SOLVING THE SYLVESTER MATRIX EQUATION WITH RESPECT TO THE GENERAL LEFT SEMI-TENSOR PRODUCT

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Abstract

We consider the Sylvester matrix equation $A \ltimes X + X \ltimes B = C$ with respect to the general left semi-tensor product where A, B, C are given real matrices of arbitrary dimensions and X is unknown. We investigate necessary/sufficient condition(s) for the Sylvester matrix equation to be solvable and uniquely solvable, concerning ranks and linear independence. Moreover, we show that the Sylvester equation can be transformed to a linear system with respect to the usual matrix product by means of the vector operator.

Keywords : Sylvester matrix equation, general left semi-tensor product, vector operator, linear system

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Sakunrat Kitsompong

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Chapter 1

Introduction

1.1 Inception and Significance of Research

Linear matrix equations arise naturally in several branches of pure/applied mathematics, e,g, differential equations, control theory, neutral network, and vibration theory; see e.g. [12, 17, 16]. The Sylvester matrix equation

$$AX + XB = C \tag{1.1.1}$$

is a famous linear matrix equation that is significantly used in stability analysis and optimal control. A usual algebraic method to solve linear matrix equations is to transform them into a linear system by means of certain kind of vectorization, namely, the column/row vector operator; see e.g. [13].

In the usual matrix multiplication, the product of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ can be defined if $n = p$, that is the matrix pair (A, B) satisfies the matching-dimension condition. For the matrices that do not satisfy matching dimension condition, Cheng [4] has introduced the semi-tensor product as a tool for multiplication between those matrices. The left-semi tensor product of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ can be defined under the factor-dimension condition namely, $n \mid p$ or $p \mid n$. In the case $n = p$, the product of $A \ltimes B$ is equal to the usual matrix product AB. The properties of left-semi tensor product contain the associative law, the left/right distributive over the matrix addition, certain identity-like properties, and the compatibility with the scalar multiplication, the transposition, the inversion and so on (see [6, 7, 8, 11, 2, 3]). The left semi-tensor product can be applied in several areas such as classical logic [10], fuzzy logic [10], Boolean networks [4, 9] dynamic system [11] and Morgan's problem [5]. The general left semi-tensor product defines product of matrices with arbitrary dimensions (see [4]). It defines product of A and B as

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}) \tag{1.1.2}$$

where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. The general left semi-tensor product turns out to possess rich algebraic properties as the usual matrix product. The general left semi-tensor product was proposed for solving decoupling problems of linear system and the problems with multi-dimensions data. In 2015, Yao et al. [18] introduced a convenient way to solve the matrix equation

$$A \ltimes X = B \tag{1.1.3}$$

where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{h\times k}$. In fact, Eq. (1.1.3) has been importantly used in Boolean network. If the equation has a solution, then the dimension of X satisfies the solvability's conditions and B must be a block Toeplitz matrix. The number of solutions depends on the greatest common divisor of n and k. If gcd$\{n,k\}=1$, then the solution has only one admissible size. Li et al. [15] have been motivated by the work of Yao and considered that solving only Eq. (1.1.3) is not sufficient. Therefore, they investigated the solvability of the system of matrix equation

$$\begin{cases} A \ltimes X = B \\ X \ltimes C = D \end{cases} \tag{1.1.4}$$

with respect to the general left semi-tensor product. In fact, if the solution of (1.1.4) exists, then B and D must be partitioned matrices so that each partition is a Toplitz matrix. The matrix equation

$$A \ltimes X \ltimes B \quad = \quad C \tag{1.1.5}$$

where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{a\times b}$ and $C \in \mathbb{R}^{h\times k}$ under the general left semi-tensor product, has been extensively used in non-linear programming, power science, parameter identification, etc. Ji [14] investigated the solvability of Eq. (1.1.5). Indeed, if Eq. (1.1.5) has solution(s), then X satisfies certain dimension conditions and C must be the block Toplitz matrix.

The previous discussion motivated us to study the Sylvester matrix equation

$$A \ltimes X + X \ltimes B = C \tag{1.1.6}$$

where $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{h\times k}, C \in \mathbb{R}^{u\times v}$ are given, and $X \in \mathbb{R}^{p\times q}$ is to be solved. The study includes Eqs. (1.1.3), (1.1.5) and the Lyapunove equation:

$$A \ltimes X + X \ltimes A^T = C$$

as special cases. We investigate the solvability and the unique solvability for Eq. (1.1.6) according to the matrix dimensions, ranks and linear independence. Moreover, we show that Eq. (1.1.6) can be transformed to a linear system, so that we can solve it via an ordinary method.

1.2 Research Objectives

1 Investigate equivalent condition(s) for the Sylvester matrix equation with respect to the general left semi-tensor product to be solvable.

2 Investigate equivalent condition(s) for the Sylvester matrix equation with respect to the general left semi-tensor product to be uniquely solvable.

3 Propose a convenient way of solving the Sylvester matrix equation with respect to the general left semi-tensor product.

1.3 Scope

We investigate the matrix equation $A \ltimes X + X \ltimes B = C$, where A, B, C are given real matrices and X is an unknown real matrix. Here, the product $\ltimes$ is the general left semi-tensor product.

1.4 Benefits

To obtain criteria and a convenient method for solving the Sylvester matrix equation with respect to the general left semi-tensor product.

1.5 Research Methodology

1) Study advanced topics in Linear Algebra and Matrix Analysis.
2) Study advanced topics in Multilinear Algebra.
3) Study topics of the left semi-tensor product from research paper.
4) Study topics of the semi-tensor product and its application from research paper.
5) Study topics of solving matrix equations under the semi-tensor product from research paper.
6) Determine the objectives and scope of the research.
7) Investigate condition(s) for solvability of the Sylvester matrix equation with respect to the general left semi-tensor product.
8) Provide the method for solving the Sylvester matrix equation with respect to the general left semi-tensor product.
9) Make a conclusion of the outcome, write the thesis and make suggestions for further studies.

Table 1.1*:* The research schedule

Activity	Time frame								
	2019		2020				2021		
	Aug. - Sep.	Oct. - Dec.	Jan. - Mar.	Aprl. - Jun.	Jul. - Sep.	Oct. - Dec.	Jan. - Mar..	Aprl. - Jun.	Jun. - Jul.
Step 1	⟷	⟷							
Step 2		⟷	⟷						
Step 3			⟷	⟷					
Step 4				⟷	⟷				
Step 5					⟷	⟷			
Step 6						⟷			
Step 7							⟷	⟷	
Step 8							⟷	⟷	
Step 9									⟷

Chapter 2

Preliminaries

For any positive integers m, n, we denote the set of $m \times n$ real matrices by $\mathbb{R}^{m\times n}$. When $n = 1$, we set $\mathbb{R}^m := \mathbb{R}^{m\times 1}$.

2.1 The Kronecker product

Definition 2.1. The Kronecker product (see e.g. [13]) of $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{h\times k}$, denoted by $A \otimes B$, is defined as the block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mh\times nk}.$$

Example 2.2. Let $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$. Then

$$A \otimes B = \begin{bmatrix} 0\begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \\ 2\begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix} & -1\begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & 2 & 1 \\ 2 & -2 & 0 & -1 & 1 & 0 \\ 6 & 4 & 2 & -3 & -2 & -1 \end{bmatrix} \in \mathbb{R}^{4\times 6}.$$

Theorem 2.3. (see e.g. [1]) Let $A \in \mathbb{R}^{m\times n}$. Then

(i) $(\alpha A) \otimes B = \alpha(A \otimes B) = A \otimes (\alpha B)$ for all $\alpha \in \mathbb{R}$ and $B \in \mathbb{R}^{p\times q}$

(ii) $(A \otimes B)^T = A^T \otimes B^T$ for $B \in \mathbb{R}^{p\times q}$

(iii) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for $B \in \mathbb{R}^{p\times q}$ and $C \in \mathbb{R}^{r\times s}$

(iv) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$ for $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{r\times s}$

(v) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$ for $B, C \in \mathbb{R}^{p\times q}$

(vi) $A \otimes 0 = 0 \otimes A$

(vii) $I_m \otimes I_n = I_{mn}$.

Definition 2.4. The (column) vector operator (see e.g. [1] [13]) vec(·) is a linear operator that turns any matrix $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ into the vector

$$V_c(A) = \begin{bmatrix} a_{11} & \dots & a_{m1} & a_{12} & \dots & a_{m2} & \dots & a_{1n} & \dots & a_{mn} \end{bmatrix}^T.$$

Example 2.5. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 5 \end{bmatrix}$. Then

$$V_c(A) = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \\ -1 \\ 3 \\ 0 \\ 1 \\ 5 \end{bmatrix}.$$

Theorem 2.6. (see e.g. [1]) Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{h\times k}$ and $X \in \mathbb{R}^{p\times q}$. Then

$$V_c(AXB) = (B^T \otimes A)V_c(X)$$

2.2 The general left semi-tensor product

Definition 2.7. The general left semi-tensor product (see e.g. [4]) of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{h\times k}$, denoted by $A \ltimes B$, is defined as

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{h}}) \in \mathbb{R}^{\frac{m\alpha}{n} \times \frac{k\alpha}{h}}$$

where α = lcm $\{n, h\}$. If $n \mid h$ or $h \mid n$, then $A \ltimes B$ becomes the left semi-tensor product (see e.g. [4])

In the case $n = h$, the product $A \ltimes B$ reduces to the usual product AB.

Example 2.8. Let $A = \begin{bmatrix} 5 & 7 \\ 2 & -1 \\ 6 & 2 \end{bmatrix} \in \mathbb{R}^{3\times 2}$ and $B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \in \mathbb{R}^{3\times 1}$. Then lcm $\{2,3\} = 6$ and

$$\begin{aligned} A \ltimes B &= (A \otimes I_{\frac{6}{2}})(B \otimes I_{\frac{6}{3}}) \\ &= (A \otimes I_3)(B \otimes I_2) \\ &= \left(\begin{bmatrix} 5 & 7 \\ 2 & -1 \\ 6 & 2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 5 & 0 & 0 & 7 & 0 & 0 \\ 0 & 5 & 0 & 0 & 7 & 0 \\ 0 & 0 & 5 & 0 & 0 & 7 \\ 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 \\ 6 & 0 & 0 & 2 & 0 & 0 \\ 0 & 6 & 0 & 0 & 2 & 0 \\ 0 & 0 & 6 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 15 & 14 \\ 7 & 15 \\ 10 & 7 \\ 6 & -2 \\ -1 & 6 \\ 4 & -1 \\ 18 & 4 \\ 2 & 18 \\ 12 & 2 \end{bmatrix} \in \mathbb{R}^{9\times 2}. \end{aligned}$$

Theorem 2.9. (see e.g. [4]) Let $A \in \mathbb{R}^{m\times n}$,$B \in \mathbb{R}^{h\times k}$ and $C \in \mathbb{R}^{u\times v}$. Then

(i) (Distributive Law)

$$\begin{cases} A \ltimes (\alpha B + \beta C) &= \alpha(A \ltimes B) + \beta(A \ltimes C) \\ (\alpha A + \beta B) \ltimes C &= \alpha(A \ltimes C) + \beta(B \ltimes C) \qquad \alpha, \beta \in \mathbb{R}. \end{cases}$$

(ii) (Associative Law)

$$(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C).$$

2.3 Solving the matrix equation $A \ltimes X = B$

2.3.1 Solutions of the matrix-vector equation $A \ltimes X = B$

Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{h\times k}$ and $X \in \mathbb{R}^{p}$.

Theorem 2.10. [18] If matrix-vector equation $A \ltimes X = B$ has a solution, then matrix B must be a block Toeplitz matrix.

Theorem 2.11. [18] The solvability of matrix-vector equation $A \ltimes X = B$ is equivalent to the solvability of the following matrix-vector equation:

$$\begin{aligned} \begin{bmatrix} \breve{A}_1 & \breve{A}_{h/m+1} & \cdots & \breve{A}_{(n/k-1)h/m+1} \end{bmatrix} Y_1 &= \breve{B}_1, \\ \begin{bmatrix} \breve{A}_2 & \breve{A}_{h/m+2} & \cdots & \breve{A}_{(n/k-1)h/m+2} \end{bmatrix} Y_2 &= \breve{B}_2, \\ &\cdots \\ \begin{bmatrix} \breve{A}_{h/m} & \breve{A}_{2h/m} & \cdots & \breve{A}_p \end{bmatrix} Y_{h/m} &= \breve{B}_{h/m}, \end{aligned} \tag{2.3.1}$$

where

$$A = [\overbrace{\breve{A}_1 \quad \breve{A}_{l_1^1} \quad \ldots \quad \breve{A}_{l_1^1+1}}^{\breve{A}_1} \underbrace{\breve{A}_{l_1^1+1} \quad \ldots \quad \breve{A}_{l_1^2} \quad \breve{A}_{l_1^2+1}}_{\breve{A}_2} \cdots \overbrace{\breve{A}_{n-l_1^1} \quad \breve{A}_{n-l_1^1+1} \quad \ldots \quad \breve{A}_n}^{\breve{A}_p}],$$

and

$$\breve{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} & b_{21} & b_{31} & \cdots & b_{h/m,1} \\ b_{h/m+1,1} & b_{h/m+1,2} & \cdots & b_{h/m+1,k} & b_{h/m2,1} & b_{h/m+3,1} & \cdots & b_{2h/m,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{h-h/m+1,1} & b_{h-h/m+1,2} & \cdots & b_{h-h/m+1,k} & b_{h-h/m2,1} & b_{h-h/m+3,1} & \cdots & b_{h,1} \end{bmatrix},$$

$\breve{B}_j$ is the $j-th$ column of $\breve{B}$.

$$\begin{aligned} \breve{B}_1 &= \begin{bmatrix} \breve{B}_1 & \breve{B}_{h/m+1} & \cdots & \breve{B}_{(l_1^1-1)h/m+1} & \breve{B}_{l_1^1 h/m+1} \end{bmatrix} \\ \breve{B}_2 &= \begin{bmatrix} \breve{B}_{k+l_2^1} & \breve{B}_{h/m-l_2^1+1} & \cdots & \breve{B}_{l_1^2-l_1^1-1)h/m-l_2^1+1} & \breve{B}_{l_1^2-l_1^1 h/m-l_2^1+1} \end{bmatrix} \\ &\vdots \\ \breve{B}_{h/m} &= \begin{bmatrix} \breve{B}_{k+h/m-l_2^1} & \breve{B}_{l_2^1+1} & \cdots & \breve{B}_{(l_1^1-2)h/m+l_2^1+1} & \breve{B}_{k-h/m+1} \end{bmatrix}, k = l_1 h/m + l_2. \end{aligned}$$

If a series of vectors $Y_1, Y_2, \ldots, Y_{h/m}$ denoted by

$$Y_s = \begin{bmatrix} y_{s1} & y_{s2} & \cdots & y_{s(n/k)} \end{bmatrix}^T, s = 1, 2, \ldots, \frac{h}{m},$$

is a solution of Eq. (2.3.1), Then

$$X = \begin{bmatrix} y_{11} & y_{21} & \cdots & y_{(h/m)1} & \ldots & y_{1(n/k)} & y_{2(n/k)} & \ldots & y_{(h/m)(n/k)} \end{bmatrix}^T$$

is a solution of matrix-vector equation.

Corollary 2.12. [18] Matrix-vector equation $A \ltimes X = B$ has a solution if and only if $\breve{A}_j, \breve{A}_{h/m+j}, \ldots, \breve{A}_{(n/k-1)h/m+j}$, and $\breve{B}_j$ are linearly dependent, $j = 1, 2, \ldots, \frac{h}{m}$. Moreover, if $\breve{A}_j, \breve{A}_{h/m+j}, \ldots, \breve{A}_{(n/k-1)h/m+j}$ are linearly independent, $j = 1, 2, \ldots, \frac{h}{m}$, then the solution of the equation would be unique.

Example 2.13. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$. Then we have $frachm = 2, k = 3$, and $k = 1 \cdot 2 + 1, \frac{n}{k} = 1, p = 2$. So

$$\breve{A}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \breve{A}_2 = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \breve{B} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & 1 & 2 \end{bmatrix}, \breve{B}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \breve{B}_2 = \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix}.$$

Solving Eq. (2.3.1), we have $Y_1 = 1, Y_2 = 2$. Thus $X = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ is a solution of the equation, and by Corollary 2.12 the solution is unique.

2.3.2 Solutions of the matrix equation $A \ltimes X = B$

Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{h \times k}$ and $X \in \mathbb{R}^{p \times q}$.

Theorem 2.14. [18] Equation $A \ltimes X = C$ with $m = h$ is equivalent to the solvability of the following equation with conventional matrix product:

$$(I_q \otimes \bar{A})V_c(X) = V_c(B), \tag{2.3.2}$$

where $\bar{A} = \begin{bmatrix} V_c(\hat{A}_1) & V_c(\hat{A}_2) & \dots & V_c(\hat{A}_p) \end{bmatrix}$.

Corollary 2.15. [18] Matrix equation $A \ltimes X = B$ has a solution if and only if the following rank condition holds:

$$\text{rank}\, \bar{A} = \text{rank} \begin{bmatrix} \bar{A} & V_c(\hat{B}_1) & V_c(\hat{B}_2) & \dots & V_c(\hat{B}_q) \end{bmatrix}$$

where $B = \begin{bmatrix} V_c(\hat{B}_1) & V_c(\hat{B}_2) & \dots & V_c(\hat{B}_q) \end{bmatrix}$.

Theorem 2.16. [18] Suppose that matrix equation $A \ltimes X = B$ has a solution in $\mathbb{R}^{p \times q}$. Split B into blocks of size $\frac{h}{m}$ by $\frac{k}{q}$, then each block is a block Toeplitz matrix. B is required to be in the form:

$$B = \begin{bmatrix} Block_{11}(B) & \dots & Block_{1q}(B) \\ \dots & \dots & \dots \\ Block_{m1}(B) & \dots & Block_{mq}(B) \end{bmatrix}$$

obtain the solutions of the equation

$$A \ltimes X = B$$

is equivalent to solve

$$A \ltimes X_i = B_i.$$

2.4 Solving the equation $A \ltimes X \ltimes B = C$

2.4.1 Solvability of the matrix-vector equation $A \ltimes X \ltimes B = C$

Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{a\times b}, C \in \mathbb{R}^{h\times k}$ and $X \in \mathbb{R}^{p}$.

Theorem 2.17. [14] If a solution of the equation $A \ltimes X \ltimes B = C$ exists, then matrix C must be a block Toplitz matrix. C has the following from:

$$C = \begin{bmatrix} Block_{11}(C) & Block_{12}(C) & \dots & Block_{1b}(C) \\ Block_{21}(C) & Block_{22}(C) & \dots & Block_{2b}(C) \\ \vdots & \vdots & \vdots & \vdots \\ Block_{m1}(C) & Block_{m2}(C) & \dots & Block_{mb}(C)) \end{bmatrix}$$

where $Block_{11}(C), Block_{12}(C), \dots, Block_{mb}(C) \in \mathbb{R}^{\frac{h}{m}\times\frac{k}{b}}$ are Toeplitz blocks of matrix C.

Theorem 2.18. [14] The solvability of the matrix-vector equation $A \ltimes X \ltimes B = C$ is equivalent to the solvability of the following matrix-vector equations:

$$\begin{gathered}
\begin{bmatrix} \breve{A}_1 & \breve{A}_{h/m+1} & \cdots & \breve{A}_{(nb/k-1)h/m+1} \end{bmatrix} Y_{11} = \breve{C}_{11}, \\
\begin{bmatrix} \breve{A}_2 & \breve{A}_{h/m+2} & \cdots & \breve{A}_{(nb/k-1)h/m+2} \end{bmatrix} Y_{21} = \breve{C}_{21}, \\
\vdots \\
\begin{bmatrix} \breve{A}_{h/m} & \breve{A}_{2h/m} & \cdots & \breve{A}_{pa} \end{bmatrix} Y_{h/m,1} = \breve{C}_{h/m,1}, \\
\begin{bmatrix} \breve{A}_1 & \breve{A}_{h/m+1} & \cdots & \breve{A}_{(nb/k-1)h/m+1} \end{bmatrix} Y_{12} = \breve{C}_{12}, \\
\vdots \\
\begin{bmatrix} \breve{A}_{h/m} & \breve{A}_{2h/m} & \cdots & \breve{A}_{pa} \end{bmatrix} Y_{h/m,2} = \breve{C}_{h/m,2}, \\
\vdots \\
\begin{bmatrix} \breve{A}_1 & \breve{A}_{h/m+1} & \cdots & \breve{A}_{(nb/k-1)h/m+1} \end{bmatrix} Y_{1b} = \breve{C}_{1b}, \\
\vdots \\
\begin{bmatrix} \breve{A}_{h/m} & \breve{A}_{2h/m} & \cdots & \breve{A}_{pa} \end{bmatrix} Y_{h/m,b} = \breve{C}_{h/m,b},
\end{gathered} \tag{2.4.1}$$

where

$$A = [\overbrace{\breve{A}_1 \quad \breve{A}_{l_1^1} \quad \dots \quad \breve{A}_{l_1^1+1}}^{\breve{A}_1} \dots \quad \breve{A}_{l_1^2} \quad \breve{A}_{l_1^2+1} \cdots \overbrace{\breve{A}_{n-l_1^1} \quad \breve{A}_{n-l_1^1+1} \quad \dots \quad \breve{A}_n}^{\breve{A}_{pa}}],$$

and

$$\breve{C}_{11} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_{h/m+1} & \dots & \tilde{C}_{(l_1^1-1)h/m+1} & \tilde{C}_{l_1^1 h/m+1} \end{bmatrix},$$

$$\breve{C}_{21} = \begin{bmatrix} \tilde{C}_{k/b+l_2^1} & \tilde{C}_{h/m+1} & \dots & \tilde{C}_{(l_1^1-1)h/m+1} & \tilde{C}_{l_1^1 h/m+1} \end{bmatrix},$$

$$\vdots$$

$$\breve{C}_{h/m,1} = \begin{bmatrix} \tilde{C}_{k/b+h/m-l_2^1} & \tilde{C}_{l_2^1+1} & \dots & \tilde{C}_{(l_1^1-2)h/m+l_2^1+1} & \tilde{C}_{k/b-h/m+1} \end{bmatrix}$$

$$\vdots$$

$$\breve{C}_{h/m,b} = \begin{bmatrix} \tilde{C}_{(b-1)h/m+k-b-k/b+2} & \tilde{C}_{bh/m+k-b-k/b+2} & \dots & \tilde{C}_{(b+l_1^1-1)h/m+k-b-k/b+2} \end{bmatrix}$$

$$\vdots$$

$$\breve{C}_{1,b} = \begin{bmatrix} \tilde{C}_{bh/m+k-b-k/b+1} & \tilde{C}_{(b-1)h/m+k-b-k/b+l_2^1+2} & \dots & \tilde{C}_{(b-3)h/m+k+2} & \tilde{C}_{(b-2)h/m+k-b+2} \end{bmatrix}$$

If a series of $Y_{sj}, s = 1, \dots, \frac{h}{m}; j = 1, \dots, b$, which denoted by

$$Y_{sj} = \begin{bmatrix} y_{s1j}, & y_{s2j}, & \dots, & y_{s\frac{nb}{k}j} \end{bmatrix}^T, s = 1, \dots, \frac{h}{m}; j = 1, \dots, b$$

is a solution of matrix-vector equation, then the column Y_g of $Y = \begin{bmatrix} Y_1, & Y_2, & \dots, & Y_b \end{bmatrix}$ can be expressed by

$$Y_g = \begin{bmatrix} y_{11g}, & y_{21g}, & \dots, & y_{1\frac{nb}{k}g}, & y_{2\frac{nb}{k}g}, & \dots, & y_{\frac{h}{m}\frac{nb}{k}g} \end{bmatrix}^T \in \mathbb{R}^{pa}$$

Partition Y as $Y = \begin{bmatrix} \hat{Y}_1^T & \hat{Y}_2^T & \dots & \hat{Y}_p^T \end{bmatrix}$, where $\hat{Y}_f^T \in \mathbb{R}^{a \times b}$ and $f = 1, \dots, p$ Then if the following system :

$$\begin{aligned} x_1 B &= \hat{Y}_1 \\ x_2 B &= \hat{Y}_2 \\ &\vdots \qquad (2.4.2) \\ x_p B &= \hat{Y}_p \qquad (2.4.3) \end{aligned}$$

has a solution $X = \begin{bmatrix} x_1 & x_2 & \dots & x_p \end{bmatrix}^T$, it is also the solution of the matrix-vector equation $A \ltimes X \ltimes B = C$.

Corollary 2.19. [14] Matrix-vector equation $A \ltimes X \ltimes B = C$ has a solution if and only if the following conditions could be satisfies simultaneously:

i) $\breve{A}_i, \breve{A}_{\frac{h}{m}+i}, \dots, \breve{A}_{(\frac{nb}{k}-1)\frac{h}{m}+i}$ and $\breve{C}_{ij}$ are linearly dependent, for $i = 1, \dots, \frac{h}{m}, j = 1, \dots, b$

ii) Eq. (2.4.3) has a solution.

Furthermore, if $\breve{A}_i, \breve{A}_{\frac{h}{m}+i}, \dots, \breve{A}_{(\frac{nb}{k}-1)\frac{h}{m}+i}$ are linearly independent for $i = 1, \dots, \frac{h}{m}$, and Eq. (2.4.3) has a solution, then the solution would be unique.

2.4.2 The matrix equation $A \ltimes X \ltimes B = C$

Let Let $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{a \times b}, C \in \mathbb{R}^{h \times k}$ and $X \in \mathbb{R}^{p \times q}$.

Theorem 2.20. [14] The solvability of the equation $A \ltimes X \ltimes B = C$ with $m = h$ is equivalent to the solvability of the following equation with conventional matrix product:

$$(B^T \otimes I_{\frac{km}{b}})(I_q \otimes \grave{A})V_c(X) = V_c(C), \tag{2.4.4}$$

where $\grave{A} = \begin{bmatrix} V_c(\tilde{A}_1) & V_c(\tilde{A}_2) & \dots & V_c(\tilde{A}_p) \end{bmatrix}$ and $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_p$ are p equal size blocks of matrix A.

Theorem 2.21. [14] If the solution of matrix equation $A \ltimes X \ltimes B = C$ exists, Then C must be a block Toplitz matrix. Moreover

$$C = \begin{bmatrix} Block_{11}(C) & Block_{12}(C) & \dots & Block_{1b}(C) \\ \vdots & \vdots & \vdots & \vdots \\ Block_{\frac{h}{m\beta}1}(C) & Block_{\frac{h}{m\beta}2}(C) & \dots & Block_{\frac{h}{m\beta}b}(C) \\ \vdots & \vdots & \vdots & \vdots \\ Block_{\frac{h}{m}-\frac{h}{m\beta}+1,1}(C) & Block_{\frac{h}{m}-\frac{h}{m\beta}+1,2}(C) & \dots & Block_{\frac{h}{m}-\frac{h}{m\beta}+1,b}(C) \\ \vdots & \vdots & \vdots & \vdots \\ Block_{\frac{h}{\beta}1}(C) & Block_{\frac{h}{\beta}2}(C) & \dots & Block_{\frac{h}{\beta}b}(C) \end{bmatrix}$$

where $Block_{11}(C), Block_{12}(C), \dots, Block_{\frac{h}{\beta}b}(C) \in \mathbb{R}^{\beta \times \frac{k}{b}}$ are Toeplitz matrix.

Chapter 3

Solving the Sylvester equation when X is a vector

In this chapter, we study on solving the Sylvester equation

$$A \ltimes X + X \ltimes B = C \tag{3.0.1}$$

where $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{h\times k}, C \in \mathbb{R}^{u\times v}$ are given, $X \in \mathbb{R}^p$ is an unknown vector. We first start from the case $m = u$, and then discuss the general case.

3.1 Case $m = u$

In this section, we study on relation between the dimensions of matrices A, B, C, conditions for solving Eq. (3.0.1) and condition for an existence of a solution. We say that Eq. (3.0.1) is well defined if

(i) the matrices $A \ltimes X$ and $X \ltimes B$ are well defined,

(ii) the dimensions of $A \ltimes X$, $X \ltimes B$ and C are the same.

Lemma 3.1. Eq. (3.0.1) is well-defined if and only if

(i) $k \mid n$ and $h \mid m$,

(ii) $v = k$, and

(iii) $p = \frac{n}{k} = \frac{m}{h}$.

Proof. The sufficient condition is quite obvious. For the necessity, suppose that Eq. (3.0.1) is well defined. Let us denote $t = \operatorname{lcm}\{n, p\}$. By definition of the general left semi-tensor product, we have that

$$A \ltimes X = (A \otimes I_{\frac{t}{n}})(X \otimes I_{\frac{t}{p}}) \in \mathbb{R}^{\frac{mt}{n}\times\frac{t}{p}} \quad \text{and} \quad X \ltimes B = (X \otimes I_h)B \in \mathbb{R}^{ph\times k}.$$

Assume that $C \in \mathbb{R}^{u\times v}$ and $m = u$. Then we have $u = m = ph$, $t = n$ and $v = \frac{t}{p} = k$. Furthermore, $p = \frac{m}{h} = \frac{n}{k}$, $\frac{m}{h}$ and $\frac{n}{k}$ must be a positive integer. Thus, if a solution of equation (3.1) exists, the solution must belong to $\mathbb{R}^p$ where $p = \frac{n}{k} = \frac{m}{h}$ and $\frac{n}{k}$ is required to be a positive integer. ☐

We obtain from the previous proof that $n = pk$. That is n is divisible by p. Then we write $X = [x_1\ x_2\ \ldots\ x_p]^T \in \mathbb{R}^p$. We split A into p^2 equal-size blocks $\hat{A}_{11}, \hat{A}_{12}, \ldots, \hat{A}_{pp}$ and split C into p equal-size blocks $\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_p$. Now we can write Eq. (3.0.1) as

$$A \ltimes X + X \ltimes B = \left[\begin{array}{c:c:c:c} \hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1p} \\ \hdashline \hat{A}_{21} & \hat{A}_{22} & \cdots & \hat{A}_{2p} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \hat{A}_{p1} & \hat{A}_{p2} & \cdots & \hat{A}_{pp} \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} + \begin{bmatrix} x_1 B \\ x_2 B \\ \vdots \\ x_p B \end{bmatrix} = \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \\ \vdots \\ \hat{C}_p \end{bmatrix}$$

where $\hat{A}_{ij}$ and $\hat{C}_i \in \mathbb{R}^{h\times k}$. By considering each $i-th$ row block of the equation, we have

$$x_1\hat{A}_{i1} + x_2\hat{A}_{i2} + \ldots + x_p\hat{A}_{ip} + x_iB = \hat{C}_i. \tag{3.1.1}$$

Hence, we can deduce the following theorem.

Theorem 3.2. Assume that Eq. (3.0.1) is well defined.

(i) Then Eq. (3.0.1) has a solution if and only if, for each $i = 1, ..., p$, the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, B, \hat{C}_i\}$ is linearly dependent in the vector space $\mathbb{R}^{h\times k}$.

(ii) If Eq. (3.0.1) has a solution, then

$$\text{rank}[\hat{A}_i \ B \ \hat{C}_i] = \text{rank}[\hat{A}_i \ B] \qquad \text{for all } i = 1, \ldots, p \tag{3.1.2}$$

where $\hat{A}_i = [\hat{A}_{i1} \ \hat{A}_{i2} \ldots \ \hat{A}_{ip}]$,

(iii) Moreover, if the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, B\}$ is linearly independent for all $i = 1, ..., p$, then the solution X is unique.

Proof. Note that the solvability of Eq. (3.0.1) means that we can write $\hat{C}_i$ in term of linear combination of $\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}$, and B, with coefficients $x_1, x_2, \ldots, x_p, x_i$. Thus the solvability of Eq.(3.0.1) is equivalent to the linearly dependence of the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, B, \hat{C}_i\}$ for all i. Now, the assertion (ii) concerning the ranks of augment matrices holds. The assertion (iii) follows from Eq.(3.1.1). ☐

Example 3.3. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 6 \\ 13 & 4 \end{bmatrix}.$$

The equation has a unique solution

$$X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We split A into 4 equal-size blocks as

$$\hat{A}_{11} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \ \hat{A}_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ \hat{A}_{21} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \ \hat{A}_{22} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$$

and split C into 2 equal-size blocks as

$$\hat{C}_1 = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}, \ \hat{C}_2 = \begin{bmatrix} 7 & 6 \\ 13 & 4 \end{bmatrix}.$$

Then we have

$$\text{rank}[\hat{A}_1\ B\ \hat{C}_1] = \text{rank}\begin{bmatrix} 0 & 1 & 1 & 1 & 3 & 0 & 5 & 3 \\ 2 & 1 & 0 & 1 & 4 & 1 & 6 & 4 \end{bmatrix} = 2,$$

$$\text{rank}[\hat{A}_1\ B] = \text{rank}\begin{bmatrix} 0 & 1 & 1 & 1 & 3 & 0 \\ 2 & 1 & 0 & 1 & 4 & 1 \end{bmatrix} = 2,$$

$$\text{rank}[\hat{A}_2\ B\ \hat{C}_2] = \text{rank}\begin{bmatrix} 1 & 2 & 0 & 2 & 3 & 0 & 7 & 6 \\ 1 & 0 & 2 & 1 & 4 & 1 & 13 & 4 \end{bmatrix} = 2,$$

$$\text{rank}[\hat{A}_2\ B] = \text{rank}\begin{bmatrix} 1 & 2 & 0 & 2 & 3 & 0 \\ 1 & 0 & 2 & 1 & 4 & 1 \end{bmatrix} = 2.$$

Thus,

$$\text{rank}[\hat{A}_1\ B\ \hat{C}_1] = \text{rank}[\hat{A}_1\ B] = 2 \text{ and } \text{rank}[\hat{A}_2\ B\ \hat{C}_2] = \text{rank}[\hat{A}_2\ B] = 2.$$

Example 3.4. Let

$$A = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We split A into 4 equal-size blocks as

$$\hat{A}_{11} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix},\ \hat{A}_{12} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix},\ \hat{A}_{21} = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix},\ \hat{A}_{22} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

and split C into 2 equal-size blocks as

$$\hat{C}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \hat{C}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we have

$$\text{rank}[\hat{A}_1\ B\ \hat{C}_1] = \text{rank}\begin{bmatrix} 0 & 2 & 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 2 & 1 & 0 & 1 \end{bmatrix} = 2,$$

$$\text{rank}[\hat{A}_1\ B] = \text{rank}\begin{bmatrix} 0 & 2 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 2 & 1 \end{bmatrix} = 2,$$

$$\text{rank}[\hat{A}_2\ B\ \hat{C}_2] = \text{rank}\begin{bmatrix} 1 & 3 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 2 & 1 & 0 & 1 \end{bmatrix} = 2,$$

$$\text{rank}[\hat{A}_2\ B] = \text{rank}\begin{bmatrix} 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 2 & 1 \end{bmatrix} = 2.$$

Thus,

$$\text{rank}[\hat{A}_1\ B\ \hat{C}_1] = \text{rank}[\hat{A}_1\ B] = 2 \text{ and } \text{rank}[\hat{A}_2\ B\ \hat{C}_2] = \text{rank}[\hat{A}_2\ B] = 2.$$

However, the equation has no solution.

Next, we apply the vector operator to Eq. (3.1.1) and obtain

$$x_1 V_c(\hat{A}_{i1}) + x_2 V_c(\hat{A}_{i2}) + \ldots + x_p V_c(\hat{A}_{ip}) + x_i V_c(B) = V_c(\hat{C}_i) \in \mathbb{R}^{hk}.$$

From this equation, we conclude the following:

Theorem 3.5. The matrix-vector equation $A \ltimes X + X \ltimes B = C$ with respect to the general left semi-tensor product is equivalent to the following matrix-vector equation

$$\tilde{A}X = \tilde{C},$$

where

$$\tilde{A} = \begin{bmatrix} V_c(\hat{A}_{11}+B) & V_c(\hat{A}_{12}) & \ldots & V_c(\hat{A}_{1p}) \\ V_c(\hat{A}_{21}) & V_c(\hat{A}_{22}+B) & \ldots & V_c(\hat{A}_{2p}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{A}_{p1}) & V_c(\hat{A}_{p2}) & \ldots & V_c(\hat{A}_{pp}+B) \end{bmatrix} \; and \; \tilde{C} = \begin{bmatrix} V_c(\hat{C}_1) \\ V_c(\hat{C}_2) \\ \vdots \\ V_c(\hat{C}_p) \end{bmatrix}.$$

Corollary 3.6. The matrix-vector equation (3.0.1) has a solution if and only if the following rank condition holds:

$$\text{rank}[\tilde{A}] = \text{rank}[\tilde{A}\ \tilde{C}]. \tag{3.1.3}$$

Example 3.7. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 6 \\ 13 & 4 \end{bmatrix}.$$

By Lemma 3.5, we calculate $p = \frac{n}{k} = \frac{4}{2} = 2$.
Write

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We split A into 4 equal-size blocks as

$$\hat{A}_{11} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \; \hat{A}_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; \hat{A}_{21} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \; \hat{A}_{22} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$$

and plit C into 2 equal-size blocks as

$$\hat{C}_1 = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}, \; \hat{C}_2 = \begin{bmatrix} 7 & 6 \\ 13 & 4 \end{bmatrix}.$$

By Theorem 3.5, we solve the equation

$$\begin{bmatrix} V_c(\hat{A}_{11}+B) & V_c(\hat{A}_{12}) \\ V_c(\hat{A}_{21}) & V_c(\hat{A}_{22}+B) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} V_c(\hat{C}_1) \\ V_c(\hat{C}_2) \end{bmatrix}.$$

Then we have

$$\begin{bmatrix} 0+3 & 1 \\ 2+4 & 0 \\ 1+0 & 1 \\ 1+1 & 1 \\ 1 & 0+3 \\ 1 & 2+0 \\ 2 & 2+4 \\ 0 & 1+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 3 \\ 4 \\ 7 \\ 13 \\ 6 \\ 4 \end{bmatrix}.$$

That is

$$\begin{bmatrix} 3 & 1 \\ 6 & 0 \\ 1 & 1 \\ 2 & 1 \\ 1 & 3 \\ 1 & 2 \\ 2 & 6 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 3 \\ 4 \\ 7 \\ 13 \\ 6 \\ 4 \end{bmatrix}.$$

Thus, $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a solution of the equation.

Example 3.8. Let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 3 & 2 \\ 2 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 6 \end{bmatrix}.$$

By Lemma 3.5, we calculate $p = \frac{n}{k} = \frac{2}{1} = 2$.

Write

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We split A into equal-size blocks as

$$\hat{A}_{11} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \hat{A}_{12} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \ \hat{A}_{21} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ \hat{A}_{22} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

and split C into 2 equal-size blocks as

$$\hat{C}_1 = \begin{bmatrix} 9 \\ 9 \end{bmatrix}, \ \hat{C}_2 = \begin{bmatrix} 9 \\ 6 \end{bmatrix}.$$

By Theorem 3.5, we solve the equation

$$\begin{bmatrix} V_c(\hat{A}_{11}+B) & V_c(\hat{A}_{12}) \\ V_c(\hat{A}_{21}) & V_c(\hat{A}_{22}+B) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} V_c(\hat{C}_1) \\ V_c(\hat{C}_2) \end{bmatrix}.$$

Then we have

$$\begin{bmatrix} 2+1 & 3 \\ 1+2 & 3 \\ 3 & 2+1 \\ 2 & 0+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 6 \end{bmatrix}.$$

That is

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \\ 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 6 \end{bmatrix}.$$

Thus, the solutions of the equation are $X = \begin{bmatrix} x_1 \\ 3-x_1 \end{bmatrix}$ where $x_1 \in \mathbb{R}$.

3.2 The general case $m \neq u$

In this section, we discuss the conditions for the solution of Eq. (3.0.1). Firstly we consider the relation between the dimensions of matrices A, B, C. Then we find the condition for dimension of a solution X.

Lemma 3.9. Eq. (3.0.1) is well defined if and only if

(i) $m \mid u$ and $k \mid n$,

(ii) $h = \frac{u}{p}$,

(iii) $k = v$,

(iv) gcd $\{k, \frac{u}{m}\} = 1$, and

(v) $p = \frac{nu}{mk}$.

Proof. Suppose that Eq. (3.0.1) is well defined. Let us denote $t =$ lcm $\{n, p\}$. By definition of the general left semi-tensor product, we have that

$$A \ltimes X = (A \otimes I_{\frac{t}{n}})(X \otimes I_{\frac{t}{p}}) \in \mathbb{R}^{\frac{mt}{n} \times \frac{t}{p}} \quad \text{and} \quad X \ltimes B = (X \otimes I_h)B \in \mathbb{R}^{ph \times k}.$$

Assume that $C \in \mathbb{R}^{u \times v}$. Then we have

$$u = \frac{mt}{n} = ph \quad \text{and} \quad v = \frac{t}{p} = k.$$

Furthermore, $\frac{u}{m} = \frac{t}{n}$ and $t = \frac{un}{m}$. Substitute $t = \frac{un}{m}$ in $\frac{t}{p} = k$. Then we obtain $p = \frac{nu}{mk}$. Since $\frac{un}{m} = t = \text{lcm}\,\{n, p\}$, we have

$$\frac{un}{m} = \text{lcm}\,\{n, \frac{nu}{mk}\}.$$

We see that $\frac{u}{m}$ is a factor of $\frac{nu}{mk}$. Thus $\frac{n}{k}$ must be a positive integer. Moreover,

$$\frac{un}{m} = \frac{n}{k} \cdot \text{lcm}\,\{k, \frac{u}{m}\}.$$

Consequently, $\gcd\{k, \frac{u}{m}\} = 1$. ☐

We obtain from the proof that $\frac{t}{p} = k$. Then we compute

$$A \ltimes X = (A \otimes I_{\frac{t}{n}})(X \otimes I_k) \qquad \text{where } t = \text{lcm}\,\{n, p\}$$

Then we split $A \otimes I_{\frac{t}{n}}$ into p^2 equal-size blocks $\hat{A}_{11}, \hat{A}_{12}, \ldots, \hat{A}_{pp}$, split $X \otimes I_k$ into p equal-size blocks $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_p$ and split C into p equal-size blocks $\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_p$. Now we can write Eq. (3.0.1) as

$$A \ltimes X = \left[\begin{array}{c:c:c:c} \hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1p} \\ \hdashline \hat{A}_{21} & \hat{A}_{22} & \cdots & \hat{A}_{2p} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \hat{A}_{p1} & \hat{A}_{p2} & \cdots & \hat{A}_{pp} \end{array}\right] \begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \vdots \\ \hat{X}_p \end{bmatrix} + \begin{bmatrix} x_1 B \\ x_2 B \\ \vdots \\ x_p B \end{bmatrix} = \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \\ \vdots \\ \hat{C}_p \end{bmatrix}$$

where $\hat{A}_{ij}, \hat{C}_i \in \mathbb{R}^{h \times k}$ and $\hat{X}_i \in \mathbb{R}^{k \times k}$. By considering each $i-th$ row block of the equation, we have

$$\hat{A}_{i1}\hat{X}_1 + \hat{A}_{i2}\hat{X}_2 + \ldots + \hat{A}_{ip}\hat{X}_p + x_i B = \hat{C}_i. \tag{3.2.1}$$

Since $\hat{A}_{ij}\hat{X}_j = x_j\hat{A}_{ij} \times I_k = x_j\hat{A}_{ij}$ for $i, j = 1 \ldots p$, Then we can write Eq. (3.2.1) as

$$x_1\hat{A}_{i1} + x_2\hat{A}_{i2} + \ldots + x_p\hat{A}_{ip} + x_i B = \hat{C}_i. \tag{3.2.2}$$

Hence, we can deduce the following theorem.

Theorem 3.10. Assume that Eq. (3.0.1) is well defined.

(i) Then Eq. (3.0.1) has a solution if and only if, for each $i = 1, ..., p$, the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, B, \hat{C}_i\}$ is linearly dependent in the vector space $\mathbb{R}^{h \times k}$.

(ii) If Eq. (3.0.1) has a solution, then

$$\text{rank}[\hat{A}_i \ B \ \hat{C}_i] = \text{rank}[\hat{A}_i \ B] \qquad \text{for all } i = 1, \ldots, p \tag{3.2.3}$$

where $\hat{A}_i = [\hat{A}_{i1} \ \hat{A}_{i2} \ldots \ \hat{A}_{ip}]$,

(iii) Moreover, if the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, B\}$ is linearly independent for all $i = 1, ..., p$, then the solution X is unique.

Proof. Note that the solvability of Eq. (3.0.1) means that we can write $\hat{C}_i$ in term of linear combination of $\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}$, and B, with coefficients $x_1, x_2, \ldots, x_p, x_i$. Thus the solvability of Eq.(3.0.1) is equivalent to the linearly dependence of the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, B, \hat{C}_i\}$ for all i. Now, the assertion (ii) concerning the ranks of augment matrices holds, The assertion (iii) follows from Eq.(3.2.2). $\square$

Example 3.11. Let Let

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 8 & 5 & 10 \\ 13 & 4 & 5 \\ 13 & 12 & 8 \\ 9 & 7 & 12 \end{bmatrix}.$$

The equation has a solution

$$X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

We split $A \otimes I_2 = \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 1 \\ 2 & 0 & 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 1 & 0 & 3 \end{bmatrix}$ into 4 equal-size blocks as

$$\hat{A}_{11} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \ \hat{A}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \ \hat{A}_{21} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \ \hat{A}_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and split C into 2 equal-size blocks as

$$\hat{C}_1 = \begin{bmatrix} 8 & 5 & 10 \\ 13 & 4 & 5 \end{bmatrix}, \ \hat{C}_2 = \begin{bmatrix} 13 & 12 & 8 \\ 9 & 7 & 12 \end{bmatrix}.$$

Then

$$\text{rank}[\hat{A}_1 \ B \ \hat{C}_1] = \text{rank}[\hat{A}_1 \ B] = 2 \text{ and } \text{rank}[\hat{A}_2 \ B \ \hat{C}_2] = \text{rank}[\hat{A}_2 \ B] = 2.$$

Example 3.12. Let Let

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 8 & 5 & 10 \\ 13 & 3 & 5 \\ 13 & 12 & 8 \\ 2 & 7 & 12 \end{bmatrix}.$$

We split $A \otimes I_2 = \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 1 \\ 2 & 0 & 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 1 & 0 & 3 \end{bmatrix}$ into 4 equal-size blocks as

$$\hat{A}_{11} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \ \hat{A}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \ \hat{A}_{21} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \ \hat{A}_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and split C into 2 equal-size blocks as

$$\hat{C}_1 = \begin{bmatrix} 8 & 5 & 10 \\ 13 & 3 & 5 \end{bmatrix}, \ \hat{C}_2 = \begin{bmatrix} 13 & 12 & 8 \\ 2 & 7 & 12 \end{bmatrix}.$$

Then

$$\text{rank}[\hat{A}_1 \ B \ \hat{C}_1] = \text{rank}[\hat{A}_1 \ B] = 2 \text{ and } \text{rank}[\hat{A}_2 \ B \ \hat{C}_2] = \text{rank}[\hat{A}_2 \ B] = 2.$$

However, the equation has no solution.

Next, we apply the vector operator to Eq. (3.1.1) and obtain

$$x_1 V_c(\hat{A}_{i1}) + x_2 V_c(\hat{A}_{i2}) + \ldots + x_p V_c(\hat{A}_{ip}) + x_i V_c(B) = V_c(\hat{C}_i) \in \mathbb{R}^{hk}.$$

From this equation, we conclude the following:

Theorem 3.13. The matrix-vector equation $A \ltimes X + X \ltimes B = C$ with respect to the general left semi-tensor product is equivalent to the following matrix-vector equation

$$\tilde{A}X = \tilde{C},$$

where

$$\tilde{A} = \begin{bmatrix} V_c(\hat{A}_{11} + B) & V_c(\hat{A}_{12}) & \ldots & V_c(\hat{A}_{1p}) \\ V_c(\hat{A}_{21}) & V_c(\hat{A}_{22} + B) & \ldots & V_c(\hat{A}_{2p}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{A}_{p1}) & V_c(\hat{A}_{p2}) & \ldots & V_c(\hat{A}_{pp} + B) \end{bmatrix} \text{ and } \tilde{C} = \begin{bmatrix} V_c(\hat{C}_1) \\ V_c(\hat{C}_2) \\ \vdots \\ V_c(\hat{C}_p) \end{bmatrix}.$$

Example 3.14. Let

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 8 & 5 & 10 \\ 13 & 4 & 5 \\ 13 & 12 & 8 \\ 9 & 7 & 12 \end{bmatrix}.$$

By Lemma 3.9, we calculate $p = \frac{nu}{mk} = \frac{3 \cdot 4}{2 \cdot 3} = 2.$ Then $t = \text{lcm}\{3, 2\} = 6.$ Write

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Compute

$$A \otimes I_{\frac{6}{3}} = A \otimes I_2 = \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

We split $A \otimes I_2$ into 4 equal-size blocks as

$$\hat{A}_{11} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \ \hat{A}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \ \hat{A}_{21} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \ \hat{A}_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and split C into into 2 equal-size blocks as

$$\hat{C}_1 = \begin{bmatrix} 8 & 5 & 10 \\ 13 & 4 & 5 \end{bmatrix},\ \hat{C}_2 = \begin{bmatrix} 13 & 12 & 8 \\ 9 & 7 & 12 \end{bmatrix}.$$

By Theorem 3.11, we solve the equation

$$\begin{bmatrix} V_c(\hat{A}_{11} + B) & V_c(\hat{A}_{12}) \\ V_c(\hat{A}_{21}) & V_c(\hat{A}_{22} + B) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} V_c(\hat{C}_1) \\ V_c(\hat{C}_2) \end{bmatrix}.$$

Then we have

$$\begin{bmatrix} 1+3 & 0 \\ 0+2 & 3 \\ 0+1 & 1 \\ 1+1 & 0 \\ 3+2 & 0 \\ 0+1 & 1 \\ 2 & 0+3 \\ 0 & 1+2 \\ 0 & 3+1 \\ 2 & 0+1 \\ 1 & 0+2 \\ 0 & 3+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ 5 \\ 4 \\ 10 \\ 5 \\ 13 \\ 9 \\ 12 \\ 7 \\ 8 \\ 12 \end{bmatrix}.$$

That is

$$\begin{bmatrix} 4 & 0 \\ 2 & 3 \\ 1 & 1 \\ 2 & 0 \\ 5 & 0 \\ 1 & 1 \\ 2 & 3 \\ 0 & 3 \\ 0 & 4 \\ 2 & 1 \\ 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ 5 \\ 4 \\ 10 \\ 5 \\ 13 \\ 9 \\ 12 \\ 7 \\ 8 \\ 12 \end{bmatrix}.$$

Thus, $X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a solution of the equation.

Chapter 4

Solving the Sylvester equation when X is a matrix

In this chapter, we study on solving the Sylvester equation

$$A \ltimes X + X \ltimes B = C \tag{4.0.1}$$

where $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{h\times k}, C \in \mathbb{R}^{u\times v}$ are given, $X \in \mathbb{R}^{p\times q}$ is an unknown matrix. We first start from the case $m = u$ and $q \mid h$, then discuss the general case.

4.1 The case $m = u$ and $q \mid h$

In this subsection, we consider conditions for Eq. (4.0.1) to be solvable and investigate the conditions for dimensions of a solution matrix X, A, B and C. We have the following lemma.

Lemma 4.1. Eq. (4.0.1) is well defined if and only if the dimensions of A, B, C, and X satisfy the following conditions

(i) $u = m = \frac{ph}{q}$,

(ii) $v = \frac{qn}{p} = k$.

Proof. Assume that $m = u$ and $q \mid h$. Denote $t = \text{lcm}\{n, p\}$. By definition of the general left semi-tensor product, we have that

$$A \ltimes X = (A \otimes I_{\frac{t}{n}})(X \otimes I_{\frac{t}{p}}) \in \mathbb{R}^{\frac{mt}{n}\times\frac{qt}{p}} \quad \text{and} \quad X \ltimes B = (X \otimes I_{\frac{h}{q}})B \in \mathbb{R}^{\frac{ph}{q}\times k}.$$

Assume that $C \in \mathbb{R}^{u\times v}$. Then we have $u = \frac{mt}{n} = \frac{ph}{q}$ and $v = \frac{qt}{p} = k$. Since $u = m$, we have $t = n$. □

Remark 4.2. If Eq. (4.0.1) has a solution, the dimensions of matrices A, B, C and X must satisfy the conditions

(i) $k \mid v$,

(ii) $\frac{m}{n} = \frac{h}{k}$.

(iii) $p = \frac{n}{\alpha}$ and $q = \frac{k}{\alpha}$, where α is a common divisor of n and k.

Since $n = t$, n is divisible by p. Assume that h is divisible by q. Then we split A into p^2 equal-size blocks $\hat{A}_{11}, \hat{A}_{12}, \ldots, \hat{A}_{pp}$, split B into q^2 equal-size blocks $\hat{B}_{11}, \hat{B}_{12}, \ldots, \hat{B}_{qq}$

and split C into pq equal-size blocks $\hat{C}_{11}, \hat{C}_{12}, \ldots, \hat{C}_{pq}$. Now we can write Eq. (4.0.1) as

$$\begin{aligned}
&A \ltimes X + X \ltimes B \\
&= \left[\begin{array}{c:c:c:c} \hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1p} \\ \hdashline \hat{A}_{21} & \hat{A}_{22} & \cdots & \hat{A}_{2p} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \hat{A}_{p1} & \hat{A}_{p2} & \cdots & \hat{A}_{pp} \end{array}\right] \begin{bmatrix} X_1 & X_2 & \ldots & X_q \end{bmatrix} + \begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^p \end{bmatrix} \left[\begin{array}{c:c:c:c} \hat{B}_{11} & \hat{B}_{12} & \cdots & \hat{B}_{1q} \\ \hdashline \hat{B}_{21} & \hat{B}_{22} & \cdots & \hat{B}_{2q} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \hat{B}_{q1} & \hat{B}_{q2} & \cdots & \hat{B}_{qq} \end{array}\right] \\
&= \left[\begin{array}{c:c:c:c} \hat{C}_{11} & \hat{C}_{12} & \cdots & \hat{C}_{1q} \\ \hdashline \hat{C}_{21} & \hat{C}_{22} & \cdots & \hat{C}_{2q} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \hat{C}_{p1} & \hat{C}_{p2} & \cdots & \hat{C}_{pq} \end{array}\right]
\end{aligned} \tag{4.1.1}$$

where $\hat{A}_{ij}, \hat{B}_{ij}$ and $\hat{C}_{ij}$ are $\in \mathbb{R}^{\frac{m}{p} \times \frac{n}{p}}$. By considering each block of the equation, we obtain

$$\hat{C}_{ij} = x_{1j}\hat{A}_{i1} + \ldots + x_{pj}\hat{A}_{ip} + x_{i1}\hat{B}_{1j} + \ldots + x_{iq}\hat{B}_{qj} \tag{4.1.2}$$

Let X^i be the i^{th} row of X, X_j be the j^{th} column of X. For each i, j, let us partition

$$\hat{A}^i = [\hat{A}_{i1} \ldots \hat{A}_{ip}], \hat{B}_j = \begin{bmatrix} \hat{B}_{1j} \\ \vdots \\ \hat{B}_{qj} \end{bmatrix}.$$

We can write Eq. (4.0.1) as

$$\hat{C}_{ij} = \hat{A}^i X_j + X^i \hat{B}_j \tag{4.1.3}$$

for each $i = 1, \ldots, p, j = 1, \ldots, q$. Thus, we can derive the following theorem.

Theorem 4.3. Assume that Eq. (4.0.1) is well defined.

(i) Then Eq. (4.0.1) has a solution if and only if, for each $i = 1, \ldots, p$ and $j = 1, \ldots, q$, the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, \hat{B}_{1j}, \hat{B}_{2j}, \ldots, \hat{B}_{qj}, \hat{C}_{ij}\}$ is linearly dependent in $\mathbb{R}^{\frac{m}{p} \times \frac{n}{p}}$.

(ii) If Eq. (4.0.1) has a solution, then

$$\text{rank}[\hat{A}^i\ \tilde{B}_j\ \hat{C}_{ij}] = \text{rank}[\hat{A}^i\ \tilde{B}_j] \qquad \textit{for all } i = 1, \ldots, p,\ j = 1, \ldots, q \tag{4.1.4}$$

where $\hat{A}^i = [\hat{A}_{i1}\ \hat{A}_{i2} \ldots\ \hat{A}_p]$ and $\tilde{B}_j = [\hat{B}_{1j}\ \hat{B}_{2j} \ldots\ \hat{B}_{qj}]$.

(iii) Moreover, if the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, \hat{B}_{1j}, \hat{B}_{2j}, \ldots, \hat{B}_{qj}\}$ is linearly independent for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$, then the solution X is unique.

Proof. We can write $\hat{C_{ij}}$ in terms of a linear combination of $\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}$, and $\hat{B}_{1j}, \hat{B}_{2j}, \ldots, \hat{B}_{qj}$, with coefficients $x_{1j}, x_{2j}, \ldots, x_{pj}, x_{i1}, x_{i2}, \ldots, x_{iq}$. Thus the solvability of Eq. (4.0.1) is equivalent to the linear dependence of the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, \hat{B}_{1j}, \hat{B}_{2j}, \ldots, \hat{B}_{qj}, \hat{C}_{ij}\}$ for all i, j. Now, the assertion (ii) concerning the ranks of augment matrices holds. The assertion (iii) follows from Eq. (4.1.2). □

Example 4.4. Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 3 \\ 0 & 3 & 3 & 1 \\ 1 & 1 & 2 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 8 & 9 & 7 \\ 5 & 4 & 8 & 13 \end{bmatrix}.$$

The equation has a unique solution $X = \begin{bmatrix} 1 & 2 \end{bmatrix}$.
We split B into 4 equal-size blocks as

$$\hat{B}_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \hat{B}_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \hat{B}_{21} = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}, \hat{B}_{22} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$$

and split C into 2 equal-size blocks as

$$\hat{C}_{11} = \begin{bmatrix} 2 & 8 \\ 5 & 4 \end{bmatrix}, \hat{C}_{12} = \begin{bmatrix} 9 & 7 \\ 8 & 13 \end{bmatrix}.$$

Then we have

$$\begin{aligned} \text{rank}[A\ \tilde{B}_1\ \hat{C}_{11}] &= \text{rank} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 3 & 2 & 8 \\ 2 & 0 & 1 & 2 & 1 & 1 & 5 & 4 \end{bmatrix} = 2, \\ \text{rank}[A\ \tilde{B}_1] &= \text{rank} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 3 \\ 2 & 0 & 1 & 2 & 1 & 1 \end{bmatrix} = 2, \\ \text{rank}[A\ \tilde{B}_2\ \hat{C}_{12}] &= \text{rank} \begin{bmatrix} 1 & 2 & 1 & 1 & 3 & 1 & 9 & 7 \\ 2 & 0 & 0 & 3 & 2 & 5 & 8 & 3 \end{bmatrix} = 2, \\ \text{rank}[A\ \tilde{B}_2] &= \text{rank} \begin{bmatrix} 1 & 2 & 1 & 1 & 3 & 1 \\ 2 & 0 & 0 & 3 & 2 & 5 \end{bmatrix} = 2. \end{aligned}$$

Thus,

$$\text{rank}[A\ \tilde{B}_1\ \hat{C}_{11}] = \text{rank}[A\ \tilde{B}_1] = 2 \text{ and } \text{rank}[A\ \tilde{B}_2\ \hat{C}_{12}] = \text{rank}[A\ \tilde{B}_2] = 2.$$

Example 4.5. Let

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}.$$

We split B into 4 equal-size blocks as

$$\hat{B}_{11} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \hat{B}_{12} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, \hat{B}_{21} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \hat{B}_{22} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

and split C into 2 equal-size blocks as

$$\hat{C}_{11} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}, \hat{C}_{12} = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}.$$

Then we have

$$\text{rank}[A\ \tilde{B}_1\ \hat{C}_{11}] = \text{rank}\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 3 & 0 & 2 & 1 & 0 & 2 & 3 & 1 \end{bmatrix} = 2,$$

$$\text{rank}[A\ \tilde{B}_1] = \text{rank}\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 3 & 0 & 2 & 1 & 0 & 2 \end{bmatrix} = 2,$$

$$\text{rank}[A\ \tilde{B}_2\ \hat{C}_{12}] = \text{rank}\begin{bmatrix} 1 & 1 & 2 & 0 & 2 & 3 & 0 & 2 \\ 3 & 0 & 3 & 1 & 1 & 1 & 3 & 1 \end{bmatrix} = 2,$$

$$\text{rank}[A\ \tilde{B}_2] = \text{rank}\begin{bmatrix} 1 & 1 & 2 & 0 & 2 & 3 \\ 3 & 0 & 3 & 1 & 1 & 1 \end{bmatrix} = 2.$$

Thus,

$$\text{rank}[A\ \tilde{B}_1\ \hat{C}_{11}] = \text{rank}[A\ \tilde{B}_1] = 2 \text{ and } \text{rank}[A\ \tilde{B}_2\ \hat{C}_{12}] = \text{rank}[A\ \tilde{B}_2] = 2.$$

However, the equation has no solution.

Next, we apply vector operator to Eq. (4.1.2) and obtain the following equation

$$V_c(\hat{C}_{ij}) = x_{1j}V_c(\hat{A}_{i1}) + \ldots + x_{pj}V_c(\hat{A}_{ip}) + x_{i1}\hat{B}_{1j}) + \ldots + x_{iq}V_c(\hat{B}_{qj})$$

Consequently, we write the matrix equation as

$$\begin{bmatrix} V_c(\hat{A}_{11}) & V_c(\hat{A}_{12}) & \cdots & V_c(\hat{A}_{1p}) \\ V_c(\hat{A}_{21}) & V_c(\hat{A}_{22)}) & \cdots & V_c(\hat{A}_{2p}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{A}_{p1}) & V_c(\hat{A}_{p2}) & \cdots & V_c(\hat{A}_{pp}) \end{bmatrix} X + X \begin{bmatrix} V_c(\hat{B}_{11}) & V_c(\hat{B}_{12}) & \cdots & V_c(\hat{B}_{1q}) \\ V_c(\hat{B}_{21}) & V_c(\hat{B}_{22}) & \cdots & V_c(\hat{B}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{B}_{q1}) & V_c(\hat{B}_{q2}) & \cdots & V_c(\hat{B}_{qq}) \end{bmatrix}$$

$$= \begin{bmatrix} V_c(\hat{C}_{11}) & V_c(\hat{C}_{12}) & \cdots & V_c(\hat{C}_{1q}) \\ V_c(\hat{C}_{21}) & V_c(\hat{C}_{22}) & \cdots & V_c(\hat{C}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{C}_{p1}) & V_c(\hat{C}_{p2}) & \cdots & V_c(\hat{C}_{pq}) \end{bmatrix}.$$

Theorem 4.6. Eq. (4.0.1) is equivalent to the following linear system:

$$(I_q \otimes \breve{A} + \breve{B}^T \otimes I_p)V_c(X) = V_c(\breve{C}) \tag{4.1.5}$$

where

$$\breve{A} = \begin{bmatrix} V_c(\hat{A}_{11}) & V_c(\hat{A}_{12}) & \cdots & V_c(\hat{A}_{1p}) \\ V_c(\hat{A}_{21}) & V_c(\hat{A}_{22)}) & \cdots & V_c(\hat{A}_{2p}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{A}_{p1}) & V_c(\hat{A}_{p2}) & \cdots & V_c(\hat{A}_{pp}) \end{bmatrix}, \quad \breve{B} = \begin{bmatrix} V_c(\hat{B}_{11}) & V_c(\hat{B}_{12}) & \cdots & V_c(\hat{B}_{1q}) \\ V_c(\hat{B}_{21}) & V_c(\hat{B}_{22}) & \cdots & V_c(\hat{B}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{B}_{q1}) & V_c(\hat{B}_{q2}) & \cdots & V_c(\hat{B}_{qq}) \end{bmatrix},$$

$$\breve{C} = \begin{bmatrix} V_c(\hat{C}_{11}) & V_c(\hat{C}_{12}) & \cdots & V_c(\hat{C}_{1q}) \\ V_c(\hat{C}_{21}) & V_c(\hat{C}_{22}) & \cdots & V_c(\hat{C}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{C}_{p1}) & V_c(\hat{C}_{p2}) & \cdots & V_c(\hat{C}_{pq}) \end{bmatrix}.$$

From Theorem 4.6, we can solve Eq. (4.0.1) by solving the linear system (4.1.5) instead.

Example 4.7. Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 3 \\ 0 & 3 & 3 & 1 \\ 1 & 1 & 2 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 8 & 9 & 7 \\ 5 & 4 & 8 & 13 \end{bmatrix}.$$

By Lemma 4.1 and Remark 4.2 , we take $\alpha = 2$, then $p = \frac{n}{\alpha} = \frac{2}{2} = 1$ and $q = \frac{k}{\alpha} = \frac{4}{2} = 2$. Write

$$X = \begin{bmatrix} x_1 & x_2 \end{bmatrix}.$$

We split B into 4 equal-size blocks as

$$\hat{B}_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \; \hat{B}_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \; \hat{B}_{21} = \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}, \; \hat{B}_{22} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$$

and split C into 2 equal-size blocks as

$$\hat{C}_{11} = \begin{bmatrix} 2 & 8 \\ 5 & 4 \end{bmatrix}, \; \hat{C}_{12} = \begin{bmatrix} 9 & 7 \\ 8 & 13 \end{bmatrix}.$$

Then we have

$$\breve{A} = \begin{bmatrix} V_c(A) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix},$$

$$\breve{B} = \left[\begin{array}{c:c} V_c(\hat{B}_{11}) & V_c(\hat{B}_{12}) \\ \hdashline V_c(\hat{B}_{21}) & V_c(\hat{B}_{22}) \end{array}\right] = \left[\begin{array}{c:c} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 2 & 3 \\ \hdashline 0 & 3 \\ 3 & 2 \\ 3 & 1 \\ 1 & 5 \end{array}\right],$$

$$\breve{C} = \left[\begin{array}{c:c} V_c(\hat{C}_{11}) & V_c(\hat{C}_{12}) \end{array}\right] = \left[\begin{array}{c:c} 2 & 1 \\ 5 & 0 \\ 8 & 1 \\ 4 & 3 \end{array}\right].$$

By Theorem 4.6, we need to solve the following linear system

$$(I_2 \otimes \breve{A} + \breve{B}^T \otimes I_1) V_c(X) = V_c(\breve{C}).$$

Then we have

$$\left(\begin{bmatrix}1 & 0\\2 & 0\\2 & 0\\0 & 0\\0 & 1\\0 & 2\\0 & 2\\0 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\1 & 1\\0 & 3\\2 & 1\\1 & 3\\0 & 2\\1 & 1\\3 & 5\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2\\5\\8\\4\\9\\8\\7\\13\end{bmatrix}.$$

That is

$$\begin{bmatrix}2 & 0\\3 & 1\\2 & 3\\2 & 1\\1 & 4\\0 & 4\\1 & 3\\3 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2\\5\\8\\4\\9\\8\\7\\13\end{bmatrix}.$$

Thus, $X = \begin{bmatrix}1 & 2\end{bmatrix}$ is a solution of the equation.

Example 4.8. Let

$$A = \begin{bmatrix}1 & 2\\1 & 2\end{bmatrix},\ B = \begin{bmatrix}1 & 2 & 3 & 4\\1 & 2 & 3 & 4\\2 & 4 & 2 & 2\\2 & 4 & 2 & 2\end{bmatrix} \text{ and } C = \begin{bmatrix}4 & 8 & 6 & 8\\4 & 8 & 6 & 8\end{bmatrix}.$$

By Lemma 4.1 and Remark 4.2 , we take $\alpha = 2$, then $p = \frac{n}{\alpha} = \frac{2}{2} = 1$ and $q = \frac{k}{\alpha} = \frac{4}{2} = 2$. Write

$$X = \begin{bmatrix}x_1 & x_2\end{bmatrix}.$$

We split B into 4 equal-size blocks as

$$\hat{B}_{11} = \begin{bmatrix}1 & 2\\1 & 2\end{bmatrix},\ \hat{B}_{12} = \begin{bmatrix}3 & 4\\3 & 4\end{bmatrix},\ \hat{B}_{21} = \begin{bmatrix}2 & 4\\2 & 4\end{bmatrix},\ \hat{B}_{22} = \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$$

and split C into 2 equal-size blocks as

$$\hat{C}_{11} = \begin{bmatrix}4 & 8\\4 & 8\end{bmatrix},\ \hat{C}_{12} = \begin{bmatrix}6 & 8\\6 & 8\end{bmatrix}.$$

Then we have

$$\breve{A} = \begin{bmatrix} V_c(A) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix},$$

$$\breve{B} = \left[\begin{array}{c:c} V_c(\hat{B}_{11}) & V_c(\hat{B}_{12}) \\ \hdashline V_c(\hat{B}_{21}) & V_c(\hat{B}_{22}) \end{array}\right] = \left[\begin{array}{c:c} 1 & 3 \\ 1 & 3 \\ 2 & 4 \\ 2 & 4 \\ \hdashline 2 & 2 \\ 2 & 2 \\ 4 & 2 \\ 4 & 2 \end{array}\right],$$

$$\breve{C} = \left[\begin{array}{c:c} V_c(\hat{C}_{11}) & V_c(\hat{C}_{12}) \end{array}\right] = \left[\begin{array}{c:c} 4 & 6 \\ 4 & 6 \\ 8 & 8 \\ 8 & 8 \end{array}\right].$$

By Theorem 4.6, we need to solve the following linear system

$$(I_2 \otimes \breve{A} + \breve{B}^T \otimes I_1)V_c(X) = V_c(\breve{C}).$$

Then we have

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 2 & 0 \\ 2 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 2 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \\ 2 & 4 \\ 3 & 2 \\ 3 & 2 \\ 4 & 2 \\ 4 & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 8 \\ 8 \\ 6 \\ 6 \\ 8 \\ 8 \end{bmatrix}.$$

That is

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 4 & 4 \\ 4 & 4 \\ 3 & 3 \\ 3 & 3 \\ 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 8 \\ 8 \\ 6 \\ 6 \\ 8 \\ 8 \end{bmatrix}.$$

Thus, the solutions of the equation are $X = \begin{bmatrix} x_1 & 2 - x_1 \end{bmatrix}$ where $x_1 \in \mathbb{R}$.

Example 4.9. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 9 & 9 & 15 \\ 9 & 7 & 4 \\ 6 & 12 & 7 \\ 5 & 13 & 7 \end{bmatrix}.$$

By Lemma 4.1 and Remark 4.2 , we take $\alpha = 1$, then $p = \frac{n}{\alpha} = \frac{2}{1} = 2$ and $q = \frac{k}{\alpha} = \frac{3}{1} = 3$. Write

$$X = \begin{bmatrix} x_1 & x_3 & x_5 \\ x_2 & x_4 & x_6 \end{bmatrix}.$$

We split A into 4 equal-size blocks as

$$\left[\begin{array}{c:c} \hat{A}_{11} & \hat{A}_{12} \\ \hdashline \hat{A}_{21} & \hat{A}_{22} \end{array}\right] = \left[\begin{array}{c:c} 1 & 2 \\ 0 & 1 \\ \hdashline 2 & 0 \\ 3 & 1 \end{array}\right],$$

split B into 9 equal-size blocks as

$$\left[\begin{array}{c:c:c} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} \\ \hdashline \hat{B}_{21} & \hat{B}_{22} & \hat{B}_{23} \\ \hdashline \hat{B}_{31} & \hat{B}_{32} & \hat{B}_{33} \end{array}\right] = \left[\begin{array}{c:c:c} 1 & 1 & 0 \\ 3 & 2 & 1 \\ \hdashline 2 & 0 & 3 \\ 2 & 1 & 0 \\ \hdashline 1 & 3 & 1 \\ 0 & 1 & 1 \end{array}\right],$$

and split C into 6 equal-size blocks as

$$\left[\begin{array}{c:c:c} \hat{C}_{11} & \hat{C}_{12} & \hat{C}_{13} \\ \hdashline \hat{C}_{21} & \hat{C}_{22} & \hat{C}_{23} \end{array}\right] = \left[\begin{array}{c:c:c} 9 & 9 & 15 \\ 9 & 7 & 4 \\ \hdashline 6 & 12 & 7 \\ 5 & 13 & 7 \end{array}\right].$$

Then we have

$$\breve{A} = \left[\begin{array}{c|c} V_c(\hat{A}_{11}) & V_c(\hat{A}_{12}) \\ \hline V_c(\hat{A}_{21}) & V_c(\hat{A}_{22}) \end{array}\right] = \left[\begin{array}{c|c} 1 & 2 \\ 0 & 1 \\ \hline 2 & 0 \\ 3 & 1 \end{array}\right],$$

$$\breve{B} = \left[\begin{array}{c|c|c} V_c(\hat{B}_{11}) & V_c(\hat{B}_{12}) & V_c(\hat{B}_{13}) \\ \hline V_c(\hat{B}_{21}) & V_c(\hat{B}_{22}) & V_c(\hat{B}_{23}) \\ \hline V_c(\hat{B}_{31}) & V_c(\hat{B}_{32}) & V_c(\hat{B}_{33}) \end{array}\right] = \left[\begin{array}{c|c|c} 1 & 1 & 0 \\ 3 & 2 & 1 \\ \hline 2 & 0 & 3 \\ 2 & 1 & 0 \\ \hline 1 & 3 & 1 \\ 0 & 1 & 1 \end{array}\right],$$

$$\breve{C} = \left[\begin{array}{c|c|c} V_c(\hat{C}_{11}) & V_c(\hat{C}_{12}) & V_c(\hat{C}_{13}) \\ \hline V_c(\hat{C}_{21}) & V_c(\hat{C}_{22}) & V_c(\hat{C}_{23}) \end{array}\right] = \left[\begin{array}{c|c|c} 9 & 9 & 15 \\ 9 & 7 & 4 \\ \hline 6 & 12 & 7 \\ 5 & 13 & 7 \end{array}\right].$$

By Theorem 4.6, we need to solve the following linear system

$$(I_3 \otimes \breve{A} + \breve{B}^T \otimes I_2)V_c(X) = V_c(\breve{C}).$$

Then we have

$$\left(\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 3 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 6 \\ 5 \\ 9 \\ 7 \\ 12 \\ 13 \\ 15 \\ 4 \\ 7 \\ 7 \end{bmatrix}.$$

That is

$$\begin{bmatrix} 2 & 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 \\ 3 & 4 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 2 & 3 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 2 & 3 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & 2 & 1 \\ 0 & 1 & 0 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 6 \\ 5 \\ 9 \\ 7 \\ 12 \\ 13 \\ 15 \\ 4 \\ 7 \\ 7 \end{bmatrix}.$$

Thus, the solution of the equation is $X = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

4.2 The general case

In this subsection, we study the conditions for the dimensions of $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{h\times k}$ *and* $C \in \mathbb{R}^{u\times v}$ and $X \in \mathbb{R}^{p\times q}$. We have the following lemma.

Lemma 4.10. Eq. (4.0.1) is well defined if and only if the dimensions of A, B, C, and X satisfy the following conditions

(i) $u = \frac{mt}{n} = \frac{ps}{q}$,

(ii) $v = \frac{tq}{p} = \frac{ks}{h}$,

where $t = \operatorname{lcm}\{n, p\}$ and $s = \operatorname{lcm}\{q, h\}$.

Proof. Denote $t = \operatorname{lcm}\{n, p\}$ and $s = \operatorname{lcm}\{q, h\}$. By definition of the general left semi-tensor product, we have that

$$A \ltimes X = (A \otimes I_{\frac{t}{n}})(X \otimes I_{\frac{t}{p}}) \in \mathbb{R}^{\frac{mt}{n}\times\frac{tq}{p}} \quad \text{and} \quad X \ltimes B = (X \otimes I_{\frac{s}{q}})(B \otimes I_{\frac{s}{h}}) \in \mathbb{R}^{\frac{ps}{q}\times\frac{ks}{h}}.$$

Assume that $C \in \mathbb{R}^{u\times v}$. Then we have $u = \frac{mt}{n} = \frac{ps}{q}$ and $v = \frac{tq}{p} = \frac{ks}{h}$. ☐

Remark 4.11. If Eq. (4.0.1) has a solution, the dimensions of matrices A, B, and C must satisfy the conditions

(i) $m \mid u$ and $k \mid v$.

(ii) $\frac{m}{n} = \frac{h}{k}$.

(iii) $p = \frac{nu}{\alpha}$ and $q = \frac{mv}{\alpha}$, where $\alpha \mid nu, \alpha \mid mv, m \mid \alpha$ and $n \mid \alpha$.

As we see in the proof that,

$$A \ltimes X = (A\otimes I_{\frac{t}{n}})(X\otimes I_{\frac{t}{p}}) \in \mathbb{R}^{\frac{mt}{n}\times\frac{tq}{p}} \quad \text{and} \quad X \ltimes B = (X\otimes I_{\frac{s}{q}})(B\otimes I_{\frac{s}{h}}) \in \mathbb{R}^{\frac{ps}{q}\times\frac{ks}{h}}.$$

Then we split $A\otimes I_{\frac{t}{n}}$ into p^2 equal-sizes blocks $\hat{A}_{11}, \hat{A}_{12}, \dots, \hat{A}_{pp} \in \mathbb{R}^{\frac{s}{q}\times\frac{t}{p}}$, split $X\otimes I_{\frac{t}{p}}$ into pq equal-sizes blocks $\hat{X}_{11}, \hat{X}_{12}, \dots, \hat{X}_{pq} \in \mathbb{R}^{\frac{t}{p}\times\frac{t}{p}}$, split $X\otimes I_{\frac{s}{q}}$ into pq equal-sizes blocks $\tilde{X}_{11}, \tilde{X}_{12}, \dots, \tilde{X}_{pq} \in \mathbb{R}^{\frac{s}{q}\times\frac{s}{q}}$,split $B\otimes I_{\frac{s}{h}}$ into q^2 equal-sizes blocks $\hat{B}_{11}, \hat{B}_{12}, \dots, \hat{B}_{qq} \in \mathbb{R}^{\frac{s}{q}\times\frac{t}{p}}$ and split C into pq equal-sizes blocks $\hat{C}_{11}, \hat{C}_{12}, \dots, \hat{C}_{pq} \in \mathbb{R}^{\frac{s}{q}\times\frac{t}{p}}$. Now we can write Eq. (4.0.1) as

$$\begin{aligned} & A \ltimes X + X \ltimes B \\ &= \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1p} \\ \hat{A}_{21} & \hat{A}_{22} & \cdots & \hat{A}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{p1} & \hat{A}_{p2} & \cdots & \hat{A}_{pp} \end{bmatrix} \begin{bmatrix} \hat{X}_1 & \hat{X}_2 & \dots & \hat{X}_q \end{bmatrix} + \begin{bmatrix} \tilde{X}^1 \\ \tilde{X}^2 \\ \vdots \\ \tilde{X}^p \end{bmatrix} \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} & \cdots & \hat{B}_{1q} \\ \hat{B}_{21} & \hat{B}_{22} & \cdots & \hat{B}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{B}_{q1} & \hat{B}_{q2} & \cdots & \hat{B}_{qq} \end{bmatrix} \\ &= \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} & \cdots & \hat{C}_{1q} \\ \hat{C}_{21} & \hat{C}_{22} & \cdots & \hat{C}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{C}_{p1} & \hat{C}_{p2} & \cdots & \hat{C}_{pq} \end{bmatrix} \end{aligned} \tag{4.2.1}$$

where $\hat{A}_{ij}, \hat{B}_{ij}$ and $\hat{C}_{ij}$ are $\in \mathbb{R}^{\frac{s}{q}\times\frac{t}{p}}$, $\hat{X}_j$ be the j^{th} column block of $X\otimes I_{\frac{t}{p}}$, $\tilde{X}^i$ be the i^{th} row block of $X\otimes I_{\frac{s}{q}}$. By considering each block of the equation, we obtain

$$\hat{C}_{ij} = \hat{A}_{i1}\hat{X}_{1j} + \ldots + \hat{A}_{ip}\hat{X}_{pj} + \hat{B}_{1j}\tilde{X}_{i1} + \ldots + \hat{B}_{qj}\tilde{X}_{iq} \tag{4.2.2}$$

Consider $\hat{A}_{ij}\hat{X}_{ij} = x_{ij}\hat{A}_{ij} \times I_{\frac{t}{p}} = x_{ij}\hat{A}_{ij}$ and $\tilde{X}_{ij}\hat{B}_{ij} = x_{ij}I_{\frac{s}{q}} \times \hat{B}_{ij} = x_{ij}\hat{B}_{ij}$. We can write Eq. (4.2.2) as

$$\hat{C}_{ij} = x_{1j}\hat{A}_{i1} + \ldots + x_{pj}\hat{A}_{ip} + x_{i1}\hat{B}_{1j} + \ldots + x_{iq}\hat{B}_{qj} \tag{4.2.3}$$

Thus, we can derive the following theorem.

Theorem 4.12. Assume that Eq. (4.0.1) is well defined.

(i) Then Eq. (4.0.1) has a solution if and only if, for each $i = 1, \dots, p$ and $j = 1, \dots, q$, the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \dots, \hat{A}_{ip}, \hat{B}_{1j}, \hat{B}_{2j}, \dots, \hat{B}_{qj}, \hat{C}_{ij}\}$ is linearly dependent in $\mathbb{R}^{\frac{s}{q}\times\frac{t}{p}}$.

(ii) If Eq. (4.0.1) has a solution, then

$$\text{rank}[\hat{A}^i\ \tilde{B}_j\ \hat{C}_{ij}] = \text{rank}[\hat{A}^i\ \tilde{B}_j] \qquad \textit{for all } i = 1, \dots, p,\ j = 1, \dots, q \tag{4.2.4}$$

where $\hat{A}^i = [\hat{A}_{i1}\ \hat{A}_{i2} \dots\ \hat{A}_p]$ and $\tilde{B}_j = [\hat{B}_{1j}\ \hat{B}_{2j} \dots\ \hat{B}_{qj}]$.

(iii) Moreover, if the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \dots, \hat{A}_{ip}, \hat{B}_{1j}, \hat{B}_{2j}, \dots, \hat{B}_{qj}\}$ is linearly independent for all $i = 1, \dots, p$ and $j = 1, \dots, q$, then the solution X is unique.

Proof. We can write $\hat{C_{ij}}$ in terms of a linear combination of $\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}$, and $\hat{B}_{1j}, \hat{B}_{2j}, \ldots, \hat{B}_{qj}$, with coefficients $x_{1j}, x_{2j}, \ldots, x_{pj}, x_{i1}, x_{i2}, \ldots, x_{iq}$. Thus the solvability of Eq. (4.0.1) is equivalent to the linear dependence of the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, \hat{B}_{1j}, \hat{B}_{2j}, \ldots, \hat{B}_{qj}, \hat{C}_{ij}\}$ for all i, j. Now, the assertion (ii) concerning the ranks of augment matrices holds. The assertion (iii) follows from Eq. (4.2.3). ☐

Example 4.13. Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 3 & 2 & 4 & 5 & 4 \\ 5 & 8 & 4 & 4 & 7 & 5 \\ 4 & 2 & 1 & 3 & 4 & 1 \\ 3 & 6 & 2 & 2 & 5 & 3 \end{bmatrix}.$$

The equation has a solution $X = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

We split $A \otimes I_2$ into 4 equal-size blocks as

$$\left[\begin{array}{c:c} \hat{A}_{11} & \hat{A}_{12} \\ \hdashline \hat{A}_{21} & \hat{A}_{22} \end{array}\right] = \left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ \hdashline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right],$$

split B into 4 equal-size blocks as

$$\left[\begin{array}{c:c} \hat{B}_{11} & \hat{B}_{12} \\ \hdashline \hat{B}_{21} & \hat{B}_{22} \end{array}\right] = \left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 & 1 & 1 \\ \hdashline 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \end{array}\right],$$

and split C into 4 equal-size blocks as

$$\left[\begin{array}{c:c} \hat{C}_{11} & \hat{C}_{12} \\ \hdashline \hat{C}_{21} & \hat{C}_{22} \end{array}\right] = \left[\begin{array}{ccc:ccc} 5 & 3 & 2 & 4 & 5 & 4 \\ 5 & 8 & 4 & 4 & 7 & 5 \\ \hdashline 4 & 2 & 1 & 3 & 4 & 1 \\ 3 & 6 & 2 & 2 & 5 & 3 \end{array}\right].$$

Then

$$\text{rank}[\hat{A}^1 \ \tilde{B}_1 \ \hat{C}_{11}] = \text{rank}[\hat{A}^1 \ \tilde{B}_1] = 2 \text{ and } \text{rank}[\hat{A}^1 \ \tilde{B}_2 \ \hat{C}_{12}] = \text{rank}[\hat{A}^1 \ \tilde{B}_2] = 2,$$

$$\text{rank}[\hat{A}^2 \ \tilde{B}_1 \ \hat{C}_{21}] = \text{rank}[\hat{A}^2 \ \tilde{B}_1] = 2 \text{ and } \text{rank}[\hat{A}^2 \ \tilde{B}_2 \ \hat{C}_{22}] = \text{rank}[\hat{A}^2 \ \tilde{B}_2] = 2.$$

Example 4.14. Let A, B be the same as Example 4.12 and

$$C = \begin{bmatrix} 5 & 3 & 2 & 4 & 5 & 4 \\ 5 & 8 & 4 & 4 & 7 & 5 \\ 4 & 2 & 4 & 3 & 4 & 1 \\ 3 & 6 & 2 & 2 & 5 & 3 \end{bmatrix}.$$

We split C into 4 equal-size blocks as

$$\left[\begin{array}{c:c} \hat{C}_{11} & \hat{C}_{12} \\ \hdashline \hat{C}_{21} & \hat{C}_{22} \end{array}\right] = \left[\begin{array}{ccc:ccc} 5 & 3 & 2 & 4 & 5 & 4 \\ 5 & 8 & 4 & 4 & 7 & 5 \\ \hdashline 4 & 2 & 4 & 3 & 4 & 1 \\ 3 & 6 & 2 & 2 & 5 & 3 \end{array}\right].$$

Then

$$\text{rank}[\hat{A}^1\ \tilde{B}_1\ \hat{C}_{11}] = \text{rank}[\hat{A}^1\ \tilde{B}_1] = 2 \text{ and } \text{rank}[\hat{A}^1\ \tilde{B}_2\ \hat{C}_{12}] = \text{rank}[\hat{A}^1\ \tilde{B}_2] = 2.$$

$$\text{rank}[\hat{A}^2\ \tilde{B}_1\ \hat{C}_{21}] = \text{rank}[\hat{A}^2\ \tilde{B}_1] = 2 \text{ and } \text{rank}[\hat{A}^2\ \tilde{B}_2\ \hat{C}_{22}] = \text{rank}[\hat{A}^2\ \tilde{B}_2] = 2.$$

However, the equation has no solution.

Next, we apply vector operator to Eq. (4.2.3) and obtain the following equation

$$V_c(\hat{C}_{ij}) = x_{1j}V_c(\hat{A}_{i1}) + \ldots + x_{pj}V_c(\hat{A}_{ip}) + x_{i1}V_c(\hat{B}_{1j}) + \ldots + x_{iq}V_c(\hat{B}_{qj})$$

Consequently, we write the matrix equation as

$$\begin{bmatrix} V_c(\hat{A}_{11}) & V_c(\hat{A}_{12}) & \cdots & V_c(\hat{A}_{1p}) \\ V_c(\hat{A}_{21}) & V_c(\hat{A}_{22}) & \cdots & V_c(\hat{A}_{2p}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{A}_{p1}) & V_c(\hat{A}_{p2}) & \cdots & V_c(\hat{A}_{pp}) \end{bmatrix} X + X \begin{bmatrix} V_c(\hat{B}_{11}) & V_c(\hat{B}_{12}) & \cdots & V_c(\hat{B}_{1q}) \\ V_c(\hat{B}_{21}) & V_c(\hat{B}_{22}) & \cdots & V_c(\hat{B}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{B}_{q1}) & V_c(\hat{B}_{q2}) & \cdots & V_c(\hat{B}_{qq}) \end{bmatrix}$$

$$= \begin{bmatrix} V_c(\hat{C}_{11}) & V_c(\hat{C}_{12}) & \cdots & V_c(\hat{C}_{1q}) \\ V_c(\hat{C}_{21}) & V_c(\hat{C}_{22}) & \cdots & V_c(\hat{C}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{C}_{p1}) & V_c(\hat{C}_{p2}) & \cdots & V_c(\hat{C}_{pq}) \end{bmatrix}.$$

Theorem 4.15. Eq. (4.0.1) is equivalent to the following linear system:

$$(I_q \otimes \breve{A} + \breve{B}^T \otimes I_p)V_c(X) = V_c(\breve{C}) \qquad (4.2.5)$$

where

$$\breve{A} = \begin{bmatrix} V_c(\hat{A}_{11}) & V_c(\hat{A}_{12}) & \cdots & V_c(\hat{A}_{1p}) \\ V_c(\hat{A}_{21}) & V_c(\hat{A}_{22}) & \cdots & V_c(\hat{A}_{2p}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{A}_{p1}) & V_c(\hat{A}_{p2}) & \cdots & V_c(\hat{A}_{pp}) \end{bmatrix}, \quad \breve{B} = \begin{bmatrix} V_c(\hat{B}_{11}) & V_c(\hat{B}_{12}) & \cdots & V_c(\hat{B}_{1q}) \\ V_c(\hat{B}_{21}) & V_c(\hat{B}_{22}) & \cdots & V_c(\hat{B}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{B}_{q1}) & V_c(\hat{B}_{q2}) & \cdots & V_c(\hat{B}_{qq}) \end{bmatrix},$$

$$\breve{C} = \begin{bmatrix} V_c(\hat{C}_{11}) & V_c(\hat{C}_{12}) & \cdots & V_c(\hat{C}_{1q}) \\ V_c(\hat{C}_{21}) & V_c(\hat{C}_{22}) & \cdots & V_c(\hat{C}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{C}_{p1}) & V_c(\hat{C}_{p2}) & \cdots & V_c(\hat{C}_{pq}) \end{bmatrix}.$$

From Theorem (4.15) we can solve Eq. (4.0.1) by solving the linear system (4.2.5) instead.

Example 4.16. Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 3 & 2 & 4 & 5 & 4 \\ 5 & 8 & 4 & 4 & 7 & 5 \\ 4 & 2 & 1 & 3 & 4 & 1 \\ 3 & 6 & 2 & 2 & 5 & 3 \end{bmatrix}.$$

By Lemma 4.10 and remark 4.11 , we take $\alpha = 6$, then $p = \frac{nu}{\alpha} = \frac{3\cdot 4}{6} = 2$ and $q = \frac{mv}{\alpha} = \frac{2\cdot 6}{2} = 2$, $t = \text{lcm}\,\{3, 2\} = 6$, $s = \text{lcm}\,\{2, 4\} = 4$.

Write

$$X = \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}.$$

Compute

$$A \otimes I_{\frac{6}{3}} = A \otimes I_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B \otimes I_{\frac{4}{4}} = B \otimes I_1 = B.$$

We split $A \otimes I_2$ into 4 equal-size blocks as

$$\left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hline \hat{A}_{21} & \hat{A}_{22} \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right],$$

split B into 4 equal-size blocks as

$$\left[\begin{array}{c|c} \hat{B}_{11} & \hat{B}_{12} \\ \hline \hat{B}_{21} & \hat{B}_{22} \end{array}\right] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \end{array}\right],$$

and split C into 4 equal-size blocks as

$$\left[\begin{array}{c|c} \hat{C}_{11} & \hat{C}_{12} \\ \hline \hat{C}_{21} & \hat{C}_{22} \end{array}\right] = \left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 4 & 5 & 4 \\ 5 & 8 & 4 & 4 & 7 & 5 \\ \hline 4 & 2 & 1 & 3 & 4 & 1 \\ 3 & 6 & 2 & 2 & 5 & 3 \end{array}\right].$$

Then we have

$$\breve{A} = \left[\begin{array}{c:c} V_c(\hat{A}_{11}) & V_c(\hat{A}_{12}) \\ \hdashline V_c(\hat{A}_{21}) & V_c(\hat{A}_{22}) \end{array}\right] = \left[\begin{array}{c:c} 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 1 & 0 \\ 1 & 0 \\ 0 & 2 \\ \hdashline 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right],$$

$$\breve{B} = \left[\begin{array}{c:c} V_c(\hat{B}_{11}) & V_c(\hat{B}_{12}) \\ \hdashline V_c(\hat{B}_{21}) & V_c(\hat{B}_{22}) \end{array}\right] = \left[\begin{array}{c:c} 2 & 0 \\ 2 & 1 \\ 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \\ \hdashline 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 2 & 2 \\ 0 & 1 \\ 1 & 1 \end{array}\right],$$

$$\breve{C} = \left[\begin{array}{c:c} V_c(\hat{C}_{11}) & V_c(\hat{C}_{12)} \\ \hdashline V_c(\hat{C}_{21}) & V_c(\hat{C}_{22}) \end{array}\right] = \left[\begin{array}{c:c} 5 & 4 \\ 5 & 4 \\ 3 & 5 \\ 8 & 7 \\ 2 & 4 \\ 4 & 5 \\ \hdashline 4 & 3 \\ 3 & 2 \\ 2 & 4 \\ 6 & 5 \\ 1 & 1 \\ 2 & 3 \end{array}\right].$$

By Theorem 4.15, we need to solve the following linear system

$$(I_2 \otimes \breve{A} + \breve{B}^T \otimes I_2)V_c(X) = V_c(\breve{C}).$$

Then we have

$$\left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&0&0\\1&0&0&0\\1&0&0&0\\0&2&0&0\\1&0&0&0\\0&0&0&0\\0&1&0&0\\1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&2\\0&0&1&0\\0&0&1&0\\0&0&0&2\\0&0&1&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix} + \begin{bmatrix}2&0&1&0\\2&0&1&0\\1&0&0&0\\3&0&2&0\\1&0&0&0\\0&0&1&0\\0&2&0&1\\0&2&0&1\\0&1&0&0\\0&3&0&2\\0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&1&0\\3&0&0&0\\1&0&2&0\\0&0&1&0\\1&0&1&0\\0&0&0&1\\0&1&0&1\\0&3&0&0\\0&1&0&2\\0&0&0&1\\0&1&0&1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}5\\5\\3\\8\\2\\4\\4\\3\\2\\6\\1\\2\\4\\4\\5\\7\\4\\5\\3\\2\\4\\5\\1\\3\end{bmatrix}.$$

That is

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 0 & 0 & 2 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 3 \\ 8 \\ 2 \\ 4 \\ 4 \\ 3 \\ 2 \\ 6 \\ 1 \\ 2 \\ 4 \\ 4 \\ 5 \\ 7 \\ 4 \\ 5 \\ 3 \\ 2 \\ 4 \\ 5 \\ 1 \\ 3 \end{bmatrix}.$$

Thus, the solution of the equation is $X = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

Example 4.17. Let

$$A = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 7 & 12 \\ 6 & 9 \\ 4 & 6 \\ 4 & 5 \end{bmatrix}.$$

By Lemma 4.10 and Remark 4.11 , we take $\alpha = 2$, then $p = \frac{nu}{\alpha} = \frac{4}{2} = 2$ and $q = \frac{mv}{\alpha} = \frac{4}{2} = 2$, $t = \text{lcm}\{1, 2\} = 2$, $s = \text{lcm}\{2, 4\} = 4$.

Write

$$X = \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}.$$

Compute

$$A \otimes I_{\frac{2}{1}} = A \otimes I_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B \otimes I_{\frac{4}{4}} = B \otimes I_1 = B.$$

We split $A \otimes I_2$ into 4 equal-size blocks as

$$\left[\begin{array}{c:c} \hat{A}_{11} & \hat{A}_{12} \\ \hdashline \hat{A}_{21} & \hat{A}_{22} \end{array}\right] = \left[\begin{array}{c:c} 2 & 0 \\ 0 & 2 \\ \hdashline 1 & 0 \\ 0 & 1 \end{array}\right],$$

split B into 4 equal-size blocks as

$$\left[\begin{array}{c:c} \hat{B}_{11} & \hat{B}_{12} \\ \hdashline \hat{B}_{21} & \hat{B}_{22} \end{array}\right] = \left[\begin{array}{c:c} 1 & 2 \\ 2 & 1 \\ \hdashline 2 & 2 \\ 1 & 3 \end{array}\right],$$

and split C into 4 equal-size blocks as

$$\left[\begin{array}{c:c} \hat{C}_{11} & \hat{C}_{12} \\ \hdashline \hat{C}_{21} & \hat{C}_{22} \end{array}\right] = \left[\begin{array}{c:c} 7 & 12 \\ 6 & 9 \\ \hdashline 4 & 6 \\ 4 & 5 \end{array}\right].$$

Then we have

$$\breve{A} = \left[\begin{array}{c:c} V_c(\hat{A}_{11}) & V_c(\hat{A}_{12}) \\ \hdashline V_c(\hat{A}_{21}) & V_c(\hat{A}_{22}) \end{array}\right] = \left[\begin{array}{c:c} 2 & 0 \\ 0 & 2 \\ \hdashline 1 & 0 \\ 0 & 1 \end{array}\right],$$

$$\breve{B} = \left[\begin{array}{c:c} V_c(\hat{B}_{11}) & V_c(\hat{B}_{12}) \\ \hdashline V_c(\hat{B}_{21}) & V_c(\hat{B}_{22}) \end{array}\right] = \left[\begin{array}{c:c} 1 & 2 \\ 2 & 1 \\ \hdashline 2 & 2 \\ 1 & 3 \end{array}\right],$$

$$\breve{C} = \left[\begin{array}{c:c} V_c(\hat{C}_{11}) & V_c(\hat{C}_{12}) \\ \hdashline V_c(\hat{C}_{21}) & V_c(\hat{C}_{22}) \end{array}\right] = \left[\begin{array}{c:c} 7 & 12 \\ 6 & 9 \\ \hdashline 4 & 6 \\ 4 & 5 \end{array}\right].$$

By Theorem 4.15, we need to solve the following linear system

$$(I_2 \otimes \breve{A} + \breve{B}^T \otimes I_2) V_c(X) = V_c(\breve{C}).$$

Then we have

$$\left(\begin{bmatrix}2&0&0&0\\0&2&0&0\\1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&2\\0&0&1&0\\0&0&0&1\end{bmatrix}+\begin{bmatrix}1&0&2&0\\2&0&1&0\\0&1&0&2\\0&2&0&1\\2&0&2&0\\1&0&3&0\\0&2&0&2\\0&1&0&3\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}7\\6\\4\\4\\12\\9\\6\\5\end{bmatrix}.$$

That is

$$\begin{bmatrix}3&0&2&0\\2&2&1&0\\1&1&0&2\\0&3&0&1\\2&0&4&0\\1&0&3&2\\0&2&1&2\\0&1&0&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}7\\6\\4\\4\\12\\9\\6\\5\end{bmatrix}.$$

Thus, the solution of the equation is $X=\begin{bmatrix}1&2\\1&1\end{bmatrix}$.

Chapter 5

Conclusions and suggestions

5.1 Conclusions

We consider the Sylvester matrix equation (4.0.1) with respect to the general left semi-tensor product where the coefficients $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{h\times k}, C \in \mathbb{R}^{u\times v}$ are real matrices of arbitrary dimensions. The study includes the equation $A \ltimes X = C$ (see [18]) , the equation $X \ltimes B = C$, and the Lyapunov equation. Our results also contain the Sylvester matrix equation with respect to the left semi-tensor product. We obtain the conforming dimensions of A, B, C, and the unknown X. We investigate necessary/sufficient condition(s) for the Sylvester matrix equation to be solvable and uniquely solvable, concerning ranks and linear independence. For matrices with arbitary dimensions, we show that the Sylvester equation can be transformed to a linear system with respect to the usual matrix product by means of the vector operator.

5.2 Suggestions

The method that we use to partition matrices in the general case of this thesis turns out to have large numbers of linear equations. The further research may work on finding a new partition's method to reduce the numbers of linear equations.

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Appendix

Appendix A

The research paper

Solving the Sylvester matrix equation with respect to the general left semi-tensor product

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Abstract

We consider the Sylvester matrix equation $A \ltimes X + X \ltimes B = C$ with respect to the general left semi-tensor product where A, B, C are given real matrices of arbitrary dimensions and X is unknown. We investigate necessary/sufficient condition(s) for the Sylvester matrix equation to be solvable and uniquely solvable, concerning ranks and linear independence. Moreover, we show that the Sylvester equation can be transformed to a linear system with respect to the usual matrix product by means of the vector operator.

Mathematics Subject Classification: 15A06, 15A24, 15A69,15B05
Keywords: Sylvester matrix equation, general left semi-tensor product, vector operator, linear system

1 Introduction

Linear matrix equations arise naturally in several branches of pure and applied mathematics such as differential equations, control theory, neutral network, and vibration theory; see e.g. [11, 15, 16]. The Sylvester matrix equation

$$AX + XB = C \tag{1.1}$$

is a famous linear matrix equation that is significantly used in stability analysis and optimal control. A usual algebraic method to solve linear matrix equations is to transform them into a

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linear system by means of certain kind of vectorization, namely, the column/row vector operator; see e.g. [12].

In the usual matrix multiplication, the product of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ can be defined if $n = p$, that is, the matrix pair (A, B) satisfies the matching-dimension condition. For the matrices that do not satisfy matching dimension condition, Cheng [3] has introduced the semi-tensor product as a tool for multiplication between those matrices. The left-semi tensor product of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ can be defined under the factor-dimension condition namely, $n \mid p$ or $p \mid n$. In the case $n = p$, the product of $A \ltimes B$ is equal to the usual matrix product AB. The properties of left-semi tensor product contain the associative law, the left/ right distributive over the matrix addition, certain identity-like properties, and the compatibility with the scalar multiplication, the transposition, the inversion and so on (see [1,2,5–7,10]). The left semi-tensor product can be applied in several areas such as classical logic [9], fuzzy logic [9], Boolean networks [3,8], dynamic systems [10] and Morgan's problem [4]. The general left semi-tensor product defines product of matrices with arbitrary dimensions (see [3]). It defines product of A and B as

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}) \tag{1.2}$$

where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. The general left semi-tensor product turns out to possess rich algebraic properties like the usual matrix product. The general left semi-tensor product was proposed for solving decoupling problems of a linear system and the problems with multidimension data. In 2015, Yao et al. [17] introduced a convenient way to solve the matrix equation

$$A \ltimes X = C \tag{1.3}$$

where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{h\times k}$. In fact, Eq. (1.3) has been importantly used in Boolean network. If the equation has a solution, then the dimension of X satisfies the solvability's conditions and B must be a block Toeplitz matrix. The number of solutions depends on the greatest common divisor of n and k. If gcd $\{n, k\} = 1$, then the solution has only one admissible size. Li et al. [14] have been motivated by the work of Yao and considered that solving only Eq.(1.3) is not sufficient. Therefore, they investigated the solvability of the system of matrix equations

$$\begin{cases} A \ltimes X = B \\ X \ltimes C = D \end{cases} \tag{1.4}$$

with respect to the general left semi-tensor product. In fact, if the solution of (1.4) exists, then B and D must be partitioned matrices so that each partition is a Toplitz matrix. The matrix equation

$$A \ltimes X \ltimes B \quad = \quad C \tag{1.5}$$

where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{a\times b}$ and $C \in \mathbb{R}^{h\times k}$ under the general left semi-tensor product, has been extensively used in non-linear programming, power science, parameter identification, etc. Ji [13] investigated the solvability of Eq. (1.5). Indeed, if Eq. (1.5) has solution(s), then X satisfies certain dimension conditions and C must be the block Toplitz matrix.

The previous discussion motivated us to study the Sylvester matrix equation

$$A \ltimes X + X \ltimes B = C \tag{1.6}$$

where $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{h\times k}, C \in \mathbb{R}^{u\times v}$ are given, and $X \in \mathbb{R}^{p\times q}$ is to be solved. The study includes Eqs. (1.3), (1.5) and the Lyapunove equation:

$$A \ltimes X + X \ltimes A^T = C$$

as special cases. We investigate the solvability and the unique solvability for Eq. (1.6) according to the matrix dimensions, ranks and linear independence. Moreover, we show that Eq. (1.6) can be transformed to a linear system, so that we can solve it via an ordinary method.

2 Preliminaries

For any positive integers m, n, we denote the set of $m \times n$ real matrices by $\mathbb{R}^{m\times n}$. When $n = 1$, we set $\mathbb{R}^m := \mathbb{R}^{m\times 1}$.

Recall that the Kronecker product (see e.g. [12]) of $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{h\times k}$, denoted by $A \otimes B$, is defined as the block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mh\times nk}.$$

Recall that the general left semi-tensor product (see e.g. [3]) of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{h\times k}$, denoted by $A \ltimes B$, is defined as

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{h}}) \in \mathbb{R}^{\frac{m\alpha}{n}\times\frac{k\alpha}{h}}$$

where $\alpha = \text{lcm}\{n, h\}$. If $n \mid h$ or $h \mid n$, then $A \ltimes B$ becomes the left semi-tensor product (see e.g. [3]) In the case $n = h$, the product $A \ltimes B$ reduces to the usual product AB.

The (column) vector operator (see e.g. [12]) vec$(\cdot)$ is a linear operator that turns any matrix $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ into the vector

$$V_c(A) = \begin{bmatrix} a_{11} & \dots & a_{m1} & a_{12} & \dots & a_{m2} & \dots & a_{1n} & \dots & a_{mn} \end{bmatrix}^T.$$

Lemma 2.1. *Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{h \times k}$ and $X \in \mathbb{R}^{p \times q}$. Then*

$$V_c(AXB) = (B^T \otimes A)V_c(X).$$

3 Solving the Sylvester equation when X is a vector

In this section, we study on solving the Sylvester equation

$$A \ltimes X + X \ltimes B = C \tag{3.1}$$

where $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{h \times k}, C \in \mathbb{R}^{u \times v}$ are given, $X \in \mathbb{R}^{p}$ is an unknown matrix. We first start from the case $m = u$, and then discuss the general case.

3.1 The case $m = u$

In this subsection, we study on relation between dimensions of matrix A, B, C, conditions for solving Eq. (3.1) and condition for an existence of a solution. We say that Eq. (3.1) is well defined if

(i) the matrices $A \ltimes X$ and$X \ltimes B$ are well defined,

(ii) the dimensions of $A \ltimes X$, $X \ltimes B$ and C are the same.

Lemma 3.1. *Eq.* (3.1) *is well-defined if and only if*

(i) $k \mid n$ *and* $h \mid m$,

(ii) $v = k$, *and*

(iii) $p = \frac{n}{k} = \frac{m}{h}$.

Proof. The sufficient condition is quite obvious. For the necessity, suppose that Eq. (3.1) is well defined. Let us denote $t = \operatorname{lcm}\{n, p\}$. By definition of the general left semi-tensor product, we have that

$$A \ltimes X = (A \otimes I_{\frac{t}{n}})(X \otimes I_{\frac{t}{p}}) \in \mathbb{R}^{\frac{mt}{n} \times \frac{t}{p}} \quad \text{and} \quad X \ltimes B = (X \otimes I_h)B \in \mathbb{R}^{ph \times k}.$$

Assume that $C \in \mathbb{R}^{u \times v}$ and $m = u$. Then we have $u = m = ph$, $t = n$ and $v = \frac{t}{p} = k$. Furthermore, $p = \frac{m}{h} = \frac{n}{k}$, $\frac{m}{h}$ and $\frac{n}{k}$ must be a positive integer. Thus, if a solution of equation (3.1) exists, the solution must belong to $\mathbb{R}^p$ where $p = \frac{n}{k} = \frac{m}{h}$ and $\frac{n}{k}$ is required to be a positive integer. ☐

We obtain from the previous proof that $n = pk$. That is n is divisible by p. Then we write $X = [x_1\ x_2\ \ldots\ x_p]^T \in \mathbb{R}^p$. We split A into p^2 equal-size blocks $\hat{A}_{11}, \hat{A}_{12}, \ldots, \hat{A}_{pp}$ and split C into p equal-size blocks $C_1, C_2, \ldots, C_p$. Now we can write Eq. (3.1) as

$$A \ltimes X + X \ltimes B = \left[\begin{array}{c:c:c:c} \hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1p} \\ \hdashline \hat{A}_{21} & \hat{A}_{22} & \cdots & \hat{A}_{2p} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \hat{A}_{p1} & \hat{A}_{p2} & \cdots & \hat{A}_{pp} \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} + \begin{bmatrix} x_1 B \\ x_2 B \\ \vdots \\ x_p B \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{bmatrix}$$

where $\hat{A}_{ij}$ and $C_i \in \mathbb{R}^{h \times k}$. By considering each $i - th$ row block of the equation, we have

$$x_1 \hat{A}_{i1} + x_2 \hat{A}_{i2} + \ldots + x_p \hat{A}_{ip} + x_i B = C_i. \tag{3.2}$$

Hence, we can deduce the following theorem.

Theorem 3.2. *Assume that Eq.* (3.1) *is well defined.*

(i) Then Eq. (3.1) *has a solution if and only if, for each* $i = 1, ..., p$, *the set* $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, B, C_i\}$ *is linearly dependent in the vector space* $\mathbb{R}^{h \times k}$.

(ii) If Eq. (3.1) *has a solution, then*

$$\operatorname{rank}[\hat{A}_i\ B\ C] = \operatorname{rank}[\hat{A}_i\ B] \qquad \textit{for all } i = 1, \ldots, p \tag{3.3}$$

where $\hat{A}_i = [\hat{A}_{i1}\ \hat{A}_{i2} \ldots\ \hat{A}_{ip}]$, *(iii) Moreover, if the set* $\{\hat{A}_{i1}, \hat{A}_{i2}, \dots, \hat{A}_{ip}, B\}$ *is linearly independent for all* $i = 1, ..., p,$ *then the solution* X *is unique.*

Proof. Note that the solvability of Eq. (3.1) means that we can write C_i in term of linear combination of $\hat{A}_{i1}, \hat{A}_{i2}, \dots, \hat{A}_{ip}$, and B, with coefficients $x_1, x_2, \dots, x_p, x_i$. Thus the solvability of Eq.(3.1) is equivalent to the linearly dependence of the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \dots, \hat{A}_{ip}, B, C_i\}$ for all i. Now, the assertion (ii) concerning the ranks of augment matrices holds, The assetion (iii) follows from Eq.(3.2). ☐

Next, we apply the vector operator to Eq. (3.1) and obtain

$$x_1 V_c(\hat{A}_{i1}) + x_2 V_c(\hat{A}_{i2}) + \ldots + x_p V_c(\hat{A}_{ip}) + x_i V_c(B) = V_c(C_i) \in \mathbb{R}^{hk}.$$

From this equation, we conclude the following:

Theorem 3.3. *The matrix-vector equation* $A \ltimes X + X \ltimes B = C$ *with respect to the general left semi-tensor product is equivalent to the following matrix-vector equation*

$$\tilde{A}X = \tilde{C},$$

where

$$\tilde{A} = \begin{bmatrix} V_c(A_{11}+B) & V_c(A_{12}) & \dots & V_c(A_{1p}) \\ V_c(A_{21}) & V_c(A_{22}+B) & \dots & V_c(A_{2p}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(A_{p1}) & V_c(A_{p2}) & \dots & V_c(A_{pp}+B) \end{bmatrix} \text{ and } \tilde{C} = \begin{bmatrix} V_c(C_1) \\ V_c(C_2) \\ \vdots \\ V_c(C_p) \end{bmatrix}.$$

Corollary 3.4. *The matrix-vector equation* (3.1) *has a solution if and only if the following rank condition holds:*

$$\text{rank}[\tilde{A}] = \text{rank}[\tilde{A}\ \tilde{C}]. \tag{3.4}$$

3.2 The general case $m \neq u$

In this subsection, we discuss the conditions for the solution of Eq. (3.1). Firstly we consider the relation between dimensions of matrices A, B, C. Then we find the condition for dimension of a solution X.

Lemma 3.5. *Eq.* (3.1) *is well defined if and only if* *(i)* $m \mid u$ *and* $k \mid n$,

(ii) $h = \frac{u}{p}$,

(iii) $k = v$,

(iv) $\gcd\{k, \frac{u}{m}\} = 1$, *and*

(v) $p = \frac{nu}{mk}$.

Proof. Suppose that Eq. (3.1) is well defined. Let us denote $t = \operatorname{lcm}\{n, p\}$. By definition of the general left semi-tensor product, we have that

$$A \ltimes X = (A \otimes I_{\frac{t}{n}})(X \otimes I_{\frac{t}{p}}) \in \mathbb{R}^{\frac{mt}{n} \times \frac{t}{p}} \quad \text{and} \quad X \ltimes B = (X \otimes I_h)B \in \mathbb{R}^{ph \times k}.$$

Assume that $C \in \mathbb{R}^{u \times v}$. Then we have

$$u = \frac{mt}{n} = ph \quad \text{and} \quad v = \frac{t}{p} = k.$$

Furthermore, $\frac{u}{m} = \frac{t}{n}$ and $t = \frac{un}{m}$. Substitute $t = \frac{un}{m}$ in $\frac{t}{p} = k$. Then we obtain $p = \frac{nu}{mk}$. Since $\frac{un}{m} = t = \operatorname{lcm}\{n, p\}$, we have

$$\frac{un}{m} = \operatorname{lcm}\{n, \frac{nu}{mk}\}.$$

We see that $\frac{u}{m}$ is a factor of $\frac{nu}{mk}$. Thus $\frac{n}{k}$ must be a positive integer. Moreover,

$$\frac{un}{m} = \frac{n}{k} \cdot \operatorname{lcm}\{k, \frac{u}{m}\}.$$

Consequently, $\gcd\{k, \frac{u}{m}\} = 1$. ☐

In the general case, it is not easy to find suitable partitions of the matrices $A \ltimes X$, $X \ltimes B$, and C in oder to get a linear system. However, for a particular case $B = 0$, we can partition $A \ltimes X$ and C to get a linear system. Indeed, if the equation $A \ltimes X = C$ has a solution, then C must be a block Toplitz matrix; see [17].

4 Solving the Sylvester matrix equation when X is a matrix

In this section, we study on solving the Sylvester equation

$$A \ltimes X + X \ltimes B = C \tag{4.1}$$

where $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{h\times k}, C \in \mathbb{R}^{u\times v}$ are given, $X \in \mathbb{R}^{p\times q}$ is an unknown matrix. We first start from the case $m = u$ and $q \mid h$, then discuss the general case.

4.1 The case $m = u$ and $q \mid h$

In this subsection, we consider conditions for Eq. (4.1) to be solvable and investigate the conditions for dimensions of a solution matrix X, A, B and C. We have the following lemma.

Lemma 4.1. *Eq.* (4.1) *is well defined if and only if the dimensions of* A, B, C*, and* X *satisfy the following conditions*

(i) $u = m = \frac{ph}{q}$,

(ii) $v = \frac{qn}{p} = k$.

Proof. Assume that $m = u$ and $q \mid h$. Denote $t = \operatorname{lcm}\{n, p\}$. By definition of the general left semi-tensor product, we have that

$$A \ltimes X = (A \otimes I_{\frac{t}{n}})(X \otimes I_{\frac{t}{p}}) \in \mathbb{R}^{\frac{mt}{n}\times\frac{qt}{p}} \quad \text{and} \quad X \ltimes B = (X \otimes I_{\frac{h}{q}})B \in \mathbb{R}^{\frac{ph}{q}\times k}.$$

Assume that $C \in \mathbb{R}^{u\times v}$. Then we have $u = \frac{mt}{n} = \frac{ph}{q}$, $t = n$ and $v = \frac{qt}{p} = k$. Since $u = m$, we have $t = n$. □

Remark 4.2. If Eq. (4.1) has a solution, the dimensions of matrices A, B, and C must satisfy the conditions

(i) $k \mid v$,

(ii) $\frac{m}{n} = \frac{h}{k}$.

(iii) $p = \frac{n}{\alpha}$ and $q = \frac{k}{\alpha}$, where α is a common divisor of n and k.

Since $n = t$, n is divisible by p. Assume that h is divisible by q. Then we split A into p^2 equal-size blocks $\hat{A}_{11}, \hat{A}_{12}, \ldots, \hat{A}_{pp}$, split B into q^2 equal-size blocks $\hat{B}_{11}, \hat{B}_{12}, \ldots, \hat{B}_{qq}$ and split C into pq equal-size blocks $\hat{C}_{11}, \hat{C}_{12}, \ldots, \hat{C}_{pq}$. Now we can write Eq. (4.1) as

$$
\begin{aligned}
& A \ltimes X + X \ltimes B \\
&= \left[\begin{array}{c:c:c:c} \hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1p} \\ \hdashline \hat{A}_{21} & \hat{A}_{22} & \cdots & \hat{A}_{2p} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \hat{A}_{p1} & \hat{A}_{p2} & \cdots & \hat{A}_{pp} \end{array}\right] \begin{bmatrix} X_1 & X_2 & \ldots & X_q \end{bmatrix} + \begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^p \end{bmatrix} \left[\begin{array}{c:c:c:c} \hat{B}_{11} & \hat{B}_{12} & \cdots & \hat{B}_{1q} \\ \hdashline \hat{B}_{21} & \hat{B}_{22} & \cdots & \hat{B}_{2q} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \hat{B}_{q1} & \hat{B}_{q2} & \cdots & \hat{B}_{qq} \end{array}\right] \\
&= \left[\begin{array}{c:c:c:c} \hat{C}_{11} & \hat{C}_{12} & \cdots & \hat{C}_{1q} \\ \hdashline \hat{C}_{21} & \hat{C}_{22} & \cdots & \hat{C}_{2q} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \hat{C}_{p1} & \hat{C}_{p2} & \cdots & \hat{C}_{pq} \end{array}\right]
\end{aligned}
\tag{4.2}
$$

where $\hat{A}_{ij}, \hat{B}_{ij}$ and $\hat{C}_{ij}$ are $\in \mathbb{R}^{\frac{m}{p} \times \frac{n}{p}}$. By considering each block of the equation, we obtain

$$
\hat{C}_{ij} = x_{1j}\hat{A}_{i1} + \ldots + x_{pj}\hat{A}_{ip} + x_{i1}\hat{B}_{1j} + \ldots + x_{iq}\hat{B}_{qj} \tag{4.3}
$$

Let X^i be the i^{th} column of X, X_j be the j^{th} row of X. For each i, j, let us partition

$$
\hat{A}^i = [\hat{A}_{i1} \ldots \hat{A}_{ip}],\, \hat{B}_j = \begin{bmatrix} \hat{B}_{1j} \\ \vdots \\ \hat{B}_{qj} \end{bmatrix}.
$$

We can write Eq. (4.1) as

$$
\hat{C}_{ij} = \hat{A}^i X_j + X^i \hat{B}_j \tag{4.4}
$$

for each $i = 1, \ldots, p, j = 1, \ldots, q$. Thus, we can derive the following theorem.

Theorem 4.3. *Assume that Eq.* (4.1) *is well defined.*

(i) *Then Eq.* (4.1) *has a solution if and only if, for each $i = 1, \ldots, p$ and $j = 1, \ldots, q$, the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, \hat{B}_{1j}, \hat{B}_{2j}, \ldots, \hat{B}_{qj}, \hat{C}_{ij}\}$ is linearly dependent in $\mathbb{R}^{\frac{m}{p} \times \frac{n}{p}}$.*

(ii) *If Eq.* (4.1) *has a solution, then*

$$
\operatorname{rank}[\hat{A}^i\ \tilde{B}_j\ \hat{C}_{ij}] = \operatorname{rank}[\hat{A}^i\ \tilde{B}_j] \qquad for\ all\ i = 1, \ldots, p,\ j = 1, \ldots, q \tag{4.5}
$$

where $\hat{A}^i = [\hat{A}_{i1}\ \hat{A}_{i2} \ldots\ \hat{A}_{p}]$ and $\tilde{B}_j = [\hat{B}_{1j}\ \hat{B}_{2j} \ldots\ \hat{B}_{qj}]$.

(iii) Moreover, if the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, \hat{B}_{1j}, \hat{B}_{2j}, \ldots, \hat{B}_{qj}\}$ *is linearly independent for all* $i = 1, \ldots, p$ *and* $j = 1, \ldots, q$*, then the solution* X *is unique.*

Proof. We can write $\hat{C}_{ij}$ in terms of a linear combination of $\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}$, and $\hat{B}_{1j}, \hat{B}_{2j}, \ldots, \hat{B}_{qj}$, with coefficients $x_{i1}, x_{i2}, \ldots, x_{ip}, x_{1j}, x_{2j}, \ldots, x_{qj}$. Thus the solvability of Eq. (4.1) is equivalent to the linear dependence of the set $\{\hat{A}_{i1}, \hat{A}_{i2}, \ldots, \hat{A}_{ip}, \hat{B}_{1j}, \hat{B}_{2j}, \ldots, \hat{B}_{qj}, \hat{C}_{ij}\}$ for all i, j. Now, the assertion (ii) concerning the ranks of augment matrices holds. The assertion (iii) follows from Eq. (4.1). ☐

Next, we apply vector operator to Eq. (4.3) and obtain the following equation

$$V_c(\hat{C}_{ij}) = x_{1j}V_c(\hat{A}_{i1}) + \ldots + x_{pj}V_c(\hat{A}_{ip}) + x_{i1}\hat{B}_{1j}) + \ldots + x_{iq}V_c(\hat{B}_{qj})$$

Consequently, we write the matrix equation as

$$\begin{bmatrix} V_c(\hat{A}_{11}) & V_c(\hat{A}_{12}) & \cdots & V_c(\hat{A}_{1p}) \\ V_c(\hat{A}_{21}) & V_c(\hat{A}_{22}) & \cdots & V_c(\hat{A}_{2p}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{A}_{p1}) & V_c(\hat{A}_{p2}) & \cdots & V_c(\hat{A}_{pp}) \end{bmatrix} X + X \begin{bmatrix} V_c(\hat{B}_{11}) & V_c(\hat{B}_{12}) & \cdots & V_c(\hat{B}_{1q}) \\ V_c(\hat{B}_{21}) & V_c(\hat{B}_{22}) & \cdots & V_c(\hat{B}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{B}_{q1}) & V_c(\hat{B}_{q2}) & \cdots & V_c(\hat{B}_{qq}) \end{bmatrix}$$
$$= \begin{bmatrix} V_c(\hat{C}_{11}) & V_c(\hat{C}_{12}) & \cdots & V_c(\hat{C}_{1q}) \\ V_c(\hat{C}_{21}) & V_c(\hat{C}_{22}) & \cdots & V_c(\hat{C}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{C}_{p1}) & V_c(\hat{C}_{p2}) & \cdots & V_c(\hat{C}_{pq}) \end{bmatrix}.$$

Theorem 4.4. *Eq.* (4.1) *is equivalent to the following linear system:*

$$(I_q \otimes \breve{A} + \breve{B}^T \otimes I_p)V_c(X) = V_c(\breve{C}) \tag{4.6}$$

where

$$\breve{A} = \begin{bmatrix} V_c(\hat{A}_{11}) & V_c(\hat{A}_{12}) & \cdots & V_c(\hat{A}_{1p}) \\ V_c(\hat{A}_{21}) & V_c(\hat{A}_{22)} & \cdots & V_c(\hat{A}_{2p}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{A}_{p1}) & V_c(\hat{A}_{p2}) & \cdots & V_c(\hat{A}_{pp}) \end{bmatrix}, \quad \breve{B} = \begin{bmatrix} V_c(\hat{B}_{11}) & V_c(\hat{B}_{12}) & \cdots & V_c(\hat{B}_{1q}) \\ V_c(\hat{B}_{21}) & V_c(\hat{B}_{22}) & \cdots & V_c(\hat{B}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{B}_{q1}) & V_c(\hat{B}_{q2}) & \cdots & V_c(\hat{B}_{qq}) \end{bmatrix},$$

$$\breve{C} = \begin{bmatrix} V_c(\hat{C}_{11}) & V_c(\hat{C}_{12}) & \cdots & V_c(\hat{C}_{1q}) \\ V_c(\hat{C}_{21}) & V_c(\hat{C}_{22}) & \cdots & V_c(\hat{C}_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ V_c(\hat{C}_{p1}) & V_c(\hat{C}_{p2}) & \cdots & V_c(\hat{C}_{pq}) \end{bmatrix}.$$

From Theorem 4.4, we can solve Eq. (4.1) by solving the linear system (4.6) instead.

4.2 The general case

In this subsection, we study the the condition for the dimensions of $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{h\times k}$ *and* $C \in \mathbb{R}^{u\times v}$ and $X \in \mathbb{R}^{p\times q}$. We have the following lemma.

Lemma 4.5. *Eq.* (4.1) *is well defined if and only if the dimensions of* A*,* B*,* C*, and* X *satisfy the following conditions*

(i) $u = \frac{mt}{n} = \frac{ps}{q}$,

(ii) $v = \frac{tq}{p} = \frac{ks}{h}$,

where $t = \text{lcm}\,\{n, p\}$ *and* $s = \text{lcm}\,\{q, h\}$.

Proof. Denote $t = \text{lcm}\,\{n, p\}$ and $s = \text{lcm}\,\{q, h\}$. By definition of the general left semi-tensor product, we have that

$$A \ltimes X = (A \otimes I_{\frac{t}{n}})(X \otimes I_{\frac{t}{p}}) \in \mathbb{R}^{\frac{mt}{n} \times \frac{tq}{p}} \quad \text{and} \quad X \ltimes B = (X \otimes I_{\frac{s}{q}})(B \otimes I_{\frac{s}{h}}) \in \mathbb{R}^{\frac{ps}{q} \times \frac{ks}{h}}.$$

Assume that $C \in \mathbb{R}^{u\times v}$. Then we have $u = \frac{mt}{n} = \frac{ps}{q}$ and $v = \frac{tq}{p} = \frac{ks}{h}$. ☐

Remark 4.6. If Eq. (4.1) has a solution, the dimensions of matrices A, B, and C must satisfy the conditions

(i) $m \mid u$ and $k \mid v$.

(ii) $\frac{m}{n} = \frac{h}{k}$.

(iii) $p = \frac{nu}{\alpha}$ and $q = \frac{mv}{\alpha}$, where $\alpha \mid nu, \alpha \mid mv, m \mid \alpha$ and $n \mid \alpha$.

In the general case, it is not easy to find partitions of the matrices $A \ltimes X, X \ltimes B$, and C in oder to get a linear system. However, for a particular case $B = 0$, we can partition $A \ltimes X$ and C to get a linear system. Indeed, if the equation $A \ltimes X = C$ has a solution, then C must be a block Toplitz matrix; see [17].

5 Conclusion

We consider the Sylvester matrix equation (4.1) with respect to the general left semi-tensor product where the coefficients $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{h\times k}, C \in \mathbb{R}^{u\times v}$ are real matrices of arbitrary dimensions. The study includes the equation $A \ltimes X = C$ ([17]) , the equation $X \ltimes B = C$, and the Lyapunov equation. Our results also contain the Sylvester matrix equation with respect to the left semi-tensor product. We obtain the conforming dimensions of A, B, C, and the unknown X. We investigate necessary/sufficient condition(s) for the Sylvester matrix equation to be solvable and uniquely solvable, concerning ranks and linear independence. For certain cases of dimensions, we show that the Sylvester equation can be transformed to a linear system with respect to the usual matrix product by means of the vector operator.

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