TWO-DIMENSIONAL LATITUDINALLY AVERAGED MATHEMATICAL MODELS FOR LONG-TERM CONTAMINATED GROUNDWATER POLLUTANT MEASUREMENT AROUND A LANDFILL

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Thesis Title	Two-dimensional Latitudinally Averaged Mathematical Models for Long-term Contaminated Groundwater Pollution Measurement around a Land fill
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Abstract

The disposal of liquid waste on a landfill is a source of polluted groundwater. Many people in rural regions depends on well water as their major supply of drinking water. This well water might be polluted by landfill groundwater. A two-dimensional mathematical model for measuring long-term polluted groundwater contamination surrounding a land fill will be presented in this thesis. A mixture of two models governs the system. The first model is a two-dimensional transient groundwater flow model that analyzes the hydraulic head of the groundwater. The second model is a transient two-dimensional advection-diffusion equation that provides the groundwater pollutant concentration. The hydraulic head and groundwater pollutant concentration will be approximated using the presented finite difference techniques. When each simulated zone becomes a hazardous zone or a protected zone, the simulations may be utilized to notify awareness. The significant groundwater quality component identified by our proposed simulations is the leachate pollutant concentration surrounding the landfill. The landfill's chloride discharge has an impact on groundwater quality. The total chloride concentration, hypochlorite concentration, chlorite concentration, chlorate concentration, and perchlorate concentration are all provided by the chloride compound dispersion model groundwater pollutant concentration. The important groundwater quality factors discovered in our simulation are pollution concentration levels surrounding the landfill and transform rate.

Keywords : Groundwater pollution, Landfill, Contamination, Finite difference method, Two-dimensional.

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Chapter 1

Introduction

1.1 Research Motivation

In the present, water is the most vital element of our life. Not only is it essential to our health, but also use it for numerous household tasks, every day we use water for cooking, cleaning, bathing and drinking, but do we think about its source?, where does its come from. There are two main source of water that is surface water and groundwater. Surface water can be found in rivers, lakes and reservoirs. Groundwater lies under the surface of the land, where it flow through the gap between rocks. The rocks store and transmit groundwater are called aquifers. Groundwater must be pumped from an aquifer to the earth's surface for use.

Figure 1.1: Picture of surface water

https://www.xylem.com/en-th/solutions/environmental-monitoring-analysis/

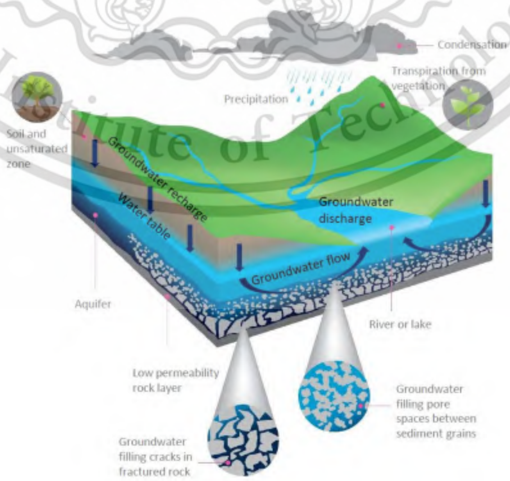

Figure 1.2: Picture of groundwater

http://www.waterpolitics.com/2019/02/13/groundwater-take-more-than-a-century-to-adapt-to-climate-change/

The question, how do we know it is safe to drink?, surface water and groundwater can be contaminated by some factor. Some contaminants occur in nature that may present a health risk if they are found in drinking water. These contaminants include uranium, radium, nitrate, chloride, bacteria and viruses. Many of these contaminants are naturally present in rock and consequently end up in the water supply.

Other sources of contamination are a result of human activities such as manufacturing, agriculture, or individual misuse. The following activities may cause harmful chemicals to enter the well water owner's water supply.

- Leakage from waste disposal, treatment, or storage sites.
- Discharges from factories, industrial sites, or sewage treatment facilities.
- Leakage from underground storage tanks

We can see that, groundwater can be contaminated from any factor and any where. In this research, we interested in the case of groundwater contaminated by the leachate from landfill, then, we need to know what the landfill and leachate is.

A landfill are the site for used to waste management purposes, such as temporary storage, consolidation and transfer, or for various stages of processing waste material, such as sorting, treatment, or recycling. Landfill used to environment protection from harmful effect of contaminated surface water infiltration into the groundwater.

Leachate are occur from water percolates through the garbage. This liquid is extremely harmful and can contaminate to any way such as air, soil and water.

Figure 1.3: Picture of landfill

https://envirotecmagazine.com/2019/01/11/ microbiologists-set-sights-on-landfill-bacteria/

Figure 1.4: Picture of leachate

https://greentechlead.com/news/ nf-energy-develops-landfill-leachate-disposal-solution-16325

The effect of drinking contaminated or dirty water causes waterborne disease. Contaminated water can cause many types of diarrheal diseases, including Cholera, and other serious illnesses, such as Guinea worm disease, Typhoid, Dysentery and high blood pressure, to avoid this problem, we used mathematical models to explain groundwater contamination.

1.2 Literature Review

France PW (1974) talking about the finite elements analysis of three-dimensional groundwater flow model. The Galerkin approach and cubic isoparametric elements are used to simulate the flow domain as these permit accurate modeling of curved boundaries.

McDonald MG and Harbaugh AW (1988) discussed a modular three-dimensional finite difference groundwater flow model. The report includes detailed explanations of physical and mathematical concepts on which the model is based and an explanation of how those concepts are incorporated in the modular structure of the computer program.

Trecott PC and Larson SP (1997) solved the three-dimensional groundwater flow equations using the strongly implicit procedure.

Bakker M (1999) presented a simulating groundwater flow in multi-aquifer systems with analytical and numerical Dupuit-models. The analytic Dupuit solutions for multi-aquifer flow are compared to exact solutions for two-dimensional flow in the vertical plane to determine the error in discharge that is introduced by adopting the Dupuit approximation.

Gardenas (2005) talking about a two-dimensional modeling approach to improve fertigation strategies and soil types on nitrate leaching potential.

K. Halil and M. Tammer Ayvaz (2005) describe a transient groundwater modeling by using spreadsheets. The models suppose variable as data in considered area to description their problem.

Husam Baalousha (2008) created groundwater modeling for groundwater management and remediation. Models are predict future behavior.

Pongnu N and Pochai N. (2017) presented the numerical simulation of groundwater measurement using alternating direction methods.

1.3 Objectives of the Study

1) To propose two-dimensional latitudinally averaged groundwater flow model that can describe groundwater hydraulic head.
2) To propose velocity potential model in two-dimensional vector fields that can explain direction of groundwater flow.
3) To propose two-dimensional latitudinally averaged groundwater pollution dispersion model that can describe concentration of groundwater pollutant.
4) To propose numerical method for solving their considered models by using the finite difference method.
5) To improve the models for trend protection of groundwater contamination.
6) To classify the consideration area into 3 classes such as the contaminated zone, the agricultural zone and the safety zone.

1.4 Scope of the Study

1) We will consider two-dimensional latitudinally averaged groundwater flow model.
2) We will consider model of velocity potential model in two-dimensional vector fields.
3) We will consider two-dimensional latitudinally averaged groundwater pollution dispersion model.
4) We will consider the models in homogeneous and heterogeneous aquifer.
5) We will propose numerical experiments by using the forward time centered.
6) We will propose numerical experiments and construct a computer program to support the numerical solution.

1.5 Research methodology

1) To study groundwater problems and effect.
2) To study about groundwater models from textbooks and related researches.
3) To propose two-dimensional latitudinally averaged groundwater flow model.
4) To propose velocity potential model in two-dimensional vector fields.
5) To propose two-dimensional latitudinally averaged groundwater pollution dispersion model.
6) To study numerical method from textbooks and related researches.
7) To propose numerical method for solving by using the finite difference methods, the forward time centered space method.
8) To write a mathematical program.
9) To create and give numerical examples to support our models.
10) To analyze the simulation results.
11) To write the thesis.

1.6 Benefits of the Study

This proposed study in thesis is to develop the mathematical models for simulation groundwater system. These simulations can represent the behaviors of hydraulic head, direction of groundwater flow and concentration of groundwater pollution. Those techniques have received acceptance that are efficient application to use in realistic scenarios. This study can take to apply for waste management in groundwater in order to prevent the groundwater pollution concentration.

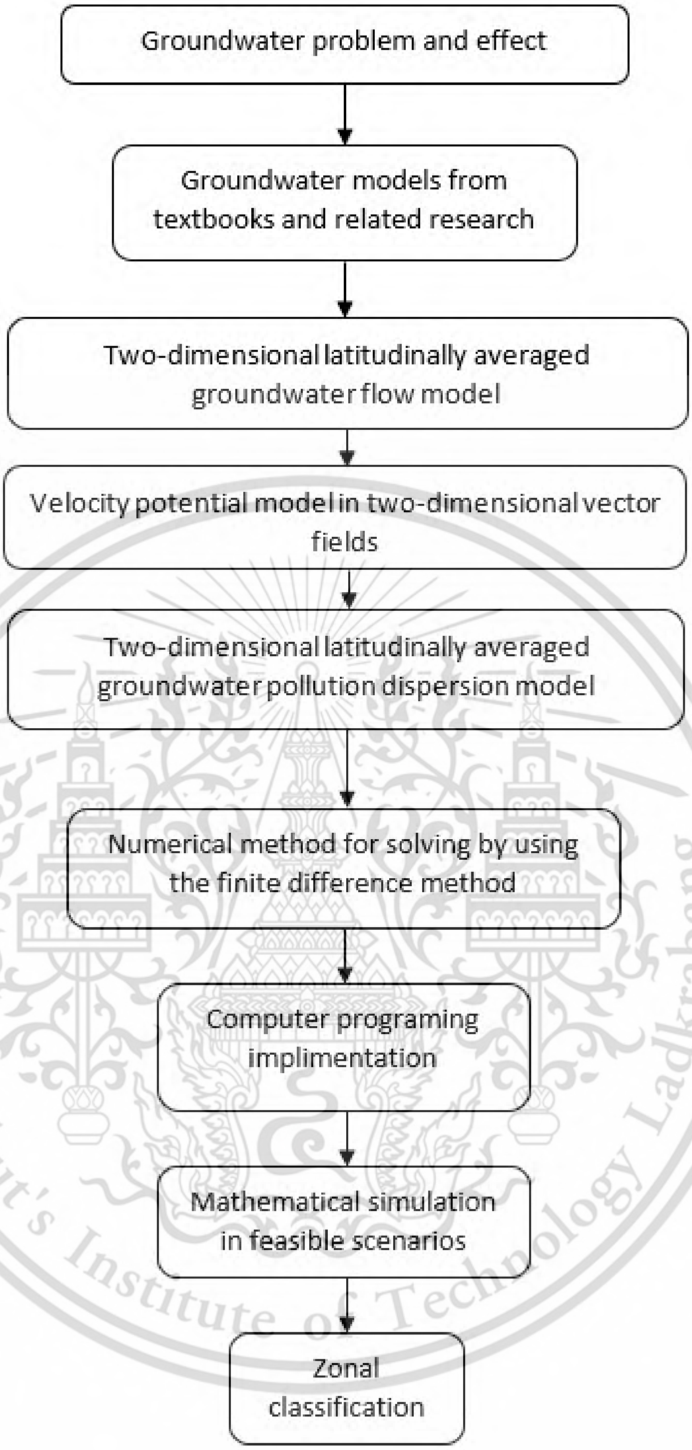

Figure 1.5: Plan of thesis

Chapter 2

Groundwater models

2.1 Mathematical models

The value of groundwater or the groundwater characteristics can be estimated by mathematical models. The groundwater model results to groundwater specific character of groundwater for prediction groundwater system. The groundwater flow through soil is governed by the Darcy's law that be described by partial differential equation.

2.1.1 A two-dimensional latitudinally averaged groundwater flow model

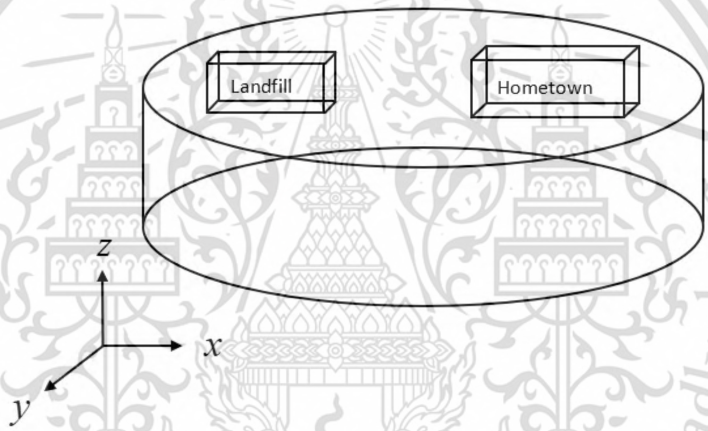

Figure 2.1: General structure of problem domain in top view.

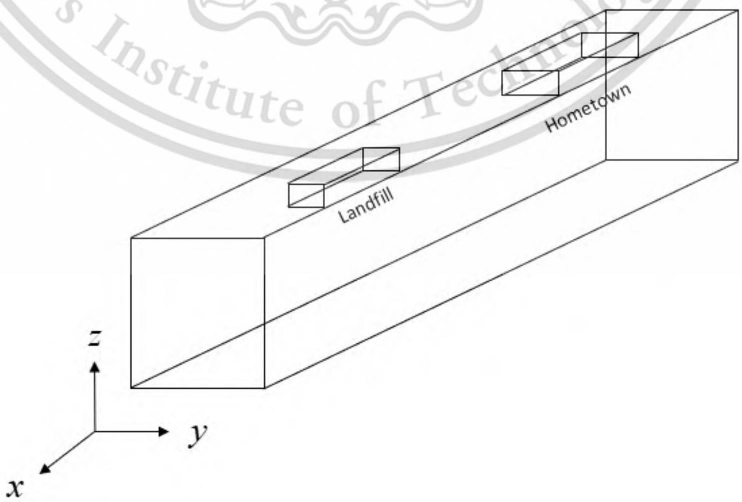

Figure 2.2: General structure of problem domain in side view.

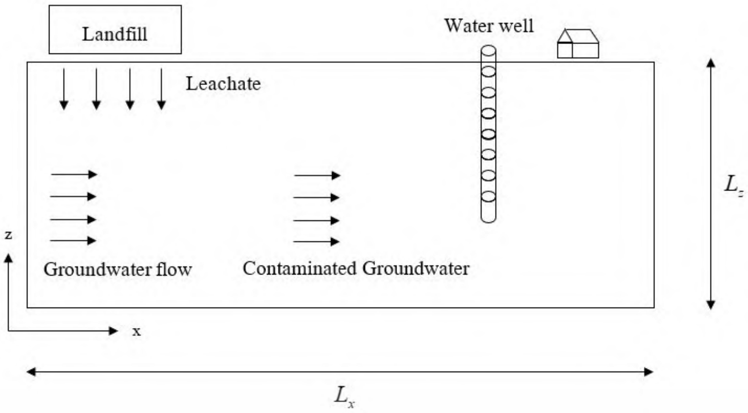

Figure 2.3: General structure of problem domain in full side view.

The governing equation of a latitudinally average integrated Darcy's flow in a two-dimensional is:

$$S\frac{\partial H(x,z,t)}{\partial t} = \frac{\partial}{\partial x}\left(K_x\frac{\partial H(x,z,t)}{\partial x}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial H(x,z,t)}{\partial z}\right), \tag{2.1}$$

where $H(x,z,t)$ is the hydraulic head(metre), S matrix of specific storage(1/metre), L_x is the considered area length, L_z is the depth of considered groundwater area, T is the stationary time of simulation as shown in Figure 3.1. The hydraulic conductivity(metre/day) component in the x, z directions are denoted by K_x, K_z, respectively. Assuming that the soil topography in the considered area is homogeneous, these mean that the hydraulic conductivity are constant.

$$S\frac{\partial H(x,z,t)}{\partial t} = K_x\frac{\partial^2 H(x,z,t)}{\partial x} + K_z\frac{\partial^2 H(x,z,t)}{\partial z}. \tag{2.2}$$

for all $(x,z,t) \in \Omega$ such that $\Omega = [0, L_x] \times [0, L_z] \times [0, T]$.

2.1.2 Groundwater flow velocity model

We can obtain that the groundwater flow velocity in x-direction is a decreasing rate of change of the hydraulic head x-direction,

$$u = -\frac{\partial H}{\partial x}. \tag{2.3}$$

Similarly, the groundwater flow velocity in z-direction is a decreasing rate of change of the hydraulic head in z-direction,

$$w = -\frac{\partial H}{\partial z}. \tag{2.4}$$

2.1.3 A two-dimensional latitudinally average groundwater pollution dispersion model

An advection-diffusion model provides a continuous description of groundwater pollutant transport in the groundwater. A two-dimensional latitudinally averaged groundwater pollution dispersion model is:

$$\frac{\partial c(x,z,t)}{\partial t} + u\frac{\partial c(x,z,t)}{\partial x} + v\frac{\partial c(x,z,t)}{\partial z} \;=\; D_x\frac{\partial^2 c(x,z,t)}{\partial x^2} + D_z\frac{\partial^2 c(x,z,t)}{\partial z^2} + Q,$$

for all $(x,z,t) \in \Omega$ such that $\Omega = [0, L_x] \times [0, L_z] \times [0, T]$, where $c(x,z,t)$ is the groundwater pollutant concentration(kg/m^3), D_x, D_z are the diffusion coefficient in x- and z-directions, $u(x,z,t)$, $v(x,z,t)$ are the groundwater flow velocity in the x- and z-directions and Q is groundwater pollutant source or sink function by contaminators.

2.2 The initial and boundary conditions

The partial differential equations are solved by using a finite difference method consisting of the applicable governing flow equation, boundary conditions, and initial conditions. The initial and boundary conditions are defined by an interpolation function of measured raw data

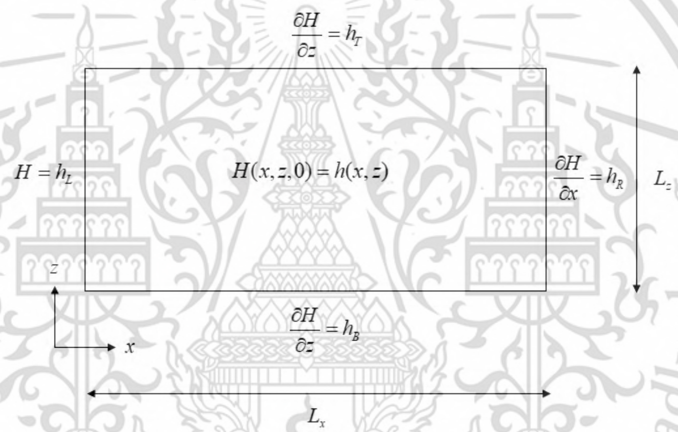

Figure 2.4: The boundary condition of groundwater flow model.

2.2.1 Initial and boundary conditions of a two-dimensional latutidinally average groundwater flow model

Initial condition If the potential hydraulic head in the area is static, the initial condition is assumed by

$$H(x,z,0) \;=\; h(x,z), \tag{2.5}$$

where $h(x,z)$ is a given potential hydraulic head function in the considered area.

Boundary conditions The top, right and bottom boundary conditions are assumed by the averaged rate of change of hydraulic head around the top, right and bottom boundaries. The left boundary condition is assumed by the interpolation function of measured raw data in the considered landfill as shown in Fig 2.1. The boundary condition, are also assumed by

$$H(0,z,t) = h_L, \text{ for all } 0 \le z \le I_z, t > 0 \tag{2.6}$$
$$\frac{\partial H(x,I_z,t)}{\partial z} = h_T, \text{ for all } 0 \le x \le I_x, t > 0 \tag{2.7}$$
$$\frac{\partial H(I_x,z,t)}{\partial x} = h_R, \text{ for all } 0 \le z \le I_z, t > 0 \tag{2.8}$$
$$\frac{\partial H(x,0,t)}{\partial z} = h_B, \text{ for all } 0 \le x \le I_x, t > 0 \tag{2.9}$$

where h_L, h_T, h_R and h_B are the boundary source of hydraulic head on the left boundary domain. The rate of change hydraulic head with respect to domain boundaries around the top, the bottom and the right bottom around the considered area as shown in Figure 2.2, respectively.

2.2.2 Initial and boundary conditions of a two-dimensional latitudinally average groundwater pollution dispersion model

Initial condition The potential groundwater pollutant concentration in the considered area is described by

$$c(x,z,0) \;=\; c_0(x,z). \tag{2.10}$$

where $c_0(x,z)$ is a averaged potential groundwater pollutant concentration function in the considered area.

Boundary conditions The left boundary condition is assumed by the interpolation function of measured raw data at the considered landfill. The top, right and bottom boundary conditions are assumed by the averaged rates of change of pollutant concentration around the top, right and bottom boundaries. The boundary conditions are also assumed by

$$c(L_z,x,t) = g_N, \text{ for all } k_1L_x \le x \le k_2L_x, t > 0 \tag{2.11}$$
$$\frac{\partial c(z,0,t)}{dx} = g_L, \text{ for all } 0 \le z \le L_z, t > 0 \tag{2.12}$$
$$\frac{\partial c(L_z,x,t)}{dz} = g_T, \text{ for all } x \in [0,k_1L_x) \cup (k_2L_x, L_x], t > 0 \tag{2.13}$$
$$\frac{\partial c(z,L_x,t)}{dx} = g_R, \text{ for all } 0 \le z \le L_z, t > 0 \tag{2.14}$$
$$\frac{\partial c(x,0,t)}{dz} = g_B, \text{ for all } 0 \le x \le L_x, t > 0 \tag{2.15}$$

where k_1L_x and k_2L_x are referred to the range area of the groundwater pollutant area source and g_L, g_T, g_R and g_B are the rate of change pollutant concentration with respect distance around the top, the bottom and the right boundaries along the considered area, respectively.

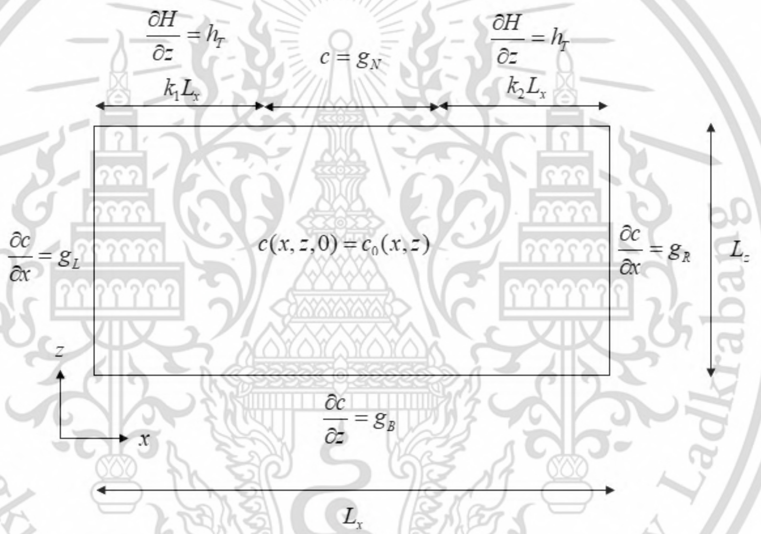

Figure 2.5: The boundary condition of groundwater pollutant dispersion model.

Chapter 3

Explicit finite difference techniques for a long-term groundwater quality assessment model

Explicit finite difference technique is proposed, the forward time central space method.

3.1 Forward Time Centered Space technique with mathematics models

The groundwater flow through soil is governed by the Darcy's law that can be described by the partial differential equation. The governing equation of Darcy's flow is:

$$S\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(K_x\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial H}{\partial z}\right), \tag{3.1}$$

This equation approximated solutions by using finite difference techniques. The following in a two-dimensional latitudinally averaged groundwater flow model (3.1) can be generated instead of using the forward time centered space technique,

$$\frac{\partial H}{\partial t} \simeq \frac{H_{i,j}^{n+1} - H_{i,j}^{n}}{\Delta t}, \tag{3.2}$$

$$\frac{\partial^2 H}{\partial x^2} \simeq \frac{H_{i,j-1}^{n} - 2H_{i,j}^{n} + H_{i,j+1}^{n}}{(\Delta x)^2}, \tag{3.3}$$

$$\frac{\partial^2 H}{\partial z^2} \simeq \frac{H_{i-1,j}^{n} - 2H_{i,j}^{n} + H_{i+1,j}^{n}}{(\Delta z)^2}. \tag{3.4}$$

Substituting (3.2)-(3.4) into (3.1). That is

$$S\left(\frac{H_{i,j}^{n+1} - H_{i,j}^{n}}{\Delta t}\right) = K_x\left(\frac{H_{i,j-1}^{n} - 2H_{i,j}^{n} + H_{i,j+1}^{n}}{(\Delta x)^2}\right) + K_z\left(\frac{H_{i-1,j}^{n} - 2H_{i,j}^{n} + H_{i+1,j}^{n}}{(\Delta z)^2}\right) \tag{3.5}$$

for all $0 \leq i \leq M_z, 0 \leq j \leq M_x$ and $0 \leq n \leq N$. Then the explicit finite difference equation becomes

$$H_{i,j}^{n+1} = \alpha H_{i,j-1}^{n} + \alpha H_{i,j+1}^{n} + (1 - 2\alpha - 2\beta)H_{i,j}^{n} + \beta H_{i-1,j}^{n} + \beta H_{i+1,j}^{n}, \tag{3.6}$$

where $\alpha = \dfrac{K_x(\Delta t)}{S(\Delta x)^2}$ and $\beta = \dfrac{K_z(\Delta t)}{S(\Delta z)^2}$.

The forward space method is introduce to the velocity potential model in two-dimension velocity field

$$u = -\frac{\partial H}{\partial x}, \tag{3.7}$$

$$v = -\frac{\partial H}{\partial z}. \tag{3.8}$$

By the forward space technique is form

$$\frac{\partial H}{\partial x} \simeq \frac{H^n_{i,j+1} - H^n_{i,j}}{\Delta x}, \tag{3.9}$$

$$\frac{\partial H}{\partial z} \simeq \frac{H^n_{i+1,j} - H^n_{i,j}}{\Delta z}. \tag{3.10}$$

Substituting (3.9) - (3.10) into (3.7) and (3.8). That is

$$u^n_{i,j} = -\frac{1}{\Delta x}(H^n_{i,j+1} - H^n_{i,j}), \tag{3.11}$$

$$v^n_{i,j} = -\frac{1}{\Delta z}(H^n_{i+1,j} - H^n_{i,j}). \tag{3.12}$$

where H is the approximated hydraulic head.

The following discretization in two-dimensional latitudinally averaged groundwater pollution dispersion model

$$\frac{\partial c}{\partial t} = -u\frac{\partial c}{\partial x} - v\frac{\partial c}{\partial z} + D_x\frac{\partial^2 c}{\partial x^2} + D_z\frac{\partial^2 c}{\partial z^2} + Q, \tag{3.13}$$

can be generated instead of using the forward time centered space technique,

$$c(x,z,t) \simeq C^n_{i,j},$$

$$\frac{\partial c}{\partial t} \simeq \frac{C^{n+1}_{i,j} - C^n_{i,j}}{\Delta t}, \tag{3.14}$$

$$\frac{\partial c}{\partial x} \simeq \frac{C^n_{i,j-1} - C^n_{i,j+1}}{2\Delta x}, \tag{3.15}$$

$$\frac{\partial c}{\partial z} \simeq \frac{C^n_{i-1,j} - C^n_{i+1,j}}{2\Delta z}, \tag{3.16}$$

$$\frac{\partial^2 c}{\partial x^2} \simeq \frac{C^n_{i,j-1} - 2C^n_{i,j} + C^n_{i,j+1}}{(\Delta x)^2}, \tag{3.17}$$

$$\frac{\partial^2 c}{\partial z^2} \simeq \frac{C^n_{i-1,j} - 2C^n_{i,j} + C^n_{i+1,j}}{(\Delta z)^2}. \tag{3.18}$$

Substituting (4.2) - (3.18) into (4.1). That is

$$\begin{aligned}\frac{C^{n+1}_{i,j} - C^n_{i,j}}{\Delta t} &= -u^n_{i,j}\left(\frac{C^n_{i+1,j} - C^n_{i-1,j}}{2\Delta x}\right) - v^n_{i,j}\left(\frac{C^n_{i,j+1} - C^n_{i,j-1}}{2\Delta z}\right)\\ &+ D_x\left(\frac{C^n_{i,j-1} - 2C^n_{i,j} + C^n_{i,j+1}}{(\Delta x)^2}\right) + D_z\left(\frac{C^n_{i-1,j} - 2C^n_{i,j} + C^n_{i+1,j}}{(\Delta z)^2}\right) + Q\end{aligned} \tag{3.19}$$

for all $0 \le i \le M_z, 0 \le j \le M_x$ and $0 \le n \le N$. Then the explicit finite difference equation becomes

$$\begin{aligned}C^{n+1}_{i,j} &= (\tau_1 + \tau_2)\,C^n_{i-1,j} + (\lambda_1 + \lambda_2)\,C^n_{i,j-1} + (1 - 2\lambda_1 - 2\tau_1)C^n_{i,j}\\ &+ (\tau_1 - \tau_2)\,C^n_{i+1,j} + (\lambda_1 - \lambda_2)\,C^n_{i,j+1} + Q\Delta t,\end{aligned} \tag{3.20}$$

where $\lambda_1 = \frac{D_x \Delta t}{(\Delta x)^2}$, $\lambda_2 = \frac{u^n_{i,j}\Delta t}{2\Delta x}$, $\tau_1 = \frac{D_z \Delta t}{(\Delta z)^2}$. and $\tau_2 = \frac{v^n_{i,j}\Delta t}{2\Delta z}$.

3.1.1 Apply their boundary conditions

The approximate of a two-dimensional latitudinally averaged groundwater flow model (3.1) having unknown value on the boundaries,

3.1.2 Forward time centered space technique with a forward space technique to approximate the boundaries solution in groundwater flow model

For $i = 0, 0 < j < M_x$ the approximated fictitious points on the boundaries, are obtained by

$$H^n_{-1,0} = H^n_{0,0} - h_B \Delta z. \tag{3.21}$$

Substituting (3.23) into (3.6), that is

$$\begin{aligned} H^{n+1}_{0,j} &= \alpha H^n_{0,j-1} + \alpha H^n_{0,j+1} + (1 - 2\alpha - \beta) H^n_{0,j} \\ &\quad + \beta H^n_{1,j} - \beta h_B \Delta z. \end{aligned} \tag{3.22}$$

For $i = 0, j = M_x$ the approximated fictitious points on the boundaries, are obtained by

$$H^n_{-1,M_x} = H^n_{0,M_x} - h_B \Delta z, \tag{3.23}$$

$$H^n_{0,M_x+1} = H^n_{0,M_x} + h_R \Delta x. \tag{3.24}$$

Substituting (3.23) - (3.24) into (3.6), that is

$$\begin{aligned} H^{n+1}_{0,M_x} &= \alpha H^n_{0,M_x-1} + (1 - 2\alpha - 2\beta) H^n_{0,M_x} - \beta h_B \Delta z \\ &\quad + \beta H^n_{1,M_x} + \alpha h_R \Delta x. \end{aligned} \tag{3.25}$$

For $0 < i < M_z, j = M_x$ the approximated fictitious points on the boundaries, are obtained by

$$H^n_{i,M_x+1} = H^n_{i,M_x} + h_R \Delta x. \tag{3.26}$$

Substituting (3.24) into (3.6), that is

$$\begin{aligned} H^{n+1}_{i,M_x} &= \alpha H^n_{i,M_x-1} + (1 - \alpha - 2\beta) H^n_{i,M_x} + \beta H^n_{i-1,M_x} \\ &\quad + \beta H^n_{i+1,M_x} + \alpha f_R(z) \Delta x. \end{aligned} \tag{3.27}$$

For $i = M_z, j = M_x$ the approximated fictitious points on the boundaries, are obtained by

$$H^n_{M_z+1,M_x} = H^n_{M_z,M_x} + h_T \Delta z, \tag{3.28}$$

$$H^n_{M_z,M_x+1} = H^n_{M_z,M_x} + h_R \Delta x. \tag{3.29}$$

Substituting (3.31) - (3.29) into (3.6), that is

$$\begin{aligned} H^{n+1}_{M_z,M_x} &= \alpha H^n_{M_z,M_x-1} + (1 - 2\alpha - 2\beta) H^n_{M_z,M_x} + \beta H^n_{M_z-1,M_x} \\ &\quad + \beta h_T \Delta z + \alpha h_R(z) \Delta x. \end{aligned} \tag{3.30}$$

For $i = M_z, 0 < j < M_x$ the approximated fictitious points on the boundaries, are obtained by

$$H^{n+1}_{M_z,j} = H^{n}_{M_z,j} + h_T \Delta z. \tag{3.31}$$

Substituting (3.31) into (3.6), that is

$$\begin{aligned} H^{n}_{M_z+1,j} &= \alpha H^{n}_{M_z,j-1} + \alpha H^{n}_{M_z,j+1} + (1-2\alpha-\beta) H^{n}_{M_z,j} \\ &\quad + \beta H^{n}_{M_z-1,j} + \beta h_T \Delta z. \end{aligned} \tag{3.32}$$

3.2 Forward time centered space technique with a forward space technique to approximate the boundaries solution in groundwater dispersion model

For $i = 0, j = 0$ the approximated fictitious points on the boundaries, are obtained by

$$C^{n}_{0,-1} = C^{n}_{0,0} - g_L \Delta x, \tag{3.33}$$

$$C^{n}_{-1,0} = C^{n}_{0,0} - g_B \Delta z. \tag{3.34}$$

Substituting (3.51) - (3.39) into (3.20), that is

$$\begin{aligned} C^{n+1}_{0,0} &= (\tau_1+\tau_2)\,(C^{n}_{0,0} - g_B \Delta z) + (\lambda_1+\lambda_2)\,(C^{n}_{0,0} - g_L \Delta x) + (1-2\lambda_1-2\tau_1) C^{n}_{0,0} \\ &\quad + (\tau_1-\tau_2)\,C^{n}_{1,0} + (\lambda_1-\lambda_2)\,C^{n}_{0,1} + Q\Delta t. \end{aligned} \tag{3.35}$$

For $i = 0, 0 < j < M_x$ the approximated fictitious points on the boundaries, are obtained by

$$C^{n}_{-1,j} = C^{n}_{0,j} - g_B \Delta z. \tag{3.36}$$

Substituting (3.39) into (3.20), that is

$$\begin{aligned} C^{n+1}_{0,j} &= (\tau_1+\tau_2)\,(C^{n}_{0,j} - g_B \Delta z) + (\lambda_1+\lambda_2)\,C^{n}_{0,j-1} + (1-2\lambda_1-2\tau_1) C^{n}_{0,j} \\ &\quad + (\tau_1-\tau_2)\,C^{n}_{1,j} + (\lambda_1-\lambda_2)\,C^{n}_{0,j+1} + Q\Delta t. \end{aligned} \tag{3.37}$$

For $i = 0, j = M_x$ the approximated fictitious points on the boundaries, are obtained by

$$C^{n}_{0,M_x+1} = C^{n}_{0,M_x} + g_R \Delta x, \tag{3.38}$$

$$C^{n}_{-1,M_x} = C^{n}_{0,M_x} - g_B \Delta z. \tag{3.39}$$

Substituting (3.43) - (3.39) into (3.20), that is

$$\begin{aligned} C^{n+1}_{0,M_x} &= (\tau_1+\tau_2)\,(C^{n}_{0,M_x} - g_B \Delta z) + (\lambda_1+\lambda_2)\,C^{n}_{0,M_x-1} + (1-2\lambda_1-2\tau_1) C^{n}_{0,M_x} \\ &\quad + (\tau_1-\tau_2)\,C^{n}_{1,M_x} + (\lambda_1-\lambda_2)\,(C^{n}_{0,M_x} + g_R \Delta x) + Q\Delta t. \end{aligned} \tag{3.40}$$

For $0 < i < M_z, j = M_x$ the approximated fictitious points on the boundaries, are obtained by

$$C^n_{i,M_x+1} = C^n_{i,M_x} + g_R\Delta x. \tag{3.41}$$

Substituting (3.43) into (3.20), that is

$$\begin{aligned} C^{n+1}_{i,M_x} &= (\tau_1+\tau_2)\,C^n_{i-1,M_x} + (\lambda_1+\lambda_2)\,C^n_{i,M_x-1} + (1-2\lambda_1-2\tau_1)C^n_{i,M_x} \\ &\quad + (\tau_1-\tau_2)\,C^n_{i+1,M_x} + (\lambda_1-\lambda_2)\,(C^n_{i,M_x}+g_R\Delta x) + Q\Delta t. \end{aligned} \tag{3.42}$$

For $i = M_z, j = M_x$ the approximated fictitious points on the boundaries, are obtained by

$$C^n_{M_z,M_x+1} = C^n_{M_z,M_x} + g_R\Delta x. \tag{3.43}$$

$$C^n_{M_z+1,M_x} = C^n_{M_z,M_x} + g_T\Delta z. \tag{3.44}$$

Substituting (3.43) - (3.48) into (3.20), that is

$$\begin{aligned} C^{n+1}_{M_z,M_x} &= (\tau_1+\tau_2)\,C^n_{M_z-1,M_x} + (\lambda_1+\lambda_2)\,C^n_{M_z,M_x-1} + (1-2\lambda_1-2\tau_1)C^n_{M_z,M_x} \\ &\quad + (\tau_1-\tau_2)\,(C^n_{M_z,M_x}+g_T\Delta z) + (\lambda_1-\lambda_2)\,(C^n_{M_z,M_x}+g_R\Delta x) + Q\Delta t. \end{aligned} \tag{3.45}$$

For $i = M_z, 0 < j < M_x$ the approximated fictitious points on the boundaries, are obtained by

$$C^n_{M_z+1,j} = C^n_{M_z,j} + g_T\Delta z. \tag{3.46}$$

Substituting (3.48) into (3.20), that is

$$\begin{aligned} C^{n+1}_{M_z,j} &= (\tau_1+\tau_2)\,C^n_{M_z-1,j} + (\lambda_1+\lambda_2)\,C^n_{M_z,j-1} + (1-2\lambda_1-2\tau_1)C^n_{M_z,j} \\ &\quad + (\tau_1-\tau_2)\,(C^n_{M_z,j}+g_T\Delta z) + (\lambda_1-\lambda_2)\,C^n_{M_z,j+1} + Q\Delta t. \end{aligned} \tag{3.47}$$

For $i = M_z, j = 0$ the approximated fictitious points on the boundaries, are obtained by

$$C^n_{M_z+1,0} = C^n_{M_z,0} + g_T\Delta z, \tag{3.48}$$

$$C^n_{M_z,-1} = C^n_{M_z,0} - g_L\Delta x. \tag{3.49}$$

Substituting (3.48) - (3.51) into (3.20), that is

$$\begin{aligned} C^{n+1}_{M_z,0} &= (\tau_1+\tau_2)\,C^n_{M_z-1,0} + (\lambda_1+\lambda_2)\,(C^n_{M_z,0}-g_L\Delta x) + (1-2\lambda_1-2\tau_1)C^n_{M_z,0} \\ &\quad + (\tau_1-\tau_2)\,(C^n_{M_z,0}+g_T\Delta z) + (\lambda_1-\lambda_2)\,C^n_{M_z,1} + Q\Delta t. \end{aligned} \tag{3.50}$$

For $0 < i < M_z, j = 0$ the approximated fictitious points on the boundaries, are obtained by

$$C^n_{i,-1} = C^n_{i,0} - g_L\Delta x. \tag{3.51}$$

Substituting (3.51) into (3.20), that is

$$\begin{aligned} C^{n+1}_{i,0} &= (\tau_1+\tau_2)\,C^n_{i-1,0} + (\lambda_1+\lambda_2)\,(C^n_{i,0}-g_L\Delta x) + (1-2\lambda_1-2\tau_1)C^n_{i,0} \\ &\quad + (\tau_1-\tau_2)\,C^n_{i+1,0} + (\lambda_1-\lambda_2)\,C^n_{i,1} + Q\Delta t. \end{aligned} \tag{3.52}$$

3.3 Numerical simulations of groundwater quality assessment using forward time centered space

The surrounding environment of a landfill its easily to contaminate by pollutant from landfill. Leachate water from a landfill can flow down and contaminate groundwater, it disperses to the community area. In this research, we consider groundwater that has been contaminated by waste material on a landfill. The simulation of the contaminated groundwater pollution model required data concerned with the velocity of the current points and any time in the domain. The governing equation of 3 models in the considered area are a two-dimensional advection-diffusion equation with time dependence. The first model is a two-dimensional groundwater flow model, it provides the hydraulic head of the groundwater. The second models is the velocity potential model, it provide the groundwater flow velocity. The third model is a two-dimensional latitudinally averaged groundwater pollutant dispersion model. The groundwater pollutant concentration is provided.

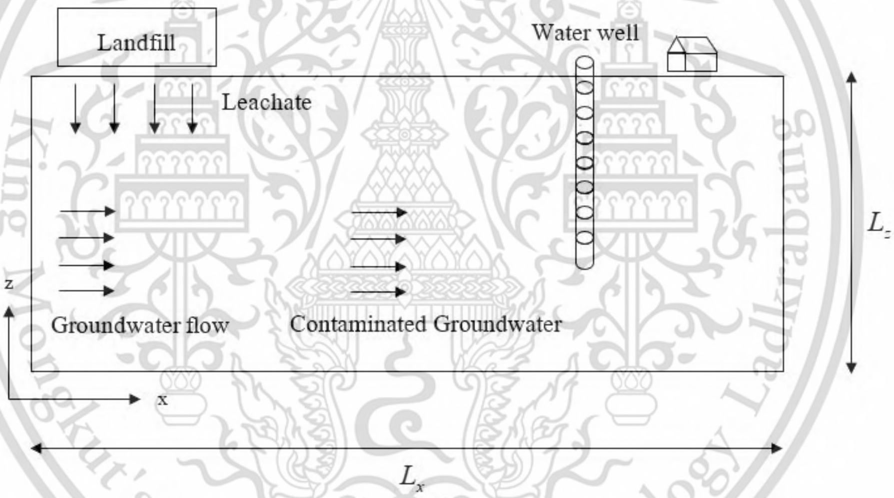

Figure 3.1: General structure of problem domain.

Suppose that the measurement of groundwater pollutant concentration C in a groundwater flow is considered in an underground area. The considered underground area has dimensions of 1.0 km of length and 0.5 km in depth, $L_x = 1.0$ km and $L_z = 0.5$ km, respectively. The simulations need to propose the measurement of the latitudinally averaged groundwater pollutant concentration in the considered area.

There is the landfill which is discharging leachate down to the considered under ground area. The landfill is aligned with longitudinal distance 0.15 km, as shown in Figure 3.1. The landfill discharges the groundwater pollutant to the underground by $c(x, 0, t) = g_N$ (kg/m^3) for all $100 \leq x \leq 250$.

Assume that the specific storage is 1 m^{-1} and the hydraulic conductivity in x- and z-direction are 15 (m/day). There is no rate of change of hydraulic head on the left, the right, the bottom of the considered domain boundaries. There is no rate of change of hydraulic head on the top boundary. We also assume that the leachate which is flowing down to the underground has a pollutant concentration of 1.0 (kg/m^3). There is no rate of change of pollutant concentration on the left, the right, the bottom domain boundaries. The related physical parameters are summarized in Table 3.1.

Table 3.1: The configuration in each simulations

	S	K_x	K_z	h_T	h_B	h_R	h_L	D_x	D_z	Q	g_N	g_T	g_B	g_R	g_L
Simulation 4.1	1	15	15	0	0	0	10	1.5	1.5	0	1	0	0	0	0
Simulation 4.2	1	15	15	0	0	0	40	1.5	1.5	0	1	0	0	0	0
Simulation 4.3	1	15	15	0	0	0	40	1.5	1.5	0	2	0	0	0	0
Simulation 4.4	1	15	15	0	0	0	40	1.5	1.5	0	4	0	0	0	0
Simulation 4.5	1	15	15	0	0	0	$0.06z + 10$	1.5	1.5	0	4	0	0	0	0

3.3.1 Simulation 1 : Low leachate pollutant release rate with low hydraulic head level

If the proposed explicit finite difference techniques for the two-dimensional groundwater flow model Eqs.(3.6), (3.27), (3.22) and Figure (3.32) are employed, we get approximated hydraulic head, as shown in Table 3.2 and Figures 3.2. If the proposed explicit finite difference technique for the groundwater flow velocity model Eqs.(3.9)-(3.10) are employed, we get the approximated groundwater flow velocity in the x- and z-direction in Table 3.3-3.4 and Figure 3.3. The approximated groundwater flow velocities are then plugged into the two-dimensional groundwater pollutant dispersion model. If the proposed explicit finite difference technique for the two-dimensional groundwater pollutant dispersion model Eqs.(3.20), (3.52), (3.42), (3.37) and (3.50) are employed, we get the approximated groundwater pollutant concentration as shown in Table 3.5 and Figures 3.4.

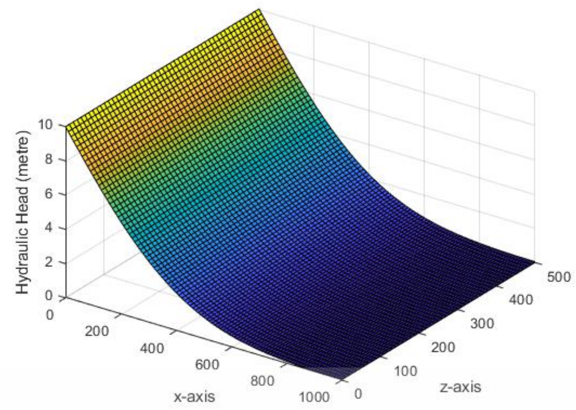

Figure 3.2: The surface plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

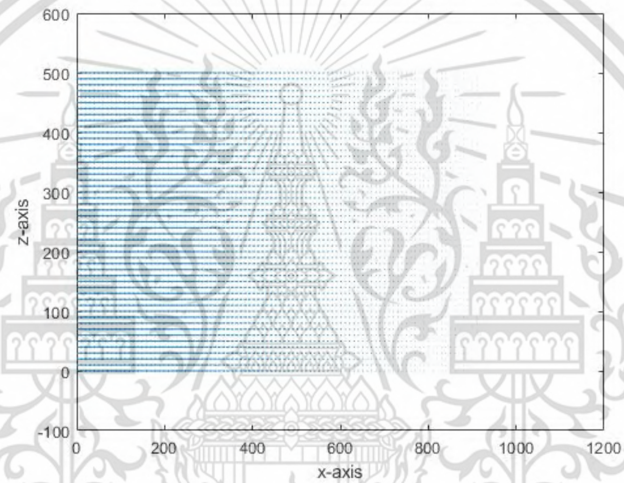

Figure 3.3: The velocity field of hydraulic head

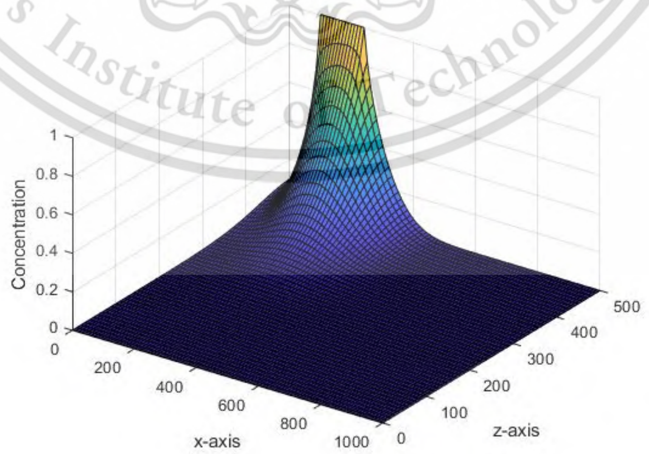

Figure 3.4: The surface plot of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

Table 3.2: The approximated hydraulic head of simulation 3.3.1 where $z = 50$ m.

			$H(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	10.0000	9.3140	8.6331	7.9622	7.3061	6.9849
10	10.0000	9.5147	9.0312	8.5512	8.0766	7.8418
15	10.0000	9.6037	9.2083	8.8149	8.4244	8.2306
20	10.0000	9.6568	9.3142	8.9729	8.6335	8.4647

Table 3.3: The approximated groundwater flow velocity in x-direction(m/day) of simulation 3.3.1 where $z = 50$ m.

			$u(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.0343	0.0341	0.0337	0.0330	0.0321	0.0316
10	0.0243	0.0242	0.0241	0.0238	0.0235	0.0233
15	0.0198	0.0197	0.0196	0.0195	0.0193	0.0192
20	0.0172	0.0171	0.0171	0.0170	0.0169	0.0167

Table 3.4: The approximated groundwater flow velocity in z-direction(m/day) of simulation 3.3.1 where $z = 50$ m.

			$v(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0	0	0	0	0	0
10	0	0	0	0	0	0
15	0	0	0	0	0	0
20	0	0	0	0	0	0

Table 3.5: The approximated groundwater pollutant concentration(kg/m^3) of simulation 3.3.1 where $z = 50$ m.

			$C(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.0148	0.0198	0.0348	0.0654	0.1153	0.1467
10	0.0683	0.0768	0.0997	0.1408	0.2006	0.2356
15	0.1324	0.1419	0.1668	0.2094	0.2689	0.3029
20	0.1945	0.2042	0.2289	0.2703	0.3269	0.3587

3.3.2 Simulation 2 : Low leachate pollutant releasing rate with a high hydraulic head level.

If the proposed explicit finite difference techniques for the two-dimensional groundwater flow model Eqs.(3.6), (3.27), (3.22) and (3.32) are employed, we get an approximated hydraulic head as shown in Table 3.6 and Figure 3.5. If the proposed explicit finite difference technique for the groundwater flow velocity model Eqs.(3.9)-(3.10) are employed, we get the approximated groundwater flow velocity in the x- and z-direction in Table 3.7-3.8 and Figure 3.6. The approximated groundwater flow velocities are then plugged into the two-dimensional groundwater pollutant dispersion model. If the proposed explicit finite difference technique for the two-dimensional groundwater pollutant dispersion model Eqs.(3.20), (3.52), (3.42), (3.37) and (3.50) are employed, we get the approximated groundwater pollutant concentration as shown in Table 3.9 and Figure 3.7.

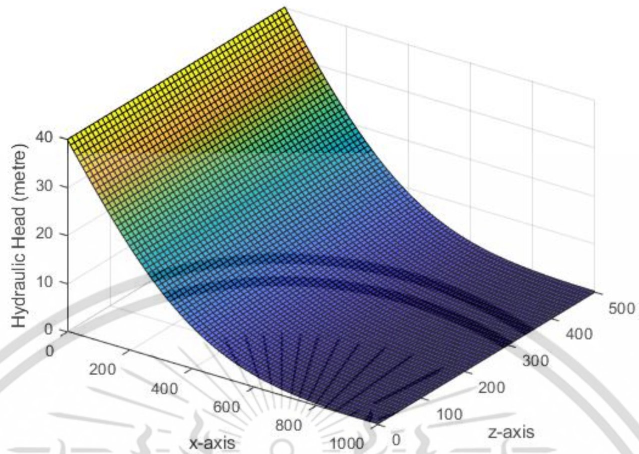

Figure 3.5: The surface and contour plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

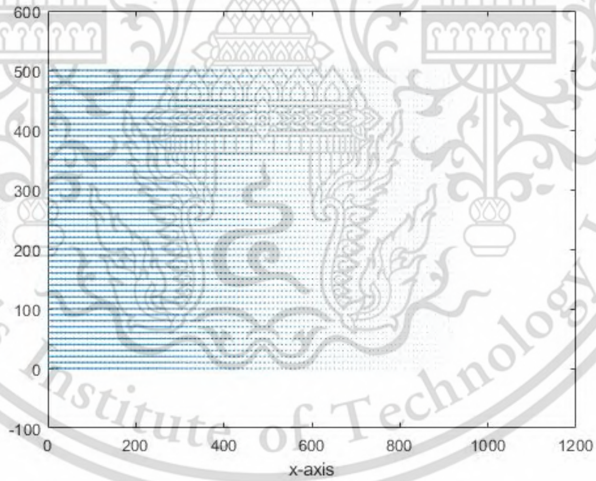

Figure 3.6: The velocity field of hydraulic head

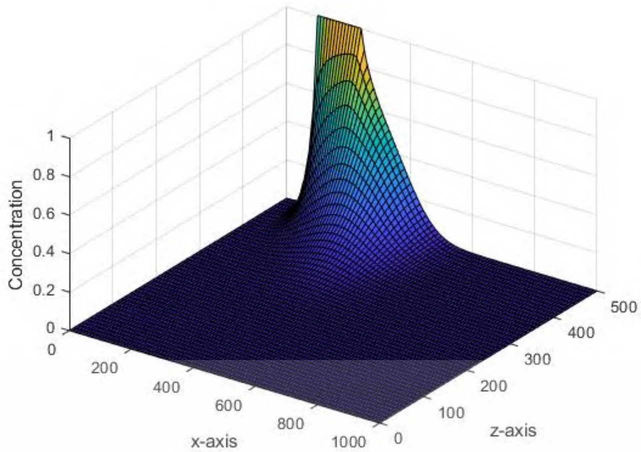

Figure 3.7: The surface of contaminated $c_{i,j}^{n}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

Table 3.6: The approximated hydraulic head(m) of simulation 3.3.2 where $z = 50$ m.

	$H(x,z,t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	40.0000	37.2561	34.5325	31.8490	29.2244	27.9397
10	40.0000	38.0587	36.1247	34.2049	32.3063	31.3670
15	40.0000	38.4147	36.8333	35.2597	33.6977	32.9223
20	40.0000	38.6272	37.2569	35.8918	34.5342	33.8590

Table 3.7: The approximated groundwater flow velocity in x-direction(m/day) of simulation 3.3.2 where $z = 50$ m.

	$u(x,z,t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	0.1371	0.1366	0.1348	0.1321	0.1285	0.1263
10	0.0970	0.0968	0.0962	0.0952	0.0939	0.0931
15	0.0792	0.0791	0.0788	0.0783	0.0775	0.0771
20	0.0686	0.0686	0.0683	0.0680	0.0675	0.0672

Table 3.8: The approximated groundwater flow velocity in z-direction(m/day) of simulation 3.3.2 where $z = 50$ m.

	$v(x,z,t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	0	0	0	0	0	0
10	0	0	0	0	0	0
15	0	0	0	0	0	0
20	0	0	0	0	0	0

Table 3.9: The approximated groundwater pollutant concentration(kg/m^3) of simulation 3.3.2 where $z = 50$ m.

			$C(x,z,t)$			
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	0.0000	0.0000	0.0002	0.0013	0.0067	0.0735
10	0.0006	0.0009	0.0023	0.0074	0.0226	0.0367
15	0.0035	0.0042	0.0076	0.0174	0.0410	0.0601
20	0.0097	0.0111	0.0164	0.0305	0.0605	0.0832

3.3.3 Simulation 3 : Medium leachate pollutant releasing rate with a high hydraulic head level.

If the proposed explicit finite difference techniques for the two-dimensional groundwater flow model Eqs.(3.6), (3.27), (3.22) and (3.32) are employed, we get an approximated hydraulic head as shown in Table 3.10 and Figure 3.8. If the proposed explicit finite difference technique for the groundwater flow velocity model Eqs.(3.9)-(3.10) are employed, we get the approximated groundwater flow velocity in the x- and z-direction in Table 3.11-3.12 and Figure 3.9. The approximated groundwater flow velocities are then plugged into the two-dimensional groundwater pollutant dispersion model. If the proposed explicit finite difference technique for the two-dimensional groundwater pollutant dispersion model Eqs.(3.20), (3.52), (3.42), (3.37) and (3.50) are employed, we get the approximated groundwater pollutant concentration as shown in Table 3.13 and Figure 3.10.

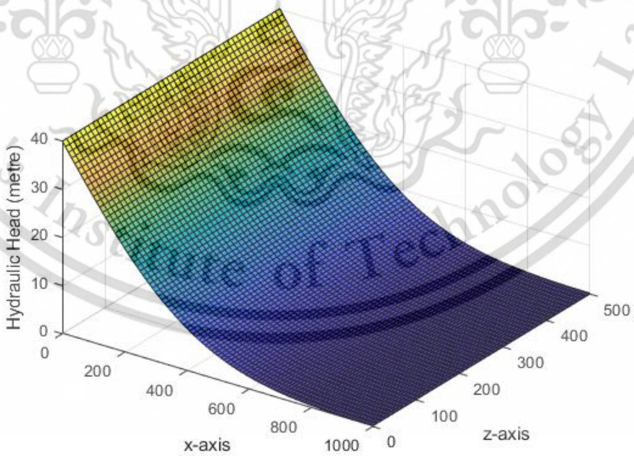

Figure 3.8: The surface plot of hydraulic head $H^n_{i,j}$ for all $(x,z,t) \in [0,1000] \times [0,500] \times [0,3600]$.

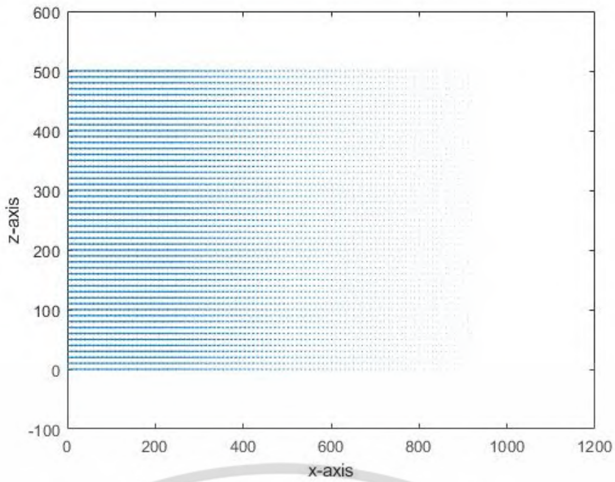

Figure 3.9: The velocity field of hydraulic head

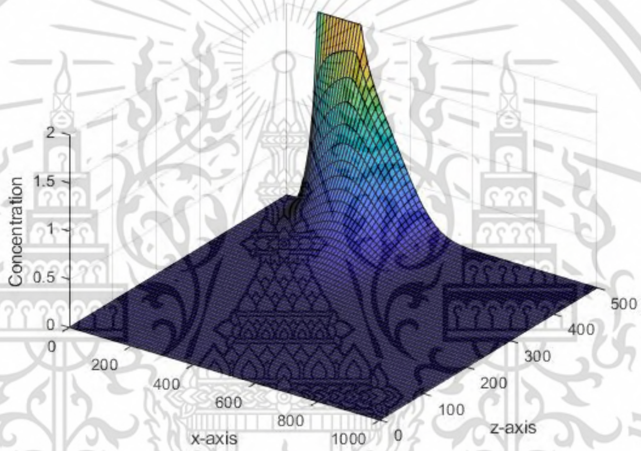

Figure 3.10: The surface and contour plot of contaminated $c_{i,j}^n$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

Table 3.10: The approximated hydraulic head(m) of simulation 3.3.3 where $z = 50$ m.

	$H(x, z, t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	40.0000	37.2561	34.5325	31.8490	29.2244	27.9397
10	40.0000	38.0587	36.1247	34.2049	32.3063	31.3670
15	40.0000	38.4147	36.8333	35.2597	33.6977	32.9223
20	40.0000	38.6272	37.2569	35.8918	34.5342	33.8590

Table 3.11: The approximated groundwater flow velocity in x-direction(m/day) of simulation 3.3.3 where $z = 50$ m.

	$u(x, z, t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	0.1371	0.1366	0.1348	0.1321	0.1285	0.1263
10	0.0970	0.0968	0.0962	0.0952	0.0939	0.0931
15	0.0792	0.0791	0.0788	0.0783	0.0775	0.0771
20	0.0686	0.0686	0.0683	0.0680	0.0675	0.0672

Table 3.12: The approximated groundwater flow velocity in z-direction(m/day) of simulation 3.3.3 where $z = 50$ m.

$v(x,z,t)$						
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	0	0	0	0	0	0
10	0	0	0	0	0	0
15	0	0	0	0	0	0
20	0	0	0	0	0	0

Table 3.13: The approximated groundwater pollutant concentration(kg/m^3) of simulation 3.3.3 where $z = 50$ m.

$C(x,z,t)$						
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	0.0000	0.0000	0.0004	0.0025	0.0134	0.0270
10	0.0012	0.0017	0.0045	0.0148	0.0453	0.0734
15	0.0069	0.0085	0.0151	0.0348	0.0819	0.1203
20	0.0194	0.0222	0.0329	0.0610	0.1209	0.1663

3.3.4 Simulation 4 : High leachate pollutant releasing rate with a high hydraulic head level.

If the proposed explicit finite difference techniques for the two-dimensional groundwater flow model Eqs.(3.6), (3.27), (3.22) and (3.32) are employed, we get an approximated hydraulic head as shown in Table 3.14 and Figure 3.11. If the proposed explicit finite difference technique for the groundwater flow velocity model Eqs.(3.9)-(3.10) are employed, we get the approximated groundwater flow velocity in the x- and z-direction in Table 3.15-3.16 and Figure . The approximated groundwater flow velocities are then plugged into the two-dimensional groundwater pollutant dispersion model. If the proposed explicit finite difference technique for the two-dimensional groundwater pollutant dispersion model Eqs.(3.20), (3.52), (3.42), (3.37) and (3.50) are employed, we get the approximated groundwater pollutant concentration as shown in Table 3.17 and Figure 3.13.

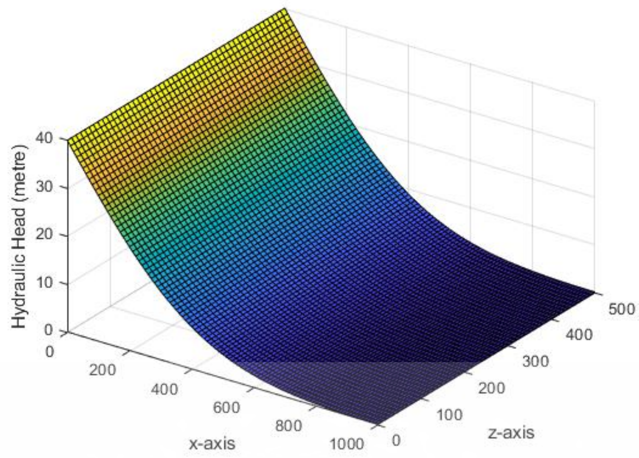

Figure 3.11: The surface plot of hydraulic head $H_{i,j}^n$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

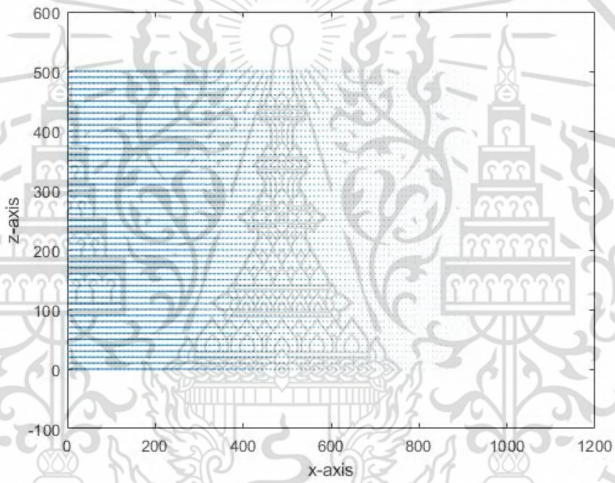

Figure 3.12: The velocity field of hydraulic head

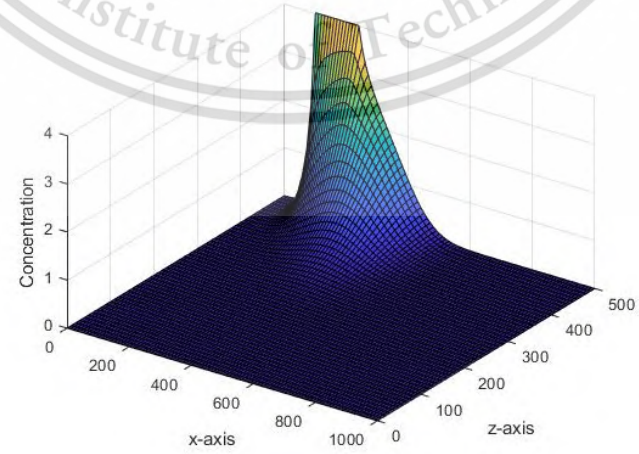

Figure 3.13: The surface plot of contaminated $c_{i,j}^n$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

Table 3.14: The approximated hydraulic head(m) of simulation 3.3.4 where $z = 50$ m.

			$H(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	40.0000	37.2561	34.5325	31.8490	29.2244	27.9397
10	40.0000	38.0587	36.1247	34.2049	32.3063	31.3670
15	40.0000	38.4147	36.8333	35.2597	33.6977	32.9223
20	40.0000	38.6272	37.2569	35.8918	34.5342	33.8590

Table 3.15: The approximated groundwater flow velocity in x-direction(m/day) of simulation 3.3.4 where $z = 50$ m.

			$u(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.1371	0.1366	0.1348	0.1321	0.1285	0.1263
10	0.0970	0.0968	0.0962	0.0952	0.0939	0.0931
15	0.0792	0.0791	0.0788	0.0783	0.0775	0.0771
20	0.0686	0.0686	0.0683	0.0680	0.0675	0.0672

Table 3.16: The approximated groundwater flow velocity in z-direction(m/day) of simulation 3.3.4 where $z = 50$ m.

			$v(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0	0	0	0	0	0
10	0	0	0	0	0	0
15	0	0	0	0	0	0
20	0	0	0	0	0	0

Table 3.17: The approximated groundwater pollutant concentration(kg/m^3) of simulation 3.3.4 where $z = 50$ m.

			$C(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.0000	0.0001	0.0007	0.0050	0.0268	0.0540
10	0.0023	0.0034	0.0090	0.0297	0.0906	0.1468
15	0.0138	0.0170	0.0303	0.0697	0.1639	0.2406
20	0.0389	0.0444	0.0658	0.1220	0.2418	0.3327

3.3.5 Simulation 5 : Practical leachate pollutant releasing rate with a high hydraulic head level.

If the proposed explicit finite difference techniques for the two-dimensional groundwater flow model Eqs.(3.6), (3.27), (3.22) and (3.32) are employed, we get an approximated hydraulic head as shown in Table 3.18 and Figure 3.14. If the proposed explicit finite difference technique for the groundwater flow velocity model Eqs.(3.9)-(3.10) are employed, we get the approximated groundwater flow velocity in the x- and z-direction in Table 3.19-3.20 and Figure 3.15. The approximated groundwater flow velocities are then plugged into the two-dimensional groundwater pollutant dispersion model. If the proposed explicit finite difference technique for the two-dimensional groundwater pollutant dispersion model Eqs.(3.20), (3.52), (3.42), (3.37) and (3.50) are employed, we get the approximated groundwater pollutant concentration as shown in Table 3.21 and Figure 3.16.

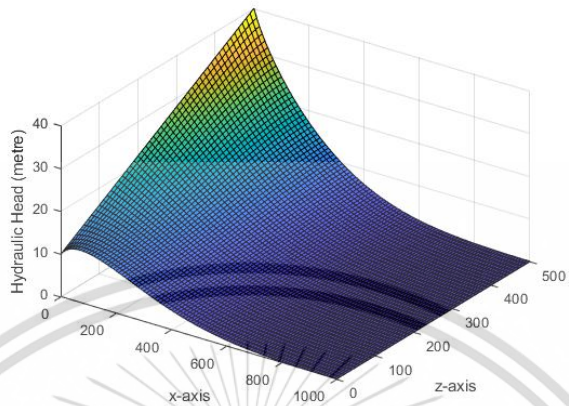

Figure 3.14: The surface plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

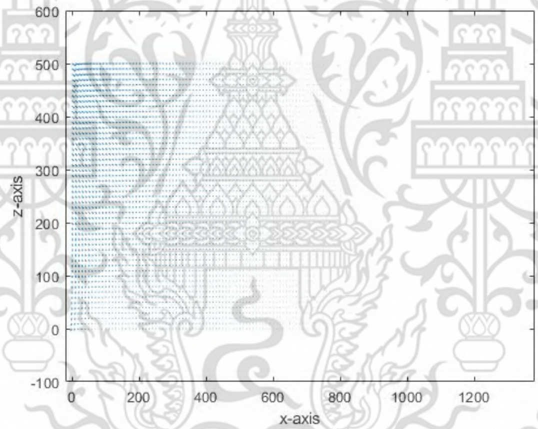

Figure 3.15: The velocity field of hydraulic head

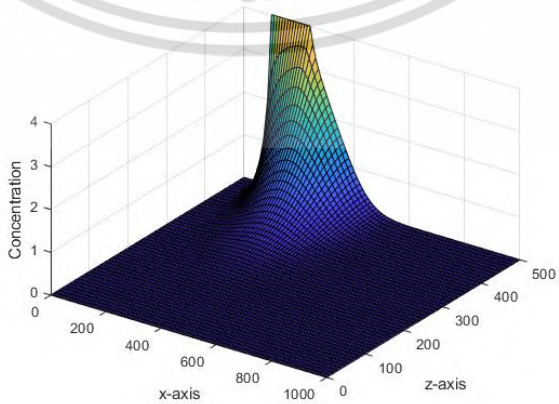

Figure 3.16: The surface plot of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

Table 3.18: The approximated hydraulic head(m) of simulation 3.3.5 where $z = 50$ m.

			$H(x,z,t)$			
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	36.4000	33.4092	30.4953	27.7080	25.0706	23.8111
10	36.4000	33.9669	31.6015	29.3447	27.2115	26.1917
15	36.4000	34.1979	32.0615	30.0295	28.1148	27.2014
20	36.4000	34.3326	32.3299	30.4300	28.6448	27.7948

Table 3.19: The approximated groundwater flow velocity in x-direction(m/day) of simulation 3.3.5 where $z = 50$ m.

			$u(x,z,t)$			
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	0.1490	0.1470	0.1411	0.1338	0.1260	0.1220
10	0.1212	0.1194	0.1143	0.1082	0.1020	0.0989
15	0.1096	0.1080	0.1030	0.0972	0.0913	0.0885
20	0.1029	0.1013	0.0964	0.0907	0.0850	0.0823

Table 3.20: The approximated groundwater flow velocity in z-direction(m/day) of simulation 3.3.5 where $z = 50$ m.

			$v(x,z,t)$			
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	-0.0600	-0.0472	-0.0366	-0.0287	-0.0228	-0.0205
10	-0.0600	-0.0473	-0.0369	-0.0291	-0.0233	-0.0211
15	-0.0600	-0.0473	-0.0369	-0.0291	-0.0234	-0.0212
20	-0.0600	-0.0473	-0.0369	-0.0291	-0.0234	-0.0212

Table 3.21: The approximated groundwater pollutant concentration(kg/m^3) of simulation 3.3.5 where $z = 50$ m.

			$C(x,z,t)$			
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	0.0000	0.0001	0.0008	0.0064	0.0360	0.0733
10	0.0008	0.0014	0.0053	0.0237	0.0885	0.1537
15	0.0038	0.0050	0.0124	0.0419	0.1311	0.2132
20	0.0093	0.0112	0.0217	0.0600	0.1669	0.2606

Chapter 4

Mathematical simulation of groundwater quality assessment models with total chloride transformation effects

The contaminated water have so much effect, it can cause many life diseases and problems. Nitrates or nitrites in water contaminates drinking water can impacts human health by decreasing blood cell ability to carry oxygen, that can be linked to blue baby syndrome [2], this is some effect of contaminated water. In this research we talking about the effect chloride and their substance.

Chloride occur naturally in groundwater but is found in greater concentrations where seawater. It generally combines with sodium, calcium, or magnesium, for example, sodium chloride (NaCl) is formed when chloride and sodium combine.

The other forms of chloride do not come from the only combination of other substances, the oxidation numbers or oxidation states is the well-know process to obtained a new form of chemical compound, for example, Chloride can be changes to hypochlorite (ClO) if the oxidation number increase by 1 or chlorite if added by 3.

oxidation state	-1	+1	+3	+5	+7
anion name	chloride	hypochlorite	chlorite	chlorate	perchlorate
formula	Cl^-	ClO^-	ClO_2^-	ClO_3^-	ClO_4^-

Figure 4.1: The other form of chloride in any state

They are substances used by the body to help it work well. Although chlorides are harmless at low levels, well water high in sodium chloride can damage plants if used for gardening or irrigation, and give drinking water an unpleasant taste. Sodium chloride is high corrosivity will also damage plumbing, appliances, and water heaters, causing toxic metals to leach into your water. It can complicate existing heart problems and contribute to high blood pressure when ingested in excess [1].

chloride	can cause heart problems and contribute to high blood pressure
hypochlorite	can cause bladder cancer
chlorite	can cause shortness of breath and other respiratory problems because of damage to the substances in blood that carry oxygen throughout the body
chlorate	chlorate is toxic, doses of a few grams of chlorate are lethal, there is a direct toxicity to the proximal renal tubule
perchlorate	at high concentrations perchlorate can affect the thyroid gland by inhibiting the uptake of iodine

Figure 4.2: Chloride compound damage to health

There are many dangers of chloride, but we can prevent amounts of them, from exceeding standards by management based on mathematical models. This is the method to measure chloride dispersion on total chloride transformation effects models. First, to apply the groundwater head, we used a hydraulic head model, followed by a groundwater velocity model based on vector in the x- and z-directions. Next, the model was used to define chloride, hypochlorite, chlorite, chlorate and perchlorate for the allocation of groundwater contamination.

4.1 Chloride Compound Dispersion Models

Leachate water infiltrates into the ground, it penetrates soil pores and becomes groundwater. Chloride is an important water contaminant. An advection-diffusion model provides a continuous description of groundwater pollutant transport. The models are used to describe amount of toxic substances in groundwater on a two-dimensional latitudinally averaged groundwater pollution dispersion model.

4.1.1 A total chloride dispersion model

The pollutant concentration measurement of total chloride in surface water can be describe by a two-dimensional advection-diffusion reaction equation.

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial z} = D_x\frac{\partial^2 c}{\partial x^2} + D_z\frac{\partial^2 c}{\partial z^2} + Q - Rc, \tag{4.1}$$

for all $(x, z, t) \in \Omega$ such that $\Omega = [0, L_x] \times [0, L_z] \times [0, T]$, where $c(x, z, t)$ is total chloride pollutant concentration of groundwater(kg/m^3), D_x, D_z are the diffusion coefficient in x- and z-directions, $u(x, z, t)$, $v(x, z, t)$ are the groundwater flow velocity in the x- and z-directions, R is transformed chloride rate and Q is groundwater pollutant source or sink function by contaminators.

4.1.1.1 Initial condition of the total chloride dispersion model

The chloride dispersion is described with conditions in the following sections, where the potential groundwater pollutant concentration in the consider area is described by

$$c(x, z, 0) = c_0(x, z) \text{ for all } (x, z) \in \Omega \tag{4.2}$$

where $c_0(x, z)$ is a averaged potential total chloride concentration in the considered area.

4.1.1.2 Boundary condition of total chloride dispersion model

The top boundary condition in some length are assumed by the function of measured raw data at the considered landfill. The top, left, right and bottom boundary conditions are assumed by the averaged rates of change of pollutant concentration around the left, right and bottom boundaries. The boundary conditions are also assumed by

$$c(L_z, x, t) = g_N, \text{for all } x \in [k_1 L_x, k_2 L_x], \tag{4.3}$$

$$\frac{\partial c(z, 0, t)}{dx} = g_L, \text{for all } z \in [0, L_z] \text{ and } t \in [0, T], \tag{4.4}$$

$$\frac{\partial c(L_z, x, t)}{dz} = g_T, \text{for all } x \in [0, k_1 L_x) \cup (k_2 L_x, L_x] \text{ and } t \in [0, T], \tag{4.5}$$

$$\frac{\partial c(z, L_x, t)}{dx} = g_R, \text{ for all } z \in [0, L_x] \text{and } t \in [0, T], \tag{4.6}$$

$$\frac{\partial c(x, 0, t)}{dz} = g_B, \text{ for all } x \in [0, L_x] \text{and } t \in [0, T], \tag{4.7}$$

where $k_1 L_x, k_2 L_x$ are referred to the range area of the total chloride pollutant area source, and g_L, g_T, g_R and g_B are the rate of change of the total chloride concentration with respect distance around the top, the bottom and the right boundaries along the considered area, respectively.

4.1.2 A hypochlorite dispersion model

The total chloride transformed to be the hypochlorite. The hypochlorite dispersion model is described by

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial z} = D1_x \frac{\partial^2 \phi}{\partial x^2} + D1_z \frac{\partial^2 \phi}{\partial z^2} + Q + R_1 Rc, \tag{4.8}$$

for all $(x, z, t) \in \Omega$ such that $\Omega = [0, L_x] \times [0, L_z] \times [0, T]$, where $\phi(x, z, t)$ is hypochlorite pollutant concentration of groundwater(kg/m^3), $D1_x, D1_z$ are the diffusion coefficient in x- and z-directions, $u(x, z, t)$, $v(x, z, t)$ are the groundwater flow velocity in the x- and z-directions, R_1 is transformed hypochlorite rate and Q is groundwater pollutant source or sink function by contaminators.

4.1.2.1 Initial condition of hypochlorite dispersion model

Dispersion of hypochlorite with following conditions, if the potential hypochlorite concentration in the consider area is described by

$$\phi(x, z, 0) = f_\phi(x, z) \text{ for all } (x, z) \in \Omega \tag{4.9}$$

where $f_\phi(x, z)$ is a averaged potential hypochlorite concentration in the considered area.

4.1.2.2 Boundary condition of hypochlorite dispersion model

The rate of change of the pollutant concentration along the domain boundaries are assumed to be:

$$\frac{\partial \phi(0,z,t)}{\partial x} = g1_L, \text{for all } z \in [0, L_z] \text{ and } t \in [0,T], \quad (4.10)$$
$$\frac{\partial \phi(x,L_z,t)}{\partial z} = g1_T, \text{for all } x \in [0, L_x] \text{ and } t \in [0,T], \quad (4.11)$$
$$\frac{\partial \phi(L_x,z,t)}{\partial x} = g1_R, \text{for all } z \in [0, L_x] \text{ and } t \in [0,T], \quad (4.12)$$
$$\frac{\partial \phi(x,0,t)}{\partial z} = g1_B, \text{for all } x \in [0, L_x] \text{ and } t \in [0,T], \quad (4.13)$$

where $g1_L, g1_T, g1_R$ and $g1_B$ are the rate of change of the hypochlorite concentration with respect distance around the top, the bottom and the right boundaries along the considered area, respectively.

4.1.3 A chlorite dispersion model

The model of chlorite dispersion model can be described by

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial z} = D2_x\frac{\partial^2 \eta}{\partial x^2} + D2_z\frac{\partial^2 \eta}{\partial z^2} + Q + R_2Rc, \quad (4.14)$$

for all $(x,z,t) \in \Omega$ such that $\Omega = [0,L_x] \times [0,L_z] \times [0,T]$, where $\eta(x,z,t)$ is chlorite pollutant concentration of groundwater(kg/m^3), $D2_x, D2_z$ are the diffusion coefficient in x- and z-directions, $u(x,z,t)$, $v(x,z,t)$ are the groundwater flow velocity in the x- and z-directions, R_2 is transformed chlorite rate and Q is groundwater pollutant source or sink function by contaminators.

4.1.3.1 Initial condition of chlorite dispersion model

Dispersion of chlorite with following conditions, if the potential chlorite concentration in the consider area is described by

$$\eta(x,z,0) = f_\eta(x,z) \text{ for all } (x,z) \in \Omega \quad (4.15)$$

where $f_\eta(x,z)$ is a averaged potential chlorite concentration in the considered area.

4.1.3.2 Boundary condition of chlorite dispersion model

Similarly, the rate of change are assumed by $g2_L$, $g2_T$, $g2_R$ and $g2_B$.

4.1.4 A chlorate dispersion model

The model of chlorate dispersion model can be described by

$$\frac{\partial \sigma}{\partial t} + u\frac{\partial \sigma}{\partial x} + v\frac{\partial \sigma}{\partial z} = D3_x\frac{\partial^2 \sigma}{\partial x^2} + D3_z\frac{\partial^2 \sigma}{\partial z^2} + Q + R_3Rc, \quad (4.16)$$

for all $(x, z, t) \in \Omega$ such that $\Omega = [0, L_x] \times [0, L_z] \times [0, T]$, where $\sigma(x, z, t)$ is chlorate pollutant concentration of groundwater(kg/m^3), $D3_x, D3_z$ are the diffusion coefficient in x- and z-directions, $u(x, z, t)$, $v(x, z, t)$ are the groundwater flow velocity in the x- and z-directions, R_3 is transformed chlorate rate and Q is groundwater pollutant source or sink function by contaminators.

4.1.4.1 Initial condition of chlorate dispersion model

Dispersion of chlorate with following conditions, if the potential chlorate concentration in the consider area is described by

$$\sigma(x, z, 0) = f_\sigma(x, z) \text{ for all } (x, z) \in \Omega \tag{4.17}$$

where $f_\sigma(x, z)$ is a averaged potential chlorate concentration in the considered area.

4.1.4.2 Boundary condition of chlorate dispersion model

Similarly, the rate of change are assumed by $g3_L$, $g3_T$, $g3_R$ and $g3_B$.

4.1.5 A perchlorate dispersion model

The model of perchlorate dispersion model can be described by

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial z} = D4_x\frac{\partial^2 \xi}{\partial x^2} + D4_z\frac{\partial^2 \xi}{\partial z^2} + Q + R_4Rc, \tag{4.18}$$

for all $(x, z, t) \in \Omega$ such that $\Omega = [0, L_x] \times [0, L_z] \times [0, T]$, where $\xi(x, z, t)$ is perchlorate pollutant concentration of groundwater(kg/m^3), $D4_x, D4_z$ are the diffusion coefficient in x- and z-directions, $u(x, z, t)$, $v(x, z, t)$ are the groundwater flow velocity in the x- and z-directions, R_4 is transformed perchlorate rate and Q is groundwater pollutant source or sink function by contaminators.

4.1.5.1 Initial condition of perchlorate dispersion model

Dispersion of perchlorate with following conditions, if the potential perchlorate concentration in the consider area is described by

$$\xi(x, z, 0) = f_\xi(x, z) \text{ for all } (x, z) \in \Omega \tag{4.19}$$

where $f_\xi(x, z)$ is a averaged potential perchlorate concentration in the considered area.

4.1.5.2 Boundary condition of perchlorate dispersion model

Similarly, the rate of change are assumed by $g4_L$, $g4_T$, $g4_R$ and $g4_B$.

4.2 Numerical simulations of total chloride dispersion model

A two-dimension hydraulic head model provides hydraulic head. Then the calculated results of the model will be input into A two-dimension groundwater velocity model which provides groundwater flow velocity. Next, input the velocity into nitrogen dispersion models that provide chloride, hypochlorite, chlorite chlorate and perchlorate concentration.

4.2.1 Simulation 6: Chloride dispersion

The application of chloride dispersion models defined grid space $\Delta x = 10$ and $\Delta z = 10$ time $\Delta t = 1$ increments in area 1×0.5. The first model, considering the hydraulic conductivity in homogeneous aquifer $K_x = K_z = 15$, the specific storage coefficient $S = 1$, source function $W = 0$. The initial $h = 0$ and boundary conditions $h_L = 0.06z + 10$, $h_T = 0$, $h_R = 0$ and $h_B = 0$. It mean that water flow on left on domain. Consider the hydraulic head of groundwater by using the explicit method approximate solution as shown in Figure 4.3 - 4.4.

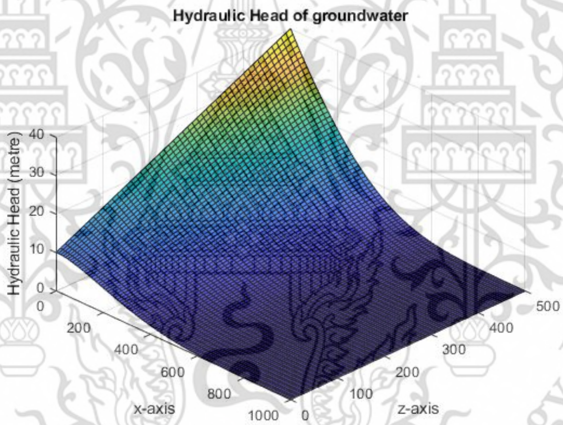

Figure 4.3: Hydraulic head of the considered area for 10 year

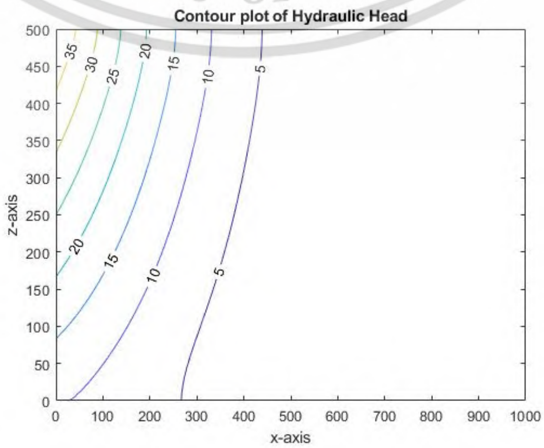

Figure 4.4: Hydraulic head of the considered area for 10 year

Next, input groundwater to approximate chloride dispersion, Organic material transformed into hypochlorite, chlorite, chlorate and lastly into perchlorate by using FTCS, considering diffusion coefficient of chloride $D_x = D_z = 1.5$ and source function of pollutant through specified $Q = 0$. At the beginning of the time, we assuming that the amount of chloride in considered area are equal to zero, i.e., $c_0 = 0$ and boundary conditions $g_L = g_T = g_R = g_B = 0$. The approximate solution as shown in Table 4.1 Figure 4.5-4.6

Table 4.1: The approximated chloride where $z = 50$ m.

	$c(x,z,t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
3	0.0000	0.0000	0.0001	0.0013	0.0123	0.0305
5	0.0000	0.0001	0.0008	0.0060	0.0326	0.0657
7	0.0002	0.0005	0.0026	0.0061	0.0285	0.0561
10	0.0012	0.0020	0.0070	0.0271	0.0909	0.1516

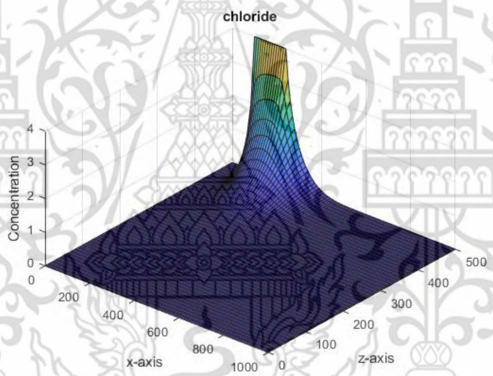

Figure 4.5: Surface plot of chloride concentration for 10 years

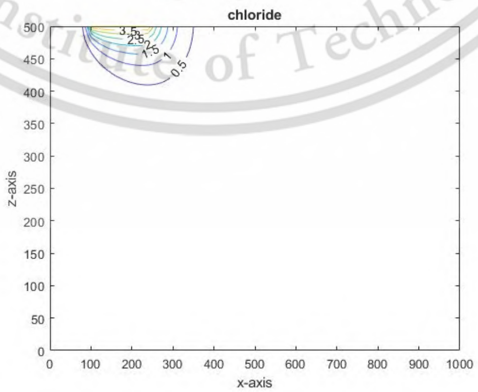

Figure 4.6: Contour plot of chloride concentration for 10 years

We set the initial and boundary conditions as the same with previous model, considering diffusion coefficient of hypochlorite $D1_x = D1_z = 2.0$ and rate of hypochlorite dispersion $R_1 = 0.25$. The approximate solution as shown in Table 4.2 and Figure 4.7-4.8.

Table 4.2: The approximated hypochlorite where $z = 50$ m.

	$\phi(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
3	0.0000	0.0000	0.0001	0.0006	0.0032	0.0064
5	0.0001	0.0002	0.0006	0.0025	0.0089	0.0154
7	0.0005	0.0008	0.0020	0.0060	0.0172	0.0273
10	0.0027	0.0034	0.0093	0.0219	0.0485	0.0687

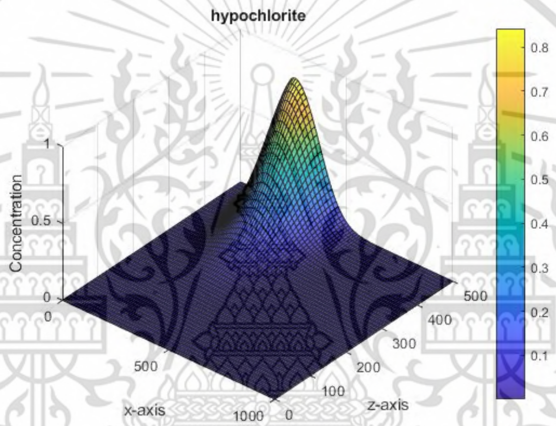

Figure 4.7: Surface plot of hypochlorite concentration for 10 years

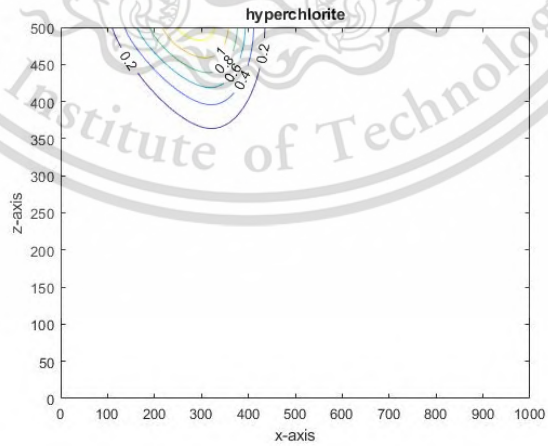

Figure 4.8: Contour plot of hypochlorite concentration for 10 years

We set the initial and boundary conditions as the same with previous model, considering diffusion coefficient of chlorite $D2_x = D2_z = 0.5$ and rate of chlorite dispersion $R_2 = 0.01$. The approximate solution as shown in Table 4.3 Figure 4.9-4.10.

Table 4.3: The approximated chlorite where $z = 50$ m.

	$\eta(x,z,t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
3	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003
5	0.0000	0.0000	0.0000	0.0001	0.0004	0.0006
7	0.0000	0.0000	0.0001	0.0002	0.0007	0.0011
10	0.0001	0.0001	0.0002	0.0006	0.0013	0.0019

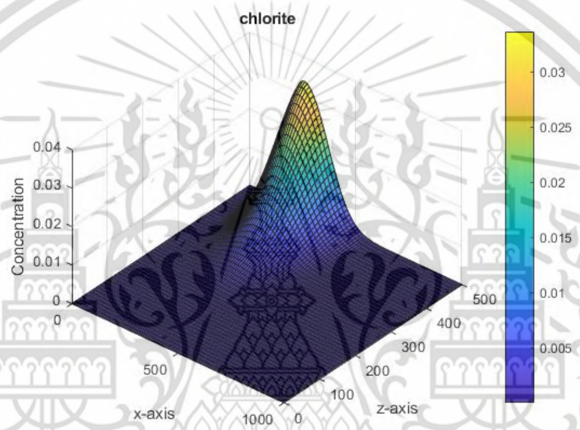

Figure 4.9: Surface plot of chlorite concentration for 10 years

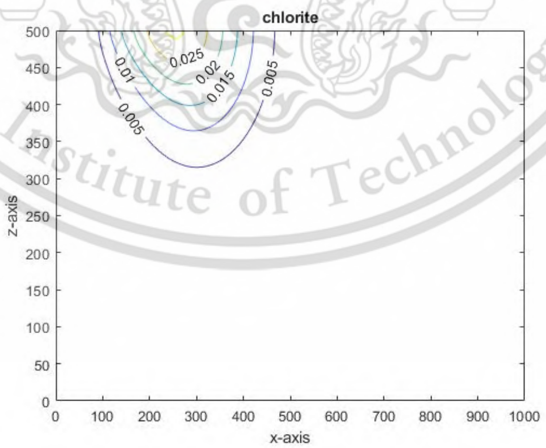

Figure 4.10: Contour plot of chlorite concentration for 10 years

We set the initial and boundary conditions as the same with previous model, considering diffusion coefficient of chlorate $D3_x = D3_z = 2.5$ and rate of chlorate dispersion $R_3 = 0.4$. The approximate solution as shown in Table 4.4 and Figure 4.11-4.12.

Table 4.4: The approximated chlorate where $z = 50$ m.

	$\sigma(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
3	0.0000	0.0000	0.0000	0.0000	0.0004	0.0011
5	0.0000	0.0000	0.0001	0.0003	0.0017	0.0039
7	0.0001	0.0001	0.0004	0.0019	0.0087	0.0171
10	0.0004	0.0004	0.0007	0.0025	0.0097	0.0184

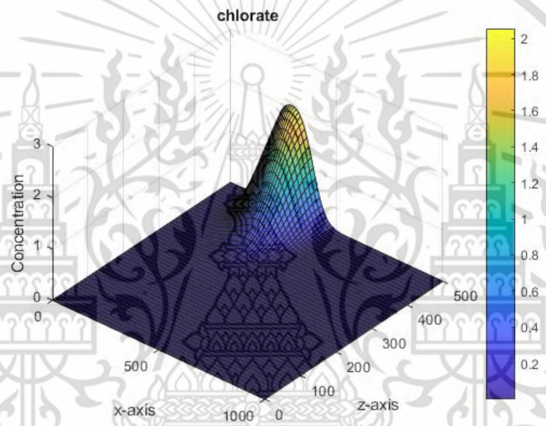

Figure 4.11: Surface plot of chlorate concentration for 10 years

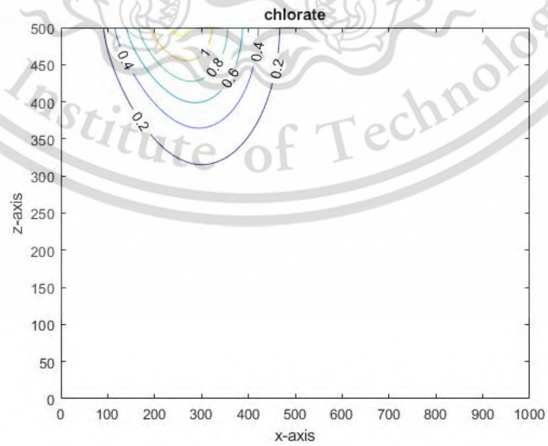

Figure 4.12: Contour plot of chlorate concentration for 10 years

We set the initial and boundary conditions as the same with previous model, considering diffusion coefficient of perchlorate $D4_x = D4_z = 1$ and rate of perchlorate dispersion $R_4 = 0.005$. The approximate solution as shown in Table 4.4 and Figure 4.13-4.14.

Table 4.5: The approximated chlorate where $z = 50$ m.

	$\xi(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
3	0.0000	0.0000	0.0000	0.0000	0.0002	0.0003
5	0.0000	0.0000	0.0000	0.0001	0.0003	0.0004
7	0.0000	0.0000	0.0001	0.0002	0.0005	0.0007
10	0.0001	0.0002	0.0003	0.0005	0.0009	0.0013

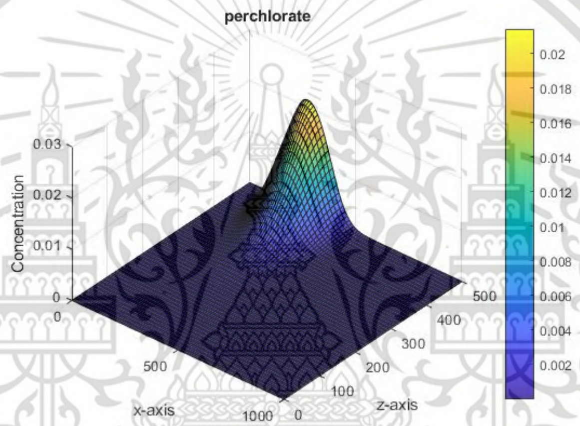

Figure 4.13: Surface plot of perchlorate concentration for 10 years

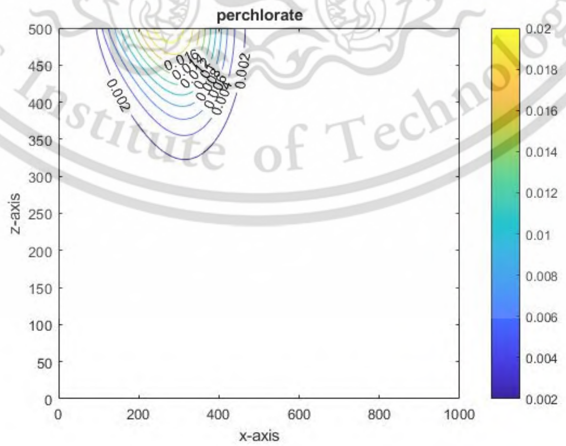

Figure 4.14: Contour plot of perchlorate concentration for 10 years

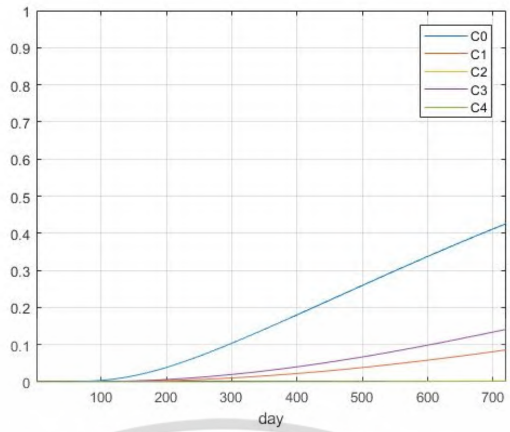

Figure 4.15: Comparison of chloride compound levels along 2 years

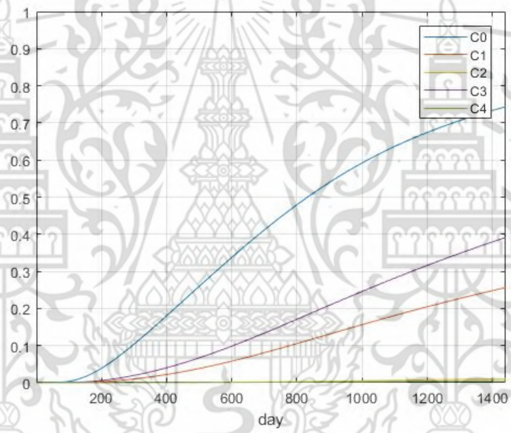

Figure 4.16: Comparison of chloride compound levels along 4 years

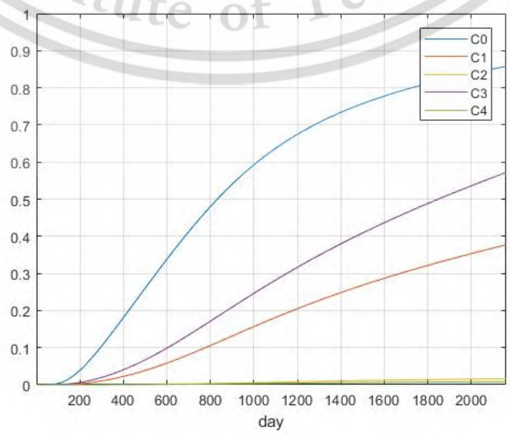

Figure 4.17: Comparison of chloride compound levels along 6 years

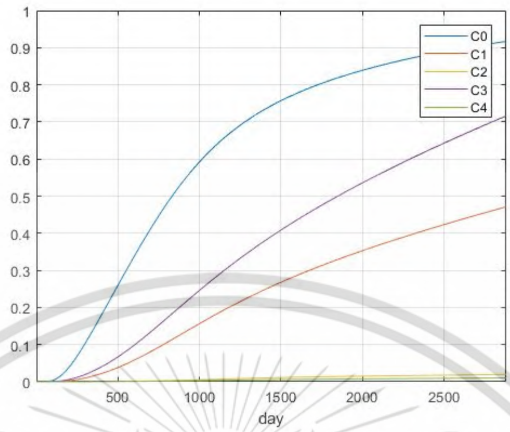

Figure 4.18: Comparison of chloride compound levels along 8 years

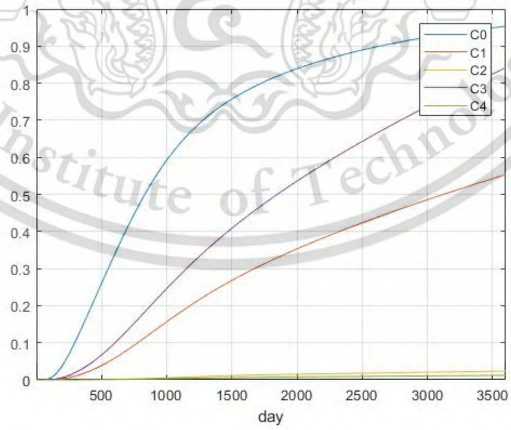

Figure 4.19: Comparison of chloride compound levels along 10 years with C0 = total chloride, C1 = hypochloride, C2 = chlorite, C3 = chlorate, C4 = perchlorate.

Chapter 5

Discussion and Conclusion

5.1 Discussion

The hydraulic head can drive the groundwater flow from the higher hydraulic head level zone to the lower zone, as shown in Tables 3.2-3.4, 3.6-3.8, 3.10-3.12, 3.14-3.16 and 3.18-3.20 . The direction of groundwater is illustrated by their velocity field, as shown in Figure 3.2-3.3, 3.5-3.6, 3.8-3.9, 3.11-3.12 and 3.14-3.15.

5.1.1 In simulation 1 : leachate pollutant release rate with low hydraulic head level

A case of low leachate pollutant release rate from a landfill with a low-level hydraulic head is considered.

5.1.2 In Simulation 2 : Low leachate pollutant releasing rate with a high hydraulic head level.

The level of the hydraulic head is increased fourfold, the overall groundwater quality becomes close to the last case.

5.1.3 In Simulation 3 : Medium leachate pollutant releasing rate with a high hydraulic head level.

In this case, the level of leachate pollutant are increasing, the overall ground-water quality becomes as poor as the pollutant concentration at the considered land-fill

5.1.4 In Simulation 4 : High leachate pollutant releasing rate with a high hydraulic head level.

If the leachate releasing rate is poor or there is a high pollutant concentration, the overall groundwater quality becomes poor as well.

5.1.5 In Simulation 5 : Practical leachate pollutant releasing rate with a high hydraulic head level.

A case of practical leachate pollutant releasing rate is also tested by using numerical data interpolation $f_L(z)$.

5.1.6 In simulation 6 : Chloride dispersion

In this simulation. we assuming that the hydraulic head drive the groundwater flow from the higher hydraulic head level zone to the lower zone, the result of simulation for 10 years have shown in Table 3.18 and Figure 4.3. The figures shown that the hydraulic head at the surface area have higher than the deep area. The hydraulic head is transformed to be the groundwater flow direction as shown in Figure 4.4. The direction have shown that groundwater flow from high to lower hydraulic head. The result be plug into the five chloride compound dispersion models. We can measure the total chloride, hypochloride, chlorite, chlorate and perchlorate pollutant levels at 10 years as shown Table 4.1-4.5 and Figure 4.5-4.13. The figure shown that the amount of groundwater changes directly,over time and the substance is less than reactant. The approximated chloride compound is compared in Figure 4.15-4.19, we can see that the simulation for 2 to 4 years tell us the trend of graph are the same and a little bit increasing, after that, for 4 to 6 years, they are slightly increase. Finally, in 6 to 10 years we obtained that the graph are stable.

5.2 Conclusion

First, three two-dimensional models of long-term contaminated groundwater pollutant measurement around a landfill are introduced. The first model is the two-dimensional transient groundwater flow model, which provides the hydraulic head. The second model is the groundwater flow velocity model, which provides the groundwater flow velocity in the x- and z-directions. The third model is the two-dimensional horizontal averaged contaminated groundwater dispersion model, which provides the groundwater pollutant concentration. All of these models can be solved by using the forward time centered space.

Second, a method to set up the initial and boundary conditions of the proposed transient groundwater flow model is proposed. The computed hydraulic head is transformed to be the groundwater flow velocity by using the second model. The results from the second model will be inputted into the third model as field data. The groundwater pollutant concentration is obtained by the third model. The hydraulic head of the first model is approximated by an explicit finite difference method. An explicit finite difference technique is used to obtain the groundwater flow velocity of the second model. A forward time centered space finite difference technique is used to approximate the groundwater pollutant concentration. A long term groundwater quality around the landfill for 5, 10, 15, and 20 years are simulated. The groundwater quality is affected by the contaminated leachate pollutant release by the landfill. The proposed simulations show that the different levels of hydraulic head have a small effect on the overall groundwater quality level. Our proposed simulations found that the main groundwater quality factor is the leachate pollutant concentration around the landfill.

Third, the groundwater pollution due to a landfill around a considered area is focused. The chloride compound dispersion model groundwater pollutant concentration such as that provides the total chloride concentration, the hypochlorite concentration, the chlorite concentration, the chlorate concentration and the perchlorate concentration. A finite different techniques are used to approximate the models. The groundwater flow model is approximated by forward time centered space finite difference technique. The groundwater flow velocity model is approximated by forward space technique, and the last model, groundwater pollution dispersion model is approximated by forward time centered space finite difference technique.

The groundwater quality is affected by the chloride release by the landfill. The simulations show that the different levels of hydraulic head have a small effect on the overall groundwater quality level. In our simulation found that the main groundwater quality factor are pollutant concentration level around the landfill and transform rate.

Furthermore, if a landfill project is proposed in a certain location, the environmental impact assessment should be included in the future simulated groundwater quality evaluation.

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Appendix

Appendix A

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A Two-dimensional Mathematical Model for Long-term Contaminated Groundwater Pollution Measurement around a Land Fill

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Abstract A source of contaminated groundwater is governed by the disposal of waste material on a land fill. There are many people in rural areas where the primary source of drinking water is well water. This well water may be contaminated with groundwater from landfills. In this research, a two-dimensional mathematical model for long-term contaminated groundwater pollution measurement around a land fill is proposed. The model is governed by a combination of two models. The first model is a transient two-dimensional groundwater flow model that provides the hydraulic head of the groundwater. The second model is a transient two-dimensional advection-diffusion equation that provides the groundwater pollutant concentration. The proposed explicit finite difference techniques are used to approximate the hydraulic head and the groundwater pollutant concentration. The simulations can be used to indicate when each simulated zone becomes a hazardous zone or a protection zone.

Keywords Groundwater Pollution, Landfill, Contamination, Finite Difference Method, Two-dimensional

1 Introduction

The importance of the utilization of groundwater resources continues to grow due to the increasing requirement for water for irrigation, drinking, commercial, agricultural and industrial proposes. From this, we can see that water become an even more important part of human life. What is the effect of contaminated water on humans? Contaminated groundwater can enter the food chain and cause many life-threatening diseases and problems. The effect of drinking contaminated or dirty water causes waterborne disease. Contaminated water can cause many types of diarrheal diseases, including Cholera, and other serious illnesses, such as Guinea worm disease, Typhoid and Dysentery. The solution to this problem is to approximate when the water from the primary source of drinking will become a hazardous zone. Hence, sustainable management planning must be developed for groundwater systems. In general, the level of pollution in groundwater can be ascertained from field data sites. However, this is rather complicated, and the results obtained tentatively deviate in measurement from one point in each time/place.

In [3], the modified MacCormack scheme improved accuracy in water quality measurement in a nonuniform water flow in a stream. A mathematical model was used to simulate a water current and the elevation in a uniform reservoir in [6]. There was some research that concerned solving the dispersal of pollutants in a river in [8]. Groundwater models and their application were represented in [1, 2]. Transient groundwater modeling using spreadsheet simulation can be solved by using the finite difference method, which was described in [1]. The results are combined with MODFLOW results, a well-known model code for groundwater modeling. The objective of [2] was to propose a simple and flexible difference method and consider the complex geometry by using grid sizes aquifer parameters, sinks, and source terms. In this research, we consider groundwater that has been contaminated by waste material on a landfill. The simulation of the contaminated groundwater pollution model required data concerned with the velocity of the current points and any time in the domain. The transient two-dimensional groundwater flow model provided the velocity field of the groundwater. We used the transient two-dimensional advection-diffusion equation to approximate the groundwater pollutant concentration. The finite difference [9, 10, 11] and finite elements methods [12, 13, 14] are the most popular numerical solution techniques. Among of these, finite difference methods, including both explicit and implicit schemes, are mostly used for two-dimensional domain such as in latitude and longitude stream.

These two mathematical models were used to simulate contaminated groundwater pollution. The first was a transient

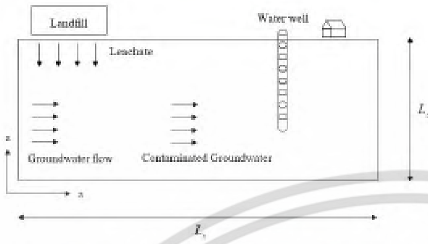

Figure 1. General structure of problem domain.

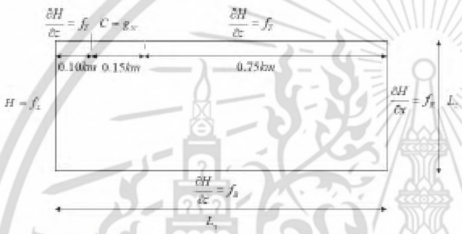

Figure 2. General structure of problem domain.

two-dimensional groundwater flow model that provided the hydraulic head of the groundwater. The second was a transient two-dimensional advection diffusion equation that gave the groundwater pollutant concentration. Both of the models were formulated in two-dimensional equations. The explicit method was used in both models. For each time, the calculated flow velocity fields of the first model were inputted into the second model as the field data.

2 Governing equation

2.1 Transient groundwater flow model

The governing equation of a latitudinally integrated Darcy's flow in a two-dimensional advection-diffusion equation [1],

$$S\frac{\partial H(x,z,t)}{\partial t} = \frac{\partial}{\partial x}\left(K_x\frac{\partial H(x,z,t)}{\partial x}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial H(x,z,t)}{\partial z}\right) \quad (2.1)$$

where $H(x,z,t)$ is the hydraulic head(metre), S matrix of specific storage(1/metre), L_x is the considered area length, L_z is the depth of considered groundwater area, T is the stationary time of simulation as shown in Figure 1. The hydraulic conductivity(metre/day) component in the x, z directions are denoted by K_x, K_z, respectively. Assuming that the soil topography in the considered area is homogeneous, these mean that the hydraulic conductivity are constant.

$$S\frac{\partial H(x,z,t)}{\partial t} = K_x\frac{\partial^2 H(x,z,t)}{\partial x} + K_z\frac{\partial^2 H(x,z,t)}{\partial z}. \quad (2.2)$$

for all $(x,z,t) \in \Omega$ such that $\Omega = [0, L_x] \times [0, L_z] \times [0, T]$.

2.2 Initial and boundary condition of transient groundwater flow model

2.2.1 The initial condition of transient groundwater flow model

The initial condition is defined by an interpolation function of measured raw data as be low

$$H(x,z,0) = h(x,z), \quad (2.3)$$

where $h(x,z)$ is a given potential hydraulic head function.

2.2.2 The boundary conditions

The top, right and bottom boundary conditions are assumed by the averaged rate of change of hydraulic head around the top, right and bottom boundaries. The left boundary condition is assumed by the interpolation function of measured raw data in the considered landfill as shown in Figure 1. The boundary condition, are also assumed by

$$h(x,z,t) = h_L(z), \quad \text{for all } z \in [0, I_z] \text{ and } x = 0, \quad (2.4)$$

$$\frac{\partial h(x,z,t)}{\partial z} = h_T(x), \quad \text{for all } x \in [0, I_x] \text{ and } z = M_z, \quad (2.5)$$

$$\frac{\partial h(x,z,t)}{\partial x} = h_R(z), \quad \text{for all } z \in [0, I_z] \text{ and } x = M_x, \quad (2.6)$$

$$\frac{\partial h(x,z,t)}{\partial z} = h_B(x), \quad \text{for all } x \in [0, I_x] \text{ and } z = 0, \quad (2.7)$$

where $h_L(z), h_T(x), h_R(z)$ and $h_B(x)$ are the boundary source of hydraulic head on the left boundary domain. The rate of change hydraulic head with respect to domain boundaries around the top, the bottom and the right bottom around the considered area as shown in Figure 3, respectively.

2.3 Groundwater flow velocity model

We can obtain that the groundwater flow velocity in x-direction is a decreasing rate of change of the hydraulic head x-direction,

$$u = -\frac{\partial H}{\partial x}. \quad (2.8)$$

Similarly, the groundwater flow velocity in z-direction is a decreasing rate of change of the hydraulic head in z-direction,

$$w = -\frac{\partial H}{\partial z}. \quad (2.9)$$

2.4 A horizontal averaged contaminated groundwater dispersion model

A horizontal averaged two-dimensional advection-diffusion equation (ADE) is expressed as follow

$$\frac{\partial c(x,z,t)}{\partial t} + u\frac{\partial c(x,z,t)}{\partial x} + w\frac{\partial c(x,z,t)}{\partial z} = D_x\frac{\partial^2 c(x,z,t)}{\partial x^2} + D_z\frac{\partial^2 c(x,z,t)}{\partial z^2} + Q, \quad (2.10)$$

for all $(x, z, t) \in \Omega$ such that $\Omega = [0, L_x] \times [0, L_z] \times [0, T]$, where $c(x, z, t)$ is the groundwater pollutant concentration(kg/m^3), D_x, D_z are the diffusion coefficient in x- and z-directions and $u(x, z, t), w(x, z, t)$ are the groundwater flow velocity in the x- and z-directions.

2.5 Initial and boundary conditions of the contaminated groundwater dispersion model

2.5.1 Initial condition of the contaminated groundwater dispersion model

The initial condition is defined by an interpolation function of the potential groundwater pollutant concentration,

$$c(x, z, 0) = c_0(x, z). \tag{2.11}$$

for all $(x, z) \in [0, L_x] \times [0, L_z]$.

2.5.2 The boundary conditions

The left boundary condition is assumed by the interpolation function of measured raw data at the considered landfill. The top, right and bottom boundary conditions are assumed by the averaged rates of change of pollutant concentration around the top, right and bottom boundaries. The boundary conditions are also assumed by

$$c(L_z, x, t) = g_N(x), \quad \text{for all } x \in [k_1 L_x, k_2 L_x], \tag{2.12}$$

$$\frac{\partial c(z, 0, t)}{dx} = g_L(z), \quad \text{for all } z \in [0, L_z] \text{ and } t \in [0, T], \tag{2.13}$$

$$\frac{\partial c(L_z, x, t)}{dz} = g_T(x), \quad \text{for all } x \in [0, k_1 L_x) \cup (k_2 L_x, L_x] \text{ and } t \in [0, T], \tag{2.14}$$

$$\frac{\partial c(z, L_x, t)}{dx} = g_R(z), \quad \text{for all } z \in [0, L_z] \text{ and } t \in [0, T], \tag{2.15}$$

$$\frac{\partial c(x, 0, t)}{dz} = g_B(x), \quad \text{for all } x \in [0, L_x] \text{ and } t \in [0, T], \tag{2.16}$$

where $k_1 L_x, k_2 L_x$ are referred to the range area of the groundwater pollutant area source, and $g_L(z), g_T(x), g_R(z)$ and $g_B(x)$ are the rate of change pollutant concentration with respect distance around the top, the bottom and the right boundaries along the considered area, respectively.

3 Numerical techniques

In this paper, we will propose finite difference methods to the transient groundwater model by using the forward time central space method. We now discretize the domain by dividing the interval $[0, L_x]$ and $[0, L_z]$ into M_x and M_z subintervals such that $M_x \Delta x = L_x, M_z \Delta z = L_z$ and the time interval $[0, T]$ into N subintervals such that $N \Delta t = T$. The grid points (x_i, z_j, t_n) are defined by $x_i = i\Delta x$ for all $i = 1, 2, ..., M_x$, $z_j = j\Delta z$ for all $j = 1, 2, ..., M_z$ and $t_n = n\Delta t$ for all $n = 1, 2, ..., T$. We can then approximate $H(x_i, z_j, t_n)$ by $H^n_{i,j}$, value of the difference approximation of $H(x, z, t)$ at point $x = i\Delta x, z = j\Delta z$ and $t = n\Delta t$, where $0 \le i \le M_x, 0 \le j \le M_z$ and $0 \le n \le N$ which M_x, M_z and N are positive integers.

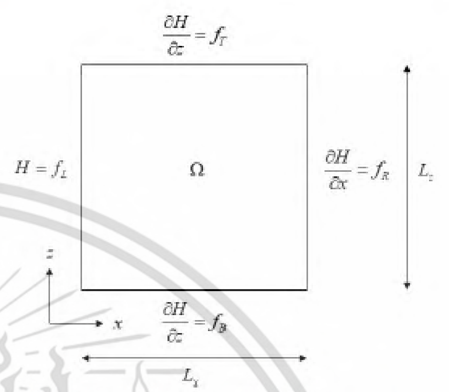

Figure 3. Boundary conditions of a transient groundwater flow model.

3.1 Explicit finite difference method for two-dimensional groundwater flow model

Taking the central difference in space and forward difference in time into scheme each terms of Eq. (2.2), we have

$$H(x, z, t) \simeq H^n_{i,j}, \tag{3.1}$$

$$\left.\frac{\partial H}{\partial t}\right|_{(x_i, z_j, t_n)} \simeq \frac{H^{n+1}_{i,j} - H^n_{i,j}}{\Delta t}, \tag{3.2}$$

$$\left.\frac{\partial^2 H}{\partial x^2}\right|_{(x_i, z_j, t_n)} \simeq \frac{H^n_{i,j-1} - 2H^n_{i,j} + H^n_{i,j+1}}{(\Delta x)^2}, \tag{3.3}$$

$$\left.\frac{\partial^2 H}{\partial z^2}\right|_{(x_i, z_j, t_n)} \simeq \frac{H^n_{i-1,j} - 2H^n_{i,j} + H^n_{i+1,j}}{(\Delta z)^2}. \tag{3.4}$$

Substituting Eqs.(3.1)-(3.4) into Eq.(2.2), we get the finite equation,

$$S\left(\frac{H^{n+1}_{i,j} - H^n_{i,j}}{\Delta t}\right) = K_x\left(\frac{H^n_{i,j-1} - 2H^n_{i,j} + H^n_{i,j+1}}{(\Delta x)^2}\right) + K_z\left(\frac{H^n_{i-1,j} - 2H^n_{i,j} + H^n_{i+1,j}}{(\Delta z)^2}\right), \tag{3.5}$$

for all $i = 1, 2, 3, ..., M_x, j = 1, 2, 3, ..., M_z$ and $n = 1, 2, 3, ..., N - 1$. Then the explicit finite difference equation becomes

$$H^{n+1}_{i,j} = \alpha H^n_{i,j-1} + \alpha H^n_{i,j+1} + (1 - 2\alpha - 2\beta)H^n_{i,j} + \beta H^n_{i-1,j} + \beta H^n_{i+1,j}, \tag{3.6}$$

where $\alpha = \dfrac{K_x(\Delta t)}{S(\Delta x)^2}$ and $\beta = \dfrac{K_z(\Delta t)}{S(\Delta z)^2}$, for all $i = 1, 2, ..., M_x - 1$ and $j = 1, 2, ..., M_z - 1$. According to the right boundary condition Eqs.(2.5) - (2.7), substituting the approximated unknown value to each boundaries,

For $j = M_x$ and $0 < i < M_z - 1$ at $t > 0$, letting that

$$H^n_{i,M_x+1} = H^n_{i,M_x} + f_R(z)\Delta x. \qquad (3.7)$$

It follow that

$$\begin{aligned} H^{n+1}_{i,M_x} &= \alpha H^n_{i,M_x-1} + (1-\alpha-2\beta)H^n_{i,M_x} + \beta H^n_{i-1,M_x} \\ &+ \beta H^n_{i+1,M_x} + \alpha f_R(z)\Delta x. \end{aligned} \qquad (3.8)$$

For $i = 0$ and $0 < j < M_x - 1$ at $t > 0$, letting that

$$H^n_{-1,j} = H^n_{0,j} - f_B(x)\Delta z. \qquad (3.9)$$

It follow that

$$\begin{aligned} H^{n+1}_{0,j} &= \alpha H^n_{0,j-1} + \alpha H^n_{0,j+1} + (1-2\alpha-\beta)H^n_{0,j} \\ &+ \beta H^n_{1,j} - \beta f_B(x)\Delta z. \end{aligned} \qquad (3.10)$$

For $i = M_z$ and $0 < j < M_x - 1$ at $t > 0$, letting that

$$H^{n+1}_{M_z,j} = H^n_{M_z,j} + f_T(x)\Delta z. \qquad (3.11)$$

It follow that

$$\begin{aligned} H^{n+1}_{M_z,j} &= \alpha H^n_{M_z,j-1} + \alpha H^n_{M_z,j+1} + (1-2\alpha-\beta)H^n_{M_z,j} \\ &+ \beta H^n_{M_z-1,j} + \beta f_T(x)\Delta z. \end{aligned} \qquad (3.12)$$

3.2 Finite difference method for groundwater flow velocity model

Taking the forward difference in space into Eq.(2.8) and Eq.(2.9), we have

$$v(x,z,t) \simeq v^n_{i,j}, \qquad (3.13)$$

$$w(x,z,t) \simeq w^n_{i,j}, \qquad (3.14)$$

$$\left.\frac{\partial H}{\partial x}\right|_{(x_i,z_j,t_n)} \simeq \frac{H^n_{i,j+1} - H^n_{i,j}}{\Delta x}, \qquad (3.15)$$

$$\left.\frac{\partial H}{\partial z}\right|_{(x_i,z_j,t_n)} \simeq \frac{H^n_{i+1,j} - H^n_{i,j}}{\Delta z}. \qquad (3.16)$$

Substituting Eqs.(3.15) - (3.16) into (2.8) and (2.9), we get the finite equation,

$$v^n_{i,j} = -\frac{1}{\Delta x}(H^n_{i,j+1} - H^n_{i,j}) \qquad (3.17)$$

and

$$w^n_{i,j} = -\frac{1}{\Delta z}(H^n_{i+1,j} - H^n_{i,j}). \qquad (3.18)$$

3.3 Explicit finite difference method for two-dimensional groundwater pollution dispersion model

In this section, the considered domain is defined in a similar grid spacing as the previous. We will employ forward time central space difference scheme (FTCS) to Eqs.(2.10)

$$c(x,z,t) \simeq C^n_{i,j}, \qquad (3.19)$$

$$\left.\frac{\partial c}{\partial t}\right|_{(x_i,z_j,t_n)} \simeq \frac{C^{n+1}_{i,j} - C^n_{i,j}}{\Delta t}, \qquad (3.20)$$

$$\left.\frac{\partial c}{\partial x}\right|_{(x_i,z_j,t_n)} \simeq \frac{C^n_{i,j-1} - C^n_{i,j+1}}{2\Delta x}, \qquad (3.21)$$

$$\left.\frac{\partial c}{\partial z}\right|_{(x_i,z_j,t_n)} \simeq \frac{C^n_{i-1,j} - C^n_{i+1,j}}{2\Delta z}, \qquad (3.22)$$

$$\left.\frac{\partial^2 c}{\partial x^2}\right|_{(x_i,z_j,t_n)} \simeq \frac{C^n_{i,j-1} - 2C^n_{i,j} + C^n_{i,j+1}}{(\Delta x)^2}, \qquad (3.23)$$

$$\left.\frac{\partial^2 c}{\partial z^2}\right|_{(x_i,z_j,t_n)} \simeq \frac{C^n_{i-1,j} - 2C^n_{i,j} + C^n_{i+1,j}}{(\Delta z)^2}. \qquad (3.24)$$

Substituting Eqs.(3.19) - (3.24) into (2.10), we get the finite equation,

$$\begin{aligned} &\frac{C^{n+1}_{i,j} - C^n_{i,j}}{\Delta t} + u^n_{i,j}\left(\frac{C^n_{i+1,j} - C^n_{i-1,j}}{2\Delta x}\right) + w^n_{i,j}\left(\frac{C^n_{i,j+1} - C^n_{i,j-1}}{2\Delta z}\right) \\ &= D_x\left(\frac{C^n_{i,j-1} - 2C^n_{i,j} + C^n_{i,j+1}}{(\Delta x)^2}\right) + D_z\left(\frac{C^n_{i-1,j} - 2C^n_{i,j} + C^n_{i+1,j}}{(\Delta z)^2}\right) \\ &+ Q, \end{aligned}$$

for all $i = 1, 2, 3, ..., M_x$, $j = 1, 2, 3, ..., M_z$ and $n = 1, 2, 3, ..., N - 1$. Then the explicit finite difference equation becomes

$$\begin{aligned} C^{n+1}_{i,j} &= (\tau_1 + \tau_2)\, C^n_{i-1,j} + (\lambda_1 + \lambda_2)\, C^n_{i,j-1} + (1 - 2\lambda_1 - 2\tau_1)C^n_{i,j} \\ &+ (\tau_1 - \tau_2)\, C^n_{i+1,j} + (\lambda_1 - \lambda_2)\, C^n_{i,j+1} + Q\Delta t, \end{aligned} \qquad (3.25)$$

where $\lambda_1 = \dfrac{D_x \Delta t}{(\Delta x)^2}$, $\lambda_2 = \dfrac{u^n_{i,j}\Delta t}{2\Delta x}$, $\tau_1 = \dfrac{D_z \Delta t}{(\Delta z)^2}$, and $\tau_2 = \dfrac{w^n_{i,j}\Delta t}{2\Delta z}$.

According to the boundary condition Eqs.(2.13)-(2.16), substituting the approximate unknown value of the each boundary,

For $j = 0$ and $0 < i < M_z - 1$ at $t > 0$, letting that

$$C^n_{i,-1} = C^n_{i,0} - g_W(z)\Delta x. \qquad (3.26)$$

$$\begin{aligned} C^{n+1}_{i,0} &= (\tau_1 + \tau_2)\, C^n_{i-1,0} + (1 - \lambda_1 - 2\tau_1 - \tau_2)C^n_{i,0} \\ &+ (\tau_1 - \tau_2)\, C^n_{i+1,0} + (\lambda_1 - \lambda_2)\, C^n_{i,1} + Q\Delta t \\ &- (\lambda_1 - \lambda_2)\, g_L(z)\Delta x. \end{aligned} \qquad (3.27)$$

For $j = M_x$ and $0 < i < M_z - 1$ at $t > 0$, letting that

$$C^n_{i,M_x+1} = C^n_{i,M_x} + g_R(z)\Delta x. \qquad (3.28)$$

It follow that

$$C^{n+1}_{i,M_z} = (\tau_1+\tau_2)\,C^n_{i-1,M_z} + (1-\lambda_1-2\tau_1-\tau_2)C^n_{i,M_z} + (\tau_1-\tau_2)\,C^n_{i+1,M_z} + (\lambda_1+\lambda_2)\,C^n_{i,M_z-1} + Q\Delta t + (\lambda_1-\lambda_2)\,g_R(z)\Delta x. \quad (3.29)$$

For $i=0$ and $0<i<M_x-1$ at $t>0$, letting that

$$C^n_{-1,j} = C^n_{0,j} - g_B(x)\Delta z. \quad (3.30)$$

It follow that

$$C^{n+1}_{0,j} = (\lambda_1+\lambda_2)\,C^n_{0,j-1} + (1-2\lambda_1-\tau_1+\tau_2)C^n_{0,j} + (\tau_1-\tau_2)\,C^n_{1,j} + (\lambda_1-\lambda_2)\,C^n_{0,j+1} + Q\Delta t - (\tau_1+\tau_2)\,g_B(x)\Delta z. \quad (3.31)$$

For $i=M_z$ and $0<j<M_x-1$ at $t>0$, letting that

$$C^n_{M_z+1,j} = C^n_{M_z,j} + g_T(x)\Delta z. \quad (3.32)$$

It follow that

$$C^{n+1}_{M_z,j} = (\tau_1+\tau_2)\,C^n_{M_z-1,j} + (1-2\lambda_1-\tau_1-\lambda_2)C^n_{M_z,j} + (\lambda_1+\lambda_2)\,C^n_{M_z,j-1} + (\lambda_1-\lambda_2)\,C^n_{M_z,j+1} + Q\Delta t + (\tau_1-\tau_2)\,g_T(x)\Delta z. \quad (3.33)$$

4 Numerical simulations

Suppose that the measurement of groundwater pollutant concentration C in a groundwater flow is considered in an underground area. The considered underground area has dimensions of 1.0 km of length and 0.5 km in depth, $L_x = 1.0$ km and $L_z = 0.5$ km, respectively. The simulations need to propose the measurement of the latitudinally averaged groundwater pollutant concentration in the considered area.

There is the landfill which is discharging leachate down to the considered under ground area. The landfill is aligned with longitudinal distance 0.15 km, as shown in Figure 2. The landfill discharges the groundwater pollutant to the underground by $C(x,0,t) = g_N(x)$ (kg/m^3) for all $0.1 \le x \le 0.25$.

Assume that the specific storage is 1 m^{-1} and the hydraulic conductivity in x- and z-direction are 15 (m/day). There is no rate of change of hydraulic head on the left, the right, the bottom of the considered domain boundaries. There is no rate of change of hydraulic head on the top boundary. We also assume that the leachate which is flowing down to the underground has a pollutant concentration of 1.0 (kg/m^3). There is no rate of change of pollutant concentration on the left, the right, the bottom domain boundaries. The related physical parameters are summarized in Table 1.

4.1 Simulation 4.1. Low leachate pollutant release rate with a low hydraulic head level.

If the proposed explicit finite difference techniques for the two-dimensional groundwater flow model Eqs.(3.6), (3.8), (3.10) and (3.12) are employed, we get approximated hydraulic head, as shown in Table 2 and Figures 4-5. If the proposed explicit finite difference technique for the groundwater flow velocity model Eqs.(3.15)-(3.16) are employed, we get the approximated groundwater flow velocity in the x- and z-direction in Table 3-4 and Figure 6. The approximated groundwater flow velocities are then plugged into the two-dimensional groundwater pollutant dispersion model. If the proposed explicit finite difference technique for the two-dimensional groundwater pollutant dispersion model Eqs.(3.25), (3.27), (3.29), (3.31) and (3.33) are employed, we get the approximated groundwater pollutant concentration as shown in Table 5 and Figures 7-8.

4.2 Simulation 4.2. Low leachate pollutant releasing rate with a high hydraulic head level.

If the proposed explicit finite difference techniques for the two-dimensional groundwater flow model Eqs.(3.6), (3.8), (3.10) and (3.12) are employed, we get an approximated hydraulic head as shown in Table 6 and Figures 9-10. If the proposed explicit finite difference technique for the groundwater flow velocity model Eqs.(3.15)-(3.16) are employed, we get the approximated groundwater flow velocity in the x- and z-direction in Table 7-8 and Figure 11. The approximated groundwater flow velocities are then plugged into the two-dimensional groundwater pollutant dispersion model. If the proposed explicit finite difference technique for the two-dimensional groundwater pollutant dispersion model Eqs.(3.25), (3.27), (3.29), (3.31) and (3.33) are employed, we get the approximated groundwater pollutant concentration as shown in Table 9 and Figures 12-13.

4.3 Simulation 4.3 Medium leachate pollutant releasing rate with a high hydraulic head level.

If the proposed explicit finite difference techniques for the two-dimensional groundwater flow model Eqs.(3.6), (3.8), (3.10) and (3.12) are employed, we get an approximated hydraulic head as shown in Table 10 and Figures 14-15. If the proposed explicit finite difference technique for the groundwater flow velocity model Eqs.(3.15)-(3.16) are employed, we get the approximated groundwater flow velocity in the x- and z-direction in Table 11-12 and Figure 16. The approximated groundwater flow velocities are then plugged into the two-dimensional groundwater pollutant dispersion model. If the proposed explicit finite difference technique for the two-dimensional groundwater pollutant dispersion model Eqs.(3.25), (3.27), (3.29), (3.31) and (3.33) are employed, we get the approximated groundwater pollutant concentration as shown in Table 13 and Figures 17-18.

4.4 Simulation 4.4. High leachate pollutant releasing rate with a high hydraulic head level.

If the proposed explicit finite difference techniques for the two-dimensional groundwater flow model Eqs.(3.6), (3.8), (3.10) and (3.12) are employed, we get an approximated hydraulic head as shown in Table 14 and Figures 19-20. If the proposed explicit finite difference technique for the groundwater flow velocity model Eqs.(3.15)-(3.16) are employed, we get the approximated groundwater flow velocity in the x- and z-direction in Table 15-16 and Figure 21. The approximated groundwater flow velocities are then plugged into the two-dimensional groundwater pollutant dispersion model. If the proposed explicit finite difference technique for the two-dimensional groundwater pollutant dispersion model Eqs.(3.25), (3.27), (3.29), (3.31) and (3.33) are employed, we get the approximated groundwater pollutant concentration as shown in Table 17 and Figures 22-23.

4.5 Simulation 4.5. Practical leachate pollutant releasing rate with a high hydraulic head level.

If the proposed explicit finite difference techniques for the two-dimensional groundwater flow model Eqs.(3.6), (3.8), (3.10) and (3.12) are employed, we get an approximated hydraulic head as shown in Table 18 and Figures 24-25. If the proposed explicit finite difference technique for the groundwater flow velocity model Eqs.(3.15)-(3.16) are employed, we get the approximated groundwater flow velocity in the x- and z-direction in Table 19-20 and Figure 26. The approximated groundwater flow velocities are then plugged into the two-dimensional groundwater pollutant dispersion model. If the proposed explicit finite difference technique for the two-dimensional groundwater pollutant dispersion model Eqs.(3.25), (3.27), (3.29), (3.31) and (3.33) are employed, we get the approximated groundwater pollutant concentration as shown in Table 21 and Figures 27-28.

Table 1. The configuration in each simulations

	S	K_x	K_z	f_T	f_B	f_R	f_L	D_x	D_z	Q	g_N	g_T	g_B	g_R	g_L
Simulation 4.1	1	15	15	0	0	0	10	1.5	1.5	0	1	0	0	0	0
Simulation 4.2	1	15	15	0	0	0	40	1.5	1.5	0	1	0	0	0	0
Simulation 4.3	1	15	15	0	0	0	40	1.5	1.5	0	2	0	0	0	0
Simulation 4.4	1	15	15	0	0	0	40	1.5	1.5	0	4	0	0	0	0
Simulation 4.5	1	15	15	0	0	0	$0.06z + 10$	1.5	1.5	0	4	0	0	0	0

Table 2. The approximated hydraulic head(m) of simulation 4.1 where $z = 50$ m.

	$H(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	10.0000	9.3140	8.6331	7.9622	7.3061	6.9849
10	10.0000	9.5147	9.0312	8.5512	8.0766	7.8418
15	10.0000	9.6037	9.2083	8.8149	8.4244	8.2306
20	10.0000	9.6568	9.3142	8.9729	8.6335	8.4647

Table 3. The approximated groundwater flow velocity in x-direction(m/day) of simulation 4.1 where $z = 50$ m.

	$u(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.0343	0.0341	0.0337	0.0330	0.0321	0.0316
10	0.0243	0.0242	0.0241	0.0238	0.0235	0.0233
15	0.0198	0.0197	0.0196	0.0195	0.0193	0.0192
20	0.0172	0.0171	0.0171	0.0170	0.0169	0.0167

Table 4. The approximated groundwater flow velocity in z-direction(m/day) of simulation 4.1 where $z = 50$ m.

	$w(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0	0	0	0	0	0
10	0	0	0	0	0	0
15	0	0	0	0	0	0
20	0	0	0	0	0	0

Table 5. The approximated groundwater pollutant concentration(kg/m^3) of simulation 4.1 where $z = 50$ m.

	$C(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.0148	0.0198	0.0348	0.0654	0.1153	0.1467
10	0.0683	0.0768	0.0997	0.1408	0.2006	0.2356
15	0.1324	0.1419	0.1668	0.2094	0.2689	0.3029
20	0.1945	0.2042	0.2289	0.2703	0.3269	0.3587

Table 6. The approximated hydraulic head(m) of simulation 4.2 where $z = 50$ m.

	$H(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	40.0000	37.2561	34.5325	31.8490	29.2244	27.9397
10	40.0000	38.0587	36.1247	34.2049	32.3063	31.3670
15	40.0000	38.4147	36.8333	35.2597	33.6977	32.9223
20	40.0000	38.6272	37.2569	35.8918	34.5342	33.8590

Table 7. The approximated groundwater flow velocity in x-direction(m/day) of simulation 4.2 where $z = 50$ m.

	$u(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.1371	0.1366	0.1348	0.1321	0.1285	0.1263
10	0.0970	0.0968	0.0962	0.0952	0.0939	0.0931
15	0.0792	0.0791	0.0788	0.0783	0.0775	0.0771
20	0.0686	0.0686	0.0683	0.0680	0.0675	0.0672

Table 8. The approximated groundwater flow velocity in z-direction(m/day) of simulation 4.2 where $z = 50$ m.

	$w(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0	0	0	0	0	0
10	0	0	0	0	0	0
15	0	0	0	0	0	0
20	0	0	0	0	0	0

Table 9. The approximated groundwater pollutant concentration(kg/m^3) of simulation 4.2 where $z = 50$ m.

	$C(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.0000	0.0000	0.0002	0.0013	0.0067	0.0735
10	0.0006	0.0009	0.0023	0.0074	0.0226	0.0367
15	0.0035	0.0042	0.0076	0.0174	0.0410	0.0601
20	0.0097	0.0111	0.0164	0.0305	0.0605	0.0832

Table 10. The approximated hydraulic head(m) of simulation 4.3 where $z = 50$ m.

	$H(x,z,t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	40.0000	37.2561	34.5325	31.8490	29.2244	27.9397
10	40.0000	38.0587	36.1247	34.2049	32.3063	31.3670
15	40.0000	38.4147	36.8333	35.2597	33.6977	32.9223
20	40.0000	38.6272	37.2569	35.8918	34.5342	33.8590

Table 11. The approximated groundwater flow velocity in x-direction(m/day) of simulation 4.3 where $z = 50$ m.

	$u(x,z,t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	0.1371	0.1366	0.1348	0.1321	0.1285	0.1263
10	0.0970	0.0968	0.0962	0.0952	0.0939	0.0931
15	0.0792	0.0791	0.0788	0.0783	0.0775	0.0771
20	0.0686	0.0686	0.0683	0.0680	0.0675	0.0672

Table 12. The approximated groundwater flow velocity in z-direction(m/day) of simulation 4.3 where $z = 50$ m.

	$w(x,z,t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
5	0	0	0	0	0	0
10	0	0	0	0	0	0
15	0	0	0	0	0	0
20	0	0	0	0	0	0

Table 13. The approximated groundwater pollutant concentration(kg/m^3) of simulation 4.3 where $z = 50$ m.

	$C(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.0000	0.0000	0.0004	0.0025	0.0134	0.0270
10	0.0012	0.0017	0.0045	0.0148	0.0453	0.0734
15	0.0069	0.0085	0.0151	0.0348	0.0819	0.1203
20	0.0194	0.0222	0.0329	0.0610	0.1209	0.1663

Table 14. The approximated hydraulic head(m) of simulation 4.4 where $z = 50$ m.

	$H(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	40.0000	37.2561	34.5325	31.8490	29.2244	27.9397
10	40.0000	38.0587	36.1247	34.2049	32.3063	31.3670
15	40.0000	38.4147	36.8333	35.2597	33.6977	32.9223
20	40.0000	38.6272	37.2569	35.8918	34.5342	33.8590

Table 15. The approximated groundwater flow velocity in x-direction(m/day) of simulation 4.4 where $z = 50$ m.

	$u(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.1371	0.1366	0.1348	0.1321	0.1285	0.1263
10	0.0970	0.0968	0.0962	0.0952	0.0939	0.0931
15	0.0792	0.0791	0.0788	0.0783	0.0775	0.0771
20	0.0686	0.0686	0.0683	0.0680	0.0675	0.0672

Table 16. The approximated groundwater flow velocity in z-direction(m/day) of simulation 4.4 where $z = 50$ m.

	$w(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0	0	0	0	0	0
10	0	0	0	0	0	0
15	0	0	0	0	0	0
20	0	0	0	0	0	0

Table 17. The approximated groundwater pollutant concentration(kg/m^3) of simulation 4.4 where $z = 50$ m.

	$C(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.0000	0.0001	0.0007	0.0050	0.0268	0.0540
10	0.0023	0.0034	0.0090	0.0297	0.0906	0.1468
15	0.0138	0.0170	0.0303	0.0697	0.1639	0.2406
20	0.0389	0.0444	0.0658	0.1220	0.2418	0.3327

Table 18. The approximated hydraulic head(m) of simulation 4.5 where $z = 50$ m.

	$H(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	36.4000	33.4092	30.4953	27.7080	25.0706	23.8111
10	36.4000	33.9669	31.6015	29.3447	27.2115	26.1917
15	36.4000	34.1979	32.0615	30.0295	28.1148	27.2014
20	36.4000	34.3326	32.3299	30.4300	28.6448	27.7948

Table 19. The approximated groundwater flow velocity in x-direction(m/day) of simulation 4.5 where $z = 50$ m.

	$u(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.1490	0.1470	0.1411	0.1338	0.1260	0.1220
10	0.1212	0.1194	0.1143	0.1082	0.1020	0.0989
15	0.1096	0.1080	0.1030	0.0972	0.0913	0.0885
20	0.1029	0.1013	0.0964	0.0907	0.0850	0.0823

Table 20. The approximated groundwater flow velocity in z-direction(m/day) of simulation 4.5 where $z = 50$ m.

	$w(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	-0.0600	-0.0472	-0.0366	-0.0287	-0.0228	-0.0205
10	-0.0600	-0.0473	-0.0369	-0.0291	-0.0233	-0.0211
15	-0.0600	-0.0473	-0.0369	-0.0291	-0.0234	-0.0212
20	-0.0600	-0.0473	-0.0369	-0.0291	-0.0234	-0.0212

Table 21. The approximated groundwater pollutant concentration(kg/m^3) of simulation 4.5 where $z = 50$ m.

	$C(x,z,t)$					
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
5	0.0000	0.0001	0.0008	0.0064	0.0360	0.0733
10	0.0008	0.0014	0.0053	0.0237	0.0885	0.1537
15	0.0038	0.0050	0.0124	0.0419	0.1311	0.2132
20	0.0093	0.0112	0.0217	0.0600	0.1669	0.2606

5 Discussion

The hydraulic head can drive the groundwater flow from the higher hydraulic head level zone to the lower zone, as shown in Tables 2-4, 6-8, 10-12, and 14-16. The direction of groundwater is illustrated by their velocity field, as shown in Fig 4-6, 9-11, 14-16, and 19-20. In Simulation 4.1, a case of low leachate pollutant release rate from a landfill with a low-level hydraulic head is considered. If the level of the hydraulic head is increased fourfold, the overall groundwater quality becomes close to the last case, as shown in Simulation 4.2. On the other hand, if the level of leachate pollutant release rate is increased, the overall groundwater quality becomes as poor as the pollutant concentration at the considered landfill, as shown in Simulation 4.3. If the leachate releasing rate is poor or there is a high pollutant concentration, the overall groundwater quality becomes poor as well, as shown in Simulation 4.4. In Simulation 4.5, a case of practical leachate pollutant releasing rate is also tested by using numerical data interpolation $f_L(x)$.

6 Conclusion

Three two-dimensional models of long-term contaminated groundwater pollutant measurement around a landfill are introduced. The first model is the two-dimensional transient groundwater flow model, which provides the hydraulic head. The second model is the groundwater flow velocity model, which provides the groundwater flow velocity in the x- and z-directions. The third model is the two-dimensional horizontal averaged contaminated groundwater dispersion model, which provides the groundwater pollutant concentration. A method to set up the initial condition and boundary conditions of the proposed transient groundwater flow model is proposed. The computed hydraulic head is transformed to be the groundwater flow velocity by using the second model. The results from the second model will be inputted into the third model as field data. The groundwater pollutant concentration is obtained by the third model. The hydraulic head of the first model is approximated by an explicit finite difference method. An explicit finite difference technique is used to obtain the groundwater flow velocity of the second model. A forward time centered space finite difference technique is used to approximate the groundwater pollutant concentration. Long term groundwater quality around the landfill for 5, 10, 15, and 20 years are simulated. The groundwater quality is affected by the contaminated leachate pollutant release by the landfill. The proposed simulations show that the different levels of hydraulic head have a small effect on the overall groundwater quality level. Our proposed simulations found that the main groundwater quality factor is the leachate pollutant concentration around the landfill. Furthermore, a three-dimensional mathematical model for contaminated groundwater pollutant measurement should be introduced for a precise simulation.

7 Acknowledgment

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Simulation 1

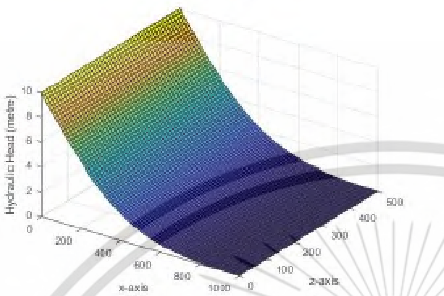

Figure 4. The surface plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

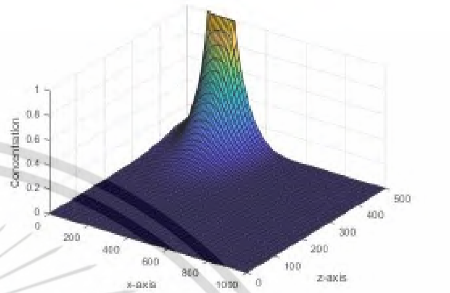

Figure 7. The surface plot of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

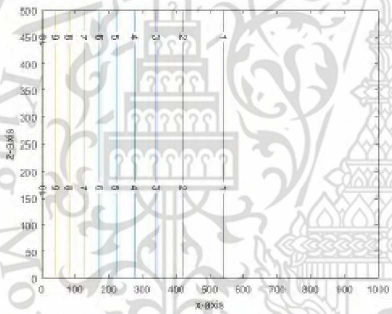

Figure 5. The contour plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

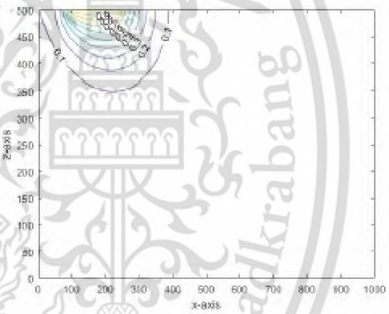

Figure 8. The contour plot of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

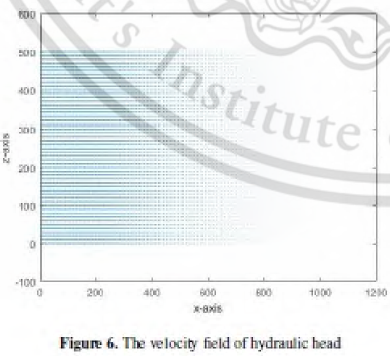

Figure 6. The velocity field of hydraulic head

Simulation 2

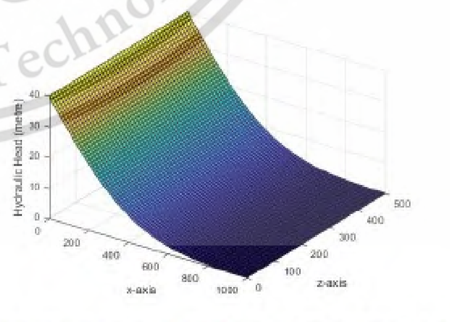

Figure 9. The surface and contour plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

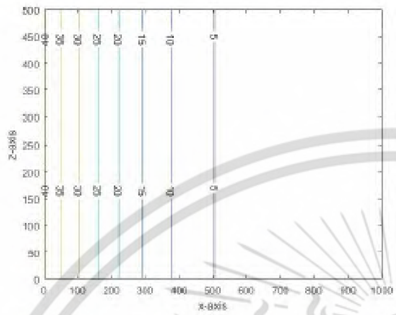

Figure 10. The surface and contour plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

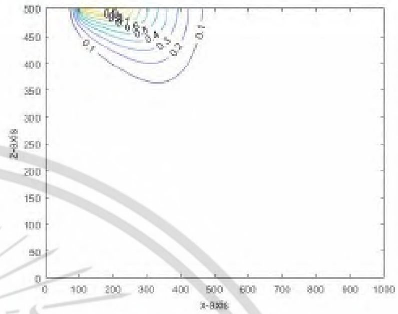

Figure 13. The contour plot of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

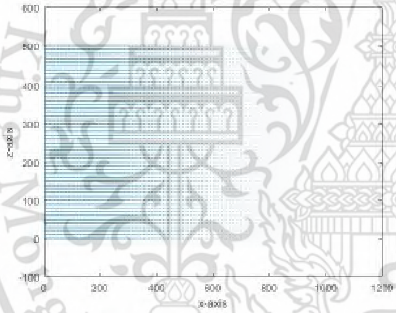

Figure 11. The velocity field of hydraulic head

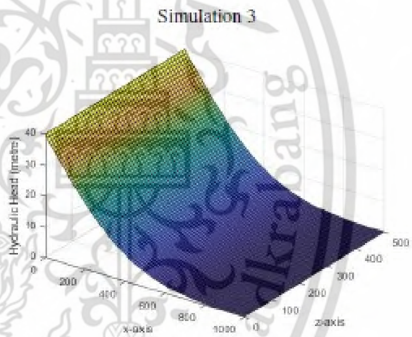

Figure 14. The surface plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

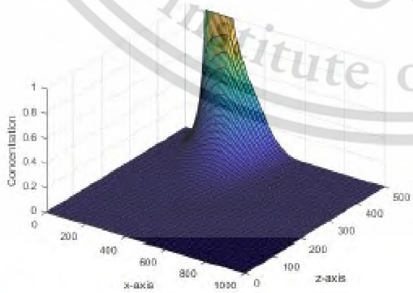

Figure 12. The surface of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

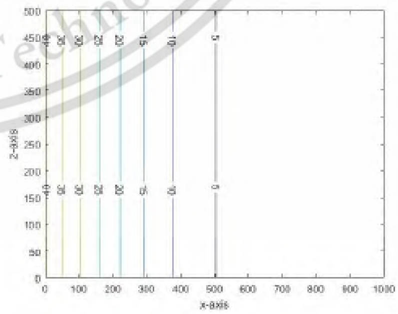

Figure 15. The contour plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

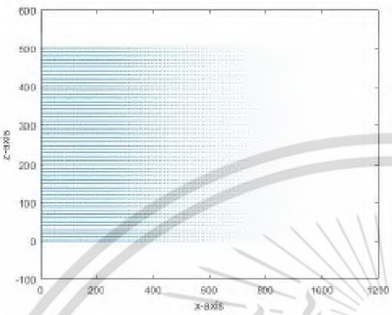

Figure 16. The velocity field of hydraulic head

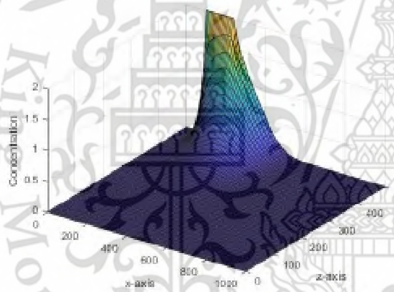

Figure 17. The surface and contour plot of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

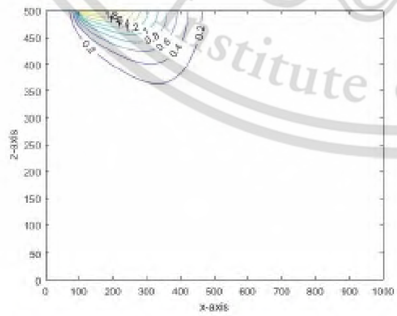

Figure 18. The surface and contour plot of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

Simulation 4

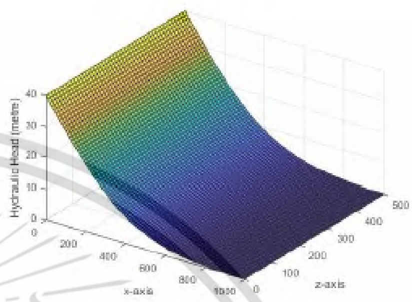

Figure 19. The surface plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

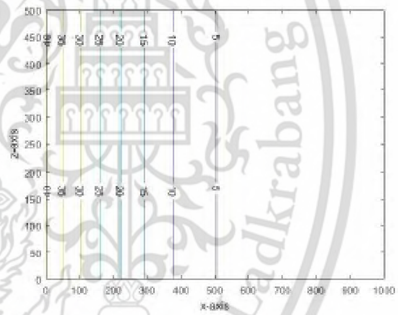

Figure 20. The contour plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

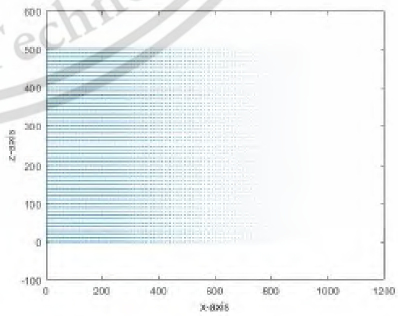

Figure 21. The velocity field of hydraulic head

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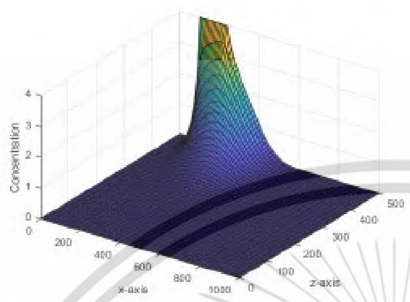

Figure 22. The surface plot of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

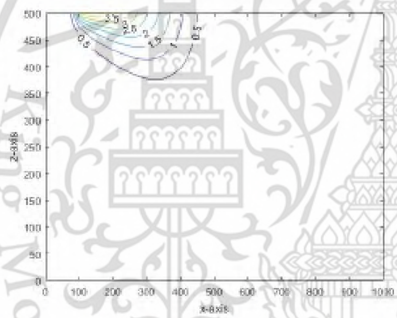

Figure 23. The contour plot of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

Simulation 5

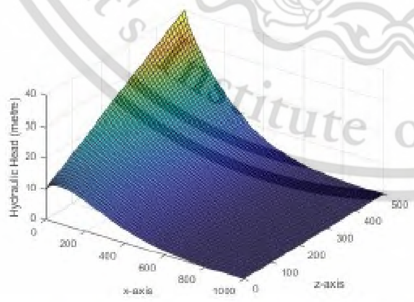

Figure 24. The surface plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

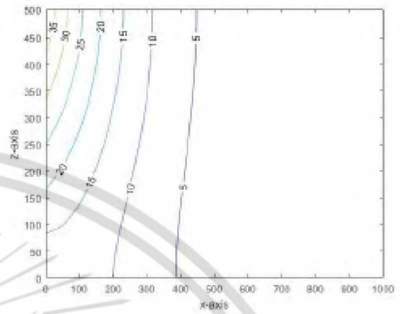

Figure 25. The contour plot of hydraulic head $H^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

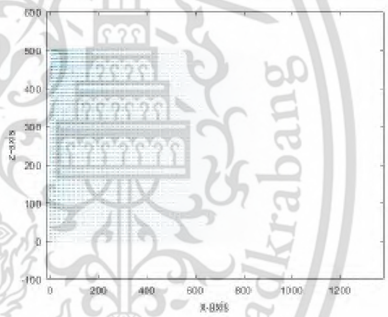

Figure 26. The velocity field of hydraulic head

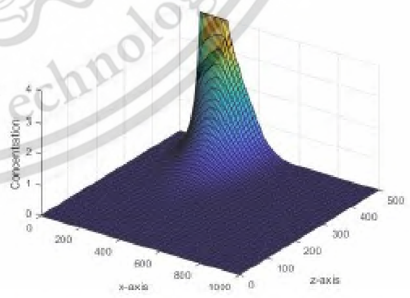

Figure 27. The surface plot of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

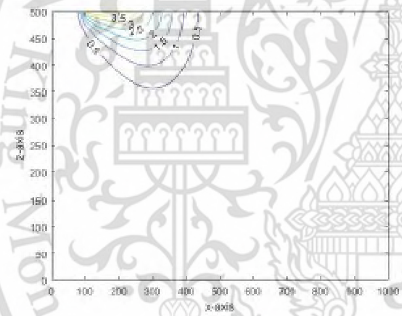

Figure 28. The contour plot of contaminated $c^n_{i,j}$ for all $(x, z, t) \in [0, 1000] \times [0, 500] \times [0, 3600]$.

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A Mathematical Model of Horizontal Averaged Groundwater Pollution Measurement with Several Substances due to Chemical Reaction

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Abstract Chloride is a well-known chemical compound that is very useful in industry and agricultural. Chloride can be transformed to hypochlorite, chlorite, chlorate and perchlorate, chloride and their substances are not dangerous if they are used in the optimal level. Groundwater containing contamination chloride and their substances impacts human health, for example, drinking water contaminated chloride exceed 250 mg/L can cause heart problems and lead to high blood pressure. To avoid this problem, we used mathematical models to explain groundwater contamination with chloride and their substances. Transient groundwater flow model provides the hydraulic head of groundwater. In this model we will get the level of groundwater, next, we need to find its velocity and direction by using the result in first model put into second model. Groundwater velocity model provides x- and z-direction vector in groundwater, after computation we will plugin the result into the last model to approximate the chloride concentration in groundwater. Groundwater contamination dispersion model provides chloride, hypochlorite, chlorite, chlorate and perchlorate concentration. The proposed explicit finite difference techniques are used to approximate the model solution. Explicit method was used to solved hydraulic head model. Forward space described groundwater velocity model. Forward time and central space used to predict transient groundwater contaminated models. The simulations can be used to indicate when each simulated zone becomes a hazardous zone or a protection zone.

Keywords Finite Difference Method, Forward Space Technique, Forward Time Centred Space Two-dimensional

1 Introduction

Nowadays, we can say that water is an important part of life, whether in daily life, agricultural and industry. The water pollution problem is created by above activity, we can found the contaminated water from the natural source, contaminated water has so much effect, it can cause many life diseases and problems. Nitrates or nitrites in water contaminates drinking water can impacts human health by decreasing blood cell ability to carry oxygen, which can be linked to blue baby syndrome [2], this is one of the effects of contaminated water. In this research we talk about the effect of chloride and their substance.

Chloride occurs naturally in groundwater but is found in greater concentrations in seawater. It generally combines with sodium, calcium, or magnesium. For example, sodium chloride (NaCl) is formed when chloride and sodium combine.

The other forms of chloride do not come from the only combination of other substances, the oxidation numbers or oxidation states is the well-know process to obtained a new form of chemical compound, for example, Chloride can be changed to hypochlorite (ClO) if the oxidation number increase by 1 or chlorite if added by 3.

oxidation state	-1	+1	+3	+5	+7
anion name	chloride	hypochlorite	chlorite	chlorate	perchlorate
formula	Cl^-	ClO^-	ClO_2^-	ClO_3^-	ClO_4^-

Figure 1. The other form of chloride in any state

They are substances used by the body to help it work well. Although chlorides are harmless at low levels, well water high in sodium chloride can damage plants if used for gardening or irrigation, and give drinking water an unpleasant taste. Sodium chloride is high corrosivity,which will also damage plumbing, appliances, and water heaters, causing toxic metals to leach into your water. It can complicate existing heart problems and contribute to high blood pressure when ingested in excess [1].

chloride	can cause heart problems and contribute to high blood pressure
hypochlorite	can cause bladder cancer
chlorite	can cause shortness of breath and other respiratory problems because of damage to the substances in blood that carry oxygen throughout the body
chlorate	chlorate is toxic, doses of a few grams of chlorate are lethal, there is a direct toxicity to the proximal renal tubule
perchlorate	at high concentrations perchlorate can affect the thyroid gland by inhibiting the uptake of iodine

Figure 2. Chloride compound damage to health

There are many dangers of chloride, but we can prevent amounts of them, from exceeding standards by management based on mathematical models. Yamashita and Sugio (2000) developed a model of advection dispersion and biochemical reactions [3]. Pochai and Kraychang (2016) solved a two mathematical models for simulating water pollutant level and pollution control in a connected reservoir system [7]. Gardenas (2005) talks about a two-dimensional modeling approach to improve fertigation strategies and soil types on nitrate leaching potential [4]. The one-dimensional advection-diffusion equation with constant coefficients have been solved by Dehghan (2004) [5]. Kewalee and Pochai (2018) described the governing equation in the air quality model in three-dimensional advection-diffusion equations with time dependence [6].

In this paper, we measured groundwater that has been contaminated by chloride on a landfill. The simulation required data concerned with the groundwater flow velocity of the current points and any time in the domain. The groundwater flow velocity can be obtained by using transient two-dimensional groundwater flow model. We used the transient two-dimensional advection diffusion equation to approximated the chloride concentration. The finite elements [19, 20, 21] and finite difference methods [16, 17, 18] are the most popular numerical solution techniques. Among of these, finite difference methods, including both explicit and implicit schemes, are mostly used for two-dimensional domain such as in latitude and longitude stream.

2 Chloride pollutant measure models

2.1 Transient groundwater flow model

The governing equation of a latitudinally integrated Darcy's flow in a two-dimensional advection-diffusion equation [8],

$$S\frac{\partial H(x,z,t)}{\partial t} = \frac{\partial}{\partial x}\left(K_x\frac{\partial H(x,z,t)}{\partial x}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial H(x,z,t)}{\partial z}\right), \tag{2.1}$$

where $H(x, z, t)$ is the hydraulic head(metre), S matrix of specific storage(1/metre), L_x is the considered area length, L_z is the depth of considered groundwater area, T is the stationary time of simulation as shown in Figure 1. The hydraulic conductivity(metre/day) component in the x, z directions are denoted by K_x, K_z, respectively. Assuming that the soil topography in the considered area is homogeneous, these mean that the hydraulic conductivity are constant.

$$S\frac{\partial H(x,z,t)}{\partial t} = K_x\frac{\partial^2 H(x,z,t)}{\partial x} + K_z\frac{\partial^2 H(x,z,t)}{\partial z}. \tag{2.2}$$

for all $(x, z, t) \in \Omega$ such that $\Omega = [0, L_x] \times [0, L_z] \times [0, T]$.

2.2 Initial and boundary condition of transient groundwater flow model

2.2.1 The initial condition of transient groundwater flow model

The initial condition is defined by an interpolation function of measured raw data as be low

$$H(x, z, 0) = h(x, z), \tag{2.3}$$

where $h(x, z)$ is a given potential hydraulic head function.

2.2.2 The boundary conditions

The top, right and bottom boundary conditions are assumed by the average rate of change of hydraulic head around the top, right and bottom boundaries. The left boundary condition is assumed by the interpolation function of measured raw data in the considered landfill as shown in Figure 1. The boundary condition, are also assumed by

$$H(x,z,t) = f_L(z), \quad \text{for all } z \in [0, I_z] \text{ and } x = 0, \tag{2.4}$$

$$\frac{\partial H(x,z,t)}{\partial z} = f_T(x), \quad \text{for all } x \in [0, I_x] \text{ and } z = M_z, \tag{2.5}$$

$$\frac{\partial H(x,z,t)}{\partial x} = f_R(z), \quad \text{for all } z \in [0, I_z] \text{ and } x = M_x, \tag{2.6}$$

$$\frac{\partial H(x,z,t)}{\partial z} = f_B(x), \quad \text{for all } x \in [0, I_x] \text{ and } z = 0, \tag{2.7}$$

where $f_L(z)$, $f_T(x)$, $f_R(z)$ and $f_B(x)$ are the boundary source of hydraulic head on the left boundary domain. The rate of change hydraulic head with respect to domain boundaries around the top, the bottom and the right bottom around the considered area are shown in Figure 2.

2.3 Groundwater flow velocity model

We can obtain that the groundwater flow velocity in x-direction is a decreasing rate of change of the hydraulic head x-direction,

$$u = -\frac{\partial H}{\partial x}. \tag{2.8}$$

Similarly, the groundwater flow velocity in z-direction is a decreasing rate of change of the hydraulic head in z-direction,

$$w = -\frac{\partial H}{\partial z}. \tag{2.9}$$

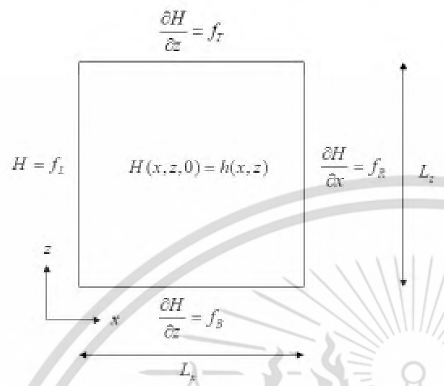

Figure 3. General structure of problem domain.

2.4 Chloride Compound Dispersion Models

2.4.1 A total chloride dispersion model

The pollutant concentration measurement of total chloride in surface water can be describe by a two-dimensional advection-diffusion reaction equation.

$$\frac{\partial c(x,z,t)}{\partial t} + u\frac{\partial c(x,z,t)}{\partial x} + w\frac{\partial c(x,z,t)}{\partial z} = D_x\frac{\partial^2 c(x,z,t)}{\partial x^2} + D_z\frac{\partial^2 c(x,z,t)}{\partial z^2} + Q - Rc, \quad (2.10)$$

for all $(x,z,t) \in \Omega$ such that $\Omega = [0, L_x] \times [0, L_z] \times [0, T]$, where $c(x,z,t)$ is total chloride pollutant concentration of groundwater(kg/m^3), $D_x and D_z$ are the diffusion coefficient in x- and z-directions, $u(x,z,t)$, $w(x,z,t)$ are the groundwater flow velocity in the x- and z-directions, and R is transformed chloride rate.

2.5 Initial condition of the total chloride dispersion model

The chloride dispersion is described with conditions in the following sections, where the potential groundwater pollutant concentration in the consider area is described by

$$c(x,z,0) = c_0(x,z). \text{ for all } (x,z) \in \Omega \quad (2.11)$$

where $c_0(x,z)$ is a averaged potential total chloride concentration in the considered area.

2.6 Boundary condition of total chloride dispersion model

The left boundary condition is assumed by the interpolation function of measured raw data at the considered landfill. The top, right and bottom boundary conditions are assumed by the averaged rates of change of pollutant concentration around the top, right and bottom boundaries. The boundary conditions are also assumed by

$$c(L_z, x, t) = g_N(x), \quad \text{for all } x \in [k_1L_x, k_2L_x], \quad (2.12)$$

$$\frac{\partial c(z,0,t)}{dx} = g_L(z), \quad \text{for all } z \in [0, L_z] \text{ and } t \in [0,T], \quad (2.13)$$

$$\frac{\partial c(L_z,x,t)}{dz} = g_T(x), \quad \text{for all } x \in [0, k_1L_x) \cup (k_2L_x, L_x] \text{ and } t \in [0,T], \quad (2.14)$$

$$\frac{\partial c(z,L_x,t)}{dx} = g_R(z), \quad \text{for all } z \in [0, L_z] \text{ and } t \in [0,T], \quad (2.15)$$

$$\frac{\partial c(x,0,t)}{dz} = g_B(x), \quad \text{for all } x \in [0, L_x] \text{ and } t \in [0,T], \quad (2.16)$$

where k_1L_x, k_2L_x are referred to as the range area of the total chloride pollutant area source, and $g_L(z), g_T(x), g_R(z)$ and $g_B(x)$ are the rate of change of the total chloride concentration with respect to distance around the top, the bottom and the right boundaries along the considered area, respectively.

2.6.1 A hypochlorite dispersion model

The total chloride is transformed to be the hypochlorite. The hypochlorite dispersion model is described by

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial z} = D1_x\frac{\partial^2 \phi}{\partial x^2} + D1_z\frac{\partial^2 \phi}{\partial z^2} + Q + R_1Rc, \quad (2.17)$$

for all $(x,z,t) \in \Omega$ such that $\Omega = [0, L_x] \times [0, L_z] \times [0, T]$, where $\phi(x,z,t)$ is hypochlorite pollutant concentration of groundwater(kg/m^3), $D1_x$, $D1_z$ are the diffusion coefficient in x- and z-directions, $u(x,z,t)$, $w(x,z,t)$ are the groundwater flow velocity in the x- and z-directions and R_1 is transformed hypochlorite rate.

2.7 Initial condition of hypochlorite dispersion model

Dispersion of hypochlorite with the following conditions, if the potential hypochlorite concentration in the consider area is described by

$$\phi(x,z,0) = f_\phi(x,z). \text{ for all } (x,z) \in \Omega \quad (2.18)$$

where $f_\phi(x,z)$ is a averaged potential hypochlorite concentration in the considered area.

2.8 Boundary condition of hypochlorite dispersion model

The rate of change of the pollutant concentration along the domain boundaries are assumed to be:

$$\frac{\partial\phi(0,z,t)}{\partial x} = g1_L(z), \text{for all } z \in [0, L_z] \text{ and } t \in [0,T], \quad (2.19)$$

$$\frac{\partial\phi(x,L_z,t)}{\partial z} = g1_T(x), \text{for all } x \in [0, L_x] \text{ and } t \in [0,T], \quad (2.20)$$

$$\frac{\partial\phi(L_x,z,t)}{\partial x} = g1_R(z), \text{for all } z \in [0, L_x] \text{ and } t \in [0,T], \quad (2.21)$$

$$\frac{\partial\phi(x,0,t)}{\partial z} = g1_B(x), \text{for all } x \in [0, L_x] \text{ and } t \in [0,T], \quad (2.22)$$

where $g1_L(z), g1_T(x), g1_R(z)$ and $g1_B(x)$ are the rate of change of the hypochlorite concentration with respect distance around the top, the bottom and the right boundaries along the considered area, respectively.

2.8.1 A chlorite dispersion model

The model of chlorite dispersion model can be described by

$$\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}+v\frac{\partial\eta}{\partial z} = D2_x\frac{\partial^2\eta}{\partial x^2}+D2_z\frac{\partial^2\eta}{\partial z^2}+Q+R_2Rc, \quad (2.23)$$

for all $(x,z,t) \in \Omega$ such that $\Omega = [0,L_x] \times [0,L_z] \times [0,T]$, where $\eta(x,z,t)$ is chlorite pollutant concentration of groundwater(kg/m^3), $D2_x, D2_z$ are the diffusion coefficient in x- and z-directions, $u(x,z,t)$, $w(x,z,t)$ are the groundwater flow velocity in the x- and z-directions, R_2 is transformed chlorite rate.

2.9 Initial condition of chlorite dispersion model

Dispersion of chlorite with the following conditions, if the potential chlorite concentration in the consider area is described by

$$\eta(x,z,0) = f_\eta(x,z). \text{ for all } (x,z) \in \Omega \quad (2.24)$$

where $f_\eta(x,z)$ is a averaged potential chlorite concentration in the considered area.

2.10 Boundary condition of chlorite dispersion model

Similarly, the rate of change are assumed by $g2_L(z)$, $g2_T(x)$, $g2_R(z)$ and $g2_B(x)$.

2.10.1 A chlorate dispersion model

The model of chlorate dispersion model can be described by

$$\frac{\partial\sigma}{\partial t}+u\frac{\partial\sigma}{\partial x}+v\frac{\partial\sigma}{\partial z} = D3_x\frac{\partial^2\sigma}{\partial x^2}+D3_z\frac{\partial^2\sigma}{\partial z^2}+Q+R_3Rc, \quad (2.25)$$

for all $(x,z,t) \in \Omega$ such that $\Omega = [0,L_x] \times [0,L_z] \times [0,T]$, where $\sigma(x,z,t)$ is chlorate pollutant concentration of groundwater(kg/m^3), $D3_x, D3_z$ are the diffusion coefficient in x- and z-directions, $u(x,z,t)$, $w(x,z,t)$ are the groundwater flow velocity in the x- and z-directions, R_3 is transformed chlorate rate.

2.11 Initial condition of chlorate dispersion model

Dispersion of chlorate with following conditions, if the potential chlorate concentration in the consider area is described by

$$\sigma(x,z,0) = f_\sigma(x,z). \text{ for all } (x,z) \in \Omega \quad (2.26)$$

where $f_\sigma(x,z)$ is a averaged potential chlorate concentration in the considered area.

2.12 Boundary condition of chlorate dispersion model

Similarly, the rate of change are assumed by $g3_L(z)$, $g3_T(x)$, $g3_R(z)$ and $g3_B(x)$.

2.12.1 A perchlorate dispersion model

The model of perchlorate dispersion model can be described by

$$\frac{\partial\xi}{\partial t}+u\frac{\partial\xi}{\partial x}+v\frac{\partial\xi}{\partial z} = D4_x\frac{\partial^2\xi}{\partial x^2}+D4_z\frac{\partial^2\xi}{\partial z^2}+Q+R_4Rc, \quad (2.27)$$

for all $(x,z,t) \in \Omega$ such that $\Omega = [0,L_x] \times [0,L_z] \times [0,T]$, where $\xi(x,z,t)$ is perchlorate pollutant concentration of groundwater(kg/m^3), $D4_x, D4_z$ are the diffusion coefficient in x- and z-directions, $u(x,z,t)$, $w(x,z,t)$ are the groundwater flow velocity in the x- and z-directions, R_4 is transformed perchlorate rate.

2.13 Initial condition of perchlorate dispersion model

Dispersion of perchlorate with following conditions, if the potential perchlorate concentration in the consider area is described by

$$\xi(x,z,0) = f_\xi(x,z). \text{ for all } (x,z) \in \Omega \quad (2.28)$$

where $f_\xi(x,z)$ is a averaged potential perchlorate concentration in the considered area.

2.14 Boundary condition of perchlorate dispersion model

Similarly, the rate of change are assumed by $g4_L(z)$, $g4_T(x)$, $g4_R(z)$ and $g4_B(x)$.

3 Numerical techniques

In this paper, we will propose finite difference methods to the transient groundwater model by using the forward time central space method. We now discretize the domain by dividing the interval $[0, L_x]$ and $[0, L_z]$ into M_x and M_z subintervals such that $M_x\Delta x = L_x, M_z\Delta z = L_z$ and the time interval $[0, T]$ into N subintervals such that $N\Delta t = T$. The grid points (x_j, z_i, t_n) are defined by $x_j = j\Delta x$ for all $j = 1, 2, ..., M_x$, $z_i = i\Delta z$ for all $i = 1, 2, ..., M_z$ and $t_n = n\Delta t$ for all $n = 1, 2, ..., T$. We can then approximate $H(x_j, z_i, t_n)$ by $H^n_{i,j}$, value of the difference approximation of $H(x, z, t)$ at point $x = j\Delta x, z = i\Delta z$ and $t = n\Delta t$, where $0 \le j \le M_x, 0 \le i \le M_z$ and $0 \le n \le N$ which M_x, M_z and N are positive integers.

3.1 Explicit finite difference method for two-dimensional groundwater flow model

Taking the central difference in space and forward difference in time into scheme each terms of Eq. (2.2), we have

$$H(x, z, t) \simeq H^n_{i,j}, \tag{3.1}$$

$$\left.\frac{\partial H}{\partial t}\right|_{(x_i,z_j,t_n)} \simeq \frac{H^{n+1}_{i,j} - H^n_{i,j}}{\Delta t}, \tag{3.2}$$

$$\left.\frac{\partial^2 H}{\partial x^2}\right|_{(x_i,z_j,t_n)} \simeq \frac{H^n_{i,j-1} - 2H^n_{i,j} + H^n_{i,j+1}}{(\Delta x)^2}, \tag{3.3}$$

$$\left.\frac{\partial^2 H}{\partial z^2}\right|_{(x_i,z_j,t_n)} \simeq \frac{H^n_{i-1,j} - 2H^n_{i,j} + H^n_{i+1,j}}{(\Delta z)^2}. \tag{3.4}$$

Substituting Eqs.(3.1)-(3.4) into Eq.(2.2), we get the finite equation,

$$S\left(\frac{H^{n+1}_{i,j} - H^n_{i,j}}{\Delta t}\right) = K_x\left(\frac{H^n_{i,j-1} - 2H^n_{i,j} + H^n_{i,j+1}}{(\Delta x)^2}\right) + K_z\left(\frac{H^n_{i-1,j} - 2H^n_{i,j} + H^n_{i+1,j}}{(\Delta z)^2}\right), \tag{3.5}$$

for all $i = 1, 2, 3, ..., M_x, j = 1, 2, 3, ..., M_z$ and $n = 1, 2, 3, ..., N - 1$. Then the explicit finite difference equation becomes

$$H^{n+1}_{i,j} = \alpha H^n_{i,j-1} + \alpha H^n_{i,j+1} + (1 - 2\alpha - 2\beta)H^n_{i,j} + \beta H^n_{i-1,j} + \beta H^n_{i+1,j}, \tag{3.6}$$

where $\alpha = \dfrac{K_x(\Delta t)}{S(\Delta x)^2}$ and $\beta = \dfrac{K_z(\Delta t)}{S(\Delta z)^2}$, for all $i = 1, 2, ..., M_x - 1$ and $j = 1, 2, ..., M_z - 1$.

The forward space technique is used to approximate fictitious points on the boundaries solution such as,

$$H^n_{i,M_x+1} = H^n_{i,M_x} + f_R(z)\Delta x, \tag{3.7}$$

$$H^n_{-1,j} = H^n_{0,j} - f_B(x)\Delta z, \tag{3.8}$$

$$H^{n+1}_{M_z,j} = H^n_{M_z,j} + f_T(x)\Delta z. \tag{3.9}$$

3.2 Finite difference method for groundwater flow velocity model

Taking the forward difference in space into Eq.(2.8) and Eq.(2.9), we have

$$v(x, z, t) \simeq v^n_{i,j}, \tag{3.10}$$

$$w(x, z, t) \simeq w^n_{i,j}, \tag{3.11}$$

$$\left.\frac{\partial H}{\partial x}\right|_{(x_i,z_j,t_n)} \simeq \frac{H^n_{i,j+1} - H^n_{i,j}}{\Delta x}, \tag{3.12}$$

$$\left.\frac{\partial H}{\partial z}\right|_{(x_i,z_j,t_n)} \simeq \frac{H^n_{i+1,j} - H^n_{i,j}}{\Delta z}. \tag{3.13}$$

Substituting Eqs.(3.12) - (3.13) into (2.8) and (2.9), we get the finite equation,

$$v^n_{i,j} = -\frac{1}{\Delta x}(H^n_{i,j+1} - H^n_{i,j}) \tag{3.14}$$

and

$$w^n_{i,j} = -\frac{1}{\Delta z}(H^n_{i+1,j} - H^n_{i,j}). \tag{3.15}$$

3.3 Explicit finite difference method for two-dimensional groundwater pollution dispersion model

In this section, the considered domain is defined in a similar grid spacing as the previous. We will employ forward time central space difference scheme (FTCS) to Eqs.(2.10)

$$c(x, z, t) \simeq C^n_{i,j}, \tag{3.16}$$

$$\left.\frac{\partial c}{\partial t}\right|_{(x_i,z_j,t_n)} \simeq \frac{C^{n+1}_{i,j} - C^n_{i,j}}{\Delta t}, \tag{3.17}$$

$$\left.\frac{\partial c}{\partial x}\right|_{(x_i,z_j,t_n)} \simeq \frac{C^n_{i,j-1} - C^n_{i,j+1}}{2\Delta x}, \tag{3.18}$$

$$\left.\frac{\partial c}{\partial z}\right|_{(x_i,z_j,t_n)} \simeq \frac{C^n_{i-1,j} - C^n_{i+1,j}}{2\Delta z}, \tag{3.19}$$

$$\left.\frac{\partial^2 c}{\partial x^2}\right|_{(x_i,z_j,t_n)} \simeq \frac{C^n_{i,j-1} - 2C^n_{i,j} + C^n_{i,j+1}}{(\Delta x)^2}, \tag{3.20}$$

$$\left.\frac{\partial^2 c}{\partial z^2}\right|_{(x_i,z_j,t_n)} \simeq \frac{C^n_{i-1,j} - 2C^n_{i,j} + C^n_{i+1,j}}{(\Delta z)^2}. \tag{3.21}$$

Substituting Eqs.(3.16) - (3.21) in chloride dispersion models,

3.4 An explicit the forward time centered space method for a total chloride model

$$C^{n+1}_{i,j} = (\tau_1 + \tau_2)\,C^n_{i-1,j} + (1 - 2\lambda_1 - 2\tau_1)C^n_{i,j} + (\tau_1 - \tau_2)\,C^n_{i+1,j} + (\lambda_1 - \lambda_2)\,C^n_{i,j+1} + (\lambda_1 + \lambda_2)\,C^n_{i,j-1} + Q\Delta t - \Delta t R C^n_{i,j}, \tag{3.22}$$

for all $i = 1, 2, 3, ..., M_x$, $j = 1, 2, 3, ..., M_z$ and $n = 1, 2, 3, ..., N - 1$.

where $\lambda_1 = \dfrac{D_x\Delta t}{(\Delta x)^2}$, $\lambda_2 = \dfrac{u^n_{i,j}\Delta t}{2\Delta x}$, $\tau_1 = \dfrac{D_z\Delta t}{(\Delta z)^2}$, and $\tau_2 = \dfrac{w^n_{i,j}\Delta t}{2\Delta z}$.

The forward space technique is used to approximate fictitious points on the boundaries solution such as,

$$C^n_{i,-1} = C^n_{i,0} - g_W(z)\Delta x, \quad (3.23)$$
$$C^n_{i,M_x+1} = C^n_{i,M_x} + g_R(z)\Delta x, \quad (3.24)$$
$$C^n_{-1,j} = C^n_{0,j} - g_B(x)\Delta z, \quad (3.25)$$
$$C^n_{M_z+1,j} = C^n_{M_z,j} + g_T(x)\Delta z. \quad (3.26)$$

3.5 An explicit the forward time centered space method for a hypochlorite

Taking the central difference in space and forward difference in time into scheme each terms of Eq. (2.18), we have

$$\phi^{n+1}_{i,j} = (\tau_3 + \tau_4)\,\phi^n_{i-1,j} + (1 - 2\lambda_3 - 2\tau_3)\phi^n_{i,j} + (\tau_3 - \tau_4)\,\phi^n_{i+1,j} + (\lambda_3 - \lambda_4)\,\phi^n_{i,j+1} + (\lambda_3 + \lambda_4)\,\phi^n_{i,j-1} + Q\Delta t + \Delta t R_1 RC^n_{i,j}, \quad (3.27)$$

for all $i = 1, 2, 3, ..., M_x$, $j = 1, 2, 3, ..., M_z$ and $n = 1, 2, 3, ..., N - 1$.

where $\lambda_3 = \dfrac{D1_x\Delta t}{(\Delta x)^2}$, $\lambda_4 = \dfrac{u^n_{i,j}\Delta t}{2\Delta x}$, $\tau_3 = \dfrac{D1_z\Delta t}{(\Delta z)^2}$, and $\tau_4 = \dfrac{w^n_{i,j}\Delta t}{2\Delta z}$.

The forward space technique is used to approximate fictitious points on the boundaries solution such as,

$$\phi^n_{i,-1} = \phi^n_{i,0} - g1_W(z)\Delta x, \quad (3.28)$$
$$\phi^n_{i,M_x+1} = \phi^n_{i,M_x} + g1_R(z)\Delta x, \quad (3.29)$$
$$\phi^n_{-1,j} = \phi^n_{0,j} - g1_B(x)\Delta z, \quad (3.30)$$
$$\phi^n_{M_z+1,j} = \phi^n_{M_z,j} + g1_T(x)\Delta z. \quad (3.31)$$

3.6 An explicit the forward time centered space method for a chlorite

Taking the central difference in space and forward difference in time into scheme each terms of Eq. (2.23), we have

$$\eta^{n+1}_{i,j} = (\tau_5 + \tau_6)\,\eta^n_{i-1,j} + (1 - 2\lambda_5 - 2\tau_5)\eta^n_{i,j} + (\tau_5 - \tau_6)\,\eta^n_{i+1,j} + (\lambda_5 - \lambda_6)\,\eta^n_{i,j+1} + (\lambda_5 + \lambda_6)\,\eta^n_{i,j-1} + Q\Delta t + \Delta t R_2 RC^n_{i,j}, \quad (3.32)$$

for all $i = 1, 2, 3, ..., M_x$, $j = 1, 2, 3, ..., M_z$ and $n = 1, 2, 3, ..., N - 1$.

where $\lambda_5 = \dfrac{D2_x\Delta t}{(\Delta x)^2}$, $\lambda_6 = \dfrac{u^n_{i,j}\Delta t}{2\Delta x}$, $\tau_5 = \dfrac{D2_z\Delta t}{(\Delta z)^2}$, and $\tau_6 = \dfrac{w^n_{i,j}\Delta t}{2\Delta z}$.

The forward space technique is used to approximate fictitious points on the boundaries solution such as,

$$\eta^n_{i,-1} = \eta^n_{i,0} - g2_W(z)\Delta x, \quad (3.33)$$
$$\eta^n_{i,M_x+1} = \eta^n_{i,M_x} + g2_R(z)\Delta x, \quad (3.34)$$
$$\eta^n_{-1,j} = \eta^n_{0,j} - g2_B(x)\Delta z, \quad (3.35)$$
$$\eta^n_{M_z+1,j} = \eta^n_{M_z,j} + g2_T(x)\Delta z. \quad (3.36)$$

3.7 An explicit the forward time centered space method for a chlorate

Taking the central difference in space and forward difference in time into scheme each terms of Eq. (2.25), we have

$$\sigma^{n+1}_{i,j} = (\tau_7 + \tau_8)\,\sigma^n_{i-1,j} + (1 - 2\lambda_7 - 2\tau_7)\sigma^n_{i,j} + (\tau_7 - \tau_8)\,\sigma^n_{i+1,j} + (\lambda_7 - \lambda_8)\,\sigma^n_{i,j+1} + (\lambda_7 + \lambda_8)\,\sigma^n_{i,j-1} + Q\Delta t + \Delta t R_3 RC^n_{i,j}, \quad (3.37)$$

for all $i = 1, 2, 3, ..., M_x$, $j = 1, 2, 3, ..., M_z$ and $n = 1, 2, 3, ..., N - 1$.

where $\lambda_7 = \dfrac{D3_x\Delta t}{(\Delta x)^2}$, $\lambda_8 = \dfrac{u^n_{i,j}\Delta t}{2\Delta x}$, $\tau_7 = \dfrac{D3_z\Delta t}{(\Delta z)^2}$, and $\tau_8 = \dfrac{w^n_{i,j}\Delta t}{2\Delta z}$.

The forward space technique is used to approximate fictitious points on the boundaries solution such as,

$$\sigma^n_{i,-1} = \sigma^n_{i,0} - g3_W(z)\Delta x, \quad (3.38)$$
$$\sigma^n_{i,M_x+1} = \sigma^n_{i,M_x} + g3_R(z)\Delta x, \quad (3.39)$$
$$\sigma^n_{-1,j} = \sigma^n_{0,j} - g3_B(x)\Delta z, \quad (3.40)$$
$$\sigma^n_{M_z+1,j} = \sigma^n_{M_z,j} + g3_T(x)\Delta z. \quad (3.41)$$

3.8 An explicit the forward time centered space method for a perchlorate

Taking the central difference in space and forward difference in time into scheme each terms of Eq. (2.27), we have

$$\xi^{n+1}_{i,j} = (\tau_9 + \tau_{10})\,\xi^n_{i-1,j} + (1 - 2\lambda_9 - 2\tau_9)\xi^n_{i,j} + (\tau_9 - \tau_{10})\,\xi^n_{i+1,j} + (\lambda_9 - \lambda_{10})\,\xi^n_{i,j+1} + (\lambda_9 + \lambda_{10})\,\xi^n_{i,j-1} + Q\Delta t + \Delta t R_4 RC^n_{i,j}, \quad (3.42)$$

for all $i = 1, 2, 3, ..., M_x$, $j = 1, 2, 3, ..., M_z$ and $n = 1, 2, 3, ..., N - 1$.

where $\lambda_9 = \dfrac{D4_x\Delta t}{(\Delta x)^2}$, $\lambda_{10} = \dfrac{u^n_{i,j}\Delta t}{2\Delta x}$, $\tau_9 = \dfrac{D4_z\Delta t}{(\Delta z)^2}$, and $\tau_{10} = \dfrac{w^n_{i,j}\Delta t}{2\Delta z}$.

The forward space technique is used to approximate fictitious points on the boundaries solution such as,

$$\xi^n_{i,-1} = \xi^n_{i,0} - g4_W(z)\Delta x, \quad (3.43)$$
$$\xi^n_{i,M_x+1} = \xi^n_{i,M_x} + g4_R(z)\Delta x, \quad (3.44)$$
$$\xi^n_{-1,j} = \xi^n_{0,j} - g4_B(x)\Delta z, \quad (3.45)$$
$$\xi^n_{M_z+1,j} = \xi^n_{M_z,j} + g4_T(x)\Delta z. \quad (3.46)$$

4 Numerical simulations

A two-dimensional hydraulic head model provides hydraulic head. The computed hydraulic head is transformed to be the groundwater flow velocity by using the second model. The results from the second model will be input into the chloride dispersion models that provide chloride, hypochlorite, chlorite, chlorate and perchlorate concentration

4.1 Numerical simulation of transient groundwater flow velocity

A chloride compound dispersion model that provide the concentration of their substance, this computation need to using the groundwater flow velocity from the first model.

The two-dimensional of groundwater flow model, considering with area domain 1.0×0.5 km. Assume that the specific storage is 1 m^{-1} and the hydraulic conductivity in x- and z-direction are 15 (m/day) and don't have source term. The grid spacing is $\Delta x = \Delta z = 10$ m, time step $\Delta t = 1$ day at time $T = 3650$ day. The initial $h = 0(m)$ and boundary conditions $h_T = h_R = h_B = 0$ and $h_L = 0.06z + 10$. Consider the hydraulic head of groundwater by using the explicit method, we get the approximated groundwater pollutant concentration as shown in Table 1 and Figures 3-4.

Table 1. The approximated hydraulic head(m) where $z = 50$ m.

	$H(x, z, t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
3	36.4000	33.0498	29.7399	26.5099	23.3977	21.8976
5	36.4000	33.7674	31.1538	28.5782	26.0593	24.8267
7	36.4000	34.1465	31.9047	29.6864	27.5034	26.4286
10	36.4000	34.4811	32.5693	30.6719	28.7960	27.8684

Next, input the hydraulic head into the second model to approximate groundwater velocity in (2.8) and (2.9) by using central space. Then, we get the approximated groundwater flow velocity as shown in Tables 2-3 and Figure 5.

Table 2. The approximated groundwater flow velocity in x-direction(m/day) where $z = 50$ m.

	$u(x, z, t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
3	0.1490	0.1470	0.1411	0.1338	0.1260	0.1220
5	0.1212	0.1194	0.1143	0.1082	0.1020	0.0989
7	0.1096	0.1080	0.1030	0.0972	0.0913	0.0885
10	0.1029	0.1013	0.0964	0.0907	0.0850	0.0823

Table 3. The approximated groundwater flow velocity in z-direction(m/day) where $z = 50$ m.

	$w(x, z, t)$					
t	$x = 0$	$x = 20$	$x = 40$	$x = 60$	$x = 80$	$x = 90$
3	-0.0600	-0.0472	-0.0366	-0.0287	-0.0228	-0.0205
5	-0.0600	-0.0473	-0.0369	-0.0291	-0.0233	-0.0211
7	-0.0600	-0.0473	-0.0369	-0.0291	-0.0234	-0.0212
10	-0.0600	-0.0473	-0.0369	-0.0291	-0.0234	-0.0212

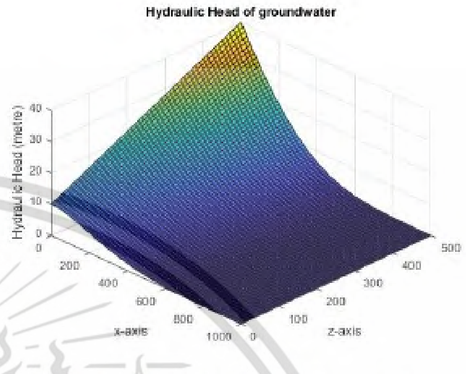

Figure 4. Hydraulic head of the considered area for 10 year

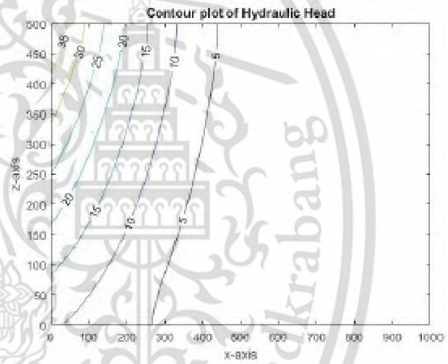

Figure 5. Hydraulic head of the considered area for 10 year

4.2 Numerical simulation of chloride dispersion

In the last model, we input the groundwater flow velocity to approximate chloride dispersion. The chemical compound of chloride can be transformed into hypochlorite, then into chlorite, chlorate, and lastly into perchlorate. At the beginning of the time, we assume that the amount of chemical compound in considered area is equal to zero, i.e., $c_0 = 0\ m$ and boundaries in considered domain suppose that there is no rate of change, then, $g_L = g_T = g_R = g_B = 0$. The diffusion coefficient in x- and z- direction in each models are different due to the mass of compound, so, we assume that the diffusion coefficient $D_x = D_z = 1.5$. The transformed rate are important factor to concentration of pollutant, its valuable between 0 to 1, then, we set $R = 0.001$. The approximate solution is shown in Table 4 and Fig 6-7.

We set the initial and boundary conditions as the same with previous model, considering diffusion coefficient of hypochlorite $D1_x = D1_z = 2.0$ and rate of hypochlorite dispersion $R_1 = 0.25$. The approximate solution is shown in Table 5 and

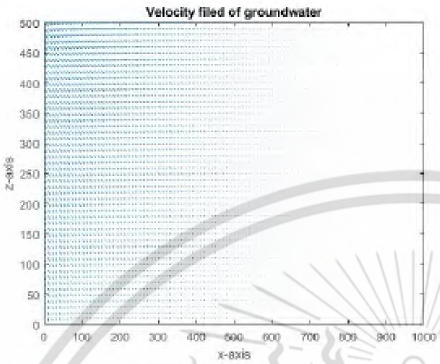

Figure 6. The groundwater flow direction for 10 years

Table 4. The approximated chloride where $z = 50$ m.

			$c(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
3	0.0000	0.0000	0.0001	0.0013	0.0123	0.0305
5	0.0000	0.0001	0.0008	0.0060	0.0326	0.0657
7	0.0002	0.0005	0.0026	0.0061	0.0285	0.0561
10	0.0012	0.0020	0.0070	0.0271	0.0909	0.1516

Fig 8-9.

Table 5. The approximated hypochlorite where $z = 50$ m.

			$\phi(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
3	0.0000	0.0000	0.0001	0.0006	0.0032	0.0064
5	0.0001	0.0002	0.0006	0.0025	0.0089	0.0154
7	0.0005	0.0008	0.0020	0.0060	0.0172	0.0273
10	0.0027	0.0034	0.0093	0.0219	0.0485	0.0687

We set the initial and boundary conditions as the same with previous model, considering diffusion coefficient of chlorite $D2_x = D2_z = 0.5$ and rate of chlorite dispersion $R_2 = 0.01$. The approximate solution as shown in Table 6 Fig 10-11.

Table 6. The approximated chlorite where $z = 50$ m.

			$\eta(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
3	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003
5	0.0000	0.0000	0.0000	0.0001	0.0004	0.0006
7	0.0000	0.0000	0.0001	0.0002	0.0007	0.0011
10	0.0001	0.0001	0.0002	0.0006	0.0013	0.0019

We set the initial and boundary conditions as the same with previous model, considering diffusion coefficient of chlorate $D3_x = D3_z = 2.5$ and rate of chlorate dispersion $R_3 = 0.4$. The approximate solution as shown in Table 7 and Fig 12-13.

We set the initial and boundary conditions as the same with previous model, considering diffusion coefficient of perchlorate $D4_x = D4_z = 1$ and rate of perchlorate dispersion $R_4 = 0.005$. The approximate solution as shown in Table 8 and Fig 14-15.

Table 7. The approximated chlorate where $z = 50$ m.

			$\sigma(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
3	0.0000	0.0000	0.0000	0.0000	0.0004	0.0011
5	0.0000	0.0000	0.0001	0.0003	0.0017	0.0039
7	0.0001	0.0001	0.0004	0.0019	0.0087	0.0171
10	0.0004	0.0004	0.0007	0.0025	0.0097	0.0184

Table 8. The approximated chlorate where $z = 50$ m.

			$\xi(x,z,t)$			
t	$x=0$	$x=20$	$x=40$	$x=60$	$x=80$	$x=90$
3	0.0000	0.0000	0.0000	0.0000	0.0002	0.0003
5	0.0000	0.0000	0.0000	0.0001	0.0003	0.0004
7	0.0000	0.0000	0.0001	0.0002	0.0005	0.0007
10	0.0001	0.0002	0.0003	0.0005	0.0009	0.0013

5 Discussion

In our simulation, we assume that the hydraulic head drive the groundwater flow from the higher hydraulic head level zone to the lower zone, the result of simulation for 10 years have been shown in Table 1 and Figure 3-4. The figures shown that the hydraulic head at the surface area is higher than the deep area. The hydraulic head is transformed to be the groundwater flow direction as shown in Fig 5. The direction has shown that groundwater flow from high to lower hydraulic head. The result has been plug into the five chloride compound dispersion models. We can measure the total chloride, hypochloride, chlorite, chlorate and perchlorate pollutant levels at 10 years as shown Table 4-8 and Fig 6-16. The figure shows that the amount of groundwater changes directly,over time the substance has been less than reactant. The approximated chloride compound is compared in Fig 16-20. In Fig 16-20, we can see that the simulation for 2 to 4 years tell us the trend of graph are the same and a little bit increasing, after that, for 4 to 6 years, they are slightly increase. Finally, in 6 to 10 years we obtained that the graph is stable.

6 Conclusion

A Mathematical model of horizontal average groundwater pollutant measurement with several substances due to chemical reaction is introduced. The first model is the two-dimensional transient groundwater flow model which provides the hydraulic head. The second is the groundwater flow velocity model which provides the groundwater flow in x- and z-directions. The third model is the two-dimensional horizontal averaged contaminated chloride dispersion model which provides the concentration of chloride and their substances. A method to set up the initial and boundary conditions of transient groundwater flow model is proposed. The computed hydraulic head is transformed to be the groundwater flow velocity by using the second model. The results in second model will be input into the last model as a field data. The concentration of chloride and their substances are obtained by the third model. The hydraulic head of the first model is approximated by an explicit finite difference method. An explicit finite difference technique is used to obtain the groundwater flow velocity of the second model. A forward time-centered space finite difference technique is used to approximate the concentration of chloride and their substances. The groundwater quality is affected by the chloride release by the landfill. The proposed simulations show that the different levels of hydraulic head have a small effect on the overall groundwater quality level. In our simulation.It is found that the main groundwater quality factor are pollutant concentration level around the landfill and transform rate.

7 Acknowledgment

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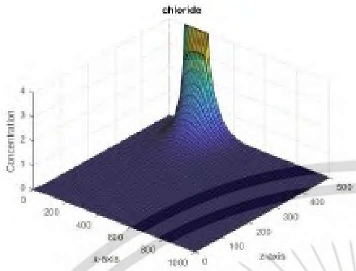

Figure 7. Surface plot of chloride concentration for 10 years

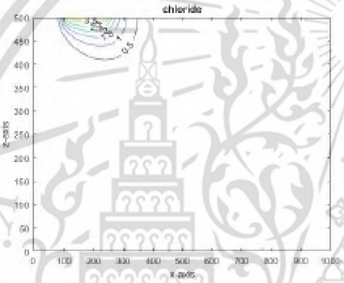

Figure 8. Contour plot of chloride concentration for 10 years

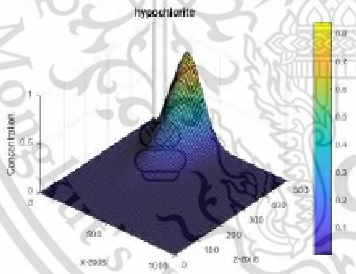

Figure 9. Surface plot of hypochlorite concentration for 10 years

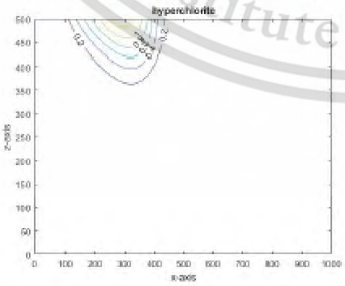

Figure 10. Contour plot of hypochlorite concentration for 10 years

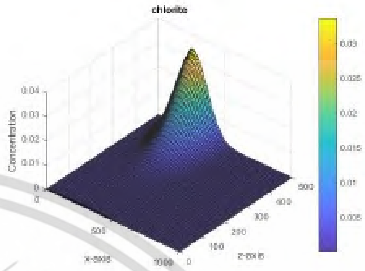

Figure 11. Surface plot of chlorite concentration for 10 years

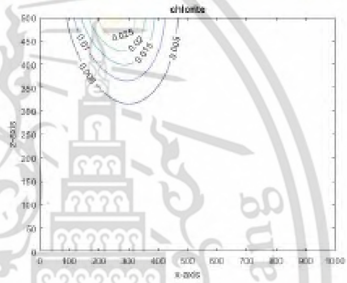

Figure 12. Contour plot of chlorite concentration for 10 years

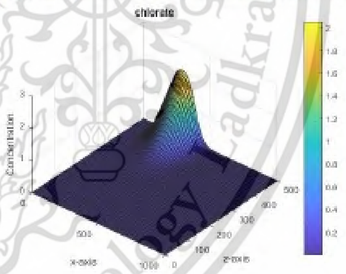

Figure 13. Surface plot of chlorate concentration for 10 years

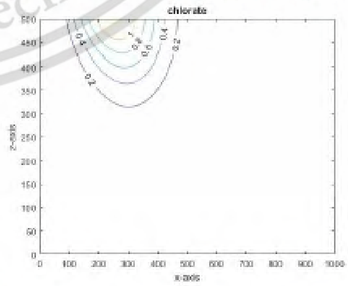

Figure 14. Contour plot of chlorate concentration for 10 years

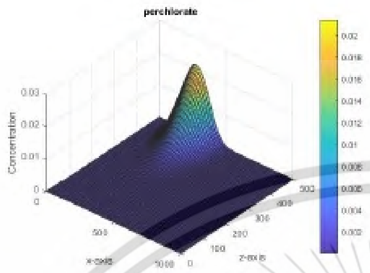

Figure 15. Surface plot of perchlorate concentration for 10 years

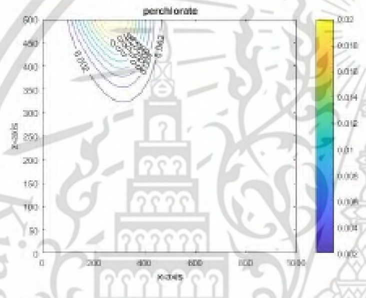

Figure 16. Contour plot of perchlorate concentration for 10 years

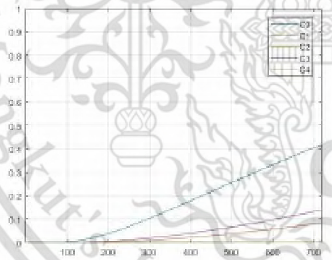

Figure 17. Comparison of chloride compound levels along 2 years

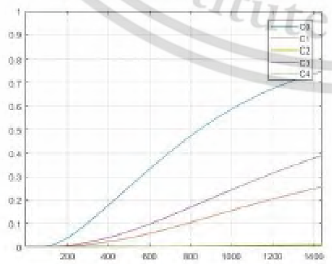

Figure 18. Comparison of chloride compound levels along 4 years

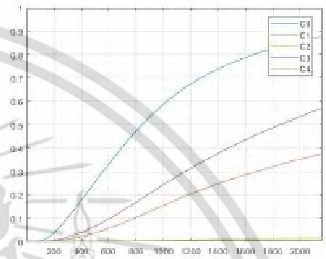

Figure 19. Comparison of chloride compound levels along 6 years

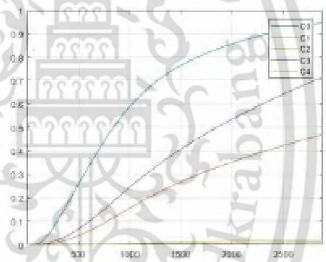

Figure 20. Comparison of chloride compound levels along 8 years

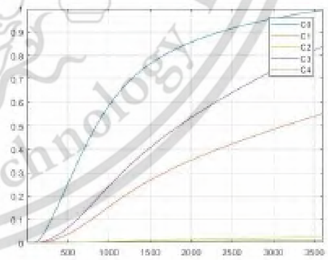

Figure 21. Comparison of chloride compound levels along 10 years with C0 = total chloride, C1 = hypochloride, C2 = chlorite, C3 = chlorate, C4 = perchlorate.

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2. Jirapud Limthanakul , Nopparat Pochai,"A Mathematical Model of Horizontal Averaged Groundwater Pollution Measurement with Several Substances due to Chemical Reaction", Mathematics and Statistics, Vol. 8, No. 6, pp. 645 - 655, 2020.