NUMERICAL COMPUTATIONS OF WATER QUALITY ASSESSMENT IN OPEN-CONNECTED RESERVOIRS

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บทคัดย่อ

ในงานวิจัยนี้นำเสนอตัวแบบเชิงคณิตศาสตร์และวิธีเชิงตัวเลขสำหรับการจำลองทิศทางการไหลของน้ำและระดับมลพิษของน้ำในอ่างเก็บน้ำเชื่อมโยงแบบเปิดที่มีการไหลของน้ำแบบไม่เอกรูป ตัวแบบจำลองเชิงคณิตศาสตร์สองตัวแบบถูกนำมาใช้ในการวัดความเข้มข้นของมลพิษเนื่องจากการปล่อยน้ำเสียของโรงงานอุตสาหกรรม อีกทั้งยังมีปัจจัยภายนอกของน้ำเข้าอันเนื่องมาจากการเปิดประตูน้ำซึ่งทำให้กระแสน้ำเปลี่ยนแปลงและนำพามลพิษเข้ามาอีกด้วย ตัวแบบแรกคือตัวแบบอุทกพลวัตที่ให้ข้อมูลสนามความเร็วและความสูงของน้ำ และตัวแบบที่สองคือตัวแบบการกระจายที่ให้สนามความเข้มข้นของมลพิษ วัตถุประสงค์ของงานวิจัยนี้เพื่อแสดงว่าประตูที่เปิดออกและช่องเชื่อมต่อระหว่างอ่างเก็บน้ำมีอิทธิพลต่อระดับผลพิษในอ่างเก็บน้ำ อีกทั้งสถานะคงตัวของตัวแบบการกระจายและวิธีการหาค่าเหมาะสมที่สุดถูกใช้เพื่อควบคุมระดับมลพิษโดยที่โรงงานอุตสาหกรรมที่อยู่รอบอ่างเก็บน้ำใช้ค่าใช้จ่ายในการบำบัดน้ำเสียต่ำที่สุด

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Abstract

In this thesis, mathematical models and numerical methods for simulating water flow directions and water pollutant levels in open–connected reservoirs with non-uniform flow are proposed. Two mathematical models are used to measure pollution caused by wastewater from industrial plants. Due to the reservoirs' open gates, the models include external factors that change the water current and contain the water pollutant falling into the reservoirs as well. The first model is the hydrodynamic model that provides the needed information for the second model regarding the water velocity field and elevation. The second model is the dispersion model that provides the information pertaining to the pollutant concentration fields. The purpose of this research is to show that the opened gates and the connected channel influence the pollutant level in the reservoirs. The steady state of the dispersion model and some optimization techniques are used to control the pollutant level with minimum costs of wastewater treatment for all industrial plants around the reservoirs.

Keywords: Hydrodynamic Model / Dispersion Model / Water Quality Treatment / Optimization

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Chapter 1

Introduction

1.1 The water pollution problem

The water pollution problem becomes one of the critical global issues which requires ongoing assessment, improvement and treatment of water resource. Water pollution refers to the condition of the property that changed from its natural state. Since there are many pollutants in the mix. In this condition, it is not suitable for the survival of aquatic life, and human consumption.

Causes that pollute the water are the microbes that cause diseases such as bacteria, viruses, fungi in the water. Death of Algae together with its decay is also a reason for sewage and water oxygen deficiency as well as chemicals and toxins from industrial. The sources of water pollution include industrial community resources, agricultural, solid waste disposal sites and etc.

Water pollution affects the people around the source because consuming water cause health problems. Water quality control is necessary for the protection of the water environment and maintenance of recognized quality of water in rivers, estuaries, and reservoirs.

Water quality measurement can be measured by sampling inspection and then evaluated in the laboratory resulting in a relatively high cost. Still, water quality cannot be measured at any position in the water sources. Mathematical modeling though is another way to measure the quality of the water sources, thanks to its lower cost and the ability to calculate the pollutant at any point.

In this thesis, we begin with modifying a mathematical model that combines two existing mathematical models: a non dimensional form of hydrodynamic model and a dispersion model. The proposed model is made to be suitable to the Rama 9 reservoir. The shallow water equation of the hydrodynamic model is assumed by averaging the equation over the depth with flat bottom topography, and discarding the term due to the Coriolis force and surface wind effect. The combination of equation with the calculated velocity is thus used in the dispersion model to approximate the concentration levels of the pollutants. To make the concentration of pollutants in the connected reservoirs lower than the standard, the wastewater treatment is always high. The Simplex method is therefore used to obtain the optimum cost of the water treatment.

1.2 Literature review

There are many methods for detecting the level of pollutants in the water, mostly conducted by a field measurement and a mathematical simulation. The shallow water mass transport's problems are presented in [1], as the method of characteristics has been reported applied. In [14], [12] and [13], the finite element method for solving steady and unsteady water pollution measurements are introduced. The various numerical techniques of solving the uniform flow of stream water quality model are presented in [4], [9] and [11]. The numerical methods of approximating the solution of the two dimensional advection-diffusion-reaction equation are proposed in [4] and [5].

Most non-uniform flow models need the input data concerned with the velocity of the current at any point and any time in the domain. The hydrodynamic model provides the velocity field and the elevation of the water. In [3], [4], [5], [9], [11], the hydrodynamic model and advection-diffusion equation are used to approximate the velocity of the water current in a bay and a channel. In [5] and [10], the results from hydrodynamic model are used as data for the non-uniform flow of the advection-diffusion-reaction equation, which provide the pollutant concentration field. The term of the friction forces occurred thanks to the drag of sides of the uniform reservoir. The theoretical solution of the model was found at the ending point of the domain and the analytical solution to check the accuracy of our approximate solution used. In [5], the Lax-Wendroff method with stability analysis to solve the two dimensional hydrodynamic model with a rectangular domain was proposed. In [18], develop mathematical models and numerical methods for approximating water flow directions and pollutant concentration level in the Rama 9 reservoirs in opened with two parallel canals and assuming bottom topography of reservoir is flat. The Lax-Wendroff method is subsequently used in non dimensional form of a shallow water equation to approximate the velocity of water and elevation of water, we use the forward difference in time and backward difference in space of advection diffusion equation. In [28] and [29], the Lax-Wendroff method for solving the dimensional form of shallow water equation in rectangular model and spherical model with Matlab program are proposed, respectively. In [30], combining two existing mathematical models, a hydrodynamic model which is used to describe the water current in an opened-closed reservoir and a dispersion model which is used to describe the diffusion of the pollutant concentration of water in an opened-closed reservoir. This is to make the proposed model suitable for the reservoir. The shallow water equation of the hydrodynamic model is assumed by averaging the equation over the depth with anisotropic bottom topography, and discarding the term regarding the Coriolis force, surface wind effect and external forces, resulting in the calculated velocity used in the dispersion model to approximate the concentration levels of the pollutants.

Determination of steady-state pollutant levels in a water reservoir causing by wastewater discharge from industrial plants and other external sources can be done accurately by field sampling of the water. However, it is difficult to get samples from every spot in the reservoir and very costly to analyze all of the samples collected. Mathematical simulation is a valuable tool that can be used to simulate the pollutant levels of the water at every spot of the whole reservoir from a relatively small set of collected samples, greatly reducing the total analytical cost. In [15], the authors propose a mathematical simulation to deal with a lake water quality problem in China. They report a good match between their calculated and measured pollutant levels. In [16], the authors present a mathematical model for analyzing the hydrodynamics of and pollutant dispersion in river-type systems. They are able to report changes in the pollutant concentration in the river with time. In a mathematical modeling study of water-quality in the Rama 9 reservoir, Pathumthani District, Thailand [18], two mathematical models are used to simulate its pollutant level. The first model is a hydrodynamic model that used the Lax-Wendroff method to provide the velocity vector of water flow and its elevation. The second model is a dispersion model that used a forward-in-time and central-space finite difference scheme to calculate the pollutant concentration. The resultant water velocity vector field, elevation, and pollutant concentration are reported in contour graphs. In [31], two dimensional hydraulic and pollution models are used to simulate the transport of the pollutant. After the pollutant level at every location has been mathematically determined, it can be input into an optimization model to find the minimum cost that an industrial plant has to expend to initially treat its wastewater to an acceptably low pollutant level before discharging it into a reservoir. Simplex optimization method is a good mathematical model for determining minimum cost. In [7], a mathematical model is proposed for optimally controlling pollutant level in wastewater discharge that would reduce initial water treatment cost to a minimum. In [32], the authors propose mathematical models and optimal control techniques for solving some problems in environmental engineering. In [27], two mathematical models are proposed: a hydrodynamic model and a steady-state pollutant dispersion model. They are used to calculate the pollutant level in a connected-pond reservoir system that has an entrance and an exit gate to open water of a canal and to determine the optimal pollutant levels in the wastewaters discharged from nearby industrial plants that would cost the plants minimally to pre-treat.

1.3 Objectives

To apply the two–dimensional of shallow water equation to the describe water flow and velocity field in an open–connected reservoirs and apply two dimensions of the convection diffusion equation of dispersion model to describe concentration of pollutant at any point in the domain with wastewater discharge from industrial plants and to approximate by finite different method (FDM). The optimization method for minimum cost of water treatment is presented.

1.4 Scope of the thesis

The scope of the thesis is restricted to the application of the finite different method to the water pollution problem in an open–connected reservoirs with flat bottom topography and an opened-closed reservoir with anisotropic bottom topography for measurement and control in the cases of unsteady and steady state.

1.5 Plan of the thesis

The thesis explains the mathematical modeling of water pollution measurement and water quality control in an open-connected reservoirs. The process of simulation discuss the flow water and pollution levels in the domain which can reduce pollution with agreed standard at the lowest cost.

The first part will study the basic knowledge about the shallow water and mathematical model for water quality, measurement and control by defining the domain of problem in thesis and domain of study case.

The second part will study the numerical method for solving two-dimensional hydrodynamic model and dispersion model

The third part is the computation of the unsteady water measurement related to the hydrodynamic model and dispersion model with flat and no flat bottom topography. Two mathematical models are used to simulate pollution due to sewage from industrial plants in an open-connected reservoirs and Rama-nine reservoirs with varied current velocity. The first is the hydrodynamic model that gives the velocity component and elevation of water. The second is the dispersion model that gives the concentration of pollution. In the process of simulation, we are using the Lax-Wendroff technique for solving the first model and Finite Different Method for second model.

The forth part is the computation of steady state two-dimensional dispersion model with the input of model with average velocity in two-dimensional hydrodynamic model. The forward, backward and central different methods applied to solve the very model.

Finally, for the part of water pollution control in the domains, the optimization method is used for calculating the lowest cost of wastewater treatment from industrial plants with concentration of pollution in domains according to the agreed standard.

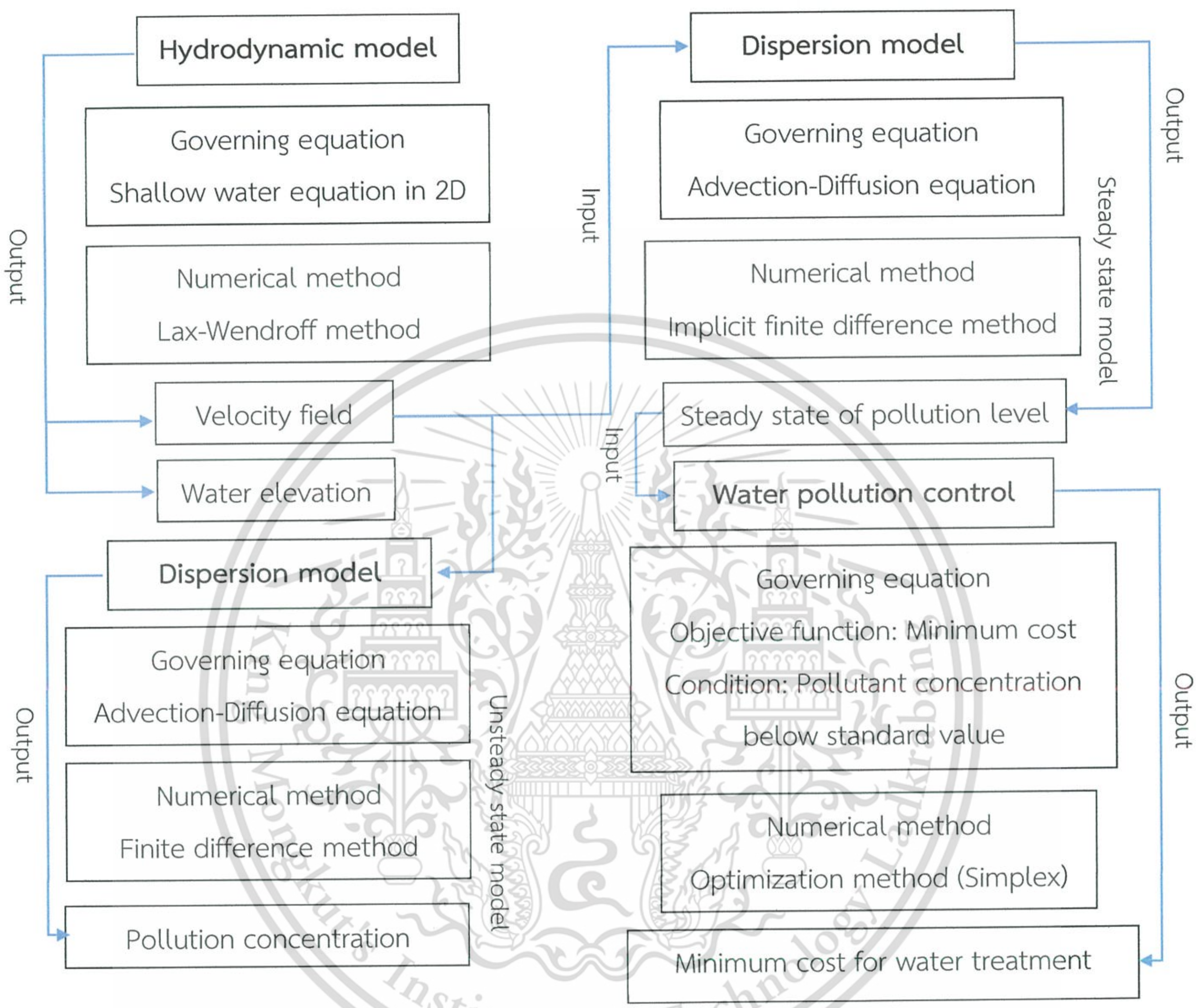


Figure 1.1 Flowchart of water quality measurement and water quality control

1.6 Expected results

The expected results of thesis are the applied model of water pollution assessment and control of water quality with optimal cost of water pollution treatment in an open–connected reservoirs.

Chapter 2

Mathematical Models of Water Quality Measurement

In this section, two mathematical models are used to simulate pollutant concentration in open-connected reservoirs with flat bottom topography and an opened-closed reservoir with anisotropic bottom topography. In the first part of the section, hydrodynamic model, including continuity equation and momentum equation, is used for describing the direction of water flow and elevation of water. In the second part of the section, dispersion model with the governing equation, advection-diffusion equation, is used to describe the dispersion of concentration in two dimension domain.

2.1 The topography of a considered reservoir

In this section, we will explain the description of the domain used in the research include open-connected reservoirs and real problem of the Rama 9 reservoirs.

2.1.1 Open-connected reservoirs

Open-connected reservoirs is the reservoir system consisting of two ponds connected by a narrow channel. One pond allows water in from a canal through an entrance gate while the other pond lets the water pass out through an exit gate shown in Figure 1.1.

2.1.2 The RAMA 9 reservoir

The Rama 9 reservoirs project is a project from the royal initiative of the King of Thailand, located in Khlong Luang, Pathumthani. It lies between the drainage canal Khlong 5 and 6. The project is a large reservoir separated into two parts as shown Figure 1.2(a)-(b).

The main objective of this project

1. to provide consumption of the community nearby.
2. to relieve wastewater condition from canals and some communities in Bangkok.
3. to be a source water storage for the flood relief in flood season.

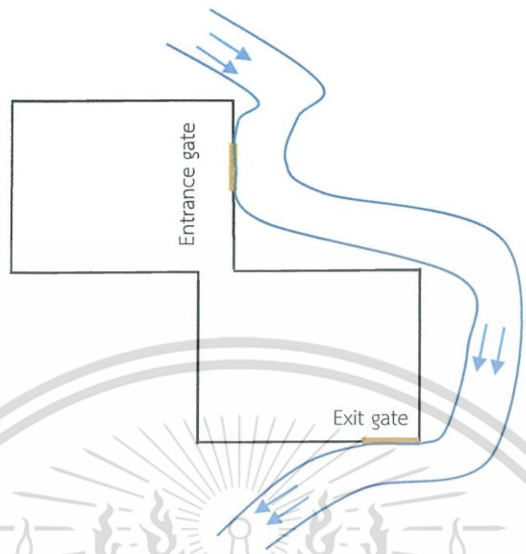
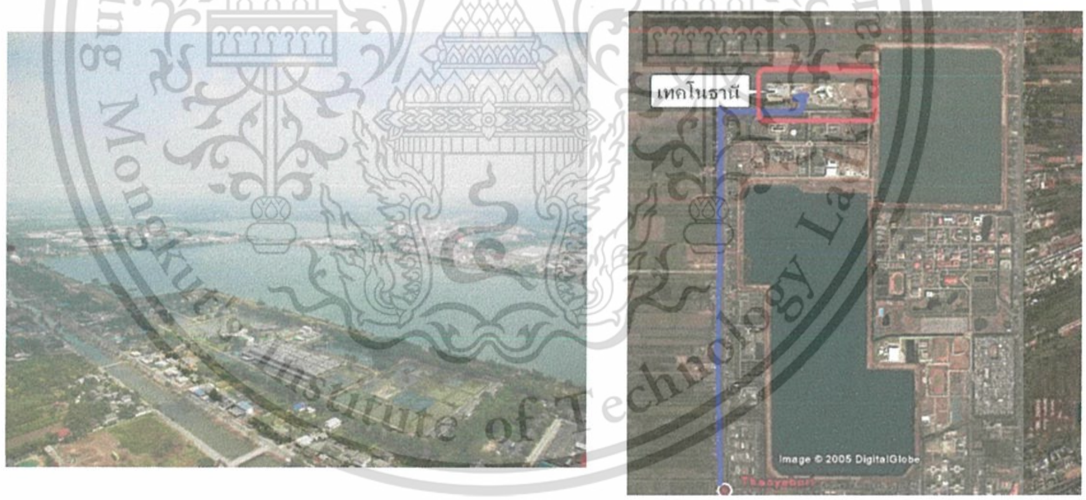


Figure 2.1 Domain of open-connected reservoirs



(a) (b)

Figure 2.2 Domain of the RAMA 9 reservoir [20]

2.2 Water quality monitoring

Water is necessary for the survival of living beings, whether human or animal. When there is growth from industrial development of the community, water, naturally, is unable to make modifications to restore itself. Water pollution in water resources occurs and causes an impact on the ecosystem, including the use of water as well. Therefore, to determine the water status in the current situation, it is important to monitor the water quality. When the facts, will lead the way in solving and preventing water quality caused by pollution. Monitoring water quality, includes the process of exploring and water quality monitoring to ensure water quality information. Determining the water samples is generally defined by 3 main points [22]:

1. The reference points include the upstream or points that are not affected by any pollution source.
2. The change checking points that are affected by the pollution from water resources.
3. The checking points at the head and bottom of the water resources before being released.

The parameters used to monitor water quality need to be specific in which case they could indicate the water quality such as color, temperature, pH, conductivity, dissolved oxygen and biochemical oxygen demand (BOD).

Biochemical oxygen demand (BOD)

In order to analyse the water quality, one of the important factors that indicates the quality of water is the quantity of dissolved oxygen (DO), but if of wastewater, biochemical oxygen demand (BOD) must be taken into account where it is very useful in designing the water treatment system and water quality control.

BOD is the amount of oxygen that microorganisms use to decompose organic substances achieved by the difference of the amount of dissolved oxygen. BOD will be an indicator of contamination of water used to make and can be used as input for the treatment of water.

The importance of the BOD

1. Used to control the contamination of water sources.
2. Used to check the quality of wastewater.
3. Used to design the wastewater treatment system.
4. Used to evaluate the ability of water to get rid of the dirt naturally.

Calculation of BOD

Determination of the BOD, Dilution BOD, which is the standard method of EPA.

1. The sample water must be given temperature up to 20 degrees Celsius.
2. Fill with the air to make oxygen saturation (about 5 - 10 minutes).
3. Fill the sample water into the BOD bottle fully, for at least 3 bottles. Be careful of the bubbles in the bottles, then close stopper tightly, and then find the DO in bottle 1, and keep bottle 2,3 at 20 ° C for 5 days.
4. After 5 days, find the dissolved oxygen in bottle 2,3.
5. Calculating BOD = D1 - D2

   D1 = Dissolved oxygen measured on day 1. (mg/l)

   D2 = Dissolved oxygen measured on day 5 (mg/l).

Water is of little dirty, when BOD is less than 7 (mg/l).

Water is of much dirty, when BOD is more than 7 (mg/l).

Wastewater treatment is the removal or destruction of contaminants in the waste water to the required standard which does not cause pollution to the environment.

Wastewater from different sources have different properties, so there are many ways for water treatment process [22].

2.3 Shallow water equations in two dimensions

The shallow water equations describe a thin layer of fluid of constant density in hydrostatic balance, bounded from below by the bottom topography and above by free surface. To derive the shallow water equations. It is necessary to start from Euler's equations without surface tension [23-25].

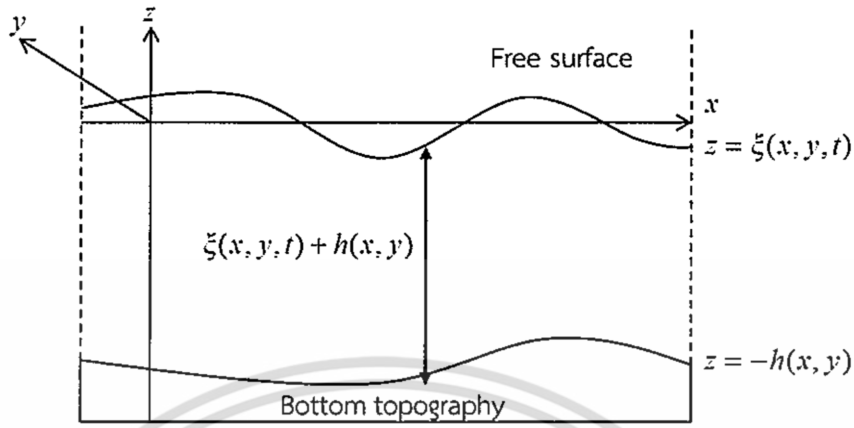


Figure 2.3 Vertical cross-section of the water in the estuary

Free surface condition: $\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \xi = w$ on $z = \xi(x, y, t)$ (2.1)

Bottom boundary condition: $\mathbf{u} \cdot \nabla (z + h(x, y)) = 0$ on $z = -h(x, y)$ (2.2)

Continuity equation: $\nabla \cdot \mathbf{u} = 0$ (2.3)

Momentum equations:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] + S_x, \tag{2.4}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right] + S_y. \tag{2.5}$$

From Eq.(2.3) continuity equation

$$\nabla \cdot \mathbf{u} = 0, \quad \text{for all} \quad -h(x,y) < z < \xi(x,y,t)$$

$$\int_{-h}^{\xi} (\nabla \cdot \mathbf{u}) dz = 0, \tag{2.6}$$

$$\int_{-h}^{\xi} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = 0, \tag{2.7}$$

$$\int_{-h}^{\xi} \frac{\partial u}{\partial x} dz + \int_{-h}^{\xi} \frac{\partial v}{\partial y} dz + \int_{-h}^{\xi} \frac{\partial w}{\partial z} dz = 0. \tag{2.8}$$

From Leibniz's rule, we have

$$\frac{\partial}{\partial x} \int_{-h}^{\xi} u dz - u\big|_{z=\xi} \frac{\partial \xi}{\partial x} + u\big|_{z=-h} \frac{\partial(-h)}{\partial x} + \frac{\partial}{\partial y} \int_{-h}^{\xi} v dz - v\big|_{z=\xi} \frac{\partial \xi}{\partial y} + v\big|_{z=-h} \frac{\partial(-h)}{\partial y}$$
$$+ w\big|_{z=\xi} - w\big|_{z=-h} = 0, \tag{2.9}$$

$$\frac{\partial}{\partial x}\int_{-h}^{\xi} udz + \frac{\partial}{\partial y}\int_{-h}^{\xi} vdz + \left(-u\big|_{z=\xi}\frac{\partial \xi}{\partial x} - v\big|_{z=\xi}\frac{\partial \xi}{\partial y} + w\big|_{z=\xi}\right)$$
$$+\left(u\big|_{z=-h}\frac{\partial(-h)}{\partial x} + v\big|_{z=-h}\frac{\partial(-h)}{\partial y} - w\big|_{z=-h}\right) = 0\,, \tag{2.10}$$

From Eq.(2.2) bottom boundary condition

$$\mathbf{u}\cdot\nabla(z+h(x,y)) = 0\,, \qquad \text{on } z=-h(x,y)$$

$$w + \mathbf{u}\cdot\nabla h(x,y) = 0\,, \tag{2.11}$$

$$w + \mathbf{u}\cdot\left(\frac{\partial h}{\partial x}\hat{i} + \frac{\partial h}{\partial y}\hat{j}\right) = 0\,, \tag{2.12}$$

$$w + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = 0\,, \tag{2.13}$$

$$-w + u\frac{\partial(-h)}{\partial x} + v\frac{\partial(-h)}{\partial y} = 0\,. \tag{2.14}$$

From Eq.(2.1) free surface condition

$$\frac{\partial \xi}{\partial t} + \mathbf{u}\cdot\nabla\xi = w\,, \qquad \text{on } z=\xi(x,y,t)$$

$$\frac{\partial \xi}{\partial t} = w - \mathbf{u}\cdot\nabla\xi\,, \tag{2.15}$$

$$= w - u\frac{\partial \xi}{\partial x} - v\frac{\partial \xi}{\partial y} - w\frac{\partial \xi}{\partial t}\,, \tag{2.16}$$

where $w\frac{\partial \xi}{\partial t} = 0$,

$$\frac{\partial \xi}{\partial t} = w - u\frac{\partial \xi}{\partial x} - v\frac{\partial \xi}{\partial y}\,. \tag{2.17}$$

Substituting Eq.(2.14) and Eq.(2.17) into Eq.(2.10), we have

$$\frac{\partial}{\partial x}\int_{-h}^{\xi} udz + \frac{\partial}{\partial y}\int_{-h}^{\xi} vdz + \frac{\partial \xi}{\partial t} = 0\,, \tag{2.18}$$

$$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x}\left[uz\right]_{-h}^{\xi} + \frac{\partial}{\partial y}\left[vz\right]_{-h}^{\xi} = 0\,, \tag{2.19}$$

$$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x}\left[\xi u + hu\right] + \frac{\partial}{\partial y}\left[\xi v + hv\right] = 0\,, \tag{2.20}$$

$$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x}\left[\xi + h\right]u + \frac{\partial}{\partial y}\left[\xi + h\right]v = 0\,. \tag{2.21}$$

We call Eq.(2.21) the continuity equation of shallow water in two dimensions, from Eqs.(2.4)-(2.5) continuity equation

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] + S_x\,, \tag{2.22}$$

$$\rho\frac{Du}{Dt} + \frac{\partial p}{\partial x} = 0\,, \tag{2.23}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial x}(\rho g\xi)=0\,, \tag{2.24}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+\rho g\frac{\partial}{\partial x}(\xi)=0\,, \tag{2.25}$$

$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+\left[\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right]+S_y\,, \tag{2.26}$$

$$\rho\frac{Dv}{Dt}+\frac{\partial p}{\partial y}=0\,, \tag{2.27}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}(\rho g\xi)=0\,, \tag{2.28}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+\rho g\frac{\partial}{\partial y}(\xi)=0\,. \tag{2.29}$$

Form Eq.(2.25) and Eq.(2.29) divided by ρ, we have momentum equation

$$\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+g\frac{\partial}{\partial x}(\xi)=0\,, \tag{2.30}$$

$$\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+g\frac{\partial}{\partial y}(\xi)=0\,. \tag{2.31}$$

2.4 A two dimensional hydrodynamic model in open-connected reservoirs with flat bottom topography

The unsteady flow of water in a two dimensional space can described by the shallow water equations, which represent mass and momentum conservation. It can be obtained by depth averaging the Navier-Stokes equations in the vertical direction. This leads to a two dimensional formulation in terms of depth averaged quantities and the water depth itself and, neglecting diffusion of momentum due to turbulence, they form the following system of equations: The continuity equation by Eq.(2.21)

$$\frac{\partial \xi}{\partial t}+\frac{\partial}{\partial x}[\xi+h]u+\frac{\partial}{\partial y}[\xi+h]v=0\,, \tag{2.32}$$

and, the momentum equations with Coriolis force term

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu+g\frac{\partial \xi}{\partial y}=0\,, \tag{2.33}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv+g\frac{\partial \xi}{\partial x}=0\,, \tag{2.34}$$

where h is the depth measured from the mean water to the bed of reservoir (m), ξ is the elevation from the mean water level to the temporary water surface (m), g is the acceleration due to gravity (m/s^2), u,v are the velocity components (m/s), f is Coriolis factor.

The governing equation of hydrodynamic describes the behavior of reservoir, averaging the equations over the depth, discarding the term of Coriolis factor, shearing stresses and surface wind and h is constant since the sea bed is a flat bottom topography [3]. We have two dimensional shallow water equation as:

$$\frac{\partial \xi}{\partial t}+\frac{\partial(h+\xi)u}{\partial x}+\frac{\partial(h+\xi)v}{\partial y}=0, \tag{2.35}$$

$$\frac{\partial u}{\partial t}+g\frac{\partial \xi}{\partial x}=0, \tag{2.36}$$

$$\frac{\partial v}{\partial t}+g\frac{\partial \xi}{\partial y}=0. \tag{2.37}$$

We assume h to be constant and $h<<\xi$, and $\xi=h+\xi$ Eqs.(2.35)–(2.37) become,

$$\frac{\partial \xi}{\partial t}+h\frac{\partial u}{\partial x}+h\frac{\partial v}{\partial y}=0, \tag{2.38}$$

$$\frac{\partial u}{\partial t}+g\frac{\partial \xi}{\partial x}=0, \tag{2.39}$$

$$\frac{\partial v}{\partial t}+g\frac{\partial \xi}{\partial y}=0. \tag{2.40}$$

We will transform Eqs.(2.38)–(2.40) into non dimensional form [3] by letting $U=u/\sqrt{gh}$, $V=v/\sqrt{gh}$, $X=x/l$, $Y=y/l$, $Z=\xi/h$, $T=t\sqrt{gh}/l$.

Since
$$Z=\frac{\xi}{h}, \xi=Zh, \tag{2.41}$$

therefore
$$\frac{\partial \xi}{\partial t}=h\frac{\partial Z}{\partial t}=h\frac{\partial Z}{\partial T}\frac{\partial T}{\partial t}=h\frac{\partial Z}{\partial T}\left(\frac{\sqrt{gh}}{l}\right), \tag{2.42}$$

$$\frac{\partial \xi}{\partial x}=h\frac{\partial Z}{\partial x}=h\frac{\partial Z}{\partial X}\frac{\partial X}{\partial x}=\frac{h}{l}\frac{\partial Z}{\partial X}, \tag{2.43}$$

$$\frac{\partial \xi}{\partial y}=h\frac{\partial Z}{\partial y}=h\frac{\partial Z}{\partial Y}\frac{\partial Y}{\partial y}=\frac{h}{l}\frac{\partial Z}{\partial Y}, \tag{2.44}$$

and
$$U=\frac{u}{\sqrt{gh}},\ u=\sqrt{gh}U, \tag{2.45}$$

therefore
$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(\sqrt{gh}\right)U=\sqrt{gh}\frac{\partial U}{\partial x}=\sqrt{gh}\frac{\partial U}{\partial X}\frac{\partial X}{\partial x}=\frac{\sqrt{gh}}{l}\frac{\partial U}{\partial X}, \tag{2.46}$$

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial t}\left(\sqrt{gh}\right)U=\sqrt{gh}\frac{\partial U}{\partial t}=\sqrt{gh}\frac{\partial U}{\partial T}\frac{\partial T}{\partial t}=\frac{gh}{l}\frac{\partial U}{\partial T}, \tag{2.47}$$

and $$V=\frac{v}{\sqrt{gh}}\,,\ v=\sqrt{gh}V\,, \tag{2.48}$$

therefore $$\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}\left(\sqrt{gh}\right)V=\sqrt{gh}\frac{\partial V}{\partial y}=\sqrt{gh}\frac{\partial V}{\partial Y}\frac{\partial Y}{\partial y}=\frac{\sqrt{gh}}{l}\frac{\partial V}{\partial Y}\,, \tag{2.49}$$

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}\left(\sqrt{gh}\right)V=\sqrt{gh}\frac{\partial V}{\partial t}=\sqrt{gh}\frac{\partial V}{\partial T}\frac{\partial T}{\partial t}=\frac{gh}{l}\frac{\partial V}{\partial Y}\,, \tag{2.50}$$

substituting Eq.(2.42), Eq.(2.46) and Eq.(2.47) into Eq.(2.38),

we have $$\frac{\partial \xi}{\partial t}+h\frac{\partial u}{\partial x}+h\frac{\partial v}{\partial y}=h\frac{\sqrt{gh}}{l}\frac{\partial Z}{\partial T}+h\frac{\sqrt{gh}}{l}\frac{\partial U}{\partial X}+h\frac{\sqrt{gh}}{l}\frac{\partial V}{\partial Y}\,, \tag{2.51}$$

$$=h\frac{\sqrt{gh}}{l}\left(\frac{\partial Z}{\partial T}+\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)=0\,, \tag{2.52}$$

$$\left(\frac{\partial Z}{\partial T}+\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)=0\,, \tag{2.53}$$

and substituting Eq.(2.43) and Eq.(2.47) into Eq.(2.39),

we have $$\frac{\partial u}{\partial t}+g\frac{\partial \xi}{\partial x}=\frac{gh}{l}\frac{\partial U}{\partial T}+\frac{gh}{l}\frac{\partial Z}{\partial X}\,, \tag{2.54}$$

$$=\frac{gh}{l}\left(\frac{\partial U}{\partial T}+\frac{\partial Z}{\partial X}\right)=0\,, \tag{2.55}$$

$$\left(\frac{\partial U}{\partial T}+\frac{\partial Z}{\partial X}\right)=0\,, \tag{2.56}$$

and substituting Eqs.(2.44) and Eq.(2.50) into Eq.(2.40),

we have $$\frac{\partial v}{\partial t}+g\frac{\partial \xi}{\partial y}=\frac{gh}{l}\frac{\partial V}{\partial T}+\frac{gh}{l}\frac{\partial Z}{\partial Y}\,, \tag{2.57}$$

$$=\frac{gh}{l}\left(\frac{\partial V}{\partial T}+\frac{\partial Z}{\partial Y}\right)=0\,, \tag{2.58}$$

$$\left(\frac{\partial V}{\partial T}+\frac{\partial Z}{\partial Y}\right)=0\,. \tag{2.59}$$

Eq.(2.53), Eq.(2.56) and Eq.(2.59) are called non dimensional from of shallow water equation. By changing variables U, V, Z to the u, v, d respectively, we see

$$\frac{\partial d}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{2.60}$$

$$\frac{\partial u}{\partial t}+\frac{\partial d}{\partial x}=0\,, \tag{2.61}$$

$$\frac{\partial v}{\partial t}+\frac{\partial d}{\partial y}=0\,. \tag{2.62}$$

2.5 A two dimensional hydrodynamic model in an opened-closed reservoir with anisotropic bottom topography

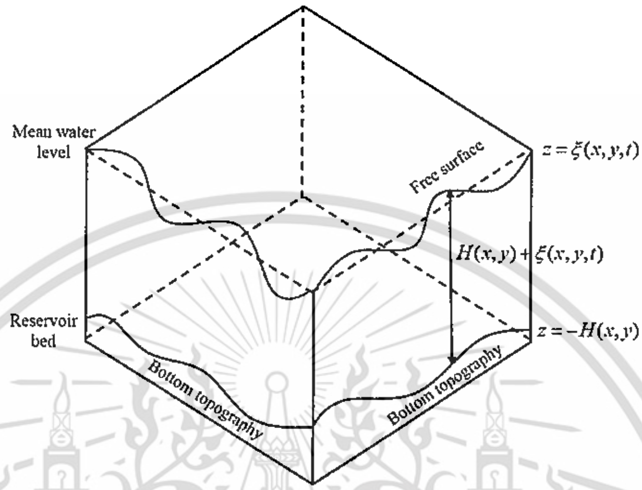


Figure 2.4 Cross-section of the reservoir

The two dimensional unsteady flow of water into and out of the reservoir could be determined by using the system of shallow water equations as the conservation of mass and conservation of momentum were taken into account. the equations of this system could be derived from depth-averaging the Navier-Stokes equations in the vertical direction, neglecting the diffusion of momentum due to turbulence and discarding the terms expressing the effects of friction, surface wind, Coriolis factor and shearing stresses. The continuity equation is then expressed as follows:

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}+\frac{\partial (vh)}{\partial y}=0, \quad (2.63)$$

and the momentum equations are expressed below:

$$\frac{\partial (uh)}{\partial t}+\frac{\partial (u^2h+\frac{1}{2}gh^2)}{\partial x}+\frac{\partial (uvh)}{\partial y}=0, \quad (2.64)$$

$$\frac{\partial (vh)}{\partial t}+\frac{\partial (uvh)}{\partial x}+\frac{\partial (v^2h+\frac{1}{2}gh^2)}{\partial y}=0, \quad (2.65)$$

where $h(x,y,t)$ is the depth measured from the mean surface of water to the reservoir bed ($h=H+\xi$) (m),

$\xi(x,y,t)$ is the elevation of surface of water from the mean water level in reservoir (sea level) (m),

$H(x,y)$ is the anisotropic bottom topography function of the reservoir (m),

$u(x,y,t)$ is velocity in x- direction (m/s),

$v(x,y,t)$ is velocity in y- direction (m/s),

g is gravitational constant 9.8 (m/s^2).

Such time t, and two space coordinates, x and y are the independent variables. Likewise, the conserved quantities are mass, which is proportional to h, and momentum, which is proportional to uh and vh. As taken with respect to the same term, the partial derivatives are grouped into vectors $(\partial x, \partial y, \partial t)$ and later rewritten as a hyperbolic partial differential equation as follows:

$$U = \begin{pmatrix} h \\ uh \\ vh \end{pmatrix}, \quad F(U) = \begin{pmatrix} uh \\ u^2 h + \frac{1}{2} g h^2 \\ uvh \end{pmatrix}, \quad G(U) = \begin{pmatrix} vh \\ uvh \\ v^2 h + \frac{1}{2} g h^2 \end{pmatrix}. \tag{2.66}$$

The hyperbolic PDE:

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial t} F(U) + \frac{\partial}{\partial t} G(U) = 0. \tag{2.67}$$

2.6 Governing equations of water quality measurement

Numerous types of water motion transport matter within natural waters. Wind energy and gravity import motion to the water that leads to mass transport. In the present context within system motion can be divided into two general categories: advection and diffusion [19].

Advection results from flow that is unidirectional and does not change the identity of the substance being transported. Advection moves matter from one position in space to another [19].

Diffusion refers to the movement of mass due to random water motion or mixing. Such transport causes the dye patch depicted to spread out and dilute over time with negligible not movement of its center of mass. Diffusion is the movement molecules from an area of higher concentration to one lower concentration [19].

Dispersion is a related process that also causes pollution to spread. Dispersion is the result of velocity differences in space [19].

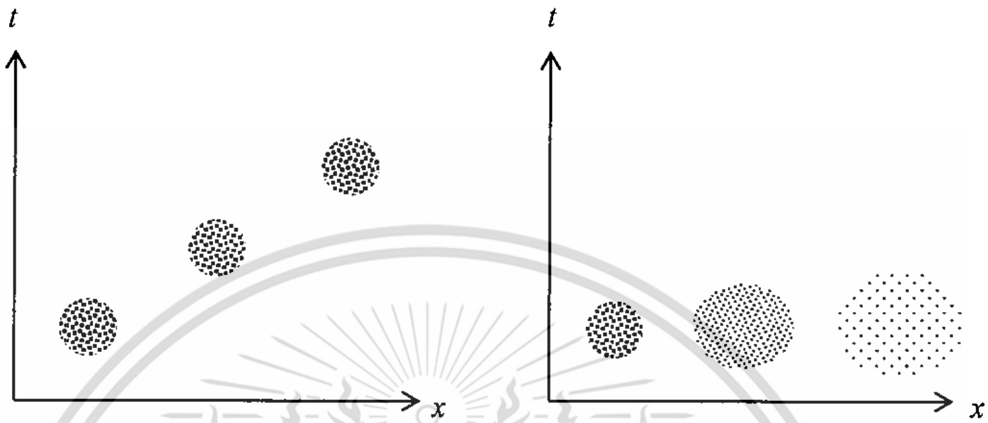


Figure 2.5 The transport of a dye patch in space and time via (a) Advection and (b) Diffusion [19]

Dispersion model (Advection-Diffusion equation)

Advection-Diffusion equation is a combination of the diffusion and advection equations, and describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes.

2.6.1 The two dimensional unsteady state of dispersion model

Mathematical model describing the pollutant concentration in the domain of two dimensional an opened-connected reservoirs where inflow from entrance gate and drain off water at exit gate with the wastewater discharged by the plants around the reservoir. The distributed pollutant process satisfies the mass transfer equation, which includes transportation and diffusion. Averaging the equation over the depth, we get the advection – diffusion equation.

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right), \quad (2.68)$$

where $C(x,y,t)$ is concentration averaged in depth at the point (x,y) at time t (kg/m^3),

D is diffusion coefficient constant (m^2/s), u,v are velocity component (m/s), with initial condition and boundary condition [18].

2.6.2 The two dimensional steady state of dispersion model

In this section, the pollution concentration in connected reservoirs from plants and outer water are considered when longtime or unchanging in time, steady state, derivative of concentration with respect to time zero. The steady state of advection-diffusion equation is below,

$$\bar{u}\frac{\partial C}{\partial x}+\bar{v}\frac{\partial C}{\partial y}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right), \tag{2.69}$$

for all $(x,y)\in\Omega\subseteq\mathbb{R}^2$,

where $\bar{u}$ and $\bar{v}$ are the average of velocities (m/s) in x- and y- directions that obtained by the hydrodynamic model, respectively.

The steady state of dispersion model has no initial condition. The concentration at boundary is $\frac{\partial C}{\partial n}=0$ [25].

Chapter 3

Numerical Methods and Examples

In this section, numerical methods are used to solve the governing equation of two mathematical models, we use the Lax-Wendroff method for solving two dimensional shallow water equation of hydrodynamic model, and explicit finite different method and implicit finite different method for solving advection-diffusion equation. In the final part of this section, the examples used to calculate water elevation, velocities and pollutant concentration in reservoir are shown.

3.1 A numerical method for two dimensional hydrodynamic model with flat bottom topography

First-order hyperbolic systems in two space dimensions

The first order system of equations in two space-dimensional

$$\frac{\partial \mathbf{u}}{\partial t} = A\frac{\partial \mathbf{u}}{\partial x} + B\frac{\partial \mathbf{u}}{\partial y}, \tag{3.1}$$

Where A, B are $n \times n$ real matrices, and $\mathbf{u}$ is an n-component column vector [26].

The Lax-Wendroff method

The region $-\infty < x, y < \infty$ and $t \geq 0$ is covered by a rectangular grid with lines parallel to the x-, y- and t-axes. A typical grid point is given by $x = lh$, $y = mh$ and $t = nk$ where h and k are the grid spacing in the distance and time coordinates respectively, and l, m and n are integers. The explicit difference methods, especially the Lax-Wendroff method, can be used to solve the initial value problem consisting of Eq.(3.1).

Define $\mathbf{u}(x, y, t) = U^n_{l,m}$ and using Taylor's theorem

$$U^{n+1}_{l,m} = U^n_{l,m} + \Delta t\left(\frac{\partial U}{\partial t}\right) + \frac{(\Delta t)^2}{2}\left(\frac{\partial^2 U}{\partial t^2}\right), \tag{3.2}$$

$$= U^n_{l,m} + \Delta t\left(A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y}\right) + \frac{(\Delta t)^2}{2}\left(\frac{\partial}{\partial t}\left(A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y}\right)\right), \tag{3.3}$$

$$= U^n_{l,m} + \Delta t\left(A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y}\right) + \frac{(\Delta t)^2}{2}\left(\frac{\partial}{\partial t}\left(A\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial t}\left(B\frac{\partial U}{\partial y}\right)\right), \tag{3.4}$$

$$= U_{l,m}^{n} + \Delta t\left(A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y}\right) + \frac{(\Delta t)^2}{2}\left(A\frac{\partial}{\partial x}\left(A\frac{\partial U}{\partial x}\right) + B\frac{\partial}{\partial y}\left(A\frac{\partial U}{\partial x}\right)\right)$$

$$+\frac{(\Delta t)^2}{2}\left(A\frac{\partial}{\partial x}\left(B\frac{\partial U}{\partial y}\right) + B\frac{\partial}{\partial y}\left(B\frac{\partial U}{\partial y}\right)\right). \tag{3.5}$$

Case I A, B depending on x, y

$$U_{l,m}^{n+1} = U_{l,m}^{n} + \Delta t\left(A\frac{\partial U}{\partial x}\right) + \Delta t\left(B\frac{\partial U}{\partial y}\right) + \frac{(\Delta t)^2}{2}\left(A\frac{\partial}{\partial x}\left(A\frac{\partial U}{\partial x}\right)\right) + \frac{(\Delta t)^2}{2}\left(B\frac{\partial}{\partial y}\left(B\frac{\partial U}{\partial y}\right)\right)$$

$$+\frac{(\Delta t)^2}{2}\left(B\frac{\partial}{\partial y}\left(A\frac{\partial U}{\partial x}\right)\right) + \frac{(\Delta t)^2}{2}\left(A\frac{\partial}{\partial x}\left(B\frac{\partial U}{\partial y}\right)\right), \tag{3.6}$$

$$= U_{l,m}^{n} + \Delta t\left(A\frac{\partial U}{\partial x}\right) + \Delta t\left(B\frac{\partial U}{\partial y}\right) + \frac{A(\Delta t)^2}{2}\left(A\frac{\partial^2 U}{\partial x^2} + \frac{\partial A}{\partial x}\frac{\partial U}{\partial x}\right) + \frac{B(\Delta t)^2}{2}\left(B\frac{\partial^2 U}{\partial y^2} + \frac{\partial B}{\partial y}\frac{\partial U}{\partial y}\right)$$

$$+\frac{B(\Delta t)^2}{2}\left(A\frac{\partial^2 U}{\partial y \partial x} + \frac{\partial A}{\partial y}\frac{\partial U}{\partial x}\right) + \frac{A(\Delta t)^2}{2}\left(B\frac{\partial^2 U}{\partial x \partial y} + \frac{\partial B}{\partial x}\frac{\partial U}{\partial y}\right), \tag{3.7}$$

$$= U_{l,m}^{n} + A\Delta t\frac{\partial U}{\partial x} + B\Delta t\frac{\partial U}{\partial y} + \frac{A^2(\Delta t)^2}{2}\frac{\partial^2 U}{\partial x^2} + \frac{A(\Delta t)^2}{2}\frac{\partial A}{\partial x}\frac{\partial U}{\partial x} + \frac{B^2(\Delta t)^2}{2}\frac{\partial^2 U}{\partial y^2}$$

$$+\frac{B(\Delta t)^2}{2}\frac{\partial B}{\partial y}\frac{\partial U}{\partial y} + \frac{BA(\Delta t)^2}{2}\frac{\partial^2 U}{\partial y \partial x} + \frac{B(\Delta t)^2}{2}\frac{\partial A}{\partial y}\frac{\partial U}{\partial x} + \frac{AB(\Delta t)^2}{2}\frac{\partial^2 U}{\partial y \partial x} + \frac{A(\Delta t)^2}{2}\frac{\partial B}{\partial x}\frac{\partial U}{\partial y}, \tag{3.8}$$

$$= U_{l,m}^{n} + A\Delta t\frac{\partial U}{\partial x} + B\Delta t\frac{\partial U}{\partial y} + \frac{A^2(\Delta t)^2}{2}\frac{\partial^2 U}{\partial x^2} + \frac{B^2(\Delta t)^2}{2}\frac{\partial^2 U}{\partial y^2} + \frac{(BA+AB)(\Delta t)^2}{2}\frac{\partial^2 U}{\partial x \partial y}$$

$$+\left[\frac{A(\Delta t)^2}{2}\frac{\partial A}{\partial x}\frac{\partial U}{\partial x} + \frac{B(\Delta t)^2}{2}\frac{\partial B}{\partial y}\frac{\partial U}{\partial y} + \frac{B(\Delta t)^2}{2}\frac{\partial A}{\partial y}\frac{\partial U}{\partial x} + \frac{A(\Delta t)^2}{2}\frac{\partial B}{\partial x}\frac{\partial U}{\partial y}\right], \tag{3.9}$$

$$= U_{l,m}^{n} + A\frac{\Delta t}{2\Delta x}\left(U_{l+1,m}^{n} - U_{l-1,m}^{n}\right) + B\frac{\Delta t}{2\Delta y}\left(U_{l,m+1}^{n} - U_{l,m-1}^{n}\right) + A^2\frac{(\Delta t)^2}{2(\Delta x)^2}\left(U_{l+1,m}^{n} - 2U_{l,m}^{n} + U_{l-1,m}^{n}\right)$$

$$+B^2\frac{(\Delta t)^2}{2(\Delta y)^2}\left(U_{l,m+1}^{n} - 2U_{l,m}^{n} + U_{l,m-1}^{n}\right) + \frac{(\Delta t)^2}{8(\Delta y)^2}(BA+AB)\left(U_{l+1,m+1}^{n} - U_{l+1,m-1}^{n} - U_{l-1,m+1}^{n} + U_{l-1,m-1}^{n}\right)$$

$$+\frac{(\Delta t)^2}{8(\Delta x)^2}\left[A\left(A_{l+1,m} - A_{l-1,m}\right)\left(U_{l+1,m}^{n} - U_{l-1,m}^{n}\right) + B\left(B_{l,m+1} - B_{l,m-1}\right)\left(U_{l,m+1}^{n} - U_{l,m-1}^{n}\right)\right]$$

$$+\frac{(\Delta t)^2}{8(\Delta x)^2}\left[B\left(A_{l,m+1} - A_{l,m-1}\right)\left(U_{l+1,m}^{n} - U_{l-1,m}^{n}\right) + A\left(B_{l+1,m} - B_{l-1,m}\right)\left(U_{l,m+1}^{n} - U_{l,m-1}^{n}\right)\right], \tag{3.10}$$

$$= U_{l,m}^{n} + A\frac{p}{2}\left(U_{l+1,m}^{n} - U_{l-1,m}^{n}\right) + B\frac{p}{2}\left(U_{l,m+1}^{n} - U_{l,m-1}^{n}\right) + A^2\frac{(p)^2}{2}\left(U_{l+1,m}^{n} - 2U_{l,m}^{n} + U_{l-1,m}^{n}\right)$$

$$+B^2\frac{(p)^2}{2}\left(U_{l,m+1}^{n} - 2U_{l,m}^{n} + U_{l,m-1}^{n}\right) + \frac{(p)^2}{8}(BA+AB)\left(U_{l+1,m+1}^{n} - U_{l+1,m-1}^{n} - U_{l-1,m+1}^{n} + U_{l-1,m-1}^{n}\right)$$

$$+\frac{(p)^2}{8}\left[A\left(A_{l+1,m} - A_{l-1,m}\right)\left(U_{l+1,m}^{n} - U_{l-1,m}^{n}\right) + B\left(B_{l,m+1} - B_{l,m-1}\right)\left(U_{l,m+1}^{n} - U_{l,m-1}^{n}\right)\right]$$

$$+\frac{(p)^2}{8}\left[B\left(A_{l,m+1}-A_{l,m-1}\right)\left(U^n_{l+1,m}-U^n_{l-1,m}\right)+A\left(B_{l+1,m}-B_{l-1,m}\right)\left(U^n_{l,m+1}-U^n_{l,m-1}\right)\right]. \quad (3.11)$$

Case II A,B are constant matrix,

from Eq.(3.9) the term $\frac{\partial A}{\partial x}=0, \frac{\partial B}{\partial y}=0, \frac{\partial A}{\partial y}=0$ and $\frac{\partial B}{\partial x}=0$,

we have

$$U^{n+1}_{l,m}=U^n_{l,m}+A\frac{\Delta t}{2\Delta x}\left(U^n_{l+1,m}-U^n_{l-1,m}\right)+B\frac{\Delta t}{2\Delta y}\left(U^n_{l,m+1}-U^n_{l,m-1}\right)+A^2\frac{(\Delta t)^2}{2(\Delta x)^2}\left(U^n_{l+1,m}-2U^n_{l,m}+U^n_{l-1,m}\right)$$
$$+\frac{(\Delta t)^2}{8(\Delta y)^2}(BA+AB)\left(U^n_{l+1,m+1}-U^n_{l+1,m-1}-U^n_{l-1,m+1}+U^n_{l-1,m-1}\right)$$
$$+B^2\frac{(\Delta t)^2}{2(\Delta y)^2}\left(U^n_{l,m+1}-2U^n_{l,m}+U^n_{l,m-1}\right), \quad (3.12)$$

$$U^{n+1}_{l,m}=U^n_{l,m}+A\frac{p}{2}\left(U^n_{l+1,m}-U^n_{l-1,m}\right)+B\frac{p}{2}\left(U^n_{l,m+1}-U^n_{l,m-1}\right)+A^2\frac{(p)^2}{2}\left(U^n_{l+1,m}-2U^n_{l,m}+U^n_{l-1,m}\right)$$
$$+\frac{(p)^2}{8}(BA+AB)\left(U^n_{l+1,m+1}-U^n_{l+1,m-1}-U^n_{l-1,m+1}+U^n_{l-1,m-1}\right)+B^2\frac{(p)^2}{2}\left(U^n_{l,m+1}-2U^n_{l,m}+U^n_{l,m-1}\right) \quad (3.13)$$

where $U^n_{l,m}=\begin{pmatrix} d^n_{l,m} \\ u^n_{l,m} \\ v^n_{l,m}\end{pmatrix}$, $A_{l,m}=\begin{bmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$, $B_{l,m}=\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}$

and $p=\frac{\Delta t}{\Delta x}=\frac{\Delta t}{\Delta y}$.

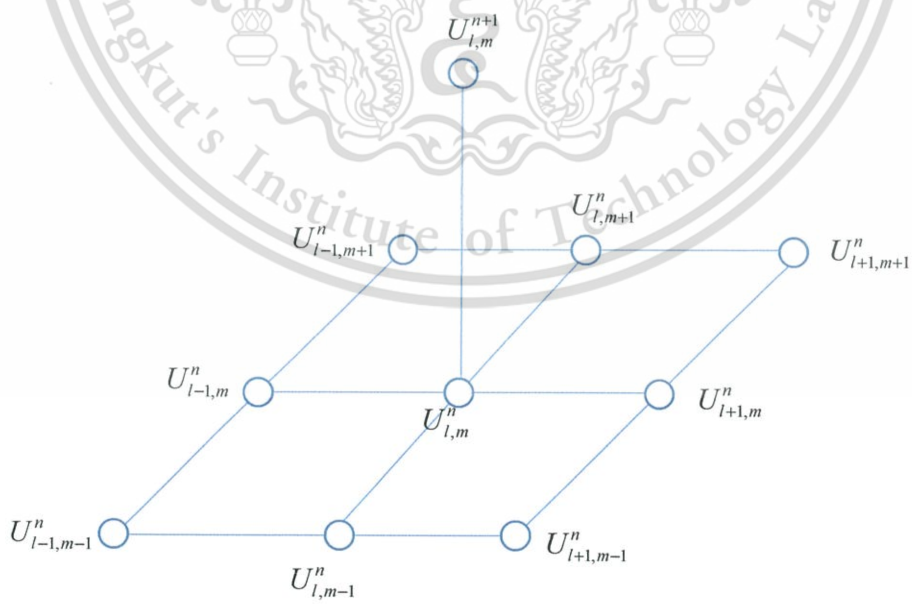


Figure 3.1 The stencil diagram of Lax-Wendroff method

3.2 A numerical method for two dimensional hydrodynamic model with anisotropic bottom topography

We will use the Lax-Wendroff method [28], [29] to compute a numerical solution of hyperbolic PDE (2.67). A regular square finite difference grid with a vector-valued solution centred in the grid cells. The domain of problem $L \times M$ dimension, l and m were subinterval, such that $l\Delta x = L$, $m\Delta y = M$ and interval time $[0,T]$, k is subintervals, such that $k\Delta t = T$, $U_{i,j}^n = U(x,y,t)$ represents a three component vector at each cell i,j with time n, where $1 \le i \le l$, $1 \le j \le m$ and $1 \le n \le k$, where $x = i\Delta x$, $y = j\Delta y$ and $t = n\Delta t$.

Step 1: Compute initial vector $U_{i,j}^n$ at centre cells.

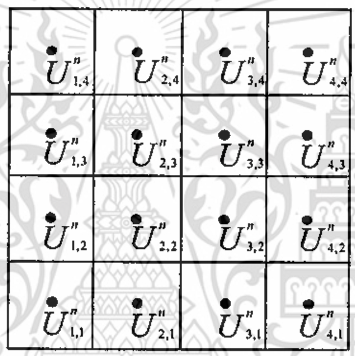
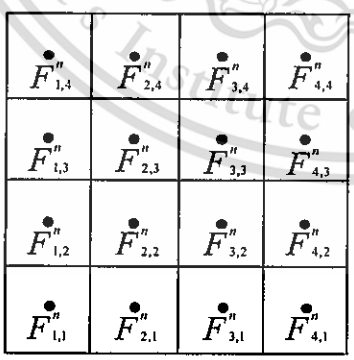
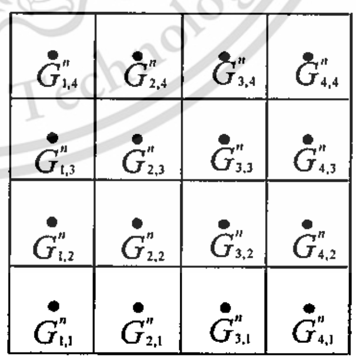


Figure 3.2 The variables represent the solution at the centres of the grids

Step 2: Take $U_{i,j}^n$ to compute vector $F_{i,j}^n$ and $G_{i,j}^n$ at centre cells.



(a)



(b)

Figure 3.3 The vector (a) F at centres of grid (b) G at centres of grid

Step 3: This stage is a half-step; it defines values of U at time step $n+\frac{1}{2}$ and the midpoints of the edges of the grid,

$$U_{i+\frac{1}{2},j}^{n+\frac{1}{2}} = \frac{1}{2}\left(U_{i+1,j}^{n} + U_{i,j}^{n}\right) - \frac{\Delta t}{2\Delta x}\left(F_{i+1,j}^{n} - F_{i,j}^{n}\right), \tag{3.14}$$

$$U_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(U_{i,j+1}^{n} + U_{i,j}^{n}\right) - \frac{\Delta t}{2\Delta y}\left(G_{i,j+1}^{n} - G_{i,j}^{n}\right). \tag{3.15}$$

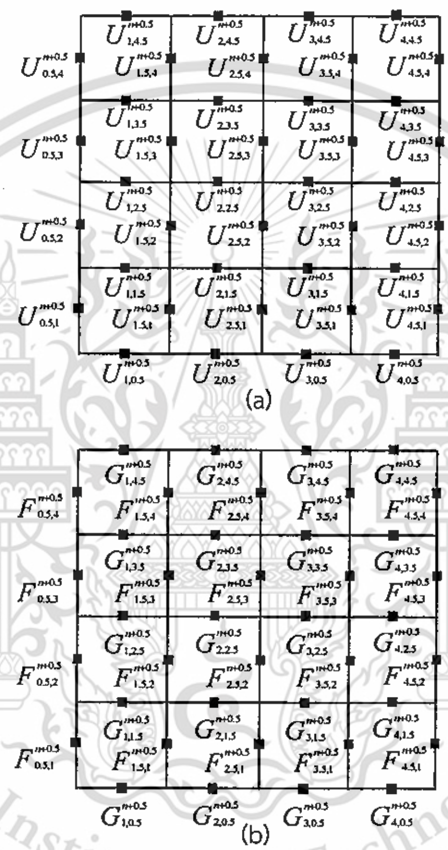


Figure 3.4 The values of vector (a) U represent the solution at the midpoints of the grids and (b) F,G at the midpoints of the grids

Step 4: Take values of U from step 3 to compute F,G at time step $n+\frac{1}{2}$ and the midpoints of the edges of the grid.

Step 5: The last step completes the time step by using the values computed in the step 1 and step 4 to compute new values at the centres of the cells,

$$U_{i,j}^{n+1} = U_{i,j}^{n} - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2},j}^{n+\frac{1}{2}} - F_{i-\frac{1}{2},j}^{n+\frac{1}{2}}\right) - \frac{\Delta t}{\Delta y}\left(G_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} - G_{i,j-\frac{1}{2}}^{n+\frac{1}{2}}\right). \quad (3.16)$$

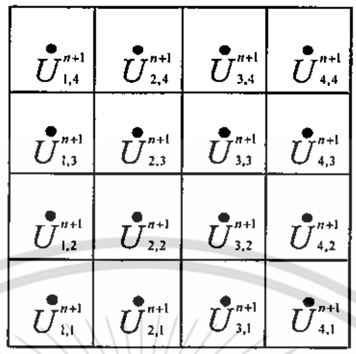


Figure 3.5 The solution U^{n+1} at centres of the grids

Use the finite difference method to compute a numerical approximation to the boundary conditions of the reservoir.

For left boundary condition, where $i=0$ and $1 \le j \le m$, therefore, $U_{0,j}^{n} = U_{1,j}^{n}$, substituting the approximate unknown vector nodes $U_{0,j}^{n}$ of left boundary into Eq.(3.14), we have

$$U_{\frac{1}{2},j}^{n+\frac{1}{2}} = \frac{1}{2}\left(U_{1,j}^{n} + U_{0,j}^{n}\right) - \frac{\Delta t}{2\Delta x}\left(F_{1,j}^{n} - F_{0,j}^{n}\right) = U_{1,j}^{n}. \quad (3.17)$$

For right boundary condition, where $i=l$ and $1 \le j \le m$, therefore, $U_{l+1,j}^{n} = U_{l,j}^{n}$, substituting the approximate unknown vector nodes $U_{l+1,j}^{n}$ of right boundary into Eq.(3.14), we have

$$U_{l+\frac{1}{2},j}^{n+\frac{1}{2}} = \frac{1}{2}\left(U_{l+1,j}^{n} + U_{l,j}^{n}\right) - \frac{\Delta t}{2\Delta x}\left(F_{l+1,j}^{n} - F_{l,j}^{n}\right) = U_{l,j}^{n}. \quad (3.18)$$

For lower boundary condition, where $1 \le i \le l$ and $j=0$, therefore, $U_{i,0}^{n} = U_{i,1}^{n}$, substituting the approximate unknown vector nodes $U_{i,0}^{n}$ of lower boundary into Eq.(3.15), we have

$$U_{i,\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(U_{i,1}^{n} + U_{i,0}^{n}\right) - \frac{\Delta t}{2\Delta y}\left(G_{i,1}^{n} - G_{i,0}^{n}\right) = U_{i,1}^{n}. \quad (3.19)$$

For upper boundary condition, where $1 \le i \le l$ and $j=m$, therefore, $U_{i,m+1}^{n} = U_{i,m}^{n}$, substituting the approximate unknown vector nodes $U_{i,m+1}^{n}$ of upper boundary into Eq.(3.15), we have

$$U_{i,m+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(U_{i,m+1}^{n} + U_{i,m}^{n}\right) - \frac{\Delta t}{2\Delta y}\left(G_{i,m+1}^{n} - G_{i,0}^{n}\right) = U_{i,m}^{n}. \tag{3.20}$$

3.3 A numerical method for two dimensional unsteady state of dispersion model

We use the forward differences in time and backward difference in space in advection diffusion equation. We can approximate $C_{l,m}^{n}$ are the values difference approximation of at point $x = l\Delta x, y = m\Delta y$ and $t = n\Delta t$ where $0 \le l \le L, 0 \le m \le M$ and $0 \le n \le N$. Taking the forward in time and backward in space in Eq.(2.68), we get the following finite difference equation [18],

$$\frac{C_{l,m}^{n+1} - C_{l,m}^{n}}{\Delta t} + u_{l,m}^{n}\left(\frac{C_{l,m}^{n} - C_{l-1,m}^{n}}{\Delta x}\right) + v_{l,m}^{n}\left(\frac{C_{l,m}^{n} - C_{l,m-1}^{n}}{\Delta y}\right) \tag{3.21}$$

$$= D\left(\frac{C_{l+1,m}^{n} - 2C_{l,m}^{n} + C_{l-1,m}^{n}}{\Delta x^2} + \frac{C_{l,m+1}^{n} - 2C_{l,m}^{n} + C_{l,m-1}^{n}}{\Delta y^2}\right),$$

$$C_{l,m}^{n+1} = D\frac{\Delta t}{\Delta x^2}C_{l+1,m}^{n} + D\frac{\Delta t}{\Delta y^2}C_{l,m+1}^{n} + \left(\frac{\Delta t}{\Delta x}u_{l,m}^{n} + D\frac{\Delta t}{\Delta x^2}\right)C_{l-1,m}^{n} + \left(\frac{\Delta t}{\Delta y}v_{l,m}^{n} + D\frac{\Delta t}{\Delta y^2}\right)C_{l,m-1}^{n}$$

$$+\left(1 - \frac{\Delta t}{\Delta x}u_{l,m}^{n} - \frac{\Delta t}{\Delta y}v_{l,m}^{n} - 2D\frac{\Delta t}{\Delta x^2} - 2D\frac{\Delta t}{\Delta y^2}\right)C_{l,m}^{n}, \tag{3.22}$$

$$C_{l,m}^{n+1} = A_1 C_{l+1,m}^{n} + A_2 C_{l,m+1}^{n} + A_3 C_{l-1,m}^{n} + A_4 C_{l,m-1}^{n} + A_5 C_{l,m}^{n}, \tag{3.23}$$

where

$$A_1 = D\frac{\Delta t}{\Delta x^2},$$

$$A_2 = D\frac{\Delta t}{\Delta y^2},$$

$$A_3 = \frac{\Delta t}{\Delta x}u_{l,m}^{n} + \frac{\Delta t}{\Delta x^2}D,$$

$$A_4 = \frac{\Delta t}{\Delta y}v_{l,m}^{n} + \frac{\Delta t}{\Delta y^2}D,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x}u_{l,m}^{n} - \frac{\Delta t}{\Delta y}v_{l,m}^{n} - 2D\frac{\Delta t}{\Delta x^2} - 2D\frac{\Delta t}{\Delta y^2},$$

3.4 A numerical method for two dimensional steady state of dispersion model

We now discretize Eq.(2.69) by dividing the $\Omega \in \mathbb{R}^2$ in x-axis and y-axis into M grids such that $M\Delta x = L_x$ where L_x is a maximum length of Ω in x-axis and N grids such that $N\Delta y = L_y$ where L_y is a maximum length of Ω in y-axis. We can then approximate $C(x_l, y_m)$ by $C_{l,m}$, values of the difference approximation of $C(x,y)$ at point $l\Delta x$ and $m\Delta y$, where $0 \le l \le M$ and $0 \le m \le N$. The grid-points (x_l, y_m) are defined by $x_l = l\Delta x$ for all $l = 0,1,2,...,M$ and $y_m = m\Delta y$ for all $m = 0,1,2,...,N$ in which M and N are positive integers. We approximate the term of Eq.(2.69) by Taylor's series expansion followed [27].

3.4.1 Backward in space finite difference scheme

Taking the backward in space technique for the first order derivatives, and the central in space technique for the second order derivatives, we get the following discretization Eq.(2.69):

$$\frac{\partial C}{\partial x} \approx \frac{C_{l,m} - C_{l-1,m}}{\Delta x}, \tag{3.24}$$

$$\frac{\partial C}{\partial y} \approx \frac{C_{l,m} - C_{l,m-1}}{\Delta y}, \tag{3.25}$$

$$\frac{\partial^2 C}{\partial x^2} \approx \frac{C_{l+1,m} - 2C_{l,m} + C_{l-1,m}}{\Delta x^2}, \tag{3.26}$$

$$\frac{\partial^2 C}{\partial y^2} \approx \frac{C_{l,m+1} - 2C_{l,m} + C_{l,m-1}}{\Delta y^2}. \tag{3.27}$$

Substituting Eqs.(3.24)-(3.27) into Eq.(2.69), we have

$$\begin{aligned} \bar{u}\left(\frac{C_{l,m} - C_{l-1,m}}{\Delta x}\right) + \bar{v}\left(\frac{C_{l,m} - C_{l,m-1}}{\Delta y}\right) \\ = D\left(\frac{C_{l+1,m} - 2C_{l,m} - C_{l-1,m}}{\Delta x^2} + \frac{C_{l,m+1} - 2C_{l,m} + C_{l,m-1}}{\Delta y^2}\right). \end{aligned} \tag{3.28}$$

We can obtain a simply form of Eq.(3.28),

$$C_{l,m} - S_1^B C_{l-1,m} - S_2^B C_{l,m-1} - S_3^B C_{l+1,m} - S_4^B C_{l,m+1} = 0, \tag{3.29}$$

where

$$S_1^B = \left[\frac{\bar{u}}{\Delta x} + \frac{D}{\Delta x^2}\right] \bigg/ \left[\frac{\bar{u}}{\Delta x} + \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{3.30}$$

$$S_2^B = \left[\frac{\bar{v}}{\Delta y} + \frac{D}{\Delta y^2}\right] \bigg/ \left[\frac{\bar{u}}{\Delta x} + \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{3.31}$$

$$S_3^B = \left[\frac{D}{\Delta x^2}\right] \bigg/ \left[\frac{\bar{u}}{\Delta x} + \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{3.32}$$

$$S_4^B = \left[\frac{D}{\Delta y^2}\right] \bigg/ \left[\frac{\bar{u}}{\Delta x} + \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right]. \tag{3.33}$$

If $C_{l,m}$ are lied on the boundary of the domain, we will approximate by using the boundary conditions and employing the backward different scheme,

$$\frac{\partial C}{\partial x} \approx \frac{C_{l,m} - C_{l-1,m}}{\Delta x}, \tag{3.34}$$

$$\frac{\partial C}{\partial y} \approx \frac{C_{l,m} - C_{l,m-1}}{\Delta y}. \tag{3.35}$$

3.4.2 Forward in space finite different scheme

Taking the forward in space technique for the first order derivatives, and the central in space technique for the second order derivatives, we get the following discretization Eq.(2.69) :

$$\frac{\partial C}{\partial x} \approx \frac{C_{l+1,m} - C_{l,m}}{\Delta x}, \tag{3.36}$$

$$\frac{\partial C}{\partial y} \approx \frac{C_{l,m+1} - C_{l,m}}{\Delta y}, \tag{3.37}$$

$$\frac{\partial^2 C}{\partial x^2} \approx \frac{C_{l+1,m} - 2C_{l,m} + C_{l-1,m}}{\Delta x^2}, \tag{3.38}$$

$$\frac{\partial^2 C}{\partial y^2} \approx \frac{C_{l,m+1} - 2C_{l,m} + C_{l,m-1}}{\Delta y^2}. \tag{3.39}$$

Substituting Eqs.(3.36)-(3.39) into Eq.(2.69), we have

$$\bar{u}\left(\frac{C_{l+1,m} - C_{l,m}}{\Delta x}\right) + \bar{v}\left(\frac{C_{l,m+1} - C_{l,m}}{\Delta y}\right) = D\left(\frac{C_{l+1,m} - 2C_{l,m} - C_{l-1,m}}{\Delta x^2} + \frac{C_{l,m+1} - 2C_{l,m} + C_{l,m-1}}{\Delta y^2}\right). \tag{3.40}$$

We can obtain a simply form of Eq.(3.40),

$$C_{l,m} - S_1^F C_{l-1,m} - S_2^F C_{l,m-1} - S_3^F C_{l+1,m} - S_4^F C_{l,m+1} = 0, \tag{3.41}$$

where

$$S_1^F = \left[\frac{D}{\Delta x^2}\right] \Big/ \left[-\frac{\bar{u}}{\Delta x} - \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \quad (3.42)$$

$$S_2^F = \left[\frac{D}{\Delta y^2}\right] \Big/ \left[-\frac{\bar{u}}{\Delta x} - \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \quad (3.43)$$

$$S_3^F = \left[-\frac{\bar{u}}{\Delta x} + \frac{D}{\Delta x^2}\right] \Big/ \left[-\frac{\bar{u}}{\Delta x} - \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \quad (3.44)$$

$$S_4^F = \left[-\frac{\bar{v}}{\Delta y} + \frac{D}{\Delta y^2}\right] \Big/ \left[-\frac{\bar{u}}{\Delta x} - \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right]. \quad (3.45)$$

If $C_{l,m}$ are lied on the boundary of the domain, we will approximate by using the boundary conditions and employing the forward different scheme,

$$\frac{\partial C}{\partial x} \approx \frac{C_{l+1,m} - C_{l,m}}{\Delta x}, \quad (3.46)$$

$$\frac{\partial C}{\partial y} \approx \frac{C_{l,m+1} - C_{l,m}}{\Delta y}. \quad (3.47)$$

3.4.3 Central in space finite different scheme

Taking the central in space technique for the first order derivatives, and the central in space technique for the second order derivatives, we get the following discretization Eq.(2.69) :

$$\frac{\partial C}{\partial x} \approx \frac{C_{l+1,m} - C_{l-1,m}}{2\Delta x}, \quad (3.48)$$

$$\frac{\partial C}{\partial y} \approx \frac{C_{l,m+1} - C_{l,m-1}}{2\Delta y}, \quad (3.49)$$

$$\frac{\partial^2 C}{\partial x^2} \approx \frac{C_{l+1,m} - 2C_{l,m} + C_{l-1,m}}{\Delta x^2}, \quad (3.50)$$

$$\frac{\partial^2 C}{\partial y^2} \approx \frac{C_{l,m+1} - 2C_{l,m} + C_{l,m-1}}{\Delta y^2}. \quad (3.51)$$

Substituting Eqs.(3.48)-(3.51) into Eq.(2.91), we have

$$\bar{u}\left(\frac{C_{l+1,m} - C_{l-1,m}}{2\Delta x}\right) + \bar{v}\left(\frac{C_{l,m+1} - C_{l,m+1}}{2\Delta y}\right) \quad (3.52)$$
$$= D\left(\frac{C_{l+1,m} - 2C_{l,m} - C_{l-1,m}}{\Delta x^2} + \frac{C_{l,m+1} - 2C_{l,m} + C_{l,m-1}}{\Delta y^2}\right).$$

We can obtain a simply form of Eq.(3.52),

$$C_{l,m} - S_1^C C_{l-1,m} - S_2^C C_{l,m-1} - S_3^C C_{l+1,m} - S_4^C C_{l,m+1} = 0, \tag{3.53}$$

where

$$S_1^C = \left[\frac{\bar{u}}{2\Delta x} + \frac{D}{\Delta x^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{3.54}$$

$$S_2^C = \left[\frac{\bar{v}}{2\Delta y} + \frac{D}{\Delta y^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{3.55}$$

$$S_3^C = \left[-\frac{\bar{u}}{2\Delta x} + \frac{D}{\Delta x^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{3.56}$$

$$S_3^C = \left[-\frac{\bar{v}}{2\Delta y} + \frac{D}{\Delta y^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right]. \tag{3.57}$$

If $C_{l,m}$ are lied on the boundary of the domain, we will approximate by using the boundary conditions and employing the central different scheme,

$$\frac{\partial C}{\partial x} \approx \frac{C_{l+1,m} - C_{l-1,m}}{2\Delta x}, \tag{3.58}$$

$$\frac{\partial C}{\partial y} \approx \frac{C_{l,m+1} - C_{l,m+1}}{2\Delta y}. \tag{3.59}$$

We can obtain the stencil diagrams of all schemes: backward, forward and central difference techniques illustrated in Figure 3.6.

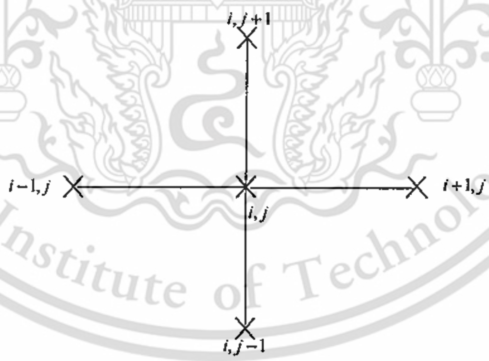


Figure 3.6 The stencil diagram of steady state of dispersion model

3.5 Examples of calculation

In this section, the first part shows examples to find the velocity and elevation of water are calculated from hydrodynamic model and pollutant concentration at any time is calculated by unsteady state dispersion model in an open reservoir and a close reservoir, the second part show example to find the steady state pollutant concentration from steady state dispersion model

Example 3.5.1. To find the velocity and water elevation in a close reservoir, when initial time of the velocity in reservoir is zero and water elevation is not the normal position or $d = x(1-x)y(1-y)$, rectangular domain $\Omega = (0,1)\times(0,1)$ in Figure 3.7 with step size $\Delta x = \Delta y = 0.25$ and $\Delta t = 0.01$, $T = 0.02$.

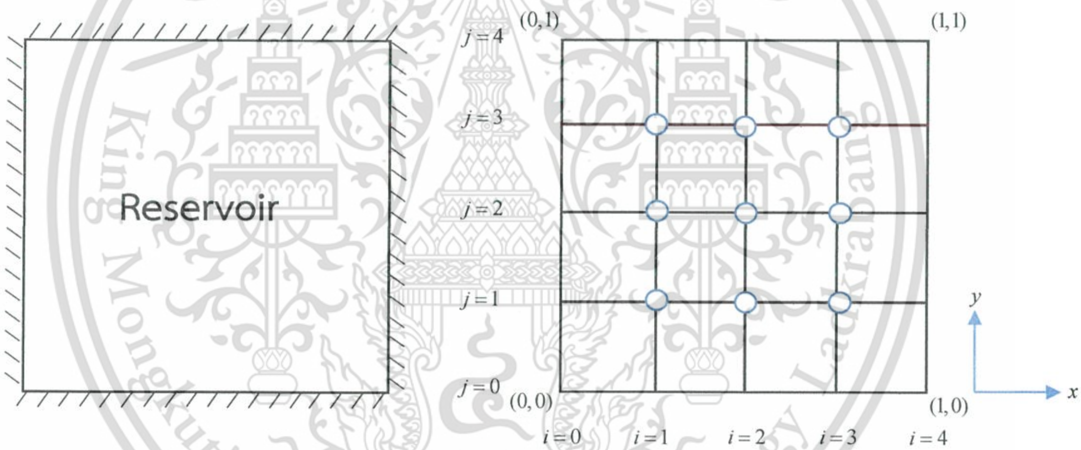


Figure 3.7 The domain and generating grid points of a close reservoir

Hydrodynamic model

Initial conditions, water elevation at initial time $d(x,y,0) = x(1-x)y(1-y)$,

velocity at initial time $u(x,y,0) = 0$, $v(x,y,0) = 0$, boundary conditions, water elevation $d(0,y,t) = d(1,y,t) = d(x,0,t) = d(x,1,t) = 0$, velocity

$v(0,y,t) = v(1,y,t) = v_y(x,0,t) = v_y(x,1,t) = 0$, $u(x,0,t) = u(x,1,t) = u_x(0,y,t) = u_x(1,y,t) = 0$, define $u(x,y,t)=u_{l,m}^{n}$, $v(x,y,t)=v_{l,m}^{n}$, and $d(x,y,t)=d_{l,m}^{n}$, with $\Delta t=0.01, \Delta x=0.25$, we have $p=\frac{\Delta t}{\Delta x}=\frac{0.01}{0.25}=0.04$.

The non dimensional form of shallow water equation in two dimensions

$$\frac{\partial d}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad (3.60)$$

$$\frac{\partial u}{\partial t}+\frac{\partial d}{\partial x}=0, \quad (3.61)$$

$$\frac{\partial v}{\partial t}+\frac{\partial d}{\partial y}=0. \quad (3.62)$$

Eqs.(3.60)-(3.62) can be written matrix form as

$$\frac{\partial U}{\partial t}=A\frac{\partial U}{\partial x}+B\frac{\partial U}{\partial y}, \quad (3.63)$$

where

$$U=U_{l,m}^{n}=\begin{bmatrix} d_{l,m}^{n} \\ u_{l,m}^{n} \\ v_{l,m}^{n} \end{bmatrix},\ A=\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B=\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad (3.64)$$

approximate differential boundary condition

$v_y(x,0,t)=0$ or $\frac{\partial v_{l,0}^{n}}{\partial y}=0$,

using forward difference

$$\frac{v_{l,1}^{n}-v_{l,0}^{n}}{\Delta y}=0,$$

$$v_{l,1}^{n}=v_{l,0}^{n}, \qquad l=0,1,2,3,4 \quad (3.65)$$

$v_y(x,1,t)=0$ or $\frac{\partial v_{l,4}^{n}}{\partial y}=0$,

using backward difference

$$\frac{v_{l,4}^{n}-v_{l,3}^{n}}{\Delta y}=0,$$

$$v_{l,4}^{n}=v_{l,3}^{n}, \qquad l=0,1,2,3,4 \quad (3.66)$$

$u_x(0,y,t)=0$ or $\frac{\partial u_{0,m}^{n}}{\partial x}=0$,

using forward difference

$$\frac{u_{1,m}^{n}-u_{0,m}^{n}}{\Delta x}=0,$$

$$u_{1,m}^{n}=u_{0,m}^{n}, \qquad m=0,1,2,3,4 \quad (3.67)$$

$u_x(1,y,t)=0$ or $\dfrac{\partial u_{4,m}^n}{\partial x}=0$,

using backward difference

$$\frac{u_{4,m}^n-u_{3,m}^n}{\Delta x}=0,$$

$$u_{4,m}^n=u_{3,m}^n, \qquad m=0,1,2,3,4 \qquad (3.68)$$

solving Eq.(3.63) of problem by Lax-Wendroff (3.13).

$l=1$, $m=1$ and $n=0$

$$U_{1,1}^1=\begin{bmatrix}d_{1,1}^0\\u_{1,1}^0\\v_{1,1}^0\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,1}^0\\u_{2,1}^0\\v_{2,1}^0\end{bmatrix}-\begin{bmatrix}d_{0,1}^0\\u_{0,1}^0\\v_{0,1}^0\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{1,2}^0\\u_{1,2}^0\\v_{1,2}^0\end{bmatrix}-\begin{bmatrix}d_{1,0}^0\\u_{1,0}^0\\v_{1,0}^0\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,1}^0\\u_{2,1}^0\\v_{2,1}^0\end{bmatrix}-2\begin{bmatrix}d_{1,1}^0\\u_{1,1}^0\\v_{1,1}^0\end{bmatrix}+\begin{bmatrix}d_{0,1}^0\\u_{0,1}^0\\v_{0,1}^0\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{1,2}^0\\u_{1,2}^0\\v_{1,2}^0\end{bmatrix}-2\begin{bmatrix}d_{1,1}^0\\u_{1,1}^0\\v_{1,1}^0\end{bmatrix}+\begin{bmatrix}d_{1,0}^0\\u_{1,0}^0\\v_{1,0}^0\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{2,2}^0\\u_{2,2}^0\\v_{2,2}^0\end{bmatrix}-\begin{bmatrix}d_{0,2}^0\\u_{0,2}^0\\v_{0,2}^0\end{bmatrix}-\begin{bmatrix}d_{2,0}^0\\u_{2,0}^0\\v_{2,0}^0\end{bmatrix}+\begin{bmatrix}d_{0,0}^0\\u_{0,0}^0\\v_{0,0}^0\end{bmatrix}\right)=\begin{bmatrix}0.035118\\-0.000937\\-0.000937\end{bmatrix}$$

$l=2$, $m=1$ and $n=0$

$$U_{2,1}^1=\begin{bmatrix}d_{2,1}^0\\u_{2,1}^0\\v_{2,1}^0\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,1}^0\\u_{3,1}^0\\v_{3,1}^0\end{bmatrix}-\begin{bmatrix}d_{1,1}^0\\u_{1,1}^0\\v_{1,1}^0\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,2}^0\\u_{2,2}^0\\v_{2,2}^0\end{bmatrix}-\begin{bmatrix}d_{2,0}^0\\u_{2,0}^0\\v_{2,0}^0\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,1}^0\\u_{3,1}^0\\v_{3,1}^0\end{bmatrix}-2\begin{bmatrix}d_{2,1}^0\\u_{2,1}^0\\v_{2,1}^0\end{bmatrix}+\begin{bmatrix}d_{1,1}^0\\u_{1,1}^0\\v_{1,1}^0\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{2,2}^0\\u_{2,2}^0\\v_{2,2}^0\end{bmatrix}-2\begin{bmatrix}d_{2,1}^0\\u_{2,1}^0\\v_{2,1}^0\end{bmatrix}+\begin{bmatrix}d_{2,0}^0\\u_{2,0}^0\\v_{2,0}^0\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^0\\u_{3,2}^0\\v_{3,2}^0\end{bmatrix}-\begin{bmatrix}d_{1,2}^0\\u_{1,2}^0\\v_{1,2}^0\end{bmatrix}-\begin{bmatrix}d_{3,0}^0\\u_{3,0}^0\\v_{3,0}^0\end{bmatrix}+\begin{bmatrix}d_{1,0}^0\\u_{1,0}^0\\v_{1,0}^0\end{bmatrix}\right)=\begin{bmatrix}0.046831\\0\\-0.001250\end{bmatrix}$$

$l=3$, $m=1$ and $n=0$

$$U_{3,1}^1=\begin{bmatrix}d_{3,1}^0\\u_{3,1}^0\\v_{3,1}^0\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,1}^0\\u_{4,1}^0\\v_{4,1}^0\end{bmatrix}-\begin{bmatrix}d_{2,1}^0\\u_{2,1}^0\\v_{2,1}^0\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^0\\u_{3,2}^0\\v_{3,2}^0\end{bmatrix}-\begin{bmatrix}d_{3,0}^0\\u_{3,0}^0\\v_{3,0}^0\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,1}^{0}\\u_{4,1}^{0}\\v_{4,1}^{0}\end{bmatrix}-2\begin{bmatrix}d_{3,1}^{0}\\u_{3,1}^{0}\\v_{3,1}^{0}\end{bmatrix}+\begin{bmatrix}d_{2,1}^{0}\\u_{2,1}^{0}\\v_{2,1}^{0}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^{0}\\u_{3,2}^{0}\\v_{3,2}^{0}\end{bmatrix}-2\begin{bmatrix}d_{3,1}^{0}\\u_{3,1}^{0}\\v_{3,1}^{0}\end{bmatrix}+\begin{bmatrix}d_{3,0}^{0}\\u_{3,0}^{0}\\v_{3,0}^{0}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{4,2}^{0}\\u_{4,2}^{0}\\v_{4,2}^{0}\end{bmatrix}-\begin{bmatrix}d_{2,2}^{0}\\u_{2,2}^{0}\\v_{2,2}^{0}\end{bmatrix}-\begin{bmatrix}d_{4,0}^{0}\\u_{4,0}^{0}\\v_{4,0}^{0}\end{bmatrix}+\begin{bmatrix}d_{2,0}^{0}\\u_{2,0}^{0}\\v_{2,0}^{0}\end{bmatrix}\right)=\begin{bmatrix}0.035118\\0.000937\\-0.000937\end{bmatrix}$$

$l=1, m=2$ and $n=0$

$$U_{1,2}^{1}=\begin{bmatrix}d_{1,2}^{0}\\u_{1,2}^{0}\\v_{1,2}^{0}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,2}^{0}\\u_{2,2}^{0}\\v_{2,2}^{0}\end{bmatrix}-\begin{bmatrix}d_{0,2}^{0}\\u_{0,2}^{0}\\v_{0,2}^{0}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{1,3}^{0}\\u_{1,3}^{0}\\v_{1,3}^{0}\end{bmatrix}-\begin{bmatrix}d_{1,1}^{0}\\u_{1,1}^{0}\\v_{1,1}^{0}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,2}^{0}\\u_{2,2}^{0}\\v_{2,2}^{0}\end{bmatrix}-2\begin{bmatrix}d_{1,2}^{0}\\u_{1,2}^{0}\\v_{1,2}^{0}\end{bmatrix}+\begin{bmatrix}d_{0,2}^{0}\\u_{0,2}^{0}\\v_{0,2}^{0}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{1,3}^{0}\\u_{1,3}^{0}\\v_{1,3}^{0}\end{bmatrix}-2\begin{bmatrix}d_{1,2}^{0}\\u_{1,2}^{0}\\v_{1,2}^{0}\end{bmatrix}+\begin{bmatrix}d_{1,1}^{0}\\u_{1,1}^{0}\\v_{1,1}^{0}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^{0}\\u_{2,3}^{0}\\v_{2,3}^{0}\end{bmatrix}-\begin{bmatrix}d_{0,3}^{0}\\u_{0,3}^{0}\\v_{0,3}^{0}\end{bmatrix}-\begin{bmatrix}d_{2,1}^{0}\\u_{2,1}^{0}\\v_{2,1}^{0}\end{bmatrix}+\begin{bmatrix}d_{0,1}^{0}\\u_{0,1}^{0}\\v_{0,1}^{0}\end{bmatrix}\right)=\begin{bmatrix}0.046831\\-0.00125\\0\end{bmatrix}$$

$l=2, m=2$ and $n=0$

$$U_{1,2}^{1}=\begin{bmatrix}d_{2,2}^{0}\\u_{2,2}^{0}\\v_{2,2}^{0}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^{0}\\u_{3,2}^{0}\\v_{3,2}^{0}\end{bmatrix}-\begin{bmatrix}d_{1,2}^{0}\\u_{1,2}^{0}\\v_{1,2}^{0}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^{0}\\u_{2,3}^{0}\\v_{2,3}^{0}\end{bmatrix}-\begin{bmatrix}d_{2,1}^{0}\\u_{2,1}^{0}\\v_{2,1}^{0}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^{0}\\u_{3,2}^{0}\\v_{3,2}^{0}\end{bmatrix}-2\begin{bmatrix}d_{2,2}^{0}\\u_{2,2}^{0}\\v_{2,2}^{0}\end{bmatrix}+\begin{bmatrix}d_{1,2}^{0}\\u_{1,2}^{0}\\v_{1,2}^{0}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^{0}\\u_{2,3}^{0}\\v_{2,3}^{0}\end{bmatrix}-2\begin{bmatrix}d_{2,2}^{0}\\u_{2,2}^{0}\\v_{2,2}^{0}\end{bmatrix}+\begin{bmatrix}d_{2,1}^{0}\\u_{2,1}^{0}\\v_{2,1}^{0}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{3,3}^{0}\\u_{3,3}^{0}\\v_{3,3}^{0}\end{bmatrix}-\begin{bmatrix}d_{1,3}^{0}\\u_{1,3}^{0}\\v_{1,3}^{0}\end{bmatrix}-\begin{bmatrix}d_{3,1}^{0}\\u_{3,1}^{0}\\v_{3,1}^{0}\end{bmatrix}+\begin{bmatrix}d_{1,1}^{0}\\u_{1,1}^{0}\\v_{1,1}^{0}\end{bmatrix}\right)=\begin{bmatrix}0.06245\\0\\0\end{bmatrix}$$

$l=3, m=2$ and $n=0$

$$U_{1,2}^{1}=\begin{bmatrix}d_{3,2}^{0}\\u_{3,2}^{0}\\v_{3,2}^{0}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,2}^{0}\\u_{4,2}^{0}\\v_{4,2}^{0}\end{bmatrix}-\begin{bmatrix}d_{2,2}^{0}\\u_{2,2}^{0}\\v_{2,2}^{0}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,3}^{0}\\u_{3,3}^{0}\\v_{3,3}^{0}\end{bmatrix}-\begin{bmatrix}d_{3,1}^{0}\\u_{3,1}^{0}\\v_{3,1}^{0}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,2}^0\\u_{4,2}^0\\v_{4,2}^0\end{bmatrix}-2\begin{bmatrix}d_{3,2}^0\\u_{3,2}^0\\v_{3,2}^0\end{bmatrix}+\begin{bmatrix}d_{2,2}^0\\u_{2,2}^0\\v_{2,2}^0\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{3,3}^0\\u_{3,3}^0\\v_{3,3}^0\end{bmatrix}-2\begin{bmatrix}d_{3,2}^0\\u_{3,2}^0\\v_{3,2}^0\end{bmatrix}+\begin{bmatrix}d_{3,1}^0\\u_{3,1}^0\\v_{3,1}^0\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{4,3}^0\\u_{4,3}^0\\v_{4,3}^0\end{bmatrix}-\begin{bmatrix}d_{2,3}^0\\u_{2,3}^0\\v_{2,3}^0\end{bmatrix}-\begin{bmatrix}d_{4,1}^0\\u_{4,1}^0\\v_{4,1}^0\end{bmatrix}+\begin{bmatrix}d_{2,1}^0\\u_{2,1}^0\\v_{2,1}^0\end{bmatrix}\right)=\begin{bmatrix}0.046853\\0.001250\\0\end{bmatrix}$$

$l=1, m=3$ and $n=0$

$$U_{1,2}^1=\begin{bmatrix}d_{1,3}^0\\u_{1,3}^0\\v_{1,3}^0\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^0\\u_{2,3}^0\\v_{2,3}^0\end{bmatrix}-\begin{bmatrix}d_{0,3}^0\\u_{0,3}^0\\v_{0,3}^0\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{1,4}^0\\u_{1,4}^0\\v_{1,4}^0\end{bmatrix}-\begin{bmatrix}d_{1,2}^0\\u_{1,2}^0\\v_{1,2}^0\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^0\\u_{2,3}^0\\v_{2,3}^0\end{bmatrix}-2\begin{bmatrix}d_{1,3}^0\\u_{1,3}^0\\v_{1,3}^0\end{bmatrix}+\begin{bmatrix}d_{0,3}^0\\u_{0,3}^0\\v_{0,3}^0\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{1,4}^0\\u_{1,4}^0\\v_{1,4}^0\end{bmatrix}-2\begin{bmatrix}d_{1,3}^0\\u_{1,3}^0\\v_{1,3}^0\end{bmatrix}+\begin{bmatrix}d_{1,2}^0\\u_{1,2}^0\\v_{1,2}^0\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{2,4}^0\\u_{2,4}^0\\v_{2,4}^0\end{bmatrix}-\begin{bmatrix}d_{0,4}^0\\u_{0,4}^0\\v_{0,4}^0\end{bmatrix}-\begin{bmatrix}d_{2,2}^0\\u_{2,2}^0\\v_{2,2}^0\end{bmatrix}+\begin{bmatrix}d_{0,2}^0\\u_{0,2}^0\\v_{0,2}^0\end{bmatrix}\right)=\begin{bmatrix}0.035118\\-0.000937\\0.000937\end{bmatrix}$$

$l=2, m=3$ and $n=0$

$$U_{1,2}^1=\begin{bmatrix}d_{2,3}^0\\u_{2,3}^0\\v_{2,3}^0\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,3}^0\\u_{3,3}^0\\v_{3,3}^0\end{bmatrix}-\begin{bmatrix}d_{1,3}^0\\u_{1,3}^0\\v_{1,3}^0\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,4}^0\\u_{2,4}^0\\v_{2,4}^0\end{bmatrix}-\begin{bmatrix}d_{2,2}^0\\u_{2,2}^0\\v_{2,2}^0\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,3}^0\\u_{3,3}^0\\v_{3,3}^0\end{bmatrix}-2\begin{bmatrix}d_{2,3}^0\\u_{2,3}^0\\v_{2,3}^0\end{bmatrix}+\begin{bmatrix}d_{1,3}^0\\u_{1,3}^0\\v_{1,3}^0\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{2,4}^0\\u_{2,4}^0\\v_{2,4}^0\end{bmatrix}-2\begin{bmatrix}d_{2,3}^0\\u_{2,3}^0\\v_{2,3}^0\end{bmatrix}+\begin{bmatrix}d_{2,2}^0\\u_{2,2}^0\\v_{2,2}^0\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{3,4}^0\\u_{3,4}^0\\v_{3,4}^0\end{bmatrix}-\begin{bmatrix}d_{1,4}^0\\u_{1,4}^0\\v_{1,4}^0\end{bmatrix}-\begin{bmatrix}d_{3,2}^0\\u_{3,2}^0\\v_{3,2}^0\end{bmatrix}+\begin{bmatrix}d_{1,2}^0\\u_{1,2}^0\\v_{1,2}^0\end{bmatrix}\right)=\begin{bmatrix}0.046831\\0\\0.001250\end{bmatrix}$$

$l=3, m=3$ and $n=0$

$$U_{1,2}^1=\begin{bmatrix}d_{3,3}^0\\u_{3,3}^0\\v_{3,3}^0\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,3}^0\\u_{4,3}^0\\v_{4,3}^0\end{bmatrix}-\begin{bmatrix}d_{2,3}^0\\u_{2,3}^0\\v_{2,3}^0\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,4}^0\\u_{3,4}^0\\v_{3,4}^0\end{bmatrix}-\begin{bmatrix}d_{3,2}^0\\u_{3,2}^0\\v_{3,2}^0\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,3}^0\\u_{4,3}^0\\v_{4,3}^0\end{bmatrix}-2\begin{bmatrix}d_{3,3}^0\\u_{3,3}^0\\v_{3,3}^0\end{bmatrix}+\begin{bmatrix}d_{2,3}^0\\u_{2,3}^0\\v_{2,3}^0\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{3,4}^0\\u_{3,4}^0\\v_{3,4}^0\end{bmatrix}-2\begin{bmatrix}d_{3,3}^0\\u_{3,3}^0\\v_{3,3}^0\end{bmatrix}+\begin{bmatrix}d_{3,2}^0\\u_{3,2}^0\\v_{3,2}^0\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{4,4}^0\\u_{4,4}^0\\v_{4,4}^0\end{bmatrix}-\begin{bmatrix}d_{2,4}^0\\u_{2,4}^0\\v_{2,4}^0\end{bmatrix}-\begin{bmatrix}d_{4,2}^0\\u_{4,2}^0\\v_{4,2}^0\end{bmatrix}+\begin{bmatrix}d_{2,2}^0\\u_{2,2}^0\\v_{2,2}^0\end{bmatrix}\right)=\begin{bmatrix}0.035118\\0.000937\\0.000937\end{bmatrix}$$

Table 3.1 The calculated water elevation and velocities at time $t=0.01$in a close reservoir of example 3.5.1

Points	$d_{l,m}^n$	$u_{l,m}^n$	$v_{l,m}^n$
$U_{1,1}^1$	0.035118	-0.000937	-0.000937
$U_{2,1}^1$	0.046831	0.000000	-0.001250
$U_{3,1}^1$	0.035118	0.000937	-0.000937
$U_{1,2}^1$	0.046831	-0.001250	0.000000
$U_{2,2}^1$	0.062450	0.000000	0.000000
$U_{3,2}^1$	0.046853	0.001250	0.000000
$U_{1,3}^1$	0.035118	-0.000937	0.000937
$U_{2,3}^1$	0.046831	0.000000	0.001250
$U_{3,3}^1$	0.035118	0.000937	0.000937

$l=1, m=1$ and $n=1$

and apply boundary condition $u_{0,1}^1=u_{1,1}^1$, $v_{1,0}^1=v_{1,1}^1$, $u_{0,2}^1=u_{1,2}^1$, $v_{2,0}^1=v_{2,1}^1$, $u_{0,0}^1=u_{1,0}^1$

$$U_{1,1}^2=\begin{bmatrix}d_{1,1}^1\\u_{1,1}^1\\v_{1,1}^1\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,1}^1\\u_{2,1}^1\\v_{2,1}^1\end{bmatrix}-\begin{bmatrix}d_{0,1}^1\\u_{0,1}^1\\v_{0,1}^1\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{1,2}^1\\u_{1,2}^1\\v_{1,2}^1\end{bmatrix}-\begin{bmatrix}d_{1,0}^1\\u_{1,0}^1\\v_{1,0}^1\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,1}^1\\u_{2,1}^1\\v_{2,1}^1\end{bmatrix}-2\begin{bmatrix}d_{1,1}^1\\u_{1,1}^1\\v_{1,1}^1\end{bmatrix}+\begin{bmatrix}d_{0,1}^1\\u_{0,1}^1\\v_{0,1}^1\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{1,2}^1\\u_{1,2}^1\\v_{1,2}^1\end{bmatrix}-2\begin{bmatrix}d_{1,1}^1\\u_{1,1}^1\\v_{1,1}^1\end{bmatrix}+\begin{bmatrix}d_{1,0}^1\\u_{1,0}^1\\v_{1,0}^1\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{2,2}^1\\u_{2,2}^1\\v_{2,2}^1\end{bmatrix}-\begin{bmatrix}d_{0,2}^1\\u_{0,2}^1\\v_{0,2}^1\end{bmatrix}-\begin{bmatrix}d_{2,0}^1\\u_{2,0}^1\\v_{2,0}^1\end{bmatrix}+\begin{bmatrix}d_{0,0}^1\\u_{0,0}^1\\v_{0,0}^1\end{bmatrix}\right)=\begin{bmatrix}-0.000074\\-0.000935\\-0.000935\end{bmatrix}$$

$l=2, m=1$ and $n=1$

and apply boundary condition $v^1_{2,0}=v^1_{2,1},\ v^1_{3,0}=v^1_{3,1},\ v^1_{1,0}=v^1_{1,1}$

$$U^2_{2,1}=\begin{bmatrix} d^1_{2,1} \\ u^1_{2,1} \\ v^1_{2,1}\end{bmatrix}+0.02\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{3,1} \\ u^1_{3,1} \\ v^1_{3,1}\end{bmatrix}-\begin{bmatrix} d^1_{1,1} \\ u^1_{1,1} \\ v^1_{1,1}\end{bmatrix}\right)+0.02\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{2,2} \\ u^1_{2,2} \\ v^1_{2,2}\end{bmatrix}-\begin{bmatrix} d^1_{2,0} \\ u^1_{2,0} \\ v^1_{2,0}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{3,1} \\ u^1_{3,1} \\ v^1_{3,1}\end{bmatrix}-2\begin{bmatrix} d^1_{2,1} \\ u^1_{2,1} \\ v^1_{2,1}\end{bmatrix}+\begin{bmatrix} d^1_{1,1} \\ u^1_{1,1} \\ v^1_{1,1}\end{bmatrix}\right)+0.0008\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix} d^1_{2,2} \\ u^1_{2,2} \\ v^1_{2,2}\end{bmatrix}-2\begin{bmatrix} d^1_{2,1} \\ u^1_{2,1} \\ v^1_{2,1}\end{bmatrix}+\begin{bmatrix} d^1_{2,0} \\ u^1_{2,0} \\ v^1_{2,0}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{3,2} \\ u^1_{3,2} \\ v^1_{3,2}\end{bmatrix}-\begin{bmatrix} d^1_{1,2} \\ u^1_{1,2} \\ v^1_{1,2}\end{bmatrix}-\begin{bmatrix} d^1_{3,0} \\ u^1_{3,0} \\ v^1_{3,0}\end{bmatrix}+\begin{bmatrix} d^1_{1,0} \\ u^1_{1,0} \\ v^1_{1,0}\end{bmatrix}\right)=\begin{bmatrix} -0.000106 \\ 0 \\ -0.001247\end{bmatrix}$$

$l=3, m=1$ and $n=1$

and apply boundary condition $u^1_{4,1}=u^1_{3,1},\ v^1_{3,0}=v^1_{3,1},\ u^1_{4,2}=u^1_{3,2},\ u^1_{4,0}=u^1_{3,0},\ v^1_{2,1}=v^1_{2,0}$

$$U^2_{3,1}=\begin{bmatrix} d^1_{3,1} \\ u^1_{3,1} \\ v^1_{3,1}\end{bmatrix}+0.02\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{4,1} \\ u^1_{4,1} \\ v^1_{4,1}\end{bmatrix}-\begin{bmatrix} d^1_{2,1} \\ u^1_{2,1} \\ v^1_{2,1}\end{bmatrix}\right)+0.02\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{3,2} \\ u^1_{3,2} \\ v^1_{3,2}\end{bmatrix}-\begin{bmatrix} d^1_{3,0} \\ u^1_{3,0} \\ v^1_{3,0}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{4,1} \\ u^1_{4,1} \\ v^1_{4,1}\end{bmatrix}-2\begin{bmatrix} d^1_{3,1} \\ u^1_{3,1} \\ v^1_{3,1}\end{bmatrix}+\begin{bmatrix} d^1_{2,1} \\ u^1_{2,1} \\ v^1_{2,1}\end{bmatrix}\right)+0.0008\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix} d^1_{3,2} \\ u^1_{3,2} \\ v^1_{3,2}\end{bmatrix}-2\begin{bmatrix} d^1_{3,1} \\ u^1_{3,1} \\ v^1_{3,1}\end{bmatrix}+\begin{bmatrix} d^1_{3,0} \\ u^1_{3,0} \\ v^1_{3,0}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{4,2} \\ u^1_{4,2} \\ v^1_{4,2}\end{bmatrix}-\begin{bmatrix} d^1_{2,2} \\ u^1_{2,2} \\ v^1_{2,2}\end{bmatrix}-\begin{bmatrix} d^1_{4,0} \\ u^1_{4,0} \\ v^1_{4,0}\end{bmatrix}+\begin{bmatrix} d^1_{2,0} \\ u^1_{2,0} \\ v^1_{2,0}\end{bmatrix}\right)=\begin{bmatrix} -0.000074 \\ 0.000935 \\ -0.000936\end{bmatrix}$$

$l=1, m=2$ and $n=1$

and apply boundary condition $u^1_{0,2}=u^1_{1,2},\ u^1_{0,3}=u^1_{1,3},\ u^1_{0,1}=u^1_{1,1}$

$$U^2_{1,2}=\begin{bmatrix} d^1_{1,2} \\ u^1_{1,2} \\ v^1_{1,2}\end{bmatrix}+0.02\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{2,2} \\ u^1_{2,2} \\ v^1_{2,2}\end{bmatrix}-\begin{bmatrix} d^1_{0,2} \\ u^1_{0,2} \\ v^1_{0,2}\end{bmatrix}\right)+0.02\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{1,3} \\ u^1_{1,3} \\ v^1_{1,3}\end{bmatrix}-\begin{bmatrix} d^1_{1,1} \\ u^1_{1,1} \\ v^1_{1,1}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{2,2} \\ u^1_{2,2} \\ v^1_{2,2}\end{bmatrix}-2\begin{bmatrix} d^1_{1,2} \\ u^1_{1,2} \\ v^1_{1,2}\end{bmatrix}+\begin{bmatrix} d^1_{0,2} \\ u^1_{0,2} \\ v^1_{0,2}\end{bmatrix}\right)+0.0008\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix} d^1_{1,3} \\ u^1_{1,3} \\ v^1_{1,3}\end{bmatrix}-2\begin{bmatrix} d^1_{1,2} \\ u^1_{1,2} \\ v^1_{1,2}\end{bmatrix}+\begin{bmatrix} d^1_{1,1} \\ u^1_{1,1} \\ v^1_{1,1}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{bmatrix}\left(\begin{bmatrix} d^1_{2,3} \\ u^1_{2,3} \\ v^1_{2,3}\end{bmatrix}-\begin{bmatrix} d^1_{0,3} \\ u^1_{0,3} \\ v^1_{0,3}\end{bmatrix}-\begin{bmatrix} d^1_{2,1} \\ u^1_{2,1} \\ v^1_{2,1}\end{bmatrix}+\begin{bmatrix} d^1_{0,1} \\ u^1_{0,1} \\ v^1_{0,1}\end{bmatrix}\right)=\begin{bmatrix} -0.000106 \\ -0.001247 \\ 0\end{bmatrix}$$

$l=2$, $m=2$ and $n=1$

$$U_{1,2}^{2}=\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^{1}\\u_{3,2}^{1}\\v_{3,2}^{1}\end{bmatrix}-\begin{bmatrix}d_{1,2}^{1}\\u_{1,2}^{1}\\v_{1,2}^{1}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^{1}\\u_{2,3}^{1}\\v_{2,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{2,1}^{1}\\u_{2,1}^{1}\\v_{2,1}^{1}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^{1}\\u_{3,2}^{1}\\v_{3,2}^{1}\end{bmatrix}-2\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}+\begin{bmatrix}d_{1,2}^{1}\\u_{1,2}^{1}\\v_{1,2}^{1}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^{1}\\u_{2,3}^{1}\\v_{2,3}^{1}\end{bmatrix}-2\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}+\begin{bmatrix}d_{2,1}^{1}\\u_{2,1}^{1}\\v_{2,1}^{1}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{3,3}^{1}\\u_{3,3}^{1}\\v_{3,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{1,3}^{1}\\u_{1,3}^{1}\\v_{1,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{3,1}^{1}\\u_{3,1}^{1}\\v_{3,1}^{1}\end{bmatrix}+\begin{bmatrix}d_{1,1}^{1}\\u_{1,1}^{1}\\v_{1,1}^{1}\end{bmatrix}\right)=\begin{bmatrix}0.000149\\-4.375\times10^{-7}\\0\end{bmatrix}$$

$l=3$, $m=2$ and $n=1$

and apply boundary condition $u_{4,2}^{1}=u_{3,2}^{1},\ u_{4,3}^{1}=u_{3,3}^{1},\ u_{4,1}^{1}=u_{3,1}^{1}$

$$U_{1,2}^{2}=\begin{bmatrix}d_{3,2}^{1}\\u_{3,2}^{1}\\v_{3,2}^{1}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,2}^{1}\\u_{4,2}^{1}\\v_{4,2}^{1}\end{bmatrix}-\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,3}^{1}\\u_{3,3}^{1}\\v_{3,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{3,1}^{1}\\u_{3,1}^{1}\\v_{3,1}^{1}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,2}^{1}\\u_{4,2}^{1}\\v_{4,2}^{1}\end{bmatrix}-2\begin{bmatrix}d_{3,2}^{1}\\u_{3,2}^{1}\\v_{3,2}^{1}\end{bmatrix}+\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{3,3}^{1}\\u_{3,3}^{1}\\v_{3,3}^{1}\end{bmatrix}-2\begin{bmatrix}d_{3,2}^{1}\\u_{3,2}^{1}\\v_{3,2}^{1}\end{bmatrix}+\begin{bmatrix}d_{3,1}^{1}\\u_{3,1}^{1}\\v_{3,1}^{1}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{4,3}^{1}\\u_{4,3}^{1}\\v_{4,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{2,3}^{1}\\u_{2,3}^{1}\\v_{2,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{4,1}^{1}\\u_{4,1}^{1}\\v_{4,1}^{1}\end{bmatrix}+\begin{bmatrix}d_{2,1}^{1}\\u_{2,1}^{1}\\v_{2,1}^{1}\end{bmatrix}\right)=\begin{bmatrix}-0.000043\\0\\-0.001250\end{bmatrix}$$

$l=1$, $m=3$ and $n=1$

and apply boundary condition $u_{0,3}^{1}=u_{1,3}^{1},\ v_{1,4}^{1}=v_{1,3}^{1},\ v_{2,3}^{1}=v_{2,4}^{1},\ u_{0,4}^{1}=u_{1,4}^{1},\ u_{0,2}^{1}=u_{1,2}^{1}$

$$U_{1,2}^{2}=\begin{bmatrix}d_{1,3}^{1}\\u_{1,3}^{1}\\v_{1,3}^{1}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^{1}\\u_{2,3}^{1}\\v_{2,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{0,3}^{1}\\u_{0,3}^{1}\\v_{0,3}^{1}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{1,4}^{1}\\u_{1,4}^{1}\\v_{1,4}^{1}\end{bmatrix}-\begin{bmatrix}d_{1,2}^{1}\\u_{1,2}^{1}\\v_{1,2}^{1}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^{1}\\u_{2,3}^{1}\\v_{2,3}^{1}\end{bmatrix}-2\begin{bmatrix}d_{1,3}^{1}\\u_{1,3}^{1}\\v_{1,3}^{1}\end{bmatrix}+\begin{bmatrix}d_{0,3}^{1}\\u_{0,3}^{1}\\v_{0,3}^{1}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{1,4}^{1}\\u_{1,4}^{1}\\v_{1,4}^{1}\end{bmatrix}-2\begin{bmatrix}d_{1,3}^{1}\\u_{1,3}^{1}\\v_{1,3}^{1}\end{bmatrix}+\begin{bmatrix}d_{1,2}^{1}\\u_{1,2}^{1}\\v_{1,2}^{1}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{2,4}^{1}\\u_{2,4}^{1}\\v_{2,4}^{1}\end{bmatrix}-\begin{bmatrix}d_{0,4}^{1}\\u_{0,4}^{1}\\v_{0,4}^{1}\end{bmatrix}-\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}+\begin{bmatrix}d_{0,2}^{1}\\u_{0,2}^{1}\\v_{0,2}^{1}\end{bmatrix}\right)=\begin{bmatrix}-0.000074\\-0.000037\\0.000935\end{bmatrix}$$

$l=2,\ m=3$ and $n=1$

and apply boundary condition $v^1_{2,4}=v^1_{2,3},\ v^1_{3,3}=v^1_{3,4},\ v^1_{1,4}=v^1_{1,3}$

$$U^2_{1,2}=\begin{bmatrix}d^1_{2,3}\\u^1_{2,3}\\v^1_{2,3}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{3,3}\\u^1_{3,3}\\v^1_{3,3}\end{bmatrix}-\begin{bmatrix}d^1_{1,3}\\u^1_{1,3}\\v^1_{1,3}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{2,4}\\u^1_{2,4}\\v^1_{2,4}\end{bmatrix}-\begin{bmatrix}d^1_{2,2}\\u^1_{2,2}\\v^1_{2,2}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{3,3}\\u^1_{3,3}\\v^1_{3,3}\end{bmatrix}-2\begin{bmatrix}d^1_{2,3}\\u^1_{2,3}\\v^1_{2,3}\end{bmatrix}+\begin{bmatrix}d^1_{1,3}\\u^1_{1,3}\\v^1_{1,3}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d^1_{2,4}\\u^1_{2,4}\\v^1_{2,4}\end{bmatrix}-2\begin{bmatrix}d^1_{2,3}\\u^1_{2,3}\\v^1_{2,3}\end{bmatrix}+\begin{bmatrix}d^1_{2,2}\\u^1_{2,2}\\v^1_{2,2}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d^1_{3,4}\\u^1_{3,4}\\v^1_{3,4}\end{bmatrix}-\begin{bmatrix}d^1_{1,4}\\u^1_{1,4}\\v^1_{1,4}\end{bmatrix}-\begin{bmatrix}d^1_{3,2}\\u^1_{3,2}\\v^1_{3,2}\end{bmatrix}+\begin{bmatrix}d^1_{1,2}\\u^1_{1,2}\\v^1_{1,2}\end{bmatrix}\right)=\begin{bmatrix}-0.000043\\-0.001250\\0\end{bmatrix}$$

$l=3,\ m=3$ and $n=1$

and apply boundary condition $u^1_{3,3}=u^1_{4,3},\ v^1_{3,4}=v^1_{3,3},\ u^1_{4,4}=u^1_{3,4},\ v^1_{2,4}=v^1_{2,3},\ u^1_{4,2}=u^1_{3,2}$

$$U^2_{1,2}=\begin{bmatrix}d^1_{3,3}\\u^1_{3,3}\\v^1_{3,3}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{4,3}\\u^1_{4,3}\\v^1_{4,3}\end{bmatrix}-\begin{bmatrix}d^1_{2,3}\\u^1_{2,3}\\v^1_{2,3}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{3,4}\\u^1_{3,4}\\v^1_{3,4}\end{bmatrix}-\begin{bmatrix}d^1_{3,2}\\u^1_{3,2}\\v^1_{3,2}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{4,3}\\u^1_{4,3}\\v^1_{4,3}\end{bmatrix}-2\begin{bmatrix}d^1_{3,3}\\u^1_{3,3}\\v^1_{3,3}\end{bmatrix}+\begin{bmatrix}d^1_{2,3}\\u^1_{2,3}\\v^1_{2,3}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d^1_{3,4}\\u^1_{3,4}\\v^1_{3,4}\end{bmatrix}-2\begin{bmatrix}d^1_{3,3}\\u^1_{3,3}\\v^1_{3,3}\end{bmatrix}+\begin{bmatrix}d^1_{3,2}\\u^1_{3,2}\\v^1_{3,2}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d^1_{4,4}\\u^1_{4,4}\\v^1_{4,4}\end{bmatrix}-\begin{bmatrix}d^1_{2,4}\\u^1_{2,4}\\v^1_{2,4}\end{bmatrix}-\begin{bmatrix}d^1_{4,2}\\u^1_{4,2}\\v^1_{4,2}\end{bmatrix}+\begin{bmatrix}d^1_{2,2}\\u^1_{2,2}\\v^1_{2,2}\end{bmatrix}\right)=\begin{bmatrix}-0.000074\\0.000935\\0.000936\end{bmatrix}$$

Table 3.2 The calculated water elevation and velocities at time $t = 0.02$ in a close reservoir of example 3.5.1

Points	$d^n_{l,m}$	$u^n_{l,m}$	$v^n_{l,m}$
$U^2_{1,1}$	-0.000074	-0.000935	-0.000935
$U^2_{2,1}$	-0.000106	0.000000	-0.001247
$U^2_{3,1}$	-0.000074	0.000935	-0.000936
$U^2_{1,2}$	-0.000106	-0.001247	0.000000
$U^2_{2,2}$	-0.000149	-4.375x10^{-7}	0.000000
$U^2_{3,2}$	-0.000043	0.000000	-0.001250
$U^2_{1,3}$	-0.000074	-0.000037	0.000935
$U^2_{2,3}$	-0.000043	-0.001250	0.000000
$U^2_{3,3}$	-0.000074	0.000935	0.000936

Example 3.5.2. To find the velocity and water elevation in an open reservoir when initial time of the velocity in reservoir is zero and water elevation is not the normal position or $d = x(1-x)y(1-y)$, with water flow pass out through the open gate $d = \sin(\pi y t)$ and calculate pollutant concentration when initial pollutant concentration in reservoir $C = 0.02$ and rate of change of pollutant concentration with respect to x at the open gate $\frac{\partial C}{\partial x} = -0.01$ rectangular domain $\Omega = (0,1)\times(0,1)$ in Figure 3.8 with step size $\Delta x = \Delta y = 0.25$ and $\Delta t = 0.01, T = 0.02$.

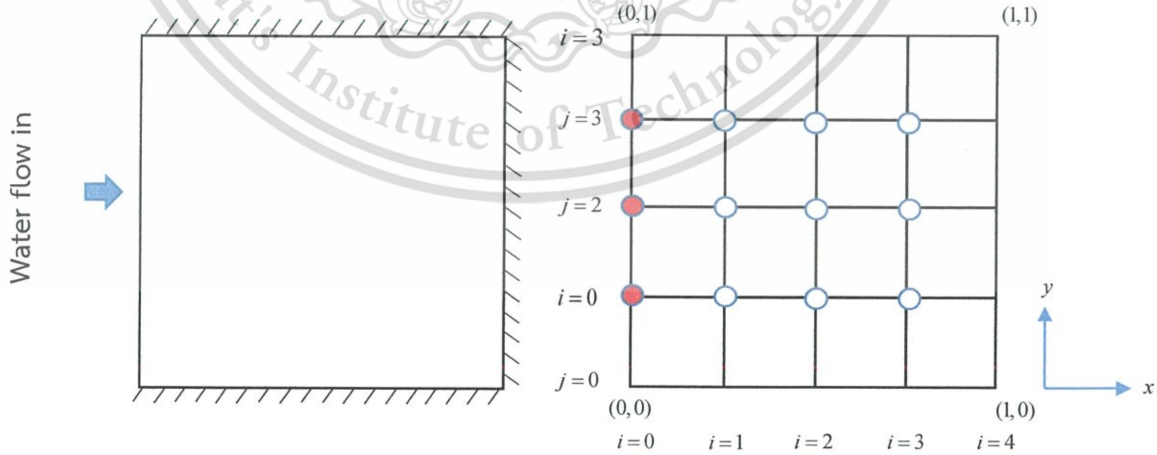


Figure 3.8 The domain and generating grid points of an open reservoir

i) Hydrodynamic model

with initial conditions, water elevation at initial time $d(x,y,0)=x(1-x)y(1-y)$, velocity at initial time $u(x,y,0)=0$, $v(x,y,0)=0$, boundary conditions, water elevation $d(0,y,t)=\sin(\pi yt)$, $d(1,y,t)=d(x,0,t)=d(x,1,t)=0$, velocity $v(0,y,t)=v(1,y,t)=v_y(x,0,t)=v_y(x,1,t)=0$, $u(x,0,t)=u(x,1,t)=u_x(0,y,t)=u_x(1,y,t)=0$, define $u(x,y,t)=u^n_{l,m}$, $v(x,y,t)=v^n_{l,m}$, and $d(x,y,t)=d^n_{l,m}$, with $\Delta t=0.01$, $\Delta x=0.25$, we have $p=\frac{\Delta t}{\Delta x}=\frac{0.01}{0.25}=0.04$, approximate differential boundary condition by Eqs.(3.65)-(3.68), since result of $n=0$ consistent with table 3.1.

For $n=1$ solving by Lax-Wendroff method (3.13)

$l=1$, $m=1$ and $n=1$

and apply boundary condition $u^1_{0,1}=u^1_{1,1}$, $v^1_{1,0}=v^1_{1,1}$, $u^1_{0,2}=u^1_{1,2}$, $v^1_{2,0}=v^1_{2,1}$, $u^1_{0,0}=u^1_{1,0}$ and $d^1_{0,1}=\sin(\pi(0.25)(0.01))$, $d^1_{0,2}=\sin(\pi(0.5)(0.01))$

$$U^2_{1,1}=\begin{bmatrix}d^1_{1,1}\\u^1_{1,1}\\v^1_{1,1}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{2,1}\\u^1_{2,1}\\v^1_{2,1}\end{bmatrix}-\begin{bmatrix}d^1_{0,1}\\u^1_{0,1}\\v^1_{0,1}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{1,2}\\u^1_{1,2}\\v^1_{1,2}\end{bmatrix}-\begin{bmatrix}d^1_{1,0}\\u^1_{1,0}\\v^1_{1,0}\end{bmatrix}\right)$$
$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{2,1}\\u^1_{2,1}\\v^1_{2,1}\end{bmatrix}-2\begin{bmatrix}d^1_{1,1}\\u^1_{1,1}\\v^1_{1,1}\end{bmatrix}+\begin{bmatrix}d^1_{0,1}\\u^1_{0,1}\\v^1_{0,1}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d^1_{1,2}\\u^1_{1,2}\\v^1_{1,2}\end{bmatrix}-2\begin{bmatrix}d^1_{1,1}\\u^1_{1,1}\\v^1_{1,1}\end{bmatrix}+\begin{bmatrix}d^1_{1,0}\\u^1_{1,0}\\v^1_{1,0}\end{bmatrix}\right)$$
$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d^1_{2,2}\\u^1_{2,2}\\v^1_{2,2}\end{bmatrix}-\begin{bmatrix}d^1_{0,2}\\u^1_{0,2}\\v^1_{0,2}\end{bmatrix}-\begin{bmatrix}d^1_{2,0}\\u^1_{2,0}\\v^1_{2,0}\end{bmatrix}+\begin{bmatrix}d^1_{0,0}\\u^1_{0,0}\\v^1_{0,0}\end{bmatrix}\right)=\begin{bmatrix}0.035050\\-0.001716\\-0.001873\end{bmatrix}$$

$l=2$, $m=1$ and $n=1$

and apply boundary condition $v^1_{2,0}=v^1_{2,1}$, $v^1_{3,0}=v^1_{3,1}$, $v^1_{1,0}=v^1_{1,1}$

$$U^2_{2,1}=\begin{bmatrix}d^1_{2,1}\\u^1_{2,1}\\v^1_{2,1}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{3,1}\\u^1_{3,1}\\v^1_{3,1}\end{bmatrix}-\begin{bmatrix}d^1_{1,1}\\u^1_{1,1}\\v^1_{1,1}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{2,2}\\u^1_{2,2}\\v^1_{2,2}\end{bmatrix}-\begin{bmatrix}d^1_{2,0}\\u^1_{2,0}\\v^1_{2,0}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,1}^1\\u_{3,1}^1\\v_{3,1}^1\end{bmatrix}-2\begin{bmatrix}d_{2,1}^1\\u_{2,1}^1\\v_{2,1}^1\end{bmatrix}+\begin{bmatrix}d_{1,1}^1\\u_{1,1}^1\\v_{1,1}^1\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{2,2}^1\\u_{2,2}^1\\v_{2,2}^1\end{bmatrix}-2\begin{bmatrix}d_{2,1}^1\\u_{2,1}^1\\v_{2,1}^1\end{bmatrix}+\begin{bmatrix}d_{2,0}^1\\u_{2,0}^1\\v_{2,0}^1\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^1\\u_{3,2}^1\\v_{3,2}^1\end{bmatrix}-\begin{bmatrix}d_{1,2}^1\\u_{1,2}^1\\v_{1,2}^1\end{bmatrix}-\begin{bmatrix}d_{3,0}^1\\u_{3,0}^1\\v_{3,0}^1\end{bmatrix}+\begin{bmatrix}d_{1,0}^1\\u_{1,0}^1\\v_{1,0}^1\end{bmatrix}\right)=\begin{bmatrix}0.046725\\0\\-0.002497\end{bmatrix}$$

$l=3, m=1$ and $n=1$

and apply boundary condition $u_{4,1}^1=u_{3,1}^1,\ v_{3,0}^1=v_{3,1}^1,\ u_{4,2}^1=u_{3,2}^1,\ u_{4,0}^1=u_{3,0}^1,\ v_{2,1}^1=v_{2,0}^1$

$$U_{3,1}^2=\begin{bmatrix}d_{3,1}^1\\u_{3,1}^1\\v_{3,1}^1\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,1}^1\\u_{4,1}^1\\v_{4,1}^1\end{bmatrix}-\begin{bmatrix}d_{2,1}^1\\u_{2,1}^1\\v_{2,1}^1\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^1\\u_{3,2}^1\\v_{3,2}^1\end{bmatrix}-\begin{bmatrix}d_{3,0}^1\\u_{3,0}^1\\v_{3,0}^1\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,1}^1\\u_{4,1}^1\\v_{4,1}^1\end{bmatrix}-2\begin{bmatrix}d_{3,1}^1\\u_{3,1}^1\\v_{3,1}^1\end{bmatrix}+\begin{bmatrix}d_{2,1}^1\\u_{2,1}^1\\v_{2,1}^1\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^1\\u_{3,2}^1\\v_{3,2}^1\end{bmatrix}-2\begin{bmatrix}d_{3,1}^1\\u_{3,1}^1\\v_{3,1}^1\end{bmatrix}+\begin{bmatrix}d_{3,0}^1\\u_{3,0}^1\\v_{3,0}^1\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{4,2}^1\\u_{4,2}^1\\v_{4,2}^1\end{bmatrix}-\begin{bmatrix}d_{2,2}^1\\u_{2,2}^1\\v_{2,2}^1\end{bmatrix}-\begin{bmatrix}d_{4,0}^1\\u_{4,0}^1\\v_{4,0}^1\end{bmatrix}+\begin{bmatrix}d_{2,0}^1\\u_{2,0}^1\\v_{2,0}^1\end{bmatrix}\right)=\begin{bmatrix}0.035718\\0.010302\\-0.010307\end{bmatrix}$$

$l=1, m=2$ and $n=1$

and apply boundary condition $u_{0,2}^1=u_{1,2}^1,\ u_{0,3}^1=u_{1,3}^1,\ u_{0,1}^1=u_{1,1}^1$ and

$d_{0,2}^1=\sin(\pi(0.25)(0.01)),\ d_{0,3}^1=\sin(\pi(0.75)(0.01)),\ d_{0,1}^1=\sin(\pi(0.25)(0.01))$

$$U_{1,2}^2=\begin{bmatrix}d_{1,2}^1\\u_{1,2}^1\\v_{1,2}^1\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,2}^1\\u_{2,2}^1\\v_{2,2}^1\end{bmatrix}-\begin{bmatrix}d_{0,2}^1\\u_{0,2}^1\\v_{0,2}^1\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{1,3}^1\\u_{1,3}^1\\v_{1,3}^1\end{bmatrix}-\begin{bmatrix}d_{1,1}^1\\u_{1,1}^1\\v_{1,1}^1\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,2}^1\\u_{2,2}^1\\v_{2,2}^1\end{bmatrix}-2\begin{bmatrix}d_{1,2}^1\\u_{1,2}^1\\v_{1,2}^1\end{bmatrix}+\begin{bmatrix}d_{0,2}^1\\u_{0,2}^1\\v_{0,2}^1\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{1,3}^1\\u_{1,3}^1\\v_{1,3}^1\end{bmatrix}-2\begin{bmatrix}d_{1,2}^1\\u_{1,2}^1\\v_{1,2}^1\end{bmatrix}+\begin{bmatrix}d_{1,1}^1\\u_{1,1}^1\\v_{1,1}^1\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^1\\u_{2,3}^1\\v_{2,3}^1\end{bmatrix}-\begin{bmatrix}d_{0,3}^1\\u_{0,3}^1\\v_{0,3}^1\end{bmatrix}-\begin{bmatrix}d_{2,1}^1\\u_{2,1}^1\\v_{2,1}^1\end{bmatrix}+\begin{bmatrix}d_{0,1}^1\\u_{0,1}^1\\v_{0,1}^1\end{bmatrix}\right)=\begin{bmatrix}0.046731\\-0.002340\\0\end{bmatrix}$$

$l=2, m=2$ and $n=1$

$$U_{1,2}^{2}=\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^{1}\\u_{3,2}^{1}\\v_{3,2}^{1}\end{bmatrix}-\begin{bmatrix}d_{1,2}^{1}\\u_{1,2}^{1}\\v_{1,2}^{1}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^{1}\\u_{2,3}^{1}\\v_{2,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{2,1}^{1}\\u_{2,1}^{1}\\v_{2,1}^{1}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,2}^{1}\\u_{3,2}^{1}\\v_{3,2}^{1}\end{bmatrix}-2\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}+\begin{bmatrix}d_{1,2}^{1}\\u_{1,2}^{1}\\v_{1,2}^{1}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^{1}\\u_{2,3}^{1}\\v_{2,3}^{1}\end{bmatrix}-2\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}+\begin{bmatrix}d_{2,1}^{1}\\u_{2,1}^{1}\\v_{2,1}^{1}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{3,3}^{1}\\u_{3,3}^{1}\\v_{3,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{1,3}^{1}\\u_{1,3}^{1}\\v_{1,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{3,1}^{1}\\u_{3,1}^{1}\\v_{3,1}^{1}\end{bmatrix}+\begin{bmatrix}d_{1,1}^{1}\\u_{1,1}^{1}\\v_{1,1}^{1}\end{bmatrix}\right)=\begin{bmatrix}0.062300\\-4.36\times10^{-7}\\0\end{bmatrix}$$

$l=3, m=2$ and $n=1$

and apply boundary condition $u_{4,2}^{1}=u_{3,2}^{1}, u_{4,3}^{1}=u_{3,3}^{1}, u_{4,1}^{1}=u_{3,1}^{1}$

$$U_{1,2}^{2}=\begin{bmatrix}d_{3,2}^{1}\\u_{3,2}^{1}\\v_{3,2}^{1}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,2}^{1}\\u_{4,2}^{1}\\v_{4,2}^{1}\end{bmatrix}-\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{3,3}^{1}\\u_{3,3}^{1}\\v_{3,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{3,1}^{1}\\u_{3,1}^{1}\\v_{3,1}^{1}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{4,2}^{1}\\u_{4,2}^{1}\\v_{4,2}^{1}\end{bmatrix}-2\begin{bmatrix}d_{3,2}^{1}\\u_{3,2}^{1}\\v_{3,2}^{1}\end{bmatrix}+\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{3,3}^{1}\\u_{3,3}^{1}\\v_{3,3}^{1}\end{bmatrix}-2\begin{bmatrix}d_{3,2}^{1}\\u_{3,2}^{1}\\v_{3,2}^{1}\end{bmatrix}+\begin{bmatrix}d_{3,1}^{1}\\u_{3,1}^{1}\\v_{3,1}^{1}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{4,3}^{1}\\u_{4,3}^{1}\\v_{4,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{2,3}^{1}\\u_{2,3}^{1}\\v_{2,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{4,1}^{1}\\u_{4,1}^{1}\\v_{4,1}^{1}\end{bmatrix}+\begin{bmatrix}d_{2,1}^{1}\\u_{2,1}^{1}\\v_{2,1}^{1}\end{bmatrix}\right)=\begin{bmatrix}0.046746\\0.002497\\0\end{bmatrix}$$

$l=1, m=3$ and $n=1$

and apply boundary condition $u_{0,3}^{1}=u_{1,3}^{1}, v_{1,4}^{1}=v_{1,3}^{1}, v_{2,3}^{1}=v_{2,4}^{1}, u_{0,4}^{1}=u_{1,4}^{1}, u_{0,2}^{1}=u_{1,2}^{1}$ and

$d_{0,3}^{1}=\sin(\pi(0.75)(0.01)), d_{0,4}^{1}=\sin(\pi(1.00)(0.01)), d_{0,2}^{1}=\sin(\pi(0.50)(0.01))$

$$U_{1,2}^{2}=\begin{bmatrix}d_{1,3}^{1}\\u_{1,3}^{1}\\v_{1,3}^{1}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^{1}\\u_{2,3}^{1}\\v_{2,3}^{1}\end{bmatrix}-\begin{bmatrix}d_{0,3}^{1}\\u_{0,3}^{1}\\v_{0,3}^{1}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d_{1,4}^{1}\\u_{1,4}^{1}\\v_{1,4}^{1}\end{bmatrix}-\begin{bmatrix}d_{1,2}^{1}\\u_{1,2}^{1}\\v_{1,2}^{1}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d_{2,3}^{1}\\u_{2,3}^{1}\\v_{2,3}^{1}\end{bmatrix}-2\begin{bmatrix}d_{1,3}^{1}\\u_{1,3}^{1}\\v_{1,3}^{1}\end{bmatrix}+\begin{bmatrix}d_{0,3}^{1}\\u_{0,3}^{1}\\v_{0,3}^{1}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d_{1,4}^{1}\\u_{1,4}^{1}\\v_{1,4}^{1}\end{bmatrix}-2\begin{bmatrix}d_{1,3}^{1}\\u_{1,3}^{1}\\v_{1,3}^{1}\end{bmatrix}+\begin{bmatrix}d_{1,2}^{1}\\u_{1,2}^{1}\\v_{1,2}^{1}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d_{2,4}^{1}\\u_{2,4}^{1}\\v_{2,4}^{1}\end{bmatrix}-\begin{bmatrix}d_{0,4}^{1}\\u_{0,4}^{1}\\v_{0,4}^{1}\end{bmatrix}-\begin{bmatrix}d_{2,2}^{1}\\u_{2,2}^{1}\\v_{2,2}^{1}\end{bmatrix}+\begin{bmatrix}d_{0,2}^{1}\\u_{0,2}^{1}\\v_{0,2}^{1}\end{bmatrix}\right)=\begin{bmatrix}0.035062\\-0.000503\\0.001873\end{bmatrix}$$

$l=2$, $m=3$ and $n=1$

and apply boundary condition $v^1_{2,4}=v^1_{2,3}$, $v^1_{3,3}=v^1_{3,4}$, $v^1_{1,4}=v^1_{1,3}$

$$U^2_{1,2}=\begin{bmatrix}d^1_{2,3}\\u^1_{2,3}\\v^1_{2,3}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{3,3}\\u^1_{3,3}\\v^1_{3,3}\end{bmatrix}-\begin{bmatrix}d^1_{1,3}\\u^1_{1,3}\\v^1_{1,3}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{2,4}\\u^1_{2,4}\\v^1_{2,4}\end{bmatrix}-\begin{bmatrix}d^1_{2,2}\\u^1_{2,2}\\v^1_{2,2}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{3,3}\\u^1_{3,3}\\v^1_{3,3}\end{bmatrix}-2\begin{bmatrix}d^1_{2,3}\\u^1_{2,3}\\v^1_{2,3}\end{bmatrix}+\begin{bmatrix}d^1_{1,3}\\u^1_{1,3}\\v^1_{1,3}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d^1_{2,4}\\u^1_{2,4}\\v^1_{2,4}\end{bmatrix}-2\begin{bmatrix}d^1_{2,3}\\u^1_{2,3}\\v^1_{2,3}\end{bmatrix}+\begin{bmatrix}d^1_{2,2}\\u^1_{2,2}\\v^1_{2,2}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d^1_{3,4}\\u^1_{3,4}\\v^1_{3,4}\end{bmatrix}-\begin{bmatrix}d^1_{1,4}\\u^1_{1,4}\\v^1_{1,4}\end{bmatrix}-\begin{bmatrix}d^1_{3,2}\\u^1_{3,2}\\v^1_{3,2}\end{bmatrix}+\begin{bmatrix}d^1_{1,2}\\u^1_{1,2}\\v^1_{1,2}\end{bmatrix}\right)=\begin{bmatrix}0.046725\\0\\0.002497\end{bmatrix}$$

$l=3$, $m=3$ and $n=1$

and apply boundary condition $u^1_{3,3}=u^1_{4,3}$, $v^1_{3,4}=v^1_{3,3}$, $u^1_{4,4}=u^1_{3,4}$, $v^1_{2,4}=v^1_{2,3}$, $u^1_{4,2}=u^1_{3,2}$

$$U^2_{1,2}=\begin{bmatrix}d^1_{3,3}\\u^1_{3,3}\\v^1_{3,3}\end{bmatrix}+0.02\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{4,3}\\u^1_{4,3}\\v^1_{4,3}\end{bmatrix}-\begin{bmatrix}d^1_{2,3}\\u^1_{2,3}\\v^1_{2,3}\end{bmatrix}\right)+0.02\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{3,4}\\u^1_{3,4}\\v^1_{3,4}\end{bmatrix}-\begin{bmatrix}d^1_{3,2}\\u^1_{3,2}\\v^1_{3,2}\end{bmatrix}\right)$$

$$+0.0008\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\left(\begin{bmatrix}d^1_{4,3}\\u^1_{4,3}\\v^1_{4,3}\end{bmatrix}-2\begin{bmatrix}d^1_{3,3}\\u^1_{3,3}\\v^1_{3,3}\end{bmatrix}+\begin{bmatrix}d^1_{2,3}\\u^1_{2,3}\\v^1_{2,3}\end{bmatrix}\right)+0.0008\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}d^1_{3,4}\\u^1_{3,4}\\v^1_{3,4}\end{bmatrix}-2\begin{bmatrix}d^1_{3,3}\\u^1_{3,3}\\v^1_{3,3}\end{bmatrix}+\begin{bmatrix}d^1_{3,2}\\u^1_{3,2}\\v^1_{3,2}\end{bmatrix}\right)$$

$$+0.0002\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\left(\begin{bmatrix}d^1_{4,4}\\u^1_{4,4}\\v^1_{4,4}\end{bmatrix}-\begin{bmatrix}d^1_{2,4}\\u^1_{2,4}\\v^1_{2,4}\end{bmatrix}-\begin{bmatrix}d^1_{4,2}\\u^1_{4,2}\\v^1_{4,2}\end{bmatrix}+\begin{bmatrix}d^1_{2,2}\\u^1_{2,2}\\v^1_{2,2}\end{bmatrix}\right)=\begin{bmatrix}0.035043\\0.001873\\0.001873\end{bmatrix}$$

Table 3.3 The calculated water elevation and velocities at time $t=0.02$ in an open reservoir of example 3.5.2

Points	$d_{l,m}^{n}$	$u_{l,m}^{n}$	$v_{l,m}^{n}$
$U_{1,1}^{2}$	0.035050	-0.001716	-0.001873
$U_{2,1}^{2}$	0.046725	0.000000	-0.002497
$U_{3,1}^{2}$	0.035718	0.010302	-0.010307
$U_{1,2}^{2}$	0.046731	-0.002340	0.000000
$U_{2,2}^{2}$	0.062300	-4.36x10^{-7}	0.000000
$U_{3,2}^{2}$	0.046746	0.002497	0.000000
$U_{1,3}^{2}$	0.035062	-0.000503	0.001873
$U_{2,3}^{2}$	0.046725	0.000000	0.002497
$U_{3,3}^{2}$	0.035043	0.001873	0.001873

ii) Unsteady state dispersion model

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right), \tag{3.69}$$

with initial conditions, pollutant concentration at initial time $C(x,y,0)=0.02$, boundary conditions, pollutant concentration $C_x(1,y,t)=C_y(x,0,t)=C_y(x,1,t)=0$, $C_x(0,y,t)=-0.01$, define $C(x,y,t)=C_{l,m}^{n}$ with $\Delta t=0.01$, $\Delta x=0.25$, and $D=0.1$, approximate differential boundary condition,

$C_y(x,0,t)=0$ or $\frac{\partial C_{l,0}^{n}}{\partial y}=0$,

using forward difference,

$$\frac{C_{l,1}^{n}-C_{l,0}^{n}}{\Delta y}=0,$$

$$C_{l,0}^{n}=C_{l,1}^{n}, \qquad l=0,1,2,3,4 \tag{3.70}$$

$C_y(x,1,t)=0$ or $\frac{\partial C_{l,4}^{n}}{\partial y}=0$,

using backward difference,

$$\frac{C_{l,4}^{n}-C_{l,3}^{n}}{\Delta y}=0,$$

$$C_{l,4}^{n}=C_{l,3}^{n}, \qquad l=0,1,2,3,4 \tag{3.71}$$

$C_x(0,y,t) = -0.01$ or $\frac{\partial C^n_{0,m}}{\partial x} = -0.01$,

using forward difference,

$$\frac{C^n_{1,m} - C^n_{0,m}}{\Delta x} = -0.01,$$

$$C^n_{0,m} = C^n_{1,m} + 0.01\Delta x, \qquad m = 0,1,2,3,4 \qquad (3.72)$$

$C_x(1,y,t) = 0$ or $\frac{\partial C^n_{4,m}}{\partial x} = 0$,

using backward difference,

$$\frac{C^n_{4,m} - C^n_{3,m}}{\Delta x} = 0,$$

$$C^n_{4,m} = C^n_{3,m}, \qquad m = 0,1,2,3,4 \qquad (3.73)$$

solving Eq.(3.69) of problem by finite difference method (3.23).

$l = 1$, $m = 1$ and $n = 0$

and apply boundary condition $C^0_{0,1} = C^0_{1,1} + 0.01(0.25)$, $C^0_{1,0} = C^0_{1,1}$

$$C^1_{1,1} = A_1C^0_{2,1} + A_2C^0_{1,2} + A_3C^0_{0,1} + A_4C^0_{1,0} + A_5C^0_{1,1}$$

$$A_1 = \frac{\Delta t D}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta t D}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x}u^0_{1,1} + \frac{\Delta t}{\Delta x^2}D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_4 = \frac{\Delta t}{\Delta y}v^0_{1,1} + \frac{\Delta t}{\Delta y^2}D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x}u^0_{1,1} - \frac{\Delta t}{\Delta y}v^0_{1,1} - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$$C^1_{1,1} = 0.02008$$

$l = 2, m = 1$ and $n = 0$

and apply boundary condition $C_{2,0}^0 = C_{2,1}^0$

$$C_{2,1}^1 = A_1 C_{3,1}^0 + A_2 C_{2,2}^0 + A_3 C_{1,1}^0 + A_4 C_{2,0}^0 + A_5 C_{2,1}^0$$

$$A_1 = \frac{\Delta t D}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta t D}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x} u_{2,1}^0 + \frac{\Delta t}{\Delta x^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_4 = \frac{\Delta t}{\Delta y} v_{2,1}^0 + \frac{\Delta t}{\Delta y^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x} u_{2,1}^0 - \frac{\Delta t}{\Delta y} v_{2,1}^0 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$$C_{2,1}^1 = 0.02$$

$l = 3, m = 1$ and $n = 0$

and apply boundary condition $C_{3,0}^0 = C_{3,1}^0,\ C_{4,1}^0 = C_{3,1}^0$

$$C_{3,1}^1 = A_1 C_{4,1}^0 + A_2 C_{3,2}^0 + A_3 C_{2,1}^0 + A_4 C_{3,0}^0 + A_5 C_{3,1}^0$$

$$A_1 = \frac{\Delta t D}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta t D}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x} u_{3,1}^0 + \frac{\Delta t}{\Delta x^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_4 = \frac{\Delta t}{\Delta y} v_{3,1}^0 + \frac{\Delta t}{\Delta y^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x} u_{3,1}^0 - \frac{\Delta t}{\Delta y} v_{3,1}^0 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$$C_{3,1}^1 = 0.02$$

$l = 1, m = 2$ and $n = 0$

and apply boundary condition $C_{0,2}^0 = C_{1,2}^0 + 0.01(0.25)$

$$C_{1,2}^1 = A_1 C_{2,2}^0 + A_2 C_{1,3}^0 + A_3 C_{0,2}^0 + A_4 C_{1,1}^0 + A_5 C_{1,2}^0$$

$$A_1 = \frac{\Delta t D}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta t D}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x} u_{1,2}^0 + \frac{\Delta t}{\Delta x^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_4 = \frac{\Delta t}{\Delta y} v_{1,2}^0 + \frac{\Delta t}{\Delta y^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x} u_{1,2}^0 - \frac{\Delta t}{\Delta y} v_{1,2}^0 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$C_{1,2}^1 = 0.02008$

$l = 2$, $m = 2$ and $n = 0$

$$C_{2,2}^1 = A_1 C_{3,2}^0 + A_2 C_{2,3}^0 + A_3 C_{1,2}^0 + A_4 C_{2,1}^0 + A_5 C_{2,2}^0$$

$$A_1 = \frac{\Delta t D}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta t D}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x} u_{2,2}^0 + \frac{\Delta t}{\Delta x^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_4 = \frac{\Delta t}{\Delta y} v_{2,2}^0 + \frac{\Delta t}{\Delta y^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x} u_{2,2}^0 - \frac{\Delta t}{\Delta y} v_{2,2}^0 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$C_{2,2}^1 = 0.02$

$l = 3$, $m = 2$ and $n = 0$

and apply boundary condition $C_{4,2}^0 = C_{3,2}^0$

$$C_{3,2}^1 = A_1 C_{4,2}^0 + A_2 C_{3,3}^0 + A_3 C_{2,2}^0 + A_4 C_{3,1}^0 + A_5 C_{3,2}^0$$

$$A_1 = \frac{\Delta t D}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta t D}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x} u_{3,2}^0 + \frac{\Delta t}{\Delta x^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_4 = \frac{\Delta t}{\Delta y} v_{3,2}^0 + \frac{\Delta t}{\Delta y^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x} u_{3,2}^0 - \frac{\Delta t}{\Delta y} v_{3,2}^0 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$C_{3,2}^1 = 0.02$

$l = 1$, $m = 3$ and $n = 0$

and apply boundary condition $C_{1,4}^0 = C_{1,3}^0$, $C_{0,3}^0 = C_{1,3}^0 + 0.01(0.25)$

$$C_{1,3}^1 = A_1 C_{2,3}^0 + A_2 C_{1,4}^0 + A_3 C_{0,3}^0 + A_4 C_{1,2}^0 + A_5 C_{1,3}^0$$

$$A_1 = \frac{\Delta t D}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta t D}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x} u_{1,3}^0 + \frac{\Delta t}{\Delta x^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_4 = \frac{\Delta t}{\Delta y} v_{1,3}^0 + \frac{\Delta t}{\Delta y^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x} u_{1,3}^0 - \frac{\Delta t}{\Delta y} v_{1,3}^0 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$C_{1,3}^1 = 0.02008$

$l = 2$, $m = 3$ and $n = 0$

and apply boundary condition $C_{2,4}^0 = C_{2,3}^0$

$$C_{2,3}^1 = A_1 C_{3,3}^0 + A_2 C_{2,4}^0 + A_3 C_{1,3}^0 + A_4 C_{2,2}^0 + A_5 C_{2,3}^0$$

$$A_1 = \frac{\Delta t D}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta t D}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x} u_{2,3}^0 + \frac{\Delta t}{\Delta x^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_4 = \frac{\Delta t}{\Delta y} v_{2,3}^0 + \frac{\Delta t}{\Delta y^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x}u_{2,3}^0 - \frac{\Delta t}{\Delta y}v_{2,3}^0 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$
$$= 1 - \frac{(0.01)}{(0.25)}(0) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$C_{2,3}^1 = 0.02$

$l = 3$, $m = 3$ and $n = 0$

and apply boundary condition $C_{4,3}^0 = C_{3,3}^0$, $C_{3,4}^0 = C_{3,3}^0$

$$C_{3,3}^1 = A_1C_{4,3}^0 + A_2C_{3,4}^0 + A_3C_{2,3}^0 + A_4C_{3,2}^0 + A_5C_{3,3}^0$$
$$A_1 = \frac{\Delta tD}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$
$$A_2 = \frac{\Delta tD}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$
$$A_3 = \frac{\Delta t}{\Delta x}u_{3,3}^0 + \frac{\Delta t}{\Delta x^2}D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$
$$A_4 = \frac{\Delta t}{\Delta y}v_{3,3}^0 + \frac{\Delta t}{\Delta y^2}D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$
$$A_5 = 1 - \frac{\Delta t}{\Delta x}u_{3,3}^0 - \frac{\Delta t}{\Delta y}v_{3,3}^0 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$
$$= 1 - \frac{(0.01)}{(0.25)}(0) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$C_{3,3}^1 = 0.02$

Table 3.4 The calculated pollutant concentration at time $t = 0.01$ in an open reservoir of example 3.5.2

Points	Pollutant concentration
$C_{1,1}^2$	0.02008
$C_{2,1}^2$	0.02000
$C_{3,1}^2$	0.02000
$C_{1,2}^2$	0.02008
$C_{2,2}^2$	0.02000
$C_{3,2}^2$	0.02000
$C_{1,3}^2$	0.02008
$C_{2,3}^2$	0.02000
$C_{3,3}^2$	0.02000

$l=1, m=1$ and $n=1$

and apply boundary condition $C_{0,1}^{1}=C_{1,1}^{1}+0.01(0.25), C_{1,0}^{1}=C_{1,1}^{1}$

$$C_{1,1}^{2}=A_1C_{2,1}^{1}+A_2C_{1,2}^{1}+A_3C_{0,1}^{1}+A_4C_{1,0}^{1}+A_5C_{1,1}^{1}$$

$$A_1=\frac{\Delta tD}{\Delta x^2}=\frac{(0.01)(0.1)}{(0.25)^2}=0.016,$$

$$A_2=\frac{\Delta tD}{\Delta y^2}=\frac{(0.01)(0.1)}{(0.25)^2}=0.016,$$

$$A_3=\frac{\Delta t}{\Delta x}u_{1,1}^{1}+\frac{\Delta t}{\Delta x^2}D=\frac{(0.01)}{(0.25)}(-0.000937)+\frac{(0.01)}{(0.25)^2}(0.1)=0.015962,$$

$$A_4=\frac{\Delta t}{\Delta y}v_{1,1}^{1}+\frac{\Delta t}{\Delta y^2}D=\frac{(0.01)}{(0.25)}(-0.000937)+\frac{(0.01)}{(0.25)^2}(0.1)=0.015962,$$

$$A_5=1-\frac{\Delta t}{\Delta x}u_{1,1}^{1}-\frac{\Delta t}{\Delta y}v_{1,1}^{1}-\frac{2D\Delta t}{\Delta x^2}-\frac{2D\Delta t}{\Delta y^2}$$

$$=1-\frac{(0.01)}{(0.25)}(-0.000937)-\frac{(0.01)}{(0.25)}(-0.000937)-\frac{2(0.1)(0.01)}{(0.25)^2}-\frac{2(0.1)(0.01)}{(0.25)^2}=0.936075,$$

$$C_{1,1}^{2}=0.0201186$$

$l=2, m=1$ and $n=1$

and apply boundary condition $C_{2,0}^{1}=C_{2,1}^{1}$

$$C_{2,1}^{2}=A_1C_{3,1}^{1}+A_2C_{2,2}^{1}+A_3C_{1,1}^{1}+A_4C_{2,0}^{1}+A_5C_{2,1}^{1}$$

$$A_1=\frac{\Delta tD}{\Delta x^2}=\frac{(0.01)(0.1)}{(0.25)^2}=0.016,$$

$$A_2=\frac{\Delta tD}{\Delta y^2}=\frac{(0.01)(0.1)}{(0.25)^2}=0.016,$$

$$A_3=\frac{\Delta t}{\Delta x}u_{2,1}^{1}+\frac{\Delta t}{\Delta x^2}D=\frac{(0.01)}{(0.25)}(0)+\frac{(0.01)}{(0.25)^2}(0.1)=0.016,$$

$$A_4=\frac{\Delta t}{\Delta y}v_{2,1}^{1}+\frac{\Delta t}{\Delta y^2}D=\frac{(0.01)}{(0.25)}(-0.00125)+\frac{(0.01)}{(0.25)^2}(0.1)=0.01595,$$

$$A_5=1-\frac{\Delta t}{\Delta x}u_{2,1}^{1}-\frac{\Delta t}{\Delta y}v_{2,1}^{1}-\frac{2D\Delta t}{\Delta x^2}-\frac{2D\Delta t}{\Delta y^2}$$

$$=1-\frac{(0.01)}{(0.25)}(0)-\frac{(0.01)}{(0.25)}(-0.00125)-\frac{2(0.1)(0.01)}{(0.25)^2}-\frac{2(0.1)(0.01)}{(0.25)^2}=0.93605,$$

$$C_{2,1}^{2}=0.0200013$$

$l=3, m=1$ and $n=1$

and apply boundary condition $C_{3,0}^{1}=C_{3,1}^{1}, C_{4,1}^{1}=C_{3,1}^{1}$

$$C_{3,1}^{2}=A_1C_{4,1}^{1}+A_2C_{3,2}^{1}+A_3C_{2,1}^{1}+A_4C_{3,0}^{1}+A_5C_{3,1}^{1}$$

$$A_1 = \frac{\Delta t D}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta t D}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x} u_{3,1}^1 + \frac{\Delta t}{\Delta x^2} D = \frac{(0.01)}{(0.25)}(0.000937) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016037,$$

$$A_4 = \frac{\Delta t}{\Delta y} v_{3,1}^1 + \frac{\Delta t}{\Delta y^2} D = \frac{(0.01)}{(0.25)}(-0.000937) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.015962,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x} u_{3,1}^1 - \frac{\Delta t}{\Delta y} v_{3,1}^1 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0.000937) - \frac{(0.01)}{(0.25)}(-0.000937) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$C_{3,1}^2 = 0.02$

$l = 1$, $m = 2$ and $n = 1$

and apply boundary condition $C_{0,2}^1 = C_{1,2}^1 + 0.01(0.25)$

$$C_{1,2}^2 = A_1 C_{2,2}^1 + A_2 C_{1,3}^1 + A_3 C_{0,2}^1 + A_4 C_{1,1}^1 + A_5 C_{1,2}^1$$

$$A_1 = \frac{\Delta t D}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta t D}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x} u_{1,2}^1 + \frac{\Delta t}{\Delta x^2} D = \frac{(0.01)}{(0.25)}(-0.00125) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.01595,$$

$$A_4 = \frac{\Delta t}{\Delta y} v_{1,2}^1 + \frac{\Delta t}{\Delta y^2} D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x} u_{1,2}^1 - \frac{\Delta t}{\Delta y} v_{1,2}^1 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(-0.00125) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.93605,$$

$C_{1,2}^2 = 0.0201186$

$l = 2$, $m = 2$ and $n = 1$

$$C_{2,2}^2 = A_1 C_{3,2}^1 + A_2 C_{2,3}^1 + A_3 C_{1,2}^1 + A_4 C_{2,1}^1 + A_5 C_{2,2}^1$$

$$A_1 = \frac{\Delta t D}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta t D}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x}u_{2,2}^1 + \frac{\Delta t}{\Delta x^2}D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_4 = \frac{\Delta t}{\Delta y}v_{2,2}^1 + \frac{\Delta t}{\Delta y^2}D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x}u_{2,2}^1 - \frac{\Delta t}{\Delta y}v_{2,2}^1 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$C_{2,2}^2 = 0.0200787$

$l=3$, $m=2$ and $n=1$

and apply boundary condition $C_{4,2}^1 = C_{3,2}^1$

$$C_{3,2}^2 = A_1C_{4,2}^1 + A_2C_{3,3}^1 + A_3C_{2,2}^1 + A_4C_{3,1}^1 + A_5C_{3,2}^1$$

$$A_1 = \frac{\Delta tD}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta tD}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x}u_{3,2}^1 + \frac{\Delta t}{\Delta x^2}D = \frac{(0.01)}{(0.25)}(0.00125) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.01605,$$

$$A_4 = \frac{\Delta t}{\Delta y}v_{3,2}^1 + \frac{\Delta t}{\Delta y^2}D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x}u_{3,2}^1 - \frac{\Delta t}{\Delta y}v_{3,2}^1 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0.00125) - \frac{(0.01)}{(0.25)}(0) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.93595,$$

$C_{3,2}^2 = 0.02$

$l=1$, $m=3$ and $n=1$

and apply boundary condition $C_{1,4}^1 = C_{1,3}^1$, $C_{0,3}^1 = C_{1,3}^1 + 0.01(0.25)$

$$C_{1,3}^2 = A_1C_{2,3}^1 + A_2C_{1,4}^1 + A_3C_{0,3}^1 + A_4C_{1,2}^1 + A_5C_{1,3}^1$$

$$A_1 = \frac{\Delta tD}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta tD}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x}u_{1,3}^1 + \frac{\Delta t}{\Delta x^2}D = \frac{(0.01)}{(0.25)}(-0.000937) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.015962,$$

$$A_4 = \frac{\Delta t}{\Delta y}v_{1,3}^1 + \frac{\Delta t}{\Delta y^2}D = \frac{(0.01)}{(0.25)}(0.000937) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016037,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x}u_{1,3}^1 - \frac{\Delta t}{\Delta y}v_{1,3}^1 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(-0.000937) - \frac{(0.01)}{(0.25)}(0.000937) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.936,$$

$C_{1,3}^2 = 0.0201186$

$l = 2, m = 3$ and $n = 1$

and apply boundary condition $C_{2,4}^1 = C_{2,3}^1$

$$C_{2,3}^2 = A_1C_{3,3}^1 + A_2C_{2,4}^1 + A_3C_{1,3}^1 + A_4C_{2,2}^1 + A_5C_{2,3}^1$$

$$A_1 = \frac{\Delta tD}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta tD}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x}u_{2,3}^1 + \frac{\Delta t}{\Delta x^2}D = \frac{(0.01)}{(0.25)}(0) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016,$$

$$A_4 = \frac{\Delta t}{\Delta y}v_{2,3}^1 + \frac{\Delta t}{\Delta y^2}D = \frac{(0.01)}{(0.25)}(0.00125) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.01605,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x}u_{2,3}^1 - \frac{\Delta t}{\Delta y}v_{2,3}^1 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0) - \frac{(0.01)}{(0.25)}(0.00125) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.93595$$

$C_{2,3}^2 = 0.0200013$

$l = 3, m = 3$ and $n = 1$

and apply boundary condition $C_{4,3}^1 = C_{3,3}^1,\ C_{3,4}^1 = C_{3,3}^1$

$$C_{3,3}^2 = A_1C_{4,3}^1 + A_2C_{3,4}^1 + A_3C_{2,3}^1 + A_4C_{3,2}^1 + A_5C_{3,3}^1$$

$$A_1 = \frac{\Delta tD}{\Delta x^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_2 = \frac{\Delta tD}{\Delta y^2} = \frac{(0.01)(0.1)}{(0.25)^2} = 0.016,$$

$$A_3 = \frac{\Delta t}{\Delta x}u_{3,3}^1 + \frac{\Delta t}{\Delta x^2}D = \frac{(0.01)}{(0.25)}(0.000937) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016037,$$

$$A_4 = \frac{\Delta t}{\Delta y}v_{3,3}^1 + \frac{\Delta t}{\Delta y^2}D = \frac{(0.01)}{(0.25)}(0.000937) + \frac{(0.01)}{(0.25)^2}(0.1) = 0.016037,$$

$$A_5 = 1 - \frac{\Delta t}{\Delta x}u_{3,3}^1 - \frac{\Delta t}{\Delta y}v_{3,3}^1 - \frac{2D\Delta t}{\Delta x^2} - \frac{2D\Delta t}{\Delta y^2}$$

$$= 1 - \frac{(0.01)}{(0.25)}(0.000937) - \frac{(0.01)}{(0.25)}(0.000937) - \frac{2(0.1)(0.01)}{(0.25)^2} - \frac{2(0.1)(0.01)}{(0.25)^2} = 0.935925,$$

$C_{3,3}^2 = 0.02$

Table 3.5 The calculated pollutant concentration at time $t = 0.02$ in an open reservoir of example 3.5.2

Points	Pollutant concentration
$C_{1,1}^2$	0.0201186
$C_{2,1}^2$	0.0200013
$C_{3,1}^2$	0.020000
$C_{1,2}^2$	0.0201186
$C_{2,2}^2$	0.0200013
$C_{3,2}^2$	0.0200000
$C_{1,3}^2$	0.0201186
$C_{2,3}^2$	0.0200013
$C_{3,3}^2$	0.0200000

Example 3.5.3. To find the pollutant concentration in a close reservoir with inflow point $C = c_1$ when unchanging in time, rectangular domain $\Omega = (0,1)\times(0,1)$ in Figure 3.9 with step size $\Delta x = \Delta y = 0.25$ and diffusion coefficient $D = 10,$ average of velocity $\bar{u} = \bar{v} = -0.025$

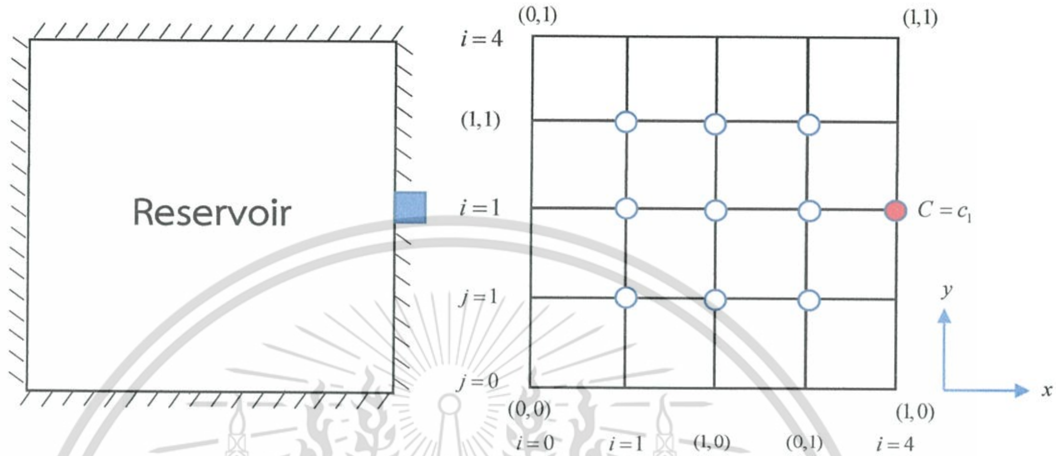


Figure 3.9 Generating grid points of a close reservoir with plant

Steady state dispersion model

$$\bar{u}\frac{\partial C}{\partial x} + \bar{v}\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right), \tag{3.74}$$

with boundary conditions, pollutant concentration

$$C(x,0) = C(x,1) = C(0,y) = 1,$$

$$C(1,y) = \begin{cases} 1 & ; \ y \neq 0.5 \\ c_1 & ; \ y = 0.5 \end{cases},$$

define $C(x,y) = C_{l,m}$ with $\Delta x = \Delta y = 0.25,\ D = 10,\ c_1 = 5,$ and $\bar{u} = \bar{v} = -0.025$, using central in space finite difference technique for solve Eq.(3.74) by Eqs.(3.53)-(3.57).

$l = 1,\ m = 1$

and apply boundary condition $C_{0,1} = C_{1,0} = 1$

$$C_{1,1} - S_1^C C_{0,1} - S_2^C C_{1,0} - S_3^C C_{2,1} - S_4^C C_{1,2} = 0.$$

$l = 2,\ m = 1$

and apply boundary condition $C_{2,0} = 1$

$$C_{2,1} - S_1^C C_{1,1} - S_2^C C_{2,0} - S_3^C C_{3,1} - S_4^C C_{2,2} = 0.$$

$l=3,\ m=1$

and apply boundary condition $C_{3,0}=C_{4,1}=1$

$$C_{3,1}-S_1^C C_{2,1}-S_2^C C_{3,0}-S_3^C C_{4,1}-S_4^C C_{3,2}=0.$$

$l=1,\ m=2$

and apply boundary condition $C_{0,2}=1$

$$C_{1,2}-S_1^C C_{0,2}-S_2^C C_{1,1}-S_3^C C_{2,2}-S_4^C C_{1,3}=0.$$

$l=2,\ m=2$

$$C_{2,2}-S_1^C C_{1,2}-S_2^C C_{2,1}-S_3^C C_{3,2}-S_4^C C_{2,3}=0.$$

$l=3,\ m=2$

and apply boundary condition $C_{4,2}=5$

$$C_{3,2}-S_1^C C_{2,2}-S_2^C C_{3,1}-S_3^C C_{4,2}-S_4^C C_{3,3}=0.$$

$l=1,\ m=3$

and apply boundary condition $C_{0,3}=C_{1,4}=1$

$$C_{1,3}-S_1^C C_{0,3}-S_2^C C_{1,2}-S_3^C C_{2,3}-S_4^C C_{1,4}=0.$$

$l=2,\ m=3$

and apply boundary condition $C_{2,4}=1$

$$C_{2,3}-S_1^C C_{1,3}-S_2^C C_{2,2}-S_3^C C_{3,3}-S_4^C C_{2,4}=0.$$

$l=3,\ m=3$

and apply boundary condition $C_{3,4}=C_{4,3}=1$

$$C_{3,3}-S_1^C C_{2,3}-S_2^C C_{3,2}-S_3^C C_{4,3}-S_4^C C_{3,4}=0.$$

Equation system of unknown nodes are

$$C_{1,1}-S_3^C C_{2,1}-S_4^C C_{1,2}=S_1^C+S_2^C, \tag{3.75}$$

$$C_{2,1}-S_1^C C_{1,1}-S_3^C C_{3,1}-S_4^C C_{2,2}=S_2^C, \tag{3.76}$$

$$C_{3,1} - S_1^C C_{2,1} - S_4^C C_{3,2} = S_2^C + S_3^C, \tag{3.77}$$

$$C_{1,2} - S_2^C C_{1,1} - S_3^C C_{2,2} - S_4^C C_{1,3} = S_1^C, \tag{3.78}$$

$$C_{2,2} - S_1^C C_{1,2} - S_2^C C_{2,1} - S_3^C C_{3,2} - S_4^C C_{2,3} = 0, \tag{3.79}$$

$$C_{3,2} - S_1^C C_{2,2} - S_2^C C_{3,1} - S_4^C C_{3,3} = 5S_3^C, \tag{3.80}$$

$$C_{1,3} - S_2^C C_{1,2} - S_3^C C_{2,3} = S_1^C + S_4^C, \tag{3.81}$$

$$C_{2,3} - S_1^C C_{1,3} - S_2^C C_{2,2} - S_3^C C_{3,3} = S_4^C, \tag{3.82}$$

$$C_{3,3} - S_1^C C_{2,3} - S_2^C C_{3,2} = S_3^C + S_4^C. \tag{3.83}$$

We have matrix form of Eqs.(3.75)-(3.83),

$$\begin{bmatrix} 1 & -S_3^C & 0 & -S_4^C & 0 & 0 & 0 & 0 & 0 \\ -S_1^C & 1 & -S_3^C & 0 & -S_4^C & 0 & 0 & 0 & 0 \\ 0 & -S_1^C & 1 & 0 & 0 & -S_4^C & 0 & 0 & 0 \\ -S_2^C & 0 & 0 & 1 & -S_3^C & 0 & -S_4^C & 0 & 0 \\ 0 & -S_2^C & 0 & -S_1^C & 1 & -S_3^C & 0 & -S_4^C & 0 \\ 0 & 0 & -S_2^C & 0 & -S_1^C & 1 & 0 & 0 & -S_4^C \\ 0 & 0 & 0 & -S_2^C & 0 & 0 & 1 & -S_3^C & 0 \\ 0 & 0 & 0 & 0 & -S_2^C & 0 & -S_1^C & 1 & -S_3^C \\ 0 & 0 & 0 & 0 & 0 & -S_2^C & 0 & -S_1^C & 1 \end{bmatrix} \begin{Bmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \\ C_{1,2} \\ C_{2,2} \\ C_{3,2} \\ C_{1,3} \\ C_{2,3} \\ C_{3,3} \end{Bmatrix} = \begin{Bmatrix} S_1^C + S_2^C \\ S_2^C \\ S_2^C + S_3^C \\ S_1^C \\ 0 \\ 5S_3^C \\ S_1^C + S_4^C \\ S_4^C \\ S_3^C + S_4^C \end{Bmatrix} \tag{3.84}$$

where

$$S_1^C = \left[\frac{\bar{u}}{2\Delta x} + \frac{D}{\Delta x^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right] = \left[\frac{-0.025}{2(0.25)} + \frac{10}{0.25^2}\right] \Big/ \left[\frac{2(10)}{0.25^2} + \frac{2(10)}{0.25^2}\right] = 0.2499,$$

$$S_2^C = \left[\frac{\bar{v}}{2\Delta y} + \frac{D}{\Delta y^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right] = \left[\frac{-0.025}{2(0.25)} + \frac{10}{0.25^2}\right] \Big/ \left[\frac{2(10)}{0.25^2} + \frac{2(10)}{0.25^2}\right] = 0.2499,$$

$$S_3^C = \left[-\frac{\bar{u}}{2\Delta x} + \frac{D}{\Delta x^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right] = \left[\frac{0.025}{2(0.25)} + \frac{10}{0.25^2}\right] \Big/ \left[\frac{2(10)}{0.25^2} + \frac{2(10)}{0.25^2}\right] = 0.2501,$$

$$S_4^C = \left[-\frac{\bar{v}}{2\Delta y} + \frac{D}{\Delta y^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right] = \left[\frac{0.025}{2(0.25)} + \frac{10}{0.25^2}\right] \Big/ \left[\frac{2(10)}{0.25^2} + \frac{2(10)}{0.25^2}\right] = 0.2501.$$

Solving matrix (3.84), we have

$$\begin{Bmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \\ C_{1,2} \\ C_{2,2} \\ C_{3,2} \\ C_{1,3} \\ C_{2,3} \\ C_{3,3} \end{Bmatrix} = \begin{Bmatrix} 1.0727 \\ 1.1927 \\ 1.3583 \\ 1.0986 \\ 1.3405 \\ 2.2413 \\ 0.9820 \\ 0.8304 \\ 1.2676 \end{Bmatrix} \quad (3.85)$$

Example 3.5.4. To find the pollutant concentration in closed reservoir, at boundary of reservoir that has the equal of pollutant concentration to 1 with inflow point $C = c_1$ and rate of change of pollutant concentration with respect to y at the open gate $\frac{\partial C}{\partial y} = c_2$ when unchanging in time, rectangular domain $\Omega = (0,1)\times(0,1)$ in Figure 3.10 with step size $\Delta x = \Delta y = 0.25$ and diffusion coefficient $D = 10$, average of velocity $\bar{u} = \bar{v} = -0.025$

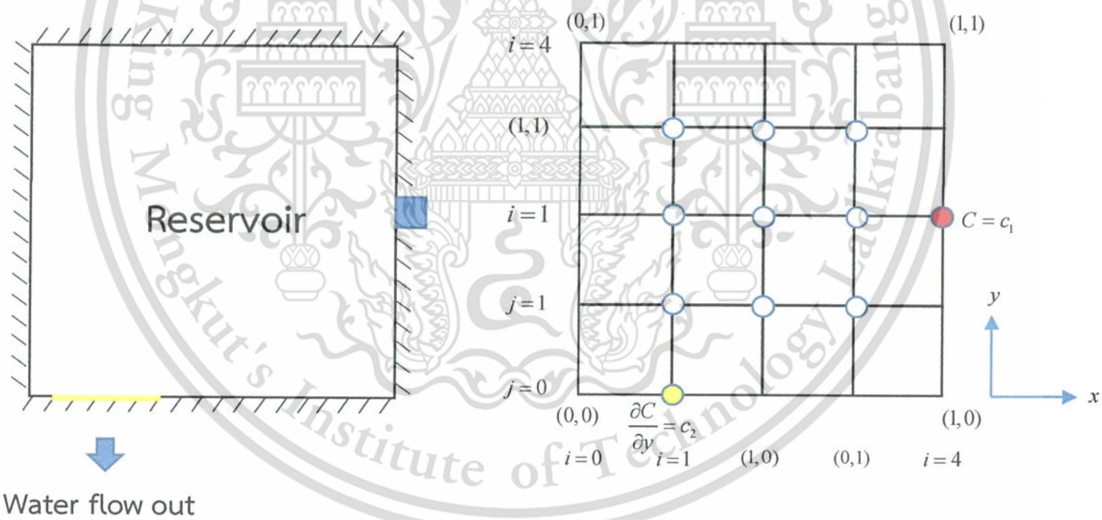


Figure 3.10 Generating grid points of an open reservoir (1) with plant

Steady state dispersion model

$$\bar{u}\frac{\partial C}{\partial x} + \bar{v}\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right), \quad (3.86)$$

with boundary conditions, pollutant concentration $C(x,0) = C(x,1) = 1$,

$$C(1,y)=\begin{cases}1 & ;\ y\neq 0.5\\ c_1 & ;\ y=0.5\end{cases},$$
$$C(x,0)=1\,;x\neq 0.25\,,$$
$$C_y(x,0)=c_2\,;x=0.25\,,$$

define $C(x,y)=C_{l,m}$ with $\Delta x=\Delta y=0.25$, $D=10$, $c_1=5$, and $\bar{u}=\bar{v}=-0.025$, using central in space finite difference technique for solve Eq.(3.86) by Eqs.(3.53)-(3.57), approximate differential boundary condition,

$C_y(0.25,0)=c_2$ or $\dfrac{\partial C_{1,0}}{\partial y}=c_2$,

using forward difference,

$$\frac{C_{1,1}-C_{1,0}}{\Delta y}=c_2\,,$$
$$C_{1,0}=C_{1,1}-c_2\Delta y\,, \tag{3.87}$$

$l=1,\ m=1$

and apply boundary condition $C_{0,1}=1,\ C_{1,0}=C_{1,1}-c_2\Delta y$

$C_{1,1}-S_1^C C_{0,1}-S_2^C C_{1,0}-S_3^C C_{2,1}-S_4^C C_{1,2}=0.$

$l=2,\ m=1$

and apply boundary condition $C_{2,0}=1$

$C_{2,1}-S_1^C C_{1,1}-S_2^C C_{2,0}-S_3^C C_{3,1}-S_4^C C_{2,2}=0.$

$l=3,\ m=1$

and apply boundary condition $C_{3,0}=C_{4,1}=1$

$C_{3,1}-S_1^C C_{2,1}-S_2^C C_{3,0}-S_3^C C_{4,1}-S_4^C C_{3,2}=0.$

$l=1,\ m=2$

and apply boundary condition $C_{0,2}=1$

$C_{1,2}-S_1^C C_{0,2}-S_2^C C_{1,1}-S_3^C C_{2,2}-S_4^C C_{1,3}=0.$

$l=2,\ m=2$

$C_{2,2}-S_1^C C_{1,2}-S_2^C C_{2,1}-S_3^C C_{3,2}-S_4^C C_{2,3}=0.$

$l=3,\ m=2$

and apply boundary condition $C_{4,2}=5$

$$C_{3,2}-S_1^C C_{2,2}-S_2^C C_{3,1}-S_3^C C_{4,2}-S_4^C C_{3,3}=0.$$

$l=1,\ m=3$

and apply boundary condition $C_{0,3}=C_{1,4}=1$

$$C_{1,3}-S_1^C C_{0,3}-S_2^C C_{1,2}-S_3^C C_{2,3}-S_4^C C_{1,4}=0.$$

$l=2,\ m=3$

and apply boundary condition $C_{2,4}=1$

$$C_{2,3}-S_1^C C_{1,3}-S_2^C C_{2,2}-S_3^C C_{3,3}-S_4^C C_{2,4}=0.$$

$l=3,\ m=3$

and apply boundary condition $C_{3,4}=C_{4,3}=1$

$$C_{3,3}-S_1^C C_{2,3}-S_2^C C_{3,2}-S_3^C C_{4,3}-S_4^C C_{3,4}=0.$$

Equation system of unknown nodes are

$$(1-S_2^C)C_{1,1}-S_3^C C_{2,1}-S_4^C C_{1,2}=S_1^C-c_2\Delta y S_2^C, \tag{3.88}$$

$$C_{2,1}-S_1^C C_{1,1}-S_3^C C_{3,1}-S_4^C C_{2,2}=S_2^C, \tag{3.89}$$

$$C_{3,1}-S_1^C C_{2,1}-S_4^C C_{3,2}=S_2^C+S_3^C, \tag{3.90}$$

$$C_{1,2}-S_2^C C_{1,1}-S_3^C C_{2,2}-S_4^C C_{1,3}=S_1^C, \tag{3.91}$$

$$C_{2,2}-S_1^C C_{1,2}-S_2^C C_{2,1}-S_3^C C_{3,2}-S_4^C C_{2,3}=0, \tag{3.92}$$

$$C_{3,2}-S_1^C C_{2,2}-S_2^C C_{3,1}-S_4^C C_{3,3}=5S_3^C, \tag{3.93}$$

$$C_{1,3}-S_2^C C_{1,2}-S_3^C C_{2,3}=S_1^C+S_4^C, \tag{3.94}$$

$$C_{2,3}-S_1^C C_{1,3}-S_2^C C_{2,2}-S_3^C C_{3,3}=S_4^C, \tag{3.95}$$

$$C_{3,3}-S_1^C C_{2,3}-S_2^C C_{3,2}=S_3^C+S_4^C. \tag{3.96}$$

We have matrix form of Eqs.(3.88)-(3.96)

$$\begin{bmatrix} 1-S_2^C & -S_3^C & 0 & -S_4^C & 0 & 0 & 0 & 0 & 0 \\ -S_1^C & 1 & -S_3^C & 0 & -S_4^C & 0 & 0 & 0 & 0 \\ 0 & -S_1^C & 1 & 0 & 0 & -S_4^C & 0 & 0 & 0 \\ -S_2^C & 0 & 0 & 1 & -S_3^C & 0 & -S_4^C & 0 & 0 \\ 0 & -S_2^C & 0 & -S_1^C & 1 & -S_3^C & 0 & -S_4^C & 0 \\ 0 & 0 & -S_2^C & 0 & -S_1^C & 1 & 0 & 0 & -S_4^C \\ 0 & 0 & 0 & -S_2^C & 0 & 0 & 1 & -S_3^C & 0 \\ 0 & 0 & 0 & 0 & -S_2^C & 0 & -S_1^C & 1 & -S_3^C \\ 0 & 0 & 0 & 0 & 0 & -S_2^C & 0 & -S_1^C & 1 \end{bmatrix} \begin{Bmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \\ C_{1,2} \\ C_{2,2} \\ C_{3,2} \\ C_{1,3} \\ C_{2,3} \\ C_{3,3} \end{Bmatrix} = \begin{Bmatrix} S_1^C - c_2 \Delta y S_2^C \\ S_2^C \\ S_2^C + S_3^C \\ S_1^C \\ 0 \\ 5S_3^C \\ S_1^C + S_4^C \\ S_4^C \\ S_3^C + S_4^C \end{Bmatrix} \tag{3.97}$$

where

$$S_1^C = \left[\frac{\bar{u}}{2\Delta x} + \frac{D}{\Delta x^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right] = \left[\frac{-0.025}{2(0.25)} + \frac{10}{0.25^2}\right] \Big/ \left[\frac{2(10)}{0.25^2} + \frac{2(10)}{0.25^2}\right] = 0.2499,$$

$$S_2^C = \left[\frac{\bar{v}}{2\Delta y} + \frac{D}{\Delta y^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right] = \left[\frac{-0.025}{2(0.25)} + \frac{10}{0.25^2}\right] \Big/ \left[\frac{2(10)}{0.25^2} + \frac{2(10)}{0.25^2}\right] = 0.2499,$$

$$S_3^C = \left[-\frac{\bar{u}}{2\Delta x} + \frac{D}{\Delta x^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right] = \left[\frac{0.025}{2(0.25)} + \frac{10}{0.25^2}\right] \Big/ \left[\frac{2(10)}{0.25^2} + \frac{2(10)}{0.25^2}\right] = 0.2501,$$

$$S_4^C = \left[-\frac{\bar{v}}{2\Delta y} + \frac{D}{\Delta y^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right] = \left[\frac{0.025}{2(0.25)} + \frac{10}{0.25^2}\right] \Big/ \left[\frac{2(10)}{0.25^2} + \frac{2(10)}{0.25^2}\right] = 0.2501.$$

Solving matrix (3.97), we have

Table 3.6 The calculated Pollutant concentration of example 3.5.4

$C_{l,m}$	$c_2 = 0$	$c_2 = -1$	$c_2 = 1$
$C_{1,1}$	1.1520	1.0454	1.2586
$C_{2,1}$	1.2641	1.2291	1.2991
$C_{3,1}$	1.3969	1.3858	1.4081
$C_{1,2}$	1.1927	1.1577	1.2277
$C_{2,2}$	1.5084	1.4861	1.5306
$C_{3,2}$	2.3246	2.3150	2.3341
$C_{1,3}$	1.1112	1.1001	1.1224
$C_{2,3}$	1.2532	1.2436	1.2627
$C_{3,3}$	1.3941	1.3893	1.3988

Example 3.5.5. To find the pollutant concentration in an open reservoir at boundary of reservoir that has the equal of pollutant concentration to 1 with inflow point $C = c_1$ and rate of change of pollutant concentration with respect to y at the open gate $\frac{\partial C}{\partial y} = c_2$ when unchanging in time, rectangular domain $\Omega = (0,1)\times(0,1)$ in Figure 3.11 with step size $\Delta x = \Delta y = 0.25$ and diffusion coefficient $D = 10$, average of velocity $\bar{u} = \bar{v} = -0.025$.

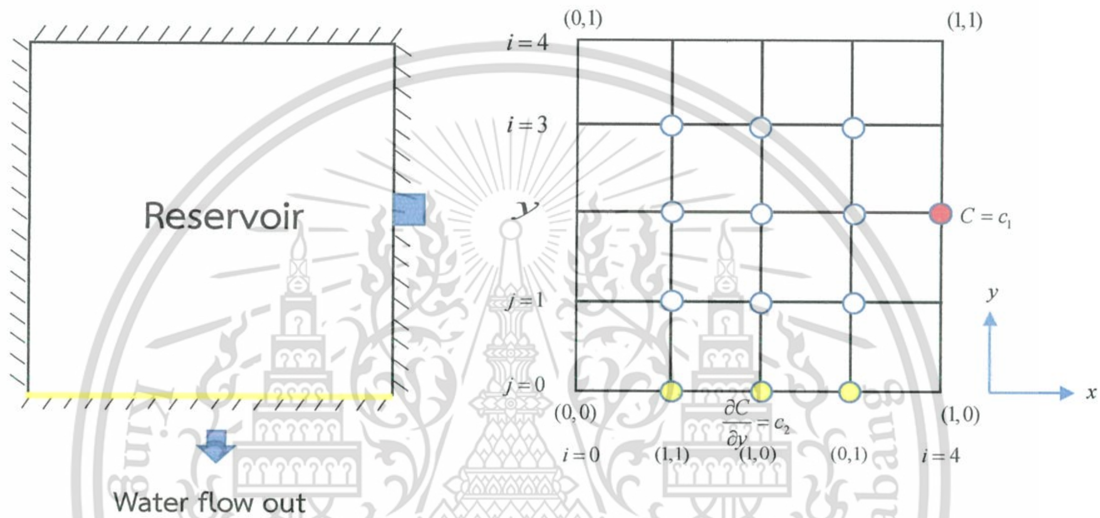


Figure 3.11 Generating grid points of an open reservoir (2) with plant

Steady state dispersion model

$$\bar{u}\frac{\partial C}{\partial x} + \bar{v}\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right), \tag{3.98}$$

with boundary conditions, pollutant concentration

$$C(x,0) = C(x,1) = 1,$$

$$C(1,y) = \begin{cases} 1 & ; \; y \neq 0.5 \\ c_1 & ; \; y = 0.5 \end{cases},$$

$$C_y(x,0) = c_2 \;,$$

define $C(x,y) = C_{l,m}$ with $\Delta x = \Delta y = 0.25$, $D = 10$, $c_1 = 5$, and $\bar{u} = \bar{v} = -0.025$, using central in space finite difference technique for solve Eq.(3.98) by Eqs.(3.53)-(3.57), approximate differential boundary condition, $C_y(x,0) = c_2$ or $\frac{\partial C_{l,0}}{\partial y} = c_2$, using forward difference,

$$\frac{C_{l,1}-C_{l,0}}{\Delta y}=c_2\,,$$

$$C_{l,0}=C_{l,1}-c_2\Delta y, \qquad l=1,2,3 \tag{3.99}$$

$l=1,\ m=1$

and apply boundary condition $C_{0,1}=1,\ C_{1,0}=C_{1,1}-c_2\Delta y$

$C_{1,1}-S_1^C C_{0,1}-S_2^C C_{1,0}-S_3^C C_{2,1}-S_4^C C_{1,2}=0.$

$l=2,\ m=1$

and apply boundary condition $C_{2,0}=C_{2,1}-c_2\Delta y$

$C_{2,1}-S_1^C C_{1,1}-S_2^C C_{2,0}-S_3^C C_{3,1}-S_4^C C_{2,2}=0.$

$l=3,\ m=1$

and apply boundary condition $C_{4,1}=1,\ C_{3,0}=C_{3,1}-c_2\Delta y$

$C_{3,1}-S_1^C C_{2,1}-S_2^C C_{3,0}-S_3^C C_{4,1}-S_4^C C_{3,2}=0.$

$l=1,\ m=2$

and apply boundary condition $C_{0,2}=1$

$C_{1,2}-S_1^C C_{0,2}-S_2^C C_{1,1}-S_3^C C_{2,2}-S_4^C C_{1,3}=0.$

$l=2,\ m=2$

$C_{2,2}-S_1^C C_{1,2}-S_2^C C_{2,1}-S_3^C C_{3,2}-S_4^C C_{2,3}=0.$

$l=3,\ m=2$

And apply boundary condition $C_{4,2}=5$

$C_{3,2}-S_1^C C_{2,2}-S_2^C C_{3,1}-S_3^C C_{4,2}-S_4^C C_{3,3}=0.$

$l=1,\ m=3$

and apply boundary condition $C_{0,3}=C_{1,4}=1$

$C_{1,3}-S_1^C C_{0,3}-S_2^C C_{1,2}-S_3^C C_{2,3}-S_4^C C_{1,4}=0.$

$l=2,\ m=3$

and apply boundary condition $C_{2,4}=1$

$$C_{2,3}-S_1^C C_{1,3}-S_2^C C_{2,2}-S_3^C C_{3,3}-S_4^C C_{2,4}=0.$$

$l=3,\ m=3$

and apply boundary condition $C_{3,4}=C_{4,3}=1$

$$C_{3,3}-S_1^C C_{2,3}-S_2^C C_{3,2}-S_3^C C_{4,3}-S_4^C C_{3,4}=0.$$

Equation system of unknown nodes are

$$(1-S_2^C)C_{1,1}-S_3^C C_{2,1}-S_4^C C_{1,2}=S_1^C-c_2\Delta y S_2^C, \quad (3.100)$$

$$(1-S_2^C)C_{2,1}-S_1^C C_{1,1}-S_3^C C_{3,1}-S_4^C C_{2,2}=-c_2\Delta y S_2^C, \quad (3.101)$$

$$(1-S_2^C)C_{3,1}-S_1^C C_{2,1}-S_4^C C_{3,2}=-c_2\Delta y S_2^C+S_3^C, \quad (3.102)$$

$$C_{1,2}-S_2^C C_{1,1}-S_3^C C_{2,2}-S_4^C C_{1,3}=S_1^C, \quad (3.103)$$

$$C_{2,2}-S_1^C C_{1,2}-S_2^C C_{2,1}-S_3^C C_{3,2}-S_4^C C_{2,3}=0, \quad (3.104)$$

$$C_{3,2}-S_1^C C_{2,2}-S_2^C C_{3,1}-S_4^C C_{3,3}=5S_3^C, \quad (3.105)$$

$$C_{1,3}-S_2^C C_{1,2}-S_3^C C_{2,3}=S_1^C+S_4^C, \quad (3.106)$$

$$C_{2,3}-S_1^C C_{1,3}-S_2^C C_{2,2}-S_3^C C_{3,3}=S_4^C, \quad (3.107)$$

$$C_{3,3}-S_1^C C_{2,3}-S_2^C C_{3,2}=S_3^C+S_4^C. \quad (3.108)$$

We have matrix form of Eqs.(3.100)-(3.108)

$$\begin{bmatrix} 1-S_2^C & -S_3^C & 0 & -S_4^C & 0 & 0 & 0 & 0 & 0 \\ -S_1^C & 1-S_2^C & -S_3^C & 0 & -S_4^C & 0 & 0 & 0 & 0 \\ 0 & -S_1^C & 1-S_2^C & 0 & 0 & -S_4^C & 0 & 0 & 0 \\ -S_2^C & 0 & 0 & 1 & -S_3^C & 0 & -S_4^C & 0 & 0 \\ 0 & -S_2^C & 0 & -S_1^C & 1 & -S_3^C & 0 & -S_4^C & 0 \\ 0 & 0 & -S_2^C & 0 & -S_1^C & 1 & 0 & 0 & -S_4^C \\ 0 & 0 & 0 & -S_2^C & 0 & 0 & 1 & -S_3^C & 0 \\ 0 & 0 & 0 & 0 & -S_2^C & 0 & -S_1^C & 1 & -S_3^C \\ 0 & 0 & 0 & 0 & 0 & -S_2^C & 0 & -S_1^C & 1 \end{bmatrix} \begin{Bmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \\ C_{1,2} \\ C_{2,2} \\ C_{3,2} \\ C_{1,3} \\ C_{2,3} \\ C_{3,3} \end{Bmatrix} = \begin{Bmatrix} S_1^C-c_2\Delta y S_2^C \\ -c_2\Delta y S_2^C \\ -c_2\Delta y S_2^C+S_3^C \\ S_1^C \\ 0 \\ 5S_3^C \\ S_1^C+S_4^C \\ S_4^C \\ S_3^C+S_4^C \end{Bmatrix} \quad (3.109)$$

where

$$S_1^C=\left[\frac{\bar{u}}{2\Delta x}+\frac{D}{\Delta x^2}\right]\Bigg/\left[\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right]=\left[\frac{-0.025}{2(0.25)}+\frac{10}{0.25^2}\right]\Bigg/\left[\frac{2(10)}{0.25^2}+\frac{2(10)}{0.25^2}\right]=0.2499,$$

$$S_2^C=\left[\frac{\bar{v}}{2\Delta y}+\frac{D}{\Delta y^2}\right]\Bigg/\left[\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right]=\left[\frac{-0.025}{2(0.25)}+\frac{10}{0.25^2}\right]\Bigg/\left[\frac{2(10)}{0.25^2}+\frac{2(10)}{0.25^2}\right]=0.2499,$$

$$S_3^C=\left[-\frac{\bar{u}}{2\Delta x}+\frac{D}{\Delta x^2}\right]\Bigg/\left[\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right]=\left[\frac{0.025}{2(0.25)}+\frac{10}{0.25^2}\right]\Bigg/\left[\frac{2(10)}{0.25^2}+\frac{2(10)}{0.25^2}\right]=0.2501,$$

$$S_4^C=\left[-\frac{\bar{v}}{2\Delta y}+\frac{D}{\Delta y^2}\right]\Bigg/\left[\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right]=\left[\frac{0.025}{2(0.25)}+\frac{10}{0.25^2}\right]\Bigg/\left[\frac{2(10)}{0.25^2}+\frac{2(10)}{0.25^2}\right]=0.2501.$$

Solving matrix (3.100), we have

Table 3.7 The calculated pollutant concentration of example 3.5.5

$C_{l,m}$	$c_2=0$	$c_2=-1$	$c_2=1$
$C_{1,1}$	1.2509	1.0538	1.4480
$C_{2,1}$	1.5027	1.2485	1.7569
$C_{3,1}$	1.6410	1.4439	1.8380
$C_{1,2}$	1.2509	1.6939	1.3382
$C_{2,2}$	1.6173	1.4986	1.7361
$C_{3,2}$	2.4215	2.3342	2.5087
$C_{1,3}$	1.1363	1.1029	1.1698
$C_{2,3}$	1.2954	1.2490	1.3417
$C_{3,3}$	1.4288	1.3954	1.4622

Example 3.5.6. To find the pollutant concentration in a close reservoir at boundary of reservoir $\frac{\partial C}{\partial n}=0$ (non-absorbing) with inflow point $C=c_1$ when unchanging in time, rectangular domain $\Omega=(0,1)\times(0,1)$, with step size $\Delta x=\Delta y=0.25$ and diffusion coefficient $D=10$, average of velocity $\bar{u}=\bar{v}=-0.025$.

Steady state dispersion model

$$\bar{u}\frac{\partial C}{\partial x}+\bar{v}\frac{\partial C}{\partial y}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right), \tag{3.110}$$

with initial conditions, pollutant concentration at initial time $C(x,y,0)=0.02$, boundary conditions, pollutant concentration $C_x(x,y,0)=C_y(x,0,t)=C_y(x,1,t)=0$, $C_x(1,y)=0; y\neq 0.5$, $C(1,y)=c_1; y=0.5$, define $C(x,y,t)=C_{l,m}^n$ in Figure 3.5.3 with $\Delta t=0.01$, $\Delta x=0.25$, and $D=0.1$, approximate differential boundary condition,

$C_y(x,0,t)=0$ or $\dfrac{\partial C_{l,0}^n}{\partial y}=0$,

using forward difference,

$$\frac{C_{l,1}^n-C_{l,0}^n}{\Delta y}=0,$$

$$C_{l,0}^n=C_{l,1}^n, \qquad l=0,1,2,3,4 \qquad (3.111)$$

$C_y(x,1,t)=0$ or $\dfrac{\partial C_{l,4}^n}{\partial y}=0$,

using backward difference,

$$\frac{C_{l,4}^n-C_{l,3}^n}{\Delta y}=0,$$

$$C_{l,4}^n=C_{l,3}^n, \qquad l=0,1,2,3,4 \qquad (3.112)$$

$C_x(0,y,t)=0$ or $\dfrac{\partial C_{0,m}^n}{\partial x}=0$,

using forward difference,

$$\frac{C_{1,m}^n-C_{0,m}^n}{\Delta x}=0,$$

$$C_{0,m}^n=C_{1,m}^n, \qquad m=0,1,2,3,4 \qquad (3.113)$$

$C_x(1,y,t)=0$ or $\dfrac{\partial C_{4,m}^n}{\partial x}=0$,

using backward difference,

$$\frac{C_{4,m}^n-C_{3,m}^n}{\Delta x}=0,$$

$$C_{4,m}^n=C_{3,m}^n, \qquad m=0,1,2,3,4 \qquad (3.114)$$

$l=1, m=1$

and apply boundary condition $C_{0,1}=C_{1,1}, C_{1,0}=C_{1,1}$

$C_{1,1}-S_1^C C_{0,1}-S_2^C C_{1,0}-S_3^C C_{2,1}-S_4^C C_{1,2}=0.$

$l=2, m=1$

and apply boundary condition $C_{2,0}=C_{2,1}$

$$C_{2,1}-S_1^C C_{1,1}-S_2^C C_{2,0}-S_3^C C_{3,1}-S_4^C C_{2,2}=0.$$

$l=3, m=1$

and apply boundary condition $C_{4,1}=C_{3,1}, C_{3,0}=C_{3,1}$

$$C_{3,1}-S_1^C C_{2,1}-S_2^C C_{3,0}-S_3^C C_{4,1}-S_4^C C_{3,2}=0.$$

$l=1, m=2$

and apply boundary condition $C_{0,2}=C_{1,2}$

$$C_{1,2}-S_1^C C_{0,2}-S_2^C C_{1,1}-S_3^C C_{2,2}-S_4^C C_{1,3}=0.$$

$l=2, m=2$

$$C_{2,2}-S_1^C C_{1,2}-S_2^C C_{2,1}-S_3^C C_{3,2}-S_4^C C_{2,3}=0.$$

$l=3, m=2$

and apply boundary condition $C_{4,2}=5$

$$C_{3,2}-S_1^C C_{2,2}-S_2^C C_{3,1}-S_3^C C_{4,2}-S_4^C C_{3,3}=0.$$

$l=1, m=3$

and apply boundary condition $C_{0,3}=C_{1,3}, C_{1,4}=C_{1,3}$

$$C_{1,3}-S_1^C C_{0,3}-S_2^C C_{1,2}-S_3^C C_{2,3}-S_4^C C_{1,4}=0.$$

$l=2, m=3$

and apply boundary condition $C_{2,4}=C_{2,3}$

$$C_{2,3}-S_1^C C_{1,3}-S_2^C C_{2,2}-S_3^C C_{3,3}-S_4^C C_{2,4}=0.$$

$l=3, m=3$

and apply boundary condition $C_{3,4}=C_{3,3}, C_{4,3}=C_{3,3}$

$$C_{3,3}-S_1^C C_{2,3}-S_2^C C_{3,2}-S_3^C C_{4,3}-S_4^C C_{3,4}=0.$$

Equation system of unknown nodes are

$$(1-S_1^C-S_2^C)C_{1,1}-S_3^C C_{2,1}-S_4^C C_{1,2}=0, \tag{3.115}$$

$$(1-S_2^C)C_{2,1}-S_1^C C_{1,1}-S_3^C C_{3,1}-S_4^C C_{2,2}=0, \tag{3.116}$$

$$(1-S_2^C-S_3^C)C_{3,1}-S_1^C C_{2,1}-S_4^C C_{3,2}=0, \tag{3.117}$$

$$(1-S_1^C)C_{1,2}-S_2^C C_{1,1}-S_3^C C_{2,2}-S_4^C C_{1,3}=0, \tag{3.118}$$

$$C_{2,2}-S_1^C C_{1,2}-S_2^C C_{2,1}-S_3^C C_{3,2}-S_4^C C_{2,3}=0, \tag{3.119}$$

$$C_{3,2}-S_1^C C_{2,2}-S_2^C C_{3,1}-S_4^C C_{3,3}=5S_3^C, \tag{3.120}$$

$$(1-S_1^C-S_4^C)_{1,3}-S_2^C C_{1,2}-S_3^C C_{2,3}=0, \tag{3.121}$$

$$(1-S_4^C)C_{2,3}-S_1^C C_{1,3}-S_2^C C_{2,2}-S_3^C C_{3,3}=0, \tag{3.122}$$

$$(1-S_3^C-S_4^C)C_{3,3}-S_1^C C_{2,3}-S_2^C C_{3,2}=0. \tag{3.123}$$

We have matrix form of Eqs.(3.106)-(3.114)

$$\begin{bmatrix} 1-S_1^C-S_2^C & -S_3^C & 0 & -S_4^C & 0 & 0 & 0 & 0 & 0 \\ -S_1^C & 1-S_2^C & -S_3^C & 0 & -S_4^C & 0 & 0 & 0 & 0 \\ 0 & -S_1^C & 1-S_2^C-S_3^C & 0 & 0 & -S_4^C & 0 & 0 & 0 \\ -S_2^C & 0 & 0 & 1-S_1^C & -S_3^C & 0 & -S_4^C & 0 & 0 \\ 0 & -S_2^C & 0 & -S_1^C & 1 & -S_3^C & 0 & -S_4^C & 0 \\ 0 & 0 & -S_2^C & 0 & -S_1^C & 1 & 0 & 0 & -S_4^C \\ 0 & 0 & 0 & -S_2^C & 0 & 0 & 1-S_1^C-S_4^C & -S_3^C & 0 \\ 0 & 0 & 0 & 0 & -S_2^C & 0 & -S_1^C & 1-S_4^C & -S_3^C \\ 0 & 0 & 0 & 0 & 0 & -S_2^C & 0 & -S_1^C & 1-S_3^C-S_4^C \end{bmatrix} \begin{Bmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \\ C_{1,2} \\ C_{2,2} \\ C_{3,2} \\ C_{1,3} \\ C_{2,3} \\ C_{3,3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5S_3^C \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.124}$$

where

$$S_1^C=\left[\frac{\bar{u}}{2\Delta x}+\frac{D}{\Delta x^2}\right]\Big/\left[\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right]=\left[\frac{-0.025}{2(0.25)}+\frac{10}{0.25^2}\right]\Big/\left[\frac{2(10)}{0.25^2}+\frac{2(10)}{0.25^2}\right]=0.2499,$$

$$S_2^C=\left[\frac{\bar{v}}{2\Delta y}+\frac{D}{\Delta y^2}\right]\Big/\left[\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right]=\left[\frac{-0.025}{2(0.25)}+\frac{10}{0.25^2}\right]\Big/\left[\frac{2(10)}{0.25^2}+\frac{2(10)}{0.25^2}\right]=0.2499,$$

$$S_3^C=\left[-\frac{\bar{u}}{2\Delta x}+\frac{D}{\Delta x^2}\right]\Big/\left[\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right]=\left[\frac{0.025}{2(0.25)}+\frac{10}{0.25^2}\right]\Big/\left[\frac{2(10)}{0.25^2}+\frac{2(10)}{0.25^2}\right]=0.2501,$$

$$S_4^C=\left[-\frac{\bar{v}}{2\Delta y}+\frac{D}{\Delta y^2}\right]\Big/\left[\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right]=\left[\frac{0.025}{2(0.25)}+\frac{10}{0.25^2}\right]\Big/\left[\frac{2(10)}{0.25^2}+\frac{2(10)}{0.25^2}\right]=0.2501.$$

Solving matrix (3.124), we have

$$\begin{Bmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \\ C_{1,2} \\ C_{2,2} \\ C_{3,2} \\ C_{1,3} \\ C_{2,3} \\ C_{3,3} \end{Bmatrix} = \begin{Bmatrix} 4.9461 \\ 4.9495 \\ 4.9549 \\ 4.9465 \\ 4.9515 \\ 4.9643 \\ 4.9461 \\ 4.9495 \\ 4.9549 \end{Bmatrix} \tag{3.125}$$

Chapter 4

Water Quality Measurement In Open-Connected Reservoirs

In this section, The Lax-Wendroff method is subsequently used in a non dimensional form of a shallow water equation to approximate the water velocity and elevation. Next, we use the forward difference in time and backward difference in space in advection-diffusion equation. Combining the equation with the calculated velocity is thus used in the unsteady state of dispersion model to approximate the concentration levels of the pollutants. The result of this section shows that the proposed model can approximate the water velocity, the elevation, and the flow pattern as well as the concentration of the pollutants in open-connected reservoirs and the RAMA 9 reservoir at any various time and position.

4.1 Water quality model in open-connected reservoirs with flat bottom topography

4.1.1 The boundary and initial condition for hydrodynamic model

There is the water flow from two parallel canals through the opened gates such that the water is drained as Figure 4.1(a). The gates are opened, so the water is going to flow into the Rama 9 reservoir. There is no drained water as Figure 4.1(b). The elevation of water on the both gates is assumed to be a wave maker function. It is assumed to be a trigonometry function $f(x,y) = 0.1\sin(\pi(x+t))$. The initial condition in the reservoir is given by the cold start technique i.e. the velocity in x- and y- directions and the elevation are assumed to be motionless.

The boundary conditions are assumed that the elevation is zero except at the both gates. There are no parallel velocities in x- and y- directions along with the boundary of reservoir.

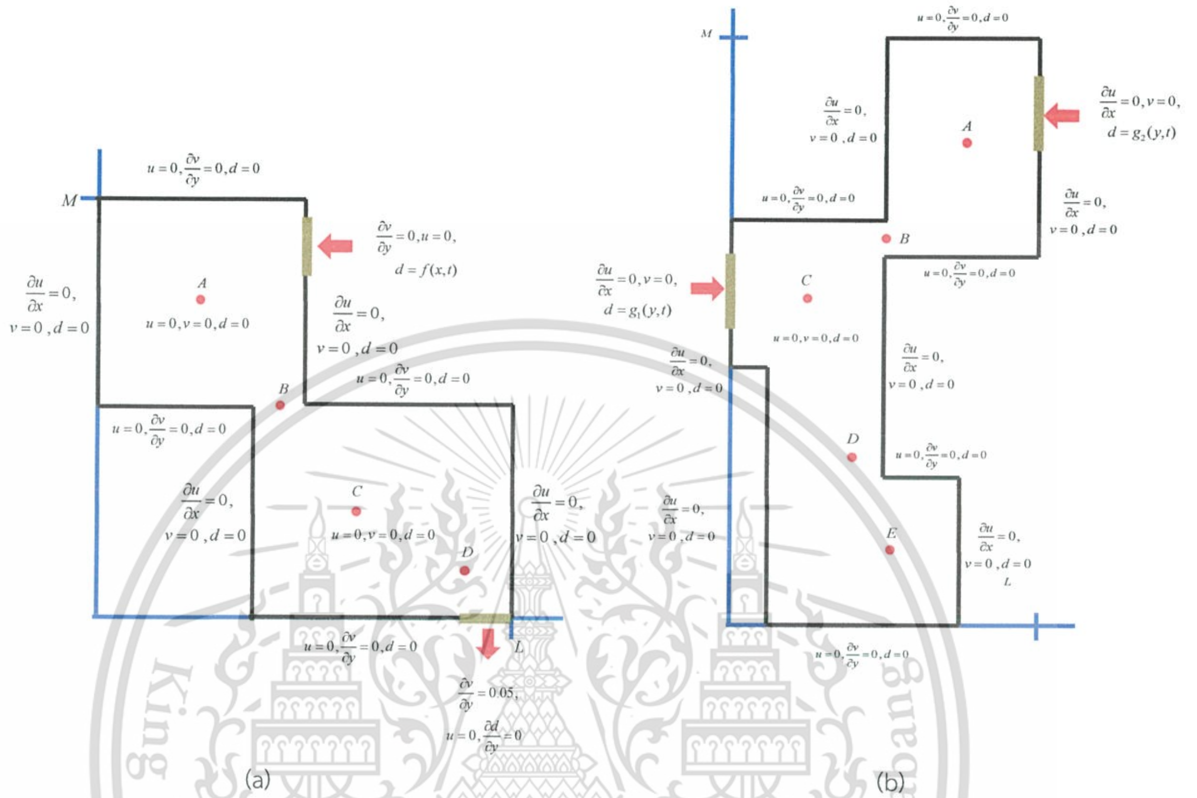
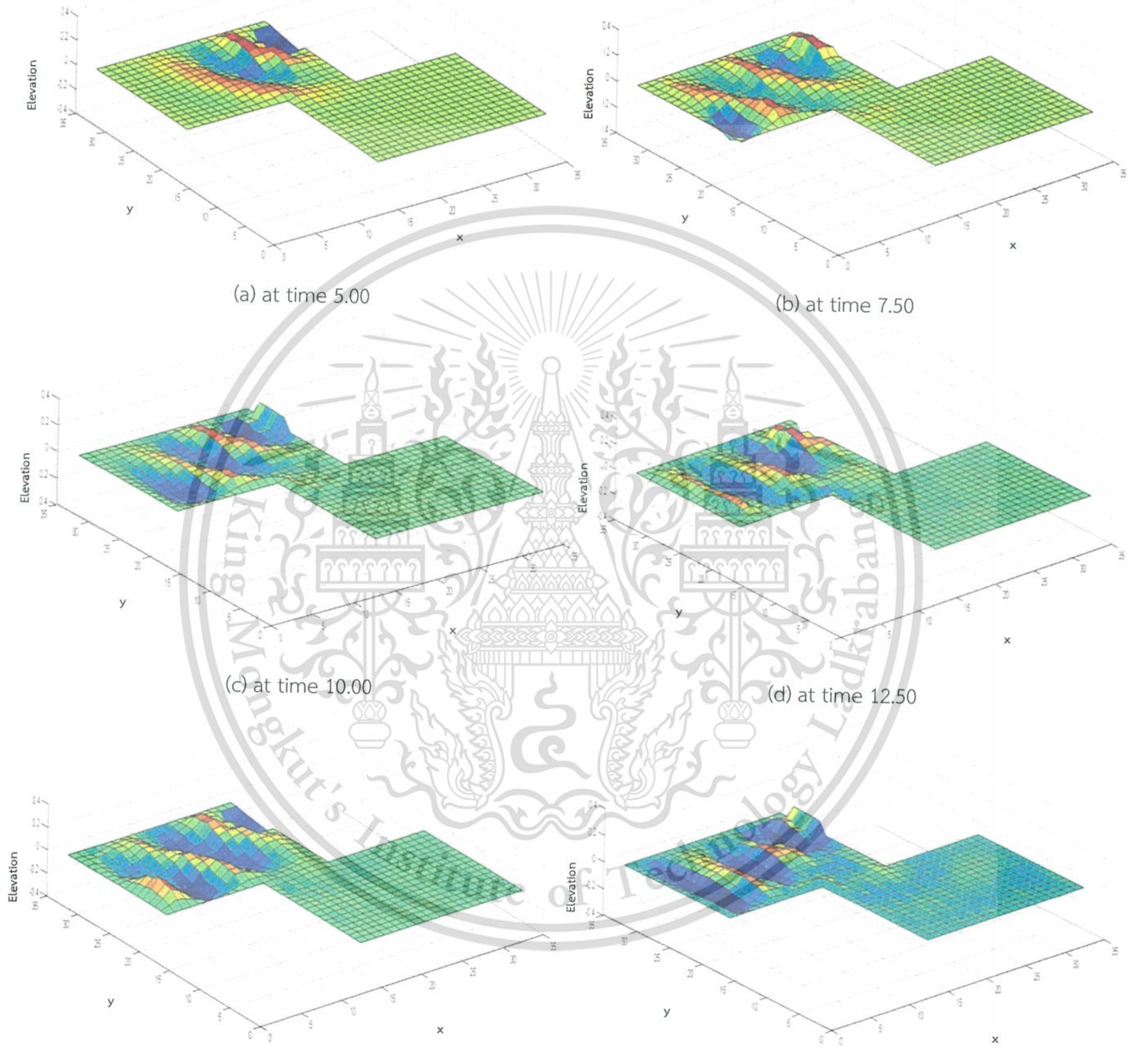


Figure 4.1 The domain problem for the hydrodynamic model of
(a) open-connected reservoirs (b) the RAMA 9 reservoir with observation points

4.1.2 Numerical results of the hydrodynamic model

The hydrodynamic model provides the elevation of water and velocity vector field. The calculated results of the hydrodynamic model are the input of the dispersion model which provides the pollutant concentration field. Firstly, the Lax-Wendroff method is subsequently used in a non dimensional form of a shallow water equation to approximate the water velocity and elevation. Define step size of x and y are 0.03125, step size of time t is 0.01 and T is 30 for open-connected reservoirs. Define step size of x is 0.03125 and y is 0.015625, step size of time t is 0.01 and T is 30 for the RAMA 9 reservoir.

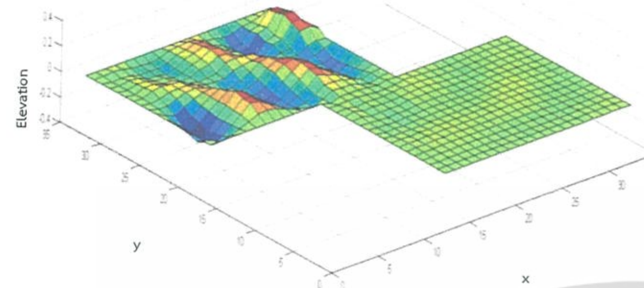


(a) at time 5.00 (b) at time 7.50

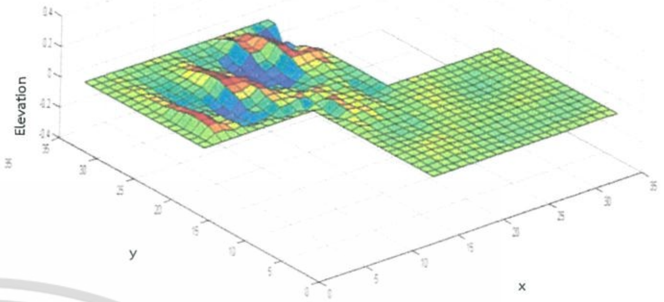
(c) at time 10.00 (d) at time 12.50

(e) at time 15.00 (f) at time 17.50

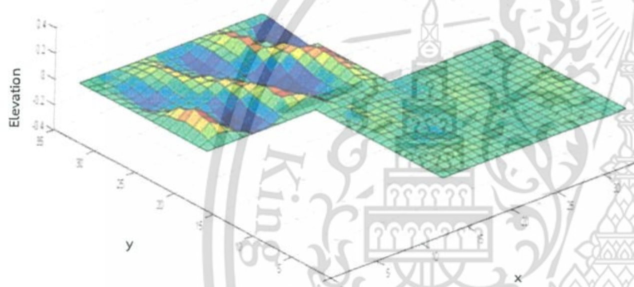
Figure 4.2 Water elevation in open-connected reservoirs measured successively at certain time spots 5.00 to 17.50 or 13.30 (sec) to 46.56 (sec)



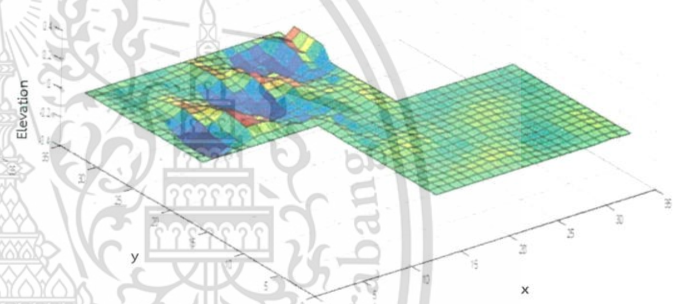
(a) at time 20.00



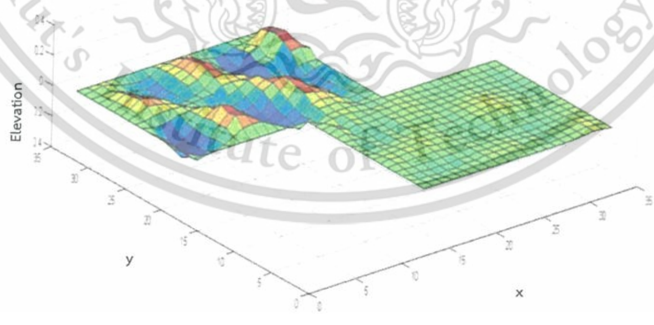
(b) at time 22.50



(c) at time 25.00



(d) at time 27.50



(e) at time 30.00

Figure 4.3 Water elevation in open-connected reservoirs measured successively at certain time spots 20.00 to 30.00 or 53.21 (sec) to 79.82 (sec)

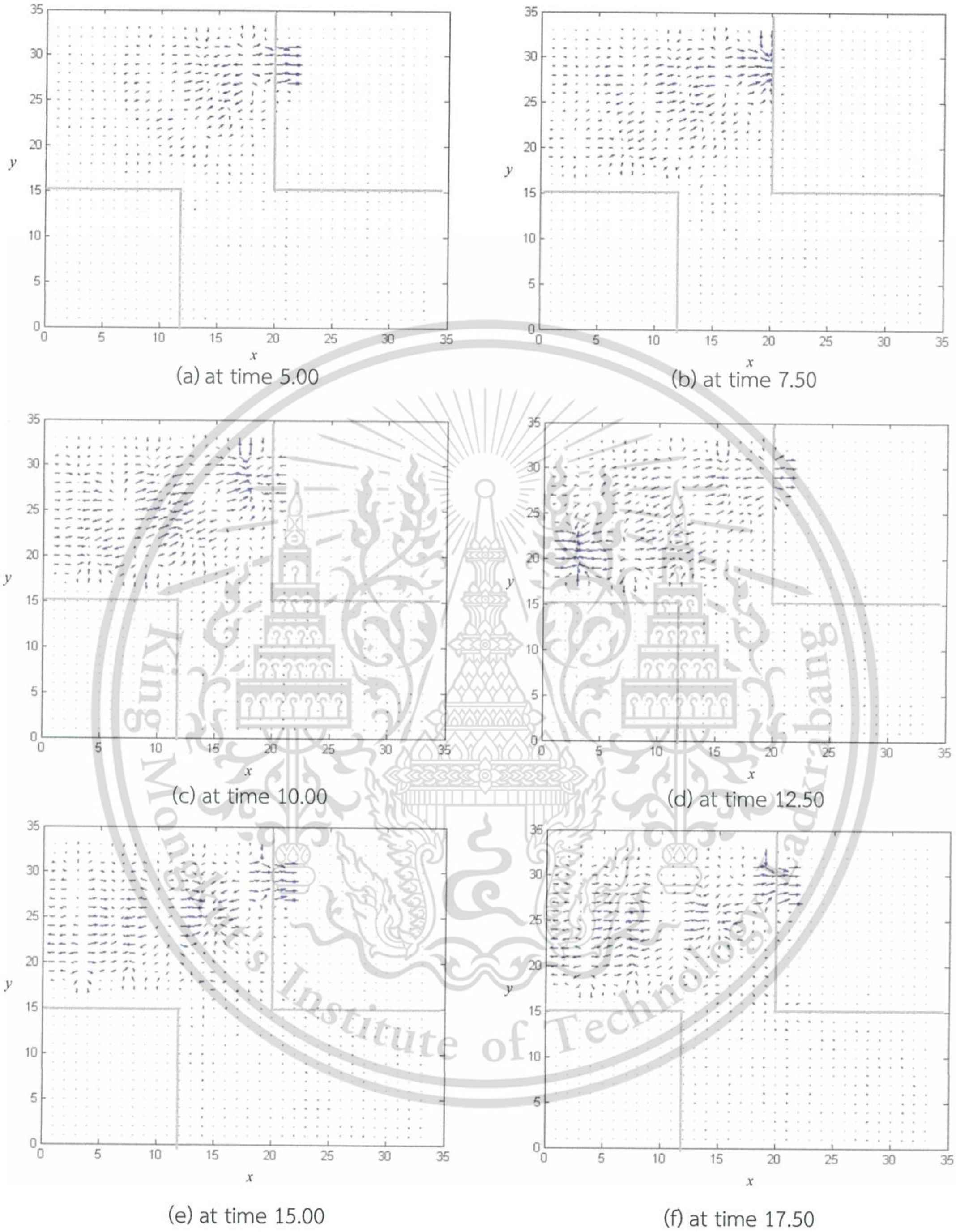
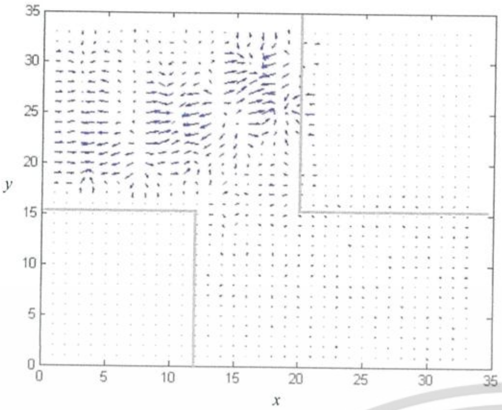
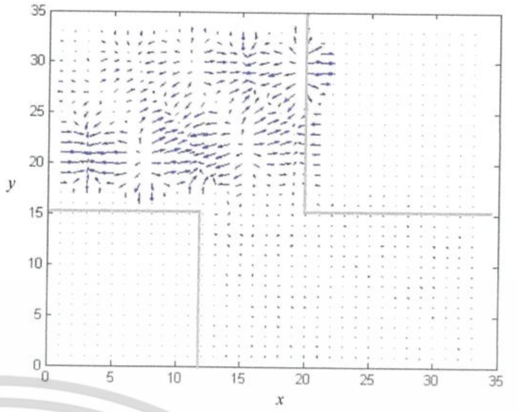


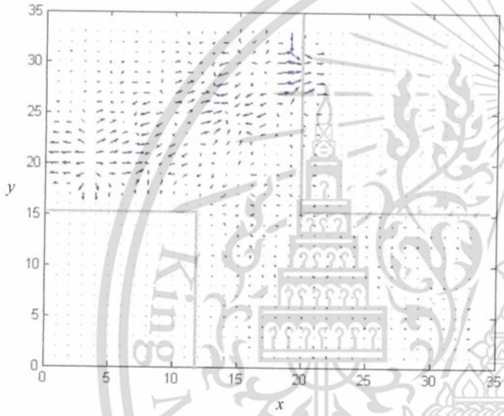
Figure 4.4 Vector field of velocities in open-connected reservoirs measured successively at certain time spots 5.00 to 17.50 or 13.30 (sec) to 46.56 (sec)



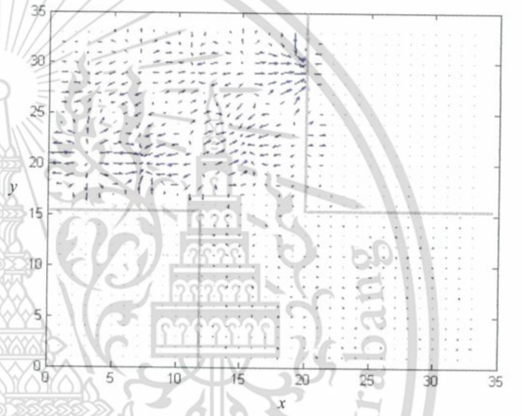
(a) at time 20.00



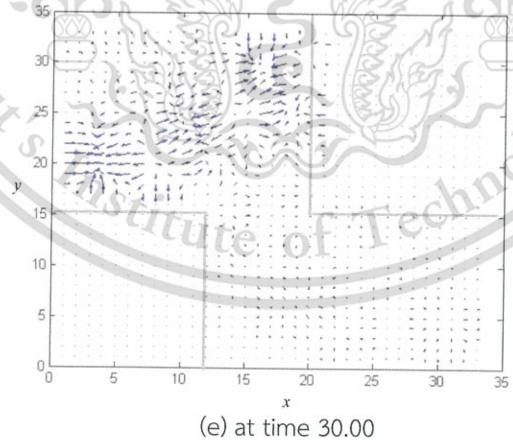
(b) at time 22.50



(c) at time 25.00

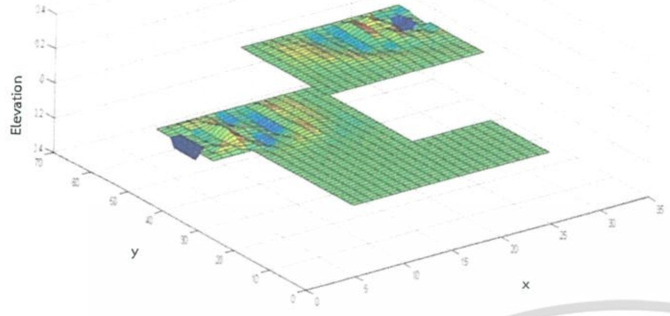


(d) at time 27.50

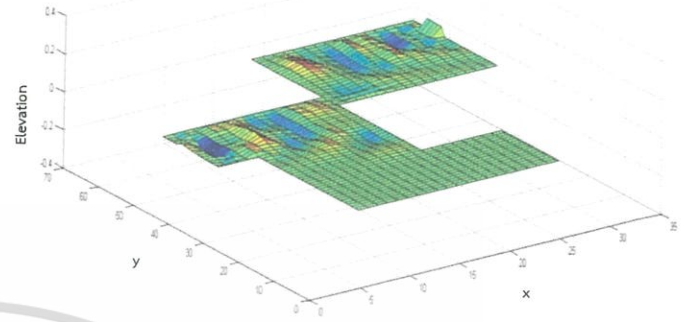


(e) at time 30.00

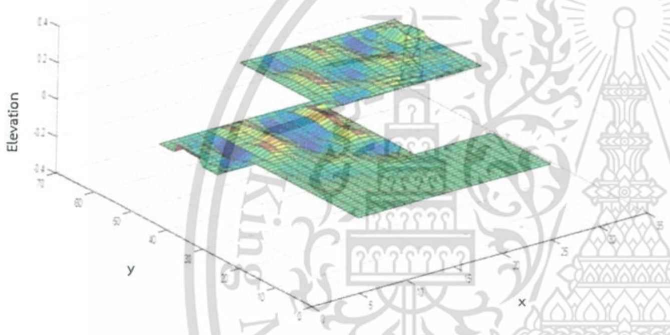
Figure 4.5 Vector field of velocities in open-connected reservoirs measured successively at certain time spots 20.00 to 30.00 or 53.21 (sec) to 79.82 (sec)



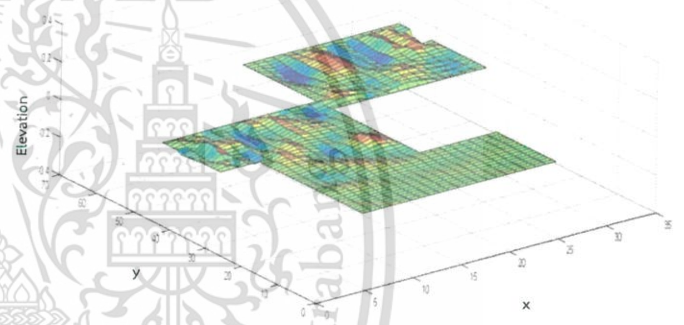
(a) at time 5.00



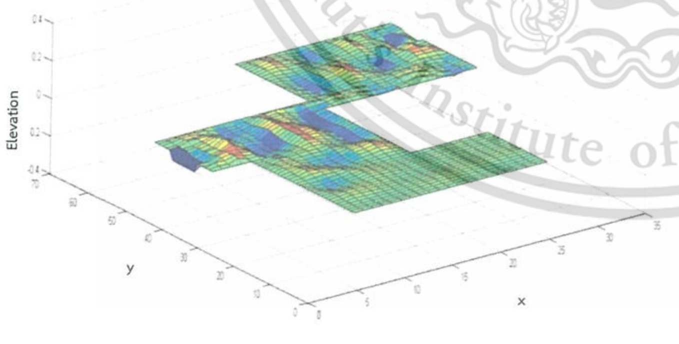
(b) at time 7.50



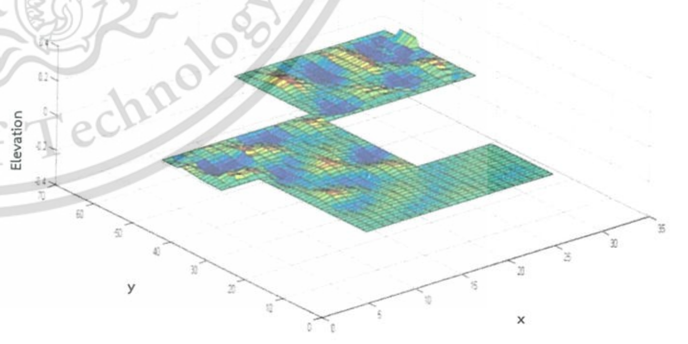
(c) at time 10.00



(d) at time 12.50

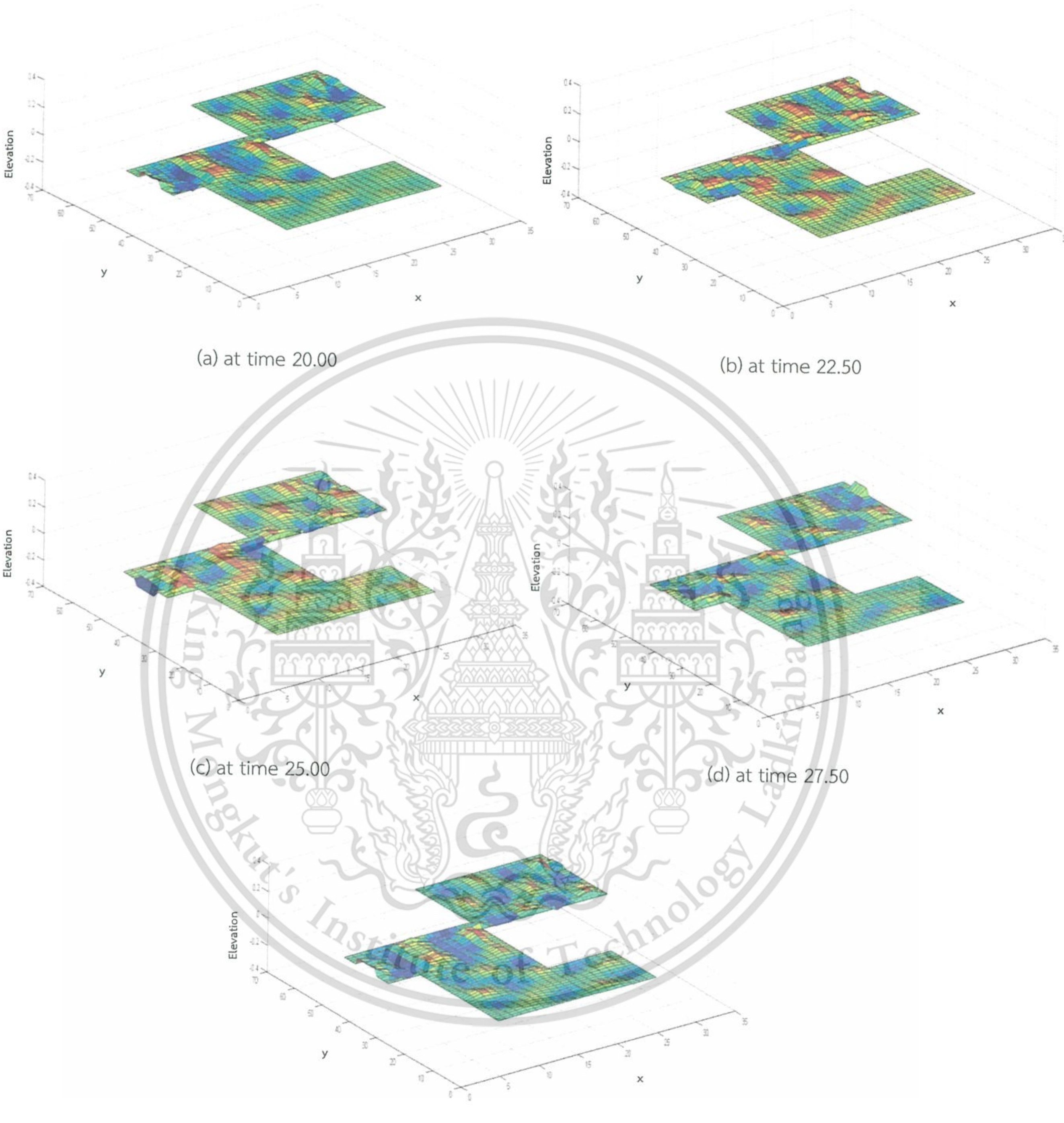


(e) at time 15.00



(f) at time 17.50

Figure 4.6 Water elevation in the RAMA 9 reservoir measured successively at certain time spots 5.00 to 17.50 or 13.30 (sec) to 46.56 (sec)



(a) at time 20.00

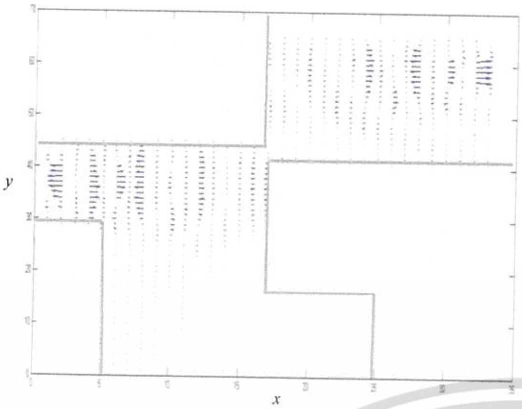
(b) at time 22.50

(c) at time 25.00

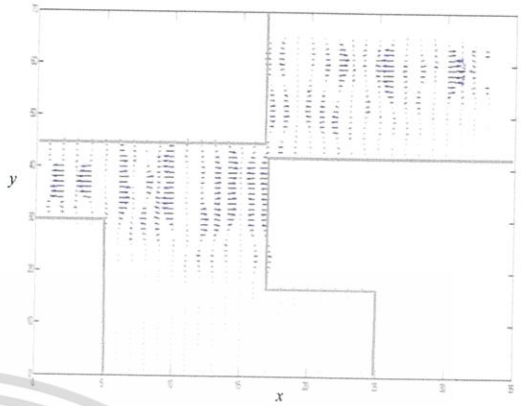
(d) at time 27.50

(e) at time 30.00

Figure 4.7 Water elevation in the RAMA 9 reservoir measured successively at certain time spots 20.00 to 30.00 or 53.21 (sec) to 79.82 (sec)



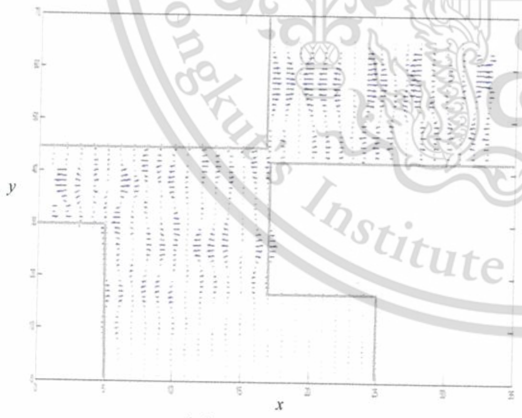
(a) at time 5.00



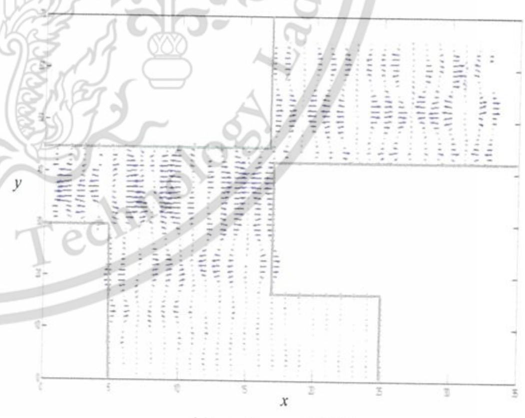
(b) at time 7.50

(c) at time 10.00

(d) at time 12.50

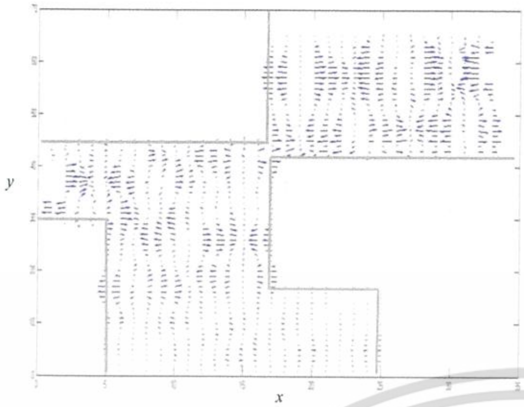


(e) at time 15.00

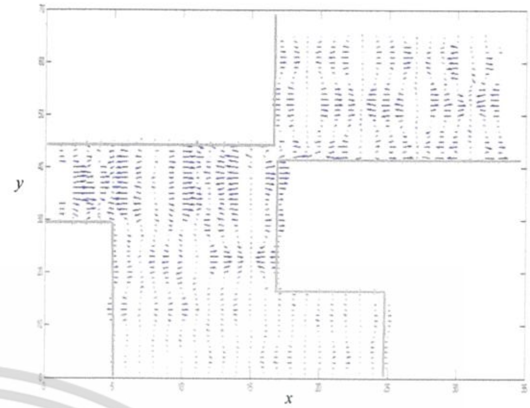


(f) at time 17.50

Figure 4.8 Vector field of velocities in the RAMA 9 reservoir measured successively at certain time spots 5.00 to 17.50 or 13.30 (sec) to 46.56 (sec)

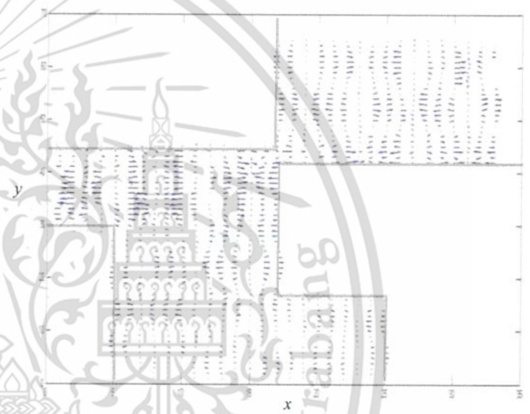


(a) at time 20.00

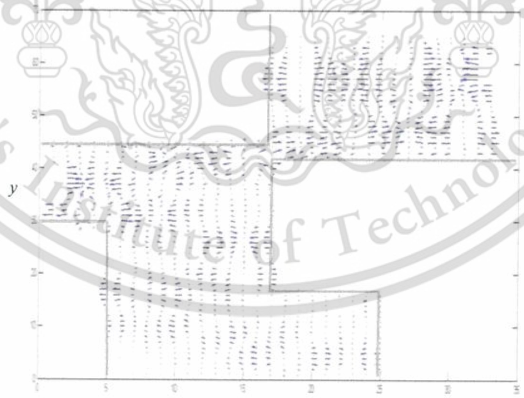


(b) at time 22.50

(c) at time 25.00



(d) at time 27.50



(e) at time 30.00

Figure 4.9 Vector field of velocities in the RAMA 9 reservoir measured successively at certain time spots 20.00 to 30.00 or 53.21 (sec) to 79.82 (sec)

4.1.3 The boundary and initial condition for unsteady state of dispersion model

The water pollutant is discharged into reservoir assumed as the exponential function, $e^{-\frac{t}{40}}$, on both opened gates as Figure 4.10(a). There is no opened gate that acts as a draining gate as Figure 4.10(b). Assume that the initial concentration of pollutant is $c = 0.02$ and there is no rate of change of pollutant concentration along the boundary.

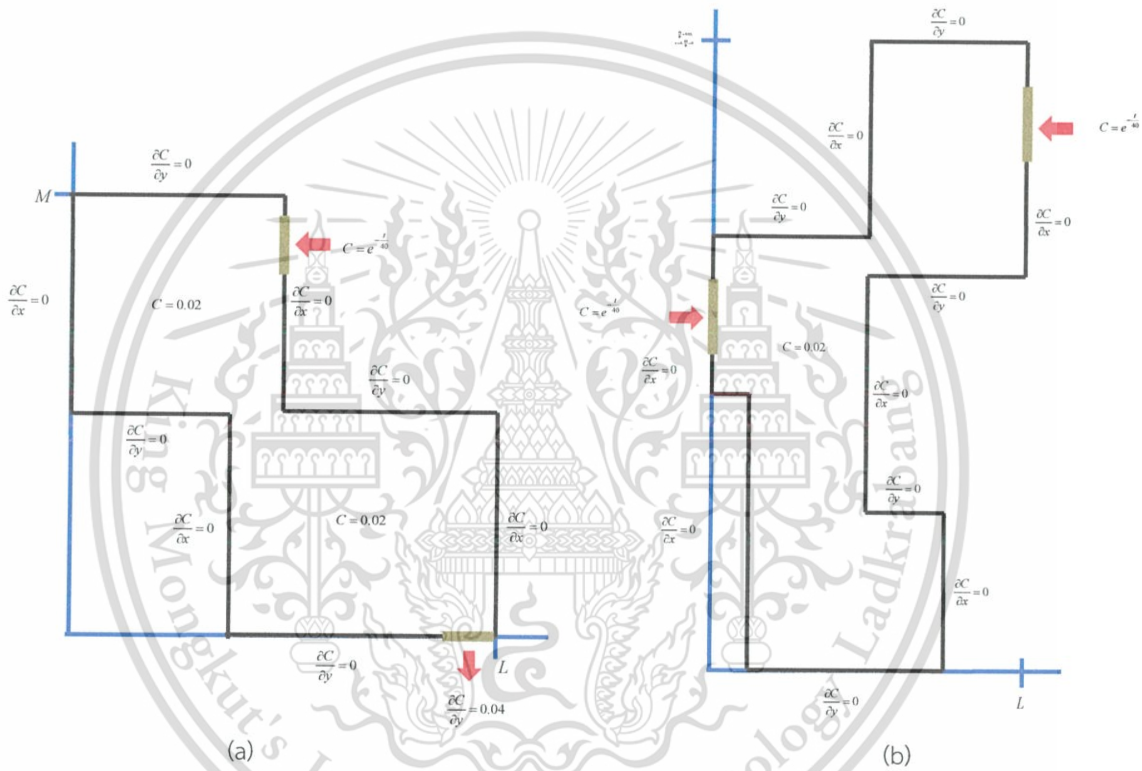
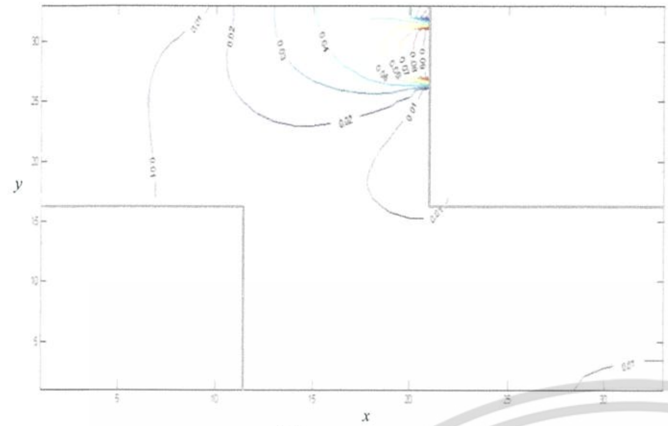


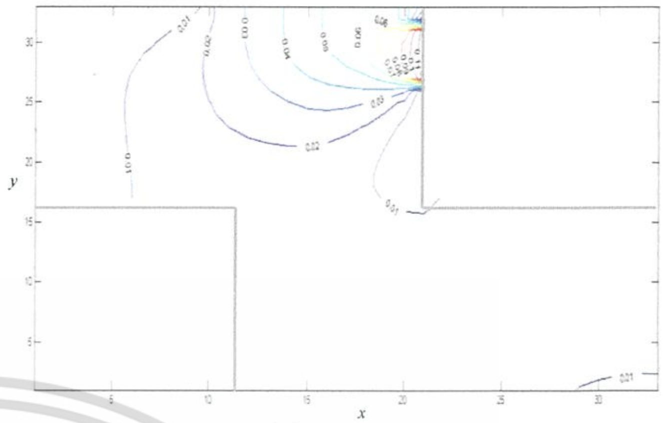
Figure 4.10 The domain problem for dispersion model of (a) open-connected reservoirs (b) the RAMA 9 reservoir

4.1.4 Numerical results of the unsteady state of dispersion model

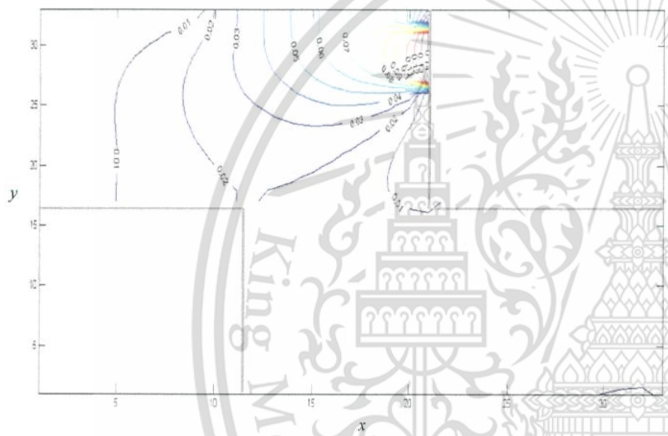
The unsteady state of dispersion model provides pollutant concentration level. Define step size of x and y are 0.03125, step size of time t is 0.01 and T is 30 for open-connected reservoirs. Define step size of x is 0.03125 and y is 0.015625, step size of time t is 0.01 and T is 30 for the RAMA 9 reservoir.



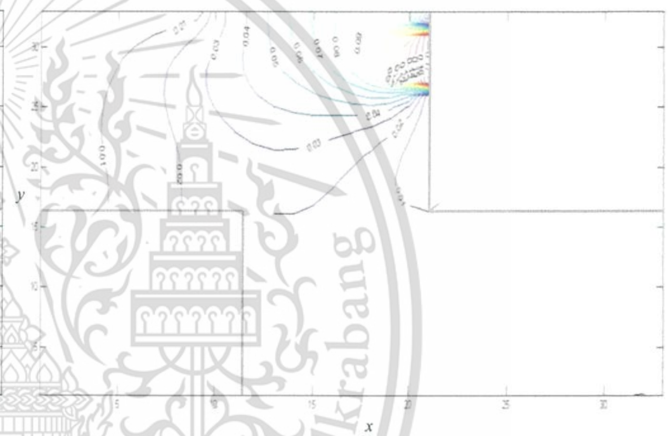
(a) at time 5.00



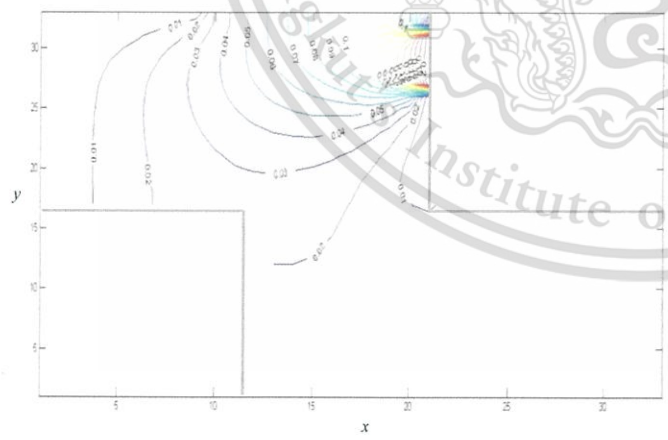
(b) at time 7.50



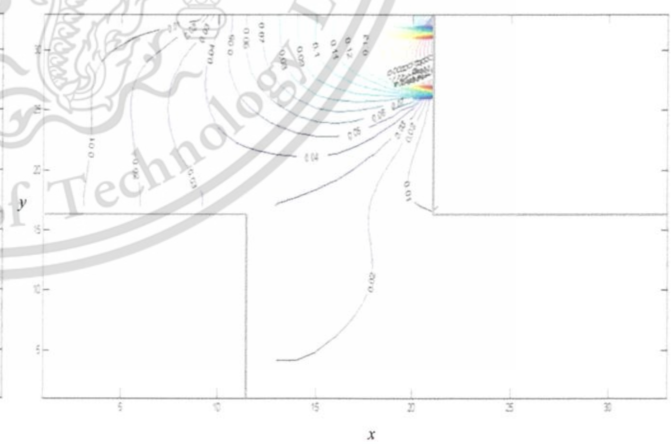
(c) at time 10.00



(d) at time 12.50

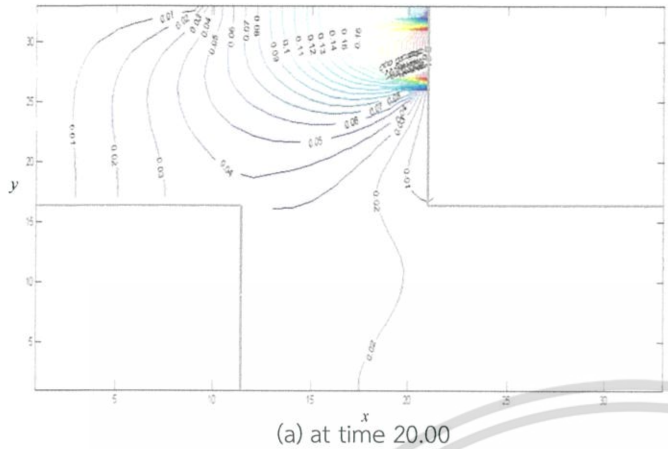


(e) at time 15.00

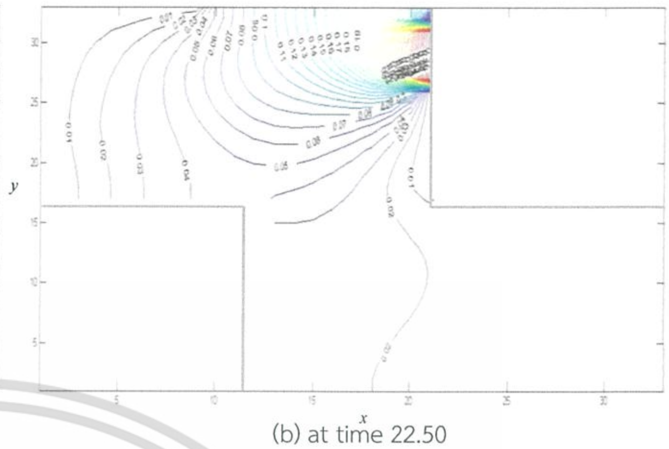


(f) at time 17.50

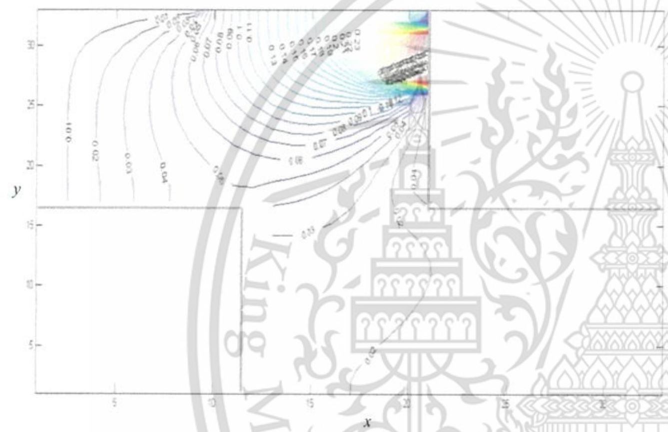
Figure 4.11 Pollutant level in open-connected reservoirs measured successively at certain time spots 5.00 to 17.50 or 13.30 (sec) to 46.56 (sec)



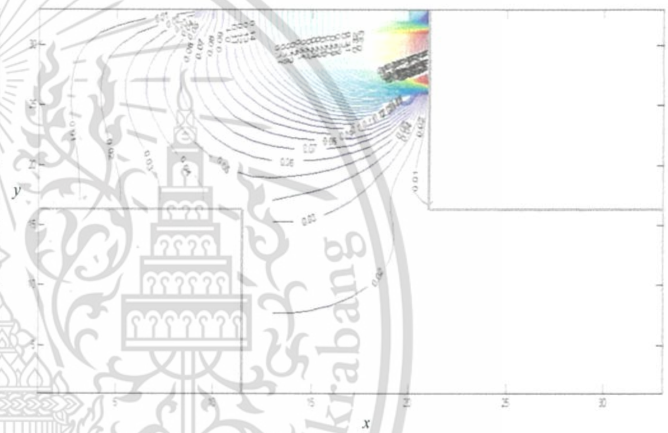
(a) at time 20.00



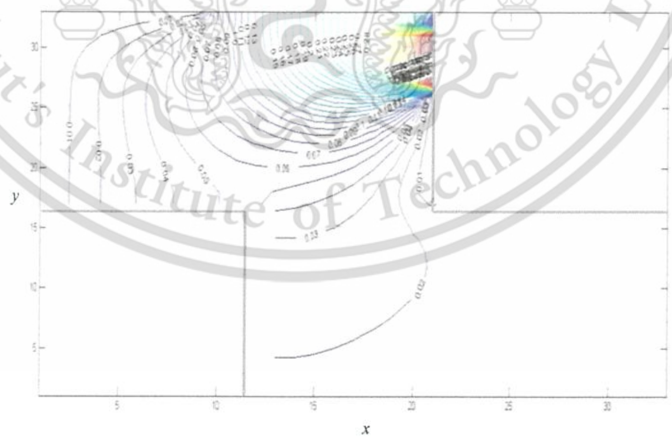
(b) at time 22.50



(c) at time 25.00

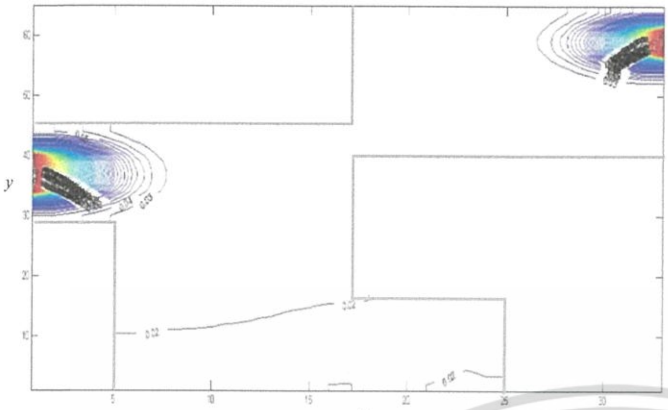


(d) at time 27.50

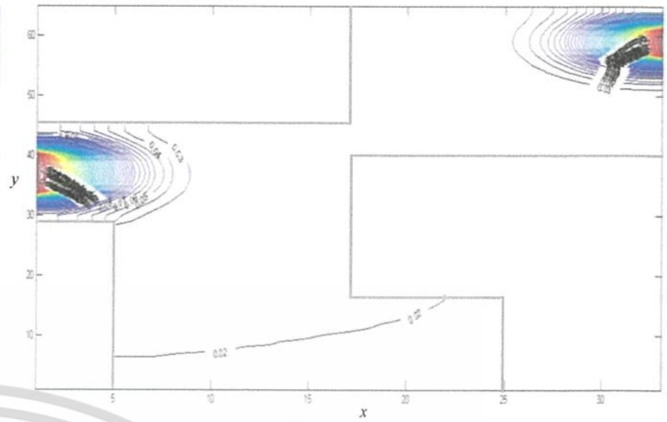


(e) at time 30.00

Figure 4.12 Pollutant level in open-connected reservoirs measured successively at certain time spots 20.00 to 30.00 or 53.21 (sec) to 79.82 (sec)

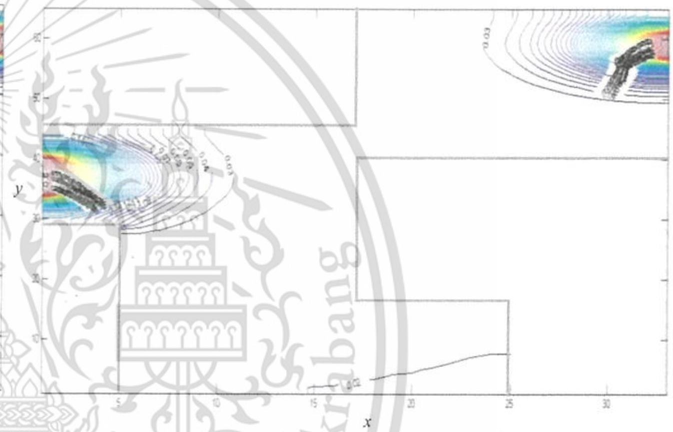


(a) at time 5.00

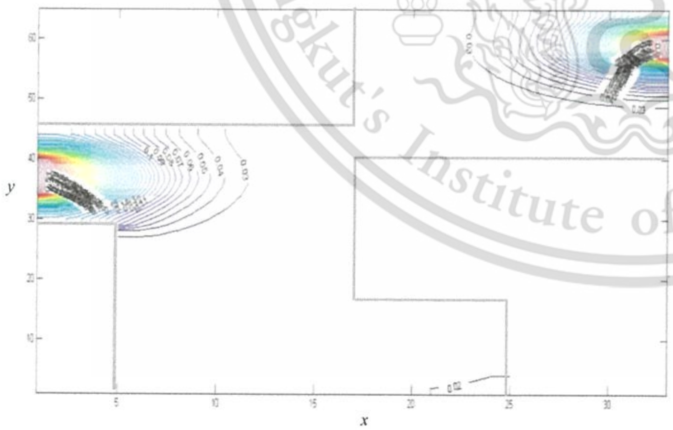


(b) at time 7.50

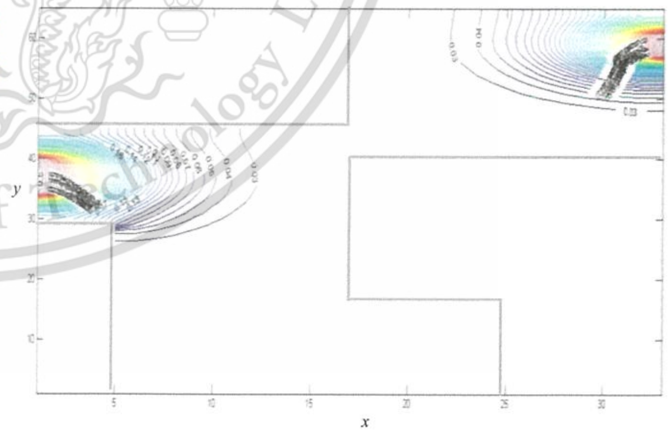
(c) at time 10.00



(d) at time 12.50

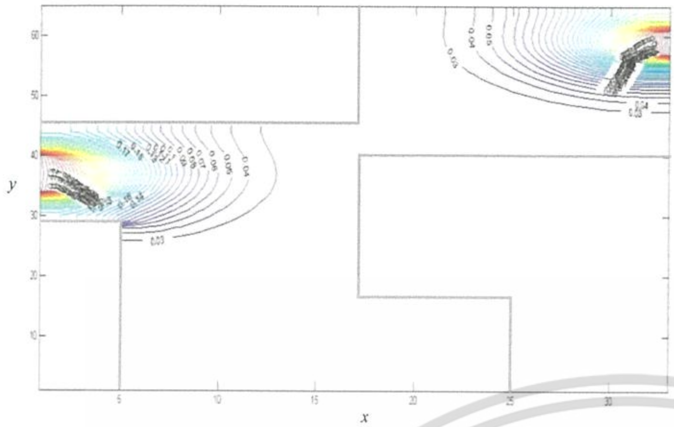


(e) at time 15.00

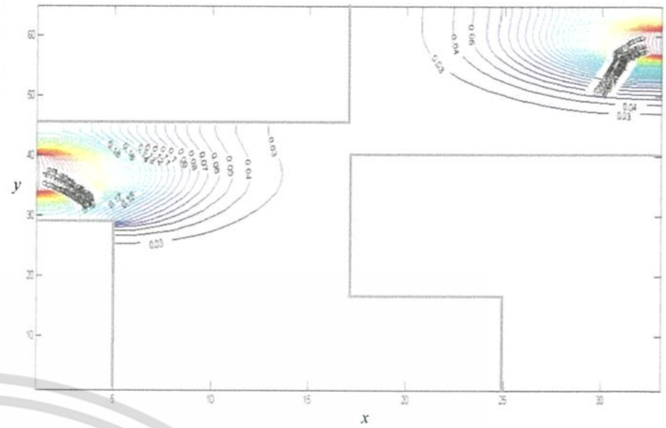


(f) at time 17.50

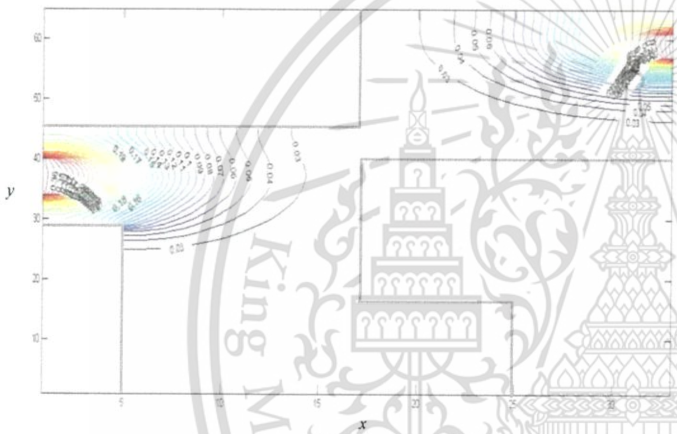
Figure 4.13 Pollutant level in the RAMA 9 reservoir measured successively at certain time spots 5.00 to 17.50 or 13.30 (sec) to 46.56 (sec)



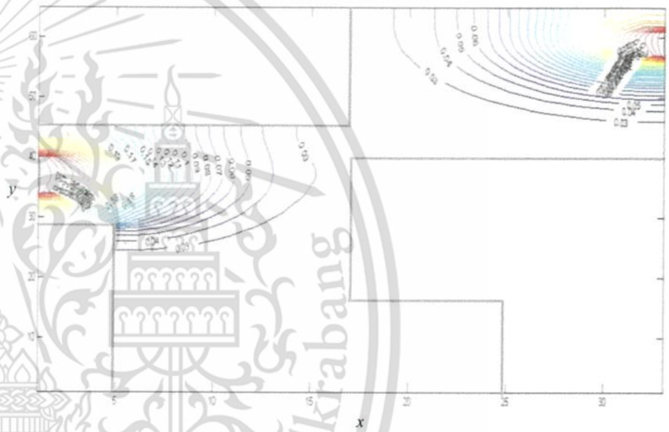
(a) at time 20.00



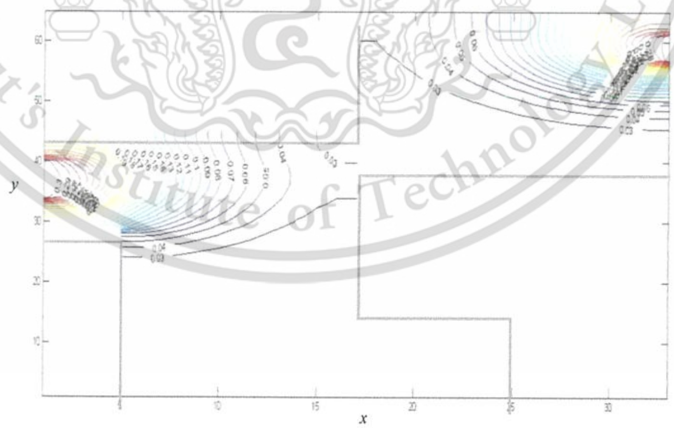
(b) at time 22.50



(c) at time 25.00



(d) at time 27.50



(e) at time 30.00

Figure 4.14 Pollutant level in the RAMA 9 reservoirs measured successively at certain time spots 20.00 to 30.00 or 53.21 (sec) to 79.82 (sec)

4.1.5 Observation points

Observation points are representing some positions in open-connected reservoirs and the RAMA 9 reservoir. These points are located at the gate, the linked reservoir and the middle reservoir. Consider the concentration charge of pollution at initial time to 2000 as shown in Figure 4.15-4.16 where x-axis is time axis and y-axis is concentration axis.

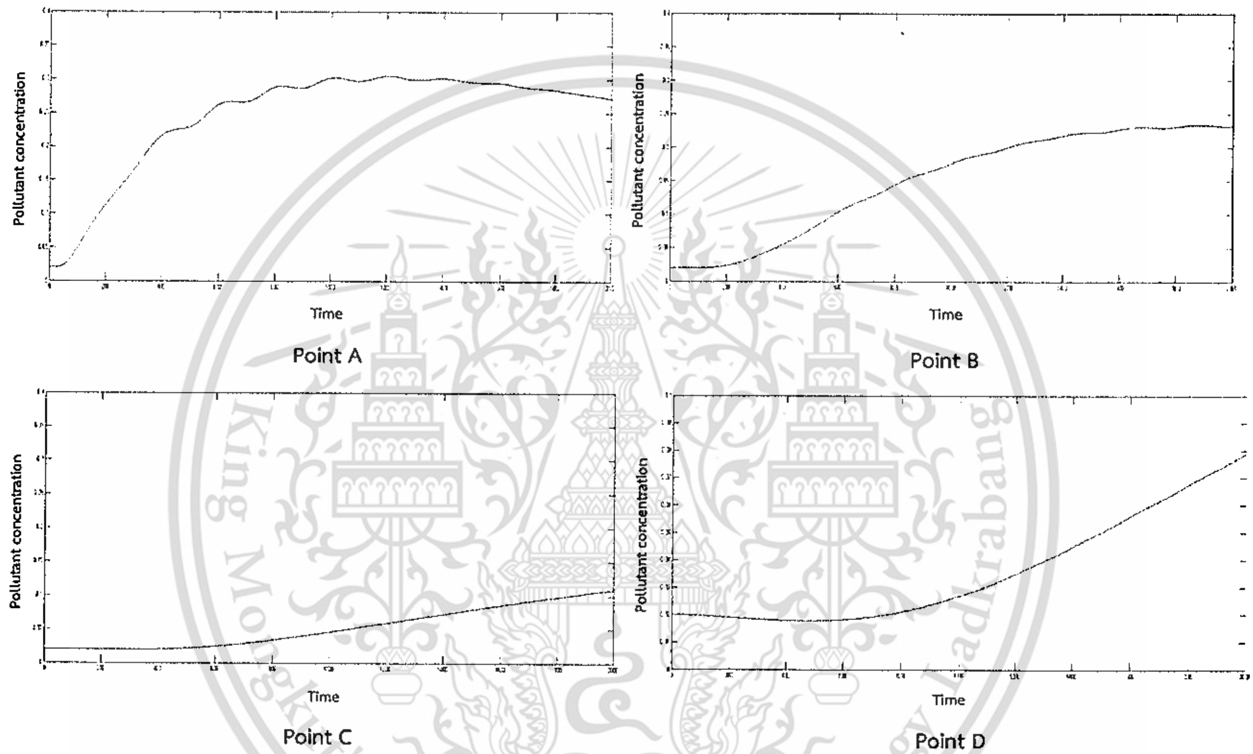
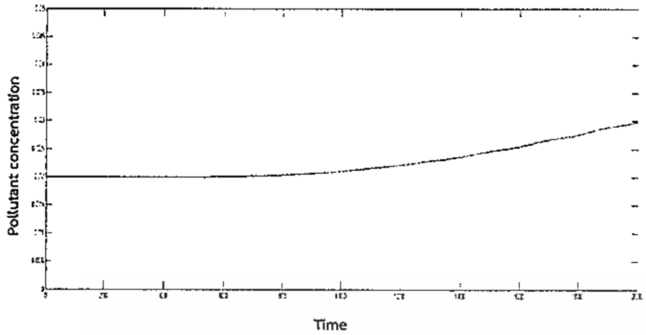
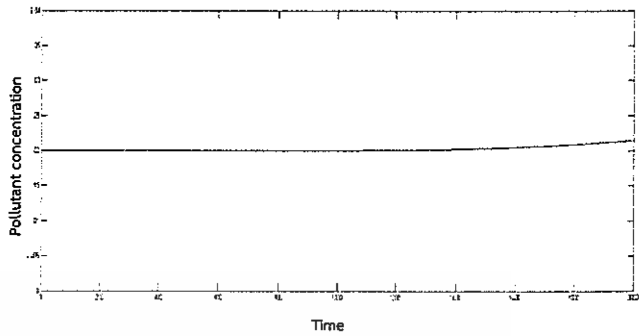


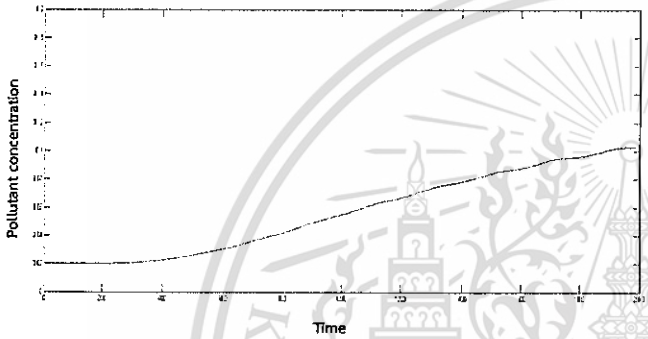
Figure 4.15 Pollutant concentration at observation points A, B, C, D at an initial time to 20.00 or 53.21 (sec) in open-connected reservoirs



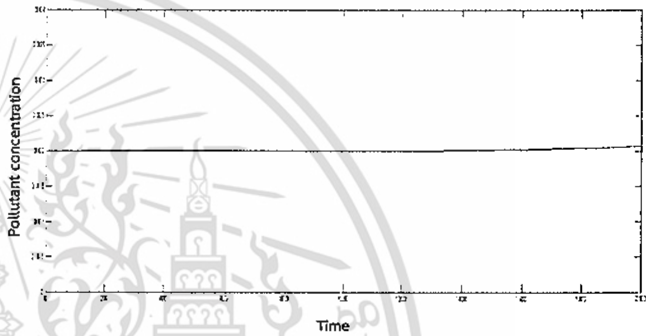
Point A



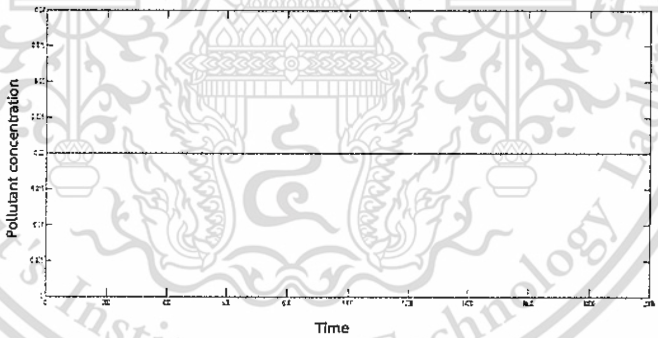
Point B



Point C



Point D



Point E

Figure 4.16 Pollutant concentration at observation points A, B, C, D and E at an initial time to 20.00 or 53.21 (sec) in the RAMA 9 reservoir

Table 4.1 Observation points in open-connected reservoirs for elevation at different times

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.000022	0.005121	0.018693	-0.046532	0.058054	-0.075299	0.091998	-0.094670	0.085147	-0.071788	0.072717	-0.082084	0.077922	-0.073851
Point B	0.000000	0.000000	0.000000	-0.000003	0.000003	0.001218	0.000472	-0.006576	0.011946	-0.012434	0.010469	-0.010245	0.010478	-0.010505
Point C	0.000000	-0.000048	-0.000743	-0.001028	-0.000021	0.001061	0.001840	0.000261	-0.005337	0.005301	-0.004236	0.003098	-0.000674	0.000036
Point D	-0.001255	-0.002501	-0.000925	0.000482	0.000549	-0.000076	-0.000365	-0.000073	0.000391	-0.001619	-0.002965	0.007028	-0.009026	0.010468

Table 4.2 Observation points in open-connected reservoirs for velocity (u) at different times

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.000011	0.004373	0.019859	-0.058009	0.081043	-0.085936	0.103255	-0.096792	-0.107876	-0.119515	0.116126	-0.088947	0.095235	-0.097334
Point B	0.000000	0.000000	0.000000	0.000027	0.000785	0.002096	-0.003860	0.014153	-0.010091	-0.009991	0.023355	-0.019409	0.021966	-0.022526
Point C	0.000000	0.000177	0.002173	0.003884	0.002205	0.001290	0.001768	0.001652	0.000562	0.002835	0.005835	-0.003674	0.008864	-0.003253
Point D	0.000014	-0.000370	-0.003565	-0.004563	-0.003285	-0.002624	-0.001854	-0.001335	-0.000341	-0.002972	-0.005271	0.004498	-0.010346	0.010503

Table 4.3 Observation points in open-connected reservoirs for velocity (v) at different times

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	-0.000144	-0.024640	-0.057304	0.122839	-0.192879	0.180976	-0.199946	0.194726	-0.229138	0.218746	-0.254176	0.251868	-0.271312	0.238285
Point B	0.000000	0.000000	0.000000	-0.000015	0.000041	0.001559	0.005172	-0.008025	0.002091	0.003510	-0.003081	0.001140	-0.002205	0.001673
Point C	0.000000	-0.000047	-0.000940	-0.001299	0.000279	0.000532	-0.004574	-0.003162	0.015790	-0.026576	0.019615	-0.014078	0.015085	-0.024868
Point D	-0.002260	-0.009154	-0.009069	-0.008032	-0.008136	-0.009156	-0.010403	-0.011505	-0.012307	-0.008339	-0.005700	-0.026443	0.007969	-0.034747

Table 4.4 Observation points in open-connected reservoirs for concentration at different times

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.049841	0.113908	0.164793	0.216161	0.229178	0.262172	0.266609	0.288353	0.286916	0.301859	0.297068	0.204881	0.299168	0.301584
Point B	0.020068	0.023934	0.036432	0.054709	0.077199	0.103739	0.123841	0.144108	0.161663	0.176936	0.189427	0.200453	0.210129	0.218047
Point C	0.019959	0.019781	0.020331	0.020331	0.021975	0.024808	0.028933	0.034201	0.039999	0.045992	0.052674	0.059276	0.066002	0.072616
Point D	0.018228	0.016987	0.016112	0.015469	0.015070	0.015027	0.015481	0.016568	0.018402	0.021021	0.024323	0.028065	0.032559	0.037299

Table 4.5 Observation points in the RAMA 9 reservoir for elevation at different times

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.000000	0.000026	-0.000857	0.002249	0.001545	0.001169	-0.000409	-0.008217	0.010252	-0.009812	0.009384	-0.009983	0.011108	-0.012955
Point B	0.000000	0.000001	0.000107	-0.000136	-0.002015	0.006549	-0.009218	0.014688	-0.016981	0.016453	-0.015764	0.013325	-0.013709	0.015262
Point C	0.000135	0.000925	-0.004922	0.007472	-0.006979	0.006503	-0.004258	0.005040	-0.009893	0.013991	-0.015930	0.017584	-0.019200	0.020685
Point D	0.000000	0.000000	0.000000	0.000045	-0.000094	-0.001296	0.002346	0.000793	-0.003498	0.005665	-0.010240	0.013108	-0.012152	0.011959
Point E	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001	0.000013	-0.000140	-0.000341	0.001431	-0.001403	0.002079	-0.003511	0.002146

Table 4.6 Observation points in the RAMA 9 reservoir for velocity (u) at different times

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	-0.000001	-0.000095	0.006525	-0.003148	-0.005862	0.006788	0.004106	-0.023559	0.029855	-0.028767	0.030651	-0.041450	0.040002	-0.036131
Point B	0.000000	0.000005	0.000758	0.001865	-0.017235	0.037167	-0.042060	0.032547	-0.039055	0.038095	-0.036498	0.037463	-0.037045	0.038587
Point C	0.001551	0.013227	-0.051944	0.073786	-0.081517	0.085095	-0.089971	0.081198	-0.065316	0.050085	-0.047510	0.045184	-0.034131	0.018782
Point D	0.000000	0.000000	0.000001	0.000160	0.000079	-0.002013	0.005828	-0.012783	0.020921	-0.024481	0.024718	-0.026681	0.028835	-0.029219
Point E	0.000000	0.000000	0.000000	0.000000	0.000000	0.000002	0.000054	-0.000047	-0.000649	0.001194	-0.000498	0.001140	-0.003342	0.002501

Table 4.7 Observation points in the RAMA 9 reservoir for velocity (v) at different times

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.000000	-0.000023	0.002607	-0.001898	-0.003150	-0.000130	0.001502	0.010142	-0.010200	0.006713	-0.002968	0.007595	-0.016144	0.015321
Point B	0.000000	0.000000	0.000000	-0.001265	0.000677	0.004376	-0.006759	0.005373	-0.003832	0.004906	-0.010695	0.013925	-0.016144	0.015321
Point C	-0.000006	-0.000538	0.001826	-0.000632	-0.001567	0.001210	-0.001585	0.005846	-0.003327	0.001224	-0.002520	0.002097	0.003363	-0.006058
Point D	0.000000	0.000000	-0.000001	-0.000109	-0.000094	0.003438	-0.005736	0.000743	0.001742	-0.006067	0.014744	-0.020454	0.020015	-0.019896
Point E	0.000000	0.000000	0.000000	0.000000	0.000000	-0.000002	-0.000034	0.000368	0.001030	-0.004173	0.004425	-0.004996	0.007380	-0.008324

Table 4.8 Observation points in the RAMA 9 reservoir for concentration at different times

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.020000	0.020000	0.020000	0.020003	0.020020	0.020065	0.020173	0.020341	0.020640	0.020991	0.021537	0.022086	0.022904	0.023580
Point B	0.020000	0.020000	0.020000	0.020000	0.0200000	0.020000	0.020001	0.020003	0.020009	0.020020	0.020043	0.020077	0.020137	0.020213
Point C	0.020002	0.020117	0.020880	0.022742	0.026187	0.030485	0.036386	0.041902	0.049083	0.054632	0.061623	0.066942	0.073592	0.078200
Point D	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020001	0.020001	0.020003	0.020005
Point E	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.0200000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000

4.1.6 Transform non dimensional results to real time (sec)

We will transform non dimensional results to real time by letting $u = U\sqrt{gh}, v = V\sqrt{gh}, x = Xl, y = Yl, \xi = Zh, t = Tl/\sqrt{gh}$. In previous section, it shows the non dimensional results of water elevation and velocities in domains calculated by non dimensional form. In this section, it shows elevation of water and velocities in table 4.9-4.14 at real time. Define $h = 1$ (m) the domain of open-connected reservoirs 500×500 (m^2) and the RAMA 9 reservoir 500×1000 (m^2).

Table 4.9 Observation points in open-connected reservoirs for elevation at real time

Time	2.66	5.32	7.98	10.64	13.30	15.96	18.62	21.29	23.59	26.61	29.28	31.93	34.59	37.25
Point A	0.000022	0.005121	0.018693	-0.046532	0.058054	-0.075299	0.091998	-0.094670	0.085147	-0.071788	0.072717	-0.082084	0.077922	-0.073851
Point B	0.000000	0.000000	0.000000	-0.000003	0.000003	0.001218	0.000472	-0.006576	0.011946	-0.012434	0.010469	-0.010245	0.010478	-0.010505
Point C	0.000000	-0.000048	-0.000743	-0.001028	-0.000021	0.001061	0.001840	0.000261	-0.005337	0.005301	-0.004236	0.003098	-0.000674	0.000036
Point D	-0.001255	-0.002501	-0.000925	0.000482	0.000549	-0.000076	-0.000365	-0.000073	0.000391	-0.001619	-0.002965	0.007028	-0.009026	0.010468

Table 4.10 Observation points in open-connected reservoirs for velocity (u) at real time

Time	2.66	5.32	7.98	10.64	13.30	15.96	18.62	21.29	23.59	26.61	29.28	31.93	34.59	37.25
Point A	0.000034	0.013696	0.062200	-0.181690	0.253834	-0.269159	0.323404	-0.303161	0.337878	-0.374331	0.363717	-0.278590	-0.278590	-0.304859
Point B	0.000000	0.000000	0.000000	0.000084	0.002459	0.006565	-0.012090	0.044329	-0.031606	-0.031293	0.073150	-0.060791	0.068800	-0.070554
Point C	0.000000	0.000554	0.006806	0.012165	0.006906	0.004040	0.005538	0.005174	0.001760	0.0088795	0.018276	-0.011507	0.027763	-0.010189
Point D	0.000043	-0.001159	-0.011166	-0.014292	-0.010289	-0.008219	-0.005807	-0.004181	-0.001068	-0.009309	-0.016509	0.014088	-0.032240	0.032896

Table 4.11 Observation points in open-connected reservoirs for velocity (v) at real time

Time	2.66	5.32	7.98	10.64	13.30	15.96	18.62	21.29	23.59	26.61	29.28	31.93	34.59	37.25
Point A	-0.000451	-0.077174	-0.179481	0.384742	-0.604115	0.566833	-0.626249	0.609900	-0717681	0.695133	-0.796103	0.788874	-0.849770	0.746331
Point B	0.000000	0.000000	0.000000	-0.000047	0.000128	0.004883	0.016199	-0.025135	0.006549	0.010993	-0.009650	0.003571	-0.006906	0.005240
Point C	0.000000	-0.000147	-0.002944	-0.004069	0.000874	0.001666	-0.014326	-0.009904	0.049456	-0.083239	0.061436	-0.044122	0.047248	-0.077889
Point D	-0.007078	-0.028671	-0.028405	-0.025157	-0.025483	-0.028677	-0.032583	-0.036034	-0.038744	-0.026119	-0.017853	-0.082822	0.024960	-0.108831

Table 4.12 Observation points in the RAMA 9 reservoir for elevation at real time

Time	2.66	5.32	7.98	10.64	13.30	15.96	18.62	21.29	23.59	26.61	29.28	31.93	34.59	37.25
Point A	0.000000	0.000026	-0.000857	0.002249	0.001545	0.001169	-0.000409	-0.008217	0.010252	-0.009812	0.009384	-0.009983	0.011108	-0.012955
Point B	0.000000	0.000001	0.000107	-0.000136	-0.002015	0.006549	-0.009218	0.014688	-0.016981	0.016453	-0.015764	0.013325	-0.013709	0.015262
Point C	0.000135	0.000925	-0.004922	0.007472	-0.006979	0.006503	-0.004258	0.005040	-0.009893	0.013991	-0.015930	0.017584	-0.019200	0.020685
Point D	0.000000	0.000000	0.000000	0.000045	-0.000094	-0.001296	0.002346	0.000793	-0.003498	0.005665	-0.010240	0.013108	-0.012152	0.011959
Point E	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001	0.000013	-0.000140	-0.000341	0.001431	-0.001403	0.002079	-0.003511	0.002146

Table 4.13 Observation points in the RAMA 9 reservoir for velocity (u) at real time

Time	2.66	5.32	7.98	10.64	13.30	15.96	18.62	21.29	23.59	26.61	29.28	31.93	34.59	37.25
Point A	-0.000003	-0.000297	0.020436	-0.009860	-0.018360	0.021261	0.012860	-0.073789	0.093509	-0.090101	0.096002	-0.129825	0.125290	-0.113166
Point B	0.000000	0.000016	0.002374	0.005841	-0.053982	0.116410	-0.131736	0.101940	-0.122324	0.119317	-0.114315	0.117338	-0.116028	0.120858
Point C	0.004857	0.041428	-0.162659	0.231105	-0.255319	0.266525	-0.281797	0.254226	-0.204576	0.156871	-0.148806	0.141520	-0.106901	0.058827
Point D	0.000000	0.000000	0.000003	0.000501	0.000247	-0.006305	0.018254	-0.040038	0.065527	-0.076677	0.077419	-0.083567	0.090314	-0.091517
Point E	0.000000	0.000000	0.000000	0.000000	0.000000	0.000006	0.000169	-0.000147	-0.002033	0.003740	-0.001560	0.003571	-0.010468	0007833

Table 4.14 Observation points in the RAMA 9 reservoir for velocity (v) at real time

Time	2.66	5.32	7.98	10.64	13.30	15.96	18.62	21.29	23.59	26.61	29.28	31.93	34.59	37.25
Point A	0.000000	-0.000072	0.008165	-0.005945	-0.009866	-0.000407	0.004704	0.031766	-0.031950	0.0210257	-0.009296	0.023788	-0.004451	0.016531
Point B	0.000000	0.000000	0.000000	-0.003962	0.002120	0.013706	-0.021170	0.016829	-0.012002	0.015366	-0.033498	0.043614	-0.050565	0.047987
Point C	-0.000019	-0.001685	0.005719	-0.001979	-0.004908	0.003790	-0.004964	0.018310	-0.010421	0.003833	-0.007893	0.006568	0.010533	-0.018974
Point D	0.000000	0.000000	-0.000003	-0.000341	-0.000294	0.010768	-0.017966	0.002327	0.005456	-0.019002	0.046180	-0.064064	0.062689	-0.062316
Point E	0.000000	0.000000	0.000000	0.000000	0.000000	-0.000006	-0.000106	0.001153	0.003226	-0.013070	0.013860	-0.016648	0.023115	-0.026072

4.2 Water quality model in an opened-closed reservoir with anisotropic bottom topography

4.2.1 The boundary and initial conditions for hydrodynamic model

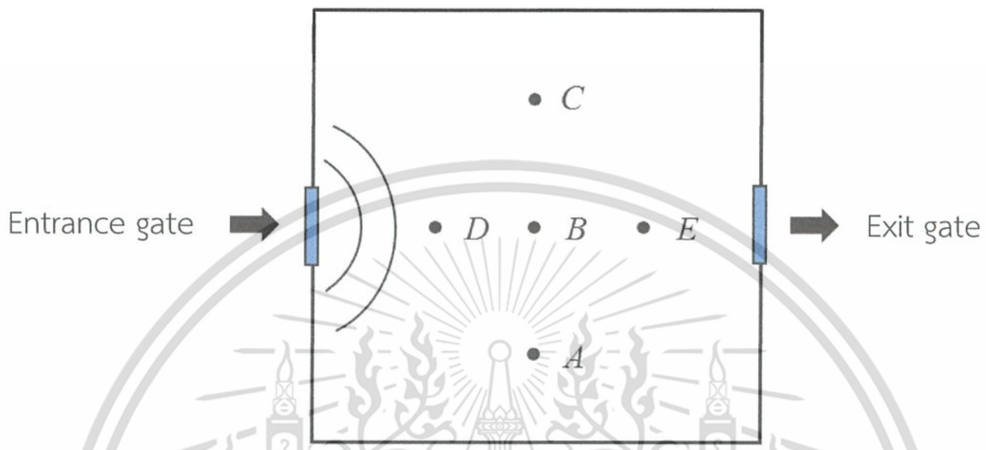


Figure 4.17 An opened-closed reservoir and observation points A, B, C, D and E

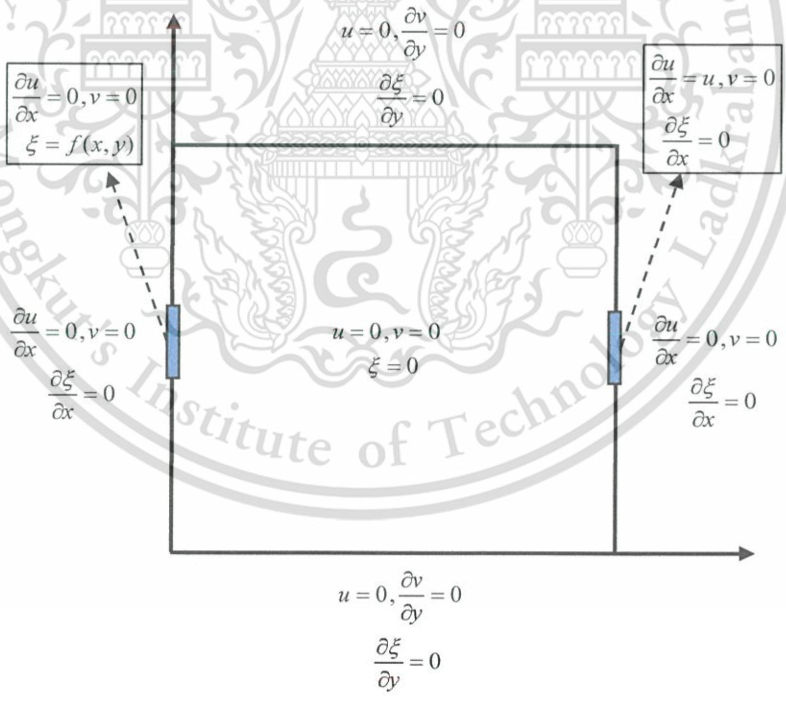


Figure 4.18 Initial condition and boundary condition of hydrodynamic model

The initial conditions of reservoir were as follows: the x and y-velocity components were zero as well as the water elevation: $u=0, v=0$ and $\xi=0$, while the boundary conditions were as follows: (i) $\frac{\partial u}{\partial x}=0, \frac{\partial v}{\partial y}=0, \xi=0$ for the horizontal edges of the rectangular reservoir; (ii) $\frac{\partial u}{\partial x}=0, v=0, \xi=0$ for the vertical edges; and (iii) $\xi=f(x,y)$ for the water flowing into the entrance gate and $\frac{\partial u}{\partial x}=u_1, \frac{\partial v}{\partial y}=0$ for the velocity of water flow at exit gate as shown in Figure 4.18.

4.2.2 Numerical results of the hydrodynamic model

In this section, various results were reported in a table, several surface and contour plots, and a comparison graph. Hydrodynamic model, calculated the velocities of water and elevation of water in an opened-closed reservoir with an empirical anisotropic bottom topography interpolated function $0.01\sin(0.01(x+y))$ as shown in Figure 4.19, using Lax-Wendroff method, when water flowed into the entrance gate by using the elevation of water $\xi=1$ (m) and discarding the drain water though the exit gate, using the rate of change of velocity u at 0.5 (m/s^2).

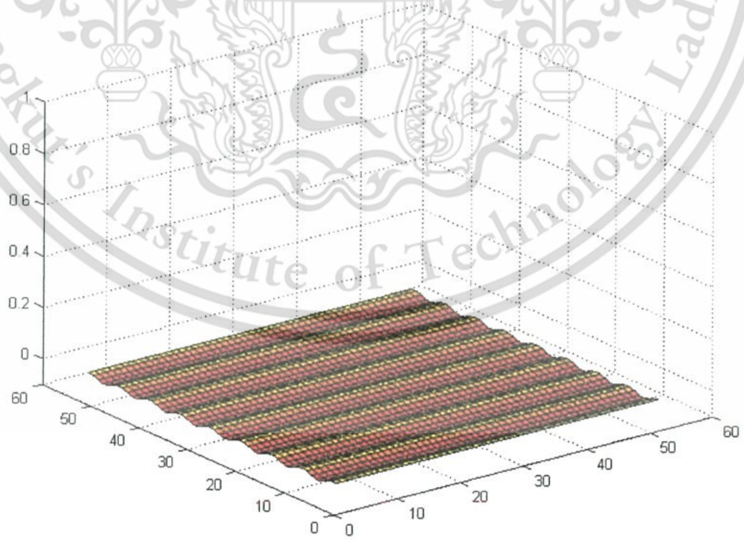
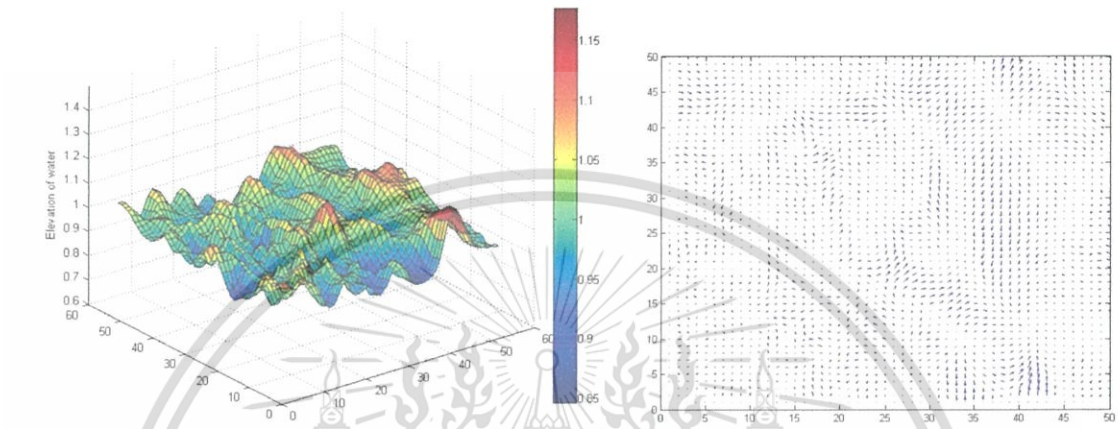


Figure 4.19 Anisotropic bottom topography surface in an opened-closed reservoir

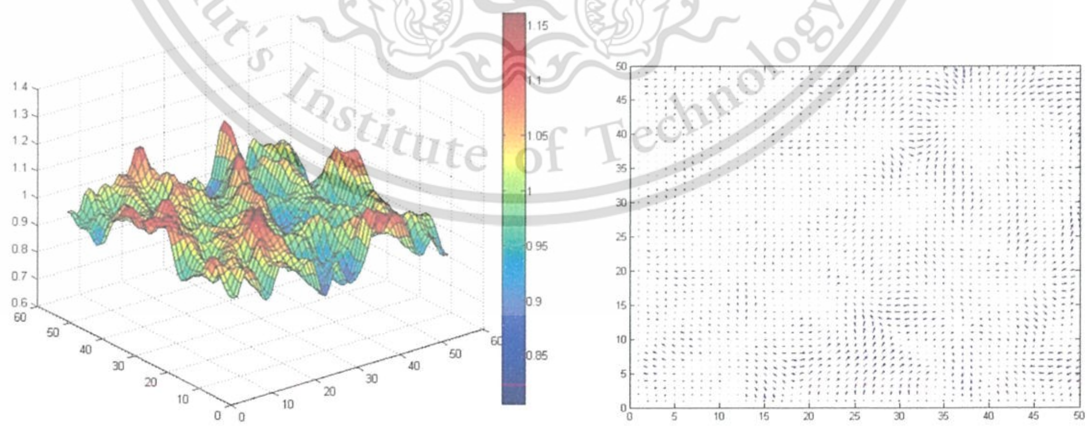
The elevation and vector field of velocities in an opened-closed reservoir for time 0 to 50 (sec), wastewater discharging from the external source every 36 (sec) into reservoir and drain water releasing though the exit gate every 36 (sec) as shown in Figure 4.20.



(a) (b)

Figure 4.20 Time 50 (sec), (a) elevation (b) vector field of velocities in opened-closed reservoir

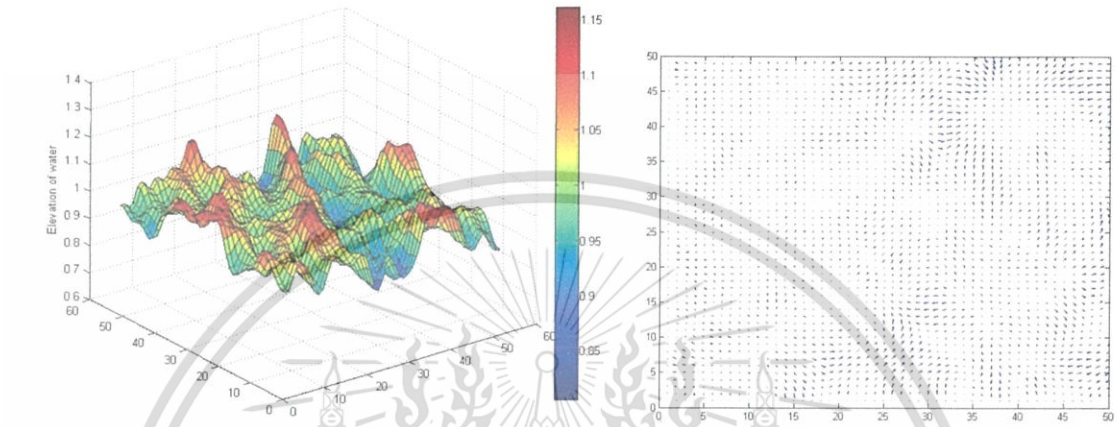
The elevation and vector field of velocities in an opened-closed reservoir for time 0 to 100 (sec), wastewater discharging from the external source every 36 (sec) into reservoir and drain water releasing though the exit gate every 36 (sec) as shown in Figure 4.21.



(a) (b)

Figure 4.21 Time 100 (sec), (a) elevation (b) vector field of velocities in an opened-closed reservoir

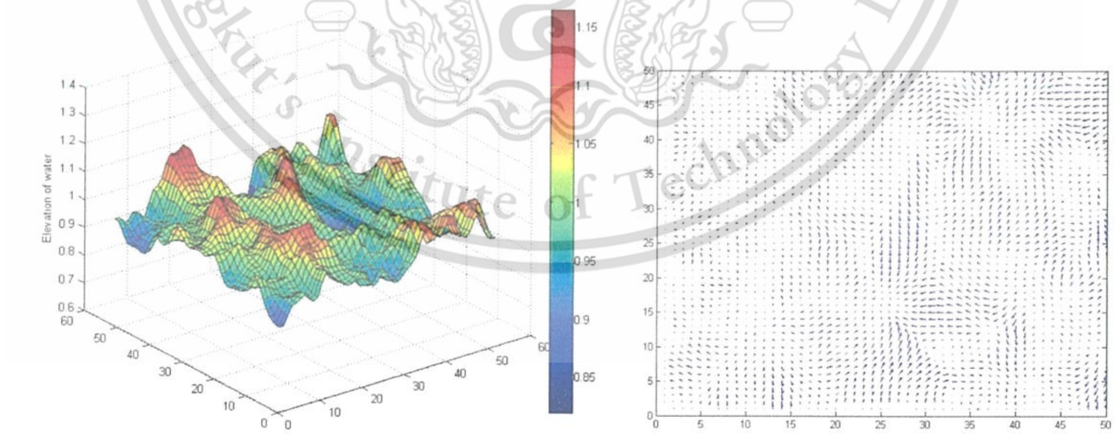
The elevation and vector field of velocities in an opened-closed reservoir for time 0 to 100 (sec), wastewater discharging from the external source every 36 (sec) into reservoir and drain water releasing though the exit gate every 72 (sec) as shown in Figure 4.22.



(a) (b)

Figure 4.22 Time 100 (sec), (a) elevation (b) vector field of velocities in an opened-closed reservoir

The elevation and vector field of velocities in an opened-closed reservoir for time 0 to 100 (sec), wastewater discharging from the external source every 72 (sec) into reservoir and drain water releasing though the exit gate every 36 (sec) as shown in Figure 4.23.



(a) (b)

Figure 4.23 Time 100 (sec), (a) elevation (b) vector field of velocities in an opened-closed reservoir

4.2.3 The boundary and initial conditions for unsteady state of dispersion model

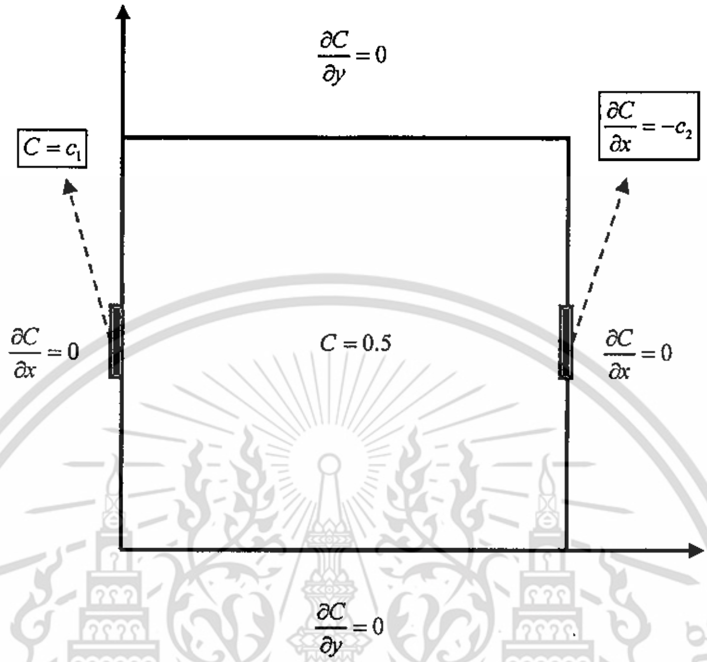


Figure 4.24 Initial condition and boundary condition of dispersion model

The water pollutant was discharged from the entrance gate into an opened-closed reservoir by assuming the pollutant concentration constant as function c_1 and this reservoir had draining water at the exit gate by assuming rate of drain of water as $\frac{\partial C}{\partial x} = -c_2$, initial pollutant concentration in reservoir was 0.5 (kg/m^3) and there was no rate of change of pollutant concentration at the boundary of reservoir as Figure 4.24.

4.2.4 Numerical results of the unsteady state of dispersion model

Dispersion model, calculated the pollutant concentration of water in an opened-closed reservoir 50×50 (m^2) by using finite difference method, step size of x and y are 1 (m), when wastewater was discharged from the external source into the reservoir and drain water was released thru the exit gate by using the rate of change of pollutant concentration with respect to x-coordinate at 0.1 (kg/m^4) and $c_1 = 0.55$ (kg/m^3) with initial pollutant concentration in this reservoir at 0.5 (kg/m^3).

The pollutant concentration in an opened-closed reservoir for time 0 to 50 (sec), wastewater discharging from the external source every 36 (sec) into reservoir and drain water releasing though the exit gate every 36 (sec) as shown in Figure 4.25-4.26.

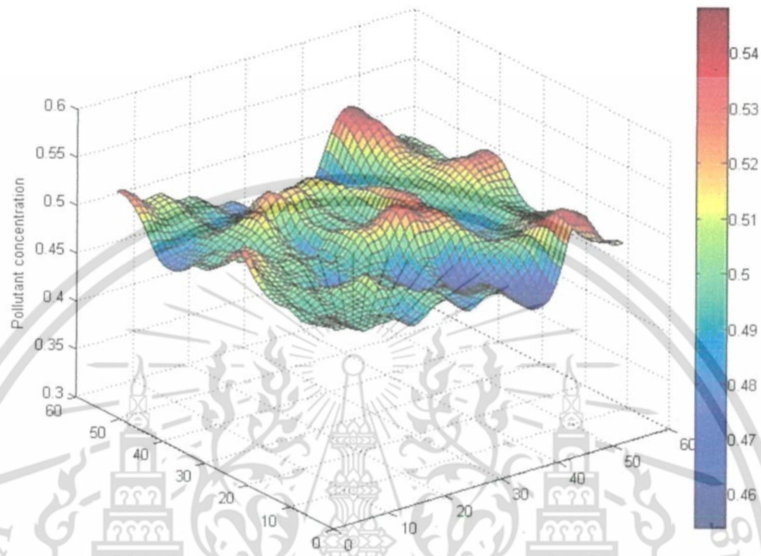


Figure 4.25 Time 50 (sec) pollutant concentration surface in an opened-closed reservoir

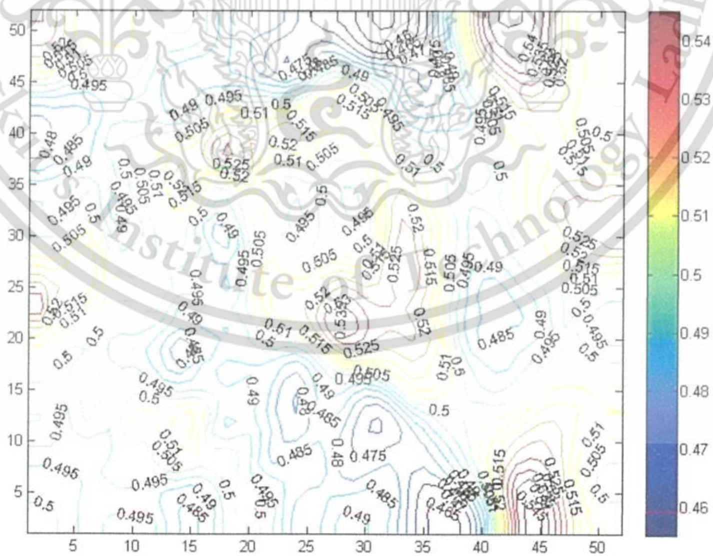


Figure 4.26 Time 50 (sec) pollutant level in an opened-closed reservoir

The pollutant concentration in an opened-closed reservoir for time 0 to 100 (sec), wastewater discharging from the external source every 36 (sec) into reservoir and drain water releasing though the exit gate every 36 (sec) as shown in Figure 4.27-4.28.

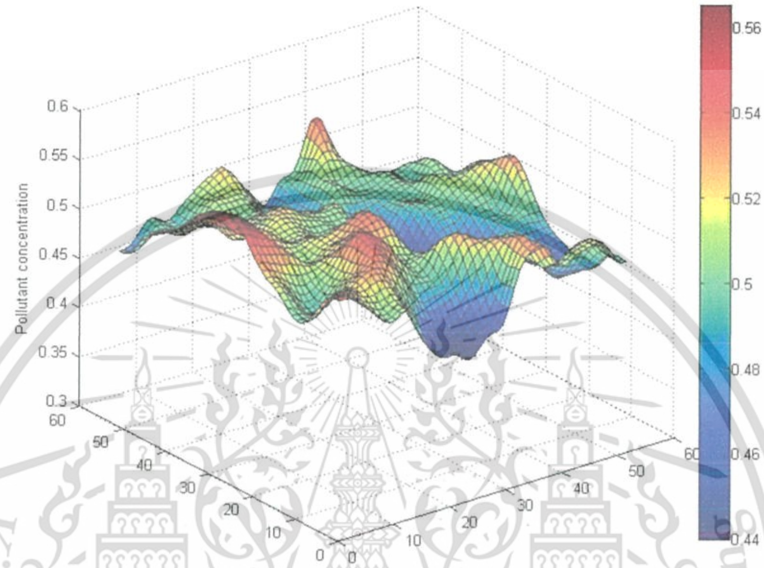


Figure 4.27 Time 100 (sec) pollutant concentration surface in an opened-closed reservoir

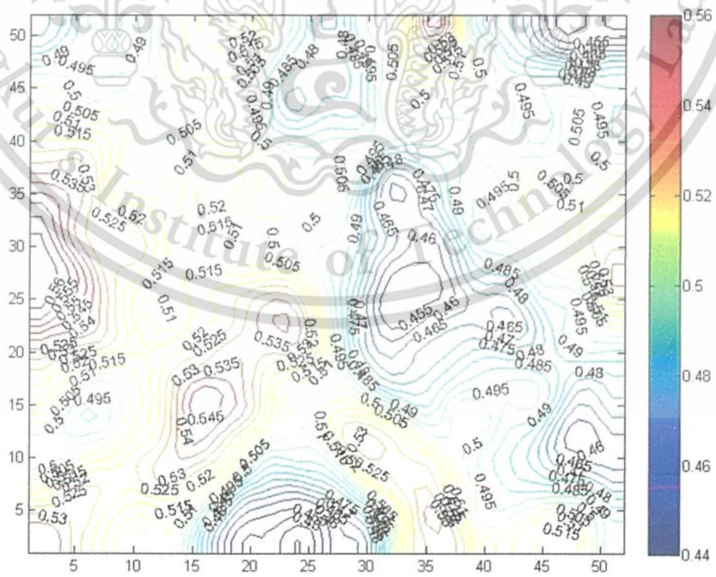


Figure 4.28 Time 100 (sec) pollutant level in an opened-closed reservoir

The pollutant concentration in an opened-closed reservoir for time 0 to 100 (sec), wastewater discharging from the external source every 36 (sec) into reservoir and drain water releasing though the exit gate every 72 (sec) as shown in Figure 4.29-4.30.

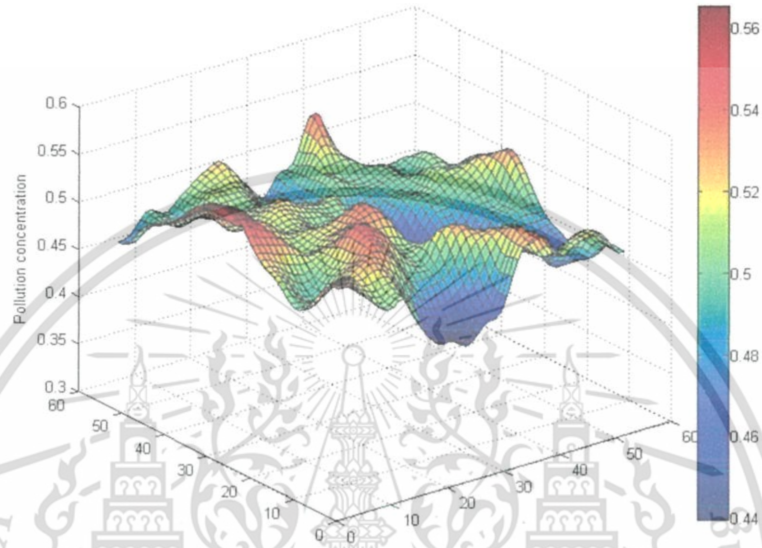


Figure 4.29 Time 100 (sec) pollutant concentration surface in an opened-closed reservoir

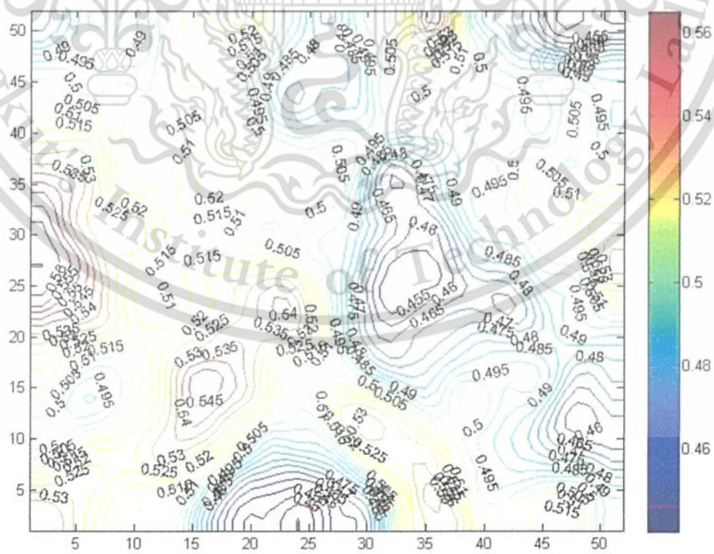


Figure 4.30 Time 100 (sec) pollutant level in an opened-closed reservoir

The pollutant concentration in an opened-closed reservoir for time 0 to 100 (sec), wastewater discharging from the external source every 72 (sec) into reservoir and drain water releasing though the exit gate every 36 (sec) as shown in Figure 4.31-4.32.

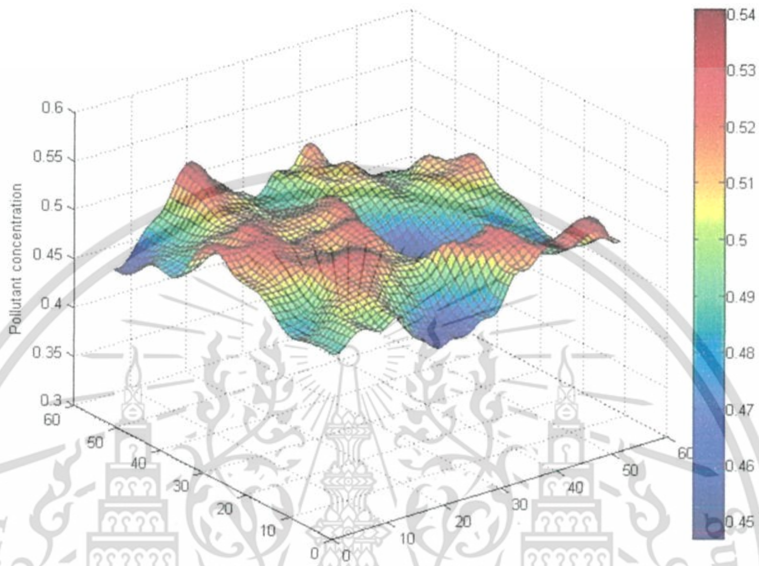


Figure 4.31 Time 100 (sec) pollutant concentration surface in an opened-closed reservoir

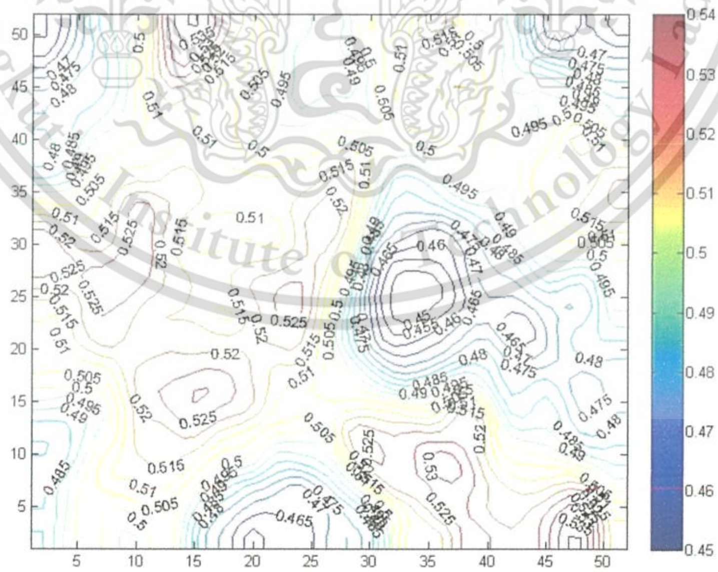


Figure 4.32 Time 100 (sec) pollutant level in an opened-closed reservoir

4.2.5 Observation points

The monitoring points in an opened-closed reservoir was used to observe the dispersion of pollutant concentration of water. In Figure 4.33 showing the comparison of pollutant concentration at monitoring point A, B and C and Figure 4.34 showing the comparison of pollutant concentration at monitoring point D and E for time 0 to 50 (sec).

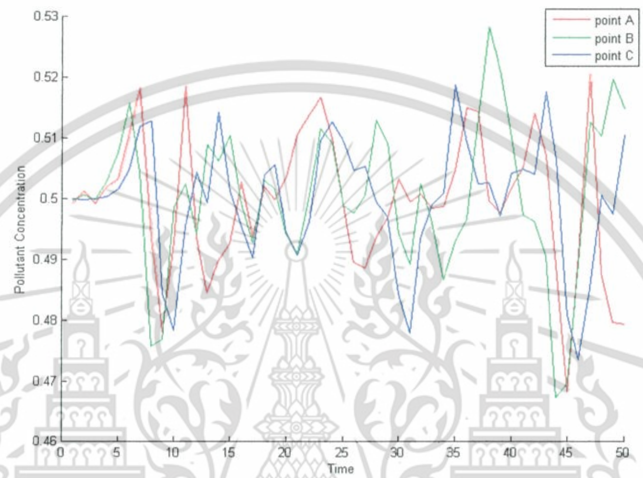


Figure 4.33 The comparison of pollutant concentration at monitoring of point A, B and C

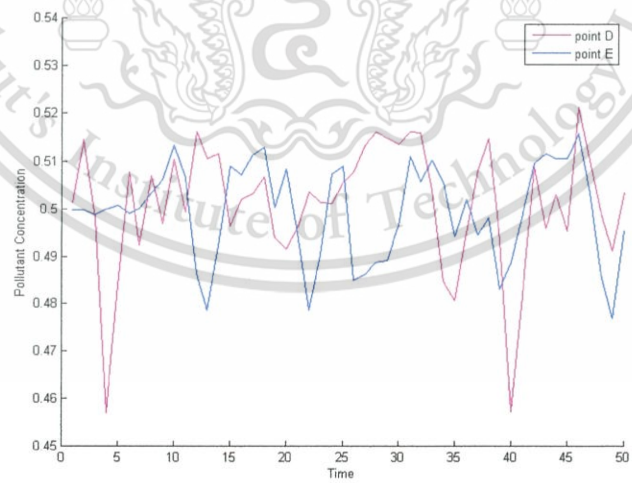


Figure 4.34 The comparison of pollutant concentration at monitoring of point D and E.

The pollutant concentration of water at monitoring point A, B, C, D and E in an opened-closed reservoir was observed in every 25 (sec) for time 0 to 200 (sec) with wastewater discharging from the external source every 36 (sec) into reservoir and drain water releasing though the exit gate every 36 (sec) in table 4.15, wastewater discharging from the external source every 36 (sec) into reservoir and drain water releasing though the exit gate every 72 (sec) in table 4.16 and wastewater discharging from external source every 72 (sec) into reservoir and drain water releasing though the exit gate every 36 (sec) in table 4.17

Table 4.15 Pollutant Concentration (kg/m^3) at observation points in reservoir case 1

Points\Time(sec)	25	50	75	100	125	150	175	200
A	0.4988	0.4794	0.4967	0.4973	0.5122	0.5187	0.4577	0.5291
B	0.4988	0.5148	0.5053	0.5177	0.4741	0.5029	0.4824	0.5098
C	0.5095	0.5106	0.4924	0.5015	0.4937	0.5508	0.4635	0.4929
D	0.5055	0.5032	0.4913	0.5083	0.5205	0.4926	0.5158	0.5115
E	0.5089	0.4953	0.5021	0.4628	0.5040	0.5078	0.5608	0.4713

Table 4.16 Pollutant Concentration (kg/m^3) at observation points in reservoir case 2

Points\Time(sec)	25	50	75	100	125	150	175	200
A	0.4988	0.4796	0.4969	0.4976	0.5126	0.5191	0.4581	0.5298
B	0.4988	0.5150	0.5055	0.5179	0.4745	0.5034	0.4828	0.5106
C	0.5095	0.5107	0.4926	0.5017	0.4940	0.5513	0.4640	0.4936
D	0.5055	0.5032	0.4915	0.5085	0.5208	0.4930	0.5163	0.5121
E	0.5089	0.4957	0.5024	0.4630	0.5047	0.5084	0.5613	0.4721

Table 4.17 Pollutant Concentration (kg/m^3) at observation points in reservoir case 3

Point\Time(sec)	25	50	75	100	125	150	175	200
A	0.4988	0.4855	0.4979	0.4982	0.5062	0.5150	0.4863	0.4968
B	0.4988	0.5084	0.4889	0.5196	0.4785	0.5247	0.4690	0.5296
C	0.5095	0.4953	0.4882	0.5073	0.4932	0.5377	0.4628	0.5097
D	0.5055	0.4911	0.4913	0.5189	0.5238	0.4882	0.5013	0.5221
E	0.5089	0.5067	0.5143	0.4597	0.4947	0.4921	0.5434	0.4835

4.3 Discussion

4.3.1 Water quality measurement in open-connected reservoirs with flat bottom topography

4.3.1.1 The numerical simulation of the hydrodynamic model

Open-connected reservoirs: Discharge water to the reservoir 1 then flow into second reservoir 2, open gate at the end of the reservoir. Figure 4.2-4.5 show elevations and velocity field at times 13.30-79.82 (sec).

the Rama 9 reservoir: Discharge water from 2 sides of the canal into the Rama 9 reservoir at initial time as shown in Figure 4.6-4.9 including the elevation and velocity field of water flow at different times.

4.3.1.2 The numerical simulation of the dispersion model

Open-connected reservoirs: Initial time discharges maximum concentration of pollution $C=1$ at door of reservoir and get lower when time increase by using exponential function $C=e^{-\frac{t}{40}}$. Pollution diffuses from reservoir 1 into reservoir 2 in Figure 4.11-4.12 show that pollution at time 13.30-79.82 (sec). Point A, B, C, and D in reservoir are observation points to see the change of the increase or decrease of the pollution with different times as shown in Figure 4.15. Pollution at point B and C is increases when time increased and point A at time 37.25 (sec), where pollution starts to decrease due to its location near the door. At initial time to 13.30 (sec), where the pollution at point D decreases and increases over time 13.30 (sec) because of the open gate at the end of the reservoir.

The Rama 9 reservoir: Initial time discharge maximum concentration of pollution $C=1$ at two gates of reservoir and get lower when time increase by using exponential function, $C=e^{-\frac{t}{40}}$. Pollution diffuses a little to reservoir because velocity is small as shown in Figure 4.13-4.14 where pollution at time 13.30-79.82 (sec). Point A, B, C, D and E in the Rama 9 reservoir are observation points as shown in Figure 4.16. At point B, D, E, pollution change slightly but points A and B pollution increases because of its location near the 2 gates.

4.3.2 Water quality measurement in an opened-closed reservoir with anisotropic bottom topography

In this research, a mathematical model to calculate the elevation of water, water current and pollutant concentration of water in an opened-closed reservoir with anisotropic bottom topography at any point and any time, anisotropic bottom topography function could be interpolated from data of reservoir bed coordinate, using cubic spline interpolate technique. When compared to other points, monitoring point B at the center of an opened-closed reservoir mostly had a high pollutant concentration, monitoring point D, near the entrance gate of reservoir, has a mostly lower pollutant concentration as shown in Figure 4.33-4.34.

4.4 Conclusion

4.4.1 Water quality measurement in open-connected reservoirs with flat bottom topography

We have revised a mathematical model that combines a non dimensional form of hydrodynamic model and a dispersion model. The model is suitable to the Rama 9 reservoir. The Lax-Wendroff method is used to solve a non dimensional form of a shallow water equation that gives the numerical solutions of the water velocities in x- and y- directions and elevation. The forward differences in time and backward difference in space with approximated velocities in x- and y- directions are also used to solve the advection-diffusion equation. The result of this research shows that the proposed model can approximate the water velocities in x- and y- directions, the elevation, and the concentration of the pollutants in the Rama 9 reservoir at any various time and position. The accuracy of approximation is within the units of centimeters and seconds. The pollutant concentrations on the floodgates are decreased at/or below initial pollutant concentration. The Rama 9 reservoir is to two flood gates. Opened both of floodgates are the sources of pollutant concentration that discharge wastewater into the reservoir. The linked gate is connection of between the north reservoir and the south reservoir. It is an obstacle of the pollutant dispersion for a short time. The linked gate can dilute the pollutant concentration of the north reservoir to the south reservoir in long period. The proposed model and numerical techniques can be applied to other water sources having non-uniformly distributed water flows.

4.4.2 Water quality measurement in an opened-closed reservoir with Anisotropic bottom topography

To conclude the numerical simulation for water-quality measurement model in an opened-closed reservoir with an empirical anisotropic bottom topography was proposed and thus the mathematical models could calculate the elevation, the velocities and the pollutant concentration of water. The very models could adjust the bottom topography according to the varying reservoir bed, simulate the wave maker function at the entrance gate of reservoir from field data by using the data interpolation in order to have a more realistic water current and water quality approximations in an opened-closed reservoir with anisotropic bottom topography.

Chapter 5
Water Quality Control
In Open-Connected Reservoirs

In this section, two mathematical models for simulating water pollutant level and pollution control in a connected reservoir system are proposed. The reservoir system is consisted of two ponds connected by a narrow channel. One pond allows water coming in from a canal through an entrance gate while the other pond lets the water flow out through an exit gate. The pond water is contaminated with wastewater released from several industrial plants located near the pond. One of the proposed models is a steady-state dispersion model simulating the pollutant level in the connected ponds. The other model is a pollution control model that determines the maximum pollutant level allowed in the wastewater released from each plant in order to achieve a specified pollution level in the ponds as well as to incur a minimum water pre-treatment cost to each plant. The simulation results of these models show that the maximum pollutant level in the two ponds could be effectively controlled at a minimum cost to each plant by optimally limiting the pollutant level in the wastewater it releases.

5.1 Steady state of dispersion model

In this section, various results are reported in a table, several surface and contour plots, and a comparison graph. Simulation runs of the dispersion model were performed with these following settings: five plants F_1, F_2, F_3, F_4 and F_5 at locations shown in Figure 4(b) discharging wastewater at various pollutant levels shown in Table 5.1; rate of change of incoming pollutant level with respect to x-coordinate at the entrance gate of c_1 (kg/m^4); rate of change of outgoing pollutant level with respect to y-coordinate at the exit gate of c_2 (kg/m^4); the same average water velocities in x- and y- directions of -0.025 (m/s), which was also employed in [3]; and diffusion coefficient of $D = 10$ (m^2/s).

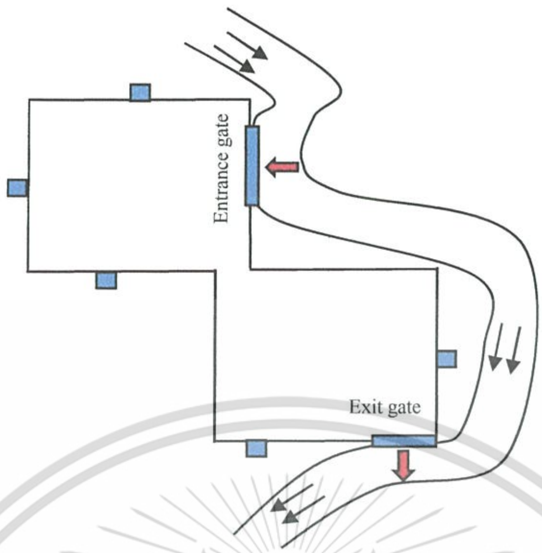


Figure 5.1 Open-connected reservoirs with openings to a canal and five plants

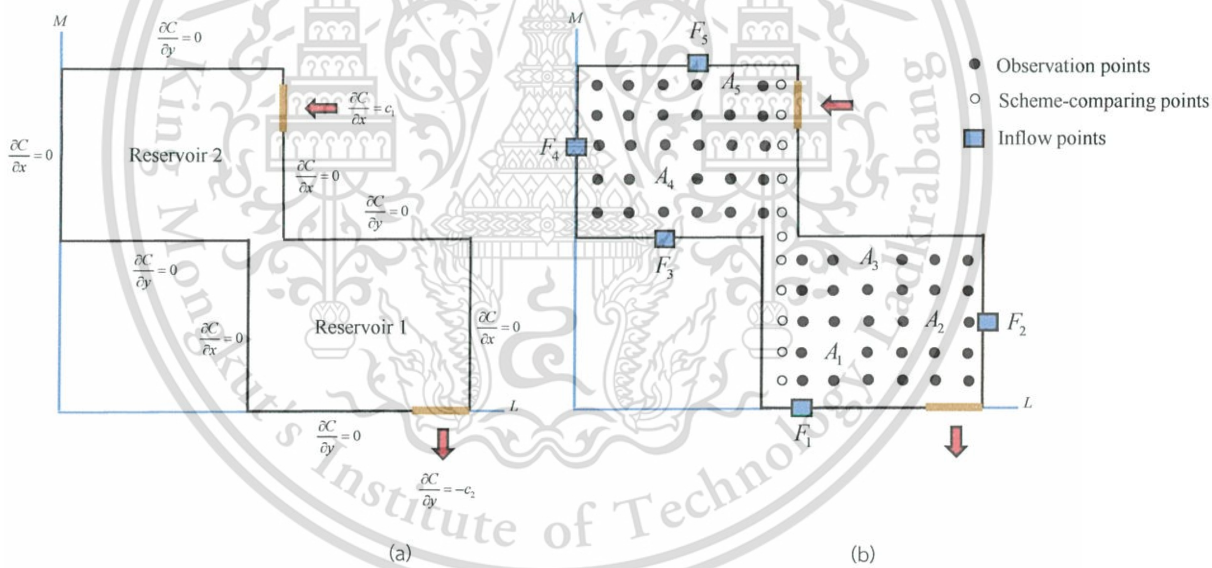


Figure 5.2 The domain of the steady-state dispersion model (a); the observation points, the scheme-comparing points, and the wastewater release points (b)

Table 5.1 shows the average pollutant concentrations in the upper pond (pond 1) and lower pond (pond 2) for 9 different combinations of $c_1, c_2, F_1, F_2, F_3, F_4$ and F_5.

Table 5.1 Average pollutant concentration in pond 1 and pond 2 for 9 Combinations of parameter settings

Case	c_1 (kg/m^4)	c_2 (kg/m^4)	Concentration in discharge (kg/m^3) from plant F_1	F_2	F_3	F_4	F_5	Average concentration in pond (kg/m^3) Pond 1	Pond 2
1	0.0100	0.0100	5.0000	5.1500	5.2000	5.6000	5.3400	5.4222	5.1850
2	0.0100	0.0500	5.0000	5.1500	5.2000	5.6000	5.3400	5.4191	5.1844
3	0.0100	0.1000	5.0000	5.1500	5.2000	5.6000	5.3400	5.4152	5.1837
4	0.0100	0.0050	5.0000	5.1500	5.2000	5.6000	5.3400	5.4226	5.1851
5	0.0100	0.0100	4.5000	4.6500	4.7000	5.1000	4.8400	4.9222	4.6850
6	0.0100	0.0500	4.5000	4.6500	4.7000	5.1000	4.8400	4.9191	4.6844
7	0.0100	0.1000	4.5000	4.6500	4.7000	5.1000	4.8400	4.9152	4.6837
8	0.0100	0.0050	4.5000	4.6500	4.7000	5.1000	4.8400	4.9226	4.6851
9	0.0100	1.0000	4.5000	4.6500	4.7000	5.1000	4.8400	4.8454	4.6706

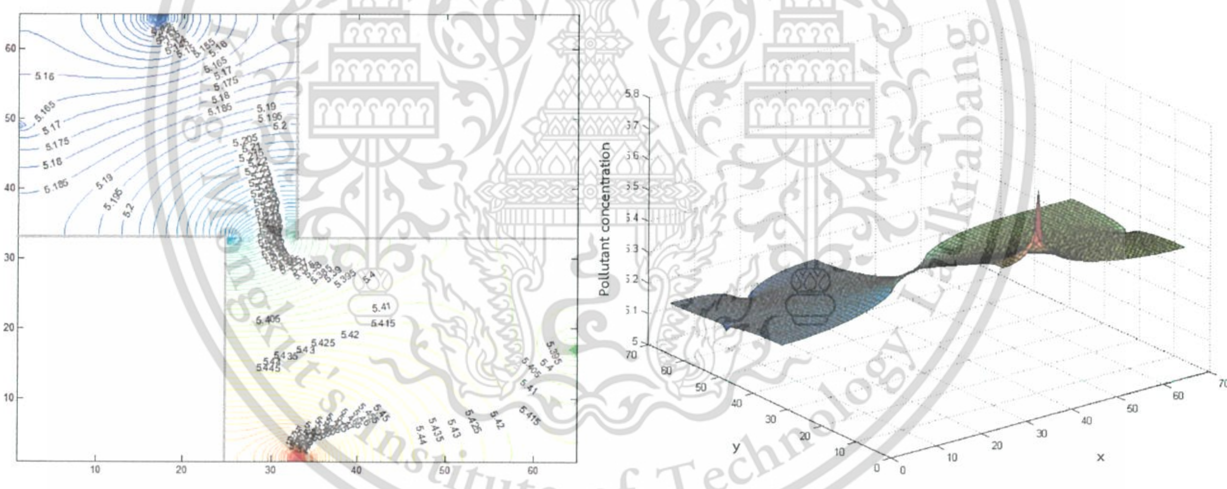


Figure 5.3 Pollutant concentration (kg/m^3) in open-connected reservoirs $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 1 in table 5.1) (a) Contour plot and (b) Surface plot

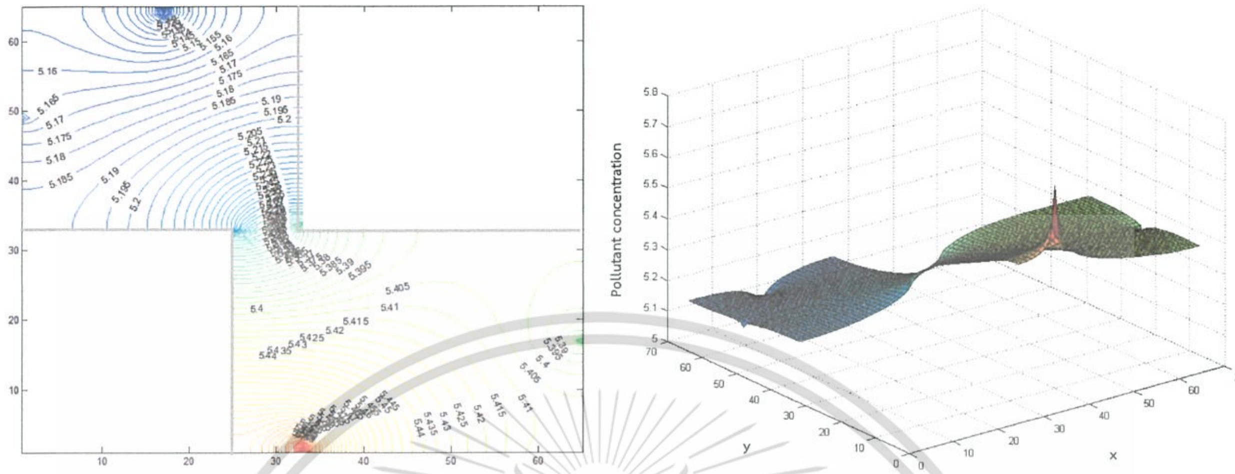


Figure 5.4 Pollutant concentration (kg/m^3) in open-connected reservoirs $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 2 in table 5.1) (a) Contour plot and (b) Surface plot

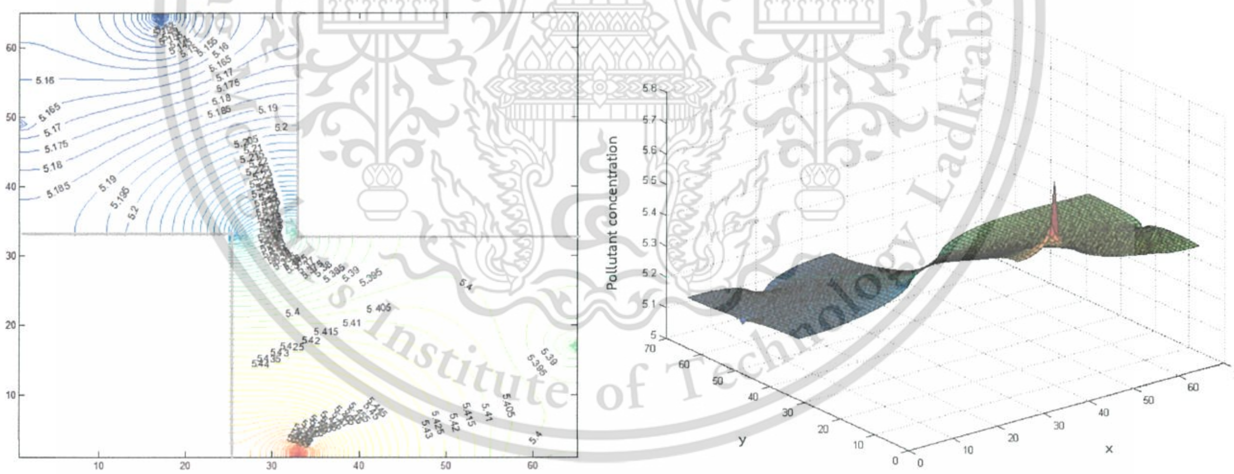


Figure 5.5 Pollutant concentration (kg/m^3) in open-connected reservoirs $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 3 in table 5.1) (a) Contour plot and (b) Surface plot

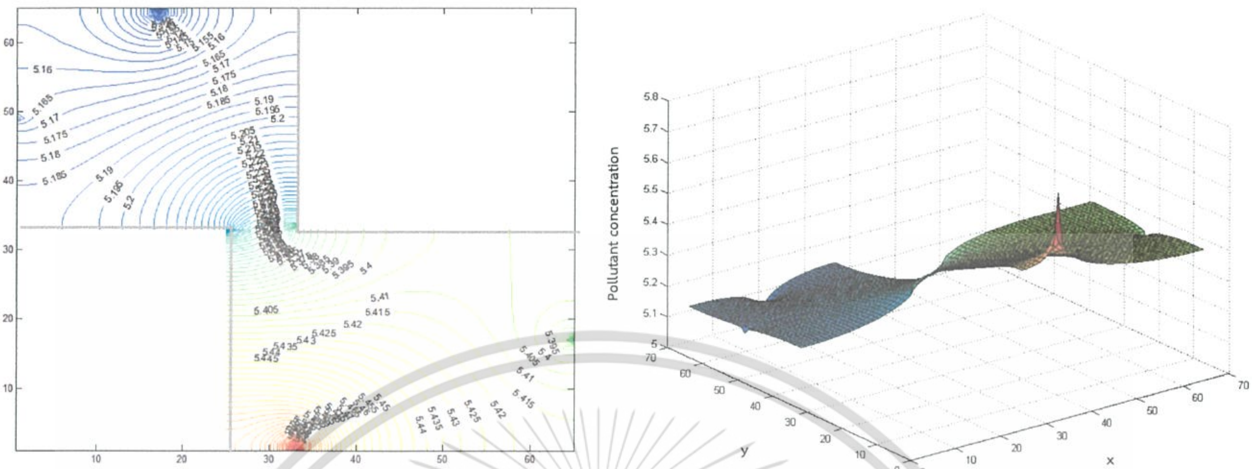


Figure 5.6 Pollutant concentration (kg/m^3) in open-connected reservoirs $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 4 in table 5.1) (a) Contour plot and (b) Surface plot

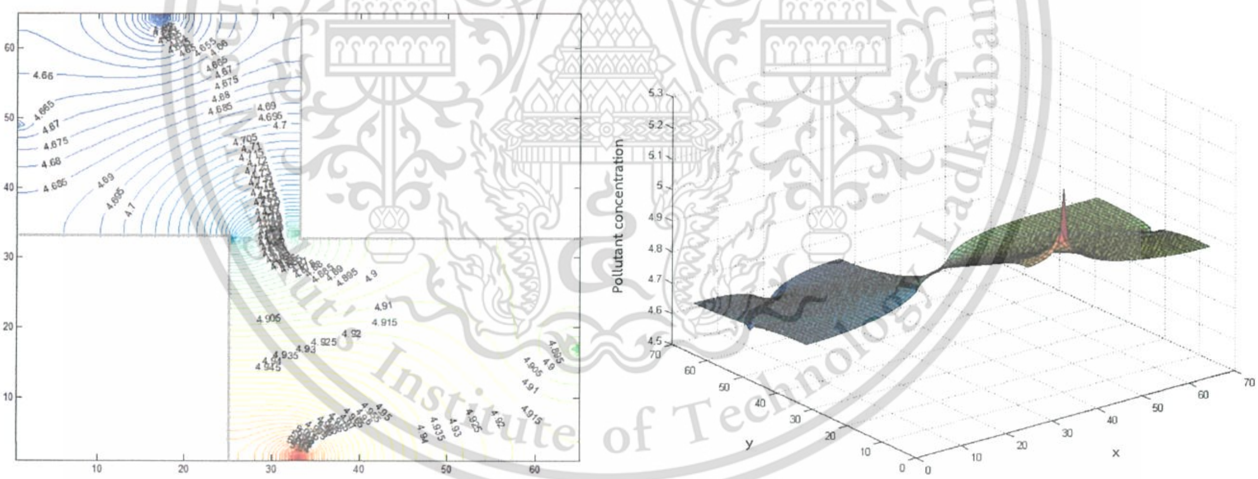


Figure 5.7 Pollutant concentration (kg/m^3) in open-connected reservoirs $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 5 in table 5.1) (a) Contour plot and (b) Surface plot

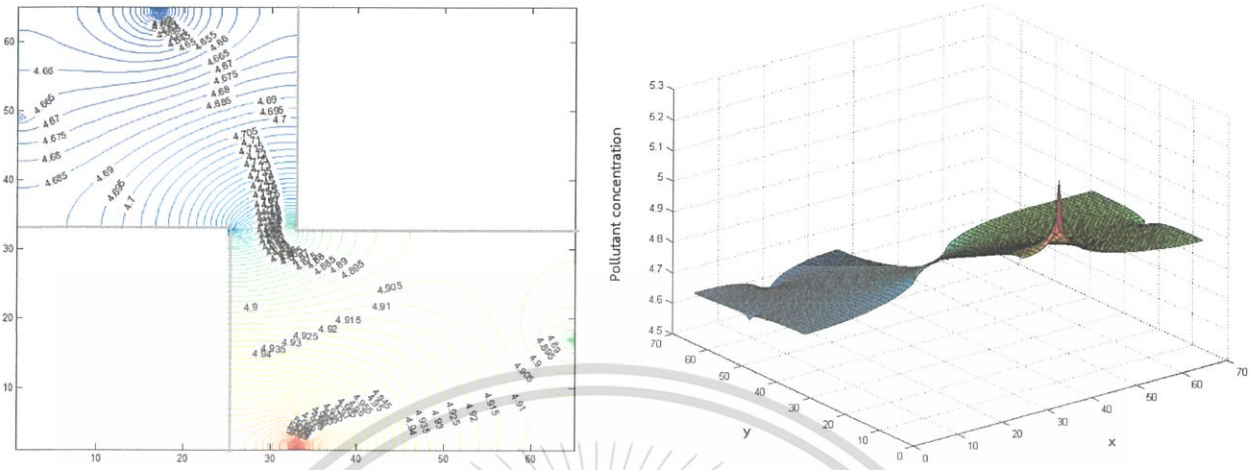


Figure 5.8 Pollutant concentration (kg/m^3) in open-connected reservoirs $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 6 in table 5.1) (a) Contour plot and (b) Surface plot

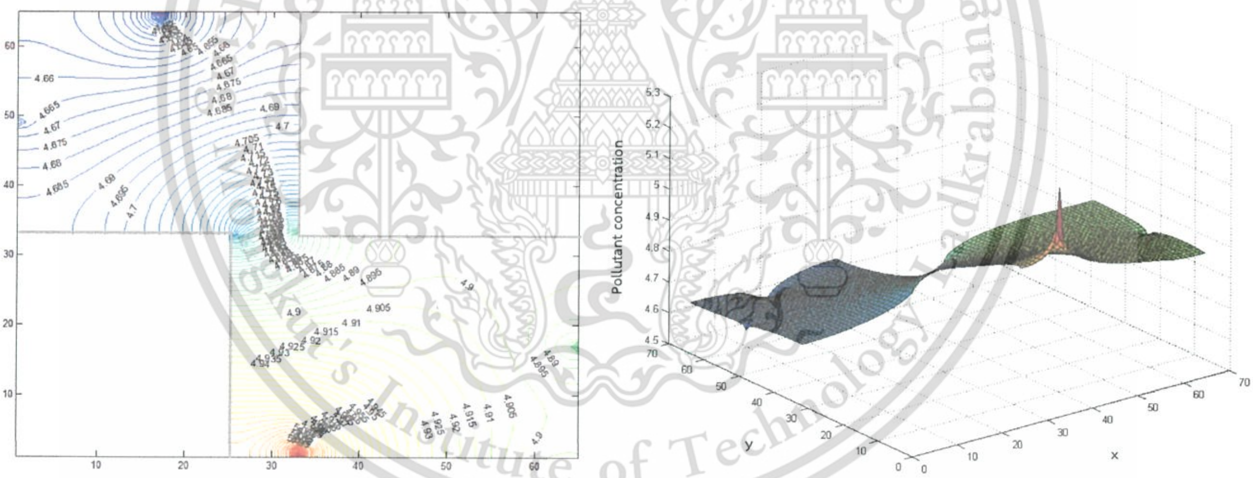


Figure 5.9 Pollutant concentration (kg/m^3) in open-connected reservoirs $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 7 in table 5.1) (a) Contour plot and (b) Surface plot

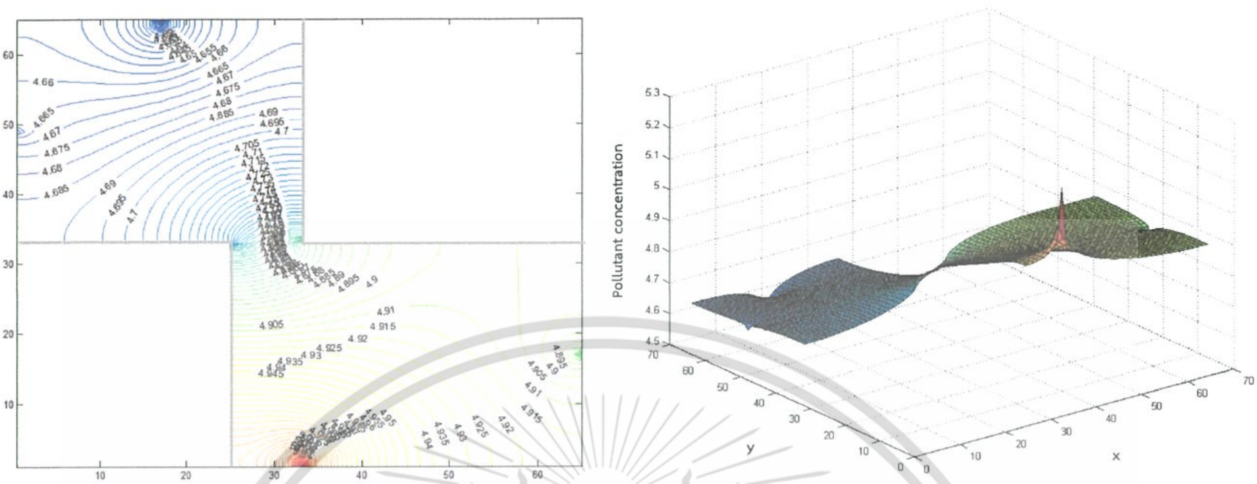


Figure 5.10 Pollutant concentration (kg/m^3) in open-connected reservoirs $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 8 in table 5.1) (a) Contour plot and (b) Surface plot

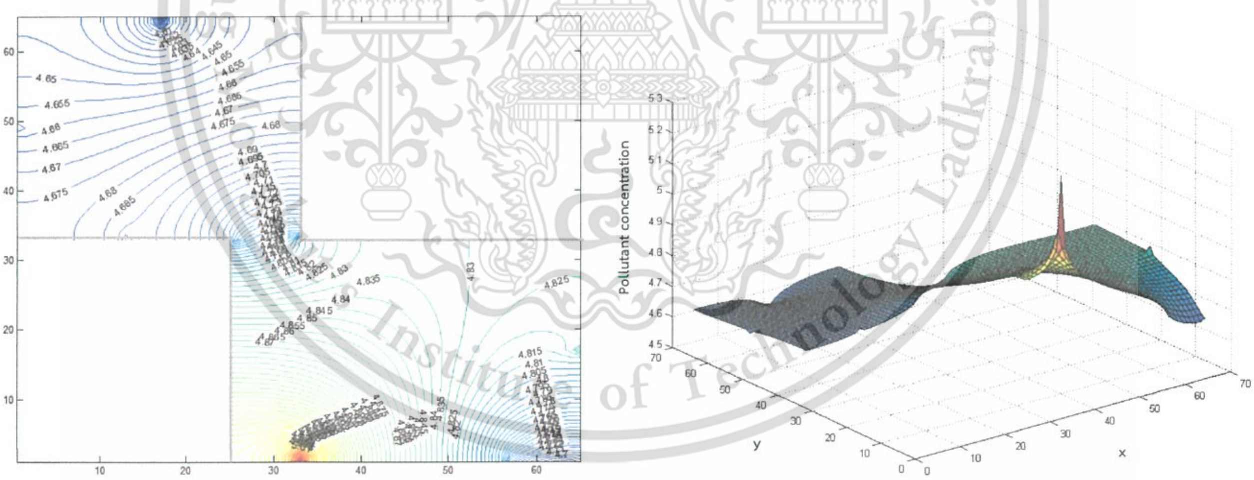


Figure 5.11 Pollutant concentration (kg/m^3) in open-connected reservoirs $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 9 in table 5.1) (a) Contour plot and (b) Surface plot

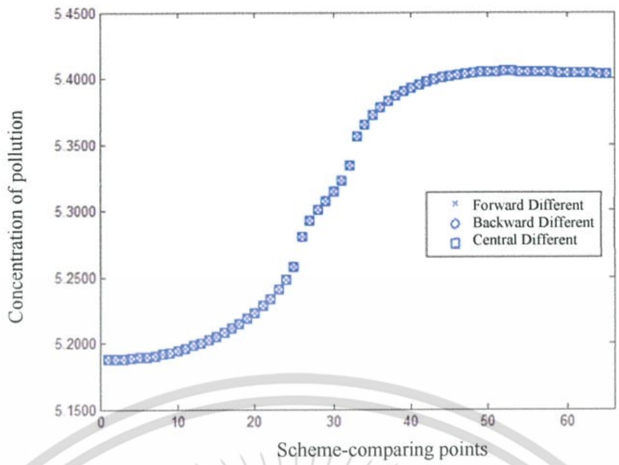


Figure 5.12 Pollutant concentration at the scheme-comparing points calculated by the backward, forward, and central difference techniques

5.2 Pollution control and cost optimization

In this section, the method of calculation of optimal cost is described. A plant had to incur a cost to pretreat its wastewater before releasing it into the pond, in order to keep the pollutant level in its wastewater discharge to stay below a certain level. The more extensive the pre-treatment, the higher the treatment cost. The criterion for acceptable level of pollutant in the released wastewater was that the pollutant levels in the pond water at all of the observation points had to be lower than a specified standard level. The higher the pollutant level in the pond water than the standard level was, the more extensive pre-treatment was, and consequently, the higher the cost to the plant was.

Let x_β be the observation points and r_α be the reduction of pollutant concentration in the wastewater at its release points. It follows that $C_\alpha - r_\alpha$ is the pollutant concentration after pre-treatment (or partial purification) at the release points. From Example 3.5.3, we have

$$[A]\{C\} = \{B\}, \tag{5.1}$$

which can be expressed as,

$$\{C\} = [A]^{-1}\{B\}. \tag{5.2}$$

Let

$$\{C\}=[D]\{B\} \tag{5.3}$$

where

$$[A]^{-1}=[D]. \tag{5.4}$$

Let $\tilde{C}_\beta$ be the pollutant concentration at an observation point x_β. Let r_α be the reduction of pollutant concentration at the wastewater release points; then, Eq.(5.3) becomes,

$$\tilde{C}_\beta = d_{\beta 1}b_1 + \ldots + d_{\beta\alpha}(b_\alpha - r_\alpha) + \ldots + d_{\beta N}b_N \tag{5.5}$$

or

$$\tilde{C}_\beta = \sum_{i=1}^{m} d_{\beta i}b_i + \sum_{j=1}^{n} d_{\beta\alpha_i}\left(b_{\alpha_j} - r_{\alpha_j}\right), \tag{5.6}$$

where m is the number of observation points and n is the number of wastewater release points $(N=m+n)$.

Let C_{ST} be the standard allowable pollutant level in the pond water. $\tilde{C}_\beta$ must be at or below this level, i.e., $\tilde{C}_\beta \le C_{ST}$ (5.7)

The objective function J is the cost of wastewater pre-treatment, so

$$J(x) = \sum_{j=1}^{m} \omega_j r_{\alpha_j}, \tag{5.8}$$

where ω_j is the cost of wastewater pre-treatment for the required reduction of pollutant concentration. The constraints are

$$\tilde{C}_\beta = \sum_{i=1}^{m} d_{\beta i}b_i + \sum_{j=1}^{n} d_{\beta\alpha_i}\left(b_{\alpha_j} - r_{\alpha_j}\right) \le C_{ST}. \tag{5.9}$$

The upper bound and lower bounds of the controls are,

$$l_{\alpha_j} \le r_{\alpha_j} \le u_{\alpha_j} \tag{5.10}$$

and the controls are non-negative

$$r_{\alpha_j} \ge 0, \tag{5.11}$$

where $l_{\alpha_j}, u_{\alpha_j}$ are the lower and upper bounds, respectively, of the reduction of pollutant level in the wastewater, specifying the minimum and maximum reduction of pollutant level in the wastewater that a plant can reduce by pre-treatment.

5.3 Numerical optimization examples

Example 5.1: Arbitrary indicators

This optimal control problem was solved by the Simplex method. There were 5 water treatment plants that discharged wastewater into the connected ponds. Plant 1 to Plant 5 had the ability to purify their wastewater such that the maximum reductions in their pollutant concentration were 1.0, 1.0, 1.5, 1.5 and 1.0 (kg/m^3) respectively, while the minimum reduction specified by the law was 0.5 (kg/m^3). The physical parameters settings were the following: diffusion coefficient of $D = 10.0$ (m^2/s) and the same velocities in the x- and y- directions of -0.025 (m/s). The standard allowable pollutant level in the pond water was 4.2 (kg/m^3). Therefore, all constraints were as follows:

$$C_{ST} = 4.2, \tag{5.12}$$

$$0.5 \le r_{\alpha_1} \le 1.0, \tag{5.13}$$

$$0.5 \le r_{\alpha_2} \le 1.0, \tag{5.14}$$

$$0.5 \le r_{\alpha_3} \le 1.5, \tag{5.15}$$

$$0.5 \le r_{\alpha_4} \le 1.5, \tag{5.16}$$

$$0.5 \le r_{\alpha_5} \le 1.0. \tag{5.17}$$

Table 5.2 Pollutant concentration at 5 observation points and 5 wastewater release points

	Pollutant concentration			
Points	Untreated Inflow	Observations	Pre-treated Inflow	Observations
A_1		5.4236		4.1160
A_2		5.3689		4.1528
A_3		5.3944		4.1276
A_4		5.1723		4.0759
A_5		5.1465		4.1825
F_1	5.0000	5.0000	4.5000	4.5000
F_2	5.1500	5.1500	4.2179	4.2179
F_3	5.2000	5.2000	3.7150	3.7150
F_4	5.6000	5.6000	4.1000	4.1000
F_5	5.3400	5.3400	4.3400	4.3400

Third column in Table 5.2 shows pollutant concentrations at observation and wastewater release points before the wastewaters were pre-treated by the plants. They were higher than the standard, C_{ST}. After pre-treatment, the concentrations at these points were lower than C_{ST}, as shown in column 5.

Table 5.3 Optimal cost of wastewater treatment

Treatment Factory	Location	Optimal Reduction of Pollutant (kg/m^3) Concentration	Cost of Treatment for Reduction by 1 (kg/m^3)	Optimal Cost of Reduction (USD)	Non-optimal Reduction of Pollutant (kg/m^3) Concentration	Non-optimal Cost of Reduction (USD)
Factory1	F_1	0.5000	40,000	20,000.00	0.8000	32,000.00
Factory2	F_2	0.9321	20,000	18,642.00	0.9500	19,000.00
Factory3	F_3	1.4849	15,000	22,273.50	1.0000	15,000.00
Factory4	F_4	1.5000	20,000	30,000.00	1.4000	28,000.00
Factory5	F_5	1.0000	60,000	60,000.00	1.1400	68,400.00
Total Cost				150,915.50		162,400.00

In Table 5.3, third column and sixth column show the extents of pollutant reduction in terms of concentration. The Non-optimal reduction was for achieving 4.2 (kg/m^3) pollutant level, C_{ST}, in the wastewater at the release points of all 5 plants, while the optimal reduction was for achieving a C_{ST} pollutant level or lower in the pond water. These reductions multiplied by the corresponding cost of treatment to each plant (column 4) gave the Non-optimal cost of reduction (column 7) and the optimal cost of reduction (column 5). It can be seen that the total cost of optimal wastewater treatment was 150,915.50 (USD), significantly lower than 162,400.00 (USD) of the Non-optimal treatment.

Example 5.2: Biochemical oxygen demand (BOD) indicators

The Pollution Control Department (PCD) has determined that wastewater discharged from industrial plants to reservoir with the BOD not exceeding 20 (kg/m^3), or depending on particular reservoirs or plants' types as agreed by the PCD, but not exceed 60 (kg/m^3), and that the water in each reservoir is not considered wastewater if of 6.5 (kg/m^3) BOD. Given that 5 industrial plants nearby the reservoir are permitted to discharge wastewater with BOD of 11.0 (kg/m^3), 10.5 (kg/m^3), 9.0 (kg/m^3), 8.0 (kg/m^3) and 10.0 (kg/m^3) respectively and that each plant could have water treated differently where, plant 1 and 5 are able to treat wastewater most with the BOD of 4.0 (kg/m^3), plant 2 and 4 with wastewater treated at BOD of 4.5 (kg/m^3) and plant 3 with the ability to treat wastewater at the BOD of 5.0 (kg/m^3). All plants, however, are to have wastewater treated with the minimum of the BOD of 3.0 (kg/m^3).

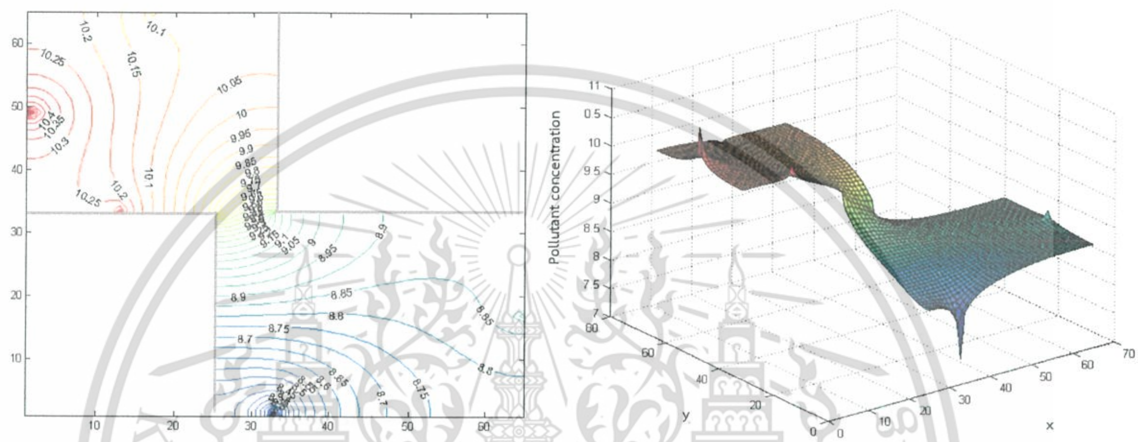


Figure 5.13 Pollutant concentration (kg/m^3) in open-connected reservoirs $\Delta x = 0.015625, \Delta y = 0.015625$, discharge wastewater at F_1 11 (kg/m^3), F_2 10.5 (kg/m^3), F_3 9.0 (kg/m^3), F_4 8.0 (kg/m^3) and F_5 10 (kg/m^3) (a) Contour plot and (b) Surface plot

Table 5.4 Optimal cost of wastewater treatment in case of discharge wastewater at F_1 11 (kg/m^3), F_2 10.5 (kg/m^3), F_3 9.0 (kg/m^3), F_4 8.0 (kg/m^3) and F_5 10 (kg/m^3)

Treatment Factory	Location of Pollutant (kg/m^3) Concentration	Optimal Reduction for Reduction by 1 (kg/m^3)	Cost of Treatment of Reduction (USD)	Optimal Cost
Factory1	F_1	4.0000	20,000.00	80,000.00
Factory2	F_2	4.5000	15,000.00	67,500.00
Factory3	F_3	3.0000	60,000.00	180,000.5
Factory4	F_4	4.2015	20,000.00	84,030.00
Factory5	F_5	4.0000	40,000.00	160,000.0
Total Cost				571,530.5

Example 5.3: There is no feasible solution

Given that 5 industrial plants nearby the reservoir are permitted to discharge wastewater with BOD of 13.0 (kg/m^3), 15.0 (kg/m^3), 15.0 (kg/m^3), 19.0 (kg/m^3) and 12.0 (kg/m^3) respectively and that each plant could have water treated differently where, plant 1 to 5 are able to treat wastewater most with the BOD of 10.0 (kg/m^3). All plants, however, are to have wastewater treated with the minimum of the BOD of 7.0 (kg/m^3) and that the water in each reservoir is not considered wastewater if there is only 6.5 (kg/m^3) of BOD. In this case, we cannot obtain any feasible solutions.

5.4 Discussion

In this study, mathematical models were proposed for determining pollution levels in the water of a connected-pond reservoir, with openings to a canal, polluted by wastewater discharges from nearby industrial plants. In the equations of the models, the parameters affecting the dispersion of pollution were water velocity and diffusion coefficient, but they did not have a significant effect in this study. More influential was the initial pollutant level in the wastewater discharge from industrial plants, as shown in Table 5.1 In Case 3, its c_2 was 10 times higher than that of Case 1, thus, making the average pollutant level in the pond water slightly but significantly lower. In Case 1 and Case 7, there c_2 were the same, but the initial pollutant level in Case 7 was reduced by 0.5 (kg/m^3). It can be seen that the average pollutant level in the pond water in this case was 5 times lower than the reduced level in Case 3. Figure 5.3-5.11 show contour and surface plots of pollutant level versus two spatial coordinates. The contour plots show different patterns of pollutant dispersion of the three cases, while the surface plots better show the different pollutant levels at various locations. Figure 5.12, shows plots of pollutant levels at scheme-comparing points calculated by 3 different finite difference schemes—backward, forward, and central. The curves of the 3 plots matched perfectly, indicating that all 3 schemes were equally valid.

Concerning the costs to the industrial plants, they were optimized with respect to the pollutant level in their wastewater discharge under the condition that the pollutant level in the pond water not exceed the acceptable standard. If every plant can control the pollutant level in its wastewater discharge to match its corresponding optimal level shown in Table 5.3, they can save 11,484.50 (USD) of their wastewater pre-treatment cost in a year.

5.5 Conclusion

In this research, we proposed a numerical simulation of water-quality control using a couple of an optimization method and an implicit finite difference technique in the two ponds with an entrance and an exit gate to open water of a canal. The simulation results of these models showed that the maximum pollutant level in the two ponds could be effectively controlled at a minimum cost to each plants by optimally limiting the pollutant level in the wastewater it released.

5.6 Future works

In the first part, this research can lead to improving hydrodynamic model by adding the colioris force to the governing equation, making it more realistic when it comes to describing the water flow in the reservoir.

In the second part, it is recommended that, in the future research, wastewater be treated within the reservoir as to make the water quality better by using pollutant removal mechanism, not only having the exit gate or wastewater treatment prior to releasing it into the reservoir.

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Appendix

The research papers

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A numerical treatment of a non-dimensional form of a water quality model in the Rama-nine reservoir

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Abstract

The purposes of this research are to develop mathematical models and numerical methods for approximating water flow directions and pollution levels in a Rama-nine reservoir, Pathumthani, Thailand with non-uniform current. The Rama-nine reservoir is opened with two parallel canals. The pollution levels in a reservoir are assessing via data collection at the real field. It is quite complex and the results obtained tentatively deviate from one point of time and position to another. There are many research works applied a mathematical model called the dispersion model to estimate the water pollutant concentration. The approximation accuracy received is seemingly unsatisfied, especially, when the water flow is not uniformly distributed.The research begins with revising a mathematical model that combines two existing mathematical models: a non-dimensional form of hydrodynamic model and a dispersion model. The model is to make suitable to the Rama-nine reservoir.The Lax-Wendroff method is subsequently used in a non-dimensional form of a shallow water equation to approximate the water velocity and elevation. Next, we use the forward differences in time and backward difference in space in advection-diffusion equation. Combined the equation with the calculated velocity is thus used in the dispersion model to approximate the concentration levels of the pollutants. The result of this research showed that the proposed

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concentration of the pollutants in a Rama-nine reservoir at any various time and position. The accuracy of approximation is within the units of centimeters and seconds. In addition, this research has shown that the proposed model can be applied to other water sources having non-uniformly distributed water flows.

1. Introduction

There are many methods for detecting the level of pollutants in the-water mostly are conducted by a field measurement and a mathematical simulation. For the shallow water mass transport problems that are presented in [1], the method of characteristics has been reported as being applied with success, but it presents in real cases some difficulties. In [3], [8], [13], [14], [15] and [16], the finite element method for solving steady and unsteady water pollution measurements are presented. The various numerical techniques to solve the uniform flow of stream water quality model are presented in [5], [11] and [12].The numerical methods to approximate the solution of the two-dimensional advection-diffusion-reaction equation are proposed in [6-10].

Most non-uniform flow models need the input data concerned with the velocity of the current at any point and any time in the domain. The hydrodynamics model provides the velocity field and tidal elevation of the water. In [4-12], the hydrodynamics model and advection-diffusion equation were used to approximate the velocity of the water current in a bay and a channel. In [7, 16], the results from hydrodynamic model are data for the non-uniform flow of the advection-diffusion-reactione quation, which provides the pollutant concentration field. The term of the friction forces due to the drag of sides of the uniform reservoir is considered. They found the theoretical solution of the model at the end point of the domain. They also use theanalytical solution to check the accuracy of our approximate solution.In [7], they propose the Lax-Wendroff method with stability analysis to solve the two-dimensional hydrodynamic model with a rectangular domain.

In this research, we begin with modifying a mathematical model that combines two existing mathematical models: a non-dimensional form of hydrodynamic model and a dispersion model. The proposed model is to make suitable to the Rama-nine reservoir.The shallow water equation of the hydrodynamic model is assumed by averaging the equation over the depth with flat bottom topography, and discarding the term due to the Coriolis force and surface wind affect. Combined the equation with the calculated velocity is thus used in the dispersion model to approximate the concentration levels of the pollutants.

2. Water-Quality Model

2.1 *The hydrodynamic model*

The unsteady flow of water in a two-dimensional space can described by the shallow water equations, which represent mass and momentum conservation. It can be obtained by depth averaging the Navier-Stokes equations in the vertical direction. This leads to a two-dimensional formulation in terms of depth averaged quantities and the water depth itself [9] and, neglecting diffusion of momentum due to turbulence, they form the following system of equations: The continuity equation

$$\frac{\partial \zeta}{\partial t}+\frac{\partial Hu}{\partial x}+\frac{\partial Hv}{\partial y}=0 \qquad \text{(a)}$$

The momentum equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv+g\frac{\partial \zeta}{\partial x}=$$
$$\frac{1}{\rho H}\left\{\frac{\partial}{\partial x}\left(\mu_\varepsilon H\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu_\varepsilon H\frac{\partial u}{\partial y}\right)\right\}+\frac{KW^2\cos\Psi}{H}-\frac{gu(u^2+v^2)^{\frac{1}{2}}}{HC^2}, \qquad \text{(b)}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu+g\frac{\partial \zeta}{\partial y}=$$
$$\frac{1}{\rho H}\left\{\frac{\partial}{\partial x}\left(\mu_\varepsilon H\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu_\varepsilon H\frac{\partial v}{\partial y}\right)\right\}+\frac{KW^2\cos\Psi}{H}-\frac{gv(u^2+v^2)^{\frac{1}{2}}}{HC^2}, \qquad \text{(c)}$$

where h is the depth measured from the mean water to the bed of reservoir, (m)
ζ is the elevation from the mean water level to the temporary water surface, (m)
$H = h + \zeta$ is total depth of the sea, (m)
g is the acceleration due to gravity, (m/s^2)
u, v are the velocity components, (m/s)
f is Coriolis factor,

ρ is density of the sea water, (kg/m^3)
μ_k is eddy viscosity,
K is the non-dimension coefficient of a superficial force due to the wind blowing on the surface,
W is wind speed, (m/s)
Ψ is the angle of the wind direction from east,
C is the Chezy coefficient of the friction on the sea bed. $(m^{1/2}/s)$

The continuity and momentum equation in (a)–(c) discarding the terms due to friction and wind [10],

$$\frac{\partial \zeta}{\partial t} + \frac{\partial Hu}{\partial x} + \frac{\partial Hv}{\partial y} = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial \zeta}{\partial y} = 0. \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial \zeta}{\partial x} = 0, \tag{3}$$

The governing equation of hydrodynamic behavior of reservoir. Averaging the equations over the depth, discarding the term of Coriolis factor, shearing stresses and surface wind and h is a constantsince the

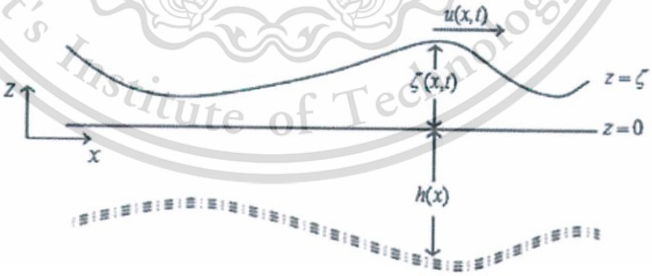


Figure 1
Vertical cross-section of the water in the estuary

sea bed is bottom topography [3]. We have two-dimension shallow water equation is:

$$\frac{\partial\zeta}{\partial t}+\frac{\partial(h+\zeta)u}{\partial x}+\frac{\partial(h+\zeta)v}{\partial y}=0, \tag{4}$$

$$\frac{\partial u}{\partial t}+g\frac{\partial\zeta}{\partial x}=0, \tag{5}$$

$$\frac{\partial v}{\partial t}+g\frac{\partial\zeta}{\partial y}=0. \tag{6}$$

We assume h is constant and $h << \zeta$, and then have $\zeta = h + \zeta$ equations (4) – (6) become.

$$\frac{\partial\zeta}{\partial t}+h\frac{\partial u}{\partial x}+h\frac{\partial v}{\partial y}=0, \tag{7}$$

$$\frac{\partial u}{\partial t}+g\frac{\partial\zeta}{\partial x}=0, \tag{8}$$

$$\frac{\partial v}{\partial t}+g\frac{\partial\zeta}{\partial y}=0. \tag{9}$$

We will transform equations (7) – (9) into non-dimensional form [2] by letting $U = u/\sqrt{gh}$

$V = v/\sqrt{gh}, X = x/l, Y = y/l, Z = \zeta/h, T = t\sqrt{gh}/l.$ Substituting into equation (7)–(9) leads to

$$\frac{\partial Z}{\partial T}+\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0, \tag{10}$$

$$\frac{\partial U}{\partial T}+\frac{\partial Z}{\partial X}=0, \tag{11}$$

$$\frac{\partial V}{\partial T}+\frac{\partial Z}{\partial Y}=0. \tag{12}$$

By changing variables *U*, *V*, *Z* to the *u*, *v*, *d* respectively, we see that

$$\frac{\partial d}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{13}$$

$$\frac{\partial u}{\partial t}+\frac{\partial d}{\partial x}=0, \tag{14}$$

$$\frac{\partial v}{\partial t}+\frac{\partial d}{\partial y}=0. \tag{15}$$

2.2 *The dispersion model*

The distributed pollutant process satisfies the mass transfer equation, which includes transportation and diffusion. Averaging the equation over the depth, we get the advection–diffusion equation,

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right), \tag{16}$$

where $C(x,y,t)(kg/m^3)$ is the concentration averaged in depth at the point (x, y) and at time t, $D(m^2/s)$ is the diffusion coefficient.

2.3 *The boundary and initial condition for hydrodynamic model*

There are the water flow form two parallel canals through the opened gates such that the water is drained as Fig. 1(a).The gates are opened, so the water is going to flow into Rama-nine reservoir. There is no drained water as Fig. 2(a). The elevation of water on the both gates is assumed to be a wave maker function. It is assumed to be a trigonometry function $f(x,t)=0.1\sin(\pi(x+t))$. The initial condition in the reservoir is given by the cold start technique i.e. the velocity in x- and y- directions and the elevationare assumed to be motionless.

The boundary conditions are assumed that the elevation is zero except at the both gates. There are no parallel velocities in x- and y-directions along with the boundary of reservoir.

2.4 *The boundary and initial condition for dispersion model*

The water pollutant is discharged into reservoir by assuming as the exponential function $e^{-\frac{t}{40}}$ on the both opened gates as Fig. 2(a). There is no opened gate that acting as a draining gate as Fig. 2(b). Assume that the initial concentration of pollutant is $c = 0.02$ and there is no rate of change of pollutant concentration along the boundary.

3. Numerical Techniques for A Water-Quality Model

The hydrodynamic model provides the elevation of water and velocity vector field. The calculated results of the hydrodynamic model are input to the dispersion model which provides the pollutant concentration field. Firstly, the Lax-Wendroff method is subsequently used in a non-dimensional form of a shallow water equation to approximate the water velocity and elevation.

3.1 *Numerical method for the hydrodynamic model*

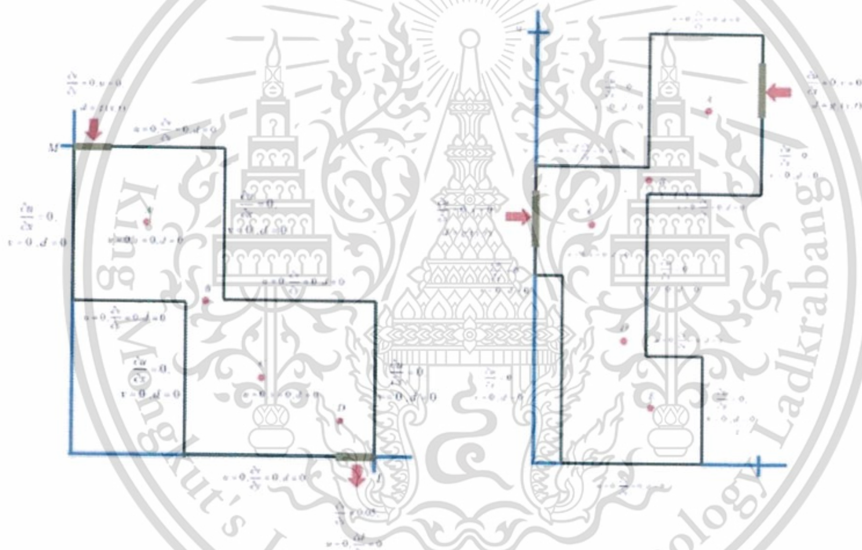
The equations (13)–(15) can be written in the matrix form

$$\frac{\partial U}{\partial t} = A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y}, \tag{17}$$

where

$$U = \begin{pmatrix} d \\ u \\ v \end{pmatrix}, A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \tag{18}$$

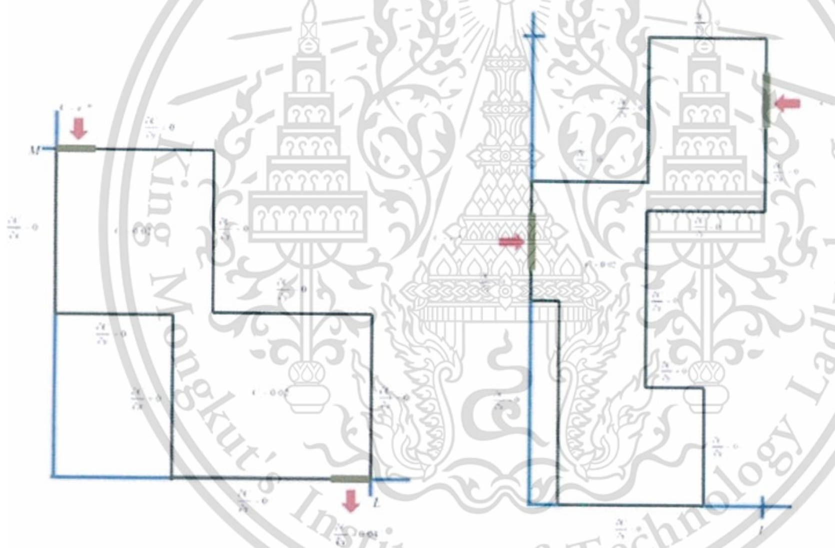
From Eq.(17) dividing the interval [0, 1] into L and M subintervals such that $L\Delta x = 1$ and $M\Delta y = 1$, and the interval $[0, T]$ into N subintervals such that $N\Delta t$ = T. We can approximate $d(x_l, y_m, t_n) = d^n_{l,m}$ are value the difference approximation of $d(x, y, t)$ at point $x = l\Delta x$, $y = m\Delta y$ and $t = n\Delta t$



(a) (b)

Figure 2

The domain problem for hydrodynamic model of (a) twin regular reservoirs (b) reservoir of RAMA 9 with observation points.



(a) (b)

Figure 3

The domain problem for dispersion model of (a) twin regular reservoirs (b) reservoir of RAMA 9.

where $0 \le l \le L$, $0 \le m \le M$, $0 \le n \le N$ and similarly defined for $u^n_{l,m}$, $v^n_{l,m}$ with Lax–Wendroff method [2], we have

$$U^{n+1}_{l,m} = U^n_{l,m} + \frac{1}{2}pA\left(U^n_{l+1,m} - U^n_{l-1,m}\right) + \frac{1}{2}pB\left(U^n_{l,m+1} - U^n_{l,m-1}\right)$$
$$+ \frac{1}{2}p^2A^2\left(U^n_{l+1,m} - 2U^n_{l,m} + U^n_{l-1,m}\right) + \frac{1}{2}p^2B^2\left(U^n_{l,m+1} - 2U^n_{l,m} + U^n_{l,m-1}\right) \quad (19)$$
$$+ \frac{1}{8}(AB + BA)\left(U^n_{l+1,m+1} - U^n_{l-1,m+1} - U^n_{l+1,m-1} + U^n_{l-1,m-1}\right),$$

where $U^n_{l,m} = \begin{pmatrix} d^n_{l,m} \\ u^n_{l,m} \\ v^n_{l,m} \end{pmatrix}$, $p = \frac{\Delta t}{\Delta x}$. A stability analysis of Lax-Wendroff scheme Eq.(19) is stable if $p|\lambda_0| \le \frac{1}{2\sqrt{2}}$ where $|\lambda_0| = \max\{|\lambda_A|, |\lambda_B|\}$ such that λ_A, λ_B are eigenvalues of matrices A, B.

3.2 *Numerical method for the dispersion model*

Secondly, we use the forward differences in time and backward difference in space in advection-diffusion equation.

We can approximate $C(x_l, y_m, t_n) = C^n_{l,m}$ are value the difference approximation of $C(x, y, t)$ at point $x = l\Delta x$, $y = m\Delta y$ and $t = n\Delta t$ where $0 \le l \le L$, $0 \le m \le M$, $0 \le n \le N$. Taking the forward in time and backward in space (17), we get the following finite difference equation,

$$\frac{C^{n+1}_{l,m} - C^n_{l,m}}{\Delta t} + u^n_{l,m}\left(\frac{C^n_{l,m} - C^n_{l-1,m}}{\Delta x}\right) + v^n_{l,m}\left(\frac{C^n_{l,m} - C^n_{l,m-1}}{\Delta y}\right)$$
$$= D\left(\frac{C^n_{l+1,m} - 2C^n_{l,m} + C^n_{l-1,m}}{(\Delta x)^2} + \frac{C^n_{l,m+1} - 2C^n_{l,m} + C^n_{l,m-1}}{(\Delta y)^2}\right), \quad (20)$$

where D is the diffusion coefficient (m^2/s)

4. Numerical Experiments

4.1 Numerical results of the hydrodynamic model

Case 1: *Twin regular reservoirs. From Fig.1(a) define step size of* $x(\Delta x) = 0.03125$, *step size of* $y(\Delta y) = 0.03125$ *and step time* $t(\Delta T) = 0.01$.

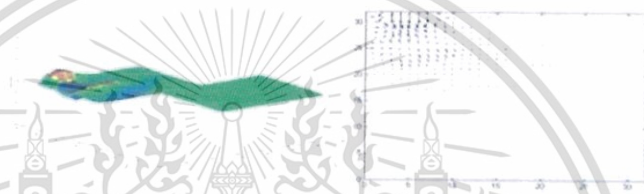


Figure 4

Time = 500 (a) elevation (b) vector field of twin regular reservoirs.

Figure 5

Time = 2000 (a) elevation (b) vector field of twin regular reservoirs.

Case 2: *Reservoir of RAMA 9. From Fig. 1(b) define step size of* $x(\Delta x) = 0.03125$, *step size of* $y(\Delta y) = 0.015625$ *and step time* $t(\Delta T) = 0.01$.

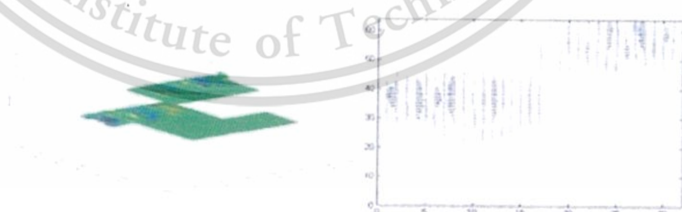


Figure 6

Time = 500 (a) elevation (b) vector field of reservoir of RAMA 9.

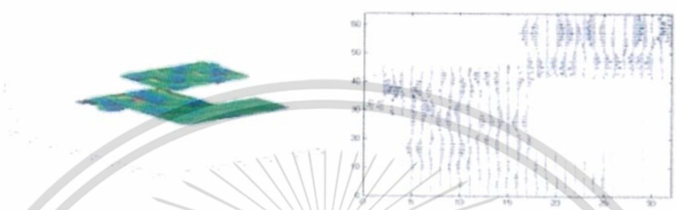


Figure 7

Time = 2000 (a) elevation (b) vector field of reservoir of RAMA 9.

4.2 *Numerical results of the dispersion model*

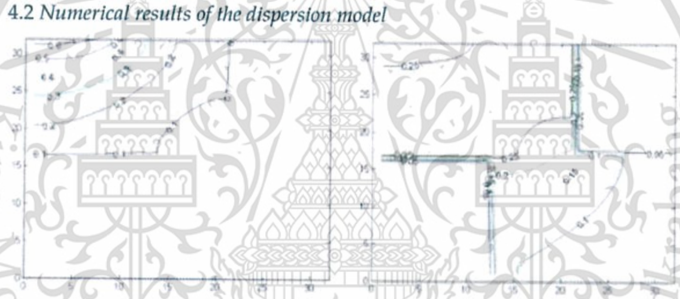


Figure 8

Concentration of pollution at time 500 and 2000 of twin regular reservoirs.

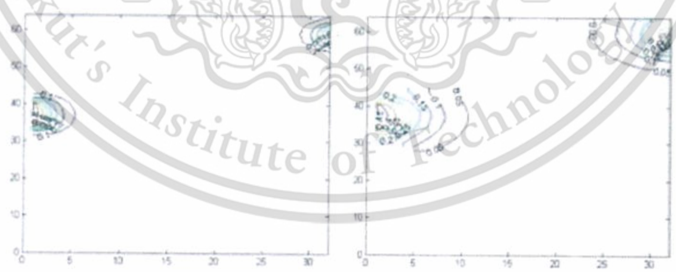


Figure 9

Concentration of pollution at time 2000 of twin regular reservoirs.

4.3 *Observation points on the Rama-nine reservoir*

Observation points are representing some positions in the Rama-nine reservoir. These points are located on the gate, the linked reservoir and middle reservoir. Consider the change of concentration of pollution at initial time to 2000 as shown in Fig.10-11 where x-axis is time axis and y-axis is concentration axis.

Figure 10

Point A, B, C and D time 0 to 2000 concentration of pollution in twin regular reservoirs.

Figure 11

Point A, B, C, D and E time 0 to 2000 concentration of pollution in reservoir of RAMA 9.

Table 1

Observation point in twin regular reservoirs for elevation at different times.

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.0498407	0.1139078	0.1647929	0.2161609	0.2291784	0.262172	0.266609	0.2883534	0.2869161	0.3018594	0.2970681	0.3048805	0.2991667	0.3015839
Point B	0.0200681	0.0239336	0.0364316	0.0547086	0.0771992	0.1037392	0.1238413	0.1441083	0.161663	0.1769358	0.1894275	0.2004532	0.2101295	0.2180475
Point C	0.0199595	0.0197814	0.0197274	0.0203309	0.0219746	0.0248085	0.0289333	0.0342006	0.0399991	0.0459919	0.0526744	0.0592764	0.0660019	0.0726162
Point D	0.018228	0.0169874	0.0161122	0.0154693	0.0150707	0.0150275	0.0154806	0.0165688	0.0184019	0.0210215	0.024323	0.0280653	0.0325593	0.0372992

Table 2

Observation point in twin regular reservoirs for velocity (u) at different times.

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.000011	0.004373	0.019859	-0.058009	0.081043	-0.085936	0.103255	-0.096792	0.107876	-0.119515	0.116126	-0.088947	0.095235	-0.097334
Point B	0.000000	0.000000	0.000000	0.000027	0.000785	0.002096	-0.003860	0.014153	-0.010091	-0.009991	0.023355	-0.019409	0.021966	-0.022526
Point C	0.000000	0.000177	0.002173	0.003884	0.002205	0.001290	0.001768	0.001652	0.000562	0.002835	0.005835	-0.003674	0.008864	-0.003253
Point B	0.000014	-0.000370	-0.003565	-0.004563	-0.003285	-0.002624	-0.001854	-0.001335	-0.000341	-0.002972	-0.005271	0.004498	-0.010346	0.010503

Table 3

Observation point in twin regular reservoirs for velocity (v) at different times.

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	-0.000144	-0.024640	-0.057304	0.122839	-0.192879	0.180976	-0.199946	0.194726	-0.229138	0.218746	-0.254176	0.251868	-0.271312	0.238285
Point B	0.000000	0.000000	0.000000	-0.000015	0.000041	0.001559	0.005172	-0.008025	0.002091	0.003510	-0.003081	0.001140	-0.002205	0.001673
Point C	0.000000	-0.000047	-0.000940	-0.001299	0.000279	0.000532	-0.004574	-0.003162	0.015790	-0.026576	0.019615	-0.014087	0.015085	-0.024868
Point B	-0.002260	-0.009154	-0.009069	-0.008032	-0.008136	-0.009156	-0.010403	-0.011505	-0.012307	-0.008339	-0.005700	-0.026443	0.007969	-0.034747

Table 4

Observation point in twin regular reservoirs for concentration at different times.

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.000022	0.005121	0.018693	-0.046532	0.058054	-0.075299	0.091998	-0.094670	0.085147	-0.071788	0.072717	-0.082084	0.077922	-0.073851
Point B	0.000000	0.000000	0.000000	-0.000003	0.000003	0.001218	0.000472	-0.006576	0.011946	-0.012434	0.010469	-0.010245	0.010478	-0.010505
Point C	0.000000	-0.000048	-0.000743	-0.001028	-0.000021	0.001061	0.001840	0.000261	-0.005337	0.005301	-0.004236	0.003098	-0.000674	0.000036
Point B	-0.001255	-0.002501	-0.000925	0.000482	0.000549	-0.000076	-0.000365	-0.000073	0.000391	-0.001619	-0.002965	0.007028	-0.009026	0.010468

Table 5

Observation point in reservoir of RAMA 9 for elevation at different times.

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.020000	0.020000	0.020000	0.020003	0.020020	0.020065	0.020173	0.020341	0.020640	0.020991	0.021537	0.022086	0.022904	0.023580
Point B	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020001	0.020003	0.020009	0.020020	0.020043	0.020077	0.020137	0.020213
Point C	0.020002	0.020117	0.020880	0.022742	0.026187	0.030485	0.036386	0.041902	0.049083	0.054632	0.061623	0.066942	0.073592	0.078200
Pont D	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020001	0.020001	0.020003	0.020005
Point E	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000	0.020000

Table 6

Observation point in reservoir of RAMA 9 for velocity (u) at different times.

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	-0.000001	-0.000095	0.006525	-0.003148	-0.005862	0.006788	0.004106	-0.023559	0.029855	-0.028767	0.030651	-0.041450	0.040002	-0.036131
Point B	0.000000	0.000005	0.000758	0.001865	-0.017235	0.037167	-0.042060	0.032547	-0.039055	0.038095	-0.036498	0.037463	-0.037045	0.038587
Point C	0.001551	0.013227	-0.051933	0.073786	-0.081517	0.085095	-0.089971	0.081168	-0.065316	0.050085	-0.047510	0.045184	-0.034131	0.018782
Pont D	0.000000	0.000000	0.000001	0.000160	0.000079	-0.002013	0.005828	-0.012783	0.020921	-0.024481	0.024718	-0.026681	0.028835	-0.029219
Point E	0.000000	0.000000	0.000000	0.000000	0.000000	0.000002	0.000054	-0.000047	-0.000649	0.001194	-0.000498	0.001140	-0.003342	0.002501

Table 7

Observation point in reservoir of RAMA 9 for velocity (v) at different times.

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.000000	-0.000023	0.002607	-0.001898	-0.003150	-0.000130	0.001502	0.010142	-0.010200	0.006713	-0.002968	0.007595	-0.001421	0.005278
Point B	0.000000	0.000000	0.000000	-0.001265	0.000677	0.004376	-0.006759	0.005373	-0.003832	0.004906	-0.010695	0.013925	-0.016144	0.015321
Point C	-0.000006	-0.000538	0.001826	-0.000632	-0.001567	0.001210	-0.001585	0.005846	-0.003327	0.001224	-0.002520	0.002097	0.003363	-0.006058
Pont D	0.000000	0.000000	-0.000001	-0.000109	-0.000094	0.003438	-0.005736	0.000743	0.001742	-0.006067	0.014744	-0.020454	0.020015	-0.019896
Point E	0.000000	0.000000	0.000000	0.000000	0.000000	-0.000002	-0.000034	0.000368	0.001030	-0.004173	0.004425	-0.004996	0.007380	-0.008324

Table 8

Observation point in reservoir of RAMA 9 for concentration at different times.

Time	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
Point A	0.000000	0.000026	-0.000857	0.002249	0.001545	0.001169	-0.000409	-0.008217	0.010252	-0.009812	0.009384	-0.009983	0.011108	-0.012955
Point B	0.000000	0.000001	0.000107	-0.000136	-0.002015	0.006549	-0.009218	0.014688	-0.016981	0.016453	-0.015764	0.013325	-0.013709	0.015262
Point C	0.000135	0.000925	-0.004922	0.007472	-0.006979	0.006503	-0.004258	0.005040	-0.009893	0.013991	-0.015930	0.017584	-0.019200	0.020685
Pont D	0.000000	0.000000	0.000000	0.000045	-0.000094	-0.001296	0.002346	0.000793	-0.003498	0.005665	-0.010240	0.013108	-0.012152	0.011959
Point E	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001	0.000013	-0.000140	-0.000341	0.001431	-0.001403	0.002079	-0.003511	0.002146

5. Discussion

5.1 *The numerical simulation of the hydrodynamic model*

Twin regular reservoirs: Discharge water to first reservoir flow into second reservoir and open gate at end of the reservoir. Fig.3-4 show elevations and velocity field at times 500s and 2000s.

Reservoir of the Rama-nine reservoir: Discharge water from canal 2 sides into the Rama-nine reservoir at initial time as shown in Fig.8-13 show elevation and velocity field of water flow at different times.

5.2 *The numerical simulation of the dispersion model*

Twin regular reservoirs: Initial time discharge maximum concentration of pollution C = 1 at door of reservoir and decreasing when time increased by exponential function $C = e^{-\frac{t}{40}}$. Pollution diffusing first reservoir into second reservoir in figure 7 and 8 show pollution at time 500s and 2000s. Point A, B, C, and D in reservoir is observation point to see the change of the increase or decrease of the pollution with different times as shown in Figure 12 and 13. Pollution at point B and C is increasing when time increased and point A at time 1400 pollution to start decreasing due to point nearly the door. At initial time to 550 the pollution at point D is decreasing and increasing over time 550 because open gate at end of the reservoir.

Reservoir of the Rama-nine reservoir: Initial time discharge maximum concentration of pollution C = 1 at two gates of reservoir and decreasing when time increased by exponential function $C = e^{-\frac{t}{40}}$. Pollution diffusing to reservoir is very little because velocity is small as shown in Figure 9 and 10 show pollution at time 500s and 2000s. Point A, B, C, D and E in reservoir of the Rama-nine reservoir is observation points as shown in Fig. 14-16. At point B, D, E has pollution change slightly but points A and B has pollution is increasing since this point nearly the two gates.

6. Conclusion

We have revised a mathematical model that combines a non-dimensional form of hydrodynamic model and a dispersion model. The model is suitable to the Rama-nine reservoir. The Lax-Wendroff method is use to solve a non-dimensional form of a shallow water equation that gives thenumerical solutions of the water velocities in x- and y-directions and elevation. The forward differences in time and backward difference in space with approximated velocities in x- and y-directions also used to solve the advection-diffusion equation. The result of this research showed that the proposed model can approximate the water velocities in x- and y-directions, the elevation, and the concentration of the pollutants in a Rama-nine reservoir at any various time and position. The accuracy of approximation is within the units of centimeters and seconds. The pollutant concentrationson the floodgates is decreased at/or below initial pollutant concentration. The Rama-ninereservoir is opened to two flood gates. The both of flood gates are the sources of pollutant concentration that discharge wastewater into the reservoir. The linked gate is connecting the north reservoir to the south reservoir. It is an obstacle of the pollutant dispersion for a short time. The linked gate can dilute the pollutant concentration of the north reservoir to the south reservoir in long period. The proposed model and numerical techniques can be applied to other water sources having non-uniformly distributed water flows.

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Numerical Treatment to a Water-Quality Measurement Model in an Opened-Closed Reservoir

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Abstract : Measuring the water quality in water sources or the Monkey Cheeks Project with opened-closed reservoir. It can be measured by the field measurement, using the water quality monitoring tools and the water quality models consist of hydrodynamic model and dispersion model, to calculate the quality of the water. Hydrodynamic model, using the shallow water equation as a governing equation, is used to describe the water current, having source of wave maker and bottom topography as the required data, bringing about the elevation and velocities of water. Dispersion model, using the advection-diffusion equation as the governing equation, is used to describe the spread of the pollutant concentration of water, having pollutant concentration at point source and calculated water velocities from the first model as the input data, bringing the time-dependent pollutant concentration of water at any point. In this research, the three-dimensional surface fitting technique is employed, the anisotropic bottom topography data is represented by a surface function in the hydrodynamic model, in order to have a more realistic water current and water quality approximations in opened-closed reservoir.

Keywords : water quality model; hydrodynamic model; dispersion model; advection-diffusion equation; opened-closed reservoir.
2010 Mathematics Subject Classification : 76R50; 39A14; 35Q30; 35L51.

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1 Introduction

There are many methods for detecting the level of pollutants in the water, mostly conducted by a field measurement and a mathematical simulation. The shallow water mass transport's problems are presented in [1], as the method of characteristics has been reported applied. In [2], [3] and [4], the finite element method for solving steady and unsteady water pollution measurements are introduced. The various numerical techniques of solving the uniform flow of stream water quality model are presented in [5], [6] and [7]. The numerical methods of approximating the solution of the two-dimensional advection-diffusion-reaction equation are proposed in [5], [8] and [9].

Most non-uniform flow models need the input data concerned with the velocity of the current at any point and any time in the domain. The hydrodynamic model provides the velocity field and the elevation of the water. In [10, 5, 8, 9, 6, 7], the hydrodynamic model and advection-diffusion equation are used to approximate the velocity of the water current in a bay and a channel. In [9] and [11], the results from hydrodynamic model are used as data for the non-uniform flow of the advection-diffusion-reaction equation, which provide the pollutant concentration field. The term of the friction forces occurred thanks to the drag of sides of the uniform reservoir. The theoretical solution of the model was found at the ending point of the domain and the analytical solution to check the accuracy of our approximate solution was used. In [9], the Lax-Wendroff method with stability analysis to solve the two-dimensional hydrodynamic model with a rectangular domain was proposed. In [12], develop mathematical models and numerical methods for approximating water flow directions and pollutant concentration level in Rama-nine reservoirs in opened with two parallel canals and assuming bottom topography of reservoir is flat. The Lax-Wendroff method is subsequently used in non-dimensional form of a shallow water equation to approximate the velocity of water and elevation of water, we use the forward difference in time and backward difference in space of advection diffusion equation. In [4] and [13], the Lax-Wendroff method for solving the dimensional form of shallow water equation in rectangular model and spherical model with Matlab program are proposed, respectively.

In this research, we begin with modifying a mathematical model, combining two existing mathematical models, a hydrodynamic model which is used to describe the water current in an opened-closed reservoir and a dispersion model which is used to describe the diffusion of the pollutant concentration of water in an opened-closed reservoir. This is to make the proposed model suitable for the reservoir. The shallow water equation of the hydrodynamic model is assumed by averaging the equation over the depth with anisotropic bottom topography, and discarding the term regarding the Coriolis force, surface wind effect and external forces, resulting in the calculated velocity used in the dispersion model to approximate the concentration levels of the pollutants.

2 Water-Quality Model

In this section, two mathematical models are described. They were used to simulate time-varying pollutant levels causing by wastewater discharges from external source into an opened-closed reservoir and drain water at the exit gate. The first model was a hydrodynamic model that determined the velocity and elevation of the water at any location in the reservoir with anisotropic bottom topography, while the second model was a pollutant dispersion model that determined the pollutant level at any point in the reservoir.

2.1 Hydrodynamic model: anisotropic bottom topography

Figure 1: Cross-section of the reservoir.

The two-dimensional unsteady flow of water into and out of the reservoir could be determined by using the system of shallow water equations as the conservation of mass and conservation of momentum were taken into account, the equations of this system could be derived from depth-averaging the Navier-Stokes equations in the vertical direction, neglecting the diffusion of momentum due to turbulence and discarding the terms expressing the effects of friction, surface wind, Coriolis factor and shearing stresses. The continuity equation is then expressed as follows:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0, \tag{2.1}$$

and the momentum equations are expressed as below:

$$\frac{\partial (uh)}{\partial t} + \frac{\partial (u^2 h + \frac{1}{2} g h^2)}{\partial x} + \frac{\partial (uvh)}{\partial y} = 0, \tag{2.2}$$

$$\frac{\partial (vh)}{\partial t} + \frac{\partial (uvh)}{\partial x} + \frac{\partial (v^2 h + \frac{1}{2} g h^2)}{\partial y} = 0, \tag{2.3}$$

where

$h(x, y, t)$ is the depth measured from the mean surface of water to the reservoir bed $h = H + \xi$ (m),

$\xi(x, y, t)$ is the elevation of surface of water from the mean water level in reservoir (sea level) (m),

$H(x, y)$ is the interpolated bottom topography function of the reservoir (m),

$u(x, y, t)$ is velocity in x direction (m/s),

$v(x, y, t)$ is velocity in y direction (m/s),

g is gravitational constant $(9.8m/s^2)$.

Such time (t), and two space coordinates, x and y are the independent variables. Likewise, the conserved quantities are mass, which is proportional to h, and momentum, which is proportional to (uh) and (vh). As taken with respect to the same term, the partial derivatives are grouped into vectors $(\partial x, \partial y, \partial t)$ and later rewritten as a hyperbolic partial differential equation as follows:

$$U = \begin{pmatrix} h \\ uh \\ vh \end{pmatrix}, F(U) = \begin{pmatrix} uh \\ u^2h + \frac{1}{2}gh^2 \\ uvh \end{pmatrix}, G(U) = \begin{pmatrix} vh \\ uvh \\ v^2h + \frac{1}{2}gh^2 \end{pmatrix}. \quad (2.4)$$

The hyperbolic PDE:

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial t}F(U) + \frac{\partial}{\partial t}G(U) = 0. \quad (2.5)$$

The initial conditions of reservoir were as follows: the x and y-velocity components

Figure 2: Opened-closed reservoir and observation points A, B, C, D and E

were zero as well as the water elevation: $u = 0, v = 0$ and $\xi = 0$, while the boundary conditions were as follows: (i) $\frac{\partial u}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$, $\xi = 0$ for the horizontal edges of the rectangular reservoir; (ii) $\frac{\partial u}{\partial x} = 0$, $v = 0$, $\xi = 0$ for the vertical edges; and (iii) $\xi = f(x, y)$ for the water flowing into the entrance gate and $\frac{\partial u}{\partial x} = u_1$, $\frac{\partial v}{\partial y} = 0$ for the velocity of water flow at exit gate as shown in Figure.2.

2.2 Dispersion model

When applying the distributed pollutant process, including the transportation and diffusion, the mass transfer equation is satisfied by averaging the equation over the depth, generating the advection-diffusion equation,

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right). \tag{2.6}$$

where $C(x, y, t)$ (kg/m^3) is the concentration averaged in depth at the displacement (x, y) and at time t . $D(m^2/s)$ is the diffusion coefficient.

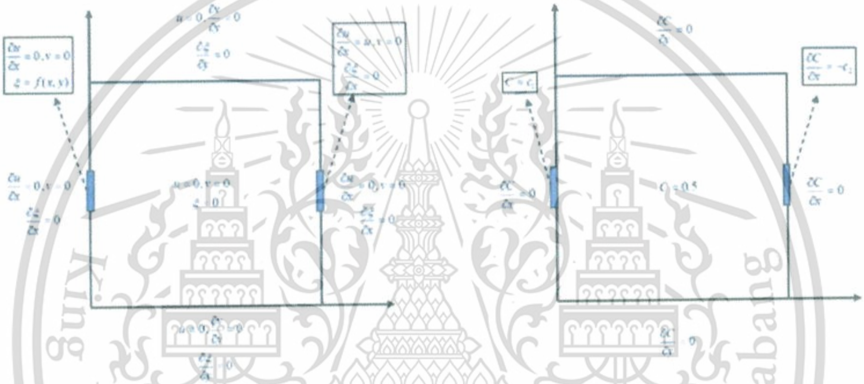


Figure 3: Initial condition and boundary condition of hydrodynamic model and dispersion model.

The water pollutant was discharged from the entrance gate into the opened-closed reservoir by assuming the pollutant concentration constant as function c_1 and this reservoir had draining water at the exit gate by assuming rate of drain of water as $\frac{\partial C}{\partial x} = -c_2$. Initial pollutant concentration in reservoir was $0.5(kg/m^3)$ and there was no rate of change of pollutant concentration at the boundary of reservoir as Figure.3.

3 Numerical Technique

3.1 Numerical method for the hydrodynamic model

We would use the Lax-Wendroff method to compute a numerical approximation to the solution of hyperbolic PDE (2.5). A regular square finite difference grid with a vector-valued solution centred in the grid cells. The domain of problem $L \times M$ dimension. l and m were subintervals, such that $l\Delta x = L$, $m\Delta y = M$ and interval time $[0, T]$, k was subintervals, such that $k\Delta t = T$,$U^n_{i,j} = U(x, y, t)$

represents a three component vector at each cell i, j with time step n , where $x = i\Delta x$, $y = j\Delta y$ and $t = k\Delta t$.

Step 1: Compute initial vector $U^n_{i,j}$ at centre cells.

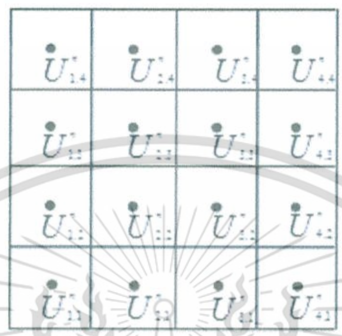


Figure 4: At the beginning of a time step, the variables represent the solution at the centres of the grids.

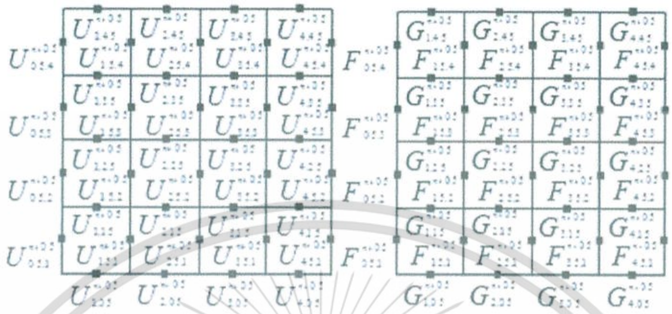
Step 2: Take $U^n_{i,j}$ to compute vector $F^n_{i,j}$ and $G^n_{i,j}$ at centre cells.

Figure 5: The vector (a) F at centres of grid (b) G at centres of grid.

Step 3: This stage is a half-step; it defines values of U at time step $n + \frac{1}{2}$ and the midpoints of the edges of the grid.

$$U^{n+\frac{1}{2}}_{i+\frac{1}{2},j} = \frac{1}{2}(U^n_{i+1,j} + U^n_{i,j}) - \frac{\Delta t}{2\Delta x}(F^n_{i+1,j} - F^n_{i,j}) \tag{3.1}$$

$$U^{n+\frac{1}{2}}_{i,j+\frac{1}{2}} = \frac{1}{2}(U^n_{i,j+1} + U^n_{i,j}) - \frac{\Delta t}{2\Delta y}(G^n_{i,j+1} - G^n_{i,j}) \tag{3.2}$$



(a) (b)

Figure 6: The values of vector (a) U represent the solution at the midpoints of the grids and (b) F, G at the midpoints of the grids.

Step 4: Take values of U from step 3 to compute F, G at time step $n + \frac{1}{2}$ and the midpoints of the edges of the grid.
Step 5: The last step completes the time step by using the values computed in the step 1 and step 4 to compute new values at the centres of the cells.

$$U_{i,j}^{n+1} = U_{i,j}^{n} - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2},j}^{n+\frac{1}{2}} - F_{i-\frac{1}{2},j}^{n+\frac{1}{2}}\right) - \frac{\Delta t}{\Delta y}\left(G_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} - G_{i,j-\frac{1}{2}}^{n+\frac{1}{2}}\right) \quad (3.3)$$

Figure 7: The solution U^{n+1} at centres of the grids.

We would use the finite difference method to compute a numerical approximation to the boundary conditions of the reservoir.
For left boundary condition, where $i = 0$ and $1 \leqslant j \leqslant m$, therefore $U_{0,j}^{n} = U_{1,j}^{n}$, substituting the approximate unknown vector nodes $U_{0,j}^{n}$ of left boundary into

(3.1), we had

$$U^{n+\frac{1}{2}}_{\frac{1}{2},j} = \frac{1}{2}(U^n_{1,j} + U^n_{0,j}) - \frac{\Delta t}{2\Delta x}(F^n_{1,j} - F^n_{0,j}) = U^n_{1,j} \tag{3.4}$$

For right boundary condition, where $i = l$ and $1 \leqslant j \leqslant m$, therefore $U^n_{l+1,j} = U^n_{l,j}$, substituting the approximate unknown vector nodes $U^n_{l+1,j}$ of right boundary into (3.1), we had

$$U^{n+\frac{1}{2}}_{l+\frac{1}{2},j} = \frac{1}{2}(U^n_{l+1,j} + U^n_{l,j}) - \frac{\Delta t}{2\Delta x}(F^n_{l+1,j} - F^n_{l,j}) = U^n_{l,j} \tag{3.5}$$

For lower boundary condition, where $1 \leqslant i \leqslant l$ and $j = 0$, therefore $U^n_{i,0} = U^n_{i,1}$, substituting the approximate unknown vector nodes $U^n_{i,0}$ of lower boundary into (3.2), we had

$$U^{n+\frac{1}{2}}_{i,\frac{1}{2}} = \frac{1}{2}(U^n_{i,1} + U^n_{i,0}) - \frac{\Delta t}{2\Delta y}(G^n_{i,1} - G^n_{i,0}) = U^n_{i,1} \tag{3.6}$$

For upper boundary condition, where $1 \leqslant i \leqslant l$ and $j = m$, therefore $U^n_{i,m+1} = U^n_{i,m}$, substituting the approximate unknown vector nodes $U^n_{i,0}$ of upper boundary into (3.2), we had

$$U^{n+\frac{1}{2}}_{i,m+\frac{1}{2}} = \frac{1}{2}(U^n_{i,m+1} + U^n_{i,m}) - \frac{\Delta t}{2\Delta y}(G^n_{i,m+1} - G^n_{i,0}) = U^n_{i,m} \tag{3.7}$$

3.2 Numerical method for the dispersion model

We used the forward differences in time and backward difference in space in advection-diffusion equation. We can approximate $C^n_{i,j}$, the value of the approximation of $C(x, y, t)$ at point $x = i\Delta x$, $y = j\Delta y$ and $t = n\Delta t$, where $1 \leq i \leq l$, $1 \leq j \leq m$ and $0 \leq n \leq k$.

$$\frac{\partial C}{\partial t} = \frac{C^{n+1}_{i,j} - C^n_{i,j}}{\Delta t}, \tag{3.8}$$

$$\frac{\partial C}{\partial x} = \frac{C^n_{i,j} - C^n_{i-1,j}}{\Delta x}, \tag{3.9}$$

$$\frac{\partial C}{\partial y} = \frac{C^n_{i,j} - C^n_{i,j-1}}{\Delta y}, \tag{3.10}$$

$$\frac{\partial^2 C}{\partial x^2} = \frac{C^n_{i+1,j} - 2C^n_{i,j} + C^n_{i-1,j}}{(\Delta x)^2}, \tag{3.11}$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{C^n_{i,j+1} - 2C^n_{i,j} + C^n_{i,j-1}}{(\Delta y)^2}. \tag{3.12}$$

Taking the forward in time and backward in space (2.6), we got the following finite difference equation,

$$\frac{C_{i,j}^{n+1}-C_{i,j}^{n}}{\Delta t}+u_{i,j}^{n}\left(\frac{C_{i,j}^{n}-C_{i-1,j}^{n}}{\Delta x}\right)+v_{i,j}^{n}\left(\frac{C_{i,j}^{n}-C_{i,j-1}^{n}}{\Delta y}\right)=,$$
$$D\left(\frac{C_{i+1,j}^{n}-2C_{i,j}^{n}+C_{i-1,j}^{n}}{(\Delta x)^2}+\frac{C_{i,j+1}^{n}-2C_{i,j}^{n}+C_{i,j-1}^{n}}{(\Delta y)^2}\right) \quad (3.13)$$

$$C_{i,j}^{n-1}=\frac{\Delta tD}{\Delta x^2}C_{i+1,j}^{n}+\frac{\Delta tD}{\Delta y^2}C_{i,j+1}^{n}+\left(\frac{\Delta t}{\Delta x}u_{i,j}^{n}+\frac{\Delta t}{\Delta x^2}D\right)C_{i-1,j}^{n}+$$
$$\left(\frac{\Delta t}{\Delta y}v_{i,j}^{n}+\frac{\Delta t}{\Delta y^2}D\right)C_{i,j-1}^{n}+\left(1-\frac{\Delta t}{\Delta x}u_{i,j}^{n}-\frac{\Delta t}{\Delta y}v_{i,j}^{n}-\frac{2D\Delta t}{\Delta x^2}-\frac{2D\Delta t}{\Delta y^2}\right)C_{i,j}^{n}. \quad (3.14)$$

where D was the diffusion coefficient (m^2/s).

If $C_{i,j}^{n}$ lay at the boundary of the opened-closed reservoir, it was calculated by applying the backward difference scheme at right boundary and top boundary, forward difference scheme at left boundary and bottom boundary. For left boundary condition, where $i=1$ and $1\le j\le m$, therefore $C_{0,j}^{n}=C_{1,j}^{n}$, substituting the approximate unknown vector nodes $C_{0,j}^{n}$ of left boundary into (3.13), we had

$$\frac{C_{1,j}^{n+1}-C_{1,j}^{n}}{\Delta t}+v_{1,j}^{n}\left(\frac{C_{1,j}^{n}-C_{1,j-1}^{n}}{\Delta y}\right)=$$
$$D\left(\frac{C_{2,j}^{n}-C_{1,j}^{n}}{(\Delta x)^2}+\frac{C_{1,j+1}^{n}-2C_{1,j}^{n}+C_{1,j-1}^{n}}{(\Delta y)^2}\right), \quad (3.15)$$

For right boundary condition, where $i=l$ and $1\le j\le m$, therefore $C_{l+1,j}^{n}=C_{l,j}^{n}$, substituting the approximate unknown vector nodes $C_{l+1,j}^{n}$ of right boundary into (3.13), we had

$$\frac{C_{l,j}^{n+1}-C_{l,j}^{n}}{\Delta t}+u_{l,j}^{n}\left(\frac{C_{l,j}^{n}-C_{l-1,j}^{n}}{\Delta x}\right)+v_{l,j}^{n}\left(\frac{C_{l,j}^{n}-C_{l,j-1}^{n}}{\Delta y}\right)$$
$$=D\left(\frac{-C_{l,j}^{n}+C_{l-1,j}^{n}}{(\Delta x)^2}+\frac{C_{l,j+1}^{n}-2C_{l,j}^{n}+C_{l,j-1}^{n}}{(\Delta y)^2}\right), \quad (3.16)$$

For lower boundary condition, where $1\le i\le l$ and $j=1$, therefore $C_{i,0}^{n}=C_{i,1}^{n}$, substituting the approximate unknown vector nodes $C_{0,j}^{n}$ of lower boundary into (3.13), we had

$$\frac{C_{i,1}^{n+1}-C_{i,1}^{n}}{\Delta t}+u_{i,1}^{n}\left(\frac{C_{i,1}^{n}-C_{i-1,1}^{n}}{\Delta x}\right)$$
$$=D\left(\frac{C_{i+1,1}^{n}-2C_{i,1}^{n}+C_{i-1,1}^{n}}{(\Delta x)^2}+\frac{C_{i,2}^{n}-C_{i,1}^{n}}{(\Delta y)^2}\right), \quad (3.17)$$

For upper boundary condition, where $1 \leq i \leq l$ and $j = m$, therefore $C^n_{i,m+1} = C^n_{i,m}$, substituting the approximate unknown vector nodes $C^n_{i,m+1}$ of left boundary into (3.13), we had

$$\frac{C^{n+1}_{i,m} - C^n_{i,m}}{\Delta t} + u^n_{i,m}\left(\frac{C^n_{i,m} - C^n_{i-1,m}}{\Delta x}\right) + v^n_{i,m}\left(\frac{C^n_{i,m} - C^n_{i,m-1}}{\Delta y}\right) = D\left(\frac{C^n_{i+1,m} - 2C^n_{i,m} + C^n_{i-1,m}}{(\Delta x)^2} + \frac{-C^n_{i,m} + C^n_{i,m-1}}{(\Delta y)^2}\right), \quad (3.18)$$

4 Numerical Experiments

In this section, various results were reported in a table, several surface and contour plots, and a comparison graph. Hydrodynamic model, calculated the velocities of water and elevation of water in opened-closed reservoir with an empirical anisotropic bottom topography interpolated function $0.01sin(0.01(x+y))$ as shown in Figure.8, using Lax-Wendroff method, when water flowed into the entrance gate by using the elevation of water $\xi = 1(m)$ and discarding drain water though the exit gate, using the rate of change of velocity u at $0.5(m/s^2)$, the results as shown in Figure.9 and Figure.10 for time $0sec$ to $50sec$. Dispersion model, calculated the pollutant concentration of water in opened-closed reservoir by using finite difference method, when wastewater was discharged from the external source into the reservoir and drain water was released thru the exit gate by using the rate of change of pollutant concentration with respect to x-coordinate at $0.1(kg/m^4)$ with initial pollutant concentration in this reservoir at $0.02(kg/m^3)$.

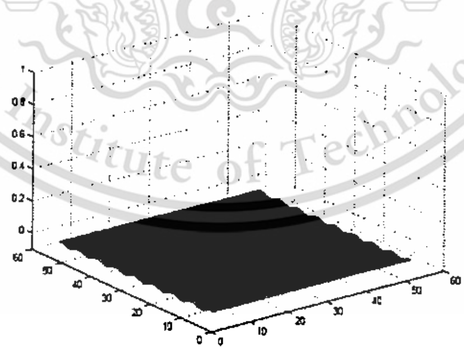


Figure 8: Anisotropic bottom topography surface in the opened-close reservoir

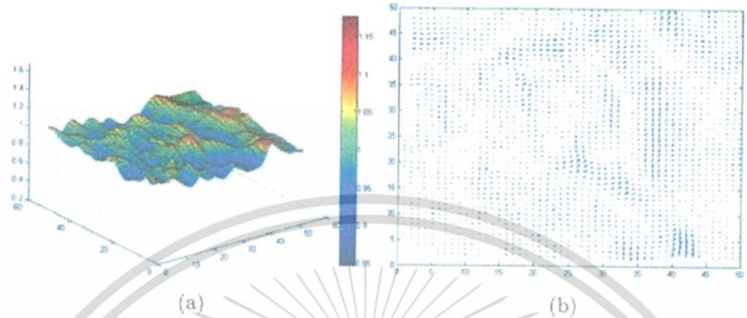


Figure 9: Time 50 sec, (a) surface plot of elevation of water (b) vector field of velocities in opened-closed reservoir.

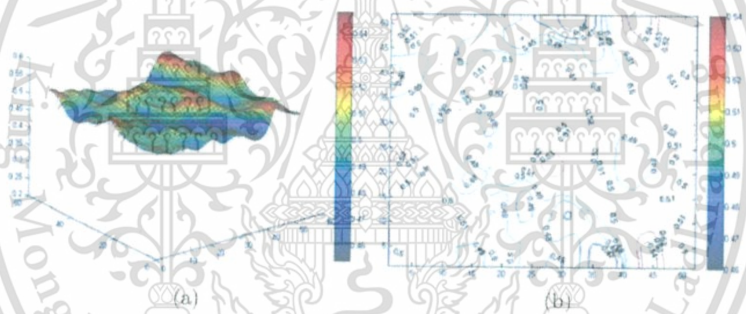
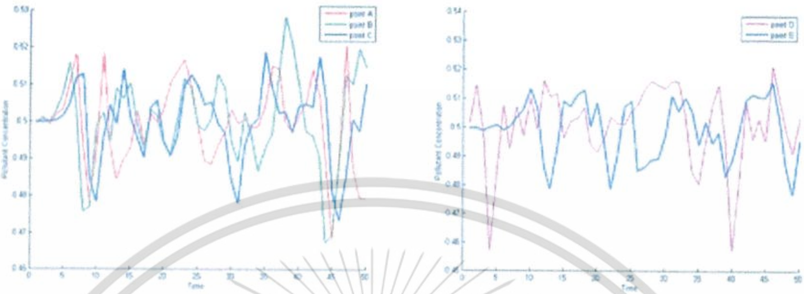


Figure 10: Time 50 sec, (a) surface plot (b) contour plot of pollutant concentration in opened-closed reservoir.

The monitoring points in opened-closed reservoir was used to observe the dispersion of pollutant concentration of water. In Figure.11(a) showing the comparison of pollutant concentration at monitoring point A, B and C and Figure.11(b) showing the comparison of pollutant concentration at monitoring point D and E for time $0sec$ to $50sec$.



(a) (b)

Figure 11: The comparison of pollutant concentration at monitoring (a) of point A, B and C and monitoring (b) of point D and E.

The pollutant concentration of water at monitoring point A, B, C, D and E in the opened-closed reservoir was observed in every $25sec$ for time $0sec$ to $200sec$ with wastewater discharging from the external source every $36sec$ into reservoir and drain water releasing though the exit gate every $36sec$ in Table.1, wastewater discharging from the external source every $36sec$ into reservoir and drain water releasing though the exit gate every $72sec$ in Table.2 and wastewater discharging from external source every $72sec$ into reservoir and drain water releasing though the exit gate every $36sec$ in Table.3.

Table 1: Pollutant Concentration (kg/m^3) at observation points in reservoir case 1.

Point\Time (sec)	25	50	75	100	125	150	175	200
A	0.4988	0.4794	0.4967	0.4973	0.5122	0.5187	0.4577	0.5291
B	0.4988	0.5148	0.5053	0.5177	0.4741	0.5029	0.4824	0.5098
C	0.5095	0.5106	0.4924	0.5015	0.4937	0.5508	0.4635	0.4929
D	0.5055	0.5032	0.4913	0.5083	0.5205	0.4926	0.5158	0.5115
E	0.5089	0.4953	0.5021	0.4628	0.5040	0.5078	0.5608	0.4713

Table 2: Pollutant Concentration (kg/m^3) at observation points in reservoir case 2.

Point\Time (sec)	25	50	75	100	125	150	175	200
A	0.4988	0.4796	0.4969	0.4976	0.5126	0.5191	0.4581	0.5298
B	0.4988	0.5150	0.5055	0.5179	0.4745	0.5034	0.4828	0.5106
C	0.5095	0.5107	0.4926	0.5017	0.4940	0.5513	0.4640	0.4936
D	0.5055	0.5032	0.4915	0.5085	0.5208	0.4930	0.5163	0.5121
E	0.5089	0.4957	0.5024	0.4630	0.5047	0.5084	0.5613	0.4721

Table 3: Pollutant Concentration (kg/m^3) at observation points in reservoir case 3.

Point\Time (sec)	25	50	75	100	125	150	175	200
A	0.4988	0.4855	0.4979	0.4982	0.5062	0.5150	0.4863	0.4968
B	0.4988	0.5084	0.4869	0.5196	0.4785	0.5247	0.4690	0.5296
C	0.5095	0.4953	0.4882	0.5073	0.4932	0.5377	0.4628	0.5097
D	0.5055	0.4911	0.4913	0.5189	0.5238	0.4882	0.5013	0.5221
E	0.5089	0.5067	0.5143	0.4597	0.4947	0.4921	0.5431	0.4835

5 Discussion and Conclusion

In this research, a mathematical model to calculate the elevation of water, water current and pollutant concentration of water in opened-closed reservoir with anisotropic bottom topography at any point and any time. anisotropic bottom topography function could be interpolated from data of reservoir bed coordinate, using cubic spline interpolate technique. When compared to other points, monitoring point B at the center of opened-closed reservoir mostly had a high pollutant concentration, monitoring point D, near the entrance gate of reservoir, has a mostly lower pollutant concentration as shown in Figure.11.

To conclude the numerical simulation for water-quality measurement model in an opened-closed reservoir with an empirical anisotropic bottom topography was proposed and thus the mathematical models could calculate the elevation, the velocities and the pollutant concentration of water. The very models could adjust the bottom topography according to the varying reservoir bed, simulate the wave maker function at the entrance gate of reservoir from field data by using the data interpolation in order to have a more realistic water current and water quality approximations in opened-closed reservoir with anisotropic bottom topography.

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