GENERAL EXACT SOLUTION FOR SYSTEMS OF COUPLED LINEAR MATRIX DIFFERENTIAL EQUATIONS

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บทคัดย่อ

ในงานวิจัยนี้เราจะพิจารณาระบบของคู่สมการเชิงอนุพันธ์เมทริกซ์เชิงเส้นโดยการประยุกต์ใช้ผลคูณโคนเนคเคอร์ และตัวดำเนินการเวกเตอร์ เราจะได้สูตรที่ชัดแจ้งของผลเฉลยทั่วไปของระบบดังกล่าวในรูปอนุกรมของเมทริกซ์ที่เกี่ยวกับฟังก์ชันเลขชี้กำลัง และฟังก์ชันไฮเพอร์โบลิก นอกจากนี้เรายังพิจารณากรณีเฉพาะต่างๆของระบบสมการนี้

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Abstract

In this paper, we investigate a system of coupled linear matrix differential equations. Applying Kronecker products and the vector operator, we obtain an explicit formula of the general solution to this system in terms of matrix series concerning exponentials and hyperbolic functions. Several special cases of this system are also discussed.

Keywords : linear matrix differential equation, Kronecker product, vector operator, matrix exponential.

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Chapter 1

Introduction

1.1 Research Motivation

Matrix differential equations bring attentions to many authors in both theory (see e.g. [2, 6, 7, 8]) and computational aspects (see e.g. [1, 9]). The theory can be applied in a broad range of scientific fields, e.g. statistics ([3, 10, 12]), game theory ([5]), ecometrics ([10, 12]), control and system theory ([4, 11]).

The simplest form of linear matrix differential equation is the equation

$$X'(t) = AX(t) \tag{1.1}$$

here, A is a given square matrix and $X(t)$ is an unknown matrix-valued function to be solved. In fact, the equation (1.1) has a unique solution given by a one-parameter matrix-valued function

$$X(t) = e^{(t-t_0)A}X(t_0). \tag{1.2}$$

A system of coupled linear matrix differential equations takes the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D, \\ Y'(t) &= EX(t)F + GY(t)H. \end{aligned} \tag{1.3}$$

Here, $A, B, C, D, E, F, G, H \in M_n(\mathbb{C})$ are given constant matrices and $X(t), Y(t)$ are unknown matrix-valued functions to be solved. One of a useful system of coupled matrix differential equations is the system

$$\begin{aligned} X'(t) &= AX(t) + BY(t), \\ Y'(t) &= CX(t) - A^T Y(t), \end{aligned}$$

in two unknown matrix-valued functions $X(t)$ and $Y(t)$. This system has been used to obtain the solution of an optimal control problem (see [1]).

In this work, we investigate a system of coupled linear matrix differential equations. We apply Kronecker products and the vector operator to reduce our complex system to the simplest form (1.1). Thus, an explicit formula of the general solution to this system is obtained in terms of matrix series concerning exponentials and hyperbolic functions. In particular, we obtain general solution of several special cases of the main system. When initial conditions are imposed to these problems, the solution of each case is unique and, in fact, easy to compute.

This work is structured as follows. We provide tools for solving a system of matrix differential equations in the next section. These includes Kronecker products, the vector operator, and functions of matrices defined by power series. Our main content, Section 3, deals with a system of two linear matrix differential equations. In Section 4, we discuss certain special cases of our main problem when initial conditions are imposed.

1.2 Objectives of the study

1) To solve a system of linear matrix differential equations.

2) To investigate initial value problems for the system of linear matrix differential equations when initial conditions are imposed.

1.3 Scopes of the study

We solve the following coupled linear matrix differential equations:

$$X'(t) = AX(t)B + CY(t)D,$$
$$Y'(t) = DX(t)C + BY(t)A,$$

where A, B, C, D, are given complex square matrices such that $DB = AD$, $BD = DA$, $AC = CB$ and $CA = BC$, and $X(t)$, $Y(t)$ are unknown matrix-valued functions. All matrices considered here are complex.

1.4 Benefits of the Study

To provide further mathematical theory for linear matrix differential equations.

1.5 Research methodology

1) Study advanced topics in matrix theory.

2) Study background in multilinear algebra.

3) Collect and study research papers and textbooks concerning Kronecker products and the vector operator.

4) Determine the objectives and scope of the research.

5) Solve a system of linear matrix differential equations by using Kronecker products and the vector operator.

6) Investigate initial value problems for the system of linear matrix differential equations when initial conditions are imposed.

7) Conclude the results, make suggestions for further works and write the thesis.

Chapter 2

Preliminaries

In this chapter, we provide basic tools for solving linear matrix differential equations. these include Kronecker product, vector operator, matrix exponential, and derivative of a matrix-valued function.

2.1 Kronecker product

Definition 2.1. Let $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{p,q}$. The Kronecker product of A and B is defined by

$$A \otimes B = [a_{ij}B]_{ij} \in M_{mp,nq}.$$

Example 2.1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Find $A \otimes B$.

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 1B & 2B \\ 3B & 4B \end{bmatrix} \\ &= \begin{bmatrix} 1\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} & 2\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ 3\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} & 4\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 5 & 6 & 10 & 12 \\ 7 & 8 & 14 & 16 \\ 15 & 18 & 20 & 24 \\ 21 & 24 & 28 & 32 \end{bmatrix}. \end{aligned}$$

Theorem 2.2. (e.g. [13]) For matrices A, B, C with appropriate sides, we have

1. $(\alpha A) \otimes B = \alpha(A \otimes B) = A \otimes (\alpha B)$ for all $\alpha \in \mathbb{F}$.
2. $(A \otimes B)(C \otimes D) = (AC \otimes BD)$.
3. $(A \otimes B)^T = A^T \otimes B^T$.
4. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.
5. $(A + B) \otimes C = A \otimes C + B \otimes C$.
6. $A \otimes (B + C) = A \otimes B + A \otimes C$.

2.2 Vector operator

Let us introduce a column-stacking operator is very useful for solving linear matrix equations.

Definition 2.2. The *vector operator* Vec : $M_{m,n}(\mathbb{C}) \to \mathbb{C}^{mn}$ is defined for each $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ by

$$\text{Vec}\, A = [a_{11} \dots a_{m1}, a_{12} \dots a_{m2}, \dots, a_{1m} \dots a_{mm}]^T$$

This operator is linear and bijective.

Example 2.3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find Vec($A$).

$$\text{Vec}(A) = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 2 \\ 5 \\ 8 \\ 3 \\ 6 \\ 9 \end{bmatrix}.$$

Theorem 2.4. (e.g. [13]) The Kronecker product and the vector operator are related by

$$\text{Vec}(AXB) = (B^T \otimes A)\,\text{Vec}\, X \tag{2.1}$$

for any matrices $A \in M_{m,n}(\mathbb{C})$, $B \in M_{p,q}(\mathbb{C})$ and $X \in M_{n,p}(\mathbb{C})$.

2.3 Norms and convergence for matrices

In this section, we recall basic knowledge on norms and convergence for matrices, which can be found in a standard textbook on matrix analysis, e.g. [15].

Definition 2.3. Let V be a vector space over the field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$). A function $\|\cdot\| : V \to \mathbb{R}$ is a norm (sometimes one says vector norm) if, for all $x, y \in V$ and all $c \in \mathbb{F}$,

(1) $\|x\| \geq 0$

(1a) $\|x\| = 0$ *if and only if* $x = 0$

(2) $\|cx\| = |c|\|x\|$

(3) $\|x + y\| \leq \|x\| + \|y\|$

Example 2.5. 1. The *max norm* ($l_\infty - norm$) on $\mathbb{C}^n$ is defined for each $x \in \mathbb{C}^n$ by

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

2. The *max norm* ($l_p - norm$) on $\mathbb{C}^n$ is defined for each $x \in \mathbb{C}^n$ by

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}, p \geq 1.$$

Definition 2.4. A metric space is a set X together with a function d (called a metric or "distance function") which assigns a real number $d(x, y)$ to every pair $x, y \in X$ satisfying the properties (or axioms):

1. $d(x, y) \geq 0$ *and* $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) + d(y, z) \geq d(x, z)$.

Definition 2.5. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in metric space (X, d). We say that the sequence $\{x_n\}_{n=1}^{\infty}$

1. is convergent to $a \in X$ if, for any $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that

$$d(x_n, a) < \epsilon, \text{ for all } n \in \mathbb{N}, n \geq n_0$$

2. is Cuachy sequence if, for any $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that

$$d(x_n, x_{n+p}) < \epsilon, \text{ for all } n, p \in \mathbb{N}, n \geq n_0$$

Definition 2.6. A metric space (X, d) is called complete if any Cuachy sequence in X is convergent.

Theorem 2.6. Every normed linear space $(X, \|\cdot\|)$ is a metric space with respect to the metric given by

$$d(x, y) = \|x - y\|, \quad x, y \in X$$

Definition 2.7. Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X is called a Cauchy sequence if and only if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall m, n \geq N,\ d(x_m, x_n) < \varepsilon$.

Theorem 2.7. In a finite-dimensional normed linear space, we have Cauchy if and only if converges.

Recall that $\mathbb{C}$ is a complete metric space. Hence $\left(\sum_{k=0}^{p} a_k z^k\right)_{p=1}^{\infty}$ converges if and only if it is a Cauchy sequence.

Definition 2.8. Let z be a complex variable.Consider the series $\sum_{k=0}^{\infty} a_k z^k$ where $a_k \in \mathbb{C}$ $\forall k = 0, 1, 2, \ldots$. This series converges to a function f on a set $U \subseteq \mathbb{C}$ if and only if $\left(\sum_{k=0}^{p} a_k z^k\right)_{p=1}^{\infty}$ converges to $f(z)\ \forall z \in U$.

Definition 2.9. The spectral radius of $A \in M_n(\mathbb{C})$ is defined by

$$\rho(A) = \max\{|\lambda| | \lambda \in \sigma(A)\}.$$

Theorem 2.8. (e.g. [15]) Let $A \in M_n$. Then $\lim_{k\to\infty} A^k = 0$ if and only if $\rho(A) < 1$.

2.4 Complex power series

In this section, we review fundamental background about complex power series. All materials in this section can be found in a standard textbook on complex analysis.

Definition 2.10. Let U be an open subset of $\mathbb{C}$. A function $f : U \to \mathbb{C}$ is said to be analytic at $z_0 \in U$ if and only if there is a neighborhood V of z_0 such that f has a Taylor expansion around z_0. That is,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \quad \forall z \in V$$

1. If $\sum a_k(z - z_0)^k$ converges for all $z \in \mathbb{C}$ such that $|z - z_0| < R$ and diverges for all $z \in \mathbb{C}$ such that $|z - z_0| > R$ then we call R the radius of convergence for this series.

2. If $\sum a_k(z - z_0)^k$ converges only for $z = z_0$, then we say that the radius of converges for this series is 0

3. If $\sum a_k(z - z_0)^k$ converges for all $z \in \mathbb{C}$, then we say that the radius of converges for the series is ∞

Example 2.9. Consider the following complex-valued functions represented by power series.

1. $e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$, $z \in \mathbb{C}$.

2. $\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$, $z \in \mathbb{C}$.

3. $\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$, $z \in \mathbb{C}$.

4. $\sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$, $z \in \mathbb{C}$.

5. $\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$, $z \in \mathbb{C}$.

6. $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$, $|z| < 1$.

7. $\log(1+z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} z^k$, $|z| < 1$.

8. $\arctan(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} z^{2k+1}$, $|z| < 1$.

2.5 Functions of a matrix defined by power series

Definition 2.11. The infinite series $\sum_{n=0}^{\infty} B_n$, converges to B if the sequence $\{S_k\}_{k=1}^{\infty}$ of partial sums, where $S_k = \sum_{n=0}^{k} B_n$, converges to B.

Next, we discuss functions of a square matrix defined by power series. Let $A \in k(\mathbb{C})$ and let f be an analytic function defined on a region containing the origin and all eigenvalues of A. Then there exists $R > 0$ such that for any $|z| < R$, we have Mclaurin series expansion

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$$

where $\alpha_n = \frac{f^{(n)}(0)}{n!}$ for any natural number n and $\alpha_0 = f(0)$.

Theorem 2.10. (e.g. [15]) Let R be the radius of convergence of a scalar power series $\sum_{k=0}^{\infty} a_k z^k$, and let $A \in M_n$ be given. The matrix power series $\sum_{k=0}^{\infty} a_k A^k$ converges if $\rho(A) < R$. This condition is satisfied if there is a matrix norm $|||\cdot|||$ on M_n such that $|||A||| < R$.

Definition 2.12. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a convergent sequence for $|z| < R$. For each $A \in M_n(\mathbb{C})$ such that $\rho(A) < R$, we define

$$f(A) = \sum_{k=0}^{\infty} a_k A^k.$$

Example 2.11. Consider the following matrix functions defined by power series.

1. $\sin A = \sum_{k=0}^{\infty} \frac{(-1)^k A^{2k+1}}{(2k+1)!}$ converges for all $A \in M_n(\mathbb{C})$.

2. $\cos A = \sum_{k=0}^{\infty} \frac{(-1)^k A^{2k}}{(2k)!}$ converges for all $A \in M_n(\mathbb{C})$.

3. $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$ converges for all $A \in M_n(\mathbb{C})$ such that $\rho(A) < 1$.

4. $\log(I + A) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} A^k$ converges for all $A \in M_n(\mathbb{C})$ such that $\rho(A) < 1$.

5. $\arctan(A) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} A^{2k+1}$ converges for all $A \in M_n(\mathbb{C})$ such that $\rho(A) < 1$.

Example 2.12. $f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is an analytic function on $\mathbb{C}$. For each $A \in M_n(\mathbb{C})$, we define

$$f(A) = e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Example 2.13. $f(z) = \sinh(z) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(1-(-1)^n)z^n}{n!}$ is an analytic function on $\mathbb{C}$. For each $A \in M_n(\mathbb{C})$, we define

$$f(A) = \sinh(A) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(1-(-1)^n)A^n}{n!}.$$

Note that

$$\begin{aligned} \sinh(A) &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{A^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{n!}\right) \\ &= \frac{e^A - e^{-A}}{2}. \end{aligned}$$

Example 2.14. $f(z) = \cosh(z) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(1+(-1)^n)z^n}{n!}$ is an analytic function on $\mathbb{C}$. For each $A \in M_n(\mathbb{C})$, we define

$$f(A) = \cosh(A) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(1+(-1)^n)A^n}{n!}.$$

Note that

$$\begin{aligned} \cosh(A) &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{A^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{n!}\right) \\ &= \frac{e^A + e^{-A}}{2}. \end{aligned}$$

Lemma 2.15 (see e.g. [13]). If $X, Y \in M_k(\mathbb{C})$ are such that $XY = YX$, then $e^{X+Y} = e^X e^Y$.

Lemma 2.16 (see e.g. [13]). Let f be analytic function on a region containing the origin and all eigenvalues of A. Then

$$f(I \otimes A) = I \otimes f(A) \quad \text{and} \quad f(A \otimes I) = f(A) \otimes I. \tag{2.2}$$

Some special cases include:

$$e^{A\otimes I} = e^A \otimes I, \qquad e^{I\otimes A} = I \otimes e^A, \tag{2.3}$$

$$\sinh(A\otimes I) = \sinh(A) \otimes I, \qquad \sinh(I\otimes A) = I \otimes \sinh(A), \tag{2.4}$$

$$\cosh(A\otimes I) = \cosh(A) \otimes I, \qquad \cosh(I\otimes A) = I \otimes \cosh(A). \tag{2.5}$$

Let A represent an $n \times n$ matrix. Define $d(\lambda) = \det(A - \lambda I)$. Thus, $d(\lambda)$ is an nth degree polynomial in λ and the characteristic equation of A is $d(\lambda) = 0$. From Chapter

6, we know that if λ_i is an eigenvalue of A, then λ_i is a root of the characteristic equation, hence

$$d(\lambda_i) = 0. \tag{2.6}$$

From the Cayley–Hamilton theorem, we know that a matrix must satisfy its own characteristic equation, hence

$$d(A) = 0. \tag{2.7}$$

Let $f(\lambda)$ represent a function of λ and suppose we wish to compute $f(A)$. It can be shown, for a large class of problems, that there exists a function $q(\lambda)$ and an $n-1$ degree polynomial $r(\lambda)$ (we assume A is of order $n \times n$) such that

$$f(\lambda) = q(\lambda)d(\lambda) + r(\lambda), \tag{2.8}$$

where $d(\lambda) = \det(A - \lambda I)$. Hence, it follows that

$$f(A) = q(A)d(A) + r(A). \tag{2.9}$$

Using (2.7), we obtain

$$f(A) = r(A)$$

Therefore, it follows that any polynomial in A may be written as a polynomial of degree $n-1$ or less.

If A is an $n \times n$ matrix, then $r(\lambda)$ will be a polynomial having the form

$$r(\lambda) = \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \cdots + \alpha_1\lambda + \alpha_0. \tag{2.10}$$

If λ_i is an eigenvalue of A, then we have, after substituting (2.6) into (2.8), that

$$f(\lambda_i) = r(\lambda_i). \tag{2.11}$$

Thus, using (2.10), Eq. (2.11) may be rewritten as

$$f(\lambda_i) = \alpha_{n-1}(\lambda_i)^{n-1} + \alpha_{n-2}(\lambda_i)^{n-2} + \cdots + \alpha_1(\lambda_i) + \alpha_0. \tag{2.12}$$

if λ_i is an eigenvalue.

If we now assume that A has distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$ (note that if the eigenvalues are distinct, there must be n of them), then (2.12) may be used to generate n simultaneous linear equations for the n unknowns $\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_1, \alpha_0$:

$$\begin{aligned} f(\lambda_1) &= r(\lambda_1) = \alpha_{n-1}(\lambda_1)^{n-1} + \alpha_{n-2}(\lambda_1)^{n-2} + \cdots + \alpha_1(\lambda_1) + \alpha_0, \\ f(\lambda_2) &= r(\lambda_2) = \alpha_{n-1}(\lambda_2)^{n-1} + \alpha_{n-2}(\lambda_2)^{n-2} + \cdots + \alpha_1(\lambda_2) + \alpha_0, \\ &\vdots \\ f(\lambda_n) &= r(\lambda_n) = \alpha_{n-1}(\lambda_n)^{n-1} + \alpha_{n-2}(\lambda_n)^{n-2} + \cdots + \alpha_1(\lambda_n) + \alpha_0. \end{aligned} \tag{2.13}$$

Note that $f(\lambda)$ and the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are assumed known; hence $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$ are known, and the only unknowns in (2.13) are $\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_1, \alpha_0$.

Theorem 2.17. (e.g. [16]) Let $f(\lambda)$ and $r(\lambda)$ be defined as in (2.8). If λ_i is an eigenvalue of multiplicity k,then

$$\begin{aligned} f(\lambda_i) &= r(\lambda_i), \\ \frac{df(\lambda_i)}{d\lambda} &= \frac{dr(\lambda_i)}{d\lambda} \\ \frac{d^2 f(\lambda_i)}{d\lambda^2} &= \frac{d^2 r(\lambda_i)}{d\lambda^2} \\ &\vdots \\ \frac{d^{k-1} f(\lambda_i)}{d\lambda^{k-1}} &= \frac{d^{k-1} r(\lambda_i)}{d\lambda^{k-1}} \end{aligned}$$

where the notation $d^n f(\lambda_i)/d\lambda^n$ denotes the nth derivative of $f(\lambda)$ with respect to λ evaluated at $\lambda = \lambda_i$.

Example 2.18. Find e^A if

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution The eigenvalues of A are $\lambda_1 = \lambda_2 = \lambda_3 = 2$. Thus,

$$\begin{aligned} f(A) &= e^A, & r(A) &= \alpha_2 A^2 + \alpha_1 A + \alpha_0 I, \\ f(\lambda) &= e^\lambda, & r(\lambda) &= \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0. \\ f'(\lambda) &= e^\lambda, & r'(\lambda) &= 2\alpha_2 \lambda + \alpha_1, \\ f''(\lambda) &= e^\lambda, & r''(\lambda) &= 2\alpha_2. \end{aligned}$$

Since $f(A) = r(A)$, we have

$$e^A = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I.$$

Since $\lambda = 2$ is an eigenvalue of multiplicity three,

$$\begin{aligned} f(2) &= r(2), \\ f'(2) &= r'(2), \\ f''(2) &= r''(2), \end{aligned}$$

we obtain,

$$\begin{aligned} e^2 &= 4\alpha_2 + 2\alpha_1 + \alpha_0, \\ e^2 &= 4\alpha_2 + 2\alpha_1, \\ e^2 &= 2\alpha_2, \end{aligned}$$

or

$$\alpha_2 = \frac{e^2}{2}, \alpha_1 = -e^2, \alpha_0 = e^2.$$

Thus

$$\begin{aligned} e^A =& \frac{e^2}{2}\begin{bmatrix} 4 & 4 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} - e^2\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} + e^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ =& \begin{bmatrix} e^2 & e^2 & \frac{e^2}{2} \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix}. \end{aligned}$$

Example 2.19. Find $\sin A$ if

$$A = \begin{bmatrix} \pi & 1 & 0 \\ 0 & \pi & 0 \\ 4 & 1 & \frac{\pi}{2} \end{bmatrix}.$$

Solution The eigenvalues of A are $\lambda_1 = \frac{\pi}{2}, \lambda_2 = \lambda_3 = \pi$. Thus,

$$\begin{aligned} f(A) =& \sin A, \quad r(A) = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I, \\ f(\lambda) =& \sin\lambda, \quad r(\lambda) = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0, \\ f'(\lambda) =& \cos\lambda, \quad r'(\lambda) = 2\alpha_2\lambda + \alpha_1. \end{aligned}$$

Since $f(A) = r(A)$, we have

$$\sin A = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I.$$

Since $\lambda = \frac{\pi}{2}$ is an eigenvalue of multiplicity 1 and $\lambda = \pi$ is an eigenvalue of multiplicity 2,it follows that

$$\begin{aligned} f(\frac{\pi}{2}) =& r(\frac{\pi}{2}), \\ f(\pi) =& r(\pi), \\ f'(\pi) =& r'(\pi), \end{aligned}$$

we obtain,

$$\begin{aligned} \sin\frac{\pi}{2} =& \alpha_2(\frac{\pi}{2})^2 + \alpha_1(\frac{\pi}{2}) + \alpha_0, \\ \sin\pi =& \alpha_2(\pi)^2 + \alpha_1(\pi) + \alpha_0, \\ \cos 2 =& 2\alpha_2(\pi) + \alpha_1, \end{aligned}$$

or simplifying

$$\begin{aligned} 4 =& \alpha_2\pi^2 + 2\alpha_1\pi + 4\alpha_0, \\ 0 =& \alpha_2\pi^2 + \alpha_1\pi + \alpha_0, \\ -1 =& 2\alpha_2\pi + \alpha_1. \end{aligned}$$

Thus,

$$\alpha_2 = (\frac{1}{\pi^2})(4-2\pi), \alpha_1 = (\frac{1}{\pi^2})(-8\pi+3\pi^2), \alpha_0 = (\frac{1}{\pi^2})(4\pi^2-\pi^3).$$

Substituting these values into $\sin A = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$, we obtain

$$\sin A = \frac{1}{\pi^2}\begin{bmatrix} 0 & -\pi^2 & 0 \\ 0 & 0 & 0 \\ -8\pi & 16-10\pi & \pi^2 \end{bmatrix}.$$

Example 2.20. Find e^A if

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution Define

$$B = At = \begin{bmatrix} t & 2t \\ 4t & 3t \end{bmatrix}.$$

The problem then reduces to finding e^B. The eigenvalues of B are $\lambda_1 = 5t, \lambda_2 = -t$. Note that the eigenvalues now depend on t.

$$f(B) = e^B, \quad r(A) = \alpha_1 B + \alpha_0 I,$$
$$f(\lambda) = e^\lambda, \quad r(\lambda) = \alpha_1 \lambda + \alpha_0.$$

Since $f(B) = r(B)$, we have

$$e^B = \alpha_1 B + \alpha_0 I.$$

Also, $f(\lambda_i) = r(\lambda_i)$; hence

$$e^{5t} = \alpha_1(5t) + \alpha_0,$$
$$e^{-t} = \alpha_1(-t) + \alpha_0.$$

Thus

$$\alpha_1 = \frac{1}{6t}(e^{5t} - e^{-t}), \alpha_0 = \frac{1}{6}(e^{5t} + 5e^{-t}).$$

Substituting these values into $e^B = \alpha_1 B + \alpha_0 I$, we obtain

$$e^{At} = e^B = \frac{1}{6t}(e^{5t} - e^{-t})\begin{bmatrix} t & 2t \\ 4t & 3t \end{bmatrix} + \frac{1}{6}(e^{5t} - e^{-t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} 2e^{5t} + 4e^{-t} & 2e^{5t} - 2e^{-t} \\ 4e^{5t} - 4e^{-t} & 4e^{5t} + 2e^{-t} \end{bmatrix}.$$

Theorem 2.21. (e.g. [16]) Let $A \in M_n(\mathbb{C})$.

1. $e^0 = I$, where 0 represents the zero matrix.

2. $(e^A)^{-1} = e^{-A}$.

3. $(e^A)^T = e^{A^T}$.

4. $(e^{cI}) = e^c I$ for any $c \in \mathbb{C}$.

5. $\overline{e^A} = e^{\overline{A}}$.

6. $(e^A)^* = e^{A^*}$.

7. $e^{SAS^{-1}} = Se^A S^{-1}$ for any invertible matrix $S \in M_n(\mathbb{C})$.

2.6 Derivative of a matrix-valued function

Definition 2.13. An $n \times n$ matrix-valued function $A(t) = [a_{ij}(t)]$ is continuous at $t = t_0$ if each of its elements $a_{ij}(t)(i, j = 1, 2, \ldots, n)$ is continuous at $t = t_0$.

Definition 2.14. An $n \times n$ matrix $A(t) = [a_{ij}(t)]$ is differentiable at $t = t_0$ if each of the elements $a_{ij}(t)(i, j = 1, 2, \ldots, n)$ is differentiable at $t = t_0$ and

$$\frac{d}{dt}A(t) = \left[\frac{d}{dt}(a_{ij}(t))\right].$$

Example 2.22. Find $A'(t)$ if

$$A(t) = \begin{bmatrix} t^2 & \sin t \\ \ln t & e^{t^2} \end{bmatrix}.$$

Solution

$$\begin{aligned} A'(t) &= \frac{d}{dt}A(t) \\ &= \begin{bmatrix} \frac{d}{dt}(t^2) & \frac{d}{dt}(\sin t) \\ \frac{d}{dt}(\ln t) & \frac{d}{dt}(e^{t^2}) \end{bmatrix} \\ &= \begin{bmatrix} 2t & \cos t \\ \frac{1}{t} & 2te^{t^2} \end{bmatrix}. \end{aligned}$$

Theorem 2.23. (e.g. [16]) If A, B are constant matrix, α is a constant and $\beta(t)$ is a scalar function. Then

1. $\frac{d}{dt}(A(t) + B(t)) = \frac{d}{dt}A(t) + \frac{d}{dt}B(t)$.

2. $\frac{d}{dt}[\alpha A(t)] = \alpha \frac{d}{dt}A(t)$.

3. $\frac{d}{dt}[\beta(t)A(t)] = \frac{d}{dt}(\beta(t))A(t) + \beta(t)\frac{d}{dt}A(t)$.

4. $\frac{d}{dt}[A(t)B(t)] = \frac{d}{dt}(A(t))B(t) + A(t)\frac{d}{dt}B(t)$.

5. $\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A$.

Chapter 3

General solutions to systems of coupled linear matrix differential equations

In this chapter, we investigate a system of coupled linear matrix differential equations of the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D, \\ Y'(t) &= DX(t)C + BY(t)A. \end{aligned} \tag{3.1}$$

Here, $A, B, C, D \in M_n(\mathbb{C})$ are given matrices and $X(t), Y(t)$ are unknown matrix-valued functions. In order to express an explicit form of the solution, we impose the conditions $DB = AD$, $BD = DA$, $AC = CB$ and $CA = BC$. Note that in general the condition $DB = AD$ does not imply $BD = DA$. To See This, considered

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Similarly, the condition $AC = CB$ does not imply $CA = BC$.

3.1 The main system

A general system of coupled linear matrix differential equations takes the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D, \\ Y'(t) &= EX(t)F + GY(t)H. \end{aligned} \tag{3.2}$$

Here, $A, B, C, D, E, F, G, H \in M_n(\mathbb{C})$ are given constant matrices and $X(t), Y(t)$ are unknown matrix-valued functions to be solved.

Lemma 3.1. The general solution of the system (3.2) can be expressed as

$$\begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix} = e^{(t-t_0)P} \begin{bmatrix} \operatorname{Vec} X(t_0) \\ \operatorname{Vec} Y(t_0) \end{bmatrix},$$

where $P = \begin{bmatrix} B^T \otimes A & D^T \otimes C \\ F^T \otimes E & H^T \otimes G \end{bmatrix}$.

Proof. We follow the idea of [7]. Applying the vector operator to the system (3.2), we obtain by Theorem 2.4 that

$$\begin{aligned} \operatorname{Vec} X'(t) &= \operatorname{Vec}(AX(t)B + CY(t)D) \\ &= \operatorname{Vec}(AX(t)B) + \operatorname{Vec}(CY(t)D) \\ &= (B^T \otimes A)\operatorname{Vec} X(t) + (D^T \otimes C)\operatorname{Vec} Y(t) \end{aligned}$$

and

$$\begin{aligned}\text{Vec}\,Y'(t) &= \text{Vec}\,(EX(t)F + GY(t)H)\\ &= \text{Vec}\,(EX(t)F) + \text{Vec}\,(GY(t)H)\\ &= (F^T \otimes E)\,\text{Vec}\,X(t) + (H^T \otimes G)\,\text{Vec}\,Y(t).\end{aligned}$$

For simplicity, let us denote

$$x(t) = \begin{bmatrix}\text{Vec}\,X(t)\\ \text{Vec}\,Y(t)\end{bmatrix}.$$

Since the vector operator is a linear isomorphism, the system (3.2) is equivalent to the following vector-matrix differential equation:

$$\begin{aligned}\begin{bmatrix}\text{Vec}\,X'(t)\\ \text{Vec}\,Y'(t)\end{bmatrix} &= \begin{bmatrix}B^T \otimes A & D^T \otimes C\\ F^T \otimes E & H^T \otimes G\end{bmatrix}\begin{bmatrix}\text{Vec}\,X(t)\\ \text{Vec}\,Y(t)\end{bmatrix}\\ x'(t) &= Px(t).\end{aligned}$$

Hence, the general exact solution is given by $x(t) = e^{(t-t_0)P}x(t_0)$. □

The following lemmas are useful for deriving explicit formula of the solution of (3.1).

Lemma 3.2. For any $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$,we have

$$e^{\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}} = \begin{bmatrix}e^A & 0\\ 0 & e^B\end{bmatrix}. \tag{3.3}$$

Proof. Using standard results in matrix analysis, we obtain

$$
\begin{aligned}
e^{\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} A^k & 0 \\ 0 & B^k \end{bmatrix} \\
&= \sum_{k=0}^{\infty} \begin{bmatrix} \frac{A^k}{k!} & 0 \\ 0 & \frac{B^k}{k!} \end{bmatrix} \\
&= \lim_{x\to\infty} \sum_{k=0}^{x} \begin{bmatrix} \frac{A^k}{k!} & 0 \\ 0 & \frac{B^k}{k!} \end{bmatrix} \\
&= \lim_{x\to\infty} \begin{bmatrix} \sum_{k=0}^{x} \frac{A^k}{k!} & 0 \\ 0 & \sum_{k=0}^{x} \frac{B^k}{k!} \end{bmatrix} \\
&= \begin{bmatrix} \lim_{x\to\infty} \sum_{k=0}^{x} \frac{A^k}{k!} & 0 \\ 0 & \lim_{x\to\infty} \sum_{k=0}^{x} \frac{B^k}{k!} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{A^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{B^k}{k!} \end{bmatrix} \\
&= \begin{bmatrix} e^A & 0 \\ 0 & e^B \end{bmatrix}.
\end{aligned}
$$

□

Lemma 3.3. For any $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$,we have

$$
e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} = \begin{bmatrix} \cosh(A) & \sinh(A) \\ \sinh(B) & \cosh(B) \end{bmatrix}. \tag{3.4}
$$

Proof. We have

$$
\begin{aligned}
e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^k \\
&= \sum_{k \text{ is even}} \frac{1}{k!} \begin{bmatrix} A^k & 0 \\ 0 & B^k \end{bmatrix} + \sum_{k \text{ is odd}} \frac{1}{k!} \begin{bmatrix} 0 & A^k \\ B^k & 0 \end{bmatrix} \\
&= \sum_{k \text{ is even}} \begin{bmatrix} \frac{A^k}{k!} & 0 \\ 0 & \frac{B^k}{k!} \end{bmatrix} + \sum_{k \text{ is odd}} \begin{bmatrix} 0 & \frac{A^k}{k!} \\ \frac{B^k}{k!} & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{k \text{ is even}} \frac{A^k}{k!} & 0 \\ 0 & \sum\limits_{k \text{ is even}} \frac{B^k}{k!} \end{bmatrix} + \begin{bmatrix} 0 & \sum\limits_{k \text{ is odd}} \frac{A^k}{k!} \\ \sum\limits_{k \text{ is odd}} \frac{B^k}{k!} & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{k \text{ is even}} \frac{1}{k!} A^k & \sum\limits_{k \text{ is odd}} \frac{1}{k!} A^k \\ \sum\limits_{k \text{ is odd}} \frac{1}{k!} B^k & \sum\limits_{k \text{ is even}} \frac{1}{k!} B^k \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \sum\limits_{k=0}^{\infty} \frac{(1+(-1)^k)A^k}{k!} & \sum\limits_{k=0}^{\infty} \frac{(1-(-1)^k)A^k}{k!} \\ \sum\limits_{k=0}^{\infty} \frac{(1-(-1)^k)B^k}{k!} & \sum\limits_{k=0}^{\infty} \frac{(1+(-1)^k)B^k}{k!} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} e^A + e^{-A} & e^A - e^{-A} \\ e^B - e^{-B} & e^B + e^{-B} \end{bmatrix} \\
&= \begin{bmatrix} \cosh(A) & \sinh(A) \\ \sinh(B) & \cosh(B) \end{bmatrix}.
\end{aligned}
$$

□

Theorem 3.4. The general solution of the system (3.1) is given by

$$
\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)}\{[\cosh(t-t_0)(D^T \otimes C)] \operatorname{Vec} X(t_0) \\
&\quad + [\sinh(t-t_0)(D^T \otimes C)] \operatorname{Vec} Y(t_0)\}
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
\operatorname{Vec} Y(t) &= e^{(t-t_0)(A^T \otimes B)}\{[\sinh(t-t_0)(C^T \otimes D)] \operatorname{Vec} X(t_0) \\
&\quad + [\cosh(t-t_0)(C^T \otimes D)] \operatorname{Vec} Y(t_0)\}.
\end{aligned} \tag{3.6}
$$

Proof. For simplicity, let us denote

$$x(t) = \begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix}.$$

By Lemma 3.1, the general exact solution of the system (3.1) is given by

$$x(t) = e^{(t-t_0)P} x(t_0).$$

In order to obtain an explicit formula of $e^{(t-t_0)P}$, denote

$$R = \begin{bmatrix} B^T \otimes A & 0 \\ 0 & A^T \otimes B \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & D^T \otimes C \\ C^T \otimes D & 0 \end{bmatrix}.$$

Observe that

$$RS = \begin{bmatrix} 0 & (B^T \otimes A)(D^T \otimes C) \\ (A^T \otimes B)(C^T \otimes D) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & B^T D^T \otimes AC \\ A^T C^T \otimes BD & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & (DB)^T \otimes AC \\ (CA)^T \otimes BD & 0 \end{bmatrix},$$

and

$$SR = \begin{bmatrix} 0 & (D^T \otimes C)(A^T \otimes B) \\ (C^T \otimes D)(B^T \otimes A) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & D^T A^T \otimes CB \\ C^T B^T \otimes DA & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & (AD)^T \otimes CB \\ (BC)^T \otimes DA & 0 \end{bmatrix}.$$

The assumptions $DB = AD, BD = DA, AC = CB$ and $CA = BC$ imply that $RS = SR$. It follows from Lemmas 2.15, 3.2 and 3.3 that

$$\begin{aligned} e^{(t-t_0)P} &= e^{(t-t_0)(R+S)} \\ &= e^{(t-t_0)R+(t-t_0)S} \\ &= e^{(t-t_0)R}e^{(t-t_0)S} \\ &= \begin{bmatrix} e^{(t-t_0)(B^T\otimes A)} & 0 \\ 0 & e^{(t-t_0)(A^T\otimes B)} \end{bmatrix} \begin{bmatrix} \cosh(t-t_0)(D^T \otimes C) & \sinh(t-t_0)(D^T \otimes C) \\ \sinh(t-t_0)(C^T \otimes D) & \cosh(t-t_0)(C^T \otimes D) \end{bmatrix} \\ &= \begin{bmatrix} e^{(t-t_0)(B^T\otimes A)} \cosh(t-t_0)(D^T \otimes C) & e^{(t-t_0)(B^T\otimes A)} \sinh(t-t_0)(D^T \otimes C) \\ e^{(t-t_0)(A^T\otimes B)} \sinh(t-t_0)(C^T \otimes D) & e^{(t-t_0)(A^T\otimes B)} \cosh(t-t_0)(C^T \otimes D) \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix} = \begin{bmatrix} e^{(t-t_0)(B^T\otimes A)} \cosh(t-t_0)(D^T \otimes C) & e^{(t-t_0)(B^T\otimes A)} \sinh(t-t_0)(D^T \otimes C) \\ e^{(t-t_0)(A^T\otimes B)} \sinh(t-t_0)(C^T \otimes D) & e^{(t-t_0)(A^T\otimes B)} \cosh(t-t_0)(C^T \otimes D) \end{bmatrix} \begin{bmatrix} \operatorname{Vec} X(t_0) \\ \operatorname{Vec} Y(t_0) \end{bmatrix}.$$

Hence, the general solution of the system (3.1) is given by (3.5) and (3.6). □

Next, we discuss some special cases of the system (3.1).

3.2 Special Cases of the main system

Corollary 3.5. Consider the following system:

$$\begin{aligned} X'(t) &= AX(t)A + CY(t)D, \\ Y'(t) &= DX(t)C + AY(t)A, \end{aligned} \tag{3.7}$$

in unknown matrix-valued functions $X(t)$ and $Y(t)$. Here, $A, C, D \in M_n(\mathbb{C})$ are given constant matrices such that $AC = CA$ and $AD = DA$. Then, the general solution of the system (3.7) is given by

$$\begin{aligned} \operatorname{Vec} X(t) = e^{(t-t_0)(A^T \otimes A)}\{&[\cosh(t-t_0)(D^T \otimes C)] \operatorname{Vec} X(t_0) \\ &+ [\sinh(t-t_0)(D^T \otimes C)] \operatorname{Vec} Y(t_0)\} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \operatorname{Vec} Y(t) = e^{(t-t_0)(A^T \otimes A)}\{&[\sinh(t-t_0)(C^T \otimes D)] \operatorname{Vec} X(t_0) \\ &+ [\cosh(t-t_0)(C^T \otimes D)] \operatorname{Vec} Y(t_0)\}. \end{aligned} \tag{3.9}$$

Proof. This is a special case of Theorem 3.4 when $A = B$. □

Corollary 3.6. Consider the following system:

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)C, \\ Y'(t) &= CX(t)C + BY(t)A, \end{aligned} \tag{3.10}$$

in which $A, B, C \in M_n(\mathbb{C})$ are such that $AC = CB$ and $CA = BC$. Then, the general solution of the system (3.10) is given by

$$\begin{aligned} \operatorname{Vec} X(t) = e^{(t-t_0)(B^T \otimes A)}\{&[\cosh(t-t_0)(C^T \otimes C)] \operatorname{Vec} X(t_0) \\ &+ [\sinh(t-t_0)(C^T \otimes C)] \operatorname{Vec} Y(t_0)\} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \operatorname{Vec} Y(t) = e^{(t-t_0)(A^T \otimes B)}\{&[\sinh(t-t_0)(C^T \otimes C)] \operatorname{Vec} X(t_0) \\ &+ [\cosh(t-t_0)(C^T \otimes C)] \operatorname{Vec} Y(t_0)\}. \end{aligned} \tag{3.12}$$

Proof. Set $C = D$ in Theorem 3.4. □

Chapter 4

Unique solutions to certain systems of matrix differential equations with initial value conditions

In this chapter, we discuss certain special cases of the system (3.1) when initial conditions are imposed.

4.1 Unique solutions for certain initial value problems

Corollary 4.1. Consider the following system:

$$\begin{aligned} X'(t) &= X(t) + CY(t)D, \\ Y'(t) &= DX(t)C + Y(t), \end{aligned} \tag{4.1}$$

subject to initial conditions $X(0) = E$ and $Y(0) = F$. Here, $C, D, E, F \in M_n(\mathbb{C})$ are given. Then, the system (4.1) has a unique solution given by

$$\operatorname{Vec} X(t) = e^t\{[\cosh t(D^T \otimes C)] \operatorname{Vec} E + [\sinh t(D^T \otimes C)] \operatorname{Vec} F\}$$

and

$$\operatorname{Vec} Y(t) = e^t\{[\sinh t(C^T \otimes D)] \operatorname{Vec} E + [\cosh t(C^T \otimes D)] \operatorname{Vec} F\}.$$

Proof. Set $A = I_n$ and $t_0 = 0$ in Corollary 3.5. By property (2.3), we have

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{t(I_n \otimes I_n)}\{[\cosh t(D^T \otimes C)] \operatorname{Vec} E + [\sinh t(D^T \otimes C)] \operatorname{Vec} F\} \\ &= e^{(tI_n \otimes I_n)}\{[\cosh t(D^T \otimes C)] \operatorname{Vec} E + [\sinh t(D^T \otimes C)] \operatorname{Vec} F\} \\ &= (e^{tI_n} \otimes I_n)\{[\cosh t(D^T \otimes C)] \operatorname{Vec} E + [\sinh t(D^T \otimes C)] \operatorname{Vec} F\} \\ &= e^t\{[\cosh(tD^T \otimes C)] \operatorname{Vec} E + [\sinh(tD^T \otimes C)] \operatorname{Vec} F\}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \operatorname{Vec} Y(t) &= e^{t(I_n \otimes I_n)}\{[\cosh t(C^T \otimes D)] \operatorname{Vec} E + [\sinh t(C^T \otimes D)] \operatorname{Vec} F\} \\ &= e^{(tI_n \otimes I_n)}\{[\cosh t(C^T \otimes D)] \operatorname{Vec} E + [\sinh t(C^T \otimes D)] \operatorname{Vec} F\} \\ &= (e^{tI_n} \otimes I_n)\{[\cosh t(C^T \otimes D)] \operatorname{Vec} E + [\sinh t(C^T \otimes D)] \operatorname{Vec} F\} \\ &= e^t\{[\cosh(tC^T \otimes D)] \operatorname{Vec} E + [\sinh(tC^T \otimes D)] \operatorname{Vec} F\}. \end{aligned}$$

□

Corollary 4.2. Consider the following system:

$$\begin{aligned} X'(t) &= AX(t)A + Y(t)D, \\ Y'(t) &= DX(t) + AY(t)A, \end{aligned} \tag{4.2}$$

subject to initial conditions $X(0) = E$ and $Y(0) = F$. Here, $A, D, E, F \in M_n(\mathbb{C})$ are such that $AD = DA$. Then, the system (4.2) has a unique solution given by

$$\operatorname{Vec} X(t) = e^{t(A^T \otimes A)} \operatorname{Vec}[E \cosh(tD) + F \sinh(tD)]$$

and

$$\operatorname{Vec} Y(t) = e^{t(A^T \otimes A)} \operatorname{Vec}[\sinh(tD)E + \cosh(tD)F].$$

Proof. Set $C = I_n$ and $t_0 = 0$ in Corollary 3.5. Using properties (2.4) and (2.5), we have

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{t(A^T \otimes A)}\{[\cosh t(D^T \otimes I_n)] \operatorname{Vec} E + [\sinh t(D^T \otimes I_n)] \operatorname{Vec} F\} \\
&= e^{t(A^T \otimes A)}\{[\cosh(tD^T \otimes I_n)] \operatorname{Vec} E + [\sinh(tD^T \otimes I_n)] \operatorname{Vec} F\} \\
&= e^{t(A^T \otimes A)}\{[\cosh(tD^T) \otimes I_n] \operatorname{Vec} E + [\sinh(tD^T) \otimes I_n] \operatorname{Vec} F\} \\
&= e^{t(A^T \otimes A)}\{\operatorname{Vec}(E \cosh(tD)) + \operatorname{Vec}(F \sinh(tD))\} \\
&= e^{t(A^T \otimes A)} \operatorname{Vec}[E \cosh(tD) + F \sinh(tD)].
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\operatorname{Vec} Y(t) &= e^{t(A^T \otimes A)}\{[\sinh t(I_n \otimes D)] \operatorname{Vec} E + [\cosh t(I_n \otimes D)] \operatorname{Vec} F\} \\
&= e^{t(A^T \otimes A)}\{[\sinh(I_n \otimes tD)] \operatorname{Vec} E + [\cosh(I_n \otimes tD)] \operatorname{Vec} F\} \\
&= e^{t(A^T \otimes A)}\{[I_n \otimes \sinh(tD)] \operatorname{Vec} E + [I_n \otimes \cosh(tD)] \operatorname{Vec} F\} \\
&= e^{t(A^T \otimes A)}\{\operatorname{Vec}(\sinh(tD)E) + \operatorname{Vec}(\cosh(tD)F)\} \\
&= e^{t(A^T \otimes A)} \operatorname{Vec}[\sinh(tD)E + \cosh(tD)F].
\end{aligned}$$

□

Corollary 4.3. Consider the following system:

$$\begin{aligned}
X'(t) &= AX(t)A + CY(t), \\
Y'(t) &= X(t)C + AY(t)A,
\end{aligned} \tag{4.3}$$

subject to initial conditions $X(0) = E$ and $Y(0) = F$. Here, $A, C, E, F \in M_n(\mathbb{C})$ are such that $AC = CA$. Then, the system (4.3) has a unique solution given by

$$\operatorname{Vec} X(t) = e^{t(A^T \otimes A)} \operatorname{Vec}[(\cosh(tC))\, E + (\sinh(tC))\, F]$$

and

$$\operatorname{Vec} Y(t) = e^{t(A^T \otimes A)} \operatorname{Vec}[E \sinh(tC) + F \cosh(tC)].$$

Proof. Set $D = I_n$ and $t_0 = 0$ in Corollary 3.5. Using properties (2.4) and (2.5), we have

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{t(A^T \otimes A)}\{[\cosh t(I_n \otimes C)] \operatorname{Vec} E + [\sinh t(I_n \otimes C)] \operatorname{Vec} F\} \\
&= e^{t(A^T \otimes A)}\{[\cosh(I_n \otimes tC)] \operatorname{Vec} E + [\sinh(I_n \otimes tC)] \operatorname{Vec} F\} \\
&= e^{t(A^T \otimes A)}\{[I_n \otimes \cosh(tC)] \operatorname{Vec} E + [I_n \otimes \sinh(tC)] \operatorname{Vec} F\} \\
&= e^{t(A^T \otimes A)}\{\operatorname{Vec}[(\cosh(tC))\, E] + \operatorname{Vec}[(\sinh(tC))\, F]\} \\
&= e^{t(A^T \otimes A)} \operatorname{Vec}[(\cosh(tC))\, E + (\sinh(tC))\, F].
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\operatorname{Vec} Y(t) &= e^{t(A^T\otimes A)}\{[\sinh t(C^T\otimes I_n)]\operatorname{Vec} E + [\cosh t(C^T\otimes I_n)]\operatorname{Vec} F\} \\
&= e^{t(A^T\otimes A)}\{[\sinh(tC^T\otimes I_n)]\operatorname{Vec} E + [\cosh(tC^T\otimes I_n)]\operatorname{Vec} F\} \\
&= e^{t(A^T\otimes A)}\{[\sinh(tC^T)\otimes I_n]\operatorname{Vec} E + [\cosh(tC^T)\otimes I_n]\operatorname{Vec} F\} \\
&= e^{t(A^T\otimes A)}\{\operatorname{Vec}(E\sinh(tC)) + \operatorname{Vec}(F\cosh(tC))\} \\
&= e^{t(A^T\otimes A)}\operatorname{Vec}[E\sinh(tC) + F\cosh(tC)].
\end{aligned}$$

□

Corollary 4.4. Consider the following system:

$$\begin{aligned}
X'(t) &= AX(t)A + Y(t), \\
Y'(t) &= X(t) + AY(t)A,
\end{aligned} \tag{4.4}$$

subject to initial conditions $X(0) = E$ and $Y(0) = F$. Here, $A, E, F \in M_n(\mathbb{C})$ are given matrices. Then, the system (4.4) has a unique solution given by

$$\operatorname{Vec} X(t) = e^{t(A^T\otimes A)}[\operatorname{Vec}((\cosh t)E + (\sinh t)F]$$

and

$$\operatorname{Vec} Y(t) = e^{t(A^T\otimes A)}[\operatorname{Vec}((\sinh t)E + (\cosh t)F].$$

Proof. Set $D = C = I_n$ and $t_0 = 0$ in Corollary 3.5. It follows from properties (2.4) and (2.5) that

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{t(A^T\otimes A)}\{[\cosh t(I_n\otimes I_n)]\operatorname{Vec} E + [\sinh t(I_n\otimes I_n)]\operatorname{Vec} F\} \\
&= e^{t(A^T\otimes A)}\{[\cosh(tI_n\otimes I_n)]\operatorname{Vec} E + [\sinh(tI_n\otimes I_n)]\operatorname{Vec} F\} \\
&= e^{t(A^T\otimes A)}\{[\cosh(tI_n)\otimes I_n]\operatorname{Vec} E + [\sinh(tI_n)\otimes I_n]\operatorname{Vec} F\} \\
&= e^{t(A^T\otimes A)}\{\operatorname{Vec}[E\cosh(tI_n)] + \operatorname{Vec}[F\sinh(tI_n)]\} \\
&= e^{t(A^T\otimes A)}\operatorname{Vec}[(\cosh t)E + (\sinh t)F].
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\operatorname{Vec} Y(t) &= e^{t(A^T\otimes A)}\{[\sinh t(I_n\otimes I_n)]\operatorname{Vec} E + [\cosh t(I_n\otimes I_n)]\operatorname{Vec} F\} \\
&= e^{t(A^T\otimes A)}\{[\sinh(tI_n\otimes I_n)]\operatorname{Vec} E + [\cosh(tI_n\otimes I_n)]\operatorname{Vec} F\} \\
&= e^{t(A^T\otimes A)}\{[\sinh(tI_n)\otimes I_n]\operatorname{Vec} E + [\cosh(tI_n)\otimes I_n]\operatorname{Vec} F\} \\
&= e^{t(A^T\otimes A)}\{\operatorname{Vec}[E\sinh(tI_n)] + \operatorname{Vec}[F\cosh(tI_n)]\} \\
&= e^{t(A^T\otimes A)}\operatorname{Vec}[(\sinh t)E + (\cosh t)F].
\end{aligned}$$

□

Corollary 4.5. Let $A, C, E, F \in M_n(\mathbb{C})$ be such that $AC = CA$. Then, the initial value problem

$$\begin{aligned} X'(t) &= AX(t)A + CY(t)C, \\ Y'(t) &= CX(t)C + AY(t)A, \\ X(0) &= E, \quad Y(0) = F \end{aligned} \tag{4.5}$$

has a unique solution given by

$$\operatorname{Vec} X(t) = e^{t(A^T \otimes A)}\{[\cosh t(C^T \otimes C)] \operatorname{Vec} E + [\sinh t(C^T \otimes C)] \operatorname{Vec} F\}$$

and

$$\operatorname{Vec} Y(t) = e^{t(A^T \otimes A)}\{[\sinh t(C^T \otimes C)] \operatorname{Vec} E + [\cosh t(C^T \otimes C)] \operatorname{Vec} F\}.$$

Proof. Set $C = D$ and $t_0 = 0$ in Corollary 3.5. □

Corollary 4.6. Given $C, D, E, F \in M_n(\mathbb{C})$, the initial value problem

$$\begin{aligned} X'(t) &= CY(t)D, \\ Y'(t) &= DX(t)C, \\ X(0) &= E, \quad Y(0) = F \end{aligned} \tag{4.6}$$

has a unique solution given by

$$\operatorname{Vec} X(t) = \cosh t(D^T \otimes C) \operatorname{Vec} E + \sinh t(D^T \otimes C) \operatorname{Vec} F$$

and

$$\operatorname{Vec} Y(t) = \sinh t(C^T \otimes D) \operatorname{Vec} E + \cosh t(C^T \otimes D) \operatorname{Vec} F.$$

Proof. Set $A = 0$ and $t_0 = 0$ in Corollary 3.5. □

Corollary 4.7. Given $A, E, F \in M_n(\mathbb{C})$, the initial value problem

$$\begin{aligned} X'(t) &= AY(t), \\ Y'(t) &= X(t)A, \\ X(0) &= E, \quad Y(0) = F \end{aligned} \tag{4.7}$$

has a unique solution given by

$$X(t) = (\cosh tA)E + (\sinh tA)F$$

and

$$Y(t) = E \sinh(tA) + F \cosh(tA).$$

Proof. Set $B = I$ and $t_0 = 0$ in Corollary 4.6 and then use properties (2.4) and (2.5). We have

$$\begin{aligned} \operatorname{Vec} X(t) &= \cosh(I \otimes tA) \operatorname{Vec} E + \sinh(I \otimes tA) \operatorname{Vec} F \\ &= (I \otimes \cosh tA) \operatorname{Vec} E + (I \otimes \sinh tA) \operatorname{Vec} F \\ &= \operatorname{Vec}\{(\cosh tA)E\} + \operatorname{Vec}\{(\sinh tA)F\} \\ &= \operatorname{Vec}\{(\cosh tA)E + (\sinh tA)F\}. \end{aligned}$$

Since the vector operator is injective, we have $X(t) = (\cosh tA)E + (\sinh tA)F$. Similarly, we get

$$\begin{aligned}
\operatorname{Vec} Y(t) &= \sinh(tA^T \otimes I)\operatorname{Vec} E + \cosh(tA^T \otimes I)\operatorname{Vec} F \\
&= (\sinh tA^T \otimes I)\operatorname{Vec} E + (\cosh tA^T \otimes I)\operatorname{Vec} F \\
&= \operatorname{Vec}\{E\sinh(tA)\} + \operatorname{Vec}\{F\cosh(tA)\} \\
&= \operatorname{Vec}\{E\sinh(tA) + F\cosh(tA)\}.
\end{aligned}$$

□

4.2 Numerical Example

Example 4.8. Consider the following system

$$\begin{aligned}
X'(t) &= AX(t)A + CY(t)C, \\
Y'(t) &= CX(t)C + AY(t)A, \\
X(0) &= E, \quad Y(0) = F,
\end{aligned}$$

when $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, F = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

Then

$$AC = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} = CA.$$

By Corollary 4.5, we have

$$\operatorname{Vec} X(t) = \begin{bmatrix} -\frac{1}{2} + \frac{3}{2}e^{18t} \\ -\frac{1}{2} + \frac{3}{2}e^{18t} \\ -\frac{1}{2} + \frac{3}{2}e^{18t} \\ -\frac{1}{2} + \frac{3}{2}e^{18t} \end{bmatrix},$$

$$\operatorname{Vec} Y(t) = \begin{bmatrix} \frac{1}{2} + \frac{3}{2}e^{18t} \\ \frac{1}{2} + \frac{3}{2}e^{18t} \\ \frac{1}{2} + \frac{3}{2}e^{18t} \\ \frac{1}{2} + \frac{3}{2}e^{18t} \end{bmatrix}.$$

Hence

$$X(t) = \begin{bmatrix} -\frac{1}{2} + \frac{3}{2}e^{18t} & -\frac{1}{2} + \frac{3}{2}e^{18t} \\ -\frac{1}{2} + \frac{3}{2}e^{18t} & -\frac{1}{2} + \frac{3}{2}e^{18t} \end{bmatrix},$$

$$Y(t) = \begin{bmatrix} \frac{1}{2} + \frac{3}{2}e^{18t} & \frac{1}{2} + \frac{3}{2}e^{18t} \\ \frac{1}{2} + \frac{3}{2}e^{18t} & \frac{1}{2} + \frac{3}{2}e^{18t} \end{bmatrix}.$$

Chapter 5

Conclusion and Suggestion

5.1 Conclusion

Let A, B, C, D be given $n \times n$ complex matrices such that

$$DB = AD,\ BD = DA,\ AC = CB \ \text{and}\ CA = BC. \tag{5.1}$$

Consider the following system of coupled linear matrix differential equations:

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D, \\ Y'(t) &= DX(t)C + BY(t)A, \end{aligned} \tag{5.2}$$

in unknown matrix-valued functions $X(t)$, $Y(t)$. The general vector solution of this system is given by the one-parameter solutions:

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)}\{[\cosh(t-t_0)(D^T \otimes C)] \operatorname{Vec} X(t_0) \\ &\quad + [\sinh(t-t_0)(D^T \otimes C)] \operatorname{Vec} Y(t_0)\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(A^T \otimes B)}\{[\sinh(t-t_0)(C^T \otimes D)] \operatorname{Vec} X(t_0) \\ &\quad + [\cosh(t-t_0)(C^T \otimes D)] \operatorname{Vec} Y(t_0)\}. \end{aligned} \tag{5.3}$$

Once we know $\operatorname{Vec} X(t)$ and $\operatorname{Vec} Y(t)$, we will obtain $X(t)$ and $Y(t)$ due to the injectivity of the vector operator.

Certain special cases for this system are also investigated. In particular, when we impose initial conditions, the solution to such system is uniquely determined. For example, when $A = B = I_n$ the condition (5.1) is fulfilled and the system is reduce to

$$\begin{aligned} X'(t) &= X(t) + CY(t)D, \\ Y'(t) &= DX(t)C + Y(t). \end{aligned} \tag{5.4}$$

When we impose the conditions $X(0) = E$ and $Y(0) = F$, the system (5.4) has a unique solution given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^t\{[\cosh t(D^T \otimes C)] \operatorname{Vec} E + [\sinh t(D^T \otimes C)] \operatorname{Vec} F\}, \\ \operatorname{Vec} Y(t) &= e^t\{[\sinh t(C^T \otimes D)] \operatorname{Vec} E + [\cosh t(C^T \otimes D)] \operatorname{Vec} F\}. \end{aligned} \tag{5.5}$$

5.2 Suggestion

We may investigate the system

$$X'(t) = AX(t)B + CY(t)D,$$
$$Y'(t) = EX(t)F + GY(t)H,$$

where A, B, C, D, E, F, G, H are given $n \times n$ complex matrices satisfying another assumptions. In addition, we may use another kind of vectorization other than the vector-operator, such as

- the diagonal extraction operator,
- the row vector-operator,
- the block-column vector-operator,
- the block-row vector-operator.

Moreover, we may use another kind of matrix product other than the Kronecker product, such as

- Hadamard product,
- Tracy-Singh product,
- Khatri-Rao product.

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Appendix

Appendix A

The research paper

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Solving Certain Systems of Coupled Linear Matrix Differential Equations

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Abstract

In this paper, we solve certain systems of coupled linear matrix differential equations. By applying Kronecker products and the vector operator, we obtain explicit formulas of solutions for these systems in terms of matrix series concerning exponentials and hyperbolic functions.

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Keywords: matrix differential equation, Kronecker product, vector operator, matrix exponential

1 Introduction

Theory of linear matrix differential equations was investigated by many authors (see, *e.g.*, [2, 6–8]). Computational aspects for this theory were discussed in [1, 9] and references therein. The theory can be applied in a broad range of scientific fields, *e.g.*, statistics [3, 10, 12], game theory [5], econometrics [10, 12], control and system theory [4, 11].

The simplest form of linear matrix differential equation is given by

$$X'(t) = AX(t). \tag{1.1}$$

Here, A is a given square matrix and $X(t)$ is an unknown matrix-valued function. In fact, (1.1) has a general solution given by

$$X(t) = e^{(t-t_0)A}X(t_0). \tag{1.2}$$

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A general system of coupled linear matrix differential equations takes the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D, \\ Y'(t) &= EX(t)F + GY(t)H. \end{aligned} \tag{1.3}$$

Here, A, B, C, D, E, F, G and H are given n-by-n complex matrices and $X(t)$, $Y(t)$ are unknown matrix-valued functions. The general solution of the system (1.3) when $E = C$, $F = D$, $G = A$ and $H = B$ was obtained in [7].

In this paper, we investigate the linear system (1.3) in a situation that $E = D$, $F = C$, $G = B$ and $H = A$. We apply Kronecker products and the vector operator to reduce our complex system to the simplest form (1.1). Then, an explicit formula of the general solution to this system is obtained in terms of matrix series concerning exponentials and hyperbolic functions. When initial conditions are imposed to these problems, the solution of each case is unique and, in fact, easy to compute.

This paper is structured as follows. We provide tools for solving a system of matrix differential equations in the next section. These includes Kronecker products, the vector operator, and functions of matrices defined by power series. Section 3 deals with a system of two linear matrix differential equations. In Section 4, we discuss certain special cases of our main problem when initial conditions are imposed.

2 Preliminaries

Denote by $M_{m,n}(\mathbb{C})$ the set of all m-by-n complex matrices and abbreviate $M_{n,n}(\mathbb{C})$ to $M_n(\mathbb{C})$ and I is an n-by-n identity matrix to I_n. Recall that the *Kronecker product* of $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$ is defined by the block matrix whose (i,j)-block is given by $A \otimes B = [a_{ij}B]$ for each $i \in \{1, 2, 3, \ldots, m\}$ and $j \in \{1, 2, 3, \ldots, n\}$. The Kronecker product satisfies the following properties:

$$\begin{aligned} \alpha(A \otimes B) &= (\alpha A) \otimes B = A \otimes (\alpha B) \\ (A \otimes B)(C \otimes D) &= AC \otimes BD. \end{aligned}$$

Let us introduce a column-stacking operator, which is very useful for solving linear matrix equations. The *vector operator* $\operatorname{Vec} : M_{m,n}(\mathbb{C}) \to \mathbb{C}^{mn}$ is defined for each $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ by

$$\operatorname{Vec} A = [a_{11} \ldots a_{m1}\ a_{12} \ldots a_{m2} \ldots a_{1n} \ldots a_{mn}]^T$$

This operator is a linear isomorphism.

Lemma 2.1 (see, *e.g.*, [13]). *For any matrices $A \in M_{m,n}(\mathbb{C})$, $B \in M_{p,q}(\mathbb{C})$ and $X \in M_{n,p}(\mathbb{C})$, we have*

$$\operatorname{Vec}(AXB) = (B^T \otimes A) \operatorname{Vec} X. \tag{2.1}$$

Next, we discuss functions of a square matrix defined by power series. Let $A \in M_k(\mathbb{C})$ and let f be an analytic function defined on a region containing the origin and all eigenvalues of A. Then, there exists the radius $R > 0$ such that for any $|z| < R$, we thus have Mclaurin series expansion

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$$

where $\alpha_n = f^{(n)}(0)/n!$ for any natural number n and $\alpha_0 = f(0)$. It follows that the following matrix series converges:

$$f(A) := \sum_{n=0}^{\infty} \alpha_n A^n.$$

For the entire function $f(z) = e^z$, we have $R = \infty$ and hence, the series

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

converges for all $A \in M_k(\mathbb{C})$. Similarly, for any $A \in M_k(\mathbb{C})$, the following matrix series converge

$$\sinh(A) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} A^{2n+1} \quad \textit{and} \quad \cosh(A) := \sum_{n=0}^{\infty} \frac{1}{(2n)!} A^{2n}.$$

Lemma 2.2 (see, *e.g.*, [13]). *If $X, Y \in M_k(\mathbb{C})$ are such that $XY = YX$, then $e^{X+Y} = e^X e^Y$.*

Lemma 2.3 (see, *e.g.*, [13]). *Let f be an analytic function on a region containing the origin and all eigenvalues of A. Then,*

$$f(I \otimes A) = I \otimes f(A) \quad \textit{and} \quad f(A \otimes I) = f(A) \otimes I. \tag{2.2}$$

Some special cases include:

$$e^{A\otimes I} = e^A \otimes I, \qquad e^{I\otimes A} = I \otimes e^A, \tag{2.3}$$

$$\sinh(A\otimes I) = \sinh(A)\otimes I, \qquad \sinh(I\otimes A) = I\otimes \sinh(A), \tag{2.4}$$

$$\cosh(A\otimes I) = \cosh(A)\otimes I, \qquad \cosh(I\otimes A) = I\otimes \cosh(A). \tag{2.5}$$

3 Solving a system of coupled linear matrix differential equations

In this section, we investigate a system of coupled linear matrix differential equations of the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D, \\ Y'(t) &= DX(t)C + BY(t)A. \end{aligned} \tag{3.1}$$

Here, $A, B, C, D \in M_n(\mathbb{C})$ are given matrices and $X(t), Y(t)$ are unknown matrix-valued functions. In order to express an explicit form of the solution, we impose the conditions $DB = AD$, $BD = DA$, $AC = CB$ and $CA = BC$. The following lemma explains how to reduce a general system (1.3) of coupled linear matrix differential equations to the simplest form (1.1).

Lemma 3.1. *The general solution of the system* (1.3) *can be expressed as*

$$\begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix} = e^{(t-t_0)P} \begin{bmatrix} \operatorname{Vec} X(t_0) \\ \operatorname{Vec} Y(t_0) \end{bmatrix},$$

where $P = \begin{bmatrix} B^T \otimes A & D^T \otimes C \\ F^T \otimes E & H^T \otimes G \end{bmatrix}$.

Proof. We follow the idea of [7]. Applying the vector operator to the system (1.3), we obtain by Lemma 2.1 that

$$\begin{aligned} \operatorname{Vec} X'(t) &= \operatorname{Vec}(AX(t)B + CY(t)D) \\ &= (B^T \otimes A)\operatorname{Vec} X(t) + (D^T \otimes C)\operatorname{Vec} Y(t) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Vec} Y'(t) &= \operatorname{Vec}(EX(t)F + GY(t)H) \\ &= (F^T \otimes E)\operatorname{Vec} X(t) + (H^T \otimes G)\operatorname{Vec} Y(t). \end{aligned}$$

For simplicity, let us denote

$$x(t) = \begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix}.$$

Since the vector operator is a linear isomorphism, the system (1.3) is equivalent to the following vector-matrix differential equation:

$$x'(t) = Px(t).$$

Hence, the general exact solution is given by $x(t) = e^{(t-t_0)P}x(t_0)$. □

The following lemmas are useful for deriving explicit formula of the solution of (3.1).

Lemma 3.2. *For any $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$, we have*

$$e^{\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}} = \begin{bmatrix} e^A & 0 \\ 0 & e^B \end{bmatrix}. \tag{3.2}$$

Proof. Using standard results in matrix analysis, we obtain

$$\begin{aligned} e^{\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^k = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} A^k & 0 \\ 0 & B^k \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{A^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{B^k}{k!} \end{bmatrix} = \begin{bmatrix} e^A & 0 \\ 0 & e^B \end{bmatrix}. \end{aligned}$$ □

Lemma 3.3. *For any $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$, we have*

$$e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} = \begin{bmatrix} \cosh(A) & \sinh(A) \\ \sinh(B) & \cosh(B) \end{bmatrix}. \tag{3.3}$$

Proof. We have

$$
\begin{aligned}
e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} &= \sum_{k \text{ is even}} \frac{1}{k!} \begin{bmatrix} A^k & 0 \\ 0 & B^k \end{bmatrix} + \sum_{k \text{ is odd}} \frac{1}{k!} \begin{bmatrix} 0 & A^k \\ B^k & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k \text{ is even}} \frac{1}{k!} A^k & \sum_{k \text{ is odd}} \frac{1}{k!} A^k \\ \sum_{k \text{ is odd}} \frac{1}{k!} B^k & \sum_{k \text{ is even}} \frac{1}{k!} B^k \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(1+(-1)^k)A^k}{k!} & \sum_{k=0}^{\infty} \frac{(1-(-1)^k)A^k}{k!} \\ \sum_{k=0}^{\infty} \frac{(1-(-1)^k)B^k}{k!} & \sum_{k=0}^{\infty} \frac{(1+(-1)^k)B^k}{k!} \end{bmatrix} \\
&= \begin{bmatrix} \cosh(A) & \sinh(A) \\ \sinh(B) & \cosh(B) \end{bmatrix}.
\end{aligned}
$$

□

Theorem 3.4. *The general solution of the system* (3.1) *is given by*

$$
\begin{aligned}
\operatorname{Vec} X(t) = e^{(t-t_0)(B^T \otimes A)} \{ & [\cosh(t-t_0)(D^T \otimes C)] \operatorname{Vec} X(t_0) \\
& + [\sinh(t-t_0)(D^T \otimes C)] \operatorname{Vec} Y(t_0) \}
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
\operatorname{Vec} Y(t) = e^{(t-t_0)(A^T \otimes B)} \{ & [\sinh(t-t_0)(C^T \otimes D)] \operatorname{Vec} X(t_0) \\
& + [\cosh(t-t_0)(C^T \otimes D)] \operatorname{Vec} Y(t_0) \}.
\end{aligned} \tag{3.5}
$$

Proof. For simplicity, let us denote

$$x(t) = \begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix}.$$

By Lemma 3.1, the general exact solution of the system (3.1) is given by

$$x(t) = e^{(t-t_0)P} x(t_0).$$

In order to obtain an explicit formula of $e^{(t-t_0)P}$, denote

$$R = \begin{bmatrix} B^T \otimes A & 0 \\ 0 & A^T \otimes B \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & D^T \otimes C \\ C^T \otimes D & 0 \end{bmatrix}.$$

Observe that

$$RS = \begin{bmatrix} 0 & (B^T \otimes A)(D^T \otimes C) \\ (A^T \otimes B)(C^T \otimes D) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & (DB)^T \otimes AC \\ (CA)^T \otimes BD & 0 \end{bmatrix}$$

and

$$SR = \begin{bmatrix} 0 & (D^T \otimes C)(A^T \otimes B) \\ (C^T \otimes D)(B^T \otimes A) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & (AD)^T \otimes CB \\ (BC)^T \otimes DA & 0 \end{bmatrix}.$$

The assumptions $DB = AD$, $BD = DA$, $AC = CB$ and $CA = BC$ imply that $RS = SR$. It follows from Lemmas 2.2, 3.2 and 3.3 that

$$\begin{aligned} e^{(t-t_0)P} &= e^{(t-t_0)R+(t-t_0)S} \\ &= e^{(t-t_0)R}e^{(t-t_0)S} \\ &= \begin{bmatrix} e^{(t-t_0)(B^T\otimes A)} & 0 \\ 0 & e^{(t-t_0)(A^T\otimes B)} \end{bmatrix} \begin{bmatrix} \cosh(t-t_0)(D^T\otimes C) & \sinh(t-t_0)(D^T\otimes C) \\ \sinh(t-t_0)(C^T\otimes D) & \cosh(t-t_0)(C^T\otimes D) \end{bmatrix} \\ &= \begin{bmatrix} e^{(t-t_0)(B^T\otimes A)}\cosh(t-t_0)(D^T\otimes C) & e^{(t-t_0)(B^T\otimes A)}\sinh(t-t_0)(D^T\otimes C) \\ e^{(t-t_0)(A^T\otimes B)}\sinh(t-t_0)(C^T\otimes D & e^{(t-t_0)(A^T\otimes B)}\cosh(t-t_0)(C^T\otimes D) \end{bmatrix}. \end{aligned}$$

Hence, the general solution of the system (3.1) is given by (3.4) and (3.5). □

Next, we discuss some special cases of the system (3.1).

Corollary 3.5. *Consider the following system:*

$$\begin{aligned} X'(t) &= AX(t)A + CY(t)D, \\ Y'(t) &= DX(t)C + AY(t)A, \end{aligned} \tag{3.6}$$

in unknown matrix-valued functions $X(t)$ and $Y(t)$. Here, $A, C, D \in M_n(\mathbb{C})$ are given constant matrices such that $AC = CA$ and $AD = DA$. Then, the general solution of the system

(3.6) *is given by*

$$\operatorname{Vec} X(t) = e^{(t-t_0)(A^T \otimes A)}\{[\cosh(t-t_0)(D^T \otimes C)] \operatorname{Vec} X(t_0) + [\sinh(t-t_0)(D^T \otimes C)] \operatorname{Vec} Y(t_0)\} \tag{3.7}$$

and

$$\operatorname{Vec} Y(t) = e^{(t-t_0)(A^T \otimes A)}\{[\sinh(t-t_0)(C^T \otimes D)] \operatorname{Vec} X(t_0) + [\cosh(t-t_0)(C^T \otimes D)] \operatorname{Vec} Y(t_0)\}. \tag{3.8}$$

Proof. This is a special case of Theorem 3.4 when $A = B$. □

Corollary 3.6. *Consider the following system:*

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)C, \\ Y'(t) &= CX(t)C + BY(t)A, \end{aligned} \tag{3.9}$$

in which $A, B, C \in M_n(\mathbb{C})$ are such that $AC = CB$ and $CA = BC$. Then, the general solution of the system (3.9) *is given by*

$$\operatorname{Vec} X(t) = e^{(t-t_0)(B^T \otimes A)}\{[\cosh(t-t_0)(C^T \otimes C)] \operatorname{Vec} X(t_0) + [\sinh(t-t_0)(C^T \otimes C)] \operatorname{Vec} Y(t_0)\} \tag{3.10}$$

and

$$\operatorname{Vec} Y(t) = e^{(t-t_0)(A^T \otimes B)}\{[\sinh(t-t_0)(C^T \otimes C)] \operatorname{Vec} X(t_0) + [\cosh(t-t_0)(C^T \otimes C)] \operatorname{Vec} Y(t_0)\}. \tag{3.11}$$

Proof. Set $C = D$ in Theorem 3.4. □

4 Systems of matrix differential equations with initial value conditions

In this section, we discuss certain special cases of the system (3.1) when initial conditions are imposed.

Corollary 4.1. *Consider the following system:*

$$\begin{aligned} X'(t) &= X(t) + CY(t)D, \\ Y'(t) &= DX(t)C + Y(t), \end{aligned} \tag{4.1}$$

subject to initial conditions $X(0) = E$ and $Y(0) = F$. Here, $C, D, E, F \in M_n(\mathbb{C})$ are given.

Then, the system (4.1) *has a unique solution given by*

$$\operatorname{Vec} X(t) = e^t\{[\cosh t(D^T \otimes C)] \operatorname{Vec} E + [\sinh t(D^T \otimes C)] \operatorname{Vec} F\}$$

and

$$\operatorname{Vec} Y(t) = e^t\{[\sinh t(C^T \otimes D)] \operatorname{Vec} E + [\cosh t(C^T \otimes D)] \operatorname{Vec} F\}.$$

Proof. Set $A = I_n$ and $t_0 = 0$ in Corollary 3.5. By property (2.3), we have

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{t(I_n \otimes I_n)}\{[\cosh t(D^T \otimes C)] \operatorname{Vec} E + [\sinh t(D^T \otimes C)] \operatorname{Vec} F\} \\
&= (e^{tI_n} \otimes I_n)\{[\cosh t(D^T \otimes C)] \operatorname{Vec} E + [\sinh t(D^T \otimes C)] \operatorname{Vec} F\} \\
&= e^t\{[\cosh t(D^T \otimes C)] \operatorname{Vec} E + [\sinh t(D^T \otimes C)] \operatorname{Vec} F\}.
\end{aligned}$$

Similarly, we get the above formula of $Y(t)$. □

Corollary 4.2. *Consider the following system:*

$$\begin{aligned}
X'(t) &= AX(t)A + Y(t)D, \\
Y'(t) &= DX(t) + AY(t)A,
\end{aligned} \tag{4.2}$$

subject to initial conditions $X(0) = E$ and $Y(0) = F$. Here, $A, D, E, F \in M_n(\mathbb{C})$ are such that $AD = DA$. Then, the system (4.2) *has a unique solution given by*

$$\operatorname{Vec} X(t) = e^{t(A^T \otimes A)} \operatorname{Vec}[E \cosh(tD) + F \sinh(tD)]$$

and

$$\operatorname{Vec} Y(t) = e^{t(A^T \otimes A)} \operatorname{Vec}[\sinh(tD)E + \cosh(tD)F].$$

Proof. Set $C = I_n$ and $t_0 = 0$ in Corollary 3.5. Using properties (2.4) and (2.5), we have

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{t(A^T \otimes A)}\{[\cosh t(D^T \otimes I_n)] \operatorname{Vec} E + [\sinh t(D^T \otimes I_n)] \operatorname{Vec} F\} \\
&= e^{t(A^T \otimes A)}\{[\cosh(tD^T) \otimes I_n] \operatorname{Vec} E + [\sinh(tD^T) \otimes I_n] \operatorname{Vec} F\} \\
&= e^{t(A^T \otimes A)}\{\operatorname{Vec}(E \cosh(tD)) + \operatorname{Vec}(F \sinh(tD))\} \\
&= e^{t(A^T \otimes A)} \operatorname{Vec}[E \sinh(tD) + F \cosh(tD)].
\end{aligned}$$

Similarly, we get the above formula of $Y(t)$. □

Corollary 4.3. *Consider the following system:*

$$\begin{aligned} X'(t) &= AX(t)A + CY(t), \\ Y'(t) &= X(t)C + AY(t)A, \end{aligned} \tag{4.3}$$

subject to initial conditions $X(0) = E$ *and* $Y(0) = F$. *Here,* $A, C, E, F \in M_n(\mathbb{C})$ *are such that* $AC = CA$. *Then, the system* (4.3) *has a unique solution given by*

$$\operatorname{Vec} X(t) = e^{t(A^T \otimes A)} \operatorname{Vec}[(\cosh(tC))\, E + (\sinh(tC))\, F]$$

and

$$\operatorname{Vec} Y(t) = e^{t(A^T \otimes A)} \operatorname{Vec}[E \sinh(tC) + F \cosh(tC)].$$

Proof. Set $D = I_n$ and $t_0 = 0$ in Corollary 3.5. Using (2.4) and (2.5), we have

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{t(A^T \otimes A)} \{[\cosh t(I_n \otimes C)] \operatorname{Vec} E + [\sinh t(I_n \otimes C)] \operatorname{Vec} F\} \\ &= e^{t(A^T \otimes A)} \{[I_n \otimes \cosh(tC)] \operatorname{Vec} E + [I_n \otimes \sinh(tC)] \operatorname{Vec} F\} \\ &= e^{t(A^T \otimes A)} \{\operatorname{Vec}[(\cosh(tC))\, E] + \operatorname{Vec}[(\sinh(tC))\, F]\} \\ &= e^{t(A^T \otimes A)} \operatorname{Vec}[(\cosh(tC))\, E + (\sinh(tC))\, F]. \end{aligned}$$

Similarly, we arrive at the above formula of $Y(t)$. □

Corollary 4.4. *Consider the following system:*

$$\begin{aligned} X'(t) &= AX(t)A + Y(t), \\ Y'(t) &= X(t) + AY(t)A, \end{aligned} \tag{4.4}$$

subject to initial conditions $X(0) = E$ *and* $Y(0) = F$. *Here,* $A, E, F \in M_n(\mathbb{C})$ *are given matrices. Then, the system* (4.4) *has a unique solution given by*

$$\operatorname{Vec} X(t) = e^{t(A^T \otimes A)} [\operatorname{Vec}((\cosh t)E + (\sinh t)F]$$

and

$$\operatorname{Vec} Y(t) = e^{t(A^T \otimes A)} [\operatorname{Vec}((\sinh t)E + (\cosh t)F].$$

Proof. Set $D = C = I_n$ and $t_0 = 0$ in Corollary 3.5. It follows from (2.4) and (2.5) that

$$\begin{aligned}\operatorname{Vec} X(t) &= e^{t(A^T\otimes A)}\{[\cosh t(I_n\otimes I_n)]\operatorname{Vec} E + [\sinh t(I_n\otimes I_n)]\operatorname{Vec} F\}\\ &= e^{t(A^T\otimes A)}\{[\cosh(tI_n)\otimes I_n]\operatorname{Vec} E + [\sinh(tI_n)\otimes I_n]\operatorname{Vec} F\}\\ &= e^{t(A^T\otimes A)}\{\operatorname{Vec}[E\cosh(tI_n)] + \operatorname{Vec}[F\sinh(tI_n)]\}\\ &= e^{t(A^T\otimes A)}\operatorname{Vec}[(\cosh t)E + (\sinh t)F].\end{aligned}$$

Similarly, we get the above formula of $Y(t)$. □

Corollary 4.5. *Let $A, C, E, F \in M_n(\mathbb{C})$ be such that $AC = CA$. Then, the initial value problem*

$$\begin{aligned}X'(t) &= AX(t)A + CY(t)C,\\ Y'(t) &= CX(t)C + AY(t)A,\\ X(0) &= E, \quad Y(0) = F\end{aligned} \tag{4.5}$$

has a unique solution given by

$$\operatorname{Vec} X(t) = e^{t(A^T\otimes A)}\{[\cosh t(C^T\otimes C)]\operatorname{Vec} E + [\sinh t(C^T\otimes C)]\operatorname{Vec} F\}$$

and

$$\operatorname{Vec} Y(t) = e^{t(A^T\otimes A)}\{[\sinh t(C^T\otimes C)]\operatorname{Vec} E + [\cosh t(C^T\otimes C)]\operatorname{Vec} F\}.$$

Proof. Set $C = D$ and $t_0 = 0$ in Corollary 3.5. □

Corollary 4.6. *Given $A, B, E, F \in M_n(\mathbb{C})$, the initial value problem*

$$\begin{aligned}X'(t) &= AY(t)B,\\ Y'(t) &= BX(t)A,\\ X(0) &= E, \quad Y(0) = F\end{aligned} \tag{4.6}$$

has a unique solution given by

$$\operatorname{Vec} X(t) = \cosh t(B^T\otimes A)\operatorname{Vec} E + \sinh t(B^T\otimes A)\operatorname{Vec} F$$

and

$$\operatorname{Vec} Y(t) = \sinh t(A^T\otimes B)\operatorname{Vec} E + \cosh t(A^T\otimes B)\operatorname{Vec} F.$$

Proof. Set $C = 0$ and $t_0 = 0$ in Corollary 3.6. □

Corollary 4.7. *Let $A, C, E, F \in M_n(\mathbb{C})$ be such that $AC = CA$. Then, the initial value problem*

$$\begin{aligned} X'(t) &= CX(t)C + AY(t)A, \\ Y'(t) &= AX(t)A + CY(t)C, \\ X(0) &= E, \quad Y(0) = F \end{aligned} \tag{4.7}$$

has a unique solution given by

$$\operatorname{Vec} X(t) = e^{t(A^T \otimes A)}\{[\cosh t(C^T \otimes C)] \operatorname{Vec} E + [\sinh t(C^T \otimes C)] \operatorname{Vec} F\}$$

and

$$\operatorname{Vec} Y(t) = e^{t(A^T \otimes A)}\{[\sinh t(C^T \otimes C)] \operatorname{Vec} E + [\cosh t(C^T \otimes C)] \operatorname{Vec} F\}.$$

Proof. Set $A = B$ and $t_0 = 0$ in Corollary 3.6. □

Corollary 4.8. *Given $A, E, F \in M_n(\mathbb{C})$, the initial value problem*

$$\begin{aligned} X'(t) &= AY(t), \\ Y'(t) &= X(t)A, \\ X(0) &= E, \quad Y(0) = F \end{aligned} \tag{4.8}$$

has a unique solution given by

$$X(t) = (\cosh tA)E + (\sinh tA)F$$

and

$$Y(t) = E \sinh(tA) + F \cosh(tA).$$

Proof. Set $B = I$ and $t_0 = 0$ in Corollary 4.6 and then use properties (2.4) and (2.5). We have

$$\begin{aligned} \operatorname{Vec} X(t) &= \cosh(I \otimes tA) \operatorname{Vec} E + \sinh(I \otimes tA) \operatorname{Vec} F \\ &= (I \otimes \cosh tA) \operatorname{Vec} E + (I \otimes \sinh tA) \operatorname{Vec} F \\ &= \operatorname{Vec}\{(\cosh tA)E\} + \operatorname{Vec}\{(\sinh tA)F\} \\ &= \operatorname{Vec}\{(\cosh tA)E + (\sinh tA)F\}. \end{aligned}$$

Since the vector operator is injective, we have $X(t) = (\cosh tA)E + (\sinh tA)F$. Similarly, we arrive at the above formula of $Y(t)$. □

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