# STABILITY ANALYSIS OF AN APPROXIMATED SINGULARLY PERTURBED NONLINEAR FUZZY CONTROL SYSTEM



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## Abstract

In this research, we study a fuzzy control system with singular perturbation in a fast-slow system. The reduced system is introduced and an approximated solution of the system is defined. Moreover, a stability analysis method for the approximated system with Tagaki-Sugeno fuzzy logic controllers is presented. In this research we provid some sufficiently stability conditions for the fuzzy control system. An example is established to explain the stability analysis method.



**Keywords:** stability analysis, Tagaki-Sugeno fuzzy logic controllers, Lyapunov function, fuzzy control system

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Preeyaporn Waree

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# Introduction

## 1.1 Research Motivation

It is natural to model dynamical control systems with uncertainty by fuzzy systems of differential equations. There are many ways to model dynamical fuzzy control systems. The most popular ones is the Takagi-Sugeno fuzzy controller (TSFC). It was introduced by Takagi Sugeno in [1,2]. Today, TSFC model is studied in several papers (see [3,4,5] and references therein) because it is wildly accepted as an advantage modeling gadget. Also, singular perturbation and stability analysis are popular topics today because they can be advantageously applied to many fields of research–such as mathematics, computer science, engineering and economics.

In this research, we consider a fuzzy control system with singular perturbation in a fast-slow system based on the TSFC model;

$$\begin{cases} \dot{x} = A(\varepsilon)x + f(x,y,\varepsilon) + B(x,\varepsilon)u \\ \dot{y} = \dfrac{1}{\varepsilon}\left[C(\varepsilon)y + g(x,y,\varepsilon)\right] \\ x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}, \tag{1.1}$$

where $x = x(t) \in \mathbb{R}^n$ and $y = y(t) \in \mathbb{R}^m$ are slow variable and fast variable, respectively. The matrixs $A(\varepsilon) \in \mathbb{R}^{n\times n}$, $B(x,\varepsilon) \in \mathbb{R}^n$, $f(\cdot,\cdot,\varepsilon)$ and $g(\cdot,\cdot,\varepsilon)$ are globally Lipschitz and uniformly bounded in $\varepsilon$. The system (1.1) is disturbed with some small parameter $\varepsilon$, $0 < \varepsilon < 1$.

In the system, $u \in R$ is the control signal fed to the process, obtained by the weighted-sum defuzzification method for TSFC.

The document is organized as follows; in section 2, we state some preliminaries such as the theory of fuzzy logic and the systems controlling nonlinear processes of TSFC, in section 3, we investigate an approximate solution of the perturbed controller system (1.1). The second, we present a stability analysis method for nonlinear processes with TSFC and proof of the stability conditions. An illustrative example is shown in section 4. The last section, we conclude the study.

## 1.2 Objectives of the study

1) To study the definite and theorems about fuzzy logic.

2) To study the possibility of bringing knowledge of fuzzy logic, to assist in the stability analysis of fuzzy control.

3) To study the sufficient conditions for the stability analysis of fuzzy control.

## 1.3 Scope of the study

This research analyzes the stability of fuzzy control of an approximated nonlinear singularly perturbed system of system (1.1).

## 1.4 Benefits of the study

1) We have necessary conditions for existence of a solution of system (1.1).

2) We have the theorem for investigation the stability of the fuzzy control system (1.1).

3) We can apply some knowledge about the stability analysis of fuzzy to other concerned projects.

# Chapter 2

# Theoretical Preliminary

In this chapter, we discusses the theoretical and research-related education system stability fuzzy T-S disturbance singularity with consist of fuzzy logic and inference, Lyapunov function, fuzzy logic control system and stable fuzzy logic controllers.

## 2.1 Fuzzy logic and inference

### 2.1.1 Fuzzy set

Fuzzy set is a generalization of crisp set in which every membership is a set membership value. This shows the level of membership in the fuzzy set.

**Definition 2.1** Let $A$ be a crisp set. A fuzzy set $\mathcal{A}$ on the crisp set $A$ is defined by

$$\mathcal{A} = \{(x, u_A(x)) | x \in A, u_A(x) \in [0,1]\}, \quad (2.1)$$

where $u_A : A \to [0,1]$ is a membership function.

Note : for convenience, sometimes we denote a fuzzy set $\mathcal{A}$ by $A$.

**Example 2.1** Given $A = \{1,2,3,4,5\}$. Defined a fuzzy set $A$ with the membership function $u_A(x)$ where $u_A(x) = \frac{1}{x}$.

Since $u_A(x) = \frac{1}{x}$, we set a table of membership value of fuzzy set as below.

**Table 2.1 Membership value of fuzzy set** $A$

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $u_A(x)$ | 1 | 0.5 | 0.33 | 0.25 | 0.2 |

Hence $A = \{(1,1),(2,0.5),(3,0.33),(4,0.25),(5,0.2)\}$.

### 2.1.2 Fuzzy number

**Definition 2.2** Let $\mathcal{A}$ be a fuzzy set with membership function $u$ and $\alpha \in [0,1]$. An $\alpha$ - cut, denoted as $[u]^{\alpha}$, is defined by

$$[u]^{\alpha} = \begin{cases} \{x \in A \mid u(x) \geq \alpha\} & ; 0 < \alpha \leq 1 \\ \overline{\{x \in A \mid u(x) > 0\}} & ; \alpha = 0 \end{cases} . \tag{2.2}$$

**Definition 2.3** Let $\mathcal{A}$ be a fuzzy set under the membership $u : \mathbb{R} \to [0,1]$. $\mathcal{A}$ is called a *fuzzy number*, if $u$ satisfies the following conditions :

1) $\exists x \in \mathbb{R}$ , $u(x) = 1$,
2) $\forall \lambda \in [0,1]$, $\forall x_1, x_2 \in \mathbb{R}$, $u(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u(x_1), u(x_2)\}$,
3) For each $\alpha \in [0,1]$ there is a closed interval $[a,b]$ such that $[u]^{\alpha} = [a,b]$.

**Definition 2.4** Let $a^L \leq a^{M_1} \leq a^{M_2} \leq a^U$. A fuzzy number $\mathcal{A}$ is called a *Trapezoidal fuzzy number*, denoted by $\langle a^L, a^M, a^U \rangle$, if the membership function $u : \mathbb{R} \to [0,1]$ is defined by

$$u(x) = \begin{cases} \dfrac{x - a^L}{a^M - a^L} & ; a^L \leq x \leq a^M \\ \dfrac{x - a^U}{a^M - a^U} & ; a^M \leq x \leq a^U \\ 0 & ; \text{otherwise} \end{cases} . \tag{2.3}$$

### 2.1.3 Linguistic variable

A linguistic variable is a variable whose values are expressed in linguistic terms . For example height is a linguistic variable such that very short (0-120 cm.), short (120-150 cm.), medium(150-170 cm.), tall(170-180 cm.) very tall(>180 cm.) are term of linguistic.

### 2.1.4 Fuzzy Inference systems

Fuzzy inference systems use Mamdani Method, parallel If-Then rules form the deducing mechanism which indicates how to project input variables onto output space. A single fuzzy If-Then rule follows the form, in this consider fuzzy logic $n$ sub conditions of rule $1,2,\ldots,q$.

Rule-1: if $x_1$ is $\tilde{a}_{11}$ and $x_2$ is $\tilde{a}_{12}$ ... and $x_n$ is $\tilde{a}_{1n}$ then $y$ is $\tilde{b}_1$,

Rule-2: if $x_1$ is $\tilde{a}_{21}$ and $x_2$ is $\tilde{a}_{22}$ ... and $x_n$ is $\tilde{a}_{2n}$ then $y$ is $\tilde{b}_2$,

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

Rule-q: if $x_1$ is $\tilde{a}_{q1}$ and $x_2$ is $\tilde{a}_{q2}$ .... and $x_n$ is $\tilde{a}_{qn}$ then $y$ is $\tilde{b}_q$,

Fact: $x_1$ is $\tilde{a}_1$ and $x_2$ is $\tilde{a}_2$ ... and $x_n$ is $\tilde{a}_n$

Conclusion: $y$ is $\tilde{b}$

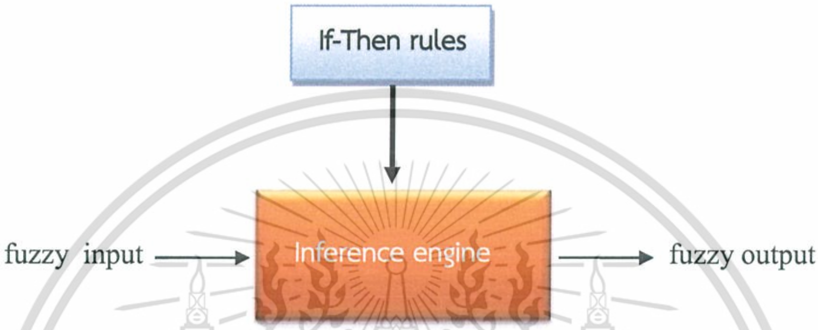


Figure 2.1 Fuzzy inference system

## 2.2 Fuzzy Logic Control System

A fuzzy logic system consist of a plant and a fuzzy logic controller (FLC) as shown in Figure 2.2.

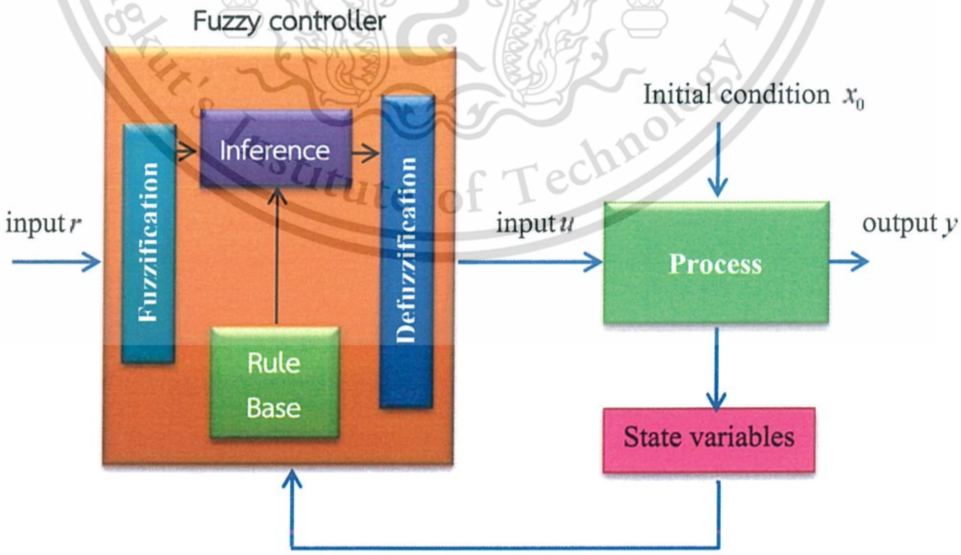


Figure 2.2 Fuzzy logic control system

Tanaka and Sugeno proposed a stability design approach which first modeled the plant by a Takagi–Sugeno (TS) fuzzy model. This fuzzy model represents the plant as a weighted sum of a set of linear state equations. An FLC is designed based on this fuzzy plant model. Then, Lyapunov's direct method can be applied to each fuzzy subsystem that is formed by each rule of the fuzzy plant model and the FLC. The stability of the whole system can be ensured if a required positive-definite matrix exists.

Let $X$ be a universe of discourse. Consider a single-input $n^{th}$-order nonlinear system of the following form

$$\dot{x} = f(x) + b(x)u$$

where $x \in X$, $x = [x_1, x_2, \ldots, x_n]^T$, is the state vector,

$f(x) = [f_1(x), f_2(x), \ldots, f_n(x)]^T$, $b(x) = [b_1(x), b_2(x), \ldots, b_n(x)]^T$ are functions describing the dynamics of the plant.

$u$ is the control input of which the value is determined by an FLC.

Next, we describe some important concepts of TSFC. The TSFC is composed of fuzzy IF-THEN rules base on the universe of discourse $X \subseteq \mathbb{R}^n$. Let $x = (x_1, \ldots, x_n) \in X$ and let $X_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ be fuzzy set that describe the linguistics terms of input variables $x_j$ in the $i^{th}$- rule, then the set of fuzzy IF–THEN rules is written as the follows:

rule $i$: If $x_1$ is $X_{i1}$ AND $x_2$ is $X_{i2}$ AND ... $x_n$ is $X_{in}$ THEN $u = u(x)$ is $\Lambda_i$, $i = 1, \ldots, m$, (2.4)

where $\Lambda_i$ is a fuzzy set describe the linguistic terms of output control variables $u$ whose membership function $\varphi_i(u) = \varphi_i(x) = \min_j \{u_{X_{ij}}(x_j)\}$. Suppose that $\varphi_i \neq 0$ for all $i = 1, \ldots, m$ and $u_i = u_i(x)$ is output control in the $i^{th}$-rule, applying the weighted-sum defuzzification method, then the control signal $u$ is given by,

$$u = \frac{\sum_{i=1}^{m} u_i \varphi_i(u_i)}{\sum_{i=1}^{m} \varphi_i(u_i)} = \frac{\sum_{i=1}^{m} u_i(x) \varphi_i(x)}{\sum_{i=1}^{m} \varphi_i(x)}. \tag{2.5}$$

## 2.3 Lyapunov Function

**Definition 2.5** Let $V:\mathbb{R}^n \to \mathbb{R}$ be a continuous scalar function. $V$ is a Lyapunov – candidate-function if it is a locally positive-definite function, i.e.

1. $V(x) > 0$ , $\forall x \in U - \{0\}$ with $U$ being a neighborhood region around $x = 0$, and
2. $V(0) = 0$.

## 2.4 Stable Fuzzy Logic Controllers

The method for stability analysis proposed in this paper is based on the following theorem. Each subsystem consist from one fuzzy rule and the process described by equation (2.4). It is proved that if each subsystem is stable in the sense of Lyapunov, under a common Lyapunov function, the overall system is also stable in sense of Lyapunov.

### 2.4.1 Globally Asymptotically Stable

**Definition 2.6** A nonlinear dynamical system $\dot{x} = f(x(t)), x(0) = x_0$, where $x(t) \subseteq \mathbb{R}^n$. Suppose $f$ has equilibrium point at $x_e$ so that $f(x_e) = 0$ then the equilibrium point of the above system is said be asymptotically stable if it is Lyapunov stable and there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|x(0) - x_e\| < \delta$, $\lim_{t\to\infty}\|x(t) - x_e\| = 0$.

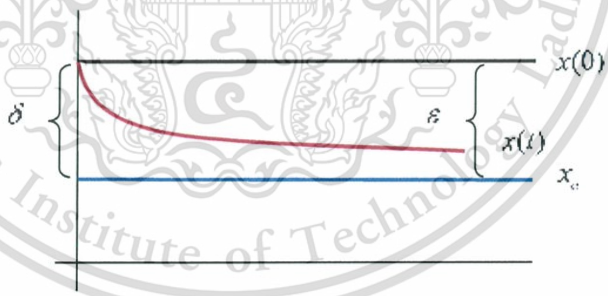


Figure 2.3 Show stable at equilibrium point

## 2.5 Banach space

Before entering the Banach space. The author will talk about. The definitions and some of the contents that are needed to describe Banach space are as follows:

**Definition 2.7** Let $X \neq \varnothing$ and $d: X \times X \to \mathbb{R}$ for all $x, y, z \in X$ with the following conditions :

1) $d(x,y) \geq 0$
2) $d(x,y) = 0 \Leftrightarrow x = y$
3) $d(x,y) = d(y,x)$
4) $d(x,y) \leq d(x,z) + d(z,y)$,

then $d$ is called distance function or metric and $(X,d)$ is metric space.

**Definition 2.8** A sequence $x_n$ in a metric space $X$ is said to be a *Cauchy sequence* if for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq N$.

**Definition 2.9** A sequence $x_n$ in a metric space $X$ converges to $x$ if for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$.

**Definition 2.10** A Metric space $(X,d)$ is said to be *complete* if every Cauchy sequence in $X$ converges ( in $X$). A complete normed vector space is called a *Banach space.*

**Definition 2.11** Let $X$ be a vector space over $\mathbb{C}$ ( or $\mathbb{R}$ ) then norm over $X$ is $\|\cdot\| : X \to \mathbb{R}$ for all $\alpha \in \mathbb{C}(\mathbb{R})$ and for all $x, y \in X$ with the following conditions:

1) $\|x\| \geq 0$
2) $\|x\| = 0 \Leftrightarrow x = 0$
3) $\|\alpha x\| = |\alpha| \|x\|$
4) $\|x + y\| \leq \|x\| + \|y\|$

and $(X, \|\cdot\|)$ is a normed space.

**Definition 2.12** A Banach space is a completes normed space with respect to the metric induced by norm, that is $d(x,y) = \|x - y\|$.

**2.5.1 Lipchitz continuous**

**Definition 2.13** A function $f : X \to Y$ from a metric space to a metric space is said to be Lipchitz continuous if there exists $L \in \mathbb{R}$ such that $d(f(u), f(v)) \leq Ld(u,v)$ for every $u, v \in X$. We call $L$ a Lipchitz constant.

**2.5.2 Fixed point**

**Definition 2.14** Let $X$ be a Banach space and $F : X \to X$ be an operator. We called that $x \in X$ is fixed point of $F$ if $F(x) = x$.

### 2.5.3 Contraction mapping

**Definition 2.15** Let $X$ be a Banach space and $F:X\to X$ be an operator. We called that $F$ is contraction. If there exist $k\in[0,1)$ such that $\|F(x)-F(y)\|<k\|x-y\|$ $\forall x,y\in X$.

Denoted that $\|\cdot\|$ is norm of $X$.

**Theorem 2.16** (Contraction Mapping Principle) If $X$ is Banach space and $F:X\to X$ is a contraction then $F$ has a unique fixed point in $X$.

### 2.5.4 Infinite-dimensional state space

We now move into the realm of infinite-dimensional dynamical systems. Therefore, in the following discussion, we shall assume that the state space $X$ is an infinite-dimensional Banach space with norm $\|\cdot\|$. The aim now is to express evolution equations in operator from as ordinary differential equations which are posed in $X$. We shall consider only problems of the type

$$\dot{x}(t)=L(x(t))+N(x(t))\ ,\quad t>0\ ,\quad u(0)=u_0\ , \tag{2.6}$$

where $L:X\supseteq D(L)\to X$ and $N:X\to X$ are linear and nonlinear operators, respectively, with $D(L)$ a linear subspace of $X$. In (2.6), the derivative is interpreted as a strong derivative, and a solution $u:[0,\infty)\to X$ is sought. The operator $L+N$ that appears in (2.6) governs the time-evolution of the infinite-dimension state vector $u(\cdot)$, and the initial-value problem (2.6) is usually called a semi linear abstract Cauchy problem (ACP).

## 2.6 Granwal's Lemma

If, for $t_0\le t\le t_1$, $x(t)\ge 0$ and $\psi(t)\ge 0$ are continuous functions such that the inequality

$$x(t)\le K+L\int_{t_0}^{t}\psi(s)x(s)ds\ .$$

Holds on $t_0\le t\le t_1$, with $K$ and $L$ positive constants, then

$$x(t)\le K\exp\left(L\int_{t_0}^{t}\psi(s)ds\right)$$

on $t_0\le t\le t_1$.

*Proof*

The inequality $$x(t) \le K + L\int_{t_0}^{t} \psi(s)x(s)ds$$

is equivalent to $$\frac{x(t)}{K + L\int_{t_0}^{t} \psi(s)x(t)ds} \le 1.$$

Multiply by $L\psi(t)$ and integrate, giving

$$\frac{x(t)}{K + L\int_{t_0}^{t} \psi(s)x(t)ds} \le 1.$$

Thus $$\ln\left( K + L\int_{t_0}^{t} \psi(s)x(s)ds \right) - \ln K \le L\int_{t_0}^{t} \psi(s)ds$$

and finally $$K + L\int_{t_0}^{t} \psi(s)x(s)ds \le K\exp\left( L\int_{t_0}^{t} \psi(s)ds \right).$$ □

## 2.7 LaSalle's invariance principle

Given a representation of the system $\dot{x} = f(x)$ where $x$ is the vector of variables, with $f(0) = 0$. If $V(x)$ can be found such that $\dot{V}(x) \le 0$,
then the set of accumulation points of any trajectory is contained $I$ where $I$ is the union of complete trajectories contained entirely in the set $x : \dot{V}(x) = 0$.

# Chapter 3

# Some approximate solution of singularly perturbed system with control

In this chapter, we consider the singularly perturbed system with control,

$$\begin{cases} \dot{x} = A(\varepsilon)x + f(x,y,\varepsilon) + B(x,\varepsilon)u \\ \dot{y} = \dfrac{1}{\varepsilon}\left[C(\varepsilon)y + g(x,y,\varepsilon)\right] \\ x(t_0) = x^0 \\ y(t_0) = y^0 \end{cases} \tag{3.1}$$

where $0 < \varepsilon << 1$ , $x = [x_1 \quad x_2 \quad \cdots \quad x_n]^T \in R^n$ and $y = [y_1 \quad y_2 \quad \cdots \quad y_m]^T \in R^m$ are the state vector, $u \in R$ is the control signal fed to the process, obtained by weighted-sum defuzzification method for Takagi-Sugino fuzzy logic control systems (T-S FLCs),

$A(\varepsilon) \in R^{n\times n}$, $C(\varepsilon) \in R^{m\times m}$ and $B(x,\varepsilon) \in R^n$,

$f(x,y,\varepsilon) = [f_1(x,y,\varepsilon) \quad f_2(x,y,\varepsilon) \quad \cdots \quad f_n(x,y,\varepsilon)]^T \in R^n$,

$g(x,y,\varepsilon) = [g_1(x,y,\varepsilon) \quad g_2(x,y,\varepsilon) \quad \cdots \quad g_m(x,y,\varepsilon)]^T \in R^m$,

$x^0 = [x_1^0 \quad x_2^0 \quad \cdots \quad x_n^0]^T \in R^n$ $y^0 = [y_1^0 \quad y_2^0 \quad \cdots \quad y_m^0]^T \in R^m$,

the symbol $\dot{x} = [\dot{x}_1 \quad \dot{x}_2 \quad \cdots \quad \dot{x}_n]^T$ and $\dot{y} = [\dot{y}_1 \quad \dot{y}_2 \quad \cdots \quad \dot{y}_m]^T$ are the derivative of $x$ and $y$ respect to the time variable $t$, respectively.

## 3.1 Some approximate solution

In this section we find some approximate solution of (3.1). We first suppose that there exists $(x^0, y^0) \in R^n \times R^m$ such that the fast equation becomes stationary i.e. $C(\varepsilon)y + g(x,y,\varepsilon) = 0$. Then system (3.1) can written to the form as follows

$$\begin{cases} \dot{x} = A(\varepsilon)x + f(x,y,\varepsilon) + B(x,\varepsilon)u \\ 0 = C(\varepsilon)y + g(x,y,\varepsilon) \\ x(t_0) = x^0 \\ y(t_0) = y^0 \end{cases} \tag{3.2}$$

Furthermore, if $\varepsilon$ is small enough, it seems reasonable to replace system (3.2) with the algebraic differential equation

$$\begin{cases} \dot{x} = A(\varepsilon)x + f(x,y,\varepsilon) + B(x,\varepsilon)u \\ 0 = Cy + g(x,y) \\ x(t_0) = x^0 \\ y(t_0) = y^0 \end{cases} \quad , \tag{3.3}$$

where $A = A(0)$, $f(x,y) = f(x,y,0)$, $B(x) = B(x,0)$, $C = C(0)$ and $g(x,y) = g(x,y,0)$.

In equation (3.3) , $0$ denotes the zero vector in $R^m$ .

**Assumption (A-1).** Suppose that the solution of $Cy + g(x,y) = 0$ with $y(t_0) = y^0$ has a unique solution $y = H(x)$ and $H$ is continuously differentiable.

By substitution $y = H(x)$ in (3.3) , we yield

$$\begin{cases} \dot{x} = A(\varepsilon)x + f(x,H(x),\varepsilon) + B(x,\varepsilon)u \\ x(t_0) = x^0 \end{cases} \tag{3.4}$$

**Definition 3.1.** An approximate mild solution on $[t_0,\infty)$ of system (3.1) is a continuous function $x:[t_0,\infty) \to \mathbb{R}^n$ satisfying the integral equation

$$x(t) = S(t-t_0)x^0 + \int_0^t S(t-s)\left[f(x(s),H(x(s),\varepsilon) + B(x(s,\varepsilon))u\right]ds \tag{3.5}$$

where $S(t) = e^{A(\varepsilon)t}$.

We will find the origin of equation (3.5) as follow:

Let $\dot{x} = Ax + Qx$,

where $Qx = f(x,H(x),\varepsilon) + B(x,\varepsilon)u$.

We have $\dot{x} - Ax = Qx$.

Let $p(t) = -A$.

Integration factor is $e^{\int p(t)dt} = e^{\int(-A)dt} = e^{-At}$.

Substituting $\dot{x}$ with $\frac{dx}{dt}$ would be,

$$\frac{dx}{dt} + p(t)x = Q(t) \ .$$

Take $e^{-At}$ , $$e^{-At}\frac{dx}{dt}+e^{-At}p(t)x=e^{-At}Q(t)$$

$$\frac{d\left(e^{-At}x(s)\right)}{dt}=e^{-At}Q(t).$$

We have, $$\int_{t_0}^{t} d\left(e^{-As}x(s)\right)=\int_{t_0}^{t} e^{-As}Q(s)ds$$

$$e^{-As}x(s)\Big|_{t_0}^{t}=\int_{t_0}^{t} e^{-As}Q(s)ds$$

$$e^{-At}x(t)-e^{-At_0}x(t_0)=\int_{t_0}^{t} e^{-As}Q(s)ds$$

$$e^{-At}x(t)=e^{-At_0}x(t_0)+\int_{t_0}^{t} e^{-As}Q(s)ds.$$

Take $e^{At}$ we have,

$$e^{At}\left(e^{-At}x(t)\right)=e^{At}\left(e^{-At_0}x(t_0)+\int_{t_0}^{t} e^{-As}Q(s)ds\right)$$

$$x(t)=e^{A(t-t_0)}x^0+e^{At}\int_{t_0}^{t} e^{-As}Q(s)ds$$

$$x(t)=e^{A(t-t_0)}x^0+\int_{t_0}^{t} e^{A(t-s)}Q(s)ds$$

$$x(t)=S(t-t_0)x^0+\int_{t_0}^{t} S(t-s)Q(s)ds,$$

where $S(t)=e^{At}$.

Thus, $x(t)=S(t-t_0)x^0+\int_{t_0}^{t} S(t-s)\left[f(x,H(x),\varepsilon)+B(x,\varepsilon)u\right]ds$ . □

Let us for convenience denote $C_n=C\left(\left[t_0,\infty\right),\mathbb{R}^n\right)$.

**Theorem 3.2** Let $H(\bullet,\varepsilon):C_n\to C_n$, $B:C_n\to C_n$ and $f(\bullet,\bullet,\varepsilon):C_n\times C_m\to C_n$ be uniformly Lipschitz continuous (with constant $L_H,L_B,L_f$). If $A$ is a generator of a fundamental matrix $S(t)$, then for every $x^0\in\mathbb{R}^n$ the semi-linear ACP (3.1) has an approximate mild solution $x\in C_n$. Moreover, the mapping $x^0\to x$ is Lipchitz continuous.

**Proof** For a given $x^0\in C_n$. Define a mapping $F_\varepsilon:C([0,\infty),\mathbb{R}^n)\to C([0,\infty),\mathbb{R}^n)$ by

$$F_\varepsilon x(t) = S(t-t_0)x^0 + \int_0^t S(t-s)\left[f(x(s), H(x(s),\varepsilon) + B(x(s),\varepsilon)u\right]ds \quad \text{for all } t\in[0,\infty)$$

Denoting by $\|x\|_\infty$ the norm of $x$ as an element of $C([0,\infty),\mathbb{R}^n)$ it follows form the definition of $F_\varepsilon$ and the uniformly Lipschitz continuous of $H, B$ and $f$ that

$$\begin{aligned}
&\|F_\varepsilon x(t) - F_\varepsilon y(t)\| \\
&= \left\| \left( S(t-t_0)x^0 + \int_0^t S(t-s)\left[f(x(s), H(x(s),\varepsilon)) + B(x(s),\varepsilon)u\right]ds \right) \right. \\
&\left. - \left( S(t-t_0)x^0 + \int_0^t S(t-s)\left[f(y(s), H(y(s),\varepsilon)) + B(y(s),\varepsilon)u\right]ds \right) \right\| \qquad (3.6) \\
&= \left\| \int_0^t S(t-s)\left(\left[f(x(s),H(x(s),\varepsilon)) - f(y(s),H(y(s),\varepsilon))\right] + \left[B(x(s),\varepsilon)u + B(y(s),\varepsilon)u\right]\right)ds \right\| \\
&\le \int_0^t \left(\|S(t-s)\| \left\|\left[f(x(s),H(x(s),\varepsilon)) - f(y(s),H(y(s),\varepsilon))\right] + \left[B(x(s),\varepsilon)u + B(y(s),\varepsilon)u\right]\right\|\right)ds \\
&\le M\int_0^t \left(L_f\left(\|x-y\| + \|Hx - Hy\|\right) + L_B\|x-y\|\|u\|\right)ds \\
&\le M\int_0^t \left(L_f\left(\|x-y\| + L_H\|x-y\|\right) + L_B\|x-y\|\|u\|\right)ds \\
&= M\left(L_f(1+L_H) + L_B\|u\|\right)\|x-y\|_\infty t, \qquad (3.7)
\end{aligned}$$

where $M$ is bound of $\|S(t)\|$ on $[0,\infty)$. Using 3.6 and 3.7 and induction on $n$, we have

$$\begin{aligned}
\|(F_\varepsilon^n x)(t) - (F_\varepsilon^n y)(t)\| &\le \frac{\left(M\left(L_f(1+L_H) + L_B\|u\|\right)\right)^n}{n!}\|x-y\|_\infty t \\
&\le \frac{\left(M\left(L_f(1+L_H) + L_B\|u\|\right)\right)^n t}{n!}\|x-y\|_\infty \qquad (3.8)
\end{aligned}$$

For $n$ sufficiently large, $\dfrac{\left(M\left(L_f(1+L_H) + L_B\|u\|\right)\right)^n t}{n!} < 1$ and by a well-known extension the contraction mapping principle, there is a unique $x \in C([0,\infty),\mathbb{R}^n)$ such that

$$x(t) = (F_\varepsilon x)(t) = S(t-s)x^0 + \int_0^t S(t-s)\left[f(x(s), H(x(s),\varepsilon) + B(x(s,\varepsilon))u\right]ds \qquad (3.9)$$

Therefore, the semi linear ACP (3.4) has a unique mild solution $x \in C([0,\infty,),\mathbb{R}^n)$. □

**Proof** We will show $\frac{\left(M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)\right)^n}{n!}<1,$

where $\left(M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)\right)^n$ is constants.

Denoted that $M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)$ is represented by $a$.

Where $n!=n(n-1)(n-2)\ldots\left(\frac{n}{2}\right)\ldots 1,$

$$n!\geq n(n-1)\ldots\left(\frac{n}{2}\right)$$

$$\geq\left(\frac{n}{2}\right)^{\frac{n}{2}}.$$

So $\left(\frac{n}{2}\right)^{\frac{n}{2}}$ is approximate of $n!$.

Thus, we will show $\frac{a^n}{\left(\frac{n}{2}\right)^{\frac{n}{2}}}<1.$

We have
$$\frac{a^n}{\left(\frac{n}{2}\right)^{\frac{n}{2}}}=\frac{\left(a^2\right)^{\left(\frac{n}{2}\right)}}{\left(\frac{n}{2}\right)^{\frac{n}{2}}}$$

$$=\left(\frac{a^2}{\left(\frac{n}{2}\right)}\right)^{\frac{n}{2}}$$

$$=\left(\frac{2a^2}{n}\right)^{\frac{n}{2}}.$$

For $a$ is constant and $n$ sufficiently large, we have $\frac{2a^2}{n}<1$.

Therefore, we have $\frac{\left(M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)\right)^n}{n!}<1.$ □

The Lipschitz continuity of the mapping $x^0 \to x$ is consequences of the following argument. Let $x$ and $y$ be the mild solution of (3.4) with the initial value $x^0$ and $y^0$, respectively. Then it follows from the definition of mild solution and the uniformly Lipschitz continuous of $H, B$ and $f$ that

$$\|x(t)-y(t)\| \le M\left\|x^0-y^0\right\| + M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)\int_0^t \|x(s)-y(s)\| ds. \tag{3.10}$$

By using Granwal's Lemma, it implies that

$$\|x(t)-y(t)\| \le M \exp\left(M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)t\right)\left\|x^0-y^0\right\|. \tag{3.11}$$

Thus,

$$\|x-y\|_\infty \le M \exp\left(M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)t\right)\left\|x^0-y^0\right\| \tag{3.12}$$

which yields the Lipschitz continuity of mapping $x^0 \to x$. □

# Chapter 4

# Stability analysis of the approximated system

In this section, we discuss the stability analysis of system (3.4), the approximated fuzzy control system of system (3.1).

Let $V(\cdot,\varepsilon):\mathbb{R}^n \to \mathbb{R}$ be a scalar function such that $V(x,\varepsilon)>0$, $\forall x \neq 0$ and its first order partial derivatives are continuous. Then the total derivative of $V=V(x,\varepsilon)$ with respect to $t$, with $x$ satisfying the approximate fuzzy control system (3.4), is

$$
\begin{aligned}
\dot{V}(x,\varepsilon) &= \frac{dV}{dt} \\
&= \sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\cdot\frac{dx_i}{dt}\right) \\
&= \sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\left(\sum_{k=1}^{n} a_{ik}(\varepsilon)x_k + f_i(x,H(x),\varepsilon)+b_i(x,\varepsilon)u\right)\right) \\
&= \sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\sum_{k=1}^{n} a_{ik}(\varepsilon)x_k\right)+\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i} f_i(x,H(x),\varepsilon)\right)+\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i} b_i(x,\varepsilon)u\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial V}{\partial x_i} a_{ik}(\varepsilon)x_k\right)+\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i} f_i(x,H(x),\varepsilon)\right)+\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i} b_i(x,\varepsilon)u\right) \\
&= \bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)+\bar{B}(x,\varepsilon)u,
\end{aligned}
$$

where $\bar{A}(x,\varepsilon)=\sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial V}{\partial x_i} a_{ik}(\varepsilon)x_k\right)$, $\bar{F}(x,H(x),\varepsilon)=\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i} f_i(x,H(x),\varepsilon)\right)$,

and $\bar{B}(x,\varepsilon)=\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i} b_i(x,\varepsilon)\right)$.

Therefore, system (3.4) can be transformed to a real-value system,

$$\dot{V}(x,\varepsilon)=\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)+\bar{B}(x,\varepsilon)u. \tag{4.1}$$

Throughout this paper we denote

$$X_i^A = \left\{ x \in X \middle| \ \varphi_i(x) \neq 0 \right\}. \tag{4.2}$$

where membership function $\varphi_i(u) = \varphi_i(\mathrm{x}) = \min_j \left\{ u_{X_{ij}}(x_j) \right\}$

and $u = \dfrac{\sum_{i=1}^{m} u_i \varphi_i(u_i)}{\sum_{i=1}^{m} \varphi_i(u_i)} = \dfrac{\sum_{i=1}^{m} u_i(x) \varphi_i(x)}{\sum_{i=1}^{m} \varphi_i(x)}$.

and partition this set by using signs of $\bar{B}(x,\varepsilon)$ as follows,

$$B^0 = \left\{ x \in X \middle| \bar{B}(x,\varepsilon) = 0 \right\},$$

$$B^+ = \left\{ x \in X \middle| \bar{B}(x,\varepsilon) > 0 \right\},$$

and $$B^- = \left\{ x \in X \middle| \bar{B}(x,\varepsilon) < 0 \right\}. \tag{4.3}$$

**Theorem 4.1** Let an approximate control system (3.4) be fed with a control signal , obtained from a weighted-sum defuzzification method for TSFC. Let $x = 0$ be an equilibrium point. Suppose that there exists a function $V(\cdot,\varepsilon): X \to \mathbb{R}$ of which $V(x,\varepsilon) > 0$, $\forall x \neq 0$ and all first-order partial derivatives are continuous, with

1) $\bar{A}(x,\varepsilon) + \bar{F}(x,H(x),\varepsilon) \leq 0$ for all $x \in B^0$,

2) $u_i \leq -\left( \dfrac{\bar{A}(x,\varepsilon) + \bar{F}(x,H(x),\varepsilon)}{\bar{B}(x,\varepsilon)} \right)$ for all $x \in X_i^A \cap B^+$, and

$u_i \geq -\left( \dfrac{\bar{A}(x,\varepsilon) + \bar{F}(x,H(x),\varepsilon)}{\bar{B}(x,\varepsilon)} \right)$ for all $x \in X_i^A \cap B^-$, $i = 1,\dots,m$,

3) $S = \left\{ x \in X \middle| \dot{V}(x,\varepsilon) = 0 \right\} = \{ x = 0 \}$.

Then the fuzzy control system is globally asymptotically stable in the sense of Lyaponov at the origin.

**Proof** We first show that $\dot{V}(x,\varepsilon)$ is negative semi-definite. Let $x_0 \in X$ and let us consider signs of $\bar{B}(x,\varepsilon)$. We partition this proof into 3 cases as follows.

Case 1 : $x_0 \in X_i^A \cap B^+$, for all $i = 1,\dots,m$, then $\bar{B}(x_0,\varepsilon) > 0$.

By condition 2), $u_i(x_0) \leq -\left( \dfrac{\bar{A}(x_0,\varepsilon) + \bar{F}(x_0,H(x_0),\varepsilon)}{\bar{B}(x_0,\varepsilon)} \right)$.

From (2.4), we have

$$u(x_0)=\frac{\sum_{i=1}^{m}u_i(x_0)\varphi_i(x_0)}{\sum_{i=1}^{m}\varphi_i(x_0)}$$

$$=\frac{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)u_i(x_0)}{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}$$

$$\leq\frac{-\left(\dfrac{\bar{A}(x_0,\varepsilon)+\bar{F}(x_0,H(x_0),\varepsilon)}{\bar{B}(x_0,\varepsilon)}\right)\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}$$

$$=-\left(\frac{\bar{A}(x_0,\varepsilon)+\bar{F}(x_0,H(x_0),\varepsilon)}{\bar{B}(x_0,\varepsilon)}\right). \tag{4.4}$$

This implies that $\dot{V}(x,\varepsilon)=\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)+\bar{B}(x,\varepsilon)u\leq 0$.

Therefore, $\dot{V}(x_0,\varepsilon)\leq 0$, $\forall x_0\in X_i^A\cap B^+$ for all $i=1,\ldots,n$.

Case 2 : $x_0\in X_i^A\cap B^-$, for all $i=1,\ldots,m$, then $\bar{B}(x_0,\varepsilon)<0$.

By condition 2), $u_i(x_0)\geq-\left(\dfrac{\bar{A}(x_0,\varepsilon)+\bar{F}(x_0,H(x_0),\varepsilon)}{\bar{B}(x_0,\varepsilon)}\right)$.

From (2.4), we have

$$u(x_0)=\frac{\sum_{i=1}^{m}u_i(x_0)\varphi_i(x_0)}{\sum_{i=1}^{m}\varphi_i(x_0)}$$

$$=\frac{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)u_i(x_0)}{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}$$

$$\geq\frac{-\left(\dfrac{\bar{A}(x_0,\varepsilon)+\bar{F}(x_0,H(x_0),\varepsilon)}{\bar{B}(x_0,\varepsilon)}\right)\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}$$

$$u(x_0)=-\left(\frac{\bar{A}(x_0,\varepsilon)+\bar{F}(x_0,H(x_0),\varepsilon)}{\bar{B}(x_0,\varepsilon)}\right). \tag{4.5}$$

This implies that $\dot{V}(x,\varepsilon)=\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)+\bar{B}(x,\varepsilon)u\leq 0$.

Therefore, $\dot{V}(x_0,\varepsilon)\leq 0$, $\forall x_0\in X_i^A\cap B^-$ for all $i=1,\ldots,n$.

Case 3 : $x_0\in B^0$, then $\bar{B}(x_0,\varepsilon)=0$. By condition 1), $\bar{A}(x_0,\varepsilon)+\bar{F}(x_0,H(x_0),\varepsilon)\leq 0$.

This implies that $\dot{V}(x_0,\varepsilon)=\bar{A}(x_0,\varepsilon)+\bar{F}(x_0,H(x_0),\varepsilon)+\bar{B}(x_0,\varepsilon)u\leq 0$.

From Case1) to Case3), we conclude that $\dot{V}(x_0,\varepsilon)\leq 0$ for all $x_0\in X$, and $V(x,\varepsilon)$ is negative semi-definite. Condition 3) of the theorem ensures the fulfillment of LaSalle's invariant set principle.

This means that LaSalle's global invariant set theorem applies, hence the equilibrium point at the origin is globally asymptotically stable. □

**Theorem 4.2** Let $S=\left\{x\in X\middle|\dot{V}(x,\varepsilon)=0\right\}$ and $S_i=\left\{x\in X\middle|\dot{V}_i(x,\varepsilon)=0\right\}$ be such that $\dot{V}_i(x,\varepsilon)=\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)+\bar{B}(x,\varepsilon)u_i(x)$, then $S\subseteq\bigcup_{i=1}^{m}S_i$.

**Proof** We will show that $S\subseteq\bigcup_{i=1}^{m}S_i$.

Note that $\dot{V}(x,\varepsilon)=\dfrac{\sum_{i=1}^{m}\dot{V}_i(x,\varepsilon)\varphi_i(x)}{\sum_{i=1}^{m}\varphi_i(x)}$ for all $x\in X$.

Let $x_0\in S$, then $\dot{V}(x_0,\varepsilon)=0$. This implies that $\sum_{i=1}^{m}\dot{V}_i(x_0,\varepsilon)\varphi_i(x_0)=0$. Following the proof of Theorem 4.1, there exists an index $k$ such that $\dot{V}_k(x_0,\varepsilon)=0$.

Thus $x_0\in\bigcup_{i=1}^{m}S_i$. □

# Chapter 5

# A demonstrative example

In this section, example is presented to show case our method of stability analysis. Let $X=[-50,50]\times[-50,50]\times[-50,50]$ and $Y=\mathbb{R}$. Consider a perturbed differential system with control $u\in\mathbb{R}$,

$$\frac{dx_1}{dt}=-\varepsilon x_1+\sigma x_2+x_4+(1+\varepsilon)u\ ,$$

$$\frac{dx_2}{dt}=-2\varepsilon x_2+x_1(\rho-x_3)-x_2\ ,$$

$$\frac{dx_3}{dt}=-(1+\varepsilon)x_3+x_1x_2-\beta x_3\ ,$$

$$\frac{dx_4}{dt}=\frac{1}{\varepsilon}\left((1+\varepsilon)x_4+(\sigma+\varepsilon)x_1\right). \tag{5.1}$$

Let $x=(x_1,x_2,x_3)$ and $y=x_4$. Then system (5.1) becomes

$$\begin{cases}\dot{x}=\begin{pmatrix}-\varepsilon & 0 & 0\\ 0 & -2\varepsilon & 0\\ 0 & 0 & -1-\varepsilon\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+\begin{pmatrix}\sigma x_2+y\\ x_1(\rho-x_3)-x_2\\ x_1x_2-\beta x_3\end{pmatrix}+\begin{pmatrix}1+\varepsilon\\ 0\\ 0\end{pmatrix}u\\ \dot{y}=\frac{1}{\varepsilon}\left((1+\varepsilon)y+(\sigma+\varepsilon)x_1\right)\\ x(t_0)=x_0\\ y(t_0)=y_0\end{cases}. \tag{5.2}$$

Then the approximate system of system (5.2) is

$$\begin{cases}\dot{x}=\begin{pmatrix}-\varepsilon & 0 & 0\\ 0 & -2\varepsilon & 0\\ 0 & 0 & -1-\varepsilon\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+\begin{pmatrix}\sigma x_2-\sigma x_1\\ x_1(\rho-x_3)-x_2\\ x_1x_2-\beta x_3\end{pmatrix}+\begin{pmatrix}1+\varepsilon\\ 0\\ 0\end{pmatrix}u\\ x(t_0)=x_0\end{cases}. \tag{5.3}$$

The approximate mild solution of system (5.2) is in the form,

$$x(t)=S(t-t_0)x^0+\int_{t_0}^{t}S(t-s)\left[\begin{pmatrix}\sigma x_2-\sigma x_1\\ x_1(\rho-x_3)-x_2\\ x_1x_2-\beta x_3\end{pmatrix}+\begin{pmatrix}1+\varepsilon\\ 0\\ 0\end{pmatrix}u\right]ds\ ,$$

which $S(t)=\exp\begin{pmatrix}-\varepsilon & 0 & 0\\ 0 & -2\varepsilon & 0\\ 0 & 0 & -1-\varepsilon\end{pmatrix}t=\begin{pmatrix}e^{-\varepsilon t} & 0 & 0\\ 0 & e^{-2\varepsilon t} & 0\\ 0 & 0 & e^{-(1+\varepsilon)t}\end{pmatrix}$.

The objective of this presentation is to find the control $u_i$ for which the approximate system (5.3) can be stabilized by the above described TSFC. By following the proof of Theorem 4.1, we can design of stable fuzzy control system as following steps.

**Step 1** : Choose a Lyapunov function $V$, calculate $\dot{V}$, $\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)$, and $\bar{B}(x,\varepsilon)$.

In this example, we choose $V(x)=x_1^2+x_2^2+x_3^2$.

From $\quad \dot{V}(x,\varepsilon)=\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)+\bar{B}(x,\varepsilon)u$

where $\quad \bar{A}(x,\varepsilon)=\sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial V}{\partial x_i}a_{ik}(\varepsilon)x_k\right)$,

$$\bar{F}(x,H(x),\varepsilon)=\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}f_i(x,H(x),\varepsilon)\right),$$

and $\quad \bar{B}(x,\varepsilon)=\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}b_i(x,\varepsilon)\right)$.

We have

$$\dot{V}(x,\varepsilon)=\sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial V}{\partial x_i}a_{ik}(\varepsilon)x_k\right)+\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}f_i(x,H(x),\varepsilon)\right)+\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}b_i(x,\varepsilon)u\right)$$

$$=\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}a_{i1}(\varepsilon)x_1+\frac{\partial V}{\partial x_i}a_{i2}(\varepsilon)x_2+\frac{\partial V}{\partial x_i}a_{i3}(\varepsilon)x_3\right)+\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}f_i(x,H(x),\varepsilon)\right)$$

$$+\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}b_i(x,\varepsilon)u\right)$$

$$\dot{V}(x,\varepsilon)=\left(\left[\frac{\partial V}{\partial x_1}a_{11}(\varepsilon)x_1+\frac{\partial V}{\partial x_1}a_{12}(\varepsilon)x_2+\frac{\partial V}{\partial x_1}a_{13}(\varepsilon)x_3\right]\right.$$

$$+\left[\frac{\partial V}{\partial x_2}a_{21}(\varepsilon)x_1+\frac{\partial V}{\partial x_2}a_{22}(\varepsilon)x_2+\frac{\partial V}{\partial x_2}a_{23}(\varepsilon)x_3\right]$$

$$\left.+\left[\frac{\partial V}{\partial x_3}a_{31}(\varepsilon)x_1+\frac{\partial V}{\partial x_3}a_{32}(\varepsilon)x_2+\frac{\partial V}{\partial x_3}a_{33}(\varepsilon)x_3\right]\right)$$

$$+\left(\frac{\partial V}{\partial x_1}f_1(x,H(x),\varepsilon)+\frac{\partial V}{\partial x_2}f_2(x,H(x),\varepsilon)+\frac{\partial V}{\partial x_3}f_3(x,H(x),\varepsilon)\right)$$

$$+\left(\frac{\partial V}{\partial x_1}b_1(x,\varepsilon)u+\frac{\partial V}{\partial x_2}b_2(x,\varepsilon)u\frac{\partial V}{\partial x_3}b_3(x,\varepsilon)u\right)$$

$$=\left(\left[(2x_1)(-\varepsilon x_1)+(2x_2)(-2\varepsilon x_2)+(2x_3)(-1-\varepsilon)x_3\right]\right)+\left(2x_1(\sigma x_2-\sigma x_1)\right.$$

$$\left.+2x_2(x_1(\rho-x_3)-x_2)+2x_3(x_1x_2-\beta x_3)\right)+\left(2x_1(1+\varepsilon)u\right)$$

$$=\left(-2\varepsilon x_1^2-4\varepsilon x_2^2-2(1+\varepsilon)x_3^2\right)+\left(2\sigma x_1x_2-2\sigma x_1^2+2\rho x_1x_2-2x_1x_2x_3-2x_2^2\right.$$

$$\left.+2x_1x_2x_3-\beta x_3^2\right)+\left(2x_1(1+\varepsilon)u\right)$$

$$=-2(\varepsilon+\sigma)x_1^2-2(2\varepsilon+1)x_2^2-2(1+\varepsilon+\beta)x_3^2+2(\sigma+\rho)x_1x_2+2(1+\varepsilon)x_1u\,.$$

Then we have

$$\dot{V}(x,\varepsilon)=-2(\varepsilon+\sigma)x_1^2-2(2\varepsilon+1)x_2^2-2(1+\varepsilon+\beta)x_3^2+2(\sigma+\rho)x_1x_2+2(1+\varepsilon)x_1u\,. \quad (5.4)$$

In this case, we have

$$\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)=-2(\varepsilon+\sigma)x_1^2-2(2\varepsilon+1)x_2^2$$

$$-2(1+\varepsilon+\beta)x_3^2+2(\sigma+\rho)x_1x_2 \quad (5.5)$$

and $$\bar{B}(x,\varepsilon)=2(1+\varepsilon)x_1\,. \quad (5.6)$$

**Step 2 :** Partition the discourse space $X$ by using $\bar{B}(x,\varepsilon)$.

In this example, we obtain

$$B^0=\{(0,x_2,x_3)\in X\,|\,x_1=0\}=\{0\}\times[-50,50]\times[-50,50]\,,$$

$$B^+=\{(x_1,x_2,x_3)\in X\,|\,x_1>0\}=(0,50]\times[-50,50]\times[-50,50]$$

and $B^{-}=\{(x_1,x_2,x_3)\in X|x_1<0\}=[-50,0)\times[-50,50]\times[-50,50]$. (5.7)

**Step 3 :** Check that $\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)\le 0$ for all $x\in B^0$. If it is not, we must choose a new Lyapunov function and do Step 1 again. Else go to the next step.

In this example, we have $B^0=\{(0,x_2,x_3)\in X|x_1=0\}$

and $\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)=-2(\varepsilon+\sigma)x_1^2-2(2\varepsilon+1)x_2^2$

$$-2(1+\varepsilon+\beta)x_3^2+2(\sigma+\rho)x_1x_2$$

$$\le 0.$$

Thus, $\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)\le 0$ for all $x\in B^0$. Then we go to the next step.

**Step 4 :** Determine the fuzzy of the linguistic terms Negative (N), Zero (Z), and Positive (P) on $X$ corresponding to state variable $x$.

In this example, we set $N=\langle -50,-50,0\rangle$, $Z=\langle -20,0,20\rangle$, and $P=\langle 0,50,50\rangle$ corresponding to the state variable $X=(x_1,x_2,x_3)$.

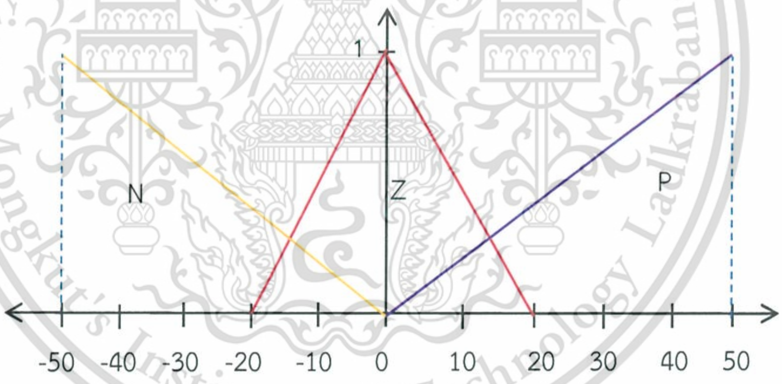


Figure 5.1 Membership functions of $X$

**Step 5 :** A fuzzy control IF-THEN rule is constructed to use as the inference engine. In this example, from (5.5) and (5.6), in the constructing fuzzy control rule we can ignore variable $x_3$ that is, it suffice to construct fuzzy rule from two variables, $x_1$ and $x_2$ only. The fuzzy control rules are illustrated in Table 5.1.

Table 5.1 The fuzzy control rules base

| Fuzzy Control Rules Base | | | |
|---|---|---|---|
| Rule | $x_1$ | $x_2$ | $u_i$ |
| 1 | P | P | $u_1$ |
| 2 | P | N | $u_2$ |
| 3 | P | Z | $u_3$ |
| 4 | N | P | $u_4$ |
| 5 | N | N | $u_5$ |
| 6 | N | Z | $u_6$ |
| 7 | Z | P | $u_7$ |
| 8 | Z | N | $u_8$ |
| 9 | Z | Z | $u_9$ |

**Step 6 :** Determine $u_i$ from each fuzzy control rule $i$ from Step 5 by using condition 2) of Theorem 4.1. In this example,

***Rule 1*** If $x_1$ is P AND $x_2$ is P. Then $X_1^A = (0,50]\times(0,50]\times[-50,50]$. Consequently, $X_1^A \cap B^+ = (0,50]\times(0,50]\times[-50,50]$ and $X_1^A \cap B^- = \varnothing$. By condition 2), we have

$$u_1(x) \le -\left(\frac{\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)}{\bar{B}(x,\varepsilon)}\right)$$

$$=\left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2. \tag{5.8}$$

From (5.8), we can choose $u_1(x) = -\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$.

***Rule 2*** If $x_1$ is P AND $x_2$ is N. Then $X_2^A = (0,50]\times[-50,0)\times[-50,50]$. Consequently, $X_2^A \cap B^+ = (0,50]\times[-50,0)\times[-50,50]$ and $X_2^A \cap B^- = \varnothing$. By condition 2), we have

$$u_2(x) \le \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2. \tag{5.9}$$

From (5.9), we can choose $u_2(x) = -1$.

*Rule 3* If $x_1$ is P AND $x_2$ is Z. Then $X_3^A=(0,50]\times(-20,20)\times[-50,50]$. Consequently, $X_3^A\cap B^+=(0,50]\times(-20,20)\times[-50,50]$ and $X_3^A\cap B^-=\varnothing$.

By condition 2), we have

$$u_3(x)\le\left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2. \tag{5.10}$$

From (5.10), we can choose

$$u_3(x)=\left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}-20\left(\frac{\sigma+\rho}{1+\varepsilon}\right).$$

*Rule 4* If $x_1$ is N AND $x_2$ is P. Then $X_4^A=[-50,0)\times(0,50]\times[-50,50]$. Consequently, $X_4^A\cap B^+=\varnothing$ and $X_4^A\cap B^-=[-50,0)\times(0,50]\times[-50,50]$. By condition 2), we have

$$u_4(x)\ge\left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2. \tag{5.11}$$

From (5.11), we can choose $u_4(x)=1$.

*Rule 5* If $x_1$ is N AND $x_2$ is N. Then $X_5^A=[-50,0)\times[-50,0)\times[-50,50]$. Consequently, $X_5^A\cap B^+=\varnothing$, and $X_5^A\cap B^-=[-50,0)\times[-50,0)\times[-50,50]$.

By condition 2), we have

$$u_5(x)\ge\left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2. \tag{5.12}$$

From (5.12), we can choose $u_5(x)=-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$.

*Rule 6* If $x_1$ is N AND $x_2$ is Z. Then $X_6^A=[-50,0)\times(-20,20)\times[-50,50]$. Consequently, $X_6^A\cap B^+=\varnothing$, and $X_6^A\cap B^-=[-50,0)\times(-20,20)\times[-50,50]$.

By condition 2), we have

$$u_6(x)\ge\left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2. \tag{5.13}$$

From (5.13), we can choose

$$u_6(x)=\left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}+20\left(\frac{\sigma+\rho}{1+\varepsilon}\right).$$

*Rule 7* If $x_1$ is Z AND $x_2$ is P. Then $X_7^A = (-20,20)\times(0,50]\times[-50,50]$. Consequently, $X_7^A \cap B^+ = (0,20)\times(0,50]\times[-50,50]$

and $X_7^A \cap B^- = (-20,0)\times(0,50]\times[-50,50]$. We next consider two cases.

*Case 1* : If $x \in X_7^A \cap B^+ = (0,20)\times(0,50]\times[-50,50]$, by condition 2),

we have $u_7(x) \le \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} - \left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$. (5.14)

From (5.14), we can choose $u_7(x) = -\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$ .

*Case 2* : If $x \in X_7^A \cap B^- = (-20,0)\times(0,50]\times[-50,50]$, by condition 2),

we have $u_7(x) \ge \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} - \left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$. (5.15)

From (5.15), we can choose $u_7(x) = -\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$.

From Case 1) and Case 2), we conclude that we can choose $u_7(x) = -\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$.

***Rule 8*** (If $x_1$ is Z AND $x_2$ is N) and ***Rule 9*** (If $x_1$ is Z AND $x_2$ is Z) are similar to ***Rule 7***, we can choose $u_8(x) = u_9(x) = u_7(x) = -\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$.

The control variables $u_i$ from each fuzzy control rule $i$ are illustrated in Table 5.2.

Table 5.2 The control variables $u_i$ from the fuzzy control rules base

| Fuzzy Control Rules Base | | | |
|---|---|---|---|
| Rule | $x_1$ | $x_2$ | $u_i$ |
| 1 | P | P | $u_1=-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$ |
| 2 | P | N | $u_2=-1$ |
| 3 | P | Z | $u_3=\left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}-20\left(\frac{\sigma+\rho}{1+\varepsilon}\right)$ |
| 4 | N | P | $u_4=1$ |
| 5 | N | N | $u_5=-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$ |
| 6 | N | Z | $u_6=\left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}+20\left(\frac{\sigma+\rho}{1+\varepsilon}\right)$ |
| 7 | Z | P | $u_7=-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$ |
| 8 | Z | N | $u_8=-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$ |
| 9 | Z | Z | $u_9=-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$ |

**Step 7** : Check that $S=\left\{x\in X\middle|\dot{V}(x,\varepsilon)=0\right\}=\{0\}$.

In this example, we follow from Rule 1) - Rule 9) as Step 6 and use Theorem 4.2, as below

***Rule 1*** Consider $S_1=\left\{x\in X\middle|\dot{V}_1(x,\varepsilon)=0\right\}$.

Since $u_1(x)=-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$, where $x_1$ is P and $x_2$ is P.

We have

$$\dot{V}_1(x,\varepsilon)=-2(\varepsilon+\sigma)x_1^2-2(2\varepsilon+1)x_2^2-2(1+\varepsilon+\beta)x_3^2+2(\sigma+\rho)x_1x_2+2(1+\varepsilon)x_1u\neq0$$

Thus $S_1=\varnothing$.

***Rule 2*** Consider $S_2=\left\{x\in X\middle|\dot{V}_2(x,\varepsilon)=0\right\}$.

Since $u_2(x)=-1$, where $x_1$ is P $x_2$ is N.

We have

$$\dot{V}_2(x,\varepsilon) = -2(\varepsilon+\sigma)x_1^2 - 2(2\varepsilon+1)x_2^2 - 2(1+\varepsilon+\beta)x_3^2 + 2(\sigma+\rho)x_1x_2 + 2(1+\varepsilon)x_1u \neq 0$$

Thus $S_2 = \varnothing$.

*Rule 3* Consider $S_3 = \left\{x \in X \middle| \dot{V}_3(x,\varepsilon) = 0\right\}$.

Since $u_3(x) = \left(\dfrac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \dfrac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \dfrac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} - 20\left(\dfrac{\sigma+\rho}{1+\varepsilon}\right)$,

where $x_1$ is P $x_2$ is Z.

We have

$$\dot{V}_3(x,\varepsilon) = -2(\varepsilon+\sigma)x_1^2 - 2(2\varepsilon+1)x_2^2 - 2(1+\varepsilon+\beta)x_3^2 + 2(\sigma+\rho)x_1x_2 + 2(1+\varepsilon)x_1u \neq 0$$

Thus $S_3 = \varnothing$ .

*Rule 4* Consider $S_4 = \left\{x \in X \middle| \dot{V}_4(x,\varepsilon) = 0\right\}$.

Since $u_4(x) = 1$, where $x_1$ is N $x_2$ is P.

We have

$$\dot{V}_4(x,\varepsilon) = -2(\varepsilon+\sigma)x_1^2 - 2(2\varepsilon+1)x_2^2 - 2(1+\varepsilon+\beta)x_3^2 + 2(\sigma+\rho)x_1x_2 + 2(1+\varepsilon)x_1u \neq 0$$

Thus $S_4 = \varnothing$.

*Rule 5* Consider $S_5 = \left\{x \in X \middle| \dot{V}_5(x,\varepsilon) = 0\right\}$.

Since $u_5(x) = -\left(\dfrac{\sigma+\rho}{1+\varepsilon}\right)x_2$, where $x_1$ is N $x_2$ is N.

We have

$$\dot{V}_5(x,\varepsilon) = -2(\varepsilon+\sigma)x_1^2 - 2(2\varepsilon+1)x_2^2 - 2(1+\varepsilon+\beta)x_3^2 + 2(\sigma+\rho)x_1x_2 + 2(1+\varepsilon)x_1u \neq 0$$

Thus $S_5 = \varnothing$ .

*Rule 6* Consider $S_6 = \left\{x \in X \middle| \dot{V}_6(x,\varepsilon) = 0\right\}$.

Since $u_6(x) = \left(\dfrac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \dfrac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \dfrac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} + 20\left(\dfrac{\sigma+\rho}{1+\varepsilon}\right)$,

where $x_1$ is N $x_2$ is Z.

We have

$$\dot{V}_6(x,\varepsilon) = -2(\varepsilon+\sigma)x_1^2 - 2(2\varepsilon+1)x_2^2 - 2(1+\varepsilon+\beta)x_3^2 + 2(\sigma+\rho)x_1x_2 + 2(1+\varepsilon)x_1u \neq 0$$

Thus $S_6 = \varnothing$ .

*Rule 7* Consider $S_7 = \left\{ x \in X \middle| \dot{V}_7(x,\varepsilon) = 0 \right\}$.

Since $u_7(x) = -\left(\dfrac{\sigma+\rho}{1+\varepsilon}\right)x_2$ , where $x_1$ is Z $x_2$ is P.

We have

$$\dot{V}_7(x,\varepsilon) = -2(\varepsilon+\sigma)x_1^2 - 2(2\varepsilon+1)x_2^2 - 2(1+\varepsilon+\beta)x_3^2 + 2(\sigma+\rho)x_1x_2 + 2(1+\varepsilon)x_1u \neq 0$$

Thus $S_7 = \varnothing$.

*Rule 8* Consider $S_8 = \left\{ x \in X \middle| \dot{V}_8(x,\varepsilon) = 0 \right\}$.

Since $u_8(x) = -\left(\dfrac{\sigma+\rho}{1+\varepsilon}\right)x_2$ , where $x_1$ is Z $x_2$ is N.

We have

$$\dot{V}_8(x,\varepsilon) = -2(\varepsilon+\sigma)x_1^2 - 2(2\varepsilon+1)x_2^2 - 2(1+\varepsilon+\beta)x_3^2 + 2(\sigma+\rho)x_1x_2 + 2(1+\varepsilon)x_1u \neq 0$$

Thus $S_8 = \varnothing$.

*Rule 9* Consider $S_9 = \left\{ x \in X \middle| \dot{V}_9(x,\varepsilon) = 0 \right\}$.

Since $u_9(x) = -\left(\dfrac{\sigma+\rho}{1+\varepsilon}\right)x_2$, where $x_1$ is Z $x_2$ is Z.

We have

$$\dot{V}_9(x,\varepsilon) = -2(\varepsilon+\sigma)x_1^2 - 2(2\varepsilon+1)x_2^2 - 2(1+\varepsilon+\beta)x_3^2 + 2(\sigma+\rho)x_1x_2 + 2(1+\varepsilon)x_1u = 0$$

Thus $S_9 = \{0\}$ .

$$\therefore \quad S \subseteq \bigcup_{i=1}^{9} S_i = \{0\}.$$

Therefore, $S_i = \varnothing$ for all $i = 1,\ldots,8$ and $S_9 = \{0\}$. Thus, we have $S \subseteq \bigcup_{i=1}^{9} S_i = \{0\}$. by Theorem 4.2, we infer that the fuzzy control system is globally asymptotically stable in the sense of Lyapunov at the origin.

# Chapter 6

# Conclusions

This article is concerned with proving of the existence and uniqueness of an approximate mild solution to a fuzzy control system with singular perturbation in a fast-slow system. Then we already provide some sufficiently stability conditions and prove asymptotically stable in the sense of Lyapunov for the fuzzy control system. Furthermore, we give the example to explain our main results. Last but not least we should be interested in studying applications of these problems. Even though it seems likely that efforts in this direction can be successful, there no guarantee for that. Therefore, we can only hope for the best and prepare the worst.

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# Appendix

# Appendix

## Stability Analysis of an Approximate System of a Nonlinear Singularly-Perturbed Fuzzy Control System

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# Stability Analysis of an Approximate System of a Nonlinear Singularly-Perturbed Fuzzy Control System

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**Abstract :** In this study, we investigated a nonlinear fuzzy control fast-slow system that is singularly perturbed. A reduced system was introduced and an approximate solution of the system was defined. Furthermore, a stability analysis of the approximate system with Takaki-Sugeno fuzzy logic controller was constructed. We provided sufficiently stability conditions and proved the approximate system to be asymptotically stable in the sense of Lyapunov. A demonstrative example is also provided to showcase the method of stability analysis.

**Keywords :** Fuzzy control system, Stability analysis, Lyapunov, Takaki-Sugeno fuzzy logic controller.
**2010 Mathematics Subject Classification :** 47H09; 47H10.

## 1 Introduction

It is quite natural to model a dynamical control system plagued with uncertainty by a fuzzy system of differential equations. There are many ways to model such a system[1-9]. The most popular way is to use the Takaki-Sugeno fuzzy

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controller (TSFC) introduced by Takaki Sugeno [8]. TSFC model has been investigated by several researchers (see [3, 5, 7, 8] and the references cited in them) because it has certain advantages as a modeling device. Singular perturbation and stability analysis are also popular research topics today because they can be advantageously applied to many fields of studies such as mathematics, computer science, engineering, and economics [10-14]. These papers are motivation of us. We attempt to combine the keywords: TSFC, Singular perturbation and stability analysis to get a new problem such that it will appear in this document.

In this paper, we consider a fuzzy control system singularly perturbed as a fast-slow system based on the TSFC model,

$$\begin{cases} \dot{x} = A(\varepsilon)x + f(x,y,\varepsilon) + B(x,\varepsilon)u \\ \dot{y} = \dfrac{1}{\varepsilon}\left[C(\varepsilon)y + g(x,y,\varepsilon)\right] \\ x(t_0) = x^0 \\ y(t_0) = y^0 \end{cases}, \tag{1.1}$$

where $x = x(t) \in \mathbb{R}^n$ and $y = y(t) \in \mathbb{R}^m$ are fast and slow variables, respectively. The matrix $A(\varepsilon) \in \mathbb{R}^{n\times n}$, $B(x,\varepsilon) \in \mathbb{R}^n$, $f(\cdot,\cdot,\varepsilon)$ and $g(\cdot,\cdot,\varepsilon)$ are globally Lipschitz and uniformly bounded in $\varepsilon$. The system (1.1) is perturbed with a small parameter $\varepsilon$, $0 < \varepsilon < 1$. In this system, $u \in \mathbb{R}$ is the control signal fed into the process, obtained from a weighted-sum defuzzification method for TSFC.

The rest of this paper is organized as follows: in section 2, we introduce some preliminaries such as the theory of fuzzy logic and the system-controlling nonlinear processes of TSFC; in section 3, we investigate an approximate solution of the perturbed controller system (1.1) and present a stability analysis method for nonlinear processes of TSFC as well as a proof of the stability conditions; a demonstrative example is shown in section 4; and the last section concludes the paper.

## 2 Preliminaries

This section describes the definitions and theorems pertaining to this research.

**Definition 2.1** Let $A$ be a crisp set. A fuzzy set $\mathcal{A}$ on the crisp set $A$ is defined by

$$\mathcal{A} = \{(x, u_{\mathcal{A}}(x)) | x \in A, u_{\mathcal{A}}(x) \in [0,1]\}, \tag{2.1}$$

where $u_{\mathcal{A}} : A \longrightarrow [0,1]$ is a membership function.

Note: for convenience, sometimes we denote a fuzzy set $\mathcal{A}$ by $A$.

**Definition 2.2** Let $\mathcal{A}$ be a fuzzy set with membership function $u$ and $\alpha \in [0,1]$. An $\alpha$-cut, denoted as $[u]^{\alpha}$, is defined by

$$[u]^{\alpha} = \begin{cases} \{x \in A \,|\, u(x) \geq \alpha\} & ; 0 < \alpha \leq 1 \\ \overline{\{x \in A \,|\, u(x) > 0\}} & ; \alpha = 0 \end{cases} \tag{2.2}$$

**Definition 2.3** Let $\mathcal{A}$ be a fuzzy set with membership function $u : \mathbb{R} \to [0,1]$. $\mathcal{A}$ is called a *fuzzy number*, if $u$ satisfies the following conditions,

1) $\exists x \in \mathbb{R}$, $u(x) = 1$,
2) $\forall \lambda \in [0,1], \forall x_1, x_2 \in \mathbb{R}$, $u(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u(x_1), u(x_2)\}$,
3) for each $\alpha \in [0,1]$ there is a closed interval $[a,b]$ such that $[\tilde{u}]^{\alpha} = [a,b]$.

**Definition 2.4** Let $a^L \leq a^{M_1} \leq a^{M_2} \leq a^U$. A fuzzy number $\mathcal{A}$ is called a *Trapezoidal fuzzy number* denoted by $\langle a^L, a^M, a^U \rangle$, if the membership function $u : \mathbb{R} \to [0,1]$ is defined by

$$u(x) = \begin{cases} \dfrac{x - a^L}{a^{M_1} - a^L} ; a^L \leq x \leq a^M \\ \dfrac{x - a^U}{a^M - a^U} ; a^M \leq x \leq a^U \\ 0 \qquad ; \text{otherwise} \end{cases} \tag{2.3}$$

Next, we describe some important concepts of TSFC. TSFC is composed of fuzzy IF-THEN rules based on the universe of discourse $X \subseteq \mathbb{R}^n$. Let $x = (x_1,...,x_n) \in X$ and let $X_{ij}$, $i = 1,...,m$, $j = 1,...,n$ be a fuzzy set that describes the linguistics terms of the input variables $x_j$ in the $i^{th}$-rule, then the set of fuzzy IF-THEN rules is written as follows: rule $i$ -If $x_1$ is $X_{i1}$ AND $x_2$ is $X_{i2}$ AND ... $x_n$ is $X_{in}$, THEN $u = u(x)$ is $\Lambda_i$, $i = 1,...,m$ (2.4) where $\Lambda_i$ is a fuzzy set describing the linguistic terms of the output control variables $u$ whose membership function $\varphi_i(u) = \varphi_i(x) = \min_j \left\{u_{\tilde{X}_{ij}}(x_j)\right\}$. Suppose that $\varphi_i \neq 0$ for all $i = 1,...,m$ and $u_i = u_i(x)$ is the output control in the $i^{th}$-rule, applying the weighted-sum defuzzification method gives us the control signal $u$ as

$$u = \frac{\sum_{i=1}^{m} u_i \varphi_i(u_i)}{\sum_{i=1}^{m} \varphi_i(u_i)} = \frac{\sum_{i=1}^{m} u_i(x)\varphi_i(x)}{\sum_{i=1}^{m} \varphi_i(x)} \qquad (2.5)$$

# 3 An approximate solution of a singularly perturbed system with control

In this section, we consider the following singularly perturbed system with control (ACP),

$$\begin{cases} \dot{x} = A(\varepsilon)x + f(x,y,\varepsilon) + B(x,\varepsilon)u \\ \dot{y} = \dfrac{1}{\varepsilon}\left[C(\varepsilon)y + g(x,y,\varepsilon)\right] \\ x(t_0) = x^0 \\ y(t_0) = y^0 \end{cases} \qquad (3.1)$$

where $0 < \varepsilon << 1$, $x = (x_1,...,x_n) \in \mathbb{R}^n$ and $y = (y_1,...,y_n) \in \mathbb{R}^m$ are state vectors, $u \in \mathbb{R}$ is the control signal fed into the process, obtained from the weighted-sum defuzzification method for Takagi-Sugino fuzzy control system,

$A(\varepsilon) \in \mathbb{R}^{n \times n}$, $C(\varepsilon) \in \mathbb{R}^{m \times m}$ and $B(x, \varepsilon) \in \mathbb{R}^{n}$,

$f(x, y, \varepsilon) = (f_1(x, y, \varepsilon), \ldots, f_n(x, y, \varepsilon)) \in \mathbb{R}^{n}$,

$g(x, y, \varepsilon) = (g_1(x, y, \varepsilon), \ldots, g_m(x, y, \varepsilon)) \in \mathbb{R}^{m}$, $x^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^{n}$

$y^0 = (y_1^0, \ldots, y_m^0) \in \mathbb{R}^{m}$, and the symbol $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_n)$ and $\dot{y} = (\dot{y}_1, \ldots, \dot{y}_m)$ are the derivatives of $x$ and $y$ respect to the time variable $t$, respectively.

First, we define an approximate solution of 3.1. Suppose that there exists an $(x^0, y^0) \in \mathbb{R}^{n} \times \mathbb{R}^{m}$ such that the fast equation becomes stationary, i.e., $C(\varepsilon)y + g(x, y, \varepsilon) = 0$, then system 3.1 can written in the following form,

$$\begin{cases} \dot{x} = A(\varepsilon)x + f(x, y, \varepsilon) + B(x, \varepsilon)u \\ 0 = C(\varepsilon)y + g(x, y, \varepsilon) \\ x(t_0) = x^0 \\ y(t_0) = y^0 \end{cases} \tag{3.2}$$

Furthermore, if $\varepsilon$ is small enough, it is reasonable to replace system 3.2 with the following algebraic differential equation,

$$\begin{cases} \dot{x} = A(\varepsilon)x + f(x, y, \varepsilon) + B(x, \varepsilon)u \\ 0 = Cy + g(x, y) \\ x(t_0) = x^0 \\ y(t_0) = y^0 \end{cases} \tag{3.3}$$

where $C = C(0)$, and $g(x, y) = g(x, y, 0)$.

**Assumption (A-1)** Suppose that the solution of $Cy + g(x, y) = 0$ with $y(t_0) = y^0$ has a unique solution $y = H(x)$ and $H$ is continuously differentiable,

By substitution $y = H(x)$ in 3.3, we obtain

$$\begin{cases} \dot{x} = A(\varepsilon)x + f(x, H(x), \varepsilon) + B(x, \varepsilon)u \\ x(t_0) = x^0 \end{cases} \tag{3.4}$$

In this paper, this system 3.4 is called an *approximate system* of system 3.1.

**Definition 3.1** An approximate mild solution on $[t_0,\infty)$ of system 3.1 is a continuous function $x:[t_0,\infty)\to\mathbb{R}^n$ satisfying the integral equation

$$x(t)=S(t-t_0)x^0+\int_{t_0}^{t}S(t-s)\left[f(x(s),H(x(s),\varepsilon)+B(x(s),\varepsilon)u\right]ds\,, \quad (3.5)$$

where $S(t)=e^{A(\varepsilon)t}$.

Let us for convenience denote $C_n=C([t_0,\infty),\mathbb{R}^n)$.

**Theorem 3.2** Let $H(\cdot,\varepsilon):C_n\to C_n$ , $B:C_n\to C_n$ and $f(\cdot,\cdot,\varepsilon):C_n\times C_m\to C_n$ be uniformly Lipschitz continuous (with constant $L_H,L_B,L_f$ ). If $A$ is the generator of a fundamental matrix $S(t)$ , then for every $x^0\in\mathbb{R}^n$ the semi-linear ACP 3.1 has an approximate mild solution $x\in C_n$. Moreover, the map $x^0\to x$ is Lipschitz continuous.

**Proof** For a given $x^0\in\mathbb{R}^n$, define a mapping $F_\varepsilon:C_n\to C_n$ by

$$F_\varepsilon x(t)=S(t-t_0)x^0+\int_{t_0}^{t}S(t-s)\left[f(x(s),H(x(s),\varepsilon)+B(x(s),\varepsilon)u\right]ds$$

for all $t\in[t_0,\infty)$. (3.6)

Denoting by $\|x\|_\infty$, the norm of $x$ is an element of $C_n$. It follows from the definition of $F_\varepsilon$ and the uniformly Lipschitz continuous of $H,B$, and $f$ that for each $t\in[t_0,\infty)$,

$$\|F_\varepsilon x(t)-F_\varepsilon y(t)\|$$

$$=\left\|\left(S(t-t_0)x^0+\int_{t_0}^{t}S(t-s)\left[f(x(s),H(x(s),\varepsilon))+B(x(s),\varepsilon)u\right]ds\right)\right.$$

$$\left.-\left(S(t-t_0)x^0+\int_{t_0}^{t}S(t-s)\left[f(y(s),H(y(s),\varepsilon))+B(y(s),\varepsilon)u\right]ds\right)\right\|$$

$$=\left\|\int_{t_0}^{t}S(t-s)\begin{pmatrix}\left[f(x(s),H(x(s),\varepsilon))-f(y(s),H(y(s),\varepsilon))\right]\\+\left[B(x(s),\varepsilon)u+B(y(s),\varepsilon)u\right]\end{pmatrix}ds\right\|$$

$$\le \int_{t_0}^{t}\left( \|S(t-s)\| \left\| \begin{matrix}[f(x(s),H(x(s),\varepsilon))-f(y(s),H(y(s),\varepsilon))] \\ +[B(x(s),\varepsilon)u+B(y(s),\varepsilon)u]\end{matrix} \right\| \right) ds$$

$$\le M\int_{t_0}^{t}\left(L_f\left(\|x-y\|+\|Hx-Hy\|\right)+L_B\|x-y\|\|u\|\right)ds$$

$$\le M\int_{t_0}^{t}\left(L_f\left(\|x-y\|+L_H\|x-y\|\right)+L_B\|x-y\|\|u\|\right)ds$$

$$= M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)\|x-y\|_\infty t, \tag{3.7}$$

where $M$ is the bound of $\|S(t)\|$ on $[0,\infty)$. Using 3.6 and 3.7 and induction on $n$ , we have

$$\|(F_\varepsilon^n x)(t)-(F_\varepsilon^n y)(t)\| \le \frac{\left(M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)\right)^n}{n!}\|x-y\|_\infty t$$

$$\le \frac{\left(M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)\right)^n t}{n!}\|x-y\|_\infty \tag{3.8}$$

For a sufficiently large $n$ , $\dfrac{\left(M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)\right)^n t}{n!}<1$ and by a well-known extension of the contraction mapping principle, there is a unique $x\in C_n$ such that

$$x(t)=(Fx)t$$

$$= S(t-s)x^0+\int_0^t S(t-s)\left[f(x(s),H(x(s),\varepsilon)+B(x(s,\varepsilon))u\right]ds \tag{3.9}$$

Therefore, the semi-linear ACP 3.4 has a unique mild solution $x\in C_n$ .

The Lipschitz continuity of the mapping $x^0 \to x$ is a consequence of the following argument. Let $x$ and $y$ be the mild solution of system 3.4 with the initial value $x^0$ and $y^0$ , respectively. It follows from the definition of a mild solution and the uniformly Lipschitz continuity of $H$ , $B$ and $f$ that

$$\|x(t)-y(t)\| \le M\|x^0-y^0\|+M\left(L_f\left(1+L_H\right)+L_B\|u\|\right)\int_0^t\|x(s)-y(s)\|ds \text{ ,} \tag{3.10}$$

and by Granwal's Lemma, this implies that

$$\left\|x(t)-y(t)\right\| \le M \exp\left(M\left(L_f\left(1+L_H\right)+L_B\left\|u\right\|\right)t\right)\left\|x^0-y^0\right\|, \tag{3.11}$$

thus

$$\left\|x-y\right\|_\infty \le M \exp\left(M\left(L_f\left(1+L_H\right)+L_B\left\|u\right\|\right)t\right)\left\|x^0-y^0\right\|. \tag{3.12}$$

demonstrating the Lipschitz continuity of mapping $x^0 \to x$. □

# 4 Stability analysis of the approximate system

In this section, we discuss the stability analysis of system 3.4, the approximate fuzzy control system of system 3.1.

Let $V(\cdot,\varepsilon):\mathbb{R}^n \to \mathbb{R}$ be a scalar function such that $V(x,\varepsilon)>0$, $\forall x \neq 0$ and its first order partial derivatives are continuous. Then the total derivative of $V=V(x,\varepsilon)$ with respect to $t$, with $x$ satisfying the approximate fuzzy control system 3.4, is

$$\begin{aligned}
\dot{V}(x,\varepsilon) = \frac{dV}{dt} &= \sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\cdot\frac{dx_i}{dt}\right) \\
&= \sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\left(\sum_{k=1}^{n} a_{ik}(\varepsilon)x_k + f_i(x,H(x),\varepsilon)+b_i(x,\varepsilon)u\right)\right) \\
&= \sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\sum_{k=1}^{n} a_{ik}(\varepsilon)x_k\right)+\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}f_i(x,H(x),\varepsilon)\right) \\
&\quad +\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}b_i(x,\varepsilon)u\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial V}{\partial x_i}a_{ik}(\varepsilon)x_k\right)+\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}f_i(x,H(x),\varepsilon)\right) \\
&\quad +\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}b_i(x,\varepsilon)u\right) \\
&= \overline{A}(x,\varepsilon)+\overline{F}(x,H(x),\varepsilon)+\overline{B}(x,\varepsilon)u,
\end{aligned}$$

where

$$\bar{A}(x,\varepsilon)=\sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial V}{\partial x_i}a_{ik}(\varepsilon)x_k\right),\bar{F}(x,H(x),\varepsilon)=\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}f_i(x,H(x),\varepsilon)\right)$$

and $\bar{B}(x,\varepsilon)=\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}b_i(x,\varepsilon)\right)$.

Therefore, system 3.4 can be transformed to a real-value system,

$$\dot{V}\left(x,\varepsilon\right)=\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)+\bar{B}(x,\varepsilon)u. \tag{4.1}$$

Throughout this paper we denote

$$X_i^A=\left\{x\in X\,\middle|\,\varphi_i(x)\neq 0\right\}. \tag{4.2}$$

and partition this set $X$ by using signs of $\bar{B}(x,\varepsilon)$ as follows,

$$B^0=\left\{x\in X\middle|\bar{B}(x,\varepsilon)=0\right\},B^+=\left\{x\in X\middle|\bar{B}(x,\varepsilon)>0\right\},$$

and

$$B^-=\left\{x\in X\middle|\bar{B}(x,\varepsilon)<0\right\}. \tag{4.3}$$

**Theorem 4.1** Let an approximate control system 3.4 be fed with a control signal $u\in\mathbb{R}$, obtained from a weighted-sum defuzzification method for TSFC. Let $x=0$ be an equilibrium point. Suppose that there exists a function $V\left(\cdot,\varepsilon\right):X\to\mathbb{R}$ of which $V(x,\varepsilon)>0$ , $\forall x\neq 0$ and all first-order partial derivatives are continuous, with

1) $\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)\leq 0$ for all $x\in B^0$,

2) $u_i\leq-\left(\frac{\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)}{\bar{B}(x,\varepsilon)}\right)$ for all $x\in X_i^A\cap B^+$, and

$u_i\geq-\left(\frac{\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)}{\bar{B}(x,\varepsilon)}\right)$ for all $x\in X_i^A\cap B^-$, $i=1,\ldots,m$,

3) $S=\left\{x\in X\middle|\dot{V}(x,\varepsilon)=0\right\}=\left\{x=0\right\}$.

Then the fuzzy control system is globally asymptotically stable in the sense of Lyponov at the origin.

**Proof** We first show that $\dot{V}(x,\varepsilon)$ is negative semi-definite. Let $x_0\in X$ and let us consider signs of $\bar{B}(x,\varepsilon)$. We partition this proof into 3 cases as follows.

*Case1)* $x_0 \in X_i^A \cap B^+$, for all $i = 1,...,m$, then $\overline{B}(x_0,\varepsilon) > 0$.

By condition 2), $u_i(x_0) \le -\left(\dfrac{\overline{A}(x_0,\varepsilon)+\overline{F}(x_0,H(x_0),\varepsilon)}{\overline{B}(x_0,\varepsilon)}\right)$.

From (2.4), we have

$$u(x_0) = \frac{\sum_{i=1}^{m} u_i(x_0)\varphi_i(x_0)}{\sum_{i=1}^{m}\varphi_i(x_0)} = \frac{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)u_i(x_0)}{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}$$

$$\le \frac{-\left(\dfrac{\overline{A}(x_0,\varepsilon)+\overline{F}(x_0,H(x_0),\varepsilon)}{\overline{B}(x_0,\varepsilon)}\right)\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}$$

$$= -\left(\frac{\overline{A}(x_0,\varepsilon)+\overline{F}(x_0,H(x_0),\varepsilon)}{\overline{B}(x_0,\varepsilon)}\right). \tag{4.4}$$

This implies that $\dot{V}(x,\varepsilon) = \overline{A}(x,\varepsilon)+\overline{F}(x,H(x),\varepsilon)+\overline{B}(x,\varepsilon)u \le 0$.

Therefore, $\dot{V}(x_0,\varepsilon) \le 0$, $\forall x^0 \in X_i^A \cap B^+$ for all $i = 1,...,n$.

*Case 2)* $x_0 \in X_i^A \cap B^-$, for all $i = 1,...,m$, then $\overline{B}(x_0,\varepsilon) < 0$.

By condition 2), $u_i(x_0) \ge -\left(\dfrac{\overline{A}(x_0,\varepsilon)+\overline{F}(x_0,H(x_0),\varepsilon)}{\overline{B}(x_0,\varepsilon)}\right)$.

From (2.4), we have

$$u(x_0) = \frac{\sum_{i=1}^{m} u_i(x_0)\varphi_i(x_0)}{\sum_{i=1}^{m}\varphi_i(x_0)} = \frac{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)u_i(x_0)}{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}$$

$$\ge \frac{-\left(\dfrac{\overline{A}(x_0,\varepsilon)+\overline{F}(x_0,H(x_0),\varepsilon)}{\overline{B}(x_0,\varepsilon)}\right)\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}{\sum_{i=1,\varphi_i\neq 0}^{m}\varphi_i(x_0)}$$

$$u(x_0) = -\left(\frac{\overline{A}(x_0,\varepsilon)+\overline{F}(x_0,H(x_0),\varepsilon)}{\overline{B}(x_0,\varepsilon)}\right). \tag{4.5}$$

This implies that $\dot{V}\left(x,\varepsilon\right) = \overline{A}(x,\varepsilon)+\overline{F}(x,H(x),\varepsilon)+\overline{B}(x,\varepsilon)u \le 0$.

Therefore, $\dot{V}(x_0,\varepsilon) \le 0$ , $\forall x^0 \in X_i^A \cap B^-$ for all $i = 1,\ldots,n$.

*Case 3* $x_0 \in B^0$, then $\overline{B}(x_0,\varepsilon) = 0$.

By condition 1), $\overline{A}(x_0,\varepsilon)+\overline{F}(x_0,H(x_0),\varepsilon) \le 0$.

This implies that $\dot{V}\left(x_0,\varepsilon\right) = \overline{A}(x_0,\varepsilon)+\overline{F}(x_0,H(x_0),\varepsilon)+\overline{B}(x_0,\varepsilon)u \le 0$.

From *Case1)* to *Case3)*, we conclude that $\dot{V}\left(x_0,\varepsilon\right) \le 0$ for all $x_0 \in X$ i.e., and $V(x,\varepsilon)$ is negative semi-definite. Condition 3) of the theorem ensures the fulfillment of LaSalle's invariant set principle. This means that LaSalle's global invariant set theorem applies; hence the equilibrium point at the origin is globally asymptotically stable. □

**Theorem 4.2** Let $S = \left\{x \in X \middle| \dot{V}(x,\varepsilon) = 0\right\}$ and $S_i = \left\{x \in X \middle| \dot{V}_i(x,\varepsilon) = 0\right\}$ be such that

$\dot{V}_i(x,\varepsilon) = \overline{A}(x,\varepsilon)+\overline{F}(x,H(x),\varepsilon)+\overline{B}(x,\varepsilon)u_i(x)$, then $S \subseteq \bigcup_{i=1}^{m} S_i$.

**Proof** We will show that $S \subseteq \bigcup_{i=1}^{m} S_i$. Note that $\dot{V}(x,\varepsilon) = \dfrac{\sum_{i=1}^{m}\dot{V}_i(x,\varepsilon)\varphi_i\left(x\right)}{\sum_{i=1}^{m}\varphi_i\left(x\right)}$ for all $x \in X$. Let $x_0 \in S$, then $\dot{V}(x_0,\varepsilon) = 0$. This implies that $\sum_{i=1}^{m}\dot{V}_i(x_0,\varepsilon)\varphi_i\left(x_0\right) = 0$. Following the proof of Theorem 4.1, there exists an index $k$ such that $\dot{V}_k(x_0,\varepsilon) = 0$. Thus $x_0 \in \bigcup_{i=1}^{m} S_i$. □

# 5 A demonstrative example

In this section, a demonstrative example is presented to showcase our method of stability analysis. Let $X=[-50,50]\times[-50,50]\times[-50,50]$. Consider a perturbed differential system with control $u\in\mathbb{R}$,

$$\begin{aligned}
\frac{dx_1}{dt} &= -\varepsilon x_1+\sigma x_2+x_4+(1+\varepsilon)u\,,\\
\frac{dx_2}{dt} &= -2\varepsilon x_2+x_1(\rho-x_3)-x_2\,,\\
\frac{dx_3}{dt} &= -(1+\varepsilon)x_3+x_1x_2-\beta x_3\,,\\
\frac{dx_4}{dt} &= \frac{1}{\varepsilon}\big((1+\varepsilon)x_4+(\sigma+\varepsilon)x_1\big).
\end{aligned} \tag{5.1}$$

Let $x=(x_1,x_2,x_3)$ and $y=x_4$, then system 5.1 becomes,

$$\begin{cases}
\dot{x}=\begin{pmatrix}-\varepsilon & 0 & 0\\ 0 & -2\varepsilon & 0\\ 0 & 0 & -1-\varepsilon\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+\begin{pmatrix}\sigma x_2+y\\ x_1(\rho-x_3)-x_2\\ x_1x_2-\beta x_3\end{pmatrix}+\begin{pmatrix}1+\varepsilon\\ 0\\ 0\end{pmatrix}u\\
\dot{y}=\frac{1}{\varepsilon}\big((1+\varepsilon)y+(\sigma+\varepsilon)x_1\big)\\
x(t_0)=x_0\\
y(t_0)=y_0
\end{cases}\quad . \tag{5.2}$$

and the approximate system of system 5.2 is,

$$\begin{cases}
\dot{x}=\begin{pmatrix}-\varepsilon & 0 & 0\\ 0 & -2\varepsilon & 0\\ 0 & 0 & -1-\varepsilon\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+\begin{pmatrix}\sigma x_2-\sigma x_1\\ x_1(\rho-x_3)-x_2\\ x_1x_2-\beta x_3\end{pmatrix}+\begin{pmatrix}1+\varepsilon\\ 0\\ 0\end{pmatrix}u\\
x(t_0)=x_0
\end{cases}\quad . \tag{5.3}$$

The approximate mild solution of system 5.2 is in the form

$$x(t)=S(t-t_0)x^0+\int_{t_0}^{t}S(t-s)\left[\begin{pmatrix}\sigma x_2-\sigma x_1\\ x_1(\rho-x_3)-x_2\\ x_1x_2-\beta x_3\end{pmatrix}+\begin{pmatrix}1+\varepsilon\\0\\0\end{pmatrix}u\right]ds,$$

where $S(t)=\exp\begin{pmatrix}-\varepsilon & 0 & 0\\ 0 & -2\varepsilon & 0\\ 0 & 0 & -1-\varepsilon\end{pmatrix}t=\begin{pmatrix}e^{-\varepsilon t} & 0 & 0\\ 0 & e^{-2\varepsilon t} & 0\\ 0 & 0 & e^{-(1+\varepsilon)t}\end{pmatrix}$.

The objective of this presentation is to find the control $u_i$ for which the approximate system 5.3 can be stabilized by the TSFC method described above. Following the proof of Theorem 4.1, we design a stable fuzzy control system as follows.

*Step1)* Choose a Lyapunov function $V$, calculate $\dot{V}$, $\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)$, and $\bar{B}(x,\varepsilon)$.

In this example, we choose $V(x)=x_1^2+x_2^2+x_3^2$. Then we have

$$\dot{V}(x,\varepsilon)=-2(\varepsilon+\sigma)x_1^2-2(2\varepsilon+1)x_2^2-2(1+\varepsilon+\beta)x_3^2 +2(\sigma+\rho)x_1x_2+2(1+\varepsilon)x_1u. \quad (5.4)$$

In this case, we have

$$\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)=-2(\varepsilon+\sigma)x_1^2-2(2\varepsilon+1)x_2^2 -2(1+\varepsilon+\beta)x_3^2+2(\sigma+\rho)x_1x_2 \quad (5.5)$$

and $\bar{B}(x,\varepsilon)=2(1+\varepsilon)x_1$. (5.6)

*Step2)* Partition the discourse space $X$ by using $\bar{B}(x,\varepsilon)$.

In this example, we obtain

$B^0=\{(0,x_2,x_3)\in X \mid x_1=0\}=\{0\}\times[-50,50]\times[-50,50]$,

$B^+=\{(x_1,x_2,x_3)\in X \mid x_1>0\}=(0,50]\times[-50,50]\times[-50,50]$ and

$B^-=\{(x_1,x_2,x_3)\in X \mid x_1<0\}=[-50,0)\times[-50,50]\times[-50,50]$. (5.7)

*Step3)* Check that $\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)\le 0$ for all $x\in B^0$. If it is not, we have to choose a new Lyapunov function and repeat Step 1. Else go to the next step.

In this example, we have $\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)\le 0$ for all $x\in B^0$ and we go to the next step.

*Step 4)* Determine the fuzzy set of the linguistic terms Negative (N), Zero (Z), and Positive (P) on $X$ corresponding to the state variable $x$.

In this example, we set $N=\langle -50,-50,0\rangle$, $Z=\langle -20,0,20\rangle$, and $P=\langle 0,50,50\rangle$ corresponding to the state variable $x=(x_1,x_2,x_3)$.

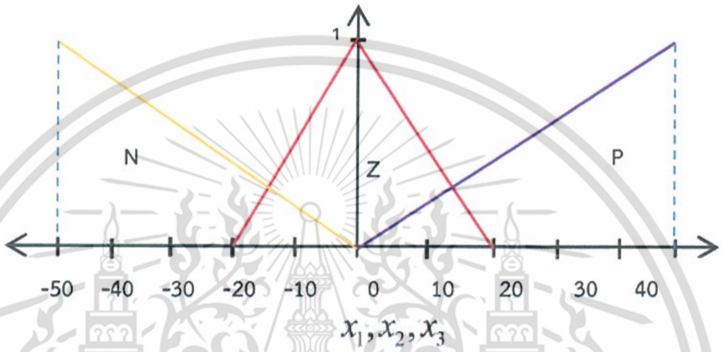


*Figure 5.1 Membership functions of $x_1, x_2$, and $x_3$*

*Step5)* A fuzzy control IF-THEN rule is constructed to be used as an inference engine.

In this example, following (5.5) and (5.6), we construct a fuzzy control rule by ignoring the variable $x_3$, that is, a fuzzy rule with two variables, $x_1$ and $x_2$ only, suffices. All fuzzy control rules are illustrated in Table 5.2.

**Fuzzy control rule base**

| Rule | $x_1$ | $x_2$ | $u_i$ |
|---|---|---|---|
| 1 | P | P | $u_1$ |
| 2 | P | N | $u_2$ |
| 3 | P | Z | $u_3$ |
| 4 | N | P | $u_4$ |
| 5 | N | N | $u_5$ |
| 6 | N | Z | $u_6$ |
| 7 | Z | P | $u_7$ |
| 8 | Z | N | $u_8$ |
| 9 | Z | Z | $u_9$ |

*Table 5.2 The fuzzy control rules of this example*

*Step6)* Determine $u_i$ from each fuzzy control rule $i$ obtained in Step 5 by using condition 2) of Theorem 4.1.

In this example,

**Rule1** If $x_1$ is P AND $x_2$ is P.then $X_1^A = (0,50]\times(0,50]\times[-50,50]$ .Consequently, $X_1^A \cap B^+ = (0,50]\times(0,50]\times[-50,50]$ and $X_1^A \cap B^- = \varnothing$. By condition 2), we have

$$u_1(x) \le -\left(\frac{\bar{A}(x,\varepsilon)+\bar{F}(x,H(x),\varepsilon)}{\bar{B}(x,\varepsilon)}\right)$$

$$=\left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2. \qquad (5.8)$$

From (5.8), we can choose $u_1(x) = -\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$.

**Rule2** If $x_1$ is P AND $x_2$ is N.then $X_2^A = (0,50]\times[-50,0)\times[-50,50]$ .Consequently, $X_2^A \cap B^+ = (0,50]\times[-50,0)\times[-50,50]$ and $X_2^A \cap B^- = \varnothing$. By condition 2), we have

$$u_2(x) \le \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1+\frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1}+\frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1}-\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2. \qquad (5.9)$$

From (5.9), we can choose $u_2(x) = -1$.

**Rule3** If $x_1$ is P AND $x_2$ is Z, then $X_3^A = (0,50] \times (-20,20) \times [-50,50]$.

Consequently, $X_3^A \cap B^+ = (0,50] \times (-20,20) \times [-50,50]$ and $X_3^A \cap B^- = \varnothing$. By condition 2), we have

$$u_3(x) \le \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} - \left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2. \tag{5.10}$$

From (5.10), we can choose

$$u_3(x) = \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} - 20\left(\frac{\sigma+\rho}{1+\varepsilon}\right).$$

**Rule4** If $x_1$ is N AND $x_2$ is P, then $X_4^A = [-50,0) \times (0,50] \times [-50,50]$.

Consequently, $X_4^A \cap B^+ = \varnothing$ and $X_4^A \cap B^- = [-50,0) \times (0,50] \times [-50,50]$.

By condition 2), we have

$$u_4(x) \ge \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} - \left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2. \tag{5.11}$$

From (5.11), we can choose $u_4(x) = 1$.

**Rule 5** If $x_1$ is N and $x_2$ is N, then $X_5^A = [-50,0) \times [-50,0) \times [-50,50]$.

Consequently, $X_5^A \cap B^+ = \varnothing$ and

$X_5^A \cap B^- = [-50,0) \times [-50,0) \times [-50,50]$. By condition 2), we have

$$u_5(x) \ge \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} - \left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2. \tag{5.12}$$

From (5.12), we can choose $u_5(x) = -\left(\dfrac{\sigma+\rho}{1+\varepsilon}\right)x_2$.

**Rule6** If $x_1$ is N AND $x_2$ is Z, then $X_6^A = [-50,0) \times (-20,20) \times [-50,50]$

Consequently, $X_6^A \cap B^+ = \varnothing$ and

$X_6^A \cap B^- = [-50,0) \times (-20,20) \times [-50,50]$. By condition 2), we have

$$u_6(x) \ge \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} - \left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2 . \qquad (5.13)$$

From (5.13), we can choose

$$u_6(x) = \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} + 20\left(\frac{\sigma+\rho}{1+\varepsilon}\right).$$

**Rule7** If $x_1$ is Z AND $x_2$ is P, then $X_7^A = (-20,20)\times(0,50]\times[-50,50]$.

Consequently, $X_7^A \cap B^+ = (0,20)\times(0,50]\times[-50,50]$ and

$X_7^A \cap B^- = (-20,0)\times(0,50]\times[-50,50]$. We next consider two cases.

*Case7.1)* If $x \in X_7^A \cap B^+ = (0,20)\times(0,50]\times[-50,50]$, by condition 2), we have

$$u_7(x) \le \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} - \left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2 . \qquad (5.14)$$

From (5.14), we can choose $u_7(x) = -\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$.

*Case7. 2)* If $x \in X_7^A \cap B^- = (-20,0)\times(0,50]\times[-50,50]$, by condition 2), we have

$$u_7(x) \ge \left(\frac{\varepsilon+\sigma}{1+\varepsilon}\right)x_1 + \frac{(2\varepsilon+1)x_2^2}{(1+\varepsilon)x_1} + \frac{(1+\varepsilon+\beta)x_3^2}{(1+\varepsilon)x_1} - \left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2 . \qquad (5.15)$$

From (5.15), we can choose $u_7(x) = -\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$.

From Case 1) and Case 2), we conclude that we can choose $u_7(x) = -\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$.

Since Rule 8 (If $x_1$ is Z AND $x_2$ is N) and Rule 9 (If $x_1$ is Z AND $x_2$ is Z) are similar to Rule 7, We can choose $u_8(x) = u_9(x) = u_7(x) = -\left(\frac{\sigma+\rho}{1+\varepsilon}\right)x_2$.

*Step7)* Check that $S = \left\{x \in X \middle| \dot{V}(x,\varepsilon) = 0\right\} = \{0\}$.

In this example, following Rule 1- Rule 9 obtained in Step 6 and Theorem 4.2, we have $S_i = \varnothing$ for all $i = 1,\dots,8$ and $S_9 = \{0\}$. Thus, we have

$$S \subseteq \bigcup_{i=1}^{9} S_i = \{0\}.$$

Therefore, by Theorem 4.1, it can be inferred that the fuzzy control system is globally asymptotically stable in the sense of Lyponov at the origin.

# 6 Conclusion

This article is concerned with proving the existence and uniqueness of an approximate system of a fuzzy control singularly perturbed fast-slow system. We provide sufficiently stable conditions and prove the fuzzy control system to be asymptotically stable in the sense of Lyapunov. Furthermore, we give a demonstrative example to showcase our main results. Last but not least, we are keenly interested in investigating its applications. Even though it seems very likely that an effort in this direction will bear successful results, there is no guarantee. We can only hope for the best and prepare for the worst.

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