THE METHODS FOR SOLVING FIXED POINT PROBLEMS IN METRIC AND HILBERT SPACES

CHINDA CHAICHUAY

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Student Name	Chinda Chaichuay
Student ID	58605003
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Thesis Advisor	Assoc. Prof. Dr.Atid Kangtunyakarn

Abstract

For the purpose of this thesis is to introduce the new split variational inequality in three Hilbert spaces. There are important tools which are used to solve classical problems will be developed. Then, we prove the convergence theorem for finding a common element of the set of solution of such problems and the sets of fixed-point of discontinuous mappings. Secondly, we introduce a new type of multi-valued mapping and g-l-graph preserving. Consequently, we prove a coincidence point theorem on complete metric spaces endowed with a directed graph. The proposed theorem can be applied to obtain the similar result in a matrix space endowed with a partial order set. Finally, we give some examples to support our main results.

Keywords : Nonspreading mapping; Pseudo-nonspreading mapping; New split variational inequalities; g-l-graph preserving; (l,g)-G contraction; Fixed point.

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Chapter 1

Introduction

1.1 Research motivation

1.1.1 Fixed point theory and the split various variational inequalities problems

Fixed point theory is a fascinating subject, with an enormous number of applications in various fields of mathematics. Moreover, fixed point techniques have been applied in such various fields as engineering, physics, economics and game theory. A fixed point of a mapping T is a point x such that $x = Tx$, where $T : X \to X$ is a nonlinear mapping. The knowledge of the existence of fixed points has relevant applications in many branches of analysis and topology. Furthermore, many mathematicians have been studying about the structure of fixed point set.

In [26], Kohsaka and Takahashi introduced the nonspreading mapping in Hilbert spaces H which is defined by the following inequality $2\|Tx-Ty\|^2 \leq \|Tx-y\|^2+\|x-Ty\|^2$, for all $x, y \in C$. Following the terminology of Browder and Petryshyn [6], in [36], Osilike and Isiogugu introduced the mapping $T : C \to C$, which is called κ-strictly pseudo-nonspreading mapping if there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 + 2\langle x - Tx, y - Ty\rangle,$$

for all $x, y \in C$. Clearly every nonspreading mapping is κ-strictly pseudo-nonspreading; see, for example, [36].

The variational inequality problem (VIP) is to find a point $\omega_* \in C$ such that

$$\langle y - \omega_*, G\omega_*\rangle \geq 0, \tag{1.1}$$

for all $y \in C$, where $G : C \to H$ is a mapping. The set of all solutions of 1.1 is denoted by $VIP(C, G)$. It is well known that numerous problems in physic, finance, optimization, minimax problem reduce to find element of 1.1. Historically the variational inequality was introduced by Stampachhia [40] in 1964. After that variational inequalities has been widely studied in the literature; see [7],[8],[9],[21],[25],[34].

For every i = 1, 2, let H_i be a real Hilbert space and C, Q be nonempty closed convex subset of H_1, and H_2, respectively. Recently, Censor [10] has introduced a new variational problem called the split inequality problem (SIP). It entails finding a solution of one variational inequality problem (VIP), the image of which, under a given bounded linear transformation, is a solution of another VIP. The split variational inequality problem is assigned to the following formula; find a point $\omega_* \in C$ such that

$$\langle f(\omega_*), x - \omega_*\rangle \geq 0, \quad for \;\; all \;\; x \in C \tag{1.2}$$

and a point $y^* = A\omega_*$ solves

$$\langle y - y^*, g(y^*) \rangle \geq 0, \quad for \;\; all \;\; y \in Q, \tag{1.3}$$

where $A : H_1 \to H_2$ is a bounded linear operator and $f : H_1 \to H_1, g : H_2 \to H_2$ are mappings. The set of all solutions of 2.3 and 2.4 is denoted by

$$\Omega = \{x \in VIP(C, f) : \quad for \;\; all \;\; x \in VIP(Q, g)\}.$$

The split variational inequality problem (SVIP) is general and should enable split minimization between two spaces so that the image of a solution point of one minimization problem is a solution point of another minimization problem. Another special case of the SVIP is the Split Feasibility Problem (SFP) which had already been studied and used in practice as a model in intensity-modulated radiation therapy (IMRT) treatment planning.

The split feasibility problem(SFP) is to find a point $x \in C$ and $Ax \in Q$. This problem was introduced by Censor and Elfving [11]. The set of all solution(SFP) is denoted by $\Gamma = \{x \in C; Ax \in Q\}$. In fact, it has been extensively investigated in the literature [13] and [12].

Over the past decades, investigations of fixed points by some iterative schemes have attracted many mathematicians.

In 2014, Bnouhachem [5] modified a projection process for finding a common solution of a system of variational inequalities, a split equilibrium problem and a hierarchical fixed-point problem in the setting of real Hilbert spaces and also proved the strong convergence theorem of the sequence x_n generated by

$$\begin{cases} u_n = \vartheta_{r_n}^{F_1}\left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right) A x_n\right); \\ z_n = P_C\left[P_C\left[u_n - \alpha_1 B_2 u_n\right] - \alpha_1 B_1 P_C\left[u_n - \alpha_2 B_2 u_n\right]\right]; \\ y_n = \beta_n \varrho x_n + \left(1 - \beta_n\right) z_n; \\ x_{n+1} = P_C\left[\alpha_n \rho U\left(x_n\right) + \left(I - \alpha_n \mu F\right) \vartheta\left(y_n\right)\right], \text{ for all } n \geq 0, \end{cases}$$

where $A : H_1 \to H_2$ is a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions, $B_i : C \to H$ is a θ_i-inverse strongly monotone mapping for each $i = 1, 2$ and $S, \vartheta : C \to C$ a nonexpansive mappings, $F : C \to C$ is a k-Lipschitzian mapping and be η-strongly monotone $U : C \to C$ be a τ-Lipschitzian mapping and positive paramiters $r_n, \alpha_n, \alpha_1, \alpha_2, \rho, \mu$, for all $n \in \mathbb{N}$.

Recently, Moudafi [32] introduced the following new split feasibility problem, which is also called *general split equality problem*:

Let H_1, H_2, H_3 be real Hilbert spaces, $C \subset H_1, Q \subset H_2$ be two nonempty closed convex sets, $A : H_1 \to H_3, B : H_2 \to H_3$ be two bounded linear operators. Moudafi study of the convergence of a relaxed alternating CQ-algorithm for solving the new split feasibility problem is to find

$$\varpi_* \in C,\ y^* \in Q \text{ such that } A\varpi_* = By^*. \tag{1.4}$$

In order to prove the weak convergence theory to solve general split equality problem (1.4) Moudafi defined the following iteration process $\{x_k\}$;

$$\begin{cases} x_{k+1} = P_C\left(x_k - \gamma_k A^*\left(Ax_k - By_k\right)\right), \\ y_{k+1} = P_Q\left(y_k + \gamma_k B^*\left(Ax_{k+1} - By_k\right)\right), \end{cases}$$

where A^*, B^* are adjoint operators of A, B respectively, proper conditions of the positive paramiter γ_k, for all $k \geq 1$. In order to avoid using the projection, Moudafi [31] introduced and studied the following problem: Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be nonlinear operators such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$, where $F(T)$ and $F(S)$ denote the sets of fixed points of T and S, respectively. If C=$F(T)$ and Q=$F(S)$, then split equality problem reduces to

$$\text{to find } x \in F(T) \text{ and } y \in F(S) \text{ such that } Ax = By, \tag{1.5}$$

which is called a *split equality fixed point problem (SEFPP)*.

Question A. Can we prove a strong convergence theorem of three Hilbert spaces by different methods from Moudafi [32] ?

1.1.2 Graph theory

Graph theoretical concepts are widely used in many real-world problem. In biology, graph theory is often used where a vertex represents regions where certain species exist and the edges represent migration path or movement between the regions. In chemistry, graph theory including study of molecules, construction of bounds and the study of atoms. In computer science, graph theory is used to represent the connecting with friends on social media, where each user is a vertex, and when users connect they create an edge.

The Banach contraction principle is the most important theorem for studying fixed point theorem. Fixed points is fundamental concepts of mathematics in many fields. For example, Nash equilibrium in economics, the theory of phase transition, renormalization group and critical phenomenon in physics. In information Technology also use fixed point for program analysis, dataflow analysis and optimization.

S.Choudhury, Metiya and Debnath [14], they propose the notation of end point of multivalued mapping in the setting of metric space endowed with graph and prove some existence results. The mapping that satisfy generalized multivalued almost G-contraction type inequalities are assumed.

Hanjing and Suantai [19], they propose a new type G-contraction multivalued mapping in a metric space endowed with a directed graph. The new theorem show that there are some fixed points of multivalued mapping will belong to the set of coincidence points of them.

From now on we will recall some mathematical background which is necessary basis for the main theorem.

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *contraction* if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$. If $k = 1$, then T is a nonexpansive mapping.

The Banach contraction principle is a very importance tool to prove existence of fixed point theorem it states as follows:

Theorem 1.1. Let X be a complete metric space and let T be a contraction of X to itself. Then T has a unique fixed point.

The Banach contraction principle theorem plays an important role in studying the existence of solutions of nonlinear integral equations, system of linear equations, nonlinear differential equations, and proving the convergence of algorithms in computational mathematics.

Let $CB(X)$ be the set of all nonempty, closed, and bounded subsets of X. A point $x \in X$ is a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$. Nadler [33] has proved a multi-valued version of the Banach contraction principle as follows:

Theorem 1.2. Let (X, d) be a complete metric space and $T : X \to CB(X)$. Assume that there exists $k \in [0, 1)$ such that $H(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

The theory of multi-valued mappings is an important role in various branches of pure and applied mathematics to solve many problems such as optimal control problem.

A partial order in a binary relation $\prec$ over the set X, that is, for all x,y and z in X it must satisfies the following conditions:

a) $x \prec x$,

b) If $x \prec y$ and $y \prec x$, then $x = y$,

c) If $x \prec y$ and $y \prec z$, then $x \prec z$.

Fixed point theorem for single and multi-valued mapping defined on partially ordered metric space received attention from many researcher (see, e.g., [39], [1]). Ran and Reurings [39] was the first to prove a theorem related to a partial ordering. They proved the following theorem.

Theorem 1.3. Let $(X, \prec)$ be a partially ordered set such that every pair $x, y \in X$ has an upper and lower bounded. Let d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:

(1) There exists a $k \in (0, 1)$ with

$$d(f(x), f(y)) \leq kd(x, y),$$

for all $x \geq y$.

(2) There exists an $x_0 \in X$ such that $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$. Then f is a Picard operator (PO), that is, f has a unique fixed point $x^* \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = x^*$.

Many research in this direction was inspired by such theorem (see, e.g., [20], [24]).

We now move on some basics and definitions in graph theory. Let $G = (V(G), E(G))$ be a directed graph, where $V(G)$ is a set of vertices of graph and E(G) is a set of its edges. We denote G^{-1} by the directed graph obtained from G by reversing the direction of edges. That is,

$$E(G^{-1}) = \{(x, y)|(x, y) \in E(G)\}.$$

Assume that G has no parallel edges. If x and y are vertices in G, then a path in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^{n}$ of $n + 1$ vertices such that $x_0 = x, x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for all $i = 1, 2, \ldots, n$. A graph G is connected if there exists a path between any two vertices of G.

In 2007, Jachymski and Jozwik [24] introduced fixed point theory by using graph strcture on metric space. At many time, many researchers are interested in studying the such theorem, for example , recently Tiammee and Suantai [44] introduce the concepts of graph-preserving multi-valued mapping and a new type of multi-valued weak G-contraction on a metric space endowed with a directed graph G and prove some coincidence point theorems for this type of multi-valued mapping and a surjective mapping $g : X \to X$ under some conditions. In the same year, Aniruth Phon-on, Areeyuth Sama-Ae et al., [38] define a new class of Reich type multi-valued contractions on a complete metric space satisfying the g-graph preserving condition and prove a fixed point theory for such mappings. See more examples [2], [18], [23].

1.2 Objectives of the study

1) To answer the question A, we have created a new tool to prove a strong convergence theorem for three Hilbert spaces to be used for finding the solution of the problem (3.1) and the fixed points problem of nonspreading and pseudo-nonspreading mappings.

2) To propose a new split variational inequality in three Hilbert spaces.

3) To define the new iterative methods for approximating the solutions of a system of variational inequalities in the setting of real Hilbert spaces.

4) To prove a strong convergence theorems for finding a common solution of a system of variational inequalities in the setting of real Hilbert spaces.

5) To introduce a new type of multi-valued mapping and g-l-graph preserving to prove a fixed point theorem on complete metric spaces endowed with a directed graph.

6) To give examples to support our main theorems.

1.3 Scopes of the study

1) Study variational inequality problems in a Hilbert space.

2) Investigate the fixed problems of nonlinear mappings, including nonspreading and pseudo-nonspreading mappings.

3) All strong convergence theorems are considered and proved in Hilbert spaces.

1.4 Research methodology

1) Study advanced topics in fixed point theory for nonspreading and pseudo-nonspreading mappings.

2) Study background in a real Hilbert space.

3) Study Graph theory.

4) Collect and study research papers concerning fixed point theorem.

5) Determine the objecttives and scope of the research.

6) Produce tools for theorem of fixed point theorem.

7) Prove a strong convergence theorem for finding a common solution of a system of variational inequalities in the setting of real Hilbert spaces.

8) Prove a fixed point theorem on complete metric spaces endowed with a directed graph.

9) Provide applications and examples to support our main theorems.

10) Conclude the result, make suggestions for further work and write the thesis.

1.5 Benefits of the study

1) Obtain new mathematical tools for the properties of a new split variational inequality in three Hilbert spaces.

2) Obtain the strong convergence theorem for finding a common solution of a system of variational inequalities in the setting of real Hilbert spaces.

3) Obtain a new type of multi-valued mapping and g-l-graph preserving to prove a fixed point theorem on complete metric spaces endowed with a directed graph.

Chapter 2

Preliminaries and Literature Reviews

In this chapter, we collect some definitions and lemmas in Hilbert spaces, which will be needed in proving our main results.

2.1 Fundamental properties in Banach and Hilbert spaces

Definition 2.1. [37] An inner product on a vector space K over the field $\mathbb{F}$ is a function $\langle\cdot,\cdot\rangle : K\times K \to \mathbb{F}$, that assigns a scalar $\langle x,y\rangle$ for every $x,y\in K$, such that for all $x,y,z\in K$ and $\alpha\in\mathbb{F}$:

1) $\langle x+z,y\rangle = \langle x,y\rangle + \langle z,y\rangle$,

2) $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$,

3) $\overline{\langle x,y\rangle} = \langle y,x\rangle$,

4) $\langle x,x\rangle > 0 \Leftrightarrow x\neq 0$, A vector space K over $\mathbb{F}$ with a specific inner product is called an inner product space. If $\mathbb{F}=\mathbb{C}$ is a complex inner product space, and if $\mathbb{F}=\mathbb{R}$, K is a real inner product space.

Theorem 2.1. [37] For an inner product space K, $x,y,z\in K$ and $\alpha\in\mathbb{F}$:

1) $\langle x,y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$,

2) $\langle x,\alpha y\rangle = \bar{\alpha}\langle x,y\rangle$,

3) $\langle x,0\rangle = \langle 0,x\rangle = 0$,

4) $\langle x,x\rangle = 0 \Leftrightarrow x = 0$,

5) If $\langle x,y\rangle = \langle x,z\rangle$ for all $x\in K$ then $y=z$.

Definition 2.2. [42] A complete inner product space is called a *Hilbert space*.

Example 2.2. Let $X = C[a,b]$, the linear space of all scalar-valued continuous functions on $[a,b]$. Then the function $\langle\cdot,\cdot\rangle : C[a,b]\times C[a,b]\to\mathbb{C}$ defined by

$$\langle f,g\rangle = \int_a^b f(t)\overline{g(t)}dt \text{ for all } f,g\in C[a,b].$$

Then $X = C[a,b]$ is a inner product space but not a Hilbert space.

Example 2.3. Let $X=\ell_2$, the set of all sequences of complex numbers $(x_1,x_2,...,x_i,...)$ with $\sum_{i=1}^{\infty}|x_i|^2<\infty$. Then the function $\langle\cdot,\cdot\rangle : \ell_2\times\ell_2\to\mathbb{C}$ defined by

$$\langle x,y\rangle = \sum_{i=1}^{\infty} x_i\bar{y}_i \text{ for all } x=\{x_i\}_{i=1}^{\infty}, y=\{y_i\}_{i=1}^{\infty}\in X.$$

Then ℓ_2 is a Hilbert space.

Example 2.4. ℓ_p^n is a finite-dimensional Banach space that is not a Hilbert space for $p \neq 2$.

Definition 2.3. [30] Let X be a linear space (or vector space) over the field $\mathbb{F}$. A *norm* on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

2) $\|\alpha x\| = |\alpha| \, \|x\|$,

3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality). The ordered pair $(X, \|\cdot\|)$ is called a *normed space or normed vector space or normed linear space*.

Definition 2.4. (**Cauchy sequence** [15]) A sequence of vectors $\{x_n\}$ in a normed space X is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n > N$.

Definition 2.5. [15] A normed space X is called *complete* if every Cauchy sequence in X converges to an element of X.

Definition 2.6. [42] A complete normed linear space is called a *Banach space*.

Example 2.5. [42] Let $X = \ell_p^n, n > 1$ and $1 \leq p < \infty$. The sequence space defined by

$$\ell_p^n = \{x : x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n\}.$$

Then ℓ_p^n is a Banach space with the norm defined by $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$.

Example 2.6. Let $X = c_{00}$, the sequence space defined by

$$c_{00} = \left\{x = \{x_i\}_{i=1}^\infty \in \ell_\infty : \{x_i\}_{i=1}^\infty \text{ has only a finite number of nonzero terms}\right\}.$$

Then c_{00} space is a normed space with norm $\|\cdot\|_\infty$ but not a Banach space.

Theorem 2.7. [15] A subset S of a normed space X is *closed* if and only if every sequence of elements of S convergent in X has its limit in S, i.e.,

$$\{x_n\} \subseteq S \text{ and } x_n \to x \text{ implies } x \in S.$$

Theorem 2.8. (**Schwarz inequality** [42]) Let K be an inner product space and let x and y be elements in K. Then the following holds:

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Definition 2.7. (**Metric projection**). The (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.9. [41] For a given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality

$$\langle y - x, z - y \rangle \geq 0, \quad \text{for all } z \in C.$$

Lemma 2.10. Let H be a real Hilbert space. Then the following identities hold:

1) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y \rangle + \|y\|^2, \forall x, y \in H;$

2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \forall x, y \in H.$

Definition 2.8 (**Strong convergence** [16]). A sequence $\{x_n\}$ of vectors in an inner product space K is called *strongly convergent* to a vector x in K if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.9 (**Weak convergence** [16]). A sequence $\{x_n\}$ of vectors in an inner product space K is called *weakly convergent* to a vector x in K if

$$\langle x_n, y \rangle \to \langle x, y \rangle \text{ as } n \to \infty \text{ for every } y \in K.$$

Remark 2.11. We represent weak and strong convergence by "$\rightharpoonup$" and "$\to$", respectively.

Theorem 2.12. [16] A strongly convergence sequence is weakly convergence (to the same limit), that is, $x_n \to x$ implies $x_n \rightharpoonup x$.

Remark 2.13. [42] If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Lemma 2.14. [42] Let $\{x_n\}$ be a Cauchy sequence of an inner product space C such that $x_n \rightharpoonup x$. Then $x_n \to x$.

Theorem 2.15. [42] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Suppose that $\{x_n\} \subset C$ and $x_n \rightharpoonup x$. Then $x \in C$.

Lemma 2.16. [35] Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{u_n\} \subset H$ with $u_n \rightharpoonup u$, the inequality

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

holds for every $v \in H$ with $v \neq u$.

Theorem 2.17. [42] Let $\{a_n\}$ be a bounded of real numbers. Then, there exists subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Lemma 2.18. [45] Let $\{\Upsilon_n\}$ be a sequence of nonnegative real numbers satisfying

$$\Upsilon_{n+1} = (1-\alpha_n)\Upsilon_n + \delta_n, \quad \text{for all } n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

1) $\sum_{n=1}^{\infty} \alpha_n = +\infty,$

2) $\limsup_{n\to+\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{+\infty} |\delta_n| < +\infty.$

Then $\lim_{n\to+\infty} \Upsilon_n = 0.$

Lemma 2.19. [46] Let $\{\Upsilon_n\}$ be a sequence of nonnegative real number satisfying

$$\Upsilon_{n+1} = (1-\alpha_n)\Upsilon_n + \alpha_n\beta_n, foralln \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions

1) $\{\alpha_n\} \subset [0,1], \ \sum_{n=1}^{+\infty} \alpha_n = +\infty;$

2) $\limsup_{n\to+\infty} \beta_n \leq 0$ or $\sum_{n=1}^{+\infty} |\alpha_n\beta_n| < +\infty.$

Then $\lim_{n\to+\infty} \Upsilon_n = 0.$

2.2 Fixed point of nonspreading mappings and κ-strictly pseudo-nonspreading mappings with some properties

Let X be a nonempty set and $T : X \to X$. We say that $x \in X$ is a fixed point of T if and only if $Tx = x$ and $F(T)$ represents the set of all fixed points of T, i.e.,

$$F(T) = \{x \in C : Tx = x\}.$$

Example 2.20. [4] Let $X = \mathbb{R}$.

1) If $T(x) = x$, then $Fix(T) = \mathbb{R}$;

2) If $T(x) = x + 9$, then $Fix(T) = \emptyset$;

3) If $T(x) = x^2 + 5x + 4$, then $Fix(T) = \{-2\}$.

Definition 2.10. Let $T : C \to C$ be a mapping. Then T is said to be

1) a *nonspreading mapping* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \forall x, y \in C. \tag{2.1}$$

2) a *κ-strictly pseudo-nonspreading mapping* if there exists $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(1-T)x - (1-T)y\|^2 + 2\langle x - Tx, y - Ty\rangle, \forall x, y \in C.$$

In 2008, Kohsaka and Takahashi [26] introduced the nonspreading mapping T in Hilbert space H as defined in (2.1) and in [36], Osilike and Isiogugu introduced the κ-strictly pseudo-nonspreading mapping, see for example, [36]. Clearly every non-spreading mapping is κ-strictly pseudo-nonspreading.

In 2009, it is shown by Iemoto and Takahashi [22] that (2.1) is equivalent to the following equation.

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \text{ for all } x, y \in C. \tag{2.2}$$

Many researcher proved the strong convergence theorem for nonspreading mapping and its generalized mappings in Hilbert spaces, see, for example, [17, 27, 28].

Theorem 2.21. [26] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a nonspreading mapping of C into itself. Then $F(T)$ is closed and convex.

Example 2.22. Let $T : [1, 100] \to [1, 100]$ be defined by $Tx = \frac{2x+7}{9}$. Then T is a non-spreading mapping.

Solution. Let $x, y \in [1, 100]$. Thus we get

$$|Tx - Ty|^2 = \left|\frac{2x+7}{9} - \frac{2y+7}{9}\right|^2 = \left|\frac{2}{9}(x-y)\right|^2 = \frac{4}{81}|x-y|^2$$

and

$$\begin{aligned} 2\langle x - Tx, y - Ty\rangle &= 2\left(x - \frac{2x+7}{9}\right)\left(y - \frac{2y+7}{9}\right) \\ &= 2\left(\frac{7x-7}{9}\right)\left(\frac{7y-7}{9}\right) \\ &= \frac{98}{81}(x-1)(y-1) \geq 0, (\text{ Since } x, y \geq 1). \end{aligned}$$

Therefore

$$\begin{aligned} |x-y|^2 + 2\langle x - Tx, y - Ty\rangle &\geq |x-y|^2 \\ &\geq \frac{4}{81}|x-y|^2 \\ &= |Tx - Ty|^2. \end{aligned}$$

Hence T is a nonspreading mapping.

2.3 Split variational inequality problems

Historically the variational inequality was introduced by Stampachhia [40] in 1964. This problem is widely used in economics, social sciences and other fields. After that variational inequalities has been widely studied in the literature; see [7],[8], [9],[21],[25],[34].

Definition 2.11. [41] Let C be a nonempty closed convex subset of H and let A be an operator of C into H. Consider the following problem: Find $x \in C$ such that

$$\langle Ax, u - x\rangle \geq 0, \quad for \;\; all \;\; u \in C.$$

Such an $x \in C$ is called *a solution of the variational inequality of A*. We will denote the solution set of the considered variational inequality by $VIP(C, A)$.

Lemma 2.23. [47] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \quad \text{if and only if} \quad u \in VIP(C, A),$$

where P_C is the metric projection of H onto C.

Theorem 2.24. [42] Let H be a real Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $\alpha > 0$ and let $A : C \to H$ be α-inverse strongly monotone. Then $VIP(C, A) \neq \emptyset$.

For every i = 1, 2, let H_i be a real Hilbert space and C, Q be nonempty closed convex subset of H_1, and H_2, respectively. Recently, Censor [10] has introduced a new variational problem called the split inequality problem (SIP). It entails finding a solution of one variational inequality problem (VIP), the image of which, under a given bounded linear transformation, is a solution of another VIP. The split variational inequality problem is assigned to the following formula; find a point $\omega_* \in C$ such that

$$\langle f(\omega_*), x - \omega_*\rangle \geq 0, \quad for \;\; all \;\; x \in C \tag{2.3}$$

and a point $y^* = A\omega_*$ solves

$$\langle y - y^*, g(y^*)\rangle \geq 0, \quad for \;\; all \;\; y \in Q, \tag{2.4}$$

where $A : H_1 \to H_2$ is a bounded linear operator and $f : H_1 \to H_1, g : H_2 \to H_2$ are mappings. The set of all solutions of 2.3 and 2.4 is denoted by

$$\Omega = \{x \in VIP(C, f) : \quad for \;\; all \;\; x \in VIP(Q, g)\}.$$

The split variational inequality problem (SVIP) is general and should enable split minimization between two spaces so that the image of a solution point of one minimization problem is a solution point of another minimization problem.

2.4 Properties of bounded linear operators in Hilbert spaces

Definition 2.12. [42] Let E and F be normed linear spaces with the same scalars, and let T be a linear mapping of E into F. Then T is called *bounded* if there exists $K \geq 0$ such that

$$\|T(x)\| \leq K\|x\| \text{ for all } x \in E.$$

Let T be a bounded linear mapping of E into F. So, we have that for $x \in E$ with $\|x\| \leq 1$,

$$\|T(x)\| \leq K, \tag{2.5}$$

where $T(x)$ is often denoted by Tx.

For a bounded linear mapping T of E into F, we define its norm by

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|. \tag{2.6}$$

For such $\|T\|$, we have the following results.

Definition 2.13. [42] Let E and F be normed linear spaces and let T be a bounded linear mapping of E into F. Then the following hold:

1) $\|Tx\| \leq \|T\|\|x\|$ for all $x \in E$,

2) $\|T\| = \sup_{\|x\|=1} \|Tx\|$.

Definition 2.14. [42] Let E and F be linear spaces with the same scalars, and let T be a mapping of E into F. Then T is called *linear* if for any $x, y \in E$ and any scalar $\alpha \in \mathbb{R}$,

$$T(x+y) = T(x) + T(y) \text{ and } T(\alpha x) = \alpha T(x).$$

In particular, for the case of $\mathbb{F} = \mathbb{R}$, T is called a *linear functional*.

Note that if $T : E \to F$ is a linear mapping, then

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \forall x, y \in E \text{ and } \alpha, \beta \in \mathbb{R}.$$

Definition 2.15. [16] Let A be a bounded operator on a Hilbert space H. If $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x \in H$, then A is called the *self-adjoint* operator.

Theorem 2.25. [16] Let T be a bounded linear self-adjoint operator on a Hilbert space H. Then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Definition 2.16. [29] A self-adjoint operator A is a strongly positive operator on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \text{ for all } x \in H.$$

Definition 2.17. [16] A self-adjoint operator A is called *positive* if $\langle Ax, x\rangle \geq 0$ for all $x \in H$.

Theorem 2.26. [15] The adjoint operator A^* of a bounded operator A is bounded. Moreover, we have $\|A\| = \|A^*\|$ and $\|A^*A\| = \|A\|^2$.

Example 2.27. Let $\mathbb{R}^2$ be the two dimensional space of real numbers with an inner product $\langle \cdot, \cdot\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\langle u, v\rangle = u \cdot v = u_1v_1 + u_2v_2$ and a usual norm $\| \cdot \| : \mathbb{R}^2 \to \mathbb{R}$ given by $\|u\| = \sqrt{u_1^2 + u_2^2}$, for all $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$. Let an operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Ax = (2x_1 - x_2, x_1 + 2x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then A is a bounded linear on $\mathbb{R}^2$.

Solution. Let $\alpha, \beta \in \mathbb{R}$ and $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Thus we derive

$$\begin{aligned}
A(\alpha x + \beta y) &= (2(\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2), (\alpha x_1 + \beta y_1) + 2(\alpha x_2 + \beta y_2)) \\
&= (2\alpha x_1 - \alpha x_2 + 2\beta y_1 - \beta y_2, \alpha x_1 + 2\alpha x_2 + \beta y_1 + 2\beta y_2) \\
&= (2\alpha x_1 - \alpha x_2, \alpha x_1 + 2\alpha x_2) + (2\beta y_1 - \beta y_2, \beta y_1 + 2\beta y_2) \\
&= \alpha (2x_1 - x_2, x_1 + 2x_2) + \beta (2y_1 - y_2, y_1 + 2y_2) \\
&= \alpha Ax + \beta Ay.
\end{aligned}$$

This implies that A is linear. Observe that

$$\begin{aligned}
\|Ax\| &= \|(2x_1 - x_2, x_1 + 2x_2)\| \\
&= \sqrt{(2x_1 - x_2)^2 + (x_1 + 2x_2)^2} \\
&= \sqrt{4x_1^2 - 4x_1x_2 + x_2^2 + x_1^2 + 4x_1x_2 + 4x_2^2} \\
&= \sqrt{5}\,\|x\| \\
&= M\,\|x\|.
\end{aligned}$$

Then A is bounded.

2.5 Graph theory foundations in Hilbert spaces

Fixed Point Theory and Graph Theory provides an intersection between the theories of fixed point theorems that give the conditions under which maps have solutions and graph theory which uses mathematical structures to illustrate the relationship between ordered pairs of objects in terms of their vertices and directed edges.

Let (X, d) be a metric space and let $CB(X)$ be the set of all nonempty closed bonded subset of X. Throughout this thesis we use

$$d(x, A) = \inf\{d(x, y) : y \in A\},$$

$$D(A, B) = \inf\{d(x, B) : x \in A\}.$$

The Hausdorff-Pompeiu metric H is a mapping defined as follows:

$$H(A, B) = \max\left\{\sup_{w\in B} d(w, A), \sup_{z\in A} d(z, B)\right\}.$$

Definition 2.18. [43] Let $G = (V(G), E(G))$ be a directed graph. A graph G is called *transitive* if for any $x, y, z \in V(G)$ with (x, y) and (y, z) are in $E(G)$, then $(x, z) \in E(G)$.

In 2008, Jachymski [23] generalized the Banach contraction principle in a complete metric space endowed with a directed graph. He also introduced a contractive-type mapping with a directed graph as follows;

Definition 2.19. Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph where $V(G) = X$ and $E(G)$ contains all loops, that is $\Delta \subseteq E(G)$. We say that a mapping $f : X \to X$ is a Banach *G-contraction* if f preserves edges of G, i.e.,

$$for\ \ any\ \ x, y \in X\ \ such\ \ that\ \ (x, y) \in E(G)\ \ implies\ \ (fx, fy) \in E(G)$$

and there exists $k \in (0, 1)$ such that

$$d(fx, fy) \leq kd(x, y)\ \ for\ \ all\ \ x, y \in X\ \ with\ \ (x, y) \in E(G).$$

Definition 2.20. Let ζ be a nonempty convex subset of a Banach space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = \zeta$ and $\mathcal{T} : \zeta \to \zeta$, then $\mathcal{T}$ is said to be *G-nonexpansive mapping* if the following conditions hold:

(i) $\mathcal{T}$ is edge-preserving, i.e., for any $x, y \in \zeta$ such that $(x, y) \in E(G) \Rightarrow (\mathcal{T}x, \mathcal{T}y) \in E(G)$,

(ii) $\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|$, where $(x, y) \in E(G)$ for any $x, y \in \zeta$.

This mapping was introduced by Tiammee et al.[43] in 2015.

Lemma 2.28. [43] Let $\mathcal{X}$ be a normed space and ζ be a subset of $\mathcal{X}$ having a property G. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = \zeta$ and $E(G)$ is a convex. Suppose $\mathcal{T} : \zeta \to \zeta$ is a G-nonexpansive mapping and $F(\mathcal{T}) \times F(\mathcal{T}) \subseteq E(G)$. Then $F(\mathcal{T})$ is closed and convex.

Definition 2.21. Let $(X, \prec)$ be a partially order set. Let $A, B \subset X$ and let $l : X \to X$ be a mapping. Then

$$A \prec B \text{ if } l(a) < l(b) \text{ for all } a \in A \text{ and } b \in B.$$

Definition 2.22. Let (X, d) be a metric space endowed with a partial order $\prec$. Let $g : X \to X$ be surjective, $l : X \to X$ be a mapping and let $T : X \to CB(X)$ is say to be *l,g-increasing* if for any $x, y \in X$,

$$g(x) < g(y) \Rightarrow Tx \prec Ty.$$

Example 2.29. Let $\mathbb{N}$ be the set of natural number and let the multi-value mapping $T : \mathbb{N} \to 2^{\mathbb{N}}$ defined by

$$Tx = \left\{x^2 + 3, x^2 + 5\right\},$$

for all $x \in \mathbb{N}$. Let the mappings $g : \mathbb{N} \to \mathbb{N}$ defined by $g(x) = x + 1$ for all $x \in \mathbb{N}$ and $l : \mathbb{N} \to \mathbb{N}$ defined by $l(x) = 2x$ for all $x \in \mathbb{N}$. Then T is l,g -increasing.

Lemma 2.30. [33] Let (X,d) be a metric space. If $A,B \in CB(X)$ and $a \in A$, Then for each $\varepsilon > 0$, there exists $b \in B$ such that

$$d(a,b) \leq H(A,B) + \varepsilon.$$

Property A. [23] For every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, If $x_n \to x$ and $(x_{n+1}, x_n) \in E(G)$, there is a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Chapter 3

Main Results

3.1 The split various variational inequality theorem

In this section, we introduce a new split variational inequality in three Hilbert spaces.

For every $i = 1, 2, 3.$ let H_i be a real Hilbert space and C_i be a nonempty closed convex subset of H_i. Let $B_i : C_i \to H_i$ be a mapping, for all $i = 1, 2, 3$ and let $A_2 : H_1 \to H_2$ and $A_3 : H_2 \to H_3$. The split various variational inequality is to find the points;

$$\begin{cases} \varpi_1^* \in C_1, \text{ such that } \langle B_1\varpi_1^*, x_1 - \varpi_1^*\rangle \geq 0, \text{ for all } x_1 \in C_1, \text{ and} \\ \varpi_2^* = A_2\varpi_1^* \in C_2, \text{ such that } \langle B_2\varpi_2^*, x_2 - \varpi_2^*\rangle \geq 0, \text{ for all } x_2 \in C_2, \text{ and} \\ \varpi_3^* = A_3\varpi_2^* \in C_3, \text{ such that } \langle B_3\varpi_3^*, x_3 - \varpi_3^*\rangle \geq 0, \text{ for all } x_3 \in C_3. \end{cases} \tag{3.1}$$

The set of the solution of (3.1) is denoted by
$\Omega = \{\varpi^* = (\varpi_1^*, \varpi_2^*, \varpi_3^*) \in C_1 \times C_2 \times C_3 : \varpi_i^* \in VIP(C_i, B_i)$, for all $i = 1, 2, 3\}$.

Moreover, we obtain the following result.

Lemma 3.1. For every $i = 1, 2, 3$, let H_i be a real Hilbert spaces and C_i be a nonempty closed convex subset of H_i. Let $B_i : C_i \to H_i$ be β_i-inverse strongly monotone mappings with $\eta = \min_{i=1,2,3}\{\beta_i\}$ and let $A_2 : H_1 \to H_2, A_3 : H_2 \to H_3$ be bounded linear operator with the adjoint operator A_2^* and A_3^*, respectively. Assume that $\overline{x}_1 \in C_1, A_2\overline{x}_1 = \overline{x}_2, A_3\overline{x}_2 = \overline{x}_3$ and $\Omega \neq \emptyset$. The following are equivalent:

(i) $\overline{x} \in \Omega,$ where $\overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) \in C_1 \times C_2 \times C_3.$

(ii) $\overline{x}_1 = P_{C_1}(I_1 - \lambda_1 B_1)(\overline{x}_1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\overline{x}_2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\overline{x}_3)),$

where $I_i : H_i \to H_i$ is an identity mappings, for all $i = 1, 2, 3,$ $\gamma_2(1 + \gamma_3) \leq \frac{1}{L}$, $L = \max\{L_1, L_2\} \leq 1$ which L_1, L_2 are spectral radius of $A_2A_2^*$ and $A_3A_3^*$, respectively, $\lambda_i \in (0, 2\eta)$, for all $i = 1, 2, 3$ and $\gamma_2, \gamma_3 \geq 0$

Proof. Let the conditions holds.
$i) \Rightarrow ii)$ Let $\overline{x} \in \Omega$ where $\overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) \in C_1 \times C_2 \times C_3$, we have

$$\overline{x}_i \in VIP(C_i, B_i), \text{ for all } i = 1, 2, 3.$$

From Lemma 2.23, we have

$$\overline{x}_i \in F(P_{C_i}(I_i - \lambda_i B_i)), \text{ for all } i = 1, 2, 3.$$

From determining the definition of $\overline{x}$, we have

$$\overline{x}_1 = P_{C_1}\left(I_1 - \lambda_1 B_1\right)\left(\overline{x}_1 - \gamma_2 A_2^*((I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2 + \gamma_3 A_3^*\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3)\right).$$

$ii) \Rightarrow i)$ Let $\overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) \in C_1 \times C_2 \times C_3$, where $\overline{x}_2 = A_2\overline{x}_1, \overline{x}_3 = A_3\overline{x}_2$ and

$$\overline{x}_1 = P_{C_1}\left(I_1 - \lambda_1 B_1\right)\left(\overline{x}_1 - \gamma_2 A_2^*((I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2 + \gamma_3 A_3^*\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3)\right).$$

Since B_i is β_i-inverse strongly monotone with $\lambda_i < 2\eta$, for all $i = 1, 2, 3$, we have $P_{C_i}(I_i - \lambda_i B_i)$ is a nonexpansive mappings, for all $i = 1, 2, 3$.

Let $w \in \Omega$ where $w = (w_1, w_2, w_3) \in C_1 \times C_2 \times C_3$ where $w_2 = A_2 w_1,\ w_3 = A_3 w_2$. From $i)$ implies $ii)$, we have

$$w_1 = P_{C_1}(I - \lambda_1 B_1)(w_1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I_2 - \lambda_2 B_2))w_2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))w_3)).$$

Put $M = (I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2 + \gamma_3 A_3^*\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3$
and $N = (I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))w_2 + \gamma_3 A_3^*\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)w_3$.
From determining the definition of $\overline{x}$ and w, we have

$$\begin{aligned}
\|\overline{x}_1 - w_1\|^2 &\le \|\overline{x}_1 - w_1 - \gamma_2 A_2^*(M - N)\|^2 \\
&= \|\overline{x}_1 - w_1\|^2 - 2\gamma_2\langle \overline{x}_1 - w_1, A_2^*(M - N)\rangle + \gamma_2^2\|A_2^*(M - N)\|^2 \\
&\le \|\overline{x}_1 - w_1\|^2 - 2\gamma_2\langle \overline{x}_2 - w_2, M - N\rangle + \gamma_2^2 L\|M - N\|^2 \\
&\le \|\overline{x}_1 - w_1\|^2 - 2\gamma_2\langle \overline{x}_2 - w_2, (I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2 + \gamma_3 A_3^*\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\rangle \\
&\qquad + \gamma_2^2 L\|(I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2 + \gamma_3 A_3^*\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\|^2 \\
&= \|\overline{x}_1 - w_1\|^2 - 2\gamma_2(\langle \overline{x}_2 - w_2, (I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2\rangle + \gamma_3\langle \overline{x}_3 - w_3, \\
&\qquad \left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\rangle) + \gamma_2^2 L\|(I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2 \\
&\qquad + \gamma_3 A_3^*\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\|^2 \\
&= \|\overline{x}_1 - w_1\|^2 + 2\gamma_2\langle w_2 - \overline{x}_2, (I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2\rangle + 2\gamma_2\gamma_3\langle w_3 - \overline{x}_3, \\
&\qquad \left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\rangle + \gamma_2^2 L(\|(I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2\|^2 \\
&\qquad + \gamma_3^2 L\|\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\|^2 + 2\gamma_3\langle (I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2, \\
&\qquad A_3^*\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\rangle) \\
&\le \|\overline{x}_1 - w_1\|^2 + 2\gamma_2\langle w_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right)\overline{x}_2 + P_{C_2}\left(I_2 - \lambda_2 B_2\right)\overline{x}_2 - \overline{x}_2, \\
&\qquad (I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2\rangle + 2\gamma_2\gamma_3\langle w_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\overline{x}_3 + P_{C_3}\left(I_3 - \lambda_3 B_3\right)\overline{x}_3 - \overline{x}_3, \\
&\qquad \left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\rangle + \gamma_2^2 L(\|(I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2\|^2 \\
&\qquad + \gamma_3^2 L\|\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\|^2 \\
&\qquad + \gamma_3\|(I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2\|^2 + \gamma_3\|A_3^*\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\|^2) \\
&\le \|\overline{x}_1 - w_1\|^2 + 2\gamma_2(\frac{1}{2}\|(I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2\|^2 - \|(I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2\|^2) \\
&\qquad + 2\gamma_2\gamma_3(\frac{1}{2}\|\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\|^2 - \|\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3\|^2) \\
&\qquad + \gamma_2^2 L(\|((I_2 - P_{C_2}\left(I_2 - \lambda_2 B_2\right))\overline{x}_2\|^2 + \gamma_3^2 L\|\left(I_3 - P_{C_3}\left(I_3 - \lambda_3 B_3\right)\right)\overline{x}_3)\|^2
\end{aligned}$$

$$+\gamma_3\|(I_2-P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\|^2+\gamma_3L\|(I_3-P_{C_3}(I_3-\lambda_3B_3))\overline{x}_3)\|^2)$$
$$=\|\overline{x}_1-w_1\|^2-\gamma_2(1-\gamma_2L(1+\gamma_3))\|(I_2-P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\|^2$$
$$-\gamma_2\gamma_3(1-\gamma_2L^2(1+\gamma_3))\|(I_3-P_{C_3}(I_3-\lambda_3B_3))\overline{x}_3)\|^2. \tag{3.2}$$

Applying above equation and Lemma 2.23, we have

$$\overline{x}_2\in F(P_{C_2}(I-\lambda_2B_2))=VIP(C_2,B_2) \text{ and } \overline{x}_3\in F(P_{C_3}(I-\lambda_3B_3))=VIP(C_3,B_3). \tag{3.3}$$

From determining the definition of $\overline{x}$ and (3.3), we have

$$\overline{x}_1\in F(P_{C_1}(I-\lambda_1B_1))=VIP(C_1,B_1).$$

Hence $\overline{x}\in\Omega$. □

Lemma 3.2. Let C be a nonempty closed convex subset of Hilbert space H. Let $\vartheta: C\to C$ be a nonspreading mapping and $\varrho: C\to C$ be κ-pseudo-nonspreding mapping with $F(\vartheta)\cap F(\varrho)\neq\emptyset$. Then $F(P_C(I-\gamma(a(I-\vartheta)+(1-a)(I-\varrho))))=F(\vartheta)\cap F(\varrho)$ for all $a\in(0,1)$ and $\gamma>0$. Moreover, if $\gamma<1-\kappa$, then

$$\|I-\gamma(a(I-\vartheta)+(1-a)(I-\varrho)))x-\varpi^*\|\leq\|\varpi^*-x\|,$$

for all $x\in C$ and $\varpi^*\in F(\varrho)\cap F(\vartheta)$.

Proof. Let $\varpi_0\in F(\varrho)\cap F(\vartheta)$, we have

$$P_C(I-\gamma(a(I-\vartheta)+(1+a)(I-\varrho)))\varpi_0=\varpi_0.$$

It follows that $\varpi_0\in F(P_C(I-\gamma(a(I-\vartheta)+(1+a)(I-\varrho))))$. Therefore

$$F(\vartheta)\cap F(\varrho)\subseteq F(P_C(I-\gamma(a(I-\vartheta)+(1+a)(I-\varrho)))).$$

Let $\varpi_0\in F(P_C(I-\gamma(a(I-\vartheta)+(1+a)(I-\varrho))))$ and $\varpi^*\in F(\vartheta)\cap F(\varrho)$. From Lemma 2.23, we have

$$\langle y-\varpi_0, a(I-\vartheta)\varpi_0+(1-a)(I-\varrho)\varpi_0\rangle\geq 0,$$

for all $y\in C$.
From determining the definition of ϱ, we have

$$\begin{aligned}\|\varpi_0-\varpi^*\|^2+\kappa\|(I-\varrho)\varpi_0\|^2 &\geq\|\varrho\varpi_0-\varpi^*\|^2\\ &=\|(I-\varrho)\varpi_0-(\varpi_0-\varpi^*)\|^2\\ &=\|(I-\varrho)\varpi_0\|^2-2\langle(I-\varrho)\varpi_0,\varpi_0-\varpi^*\rangle\\ &\quad+\|\varpi_0-\varpi^*\|^2.\end{aligned}\tag{3.4}$$

The result of the calculation from the inequality (3.4), we get

$$\langle(I-\varrho)\varpi_0,\varpi_0-\varpi^*\rangle\geq\left(\frac{1-\kappa}{2}\right)\|(I-\varrho)\varpi_0\|^2. \tag{3.5}$$

Assume that $\varpi_0 \neq \vartheta\varpi_0$, we have $\|(I-\vartheta)\,\varpi_0\| > 0$. Using the same method as (3.5) and definitions of ϑ, we get

$$\langle (I-\vartheta)\,\varpi_0, \varpi_0 - \varpi^*\rangle \geq \frac{1}{2}\,\|(I-\vartheta)\,\varpi_0\|^2\,. \tag{3.6}$$

From (3.5) and $a \in (0,1)$, we obtain

$$\begin{aligned}\langle \varpi^* - \varpi_0, a\,(I-\vartheta)\,\varpi_0\rangle &= \langle \varpi^* - \varpi_0, a\,(I-\vartheta)\,\varpi_0 + (1-a)\,(I-\varrho)\,\varpi_0\rangle \\ &\quad - (1-a)\,\langle \varpi^* - \varpi_0, (I-\varrho)\,\varpi_0\rangle \\ &\geq (1-a)\,\langle \varpi_0 - \varpi^*, (I-\varrho)\,\varpi_0\rangle. \end{aligned} \tag{3.7}$$

From (3.7), we have

$$\langle \varpi^* - \varpi_0, (I-\vartheta)\,\varpi_0\rangle \geq 0.$$

From above and (3.6), we have

$$0 \leq \langle \varpi^* - \varpi_0, (I-\vartheta)\,\varpi_0\rangle \leq -\frac{1}{2}\,\|(I-\vartheta)\,\varpi_0\|^2\,.$$

Thus, $\|(I-\vartheta)\,\varpi_0\| \leq 0$. This is a contradiction.
Thus, we have $\varpi_0 = \vartheta\varpi_0$ and it implies that

$$\varpi_0 \in F(\vartheta). \tag{3.8}$$

Similarly, by using the same technique as (3.8), we have

$$\varpi_0 \in F(\varrho). \tag{3.9}$$

From (3.8) and (3.9), we have

$$F(P_C(I - \gamma(a(I-\vartheta) + (1+a)(I-\varrho)))) \subseteq F(\varrho) \cap F(\vartheta).$$

Let $\varpi^* \in F(\varrho) \cap F(\vartheta)$ and $x \in C$, we have

$$\begin{aligned}&\|(I - \gamma(a(I-\vartheta) + (1-a)(I-\varrho)))x - \varpi^*\|^2 \\ &= \|(I - \gamma(a(I-\vartheta) + (1-a)(I-\varrho)))x \\ &\qquad - (I - \gamma\,(a\,(I-\vartheta) + (1-a)\,(I-\varrho)))\,\varpi^*\|^2 \\ &= \|x - \varpi^* - \gamma(a((I-\vartheta)x - (I-\vartheta)\varpi^*) \\ &\qquad + (1-a)((I-\varrho)x - (I-\varrho)\varpi^*))\|^2 \\ &= \|x - \varpi^*\|^2 - 2\gamma\langle a((I-\vartheta)x - (I-\vartheta)\varpi^*) \\ &\qquad + (1-a)((I-\varrho)x - (I-\varrho)\varpi^*), x - \varpi^*\rangle \\ &\qquad + \gamma^2\|a((I-\vartheta)x - (I-\vartheta)\varpi^*) \\ &\qquad + (1-a)((I-\varrho)x - (\mathcal{I}-\varrho)\varpi^*)\|^2 \\ &\leq \|x - \varpi^*\|^2 - 2\gamma a\langle (I-\vartheta)x - (I-\vartheta)\varpi^*, x - \varpi^*\rangle \end{aligned}$$

$$
\begin{aligned}
&- 2\gamma(1-a)\langle (I-\varrho)x-(I-\varrho)\varpi^*, x-\varpi^*\rangle \\
&+ \gamma^2 a\|(I-\vartheta)x-(I-\vartheta)\varpi^*\|^2 \\
&+ (1-a)\gamma^2\|(I-\varrho)x-(I-\varrho)\varpi^*\|^2 \\
\leq\ & \|x-\varpi^*\|^2 - 2\gamma a\frac{\|(I-T)x\|^2}{2} \\
&- 2\gamma(1-a)(1-\kappa)\frac{\|(I-\varrho)x\|^2}{2} \\
&+ \gamma^2 a\|(I-\vartheta)x\|^2 + (1-a)\gamma^2\|(I-\varrho)x\|^2 \\
\leq\ & \|x-\varpi^*\|^2.
\end{aligned}
$$

□

Theorem 3.3. For every $i=1,2,3$, let C_i be a closed convex subset of a real Hilbert space H_i. Let $B_i : C_i \to H_i$ be β_i-inverse strongly monotone mappings with $\eta = \min_{i=1,2,3}\{\beta_i\}$ and let $A_2 : H_1 \to H_2, A_3 : H_2 \to H_3$ be bounded linear operator with the adjoint operator A_2^* and A_3^*, respectively. Assume that $\overline{x}_1 \in H_1, \overline{x}_2 = A_2\overline{x}_1, \overline{x}_3 = A_3\overline{x}_2$ and $\Omega \neq \emptyset$. Let $\vartheta, \varrho : C_1 \to C_1$ be nonspreading and κ-strictly pseudo-nonspreading mappings, respectively. Assume that $\Omega \cap F(\vartheta) \cap F(\varrho) \neq \emptyset$ and let the sequence $\{x_n\}$ generated by $u, x_1 \in C_1$, and

$$
\begin{aligned}
x_{n+1} \;=\; & \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}\left(I_1 - \lambda_n\left(a\left(I_1-\vartheta\right)+(1-a)\left(I_1-\varrho\right)\right)\right)x_n \\
& +\delta_n P_{C_1}(I_1-\lambda_1 B_1)(x_n^1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I-\lambda_2 B_2))x_n^2 \\
& +\gamma_3 A_3^*(I_3 - P_{C_3}(I-\lambda_3 B_3))x_n^3)),
\end{aligned}
$$

for all $n \geq 1$ and $a \in (0,1)$, $I_i : H_i \to H_i$ is an identity mappings, for all $i = 1,2,3$, where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\} \subseteq [0,1]$ and $\alpha_n+\beta_n+\gamma_n+\delta_n = 1$ and $x_n = x_n^1, x_n^2 = A_2x_n^1, x_n^3 = A_3x_n^2$, for all $n \in \mathbb{N}, 0 < \lambda_i < 2\eta$, for all $i = 1,2,3$ and $\gamma_j > 0$, for all $j = 2,3$. Suppose that the conditions $i)$ to $v)$ is true;

(i) $\lim_{n+\to\infty} \alpha_n = 0, \sum_{n=1}^{+\infty} \alpha_n = +\infty$;

(ii) $\gamma_2\left(1+\gamma_3\right) \leq \dfrac{1}{L}$, where $L = \max\{L_{A_1}, L_{A_2}\} \leq 1$ where $L_{A_1} L_{A_2}$ are spectral radius of $A_2A_2^*, A_3A_3^*$, respectively;

(iii) $0 < a \leq \beta_n, \gamma_n, \delta_n \leq b < 1,$ for some $a, b > 0,$ for all $n \in \mathbb{N}$;

(iv) $\sum_{n=1}^{+\infty} \lambda_n < +\infty$ and $0 < \lambda_n < 1-\kappa,$ for all $n \in \mathbb{N}$;

(v) $\sum_{n=1}^{+\infty} |\alpha_{n+i} - \alpha_n|, \sum_{n=1}^{+\infty} |\beta_{n+i} - \beta_n|, \sum_{n=1}^{+\infty} |\gamma_{n+1} - \gamma_n| < +\infty.$

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\Omega\cap F(\vartheta)\cap F(\varrho)}u$.

Proof. Put $M = a\left(I-\vartheta\right)+(1-a)\left(I-\varrho\right)$ and $u_n = P_{C_1}(I-\lambda_1 B_1)(x_n^1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I-\lambda_2 B_2))x_n^2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I-\lambda_n B_3))x_n^3))$. So, we can rewrite x_n as follows;

$$
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}\left(I - \lambda_n M\right)x_n + \delta_n u_n, \tag{3.10}
$$

for all $n \geq 1$.

From determining the definition of u_n put $w_n = (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_n^2 + \gamma_3 A_3^* (I_3 - P_{C_3}(I - \lambda_3 B_3))\, x_n^3$ and $z_n = (I_3 - P_{C_3}(I - \lambda_3 B_3))\, x_n^3$, we have

$$u_n = P_{C_1}(I_1 - \lambda_1 B_1)(x_n - \gamma_2 A_2^* w_n).$$

For every $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|u_n - u_{n-1}\|^2 &\leq \|x_n - \gamma_2 A_2^* w_n - x_{n-1} + \gamma_2 A_2^* w_{n-1}\|^2 \\
&= \|x_n - x_{n-1} - \gamma_2 A_2^* (w_n - w_{n-1})\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\gamma_2 \langle A_2 x_n - A_2 x_{n-1}, w_n - w_{n-1}\rangle + \gamma_2^2 \|A_2^*(w_n - w_{n-1})\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\gamma_2 \langle x_n^2 - x_{n-1}^2, (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_n^2 \\
&\qquad + \gamma_3 A_3^* z_n - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2 - \gamma_3 A_3^* z_{n-1}\rangle \\
&\qquad + \gamma_2^2 \|A_2^* (w_n - w_{n-1})\|^2 \\
&= \|x_n - x_{n-1}\|^2 + 2\gamma_2 \langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_n^2 \\
&\qquad - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2\rangle + 2\gamma_2\gamma_3 \langle x_{n-1}^3 - x_n^3, z_n \\
&\qquad - z_{n-1}\rangle + \gamma_2^2 \|A_2^*(w_n - w_{n-1})\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 + 2\gamma_2 \langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_n^2 \\
&\qquad - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2\rangle + 2\gamma_2\gamma_3 \langle x_{n-1}^3 - x_n^3, z_n - z_{n-1}\rangle \\
&\qquad + \gamma_2^2 L \|(I_2 - P_{C_2}(I_2 - \lambda_2 B_2)) x_n^2 - (I_2 - P_{C_2}(I_2 - \lambda_2 B_2)) x_{n-1}^2 \\
&\qquad + \gamma_3 A_3^*(z_n - z_{n-1})\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 + 2\gamma_2 \langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_n^2 \\
&\qquad - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2\rangle + 2\gamma_2\gamma_3 \langle x_{n-1}^3 - x_n^3, z_n - z_{n-1}\rangle \\
&\qquad + \gamma_2^2 L (\|(I_2 - P_{C_2}(I - \lambda_2 B_2)) x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_{n-1}^2\|^2 \\
&\qquad + \gamma_3^2 L \|z_n - z_{n-1}\|^2 + 2\gamma_3 \langle (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_n^2 \\
&\qquad - (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_{n-1}^2, A^*(z_n - z_{n-1})\rangle) \\
&\leq \|x_n - x_{n-1}\|^2 + 2\gamma_2 \langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2\rangle \\
&\qquad + 2\gamma_2\gamma_3 \langle x_{n-1}^3 - x_n^3, (I_3 - P_{C_3}(I - \lambda_3 B_3)) x_n^3 - (I_3 - P_{C_3}(I - \lambda_3 B_3))\, x_{n-1}^3\rangle \\
&\qquad + \gamma_2^2 L (\|(I_2 - P_{C_2}(I - \lambda_2 B_2)) x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_{n-1}^2\|^2 + \gamma_3^2 L \|z_n - z_{n-1}\|^2 \\
&\qquad + \gamma_3 \|(I_2 - P_{C_2}(I - \lambda_2 B_2)) x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_{n-1}^2\|^2 + \gamma_3 L \|z_n - z_{n-1}\|^2) \\
&\leq \|x_n - x_{n-1}\|^2 + 2\gamma_2 (- \left\|(I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2\right\|^2 \\
&\qquad + \frac{1}{2} \left\|(I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2\right\|^2) \\
&\qquad + 2\gamma_2\gamma_3 (- \|z_n - z_{n-1}\|^2 + \frac{1}{2} \|z_n - z_{n-1}\|^2) \\
&\qquad + \gamma_2^2 L (\|(I_2 - P_{C_2}(I - \lambda_2 B_2)) x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_{n-1}^2\|^2 + \gamma_3^2 L \|z_n - z_{n-1}\|^2 \\
&\qquad + \gamma_3 \|(I_2 - P_{C_2}(I - \lambda_2 B_2)) x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2)) x_{n-1}^2\|^2 + \gamma_3 L \|z_n - z_{n-1}\|) \\
&= \|x_n - x_{n-1}\|^2 - \gamma_2 \left\|(I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - \gamma_2\gamma_3 \left\|z_n - z_{n-1}\right\|^2 + \gamma_2^2 L\|(I_2 - P_{C_2}(I-\lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I-\lambda_2 B_2))x_{n-1}^2\|^2 \\
&\quad + \gamma_2^2\gamma_3^2 L^2\|z_n - z_{n-1}\|^2 + \gamma_2^2\gamma_3 L\|(I_2 - P_{C_2}(I-\lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I-\lambda_2 B_2))x_{n-1}^2\|^2 \\
&\quad + \gamma_2^2\gamma_3 L^2\|z_n - z_{n-1}\| \\
&= \left\|x_n - x_{n-1}\right\|^2 - \gamma_2(1-\gamma_2 L\left(1+\gamma_3\right))\|\left(I_2 - P_{C_2}\left(I-\lambda_2 B_2\right)\right)x_n^2 \\
&\quad - \left(I_2 - P_{C_2}\left(I-\lambda_2 B_2\right)\right)x_{n-1}^2\|^2 - \gamma_2\gamma_3(1-\gamma_2 L^2\left(1+\gamma_3\right))\left\|z_n - z_{n-1}\right\|^2 \\
&\le \left\|x_n - x_{n-1}\right\|^2 . \qquad (3.11)
\end{aligned}
$$

Let $\varpi^* \in \Omega \cap F(\varrho) \cap F(\vartheta)$. From Lemma 3.2 and utilization of (3.2), we have

$$
\begin{aligned}
\|x_{n+1} - \varpi^*\| &\le \alpha_n \|u - \varpi^*\| + \beta_n \|x_n - \varpi^*\| + \gamma_n \|P_{C_1}(I - \lambda_n M)x_n - \varpi^*\| \\
&\quad + \delta_n \|u_n - \varpi^*\| \\
&\le \alpha_n \|u - \varpi^*\| + (1-\alpha_n)\|x_n - \varpi^*\| \\
&\le \widetilde{M}, \qquad (3.12)
\end{aligned}
$$

where $\widetilde{M} = \max\{\|u - \varpi^*\|, \|x_1 - \varpi^*\|\}$. By induction we can conclulde that the sequence $\{x_n\}$ is bounded and so are $\{u_n\}$ and $\{P_{C_1}(I - \lambda_n M)x_n\}$.
From determining the definition of x_n and (3.11), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n u + \beta_n x_n + \gamma_n P_{C1}(I - \lambda_n M)x_n + \delta_n u_n \\
&\quad - \alpha_{n-1} u - \beta_{n-1} x_{n-1} - \gamma_{n-1} P_{C1}(I - \lambda_{n-1} M)x_{n-1} - \delta_{n-1} u_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}|\,\|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\,\|x_{n-1}\| + \delta_n \|u_n - u_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}|\,\|u_{n-1}\| + \gamma_n \|P_{C_1}(I - \lambda_n M)x_n - P_{C_1}(I - \lambda_{n-1} M)x_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\,\|P_{C_1}(I_1 - \lambda_{n-1} M)x_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}|\,\|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\,\|x_{n-1}\| + \delta_n \|u_n - u_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}|\,\|u_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| + \|\lambda_{n-1} M x_{n-1} - \lambda_n M x_n\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\,\|P_{C_1}(I_1 - \lambda_{n-1} M)x_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}|\,\|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\,\|x_{n-1}\| + \delta_n \|x_n - x_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}|\,\|u_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| + \lambda_{n-1}\|M x_{n-1}\| + \lambda_n \|M x_n\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\,\|P_{C_1}(I_1 - \lambda_{n-1} M)x_{n-1}\| \\
&= (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\,\|u\| + |\beta_n - \beta_{n-1}|\,\|x_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}|\,\|u_{n-1}\| + \lambda_{n-1}\|M x_{n-1}\| + \lambda_n \|M x_n\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\,\|P_{C_1}(I_1 - \lambda_{n-1} M)x_{n-1}\| .
\end{aligned}
$$

From the conditions $i), iv), v)$ and Lemma 2.18, we have

$$
\lim_{n\to+\infty} \|x_{n+1} - x_n\| = 0. \qquad (3.13)
$$

Applying (3.2) and the definition of x_n, we have

$$
\|x_{n+1} - \varpi^*\|^2 \le \alpha_n \|u - \varpi^*\|^2 + \beta_n \|x_n - \varpi^*\|^2 + \gamma_n \|P_{C_1}(I - \lambda_n M)x_n - \varpi^*\|^2
$$

$$
\begin{aligned}
& +\delta_n \|u_n - \varpi^*\|^2 - \gamma_n\beta_n \|P_{C_1}(I-\lambda_n M)x_n - x_n\|^2 - \delta_n\beta_n \|u_n - x_n\|^2 \\
\leq \quad & \alpha_n \|u-\varpi^*\|^2 + \|x_n - \varpi^*\|^2 - \gamma_n\beta_n \|P_{C1}(I-\lambda_n M)x_n - x_n\|^2 \\
& -\delta_n\beta_n \|u_n - x_n\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\gamma_n\beta_n\|P_{C_1}(I-\lambda_n M)x_n - x_n\|^2 + \delta_n\beta_n \|u_n - x_n\|^2 \leq (\|x_{n+1}-\varpi^*\| + \|x_n - \varpi^*\|)\|x_{n+1}-x_n\| \\
+ \alpha_n \|u - \varpi^*\|^2.
\end{aligned}
$$

From the conditions i), iii) and (3.13) we can conclude the following results

$$
\lim_{n\to+\infty} \|u_n - x_n\| = \lim_{n\to+\infty} \|P_{C1}(I-\lambda_n M)x_n - x_n\| = 0. \tag{3.14}
$$

Next, we show that

$$
\limsup_{n\to+\infty} \langle u - z_0, z_0 - x_n\rangle \leq 0, \tag{3.15}
$$

where $z_0 = P_{\Omega\cap F(\varrho)\cap F(\vartheta)}u$. In order to prove this we may assume that

$$
\limsup_{n\to+\infty} \langle u - z_0, x_n - z_0\rangle = \lim_{k\to+\infty} \langle u - z_0, x_{n_k} - z_0\rangle, \tag{3.16}
$$

where $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Since $\{x_n\}$ is bounded, we may assume that $x_{n_k} \rightharpoonup q$ as $k \to +\infty$.

Assume that $q \notin F(\varrho)\cap F(\vartheta)$. From Lemma 3.2, we have $q \notin F(P_{C_1}(I-\lambda_{n_k}M))$. By using properties of Opial's condition and (3.14), we have

$$
\begin{aligned}
\liminf_{k\to+\infty} \|x_{n_k} - q\| \quad & < \quad \liminf_{k\to+\infty} \|x_{n_k} - P_{C_1}(I-\lambda_{n_k}M)q\| \\
& \leq \quad \liminf_{k\to+\infty} (\|x_{n_k} - P_{C_1}(I-\lambda_{n_k}M)x_{n_k}\| \\
& \qquad + \|P_{C_1}(I-\lambda_{n_k}M)x_{n_k} - P_{C_1}(I-\lambda_{n_k}M)q\|) \\
& \leq \quad \liminf_{k\to+\infty} (\|x_{n_k} - q\| + \lambda_{n_k}\|Mx_{n_k} - Mq\|) \\
& \leq \quad \liminf_{k\to+\infty} \|x_{n_k} - q\|.
\end{aligned}
$$

This is a contradiction. Therefore $q \in F(\varrho)\cap F(\vartheta)$.

Assume $q \notin \Omega$. From Lemma 3.1, we have

$$
q \neq P_{C_1}(I-\lambda_1 B_1)(q - \gamma_2 A_2^*((I_2 - P_{C_2}(I-\lambda_2 B_2))A_2 q + \gamma_3 A_3^*(I_3 - P_{C_3}(I-\lambda_3 B_3))A_3 A_2 q)).
$$

By using properties of Opial's condition, the definition of u_n and (3.14), we have

$$
\begin{aligned}
\liminf_{k\to+\infty} \|x_{n_k} - q\| \quad & < \quad \liminf_{k\to+\infty} \|x_{n_k} - P_{C_1}(I-\lambda_1 B_1)(q - \gamma_2 A_2^*((I_2 - P_{C_2}(I-\lambda_2 B_2))A_2 q \\
& \qquad + \gamma_3 A_3^*(I_3 - P_{C_3}(I-\lambda_3 B_3))A_3 A_2 q))\| \\
& \leq \quad \liminf_{k\to+\infty} (\|x_{n_k} - u_{n_k}\| \\
& \qquad + \|u_{n_k} - P_{C_1}(I-\lambda_1 B_1)(q - \gamma_2 A_2^*((I_2 - P_{C_2}(I-\lambda_2 B_2))A_2 q
\end{aligned}
$$

$$+\gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))A_3A_2q))\|)$$
$$\leq \liminf_{k\to+\infty} \|x_{n_k} - q\|.$$

This is a contradiction. Then $q \in \Omega$. Therefore $q \in \Omega \cap F(\varrho) \cap F(\vartheta)$.
From (3.16) and the well-known properties of metric projection, we have

$$\limsup_{n\to\infty} \langle u - z_0, x_n - z_0\rangle \leq 0.$$

From determining the definition of x_n, we can conclude that

$$\|x_{n+1} - z_0\|^2 \leq (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0\rangle,$$

where $z_0 = P_{\Omega\cap F(\varrho)\cap F(\vartheta)}u$. From Lemma 2.19, we can conclude that the sequence $\{x_n\}$ converses strongly to $z_0 = P_{\Omega\cap F(\varrho)\cap F(\vartheta)}u$. □

The following results is obtained directly from the main theorem.

Corollary 3.4. For every $i = 1, 2, 3$, let C_i be a closed convex subset of a real Hilbert space H_i. Let $B_i : C_i \to H_i$ be β_i-inverse strongly monotone mappings with $\eta = \min_{i=1,2,3}\{\beta_i\}$ and let $A_2 : H_1 \to H_2, A_3 : H_2 \to lH_3$ be bounded linear operator with the adjoint operator A_2^* and A_3^*, respectively. Assume that $\overline{x}_1 \in H_1, \overline{x}_2 = A_2\overline{x}_1, \overline{x}_3 = A_3\overline{x}_2$ and $\Omega \neq \emptyset$. Let $\vartheta, \varrho : C_1 \to C_1$ be nonspreading mappings, respectively. Assume that $\Omega \cap F(\vartheta) \cap F(\varrho) \neq \emptyset$ and let the sequence $\{x_n\}$ generated by $u, x_1 \in C_1$, and

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}(I_1 - \lambda_n (a(I_1 - \vartheta) + (1-a)(I_1 - \varrho)))x_n \\ &\quad + \delta_n P_{C_1}(I_1 - \lambda_1 B_1)(x_n^1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 \\ &\quad + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))x_n^3)), \end{aligned}$$

for all $n \geq 1$ and $a \in (0,1)$, $I_i : H_i \to \mathcal{H}_i$ is an identity mappings, for all $i = 1, 2, 3$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and $x_n = x_n^1, x_n^2 = A_2 x_n^1, x_n^3 = A_3 x_n^2$, for all $n \in \mathbb{N}, 0 < \lambda_i < 2\eta$, for all $i = 1, 2, 3$ and $\gamma_j > 0$, for all $j = 2, 3$. Suppose that the conditions $i)$ to $v)$ is true;

(i) $\lim_{n\to+\infty} \alpha_n = 0, \sum_{n=1}^{+\infty} \alpha_n = +\infty$;

(ii) $\gamma_2(1 + \gamma_3) \leq \frac{1}{L}$, where $L = \max\{L_{A_1}, L_{A_2}\} \leq 1$,
where L_{A_1}, L_{A_2} are spectral radius of $A_2A_2^*, A_3A_3^*$, respectively;

(iii) $0 < a \leq \beta_n, \gamma_n, \delta_n \leq b < 1$, for some $a, b > 0$, for all $n \in \mathbb{N}$;

(iv) $\sum_{n=1}^{+\infty} \lambda_n < +\infty$ and $0 < \lambda_n < 1 - \kappa$, for all $n \in \mathbb{N}$;

(v) $\sum_{n=1}^{+\infty} |\alpha_{n+i} - \alpha_n|, \sum_{n=1}^{+\infty} |\beta_{n+i} - \beta_n|, \sum_{n=1}^{+\infty} |\gamma_{n+1} - \gamma_n| < +\infty$.

Then the sequence $\{x_n\}$ converses strongly to $z_0 = P_{\Omega\cap F(\vartheta)\cap F(\varrho)}u$.

3.2 Coincidence point theory in metric spaces endowed with graph

Coincidence theory is a generalization of fixed point theory. There are many researchs combining fixed point theory and graph theory. In this section we introduce a new type of multi-valued mapping and g-l-graph preserving to prove a fixed point theorem on complete metric spaces endowed with a directed graph. The proof of this theorem using different techniques from the research of [44] and [38].

The following definitions are important for our main theorem.

Definition 3.1. Let X be a nonempty set and $G = (V(G), E(G))$ be a graph such that $V(G) = X$ and let $g, l : X \to X$ be mappings. The multi-valued mapping $T : X \to CB(X)$ is called *g-l-graph preserving* if for every $x, y \in X$ such that

$$(g(x), g(y)) \in E(G) \Rightarrow (l(u), l(v)) \in E(G),$$

for all $u \in Tx$ and $v \in Ty$.

By motivated from Suantai and Tiammee [44], we give an example of the such mapping.

Example 3.5. Let $\mathbb{N}$ be the set of natural number and let $G = (\mathbb{N}, E(G))$ and $E(G) = \{(2n-1, 2n+1) : n \in \mathbb{N}\} \cup \{(2n, 2n+2) : n > 1\} \cup \{(2n, 2n+4) : n > 1\} \cup \{(2n, 2n) : n > 1\} \cup (1,1) \cup (6,4) \cup (8,6)$. Defined $T : \mathbb{N} \to CB(\mathbb{N})$ by

$$T(x) = \begin{cases} \{2k, 2k+2\} & \text{if } x = 2k-1, \forall k \in \mathbb{N}, \\ \{1\} & \text{if } x = 2k, \forall k \in \mathbb{N}, \end{cases}$$

$g : \mathbb{N} \to \mathbb{N}$ be defined by

$$g(x) = \begin{cases} 2k & \text{if } x = 2k+2, \forall k \in \mathbb{N}, \\ 2k-1 & \text{if } x = 2k+1, \forall k \in \mathbb{N}, \\ 2 & \text{if } x = 1, 2, \end{cases}$$

and $l : \mathbb{N} \to \mathbb{N}$ be defined by

$$l(x) = \begin{cases} 2k+2 & \text{if } x = 2k, \forall k \in \mathbb{N}, \\ 3k-2 & \text{if } x = 2k-1, \forall k \in \mathbb{N}, \end{cases}$$

Then T is g-l-graph preserving.

Solution Let $(g(x), g(y)) \in E(G)$.

If $(g(x), g(y)) = (2k-1, 2k+1), \forall k \in \mathbb{N}$, then $(x, y) = (2k+1, 2k+3)$. From the definition of T, we have $Tx = \{2k+2, 2k+4\}, Ty = \{2k+4, 2k+6\}$. From the definition of l, we have

$$(l(2k+2), l(2k+4)) = (2k+4, 2k+6) \in E(G),$$

$$(l(2k+2), l(2k+6)) = (2k+4, 2k+8) \in E(G),$$

$$(l(2k+4), l(2k+4)) = (2k+6, 2k+6) \in E(G),$$

$$(l(2k+4), l(2k+6)) = (2k+6, 2k+8) \in E(G).$$

If $(g(x),g(y))=(2k,2k+2)$ or $(2k,2k+4)$ or $(2k,2k)$ $,\forall k\in\mathbb{N}$, then $Tx=\{1\},Ty=\{1\}$. From the definition of l, we have $(l(1),l(1))=(1,1)\in E(G)$.

If $(g(x),g(y))=(1,1)$, then $(x,y)=(3,3)$. It follows that $Tx=Ty=\{4,6\}$. From the definition of l, we have

$$(l(4),l(4))=(6,6)\in E(G),$$
$$(l(4),l(6))=(6,8)\in E(G),$$
$$(l(6),l(4))=(8,6)\in E(G),$$
$$(l(6),l(6))=(8,8)\in E(G).$$

If $(g(x),g(y))=(8,6)$ or $(6,4)$, then $Tx=Ty=\{1\}$. From the definition of l, we have $(l(1),l(1))=(1,1)\in E(G)$. Hence T is g-l-graph preserving.

Remark 3.6. If $l=I$, where I is an identity mapping, then g-l-graph preserving is reduced to g-graph preserving, see [44].

Definition 3.2. Let (X,d) be a metric spaces, $G=(V(G),E(G))$ be a direct graph such that $V(G)=X$ and the mappings $g,l:X\to X$. Then $T:X\to CB(X)$ is said to be *(l,g)-G contraction* if there exists $0<\alpha<\beta$ with $\alpha+\beta<\frac{1}{2}$ and $L\geq 0$ with

$$H(Tx,Ty)\leq \alpha d(g(x),g(y))+\beta d(l(u),l(v))+LD(g(y),Tx),$$

for all $x,y\in X$ and $u\in Tx,v\in Ty$ such that $(g(x),g(y))\in E(G)$ and $(l(u),l(v))\in E(G)$.

Example 3.7. Let $\mathbb{N}$ be the set of natural number and let $G=(\mathbb{N},E(G))$ be a directed graph where $E(G)=\{(2n,2n+2):n\in\mathbb{N}\}\cup\{(2n-1,2n+1):n\in\mathbb{N}\}\cup\{(2n-1,2n+3):n\in\mathbb{N}\}\cup$ $\{(2n-1,2n-1):n\in\mathbb{N}\}$. Defined $T:\mathbb{N}\to CB(\mathbb{N})$ by

$$T(x)=\begin{cases}\{2k-1,2k+1\} & \text{if } x=2k,\forall k\in\mathbb{N},\\ \{3\} & \text{if } x=2k-1,\forall k\in\mathbb{N},\end{cases}$$

$g:\mathbb{N}\to\mathbb{N}$ be defined by

$$g(x)=\begin{cases}2k+1 & \text{if } x=2k-1,\forall k\in\mathbb{N},\\ 2k+2 & \text{if } x=2k,\forall k\in\mathbb{N},\end{cases}$$

and $l:\mathbb{N}\to\mathbb{N}$ be defined by

$$l(x)=\begin{cases}2k+4 & \text{if } x=2k+2,\forall k\in\mathbb{N},\\ 2k+3 & \text{if } x=2k+1,\forall k\in\mathbb{N},\\ 2 & \text{if } x=1,2.\end{cases}$$

Then T is (l,g)-G contraction.

Solution Let $(g(x),g(y))\in E(G)$.

If $(g(x),g(y))=(2n,2n+2)\in E(G)$ for $n\in\mathbb{N}$, then $x=2(n-1),y=2n$ and $Tx=\{2n-3,2n-1\},Ty=\{2n-1,2n+1\}$ for $n\in\mathbb{N}$. Then $H(Tx,Ty)=2$ and

$D\left(g\left(y\right),Tx\right)=3.$

Let $u\in Tx$.

If $u=2n-3$,then $l(u)=2n-1$ for $n\in\mathbb{N}$,

If $u=2n-1$,then $l(u)=2n+1$ for $n\in\mathbb{N}$.

Let $v\in Ty$.

If $v=2n-1$,then $l(v)=2n+1$ for $n\in\mathbb{N}$,

If $v=2n+1$,then $l(v)=2n+3$ for $n\in\mathbb{N}$.

It implies that $\left(l\left(u\right),l\left(v\right)\right)=(2n-1,2n+1)$ or $(2n-1,2n+3)$ or $(2n+1,2n+1)$ or $(2n+1,2n+3)\in E\left(G\right)$.

Put $\alpha=\dfrac{1}{8},\beta=\dfrac{2}{8},L=2$, we have

$$H\left(Tx,Ty\right)<\alpha d\left(g\left(x\right),g\left(y\right)\right)+\beta d\left(l\left(u\right),l\left(v\right)\right)+LD\left(g\left(y\right),Tx\right),$$

If $\left(g\left(x\right),g\left(y\right)\right)=(2n-1,2n+1)$ or $(2n-1,2n+3)$ or $(2n-1,2n-1)\in E\left(G\right)$, we get $Tx=\{3\}=Ty$. If $u=v=3$, then $l(3)=5$. Therefore $(l(3),l(3))\in E(G)$. Then $H\left(Tx,Ty\right)=0$. It is obvious that,

$$H\left(Tx,Ty\right)\leq\alpha d\left(g\left(x\right),g\left(y\right)\right)+\beta d\left(l\left(u\right),l\left(v\right)\right)+LD\left(g\left(y\right),Tx\right),$$

where $x,y=2n-1$ for all $n\in\mathbb{N}$.

Hence T is (l, g)-G contraction.

Theorem 3.8. Let (X,d) be a complete metric space and $G=(V(G),E(G))$ be a directed graph with $V(G)=X$. Let $g:X\to X$ be a surjective mapping and let $l:X\to X$ be a nonexpansive mapping. Suppose that the multi-value mapping $T:X\to CB\left(X\right)$ is satisfied the following properties:

1) T is a g-l-graph preserving;

2) T is (l,g)-G contraction;

3) X has Property A;

4) there exists $x_0\in X$ such that $\left(g\left(x_0\right),y\right)\in E\left(G\right)$ for some $y\in Tx_0$.

5) $H\left(Tx,Ty\right)\leq d\left(g\left(x\right),g\left(y\right)\right)+d\left(l\left(g\left(x\right)\right),l\left(g\left(y\right)\right)\right)+D\left(g\left(y\right),Tx\right)$ for all $x,y\in X$ such that $\left(g\left(x\right),g\left(y\right)\right)\in E\left(G\right)$ and $\left(l\left(g\left(x\right)\right),l\left(g\left(y\right)\right)\right)\in E\left(G\right)$.

Then there exists $u\in X$ such that $g(u)\in Tu$.

Proof. Since g is a surjection, there exists $x_1\in X$ such that $g(x_1)\in Tx_0$. From 4), we have

$$\left(g\left(x_0\right),g\left(x_1\right)\right)\in E\left(G\right). \tag{3.17}$$

Put $n_1\in\mathbb{N}$, we have

$$\left(\alpha+\beta\right)^{n_1}<\left(\beta-\alpha\right)d\left(g\left(x_0\right),g\left(x_1\right)\right). \tag{3.18}$$

By Lemma 2.30, there is $g(x_2)\in Tx_1$ such that

$$d\left(g\left(x_1\right)\right)g\left(x_2\right))\leq H\left(Tx_0,Tx_1\right)+\left(\alpha+\beta\right)^{n_1}. \tag{3.19}$$

Since $(g(x_0), g(x_1) \in E(G), g(x_1) \in Tx_0, g(x_2) \in Tx_1$ and T is g-l-graph preserving, we have $(l(g(x_1)), l(g(x_2))) \in E(G)$.
From (3.18) and (3.19), we have

$$\begin{aligned} d(g(x_1)) g(x_2)) &\leq H(Tx_0, Tx_1) + (\alpha+\beta)^{n_1} \\ &\leq \alpha d(g(x_0), g(x_1)) + \beta d(l(g(x_1)), l(g(x_2))) \\ &\quad + LD(g(x_1), Tx_0) + (\alpha+\beta)^{n_1} \\ &\leq \alpha d(g(x_0), g(x_1)) + \beta d(g(x_1), g(x_2)) + (\beta-\alpha) d(g(x_0), g(x_1)). \end{aligned}$$

It implies that

$$d(g(x_1), g(x_2)) \leq \frac{\beta}{1-\beta} d(g(x_0), g(x_1)).$$

By using the same method as (3.17), we have

$$(g(x_1), g(x_2)) \in E(G).$$

Choose $n_2 > n_1$ such that

$$(\alpha+\beta)^{n_2} < (\beta-\alpha) d(g(x_1), g(x_2)). \tag{3.20}$$

From Lemma 2.30, there exists $g(x_3) \in Tx_2$ such that

$$d(g(x_2), g(x_3)) \leq H(Tx_1, Tx_2) + (\alpha+\beta)^{n_2} \tag{3.21}$$

Since $(g(x_1), g(x_2)) \in E(G)$, $g(x_2) \in Tx_1, g(x_3) \in Tx_2$ and T is g-l-graph preserving, we have

$$(l(g(x_2)), l(g(x_3))) \in E(G).$$

From (3.20) and (3.21), we have

$$\begin{aligned} d(g(x_2), g(x_3)) &\leq H(Tx_1, Tx_2) + (\alpha+\beta)^{n_2} \\ &\leq \alpha d(g(x_1), g(x_2)) + \beta d(l(g(x_2)), l(g(x_3))) + LD(g(x_2), Tx_1) \\ &\quad + (\beta-\alpha) d(g(x_1)), g(x_2)) \\ &\leq \alpha d(g(x_1), g(x_2)) + \beta d(g(x_2), g(x_3)) + (\beta-\alpha) d(g(x_1)), g(x_2)). \end{aligned}$$

It implies that

$$d(g(x_2), g(x_3)) \leq \frac{\beta}{1-\beta} d(g(x_1), g(x_2)).$$

Continuous on this way, for every $k \in \mathbb{N}$, we have $g(x_{k+1}) \in Tx_k$ with

$$d(g(x_k), g(x_{k+1})) \leq \frac{\beta}{1-\beta} d(g(x_{k-1}), g(x_k)) \tag{3.22}$$

and $(g(x_{k-1}), g(x_k)) \in E(G)$, $(l(g(x_k)), l(g(x_{k+1}))) \in E(G)$.

From (3.22), we have

$$
\begin{aligned}
d\left(g\left(x_{k}\right), g\left(x_{k+1}\right)\right) &\leq \frac{\beta}{1-\beta} d\left(g\left(x_{k-1}\right), g\left(x_{k}\right)\right) \\
&\leq a\left(a d\left(g\left(x_{k-2}\right), g\left(x_{k-1}\right)\right)\right) \\
&= a^{2} d\left(g\left(x_{k-2}\right), g\left(x_{k-1}\right)\right) \\
&\vdots \\
&\leq a^{k} d\left(g\left(x_{0}\right), g\left(x_{1}\right)\right), \qquad (3.23)
\end{aligned}
$$

where $a=\dfrac{\beta}{1-\beta}$ for all $k \in \mathbb{N}$.
For every $n, k \in \mathbb{N}$ and (3.23), we have

$$
\begin{aligned}
d\left(g\left(x_{n+k}\right), g\left(x_{n}\right)\right) &\leq \sum_{j=n}^{n+k-1} d\left(g\left(x_{j+1}\right), g\left(x_{j}\right)\right) \\
&\leq \sum_{j=n}^{n+k-1} a^{j} d\left(g\left(x_{0}\right), g\left(x_{1}\right)\right) \\
&\leq \frac{a^{n}}{1-a} d\left(g\left(x_{0}\right), g\left(x_{1}\right)\right). \qquad (3.24)
\end{aligned}
$$

Since $\lim_{n \to \infty} a^{n}=0$ and (3.24), we can conclude that the sequence $\{g(x_n)\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $u \in X$ such that

$$\lim_{n \to \infty} g\left(x_{n}\right)=g(u). \qquad (3.25)$$

From the property A, there is a subsequence $g(x_{k_n})$ of $g(x_n)$ such that

$$\left(g\left(x_{k_{n}}\right), g(u)\right) \in E(G). \qquad (3.26)$$

Since l is a nonexpansive mapping and (3.25), we have

$$\lim_{n \to \infty} l(g\left(x_{n}\right))=l(g(u)). \qquad (3.27)$$

From the property A, without loss of generality, there exists a subsequence $l\left(g\left(x_{k_{n}}\right)\right)$ of $l\left(g\left(x_{n}\right)\right)$ such that

$$(l(g\left(x_{k_{n}}\right)), l(g(u))) \in E(G). \qquad (3.28)$$

From the condition 5), we get

$$
\begin{aligned}
D(g(u), T u) &\leq d\left(g(u), g\left(x_{k_{n}+1}\right)\right)+D\left(g\left(x_{k_{n}+1}\right), T u\right) \\
&\leq d\left(g(u), g\left(x_{k_{n}+1}\right)\right)+H\left(T x_{k_{n}}, T u\right) \\
&\leq d\left(g(u), g\left(x_{k_{n}+1}\right)\right)+d\left(g\left(x_{k_{n}}\right), g(u)\right)+d\left(l\left(g\left(x_{k_{n}}\right)\right), l(g(u))\right)+D\left(g(u), T x_{k_{n}}\right) \\
&\leq d\left(g(u), g\left(x_{k_{n}+1}\right)\right)+d\left(g\left(x_{k_{n}}\right), g(u)\right)+d\left(g\left(x_{k_{n}}\right), g(u)\right)+d\left(g(u), g(x_{k_{n}+1})\right).
\end{aligned}
$$

Since $\lim_{n \to \infty} g\left(x_{n}\right)=g(u)$, we have $D(g(u), Tu)=0$. Since Tu is a closed set, we have $g(u) \in Tu$. This complete the proof. □

Corollary 3.9. Let (X, d) be a metric space endowed with a partial order $\prec$. Let $g : X \to X$ be a surjective mapping, $l : X \to X$ be a nonexpansive mapping and let $T : X \to CB(X)$ be a multi-valued mapping. Suppose the following conditions hold:

1) T is an l,g-increasing,

2) There exists $x_0 \in X$ and $u \in Tx_0$ such that $g(x_0) \prec u$,

3) For every sequence $\{x_k\}$ such $g(x_k) \prec g(x_{k+1})$ or all $k \in \mathbb{N}$ and $g(x_k)$ converse to $g(x)$ for some $x \in X$ such that $g(x_k) \prec g(x)$,

4) For every sequence $\{x_k\}$ such $l(g(x_k)) \prec l(g(x_{k+1}))$ for all $k \in \mathbb{N}$ and $l(g(x_k))$ converse to $l(g(x))$ for some $x \in X$ such that $l(g(x_k)) \prec l(g(x))$,

5) There exists $0 < \alpha < \beta$ with $\alpha + \beta < \frac{1}{2}$ and $L > 0$ such that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta d(l(u), l(v)) + LD(g(y), Tx),$$

for all $x, y \in X$ and $u \in Tx, u \in Ty$ with $g(x) \prec g(y)$ and $l(u) \prec l(v)$
and

$$H(Tx, Ty) \leq d(g(x), g(y)) + d(l(g(x)), l(g(y))) + D(g(y), Tx),$$

for all $x, y \in X$ with $g(x) \prec g(y)$ and $l(g(x)) \prec l(g(y))$,

6) The metric d is complete.

Then there exists $u \in X$ such that $u \in Tu$.

Proof. Let $G = (V(G), E(G)$ where $X = V(G)$ and $E(G) = \{(x, y) : x \prec y, \forall x, y \in X\}$. First, we show that T is g-l-graph preserving. Let $(g(x), g(y)) \in E(G)$, we have $Tx \prec Ty$. For every $u \in Tx$ and $v \in Ty$, we have $l(u) \prec l(v)$. Then $(l(u), l(v)) \in E(G)$. Then T is g-l-graph preserving. It is easy to see that the conditions 2)-5) satisfying conditions 2)-4) in theorem 3.8. From theorem 3.8, we can concluded the desired result. □

Chapter 4

Applications and Some Examples

We have applied the problem (3.1) for the various fixed point problems in three Hilbert spaces as follows:

For every $i = 1,2,3$ let H_i be a real Hilbert space and C_i be a nonempty closed convex subset of H_i. Let $\vartheta_i : C_i \to C_i$ be a mapping, for all $i = 1,2,3$ and let $A_2 : H_1 \to H_2$ and $A_3 : H_2 \to H_3$. The fixed points problem in three Hilbert spaces is to find the point;

$$\begin{cases} \varpi_1^* \in C_1, \text{ such that } \varpi_1^* \in F(\vartheta_1) \text{ and} \\ \varpi_2^* = A_2\varpi_1^* \in C_2, \text{ such that } \varpi_2^* \in F(\vartheta_2) \text{ and} \\ \varpi_3^* = A_3\varpi_2^* \in C_3, \text{ such that } \varpi_3^* \in F(\vartheta_3). \end{cases} \tag{4.1}$$

The set of the solution of (4.1) is denoted by $\underline{\Omega} = \{\varpi^* = (\varpi_1^*, \varpi_2^*, \varpi_3^*) \in C_1 \times C_2 \times C_3 : \varpi_i^* \in F(\vartheta_i)$, for all $i = 1,2,3\}$. It is clear that $VIP(C, I - T) = F(\vartheta)$, where $\vartheta : C \to C$ is a nonexpansive mapping with $F(\vartheta) \neq \emptyset$. By leveraging Lemma 3.1 and such knowledge, we have the following results;

Lemma 4.1. For every $i = 1, 2, 3$, let H_i be a real Hilbert spaces and C_i be a nonempty closed convex subset of H_i. Let $\vartheta_i : C_i \to C_i$ be nonexpansive mappings and let $A_2 : H_1 \to H_2, A_3 : H_2 \to H_3$ be bounded linear operator with the adjoint operator A_2^* and A_3^*, respectively. Assume that $\overline{x}_1 \in C_1, A_2\overline{x}_1 = \overline{x}_2, A_3\overline{x}_2 = \overline{x}_3$ and $\underline{\Omega} \neq \emptyset$. The following are equivalent:

(i) $\overline{x} \in \underline{\Omega}$, where $\overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) \in C_1 \times C_2 \times C_3$.

(ii) $\overline{x}_1 = P_{C_1}(I_1 - \lambda_1(I_1 - \vartheta_1))(\overline{x}_1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I_2 - \lambda_2(I_2 - \vartheta_2)))\overline{x}_2$
$+\gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3(I_3 - \vartheta_3)))\overline{x}_3))$,

where $I_i : H_i \to H_i$ is an identity mappings, for all $i = 1,2,3$, $\gamma_2(1+\gamma_3) \leq \frac{1}{L}$, $L = \max\{L_1, L_2\} \leq 1$ which L_1, L_2 are spectral radius of $A_2A_2^*$ and $A_3A_3^*$, respectively, $\lambda_i \in (0,1)$, for all $i = 1,2,3$ and $\gamma_2, \gamma_3 \geq 0$

Proof. Since $F(\vartheta_i) = VIP(C, I_i - \vartheta_i)$ for all $i = 1,2,3$ and $(I_i - \vartheta_i)$ is $\frac{1}{2}$-inverse strongly monotone and Lemma 3.1, we can summarize the result of Lemma 4.1. □

The direct benefits of Lemma 4.1, we get Corollary 4.2;

Corollary 4.2. For every $i = 1, 2, 3$, let C_i be a closed convex subset of a real Hilbert space H_i. Let $\vartheta_i : C_i \to C_i$ be a nonexpansive mappings and let $A_2 : H_1 \to H_2, A_3 : H_2 \to H_3$ be bounded linear operator with the adjoint operator A_2^* and A_3^*, respectively.

Assume that $\overline{x}_1 \in H_1, \overline{x}_2 = A_2\overline{x}_1, \overline{x}_3 = A_3\overline{x}_2$. Let $\vartheta, \varrho : C_1 \to C_1$ be nonspreading and κ-strictly pseudo-nonspreading mappings, respectively. Assume that $\Omega \cap F(\vartheta) \cap F(\varrho) \neq \emptyset$ and let the sequence $\{x_n\}$ generated by $u, x_1 \in C_1$, and

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}(I_1 - \lambda_n (a(I_1 - \vartheta) + (1-a)(I_1 - \varrho)))x_n \\ &\quad + \delta_n P_{C_1}(I_1 - \lambda_1(I_1 - \vartheta_1))(x_n^1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2(I_2 - \vartheta_2)))x_n^2 \\ &\quad + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3(I_3 - \vartheta_3)))x_n^3)), \end{aligned}$$

for all $n \geq 1$ and $a \in (0,1)$, $I_i : H_i \to H_i$ is an identity mappings, for all $i = 1,2,3$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and $x_n = x_n^1, x_n^2 = A_2 x_n^1, x_n^3 = A_3 x_n^2$, for all $n \in \mathbb{N}, 0 < \lambda_i < 1$, for all $i = 1,2,3$ and $\gamma_j > 0$, for all $j = 2,3$. Suppose that the conditions $i)$ to $v)$ is true;

(i) $\lim_{n\to+\infty} \alpha_n = 0, \sum_{n=1}^{+\infty} \alpha_n = +\infty$;

(ii) $\gamma_2(1+\gamma_3) \leq \frac{1}{L}$, where $L = \max\{L_{A_1}, L_{A_2}\} \leq 1$, where L_{A_1}, L_{A_2} are spectral radius of $A_2A_2^*, A_3A_3^*$, respectively;

(iii) $0 < a \leq \beta_n, \gamma_n, \delta_n \leq b < 1$, for some $a, b > 0$, for all $n \in \mathbb{N}$;

(iv) $\sum_{n=1}^{+\infty} \lambda_n < +\infty$ and $0 < \lambda_n < 1 - \kappa$, for all $n \in \mathbb{N}$;

(v) $\sum_{n=1}^{+\infty} |\alpha_{n+i} - \alpha_n|, \sum_{n=1}^{+\infty} |\beta_{n+i} - \beta_n|, \sum_{n=1}^{+\infty} |\gamma_{n+1} - \gamma_n| < +\infty$.

Then the sequence $\{x_n\}$ converses strongly to $z_0 = P_{\Omega \cap F(T) \cap F(S)} u$.

To support theorem 3.8, we give the following example.

Example 4.3. Let $X = \{3,4,5,6,7,8,\}$ and $d(x,y) = |x-y|$ for all $x, y \in X$, where $E(G) = \{(4,6),(6,8)\} \cup \{(3,5),(5,7),(3,7)\} \cup \{(3,3),(5,5),(6,6),(7,7)\} \cup \{(6,7),(7,6),(5,3),(7,3),(7,5)\}$. Defined the multi-valued mapping $T : X \to CB(X)$ by

$$T(x) = \begin{cases} \{3,5\} & \text{if } x = 4, \\ \{5,7\} & \text{if } x = 6, \\ \{5\} & \text{if } x = 3,5,7,8, \end{cases}$$

the mapping $g : X \to X$ be defined by

$$g(x) = \begin{cases} 3 & \text{if } x = 7, \\ 5 & \text{if } x = 3, \\ 7 & \text{if } x = 5, \\ 4 & \text{if } x = 8, \\ 6 & \text{if } x = 4, \\ 8 & \text{if } x = 6, \end{cases}$$

and the mapping $l : X \to X$ be defined by

$$l(x) = \begin{cases} 6 & \text{if } x = 3,4,6,7,8, \\ 7 & \text{if } x = 5. \end{cases}$$

Then there exists $3 \in X$ such that $g(3) \in T3$.

Solution It is easy to see that l is a nonexpansive mapping and g is surjective mapping.

First, we show that T is g-l graph preserving. Let $(g(x), g(y)) \in E(G)$.

if $(4,6),(6,8) \in E(G)$, then $x = 8,4$ and $y = 4,6$. From the definition of T, we have $T8 = \{5\}, T4 = \{3,5\}$ and $T6 = \{5,7\}$.

From $T8, T4$ and the definition of l, we have

$(l(5), l(3)) = (7,6) \in E(G)$,

$(l(5), l(5)) = (7,7) \in E(G)$.

From $T4, T6$ and the definition of l, we have

$(l(3), l(5)) = (6,7) \in E(G)$,

$(l(3), l(7)) = (6,6) \in E(G)$,

$(l(5), l(5)) = (7,7) \in E(G)$,

$(l(5), l(7)) = (7,6) \in E(G)$.

If $(g(x), g(y)) = (3,5),(5,7),(3,7),(3,3),(5,5),(7,7),(5,3),(7,3),(7,5)$, then $x, y = ,3,5,7$. From the definition of T, we have $Tx = Ty = \{5\}$ where $x, y = 3,5,7$. It follows that $(l(5), l(5)) = (7,7) \in E(G)$.

If $(g(x), g(y)) = (6,7)$, then $x = 4$ and $y = 5$. From the definition of T, we have $T4 = \{3,5\}$ and $T5 = \{5\}$.

From $T4, T5$ and the definition of l, we have

$(l(3), l(5)) = (6,7) \in E(G)$,

$(l(5), l(5)) = (7,7) \in E(G)$.

If $(g(x), g(y)) = (7,6)$, then $x = 5$ and $y = 4$. From the definition of T, we have $T5 = \{5\}$ and $T4 = \{3,5\}$.

From $T5, T4$ and the definition of l, we have

$(l(5), l(3)) = (7,6) \in E(G)$,

$(l(5), l(5)) = (7,7) \in E(G)$.

If $(g(x), g(y)) = (6,6)$, then $x = y = 4$. From the definition of T, we have $T4 = \{3,5\}$.

From $T4$ and the definition of l, we have

$(l(3), l(3)) = (6,6) \in E(G)$,

$(l(3), l(5)) = (6,7) \in E(G)$.

$(l(5), l(3)) = (7,6) \in E(G)$,

$(l(5), l(5)) = (7,7) \in E(G)$.

Hence T is g-l graph preserving.

Next, we show that T is (l,g)-G contractive mapping. Let $(g(x), g(y)) \in E(G)$.
Put $\alpha = \frac{1}{8}, \beta = \frac{2}{8}$ and $L = 3$.
If $(4,6) \in E(G)$, then $x = 8$ and $y = 4$. $T8 = \{5\}$ and $T4 = \{3,5\}$.
It follows that

$$H(T8, T4) = 2,$$

and

$$D(g(4), T8) = 1.$$

It is easy to conclude that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta d(l(u), l(v)) + LD(g(y), Tx),$$

where $x = 8$ and $y = 4$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.
If $(6,8) \in E(G)$, then $x = 4$ and $y = 6$. Then $T4 = \{3,5\}$ and $T6 = \{5,7\}$.
It follows that $H(T4, T6) = 2$ and $D(g(6), T4) = 3$.
It is easy to conclude that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta d(l(u), l(v)) + LD(g(y), Tx),$$

where $x = 4$ and $y = 6$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.
If $(3,5), (5,7), (3,7), (3,3), (5,5), (7,7), (5,3), (7,3), (7,5) \in E(G)$. Then $Tx = Ty = \{5\}$ where $x, y = 3, 5, 7$. It follows that $H(Tx, Ty) = 0$. It is easy to conclude that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta d(l(u), l(v)) + LD(g(y), Tx),$$

where $x, y = 3, 5, 7$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.
If $(6,7) \in E(G)$, then $x = 4$ and $y = 5$. It follows that $T4 = \{3,5\}$ and $T5 = \{5\}$.
It implies that $H(T4, T5) = 2$ and $D(g(5), T4) = 2$. It is easy to conclude that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta d(l(u), l(v)) + LD(g(y), Tx),$$

where $x = 4$ and $y = 5$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.
If $(7,6) \in E(G)$, then $x = 5$ and $y = 4$. It follows that $T5 = \{5\}$ and $T4 = \{3,5\}$.
It follows that $H(T5, T4) = 2$ and $D(g(4), T5) = 1$. It is easy to conclude that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta d(l(u), l(v)) + LD(g(y), Tx),$$

where $x = 5$ and $y = 4$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.

If $(6,6) \in E(G)$, then $x = y = 4$. It follows that $T4 = \{3, 5\}$.
It follows that $H(T4, T4) = 0$ and $D(g(4), T4) = 1$. It is easy to conclude that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta d(l(u), l(v)) + LD(g(y), Tx),$$

where $x = y = 4$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.
Hence T is (l, g)-G contraction.
Let $(g(x), g(y)) \in E(G)$.
If $(4,6) \in E(G)$, then $x = 8$ and $y = 4$. It follows that $H(T8, T4) = 2$, $D(g(4), T8) = 1$.
It is easy to see that

$$H(Tx, Ty) \leq d(g(x), g(y)) + d(l(g(x)), l(g(y))) + D(g(y), Tx),$$

where $x = 8$ and $y = 4$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.
If $(6,8) \in E(G)$, then $x = 4$ and $y = 6$. It follows that $H(T4, T6) = 2$ and $D(g(6), T4) = 3$.
It is easy to see that

$$H(Tx, Ty) \leq d(g(x), g(y)) + d(l(g(x)), l(g(y))) + D(g(y), Tx),$$

where $x = 4$ and $y = 6$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.
If $(3,5), (5,7), (3,7), (3,3), (5,5), (7,7), (5,3), (7,3), (7,5) \in E(G)$. Then $Tx = Ty = \{5\}$ for all $x, y = 3, 7, 5$. It follows that

$$H(Tx, Ty) \leq d(g(x), g(y)) + d(l(g(x)), l(g(y))) + D(g(y), Tx),$$

where $x, y = 3, 5, 7$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.
If $(6,7) \in E(G)$, then $x = 4$ and $y = 5$. Then $H(T4, T5) = 2$ and $D(g(5), T4) = 2$.
It is easy to see that

$$H(Tx, Ty) \leq d(g(x), g(y)) + d(l(g(x)), l(g(y))) + D(g(y), Tx),$$

where $x = 4$ and $y = 5$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.
If $(7,6) \in E(G)$, then $x = 5$ and $y = 4$. It follow that $H(T5, T4) = 2$ and $D(g(4), T5) = 1$.
It is easy to see that

$$H(Tx, Ty) \leq d(g(x), g(y)) + d(l(g(x)), l(g(y))) + D(g(y), Tx),$$

where $x = 5$ and $y = 4$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.
If $(6,6) \in E(G)$, then $x = y = 4$. It follow that $H(T4, T4) = 0$ and $D(g(4), T4) = 1$.
It is easy to see that

$$H(Tx, Ty) \leq d(g(x), g(y)) + d(l(g(x)), l(g(y))) + D(g(y), Tx),$$

where $x = y = 4$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.
Hence the condition 5) in Theorem 3.8 is satisfied.
Since $5 \in T7$, then $(g(7), 5) = (3, 5) \in E(G)$. Then condition 4) in Theorem 3.8 is satisfied.
It is easy to see that X has Property A. From Theorem 3.8, then there exists $3 \in X$ such that $g(3) \in T3$.

Chapter 5

Conclusions

In this chapter, we conclude all main theorems and corollaries obtained in this thesis.

The first problem, we have a new split variational inequality in three Hilbert spaces. The convergence theorem to find a common element of the set of solutions of these problems and the sets of fixed points of discontinuous mappings are proved as follows;

5.1 The split various variational inequality theorem

(1) For every $i = 1, 2, 3$, let C_i be a closed convex subset of a real Hilbert space H_i. Let $B_i : C_i \to H_i$ be β_i-inverse strongly monotone mappings with $\eta = \min_{i=1,2,3}\{\beta_i\}$ and let $A_2 : H_1 \to H_2, A_3 : H_2 \to H_3$ be bounded linear operator with the adjoint operator A_2^* and A_3^*, respectively. Assume that $\overline{x}_1 \in H_1, \overline{x}_2 = A_2\overline{x}_1, \overline{x}_3 = A_3\overline{x}_2$ and $\Omega \neq \emptyset$. Let $\vartheta, \varrho : C_1 \to C_1$ be nonspreading and κ-strictly pseudo-nonspreading mappings, respectively. Assume that $\Omega \cap F(\vartheta) \cap F(\varrho) \neq \emptyset$ and let the sequence $\{x_n\}$ generated by $u, x_1 \in C_1$, and

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}\left(I_1 - \lambda_n\left(a\left(I_1 - \vartheta\right) + (1-a)\left(I_1 - \varrho\right)\right)\right) x_n \\ &\quad + \delta_n P_{C_1}(I_1 - \lambda_1 B_1)(x_n^1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 \\ &\quad + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))x_n^3)), \end{aligned}$$

for all $n \geq 1$ and $a \in (0, 1)$, $I_i : H_i \to H_i$ is an identity mappings, for all $i = 1, 2, 3$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and $x_n = x_n^1, x_n^2 = A_2 x_n^1, x_n^3 = A_3 x_n^2$, for all $n \in \mathbb{N}, 0 < \lambda_i < 2\eta$, for all $i = 1, 2, 3$ and $\gamma_j > 0$, for all $j = 2, 3$. Suppose that the conditions i) to v) is true;

(i) $\lim_{n+\to\infty} \alpha_n = 0, \sum_{n=1}^{+\infty} \alpha_n = +\infty$;

(ii) $\gamma_2(1 + \gamma_3) \leq \frac{1}{L}$, where $L = \max\{L_{A_1}, L_{A_2}\} \leq 1$ where $L_{A_1} L_{A_2}$ are spectral radius of $A_2A_2^*, A_3A_3^*$, respectively;

(iii) $0 < a \leq \beta_n, \gamma_n, \delta_n \leq b < 1,$ for some $a, b > 0,$ for all $n \in \mathbb{N}$;

(iv) $\sum_{n=1}^{+\infty} \lambda_n < +\infty$ and $0 < \lambda_n < 1 - \kappa,$ for all $n \in \mathbb{N}$;

(v) $\sum_{n=1}^{+\infty} |\alpha_{n+i} - \alpha_n|, \sum_{n=1}^{+\infty} |\beta_{n+i} - \beta_n|, \sum_{n=1}^{+\infty} |\gamma_{n+1} - \gamma_n| < +\infty.$

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\Omega \cap F(\vartheta) \cap F(\varrho)} u$.

(2) For every $i = 1,2,3$, let C_i be a closed convex subset of a real Hilbert space H_i. Let $B_i : C_i \to H_i$ be β_i-inverse strongly monotone mappings with $\eta = \min_{i=1,2,3}\{\beta_i\}$ and let $A_2 : H_1 \to H_2, A_3 : H_2 \to H_3$ be bounded linear operator with the adjoint operator A_2^* and A_3^*, respectively. Assume that $\bar{x}_1 \in H_1, \bar{x}_2 = A_2\bar{x}_1, \bar{x}_3 = A_3\bar{x}_2$ and $\Omega \neq \emptyset$. Let $\vartheta, \varrho : C_1 \to C_1$ be nonspreading mappings, respectively. Assume that $\Omega \cap F(\vartheta) \cap F(\varrho) \neq \emptyset$ and let the sequence $\{x_n\}$ generated by $u, x_1 \in C_1$, and

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}\left(I_1 - \lambda_n\left(a\left(I_1 - \vartheta\right) + (1-a)\left(I_1 - \varrho\right)\right)\right)x_n \\ &\quad + \delta_n P_{C_1}(I_1 - \lambda_1 B_1)(x_n^1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 \\ &\quad + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))x_n^3)), \end{aligned}$$

for all $n \geq 1$ and $a \in (0,1)$, $I_i : H_i \to H_i$ is an identity mappings, for all $i = 1,2,3$, where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\} \subseteq [0,1]$ and $\alpha_n+\beta_n+\gamma_n+\delta_n = 1$ and $x_n = x_n^1, x_n^2 = A_2x_n^1, x_n^3 = A_3x_n^2$, for all $n \in \mathbb{N}, 0 < \lambda_i < 2\eta$, for all $i = 1,2,3$ and $\gamma_j > 0$, for all $j = 2,3$. Suppose that the conditions $i)$ to $v)$ is true;

(i) $\lim_{n\to+\infty} \alpha_n = 0, \sum_{n=1}^{+\infty} \alpha_n = +\infty$;

(ii) $\gamma_2(1+\gamma_3) \leq \dfrac{1}{L}$, where $L = \max\{L_{A_1}, L_{A_2}\} \leq 1$,
where L_{A_1}, L_{A_2} are spectral radius of $A_2A_2^*, A_3A_3^*$, respectively;

(iii) $0 < a \leq \beta_n, \gamma_n, \delta_n \leq b < 1$, for some $a, b > 0$, for all $n \in \mathbb{N}$;

(iv) $\sum_{n=1}^{+\infty} \lambda_n < +\infty$ and $0 < \lambda_n < 1 - \kappa$, for all $n \in \mathbb{N}$;

(v) $\sum_{n=1}^{+\infty} |\alpha_{n+i} - \alpha_n|, \sum_{n=1}^{+\infty} |\beta_{n+i} - \beta_n|, \sum_{n=1}^{+\infty} |\gamma_{n+1} - \gamma_n| < +\infty$.

Then the sequence $\{x_n\}$ converses strongly to $z_0 = P_{\Omega\cap F(\vartheta)\cap F(\varrho)}u$.

The second problem, we have created a new type of multi-valued mapping and g-l-graph preserving to prove a fixed point theorem on complete metric spaces endowed with a directed graph and give an example to support our main theorems. The proof of our main theorem using different techniques from the research of [44] and [38].

5.2 Coincidence point theory in metric spaces endowed with graph

(1) Let (X,d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph with $V(G) = X$. Let $g : X \to X$ be a surjective mapping and let $l : X \to X$ be a nonexpansive mapping. Suppose that the multi-value mapping $T : X \to CB(X)$ is satified the following properties:

1) T is a g-l-graph preserving;

2) T is (l,g)-G contraction;

3) X has Property A;

4) there exists $x_0 \in X$ such that $(g(x_0), y) \in E(G)$ for some $y \in Tx_0$.

5) $H(Tx, Ty) \leq d(g(x), g(y)) + d(l(g(x)), l(g(y))) + D(g(y), Tx)$ for all $x, y \in X$ such that $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.

Then there exists $u \in X$ such that $g(u) \in Tu$.

(2) Let (X, d) be a metric space endowed with a partial order $\leq$. Let $g : X \to X$ be a surjetive mapping, $l : X \to X$ be a nonexpansive mapping and let $T : X \to CB(X)$ be a multi-valued mapping. Suppose the following conditions hold:

1) T is an l,g-increasing,

2) There exists $x_0 \in X$ and $u \in Tx_0$ such that $g(x_0) < u$,

3) For every sequence $\{x_k\}$ such $g(x_k) < g(x_{k+1})$ or all $k \in \mathbb{N}$ and $g(x_k)$ converse to $g(x)$ for some $x \in X$ such that $g(x_k) < g(x)$,

4) For every sequence $\{x_k\}$ such $l(g(x_k)) < l(g(x_{k+1}))$ for all $k \in \mathbb{N}$ and $l(g(x_k))$ converse to $l(g(x))$ for some $x \in X$ such that $l(g(x_k)) < l(g(x))$,

5) There exists $0 < \alpha < \beta$ with $\alpha + \beta < \dfrac{1}{2}$ and $L > 0$ such that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta d(l(u), l(v)) + LD(g(y), Tx),$$

for all $x, y \in X$ and $u \in Tx, u \in Ty$ with $g(x) < g(y)$ and $l(u) < l(v)$

and

$$H(Tx, Ty) \leq d(g(x), g(y)) + d(l(g(x)), l(g(y))) + D(g(y), Tx),$$

for all $x, y \in X$ with $g(x) < g(y)$ and $l(g(x)) < l(g(y))$,

6) The metric d is complete.

Then there exists $u \in X$ such that $u \in Tu$.

5.3 Some Examples

We have an example to support the Theorem 3.8 as follows:

Let $X = \{3, 4, 5, 6, 7, 8, \}$ and $d(x, y) = |x - y|$ for all $x, y \in X$, where $E(G) = \{(4, 6), (6, 8)\} \cup \{(3, 5), (5, 7), (3, 7)\} \cup \{(3, 3), (5, 5), (6, 6), (7, 7)\} \cup \{(6, 7), (7, 6), (5, 3), (7, 3), (7, 5)\}$.

Defined the multi-valued mapping $T : X \to CB(X)$ by

$$T(x) = \begin{cases} \{3, 5\} & \text{if } x = 4, \\ \{5, 7\} & \text{if } x = 6, \\ \{5\} & \text{if } x = 3, 5, 7, 8, \end{cases}$$

the mapping $g : X \to X$ be defined by

$$g(x) = \begin{cases} 3 & \text{if } x = 7, \\ 5 & \text{if } x = 3, \\ 7 & \text{if } x = 5, \\ 4 & \text{if } x = 8, \\ 6 & \text{if } x = 4, \\ 8 & \text{if } x = 6, \end{cases}$$

and the mapping $l : X \to X$ be defined by

$$l(x) = \begin{cases} 6 & \text{if } x = 3, 4, 6, 7, 8, \\ 7 & \text{if } x = 5. \end{cases}$$

Then there exists $3 \in X$ such that $g(3) \in T3$.

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Appendix

Appendix A

The research papers

axioms

MDPI

Article

The Split Various Variational Inequalities Problems for Three Hilbert Spaces

Chinda Chaichuay † and Atid Kangtunyakarn *,†

Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand; kcchinda3@gmail.com

* Correspondence: beawrock@hotmail.com

† These authors contributed equally to this work.

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Abstract: There are many methods for finding a common solution of a system of variational inequalities, a split equilibrium problem, and a hierarchical fixed-point problem in the setting of real Hilbert spaces. They proved the strong convergence theorem. Many split feasibility problems are generated in real Hillbert spaces. The open problem is proving a strong convergence theorem of three Hilbert spaces with different methods from the lasted method. In this research, a new split variational inequality in three Hilbert spaces is proposed. Important tools which are used to solve classical problems will be developed. The convergence theorem for finding a common element of the set of solution of such problems and the sets of fixed-points of discontinuous mappings has been proved.

Keywords: nonspreading; pseudo-nonspreading; new split variational inequalities

MSC: 47H10; 47J20; 49J40

1. Introduction

We use the following symbols throughout this paper: let H be a real Hilbert space and C be a nonempty closed convex subset of H. We also use the symbols \\ $\longrightarrow$ // and \\ $\rightharpoonup$ //, which represent strong and weak convergence, respectively. The variational inequality problem (VIP) is a well known problem. That is to find a point ϖ_* such that

$$\langle y - \varpi_*, G\varpi_* \rangle \geq 0, \text{ for all } y \in C, \tag{1}$$

where $G : C \longrightarrow H$ is a mapping. The set of all solutions of (1) is denoted by $Var(C, G)$.

The variational inequality problem has been applied in various fields such as industry, finance, economics, social, ecology, regional, pure and applied sciences; see [1–3]. For every $i = 1, 2$, let H_i be a real Hilbert space and C, Q be nonempty closed convex subset of H_1, and H_2, respectively. Recently, Censor [4] has introduced a new variational problem called the split inequality problem (SIP). It entails finding a solution of one variational inequality problem (VIP), the image of which, under a given bounded linear transformation, is a solution of another VIP.

The split variational inequality problem is assigned to the following formula; find a point $\varpi_* \in C$ such that

$$\langle f(\varpi_*), x - \varpi_* \rangle \geq 0, \text{ for all } x \in C \tag{2}$$

and a point $y^* = A\varpi_*$ solves

$$\langle y - y^*, g(y^*)\rangle \geq 0, \text{ for all } y \in Q, \tag{3}$$

where $A : H_1 \to H_2$ is a bounded linear operator and $f : H_1 \to H_1, g : H_2 \to H_2$ are mappings. The set of all solutions of (2) and (3) is denoted by

$$\Omega = \{x \in Var\,(C, f) : \text{ for all } x \in Var\,(Q, g)\}\,.$$

The split variational inequality problem can be applied to model in intensity-modulated radiation therapy (IMRT) treatment planning.

There are also a lot of authors who have introduced convergence theorem related to the split variational inequality and fixed point problems; see [5–7] for an example. In [8], they have studied the Mann implicit iterations for strongly accreative and strongly pseudo-contractive mappings and found that this implicit scheme gives a better convergence rate estimate.

The following definitions are important tools used in this research. A mapping ϑ of H into itself is called nonexpansive if $\|\vartheta x - \vartheta y\| \leq \|x - y\|$, for all $x, y \in H$. We denote by $F(\vartheta)$ the set of fixed points of ϑ (i.e., $F(\vartheta) = \{x \in C : \vartheta x = x\}$). A nonexpansive mapping ϑ is equivalent to the following inequality;

$$\langle (I - \vartheta)\,x - (I - \vartheta)\,y, \vartheta y - \vartheta x\rangle \leq \frac{1}{2}\|(I - \vartheta)x - (I - \vartheta)\,y\|^2,$$

for all $x, y \in H$. From the equation above if $y \in F(\vartheta)$ and $x \in H$, we can conclude that;

$$\langle (I - \vartheta)\,x, y - \vartheta x\rangle \leq \frac{1}{2}\,\|(I - \vartheta)\,x\|^2.$$

A mapping A of C into H is called inverse strongly monotonic, if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2,$$

for all $x, y \in C$. In [9], Kohsaka and Takahashi introduced the nonspreading mapping in Hilbert spaces H which is defined by the following inequality $2\|\vartheta x - \vartheta y\|^2 \leq \|\vartheta x - y\|^2 + \|x - \vartheta y\|^2$, for all $x, y \in C$.

Following the terminology of Browder and Petryshyn [10], in [11], Osilike and Isiogugu introduced the mapping $\vartheta : C \to C$, which is called $\kappa-$strictly pseudo-nonspreading mapping if there exists $\kappa \in [0, 1)$ such that

$$\|\vartheta x - \vartheta y\|^2 \leq \|x - y\|^2 + \kappa\|(I - \vartheta)x - (I - \vartheta)y\|^2 + 2\langle x - \vartheta x, y - \vartheta y\rangle,$$

for all $x, y \in C$. Clearly every nonspreading mapping is κ-strictly pseudo-nonspreading; see, for example, [11].

In [12], Bnouhachem modified a projection process for finding a common solution of a system of variational inequalities, a split equilibrium problem and a hierarchical fixed-point problem in the setting of real Hilbert spaces and also proved the strong convergence theorem of the sequence $\{x_n\}$ generated by

$$\begin{cases} u_n = \vartheta_{r_n}^{F_1}\left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right)Ax_n\right); \\ z_n = P_C\left[P_C\left[u_n - \alpha_1 B_2 u_n\right] - \alpha_1 B_1 P_C\left[u_n - \alpha_2 B_2 u_n\right]\right]; \\ y_n = \beta_n \varrho x_n + (1 - \beta_n)\,z_n; \\ x_{n+1} = P_C\left[\alpha_n \rho U\,(x_n) + (I - \alpha_n \mu F)\,\vartheta\,(y_n)\right], \text{ for all } n \geq 0, \end{cases}$$

where $A : H_1 \to H_2$ is a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions; $B_i : C \to H$ is a θ_i-inverse strongly monotonic mapping for each $i = 1, 2$ and $S, \vartheta : C \to C$ nonexpansive mapping; $F : C \to C$ is a k-Lipschitzian mapping and is η-strongly monotonic; $U : C \to C$ is a τ-Lipschitzian mapping; and the positive parameters are $r_n, \alpha_n, \alpha_1, \alpha_2, \rho, \mu$, for all $n \in \mathbb{N}$.

Let C and Q be nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is formulated as :

$$\text{to find } x^* \in C \text{ such that } Ax^* \in Q, \tag{4}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP are also applied in [13,14].

Recently, Moudafi [15] introduced the following new split feasibility problem, which is also called general split equality problem:

Let H_1, H_2, H_3 be real Hilbert spaces, $C \subset H_1, Q \subset H_2$ be two nonempty closed convex sets and $A : H_1 \to H_3, B : H_2 \to H_3$ be two bounded linear operators. Moudafi studied the convergence of a relaxed alternating CQ-algorithm for solving the new split feasibility problem, aiming to find

$$\varpi_* \in C,\ y^* \in Q \text{ such that } A\varpi_* = By^*. \tag{5}$$

In order to prove the weak convergence theory to solve general split equality problem (5), Moudafi defined the following iteration process $\{x_k\}$:

$$\begin{cases} x_{k+1} = P_C\left(x_k - \gamma_k A^*\left(Ax_k - By_k\right)\right), \\ y_{k+1} = P_Q\left(y_k + \gamma_k B^*\left(Ax_{k+1} - By_k\right)\right), \end{cases}$$

where A^*, B^* are adjoint operators of A, B respectively, proper conditions of the positive paramiter γ_k, for all $k \geq 1$. In order to avoid using the projection, Moudafi [16] introduced and studied the following problem: Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be nonlinear operators such that $\mathrm{Fix}(T) \neq \emptyset$ and $\mathrm{Fix}(S) \neq \emptyset$, where $\mathrm{Fix}(T)$ and $\mathrm{Fix}(S)$ denote the sets of fixed points of T and S, respectively. If $C = \mathrm{Fix}(T)$ and $Q = \mathrm{Fix}(S)$; then the split equality problem reduces to

$$\text{to find } x \in \mathrm{Fix}(T) \text{ and } y \in \mathrm{Fix}(S) \text{ such that } Ax = By, \tag{6}$$

which is called a split equality fixed point problem (SEFPP).

Denote by Γ the solution set of split equality fixed point problem (6). There were recently SEFPP research articles in [17,18].

Question A. Can we prove a strong convergence theorem of three Hilbert spaces by different methods from Moudafi [15]?

For every $i = 1, 2, 3$, let H_i be a real Hilbert space and C_i be a nonempty closed convex subset of H_i. Let $B_i : C_i \to H_i$ be a mapping, for all $i = 1, 2, 3$, and let $A_2 : H_1 \to H_2$ and $A_3 : H_2 \to H_3$. The split various variational inequality is to find the points

$$\begin{cases} \varpi_1^* \in C_1, \text{ such that } \langle B_1\varpi_1^*, x_1 - \varpi_1^* \rangle \geq 0, \text{ for all } x_1 \in C_1, \text{ and} \\ \varpi_2^* = A_2\varpi_1^* \in C_2, \text{ such that } \langle B_2\varpi_2^*, x_2 - \varpi_2^* \rangle \geq 0, \text{ for all } x_2 \in C_2, \text{ and} \\ \varpi_3^* = A_3\varpi_2^* \in C_3, \text{ such that } \langle B_3\varpi_3^*, x_3 - \varpi_3^* \rangle \geq 0, \text{ for all } x_3 \in C_3. \end{cases} \tag{7}$$

The set of the solutions of (7) is denoted by $\Omega = \{\varpi^* = (\varpi_1^*, \varpi_2^*, \varpi_3^*) \in C_1 \times C_2 \times C_3 : \varpi_i^* \in Var(C_i, B_i), \text{ for all } i = 1,2,3\}$.

To answer question A, we have created a new tool to prove a strong convergence theorem for three Hilbert spaces to be used for finding the solution of the problem (7) and the fixed points problem of nonspreading and pseudo-nonspreading mappings. Preliminaries In this section, we collect some definitions and lemmas in Hilbert space, which will be needed for proving our main results. More properties of Hilbert space can be found in [19].

Definition 1. *The (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property*

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemmas 1 and 2 are properties of P_C.

Lemma 1. *([20]) For a given $x \in H$ and $y \in C$, $P_C x = y$ if and only if there holds the inequality $\langle y - x, z - y\rangle \geq 0$, for all $z \in C$.*

Lemma 2. *([21]) Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$, $u = P_C(I - \lambda A)u$ if and only if $u \in Var(C, A)$, where P_C is the metric projection of H onto C.*

Lemma 3. *([22]) Let $\{Y_n\}$ be a sequence of nonnegative real numbers satisfying $Y_{n+1} = (1 - \alpha_n)Y_n + \delta_n$, for all $n \geq 0$,*

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

(2) $\limsup_{n \to +\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{+\infty} |\delta_n| < +\infty$.

Then $\lim_{n \to +\infty} Y_n = 0$.

Lemma 4. *([23]) Let $\{Y_n\}$ be a sequence of nonnegative real number satisfying,*

$$Y_{n+1} = (1 - \alpha_n)Y_n + \alpha_n \beta_n, \text{for all } n \geq 0$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions

(1) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{+\infty} \alpha_n = +\infty$;

(2) $\limsup_{n \to +\infty} \beta_n \leq 0$ *or* $\sum_{n=1}^{+\infty} |\alpha_n \beta_n| < +\infty$.

Then $\lim_{n \to +\infty} Y_n = 0$.

Lemma 5. *For every $i = 1,2,3$, let H_i be a real Hilbert spaces and C_i be a nonempty closed convex subset of H_i. Let $B_i : C_i \to H_i$ be β_i-inverse strongly monotonic mappings with $\eta = \min_{i=1,2,3}\{\beta_i\}$ and let $A_2 : H_1 \to H_2$, $A_3 : H_2 \to H_3$ be bounded linear operators with the adjoint operator A_2^* and A_3^*, respectively. Assume that $\overline{x}_1 \in C_1$, $A_2\overline{x}_1 = \overline{x}_2$, $A_3\overline{x}_2 = \overline{x}_3$ and $\Omega \neq \emptyset$. The following are equivalent:*

(i) $\overline{x} \in \Omega$, *where* $\overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) \in C_1 \times C_2 \times C_3$.
(ii) $\overline{x}_1 = P_{C_1}(I_1 - \lambda_1 B_1)(\overline{x}_1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\overline{x}_2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\overline{x}_3))$,

where $I_i : H_i \to H_i$ *is an identity mapping, for all* $i = 1, 2, 3$, $\gamma_2(1 + \gamma_3) \leq \frac{1}{L}$, $L = \max\{L_1, L_2\} \leq 1$ *which* L_1, L_2 *are spectral radii of* $A_2 A_2^*$ *and* $A_3 A_3^*$, *respectively,* $\lambda_i \in (0, 2\eta)$, *for all* $i = 1, 2, 3$ *and* $\gamma_2, \gamma_3 \geq 0$.

Proof. Let the conditions hold.
$(i) \Rightarrow (ii)$ Let $\overline{x} \in \Omega$ where $\overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) \in C_1 \times C_2 \times C_3$; we have

$$\overline{x}_i \in Var(C_i, B_i), \text{ for all } i = 1, 2, 3.$$

From Lemma 2, we have

$$\overline{x}_i \in F(P_{C_i}(I_i - \lambda_i B_i)), \text{ for all } i = 1, 2, 3.$$

From determining the definition of $\overline{x}$, we have

$$\overline{x}_1 = P_{C_1}(I_1 - \lambda_1 B_1)(\overline{x}_1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\overline{x}_2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\overline{x}_3)).$$

$(ii) \Rightarrow (i)$ Let $\overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) \in C_1 \times C_2 \times C_3$, where $\overline{x}_2 = A_2 \overline{x}_1, \overline{x}_3 = A_3 \overline{x}_2$ and

$$\overline{x}_1 = P_{C_1}(I_1 - \lambda_1 B_1)(\overline{x}_1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\overline{x}_2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\overline{x}_3)).$$

Since B_i is β_i-inverse strongly monotonic with $\lambda_i < 2\eta$, for all $i = 1, 2, 3$, we have $P_{C_i}(I_i - \lambda_i B_i)$ which is a nonexpansive mapping, for all $i = 1, 2, 3$.

Let $w \in \Omega$ where $w = (w_1, w_2, w_3) \in C_1 \times C_2 \times C_3$ where $w_2 = A_2 w_1$, $w_3 = A_3 w_2$. From (i) implies (ii), we have

$$w_1 = P_{C_1}(I - \lambda_1 B_1)(w_1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I_2 - \lambda_2 B_2))w_2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))w_3)).$$

Put $M = (I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\overline{x}_2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\overline{x}_3$ and $N = (I_2 - P_{C_2}(I_2 - \lambda_2 B_2))w_2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))w_3$.

From determining the definition of $\overline{x}$ and w, we have

$$
\begin{aligned}
\|\overline{x}_1 - w_1\|^2 &\leq \|\overline{x}_1 - w_1 - \gamma_2 A_2^*(M-N)\|^2 \\
&= \|\overline{x}_1 - w_1\|^2 - 2\gamma_2\langle \overline{x}_1 - w_1, A_2^*(M-N)\rangle + \gamma_2^2\|A_2^*(M-N)\|^2 \\
&\leq \|\overline{x}_1 - w_1\|^2 - 2\gamma_2\langle \overline{x}_2 - w_2, M-N\rangle + \gamma_2^2 L\|M-N\|^2 \\
&\leq \|\overline{x}_1 - w_1\|^2 - 2\gamma_2\langle \overline{x}_2 - w_2, (I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2 + \gamma_3 A_3^*\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\rangle \\
&\quad +\gamma_2^2 L\|(I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2 + \gamma_3 A_3^*\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\|^2 \\
&= \|\overline{x}_1 - w_1\|^2 - 2\gamma_2(\langle \overline{x}_2 - w_2, (I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\rangle + \gamma_3\langle \overline{x}_3 - w_3, \\
&\quad \left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\rangle) + \gamma_2^2 L\|(I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2 \\
&\quad +\gamma_3 A_3^*\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\|^2 \\
&= \|\overline{x}_1 - w_1\|^2 + 2\gamma_2\langle w_2 - \overline{x}_2, (I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\rangle + 2\gamma_2\gamma_3\langle w_3 - \overline{x}_3, \\
&\quad \left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\rangle + \gamma_2^2 L(\|(I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\|^2 \\
&\quad +\gamma_3^2 L\|\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\|^2 + 2\gamma_3\langle (I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2, \\
&\quad A_3^*\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\rangle) \\
&\leq \|\overline{x}_1 - w_1\|^2 + 2\gamma_2\langle w_2 - P_{C_2}(I_2-\lambda_2B_2)\overline{x}_2 + P_{C_2}(I_2-\lambda_2B_2)\overline{x}_2 - \overline{x}_2, \\
&\quad (I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\rangle + 2\gamma_2\gamma_3\langle w_3 - P_{C_3}(I_3-\lambda_3B_3)\overline{x}_3 + P_{C_3}(I_3-\lambda_3B_3)\overline{x}_3 - \overline{x}_3, \\
&\quad \left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\rangle + \gamma_2^2 L(\|(I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\|^2 \\
&\quad +\gamma_3^2 L\|\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\|^2 \\
&\quad +\gamma_3\|(I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\|^2 + \gamma_3\|A_3^*\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\|^2) \qquad (8)\\
&\leq \|\overline{x}_1 - w_1\|^2 + 2\gamma_2(\frac{1}{2}\|(I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\|^2 - \|(I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\|^2) \\
&\quad +2\gamma_2\gamma_3(\frac{1}{2}\|\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\|^2 - \|\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3\|^2) \\
&\quad +\gamma_2^2 L(\|((I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\|^2 + \gamma_3^2 L\|\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3)\|^2 \\
&\quad +\gamma_3\|(I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\|^2 + \gamma_3 L\|\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3)\|^2) \\
&= \|\overline{x}_1 - w_1\|^2 - \gamma_2(1-\gamma_2L(1+\gamma_3))\|(I_2 - P_{C_2}(I_2-\lambda_2B_2))\overline{x}_2\|^2 \\
&\quad -\gamma_2\gamma_3(1-\gamma_2L^2(1+\gamma_3))\|\left(I_3 - P_{C_3}(I_3-\lambda_3B_3)\right)\overline{x}_3)\|^2.
\end{aligned}
$$

By applying above equation and Lemma 2, we have

$$\overline{x}_2 \in F\left(P_{C_2}(I-\lambda_2B_2)\right) = Var(C_2, B_2) \text{ and } \overline{x}_3 \in F\left(P_{C_3}(I-\lambda_3B_3)\right) = Var(C_3, B_3). \qquad (9)$$

From determining the definitions of $\overline{x}$ and (9), we have

$$\overline{x}_1 \in F\left(P_{C_1}(I-\lambda_1B_1)\right) = Var(C_1, B_1).$$

Hence $\overline{x} \in \Omega$. $\square$

Lemma 6. *Let C be a nonempty closed convex subset of Hilbert space H. Let $\vartheta : C \to C$ be a nonspreading mapping and $\varrho : C \to C$ be κ-pseudo-nonspreding mapping with $F(\vartheta) \cap F(\varrho) \neq \emptyset$. Then $F(P_C(I - \gamma(a(I-\vartheta) + (1-a)(I-\varrho)))) = F(\vartheta) \cap F(\varrho)$ for all $a \in (0,1)$ and $\gamma > 0$. Moreover, if $\gamma < 1-\kappa$, then*

$$\|I - \gamma(a(I-\vartheta) + (1-a)(I-\varrho)))x - \varpi^*\| \leq \|\varpi^* - x\|,$$

for all $x \in C$ *and* $\varpi^* \in F(\varrho) \cap F(\vartheta)$.

Proof. Let $\varpi_0 \in F(\varrho) \cap F(\vartheta)$, we have

$$P_C(I - \gamma(a\,(I-\vartheta) + (1+a)\,(I-\varrho)))\varpi_0 = \varpi_0.$$

It follows that $\varpi_0 \in F(P_C(I - \gamma(a\,(I-\vartheta) + (1+a)\,(I-\varrho))))$. Therefore

$$F(\vartheta) \cap F(\varrho) \subseteq F(P_C(I - \gamma(a\,(I-\vartheta) + (1+a)\,(I-\varrho))))$$

Let $\varpi_0 \in F(P_C(I - \gamma(a\,(I-\vartheta) + (1+a)\,(I-\varrho))))$ and $\varpi^* \in F(\vartheta) \cap F(\varrho)$. From Lemma 2, we have

$$\langle y - \varpi_0, a\,(I-\vartheta)\,\varpi_0 + (1-a)\,(I-\varrho)\,\varpi_0\rangle \geq 0,$$

for all $y \in C$.

From determining the definition of ϱ, we have

$$\begin{aligned} \|\varpi_0 - \varpi^*\|^2 + \kappa\,\|(I-\varrho)\,\varpi_0\|^2 &\geq \|\varrho\varpi_0 - \varpi^*\|^2 \\ &= \|(I-\varrho)\,\varpi_0 - (\varpi_0 - \varpi^*)\|^2 \\ &= \|(I-\varrho)\,\varpi_0\|^2 - 2\langle (I-\varrho)\,\varpi_0, \varpi_0 - \varpi^*\rangle + \|\varpi_0 - \varpi^*\|^2 \end{aligned} \tag{10}$$

From the result of the calculation from the inequality (10), we get

$$\langle (I-\varrho)\,\varpi_0, \varpi_0 - \varpi^*\rangle \geq \left(\frac{1-\kappa}{2}\right)\|(I-\varrho)\,\varpi_0\|^2 \tag{11}$$

Assume that $\varpi_0 \neq \vartheta\varpi_0$; then we have $\|(I-\vartheta)\,\varpi_0\| > 0$. Using the same method as (11) and definitions of ϑ, we get

$$\langle (I-\vartheta)\,\varpi_0, \varpi_0 - \varpi^*\rangle \geq \frac{1}{2}\,\|(I-\vartheta)\,\varpi_0\|^2 \tag{12}$$

From (11) and $a \in (0,1)$, we obtain

$$\begin{aligned} \langle \varpi^* - \varpi_0, a\,(I-\vartheta)\,\varpi_0\rangle &= \langle \varpi^* - \varpi_0, a\,(I-\vartheta)\,\varpi_0 + (1-a)\,(I-\varrho)\,\varpi_0\rangle \\ &\quad - (1-a)\,\langle \varpi^* - \varpi_0, (I-\varrho)\,\varpi_0\rangle \\ &\geq (1-a)\,\langle \varpi_0 - \varpi^*, (I-\varrho)\,\varpi_0\rangle. \end{aligned} \tag{13}$$

From (13), we have

$$\langle \varpi^* - \varpi_0, (I-\vartheta)\,\varpi_0\rangle \geq 0.$$

From above and (12), we have

$$0 \leq \langle \varpi^* - \varpi_0, (I-\vartheta)\,\varpi_0\rangle \leq -\frac{1}{2}\,\|(I-\vartheta)\,\varpi_0\|^2.$$

Thus, $\|(I-\vartheta)\,\varpi_0\| \leq 0$. This is a contradiction.

Thus, we have $\varpi_0 = \vartheta\varpi_0$ and it implies that

$$\varpi_0 \in F(\vartheta). \tag{14}$$

Similarly, by using the same technique as (14), we have

$$\varpi_0 \in F(\varrho). \tag{15}$$

From (14) and (15), we have

$$F(P_C(I - \gamma(a(I - \vartheta) + (1 + a)(I - \varrho)))) \subseteq F(\varrho) \cap F(\vartheta).$$

Let $\varpi^* \in F(\varrho) \cap F(\vartheta)$ and $x \in C$; we have

$$\begin{aligned}
\|(I - \gamma(a(I - \vartheta) + (1 - a)(I - \varrho)))x - \varpi^*\|^2 &= \|(I - \gamma(a(I - \vartheta) + (1 - a)(I - \varrho)))x \\
&\quad -(I - \gamma(a(I - \vartheta) + (1 - a)(I - \varrho)))\varpi^*\|^2 \\
&= \|x - \varpi^* - \gamma(a((I - \vartheta)x - (I - \vartheta)\varpi^*) \\
&\quad +(1 - a)((I - \varrho)x - (I - \varrho)\varpi^*))\|^2 \\
&= \|x - \varpi^*\|^2 - 2\gamma\langle a((I - \vartheta)x - (I - \vartheta)\varpi^*) \\
&\quad +(1 - a)((I - \varrho)x - (I - \varrho)\varpi^*), x - \varpi^*\rangle \\
&\quad +\gamma^2\|a((I - \vartheta)x - (I - \vartheta)\varpi^*) \\
&\quad +(1 - a)((I - \varrho)x - (I - \varrho)\varpi^*)\|^2 \\
&\leq \|x - \varpi^*\|^2 - 2\gamma a\langle(I - \vartheta)x - (I - \vartheta)\varpi^*, x - \varpi^*\rangle \\
&\quad -2\gamma(1 - a)\langle(I - \varrho)x - (I - \varrho)\varpi^*, x - \varpi^*\rangle \\
&\quad +\gamma^2 a\|(I - \vartheta)x - (I - \vartheta)\varpi^*\|^2 \\
&\quad +(1 - a)\gamma^2\|(I - \varrho)x - (I - \varrho)\varpi^*\|^2 \\
&\leq \|x - \varpi^*\|^2 - 2\gamma a\frac{\|(I - T)x\|^2}{2} \\
&\quad -2\gamma(1 - a)(1 - \kappa)\frac{\|(I - \varrho)x\|^2}{2} \\
&\quad +\gamma^2 a\|(I - \vartheta)x\|^2 + (1 - a)\gamma^2\|(I - \varrho)x\|^2 \\
&\leq \|x - \varpi^*\|^2.
\end{aligned}$$

□

2. The Split Various Variational Inequality Theorem

Theorem 7. *For every $i = 1,2,3$, let C_i be a closed convex subset of a real Hilbert space H_i. Let $B_i : C_i \to H_i$ be β_i-inverse strongly monotonic mappings with $\eta = \min_{i=1,2,3}\{\beta_i\}$ and let $A_2 : H_1 \to H_2$, $A_3 : H_2 \to H_3$ be a bounded linear operator with the adjoint operator A_2^* and A_3^*, respectively. Assume that $\bar{x}_1 \in H_1, \bar{x}_2 = A_2\bar{x}_1, \bar{x}_3 = A_3\bar{x}_2$ and $\Omega \neq \emptyset$. Let $\vartheta, \varrho : C \to C$ be nonspreading and κ-strictly pseudo-nonspreading mappings, respectively. Assume that $\Omega \cap F(\vartheta) \cap F(\varrho) \neq \emptyset$ and let the sequence $\{x_n\}$ generated by $u, x_1 \in C$, and*

$$\begin{aligned}
x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}(I_1 - \lambda_n(a(I_1 - \vartheta) + (1 - a)(I_1 - \varrho)))x_n \\
&\quad +\delta_n P_{C_1}(I_1 - \lambda_1 B_1)(x_n^1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))x_n^3)),
\end{aligned}$$

for all $n \geq 1$ and $a \in (0,1)$, $I_i : H_i \to H_i$ are identity mappings, for all $i = 1,2,3$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and $x_n = x_n^1, x_n^2 = A_2x_n^1, x_n^3 = A_3x_n^2$, for all $n \in \mathbb{N}, 0 < \lambda_i < 2\eta$, for all $i = 1,2,3$ and $\gamma_j > 0$, for all $j = 2,3$. Suppose that the conditions (i)–(v) are true;

(i) $\lim_{n+\to\infty} \alpha_n = 0, \sum_{n=1}^{+\infty} \alpha_n = +\infty$;

(ii) $\gamma_2(1+\gamma_3) \leq \frac{1}{L}$, *where* $L = \max\{L_{A_1}, L_{A_2}\} \leq 1$ *where* $L_{A_1} L_{A_2}$ *are spectral radius of* $A_2A_2^*, A_3A_3^*$, *respectively;*

(iii) $0 < a \leq \beta_n, \gamma_n, \delta_n \leq b < 1$, *for some* $a, b > 0$, *for all* $n \in \mathbb{N}$;

(iv) $\sum_{n=1}^{+\infty} \lambda_n < +\infty$ *and* $0 < \lambda_n < 1 - \kappa$, *for all* $n \in \mathbb{N}$;

(v) $\sum_{n=1}^{+\infty} |\alpha_{n+i} - \alpha_n|, \sum_{n=1}^{+\infty} |\beta_{n+i} - \beta_n|, \sum_{n=1}^{+\infty} |\gamma_{n+1} - \gamma_n| < +\infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $z_0 = P_{\Omega\cap F(\vartheta)\cap F(\varrho)}u$.

Proof. Put $M = a(I-\vartheta) + (1-a)(I-\varrho)$ and $u_n = P_{C_1}(I-\lambda_1B_1)(x_n^1 - \gamma_2A_2^*((I_2 - P_{C_2}(I-\lambda_2B_2))x_n^2 + \gamma_3A_3^*(I_3 - P_{C_3}(I-\lambda_nB_3))x_n^3))$. Thus, we can rewrite x_n as follows:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}(I-\lambda_n M)x_n + \delta_n u_n, \tag{16}$$

for all $n \geq 1$.

From determining the definition of u_n put $w_n = (I_2 - P_{C_2}(I-\lambda_2B_2))x_n^2 + \gamma_3A_3^*(I_3 - P_{C_3}(I-\lambda_3B_3))x_n^3$ and $z_n = (I_3 - P_{C_3}(I-\lambda_3B_3))x_n^3$, we have

$$u_n = P_{C_1}(I_1 - \lambda_1B_1)(x_n - \gamma_2A_2^*w_n).$$

For every $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|u_n - u_{n-1}\|^2 \leq\ & \|x_n - \gamma_2A_2^*w_n - x_{n-1} + \gamma_2A_2^*w_{n-1}\|^2 \\
=\ & \|x_n - x_{n-1} - \gamma_2A_2^*(w_n - w_{n-1})\|^2 \\
=\ & \|x_n - x_{n-1}\|^2 - 2\gamma_2\langle A_2x_n - A_2x_{n-1}, w_n - w_{n-1}\rangle + \gamma_2^2\|A_2^*(w_n - w_{n-1})\|^2 \\
=\ & \|x_n - x_{n-1}\|^2 - 2\gamma_2\langle x_n^2 - x_{n-1}^2, (I_2 - P_{C_2}(I-\lambda_2B_2))x_n^2 \\
& +\gamma_3A_3^*z_n - (I_2 - P_{C_2}(I-\lambda_2B_2))x_{n-1}^2 - \gamma_3A_3^*z_{n-1}\rangle \\
& +\gamma_2^2\|A_2^*(w_n - w_{n-1})\|^2 \\
=\ & \|x_n - x_{n-1}\|^2 + 2\gamma_2\langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I-\lambda_2B_2))x_n^2 \\
& -(I_2 - P_{C_2}(I-\lambda_2B_2))x_{n-1}^2\rangle + 2\gamma_2\gamma_3\langle x_{n-1}^3 - x_n^3, z_n \\
& -z_{n-1}\rangle + \gamma_2^2\|A_2^*(w_n - w_{n-1})\|^2 \\
\leq\ & \|x_n - x_{n-1}\|^2 + 2\gamma_2\langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I-\lambda_2B_2))x_n^2 \\
& -(I_2 - P_{C_2}(I-\lambda_2B_2))x_{n-1}^2\rangle + 2\gamma_2\gamma_3\langle x_{n-1}^3 - x_n^3, z_n - z_{n-1}\rangle \\
& +\gamma_2^2L\left\|(I_2 - P_{C_2}(I_2-\lambda_2B_2))x_n^2 - (I_2 - P_{C_2}(I_2-\lambda_2B_2))x_{n-1}^2 + \gamma_3A_3^*(z_n - z_{n-1}\right\|^2 \\
\leq\ & \|x_n - x_{n-1}\|^2 + 2\gamma_2\langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I-\lambda_2B_2))x_n^2 \\
& -(I_2 - P_{C_2}(I-\lambda_2B_2))x_{n-1}^2\rangle + 2\gamma_2\gamma_3\langle x_{n-1}^3 - x_n^3, z_n - z_{n-1}\rangle \\
& +\gamma_2^2L(\|(I_2 - P_{C_2}(I-\lambda_2B_2))x_n^2 - (I_2 - P_{C_2}(I-\lambda_2B_2))x_{n-1}^2\|^2 \\
& +\gamma_3^2L\|z_n - z_{n-1}\|^2 + 2\gamma_3\langle(I_2 - P_{C_2}(I-\lambda_2B_2))x_n^2 \\
& -(I_2 - P_{C_2}(I-\lambda_2B_2))x_{n-1}^2, A^*(z_n - z_{n-1})\rangle)
\end{aligned}$$

$$
\begin{aligned}
&\leq \|x_n - x_{n-1}\|^2 + 2\gamma_2 \langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2 \rangle \\
&\quad + 2\gamma_2\gamma_3 \langle x_{n-1}^3 - x_n^3, (I_3 - P_{C_3}(I - \lambda_3 B_3))x_n^3 - (I_3 - P_{C_3}(I - \lambda_3 B_3))\, x_{n-1}^3 \rangle \\
&\quad + \gamma_2^2 L(\|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 + \gamma_3^2 L\|z_n - z_{n-1}\|^2 \\
&\quad + \gamma_3\|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 + \gamma_3 L\|z_n - z_{n-1}\|^2) \\
&\leq \|x_n - x_{n-1}\|^2 + 2\gamma_2(-\left\|(I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2\right\|^2 \\
&\quad + \frac{1}{2}\left\|(I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2\right\|^2) \\
&\quad + 2\gamma_2\gamma_3(-\|z_n - z_{n-1}\|^2 + \frac{1}{2}\|z_n - z_{n-1}\|^2) \\
&\quad + \gamma_2^2 L(\|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 + \gamma_3^2 L\|z_n - z_{n-1}\|^2 \qquad (17) \\
&\quad + \gamma_3\|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 + \gamma_3 L\|z_n - z_{n-1}\|) \\
&= \|x_n - x_{n-1}\|^2 - \gamma_2\left\|(I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2\right\|^2 \\
&\quad - \gamma_2\gamma_3\|z_n - z_{n-1}\|^2 + \gamma_2^2 L\|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 \\
&\quad + \gamma_2^2\gamma_3^2 L^2\|z_n - z_{n-1}\|^2 + \gamma_2^2\gamma_3 L\|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 \\
&\quad + \gamma_2^2\gamma_3 L^2\|z_n - z_{n-1}\| \\
&= \|x_n - x_{n-1}\|^2 - \gamma_2(1 - \gamma_2 L(1 + \gamma_3))\|(I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_n^2 \\
&\quad - (I_2 - P_{C_2}(I - \lambda_2 B_2))\, x_{n-1}^2\|^2 - \gamma_2\gamma_3(1 - \gamma_2 L^2(1 + \gamma_3))\|z_n - z_{n-1}\|^2 \\
&\leq \|x_n - x_{n-1}\|^2.
\end{aligned}
$$

Let $\varpi^* \in \Omega \cap F(\varrho) \cap F(\vartheta)$. From Lemma 6 and utilization of (8), we have

$$
\begin{aligned}
\|x_{n+1} - \varpi^*\| &\leq \alpha_n\|u - \varpi^*\| + \beta_n\|x_n - \varpi^*\| + \gamma_n\|P_{C_1}(I - \lambda_n M)\, x_n - \varpi^*\| \\
&\quad + \delta_n\|u_n - \varpi^*\| \\
&\leq \alpha_n\|u - \varpi^*\| + (1 - \alpha_n)\|x_n - \varpi^*\| \\
&\leq \widetilde{M},
\end{aligned}
\qquad (18)
$$

where $\widetilde{M} = \max\{\|u - \varpi^*\|, \|x_1 - \varpi^*\|\}$. By induction we can conclude that the sequence $\{x_n\}$ is bounded and so are $\{u_n\}$ and $\{P_{C_1}(I - \lambda_n M)\, x_n\}$.

From determining the definition of x_n and (17), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n u+\beta_n x_n+\gamma_n P_{C1}(I-\lambda_n M)x_n+\delta_n u_n\\&\quad -\alpha_{n-1}u-\beta_{n-1}x_{n-1}-\gamma_{n-1}P_{C1}(I-\lambda_{n-1}M)x_{n-1}-\delta_{n-1}u_{n-1}\|\\&\leq |\alpha_n-\alpha_{n-1}|\,\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\,\|x_{n-1}\|+\delta_n\|u_n-u_{n-1}\|\\&\quad +|\delta_n-\delta_{n-1}|\,\|u_{n-1}\|+\gamma_n\|P_{C_1}(I-\lambda_n M)x_n-P_{C_1}(I-\lambda_{n-1}M)x_{n-1}\|\\&\quad +|\gamma_n-\gamma_{n-1}|\,\|P_{C_1}(I_1-\lambda_{n-1}M)x_{n-1}\|\\&\leq |\alpha_n-\alpha_{n-1}|\,\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\,\|x_{n-1}\|+\delta_n\|u_n-u_{n-1}\|\\&\quad +|\delta_n-\delta_{n-1}|\,\|u_{n-1}\|+\gamma_n\|x_n-x_{n-1}\|+\|\lambda_{n-1}Mx_{n-1}-\lambda_n Mx_n\|\\&\quad +|\gamma_n-\gamma_{n-1}|\,\|P_{C_1}(I_1-\lambda_{n-1}M)x_{n-1}\|\\&\leq |\alpha_n-\alpha_{n-1}|\,\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\,\|x_{n-1}\|+\delta_n\|x_n-x_{n-1}\|\\&\quad +|\delta_n-\delta_{n-1}|\,\|u_{n-1}\|+\gamma_n\|x_n-x_{n-1}\|+\lambda_{n-1}\|Mx_{n-1}\|+\lambda_n\|Mx_n\|\\&\quad +|\gamma_n-\gamma_{n-1}|\,\|P_{C_1}(I_1-\lambda_{n-1}M)x_{n-1}\|\\&= (1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\,\|u\|+|\beta_n-\beta_{n-1}|\,\|x_{n-1}\|\\&\quad +|\delta_n-\delta_{n-1}|\,\|u_{n-1}\|+\lambda_{n-1}\|Mx_{n-1}\|+\lambda_n\|Mx_n\|\\&\quad +|\gamma_n-\gamma_{n-1}|\,\|P_{C_1}(I_1-\lambda_{n-1}M)x_{n-1}\|.\end{aligned}$$

From the conditions (i), (iv), (v) and Lemma 3, we have

$$\lim_{n\to+\infty}\|x_{n+1}-x_n\|=0 \tag{19}$$

Applying (8) and the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-\varpi^*\|^2 &\leq \alpha_n\|u-\varpi^*\|^2+\beta_n\|x_n-\varpi^*\|^2+\gamma_n\|P_{C_1}(I-\lambda_n M)x_n-\varpi^*\|^2\\&\quad +\delta_n\|u_n-\varpi^*\|^2-\gamma_n\beta_n\|P_{C_1}(I-\lambda_n M)x_n-x_n\|^2-\delta_n\beta_n\|u_n-x_n\|^2\\&\leq \alpha_n\|u-\varpi^*\|^2+\|x_n-\varpi^*\|^2-\gamma_n\beta_n\|P_{C1}(I-\lambda_n M)x_n-x_n\|^2\\&\quad -\delta_n\beta_n\|u_n-x_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\gamma_n\beta_n\|P_{C_1}(I-\lambda_n M)x_n-x_n\|^2+\delta_n\beta_n\|u_n-x_n\|^2 &\leq (\|x_{n+1}-\varpi^*\|+\|x_n-\varpi^*\|)\|x_{n+1}-x_n\|\\&\quad +\alpha_n\|u-\varpi^*\|^2.\end{aligned}$$

From the conditions *(i)*, *(iii)* and (19) we can conclude the following results

$$\lim_{n\to+\infty}\|u_n-x_n\|=\lim_{n\to+\infty}\|P_{C1}(I-\lambda_n M)x_n-x_n\|=0. \tag{20}$$

Next, we show that

$$\limsup_{n\to+\infty}\langle u-z_0, z_0-x_n\rangle\leq 0, \tag{21}$$

where $z_0 = P_{\Omega \cap F(\varrho) \cap F(\vartheta)} u$. In order to prove this we may assume that

$$\limsup_{n \to +\infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{k \to +\infty} \langle u - z_0, x_{n_k} - z_0 \rangle, \tag{22}$$

where $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Since $\{x_n\}$ is bounded, we may assume that $x_{n_k} \rightharpoonup q$ as $k \to +\infty$. Assume that $q \notin F(\varrho) \cap F(\vartheta)$. From Lemma 6, we have $q \notin F\left(P_{C_1}\left(I - \lambda_{n_k} M\right)\right)$. By using properties of Opial's condition and (20), we have

$$\begin{aligned}
\liminf_{k \to +\infty} \|x_{n_k} - q\| &< \liminf_{k \to +\infty} \|x_{n_k} - P_{C_1}\left(I - \lambda_{n_k} M\right) q\| \\
&\leq \liminf_{k \to +\infty} (\|x_{n_k} - P_{C_1}\left(I - \lambda_{n_k} M\right) x_{n_k}\| \\
&\quad + \|P_{C_1}\left(I - \lambda_{n_k} M\right) x_{n_k} - P_{C_1}\left(I - \lambda_{n_k} M\right) q\|) \\
&\leq \liminf_{k \to +\infty} (\|x_{n_k} - q\| + \lambda_{n_k} \|M x_{n_k} - M q\|) \\
&\leq \liminf_{k \to +\infty} \|x_{n_k} - q\|.
\end{aligned}$$

This is a contradiction. Therefore $q \in F(\varrho) \cap F(\vartheta)$.
Assume $q \notin \Omega$. From Lemma 5, we have

$$q \neq P_{C_1}\left(I - \lambda_1 B_1\right)\left(q - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))A_2 q + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))A_3 A_2 q)\right).$$

By using properties of Opial's condition, and the definitions of u_n and (20), we have

$$\begin{aligned}
\liminf_{k \to +\infty} \|x_{n_k} - q\| &< \liminf_{k \to +\infty} \|x_{n_k} - P_{C_1}\left(I - \lambda_1 B_1\right)\left(q - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))A_2 q\right. \\
&\quad \left. + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))A_3 A_2 q)\right)\| \\
&\leq \liminf_{k \to +\infty} (\|x_{n_k} - u_{n_k}\| \\
&\quad + \|u_{n_k} - P_{C_1}\left(I - \lambda_1 B_1\right)\left(q - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))A_2 q\right. \\
&\quad \left. + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))A_3 A_2 q)\right)\|) \\
&\leq \liminf_{k \to +\infty} \|x_{n_k} - q\|.
\end{aligned}$$

This is a contradiction. Then $q \in \Omega$. Therefore $q \in \Omega \cap F(\varrho) \cap F(\vartheta)$.
From (22) and the well-known properties of metric projection, we have

$$\limsup_{n \to +\infty} \langle u - z_0, x_n - z_0 \rangle \leq 0.$$

From determining the definition of x_n, we can conclude that

$$\|x_{n+1} - z_0\|^2 \leq (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle,$$

where $z_0 = P_{\Omega \cap F(\varrho) \cap F(\vartheta)} u$. From Lemma 4, we can conclude that the sequence $\{x_n\}$ converses strongly to $z_0 = P_{\Omega \cap F(\varrho) \cap F(\vartheta)} u$. □

The following results were obtained directly from the main theorem.

Corollary 8. *For every $i = 1,2,3$, let C_i be a closed convex subset of a real Hilbert space H_i. Let $B_i : C_i \to H_i$ be β_i-inverse strongly monotonic mappings with $\eta = \min_{i=1,2,3}\{\beta_i\}$ and let $A_2 : H_1 \to H_2, A_3 : H_2 \to lH_3$ be a bounded linear operator with the adjoint operator A_2^* and A_3^*, respectively. Assume that $\bar{x}_1 \in H_1, \bar{x}_2 = A_2\bar{x}_1, \bar{x}_3 = A_3\bar{x}_2$ and $\Omega \neq \emptyset$. Let $\vartheta, \varrho : C \to C$ be nonspreading mappings, respectively. Assume that $\Omega \cap F(\vartheta) \cap F(\varrho) \neq \emptyset$ and let the sequence $\{x_n\}$ generated by $u, x_1 \in C$, and*

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}\left(I_1 - \lambda_n\left(a\left(I_1 - \vartheta\right) + (1-a)\left(I_1 - \varrho\right)\right)\right)x_n \\ &\quad + \delta_n P_{C_1}(I_1 - \lambda_1 B_1)(x_n^1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))x_n^3)), \end{aligned}$$

for all $n \geq 1$ and $a \in (0,1)$, $I_i : H_i \to H_i$ are identity mappings, for all $i = 1,2,3$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and $x_n = x_n^1, x_n^2 = A_2 x_n^1, x_n^3 = A_3 x_n^2$, for all $n \in \mathbb{N}, 0 < \lambda_i < 2\eta$, for all $i = 1,2,3$ and $\gamma_j > 0$, for all $j = 2,3$. Suppose that the conditions (i) to (v) are true;

(i) $\lim_{n\to+\infty}\alpha_n = 0, \sum_{n=1}^{+\infty}\alpha_n = +\infty$;

(ii) $\gamma_2(1+\gamma_3) \leq \frac{1}{L}$, *where* $L = \max\{L_{A_1}, L_{A_2}\} \leq 1$, *where L_{A_1}, L_{A_2} are spectral radius of $A_2A_2^*, A_3A_3^*$, respectively;*

(iii) $0 < a \leq \beta_n, \gamma_n, \delta_n \leq b < 1$, *for some $a, b > 0$, for all $n \in \mathbb{N}$;*

(iv) $\sum_{n=1}^{+\infty}\lambda_n < +\infty$ *and* $0 < \lambda_n < 1-\kappa$, *for all $n \in \mathbb{N}$;*

(v) $\sum_{n=1}^{+\infty}|\alpha_{n+i} - \alpha_n|, \sum_{n=1}^{+\infty}|\beta_{n+i} - \beta_n|, \sum_{n=1}^{+\infty}|\gamma_{n+1} - \gamma_n| < +\infty$.

Then the sequence $\{x_n\}$ converses strongly to $z_0 = P_{\Omega\cap F(\vartheta)\cap F(\varrho)}u$.

3. Application

We have applied the problem (7) for the various fixed point problems in three Hilbert spaces as follows:

For every $i = 1,2,3$, let H_i be a real Hilbert space and C_i be a nonempty closed convex subset of H_i. Let $\vartheta_i : C_i \to C_i$ be a mapping, for all $i = 1,2,3$ and let $A_2 : H_1 \to H_2$ and $A_3 : H_2 \to H_3$. The fixed points problem in three Hilbert spaces is meant to find the point

$$\begin{cases} \varpi_1^* \in C_1, \text{ such that } \varpi_1^* \in F(\vartheta_1) \text{ and} \\ \varpi_2^* = A_2\varpi_1^* \in C_2, \text{ such that } \varpi_2^* \in F(\vartheta_2) \text{ and} \\ \varpi_3^* = A_3\varpi_2^* \in C_3, \text{ such that } \varpi_3^* \in F(\vartheta_3). \end{cases} \tag{23}$$

The set of the solutions of (23) is denoted by $\Omega = \{\varpi^* = (\varpi_1^*, \varpi_2^*, \varpi_3^*) \in C_1 \times C_2 \times C_3 : \varpi_i^* \in F(\vartheta_i)$, for all $i = 1,2,3\}$. It is clear that $Var(C, I-T) = F(\vartheta)$, where $\vartheta : C \to C$ is a nonexpansive mapping with $F(\vartheta) \neq \emptyset$. By leveraging Lemma 5 and such knowledge, we have the following results:

Lemma 9. *For every $i = 1,2,3$, let H_i be a real Hilbert spaces and C_i be a nonempty closed convex subset of H_i. Let $\vartheta_i : C_i \to C_i$ be nonexpansive mappings and let $A_2 : H_1 \to H_2, A_3 : H_2 \to H_3$ be a bounded linear operator with the adjoint operator A_2^* and A_3^*, respectively. Assume that $\bar{x}_1 \in C_1, A_2\bar{x}_1 = \bar{x}_2, A_3\bar{x}_2 = \bar{x}_3$ and $\Omega \neq \emptyset$. The following are equivalent:*

(i) $\bar{x} \in \Omega$, *where* $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in C_1 \times C_2 \times C_3$.

(ii) $\bar{x}_1 = P_{C_1}(I_1 - \lambda_1(I_1 - \vartheta_1))(\bar{x}_1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I_2 - \lambda_2(I_2 - \vartheta_2)))\bar{x}_2$
$+\gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3(I_3 - \vartheta_3)))\bar{x}_3))$,

where $I_i : H_i \to H_i$ is an identity mapping, for all $i = 1,2,3$, $\gamma_2(1+\gamma_3) \leq \frac{1}{L}$, $L = \max\{L_1, L_2\} \leq 1$ which L_1, L_2 are spectral radii of $A_2A_2^*$ and $A_3A_3^*$, respectively, $\lambda_i \in (0,1)$, for all $i = 1,2,3$ and $\gamma_2, \gamma_3 \geq 0$

Proof. Given $F(\vartheta_i) = Var(C, I_i - \vartheta_i)$ for all $i = 1, 2, 3$; $(I_i - \vartheta_i)$—a $\frac{1}{2}$-inverse strongly monotonic; and Lemma 5, we can summarize the result of Lemma 9. □

As the direct benefits of Lemma 9, we get Corollary 10.

Corollary 10. *For every $i = 1, 2, 3$, let C_i be a closed convex subset of a real Hilbert space H_i. Let $\vartheta_i : C_i \to C_i$ be a nonexpansive mapping and let $A_2 : H_1 \to H_2, A_3 : H_2 \to H_3$ be a bounded linear operator with the adjoint operator A_2^* and A_3^*, respectively. Assume that $\overline{x}_1 \in H_1, \overline{x}_2 = A_2\overline{x}_1, \overline{x}_3 = A_3\overline{x}_2$. Let $\vartheta, \varrho : C \to C$ be nonspreading and κ-strictly pseudo-nonspreading mappings, respectively. Assume that $\underline{\Omega} \cap F(\vartheta) \cap F(\varrho) \neq \emptyset$ and let the sequence $\{x_n\}$ generated by $u, x_1 \in C$, and*

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}\left(I_1 - \lambda_n\left(a\left(I_1 - \vartheta\right) + (1-a)\left(I_1 - \varrho\right)\right)\right) x_n \\ &\quad + \delta_n P_{C_1}(I_1 - \lambda_1(I_1 - \vartheta_1))(x_n^1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2(I_2 - \vartheta_2)))x_n^2 \\ &\quad + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3(I_3 - \vartheta_3)))x_n^3)), \end{aligned}$$

for all $n \geq 1$ and $a \in (0,1)$, $I_i : H_i \to H_i$ is an identity mapping, for all $i = 1, 2, 3$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and $x_n = x_n^1, x_n^2 = A_2 x_n^1, x_n^3 = A_3 x_n^2$, for all $n \in \mathbb{N}, 0 < \lambda_i < 1$, for all $i = 1, 2, 3$ and $\gamma_j > 0$, for all $j = 2, 3$. Suppose that the conditions (i) to (v) are true;

(i) $\lim_{n \to +\infty} \alpha_n = 0, \sum_{n=1}^{+\infty} \alpha_n = +\infty$;

(ii) $\gamma_2(1 + \gamma_3) \leq \frac{1}{L}$, *where* $L = \max\{L_{A_1}, L_{A_2}\} \leq 1$, *where* L_{A_1}, L_{A_2} *are spectral radii of* $A_2 A_2^*, A_3 A_3^*$, *respectively;*

(iii) $0 < a \leq \beta_n, \gamma_n, \delta_n \leq b < 1$, *for some* $a, b > 0$, *for all* $n \in \mathbb{N}$;

(iv) $\sum_{n=1}^{+\infty} \lambda_n < +\infty$ *and* $0 < \lambda_n < 1 - \kappa$, *for all* $n \in \mathbb{N}$;

(v) $\sum_{n=1}^{+\infty} |\alpha_{n+i} - \alpha_n|, \sum_{n=1}^{+\infty} |\beta_{n+i} - \beta_n|, \sum_{n=1}^{+\infty} |\gamma_{n+1} - \gamma_n| < +\infty$.

Then the sequence $\{x_n\}$ converses strongly to $z_0 = P_{\Omega \cap F(T) \cap F(S)} u$.

4. Conclusions

We have proposed a new split variational inequality in three Hilbert spaces. The convergence theorem for finding a common element of the set of solutions of such problems and the sets of fixed-points of discontinuous mappings are proved.

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A new technique for coincidence point theory in metric spaces endowed with graph

Chinda Chaichuay
Atid Kangtunyakarn*
Department of Mathematics
Faculty of Science
King Mongkut's Institute of Technology Ladkrabang
Bangkok 10520
Thailand

and

Centre of Excellence in Mathematics
Commission on Higher Education (CHE)
Si Ayutthaya Road
Bangkok 10400
Thailand

Abstract

Coincidence theory is a generalization of fixed point theory. There are many researchs combining fixed point theory and graph theory. In this paper, a new type of multi-valued mapping and g-l-graph preserving is proposed to prove a coincidence point theorem on complete metric spaces endowed with a directed graph. Supported examples of there main theorems are also introduced. Main results are sufficiently conditions for finding a vertex in the directed graph such that itself image of the defined surjective mapping is contained in the defined multivalued mapping. The proposed theorem can be apply use to obtain the similar result in a matrix space endowed with a partial order set.

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*E-mail: **beawrock@hotmail.com**

1. Introduction

Graph theoretical concepts are widely used in many real-world problem. In biology, graph theory is often used where a vertex represents regions where certain species exist and the edges represent migration path or movement between the regions. In chemistry, graph theory including study of molecules, construction of bounds and the study of atoms. In computer science, graph theory is used to represent the connecting with friends on social media, where each user is a vertex, and when users connect they create an edge.

The Banach contraction principle is the most important theorem for studying fixed point theorem. Fixed points is fundamental concepts of mathematics in many fields. For example, Nash equilibrium in economics, the theory of phase transition, renormalization group and critical phenomenon in physics. In information Technology also use fixed point for program analysis, dataflow analysis and optimization.

S.Choudhury, Metiya and Debnath [1], they propose the notation of end point of multivalued mapping in the setting of metric space endowed with graph and prove some existence results. The mapping that satisfy generalized multivalued almost G-contraction type inequalities are assumed.

Hanjing and Suantai [2], they propose a new type G-contraction multivalued mapping in a metric space endowed with a directed graph. The new theorem show that there are some fixed points of multivalued mapping will belong to the set of coincidence points of them.

From now on we will recall some mathematical background which is necessary basis for the main theorem.

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a contraction if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$. If $k = 1$, then T is a nonexpansive mapping.

The Banach contraction principle is a very importance tool to prove existence of fixed point theorem it states as follows:

Theorem 1.1 : *Let X be a complete metric space and let T be a contraction of X to itself. Then T has a unique fixed point.*

The Banach contraction principle theorem plays an important role in studying the existence of solutions of nonlinear integral equations, system of linear equations, nonlinear differential equations, and proving the convergence of algorithms in computational mathematics.

Let $C(X)$ be the set of all nonempty, closed, and bounded subsets of X. A point $x \in X$ is a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$. Nadler [3] has proved a multi-valued version of the Banach contraction principle as follows:

Theorem 1.2 : *Let (X, d) be a complete metric space and $T : X \to CB(X)$. Assume that there exists $k \in [0, 1)$ such that $H(Tx, Ty) \le kd(x, y)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.*

The theory of multi-valued mappings is an important role in various branches of pure and applied mathematics to solve many problems such as optimal control problem.

A partial order in a binary relation $\prec$ over the set X, that is, for all x, y and z in X it must satisfies the following conditions:

a) $x \prec x$,
b) If $x \prec y$ and $y \prec x$, then $x = y$,
c) If $x \prec y$ and $y \prec z$, then $x \prec z$.

Fixed point theorem for single and multi-valued mapping defined on partially ordered metric space received attention from many researcher (see, e.g., [4], [5]). Ran and Reurings [4] was the first to prove a theorem related to a partial ordering. They proved the following theorem.

Theorem 1.3 : Let $(X, \prec)$ be a partially ordered set such that every pair x, $y \in X$ has an upper and lower bounded. Let d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:

(1) There exists a $k \in (0, 1)$ with
$$d(f(x), f(y)) \le kd(x, y),$$
for all $x \ge y$.

(2) There exists an $x_0 \in X$ such that $x_0 \le f(x_0)$ or $x_0 \ge f(x_0)$. Then f is a Picard operator (PO), that is, f has a unique fixed point $x^* \in X$ and for each $x \in X, \lim_{n\to\infty} f^n(x) = x^*$.

Many research in this direction was inspired by such theorem (see, e.g., [7],[8]).

Let $G = (V(G), E(G))$ be a directed graph with $V(G)$ is a set of vertices of the graph and $E(G)$ be the set of its edges. Assume that G has no parallel edges. If x and y are vertices in G, then a path in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for all $i = 1, 2, \ldots, n$. A graph G is connected if there exists a path between any two vertices of G.

In 2007, Jachymski and Jozwik [8] introduced fixed point theory by using graph strcture on metric space. At many time, many researchers are interested in studying the such theorem, for example, recently Tiammee and Suantai [9] introduce the concepts of graph-preserving multi-valued mapping and a new type of multi-valued weak G-contraction on a metric space endowed with a directed graph G and prove some coincidence point theorems for this type of multi-valued mapping and a surjective mapping $g : X \to X$ under some conditions. In the same year, Aniruth Phon-on, Areeyuth Sama-Ae et al., [10] define a new class of Reich type multi-valued contractions on a complete metric space satisfying the g-graph preserving condition and prove a fixed point theory for such mappings. See more examples [11], [12], [13],.

In this paper we introduce a new type of multi-valued mapping and g-l-graph preserving to prove a fixed point theorem on complete metric spaces endowed with a directed graph and give an example to support our main theorem. The proof of our main theorem using different techniques from the research of [9] and [10].

2. Preliminaries

Let (X, d) be a metric space and let $CB(X)$ be the set of all nonempty closed bonded subset of X. Throughout this paper we use

$$d(x,A) = \inf\{d(x,y) : y \in A\},$$

$$D(A,B) = \inf\{d(x,B) : x \in A\}.$$

The Hausdorff-Pompeiu metric H is a mapping defined as follows:

$$H(A,B) = \max\left\{\sup_{w\in B} d(w,A), \sup_{z\in A} d(z,B)\right\}.$$

The following definitions are important for our main theorem.

Definitiion 2.1 : Let X be a nonempty set and $G = (V(G), E(G))$ be a graph such that $V(G) = X$ and let g, $l : X \to X$ be mappings. The multi-valued mapping $T : X \to CB(X)$ is called *g-l-graph preserving* if for every $x, y \in X$ such that

$$\left(g(x), g(y)\right) \in E(G) \Rightarrow \left(l(u), l(v)\right) \in E(G),$$

for all $u \in Tx$ and $v \in Ty$.

By motivated from Suantai and Tiammee [9], we give an example of the such mapping.

Example 2.1 : Let $\mathbb{N}$ be the set of natural number and let $G = (\mathbb{N}, E(G))$ and $E(G) = \{(2n-1, 2n+1) : n \in \mathbb{N}\} \cup \{(2n, 2n+2) : n > 1\} \cup \{(2n, 2n+4) : n > 1\} \cup \{(2n, 2n) : n > 1\} \cup (1,1) \cup (6,4) \cup (8,6)$. Defined $T : \mathbb{N} \to CB(\mathbb{N})$ by

$$T(x) = \begin{cases} \{2k, 2k+2\} & \text{if } x = 2k-1, \forall k \in \mathbb{N}, \\ \{1\} & \text{if } x = 2k, \forall k \in \mathbb{N}, \end{cases}$$

$g : \mathbb{N} \to \mathbb{N}$ be defined by

$$g(x) = \begin{cases} 2k & \text{if } x = 2k+2, \forall k \in \mathbb{N}, \\ 2k-1 & \text{if } x = 2k+1, \forall k \in \mathbb{N}, \\ 2 & \text{if } x = 1, 2, \end{cases}$$

and $l : \mathbb{N} \to \mathbb{N}$ be defined by

$$l(x) = \begin{cases} 2k+2 & \text{if } x = 2k, \forall k \in \mathbb{N}, \\ 3k-2 & \text{if } x = 2k-1, \forall k \in \mathbb{N}, \end{cases}$$

Then T is g-l-graph preserving.

Solution : Let $(g(x), g(y)) \in E(G)$.

If $(g(x), g(y)) = (2k - 1, 2k + 1)$, $\forall k \in \mathbb{N}$, then $(x, y) = (2k + 1, 2k + 3)$. From the definition of T, we have $Tx = \{2k + 2, 2k + 4\}$, $T_y = \{2k + 4, 2k + 6\}$. From the definition of l, we have

$$\begin{aligned} \left(l(2k+2), l(2k+4)\right) &= \left(2k+4, 2k+6\right) \in E(G), \\ \left(l(2k+2), l(2k+6)\right) &= \left(2k+4, 2k+8\right) \in E(G), \\ \left(l(2k+4), l(2k+4)\right) &= \left(2k+6, 2k+6\right) \in E(G), \\ \left(l(2k+4), l(2k+6)\right) &= \left(2k+6, 2k+8\right) \in E(G). \end{aligned}$$

If $(g(x), g(y)) = (2k, 2k+2)$ or $(2k, 2k+4)$ or $(2k, 2k), \forall k \in \mathbb{N}$, then $Tx = \{1\}$, $Ty = \{1\}$. From the definition of l, we have $(l(1), l(1)) = (1, 1) \in E(G)$.

If $(g(x), g(y)) = (1, 1)$, then $(x, y) = (3, 3)$. It follows that $Tx = Ty = \{4, 6\}$.

From the definition of l, we have

$$\begin{aligned}
(l(4), l(4)) &= (6,6) \in E(G), \\
(l(4), l(6)) &= (6,8) \in E(G), \\
(l(6), l(4)) &= (8,6) \in E(G), \\
(l(6), l(6)) &= (8,8) \in E(G).
\end{aligned}$$

If $(g(x), g(y)) = (8, 6)$ or $(6, 4)$, then $Tx = Ty = \{1\}$. From the definition of l, we have $(l(1), l(1)) = (1, 1) \in E(G)$. Hence T is g-l-graph preserving.

Remark 2.1 : If $l = I$, where I is an identity mapping, then g-l-graph preserving is redued to g-graph preserving, see [9].

Definition 2.2 : Let (X, d) be a metric spaces, $G = (V(G), E(G))$ be a direct graph such that $V(G) = X$ and the mappings $g, l : X \to X$. Then $T : X \to CB(X)$ is said to be (l, g)-G *contraction* if there exists $0 < \alpha < \beta$ with $\alpha + \beta < \frac{1}{2}$ and $L \geq 0$ with

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta d(l(u), l(v)) + LD(g(y), Tx),$$

for all $x, y \in X$ and $u \in Tx$, $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.

Example 2.2 : Let $\mathbb{N}$ be the set of natural number and let $G = (\mathbb{N}, E(G))$ be a directed graph where $E(G) = \{(2n, 2n+2) : n \in \mathbb{N}\} \cup \{(2n-1, 2n+1) : n \in \mathbb{N}\} \cup \{(2n-1, 2n+3) : n \in \mathbb{N}\} \cup \{(2n-1, 2n-1) : n \in \mathbb{N}\}$. Defined $T : \mathbb{N} \to CB(\mathbb{N})$ by

$$T(x) = \begin{cases} \{2k-1, 2k+1\} & \text{if } x = 2k, \forall k \in \mathbb{N}, \\ \{3\} & \text{if } x = 2k-1, \forall k \in \mathbb{N}, \end{cases}$$

$g : \mathbb{N} \to \mathbb{N}$ be defined by

$$g(x) = \begin{cases} 2k+1 & \text{if } x = 2k-1, \forall k \in \mathbb{N}, \\ 2k+2 & \text{if } x = 2k, \forall k \in \mathbb{N}, \end{cases}$$

and $l : \mathbb{N} \to \mathbb{N}$ be defined by

$$l(x) = \begin{cases} 2k+4 & \text{if } x = 2k+2, \forall k \in \mathbb{N}, \\ 2k+3 & \text{if } x = 2k+1, \forall k \in \mathbb{N}, \\ 2 & \text{if } x = 1, 2. \end{cases}$$

Then T is (l, g)-G contraction.

Solution : Let $(g(x), g(y)) \in E(G)$.

If $(g(x), g(y)) = (2n, 2n+2) \in E(G)$ for $n \in \mathbb{N}$, then $x = 2(n-1)$, $y = 2n$ and $Tx = \{2n-3, 2n-1\}$, $Ty = \{2n-1, 2n+1\}$ for $n \in \mathbb{N}$. Then $H(Tx, Ty) = 2$ and $D(g(y), Tx) = 3$.

Let $u \in Tx$.

If $u = 2n-3$, then $l(u) = 2n-1$ for $n \in \mathbb{N}$,
If $u = 2n-1$, then $l(u) = 2n+1$ for $n \in \mathbb{N}$.

Let $v \in Ty$.

If $v = 2n-1$, then $l(v) = 2n+1$ for $n \in N$,
If $v = 2n+1$, then $l(v) = 2n+3$ for $n \in N$.

It implies that $(l(u), l(v)) = (2n-1, 2n+1)$ or $(2n-1, 2n+3)$ or $(2n+1, 2n+1)$ or $(2n+1, 2n+3) \in E(G)$.

Put $\alpha = \frac{1}{8}, \beta = \frac{2}{8}, L = 2, L = 2$, we have

$$H(Tx, Ty) < \alpha d(g(x), g(y)) + \beta d(l(u), l(v)) + LD(g(y), Tx),$$

If $(g(x), g(y)) = (2n-1, 2n+1)$ or $(2n-1, 2n+3)$ or $(2n-1, 2n-1) \in E(G)$, we get $Tx = \{3\} = Ty$. If $u = v = 3$, then $l(3) = 5$. Therefore $(l(3), l(3)) \in E(G)$. Then $H(Tx, Ty) = 0$. It is obvious that,

$$H(Tx, Ty) \le \alpha d(g(x), g(y)) + \beta d(l(u), l(v)) + LD(g(y), Tx),$$

where $x, y = 2n-1$ for all $n \in \mathbb{N}$.

Hence T is (l, g)-G contraction.

Lemma 2.2 : [3] *Let (X, d) be a metric space. If $A, B \in CB(X)$ and $a \in A$, Then for each $\varepsilon > 0$, there exists $b \in B$ such that*

$$d(a, b) \le H(A, B) + \varepsilon.$$

Property A : [13] For every sequence $\{x_n\}_{n \in \mathbb{N}}$ in X, If $x_n \to x$ and $(x_{n+1}, x_n) \in E(G)$, there is a subsequence $\{x_{k_n}\}_{n \in \mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

3. Main Results

Let (X, d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph with $V(G) = X$. Let $g : X \to X$ be a surjective mapping and let $l : X \to X$ be a nonexpansive mapping. Suppose that the multi-value mapping $T : X \to CB(X)$ is satified the following properties:

1) T is a g-l-graph preserving;
2) T is (l,g)-G contraction;
3) X has Property A;
4) there exists $x_0 \in X$ such that $C(g(x_0), y) \in E(G)$ for some $y \in Tx_0$.
5) $H(Tx, Ty) \leq d(g(x), g(y)) + d(l(g(x)), l(g(y))) + D(g(y), Tx)$, for all $x, y \in X$ such that $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.

Then there exists $u \in X$ such that $g(u) \in Tu$.

Proof : Since g is a surjection, there exists $x_1 \in X$ such that $g(x_1) \in Tx_0$. From 4), we have

$$\big(g(x_0), g(x_1)\big) \in E(G). \tag{3.1}$$

Put $n_1 \in \mathbb{N}$, we have

$$(\alpha + \beta)^{n_1} < (\beta - \alpha) d\big(g(x_0), g(x_1)\big). \tag{3.2}$$

By Lemma 2.2, there is $g(x_2) \in Tx_1$ such that

$$d(g(x_1)) g(x_2)) \leq H(Tx_0, Tx_1) + (\alpha + \beta)^{n_1}. \tag{3.3}$$

Since $(g(x_0), g(x_1) \in E(G), g(x_1) \in Tx_0, g(x_2) \in Tx_1$ and T is g-l-graph preserving, we have $(l(g(x_1)), l(g(x_2))) \in E(G)$.

From (3.2) and (3.3), we have

$$\begin{aligned} d(g(x_1)) g(x_2)) &\leq H(Tx_0, Tx_1) + (\alpha + \beta)^{n_1} \\ &\leq \alpha d(g(x_0), g(x_1)) + \beta d(l(g(x_1)), l(g(x_2))) \\ &\quad + LD(g(x_1), Tx_0) + (\alpha + \beta)^{n_1} \\ &\leq \alpha d(g(x_0), g(x_1)) + \beta d(g(x_1), g(x_2)) \\ &\quad + (\beta - \alpha) d(g(x_0), g(x_1)). \end{aligned}$$

It implies that

$$d(g(x_1), g(x_2)) \leq \frac{\beta}{1 - \beta} d(g(x_0), g(x_1)).$$

By using the same method as (3.1), we have

$$(g(x_1), g(x_2)) \in E(G).$$

Choose $n_2 > n_1$ such that

$$(\alpha+\beta)^{n_2} < (\beta-\alpha)d\big(g(x_1), g(x_2)\big). \tag{3.4}$$

From Lemma 2.2, there exists $g(x_3) \in Tx_2$ such that

$$d(g(x_2), g(x_3)) \leq H(Tx_1, Tx_2) + (\alpha+\beta)^{n_2} \tag{3.5}$$

Since $(g(x_1), g(x_2)) \in E(G)$, $g(x_2) \in Tx_1, g(x_3) \in Tx_2$ and T is g-l-graph preserving, we have

$$\big(l(g(x_2)), l(g(x_3))\big) \in E(G).$$

From (3.4) and (3.5), we have

$$\begin{aligned} d(g(x_2), g(x_3)) &\leq H(Tx_1, Tx_2) + (\alpha+\beta)^{n_2} \\ &\leq \alpha d(g(x_1), g(x_2)) + \beta d(l(g(x_2)), l(g(x_3))) \\ &\quad + LD(g(x_2), Tx_1) + (\beta-\alpha)d(g(x_1)), g(x_2)) \\ &\leq \alpha d(g(x_1), g(x_2)) + \beta d(g(x_2), g(x_3)) \\ &\quad + (\beta-\alpha)d(g(x_1)), g(x_2)). \end{aligned}$$

It implies that

$$d(g(x_2), g(x_3)) \leq \frac{\beta}{1-\beta} d(g(x_1), g(x_2)).$$

Continuous on this way, for every $k \in \mathbb{N}$, we have $g(x_{k+1}) \in Tx_k$ with

$$d(g(x_k), g(x_{k+1})) \leq \frac{\beta}{1-\beta} d(g(x_{k-1}), g(x_k)) \tag{3.6}$$

and $(g(x_{k-1}), g(x_k)) \in E(G)$, $(l(g(x_k)), l(g(x_{k+1}))) \in E(G)$. .

From (3.6), we have

$$\begin{aligned} d\big(g(x_k), g(x_{k+1})\big) &\leq \frac{\beta}{1-\beta} d\big(g(x_{k-1}), g(x_k)\big) \\ &\leq a\Big(ad\big(g(x_{k-2}), g(x_{k-1})\big)\Big) \\ &= a^2 d\big(g(x_{k-2}), g(x_{k-1})\big) \\ &\vdots \\ &\leq a^k d\big(g(x_0), g(x_1)\big), \end{aligned} \tag{3.7}$$

where $a = \frac{\beta}{1-\beta}$ for all $k \in \mathbb{N}$.

For every $n, k \in \mathbb{N}$ and (3.7), we have

$$\begin{aligned} d\left(g(x_{n+k}), g(x_n)\right) &\le \sum_{j=n}^{n+k-1} d\left(g(x_{j+1}), g(x_j)\right) \\ &\le \sum_{j=n}^{n+k-1} a^j d\left(g(x_0), g(x_1)\right) \qquad (3.8) \\ &\le \frac{a^n}{1-a} d\left(g(x_0), g(x_1)\right). \end{aligned}$$

Since $\lim_{n\to\infty} a^n = 0$ and (3.8), we can conclude that the sequence $\{g(x_n)\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} g(x_n) = g(u). \qquad (3.9)$$

From the property A, there is a subsequence $g(x_{k_n})$ of $g(x_n)$ such that

$$(g(x_{k_n}), g(u)) \in E(G). \qquad (3.10)$$

Since l is a nonexpansive mapping and (3.9), we have

$$\lim_{n\to\infty} l(g(x_n)) = l(g(u)). \qquad (3.11)$$

From the property A, without loss of generality, there exists a subsequence $l(g(x_{k_n}))$ of $l(g(x_n))$ such that

$$(l(g(x_{k_n})), l(g(u))) \in E(G). \qquad (3.12)$$

From the condition 5), we get

$$\begin{aligned} D(g(u), Tu) &\le d\left(g(u), g(x_{k_n+1})\right) + D\left(g(x_{k_n+1}), Tu\right) \\ &\le d\left(g(u), g(x_{k_n+1})\right) + H\left(Tx_{k_n}, Tu\right) \\ &\le d\left(g(u), g(x_{k_n+1})\right) + d\left(g(x_{k_n}), g(u)\right) \\ &\quad + d\left(l\left(g(x_{k_n})\right), l\left(g(u)\right)\right) + D\left(g(u), Tx_{k_n}\right) \\ &\le d\left(g(u), g(x_{k_n+1})\right) + d\left(g(x_{k_n}), g(u)\right) \\ &\quad + d\left(g(x_{k_n}), g(u)\right) + d\left(g(u), g(x_{k_n+1})\right). \end{aligned}$$

Since $\lim_{n\to\infty} g(x_n) = g(u)$, we have $D(g(u), Tu) = 0$. Since Tu is a closed set, we have $g(u) \in Tu$. This complete the proof. □

Definition 3.1 : Let $(X, \prec)$ be a partially order set. Let $A, B \subset X$ and let $l : X \to X$ be a mapping. Then $A \prec B$ if $l(a) < l(b)$ for all $a \in A$ and $b \in B$.

Definition 3.2 : Let (X, d) be a metric space endowed with a partial order $\prec$. Let $g : X \to X$ be surjective, $l : X \to X$ be a mapping and let $T : X \to CB(X)$ is say to be *l,g-increasing* if for any $x, y \in X$,

$$g(x) < g(y) \Rightarrow Tx \prec Ty.$$

Example 3.1 : Let $\mathbb{N}$ be the set of natural number and let the multi-value mapping $T : \mathbb{N} \to 2^{\mathbb{N}}$ defined by

$$Tx = \{x^2 + 3, x^2 + 5\},$$

for all $x \in \mathbb{N}$. Let the mappings $g : \mathbb{N} \to \mathbb{N}$ defined by $g(x) = x + 1$ for all $x \in \mathbb{N}$ and $l : \mathbb{N} \to \mathbb{N}$ defined by $l(x) = 2x$ for all $x \in \mathbb{N}$. Then T is l, g -increasing.

Corollary 3.2 : *Let (X, d) be a metric space endowed with a partial order $\prec$. Let $g : X \to X$ be a surjetive mapping, $l : X \to X$ be a nonexpansive mapping and let $T : X \to CB(X)$ be a multi-valued mapping. Suppose the following conditions hold:*

1) *T is an l,g-increasing,*
2) *There exists $x_0 \in X$ and $u \in Tx_0$ such that $g(x_0) < u$,*
3) *For every sequence $\{x_k\}$ such $g(x_k) < g(x_{k+1})$ or all $k \in \mathbb{N}$ and $g(x_k)$ converse to $g(x)$ for some $x \in X$ such that $g(x_k) < g(x)$,*
4) *For every sequence $\{x_k\}$ such $l(g(x_k)) < l(g(x_{k+1}))$ for all $k \in \mathbb{N}$ and $l(g(x_k)$ converse to $l(g(x)$ for some $x \in X$ such that $l(g(x_k)) < l(g(x))$,*
5) *There exists $0 < \alpha < \beta$ with $\alpha + \beta < \frac{1}{2}$ and $L > 0$ such that*

$$H(Tx, Ty) \le \alpha d\big(g(x), g(y)\big) + \beta d\big(l(u), l(v)\big) + LD\big(g(y), Tx\big),$$

for all $x, y \in X$ and $u \in Tx$, $u \in Ty$ with $g(x) < g(y)$ and $l(u) < l(v)$ and

$$H(Tx, Ty) \le d\big(g(x), g(y)\big) + d\big(l(g(x)), l(g(y))\big) + D\big(g(y), Tx\big),$$

for all $x, y \in X$ with $g(x) < g(y)$ and $l(g(X)) < l(g(y))$,

6) *The metric d is complete.*

Then there exists $u \in X$ such that $u \in Tu$.

***Proof*:** Let $G = (V(G), E(G)$ where $X = V(G)$ and $E(G) = \{(x, y) : x < y, \forall x, y \in X\}$. First, we show that T is g-l-graph preserving. Let $(g(x), g(y)) \in E(G)$, we have $Tx \prec Ty$. For every $u \in Tx$ and $v \in Ty$, we have $l(u) < l(v)$. Then $(l(u), l(v) \in E(G))$. Then T is g-l-graph preserving. It is easy to see that the conditions 2)-5) satisfying conditions 2)-4) in theorem 3.1. From theorem 3.1, we can concluded the desired result. □

To support our main theorem we give the following example.

Example 3.2 : Let X = {3, 4, 5, 6, 7, 8} and $d(x, y) = |x - y|$ for all $x, y \in X$, where $E(G) = \{(4,6),(6,8)\} \cup \{(3,5),(5,7),(3,7)\} \cup \{(3,3),(5,5),(6,6),(7,7)\} \cup \{(6,7),(7,6),(5,3),(7,3),(7,5)\}$. Defined the multi-valued mapping $T : X \to CB(X)$ by

$$T(x) = \begin{cases} \{3,5\} & \text{if } x = 4, \\ \{5,7\} & \text{if } x = 6, \\ \{5\} & \text{if } x = 3,5,7,8, \end{cases}$$

the mapping $g : X \to X$ be defined by

$$g(x) = \begin{cases} 3 & \text{if } x = 7, \\ 5 & \text{if } x = 3, \\ 7 & \text{if } x = 5, \\ 4 & \text{if } x = 8, \\ 6 & \text{if } x = 4, \\ 8 & \text{if } x = 6, \end{cases}$$

and the mapping $l : X \to X$ be defined by

$$i(x) = \begin{cases} 6 & \text{if } x = 3,4,6,7,8, \\ 7 & \text{if } x = 5. \end{cases}$$

Then there exists $3 \in X$ such that $g(3) \in T3$.

Solution : It is easy to see that l is a nonexpansive mapping and g is surjective mapping.

First, we show that T is g-l graph preserving. Let $(g(x), g(y)) \in E(G)$. If $(4, 6), (6, 8) \in E(G)$, then $x = 8, 4$ and $y = 4, 6$. From the definition of T, we have

$$T8 = \{5\}, T4 = \{3, 5\} \text{ and } T6 = \{5, 7\}.$$

From $T8$; $T4$ and the definition of l, we have

$$(l(5), l(3)) = (7, 6) \in E(G),$$
$$(l(5), l(5)) = (7, 7) \in E(G).$$

From $T4$,$T6$ and the definition of l, we have

$$(l(3), l(5)) = (6, 7) \in E(G),$$
$$(l(3), l(7)) = (6, 6) \in E(G),$$
$$(l(5), l(5)) = (7, 7) \in E(G),$$
$$(l(5), l(7)) = (7, 6) \in E(G).$$

If $(g(x), g(y)) = (3, 5), (5, 7), (3, 7), (3, 3), (5, 5), (7, 7), (5, 3), (7, 3), (7, 5)$, then $x, y =, 3, 5, 7$. From the definition of T, we have $Tx = Ty = \{5\}$ where $x, y = 3, 5, 7$. It follows that $(l(5), l(5)) = (7, 7) \in E(G)$.

If $(g(x), g(y)) = (6, 7)$, then $x = 4$ and $y = 5$. From the definition of T, we have $T4 = \{3, 5\}$ and $T5 = \{5\}$.

From $T4$, $T5$ and the definition of l, we have

$$(l(3), l(5)) = (6, 7) \in E(G),$$
$$(l(5), l(5)) = (7, 7) \in E(G).$$

If $(g(x), g(y)) = (7, 6)$, then $x = 5$ and $y = 4$. From the definition of T, we have $T5 = \{5\}$ and $T4 = \{3, 5\}$.

From $T5$, $T4$ and the definition of l, we have

$$(l(5), l(3)) = (7, 6) \in E(G),$$
$$(l(5), l(5)) = (7, 7) \in E(G).$$

If $(g(x), g(y)) = (6, 6)$, then $x = y = 4$. From the definition of T, we have $T4 = \{3, 5\}$.

From $T4$ and the definition of l, we have

$$(l(3), l(3)) = (6, 6) \in E(G),$$
$$(l(3), l(5)) = (6, 7) \in E(G).$$
$$(l(5), l(3)) = (7, 6) \in E(G),$$
$$(l(5), l(5)) = (7, 7) \in E(G).$$

Hence T is g-l graph preserving.

Next, we show that T is (l, g)-G contractive mapping. Let $(g(x), g(y)) \in E(G)$.

Put $\alpha = \frac{1}{8}, \beta = \frac{2}{8}$ and $L = 3$.

If $(4, 6) \in E(G)$, then $x = 8$ and $y = 4$. $T8 = \{5\}$ and $T4 = \{3, 5\}$.

It follows that

$$H(T8, T4) = 2,$$

and

$$D(g(4), T8) = 1.$$

It is easy to conclude that

$$H(Tx,Ty) \le \alpha d\big(g(x),g(y)\big)+\beta d\big(l(u),l(v)\big)+LD\big(g(y),Tx\big),$$

where $x = 8$ and $y = 4$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.

If $(6, 8) \in E(G)$, then $x = 4$ and $y = 6$. Then $T4 = \{3, 5\}$ and $T6 = \{5, 7\}$.

It follows that $H(T4, T6) = 2$ and $D(g(6), T4) = 3$.

It is easy to conclude that

$$H(Tx,Ty) \le \alpha d(g(x),g(y))+\beta d(l(u),l(v))+LD(g(y),Tx),$$

where $x = 4$ and $y = 6$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.

If $(3,5),(5,7),(3,7),(3,3),(5,5),(7,7),(5,3),(7,3),(7,5) \in E(G)$. Then $Tx = Ty = \{5\}$ where $x, y = 3, 5, 7$. It follows that $G(Tx, Ty) = 0$. It is easy to conclude that

$$H(Tx,Ty) \le \alpha d(g(x),g(y))+\beta d(l(u),l(v))+LD(g(y),Tx),$$

where $x, y = 3, 5, 7$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.

If $(6, 7) \in E(G)$, then $x = 4$ and $y = 5$. It follows that $T4 = \{3, 5\}$ and $T5 = \{5\}$.

It implies that $H(T4, T5) = 2$ and $D(g(5), T4) = 2$. It is easy to conclude that

$$H(Tx,Ty) \le \alpha d(g(x),g(y))+\beta d(l(u),l(v))+LD(g(y),Tx),$$

where $x = 4$ and $y = 5$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.

If $(7, 6) \in E(G)$, then $x = 5$ and $y = 4$. It follows that $T5 = \{5\}$ and $T4 = \{3, 5\}$.

It follows that $H(T5, T4) = 2$ and $D(g(4), T5) = 1$. It is easy to conclude that

$$H(Tx,Ty) \le \alpha d(g(x),g(y))+\beta d(l(u),l(v))+LD(g(y),Tx),$$

where $x = 5$ and $y = 4$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.

If $(6, 6) \in E(G)$, then $x = y = 4$. It follows that $T4 = \{3, 5\}$.

It follows that $H(T4, T4) = 0$ and $D(g(4), T4) = 1$. It is easy to conclude that

$$H(Tx,Ty) \le \alpha d(g(x),g(y))+\beta d(l(u),l(v))+LD(g(y),Tx),$$

where $x = y = 4$ with $u \in Tx$ and $v \in Ty$ such that $(g(x), g(y)) \in E(G)$ and $(l(u), l(v)) \in E(G)$.

Hence T is (l, g)-G contraction.

Let $(g(x), g(y)) \in E(G)$.

If $(4, 6) \in E(G)$, then $x = 8$ and $y = 4$. It follows that $H(T8, T4) = 2$ and $D(g(4), T8) = 1$.

It is easy to see that

$$H(Tx,Ty) \leq d\big(g(x),g(y)\big)+d\big(l(g(x)),l(g(y))\big)+D(g(y),Tx),$$

where $x = 8$ and $y = 8$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.

If $(6, 8) \in E(G)$, then $x = 4$ and $y = 6$. It follows that $H(T4, T6) = 2$ and $D(g(6), T4) = 3$.

It is easy to see that

$$H(Tx,Ty) \leq d\big(g(x),g(y)\big)+d\big(l(g(x)),l(g(y))\big)+D(g(y),Tx),$$

where $x = 4$ and $y = 6$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.

If $(3,5),(5,7),(3,7),(3,3),(5,5),(7,7),(5,3),(7,3),(7,5) \in E(G)$. Then $Tx = Ty = \{5\}$ for all $x, y = 3, 7, 5$. It follows that

$$H(Tx,Ty) \leq d(g(x),g(y))+d\big(l(g(x)),l(g(y))\big)+D\big(g(y),Tx\big),$$

where $x, y = 3, 5, 7$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.

If $(6, 7) \in E(G)$, then $x = 4$ and $y = 5$. It follows that $H(T4, T5) = 2$ and $D(g(4), T4) = 2$

It is easy to see that

$$H(Tx,Ty) \leq d(g(x),g(y))+d\big(l(g(x)),l(g(y))\big)+D\big(g(y),Tx\big),$$

where $x = 4$ and $y = 5$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.

If $(7, 6) \in E(G)$, then $x = 5$ and $y = 4$. It follows that $H(T4, T5) = 2$ and $D(g(4), T4) = 1$

It is easy to see that

$$H(Tx,Ty) \leq d(g(x),g(y))+d\big(l(g(x)),l(g(y))\big)+D\big(g(y),Tx\big),$$

where $x = 5$ and $y = 4$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.

If $(6, 6) \in E(G)$, then $x = y = 4$. It follows that $H(T4, T4) = 0$ and $D(g(4), T4) = 1$.

It is easy to see that

$$H(Tx,Ty) \leq d(g(x),g(y))+d\big(l(g(x)),l(g(y))\big)+D(g(y),Tx),$$

where $x = y = 4$ with $(g(x), g(y)) \in E(G)$ and $(l(g(x)), l(g(y))) \in E(G)$.

Hence the condition 5) in Theorem 3.1 is satisfied.

Since $5 \in T7$, then $(g(7), 5) = (3, 5) \in E(G)$. Then condition 4) in Theorem 3.1 is satisfied. It is easy to see that X has Property A. From Theorem 3.1, then there exists $3 \in X$ such that $g(3) \in T3$.

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Author Biography

Name Mr. Chinda Chaichuay

Date of Birth 7^{th} August 1971

Address 132, Soi Ramkhamhaeng 39, Ramkhamhaeng Road, Wangthonglang, Bangkok, 10310, Thailand.

Education (1994) Bachelor of Science in Mathematics (2nd Class Hons).
Ramkhamhaeng University
(1999) Master of Science in Applied Mathematics
King Mongkut's Institute of Technology Ladkrabang
(2021) Doctor of Philosophy in Applied Mathematics
King Mongkut's Institute of Technology Ladkrabang

Academic Publications

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