for supporting Theorem 3.1.

Example 4.1 Let $\mathbb{R}$ be the set of real numbers and let A be a mapping from [0, 100] to $\mathbb{R}$ defined by $Ax = \frac{x-1}{5}$, $\forall x \in [0, 100]$ and T be a mapping from [0, 100] into itself defined by $Tx = \frac{x+1}{2}$, $\forall x \in [0, 100]$. Let $F : [0, 100] \times [0, 100] \to \mathbb{R}$ defined by

$$F(x, y) = -(x - 1)^2 + (y - 1)^2, \forall x, y \in \mathbb{R}.$$

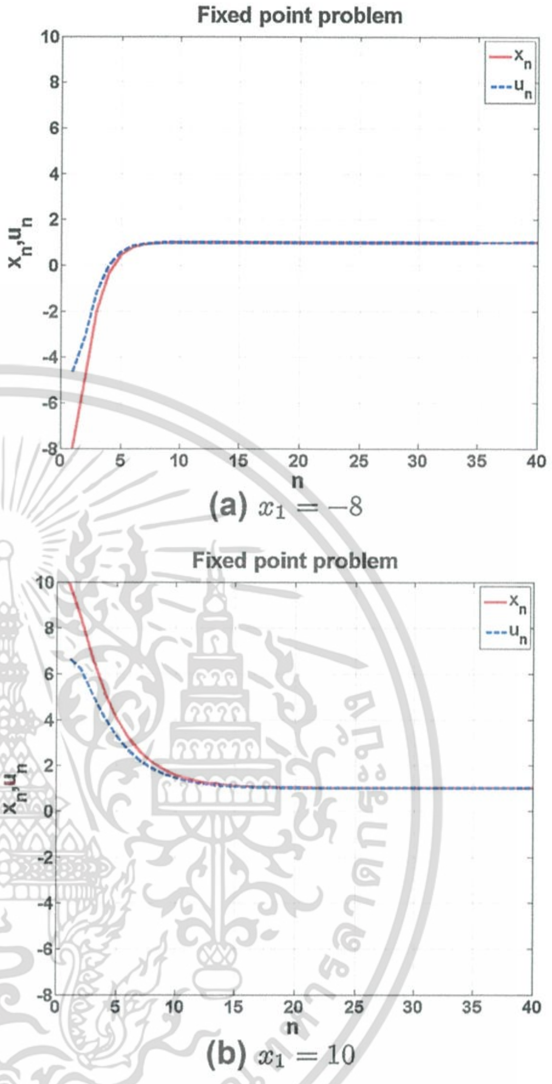

Fig. 1 The convergence comparison of the sequences $\{x_n\}$ and $\{u_n\}$ with different initial values x_1

By the definition of F, we have

$$\begin{aligned}
0 &\leq F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \\
&= -(u_n - 1)^2 + (y-1)^2 + \frac{1}{r_n}(y - u_n)(u_n - x_n) \\
&= -(u_n - 1)^2 + (y-1)^2 + \frac{1}{r_n}\left(yu_n - yx_n - u_n^2 + u_nx_n\right) \\
&\Leftrightarrow \\
0 &\leq r_n\big(-(u_n-1)^2 + (y-1)^2\big) + \left(yu_n - yx_n - u_n^2 + u_nx_n\right) \\
&= 2r_nu_n - u_n^2 - r_nu_n^2 + u_nx_n + (-2r_n + u_n - x_n)\,y + r_ny^2.
\end{aligned}$$

Table 1 The values of $\{x_n\}$ and $\{u_n\}$ where $n = 40$

n	$x_1 = -8$		$x_1 = 10$	
	x_n	u_n	x_n	u_n
1	−8.0000	−4.7273	10.0000	6.7273
2	−5.0000	−3.2000	8.4618	6.2233
3	−2.0000	−1.1724	6.6610	5.0994
4	−0.3333	0.0175	5.1801	4.0800
⋮	⋮	⋮	⋮	⋮
20	1.0000	1.0000	1.0191	1.0147
⋮	⋮	⋮	⋮	⋮
37	1.0000	1.0000	1.0001	1.0000
38	1.0000	1.0000	1.0000	1.0000
39	1.0000	1.0000	1.0000	1.0000
40	1.0000	1.0000	1.0000	1.0000

Let $G(y) = r_n y^2 + (-2r_n + u_n - x_n)y + 2r_n u_n - u_n^2 - r_n u_n^2 + u_n x_n$ which is a quadratic function of y with coefficient $a = r_n$, $b = -2r_n + u_n - x_n$, and $c = 2r_n u_n - u_n^2 - r_n u_n^2 + u_n x_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (-2r_n + u_n - x_n)^2 - 4r_n\left(2r_n u_n - u_n^2 - r_n u_n^2 + u_n x_n\right) \\
&= 4r_n^2 - 4r_n u_n - 8r_n^2 u_n + u_n^2 + 4r_n u_n^2 + 4r_n^2 u_n^2 + 4r_n x_n - 2u_n x_n - 4r_n u_n x_n + x_n^2 \\
&= (-2r_n + u_n + 2r_n u_n - x_n)^2 .
\end{aligned}$$

We know that $G(y) \geq 0$, $\forall y \in [0, 100]$. If it has most one solution in $\mathbb{R}$, then $\Delta \leq 0$. So we obtain

$$u_n = \frac{2r_n + x_n}{1 + 2r_n}. \tag{4.1}$$

Let $x_1 \in \mathbb{R}$ and $\{x_n\}$ generated by (3.14), where $\lambda = 1$, $r_n = \frac{n+1}{7n}$, $\alpha_n = \frac{n+1}{3n}$, $\beta_n = \frac{7n-3}{15n}$ and $\gamma_n = \frac{3n-2}{15n}$ for all $n \in \mathbb{N}$. By the definition of F, A and T, we have $1 \in F(T) \cap EP(F) \cap VI(C, A)$. Therefore the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to 1. They are rewritten as follows:

$$\begin{cases} u_n = \frac{2r_n + x_n}{1 + 2r_n}, \\ x_{n+1} = \left(\frac{n+1}{3n}\right) x_n + \left(\frac{7n-3}{15n}\right) P_{[0,100]}(I - A)u_n + \left(\frac{3n-2}{15n}\right) T P_{[0,100]}(I - A)x_n, \ \forall\, n \geq 1. \end{cases}$$

The following table shows the values of sequences $\{x_n\}$ and $\{u_n\}$ where $x_1 = -8$ and $x_1 = 10$ and $n = 40$.

The Fig. 1 is the values of sequences $\{x_n\}$ and $\{u_n\}$ that correspond to the Table 1.

5 Conclusion

1. The sequences $\{x_n\}$ and $\{u_n\}$ in Table 1 and Fig. 1 converge to 1, where $1 \in F(T) \cap EP(F) \cap VI(C, A)$.
2. Theorem 3.1 assure the convergence of $\{x_n\}$ and $\{u_n\}$ in the Example 4.1.

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The modified split generalized equilibrium problem for quasi-nonexpansive mappings and applications

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Abstract

In this paper, we introduce a new problem, the modified split generalized equilibrium problem, which extends the generalized equilibrium problem, the split equilibrium problem and the split variational inequality problem. We introduce a new method of an iterative scheme $\{x_n\}$ for finding a common element of the set of solutions of variational inequality problems and the set of common fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of the modified split generalized equilibrium problem without assuming a demicloseness condition and $T_\omega := (1-\omega)I + \omega T$, where T is a quasi-nonexpansive mapping and $\omega \in (0, \frac{1}{2})$; a difficult proof in the framework of Hilbert space. In addition, we give a numerical example to support our main result.

MSC: 47B40; 47H10; 47J20

Keywords: The modified split generalized equilibrium problem; Quasi-nonexpansive mapping; Variational Inequality problem; Fixed Point problem

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. The set of fixed points of T is denoted by $F(T)$. The mapping $T : C \to C$ is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \le \|x - p\|,$$

for all $x \in C$ and $p \in F(T)$.

Definition 1.1 ([1]) Let $T : H \to H$. Then the following are equivalent:

1. T is firmly nonexpansive,
2. $\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle$, $\forall x, y \in H$,
3. $\langle Tx - Ty, (I - T)x - (I - T)y\rangle \ge 0$, $\forall x, y \in H$.

Let $A : C \to H$ be a mapping. The *variational inequality* is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \ge 0, \tag{1.1}$$

for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C,A)$. A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$. They have been investigated in the literature; see, for example, [2, 3]. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(F)$. Equilibrium problems were introduced by [4] in 1994 and included many well-known problems such as variational inequality, optimization problem, nonexpansive mapping and fixed point problem; see, for example, [5–8].

Let F be a function of $C \times C$ into $\mathbb{R}$ and let $f : H \to H$ be a mapping. The *generalized equilibrium problem* is to find $x \in C$ such that

$$F(x,y) + \langle f(x), y - x\rangle \geq 0, \tag{1.3}$$

for all $y \in C$. The set of solutions of (1.3) is denoted by $EP(F,f)$. When $f \equiv 0$, $EP(F,f)$ is denoted by $EP(F)$ and $F \equiv 0$, $EP(F,f)$ is denoted by $VI(C,f)$.

Throughout this section, let H_1, H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator.

In 1994, Censor and Elfving [9] introduced the *split feasibility problem* (in short, SFP) which is to find a point $x \in C$ such that $Ax \in Q$. The set of all solutions of split feasibility problem is denoted by $\varphi = \{x \in C : Ax \in Q\}$.

To solve the SFP, Byrne [10] introduced *CQ* algorithm whose sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_{C_1}\big(x_n - \gamma A^*(I - P_{C_2})Ax_n\big),$$

where the initial $x_0 \in H_1$ and $\gamma \in (0, 2/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Then the *CQ* algorithm converges to a solution of the SFP, whenever solutions exist. If there are no solutions of the SFP, the *CQ* algorithm converges to a minimizer of the function

$$\frac{1}{2}\big\|(I - P_{C_2})Ax\big\|^2,$$

whenever such minimizers exist.

Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two nonlinear operators. The *split common fixed points problem* (SCFPP) [11, 12] is to find a point x^* such that

$$x^* \in F(U) \quad \text{and} \quad Ax^* \in F(T).$$

The solution set of SCFPP is denoted by $\Phi = \{p^* \in F(U) : Ap^* \in F(T)\}$. The split common fixed point problem is a generalization of the split feasibility problem.

In 2017, Wang [13] introduced a new method for solving SCFPP as follows:

$$x_{n+1} = x_n - \rho_n\big((I-U)x_n + A^*(I-T)Ax_n\big),$$

where $\rho_n \subset (0,\infty)$ is chosen such that

$$\rho_n = \frac{\|(I-U)x_n\|^2 + \|(I-T)Ax_n\|^2}{\|(I-U)x_n + A^*(I-T)Ax_n\|^2} \tag{1.4}$$

and U and T are firmly quasi-nonexpansive mappings. Then the sequence $\{x_n\}$ converges weakly to z, where $z = \lim_{n\to\infty} P_\Phi x_n$.

Censor et al. [11, 14] introduced the prototypical *split inverse problem* (SIP) which is a generalization of the split common fixed points problem. In this, there are given two vector spaces X and Y and a linear operator $A : X \to Y$. In addition, two inverse problems are involved. The first one, denoted IP_1, is formulated in the space X and the second one, denoted IP_2, is formulated in the space Y. Given these data, the split inverse problem is formulated as follows:

$$\text{find a point } x^* \in X \text{ that solves } \mathrm{IP}_1, \tag{1.5}$$

and such that

$$\text{find a point } y^* \in Y \text{ that solves } \mathrm{IP}_2. \tag{1.6}$$

This problem is used in many modeling arising in sensor networks, radiation therapy treatment planning, color imaging, etc.

The *split equilibrium problem* (SEP) [12] is to find $\widehat{x} \in C$ such that

$$F_1(\widehat{x}, x) \ge 0, \quad \forall x \in C, \tag{1.7}$$

and such that

$$\widehat{y} = A\widehat{x} \in Q \quad \text{solves } F_2(\widehat{y}, y) \ge 0, \quad \forall y \in Q, \tag{1.8}$$

where $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions. If we consider only problem (1.7), it is the equilibrium problem and we denoted its solution set by $EP(F_1)$. The solution set of SEP is denoted by $\Gamma = \{\widehat{p} \in EP(F_1) : A\widehat{p} \in EP(F_2)\}$. SEP is reduced to $EP(F)$, where $H_1 \equiv H_2$, $F_1 \equiv F_2$ and $A \equiv I$. $EP(F)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc.

The *split variational inequality problems* (in short, SVIP) were introduced and studied by Cencor et al. [11]: find $\overline{x} \in C$ such that

$$\langle f_1(\overline{x}), x - \overline{x}\rangle \ge 0, \quad \forall x \in C, \tag{1.9}$$

and such that

$$\bar{y} = A\bar{x} \in Q \quad \text{solves } \langle f_2(\bar{y}), y - \bar{y}\rangle \geq 0, \quad \forall y \in Q, \tag{1.10}$$

where $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings. The solution set of SVIP is denoted by $\Psi = \{\overline{p} \in VI(C,f_1) : A\overline{p} \in VI(Q,f_2)\}$. The split variational inequality problems have already been studied and used in practice as a model in intensity-modulated radiation therapy (IMRT) treatment planning; see, for example, [15] and the modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see, for example, [16, 17].

By investigating SEP and SVIP, we introduce the *modified split generalized equilibrium problem* (MSGEP) which is to find $x^* \in C$ such that

$$F_1(x^*, x) + \langle f_1(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C, \tag{1.11}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves } F_2(y^*, y) + \langle f_2(y^*), y - y^*\rangle \geq 0, \quad \forall y \in Q, \tag{1.12}$$

where $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifunctions and $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings. The solution set of MSGEP is denoted by $\Omega = \{p^* \in EP(F_1,f_1) : Ap^* \in EP(F_2,f_2)\}$.

Remark 1.1

1. If we put $f_1 \equiv f_2 \equiv 0$ in MSGEP then the MSGEP is reduced to SEP.
2. If we put $F_1 \equiv F_2 \equiv 0$ in MSGEP then the MSGEP is reduced to SVIP.
3. In the case of bifunctions F_1 and F_2 are according to (A1)–(A4). From (1.11), (1.12) and Lemma 2.2, we have $x^* \in F(T_r^{F_1}(I - rf_1))$ and $Ax^* \in F(T_s^{F_2}(I - sf_2))$, for all $r, s > 0$. So, MSGEP can be viewed as SCFPP.

MSGEP is a generalization of the generalized equilibrium problem, the split equilibrium problem and the split variational inequality problem. So, this problem can be used in sensor networks, data compression, practice as a model in intensity-modulated radiation therapy (IMRT) treatment planning, robustness to marginal changes and equilibrium stability etc.

Example 1.2 Let $H_1 = [0,6]$, $H_2 = [0,18]$, $C = [2,5]$ and $Q = [6,10]$. Let $A : H_1 \to H_2$ be defined by $Ax = 3x$ for all $x \in H_1$. Let the mapping $F_1 : C \times C \to \mathbb{R}$ be defined by

$$F_1(x^*, x) = -(x^* - 2)^2 + (x - 2)^2, \quad \forall x, y \in C,$$

and $F_2 : Q \times Q \to \mathbb{R}$ be defined by

$$F_2(y^*, y) = -(y^* - 6)^2 + (y - 6)^2, \quad \forall x, y \in Q.$$

Let the mapping $f_1 : C \to H_1$ be defined by $f_1 x = \frac{x-2}{9}$, $\forall x \in C$ and the mapping $f_2 : Q \to H_2$ be defined by $f_2 x = \frac{x-6}{7}$, $\forall x \in Q$.

Then $2 \in \Omega$. Therefore 2 is a solution of MSGEP.

In 2012, Tain and Jin [18] introduced iterative algorithms involving a quasi-nonexpansive mapping. They generated the iterative as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T_\omega x_n,$$

where A is a bounded linear operator on H, T is a quasi-nonexpansive mapping on H, f is a contraction with coefficient a under suitable conditions of the parameters α_n, γ and ω. By assuming $\omega \in (0, \frac{1}{2})$, $T_\omega := (1-\omega)I + \omega T$ and T is demiclosed on H.

Motivated by SFP and SVIP, we introduced a new problem, the modified split generalized equilibrium problem, which extends the generalized equilibrium problem, the split equilibrium problem and the split variational inequality problem. Many authors proved strong convergence theorem involving a quasi-nonexpansive mapping T by assuming $T_\omega := (1-\omega)I + \omega T$ and T is demiclosed on H; a difficult proof. Motivated by [19], we introduced Remark 2.5 and [11, 12] and [18], we introduce a new method of iterative scheme $\{x_n\}$ for finding a common element of the set of solutions of variational inequality problems and the set of common fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of the modified split generalized equilibrium problem without the condition above in the framework of a Hilbert space.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Throughout this paper, we use the notations of weak and strong convergence by "$\rightharpoonup$" and "$\to$", respectively. Recall that H satisfies *Opial's condition* [20], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|$, holds for every $y \in H$ with $y \neq x$.

For solving the equilibrium problem, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfy the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$,

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$,

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$,

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.1 ([4]) *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4). *Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.2 ([21]) *Assume that $F : C \times C \to \mathbb{R}$ satisfies* (A1)–(A4). *For $r > 0$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(1) T_r *is single-valued,*
(2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \le \langle T_r(x) - T_r(y), x - y\rangle,$$

(3) $F(T_r) = EP(F)$,
(4) $EP(F)$ *is closed and convex.*

Lemma 2.3 ([22]) *Let* H *be a real Hilbert space, let* C *be a nonempty closed convex subset of* H *and let* A *be a mapping of* C *into* H. *Let* $u \in C$. *Then, for* $\lambda > 0$,

$$u = P_C(I - \lambda A)u \quad \Leftrightarrow \quad u \in VI(C, A),$$

where P_C *is the metric projection of* H *onto* C.

Lemma 2.4 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{T_i\}_{i=1}^N$ *be a finite family of quasi-nonexpansive mappings of* C *into* H *with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and let* $0 < a_i < 1$ *with* $\sum_{i=1}^N a_i = 1$. *Then*

$$\bigcap_{i=1}^N F(T_i) = VI\left(C, \sum_{i=1}^N a_i(I - T_i)\right).$$

Proof In this lemma, we show that $\bigcap_{i=1}^N F(T_i) = \bigcap_{i=1}^N VI(C, I - T_i)$ and $\bigcap_{i=1}^N VI(C, I - T_i) = VI(C, \sum_{i=1}^N a_i(I - T_i))$. Lastly, we have

$$\bigcap_{i=1}^N F(T_i) = VI\left(C, \sum_{i=1}^N a_i(I - T_i)\right).$$

To start with, it is easy to see that $\bigcap_{i=1}^N F(T_i) \subseteq \bigcap_{i=1}^N VI(C, I - T_i)$. Next, we show that $\bigcap_{i=1}^N VI(C, I - T_i) \subseteq \bigcap_{i=1}^N F(T_i)$. Let $u \in \bigcap_{i=1}^N VI(C, I - T_i)$ and $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. So, we get $u \in VI(C, I - T_i)$, $\forall i = 1, 2, \ldots, N$. We may write

$$\langle u - v, (I - T_i)u\rangle \le 0, \quad \forall v \in C. \tag{2.1}$$

There exists $v^* \in C$ such that $v^* = T_i v^*$, $\forall i = 1, 2, \ldots, N$. Since T_i is a quasi-nonexpansive mapping, $\forall i = 1, 2, \ldots, N$, it follows that

$$\begin{aligned} \|T_i u - v^*\|^2 &= \|(u - v^*) - (I - T_i)u\|^2 \\ &= \|u - v^*\|^2 - 2\langle u - v^*, (I - T_i)u\rangle + \|(I - T_i)u\|^2 \\ &\le \|u - v^*\|^2. \end{aligned} \tag{2.2}$$

By using (2.1) and (2.2), we conclude that

$$\|(I - T_i)u\|^2 \le 2\langle u - v^*, (I - T_i)u\rangle \le 0.$$

It implies that $u \in \bigcap_{i=1}^N F(T_i)$. Therefore $\bigcap_{i=1}^N VI(C, I-T_i) \subseteq \bigcap_{i=1}^N F(T_i)$. Hence

$$\bigcap_{i=1}^N F(T_i) = \bigcap_{i=1}^N VI(C, I-T_i).$$

After that, we show $\bigcap_{i=1}^N VI(C, I-T_i) = VI(C, \sum_{i=1}^N a_i(I-T_i))$ where $0 < a_i < 1$ and $\sum_{i=1}^N a_i = 1$. Observe that

$$\begin{aligned}
& u \in \bigcap_{i=1}^N VI(C, I-T_i) \\
\Leftrightarrow\quad & u \in VI(C, I-T_i), \quad \forall i = 1,2,\ldots,N \\
\Leftrightarrow\quad & \langle (I-T_i)u, v-u \rangle \ge 0, \quad \forall v \in C \text{ and } \forall i = 1,2,\ldots,N \\
\Leftrightarrow\quad & \sum_{i=1}^N a_i \langle (I-T_i)u, v-u \rangle \ge 0, \quad \forall v \in C \\
\Leftrightarrow\quad & \left\langle \sum_{i=1}^N a_i(I-T_i)u, v-u \right\rangle \ge 0, \quad \forall v \in C \\
\Leftrightarrow\quad & u \in VI\left(C, \sum_{i=1}^N a_i(I-T_i)\right).
\end{aligned}$$

Therefore $\bigcap_{i=1}^N VI(C, I-T_i) = VI(C, \sum_{i=1}^N a_i(I-T_i))$. Hence $\bigcap_{i=1}^N F(T_i) = VI(C, \sum_{i=1}^N a_i(I-T_i))$. □

Remark 2.5 From Lemma 2.3 and Lemma 2.4, we have

$$\bigcap_{i=1}^N F(T_i) = VI\left(C, \sum_{i=1}^N a_i(I-T_i)\right) = F\left(P_C\left(I - \lambda\left(\sum_{i=1}^N a_i(I-T_i)\right)\right)\right),$$

for all $\lambda > 0$ and $0 < a_i < 1$ with $\sum_{i=1}^N a_i = 1$.

Lemma 2.6 ([23]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1-\alpha_n)s_n + \delta_n, \quad \forall n \ge 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

$$(1) \quad \sum_{n=1}^\infty \alpha_n = \infty, \qquad (2) \quad \limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0 \quad or \quad \sum_{n=1}^\infty |\delta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

3 Main results

Lemma 3.1 *Let C and Q be nonempty closed convex subsets of a real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $F_1 : C \times C \to \mathbb{R}$ and*

$F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)–(A4). *Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Then*

1. *$T_r^{F_1}(I - rf_1)$ and $T_s^{F_2}(I - sf_2)$ are nonexpansive mapping,*
2. $$\begin{aligned}&\big\| T_r^{F_1}(I - rf_1)\big(p + \gamma A^*\big(T_s^{F_2}(I - sf_2) - I\big)Ap\big) \\ &\quad - T_r^{F_1}(I - rf_1)\big(q + \gamma A^*\big(T_s^{F_2}(I - sf_2) - I\big)Aq\big)\big\|^2 \\ &\leq \|p - q\|^2 + \gamma(\gamma L - 1)\big\|\big(T_s^{F_2}(I - sf_2) - I\big)Ap - \big(T_s^{F_2}(I - sf_2) - I\big)Aq\big\|^2,\end{aligned}$$

for all $p, q \in C$, where $r \in (0, 2\rho)$, $s \in (0, 1)$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A, $T_r^{F_1} : H_1 \to C$ defined by

$$T_r^{F_1}(x) = \left\{z \in C : F_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\},$$

for all $x \in H_1$ and $T_s^{F_2} : H_2 \to Q$ defined by

$$T_s^{F_2}(\overline{x}) = \left\{\overline{z} \in Q : F_2(\overline{z}, y) + \frac{1}{s}\langle y - \overline{z}, \overline{z} - \overline{x}\rangle \geq 0, \forall y \in Q\right\},$$

for all $\overline{x} \in H_2$.

Proof Let $p, q \in C$. First, we show 1 is true. Since f_1 is a ρ-inverse strongly monotone mapping and $r \in (0, 2\rho)$, we obtain

$$\begin{aligned}\big\| T_r^{F_1}(I - rf_1)p - T_r^{F_1}(I - rf_1)q\big\|^2 &\leq \|p - q\|^2 - 2r\langle p - q, f_1p - f_1q\rangle + r^2\|f_1p - f_1q\|^2 \\ &\leq \|p - q\|^2 + r(r - 2\rho)\|f_1p - f_1q\|^2 \\ &\leq \|p - q\|^2.\end{aligned}$$

Thus $T_r^{F_1}(I - rf_1)$ is a nonexpansive mapping. Since f_2 is a firmly nonexpansive mapping and $s \in (0, 1)$, we get

$$\begin{aligned}\big\| T_s^{F_2}(I - sf_2)\overline{p} - T_s^{F_2}(I - sf_2)\overline{q}\big\|^2 &\leq \|\overline{p} - \overline{q}\|^2 - 2s\langle \overline{p} - \overline{q}, f_2\overline{p} - f_2\overline{q}\rangle + s^2\|f_2\overline{p} - f_2\overline{q}\|^2 \\ &\leq \|\overline{p} - \overline{q}\|^2 - s(2 - s)\|f_2\overline{p} - f_2\overline{q}\|^2 \\ &\leq \|\overline{p} - \overline{q}\|^2,\end{aligned}$$

for all $\overline{p}, \overline{q} \in Q$. Therefore $T_s^{F_2}(I - sf_2)$ is a nonexpansive mapping.

Next, we show 2 is true. From Lemma 3.1(1), we have

$$\begin{aligned}&\big\| T_r^{F_1}(I - rf_1)\big(p + \gamma A^*\big(T_s^{F_2}(I - sf_2) - I\big)Ap\big) \\ &\quad - T_r^{F_1}(I - rf_1)\big(q + \gamma A^*\big(T_s^{F_2}(I - sf_2) - I\big)Aq\big)\big\|^2 \\ &\leq \big\|(p - q) + \gamma\big(A^*\big(T_s^{F_2}(I - sf_2) - I\big)Ap - A^*\big(T_s^{F_2}(I - sf_2) - I\big)Aq\big)\big\|^2 \\ &\leq \|p - q\|^2 + 2\gamma\big\langle Ap - Aq, \big(T_s^{F_2}(I - sf_2) - I\big)Ap - \big(T_s^{F_2}(I - sf_2) - I\big)Aq\big\rangle \\ &\quad + \gamma^2L\big\|\big(T_s^{F_2}(I - sf_2) - I\big)Ap - \big(T_s^{F_2}(I - sf_2) - I\big)Aq\big\|^2. \qquad (3.1)\end{aligned}$$

From the property of $T_s^{F_2}$, we get

$$\begin{aligned}
&\|(I - sf_2)Ap - (I - sf_2)Aq\|^2 \\
&\quad \geq \|T_s^{F_2}(I - sf_2)Ap - T_s^{F_2}(I - sf_2)Aq - (Ap - Aq) + (Ap - Aq)\|^2 \\
&\quad = \|(T_s^{F_2}(I - sf_2) - I)Ap - (T_s^{F_2}(I - sf_2) - I)Aq\|^2 \\
&\qquad + 2\langle (T_s^{F_2}(I - sf_2) - I)Ap - (T_s^{F_2}(I - sf_2) - I)Aq, Ap - Aq\rangle \\
&\qquad + \|Ap - Aq\|^2.
\end{aligned} \tag{3.2}$$

We have

$$\begin{aligned}
&\|(I - sf_2)Ap - (I - sf_2)Aq\|^2 \\
&\quad = \|Ap - Aq\|^2 - 2s\langle Ap - Aq, f_2Ap - f_2Aq\rangle \\
&\qquad + s^2\|f_2Ap - f_2Aq\|^2.
\end{aligned} \tag{3.3}$$

From (3.2), (3.3) and the property of firmly nonexpansive mapping, we get

$$\begin{aligned}
&2\langle (T_s^{F_2}(I - sf_2) - I)Ap - (T_s^{F_2}(I - sf_2) - I)Aq, Ap - Aq\rangle \\
&\quad \leq -\|(T_s^{F_2}(I - sf_2) - I)Ap - (T_s^{F_2}(I - sf_2) - I)Aq\|^2 \\
&\qquad - 2s\langle Ap - Aq, f_2Ap - f_2Aq\rangle + s^2\|f_2Ap - f_2Aq\|^2 \\
&\quad \leq -\|(T_s^{F_2}(I - sf_2) - I)Ap - (T_s^{F_2}(I - sf_2) - I)Aq\|^2.
\end{aligned}$$

That is,

$$\begin{aligned}
&2\gamma\langle (T_s^{F_2}(I - sf_2) - I)Ap - (T_s^{F_2}(I - sf_2) - I)Aq, Ap - Aq\rangle \\
&\quad \leq -\gamma\|(T_s^{F_2}(I - sf_2) - I)Ap - (T_s^{F_2}(I - sf_2) - I)Aq\|^2.
\end{aligned} \tag{3.4}$$

Substituting (3.4) in (3.1), we obtain

$$\begin{aligned}
&\|T_r^{F_1}(I - rf_1)(p + \gamma A^*(T_s^{F_2}(I - sf_2) - I)Ap) \\
&\qquad - T_r^{F_1}(I - rf_1)(q + \gamma A^*(T_s^{F_2}(I - sf_2) - I)Aq)\|^2 \\
&\quad \leq \|p - q\|^2 - \gamma\|(T_s^{F_2}(I - sf_2) - I)Ap - (T_s^{F_2}(I - sf_2) - I)Aq\|^2 \\
&\qquad + \gamma^2L\|(T_s^{F_2}(I - sf_2) - I)Ap - (T_s^{F_2}(I - sf_2) - I)Aq\|^2 \\
&\quad = \|p - q\|^2 + \gamma(\gamma L - 1)\|(T_s^{F_2}(I - sf_2) - I)Ap - (T_s^{F_2}(I - sf_2) - I)Aq\|^2.
\end{aligned}$$ □

Lemma 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then*

$$\|(I - T)x\|^2 \leq 2\langle x - z, (I - T)x\rangle, \quad \forall x \in C.$$

Proof Let $x \in C$ and $z \in F(T)$. Since T is a quasi-nonexpansive mapping, we get

$$\begin{aligned}\|Tx - z\|^2 &= \|(x-z) - (I-T)x\|^2 \\ &= \|x-z\|^2 - 2\langle x-z, (I-T)x\rangle + \|(I-T)x\|^2 \\ &\le \|x-z\|^2.\end{aligned}$$

We can conclude that

$$\|(I-T)x\|^2 \le 2\langle x-z, (I-T)x\rangle. \qquad \square$$

Lemma 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Then*

$$\left\|P_C\left(I - \bar{\lambda}\left(\sum_{i=1}^N k_i(I-T_i)\right)\right)x - z\right\|^2 \le \|x-z\|^2,$$

for all $x \in C$, where $0 < k_i < 1$ with $\sum_{i=1}^N k_i = 1$ and $0 < \bar{\lambda} < 1$.

Proof Let $x \in C$ and $z \in \bigcap_{i=1}^N F(T_i)$. From Remark 2.5 and $z \in \bigcap_{i=1}^N F(T_i)$, we have $z \in F(P_C(I - \bar{\lambda}(\sum_{i=1}^N k_i(I-T_i))))$ and $z = T_i z$, $\forall i = 1, 2, \ldots, N$. Since P_C is nonexpansive mapping, $0 < \bar{\lambda} < 1$ and Lemma 3.2, we have

$$\begin{aligned}&\left\|P_C\left(I - \bar{\lambda}\left(\sum_{i=1}^N k_i(I-T_i)\right)\right)x - z\right\|^2 \\ &\quad= \left\|P_C\left(I - \bar{\lambda}\left(\sum_{i=1}^N k_i(I-T_i)\right)\right)x - P_C\left(I - \bar{\lambda}\left(\sum_{i=1}^N k_i(I-T_i)\right)\right)z\right\|^2 \\ &\quad\le \|x-z\|^2 - 2\bar{\lambda}\sum_{i=1}^N k_i\langle x-z, (I-T_i)x\rangle + \bar{\lambda}^2\sum_{i=1}^N k_i\|(I-T_i)x\|^2 \\ &\quad\le \|x-z\|^2 - \bar{\lambda}\sum_{i=1}^N k_i\|(I-T_i)x\|^2 + \bar{\lambda}^2\sum_{i=1}^N k_i\|(I-T_i)x\|^2 \\ &\quad\le \|x-z\|^2.\end{aligned} \tag{3.5}$$

Next, we prove a strong convergence theorem for solving the modified split generalized equilibrium problem (MSGEP).

Theorem 3.4 *Let C and Q be nonempty closed convex subsets of a real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying* (A1)–(A4). *Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathcal{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}$, $\{u_n\}$ and*

$\{y_n\}$ *be sequences generated by*

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)(x_n + \gamma A^*(T_s^{F_2}(I - sf_2) - I)Ax_n), \\ y_n = P_C(I - d_1D_1)(au_n + (1-a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(\sum_{i=1}^N k_i(I - T_i)))y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.6}$$

where $d_1 \in (0, 2\alpha)$, $d_2 \in (0, 2\beta)$, $r \in (0, 2\rho)$, $s \in (0,1)$, $a \in [0,1]$, $0 < k_i < 1$ *with* $\sum_{i=1}^N k_i = 1$, $\gamma \in (0, 1/L)$, *L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0,1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$. *Suppose the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $0 < c \le \beta_n, \gamma_n \le d < 1$ *for some* $c, d > 0$ *for all* $n \ge 1$,
(iii) $\sum_{n=1}^\infty \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,
(iv) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *converge strongly to* $z = P_{\mathcal{F}}u$.

Proof Let $x, y \in C$ and $z \in \mathcal{F}$. First, we show that $(I - d_1D_1)$ is a nonexpansive mapping. Since D_1 is an α-inverse strongly monotone mapping, we obtain

$$\begin{aligned} \|(I - d_1D_1)x - (I - d_1D_1)y\|^2 &= \|x - y\|^2 - 2d_1\langle x - y, D_1x - D_1y\rangle + d_1^2\|D_1x - D_1y\|^2 \\ &\le \|x - y\|^2 + d_1(d_1 - 2\alpha)\|D_1x - D_1y\|^2 \le \|x - y\|^2. \end{aligned}$$

Thus $(I - d_1D_1)$ is a nonexpansive mapping. By using the same method as above, we see that $(I - d_2D_2)$ is a nonexpansive mapping. Since f_1 is a ρ-inverse strongly monotone mapping and f_2 is a firmly nonexpansive mapping. From Lemma 3.1(1), we have $(T_r^{F_1}(I - rf_1))$ and $(T_s^{F_2}(I - sf_2))$ are nonexpansive mappings. Since $z \in \bigcap_{i=1}^N F(T_i)$ and Lemma 3.3, we have

$$\left\| P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_n - z\right\|^2 \le \|y_n - z\|^2. \tag{3.7}$$

Since $z \in VI(C, D_1)$ and $z \in VI(C, D_2)$ and using the property of $(I - d_1D_1)$ and $(I - d_2D_2)$, we get

$$\begin{aligned} \|y_n - z\|^2 &= \|P_C(I - d_1D_1)(au_n + (1-a)P_C(I - d_2D_2)u_n) - P_C(I - d_1D_1)z\|^2 \\ &\le a\|u_n - z\|^2 + (1-a)\|P_C(I - d_2D_2)u_n - z\|^2 \qquad (3.8) \\ &\le \|u_n - z\|^2. \qquad (3.9) \end{aligned}$$

Since $z \in \Omega$, we have $z = T_r^{F_1}(I - rf_1)z$ and $Az = T_s^{F_2}(I - sf_2)Az$. From Lemma 3.1(2) and $\gamma \in (0, 1/L)$, we obtain

$$\begin{aligned} \|u_n - z\|^2 &= \|T_r^{F_1}(I - rf_1)(x_n + \gamma A^*(T_s^{F_2}(I - sf_2) - I)Ax_n) - T_r^{F_1}(I - rf_1)z\|^2 \\ &\le \|x_n - z\|^2 + \gamma(L\gamma - 1)\|(T_s^{F_2}(I - sf_2) - I)Ax_n\|^2 \qquad (3.10) \\ &\le \|x_n - z\|^2. \qquad (3.11) \end{aligned}$$

Using the definition of x_n, (3.7), (3.9) and (3.11), we get

$$\begin{aligned}\|x_{n+1} - z\| &= \left\| \alpha_n(u - z) + \beta_n(x_n - z) \right. \\ &\quad \left. + \gamma_n \left(P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i(I - T_i) \right) \right) y_n - z \right) \right\| \\ &\le \alpha_n \|u - z\| + \beta_n \|x_n - z\| + \gamma_n \|y_n - z\| \\ &\le \alpha_n \|u - z\| + \beta_n \|x_n - z\| + \gamma_n \|u_n - z\| \\ &\le \alpha_n \|u - z\| + (1 - \alpha_n)\|x_n - z\|.\end{aligned}$$

Using induction, we can conclude that

$$\|x_n - z\| \le \max\{\|u - z\|, \|x_1 - z\|\}$$

for all $n \ge 1$. This implies that the sequence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{u_n\}$. From Lemma 3.1 (2) and $\gamma \in (0, 1/L)$, we obtain

$$\begin{aligned}&\|u_n - u_{n-1}\|^2 \\ &\quad = \left\| T_r^{F_1}(I - rf_1)\left(x_n + \gamma A^*\left(T_s^{F_2}(I - sf_2) - I\right)Ax_n\right) \right. \\ &\qquad \left. - T_r^{F_1}(I - rf_1)\left(x_{n-1} + \gamma A^*\left(T_s^{F_2}(I - sf_2) - I\right)Ax_{n-1}\right) \right\|^2 \\ &\quad \le \|x_n - x_{n-1}\|^2 + \gamma(\gamma L - 1)\left\| \left(T_s^{F_2}(I - sf_2) - I\right)Ax_n - \left(T_s^{F_2}(I - sf_2) - I\right)Ax_{n-1} \right\|^2 \\ &\quad \le \|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.12}$$

Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. According to Eq. (3.12), we have

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &\quad = \left\| \left(\alpha_n u + \beta_n x_n + \gamma_n P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i(I - T_i) \right) \right) y_n \right) \right. \\ &\qquad \left. - \left(\alpha_{n-1} u + \beta_{n-1} x_{n-1} + \gamma_{n-1} P_C \left(I - \lambda_{n-1} \left(\sum_{i=1}^{N} k_i(I - T_i) \right) \right) y_{n-1} \right) \right\| \\ &\quad \le |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n \|y_n - y_{n-1}\| \\ &\qquad + \lambda_n \left\| \left(\sum_{i=1}^{N} k_i(I - T_i) \right) y_n - \left(\sum_{i=1}^{N} k_i(I - T_i) \right) y_{n-1} \right\| \\ &\qquad + |\lambda_n - \lambda_{n-1}| \left\| \left(\sum_{i=1}^{N} k_i(I - T_i) \right) y_{n-1} \right\| \\ &\qquad + |\gamma_n - \gamma_{n-1}| \left\| P_C \left(I - \lambda_{n-1} \left(\sum_{i=1}^{N} k_i(I - T_i) \right) \right) y_{n-1} \right\| \\ &\quad \le (1 - \alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\|\end{aligned}$$

$$\begin{aligned}
&+ \lambda_n \left\| \left(\sum_{i=1}^{N} k_i(I - T_i) \right) y_n - \left(\sum_{i=1}^{N} k_i(I - T_i) \right) y_{n-1} \right\| \\
&+ |\lambda_n - \lambda_{n-1}| \left\| \left(\sum_{i=1}^{N} k_i(I - T_i) \right) y_{n-1} \right\| \\
&+ |\gamma_n - \gamma_{n-1}| \left\| P_C \left(I - \lambda_{n-1} \left(\sum_{i=1}^{N} k_i(I - T_i) \right) \right) y_{n-1} \right\| \\
\le{}& (1 - \alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M + |\beta_n - \beta_{n-1}|M + \lambda_n M \\
&+ |\lambda_n - \lambda_{n-1}|M + |\gamma_n - \gamma_{n-1}|M,
\end{aligned}$$

where

$$\begin{aligned}
M := \max_{n \in \mathbb{N}} \Bigg\{ & \|u\|, \|x_n\|, \left\| \left(\sum_{i=1}^{N} k_i(I - T_i) \right) y_{n+1} - \left(\sum_{i=1}^{N} k_i(I - T_i) \right) y_n \right\|, \\
& \left\| \left(\sum_{i=1}^{N} k_i(I - T_i) \right) y_n \right\|, \left\| P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i(I - T_i) \right) \right) y_n \right\| \Bigg\}.
\end{aligned}$$

From condition (i), (iii), (iv) and Lemma 2.6, we have

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.13}$$

According to Eqs. (3.7), (3.9) and (3.10), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 \le{}& \alpha_n \|u - z\|^2 + \gamma_n \left\| P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i(I - T_i) \right) \right) y_n - z \right\|^2 \\
&+ \beta_n \|x_n - z\|^2 - \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i(I - T_i) \right) \right) y_n \right\|^2 \\
\le{}& \alpha_n \|u - z\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \|y_n - z\|^2 \\
&- \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i(I - T_i) \right) \right) y_n \right\|^2 && (3.14) \\
\le{}& \alpha_n \|u - z\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \|u_n - z\|^2 \\
&- \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i(I - T_i) \right) \right) y_n \right\|^2 && (3.15) \\
\le{}& \alpha_n \|u - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 + \gamma_n \gamma (L\gamma - 1) \left\| \left(T_s^{F_2}(I - sf_2) - I \right) A x_n \right\|^2 \\
&- \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i(I - T_i) \right) \right) y_n \right\|^2 .
\end{aligned}$$

This implies that

$$\begin{aligned}
&\gamma_n \gamma (1 - L\gamma) \left\| \left(T_s^{F_2}(I - sf_2) - I \right) A x_n \right\|^2 \\
&\quad \le \alpha_n \|u - z\|^2 + \|x_n - x_{n+1}\| \left(\|x_n - z\| + \|x_{n+1} - z\| \right).
\end{aligned}$$

By using condition (i) and (3.13), we have

$$\lim_{n\to\infty} \left\| \left(T_s^{F_2}(I - sf_2) - I\right)Ax_n \right\| = 0. \tag{3.16}$$

By using the same method as (3.16), we have

$$\lim_{n\to\infty} \left\| x_n - P_C\left(I - \lambda_n\left(\sum_{i=1}^{N} k_i(I - T_i)\right)\right)y_n \right\| = 0. \tag{3.17}$$

Let $M_n = x_n + \gamma A^*(T_s^{F_2}(I - sf_2) - I)Ax_n$. Applying the inequality (3.11), we have

$$\|M_n - z\| \le \|x_n - z\|. \tag{3.18}$$

Using the property of inverse strongly monotone operators and (3.18), we have

$$\begin{aligned} \|u_n - z\|^2 &= \left\| T_r^{F_1}(I - rf_1)M_n - T_r^{F_1}(I - rf_1)z \right\|^2 \\ &\le \left\| (I - rf_1)M_n - (I - rf_1)z \right\|^2 \\ &= \|M_n - z\|^2 - 2r\langle M_n - z, f_1M_n - f_1z\rangle + r^2\|f_1M_n - f_1z\|^2 \\ &\le \|x_n - z\|^2 + r(r - 2\rho)\|f_1M_n - f_1z\|^2. \end{aligned} \tag{3.19}$$

Substituting (3.19) in (3.15), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 \\ &\quad + \gamma_n\left(\|x_n - z\|^2 + r(r - 2\rho)\|f_1M_n - f_1z\|^2\right) \\ &\le \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 + \gamma_n r(r - 2\rho)\|f_1M_n - f_1z\|^2. \end{aligned}$$

That is,

$$\gamma_n r(2\rho - r)\|f_1M_n - f_1z\|^2 \le \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - z\| + \|x_{n+1} - z\|\right).$$

According to condition (i) and (3.13), we get

$$\lim_{n\to\infty} \|f_1M_n - f_1z\| = 0. \tag{3.20}$$

By the property of firmly nonexpansive mappings, we have

$$\begin{aligned} \|u_n - z\|^2 &= \left\| T_r^{F_1}(I - rf_1)M_n - T_r^{F_1}(I - rf_1)z \right\|^2 \\ &\le \left\langle u_n - z, (I - rf_1)M_n - (I - rf_1)z \right\rangle \\ &= \frac{1}{2}\left(\|u_n - z\|^2 + \left\|(I - rf_1)M_n - (I - rf_1)z\right\|^2 \right. \\ &\quad \left. - \left\|(u_n - z) - \left((I - rf_1)M_n - (I - rf_1)z\right)\right\|^2\right). \end{aligned} \tag{3.21}$$

That is,

$$\begin{aligned}\|u_n - z\|^2 &\le \left\|(I - rf_1)M_n - (I - rf_1)z\right\|^2 - \left\|(u_n - M_n) + r(f_1M_n - f_1z)\right\|^2 \\ &\le \|M_n - z\|^2 - \left(\|u_n - M_n\|^2 + 2r\langle u_n - M_n, f_1M_n - f_1z\rangle \right.\\ &\quad \left. + r^2\|f_1M_n - f_1z\|^2\right) \\ &\le \|M_n - z\|^2 - \|u_n - M_n\|^2 + 2r\|u_n - M_n\|\|f_1M_n - f_1z\| \\ &\quad - r^2\|f_1M_n - f_1z\|^2. \end{aligned} \tag{3.22}$$

Substituting (3.22) in (3.15), we get

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left(\|M_n - z\|^2 - \|u_n - M_n\|^2\right.\\ &\quad \left. + 2r\|u_n - M_n\|\|f_1M_n - f_1z\| - r^2\|f_1M_n - f_1z\|^2\right) \\ &\le \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - \gamma_n\|u_n - M_n\|^2 \\ &\quad + 2r\gamma_n\|u_n - M_n\|\|f_1M_n - f_1z\|. \end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n\|u_n - M_n\|^2 &\le \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - z\| + \|x_{n+1} - z\|\right) \\ &\quad + 2r\gamma_n\|u_n - M_n\|\|f_1M_n - f_1z\|. \end{aligned}$$

From condition (i), (3.13) and (3.20), we ensure that

$$\lim_{n\to\infty} \|u_n - M_n\| = 0. \tag{3.23}$$

From (3.16) and (3.23), we also have

$$\begin{aligned}\|u_n - x_n\| &\le \|u_n - M_n\| + \|M_n - x_n\| \\ &= \|u_n - M_n\| + \left\|x_n + \gamma A^*\left(T_s^{F_2}(I - sf_2) - I\right)Ax_n - x_n\right\| \\ &\le \|u_n - M_n\| + \gamma\|A\|\left\|\left(T_s^{F_2}(I - sf_2) - I\right)Ax_n\right\|. \end{aligned}$$

Then we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.24}$$

By using the same method as (3.19), we have

$$\left\|P_C(I - d_2D_2)u_n - z\right\|^2 \le \|x_n - z\|^2 + d_2(d_2 - 2\beta)\|D_2u_n - D_2z\|^2. \tag{3.25}$$

Substituting (3.8) and (3.25) in (3.14), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left(a\|u_n - z\|^2\right.\\ &\quad \left. + (1 - a)\left\|P_C(I - d_2D_2)u_n - z\right\|^2\right) \end{aligned}$$

$$\begin{aligned}&\le \alpha_n\|u-z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\ &\quad + \gamma_n(1-a)d_2(d_2-2\beta)\|D_2u_n - D_2z\|^2.\end{aligned}$$

We can conclude that

$$\begin{aligned}&\gamma_n(1-a)d_2(2\beta-d_2)\|D_2u_n - D_2z\|^2 \\ &\quad \le \alpha_n\|u-z\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - z\| + \|x_{n+1} - z\|\right).\end{aligned}$$

According to condition (i) and (3.13), we get

$$\lim_{n\to\infty}\|D_2u_n - D_2z\| = 0. \tag{3.26}$$

Since P_C is a firmly nonexpansive mapping and using the same method as (3.21), we get

$$\begin{aligned}&\left\|P_C(I-d_2D_2)u_n - z\right\|^2 \\ &\quad \le \frac{1}{2}\Big(\left\|P_C(I-d_2D_2)u_n - z\right\|^2 + \left\|(I-d_2D_2)u_n - (I-d_2D_2)z\right\|^2 \\ &\qquad - \left\|P_C(I-d_2D_2)u_n - z - (I-d_2D_2)u_n + (I-d_2D_2)z\right\|^2\Big).\end{aligned}$$

That is,

$$\begin{aligned}\left\|P_C(I-d_2D_2)u_n - z\right\|^2 &\le \|u_n - z\|^2 - \left\|\left(P_C(I-d_2D_2)u_n - u_n\right) + d_2(D_2u_n - D_2z)\right\|^2 \\ &\le \|x_n - z\|^2 - \left\|P_C(I-d_2D_2)u_n - u_n\right\|^2 \\ &\quad + 2d_2\left\|P_C(I-d_2D_2)u_n - u_n\right\|\|D_2u_n - D_2z\| \\ &\quad - d_2^2\|D_2u_n - D_2z\|^2.\end{aligned} \tag{3.27}$$

Substituting (3.8) and (3.27) in (3.14), we have

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &\quad \le \alpha_n\|u-z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left(a\|u_n - z\|^2 + (1-a)\left\|P_C(I-d_2D_2)u_n - z\right\|^2\right) \\ &\quad \le \alpha_n\|u-z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\Big(a\|x_n - z\|^2 + (1-a)\big(\|x_n - z\|^2 \\ &\qquad - \left\|P_C(I-d_2D_2)u_n - u_n\right\|^2 + 2d_2\left\|P_C(I-d_2D_2)u_n - u_n\right\|\|D_2u_n - D_2z\| \\ &\qquad - d_2^2\|D_2u_n - D_2z\|^2\big)\Big) \\ &\quad \le \alpha_n\|u-z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \gamma_n(1-a)\left\|P_C(I-d_2D_2)u_n - u_n\right\|^2 \\ &\qquad + 2d_2\gamma_n(1-a)\left\|P_C(I-d_2D_2)u_n - u_n\right\|\|D_2u_n - D_2z\|.\end{aligned}$$

Therefore

$$\begin{aligned}&\gamma_n(1-a)\left\|P_C(I-d_2D_2)u_n - u_n\right\|^2 \\ &\quad \le \alpha_n\|u-z\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - z\| + \|x_{n+1} - z\|\right) \\ &\qquad + 2d_2\gamma_n(1-a)\left\|P_C(I-d_2D_2)u_n - u_n\right\|\|D_2u_n - D_2z\|.\end{aligned}$$

From condition (i), (3.13) and (3.26), we get

$$\lim_{n\to\infty} \left\| P_C(I - d_2D_2)u_n - u_n \right\| = 0. \tag{3.28}$$

Let $k_n = au_n + (1-a)P_C(I - d_2D_2)u_n$. By using the same method as (3.19), we have

$$\|y_n - z\|^2 \le \|x_n - z\|^2 + d_1(d_1 - 2\alpha)\|D_1k_n - D_1z\|^2. \tag{3.29}$$

Substituting (3.29) in (3.14), we have

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &\quad \le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left(\|x_n - z\|^2 + d_1(d_1 - 2\alpha)\|D_1k_n - D_1z\|^2\right) \\ &\quad \le \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 + d_1(d_1 - 2\alpha)\gamma_n\|D_1k_n - D_1z\|^2. \end{aligned}$$

This implies that

$$d_1(2\alpha - d_1)\gamma_n\|D_1k_n - D_1z\|^2 \le \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - z\| + \|x_{n+1} - z\|\right).$$

According to condition (i) and (3.13), we have

$$\lim_{n\to\infty} \|D_1k_n - D_1z\| = 0. \tag{3.30}$$

By using the same method as (3.21), we have

$$\begin{aligned} \|y_n - z\|^2 \le \frac{1}{2}\Big(&\|y_n - z\|^2 + \left\|(I - d_1D_1)k_n - (I - d_1D_1)z\right\|^2 \\ &- \left\|(y_n - k_n) + d_1(D_1k_n - D_1z)\right\|^2\Big). \end{aligned}$$

That is,

$$\begin{aligned} \|y_n - z\|^2 &\le \|k_n - z\|^2 - \big(\|y_n - k_n\|^2 + 2d_1\langle y_n - k_n, D_1k_n - D_1z\rangle \\ &\quad + d_1^2\|D_1k_n - D_1z\|^2\big) \\ &\le \|x_n - z\|^2 - \|y_n - k_n\|^2 + 2d_1\|y_n - k_n\|\|D_1k_n - D_1z\| \\ &\quad - d_1^2\|D_1k_n - D_1z\|^2. \end{aligned} \tag{3.31}$$

Substituting (3.31) in (3.14), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\big(\|x_n - z\|^2 - \|y_n - k_n\|^2 \\ &\quad + 2d_1\|y_n - k_n\|\|D_1k_n - d_1z\| - d_1^2\|D_1k_n - D_1z\|^2\big) \\ &\le \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - \gamma_n\|y_n - k_n\|^2 \\ &\quad + 2\gamma_n d_1\|y_n - k_n\|\|D_1k_n - D_1z\|. \end{aligned} \tag{3.32}$$

This implies that

$$\gamma_n\|y_n - k_n\|^2 \le \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|\big(\|x_n - z\| + \|x_{n+1} - z\|\big) + 2\gamma_n d_1\|y_n - k_n\|\|D_1 k_n - D_1 z\|.$$

According to condition (i), (3.13) and (3.30), we get

$$\lim_{n\to\infty}\|y_n - k_n\| = 0. \tag{3.33}$$

From (3.28) and (3.33)

$$\begin{aligned}\|y_n - u_n\| &\le \|y_n - k_n\| + \|k_n - u_n\| \\ &\le \|y_n - k_n\| + (1-a)\big\|P_C(I - d_2D_2)u_n - u_n\big\|,\end{aligned}$$

we conclude that

$$\lim_{n\to\infty}\|y_n - u_n\| = 0. \tag{3.34}$$

By (3.24) and (3.34), we also conclude that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.35}$$

Afterward, we show that $\limsup_{n\to\infty}\langle u - z, x_n - z\rangle \le 0$, where $z = P_{\mathcal{F}}u$. To show this, choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z, x_n - z\rangle = \lim_{j\to\infty}\langle u - z, x_{n_j} - z\rangle. \tag{3.36}$$

Without loss of generality, we may assume that $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$. From (3.35), we obtain $y_{n_j} \rightharpoonup \omega$ as $j \to \infty$. From Lemma 2.3, we have $VI(C, D_1) = F(P_C(I - d_1D_1))$. Assume that $\omega \notin VI(C, D_1)$, we have $\omega \neq P_C(I - d_1D_1)\omega$. Using Opial's condition, (3.33), we obtain

$$\begin{aligned}\liminf_{j\to\infty}\|y_{n_j} - \omega\| &< \liminf_{j\to\infty}\big\|y_{n_j} - P_C(I - d_1D_1)\omega\big\| \\ &\le \liminf_{j\to\infty}\big(\big\|P_C(I - d_1D_1)k_{n_j} - P_C(I - d_1D_1)y_{n_j}\big\| \\ &\quad + \big\|P_C(I - d_1D_1)y_{n_j} - P_C(I - d_1D_1)\omega\big\|\big) \\ &\le \liminf_{j\to\infty}\big(\|k_{n_j} - y_{n_j}\| + \|y_{n_j} - \omega\|\big) \\ &\le \liminf_{j\to\infty}\|y_{n_j} - \omega\|.\end{aligned}$$

This is a contradiction, so we have

$$\omega \in VI(C, D_1). \tag{3.37}$$

From (3.24), we have $u_{n_j} \rightharpoonup \omega$ as $j \to \infty$. By (3.28) and using the same method as (3.37), we obtain

$$\omega \in VI(C, D_2). \tag{3.38}$$

Next, we show that $\omega \in \bigcap_{i=1}^N F(T_i)$. From Lemma 2.5, we have

$$\bigcap_{i=1}^N F(T_i) = F\left(P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)\right).$$

Assume that $\omega \notin \bigcap_{i=1}^N F(T_i)$, and that $\omega \neq P_C(I - \lambda_{n_j}(\sum_{i=1}^N k_i(I - T_i)))\omega$. Using Opial's condition, (3.17) and (3.35), we obtain

$$\begin{aligned}
&\liminf_{j\to\infty} \|x_{n_j} - \omega\| \\
&\quad < \liminf_{j\to\infty} \left\| x_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)\omega \right\| \\
&\quad \leq \liminf_{j\to\infty} \left(\left\| x_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right) y_{n_j} \right\| \right. \\
&\qquad + \left\| P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right) y_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right) x_{n_j} \right\| \\
&\qquad \left. + \left\| P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right) x_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right) \omega \right\| \right) \\
&\quad \leq \liminf_{j\to\infty} \left(\|y_{n_j} - x_{n_j}\| + \lambda_{n_j} \left\| \left(\sum_{i=1}^N k_i(I - T_i)\right) y_{n_j} - \left(\sum_{i=1}^N k_i(I - T_i)\right) x_{n_j} \right\| \right. \\
&\qquad \left. + \|x_{n_j} - \omega\| + \lambda_{n_j} \left\| \left(\sum_{i=1}^N k_i(I - T_i)\right) x_{n_j} - \left(\sum_{i=1}^N k_i(I - T_i)\right) \omega \right\| \right) \\
&\quad \leq \liminf_{j\to\infty} \|x_{n_j} - \omega\|.
\end{aligned}$$

This is a contradiction, so we have

$$\omega \in \bigcap_{i=1}^N F(T_i). \tag{3.39}$$

After that, we show that $\omega \in \Omega$. Assume $\omega \notin EP(F_1, f_1)$. Since $EP(F_1, f_1) = F(T_r^{F_1}(I - rf_1))$, we obtain $\omega \neq T_r^{F_1}(I - rf_1)\omega$. Using Opial's condition and (3.23), we get

$$\begin{aligned}
\liminf_{j\to\infty} \|u_{n_j} - \omega\| &< \liminf_{j\to\infty} \left\| u_{n_j} - T_r^{F_1}(I - rf_1)\omega \right\| \\
&\leq \liminf_{j\to\infty} \left(\left\| T_r^{F_1}(I - rf_1)M_{n_j} - T_r^{F_1}(I - rf_1)u_{n_j} \right\| \right. \\
&\quad \left. + \left\| T_r^{F_1}(I - rf_1)u_{n_j} - T_r^{F_1}(I - rf_1)\omega \right\| \right)
\end{aligned}$$

$$\leq \liminf_{j\to\infty}\left(\|M_{n_j} - u_{n_j}\| + \|u_{n_j} - \omega\|\right)$$
$$\leq \liminf_{j\to\infty}\|u_{n_j} - \omega\|.$$

This is a contradiction, so we have

$$\omega \in EP(F_1,f_1). \tag{3.40}$$

Next, we show that $A\omega \in EP(F_2,f_2)$. Since A is bounded linear operator so that $Ax_{n_j} \rightharpoonup A\omega$ as $j \to \infty$. Assume $A\omega \notin EP(F_2,f_2)$. Since $EP(F_2,f_2) = F(T_s^{F_2}(I - sf_2))$, we obtain $A\omega \neq T_s^{F_2}(I - sf_s)A\omega$. Using Opial's condition and (3.16), we have

$$A\omega \in EP(F_2,f_2). \tag{3.41}$$

We can conclude that $\omega \in \Omega$. Therefore $\omega \in \mathcal{F}$. Since $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z, x_n - z\rangle &= \lim_{j\to\infty}\langle u - z, x_{n_j} - z\rangle \\ &= \langle u - z, \omega - z\rangle \leq 0.\end{aligned} \tag{3.42}$$

Finally, we show that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}u$. By (3.7), (3.9) and (3.11), we get

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \left\|\alpha_n(u - z) + \beta_n(x_n - z) + \gamma_n\left(P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_n - z\right)\right\|^2 \\ &\leq \left\|\beta_n(x_n - z) + \gamma_n\left(P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_n - z\right)\right\|^2 \\ &\quad + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\leq \left(\beta_n\|x_n - z\| + \gamma_n\|u_n - z\|\right)^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle.\end{aligned}$$

According to condition (i), (3.42) and Lemma 2.6, we can conclude that $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}u$. By (3.24) and (3.35), we have $\{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathcal{F}}u$. This completes the proof. □

These results are directly proved from Theorem 3.4. Therefore, we omit the proof.

Corollary 3.5 *Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying* (A1)–(A4). *Let T be a quasi-nonexpansive mapping of C into itself. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathcal{F} = VI(C,D_1) \cap VI(C,D_2) \cap F(T) \cap \Omega \neq \emptyset$. For* *given* $x_1, u \in C$, *and let* $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)(x_n + \gamma A^*(T_s^{F_2}(I - sf_2) - I)Ax_n), \\ y_n = P_C(I - d_1D_1)(au_n + (1-a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(I - T))y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha)$, $d_2 \in (0, 2\beta)$, $r \in (0, 2\rho)$, $s \in (0, 1)$, $a \in [0, 1]$, $\gamma \in (0, 1/L)$, L *is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of* A. *Also* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0, 1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$. *Suppose the conditions* (i)–(iv) *of Theorem* 3.4 *hold. Then* $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *converge strongly to* $z = P_{\mathcal{F}}u$.

Corollary 3.6 *Let* C *be nonempty closed convex subset of a real Hilbert space* H_1. *Let* $D_1, D_2 : C \to H_1$ *be* α, β*-inverse strongly monotone mappings, respectively. Let* $F_1 : C \times C \to \mathbb{R}$ *be the bifunction satisfying* (A1)–(A4). *Let* $\{T_i\}_{i=1}^N$ *be a finite family of quasi-nonexpansive mappings of* C *into itself with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Let* $f_1 : H_1 \to H_1$ *be a* ρ*-inverse strongly monotone mapping. Assume* $\mathcal{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap EP(F_1, f_1) \neq \emptyset$. *For given* $x_1, u \in C$ *and let* $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)x_n, \\ y_n = P_C(I - d_1D_1)(au_n + (1-a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(\sum_{i=1}^N k_i(I - T_i)))y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha)$, $d_2 \in (0, 2\beta)$, $r \in (0, 2\rho)$, $a \in [0, 1]$, $0 < k_i < 1$ *with* $\sum_{i=1}^N k_i = 1$. *Also* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0, 1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$. *Suppose the conditions* (i)–(iv) *of Theorem* 3.4 *hold. Then* $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *converge strongly to* $z = P_{\mathcal{F}}u$.

Corollary 3.7 *Let* C *and* Q *be nonempty closed convex subsets of a real Hilbert space* H_1 *and* H_2, *respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Let* $D_1, D_2 : C \to H_1$ *be* α, β*-inverse strongly monotone mappings, respectively. Let* $F_1 : C \times C \to \mathbb{R}$ *and* $F_2 : Q \times Q \to \mathbb{R}$ *be the bifunctions satisfying* (A1)–(A4). *Let* $\{T_i\}_{i=1}^N$ *be a finite family of quasi-nonexpansive mappings of* C *into itself with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Assume* $\mathcal{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Gamma \neq \emptyset$. *For given* $x_1, u \in C$ *and let* $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} u_n = T_r^{F_1}(x_n + \gamma A^*(T_s^{F_2} - I)Ax_n), \\ y_n = P_C(I - d_1D_1)(au_n + (1-a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(\sum_{i=1}^N k_i(I - T_i)))y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha)$, $d_2 \in (0, 2\beta)$, $a \in [0, 1]$, $0 < k_i < 1$ *with* $\sum_{i=1}^N k_i = 1$, $\gamma \in (0, 1/L)$, L *is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of* A. *Also* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0, 1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$. *Suppose the conditions* (i)–(iv) *of Theorem* 3.4 *hold. Then* $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *converge strongly to* $z = P_{\mathcal{F}}u$.

Remark 3.8 If we take $N = 1$ in Theorem 3.4, we have a strong convergence for finding a common element of the set of solutions of variational inequality problems and the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of the modified split generalized equilibrium problem. From previous result, we can apply by using the same method as Theorem 4.5 in [24]. We have a strong convergence for finding a common element of the set of solutions of variational inequality problems and the set of fixed points of a finite family of nonspreading mappings and the set of solutions of the modified split generalized equilibrium problem. By using our main result, Theorem 3.4 reduces to the Corollary 3.6, the solution of the generalized equilibrium problem and Corollary 3.7, the split equilibrium problem. All theorems are found as regards the solution of common fixed points of a finite family of quasi-nonexpansive mappings without assuming $T_\omega := (1-\omega)I + \omega T$ and T is demiclosed; a difficult proof in a framework of Hilbert space.

4 Application

The following knowledge is used to prove Theorem 4.4. A mapping $T : C \to C$ is called nonspreading if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C. \tag{4.1}$$

Such a mapping is defined by Kohsaka and Takahashi [25].

In 2009, Iemoto and Takahashi [26] proved that (4.1) is equivalent to

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C. \tag{4.2}$$

Remark 4.1 A nonspreading mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive mapping T.

Lemma 4.2 ([25]) *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.*

In 2009, Kangtunyakarn and Suantai[27] introduced the S-mapping generated by $T_1, T_2, T_3, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$ as follows.

Definition 4.1 Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, \dots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\vdots\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called an *S-mapping* generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 4.3 ([28]) *Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \dots, N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \dots, N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in [0,1)$ for all $j = 1, 2, \dots, N$. Let S be the mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a quasi-nonexpansive mapping.*

By using these results, we obtain the following theorems.

Theorem 4.4 *Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying* (A1)–(A4). *Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \dots, N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \dots, N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in [0,1)$ for all $j = 1, 2, \dots, N$. Let S be the mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathcal{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)(x_n + \gamma A^*(T_s^{F_2}(I - sf_2) - I)Ax_n), \\ y_n = P_C(I - d_1 D_1)(au_n + (1-a)P_C(I - d_2 D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(I - S))y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha)$, $d_2 \in (0, 2\beta)$, $r \in (0, 2\rho)$, $s \in (0,1)$, $a \in [0,1]$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions (i)–(iv) *of Theorem* 3.4 *hold. Then $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.*

Proof By using Corollary 3.5 and Lemma 4.3, we obtain the conclusion. □

5 Example and numerical results

In this section, an example is given for supporting Theorem 3.4. In Example 5.1, we only instance an example in infinite dimensional Hilbert space for supporting Theorem 3.4. We omit the computer programming.

Example 5.1 Let $H_1 = H_2 = C = Q = \ell_2$ be the linear space whose elements consist of all 2-summable sequences $(x_1, x_2, \dots, x_j, \dots)$ of scalars, i.e.,

$$\ell_2 = \left\{ x : x = (x_1, x_2, \dots, x_j, \dots) \text{ and } \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\},$$

with an inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j$ where $x = \{x_j\}_{j=1}^{\infty}$, $y = \{y_j\}_{j=1}^{\infty} \in \ell_2$ and a norm $\|\cdot\| : \ell_2 \to \mathbb{R}$ defined by $\|x\|_2 = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$ where $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. Let the mapping $A : \ell_2 \to \ell_2$ be defined by $Ax = (\frac{x_1}{3}, \frac{x_2}{3}, \dots, \frac{x_j}{3}, \dots)$ for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$ and $A^* : \ell_2 \to \ell_2$ be defined by $A^*z = (\frac{z_1}{3}, \frac{z_2}{3}, \dots, \frac{z_j}{3}, \dots)$ for all $z = \{z_j\}_{j=1}^{\infty} \in \ell_2$. Let D_1, D_2 :

$\ell_2 \to \ell_2$ be defined by $D_1x = (\frac{x_1}{6}, \frac{x_2}{6}, \ldots, \frac{x_j}{6}, \ldots)$ and $D_2x = (\frac{x_1}{5}, \frac{x_2}{5}, \ldots, \frac{x_j}{5}, \ldots)$, $\forall x = \{x_j\}_{j=1}^\infty \in \ell_2$, respectively. Let the mapping $T_i : \ell_2 \to \ell_2$ be defined by $T_ix = (\frac{3ix_1}{5i+1}, \frac{3ix_2}{5i+1}, \ldots, \frac{3ix_j}{5i+1}, \ldots)$, $\forall x = \{x_j\}_{j=1}^\infty \in \ell_2$ and $k_i = \frac{6}{7^i} + \frac{1}{N7^N}$ for every $i = 1, 2, \ldots, N$. Let the mapping $F_1, F_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F_1(x,y) = -x^2 + y^2, \quad \forall x = \{x_j\}_{j=1}^\infty, y = \{y_j\}_{j=1}^\infty \in \ell_2,$$

and

$$F_2(x,y) = -2x^2 + xy + y^2, \quad \forall x = \{x_j\}_{j=1}^\infty, y = \{y_j\}_{j=1}^\infty \in \ell_2.$$

Let the mapping $f_1 : \ell_2 \to \ell_2$ be defined by $f_1x = (\frac{x_1}{5}, \frac{x_2}{5}, \ldots, \frac{x_j}{5}, \ldots)$, $\forall x = \{x_j\}_{j=1}^\infty \in \ell_2$ and the mapping $f_2 : \ell_2 \to \ell_2$ be defined by $f_2x = (\frac{x_1}{7}, \frac{x_2}{7}, \ldots, \frac{x_j}{7}, \ldots)$, $\forall x = \{x_j\}_{j=1}^\infty \in \ell_2$. Let $r = 1$ and $s = 0.5$. Since $L = \frac{1}{9}$, we choose $\gamma = 0.5$. Let $x_1 = (x_1^1, x_1^2, \ldots, x_1^j, \ldots)$ and $u = (u_1, u_2, \ldots, u_j, \ldots) \in \ell_2$ and let the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be generated by (3.6) as follows:

$$\begin{cases} u_n = T_1^{F_1}(I - f_1)(x_n + 0.5A^*(T_{0.5}^{F_2}(I - 0.5f_2) - I)Ax_n), \\ y_n = (I - D_1)(0.5u_n + 0.5(I - D_2)u_n), \\ x_{n+1} = \frac{1}{2n}u + \frac{7n-4}{12n}x_n + \frac{5n-2}{12n}(y_n - ((\frac{1}{2n^2})(\sum_{i=1}^N (\frac{6}{7^i} + \frac{1}{N7^N})(y_n - T_iy_n)))), \end{cases}$$

for all $n \geq 1$, where $x_n = (x_n^1, x_n^2, \ldots, x_n^j, \ldots)$, $y_n = (y_n^1, y_n^2, \ldots, y_n^j, \ldots)$ and $u_n = (u_n^1, u_n^2, \ldots, u_n^j, \ldots)$. It easy to see that $D_1, D_2, T_i, F_1, F_2, f_1$ and f_2 satisfy Theorem 3.4. Moreover, we have $VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega = \{0\}$, where $\rho = d_1 = d_2 = 1$. From Theorem 3.4, we can conclude that the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to 0.

In Example 5.2, we give computer programming to support our main result.

Example 5.2 Let $H_1 = H_2 = C = Q = \mathbb{R}^2$ be the two-dimensional Euclidean space of the real number with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\langle x, y \rangle = x \cdot y = x_1y_1 + x_2y_2$ where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and a usual norm $\| \cdot \| : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$ where $x = (x_1, x_2) \in \mathbb{R}^2$. Let the mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Ax = (2x_1 - x_2, x_1 + 2x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $A^* : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $A^*z = (2z_1 - z_2, 2z_2 - z_1)$ for all $z = (z_1, z_2) \in \mathbb{R}^2$. Let $D_1, D_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $D_1x = (\frac{x_1}{6}, \frac{x_2}{6})$ and $D_2x = (\frac{x_1}{2}, \frac{x_2}{3})$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$, respectively. Let the mapping $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T_ix = (\frac{3ix_1}{3i+1}, \frac{3ix_2}{3i+2})$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$ and $k_i = \frac{6}{7^i} + \frac{1}{N7^N}$ for every $i = 1, 2, \ldots, N$. Let the mapping $F_1, F_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F_1(x,y) = -x^2 + y^2, \quad \forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2,$$

and

$$F_2(x,y) = -2x^2 + xy + y^2, \quad \forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

Let the mapping $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_1x = (\frac{x_1}{5}, \frac{x_2}{5})$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$ and the mapping $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_2x = (\frac{x_1}{7}, \frac{x_2}{7})$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$. Let $r = 1$ and $s = 0.5$, the sequences $z_n = (z_n^1, z_n^2)$, $x_n = (x_n^1, x_n^2)$, $u_n = (u_n^1, u_n^2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. By the definition of f_1 and f_2, we get

$$\begin{aligned} 0 &\le F_1(z_n, y) + \langle f_1(z_n), y - z_n\rangle + \frac{1}{r}\langle y - z_n, z_n - x_n\rangle \\ &= -(z_n^1)^2 - (z_n^2)^2 + (y_1)^2 + (y_2)^2 + \frac{1}{5}z_n^1(-z_n^1 + y_1) + \frac{1}{5}z_n^2(-z_n^2 + y_2) \\ &\quad + (y_1 - z_n^1)(z_n^1 - x_n^1) + (y_2 - z_n^2)(z_n^2 - x_n^2) \\ &= \left((y_1)^2 + \left(-x_n^1 + \frac{6}{5}z_n^1\right)y_1 + x_n^1 z_n^1 - \frac{11}{5}(z_n^1)^2\right) \\ &\quad + \left((y_2)^2 + \left(-x_n^2 + \frac{6}{5}z_n^2\right)y_2 + x_n^2 z_n^2 - \frac{11}{5}(z_n^2)^2\right) \\ &= G_1(y_1) + G_2(y_2). \end{aligned}$$

Let $G_1(y_1) = (y_1)^2 + (-x_n^1 + \frac{6}{5}z_n^1)y_1 + x_n^1 z_n^1 - \frac{11}{5}(z_n^1)^2$ and $G_2(y_2) = (y_2)^2 + (-x_n^2 + \frac{6}{5}z_n^2)y_2 + x_n^2 z_n^2 - \frac{11}{5}(z_n^2)^2$. $G_1(y_1)$ and $G_2(y_2)$ are quadratic functions with coefficients $a_1 = 1$, $b_1 = -x_n^1 + \frac{6}{5}z_n^1$, and $c_1 = x_n^1 z_n^1 - \frac{11}{5}(z_n^1)^2$ of $G_1(y_1)$ and coefficients $a_2 = 1$, $b_2 = -x_n^2 + \frac{6}{5}z_n^2$, and $c_2 = x_n^2 z_n^2 - \frac{11}{5}(z_n^2)^2$ of $G_2(y_2)$, respectively. Determine the discriminant Δ_1 of G_1 as follows:

$$\begin{aligned} \Delta_1 &= b_1^2 - 4a_1c_1 \\ &= \left(-x_n^1 + \frac{6}{5}z_n^1\right)^2 - 4(1)\left(x_n^1 z_n^1 - \frac{11}{5}(z_n^1)^2\right) = \frac{1}{25}(5x_n^1 - 16z_n^1)^2. \end{aligned}$$

We know that $G_1(y_1) \ge 0$, $\forall y \in \mathbb{R}$. If it has most one solution in $\mathbb{R}$, then $\Delta_1 \le 0$, so we obtain $z_n^1 = \frac{5x_n^1}{16}$. Next, we determine the discriminant Δ_2 of G_2 by using the same method as above, we obtain $z_n^2 = \frac{5x_n^2}{16}$. That is $T_r^{F_1}(I - rf_1)z_n = (\frac{5x_n^1}{16}, \frac{5x_n^2}{16})$. After that, we find the solution of $u_n = (u_n^1, u_n^2)$ in this inequality $0 \le F_2(u_n, y) + \langle f_2(u_n), y - u_n\rangle + \frac{1}{s}\langle y - u_n, u_n - x_n\rangle$. By using the same method as $z_n = (z_n^1, z_n^2)$, we obtain

$$u_n = (u_n^1, u_n^2) = \left(\frac{7x_n^1}{51}, \frac{7x_n^2}{51}\right). \tag{5.1}$$

That is, $T_s^{F_2}(I - sf_2)u_n = (\frac{7x_n^1}{51}, \frac{7x_n^2}{51})$.

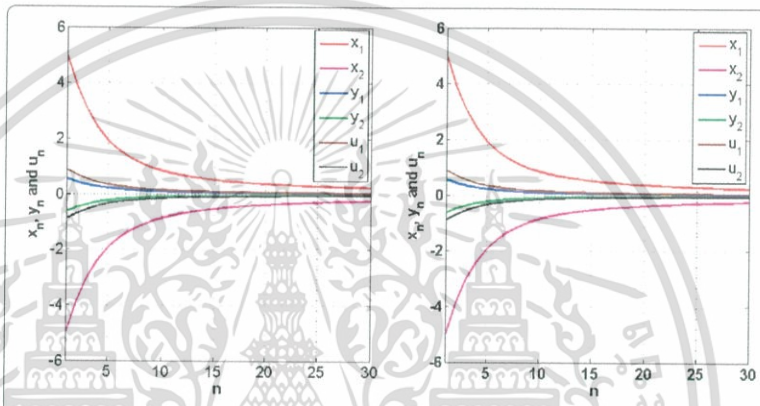
Let $x_1 = (x_1^1, x_1^2)$ and $u = (u_1, u_2) \in \mathbb{R}^2$. The sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are generated by (3.6), where $k_i = \frac{6}{7^i} + \frac{1}{N7^N}$, $d_1 = 1$, $d_2 = 1$, $a = 0.5$, $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{7n-4}{12n}$, $\gamma_n = \frac{5n-2}{12n}$ and $\lambda_n = \frac{1}{2n^2}$ for all $n \in \mathbb{N}$. Since $L = 5$, we choose $\gamma = 0.1$. From the definition of D_1, D_2, T_i, F_1, F_2, f_1 and f_2, we have $VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega = \{0\}$. From Theorem 3.4, we can conclude that the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to 0. We can rewrite (3.6) as follows:

$$\begin{cases} u_n = T_1^{F_1}(I - f_1)(x_n + 0.1A^*(T_{0.5}^{F_2}(I - 0.5f_2) - I)Ax_n), \\ y_n = (I - D_1)(0.5u_n + 0.5(I - D_2)u_n), \\ x_{n+1} = \frac{1}{2n}u + \frac{7n-4}{12n}x_n + \frac{5n-2}{12n}(y_n - ((\frac{1}{2n^2})(\sum_{i=1}^N(\frac{6}{7^i} + \frac{1}{N7^N})(y_n - T_i y_n)))), \end{cases}$$

for all $n \ge 1$, where $x_n = (x_n^1, x_n^2)$, $y_n = (y_n^1, y_n^2)$ and $u_n = (u_n^1, u_n^2)$.

Table 1 The values of $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ where $u = (5,-5)$, $x_1 = (5,-5)$ and $n = 30$

	$N = 1$			$N = 20$		
n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$	$u_n = (u_n^1, u_n^2)$	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$	$u_n = (u_n^1, u_n^2)$
1	(5.0000, −5.0000)	(0.5553, −0.6170)	(0.8885, −0.8885)	(5.0000, −5.0000)	(0.5553, −0.6170)	(0.8885, −0.8885)
2	(3.8715, −3.8850)	(0.4300, −0.4794)	(0.6879, −0.6903)	(3.8700, −3.8833)	(0.4298, −0.4792)	(0.6877, −0.6901)
⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	(0.5189, −0.5274)	(0.0576, −0.0651)	(0.0922, −0.0937)	(0.5189, −0.5274)	(0.0576, −0.0651)	(0.0922, −0.0937)
⋮	⋮	⋮	⋮	⋮	⋮	⋮
29	(0.2485, −0.2522)	(0.0276, −0.0311)	(0.0442, −0.0448)	(0.2485, −0.2522)	(0.0276, −0.0311)	(0.0442, −0.0448)
30	(0.2397, −0.2432)	(0.0266, −0.0300)	(0.0426, −0.0432)	(0.2397, −0.2432)	(0.0266, −0.0300)	(0.0426, −0.0432)



(a) $N = 1$ (b) $N = 20$

Figure 1 The convergence comparison with different values N

Table 1 shows the values of sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ where $u = (5,-5)$, $x_1 = (5,-5)$ and $n = 30$.

6 Conclusion

1. Example 5.1 is an example in infinite dimensional Hilbert space for supporting Theorem 3.4
2. Table 1 and Fig. 1 in Example 5.2 show that the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge to 0, where $\{0\} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^{N} F(T_i) \cap \Omega$.
3. Theorem 3.4 guarantees the convergence of $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ in Example 5.1 and Example 5.2.
4. By using the concept of Picard iteration, Wang [13] defined the iterative scheme $\{x_n\}$ for solving SCFPP as follows:

$$\begin{aligned} x_{n+1} &= x_n - \rho_n\big((I-U)x_n + A^*(I-T)Ax_n\big) \\ &= \big(I - \rho_n\big((I-U) + A^*(I-T)A\big)\big)x_n, \end{aligned} \tag{6.1}$$

where ρ_n is according to (1.4) and U and T are firmly quasi-nonexpansive mappings. Then the sequence $\{x_n\}$ converges weakly to z, where $z = \lim_{n\to\infty} P_\Phi x_n$.

In Theorem 3.4, we use the concept of Halpern iteration and suitable conditions of the parameters $d_1, d_2, r, s, a, \gamma, L, \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$, the sequence $\{x_n\}$ defined by (3.6) converges strongly to $z = P_{\mathcal{F}}u$, which is a different method from (6.1).

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Competing interests
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Authors' contributions
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