SOLVING WEIGHTED MEAN EQUATIONS IN LINEATED SYMMETRIC SPACES

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE (APPLIED MATHEMATICS)

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE

KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

2017

KMITL-2017-SC-M-001-017

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บทคัดย่อ

ในงานวิจัยนี้ เราพัฒนาทฤษฎีของค่าเฉลี่ยในปริภูมิสมมาตรที่เป็นเส้น อันดับแรก เรายกตัวอย่างพื้นฐานของปริภูมิสมมาตรที่เป็นเส้น ต่อจากนั้นเราศึกษาสมบัติของค่าเฉลี่ยแบบถ่วงน้ำหนักในปริภูมิสมมาตรที่เป็นเส้นซึ่งน้ำหนักเป็นจำนวนจริงใดๆ ยิ่งกว่านั้น เราหาผลเฉลยของสมการค่าเฉลี่ยแบบถ่วงน้ำหนักในปริภูมิดังกล่าว

คำสำคัญ : ควอซีกรุปแบบสะท้อน ค่าเฉลี่ยเรขาคณิตแบบถ่วงน้ำหนัก สาทิสสัณฐานแบบ s ปริภูมิสมมาตรที่เป็นเส้น

Thesis Title	Solving Weighted Mean Equations in Lineated Symmetric Spaces
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Abstract

In this paper, we developed further theory of means in a lineated symmetric space. First, we illustrate fundamental examples of lineated symmetric spaces. Then we investigate properties of weighted means in which weights are arbitrary real numbers. Moreover, we solve certain weighted mean equations in that spaces.

Keywords : reflection quasigroup, weighted geometric mean, $\mathfrak{s}$–homomorphism, lineated symmetric space.

Acknowledgements

Immeasurable appreciation and deepest gratitude for the advices, help and support are extended to the following persons who have contributed to making this thesis possible.

Firstly, I would like to express my sincere gratitude to my advisor, Asst.Prof.Dr Pattrawut Chansangiam, for the continuous support of my M.Sc. study, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of study and writing of this thesis. I could not have imagined having a better advisor and mentor for my M.Sc. study.

Besides my advisor, I would like to thank the rest of my thesis committee: Asst.Prof.Dr. Keng Wiboonton, Asst.Prof.Dr. Decha Samana, and Dr. Sukrawan Mavecha, for their insightful comments.

Finally, I must express my very profound gratitude to my family for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them. Thank you.

Achiraya Ngodngam

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Chapter 1

Introduction

1.1 Inception and importance

The concept of mean or midpoint shows up in many areas of pure and applied mathematics. One of the familiar mean between positive real numbers is the geometric mean $a\#b = \sqrt{ab}$, which is the solution of the algebraic equation $x^2 = ab$. This kind of mean can be extended to matrices as follows. For positive definite matrices A and B of same size, their geometric mean is given by

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

Here $X^{1/2}$ is the unique positive square root of a positive definite matrix X. See more information about positive definite matrices in e.g. [4]. This definition was exhibited in [3] by Ando, and in fact, it is equivalent to that was given in [13]. The fundamental properties of geometric mean for matrices were established in [1]. For any $t \in [0,1]$, the t-weighted geometric mean of positive definite matrices A and B is defined by

$$A\#_tB = A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2}.$$

Note that when $AB = BA$, we have $A\#_tB = A^{1-t}B^t$ and, in particular, $A\#B = (AB)^{1/2}$. More generally, in a uniquely 2-divisible twisted of a group, the geometric mean of two elements a and b can be defined directly by

$$a\#b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}.$$

In the literature, there are many works concerning axiomatized means, see e.g. [2, 6, 7, 11, 12, 14]. Let us focus on the concept of reflection quasigroups, introduced by Lawson and Lim [9]. A reflection quasigroup is a set X together with an assignment $x \mapsto S(x)$, called the symmetry or the point reflection such that the following conditions hold for all $a, b, c \in X$:

(M1) $S_aS_a(b) = b$;

(M2) $S_aS_a(b) = b$;

(M3) $S_aS_b(c) = S_{S_ab}S_a(c)$;

(M4) the equation $S_x(a) = b$ has a unique solution $x \in X$.

The unique solution of the equation $S_x(a) = b$ will be called the (geometric) mean or the midpoint of a and b, denoted by $a\#b$.

Let $\mathbb{D}$ be the set of dyadic rationals in $\mathbb{R}$. Then $\mathbb{D}$ is a reflection quasigroup with respect to the symmetry $S_a(b) = 2a - b$ for each $a, b \in \mathbb{D}$. A function f between two reflection quasigroup is called a symmetry homomorphism or an $\mathfrak{s}$–homomorphism if $f(S_a(b)) = S_{f(a)}(f(b))$ for all a, b. For any reflection quasigroup X and two elements $x, y \in X$, there exists a unique $\mathfrak{s}$–homomorphism $\gamma : \mathbb{D} \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. The function γ is automatically mean–preserving, i.e. $\gamma(a\#b) = \gamma(a)\#\gamma(b)$ for all $a, b \in \mathbb{D}$. See [9] for more information. So we can define the t-weighted mean $x\#_t y = \gamma(t)$, where γ is the unique dyadic geodesic with $\gamma(0) = x$ and $\gamma(1) = y$.

In a reflection quasigroup X,the satisfy the following properties hold for all $u, v, w, z \in X$, and all $r, s, t \in \mathbb{D}$:

(1) (idempotency) $u\#_t u = u$;

(2) (commutivity) $u\#_t v = v\#_{1-t} u$;

(3) (exponential law) $u\#_r(u\#_s v) = u\#_{rs} v$;

(4) (affine change of parameter) $(u\#_r v)\#_t(u\#_s v) = u\#_{(1-t)r+ts} v$;

(5) (limited mediality) $(u\#_t v)\#(w\#_t z) = m$, provided that $u\#w = m = v\#z$;

(6) (cancellativity) $u\#_t v = u\#_t w$ for $t \neq 0$ implies $v = w$.

We consider a topological version of a reflection quasigroup. A lineated symmetric space consists of a set X satisfying the properties (M1)-(M3) together with a Hausdorff topology such that

1) The map $(x, y) \mapsto S_x(y) : X \times X \to X$ is continuous.

2) For $x, y \in X$, there is a unique continuous $\mathfrak{s}$–homomorphism $\alpha_{x,y} : \mathbb{R} \to X$ such that $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$.

3) The map$(t, x, y) \mapsto x\#_t y : \mathbb{R} \times X \times X \to X$ is continuous.

The image $\alpha_{x,y}(t)$ is also denoted $x\#_t y$, and is called the t-weighted mean of x and y. Certain mean equations were investigated in [9].

1.2 Objectives

The objectives of this research are as follows:

1) Provide fundamental examples of lineated symmetric spaces.

2) Investigate properties of weighted means in which weights are arbitrary real numbers.

3) Solve certain weighted means equations in lineated symmetric spaces.

1.3 Scopes of the study

In this paper, we develop further theory of means in a lineated symmetric space. First, we provide fundamental examples of lineated symmetric spaces. Then we investigate properties of weighted means in which weights are arbitrary real numbers. Moreover, we solve the following certain weighted mean equations in lineated symmetric spaces:

- $a\#_t x = b$
- $(a\#_m x)\#_s a = b$
- $(x\#_m a)\#_s x = b$
- $x\#_s(x\#_t a\#_t x)\#_s x = b$
- $a\#_s(a\#_t x\#_t a)\#_s a = b$.

1.4 Benefits

To obtain further theory of weighted means.

1.5 Research methodology

1) Study background in matrix theory.
2) Study background in quasigroup theory.
3) Study the geometric mean for real numbers and matrices.
4) Study research papers about reflection quasigroups and lineated symmetric spaces.
5) Determine objectives and scope of the research.
6) Provide fundamental examples of lineated symmetric spaces.
7) Investigate properties of weighted means in lineated symmetric space.
8) Solving weighted mean equation in lineated symmetric spaces.
9) Conclude the results and write the thesis.

Table 1.1: The research schedule

Activity	Time frame							
	2016				2017			
	Jan.-Mar.	Apr.-Jun.	Jul.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.	Jul.-Sep.	Oct.-Dec.
Step 1	⟷							
Step 2		⟷						
Step 3		⟷						
Step 4			⟷					
Step 5				⟷				
Step 6					⟷			
Step 7						⟷		
Step 8							⟷	
Step 9								⟷

Chapter 2

Preliminaries

In this chapter, we provide preliminary results on theory of (geometric) mean. This chapter includes geometric mean for matrices, geometric mean in uniquely 2-divisible twisted subgroups of a group, reflection quasigroup and lineated symmetric spaces.

2.1 Geometric mean for matrices

Definition 2.1. A Hermitian matrix $A \in M_n(\mathbb{C})$ is said to be positive definite if $x^*Ax > 0$ for all $x \in \mathbb{C}^n - \mathbf{0}$.

Example 2.2. $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is positive definite.

Proof. Clearly, $A^* = A^T = A$. Let $z = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 - \mathbf{0}$. Then

$$z^*Az = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (x+2y)^2 + y^2 \geq 0.$$

If $z^*Az = 0$,then $x = 0$ and $y = 0$. Since x and y are not concurrent zero, we have $z^*Az > 0$. Therefore, A is positive definite. □

Theorem 2.3. ([4]). Let A be a positive definite matrix and $k \in \mathbb{N}$. Then there exists a unique positive definite matrix B such that $B^k = A$.

Definition 2.4. We call the matrix B in Theorem 2.3 the positive k-root of A, denoted by $A^{1/k}$.

Example 2.5. Find $A^{1/2}$ where $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$.

Solution. Write $A = UDU^*$ where U is unitary matrix and D is diagonal matrix.
Then all eigenvalues of A are $\lambda_1 = 9$ and$\lambda_2 = 1$.
$x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigenvector that corresponding with $\lambda_1 = 9$.
$x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is eigenvector that corresponding with $\lambda_2 = 1$.
Then $D = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$.
Hence, $A^{1/2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

The concept of geometric mean for positive real numbers can be extended to positive definite matrices as follows.

Definition 2.6. Let A and B be positive definite matrices of the same size. Then the geometric mean of A and B is defined by

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

Example 2.7. Find $A\#B$ where $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$.

Solution

$$\begin{aligned} A\#B &= A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \\ &= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^{1/2} \left(\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^{-1/2} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^{-1/2} \right)^{1/2} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^{1/2} \\ &= \begin{bmatrix} 1.7889 & 2.2361 \\ 2.2361 & 4.4721 \end{bmatrix}. \end{aligned}$$

2.2 Geometric mean in uniquely 2-divisible twisted subgroups of a group

We shall explain how to define the geometric mean in twisted subgroups of a group.

Definition 2.8. A subset K of a group G is termed a twisted subgroup if it fulfills the following conditions:

1) The identity element belongs to K.

2) For all $x \in K$, we have $x^{-1} \in K$.

3) For all $x, y \in K$,we have $xyx \in K$.

Example 2.9. Recall that $M_n(\mathbb{R})$ is a group under matrix addition. Then $M_n(\mathbb{R})$ is a twisted subgroup of itself.

Proof. Clearly, $\mathbf{0} \in M_n(\mathbb{R})$. For each $A, B \in M_n(\mathbb{R})$, we have $-A, A+B+A \in M_n(\mathbb{R})$. □

Example 2.10. Recall that the general linear group $GL_n(\mathbb{R})$ is a group under usual matrix multiplication. Consider the set $P_n(\mathbb{R})$ of n-by-n positive definite matrices. Then $P_n(\mathbb{R})$ is a twisted subgroup of $GL_n(\mathbb{R})$

Proof. Clearly, $I \in P_n(\mathbb{R})$. For each $A \in P_n(\mathbb{R})$, we have $A^{-1} \in P_n(\mathbb{R})$. For each $A, B \in P_n(\mathbb{R})$, we have $ABA = A^*BA \in P_n(\mathbb{R})$. Here, A^* denotes the conjugate transpose of A. □

Definition 2.11. A twisted subgroup A of a group G is said to be uniquely 2-divisible if every element in A has a unique square root in A.

Example 2.12. By Theorem 2.3, $P_n(\mathbb{R})$ is a uniquely 2-divisible twisted subgroup of the general linear group $GL_n(\mathbb{R})$.

Let A be a twisted subgroup of a group. For each $a, b \in A$, the geometric mean of a and b can be defined by

$$a\#b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}.$$

The core operation of two elements a and b in A is defined by $a \bullet b = ab^{-1}a$.

2.3 Reflection quasigroup

The fundamental idea of a symmetric space is that of a space endowed with a "canonical reflection" S_x through each point x, called a symmetry or point reflection. The symmetry sending x to y ought to be the point symmetry S_m through the midpoint m of x and y. The "midpoint" or "geometric mean" of point x and y should be the point through that on reflects x to y.

Definition 2.13. A quasigroup is a pair consists of a set Q and a binary operation $*$ on Q, such that for each a and b in Q, there exist unique elements x and y in Q for which

1) $a * x = b$ (left quasigroup),

2) $y * a = b$ (right quasigroup).

Definition 2.14. Let $(Q, *)$ be a quasigroup. A subset $A \subseteq Q$ is called a subquasigroup of Q if $(A, *|_A)$ is a quasigroup.

Definition 2.15. A reflection quasigroup, a dyadic symmetric set or a dyadic symset for short, is a set X together with an assignment $a \mapsto S_a$ satisfying the following conditions for all $a, b, c \in X$:

(M1) $S_a(a) = a$

(M2) $S_aS_a(b) = b$

(M3) $S_aS_b(c) = S_{S_ab}S_a(c)$

(M4) the equation $S_x(a) = b$ has a unique solution $x \in X$.

Each S_a is called the symmetry or the point reflection through a. The unique solution of the equation $S_x(a) = b$ will be called the (geometric) mean or the midpoint of a and b, denoted by $a\#b$.

From the above definition, we can set $a \bullet b = S_a(b)$ for each $a, b \in X$. Then a groupoid $(X, \bullet)$ is a reflection quasigroup if and only if it is a right quasigroup satisfies the following conditions for all $a, b, c \in X$:

(M1) $a \bullet a = a$

(M2) $a \bullet (a \bullet b) = b$

(M3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c)$.

Example 2.16. Recall that $(\mathbb{R}, +)$ is an additive group. The natural core operation on $\mathbb{R}$ is defined for each $x, y \in \mathbb{R}$ by

$$x \bullet y = 2x - y.$$

Then $(\mathbb{R}, \bullet)$ is a dyadic symmetric set.

Proof. For any $x, y, z \in \mathbb{R}$, we have

(M1) $x \bullet x = 2x - x = x$;

(M2) $x \bullet (x \bullet y) = x \bullet (2x - y) = 2x - (2x - y) = y$;

(M3) $x \bullet (y \bullet z) = x \bullet (2y - z) = 2x - (2y - z) = 2(2x - y) - (2x - z) = (x \bullet y) \bullet (x \bullet z)$.

(M4) (Existence) We have $\frac{a+b}{2} \bullet a = 2\frac{a+b}{2} - a = b$. This shows the existence of a solution for the equation $x \bullet a = b$.

(Uniqueness) Suppose there exists an element $y \in X$ such that $y \bullet a = b$. Then

$$x \bullet a = b$$
$$2x - a = 2y - a$$
$$x = y$$

This shows the uniqueness of a solution for the equation $x \bullet a = b$.

Therefore, $(\mathbb{R}, \bullet)$ is a dyadic symmetric set. □

Observe that the symmetries in a symmetric space are typically required to have isolated fixed points. Note also that Axiom (M4) ensures that each symmetry has only one fixed point. More precisely, the condition $S_x(a) = a$ holds if and only if $x = a$.

Theorem 2.17. ([9, Proposition1.3]). A twisted subgroup of a group is a reflection quasigroup if and only if it is uniquely 2-divisible.

Note that if X is a uniquely 2-divisible twisted subgroup of a group, $\bullet$ is a core operation, and ϵ is the identity, then

$$x = (\epsilon \# x) \bullet \epsilon = (\epsilon \# x)\epsilon^{-1}(\epsilon \# x) = (\epsilon \# x)^2$$

Thus, $x^{1/2} = \epsilon \# x$. By definition of the core operation $\bullet$, we also have $x \bullet \epsilon = x^2$ and $\epsilon \bullet x = x^{-1}$.

Let $(X, \bullet)$ be an arbitrary reflection quasigroup and fix an element $\epsilon \in X$, called a base point of X. For each $x \in X$ and $t \in \mathbb{D}$, we define

$$x^t = \epsilon \#_t x.$$

Example 2.18. ([9, Example 3.1]). Recall that the set

$$\mathbb{D} = \{\frac{a}{2^b} : a \in \mathbb{Z}, b \in \mathbb{N}\}$$

of dyadic rationals is a group under usual addition. It is a uniquely 2-divisible subgroup of itself, and thus, a reflection quasigroup by Theorem 2.17. The reflection operation is given by

$$r \bullet s = 2r - s$$

The corresponding geometric mean operation is the familiar arithmetic mean

$$r\#s = \frac{r+s}{2}.$$

Definition 2.19. Let $(X, \bullet_X)$ and $(Y, \bullet_Y)$ be reflection quasigroups with associated mean operations $\#_X$ and $\#_Y$, respectively. A function $f : X \to Y$ is called a $\bullet$–homomorphism or an $\mathfrak{s}$–homomorphism if $f(a \bullet_X b) = f(a) \bullet_Y f(b)$ for all $a, b \in X$. It is called a #–homomorphism if $f(a\#_X b) = f(a)\#_Y f(b)$ for all $a, b \in X$.

Theorem 2.20. ([8, Corollary 5.8]). Let $(X, \bullet)$ be a reflection quasigroup and let $x, y \in X$. Then there is a unique $\bullet$-homomorphism (and also #-homomorphism) γ from the dyadic line $\mathbb{D}$ to X such that $\gamma(0) = x$ and $\gamma(1) = y$. For the particular case that when x is chosen to be a base point of X, then the function γ is given by $\gamma(t) = y^t$.

Definition 2.21. A $\bullet$-homomorphism from $\mathbb{D}$ into a reflection quasigroup is called a dyadic geodesic.

Remark 2.22. ([9, Remark 3.3]). A function $\beta : \mathbb{D} \to \mathbb{D}$ is a $\bullet$-homomorphism if and only if it is of the form $\beta(t) = at + b$ for some $a, b \in \mathbb{D}$. This map preserves midpoints, hence is a #-homomorphism, and thus a $\bullet$-homomorphisms.

From Theorem 2.20, one can define weighted geometric means in any pointed reflection quasigroup as follows.

Definition 2.23. Let $(X, \bullet)$ be a reflection quasigroup and let $x, y \in X$. For each $t \in \mathbb{D}$, we define the t-weighted mean of x and y to be $x\#_t y = \gamma(t)$, where γ is the dyadic geodesic such that $\gamma(0) = x$ and $\gamma(1) = y$.

Theorem 2.24. ([9, Theorem 3.10]). In a reflection quasigroup $(X, \bullet)$, the following properties hold for all $u, v, w, z \in X$ and $r, s, t \in \mathbb{D}$:

(1) (idempotency) $u\#_t u = u$;

(2) (commutivity) $u\#_t v = v\#_{1-t} u$;

(3) (exponential law) $u\#_r(u\#_s v) = u\#_{rs} v$;

(4) (affine change of parameter) $(u\#_r v)\#_t(u\#_s v) = u\#_{(1-t)r+ts} v$;

(5) (limited mediality) $(u\#_t v)\#(w\#_t z) = m$, provided that $u\#w = m = v\#z$;

(6) (cancellativity) $u\#_t v = u\#_t w$ for $t \neq 0$ implies $v = w$.

2.4 Lineated symmetric spaces

Definition 2.25. ([6]). A symmetric set, or symset for short, is a groupoid $(X, \bullet)$ satisfying the axioms (M1)-(M3) in Definition 2.15.

Definition 2.26. A lineated symmetric space consists of a symset $(X, \bullet)$ which is also a Hausdorff topological space such that the following hold:

1) The map $(x, y) \mapsto x \bullet y : X \times X \to X$ is continuous.

2) For $x, y \in X$,there exists a unique continuous $\mathfrak{s}-$ homomorphism $\alpha_{x,y} : \mathbb{R} \to X$ such that $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$. Here, we use the natural core operation on $\mathbb{R}$, namely, $a \bullet b = 2a - b$ for each $a, b \in \mathbb{R}$.

3) The map $(t, x, y) \mapsto x\#_t y : \mathbb{R} \times X \times X \to X$ is continuous.

We call $\alpha_{x,y}(t)$ the $t-$weighted mean of x and y, also denoted by $x\#_t y$

Theorem 2.27. ([10, Proposition 3.3]). If $(X, \bullet)$ is a lineated symmetric space, then X is a dyadic symset.

Theorem 2.28. ([10, Theorem 3.10]). Let $(X, \bullet, \varepsilon)$ be a pointed dyadic symset equipped with a Hausdorff topology such that

(i) the map $(x, y) \mapsto x \bullet y : X \times X \to X$ is continuous;

(ii) the map $(q, x) \mapsto x^q : \mathbb{D} \times X \to X$ can be extended a continuous map $(x, y) \mapsto x^t : \mathbb{R} \times X \to X$.

Then X is a pointed lineated symmetric space.

Chapter 3

Fundamental Example of Lineated Symmetric Spaces

In this chapter, we provide four fundamental example of lineated symmetric spaces.

3.1 Auxiliary results

Lemma 3.1. ([9]) Let $A, B \in P_n(\mathbb{R})$. Then the Riccati equation

$$XA^{-1}X = B$$

has a unique positive solution $X = A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$.

Lemma 3.2. ([5, p.430]). For any $X, Y \in M_n(\mathbb{R})$, we have

$$\|e^{X+Y} - e^X\| \leq \|Y\|e^{\|X\|}e^{\|Y\|}.$$

Here, the matrix norm $\| \bullet \|$ is defined by

$$\|A\| = (\sum_{i,j=1}^{n} a_{ij}{}^2)^{1/2}$$

where $A = [a_{ij}] \in M_n(\mathbb{R})$.

Lemma 3.3. The map $\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R}), A \mapsto e^A$ is continuous.

Proof. To show that exp is continuous, let $(A_n)_{n=1}^{\infty}$ be a sequence in $M_n(\mathbb{R})$ such that $A_n \to A \in M_n(\mathbb{R})$. Then by Lemma 3.2, we have

$$\|e^{A_n} - e^A\| \leqslant \|A_n - A\| \cdot e^{\|A\|} \cdot e^{\|A_n - A\|} \to 0 \cdot e^{\|A\|} \cdot e^0 = 0.$$

Hence, $e^{A_n} \to e^A$. That is the map exp is continuous. □

Lemma 3.4. ([5, p.478]) The map $\log : P_n(\mathbb{R}) \to M_n(\mathbb{R}), A \mapsto \log A$ is continuous.

3.2 Fundamental example of lineated symmetric spaces

Example 3.5. Recall that $(\mathbb{R}, +)$ is an additive group. The natural core operation on $\mathbb{R}$ is defined for each $x, y \in \mathbb{R}$ by

$$x \bullet y = 2x - y.$$

Then $(\mathbb{R}, \bullet)$ is a lineated symmetric space.

Proof. For any $x, y, z \in \mathbb{R}$, we have

$$\begin{aligned} x \bullet x &= 2x - x \\ &= x, \\ x \bullet (x \bullet y) &= x \bullet (2x - y) \\ &= 2x - (2x - y) \\ &= y \end{aligned}$$

and

$$\begin{aligned} x \bullet (y \bullet z) &= x \bullet (2y - z) \\ &= 2x - (2y - z) \\ &= 2(2x - y) - (2x - z) \\ &= (x \bullet y) \bullet (x \bullet z). \end{aligned}$$

Thus, $(\mathbb{R}, \bullet)$ is a symset. We equip $\mathbb{R}$ with the usual topology, namely, the metric topology on $\mathbb{R}$ defined by

$$d(x, y) = |x - y|, \qquad x, y \in \mathbb{R}.$$

This topology is known to be Hausdorff.

Next, we will show that $(\mathbb{R}, \bullet)$ is a lineated symmetric space. Since $x \bullet y$ is a polynomial function, we have $(x, y) \mapsto x \bullet y : X \times X \to X$ is continuous. Let $x, y \in \mathbb{R}$. Define $\alpha_{x,y} : (\mathbb{R}, \bullet) \to (\mathbb{R}, \bullet)$ by

$$\alpha_{x,y}(t) = (1 - t)x + ty.$$

For any $a, b \in \mathbb{R}$, we have

$$\begin{aligned} \alpha_{x,y}(a) \bullet \alpha_{x,y}(b) &= [(1 - a)x + ay] \bullet [(1 - b)x + by] \\ &= 2(x - ax + ay) - (x - bx + by) \\ &= [1 - (2a - b)]x + (2a - b)y \\ &= [1 - (a \bullet b)]x + (a \bullet b)y \\ &= \alpha_{x,y}(a \bullet b). \end{aligned}$$

Hence, $\alpha_{x,y}$ is an $\mathfrak{s}$–homomorphism. For uniqueness, let $\beta : (\mathbb{R}, \bullet) \to (\mathbb{R}, \bullet)$ be a continuous $\mathfrak{s}$–homomorphism such that $\beta(0) = x$ and $\beta(1) = y$. We shall consider the restriction $\beta|_{\mathbb{D}}$. Note that

$$\begin{aligned} \beta(\frac{1}{2}) &= \beta(0 \# 1) \\ &= \beta(0) \# \beta(1) \\ &= x \# y \\ &= \frac{x + y}{2} \end{aligned}$$

and

$$\begin{aligned}\beta(\frac{3}{4}) &= \beta(\frac{1}{2}\#1)\\ &= \beta(\frac{1}{2})\#\beta(1)\\ &= \frac{\frac{x+y}{2}+y}{2} \in \mathbb{D}.\end{aligned}$$

Similarly, for any $t \in \mathbb{D}$, we have $\beta(t) \in \mathbb{D}$. Thus, Range $\beta|_{\mathbb{D}} \subseteq \mathbb{D}$. By Remark 2.22, $\beta|_{\mathbb{D}}$ must be of the form $\beta(t) = at + b$ for some $a, b \in \mathbb{D}$. Since both $\alpha_{x,y}$ and β are continuous on $\mathbb{R}$, $\alpha_{x,y} = \beta$ on $\mathbb{D}$, and $\mathbb{D}$ is dense in $\mathbb{R}$, we conclude $\alpha_{x,y} = \beta$ on $\mathbb{R}$. Thus, for $x, y \in \mathbb{R}$, there exists a unique continuous $\mathfrak{s}$−homomorphism $\alpha_{x,y} : \mathbb{R} \to \mathbb{R}$ such that $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$. Since $(t, x, y) \mapsto x\#_t y = \alpha_{x,y}(t) = (1-t)x + ty$ is a polynomial, this map is continuous. Therefore, $(\mathbb{R}, \bullet)$ is a lineated symmetric space. □

Example 3.6. Recall that $(M_n(\mathbb{R}), +)$ is an additive group. Its natural core operation is defined by

$$A \odot B = 2A - B.$$

Then $(M_n(\mathbb{R}), \odot)$ is a lineated symmetric space.

Proof. For any $A, B, C \in M_n(\mathbb{R})$, we have

$$\begin{aligned}A \odot A &= 2A - A\\ &= A,\\ A \odot (A \odot B) &= A \odot (2A - B)\\ &= 2A - (2A - B)\\ &= B\end{aligned}$$

and

$$\begin{aligned}A \odot (B \odot C) &= A \odot (2B - C)\\ &= 2A - (2B - C)\\ &= 2(2A - B) - (2A - C)\\ &= (A \odot B) \odot (A \odot C).\end{aligned}$$

Thus, $(M_n(\mathbb{R}), \odot)$ is a symset. We equip $M_n(\mathbb{R})$ with the usual topology defined by the metric $d : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}$,

$$d(A, B) = (\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij} - b_{ij}|^2)^{1/2}, A = [a_{ij}], B = [b_{ij}].$$

This topology is known to be Hausdorff.

Next, we will show that $(M_n(\mathbb{R}), \odot)$ is a lineated symmetric space. Clearly, the map $(A, B) \mapsto A \odot B = 2A - B$ is continuous. Let $A, B \in M_n(\mathbb{R})$. Define $\alpha_{A,B} : (\mathbb{R}, \bullet) \to (M_n(\mathbb{R}), \odot)$ by

$$\alpha_{A,B}(t) = (1-t)A + tB.$$

Here, the operation $\bullet$ is defined on $\mathbb{R}$ as in Example 3.5. Then, $\alpha_{A,B}(0) = A$ and $\alpha_{A,B}(1) = B$. For any $s, w \in \mathbb{R}$, we have

$$\begin{aligned}
\alpha_{A,B}(s) \odot \alpha_{A,B}(w) &= [(1-s)A + sB] \odot [(1-w)A + wB] \\
&= 2(A - sA + sB) - (A - wA + wB) \\
&= [1 - (2s - w)]A + (2s - w)B \\
&= [1 - (s \odot w)]A + (s \odot w)B \\
&= \alpha_{A,B}(s \odot w).
\end{aligned}$$

Hence, $\alpha_{A,B}$ is an $\mathfrak{s}$–homomorphism. Let $\beta : (\mathbb{R}, \bullet) \to (M_n(\mathbb{R}), \odot)$ be a continuous $\mathfrak{s}$–homomorphism such that $\beta(0) = A$ and $\beta(1) = B$. For each $X \in M_n(\mathbb{R})$, we define $P_{ij}(X) = X_{ij}$, the (i, j)-th entry of X.

$$(\mathbb{R}, \bullet) \xrightarrow{\ \beta\ } (M_n(\mathbb{R}), \odot) \xrightarrow{\ P_{ij}\ } (\mathbb{R}, \bullet)$$

For any $t, s \in \mathbb{R}$, we have

$$\begin{aligned}
(P_{ij} \circ \beta)(t \bullet s) &= P_{ij}(\beta(t) \odot \beta(s)) \\
&= P_{ij}(2\beta(t) - \beta(s)) \\
&= (2\beta(t) - \beta(s))_{ij} \\
&= 2\beta(t)_{ij} - \beta(s)_{ij} \\
&= \beta(t)_{ij} \bullet \beta(s)_{ij} \\
&= (P_{ij} \circ \beta)(t) \bullet (P_{ij} \circ \beta)(s)
\end{aligned}$$

Note also that

$$\begin{aligned}
(P_{ij} \circ \beta)(0) &= P_{ij}(\beta(0)) \\
&= P_{ij}(A) \\
&= A_{ij},
\end{aligned}$$

and

$$\begin{aligned}
(P_{ij} \circ \beta)(1) &= P_{ij}(\beta(1)) \\
&= P_{ij}(\beta) \\
&= B_{ij}.
\end{aligned}$$

Hence, $P_{ij} \circ \beta$ is a continuous $\mathfrak{s}$–homomorphism from $(\mathbb{R}, \bullet)$ to itself such that $(P_{ij} \circ \beta)(0) = A_{ij}$ and $(P_{ij} \circ \beta)(1) = B_{ij}$. By Example 3.5, $P_{ij} \circ \beta$ must be of the form

$$(P_{ij} \circ \beta)(t) = (1-t)A_{ij} + tB_{ij}.$$

Since the (i,j)–th entry of $\beta(t)$ is given by $(P_{ij} \circ \beta)(t)$, we have

$$\begin{aligned}\beta(t) &= [(P_{ij} \circ \beta)(t)]_{ij} \\ &= [(1-t)A_{ij} + tB_{ij}]_{ij} \\ &= (1-t)A + tB \\ &= \alpha_{A,B}(t).\end{aligned}$$

Thus, for any $A, B \in M_n(\mathbb{R})$, there is a unique continuous $\mathfrak{s}$–homomorphism $\alpha_{A,B} : \mathbb{R} \to M_n(\mathbb{R})$ such that $\alpha_{A,B}(0) = A$ and $\alpha_{A,B}(1) = B$. Since the matrix addition and the scalar multiplication are continuous, the map $(t, A, B) \mapsto (1-t)A + tB$ is continuous. Therefore, $(M_n(\mathbb{R}), \odot)$ is a lineated symmetric space. □

Example 3.7. Recall that the positive real numbers $\mathbb{R}^+$ is a group under multiplication. Its natural core operation is defined for each $x, y \in \mathbb{R}^+$ by

$$x \otimes y = xy^{-1}x = \frac{x^2}{y}.$$

Then $(\mathbb{R}^+, \otimes)$ is a lineated symmetric space.

Proof. For any $x, y, z \in \mathbb{R}^+$, we have

$$\begin{aligned}x \otimes x &= \frac{x^2}{x} \\ &= x, \\ x \otimes (x \otimes y) &= x \otimes (\frac{x^2}{y}) \\ &= \frac{x^2}{\frac{x^2}{y}} \\ &= y,\end{aligned}$$

and

$$\begin{aligned}x \otimes (y \otimes z) &= x \otimes (\frac{y^2}{z}) \\ &= \frac{x^2}{\frac{y^2}{z}} \\ &= \frac{\frac{x^4}{y^2}}{\frac{x^2}{z}} \\ &= (\frac{x^2}{y}) \otimes (\frac{x^2}{z}) \\ &= (x \otimes y) \otimes (x \otimes z).\end{aligned}$$

Thus, $(\mathbb{R}^+, \otimes)$ is a symset. We equip $\mathbb{R}^+$ with the subspace topology inherited from $\mathbb{R}$. Since $\mathbb{R}$ is Hausdorff, the subspace $\mathbb{R}^+$ is also Hausdorff.

Next, we will show that $(\mathbb{R}^+, \otimes)$ is a lineated symmetric space. Clearly, the map $(x, y) \mapsto x \otimes y$ is continuous. Let $x, y \in \mathbb{R}^+$. Define $\alpha_{x,y} : (\mathbb{R}, \bullet) \to (\mathbb{R}^+, \odot)$ by

$$\alpha_{x,y}(t) = x^{1-t}y^t.$$

Then, $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$. For any $a, b \in \mathbb{R}$, we get

$$\begin{aligned}
\alpha_{x,y}(a) \otimes \alpha_{x,y}(b) &= [x^{1-a}y^a] \otimes [x^{1-b}y^b] \\
&= \frac{(x^{1-a}y^a)^2}{x^{1-b}y^b} \\
&= x^{1-2a+b}y^{2a-b} \\
&= x^{1 \otimes (a \bullet b)}y^{a \bullet b} \\
&= \alpha_{x,y}(a \bullet b).
\end{aligned}$$

Hence, $\alpha_{x,y}$ is an $\mathfrak{s}$–homomorphism. For uniqueness, let $\beta : (\mathbb{R}, \bullet) \to (\mathbb{R}^+, \otimes)$ be a continuous $\mathfrak{s}$–homomorphism such that $\beta(0) = x$ and $\beta(1) = y$. Consider the function $G : \mathbb{R}^+ \to \mathbb{R}$ defined by $G(x) = \log x$.

$$(\mathbb{R}, \bullet) \xrightarrow{\beta} (M_n(\mathbb{R}^+), \otimes) \xrightarrow{G} (\mathbb{R}, \bullet)$$

Clearly, G is continuous. For any $x, y \in \mathbb{R}$, we obtain

$$\begin{aligned}
(G \circ \beta)(x \bullet y) &= G(\beta(x) \otimes \beta(y)) \\
&= \log(\frac{\beta(x)^2}{\beta(y)}) \\
&= \log \beta(x)^2 - \log \beta(y) \\
&= 2\log \beta(x) - \log \beta(y) \\
&= \log \beta(x) \bullet \log \beta(y) \\
&= (G \circ \beta)(x) \bullet (G \circ \beta)(y).
\end{aligned}$$

Note that

$$\begin{aligned}
(G \circ \beta)(0) &= G(\beta(0)) \\
&= G(x) \\
&= \log x,
\end{aligned}$$

and

$$\begin{aligned}
(G \circ \beta)(1) &= G(\beta(1)) \\
&= G(y) \\
&= \log y.
\end{aligned}$$

Hence, $G \circ \beta$ is a continuous $\mathfrak{s}$–homomorphism such that $(G \circ \beta)(0) = \log x$ and $(G \circ \beta)(1) = \log y$. By Example 3.5, $G \circ \beta$ must be of the form $(G \circ \beta)(t) = (1-t)\log(x) + t\log(y)$.

It follows that

$$\begin{aligned}\log(\beta(t)) &= (1-t)\log(x) + t\log(y)\\ e^{\log(\beta(t))} &= e^{(1-t)\log(x)+t\log(y)} = e^{\log(x^{1-t})+\log(y^t}\\ \beta(t) &= e^{\log(x^{1-t})}e^{\log(y^t)} = \alpha_{x,y}(t).\end{aligned}$$

Thus, for $x, y \in \mathbb{R}^+$, there exists a unique continuous $\mathfrak{s}$–homomorphism $\alpha_{x,y} : \mathbb{R} \to \mathbb{R}^+$ such that $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$. The map $(t, x, y) \mapsto x\#_t y = \alpha_{x,y}(t) = x^{1-t}y^t$ is clearly continuous. Therefore, $(\mathbb{R}^+, \otimes)$ is a lineated symmetric space. ☐

Example 3.8. Recall that $P_n(\mathbb{R})$ is a group under multiplication. Its natural core operation is defined for each $A, B \in P_n(\mathbb{R})$ by

$$A \boxtimes B = AB^{-1}A.$$

Then $(P_n(\mathbb{R}), \boxtimes)$ is a lineated symmetric space.

Proof. For any $A, B, C \in P_n(\mathbb{R})$, we have

(M1)

$$\begin{aligned}A \boxtimes A &= AA^{-1}A\\ &= IA\\ &= A,\end{aligned}$$

(M2)

$$\begin{aligned}A \boxtimes (A \boxtimes B) &= A \boxtimes (AB^{-1}A)\\ &= A(AB^{-1}A)^{-1}A\\ &= AA^{-1}BA^{-1}A\\ &= IBI\\ &= B,\end{aligned}$$

(M3)

$$\begin{aligned}A \boxtimes (B \boxtimes C) &= A \boxtimes (BC^{-1}B)\\ &= A(BC^{-1}B)^{-1}A\\ &= AB^{-1}CB^{-1}A\\ &= AB^{-1}ICIB^{-1}A\\ &= AB^{-1}AA^{-1}CA^{-1}AB^{-1}A\\ &= (AB^{-1}A)(AC^{-1}A)^{-1}(AB^{-1}A)\\ &= (A \boxtimes B) \boxtimes (A \boxtimes C)\end{aligned}$$

and (M4) Lemma 3.1 tells us that the equation

$$X \boxtimes A = XA^{-1}X = B$$

has a unique solution $X = A\#B$. Thus, $(X, \boxtimes)$ is a dyadic symmetric set. We equip $P_n(\mathbb{R})$ with the subspace topology inherited from $M_n(\mathbb{R})$. Since $M_n(\mathbb{R})$ is Haudorff, $P_n(\mathbb{R})$ is also Haudorff.

Next, we will show that $(P_n(\mathbb{R}), \boxtimes)$ is a lineated symmetric space. Since the maps $B \mapsto B^{-1}$ and $(A, B) \mapsto ABA$ are continuous, we have $(A, B) \mapsto A \bullet B = AB^{-1}A$ is continuous. Consider the map $(q, A) \mapsto A^q : \mathbb{D} \times P_n(\mathbb{R}) \to P_n(\mathbb{R})$. We extend this map to the map $\Phi : (x, A) \mapsto A^x : \mathbb{R} \times P_n(\mathbb{R}) \to P_n(\mathbb{R})$. We shall show that the map Φ is continuous. Note that for each $x \in \mathbb{R}$ and $A \in P_n(\mathbb{R})$, we have

$$\begin{aligned} A^x &= e^{\log A^x} \\ &= e^{x \log A}. \end{aligned}$$

Hence, Φ is the composition between the logarithm map, the map $A \mapsto xA$, and the exponential map. The map $A \mapsto xA$ is known to be continuous. The exponential map and the logarithm map are continuous by Lemma 3.3 and Lemma 3.4, respectively. Hence, Φ is continuous. Therefore, $P_n(\mathbb{R})$ is a lineated symmetric space by Theorem 2.28. □

Proposition 3.9. The weighted mean $\#_t$ on $P_n(\mathbb{R})$ is defined by

$$A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \qquad A, B \in P_n(\mathbb{R}).$$

Proof. We shall show that the map $\alpha_{A,B} : \mathbb{R} \to P_n(\mathbb{R}), \alpha_{A,B} = A\#_t B$ is an $\mathfrak{s}$-homomorphism. Let $s, t \in \mathbb{R}$. Then

$$\begin{aligned} \alpha_{A,B}(s) \boxtimes \alpha_{A,B}(t) &= [A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2}] \boxtimes [A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}] \\ &= [A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2}][A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}]^{-1}[A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2}] \\ &= A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2}A^{-1/2}(A^{-1/2}BA^{-1/2})^{-t}A^{-1/2}A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2} \\ &= A^{1/2}(A^{-1/2}BA^{-1/2})^s(A^{-1/2}BA^{-1/2})^{-t}(A^{-1/2}BA^{-1/2})^s A^{1/2} \\ &= A^{1/2}(A^{-1/2}BA^{-1/2})^{2s-w}A^{1/2} \\ &= \alpha_{A,B}(2s-t) \\ &= \alpha_{A,B}(s \bullet t). \end{aligned}$$

Clearly, $t \mapsto A\#_t B$ is continuous. By (3.9), we have $\alpha_{A,B}(0) = A$ and $\alpha_{A,B}(1) = B$. By Example 3.8, the weighted mean $\#_t$ on $P_n(\mathbb{R})$ is defined by (3.9).

□

Chapter 4

Properties of Weighted Means in Lineated Symmetric Spaces

In this chapter, we establish properties of weighted means in lineated symmetric spaces

4.1 Auxiliary results

Lemma 4.1. Let $(X, \bullet)$ be a reflection quasigroup, and let m $\in X$. Then

$$X_m := \{(a, b) \in X \times X | a\#b = m\}$$

is a subquasigroup of the product quasigroup $X \times X$.

Proof. Assume $(X, \bullet_X)$ and $(Y, \bullet_Y)$ are quasigroup. First of all, We must show that $(X \times Y, \star)$ is quasigroup. Let (a_1, a_2), $(b_1, b_2) \in X \times Y$. Since $(X, \bullet_X)$ is a quasigroup, then there is x_1, $x_2 \in X$ sush that $a_1 \bullet_X x_1 = b_1$ and $x_2 \bullet_X a_1 = b_1$. Since $(Y, \bullet_Y)$ is a quasigroup, then there is y_1, $y_2 \in$ Y such that $a_2 \bullet_Y Y_1 = b_2$ and $Y_2 \bullet_Y a_2 = b_2$. Then

$$\begin{aligned}(a_1, a_2) \star (x_1, y_1) &= (a_1 \bullet_X x_1, a_2 \bullet_Y y_1) \\ &= (b_1, b_2),\end{aligned}$$

and

$$\begin{aligned}(x_2, y_2) \star (a_1, a_2) &= (x_2 \bullet_X a_1, y_2 \bullet_Y a_2) \\ &= (b_1, b_2).\end{aligned}$$

We will show uniqueness. Suppose $(a, b) \star (x_1', y_1') = (b_1, b_2)$ where $(x_1', y_1') \in X \times Y$. Then

$$(a_1 \bullet_X x_1', a_2 \bullet_Y y_1') = (b_1, b_2)$$

so

$$a_1 \bullet_X x_1' = b_1 \tag{4.1}$$

and

$$a_2 \bullet_Y y_1' = b_2. \tag{4.2}$$

Since (4.1) and $a_1 \bullet_X x_1 = b_1$, then $x_1 = x_1'$. Since (4.2) and $a_2 \bullet_Y y_1 = b_2$, then $y_1 = y_1'$. Hence, $(x_1, y_1) = (x_1', y_1')$. Suppose $(x_2', y_2') \star (a_1, a_2) = (b_1, b_2)$ where $(x_2', y_2') \in X \times Y$. Then

$$(x_2' \bullet_X a_1, y_2' \bullet_Y a_2) = (b_1, b_2)$$

so

$$(a_1 \bullet_X x_1', a_2 \bullet_Y y_1') = (b_1, b_2).$$

This is

$$x_2' \bullet_X a_1 = b_1 \tag{4.3}$$

and

$$y_2' \bullet_Y a_2 = b_2. \tag{4.4}$$

Since (4.3) and $x_2 \bullet_X a_1 = b_1$, then $x_2 = x_2'$. Since (4.4) and $y_2 \bullet_Y a_2 = b_2$, we have $y_2 = y_2'$. Hence, $(x_2, y_2) = (x_2', y_2')$. Thus, $(X \times Y, \star)$ is quasigroup.

Next, Recall that $X_m := \{(a,b) \in X \times X | a\#b = m\}$. Must show that X_m is a subquasigroup of the product quasigroup $X \times X$. Since $(m,m) \in X_m$, we have $X_m \neq \emptyset$. For each $(a,b),(c,d) \in X_m$, we have

$$(a,b) \bullet (x,y) = (c,d) \tag{4.5}$$

and

$$(x', y') \bullet (a,b) = (c,d). \tag{4.6}$$

By (4.5), $a \bullet x = c$ and $b \bullet y = d$. Hence, $x = c\#a$ and $y = d\#b$. Consider

$$\begin{aligned} x\#y &= (c\#a)\#(d\#b) \\ &= (c\#d)\#(a\#b) \\ &= m\#m \\ &= m. \end{aligned}$$

Thus, $(x,y) \in X_m$. By (4.6), $x' \bullet a = c$ and $y' \bullet b = d$, we have $x' = a\#c$ and $y' = b\#d$. Consider

$$\begin{aligned} x'\#y' &= (a\#c)\#(b\#d) \\ &= (a\#b)\#(c\#d) \\ &= m\#m \\ &= m. \end{aligned}$$

Thus, $(x', y') \in X_m$.

Finally, we will show the uniqueness. Suppose $(a,b) \bullet (\bar{x}, \bar{y}) = (c,d)$ where $(\bar{x}, \bar{y}) \in X_m$. Then $a \bullet \bar{x} = c$ and $b \bullet \bar{y} = d$. Hence, $\bar{x} = c\#a = x$ and $\bar{y} = d\#b = y$.

Suppose $(x'', y'') \bullet (a,b) = (c,d)$ where $(x'', y'') \in X_m$. Then $x'' \bullet a = c$ and $y'' \bullet b = d$. Hence, $x'' = a\#c = x'$ and $y'' = b\#d = y'$.Therefore, X_m is a subquasigroup of the product quasigroup $X \times X$. □

Lemma 4.2. ([5, Proposition 3.7]). Let X be a lineated symmetric space. Then

$$x \bullet (y\#_t z) = (x \bullet y)\#_t(x \bullet z);$$

in particular, $(y\#_t z)^{-1} = y^{-1}\#_t z^{-1}$ for all $t \in \mathbb{D}$ and $x, y, z \in X$.

4.2 Properties of Weighted Means in Lineated Symmetric Spaces

Definition 4.3. A pair (X, ε) is called a pointed lineated symmetric space if X is a lineated symmetric space and ε is an element of X, called a base point.

Definition 4.4. For each $x \in X$ and $t \in \mathbb{R}$, we define

$$x^t = \varepsilon \#_t x.$$

Remark 4.5. The map $\alpha_{x,y}$ in Definition 2.26 is automatically a $\bullet$–homomorphism and a #–homomorphism due to Theorem 2.20 and continuity. More precisely, for each $x, y \in X$ and $t, s \in \mathbb{R}$, we have

$$x \#_{t \bullet s} y = (x \#_t y) \bullet (x \#_s y).$$

Theorem 4.6. Let X be a lineated symmetric space and let $x, y \in X$. Then the map $\Phi_{x,y} : \mathbb{R} \to X$ defined by $\Phi_{x,y}(t) = x \#_t y$ is a #–homomorphism. In particular, the map $t \mapsto y^t : \mathbb{R} \to X$ is a #-homomorphism for each $y \in X$.

Proof. Let $t, s \in \mathbb{R}$. Then, there are two sequences (t_n) and (s_n) in $\mathbb{D}$ such that $t_n \to t$ and $s_n \to s$. By continuity, we have $t_n \bullet s_n \to t \bullet s$ and hence

$$\begin{aligned}
\Phi_{x,y}(t \# s) &= x \#_{t \# s} y \\
&= \lim_{x \to \infty} x \#_{t_n \# s_n} y \\
&= \lim_{x \to \infty} \gamma(t_n \# s_n) \\
&= \lim_{x \to \infty} [\gamma(t_n) \# \gamma(s_n)] \\
&= \lim_{x \to \infty} \gamma(t_n) \# \lim_{x \to \infty} \gamma(s_n) \\
&= (\lim_{x \to \infty} x \#_{t_n} y) \# (\lim_{x \to \infty} x \#_{s_n} y) \\
&= (x \#_t y) \# (x \#_s y) \\
&= \Phi_{x,y}(t) \# \Phi_{x,y}(s).
\end{aligned}$$

Thus, $\Phi_{x,y}$ is a #–homomorphism.

When we consider the pointed lineated symmetric space X with base point $\varepsilon = x$, we get

$$\Phi_{x,y}(t) = x \#_t y = y^t.$$

□

Theorem 4.7. Every lineated symmetric space X satisfies the following properties for all $x, y \in X$ and $r, s \in \mathbb{R}$

(1) $x\#_0 y = x$;

(2) $x\#_1 y = y$;

(3) $\varepsilon\#_{-1} x = \varepsilon \bullet x$;

(4) $x\#_2 y = y \bullet x$;

(5) $x^r \# x^s = x^{\frac{r+s}{2}}$.

Proof. (1) $x\#_0 y = \alpha_{x,y}(0) = x$.

(2) $x\#_1 y = \alpha_{x,y}(1) = y$.

(3) $\varepsilon\#_{-1} x = \alpha_{\varepsilon,x}(-1) = \alpha_{\varepsilon,x}(0 \bullet 1) = \alpha_{\varepsilon,x}(0) \bullet \alpha_{\varepsilon,x}(1) = \varepsilon \bullet x$.

(4) $x\#_2 y = x\#_{1\bullet 0} y = \alpha_{x,y}(1 \bullet 0) = \alpha_{x,y}(1) \bullet \alpha_{x,y}(0) = y \bullet x$.

(5) Let $\varepsilon \in X$ be a base point. Since $\alpha_{x,y}$ is a $\#-$homomorphism (Theorem 4.6), we have

$$\begin{aligned} x^r \# x^s &= (\varepsilon\#_r x)\#(\varepsilon\#_s x) \\ &= \alpha_{\varepsilon,x}(r)\#\alpha_{\varepsilon,x}(s) \\ &= \alpha_{\varepsilon,x}(r\#s) \\ &= \varepsilon\#_{r\#s} x \\ &= x^{r\#s} \\ &= x^{\frac{r+s}{2}}. \end{aligned}$$

□

Theorem 4.8. Let X be a lineated symmetric space. Then the following properties hold for all $x, y, z, w \in X$ and $r, s, t \in \mathbb{R}$:

(1) (idempotency) $x\#_t x = x$;

(2) (commutativity) $x\#_t y = y\#_{1-t} x$;

(3) (limited mediality) If $x\#w = m = y\#z$, then $(x\#_t y)\#(w\#_t z) = m$;

(4) (affine change of parameter) $(x\#_r y)\#_t(x\#_s y) = x\#_{(1-t)r+ts} y$;

(5) (exponential law) $x\#_r(x\#_s y) = x\#_{rs} y$.

Proof. (1) Let $t \in \mathbb{R}$. Since $\mathbb{D}$ is dense in $\mathbb{R}$, there is a sequence (t_n) in $\mathbb{D}$ such that $t_n \to t$. By continuity, $x\#_{t_n} x \to x\#_t x$. By Theorem 2.24(1), $x\#_{t_n} x = x$. Since X is Hausdorff, we conclude $x\#_t x = x$.

(2) Let $t \in \mathbb{R}$. Since $\mathbb{D}$ is dense in $\mathbb{R}$, there is a sequence (t_n) in $\mathbb{D}$ such that $t_n \to t$. By continuity, $x\#_{t_n} y \to x\#_t y$ and $y\#_{1-t_n} x \to y\#_{1-t} x$. By Theorem 2.24(2), $x\#_{t_n} y = y\#_{1-t_n} x$. Since X is Hausdorff, we conclude $x\#_t y = y\#_{1-t} x$.

(3) Suppose $x\#w = m = y\#z$. Let $t \in \mathbb{R}$. Then there is a sequence (t_n) in $\mathbb{D}$ such that $t_n \to t$. We have

$$(x, w), (y, z) \in \{(a, b) \in X \times X \mid a\#b = m\} := X_m.$$

By the Lemma 4.1, X_m is a subquasigroup of $X \times X$ and thus, a reflection quasigroup. For each $n \in \mathbb{N}$, there exists a dyadic geodesic γ_n such that $\gamma_n(t_n) = (x, w)\#_{t_n}(y, z)$ in X_m. Then

$$\begin{aligned}
m &= (x, w)\#_t(y, z) \\
&= \lim_{x\to\infty} (x, w)\#_{t_n} \lim_{x\to\infty} (y, z) \\
&= \lim_{x\to\infty} (x\#_{t_n} y, w\#_{t_n} z) \\
&= \lim_{x\to\infty} (x\#_{t_n} y)\#(w\#_{t_n} z) \\
&= (\lim_{x\to\infty} x\#_{t_n} y)\#(\lim_{x\to\infty} w\#_{t_n} z) \\
&= (x\#_t y)\#(w\#_t z).
\end{aligned}$$

Therefore, $(x\#_t y)\#(w\#_t z) = m$.

(4) Let $t, r, s \in \mathbb{R}$. Since $\mathbb{D}$ is dense in $\mathbb{R}$, there are sequences (t_n), (r_n) and (s_n) in $\mathbb{D}$ such that $t_n \to t$, $r_n \to r$ and $s_n \to s$. By continuity, we have $x\#_{r_n} y \to x\#_r y$ and $x\#_{s_n} y \to x\#_s y$. It follows that $(x\#_{r_n} y)\#_{t_n}(x\#_{s_n} y) \to (x\#_r y)\#_t(x\#_s y)$ and hence

$$x\#_{(1-t_n)r_n + t_n s_n} y \to x\#_{(1-t)r+ts} y.$$

By Theorem 2.24(5), we have

$$(x\#_{r_n} y)\#_{t_n}(x\#_{s_n} y) = x\#_{(1-t_n)r_n + t_n s_n} y.$$

Since X is Hausdorff, we get $(x\#_r y)\#_t(x\#_s y) = x\#_{(1-t)r+ts} y$.

(5) Let $r, s \in \mathbb{R}$. Since $\mathbb{D}$ is dense in $\mathbb{R}$, there are sequence (r_n) and (s_n) in $\mathbb{D}$ such that $r_n \to r$ and $s_n \to s$. Then $r_n s_n \to rs$. By continuity, we have $x\#_{r_n}(x\#_{s_n} y) \to x\#_r(x\#_s y)$ and $x\#_{r_n s_n} y \to x\#_{rs} y$. By Theorem 2.24(5), we have

$$x\#_{r_n}(x\#_{s_n} y) = x\#_{r_n s_n} y.$$

Since X is Hausdorff, we have $x\#_r(x\#_s y) = x\#_{rs} y$. □

Theorem 4.9. Let X be a lineated symmetric space. Then $x \bullet (y\#_t z) = (x \bullet y)\#_t(x \bullet z)$.

Proof. let $x, y, z \in X$ and $t \in \mathbb{R}$. Since $\mathbb{D}$ is dense in $\mathbb{R}$, there is a sequence (t_n) in $\mathbb{D}$ such that $t_n \to t$. By continuity, we have $x \bullet (y\#_{t_n} z) \to x \bullet (y\#_t z)$ and $(x \bullet y)\#_{t_n}(x \bullet z) \to (x \bullet y)\#_t(x \bullet z)$. By Lemma 4.2, we have

$$x \bullet (y\#_{t_n} z) = (x \bullet y)\#_{t_n}(x \bullet z).$$

Since X is Hausdorff, we have $x \bullet (y\#_t z) = (x \bullet y)\#_t(x \bullet z)$. □

Corollary 4.10. Let X be a lineated symmetric space. Then $(y\#_t z)^{-1} = y^{-1}\#_t z^{-1}$ for all $t \in \mathbb{R}$.

Proof. Let $y, z \in X$, y is a base point and $t \in \mathbb{R}$. Since $\mathbb{D}$ is dense in $\mathbb{R}$, there is a sequence (t_n) in $\mathbb{D}$ such that $t_n \to t$. By continuity,

$$\begin{aligned}(y\#_{t_n} z)^{-1} &= y\#_{-1}(y\#_{t_n} z)\\ &= y \bullet (y\#_{t_n} z) \to y \bullet (y\#_t z)\\ &= (y\#_t z)^{-1}\end{aligned}$$

and

$$\begin{aligned}y^{-1}\#_{t_n} z^{-1} &= (y\#_{-1} y)\#_{t_n}(y\#_{-1} z)\\ &= (y \bullet y)\#_{t_n}(y \bullet z) \to (y \bullet y)\#_t(y \bullet z)\\ &= y^{-1}\#_t z^{-1}.\end{aligned}$$

By Lemma 4.2, we have

$$(y\#_{t_n} z)^{-1} = y^{-1}\#_{t_n} z^{-1}.$$

Since X is Hausdorff, we have $(y\#_t z)^{-1} = y^{-1}\#_t z^{-1}$. □

Theorem 4.11. The following hold in a lineated symmetric space X:

(1) For each $w \in X$ and $t \in \mathbb{R} - \{0\}$, the map $x \mapsto x\#_t w$ is bijective.

(2) For each $w \in X$ and $t \in \mathbb{R} - \{1\}$, the map $x \mapsto w\#_t x$ is bijective.

Proof. (1) For each $z \in X$, there exists $a\#_t z \in X$ such that

$$\begin{aligned}a\#_t(a\#_{\frac{1}{t}} z) &= a\#_{t(\frac{1}{t})} z\\ &= a\#_1 z\\ &= z\end{aligned}$$

by Theorem 4.8 (5). Hence, $x \mapsto w\#_t x$ is surjective. Let $w, y, z \in X$ be such that $w\#_t y = w\#_t z$. By Theorem 4.8 (5), we get

$$\begin{aligned}y &= w\#_1 y\\ &= w\#_{\frac{1}{t}}(w\#_t y)\\ &= w\#_1 z\\ &= z.\end{aligned}$$

Hence, $x \mapsto w\#_t x$ is injective. Thus, $x \mapsto w\#_t x$ is bijective.

(2) For each $z \in X$, there exists $z\#_t a \in X$ such that

$$\begin{aligned}(a\#_{\frac{1}{1-t}}z)\#_t a &= a\#_{1-t}(a\#_{\frac{1}{1-t}}z)\\ &= a\#_1 z\\ &= z\end{aligned}$$

by Theorem 4.8 (5). Hence, $x \mapsto w\#_t x$ is surjective. Let $w, y, z \in X$ be such that $y\#_t w = z\#_t w$. By Theorem 4.8 (5), we get

$$\begin{aligned}y &= w\#_1 y\\ &= w\#_{\frac{1}{1-t}}(w\#_{1-t}y)\\ &= w\#_{\frac{1}{1-t}}(w\#_{1-t}z)\\ &= w\#_1 z\\ &= z.\end{aligned}$$

Hence, $x \mapsto w\#_t x$ is injective. Thus, $x \mapsto w\#_t x$ is bijective. □

Theorem 4.12. Let X be a lineated symmetric space X. Then for all $x, y, z \in \mathbb{R}$:

(1) (left cancellability) $x\#_t y = x\#_t z$ for some $t \neq 1$ implies $y = z$.

(2) (right cancellability) $y\#_t x = z\#_t x$ for some $t \neq 0$ implies $y = z$.

Proof. (1) It follows from the injectivity of the map $x \mapsto w\#_t x$ for each $w \in X$ and $t \neq 1$.

(2) It follows from the injectivity of the map $x \mapsto x\#_t w$ for each $w \in X$ and $t \neq 0$. □

Chapter 5

Weighted Mean Equations in Lineated Symmetric Spaces

In this chapter, we investigate certain weighted mean equations in lineated symmetric spaces. Every equation considered here is shown to have unique solution in an explicit form.

Lemma 5.1. Let X be a lineated symmetric space. For any $a, b \in X$ and $s, t \in \mathbb{R}$, we have

$$(a\#_t b)\#_s a = a\#_t(b\#_s a).$$

Proof. By Theorem 4.8, we have

$$\begin{aligned}(a\#_t b)\#_s a &= (a\#_s a)\#_t(b\#_s a)\\ &= a\#_t(b\#_s a).\end{aligned}$$

□

Theorem 5.2. Let X be a lineated symmetric space and let $a, b \in X$ and $t \in \mathbb{R} - \{0\}$. Then the equation $a\#_t x = b$ has a unique solution $x = a\#_{\frac{1}{t}} b$.

Proof. (Existence) By Theorem 4.8, we have

$$\begin{aligned}a\#_t(a\#_{\frac{1}{t}} b) &= a\#_1 b\\ &= b.\end{aligned}$$

(Uniqueness) Suppose there exists an element $y \in X$ such that $a\#_t y = b$. Then

$$\begin{aligned}a\#_{\frac{1}{t}} b &= a\#_{\frac{1}{t}}(a\#_t y)\\ &= a\#_1 y\\ &= y.\end{aligned}$$

□

Corollary 5.3. Let X be a lineated symmetric space and let $a, b \in X$ and $m, s \in \mathbb{R} - \{0, 1\}$. Then, the equation $(a\#_m x)\#_s a = b$ has a unique solution $x = a\#_{\frac{1}{m(1-s)}} b$.

Proof. By Theorem 4.8, we have

$$\begin{aligned}b &= (a\#_m x)\#_s a\\ &= a\#_{1-s}(a\#_m x)\\ &= a\#_{m(1-s)} x.\end{aligned}$$

Hence, $a\#_{m(1-s)} x = b$. It follows from Theorem 5.2 that $x = a\#_{\frac{1}{m(1-s)}} b$ is the unique solution of this equation. □

Corollary 5.4. Let X be a lineated symmetric space and let $a, b \in X$ and $m, s \in \mathbb{R}$ such that $m(1-s) \neq 1$. Then the equation $(x\#_m a)\#_s x = b$ has a unique solution $x = a\#_{\frac{1}{1-m(1-s)}} b$.

Proof. Using Theorem 4.8, we get

$$\begin{aligned} b &= (x\#_m a)\#_s x \\ &= x\#_{1-s}(x\#_m a) \\ &= x\#_{m(1-s)} a. \end{aligned}$$

Hence, $a\#_{1-m(1-s)} x = b$. By Theorem 5.2, we conclude that $x = a\#_{\frac{1}{1-m(1-s)}} b$ is the unique solution of this equation. □

Theorem 5.5. Let X be a lineated symmetric space and let $a, b \in X$ and $t \in \mathbb{R}$. Then the geometric mean $x = a\#b$ is the unique solution of

$$(x\#_t a)\#(x\#_t b) = x.$$

Proof. (Uniqueness) Suppose $(x\#_t a)\#(x\#_t b) = x$. Then

$$\begin{aligned} x\#_t b &= x \bullet (x\#_t a) \\ &= (x \bullet x)\#_t(x \bullet a) \\ &= x\#_t(x \bullet a). \end{aligned}$$

By Theorem 4.12, we have $b = x \bullet a$. Hence, $x = a\#b$.

(Existence) Consider $x = a\#b$. We have $a\#b = x = x\#x$. By Theorem 4.8, $(x\#_t a)\#(x\#_t b) = x$.

□

Theorem 5.5 is a generalization of Theorem 4.1 of [9].

Theorem 5.6. Let X be a lineated symmetric space and $t, s \in \mathbb{R}$ such that $ts \neq 1$. The weighted mean $a\#_{\frac{1}{1-ts}} b$ is the unique solution of the equation

$$x\#_t(x\#_s a) = b.$$

Proof. (Uniqueness) Suppose $x\#_t(x\#_s a) = b$. Then, $x\#_{ts} a = b$. Hence, $a\#_{1-ts} x = b$. By Theorem 5.2, $x = a\#_{\frac{1}{1-ts}} b$.

(Existence) Consider $x = a\#_{\frac{1}{1-ts}} b$. We have

$$\begin{aligned} x\#_t(x\#_s a) &= (a\#_{\frac{1}{1-ts}} b)\#_t((a\#_{\frac{1}{1-ts}} b)\#_s a) \\ &= (a\#_{\frac{1}{1-ts}} b)\#_{ts} a \\ &= a\#_{1-ts}(a\#_{\frac{1}{1-ts}} b) \\ &= a\#_1 b \\ &= b. \end{aligned}$$

□

Theorem 5.7. Let X be a lineated symmetric space and $t \in \mathbb{R} - \{2\}$. The weighted mean $a\#_{\frac{1}{2-t}} b$ is the unique solution of the equation

$$(a\#_t x)\#b = x.$$

Proof. Assume that $(a\#_t x)\#b = x$. Then

$$\begin{aligned} b &= x \bullet (a\#_t x) \\ &= (x \bullet a)\#_t (x \bullet x) \\ &= (x \bullet a)\#_t x. \end{aligned}$$

Set $y = x \bullet a$. Then $x = y\#a$ and

$$\begin{aligned} b &= y\#_t (y\#a) \\ &= y\#_t (y\#_{\frac{1}{2}} a) \\ &= y\#_{\frac{t}{2}} a \\ &= a\#_{1-\frac{t}{2}} y. \end{aligned}$$

Hence, the equation $(a\#_t x)\#b = x$ is transformed to $a\#_{1-\frac{t}{2}} y = b$. The latter equation has a unique solution $y = a\#_{\frac{1}{1-\frac{t}{2}}} b$. Thus, the equation $(a\#_t x)\#b = x$ has a unique solution

$$\begin{aligned} x &= y\#a \\ &= (a\#_{1-\frac{t}{2}} b)\#a \\ &= a\#_{\frac{1}{2}} (a\#_{1-\frac{t}{2}} b) \\ &= (a\#_{2(1-\frac{t}{2})} b) \\ &= a\#_{\frac{1}{2-t}} b. \end{aligned}$$

□

Chapter 6

Conclusions and Suggestions

6.1 Conclusions

The following examples are lineated symmetric spaces:

- The real numbers $\mathbb{R}$ with the core operation $x \bullet y = 2x - y$. Here $x\#_t y = (1-t)x + ty$.
- The set of square real matrices $M_n(\mathbb{R})$ with the core operation $A \odot B = 2A - B$. Here $A\#_t B = (1-t)A + tB$.
- The positive real number $\mathbb{R}^+$ with the core operation $x \otimes y = x^2/y$. Here $x\#_t y = x^{1-t}y^t$.
- The set of positive definite matrices $P_n(\mathbb{R})$ with the core operation $A \boxtimes B = AB^{-1}A$. Here $A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$.

Weighted means in lineated symmetric spaces have the following properties:

- the map $\Phi_{x,y} : \mathbb{R} \to X$ defined by $\Phi_{x,y}(t) = x\#_t y$ is a #−homomorphism; in particular, this holds for $t \mapsto y^t : \mathbb{R} \to X$ for all $x, y \in X$ and $t \in \mathbb{R}$;
- $x\#_0 y = x$;
- $x\#_1 y = y$;
- $\varepsilon\#_{-1} x = \varepsilon \bullet x$;
- $x\#_2 y = y \bullet x$;
- $x^r \# x^s = x^{\frac{r+s}{2}}$;
- (idempotency) $x\#_t x = x$;
- (commutativity) $x\#_t y = y\#_{1-t} x$;
- (limited mediality) $(x\#_t y)\#(w\#_t z) = m$, provided that $x\#w = m = y\#z$;
- (affine change of parameter) $(x\#_r y)\#_t(x\#_s y) = x\#_{(1-t)r+ts} y$;
- (exponential law) $x\#_r(x\#_s y) = x\#_{rs} y$;
- $x \bullet (y\#_t z) = (x \bullet y)\#_t(x \bullet z)$;
- $(y\#_t z)^{-1} = y^{-1}\#_t z^{-1}$ for all $t \in \mathbb{R}$;
- For each $w \in X$ and $t \in \mathbb{R} - \{0\}$, the map $x \mapsto x\#_t w$ is bijective;
- For each $w \in X$ and $t \in \mathbb{R} - \{1\}$, the map $x \mapsto w\#_t x$, is bijective;

- (left cancellability) $x\#_t y = x\#_t z$ for some $t \neq 1$ implies $y = z$;
- (right cancellability) $y\#_t x = z\#_t x$ for some $t \neq 0$ implies $y = z$.

Weighted mean equations in lineated symmetric spaces have the following solutions:

- The equation $a\#_t x = b$ has a unique solution $x = a\#_{\frac{1}{t}} b$.
- The geometric mean $x = a\#b$ is the unique solution of $(x\#_t a)\#(x\#_t b) = x$.
- The weighted mean $a\#_{\frac{1}{1-ts}} b$ is the unique solution of the equation $x\#_t(x\#_s a) = b$.
- The weighted mean $a\#_{\frac{1}{2-t}} b$ is the unique solution of the equation $(a\#_t x)\#b = x$.

6.2 Suggestions

We may consider systems of weighted mean equations in lineated symmetric spaces.

References

[1] T. Ando: *Concavity of certain maps on positive definite matrices and applications to Hadamard products,* Linear Algebra and Its Applications, 26 (1979), 203-241.

[2] T. Ando: *Topics on Operator Inequalities,* Lecture Note Hokkaido Univ., Sapporo, 1978.

[3] T. Ando: *Topics on Operator Inequalities,* Hokkaido Univ., Sapporo, 1979.

[4] R. A. Horn and C. R. Johnson: *Matrix Analysis,* Cambridge University Press, 1990.

[5] R. A. Horn and C. R. Johnson: *Topic sin Matrix Analysis,* Cambridge University Press, 1991.

[6] F. Kubo and T. Ando: *Mean of Positive Linear Operator,* Math Ann, 246 (1980), 205-224.

[7] J. D. Lawson and Y. Lim: *The Geometric Mean, Matrices, Metrics, and More,* Amer. Math., 108 (2001), 797-812.

[8] J. D. Lawson and Y. Lim: *Symmetric sets with midpoints and algebraically equivalent theories,* Results in Mathematics 46 (2004), 37-56.

[9] J. D. Lawson and Y. Lim: *Geometric means and reflection quasigroups,* 14 (2006), 43-59.

[10] J. D. Lawson and Y. Lim: *Symmetric Space with Convex Metrics,* Forum Mathematicum, 2007.

[11] Y. Lim: *Geometric Mean on Symmetric Cones,* Arch, 75 (2000), 39-45.

[12] M. Moakher: *A Differential Geometric Approach to The Arithmatic and Geometric Means of Operators in Some Symmetric Spaces,* Preprint.

[13] W. Pusz and S. Woronowicz, *Functional calculus for sesquilinear forms and the purification map,* Reports on Mathematical Physics, 8 (1975), 159-170.

[14] A. Thompson: *Geometric Means of Eillipsoids and Other Convex Bodies,* III International Conference in "Stochastic Geometric, Convex Bodies and Empirical Measures", Part II (Mazara del Vallo, 1999). Rend. Circ. Mat. Palermo(2) Soppl, 65 (2000), 179-191.

Appendix

Appendix A

The research paper

G_001_OF: WEIGHTED MEANS AND SOLVING MEAN EQUATIONS IN LINEATED SYMMETRIC SPACES

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Abstract: In this paper, we develop further theory of means in a lineated symmetric space. First, we investigate properties of weighted means in which weights are arbitrary real numbers. Then, we provide fundamental examples of lineated symmetric spaces. Moreover, we investigate certain weighted mean equations in lineated symmetric spaces.

Keywords: reflection quasigroup, weighted geometric mean, s–homomorphism, lineated symmetric space.

Introduction: The concept of geometric mean $a\#b = \sqrt{ab}$ for positive real numbers a and b, as the solution of the equation $x^2 = ab$, can be extended to positive definite matrices as follows. For positive definite matrices A and B of the same size, their geometric mean is given by

$$A\#B = A^{1/2}\,(A^{-1/2}\,BA^{-1/2})^{1/2}A^{1/2}.$$

This definition was exhibited in [2] by Ando, and in fact, it is equivalent to that was given in [11]. Here, $A^{1/2}$ is the unique positive square root of A. See more information about positive definite matrices in e.g. [12]. Fundamental properties of geometric mean for matrices were established in [1]. For any $t \in [0,1]$, the t-weighted geometric mean of positive definite matrices A and B is defined by

$$A\#_tB = A^{1/2}\,(A^{-1/2}\,BA^{-1/2})^tA^{1/2}.$$

Note that when $AB = BA$, we have $A\#_tB = A^{1-t}B^t$ and, in particular, $A\#B = (AB)^{1/2}$. More generally, in a uniquely 2-divisible twisted subgroup of a group, the geometric mean of two elements a and b can be defined by

$$a\#b = a^{1/2}\,(a^{-1/2}\,ba^{-1/2})^{1/2}a^{1/2}.$$

In the literature, there are many works concerning axiomatized means, see e.g. [3,4,8,9,10,13]. The fundamental idea of a symmetric space is that of a space endowed with a "canonical reflection" S_x through each point x, called a symmetry or point reflection. The symmetry that sends x to y ought to be the point symmetry S_m through the midpoint m of x and y. The "midpoint" or "geometric mean" of point x and y should be the point through that on reflects x to y.

Let $(X, \bullet)$ be a binary system. We define $S_x: X \to X$ by $S_x(y) = x \bullet y$ and view S_x as the symmetry or point reflection through x. Recall that a function $f: X \to Y$ between two binary systems is a homomorphism if $f(u \bullet v) = f(u) \bullet f(v)$ for all $u, v \in X$, and an isomorphism if it is additionally a bijection.

Definition 1.1. A *reflection quasigroup*, a *dyadic symmetric set* or a *dyadic symset* for short, is a groupoid $(X, \bullet)$ satisfying the following conditions for all $a, b, c \in X$:

(M1) $a \bullet a = a \quad (S_a(a) = a)$;
(M2) $a \bullet (a \bullet b) = b \quad (S_aS_a = id_X)$;
(M3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c) \quad (S_aS_b = S_{S_ab})$;
(M4) the equation $x \bullet a = b$ $(S_x(a) = b)$ has a unique solution $x \in X$, called the (*geometric*) *mean*, or *midpoint* of a and b, and denoted by $a\#b$.

If (M1)-(M3) are satisfied, we call $(X, \bullet)$ a symset.

We observe that while the symmetries in a symmetric space are typically required to have isolated fixed points, the axiom (M4) ensures that each symmetry has only one fixed point. More precisely, the condition $S_x(a) = a$ holds if and only if $x = a$.

Theorem 1.2 ([6, Proposition 1.3]). Let A be a twisted subgroup of a group G. Then, $(A, \bullet)$ is a reflection quasigroup if and only if A is uniquely 2-divisible.

Hence, a uniquely 2-divisible twisted subgroup of a group is an example of a reflection quasigroup.

A pair (X, ε) is called a pointed reflection quasigroup if X is a reflection quasigroup and ε is a fixed element of X, called a base point. By the defining equation of $\varepsilon\#x$, we have $x = (\varepsilon\#x) \bullet \varepsilon = (\varepsilon\#x)\varepsilon^{-1}(\varepsilon\#x) = (\varepsilon\#x)^2$.Thus, $x^{1/2} = \varepsilon \# x$. By definition of the core operation $\bullet$, we also have $x \bullet \varepsilon = x^2$ and $\varepsilon \bullet x = x^{-1}$.

Example 1.3 ([6, Example 3.1]). Recall that the set

$$\mathbb{D} = \left\{\frac{a}{2^b}:\ a \in \mathbb{Z}, b \in \mathbb{N}\right\}$$

of dyadic rationals is a group under usual addition. It is a uniquely 2-divisible subgroup of itself, and thus, a reflection quasigroup by Theorem 1.2. The reflection operation is given by

$$r \bullet s = 2r - s.$$

The corresponding geometric mean operation is the familiar midpoint operation (indeed, the arithmetic mean)

$$r\#s = \frac{r+s}{2}.$$

Theorem 1.4 ([5, Corollary 5.8]). Let $(X, \bullet)$ be a reflection quasigroup and let $x, y \in X$. Then, there exists a unique $\bullet$-homomorphism (and also #-homomorphism) γ from the dyadic line $(\mathbb{D}, \bullet)$ to X such that $\gamma(0) = x$ and $\gamma(1) = y$. For the particular case that $x = \varepsilon$ in a pointed reflection quasigroup, γ is given by

$$\gamma(t) = y^t.$$

Remark 1.5 ([6, Remark 3.3]). A function $\beta: \mathbb{D} \longrightarrow \mathbb{D}$ is a $\bullet$-homomorphism if and only if it is of the form $\beta(t) = at + b$ for some $a, b \in \mathbb{D}$. This map preserves midpoints, hence, are #-homomorphisms, and thus, $\bullet$-homomorphisms.

From Theorem 1.4, one can define weighted geometric means in any reflection quasigroup as follows.

Definition 1.6. Let $(X, \bullet)$ be a reflection quasigroup and let $x, y \in X$. For each $t \in \mathbb{D}$, we define the t-weighted mean of x and y to be $x\#_t y = \gamma(t)$, where γ is the unique dyadic geodesic such that $\gamma(0) = x$ and $\gamma(1) = y$.

Theorem 1.7 ([6, Theorem 3.10]). In a reflection quasigroup $(X, \bullet)$, the weighted means satisfy the following properties for all $a, b, c, d \in X$ and $r, s, t \in \mathbb{D}$:

(0) $a\#_0 b = a,\ a\#_1 b = b$;
(1) (idempotency) $a\#_t a = a$;
(2) (commutativity) $a\#_t b = b\#_{1-t} a$;
(3) (limited mediality)If $a\#c = m = b\#d$, then $(a\#_t b)\#(a\#_t d) = m$;
(4) (affine change of parameter) $(a\#_r b)\#_t(a\#_s b) = a\#_{(1-t)r+ts} b$;
(5) (exponential law) $a\#_r(a\#_s b) = a\#_{rs} b$;
(6) (cancellability) $a\#_t b = a\#_t c$ for $t \neq 0$ implies $b=c$.

We consider a topological version of a reflection quasigroup as follows.

Definition 1.8. A *lineated symmetric space* consists of a symset $(X, \bullet)$ equipped with a Hausdorff topology such that

(i) the map $(x, y) \mapsto x \bullet y : X \times X \to X$ is continuous;
(ii) for each $x, y \in X$, there exists a unique continuous ร-homomorphism $\alpha_{x,y}: \mathbb{R} \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. The image $\alpha_{x,y}(t)$ is also denoted by $x\#_t y$, and is called the t-

weighted mean of x and y. Here, we equip $\mathbb{R}$ with the natural core operation $a \bullet b = 2a - b$, $a, b \in \mathbb{R}$;

(iii) the map $(t, x, y) \longmapsto x\#_t y: \mathbb{R} \times X \times X \to X$ is continuous.

Theorem 1.9 ([7, Proposition 3.3]). If X is a lineated symmetric space, then $(X, \bullet)$ is a dyadic symset.

In this paper, we develop further theory of means in a lineated symmetric space. First, we investigate properties of weighted means in which weights are arbitrary real numbers. Then, we provide fundamental examples of lineated symmetric spaces. Moreover, we show that each of the following weighted mean equations has a unique solution:

- $a\#_t x = b$
- $(x\#_t a)\#(x\#_t b) = x$
- $x\#_t(x\#_s a) = b$
- $(a\#_t x)\#b = x.$

Explicit formula of such solutions are presented.

The paper is organized as follows. In the next section, we establish properties of weighted means on lineated symmetric spaces and provide three fundamental examples of those spaces. In Section 3, we investigate certain weighted mean equations in lineated symmetric spaces.

Properties of weighted means and examples of lineated symmetric spaces: In this section, we provide properties of weighted means and three fundamental examples of lineated symmetric spaces.

A pair (X, ε) is called a pointed lineated symmetric space if X is a lineated symmetric space and ε is an element of X, called a base point. For each $x \in X$ and $t \in \mathbb{R}$, we define

$$x^t = \varepsilon\#_t x.$$

Note that the map $\alpha_{x,y}$ in Definition 1.8 is automatically a $\bullet$−homomorphism and a #−homomorphism due to Theorem 1.4 and continuity. More precisely, for each $x, y \in X$ and $t, s \in \mathbb{R}$, we have $x\#_{t\bullet s}y = (x\#_t y) \bullet (x\#_s y)$.

Theorem 2.1 Let X be a lineated symmetric space. Then, the following properties hold for all $x, y, z, w \in X$ and $r, s, t \in \mathbb{R}$:

1) the map $\Phi_{x,y}: \mathbb{R} \to X$ defined by $\Phi_{x,y}(t) = x\#_t y$ is a $\bullet$− and #−homomorphism; in particular, this holds for $t \mapsto y^t: \mathbb{R} \to X$;
2) $x\#_0 y = x,\ x\#_1 y = y$;
3) (idempotency) $x\#_t x = x$;
4) (commutativity) $x\#_t y = y\#_{1-t}x$;
5) (limited mediality) If $x\#w = m = y\#z$, then $(x\#_t y)\#(w\#_t z) = m$;
6) (affine change of parameter) $(x\#_r y)\#_t(x\#_s y) = x\#_{(1-t)r+ts}y$;
7) (exponential law) $x\#_r(x\#_s y) = x\#_{rs}y$;
8) $x \bullet (y\#_t z) = (x \bullet y)\#_t(x \bullet z)$; in particular, $(y\#_t z)^{-1} = y^{-1}\#_t z^{-1}$;
9) The map $x \longmapsto x\#_t w,\ t \neq 1$, and $x \longmapsto w\#_t x,\ t \neq 0$ are bijective;
10) (left cancellability) $x\#_t y = x\#_t z$ for $t \neq 0$ implies $y = z$;
11) (right cancellability) $y\#_t x = z\#_t x$ for $t \neq 0$ implies $y = z$;
12) $x^r\#x^s = x^{\frac{r+s}{2}}$.

<u>Proof</u> We shall prove only the properties 1), 4), 5), 9), 10) and 12). The others can be similarly proven. First, note that X is a dyadic symset by Theorem 1.9.

1) Let $t, s \in \mathbb{R}$. Then, there are two sequences (t_n) and (s_n) in $\mathbb{D}$ such that $t_n \longrightarrow t$ and $s_n \longrightarrow s$. By continuity, we have $t_n \bullet s_n \longrightarrow t \bullet s$ and hence

$$\Phi_{x,y}(t \bullet s) = x\#_{t\bullet s}y = \lim_{n\to\infty} x\#_{t_n\bullet s_n}y = \lim_{n\to\infty} \gamma(t_n \bullet s_n) = \lim_{n\to\infty} [\gamma(t_n) \bullet \gamma(s_n)]$$

$$= \lim_{n\to\infty} \gamma(t_n) \bullet \lim_{n\to\infty} \gamma(s_n) = \left(\lim_{n\to\infty} x\#_{t_n}y\right) \bullet \left(\lim_{n\to\infty} x\#_{s_n}y\right) = (x\#_t y) \bullet (x\#_s y)$$

$= \Phi_{x,y}(t) \bullet \Phi_{x,y}(s)$.

Thus, $\Phi_{x,y}$ is $\bullet$-homomorphism. Similarly, we can conclude that $\Phi_{x,y}$ is #-homomorphism.

When we consider the pointed lineated symmetric space (X, x), we get

$$\Phi_{x,y}(t) = x\#_t y = y^t.$$

4) Let $t \in \mathbb{R}$. Since $\mathbb{D}$ is dense in $\mathbb{R}$, there is a sequence (t_n) in $\mathbb{D}$ such that $t_n \to t$. By continuity, $x\#_{t_n} y \to x\#_t y$ and $y\#_{1-t_n} x \to y\#_{1-t} x$. By Proposition 3.7 of [6], $x\#_{t_n} y = y\#_{1-t_n} x$. Since X is Hausdorff, we conclude $x\#_t y = y\#_{1-t} x$.

5) Suppose $x\#w = m = y\#z$. Let $t \in \mathbb{R}$. Then, there is a sequence (t_n) in $\mathbb{D}$ such that $t_n \longrightarrow t$. By Lemma 3.8 of [6], we have

$$(x, w),\ (y, z) \in \{(a, b) \in X \times X \mid a\#b = m\} \coloneqq X_m.$$

By the same lemma, X_m is a subquasigroup of $X \times X$ and thus, a reflection quasigroup. For each $n \epsilon \mathbb{N}$, there exists a dyadic geodesic γ_n such that $\gamma_n(t_n) = (x, w)\#_{t_n}(y, z)$ in X_m. Then,

$$m = (x, w)\#_t(y, z) = \lim_{n\to\infty}(x, w)\#_{t_n} \lim_{n\to\infty}(y, z) = \lim_{n\to\infty}(x\#_{t_n} y, w\#_{t_n} z)$$

$$= \lim_{n\to\infty}(x\#_{t_n} y)\#(w\#_{t_n} z) = \left(\lim_{n\to\infty} x\#_{t_n} y\right)\#\left(\lim_{n\to\infty} w\#_{t_n} z\right) = (x\#_t y)\#(w\#_t z).$$

Therefore, $(x\#_t y)\#(w\#_t z) = m$.

9) Let $t \in \mathbb{R} - \{0\}$. For each $z \in X$, there exists $a\#_t z \in X$ such that

$$a\#_t\left(a\#_{\frac{1}{t}} z\right) = a\#_{t\left(\frac{1}{t}\right)} z = a\#_1 z = z$$

by 7). Hence, $x \longmapsto w\#_t x$ is surjective. Let $w, y, z \in X$ be such that $w\#_t y = w\#_t z$. By 7), we get

$$y = w\#_1 y = w\#_{\frac{1}{t}}(w\#_t y) = w\#_{\frac{1}{t}}(w\#_t z) = z.$$

Hence, $x \longmapsto w\#_t x$ is injective. Thus, $x \longmapsto w\#_t x$ is bijective. By 4), the map $x \longmapsto x\#_t w,\ t \neq 1$ is also bijective.

10) It follows from the injectivity of the map $x \longmapsto w\#_t x$ for each $w \in X$ and $t \neq 0$.

12) Let $r, s \in \mathbb{R}$. When the base point $\varepsilon = x$, the unique continuous $\mathfrak{s}$-homomorphism α is given by $\alpha(t) = y^t$. Since α is a #-homomorphism, we have

$$x^r\#x^s = \alpha(r)\#\alpha(s) = \alpha(r\#s) = x^{r\#s} = x^{\frac{r+s}{2}}.$$

Next, we provide fundamental examples of lineated symmetric spaces.

Example 2.2. Recall that $(\mathbb{R}, +)$ is an additive group. The natural core operation on $\mathbb{R}$ is defined for each $x, y \in \mathbb{R}$ by

$$x \bullet y = 2x - y.$$

It is straightforward to verify that $(\mathbb{R}, \bullet)$ is a symset. We equip $\mathbb{R}$ with the usual topology, which is known to be Hausdorff.

Next, we will show that $(\mathbb{R}, \bullet)$ is a lineated symmetric space. Since $x \bullet y$ is a polynomial function, we have $(x, y) \longmapsto x \bullet y: X \times X \to X$ is continuous. Let $x, y \in \mathbb{R}$ and let $\alpha_{x,y}: (\mathbb{R}, \bullet) \longrightarrow (\mathbb{R}, \bullet)$ be defined by

$$\alpha_{x,y}(t) = (1 - t)x + ty.$$

For any $a, b \in \mathbb{R}$, we have

$$\begin{aligned}\alpha_{x,y}(a) \bullet \alpha_{x,y}(b) &= [(1-a)x + ay] \bullet [(1-b)x + by] \\ &= 2(x - ax + ay) - (x - bx + by) \\ &= [1 - (2a - b)]x + (2a - b)y \\ &= [1 - (a \bullet b)]x + (a \bullet b)y \\ &= \alpha_{x,y}(a \bullet b).\end{aligned}$$

Hence, $\alpha_{x,y}$ is an $\mathfrak{s}$-homomorphism. For uniqueness, let $\beta: (\mathbb{R}, \bullet) \to (\mathbb{R}, \bullet)$ be a continuous $\mathfrak{s}$-homomorphism such that $\beta(0) = x$ and $\beta(1) = y$. We shall consider the restriction $\beta|_{\mathbb{D}}$. Note that

$$\beta\left(\frac{1}{2}\right) = \beta(0\#1) = \beta(0)\#\beta(1) = x\#y = \frac{x+y}{2},$$
$$\beta\left(\frac{3}{4}\right) = \beta\left(\frac{1}{2}\#1\right) = \beta\left(\frac{1}{2}\right)\#\beta(1) = \frac{\frac{x+y}{2}+y}{2} \in \mathbb{D}.$$

Similarly, for any $t \in \mathbb{D}$, we have $\beta(t) \in \mathbb{D}$. Thus, Range $(\beta|_{\mathbb{D}}) \subseteq \mathbb{D}$. By Remark 3.3 of [6], $\beta|_{\mathbb{D}}$ must be of the form $\beta(t) = at + b$, $t \in \mathbb{D}$ for some $a, b \in \mathbb{D}$. Since both α and β are continuous on $\mathbb{R}$, $\alpha = \beta$ on $\mathbb{D}$, and $\mathbb{D}$ is dense in $\mathbb{R}$, we conclude $\alpha = \beta$ on $\mathbb{R}$. Thus, for $x, y \in \mathbb{R}$, there exists a unique continuous $\mathfrak{s}$-homomorphism $\alpha_{x,y}: \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

Since $(t, x, y) \mapsto x\#_t y = \alpha_{x,y}(t) = (1-t)x + ty$ is a polynomial, this map is continuous. Therefore, $(\mathbb{R}, \bullet)$ is a lineated symmetric space.

Example 2.3. Recall that $(M_n(\mathbb{R}), +)$ is an additive group. Its natural core operation is defined by

$$A \odot B = 2A - B.$$

It is straightforward to show that $(M_n(\mathbb{R}), \odot)$ is a symset. We equip $M_n(\mathbb{R})$ with the usual topology, which is Hausdorff.

Next, we will show that $(M_n(\mathbb{R}), \odot)$ is a lineated symmetric space. Clearly, the map $(A, B) \mapsto A \odot B = 2A - B$ is continuous. Let $A, B \in M_n(\mathbb{R})$ and let $\alpha_{A,B}: (\mathbb{R}, \bullet) \to (M_n(\mathbb{R}), \odot)$ be defined by

$$\alpha_{A,B}(t) = (1-t)A + tB.$$

Here, the operation $\bullet$ is defined on $\mathbb{R}$ as in Example 2.2. Then, $\alpha_{A,B}(0) = A$ and $\alpha_{A,B}(1) = B$. For any $s, w \in \mathbb{R}$, we have

$$\begin{aligned}\alpha_{A,B}(s) \odot \alpha_{A,B}(w) &= [(1-s)A + sB] \odot [(1-w)A + wB] \\ &= 2(A - sA + sB) - (A - wA + wB) \\ &= [1 - (2s - w)]A + (2s - w)B \\ &= [1 - (s \bullet w)]A + (s \bullet w)B \\ &= \alpha_{A,B}(s \bullet w).\end{aligned}$$

Hence, $\alpha_{A,B}$ is an $\mathfrak{s}$-homomorphism.

Let $\beta: (\mathbb{R}, \bullet) \to (M_n(\mathbb{R}), \odot)$ be a continuous $\mathfrak{s}$-homomorphism such that $\beta(0) = x$ and $\beta(1) = y$. For each $A \in M_n(\mathbb{R})$, we define $P_{ij}(A) = A_{ij}$, the (i,j)-th entry of A.

$$(\mathbb{R}, \bullet) \xrightarrow{\beta} (M_n(\mathbb{R}), \odot) \xrightarrow{P_{ij}} (\mathbb{R}, \bullet)$$

For any $t, s \in \mathbb{R}$, we have

$$\begin{aligned}(P_{ij} \circ \beta)(t \bullet s) &= P_{ij}(\beta(t) \odot \beta(s)) = P_{ij}(2\beta(t) - \beta(s)) = (2\beta(t) - \beta(s))_{ij} \\ &= 2\beta(t)_{ij} - \beta(s)_{ij} = \beta(t)_{ij} \bullet \beta(s)_{ij} = (P_{ij} \circ \beta)(t) \bullet (P_{ij} \circ \beta)(s).\end{aligned}$$

Note also that $(P_{ij}o\beta)(0) = P_{ij}(\beta(0)) = A_{ij}$, and $(P_{ij}o\beta)(1) = P_{ij}(\beta(1)) = B_{ij}$. Hence, $P_{ij}o\beta$ is a continuous ร-homomorphism from $(\mathbb{R}, \bullet)$ to itself such that $(P_{ij}o\beta)(0) = A_{ij}$ and $(P_{ij}o\beta)(1) = B_{ij}$. By Example 2.2, $P_{ij}o\beta$ must be of the form

$$(P_{ij}o\beta)(t) = (1-t)A_{ij} + tB_{ij}.$$

Since the (i,j)-th entry of $\beta(t)$ is given by $(P_{ij}o\beta)(t)$, we have

$$\beta(t) = \left[(P_{ij}o\beta)(t)\right]_{ij} = \left[(1-t)A_{ij} + tB_{ij}\right]_{ij} = (1-t)A + tB = \alpha(t).$$

Thus, for any $A, B \in M_n(\mathbb{R})$, there is a unique continuous ร-homomorphism $\alpha_{x,y}: \mathbb{R} \to M_n(\mathbb{R})$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

Since the matrix addition and the scalar multiplication are continuous, the map $(t, A, B) \mapsto (1-t)A + tB$ is continuous. Therefore, $(M_n(\mathbb{R}), \odot)$ is a lineated symmetric space.

Example 2.4. Recall that the positive real numbers $\mathbb{R}^+$ is a group under multiplication. Its natural core operation is defined for each $x, y \in \mathbb{R}$ by

$$x \otimes y = xy^{-1}x = \frac{x^2}{y}.$$

It is easy to prove that $(\mathbb{R}^+, \otimes)$ is a symset. We equip $\mathbb{R}^+$ with the subspace topology inherited from $\mathbb{R}$. Since $\mathbb{R}$ is Hausdorff, the subspace $\mathbb{R}^+$ is also Hausdorff.

Next, we will show that $(\mathbb{R}^+, \otimes)$ is a lineated symmetric space. Clearly, the map $(x, y) \mapsto x \otimes y$ is continuous. Let $x, y \in \mathbb{R}^+$ and let $\alpha_{x,y}: (\mathbb{R}, \bullet) \to (\mathbb{R}^+, \otimes)$ be defined by

$$\alpha_{x,y}(t) = x^{1-t}y^t.$$

Then, $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$. For any $a, b \in \mathbb{R}$, we get

$$\alpha_{x,y}(a) \otimes \alpha_{x,y}(b) = [x^{1-a}y^a] \otimes [x^{1-b}y^b] = \frac{(x^{1-a}y^a)^2}{x^{1-b}y^b} = x^{1-2a+b}y^{2a-b}$$
$$= x^{1\otimes(a\bullet b)}y^{a\bullet b} = \alpha_{A,B}(x \bullet y).$$

Hence, $\alpha_{x,y}$ is an ร-homomorphism. For uniqueness, let $\beta: (\mathbb{R}, \bullet) \to (\mathbb{R}^+, \otimes)$ be a continuous ร-homomorphism such that $\beta(0) = x$ and $\beta(1) = y$. Consider the function $G: \mathbb{R}^+ \to \mathbb{R}$ defined by $G(x) = \log x$.

$$(\mathbb{R}, \bullet) \xrightarrow{\beta} (\mathbb{R}^+, \otimes) \xrightarrow{G} (\mathbb{R}, \bullet)$$

Clearly, G is continuous. For any $x, y \in \mathbb{R}$, we obtain

$$(Go\beta)(x \bullet y) = G(\beta(x) \otimes \beta(y)) = \log\left(\frac{\beta(x)^2}{\beta(y)}\right) = \log\beta(x)^2 - \log\beta(y)$$
$$= 2\log\beta(x) - \log\beta(y) = \log\beta(x) \bullet \log\beta(y)$$
$$= (Go\beta)(x) \bullet (Go\beta)(y).$$

Note that $(Go\beta)(0) = G(\beta(0)) = \log(x)$, and $(Go\beta)(1) = G(\beta(1))\log(y)$. Hence, $Go\beta$ is a continuous ร-homomorphism such that $(Go\beta)(0) = \log(x)$ and $(Go\beta)(1) = \log(y)$. By Example 2.2, $Go\beta$ must be of the form $(Go\beta)(t) = (1-t)\log(x) + t\log(y)$. It follows that

$$\log(\beta(t)) = (1-t)\log(x) + t\log(y)$$
$$e^{\log(\beta(t))} = e^{(1-t)\log(x) + t\log(y)} = e^{\log(x^{1-t}) + \log(y^t)}$$
$$\beta(t) = e^{\log(x^{1-t})}e^{\log(y^t)} = \alpha(t).$$

Thus, for $x, y \in \mathbb{R}^+$, there exists a unique continuous s-homomorphism $\alpha_{x,y}: \mathbb{R} \to \mathbb{R}^+$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

The map $(t, x, y) \longmapsto x\#_t y = \alpha_{x,y}(t) = x^{1-t}y^t$ is clearly continuous. Therefore, $(\mathbb{R}^+, \otimes)$ is a lineated symmetric space.

Weighted mean equations in lineated symmetric spaces: In this section, we investigate certain weighted mean equations in lineated symmetric spaces. Every equation considered here is shown to have unique solution in an explicit form.

Lemma 3.1. Let X be a lineated symmetric space. For any $a, b \in X$ and $s, t \in \mathbb{R}$, we have

$$(a\#_t b)\#_s a = a\#_t(b\#_s a).$$

Proof By Theorem 2.1, we have $(a\#_t b)\#_s a = (a\#_s a)\#_t(b\#_s a) = a\#_t(b\#_s a)$.

Theorem 3.2. Let X be a lineated symmetric space and let $a, b \in X$ and $t \in \mathbb{R} - \{0\}$. Then, the equation $a\#_t x = b$ has a unique solution $x = a\#_{\frac{1}{t}} b$.

Proof (Existence) By Theorem 2.1, we have $a\#_t\left(a\#_{\frac{1}{t}} b\right) = a\#_1 b = b$.

(Uniqueness) Suppose there exists an element $y \in X$ such that $a\#_t y = b$.

Then, $a\#_{\frac{1}{t}} b = a\#_{\frac{1}{t}}(a\#_t y) = a\#_1 y = y$.

Corollary 3.3. Let X be a lineated symmetric space and let $a, b \in X$ and $m, s \in \mathbb{R} - \{0,1\}$. Then, the equation $(a\#_m x)\#_s a = b$ has a unique solution $x = a\#_{\frac{1}{m(1-s)}} b$.

Proof By Theorem 2.1, we have $b = (a\#_m x)\#_s a = a\#_{1-s}(a\#_m x) = a\#_{m(1-s)} x$. Hence, $a\#_{m(1-s)} x = b$. It follows from Theorem 3.2 that $x = a\#_{\frac{1}{m(1-s)}} b$ is the unique solution of this equation.

Corollary 3.4. Let X be a lineated symmetric space and let $a, b \in X$ and $m, s \in \mathbb{R} - \{0,1\}$. Then, the equation $(x\#_m a)\#_s x = b$ has a unique solution $x = a\#_{\frac{1}{1-m(1-s)}} b$.

Proof Using Theorem 2.1, we get

$$b = (x\#_m a)\#_s x = x\#_{1-s}(x\#_m a) = x\#_{m(1-s)} a.$$

Hence, $a\#_{1-m(1-s)} x = b$. By Theorem 3.2, we conclude that $x = a\#_{\frac{1}{1-m(1-s)}} b$ is the unique solution of this equation.

Theorem 3.5. Let X be a lineated symmetric space and let $a, b \in X$ and $t \in \mathbb{R}$. Then, the geometric mean $x = a\#b$ is the unique solution of

$$(x\#_t a)\#(x\#_t b) = x.$$

Proof (Uniqueness) Suppose $(x\#_t a)\#(x\#_t b) = x$. Then,

$$x\#_t b = x \bullet (x\#_t a) = (x \bullet x)\#_t(x \bullet a) = x\#_t(x \bullet a).$$

By the property10) in Theorem 2.1, we have $b = x \bullet a$. Hence, $x = a\#b$.

(Existence) Consider $x = a\#b$. We have $a\#b = x = x\#x$.

By the property 5) in Theorem 2.1, $(x\#_t a)\#(x\#_t b) = x$.

Theorem 3.5 is a generalization of Theorem 4.1 of [6].

Theorem 3.6. Let X be a lineated symmetric space and $t, s \in \mathbb{R}$ such that $ts \neq 1$. The weighted mean $a\#_{\frac{1}{1-ts}} b$ is the unique solution of the equation

$$x\#_t(x\#_s a) = b.$$

Proof (Uniqueness) Suppose $x\#_t(x\#_s a) = b$. Then, $x\#_{ts} a = b$.

Hence, $a\#_{1-ts} x = b$. By Theorem 3.2, $x = a\#_{\frac{1}{1-ts}} b$ is the unique solution of $x\#_t(x\#_s a) = b$.

(Existence) Consider $x = a\#_{\frac{1}{1-ts}} b$. We have

$$x\#_t(x\#_s a) = \left(a\#_{\frac{1}{1-ts}} b\right)\#_t\left(\left(a\#_{\frac{1}{1-ts}} b\right)\#_s a\right) = \left(a\#_{\frac{1}{1-ts}} b\right)\#_{ts} a$$
$$= a\#_{1-ts}\left(a\#_{\frac{1}{1-ts}} b\right) = a\#_1 b = b.$$

Theorem 3.7. Let X be a lineated symmetric space and $t \in \mathbb{R} - \{2\}$. The weighted mean $a\#_{\frac{1}{2-t}} b$ is the unique solution of the equation

$$(a\#_t x)\#b = x.$$

<u>Proof</u> Assume that $(a\#_t x)\#b = x$. Then,

$$b = x \bullet (a\#_t x) = (x \bullet a)\#_t(x \bullet x) = (x \bullet a)\#_t x.$$

Set $y = x \bullet a$. Then, $x = y\#a$ and

$$b = y\#_t(y\#a) = y\#_t\left(y\#_{\frac{1}{2}}a\right) = y\#_{\frac{t}{2}}a = a\#_{1-\frac{t}{2}}y.$$

Hence, the equation $(a\#_t x)\#b = x$ is transformed to $a\#_{\frac{1}{1-\frac{t}{2}}} y = b$. The latter equation has a unique solution $y = a\#_{\frac{1}{1-\frac{t}{2}}} b$. Thus, the equation $(a\#_t x)\#b = x$ has a unique solution

$$x = y\#a = \left(a\#_{1-\left(\frac{t}{2}\right)} b\right)\#a = a\#_{\frac{1}{2}}\left(a\#_{1-\left(\frac{t}{2}\right)} b\right) = \left(a\#_{2\left(1-\left(\frac{t}{2}\right)\right)} b\right) = a\#_{\frac{1}{2-t}} b.$$

References:

1. Ando T. Concavity of Certain Maps on Positive Definite Matrices and Applications to Hadamard Products. Linear Algebra and Its Application. 1979; 26: 203-241.
2. Ando T. Topics on Operator Inequalities. Hokkadido Univ., Sapporo.
3. Ando T, Topics on Operator Inequalities. Lecture Note Hokkaido Univ., Sapporo, 1978.
4. Kubo F, Ando T. Mean of Positive Linear Operator. Math Ann. 1980; 246: 205-224.
5. Lawson J, Lim Y. Symmetric Sets with Midpoints and Algebraically Equivalent Theories. Results in Mathematics. 2004; 46: 37 – 56.
6. Lawson J, Lim Y. Geometric Means and Reflection Quasigroups. Quasigroup and Related Systems. 2006; 14: 43-59.
7. Lawson J, Lim Y. Symmetric Space with Convex Metrics. Forum Mathematicum. 2007; 39 pages.
8. Lawson J, Lim Y. The Geometric Mean, Matrices, Metrics, and More. Amer. Math. 2001; 108: 797-812.
9. Lim Y. Geometric Mean on Symmetric Cones. Arch. 2000; 75: 39-45.
10. Moakher M. A Differential Geometric Approach to The Arithmatic and Geometric Means of Operators in Some Symmetric Spaces. Preprint.
11. Pusz W, Woronowicz S. Functional Calculus for Sesquilinear Forms and The Purification Map. Reports on Mathematical Physics.1975; 8: 159-170.
12. Roger A Horn and Charles R Johnson. Matrix Analysis Cambridge University Press., Cambridge, 1990.
13. Thompson A. Geometric Means of Eillipsoids and Other Convex Bodies. III International Conference in "Stochastic Geometric, Convex Bodies and Empirical Measures", Part II (Mazara del Vallo, 1999). Rend. Circ. Mat. Palermo(2) Soppl. 2000; 65: 179-191.

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Scholarship(s)	1. Scholarships for Graduate Student, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang.

Academic Publication(s)

1. A. Ngodngam and P. Chansangiam "Weighted Means and Solving Mean Equations in Lineated Symmetric Spaces." The 43^{rd} Congress on Science and Technology of Thailand, Chulalongkorn University, Bangkok, Thailand, 17-19 October 2017. pp.650-657.