EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR SYSTEMS OF COUPLED LINEAR MATRIX EQUATIONS

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE (APPLIED MATHEMATICS)

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE
KING MONGKUT'S INSTITUTE OF TECHNOLOGY LADKRABANG

2017
KMITL-2017-SC-M-001-043

Faculty of Science

King Mongkut's Institute of Technology Ladkrabang

Thesis Certification

Thesis Title "EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR SYSTEMS OF COUPLED LINEAR MATRIX EQUATIONS"

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Examination Date 14th July 2017 **Time** 10.00 a.m. - 13.00 p.m.

Place Faculty of Science room 301

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Date 19 Jul 2017

หัวข้อวิทยานิพนธ์	การมีจริงและความเป็นได้อย่างเดียวของผลเฉลยสำหรับระบบสมการเมทริกซ์เชิงเส้นแบบควบคู่
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บทคัดย่อ

ในงานวิจัยนี้เราพิจารณาศึกษาระบบสมการเมทริกซ์เชิงเส้นแบบควบคู่ในรูปแบบของ

$$AXB + CYD = E,$$
$$CXD + AYB = F$$

เมื่อ A, B, C, D เป็นเมทริกซ์เชิงซ้อน และ X, Y เป็นเมทริกซ์ที่ไม่ทราบค่า เราได้เกณฑ์หลายประการสำหรับการมีจริงและความเป็นได้อย่างเดียวของผลเฉลยสำหรับระบบสมการนี้และกรณีเฉพาะที่น่าสนใจ เกณฑ์เหล่านี้อาศัยเรื่องผลคูณโครเนคเคอร์ ตัวดำเนินการเวกเตอร์ ตัวผกผันมัวร์-เพนโรส และค่าลำดับชั้น นอกจากนี้ได้นำเสนอสูตรที่ชัดแจ้งของผลเฉลยดังกล่าว

คำสำคัญ : สมการเมทริกซ์เชิงเส้น ผลคูณโครเนคเคอร์ ตัวดำเนินการเวกเตอร์ ตัวผกผันมัวร์-เพนโรส

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Abstract

In this paper, we investigate a system of coupled linear matrix equations of the form

$$AXB + CYD = E,$$
$$CXD + AYB = F,$$

where A, B, C, D, E, F are complex matrices and X, Y are unknown complex matrices. We obtain several criterions for existence and uniqueness of the system and its interesting special cases. These criterions rely on Kronecker product, vector operator, Moore Penrose inverses, and ranks. Moreover, explicit formulas of solutions are presented.

Keywords : linear matrix equation, Kronecker product, vector operator, Moore - Penrose inverse

Acknowledgements

Immeasurable appreciation and deepest gratitude for the advices, help and support are extended to the following persons who have contributed to making this thesis possible.

Firstly, I would like to express my sincere gratitude to my advisor, Asst.Prof.Dr Pattrawut Chansangiam, for the continuous support of my M.Sc. study, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of study and writing of this thesis. I could not have imagined having a better advisor and mentor for my M.Sc. study.

Besides my advisor, I would like to thank the rest of my thesis committee: Assoc.Prof.Dr. Wicharn Lewkeeratiyutkul, Asst.Prof.Dr. Decha Samana, and Dr. Sukrawan Mavecha, for their insightful comments.

Finally, I must express my very profound gratitude to my family for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them. Thank you.

Thitison Nuchniyom

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Chapter 1

Introduction

1.1 Inception and importance

Theory of linear matrix equations can be applied in many subareas of science and engineering such as system and control theory, image processing, transportation problems and quantum mechanics (see, e.g., [1, 4, 5, 12]). The current research of linear matrix equations can be classified into three topics. The first one is to investigate necessary and sufficient conditions for existence and uniqueness of certain linear matrix equations and then derive exact formulas for solutions (see, e.g., [2, 3, 8, 13, 17]). The second one is to discuss least-squares solutions for linear matrix equations. (see, e.g., [9, 15]). The last one is to establish iterative processes for solving linear matrix equations and deduce their convergence analysis (see, e.g., [6, 7, 14, 16]).

A general system of coupled linear matrix equations takes the form

$$\begin{aligned} A_1XA_2 + B_1YB_2 &= E, \\ C_1XC_2 + D_1YD_2 &= F, \end{aligned} \tag{1.1}$$

where A_i, B_i, C_i, D_i, E, F are given matrices for $i = 1, 2$, and X, Y are unknown matrices. The existence and uniqueness of solutions of (1.1) when every mentioned matrix is square and $C_iD_i = D_iC_i$ for $i = 1, 2$, was given in [8].

In this paper, we investigate existence and uniqueness of certain systems of coupled linear matrix equations. Our main system to consider takes the form

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned} \tag{1.2}$$

where A, B, C, D, E and F are complex matrices and X and Y are unknown complex matrices. We use the Kronecker products and vector operator to reduce the system (1.2) to a simple vector-matrix equation. Then we obtain necessary and sufficient conditions for existence and uniqueness of the system (1.2). These conditions rely on Kronecker products, vector operator, Moore-Penrose inverses and ranks. An interesting special case of (1.2) is given by

$$\begin{aligned} AX + YB &= E, \\ XB + AY &= F \end{aligned} \tag{1.3}$$

which is known as a system of Sylvester matrix equations. Finally, we obtain explicit formulas of solutions in system (1.2). Certain interesting special cases of (1.2) are also investigated. Note that our result includes (1.3) as a special case.

1.2 Objectives

1) To investigate existence and uniqueness of solutions for certain systems of coupled linear matrix equations.

2) To derive explicit formulas of solutions for systems of coupled linear matrix equations.

1.3 Scope of the study

Our main system to consider takes the form

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned}$$

where A, B, C, D, E, F are rectangular complex matrices and X, Y are unknown complex matrices. We also discuss certain special cases of the main system.

1.4 Benefits

To obtain criteria for solvability and unique solvability for system of coupled linear matrix equations.

1.5 Research methodology

1) Study advanced topics in matrix theory and applications and multilinear algebra.

2) Study research papers and textbooks concerning Kronecker product and linear matrix equations.

3) Determine the objectives and scope of the research.

4) Investigate existence and uniqueness of solutions for a system of coupled linear matrix equations.

5) Discuss certain special cases of the main system.

6) Provide numerical examples.

7) Conclude the results, make suggestions for further works and write the thesis.

Table 1.1: The research schedule

Activity	Time frame					
	2016				2017	
	Jan.-Mar.	Apr.-Jun.	Jul.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.
Step 1	⟷	⟷				
Step 2		⟷	⟷			
Step 3			⟷			
Step 4			⟷	⟷		
Step 5				⟷	⟷	
Step 6					⟷	
Step 7						⟷

Chapter 2

Preliminaries

In this chapter, we provide basic tools for solving linear matrix equations. These tools include Kronecker product, vector operator, Moore-Penrose inverses, and block matrix technique.

Let $M_{m,n}(\mathbb{C})$ be the set of m-by-n complex matrices. For simplicity, abbreviate $M_{n,n}(\mathbb{C})$ to $M_n(\mathbb{C})$. For each complex matrix, denote by A^* the conjugate transpose of A.

2.1 Kronecker product

Let $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$. The Kronecker product of A and B defined by

$$\begin{aligned} A \otimes B &= [a_{ij}B] \\ &= \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in M_{mp,nq}(\mathbb{C}). \end{aligned}$$

Example 2.1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$. Then the Kronecker product of $A \otimes B$ is

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 1B & 2B & 3B \\ 3B & 2B & 1B \end{bmatrix} \\ &= \begin{bmatrix} 1\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} & 2\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} & 3\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \\ 3\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} & 2\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} & 1\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 4 & 2 & 6 & 3 \\ 2 & 3 & 4 & 6 & 6 & 9 \\ 6 & 3 & 4 & 2 & 2 & 1 \\ 6 & 9 & 4 & 6 & 2 & 3 \end{bmatrix}. \end{aligned}$$

Lemma 2.2 (see, e.g., [10]). Let A, B, C and D be complex matrices with appropriated sizes. Then the following properties hold (provided that every operation is well-defined):

(1) Compatibility with addition: $(A+B)\otimes C = (A\otimes C)+(B\otimes C)$ and $A\otimes(B+C) = (A\otimes B)+(A\otimes C)$.

(2) Compatibility with scalar multiplication: $A\otimes(\alpha B) = \alpha(A\otimes B) = (\alpha A)\otimes B$ for any $\alpha\in\mathbb{C}$.

(3) Compatibility with transpose: $(A\otimes B)^T = A^T\otimes B^T$.

(4) Mixed product property: $(A\otimes B)(C\otimes D) = AC\otimes BD$.

(5) Compatibility with inverse: $A\otimes B$ is invertible if and only if both A and B are invertible, in which case $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$.

(6) $\operatorname{rank}(A\otimes B) = (\operatorname{rank} A)(\operatorname{rank} B)$.

2.2 Vector operator

The vector operator is a column-stacking operator assigned to a matrix $A\in M_{m,n}(\mathbb{C})$ by

$$\operatorname{Vec} A = [a_{11}, a_{12}, \ldots, a_{m1}, a_{12}, a_{22}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn}]^T \in \mathbb{C}^{mn}.$$

This operator is clearly linear and bijective.

Example 2.3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$. Then

$$\operatorname{Vec} A = \begin{bmatrix} 1 \\ 6 \\ 2 \\ 5 \\ 3 \\ 4 \end{bmatrix}.$$

Lemma 2.4 (see, e.g., [10]). The vector operator concerned Kronecker product as follows:

$$\operatorname{Vec}(AXB) = (B^T\otimes A)\operatorname{Vec} X,$$

provided that every operation is well-defined.

2.3 Moore-Penrose inverse and linear equations

A matrix $A\in M_{m,n}(\mathbb{C})$ possess its Moore-Penrose inverse, which is the matrix $A^\dagger\in M_{n,m}(\mathbb{C})$ satisfying the following conditions:

(i) $AA^\dagger A = A$,

(ii) $A^\dagger A A^\dagger = A^\dagger$,

(iii) $(AA^\dagger)^* = AA^\dagger$,

(iv) $(A^\dagger A)^* = A^\dagger A$.

If $A \in M_n(\mathbb{C})$ is invertible, then $A^\dagger = A^{-1}$. A simple and accurate way to compute Moore-Penrose inverse is to use the singular value decomposition. Indeed, for any $A \in M_{m,n}(\mathbb{C})$ we can decompose

$$A = U\Sigma V^*,$$

where $U \in M_m(\mathbb{C})$ is unitary, $V \in M_n(\mathbb{C})$ is unitary and $\Sigma = [d_{ij}] \in M_{m,n}(\mathbb{C})$ is a rectangular diagonal matrix with nonnegative entries. Then

$$A^\dagger = V\Sigma^\dagger U^*,$$

where $\Sigma^\dagger = [d_{ij}^\dagger]$ is defined by $d_{ij}^\dagger = \begin{cases} d_{ij}^{-1}, & d_{ij} \neq 0 \\ 0, & d_{ij} = 0. \end{cases}$

Example 2.5. Find the singular value decomposition of A, where $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

<u>Solution.</u> The column of U are eigenvector of AA^*, and the column of V are eigenvector of A^*A. The element on the diagonal of Σ are the square roots of eigenvalue of both AA^* and A^*A

$$A^*A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

We now compute the determinant of

$$\begin{aligned} \det(\lambda I - A^*A) &= \begin{vmatrix} \lambda - 1 & 1 & 0 \\ 1 & \lambda - 2 & 1 \\ 0 & 1 & \lambda - 1 \end{vmatrix} \\ &= (\lambda-1)^2(\lambda-2) - (\lambda-1) - (\lambda-1) \\ &= (\lambda-1)(\lambda^2 - 3\lambda + 2 - 2) \\ &= (\lambda-1)(\lambda-0)(\lambda-3) \end{aligned}$$

$$\therefore\ \sigma(A^*A) = \{1, 3, 0\}$$

when $\lambda_1 = 1$:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ x_1 - y_1 + z_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ y_1 = 0$, Choose $x_1 = 1$. Then $z_1 = -1$ and $x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

when $\lambda_2 = 3$:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x_2 + y_2 \\ x_2 + y_2 + z_2 \\ y_2 + 2z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Setting $y_2 = 2$ yields $x_2 = -1$ and $z_2 = -1$. Hence, $y = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

when $\lambda_3 = 0$:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -x_3 + y_3 \\ x_3 - 2y_3 + z_3 \\ y_3 - z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Setting $y_3 = 1$ yields $x_3 = 1$ and $z_3 = 1$. Hence, $z = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\therefore V = \begin{bmatrix} \frac{1}{\|x\|}x & \frac{1}{\|y\|}y & \frac{1}{\|z\|}z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}.$$

Now,

$$AA^* = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\therefore \ \sigma(AA^*) = \{1, 3\}$$

when $\sigma_1 = \sqrt{1} = 1$:

$$u_1 = \frac{1}{\sigma_1} A \frac{x}{\|x\|} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

when $\sigma_2 = \sqrt{3}$:

$$u_2 = \frac{1}{\sigma_2} A \frac{y}{\|y\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\therefore U = [u_1 \ \ u_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

<u>Check</u>

$$\begin{aligned} U\Sigma V^* &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \sqrt{2} & \frac{-1}{\sqrt{2}} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} & 0 \\ 0 & -\sqrt{2} & \sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = A \end{aligned}$$

Then

$$
\begin{aligned}
A^\dagger &= V\Sigma^\dagger U^* \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} \\ 0 & \frac{2}{\sqrt{18}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} -2 & -1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}.
\end{aligned}
$$

Lemma 2.6 (see, e.g., [10]). Let $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$. Then

(i) $(A^\dagger)^T = (A^T)^\dagger$,

(ii) $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

Lemma 2.7 (see, e.g., [10]). Let $A \in M_{m,n}(\mathbb{C})$ and $b \in \mathbb{C}^m$. The following statements are equivalent:

(i) The vector-matrix equation $Ax = b$ has a solution x.

(ii) $\operatorname{rank}[A \mid b] = \operatorname{rank} A$.

(iii) $AA^\dagger b = b$.

In the above case, the general solution is given by $x = A^\dagger b + (I - A^\dagger A)q$, where $q \in \mathbb{C}^m$ is arbitrary.

2.4 Block matrix technique

The following fact is well known.

Lemma 2.8 (see, e.g., [11]). If $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A, B, C, D \in M_n(\mathbb{C})$, then

$$\det(T) = \det(AD - BC)$$

whenever at least one of the blocks A,B,C,D is equal to 0.

Lemma 2.9 (see, e.g., [11]). Let $A, B, C, D \in M_n(\mathbb{C})$, where A is invertible. Then $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is invertible if and only if $D - CA^{-1}B$ is invertible.

Proof. One can observe the following matrix decomposition:

$$\begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

Taking determinant and applying Lemma 2.8 yield

$$\begin{aligned} \det \begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix} \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \det \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}, \\ \det(I_n) \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \det(A) \det(D - CA^{-1}B). \end{aligned}$$

Therefore, $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is invertible if and only if $D - CA^{-1}B$ is invertible. □

Chapter 3

Existence of a Solution for Systems of Coupled Linear Matrix Equations

In this chapter, we investigate existence of solution for the main system

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned}$$

and its interesting special cases.

3.1 The main system

Theorem 3.1. Let $A, C \in M_{m,n}(\mathbb{C})$, $B, D \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{m,q}(\mathbb{C})$. Denote

$$H = \begin{bmatrix} B^T \otimes A & D^T \otimes C \\ D^T \otimes C & B^T \otimes A \end{bmatrix}, \quad P = B^T \otimes A + D^T \otimes C \ \text{ and } \ Q = B^T \otimes A - D^T \otimes C.$$

Consider the following coupled linear matrix equations

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F. \end{aligned} \tag{3.1}$$

Then, the following statements are equivalent:

(i) The system (3.1) has a solution.

(ii) $\operatorname{rank} \left[\begin{array}{cc|c} B^T \otimes A & D^T \otimes C & \operatorname{Vec} E \\ D^T \otimes C & B^T \otimes A & \operatorname{Vec} F \end{array}\right] = \operatorname{rank} P + \operatorname{rank} Q.$

(iii) $(2I_{mq} - (PP^\dagger + QQ^\dagger)) \operatorname{Vec} E = (PP^\dagger - QQ^\dagger) \operatorname{Vec} F$ and $(2I_{mq} - (PP^\dagger + QQ^\dagger)) \operatorname{Vec} F = (PP^\dagger - QQ^\dagger) \operatorname{Vec} E.$

In the above case, the general solution is given by

$$\begin{aligned} \operatorname{Vec} X &= \frac{1}{2}\left[(P^\dagger + Q^\dagger)\operatorname{Vec} E + (P^\dagger - Q^\dagger)\operatorname{Vec} F + (2I_{np} - (P^\dagger P + Q^\dagger Q))q_1 - (P^\dagger P - Q^\dagger Q)q_2\right], \\ \operatorname{Vec} Y &= \frac{1}{2}\left[(P^\dagger - Q^\dagger)\operatorname{Vec} E + (P^\dagger + Q^\dagger)\operatorname{Vec} F - (P^\dagger P - Q^\dagger Q)q_1 + (2I_{np} - (P^\dagger P + Q^\dagger Q))q_2\right], \end{aligned}$$

where $q_1, q_2 \in \mathbb{C}^{np}$ are arbitrary.

Proof. Taking the vector operator to (3.1) and then using Lemma 2.4, we get

$$\begin{aligned} (B^T \otimes A)\operatorname{Vec} X + (D^T \otimes C)\operatorname{Vec} Y &= \operatorname{Vec} E, \\ (D^T \otimes C)\operatorname{Vec} X + (B^T \otimes A)\operatorname{Vec} Y &= \operatorname{Vec} F. \end{aligned}$$

For convenience, let us denote

$$x = \begin{bmatrix} \operatorname{Vec} X \\ \operatorname{Vec} Y \end{bmatrix}, \quad b = \begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{mq} & -I_{mq} \\ I_{mq} & I_{mq} \end{bmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{np} & I_{np} \\ -I_{np} & I_{np} \end{bmatrix}.$$

The system (3.1) is equivalent to the vector-matrix equation $Hx = b$ due to the injectivity of the vector operator. One can decompose H as follows:

$$\begin{aligned} H &= U \begin{bmatrix} B^T \otimes A + D^T \otimes C & 0 \\ 0 & B^T \otimes A - D^T \otimes C \end{bmatrix} V \\ &= U \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} V. \end{aligned} \tag{3.2}$$

Since U and V are invertible, we have

$$\operatorname{rank} H = \operatorname{rank} \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} = \operatorname{rank} P + \operatorname{rank} Q.$$

Hence, the statements (i) and (ii) are equivalent. On the other hand, according to Lemma 2.7, the system (3.1) is consistent if and only if $HH^\dagger b = b$. One can compute

$$\begin{aligned} HH^\dagger b &= U \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} VV^* \begin{bmatrix} P^\dagger & 0 \\ 0 & Q^\dagger \end{bmatrix} U^* \begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} PP^\dagger + QQ^\dagger & PP^\dagger - QQ^\dagger \\ PP^\dagger - QQ^\dagger & PP^\dagger + QQ^\dagger \end{bmatrix} \begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} (PP^\dagger + QQ^\dagger) \operatorname{Vec} E + (PP^\dagger - QQ^\dagger) \operatorname{Vec} F \\ (PP^\dagger - QQ^\dagger) \operatorname{Vec} E + (PP^\dagger + QQ^\dagger) \operatorname{Vec} F \end{bmatrix}. \end{aligned}$$

Hence, the statements (i) and (iii) are equivalent.

If the solution x exists, Lemma 2.7 ensures that it must be in the form

$$x = H^\dagger b + (I_{2np} - H^\dagger H) q,$$

where $q \in \mathbb{C}^{2np}$ is arbitrary. Setting $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ when $q_1, q_2 \in \mathbb{C}^{np}$, we have

$$\begin{aligned} x &= V^* \begin{bmatrix} P^\dagger & 0 \\ 0 & Q^\dagger \end{bmatrix} U^* \begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix} + \left(\begin{bmatrix} I_{np} & 0 \\ 0 & I_{np} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} P^\dagger P + Q^\dagger Q & P^\dagger P - Q^\dagger Q \\ P^\dagger P - Q^\dagger Q & P^\dagger P + Q^\dagger Q \end{bmatrix} \right) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} (P^\dagger + Q^\dagger) \operatorname{Vec} E + (P^\dagger - Q^\dagger) \operatorname{Vec} F \\ (P^\dagger - Q^\dagger) \operatorname{Vec} E + (P^\dagger + Q^\dagger) \operatorname{Vec} F \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \left(2I_{np} - (P^\dagger P + Q^\dagger Q)\right) q_1 - (P^\dagger P - Q^\dagger Q) q_2 \\ -(P^\dagger P - Q^\dagger Q) q_1 + \left(2I_{np} - (P^\dagger P + Q^\dagger Q)\right) q_2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} (P^\dagger + Q^\dagger) \operatorname{Vec} E + (P^\dagger - Q^\dagger) \operatorname{Vec} F + \left(2I_{np} - (P^\dagger P + Q^\dagger Q)\right) q_1 - (P^\dagger P - Q^\dagger Q) q_2 \\ (P^\dagger - Q^\dagger) \operatorname{Vec} E + (P^\dagger + Q^\dagger) \operatorname{Vec} F - (P^\dagger P - Q^\dagger Q) q_1 + \left(2I_{np} - (P^\dagger P + Q^\dagger Q)\right) q_2 \end{bmatrix}. \end{aligned}$$

Therefore, the general (vector) solution of system (3.1) is given by

$$\begin{aligned} \operatorname{Vec} X &= \frac{1}{2} \left[(P^\dagger + Q^\dagger) \operatorname{Vec} E + (P^\dagger - Q^\dagger) \operatorname{Vec} F + (2I_{np} - (P^\dagger P + Q^\dagger Q)) q_1 - (P^\dagger P - Q^\dagger Q) q_2 \right], \\ \operatorname{Vec} Y &= \frac{1}{2} \left[(P^\dagger - Q^\dagger) \operatorname{Vec} E + (P^\dagger + Q^\dagger) \operatorname{Vec} F - (P^\dagger P - Q^\dagger Q) q_1 + (2I_{np} - (P^\dagger P + Q^\dagger Q)) q_2 \right]. \end{aligned}$$

□

3.2 Certain special cases of the main system

Theorem 3.2. Let $A \in M_{m,n}(\mathbb{C})$, $B, D \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{m,q}(\mathbb{C})$. Consider the following coupled linear matrix equations

$$\begin{aligned} AXB + AYD &= E, \\ AXD + AYB &= F. \end{aligned} \tag{3.3}$$

Then, the following statements are equivalent:

(i) The system (3.3) has a solution.

(ii) $\operatorname{rank}\left[\begin{array}{cc|c} B^T \otimes A & D^T \otimes A & \operatorname{Vec} E \\ D^T \otimes A & B^T \otimes A & \operatorname{Vec} F \end{array}\right] = (\operatorname{rank} A)\left(\operatorname{rank}(B+D) + \operatorname{rank}(B-D)\right)$.

(iii) The following two conditions hold:

$$AA^{\dagger}\left[(E+F)(B+D)^{\dagger}(B+D) + (E-F)(B-D)^{\dagger}(B-D)\right] = 2E, \tag{3.4}$$

$$AA^{\dagger}\left[(E+F)(B+D)^{\dagger}(B+D) - (E-F)(B-D)^{\dagger}(B-D)\right] = 2F. \tag{3.5}$$

In the above case, the general solution is given by

$$\begin{aligned} X = {} & Q_1 + \frac{1}{2}\Big[A^{\dagger}\left\{(E+F)(B+D)^{\dagger} + (E-F)(B-D)^{\dagger}\right\} \\ & - A^{\dagger}A\left\{(Q_1+Q_2)(B+D)(B+D)^{\dagger} + (Q_1-Q_2)(B-D)(B-D)^{\dagger}\right\}\Big], \end{aligned} \tag{3.6}$$

$$\begin{aligned} Y = {} & Q_2 + \frac{1}{2}\Big[A^{\dagger}\left\{(E+F)(B+D)^{\dagger} - (E-F)(B-D)^{\dagger}\right\} \\ & - A^{\dagger}A\left\{(Q_1+Q_2)(B+D)(B+D)^{\dagger} - (Q_1-Q_2)(B-D)(B-D)^{\dagger}\right\}\Big], \end{aligned} \tag{3.7}$$

where Q_1, $Q_2 \in M_{n,p}(\mathbb{C})$ are arbitrary.

Proof. Denote $R = (B+D)^T \otimes A$ and $S = (B-D)^T \otimes A$. In the viewpoint of Theorem 3.1 when $C = A$, the existence of a solution of the system (3.3) is equivalent to any of the following statements:

(ii)′ $\operatorname{rank}\left[\begin{array}{cc|c} B^T \otimes A & D^T \otimes A & \operatorname{Vec} E \\ D^T \otimes A & B^T \otimes A & \operatorname{Vec} F \end{array}\right] = \operatorname{rank} R + \operatorname{rank} S$.

(iii)′ The following two conditions hold:

$$(2I_{mq} - (RR^{\dagger} + SS^{\dagger}))\operatorname{Vec} E = (RR^{\dagger} - SS^{\dagger})\operatorname{Vec} F, \tag{3.8}$$

$$(2I_{mq} - (RR^{\dagger} + SS^{\dagger}))\operatorname{Vec} F = (RR^{\dagger} - SS^{\dagger})\operatorname{Vec} E. \tag{3.9}$$

By Lemma 2.2, we have

$$\operatorname{rank} R = (\operatorname{rank} A)(\operatorname{rank}(B+D)), \qquad \operatorname{rank} S = (\operatorname{rank} A)(\operatorname{rank}(B-D)).$$

We obtain

$$\operatorname{rank} R + \operatorname{rank} S = (\operatorname{rank} A)\left(\operatorname{rank}(B+D) + \operatorname{rank}(B-D)\right).$$

Hence, the statement (ii)′ becomes the statement (ii). Now, we shall show that the equation (3.8) is reduced to (3.4). Indeed, note that by Lemma 2.6, we have

$$R^\dagger = ((B+D)^\dagger)^T \otimes A^\dagger, \qquad S^\dagger = ((B-D)^\dagger)^T \otimes A^\dagger.$$

It follows that

$$\begin{aligned} RR^\dagger &= \left((B+D)^\dagger(B+D)\right)^T \otimes AA^\dagger, \\ SS^\dagger &= \left((B-D)^\dagger(B-D)\right)^T \otimes AA^\dagger. \end{aligned}$$

Hence,

$$\begin{aligned} (RR^\dagger + SS^\dagger)\,\mathrm{Vec}\,E &= \mathrm{Vec}\left[AA^\dagger E\left\{(B+D)^\dagger(B+D) + (B-D)^\dagger(B-D)\right\}\right], \\ (RR^\dagger - SS^\dagger)\,\mathrm{Vec}\,F &= \mathrm{Vec}\left[AA^\dagger F\left\{(B+D)^\dagger(B+D) - (B-D)^\dagger(B-D)\right\}\right]. \end{aligned}$$

Now, substitute $(RR^\dagger + SS^\dagger)\,\mathrm{Vec}\,E$ and $(RR^\dagger - SS^\dagger)\,\mathrm{Vec}\,F$ into (3.8). We obtain

$$\begin{aligned} 2E - AA^\dagger E\Big[(B+D)^\dagger(B+D) + (B-D)^\dagger(B-D)\Big] & \\ = AA^\dagger F\left[(B+D)^\dagger(B+D) - (B-D)^\dagger(B-D)\right] & \\ 2E = AA^\dagger\left[(E+F)(B+D)^\dagger(B+D) + (E-F)(B-D)^\dagger(B-D)\right], & \end{aligned}$$

which is equivalent to (3.4). Likewise,

$$\begin{aligned} 2F - AA^\dagger F\Big[(B+D)^\dagger(B+D) - (B-D)^\dagger(B-D)\Big] & \\ = AA^\dagger E\left[(B+D)^\dagger(B+D) + (B-D)^\dagger(B-D)\right] & \\ 2F = AA^\dagger\left[(E+F)(B+D)^\dagger(B+D) - (E-F)(B-D)^\dagger(B-D)\right], & \end{aligned}$$

which is equivalent to (3.5). Hence, the statement (iii)′ change to the statement (iii).

From the general solution of Theorem 3.1 setting $P = R$, $Q = S$, $P^\dagger = R^\dagger$ and $Q^\dagger = S^\dagger$, we get

$$\begin{aligned} \mathrm{Vec}\,X &= \frac{1}{2}\Big[(R^\dagger + S^\dagger)\,\mathrm{Vec}\,E + (R^\dagger - S^\dagger)\,\mathrm{Vec}\,F \\ &\quad + (2I_{np} - (R^\dagger R + S^\dagger S))q_1 - (R^\dagger R - S^\dagger S)q_2\Big], \\ \mathrm{Vec}\,Y &= \frac{1}{2}\Big[(R^\dagger - S^\dagger)\,\mathrm{Vec}\,E + (R^\dagger + S^\dagger)\,\mathrm{Vec}\,F \\ &\quad - (R^\dagger R - S^\dagger S)q_1 + (2I_{np} - (R^\dagger R + S^\dagger S))q_2\Big], \end{aligned}$$

where $q_1, q_2 \in \mathbb{C}^{np}$ are arbitrary. To obtain a formula of the general solution, note that Theorem 3.2, Lemma 2.2 and Lemma 2.6 together imply that

$$\begin{aligned} \mathrm{Vec}\,X &= \frac{1}{2}\Big[K_1\,\mathrm{Vec}\,E + K_2\,\mathrm{Vec}\,F + K_3 q_1 + K_4 q_2\Big], \\ \mathrm{Vec}\,Y &= \frac{1}{2}\Big[K_2\,\mathrm{Vec}\,E + K_1\,\mathrm{Vec}\,F + K_4 q_1 + K_3 q_2\Big], \end{aligned}$$

where $q_1, q_2 \in \mathbb{C}^{np}$ are arbitrary and

$$\begin{aligned} K_1 &= \left((B+D)^\dagger + (B-D)^\dagger\right)^T \otimes A^\dagger, \\ K_2 &= \left((B+D)^\dagger - (B-D)^\dagger\right)^T \otimes A^\dagger, \\ K_3 &= 2I_{np} - \left[\left((B+D)(B+D)^\dagger + (B-D)(B-D)^\dagger\right)^T \otimes A^\dagger A\right], \\ K_4 &= -\left((B+D)(B+D)^\dagger - (B-D)(B-D)^\dagger\right)^T \otimes A^\dagger A. \end{aligned}$$

Since the vector operator is bijective, we can write $q_1 = \operatorname{Vec} Q_1$ and $q_2 = \operatorname{Vec} Q_2$ for some unique $Q_1, Q_2 \in M_{n,p}(\mathbb{C})$, respectively. Then, by Lemmas 2.4 and 2.6, we have

$$\begin{aligned} K_1 \operatorname{Vec} E &= \left[\left((B+D)^\dagger + (B-D)^\dagger\right)^T \otimes A^\dagger\right] \operatorname{Vec} E \\ &= \operatorname{Vec}\left[A^\dagger E\left((B+D)^\dagger + (B-D)^\dagger\right)\right]. \\ K_2 \operatorname{Vec} F &= \left[\left((B+D)^\dagger - (B-D)^\dagger\right)^T \otimes A^\dagger\right] \operatorname{Vec} F \\ &= \operatorname{Vec}\left[A^\dagger F\left((B+D)^\dagger - (B-D)^\dagger\right)\right]. \\ K_3 q_1 &= \left(2I_{np} - \left[\left((B+D)(B+D)^\dagger + (B-D)(B-D)^\dagger\right)^T \otimes A^\dagger A\right]\right) \operatorname{Vec} Q_1 \\ &= 2\operatorname{Vec} Q_1 - \left[\left((B+D)(B+D)^\dagger + (B-D)(B-D)^\dagger\right)^T \otimes A^\dagger A\right] \operatorname{Vec} Q_1 \\ &= \operatorname{Vec}\left[2Q_1 - A^\dagger A Q_1\left((B+D)(B+D)^\dagger + (B-D)(B-D)^\dagger\right)\right]. \\ K_4 q_2 &= \left[-\left((B+D)(B+D)^\dagger - (B-D)(B-D)^\dagger\right)^T \otimes A^\dagger A\right] \operatorname{Vec} Q_2 \\ &= \operatorname{Vec}\left[-A^\dagger A Q_2\left((B+D)(B+D)^\dagger - (B-D)(B-D)^\dagger\right)\right]. \end{aligned}$$

Substituting $K_1 \operatorname{Vec} E$, $K_2 \operatorname{Vec} F$, $K_3 q_1$ and $K_4 q_2$ into

$$\operatorname{Vec} X = \frac{1}{2}\left[K_1 \operatorname{Vec} E + K_2 \operatorname{Vec} F + K_3 q_1 + K_4 q_2\right],$$

we get

$$\begin{aligned} \operatorname{Vec} X &= \frac{1}{2}\Big(\operatorname{Vec}\left[A^\dagger E\left((B+D)^\dagger + (B-D)^\dagger\right)\right] \\ &\quad + \operatorname{Vec}\left[A^\dagger F\left((B+D)^\dagger - (B-D)^\dagger\right)\right] \\ &\quad + \operatorname{Vec}\left[2Q_1 - A^\dagger A Q_1\left((B+D)(B+D)^\dagger + (B-D)(B-D)^\dagger\right)\right] \\ &\quad + \operatorname{Vec}\left[-A^\dagger A Q_2\left((B+D)(B+D)^\dagger - (B-D)(B-D)^\dagger\right)\right]\Big) \\ &= \operatorname{Vec}\Big(\frac{1}{2}\Big[A^\dagger E\left((B+D)^\dagger + (B-D)^\dagger\right) + A^\dagger F\left((B+D)^\dagger - (B-D)^\dagger\right) \\ &\quad + 2Q_1 - A^\dagger A Q_1\left((B+D)(B+D)^\dagger + (B-D)(B-D)^\dagger\right) \\ &\quad - A^\dagger A Q_2\left((B+D)(B+D)^\dagger - (B-D)(B-D)^\dagger\right)\Big]\Big). \end{aligned}$$

Since the vector operator is linear and bijective, we obtain

$$\begin{aligned} X &= \frac{1}{2}\Big[A^{\dagger}E\left((B+D)^{\dagger}+(B-D)^{\dagger}\right)+A^{\dagger}F\left((B+D)^{\dagger}-(B-D)^{\dagger}\right) \\ &\quad +2Q_1-A^{\dagger}AQ_1\left((B+D)(B+D)^{\dagger}+(B-D)(B-D)^{\dagger}\right) \\ &\quad -A^{\dagger}AQ_2\left((B+D)(B+D)^{\dagger}-(B-D)(B-D)^{\dagger}\right)\Big] \\ &= Q_1+\frac{1}{2}\Big[A^{\dagger}\left\{(E+F)(B+D)^{\dagger}+(E-F)(B-D)^{\dagger}\right\} \\ &\quad -A^{\dagger}A\left\{(Q_1+Q_2)(B+D)(B+D)^{\dagger}+(Q_1-Q_2)(B-D)(B-D)^{\dagger}\right\}\Big], \end{aligned}$$

which can be reformed to the desired formula (3.6). Similarly,

$$\begin{aligned} K_2\operatorname{Vec}E &= \left[\left((B+D)^{\dagger}-(B-D)^{\dagger}\right)^T\otimes A^{\dagger}\right]\operatorname{Vec}E \\ &= \operatorname{Vec}\left[A^{\dagger}E\left((B+D)^{\dagger}-(B-D)^{\dagger}\right)\right]. \end{aligned}$$

$$\begin{aligned} K_1\operatorname{Vec}F &= \left[\left((B+D)^{\dagger}+(B-D)^{\dagger}\right)^T\otimes A^{\dagger}\right]\operatorname{Vec}F \\ &= \operatorname{Vec}\left[A^{\dagger}F\left((B+D)^{\dagger}+(B-D)^{\dagger}\right)\right]. \\ K_4q_1 &= \left[-\left((B+D)(B+D)^{\dagger}-(B-D)(B-D)^{\dagger}\right)^T\otimes A^{\dagger}A\right]\operatorname{Vec}Q_1 \\ &= \operatorname{Vec}\left[-A^{\dagger}AQ_1\left((B+D)(B+D)^{\dagger}-(B-D)(B-D)^{\dagger}\right)\right]. \end{aligned}$$

$$\begin{aligned} K_3q_2 &= \left(2I_{np}-\left[\left((B+D)(B+D)^{\dagger}+(B-D)(B-D)^{\dagger}\right)^T\otimes A^{\dagger}A\right]\right)\operatorname{Vec}Q_2 \\ &= 2I_{np}\operatorname{Vec}Q_2-\left[\left((B+D)(B+D)^{\dagger}+(B-D)(B-D)^{\dagger}\right)^T\otimes A^{\dagger}A\right]\operatorname{Vec}Q_2 \\ &= \operatorname{Vec}\left[2Q_2-A^{\dagger}AQ_2\left((B+D)(B+D)^{\dagger}+(B-D)(B-D)^{\dagger}\right)\right]. \end{aligned}$$

Substituting $K_2\operatorname{Vec}E$, $K_1\operatorname{Vec}F$, K_4q_1 and K_3q_2 into

$$\operatorname{Vec}Y = \frac{1}{2}\Big[K_2\operatorname{Vec}E+K_1\operatorname{Vec}F+K_4q_1+K_3q_2\Big],$$

we obtain

$$\begin{aligned} \operatorname{Vec}Y &= \frac{1}{2}\Big(\operatorname{Vec}\left[A^{\dagger}E\left((B+D)^{\dagger}-(B-D)^{\dagger}\right)\right] \\ &\quad +\operatorname{Vec}\left[A^{\dagger}F\left((B+D)^{\dagger}+(B-D)^{\dagger}\right)\right] \\ &\quad +\operatorname{Vec}\left[-A^{\dagger}AQ_1\left((B+D)(B+D)^{\dagger}-(B-D)(B-D)^{\dagger}\right)\right] \\ &\quad +\operatorname{Vec}\left[2Q_2-A^{\dagger}AQ_2\left((B+D)(B+D)^{\dagger}+(B-D)(B-D)^{\dagger}\right)\right]\Big). \\ Y &= Q_2+\frac{1}{2}\Big[A^{\dagger}\left\{(E+F)(B+D)^{\dagger}-(E-F)(B-D)^{\dagger}\right\} \\ &\quad -A^{\dagger}A\left\{(Q_1+Q_2)(B+D)(B+D)^{\dagger}-(Q_1-Q_2)(B-D)(B-D)^{\dagger}\right\}\Big]. \end{aligned}$$

Therefore, we get the formula of Y. □

Corollary 3.3. Let $B, D \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{n,q}(\mathbb{C})$. Consider the following system

$$\begin{aligned} XB+YD &= E, \\ XD+YB &= F. \end{aligned} \tag{3.10}$$

Then, the following statements are equivalent:

(i) The system (3.10) has a solution.

(ii) $\operatorname{rank}\left[\begin{array}{cc|c} B^T\otimes I_n & D^T\otimes I_n & \operatorname{Vec} E \\ D^T\otimes I_n & B^T\otimes I_n & \operatorname{Vec} F\end{array}\right] = n\left(\operatorname{rank}(B+D)+\operatorname{rank}(B-D)\right).$

(iii) The following two conditions hold:

$$\left[(E+F)(B+D)^\dagger(B+D)+(E-F)(B-D)^\dagger(B-D)\right] = 2E,$$
$$\left[(E+F)(B+D)^\dagger(B+D)-(E-F)(B-D)^\dagger(B-D)\right] = 2F.$$

In the above case, the general solution is given by

$$\begin{aligned} X &= Q_1+\frac{1}{2}\Big[\left\{(E+F)(B+D)^\dagger+(E-F)(B-D)^\dagger\right\} \\ &\quad -\left\{(Q_1+Q_2)(B+D)(B+D)^\dagger+(Q_1-Q_2)(B-D)(B-D)^\dagger\right\}\Big], \\ Y &= Q_2+\frac{1}{2}\Big[\left\{(E+F)(B+D)^\dagger-(E-F)(B-D)^\dagger\right\} \\ &\quad -\left\{(Q_1+Q_2)(B+D)(B+D)^\dagger-(Q_1-Q_2)(B-D)(B-D)^\dagger\right\}\Big], \end{aligned}$$

where $Q_1,\ Q_2\in M_{n,p}(\mathbb{C})$ are arbitrary.

Proof. This is a special case of Theorem 3.2 by setting $A=I_n$. We obtain

$$(\operatorname{rank} I_n)\left(\operatorname{rank}(B+D)+\operatorname{rank}(B-D)\right) = (n)\left(\operatorname{rank}(B+D)+\operatorname{rank}(B-D)\right).$$

Thus, the statements (i) and (ii) are equivalent. Next,

$$\begin{aligned} (I_n)\left[(E+F)(B+D)^\dagger(B+D)+(E-F)(B-D)^\dagger(B-D)\right] &= 2E \\ \left[(E+F)(B+D)^\dagger(B+D)+(E-F)(B-D)^\dagger(B-D)\right] &= 2E \end{aligned}$$

and

$$\begin{aligned} (I_n)\left[(E+F)(B+D)^\dagger(B+D)-(E-F)(B-D)^\dagger(B-D)\right] &= 2F \\ \left[(E+F)(B+D)^\dagger(B+D)-(E-F)(B-D)^\dagger(B-D)\right] &= 2F. \end{aligned}$$

Hence, the statements (i) and (iii) are equivalent.

The general solution of a system (3.10) is as below

$$\begin{aligned} X &= Q_1+\frac{1}{2}\Big[(I_n)\left\{(E+F)(B+D)^\dagger+(E-F)(B-D)^\dagger\right\} \\ &\quad -(I_n)\left\{(Q_1+Q_2)(B+D)(B+D)^\dagger+(Q_1-Q_2)(B-D)(B-D)^\dagger\right\}\Big], \\ &= Q_1+\frac{1}{2}\Big[\left\{(E+F)(B+D)^\dagger+(E-F)(B-D)^\dagger\right\} \\ &\quad -\left\{(Q_1+Q_2)(B+D)(B+D)^\dagger+(Q_1-Q_2)(B-D)(B-D)^\dagger\right\}\Big] \end{aligned}$$

and

$$\begin{aligned} Y &= Q_2+\frac{1}{2}\Big[(I_n)\left\{(E+F)(B+D)^\dagger-(E-F)(B-D)^\dagger\right\} \\ &\quad -(I_n)\left\{(Q_1+Q_2)(B+D)(B+D)^\dagger-(Q_1-Q_2)(B-D)(B-D)^\dagger\right\}\Big] \\ &= Q_2+\frac{1}{2}\Big[\left\{(E+F)(B+D)^\dagger-(E-F)(B-D)^\dagger\right\} \\ &\quad -\left\{(Q_1+Q_2)(B+D)(B+D)^\dagger-(Q_1-Q_2)(B-D)(B-D)^\dagger\right\}\Big]. \end{aligned}$$

Therefore, we get the formulas of X and Y. □

Theorem 3.4. Let $A, C \in M_{m,n}(\mathbb{C})$, $B \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{m,q}(\mathbb{C})$. Consider the following coupled linear matrix equations

$$\begin{aligned} AXB + CYB &= E, \\ CXB + AYB &= F. \end{aligned} \tag{3.11}$$

Then, the following statements are equivalent:

(i) The system (3.11) has a solution.

(ii) $\operatorname{rank}\left[\begin{array}{cc|c} B^T \otimes A & B^T \otimes C & \operatorname{Vec} E \\ B^T \otimes C & B^T \otimes A & \operatorname{Vec} F \end{array}\right] = (\operatorname{rank} B)\,(\operatorname{rank}(A+C) + \operatorname{rank}(A-C))$.

(iii) The following two conditions hold:

$$[(A+C)(A+C)^\dagger(E+F) + (A-C)(A-C)^\dagger(E+F)]\, B^\dagger B = 2E, \tag{3.12}$$

$$[(A+C)(A+C)^\dagger(E+F) - (A-C)(A-C)^\dagger(E+F)]\, B^\dagger B = 2F. \tag{3.13}$$

In the above case, the general solution is given by

$$\begin{aligned} X = Q_1 + \frac{1}{2}\Big[&\{(A+C)^\dagger(E+F) + (A-C)^\dagger(E-F)\}\, B^\dagger \\ &- \{(A+C)^\dagger(A+C)(Q_1+Q_2) + (A-C)^\dagger(A-C)(Q_1-Q_2)\}\, BB^\dagger\Big], \end{aligned} \tag{3.14}$$

$$\begin{aligned} Y = Q_2 + \frac{1}{2}\Big[&\{(A+C)^\dagger(E+F) - (A-C)^\dagger(E-F)\}\, B^\dagger \\ &- \{(A+C)^\dagger(A+C)(Q_1+Q_2) - (A-C)^\dagger(A-C)(Q_1-Q_2)\}\, BB^\dagger\Big], \end{aligned} \tag{3.15}$$

where Q_1, $Q_2 \in M_{n,p}(\mathbb{C})$ are arbitrary.

Proof. The idea of proof is similar to that of Theorem 3.2. Denote $T = B^T \otimes (A+C)$ and $V = B^T \otimes (A-C)$, the existence of a solution of the system (3.11) is equivalent to the following statements:

(ii)′ $\operatorname{rank}\left[\begin{array}{cc|c} B^T \otimes A & B^T \otimes C & \operatorname{Vec} E \\ B^T \otimes C & B^T \otimes A & \operatorname{Vec} F \end{array}\right] = \operatorname{rank} T + \operatorname{rank} V$.

(iii)′ The following two conditions hold:

$$(2I_{mq} - (TT^\dagger + VV^\dagger))\operatorname{Vec} E = (TT^\dagger - VV^\dagger)\operatorname{Vec} F, \tag{3.16}$$

$$(2I_{mq} - (TT^\dagger + VV^\dagger))\operatorname{Vec} F = (TT^\dagger - VV^\dagger)\operatorname{Vec} E. \tag{3.17}$$

We have

$$\operatorname{rank} T = (\operatorname{rank} B)(\operatorname{rank}(A+C)), \qquad \operatorname{rank} V = (\operatorname{rank} B)(\operatorname{rank}(A-C)).$$

Hence, the statements (ii)′ change to the statements (ii). Next, we will show that the equation (3.16) turn into the equation (3.12). We have

$$T^\dagger = (B^\dagger)^T \otimes (A+C)^\dagger, \qquad V^\dagger = (B^\dagger)^T \otimes (A-C)^\dagger.$$

Then,

$$TT^{\dagger} = (B^{\dagger}B)^T \otimes \left((A+C)(A+C)^{\dagger}\right),$$
$$VV^{\dagger} = (B^{\dagger}B)^T \otimes \left((A-C)(A-C)^{\dagger}\right).$$

We get

$$(TT^{\dagger} + VV^{\dagger})\operatorname{Vec} E = \operatorname{Vec}\left[\left((A+C)(A+C)^{\dagger} + (A-C)(A-C)^{\dagger}\right) EB^{\dagger}B\right],$$
$$(TT^{\dagger} - VV^{\dagger})\operatorname{Vec} F = \operatorname{Vec}\left[\left((A+C)(A+C)^{\dagger} - (A-C)(A-C)^{\dagger}\right) FB^{\dagger}B\right].$$

Substituting $(TT^{\dagger} + VV^{\dagger})\operatorname{Vec} E$ and $(TT^{\dagger} - VV^{\dagger})\operatorname{Vec} F$ into (3.16), we get

$$\operatorname{Vec}\left[2E - ((A+C)(A+C)^{\dagger} + (A-C)(A-C)^{\dagger})EB^{\dagger}B\right]$$
$$= \operatorname{Vec}\left[((A+C)(A+C)^{\dagger} - (A-C)(A-C)^{\dagger})FB^{\dagger}B\right]$$

$$2E - ((A+C)(A+C)^{\dagger} + (A-C)(A-C)^{\dagger})EB^{\dagger}B$$
$$= ((A+C)(A+C)^{\dagger} - (A-C)(A-C)^{\dagger})FB^{\dagger}B$$

Thus, we obtain

$$2E = \left[(A+C)(A+C)^{\dagger}(E+F) + (A-C)(A-C)^{\dagger}(E-F)\right]B^{\dagger}B,$$

which is equivalent to (3.12). Likewise, the conditions (3.17) and (3.13) are equivalent.

To obtain a formulas of the general solution, note that Theorem 3.4, Lemma 2.2 and Lemma 2.6 together imply that

$$\operatorname{Vec} X = \frac{1}{2}\left[J_1 \operatorname{Vec} E + J_2 \operatorname{Vec} F + J_3 q_1 + J_4 q_2\right],$$
$$\operatorname{Vec} Y = \frac{1}{2}\left[J_2 \operatorname{Vec} E + J_1 \operatorname{Vec} F + J_4 q_1 + J_3 q_2\right],$$

where $q_1, q_2 \in \mathbb{C}^{np}$ are arbitrary and

$$J_1 = (B^{\dagger})^T \otimes \left\{(A+C)^{\dagger} + (A-C)^{\dagger}\right\},$$
$$J_2 = (B^{\dagger})^T \otimes \left\{(A+C)^{\dagger} - (A-C)^{\dagger}\right\},$$
$$J_3 = 2I_{np} - \left[(BB^{\dagger})^T \otimes \left\{(A+C)^{\dagger}(A+C) + (A-C)^{\dagger}(A-C)\right\}\right],$$
$$J_4 = -(BB^{\dagger})^T \otimes \left\{(A+C)^{\dagger}(A+C) - (A-C)^{\dagger}(A-C)\right\}.$$

Write $q_1 = \operatorname{Vec} Q_1$ and $q_2 = \operatorname{Vec} Q_2$ for some unique $Q_1, Q_2 \in M_{n,p}(\mathbb{C})$, respectively. By Lemma 2.4 and Lemma 2.6, we obtain

$$J_1 \operatorname{Vec} E = \left[(B^{\dagger})^T \otimes \left((A+C)^{\dagger} + (A-C)^{\dagger}\right)\right] \operatorname{Vec} E$$
$$= \operatorname{Vec}\left[((A+C)^{\dagger} + (A-C)^{\dagger})EB^{\dagger}\right].$$
$$J_2 \operatorname{Vec} F = \left[(B^{\dagger})^T \otimes \left((A+C)^{\dagger} - (A-C)^{\dagger}\right)\right] \operatorname{Vec} F$$
$$= \operatorname{Vec}\left[((A+C)^{\dagger} - (A-C)^{\dagger})FB^{\dagger}\right].$$

$$\begin{aligned} J_3 q_1 &= \Big(2I_{np} - \Big[(BB^\dagger)^T \otimes \big((A+C)^\dagger(A+C) + (A-C)^\dagger(A-C)\big)\Big]\Big) \operatorname{Vec} Q_1 \\ &= 2\operatorname{Vec} Q_1 - \Big[(BB^\dagger)^T \otimes \big((A+C)^\dagger(A+C) + (A-C)^\dagger(A-C)\big)\Big] \operatorname{Vec} Q_1 \\ &= \operatorname{Vec}\left[2Q_1 - ((A+C)^\dagger(A+C) + (A-C)^\dagger(A-C))Q_1 BB^\dagger\right]. \\ J_4 q_2 &= \Big[-(BB^\dagger)^T \otimes \big((A+C)^\dagger(A+C) - (A-C)^\dagger(A-C)\big)\Big] \operatorname{Vec} Q_2 \\ &= \operatorname{Vec}\Big[((A+C)^\dagger(A+C) - (A-C)^\dagger(A-C))Q_2 BB^\dagger\Big]. \end{aligned}$$

Substituting $J_1 \operatorname{Vec} E,\ \ J_2 \operatorname{Vec} F,\ \ J_3 q_1$ and $J_4 q_2$ into

$$\operatorname{Vec} X = \frac{1}{2}\Big[J_1 \operatorname{Vec} E + J_2 \operatorname{Vec} F + J_3 q_1 + J_4 q_2\Big],$$

we get

$$\begin{aligned} \operatorname{Vec} X &= \frac{1}{2}\Big(\operatorname{Vec}\Big[((A+C)^\dagger + (A-C)^\dagger)EB^\dagger\Big] \\ &\quad + \operatorname{Vec}\Big[((A+C)^\dagger - (A-C)^\dagger)FB^\dagger\Big] \\ &\quad + \operatorname{Vec}\left[2Q_1 - ((A+C)^\dagger(A+C) + (A-C)^\dagger(A-C))Q_1 BB^\dagger\right] \\ &\quad + \operatorname{Vec}\Big[((A+C)^\dagger(A+C) - (A-C)^\dagger(A-C))Q_2 BB^\dagger\Big]\Big) \\ &= \operatorname{Vec}\Big(\frac{1}{2}\Big[((A+C)^\dagger + (A-C)^\dagger)EB^\dagger + ((A+C)^\dagger - (A-C)^\dagger)FB^\dagger \\ &\quad + 2Q_1 - ((A+C)^\dagger(A+C) + (A-C)^\dagger(A-C))Q_1 BB^\dagger \\ &\quad - ((A+C)^\dagger(A+C) - (A-C)^\dagger(A-C))Q_2 BB^\dagger\Big]\Big). \end{aligned}$$

Since the vector operator is linear and bijective, we obtain

$$\begin{aligned} X &= \frac{1}{2}\Big[((A+C)^\dagger + (A-C)^\dagger)EB^\dagger + ((A+C)^\dagger - (A-C)^\dagger)FB^\dagger \\ &\quad + 2Q_1 - ((A+C)^\dagger(A+C) + (A-C)^\dagger(A-C))Q_1 BB^\dagger \\ &\quad - ((A+C)^\dagger(A+C) - (A-C)^\dagger(A-C))Q_2 BB^\dagger\Big] \\ &= Q_1 + \frac{1}{2}\Big[\left\{(A+C)^\dagger(E+F) + (A-C)^\dagger(E-F)\right\} B^\dagger \\ &\quad - \left\{(A+C)^\dagger(A+C)(Q_1+Q_2) + (A-C)^\dagger(A-C)(Q_1-Q_2)\right\} BB^\dagger\Big]. \end{aligned}$$

Likewise,

$$\begin{aligned} J_2 \operatorname{Vec} E &= \Big[(B^\dagger)^T \otimes \big((A+C)^\dagger - (A-C)^\dagger\big)\Big] \operatorname{Vec} E \\ &= \operatorname{Vec}\Big[((A+C)^\dagger - (A-C)^\dagger)EB^\dagger\Big]. \\ J_1 \operatorname{Vec} F &= \Big[(B^\dagger)^T \otimes \big((A+C)^\dagger + (A-C)^\dagger\big)\Big] \operatorname{Vec} F \\ &= \operatorname{Vec}\Big[((A+C)^\dagger + (A-C)^\dagger)FB^\dagger\Big]. \end{aligned}$$

$$\begin{aligned} J_4q_1 &= \Big[-(BB^\dagger)^T \otimes \big((A+C)^\dagger(A+C)-(A-C)^\dagger(A-C)\big)\Big] \operatorname{Vec} Q_1 \\ &= \operatorname{Vec}\Big[\big((A+C)^\dagger(A+C)-(A-C)^\dagger(A-C)\big)Q_1BB^\dagger\Big]. \\ J_3q_2 &= \Big(2I_{np} - \Big[(BB^\dagger)^T \otimes \big((A+C)^\dagger(A+C)+(A-C)^\dagger(A-C)\big)\Big]\Big) \operatorname{Vec} Q_2 \\ &= 2\operatorname{Vec} Q_2 - \Big[(BB^\dagger)^T \otimes \big((A+C)^\dagger(A+C)+(A-C)^\dagger(A-C)\big)\Big] \operatorname{Vec} Q_2 \\ &= \operatorname{Vec}\left[2Q_2 - \big((A+C)^\dagger(A+C)+(A-C)^\dagger(A-C)\big)Q_2BB^\dagger\right]. \end{aligned}$$

Now, substituting $J_2 \operatorname{Vec} E$, $J_1 \operatorname{Vec} F$, J_4q_1 and J_3q_2 into

$$\operatorname{Vec} Y = \frac{1}{2}\Big[J_2 \operatorname{Vec} E + J_1 \operatorname{Vec} F + J_4q_1 + J_3q_2\Big],$$

we obtain

$$\begin{aligned} \operatorname{Vec} Y = \frac{1}{2}\Big(&\operatorname{Vec}\Big[\big((A+C)^\dagger - (A-C)^\dagger\big)EB^\dagger\Big] \\ &+ \operatorname{Vec}\Big[\big((A+C)^\dagger + (A-C)^\dagger\big)FB^\dagger\Big] \\ &+ \operatorname{Vec}\Big[\big((A+C)^\dagger(A+C)-(A-C)^\dagger(A-C)\big)Q_1BB^\dagger\Big] \\ &+ \operatorname{Vec}\left[2Q_2 - \big((A+C)^\dagger(A+C)+(A-C)^\dagger(A-C)\big)Q_2BB^\dagger\right]\Big). \end{aligned}$$

Therefore, we get the formula of Y as below

$$\begin{aligned} Y = Q_2 + \frac{1}{2}\Big[&\left\{(A+C)^\dagger(E+F) - (A-C)^\dagger(E-F)\right\}B^\dagger \\ &- \left\{(A+C)^\dagger(A+C)(Q_1+Q_2) - (A-C)^\dagger(A-C)(Q_1-Q_2)\right\}BB^\dagger\Big]. \end{aligned}$$

□

Corollary 3.5. Let $A, C \in M_{m,n}(\mathbb{C})$ and $E, F \in M_{m,p}(\mathbb{C})$. Consider the following coupled linear matrix equations

$$\begin{aligned} AX + CY &= E, \\ CX + AY &= F. \end{aligned} \tag{3.18}$$

Then, the following statements are equivalent:

(i) The system (3.18) has a solution.

(ii) $\operatorname{rank}\left[\begin{array}{cc|c} I_p \otimes A & I_p \otimes C & \operatorname{Vec} E \\ I_p \otimes C & I_p \otimes A & \operatorname{Vec} F \end{array}\right] = p\left(\operatorname{rank}(A+C) + \operatorname{rank}(A-C)\right).$

(iii) The following two conditions hold:

$$\begin{aligned} \left[(A+C)(A+C)^\dagger(E+F) + (A-C)(A-C)^\dagger(E+F)\right] &= 2E, \\ \left[(A+C)(A+C)^\dagger(E+F) - (A-C)(A-C)^\dagger(E+F)\right] &= 2F. \end{aligned}$$

In the above case, the general solution is given by

$$\begin{aligned} X &= Q_1+\frac{1}{2}\Big[\left\{(A+C)^{\dagger}(E+F)+(A-C)^{\dagger}(E-F)\right\} \\ &\quad -\left\{(A+C)^{\dagger}(A+C)(Q_1+Q_2)+(A-C)^{\dagger}(A-C)(Q_1-Q_2)\right\}\Big], \\ Y &= Q_2+\frac{1}{2}\Big[\left\{(A+C)^{\dagger}(E+F)-(A-C)^{\dagger}(E-F)\right\} \\ &\quad -\left\{(A+C)^{\dagger}(A+C)(Q_1+Q_2)-(A-C)^{\dagger}(A-C)(Q_1-Q_2)\right\}\Big], \end{aligned}$$

where Q_1, $Q_2 \in M_{n,p}(\mathbb{C})$ are arbitrary.

Proof. It follows from Theorem 3.4 by setting $B = I_p$ that

$$(I_p)\,(\text{rank}(A+C)+\text{rank}(A-C)) = (p)\,(\text{rank}(A+C)+\text{rank}(A-C))\,.$$

Thus, the statements (i) and (ii) are equivalent. Next,

$$\begin{aligned} \left[(A+C)(A+C)^{\dagger}(E+F)+(A-C)(A-C)^{\dagger}(E+F)\right](I_p) &= 2E \\ \left[(A+C)(A+C)^{\dagger}(E+F)+(A-C)(A-C)^{\dagger}(E+F)\right] &= 2E \end{aligned}$$

and

$$\begin{aligned} \left[(A+C)(A+C)^{\dagger}(E+F)-(A-C)(A-C)^{\dagger}(E+F)\right](I_p) &= 2F \\ \left[(A+C)(A+C)^{\dagger}(E+F)-(A-C)(A-C)^{\dagger}(E+F)\right] &= 2F. \end{aligned}$$

Hence, the statements (i) and (iii) are equivalent. The general solution of a system (3.18) is as below

$$\begin{aligned} X &= Q_1+\frac{1}{2}\Big[\left\{(A+C)^{\dagger}(E+F)+(A-C)^{\dagger}(E-F)\right\}(I_p) \\ &\quad -\left\{(A+C)^{\dagger}(A+C)(Q_1+Q_2)+(A-C)^{\dagger}(A-C)(Q_1-Q_2)\right\}(I_p)\Big] \\ &= Q_1+\frac{1}{2}\Big[\left\{(A+C)^{\dagger}(E+F)+(A-C)^{\dagger}(E-F)\right\} \\ &\quad -\left\{(A+C)^{\dagger}(A+C)(Q_1+Q_2)+(A-C)^{\dagger}(A-C)(Q_1-Q_2)\right\}\Big] \end{aligned}$$

and

$$\begin{aligned} Y &= Q_2+\frac{1}{2}\Big[\left\{(A+C)^{\dagger}(E+F)-(A-C)^{\dagger}(E-F)\right\}(I_p) \\ &\quad -\left\{(A+C)^{\dagger}(A+C)(Q_1+Q_2)-(A-C)^{\dagger}(A-C)(Q_1-Q_2)\right\}(I_p)\Big] \\ &= Q_2+\frac{1}{2}\Big[\left\{(A+C)^{\dagger}(E+F)-(A-C)^{\dagger}(E-F)\right\} \\ &\quad -\left\{(A+C)^{\dagger}(A+C)(Q_1+Q_2)-(A-C)^{\dagger}(A-C)(Q_1-Q_2)\right\}\Big]. \end{aligned}$$

Therefore, we get the formulas of X and Y.

□

Corollary 3.6. Let $A \in M_{m,n}(\mathbb{C})$, $D \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{m,q}(\mathbb{C})$. Consider the following coupled linear matrix equations

$$\begin{aligned} AXA+AYD &= E, \\ AXD+AYA &= F. \end{aligned} \tag{3.19}$$

Then, the following statements are equivalent:

(i) The system (3.19) has a solution.

(ii) rank $\begin{bmatrix} A^T \otimes A & D^T \otimes A & \mathrm{Vec}\, E \\ D^T \otimes A & A^T \otimes A & \mathrm{Vec}\, F \end{bmatrix} = (\mathrm{rank}\, A)\,(\mathrm{rank}(A+D) + \mathrm{rank}(A-D)).$

(iii) The following two conditions hold:

$$AA^\dagger\left[(E+F)(A+D)^\dagger(A+D) + (E-F)(A-D)^\dagger(A-D)\right] = 2E, \tag{3.20}$$

$$AA^\dagger\left[(E+F)(A+D)^\dagger(A+D) - (E-F)(A-D)^\dagger(A-D)\right] = 2F. \tag{3.21}$$

In the above case, the general solution is given by

$$\begin{aligned} X = {} & Q_1 + \frac{1}{2}\Big[A^\dagger\left\{(E+F)(A+D)^\dagger + (E-F)(A-D)^\dagger\right\} \\ & - A^\dagger A\left\{(Q_1+Q_2)(A+D)(A+D)^\dagger + (Q_1-Q_2)(A-D)(A-D)^\dagger\right\}\Big], \end{aligned} \tag{3.22}$$

$$\begin{aligned} Y = {} & Q_2 + \frac{1}{2}\Big[A^\dagger\left\{(E+F)(A+D)^\dagger - (E-F)(A-D)^\dagger\right\} \\ & - A^\dagger A\left\{(Q_1+Q_2)(A+D)(A+D)^\dagger - (Q_1-Q_2)(A-D)(A-D)^\dagger\right\}\Big], \end{aligned} \tag{3.23}$$

where $Q_1, Q_2 \in M_{n,p}(\mathbb{C})$ are arbitrary. □

Proof. It follows from Theorem 3.2 by setting $B = A$.

Corollary 3.7. Let $A, C \in M_{m,n}(\mathbb{C})$, $B \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{m,q}(\mathbb{C})$. Consider the following coupled linear matrix equations

$$\begin{aligned} AXB + BYB &= E, \\ BXB + AYB &= F. \end{aligned} \tag{3.24}$$

Then, the following statements are equivalent:

(i) The system (3.24) has a solution.

(ii) rank $\begin{bmatrix} B^T \otimes A & B^T \otimes B & \mathrm{Vec}\, E \\ B^T \otimes B & B^T \otimes A & \mathrm{Vec}\, F \end{bmatrix} = (\mathrm{rank}\, B)\,(\mathrm{rank}(A+B) + \mathrm{rank}(A-B)).$

(iii) The following two conditions hold:

$$\left[(A+B)(A+B)^\dagger(E+F) + (A-B)(A-B)^\dagger(E+F)\right] B^\dagger B = 2E, \tag{3.25}$$

$$\left[(A+B)(A+B)^\dagger(E+F) - (A-B)(A-B)^\dagger(E+F)\right] B^\dagger B = 2F. \tag{3.26}$$

In the above case, the general solution is given by

$$\begin{aligned} X = {} & Q_1 + \frac{1}{2}\Big[\left\{(A+B)^\dagger(E+F) + (A-B)^\dagger(E-F)\right\} B^\dagger \\ & - \left\{(A+B)^\dagger(A+B)(Q_1+Q_2) + (A-B)^\dagger(A-B)(Q_1-Q_2)\right\} BB^\dagger\Big], \end{aligned} \tag{3.27}$$

$$\begin{aligned} Y = {} & Q_2 + \frac{1}{2}\Big[\left\{(A+B)^\dagger(E+F) - (A-B)^\dagger(E-F)\right\} B^\dagger \\ & - \left\{(A+B)^\dagger(A+B)(Q_1+Q_2) - (A-B)^\dagger(A-B)(Q_1-Q_2)\right\} BB^\dagger\Big], \end{aligned} \tag{3.28}$$

where $Q_1,\ Q_2 \in M_{n,p}(\mathbb{C})$ are arbitrary.

Proof. It follows from Theorem 3.4 by setting $C = B$. $\square$

Chapter 4

Uniqueness of a Solution for Systems of Coupled Linear Matrix Equations

In this chapter, we investigate uniqueness of solution for main the system

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned}$$

and its interesting special cases. Exact formula of the unique solution is explicitly presented.

4.1 The main system

Theorem 4.1. Let $A, B, C, D, E, F \in M_n(\mathbb{C})$. Denote $P = B^T \otimes A + D^T \otimes C$ and $Q = B^T \otimes A - D^T \otimes C$. Consider the following coupled linear matrix equations

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F. \end{aligned} \tag{4.1}$$

This system has a unique solution if and only if P and Q are invertible. In this case, the unique solution is given by

$$\begin{aligned} \operatorname{Vec} X &= \frac{1}{2}\Big[P^{-1}(\operatorname{Vec} E + \operatorname{Vec} F) + Q^{-1}(\operatorname{Vec} E - \operatorname{Vec} F)\Big], \\ \operatorname{Vec} Y &= \frac{1}{2}\Big[P^{-1}(\operatorname{Vec} E + \operatorname{Vec} F) - Q^{-1}(\operatorname{Vec} E - \operatorname{Vec} F)\Big]. \end{aligned}$$

Proof. From the proof of Theorem 3.1, the system (4.1) is equivalent to the vector-matrix equation $Hx = b$, where

$$H = \begin{bmatrix} B^T \otimes A & D^T \otimes C \\ D^T \otimes C & B^T \otimes A \end{bmatrix}, \quad x = \begin{bmatrix} \operatorname{Vec} X \\ \operatorname{Vec} Y \end{bmatrix}, \quad b = \begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix}.$$

The decomposition

$$H = \frac{1}{2}\begin{bmatrix} I_{mq} & -I_{mq} \\ I_{mq} & I_{mq} \end{bmatrix} \begin{bmatrix} B^T \otimes A + D^T \otimes C & 0 \\ 0 & B^T \otimes A - D^T \otimes C \end{bmatrix} \begin{bmatrix} I_{np} & I_{np} \\ -I_{np} & I_{np} \end{bmatrix},$$

says that the system (4.1) has a unique solution if and only if both P and Q are invertible. To obtain the unique solution of (4.1), we substitute $P^\dagger = P^{-1}$ and $Q^\dagger = Q^{-1}$ into the general solution of Theorem 3.1. We obtain

$$\begin{aligned} \operatorname{Vec} X &= \frac{1}{2}\Big[(P^{-1} + Q^{-1}) \operatorname{Vec} E + (P^{-1} - Q^{-1}) \operatorname{Vec} F \\ &\quad + (2I_{n^2} - (P^{-1}P + Q^{-1}Q))q_1 - (P^{-1}P - Q^{-1}Q)q_2\Big] \\ &= \frac{1}{2}\Big[P^{-1} \operatorname{Vec} E + Q^{-1} \operatorname{Vec} E + P^{-1} \operatorname{Vec} F - Q^{-1} \operatorname{Vec} F\Big] \\ &= \frac{1}{2}\Big[P^{-1}(\operatorname{Vec} E + \operatorname{Vec} F) + Q^{-1}(\operatorname{Vec} E - \operatorname{Vec} F)\Big]. \end{aligned}$$

Likewise,

$$\begin{aligned}\operatorname{Vec} Y &= \frac{1}{2}\Big[(P^{-1}-Q^{-1})\operatorname{Vec} E+(P^{-1}+Q^{-1})\operatorname{Vec} F \\ &\quad -(P^{-1}P-Q^{-1}Q)q_1+(2I_{n^2}-(P^{-1}P+Q^{-1}Q))q_2\Big] \\ &= \frac{1}{2}\Big[P^{-1}\operatorname{Vec} E-Q^{-1}\operatorname{Vec} E+P^{-1}\operatorname{Vec} F+Q^{-1}\operatorname{Vec} F\Big] \\ &= \frac{1}{2}\Big[P^{-1}(\operatorname{Vec} E+\operatorname{Vec} F)-Q^{-1}(\operatorname{Vec} E-\operatorname{Vec} F)\Big].\end{aligned}$$

□

Theorem 4.2. Let $A, C \in M_n(\mathbb{C})$, $B, D \in M_p(\mathbb{C})$ and $E, F \in M_{n,p}(\mathbb{C})$ be such that A and B are invertible. Then the system (4.1) has a unique solution if and only if

$$B^T \otimes A-((DB^{-1}D)^T \otimes CA^{-1}C)$$

is invertible.

Proof. From the proof of Theorem 3.1, we see that the system (4.1) is equivalent to vector-matrix equation

$$\begin{bmatrix} B^T \otimes A & D^T \otimes C \\ D^T \otimes C & B^T \otimes A \end{bmatrix}\begin{bmatrix}\operatorname{Vec} X \\ \operatorname{Vec} Y\end{bmatrix}=\begin{bmatrix}\operatorname{Vec} E \\ \operatorname{Vec} F\end{bmatrix}. \tag{4.2}$$

Then the system (4.2) has a unique solution if and only if

$$\begin{bmatrix} B^T \otimes A & D^T \otimes C \\ D^T \otimes C & B^T \otimes A \end{bmatrix} \tag{4.3}$$

is invertible. Since A and B are invertible, so is $B^T \otimes A$ by Lemma 2.2. Hence, Lemma 2.9 implies that the invertibility of block matrices (4.3) is equivalent to the invertibility of

$$(B^T \otimes A)(B^T \otimes A)-(B^T \otimes A)(D^T \otimes C)(B^T \otimes A)^{-1}(D^T \otimes C),$$

which can be reduced to that of $B^T \otimes A-((DB^{-1}D)^T \otimes CA^{-1}C)$ by Lemma 2.2. □

4.2 Certain special cases of the main system

Corollary 4.3. Let $A, B, D, E, F \in M_n(\mathbb{C})$. Consider the following coupled linear matrix equations

$$\begin{aligned} AXB+AYD &= E, \\ AXD+AYB &= F. \end{aligned} \tag{4.4}$$

Then, the above system has a unique solution if and only if $A, B+D$ and $B-D$ are invertible. In this case, the unique solution is given by

$$\begin{aligned} X &= \frac{1}{2}A^{-1}\Big[(E+F)(B+D)^{-1}+(E-F)(B-D)^{-1}\Big], \\ Y &= \frac{1}{2}A^{-1}\Big[(E+F)(B+D)^{-1}-(E-F)(B-D)^{-1}\Big]. \end{aligned}$$

Proof. In the viewpoint of Theorem 4.1 when $C = A$, the system (4.4) is equivalent to vector-matrix equation

$$\begin{bmatrix} B^T \otimes A & D^T \otimes A \\ D^T \otimes A & B^T \otimes A \end{bmatrix} \begin{bmatrix} \text{Vec}\, X \\ \text{Vec}\, Y \end{bmatrix} = \begin{bmatrix} \text{Vec}\, E \\ \text{Vec}\, F \end{bmatrix}.$$

Then the system (4.4) has a unique solution if and only if

$$\begin{bmatrix} B^T \otimes A & D^T \otimes A \\ D^T \otimes A & B^T \otimes A \end{bmatrix}$$

is invertible. From the decomposition (3.2), this condition is equivalent to the invertibility of the following matrix

$$\begin{bmatrix} (B+D)^T \otimes A & 0 \\ 0 & (B-D)^T \otimes A \end{bmatrix},$$

which is also equivalent to the invertibility of

$$(B+D)^T \otimes A \quad \text{and} \quad (B-D)^T \otimes A.$$

By Lemma 2.2, the desired condition holds if and only if A, $B+D$ and $B-D$ are invertible.

The formula of the unique solution can be obtained by substituting $A^\dagger = A^{-1}$, $(B+D)^\dagger = (B+D)^{-1}$ and $(B-D)^\dagger = (B-D)^{-1}$ in (3.6) and (3.7). Indeed, we get

$$\begin{aligned} X &= Q_1 + \frac{1}{2}\Big[A^{-1}\left\{(E+F)(B+D)^{-1} + (E-F)(B-D)^{-1}\right\} \\ &\quad - A^{-1}A\left\{(Q_1+Q_2)(B+D)(B+D)^{-1} + (Q_1-Q_2)(B-D)(B-D)^{-1}\right\}\Big], \\ &= \frac{1}{2}A^{-1}\Big[(E+F)(B+D)^{-1} + (E-F)(B-D)^{-1}\Big]. \\ Y &= Q_2 + \frac{1}{2}\Big[A^{-1}\left\{(E+F)(B+D)^{-1} - (E-F)(B-D)^{-1}\right\} \\ &\quad - A^{-1}A\left\{(Q_1+Q_2)(B+D)(B+D)^{-1} - (Q_1-Q_2)(B-D)(B-D)^{-1}\right\}\Big], \\ &= \frac{1}{2}A^{-1}\Big[(E+F)(B+D)^{-1} - (E-F)(B-D)^{-1}\Big]. \end{aligned}$$

□

Corollary 4.3 was obtained in [8, Theorem 4.11] under the restrict condition $AB = BA$.

Corollary 4.4. Let $A \in M_n(\mathbb{C})$, $B, D \in M_p(\mathbb{C})$ and $E, F \in M_{n,p}(\mathbb{C})$ be such that A and B are invertible. Then the system (4.4) has a unique solution if and only if $B - DB^{-1}D$ is invertible.

Proof. The idea of proof is similar to the proof of Theorem 4.2. The system (4.4) is equivalent to vector-matrix equation

$$\begin{bmatrix} B^T \otimes A & D^T \otimes A \\ D^T \otimes A & B^T \otimes A \end{bmatrix} \begin{bmatrix} \text{Vec}\, X \\ \text{Vec}\, Y \end{bmatrix} = \begin{bmatrix} \text{Vec}\, E \\ \text{Vec}\, F \end{bmatrix}.$$

The above system has a unique solution if and only if

$$\begin{bmatrix} B^T \otimes A & D^T \otimes A \\ D^T \otimes A & B^T \otimes A \end{bmatrix} \tag{4.5}$$

is invertible. Since A and B are invertible, so is $B^T \otimes A$. Then Lemma 2.9 implies that the invertibility of block matrices (4.5) is equivalent to the invertibility of

$$(B^T \otimes A) - \left[(D^T \otimes A)(B^T \otimes A)^{-1}(D^T \otimes A)\right],$$

which can be reduced to that $B - DB^{-1}D$ by Lemma 2.9. □

Corollary 4.5. Let $B, D, E, F \in M_n(\mathbb{C})$. Consider the following system

$$\begin{aligned} XB + YD &= E, \\ XD + YB &= F. \end{aligned} \tag{4.6}$$

This system has a unique solution if and only if $B + D$ and $B - D$ are invertible. In this case, the unique solution is given by

$$\begin{aligned} X &= \frac{1}{2}\left[(E+F)(B+D)^{-1} + (E-F)(B-D)^{-1}\right], \\ Y &= \frac{1}{2}\left[(E+F)(B+D)^{-1} - (E-F)(B-D)^{-1}\right]. \end{aligned}$$

Proof. The idea of proof is similar to that of Corollary 4.3. In this case, we substituting $A = I_n$. Then the system (4.6) is equivalent to vector-matrix equation

$$\begin{bmatrix} B^T \otimes I_n & D^T \otimes I_n \\ D^T \otimes I_n & B^T \otimes I_n \end{bmatrix} \begin{bmatrix} \operatorname{Vec} X \\ \operatorname{Vec} Y \end{bmatrix} = \begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix}.$$

Hence, the system (4.4) has a unique solution if and only if

$$\begin{bmatrix} B^T \otimes I_n & D^T \otimes I_n \\ D^T \otimes I_n & B^T \otimes I_n \end{bmatrix}$$

is invertible. From the decomposition (3.2), this condition is equivalent to the invertibility of the following matrix

$$\begin{bmatrix} (B+D)^T \otimes I_n & 0 \\ 0 & (B-D)^T \otimes I_n \end{bmatrix},$$

which is also equivalent to the invertibility of

$$(B+D)^T \otimes I_n \quad \text{and} \quad (B-D)^T \otimes I_n.$$

Since I_n is invertible, Lemma 2.2 implies that the desired condition holds if and only if $B + D$ and $B - D$ are invertible.

The formula of the unique solution can be obtained by substituting $A^\dagger = I_n$, $(B+D)^\dagger = (B+D)^{-1}$ and $(B-D)^\dagger = (B-D)^{-1}$ in (3.6) and (3.7). We get

$$\begin{aligned} X &= Q_1 + \frac{1}{2}\Big[(I_n)\left\{(E+F)(B+D)^{-1} + (E-F)(B-D)^{-1}\right\} \\ &\quad - (I_n)\left\{(Q_1+Q_2)(B+D)(B+D)^{-1} + (Q_1-Q_2)(B-D)(B-D)^{-1}\right\}\Big] \\ &= \frac{1}{2}\Big[(E+F)(B+D)^{-1} + (E-F)(B-D)^{-1}\Big]. \\ Y &= Q_2 + \frac{1}{2}\Big[(I_n)\left\{(E+F)(B+D)^{-1} - (E-F)(B-D)^{-1}\right\} \\ &\quad - (I_n)\left\{(Q_1+Q_2)(B+D)(B+D)^{-1} - (Q_1-Q_2)(B-D)(B-D)^{-1}\right\}\Big] \\ &= \frac{1}{2}\Big[(E+F)(B+D)^{-1} - (E-F)(B-D)^{-1}\Big]. \end{aligned}$$

□

Corollary 4.6. Let $B, D \in M_p(\mathbb{C})$ and $E, F \in M_{n,p}(\mathbb{C})$ be such that B is invertible. Then the system (4.6) has a unique solution if and only if $B - DB^{-1}D$ is invertible.

Proof. The idea of proof is similar to that of Corollary 4.4. In this case, we set $A = I_n$. Then the system (4.6) is equivalent to vector-matrix equation

$$\begin{bmatrix} B^T \otimes I_n & D^T \otimes I_n \\ D^T \otimes I_n & B^T \otimes I_n \end{bmatrix} \begin{bmatrix} \operatorname{Vec} X \\ \operatorname{Vec} Y \end{bmatrix} = \begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix}.$$

The system (4.6) has a unique solution if and only if

$$\begin{bmatrix} B^T \otimes I_n & D^T \otimes I_n \\ D^T \otimes I_n & B^T \otimes I_n \end{bmatrix} \tag{4.7}$$

is invertible. Since B is invertible, so is $B^T \otimes I_n$. Hence, Lemma 2.9 implies that the invertibility of block matrices (4.7) is equivalent to the invertibility of

$$(B^T \otimes I_n) - \Big[(D^T \otimes I_n)(B^T \otimes I_n)^{-1}(D^T \otimes I_n)\Big],$$

which can be reduced to that $B - DB^{-1}D$ by Lemma 2.9. □

Corollary 4.7. Let $A, B, C, E, F \in M_n(\mathbb{C})$. Consider the following coupled linear matrix equations

$$\begin{aligned} AXB + CYB &= E, \\ CXB + AYB &= F. \end{aligned} \tag{4.8}$$

Then, the above system has a unique solution if and only if $B, A+C$ and $A-C$ are invertible. In this case, the unique solution is given by

$$\begin{aligned} X &= \frac{1}{2}\Big[(A+C)^{-1}(E+F) + (A-C)^{-1}(E-F)\Big]B^{-1}, \\ Y &= \frac{1}{2}\Big[(A+C)^{-1}(E+F) - (A-C)^{-1}(E-F)\Big]B^{-1}. \end{aligned}$$

Proof. The criterion for uniqueness of solution follows from Theorem 4.1 by setting $D = B$. The system (4.8) is equivalent to vector-matrix equation

$$\begin{bmatrix} B^T \otimes A & B^T \otimes C \\ B^T \otimes C & B^T \otimes A \end{bmatrix} \begin{bmatrix} \mathrm{Vec}\, X \\ \mathrm{Vec}\, Y \end{bmatrix} = \begin{bmatrix} \mathrm{Vec}\, E \\ \mathrm{Vec}\, F \end{bmatrix}.$$

Then the system (4.8) has a unique solution if and only if

$$\begin{bmatrix} B^T \otimes A & B^T \otimes C \\ B^T \otimes C & B^T \otimes A \end{bmatrix}$$

is invertible. From the decomposition (3.2), this condition is equivalent to the invertibility of the following matrix

$$\begin{bmatrix} B^T \otimes (A+C) & 0 \\ 0 & B^T \otimes (A-C) \end{bmatrix},$$

which is also equivalent to the invertibility of

$$B^T \otimes (A+C) \quad \text{and} \quad B^T \otimes (A-C).$$

By Lemma2.2, the desired condition holds if and only if B, $A+C$ and $A-C$ are invertible.

The formula of the unique solution can be derived from (3.14) and (3.15) by substituting $B^\dagger = B^{-1}$, $(A+C)^\dagger = (A+C)^{-1}$ and $(A-C)^\dagger = (A-C)^{-1}$. We obtain

$$\begin{aligned} X &= Q_1 + \frac{1}{2}\Big[\left\{(A+C)^{-1}(E+F) + (A-C)^{-1}(E-F)\right\} B^{-1} \\ &\quad - \left\{(A+C)^{-1}(A+C)(Q_1+Q_2) + (A-C)^{-1}(A-C)(Q_1-Q_2)\right\} BB^{-1}\Big] \\ &= \frac{1}{2}\Big[(A+C)^{-1}(E+F) + (A-C)^{-1}(E-F)\Big]B^{-1}. \\ Y &= Q_2 + \frac{1}{2}\Big[\left\{(A+C)^{-1}(E+F) - (A-C)^{-1}(E-F)\right\} B^{-1} \\ &\quad - \left\{(A+C)^{-1}(A+C)(Q_1+Q_2) - (A-C)^{-1}(A-C)(Q_1-Q_2)\right\} BB^{-1}\Big] \\ &= \frac{1}{2}\Big[(A+C)^{-1}(E+F) - (A-C)^{-1}(E-F)\Big]B^{-1}. \end{aligned}$$

□

Corollary 4.7 under the restrict condition $AB = BA$ was obtained in [8, Theorem 4.10].

Corollary 4.8. Let $A, C \in M_n(\mathbb{C})$, $B \in M_p(\mathbb{C})$ and $E, F \in M_{n,p}(\mathbb{C})$. If A and B are invertible, then the system (4.8) has a unique solution if and only if $A - CA^{-1}C$ is invertible.

Proof. The proof is similar to that of Theorem 4.2. The system (4.8) is equivalent to vector-matrix equation

$$\begin{bmatrix} B^T \otimes A & B^T \otimes C \\ B^T \otimes C & B^T \otimes A \end{bmatrix} \begin{bmatrix} \mathrm{Vec}\, X \\ \mathrm{Vec}\, Y \end{bmatrix} = \begin{bmatrix} \mathrm{Vec}\, E \\ \mathrm{Vec}\, F \end{bmatrix}.$$

The system (4.8) has a unique solution if and only if

$$\begin{bmatrix} B^T \otimes A & B^T \otimes C \\ B^T \otimes C & B^T \otimes A \end{bmatrix} \tag{4.9}$$

is invertible. Since A and B are invertible, so is $B^T \otimes A$. Then Lemma 2.9 implies that the invertibility of block matrices (4.9) is equivalent to the invertibility of

$$(B^T \otimes A) - \Big[(B^T \otimes C)(B^T \otimes A)^{-1}(B^T \otimes C)\Big],$$

which can be reduced to that $A - CA^{-1}C$ by Lemma 2.9. □

Corollary 4.9. Let $A, C, E, F \in M_n(\mathbb{C})$. Consider the following coupled linear matrix equations

$$\begin{aligned} AX + CY &= E, \\ CX + AY &= F. \end{aligned} \tag{4.10}$$

This system has a unique solution if and only if $A + C$ and $A - C$ are invertible. In this case, the unique solution is given by

$$\begin{aligned} X &= \frac{1}{2}\Big[(A+C)^{-1}(E+F) + (A-C)^{-1}(E-F)\Big], \\ Y &= \frac{1}{2}\Big[(A+C)^{-1}(E+F) - (A-C)^{-1}(E-F)\Big]. \end{aligned}$$

Proof. From Corollary 4.7, we set $B = I_p$. □

Corollary 4.10. Let $A, E, F \in M_n(\mathbb{C})$. Consider the following coupled linear matrix equations

$$\begin{aligned} AX + Y &= E, \\ X + AY &= F. \end{aligned} \tag{4.11}$$

The following statements are equivalent:

(i) The system (4.11) has a unique solution.

(ii) $A^2 - I_n$ is invertible.

(iii) 1 and -1 are not eigenvalues of A.

In this case, the unique solution is given by

$$\begin{aligned} X &= \frac{1}{2}\Big[(A+I_n)^{-1}(E+F) + (A-I_n)^{-1}(E-F)\Big], \\ Y &= \frac{1}{2}\Big[(A+I_n)^{-1}(E+F) - (A-I_n)^{-1}(E-F)\Big]. \end{aligned}$$

Proof. It follows directly from Theorem 4.1 by setting $B = C = D = I_n$. Note that the statements (ii) and (iii) are equivalent since $A^2 - I_n = (A - I_n)(A + I_n)$.

The formula of the unique solution can be obtained by substituting $B = C = D = I_n$. Indeed, we get

$$\begin{aligned}
\operatorname{Vec} X &= \frac{1}{2}\Big[(I_n \otimes (A+I_n))^{-1}(\operatorname{Vec} E + \operatorname{Vec} F) + (I_n \otimes (A-I_n))^{-1}(\operatorname{Vec} E - \operatorname{Vec} F)\Big] \\
&= \frac{1}{2}\Big[(I_n \otimes (A+I_n))^{-1}\operatorname{Vec}(E+F) + (I_n \otimes (A-I_n))^{-1}\operatorname{Vec}(E-F)\Big] \\
&= \operatorname{Vec}\Big[\frac{1}{2}\Big[(A+I_n)^{-1}(E+F) + (A-I_n)^{-1}(E-F)\Big]\Big].
\end{aligned}$$

The injectivity of the vector operator implies that

$$X = \frac{1}{2}\Big[(A+I_n)^{-1}(E+F) + (A-I_n)^{-1}(E-F)\Big].$$

Likewise,

$$\begin{aligned}
\operatorname{Vec} Y &= \frac{1}{2}\Big[(I_n \otimes (A+I_n))^{-1}(\operatorname{Vec} E + \operatorname{Vec} F) - (I_n \otimes (A-I_n))^{-1}(\operatorname{Vec} E - \operatorname{Vec} F)\Big] \\
&= \frac{1}{2}\Big[(I_n \otimes (A+I_n))^{-1}\operatorname{Vec}(E+F) - (I_n \otimes (A-I_n))^{-1}\operatorname{Vec}(E-F)\Big] \\
&= \operatorname{Vec}\Big[\frac{1}{2}\Big[(A+I_n)^{-1}(E+F) - (A-I_n)^{-1}(E-F)\Big]\Big].
\end{aligned}$$

Hence,

$$Y = \frac{1}{2}\Big[(A+I_n)^{-1}(E+F) - (A-I_n)^{-1}(E-F)\Big].$$

by the injectivity of Vec. □

Chapter 5

Numerical Examples

In this chapter, we provide numerical examples for the main results obtained in Sections 3.1 and 4.1. We use MATLAB for matrix computations, such as basic matrix operations, ordinary inverses, Moore-Penrose inverses, Kronecker products, vector operators, and computing ranks.

Example 5.1. Consider the following coupled linear matrix equations :

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned} \tag{5.1}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad F = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

We shall investigate the existence of solution for system (5.1). We have

$$\begin{aligned} P &= B^T \otimes A + D^T \otimes C \\ &= \begin{bmatrix} 3 & -3 & 4 & -1 & -12 & 3 \\ -3 & 7 & -2 & 3 & 6 & -9 \end{bmatrix}, \\ Q &= B^T \otimes A - D^T \otimes C \\ &= \begin{bmatrix} -7 & -5 & -6 & -3 & -18 & 9 \\ -1 & 9 & 0 & 5 & 0 & -15 \end{bmatrix}. \end{aligned}$$

Then $\operatorname{rank} P = 2$ and $\operatorname{rank} Q = 2$. On the other hand

$$\begin{aligned} &\operatorname{rank} \left[\begin{array}{cc|c} B^T \otimes A & D^T \otimes C & \operatorname{Vec} E \\ D^T \otimes C & B^T \otimes A & \operatorname{Vec} F \end{array}\right] \\ &= \operatorname{rank} \left[\begin{array}{cccccccccccc|c} -2 & -4 & -1 & -2 & 3 & 6 & 5 & 1 & 5 & 1 & -15 & -3 & 1 \\ -2 & 8 & -1 & 4 & 3 & -12 & -1 & -1 & -1 & -1 & 3 & 3 & 3 \\ 5 & 1 & 5 & 1 & -15 & -3 & -2 & -4 & -1 & -2 & 3 & 6 & 2 \\ -1 & -1 & -1 & -1 & 3 & 3 & -2 & 8 & -1 & 4 & 3 & -12 & -1 \end{array}\right] \\ &= 4. \end{aligned}$$

Hence, the statement (ii) in Theorem 3.1 is satisfied. We conclude that the system (5.1) has a solution.

Alternatively, we can check the existence of (5.1) from the statement (iii) of

Theorem 3.1. Indeed, we have

$$P^\dagger = \begin{bmatrix} 0.0091 & -0.0091 \\ 0.0264 & 0.0569 \\ 0.3000 & 0.0117 \\ 0.0147 & 0.0269 \\ -0.0899 & -0.0351 \\ -0.0422 & -0.0808 \end{bmatrix} \quad \text{and} \quad Q^\dagger = \begin{bmatrix} -0.0181 & -0.0133 \\ 0.0002 & 0.0272 \\ -0.0144 & -0.0081 \\ -0.0004 & 0.0148 \\ 0.0431 & 0.0244 \\ 0.0012 & -0.0445 \end{bmatrix}.$$

It follows that

$$\begin{aligned} (2I_{mq} - (PP^\dagger + QQ^\dagger))\,\mathrm{Vec}\,E &= 0, \\ (PP^\dagger - QQ^\dagger)\,\mathrm{Vec}\,F &= 0, \\ (2I_{mq} - (PP^\dagger + QQ^\dagger))\,\mathrm{Vec}\,F &= 0, \\ (PP^\dagger - QQ^\dagger)\,\mathrm{Vec}\,E &= 0. \end{aligned}$$

Hence, the statement (iii) is satisfied. We get the general solution of (5.1) as follows :

$$\mathrm{Vec}\,X = \frac{1}{2}\left[(P^\dagger + Q^\dagger)\,\mathrm{Vec}\,E + (P^\dagger - Q^\dagger)\,\mathrm{Vec}\,F + (2I_{np} - (P^\dagger P + Q^\dagger Q))Q_1 - (P^\dagger P - Q^\dagger Q)Q_2\right]$$

$$= \begin{bmatrix} 1.8050q_1 + 0.1203q_2 - 0.1636q_3 + 0.0486q_4 + 0.4908q_5 - 0.1458q_6 + 0.0852w_1 + 0.0626w_2 + 0.0538w_3 + 0.0246w_4 - 0.1615w_5 - 0.0737w_6 - 0.0258 \\ 0.1203q_1 + 1.4369q_2 + 0.0095q_3 - 0.2798q_4 - 0.0285q_5 + 0.8394q_6 + 0.0626w_1 - 0.0751w_2 + 0.0067w_3 - 0.0088w_4 - 0.0202w_5 + 0.0264w_6 + 0.3018 \\ 0.0095q_2 - 0.1636q_1 + 1.8172q_3 - 0.0075q_4 + 0.5483q_5 + 0.0225q_6 + 0.0538w_1 + 0.0067w_2 - 0.0103w_3 - 0.0027w_4 + 0.0310w_5 + 0.0080w_6 + 0.0951 \\ 0.0486q_1 - 0.2798q_2 - 0.0075q_3 + 1.8586q_4 + 0.0225q_5 + 0.4243q_6 + 0.0246w_1 - 0.0088w_2 - 0.0027w_3 + 0.0093w_4 + 0.0080w_5 - 0.0279w_6 + 0.1578 \\ 0.4908q_1 - 0.0285q_2 + 0.5483q_3 + 0.0225q_4 + 0.3551q_5 - 0.0675q_6 - 0.1615w_1 - 0.0202w_2 + 0.0310w_3 + 0.0080w_4 - 0.0929w_5 - 0.0239w_6 - 0.2854 \\ 0.8394q_2 - 0.1458q_1 + 0.0225q_3 + 0.4243q_4 - 0.0675q_5 + 0.7272q_6 - 0.0737w_1 + 0.0264w_2 + 0.0080w_3 - 0.0279w_4 - 0.0239w_5 + 0.0838w_6 - 0.4734 \end{bmatrix}$$

and

$$\mathrm{Vec}\,Y = \frac{1}{2}\left[(P^\dagger - Q^\dagger)\,\mathrm{Vec}\,E + (P^\dagger + Q^\dagger)\,\mathrm{Vec}\,F - (P^\dagger P - Q^\dagger Q)Q_1 + (2I_{np} - (P^\dagger P + Q^\dagger Q))Q_2\right]$$

$$= \begin{bmatrix} 0.0852q_1 + 0.0626q_2 + 0.0538q_3 + 0.0246q_4 - 0.1615q_5 - 0.0737q_6 + 1.8050w_1 + 0.1203w_2 - 0.1636w_3 + 0.0486w_4 + 0.4908w_5 - 0.1458w_6 + 0.0441 \\ 0.0626q_1 - 0.0751q_2 + 0.0067q_3 - 0.0088q_4 - 0.0202q_5 + 0.0264q_6 + 0.1203w_1 + 1.4369w_2 + 0.0095w_3 - 0.2798w_4 - 0.0285w_5 + 0.8394w_6 + 0.0844 \\ 0.0538q_1 + 0.0067q_2 - 0.0103q_3 - 0.0027q_4 + 0.0310q_5 + 0.0080q_6 - 0.1636w_1 + 0.0095w_2 + 1.8172w_3 - 0.0075w_4 + 0.5483w_5 + 0.0225w_6 + 0.1315 \\ 0.0246q_1 - 0.0088q_2 - 0.0027q_3 + 0.0093q_4 + 0.0080q_5 - 0.0279q_6 + 0.0486w_1 - 0.2798w_2 - 0.0075w_3 + 1.8586w_4 + 0.0225w_5 + 0.4243w_6 + 0.0383 \\ 0.0310q_3 - 0.0202q_2 - 0.1615q_1 + 0.0080q_4 - 0.0929q_5 - 0.0239q_6 + 0.4908w_1 - 0.0285w_2 + 0.5483w_3 + 0.0225w_4 + 0.3551w_5 - 0.0675w_6 - 0.3945 \\ 0.0264q_2 - 0.0737q_1 + 0.0080q_3 - 0.0279q_4 - 0.0239q_5 + 0.0838q_6 - 0.1458w_1 + 0.8394w_2 + 0.0225w_3 + 0.4243w_4 - 0.0675w_5 + 0.7272w_6 - 0.1150 \end{bmatrix},$$

where $Q_1 = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{bmatrix}^T$ and $Q_2 = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{bmatrix}^T$.

Example 5.2. Consider the following coupled linear matrix equations :

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned} \tag{5.2}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \\ 1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & -2 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}.$$

We shall investigate the existence of solution for system (5.2). We have

$$P = B^T \otimes A + D^T \otimes C = \begin{bmatrix} -4 & 19 \\ 7 & 18 \\ 11 & 7 \\ 6 & -6 \\ -8 & -2 \\ 6 & 2 \\ 7 & 5 \\ -8 & 11 \\ 19 & 9 \end{bmatrix},$$

$$Q = B^T \otimes A - D^T \otimes C = \begin{bmatrix} 8 & -11 \\ -11 & -6 \\ 5 & 1 \\ -2 & 14 \\ 4 & 14 \\ 10 & 6 \\ 3 & 15 \\ -2 & 19 \\ 21 & 11 \end{bmatrix}.$$

Then rank $P = 2$ and rank $Q = 2$. On the other hand

$$
\begin{aligned}
\operatorname{rank}\left[\begin{array}{cc|c} B^T \otimes A & D^T \otimes C & \operatorname{Vec} E \\ D^T \otimes C & B^T \otimes A & \operatorname{Vec} F \end{array}\right] &= \operatorname{rank}\left[\begin{array}{cccc|c}
2 & 4 & -6 & 15 & 1 \\
-2 & 6 & 9 & 12 & 4 \\
8 & 4 & 3 & 3 & 7 \\
2 & 4 & 4 & -10 & 2 \\
-2 & 6 & -6 & -8 & 5 \\
8 & 4 & -2 & -2 & 8 \\
5 & 10 & 2 & -5 & 3 \\
-5 & 15 & -3 & -4 & 6 \\
20 & 10 & -1 & -1 & 9 \\
-6 & 15 & 2 & 4 & 1 \\
9 & 12 & -2 & 6 & 2 \\
3 & 3 & 8 & 4 & 3 \\
4 & -10 & 2 & 4 & 1 \\
-6 & -8 & -2 & 6 & 2 \\
-2 & -2 & 8 & 4 & 3 \\
2 & -5 & 5 & 10 & 1 \\
-3 & -4 & -5 & 15 & 2 \\
-1 & -1 & 20 & 10 & 3
\end{array}\right] \\
&= 5.
\end{aligned}
$$

Hence, the statement (ii) in Theorem 3.1 is not satisfied. We conclude that the system (5.2) has no a solution.

Example 5.3. Consider the following coupled linear matrix equations :

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned} \qquad (5.3)$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & -4 \\ 3 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 2 \\ 3 & 7 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & -5 \\ -1 & 9 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We can check the existence of (5.3) from the statement (iii) of Theorem 3.1. Indeed, we have

$$\begin{aligned} P &= B^T \otimes A + D^T \otimes C \\ &= \begin{bmatrix} -6 & -8 & 4 & 4 \\ -2 & 27 & 0 & -19 \\ 1 & -18 & -12 & 12 \\ -19 & -19 & 24 & 75 \end{bmatrix} \end{aligned}$$

and

$$P^{-1} = \begin{bmatrix} -0.1360 & -0.0790 & -0.0537 & -0.0042 \\ -0.0435 & 0.0355 & 0.0062 & 0.0103 \\ 0.0064 & -0.0537 & -0.0826 & -0.0007 \\ -0.0475 & 0.0062 & 0.0144 & 0.0151 \end{bmatrix}.$$

Likewise,

$$\begin{aligned} Q &= B^T \otimes A - D^T \otimes C \\ &= \begin{bmatrix} -4 & -12 & 2 & 8 \\ -8 & 13 & 6 & -5 \\ -9 & 2 & 6 & -24 \\ 11 & 51 & -30 & -51 \end{bmatrix}, \end{aligned}$$

$$Q^{-1} = \begin{bmatrix} -0.1898 & -0.0743 & 0.0075 & -0.0260 \\ -0.0297 & 0.0446 & -0.0238 & 0.0022 \\ -0.1662 & -0.0035 & 0.0289 & -0.0393 \\ 0.0272 & 0.0307 & -0.0392 & 0.0001 \end{bmatrix}.$$

Since P and Q are invertible, we can substituted $P^{\dagger} = P^{-1}$ and $Q^{\dagger} = Q^{-1}$ into the statement (iii) in Theorem 3.1. We get

$$\begin{aligned}
(2I_{mq} - (PP^{\dagger} + QQ^{\dagger}))\,\mathrm{Vec}\,E &= 0,\\
(PP^{\dagger} - QQ^{\dagger})\,\mathrm{Vec}\,F &= 0,\\
(2I_{mq} - (PP^{\dagger} + QQ^{\dagger}))\,\mathrm{Vec}\,F &= 0,\\
(PP^{\dagger} - QQ^{\dagger})\,\mathrm{Vec}\,E &= 0.
\end{aligned}$$

Hence, the statement (iii) is satisfied. We conclude that the system (5.3) has a solution. Next, we know that the uniqueness of solution for the system (5.3) from P and Q are invertible. Then the unique solution of (5.3) as follows :

$$\begin{aligned}
\mathrm{Vec}\,X &= \frac{1}{2}\Big[P^{-1}(\mathrm{Vec}\,E + \mathrm{Vec}\,F) + Q^{-1}(\mathrm{Vec}\,E - \mathrm{Vec}\,F)\Big]\\
&= \begin{bmatrix} -0.2756\\ 0.0266\\ -0.1787\\ 0.0129 \end{bmatrix},\\
X &= \begin{bmatrix} -0.2756 & -0.1787\\ 0.0266 & 0.0129 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Vec}\,Y &= \frac{1}{2}\Big[P^{-1}(\mathrm{Vec}\,E + \mathrm{Vec}\,F) - Q^{-1}(\mathrm{Vec}\,E - \mathrm{Vec}\,F)\Big]\\
&= \begin{bmatrix} -0.4947\\ 0.0986\\ -0.2748\\ 0.0354 \end{bmatrix},\\
Y &= \begin{bmatrix} -0.4947 & -0.2748\\ 0.0986 & 0.0354 \end{bmatrix}.
\end{aligned}$$

Example 5.4. Consider the following coupled linear matrix equations :

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned} \tag{5.4}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}, \ B = \begin{bmatrix} -5 & 1 \\ 15 & -3 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}, \ E = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \ F = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

We will investigate the uniqueness of solution for the system (5.4) from P and Q are invertible. We get

$$\begin{aligned} P &= B^T \otimes A + D^T \otimes C \\ &= \begin{bmatrix} -3 & -8 & 21 & 36 \\ 8 & 13 & -6 & -21 \\ 1 & 2 & 7 & 4 \\ -1 & -2 & 18 & 21 \end{bmatrix}, \\ Q &= B^T \otimes A - D^T \otimes C \\ &= \begin{bmatrix} -7 & -12 & 9 & 24 \\ 2 & 7 & -24 & -39 \\ 1 & 2 & -13 & -16 \\ -1 & -2 & -12 & -9 \end{bmatrix}. \end{aligned}$$

We see that P and Q are not invertible, by Theorem 4.1 it is not true that this system has a unique solution. That is, this system has no solution or infinitely many solutions.

Next, we can check it by using Theorem 3.1 from the statement (ii). Indeed, we get

$$\operatorname{rank} P = \operatorname{rank} \begin{bmatrix} -3 & -8 & 21 & 36 \\ 8 & 13 & -6 & -21 \\ 1 & 2 & 7 & 4 \\ -1 & -2 & 18 & 21 \end{bmatrix} = 3$$

and

$$\operatorname{rank} Q = \operatorname{rank} \begin{bmatrix} -7 & -12 & 9 & 24 \\ 2 & 7 & -24 & -39 \\ 1 & 2 & -13 & -16 \\ -1 & -2 & -12 & -9 \end{bmatrix} = 3.$$

In other direction,

$$\operatorname{rank}\left[\begin{array}{cc|c} B^T\otimes A & D^T\otimes C & \operatorname{Vec} E \\ D^T\otimes C & B^T\otimes A & \operatorname{Vec} F \end{array}\right]$$

$$= \operatorname{rank}\left[\begin{array}{cccccccc|c} -5 & -10 & 15 & 30 & 2 & 2 & 6 & 6 & 2 \\ 5 & 10 & -15 & -30 & 3 & 3 & 9 & 9 & -1 \\ 1 & 2 & -3 & -6 & 0 & 0 & 10 & 10 & -1 \\ -1 & -2 & 3 & 6 & 0 & 0 & 15 & 15 & 2 \\ 2 & 2 & 6 & 6 & -5 & -10 & 15 & 30 & 1 \\ 3 & 3 & 9 & 9 & 5 & 10 & -15 & -30 & 0 \\ 0 & 0 & 10 & 10 & 1 & 2 & -3 & -6 & 2 \\ 0 & 0 & 15 & 15 & -1 & -2 & 3 & 6 & 1 \end{array}\right]$$

$$= \quad 7.$$

Hence, the statement (ii) in Theorem 3.1 is not satisfied. We conclude that the system (5.4) has no a solution.

Example 5.5. Consider the following coupled linear matrix equations :

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned} \tag{5.5}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & -4 \\ 3 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 2 \\ 3 & 7 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & -5 \\ -1 & 9 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We will investigate the uniqueness of solution for the system (5.5). In deed, we get

$$A^{-1} = \begin{bmatrix} 0.6667 & 0.3333 \\ 0.1667 & -0.1667 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} -0.1111 & 0.1481 \\ -0.1111 & -0.1852 \end{bmatrix}.$$

By Theorem 4.2, the system (5.5) has a unique solution if and only if

$$B^T \otimes A - ((DB^{-1}D)^T \otimes CA^{-1}C)$$

is invertible. Next, we will find inverse of it. We obtain

$$\left[B^T \otimes A - ((DB^{-1}D)^T \otimes CA^{-1}C)\right]^{-1} = \begin{bmatrix} -0.1629 & -0.0767 & -0.0231 & -0.0151 \\ -0.0366 & 0.0401 & -0.0088 & 0.0063 \\ -0.0799 & -0.0286 & -0.0269 & -0.0200 \\ -0.0102 & 0.0184 & -0.0124 & 0.0076 \end{bmatrix}.$$

Hence, we conclude that the system 5.5 has a unique solution.

Chapter 6

Conclusions

The summary of the main results for this thesis is as follows. Consider the following system of coupled linear matrix equations:

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned} \tag{6.1}$$

where $A, C \in M_{m,n}(\mathbb{C})$, $B, D \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{m,q}(\mathbb{C})$.

1. The existence criteria of the system are

(ii) $\operatorname{rank}\left[\begin{array}{cc|c} B^T \otimes A & D^T \otimes C & \operatorname{Vec} E \\ D^T \otimes C & B^T \otimes A & \operatorname{Vec} F \end{array}\right] = \operatorname{rank} P + \operatorname{rank} Q.$

(iii) $(2I_{mq} - (PP^\dagger + QQ^\dagger)) \operatorname{Vec} E = (PP^\dagger - QQ^\dagger) \operatorname{Vec} F$ and
$(2I_{mq} - (PP^\dagger + QQ^\dagger)) \operatorname{Vec} F = (PP^\dagger - QQ^\dagger) \operatorname{Vec} E.$

In this case, the general solution is given by

$$\begin{aligned} \operatorname{Vec} X &= \frac{1}{2}\left[(P^\dagger + Q^\dagger)\operatorname{Vec} E + (P^\dagger - Q^\dagger)\operatorname{Vec} F + (2I_{np} - (P^\dagger P + Q^\dagger Q))q_1 - (P^\dagger P - Q^\dagger Q)q_2\right], \\ \operatorname{Vec} Y &= \frac{1}{2}\left[(P^\dagger - Q^\dagger)\operatorname{Vec} E + (P^\dagger + Q^\dagger)\operatorname{Vec} F - (P^\dagger P - Q^\dagger Q)q_1 + (2I_{np} - (P^\dagger P + Q^\dagger Q))q_2\right], \end{aligned}$$

where $P = B^T \otimes A + D^T \otimes C$, $Q = B^T \otimes A - D^T \otimes C$ and $q_1, q_2 \in \mathbb{C}^{np}$ are arbitrary.

2. When $C = A$, the system has a solution if and only if one of the following statements holds:

(ii) $\operatorname{rank}\left[\begin{array}{cc|c} B^T \otimes A & D^T \otimes A & \operatorname{Vec} E \\ D^T \otimes A & B^T \otimes A & \operatorname{Vec} F \end{array}\right] = (\operatorname{rank} A)\,(\operatorname{rank}(B+D) + \operatorname{rank}(B-D)).$

(iii) The following two conditions hold:

$$\begin{aligned} AA^\dagger\left[(E+F)(B+D)^\dagger(B+D) + (E-F)(B-D)^\dagger(B-D)\right] &= 2E, \\ AA^\dagger\left[(E+F)(B+D)^\dagger(B+D) - (E-F)(B-D)^\dagger(B-D)\right] &= 2F. \end{aligned}$$

In this case, the general solution is given by

$$\begin{aligned} X = Q_1 + \frac{1}{2}\Big[&A^\dagger\left\{(E+F)(B+D)^\dagger + (E-F)(B-D)^\dagger\right\} \\ &- A^\dagger A\left\{(Q_1+Q_2)(B+D)(B+D)^\dagger + (Q_1-Q_2)(B-D)(B-D)^\dagger\right\}\Big], \\ Y = Q_2 + \frac{1}{2}\Big[&A^\dagger\left\{(E+F)(B+D)^\dagger - (E-F)(B-D)^\dagger\right\} \\ &- A^\dagger A\left\{(Q_1+Q_2)(B+D)(B+D)^\dagger - (Q_1-Q_2)(B-D)(B-D)^\dagger\right\}\Big], \end{aligned}$$

where Q_1, $Q_2 \in M_{n,p}(\mathbb{C})$ are arbitrary.

3. When A, B, C, D, E and F are square matrices, this system has a unique solution if and only if P and Q are invertible. In this case, the unique solution is given by

$$\begin{aligned}\operatorname{Vec} X &= \frac{1}{2}\Big[P^{-1}(\operatorname{Vec} E + \operatorname{Vec} F) + Q^{-1}(\operatorname{Vec} E - \operatorname{Vec} F)\Big],\\ \operatorname{Vec} Y &= \frac{1}{2}\Big[P^{-1}(\operatorname{Vec} E + \operatorname{Vec} F) - Q^{-1}(\operatorname{Vec} E - \operatorname{Vec} F)\Big].\end{aligned}$$

4. When $A, C \in M_n(\mathbb{C})$, $B, D \in M_p(\mathbb{C})$, $E, F \in M_{n,p}(\mathbb{C})$ and A, B are invertible, the system (6.1) has a unique solution if and only if

$$B^T \otimes A - ((DB^{-1}D)^T \otimes CA^{-1}C)$$

is invertible.

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Appendix

Appendix A

The research paper

Existence and Uniqueness of Solutions for Systems of Coupled Linear Matrix Equations

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Abstract

We investigate systems of coupled linear matrix equations. Criterions for existence and uniqueness of solutions for such systems are obtained in terms of Kronecker products, vector operator, Moore-Penrose inverses and ranks. Moreover, explicit formulas of solutions are presented.

Mathematics Subject Classification: 15A09, 15A24, 15A69
Keywords: linear matrix equation, Kronecker product, vector operator, Moore-Penrose inverse

1 Introduction

Theory of linear matrix equations can be applied in many subareas of science and engineering such as system and control theory, image processing, transportation problems and quantum mechanics (see, e.g., [1, 4, 5, 11]). The current research of linear matrix equations can be classified into three topics. The first one is to investigate necessary and sufficient conditions for existence and uniqueness of certain linear matrix equations and then derive exact formulas for solutions (see, e.g., [2, 3, 8, 12, 16]). The second one is to discuss least-squares solutions for linear matrix equations. (see, e.g., [9, 14]). The last one is to establish iterative processes for solving linear matrix equations and deduce their convergence analysis (see, e.g., [6, 7, 13, 15]).

In this paper, we investigate existence and uniqueness of certain systems of coupled linear matrix equations. Our main system to consider takes the form

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned} \tag{1.1}$$

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Annual Pure and Applied Mathematics Conference 2017 (APAM 2017)
Chulalongkorn University, 28-30 June, 2017

where A, B, C, D, E and F are complex matrices and X and Y are unknown complex matrices. We use the Kronecker products and vector operator to reduce the system (1.1) to a simple vector-matrix equation. Then we obtain necessary and sufficient conditions for existence and uniqueness of the system (1.1). These conditions rely on Kronecker products, vector operator, Moore-Penrose inverses and ranks. An interesting special case of (1.1) is given by

$$\begin{aligned} AX + YB &= E, \\ XB + AY &= F \end{aligned} \tag{1.2}$$

which is known as a system of Sylvester matrix equations. Finally, we obtain explicit formulas of solutions in system (1.1). Certain interesting special cases of (1.1) are also investigated. Note that our result includes (1.2) as a special case.

2 Preliminaries

In this section, we provide basic tools for solving linear matrix equations. These tools include Kronecker product, vector operator and Moore-Penrose inverses. Let $M_{m,n}(\mathbb{C})$ be the set of m-by-n complex matrices. For simplicity, abbreviate $M_{n,n}(\mathbb{C})$ to $M_n(\mathbb{C})$.

Let $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$. The Kronecker product of A and B defined by

$$\begin{aligned} A \otimes B &= [a_{ij}B] \\ &= \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in M_{mp,nq}(\mathbb{C}). \end{aligned}$$

Lemma 2.1 (see, e.g., [10]). *The Kronecker product satisfies the following properties (provided that every operation is well-defined):*

(1) Compatibility with addition: $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$ *and* $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$.

(2) Compatibility with scalar multiplication: $A \otimes (\alpha B) = \alpha(A \otimes B) = (\alpha A) \otimes B$ *for any* $\alpha \in \mathbb{C}$.

(3) Compatibility with transpose: $(A \otimes B)^T = A^T \otimes B^T$.

(4) Mixed product property: $(A \otimes B)(C \otimes D) = AC \otimes BD$.

(5) Compatibility with inverse: $A \otimes B$ *is invertible if and only if both* A *and* B *are invertible, in which case* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

(6) $\operatorname{rank}(A \otimes B) = (\operatorname{rank} A)(\operatorname{rank} B)$.

The vector operator is a column-stacking operator assigned to a matrix $A \in M_{m,n}(\mathbb{C})$ by

$$\operatorname{Vec} A = [a_{11}, a_{12}, \ldots, a_{m1}, a_{12}, a_{22}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn}]^T \in \mathbb{C}^{mn}.$$

This operator is clearly a linear isomorphism.

Lemma 2.2 (see, e.g., [10]). *The vector operator concerned Kronecker product as follows:*

$$\operatorname{Vec}(AXB) = (B^T \otimes A) \operatorname{Vec} X,$$

provided that every operation is well-defined.

A matrix $A \in M_{m,n}(\mathbb{C})$ possess its Moore-Penrose inverse, which is the matrix $A^\dagger \in M_{n,m}(\mathbb{C})$ and A^* denote the Hermitian of matrix A satisfying the following conditions:

(i) $AA^\dagger A = A$,

(ii) $A^\dagger AA^\dagger = A^\dagger$,

(iii) $(AA^\dagger)^* = AA^\dagger$,

(iv) $(A^\dagger A)^* = A^\dagger A$.

If $A \in M_n(\mathbb{C})$ is invertible, then $A^\dagger = A^{-1}$. A simple and accurate way to compute Moore-Penrose inverse is to use the singular value decomposition. Indeed, for any $A \in M_{m,n}(\mathbb{C})$ we can decompose

$$A = U\Sigma V^*,$$

where $U \in M_m(\mathbb{C})$ is unitary, $V \in M_n(\mathbb{C})$ is unitary and $\Sigma = [d_{ij}] \in M_{m,n}(\mathbb{C})$ is a diagonal matrix with nonnegative entries. Then,

$$A^\dagger = V\Sigma^\dagger U^*,$$

where $\Sigma^\dagger = [d^\dagger_{ij}]$ is defined by $d^\dagger_{ij} = \begin{cases} d_{ij}^{-1}, & d_{ij} \neq 0 \\ 0, & d_{ij} = 0. \end{cases}$

Lemma 2.3 (see, e.g., [10]). *Let $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$. Then,*

(i) $(A^\dagger)^T = (A^T)^\dagger$,

(ii) $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

Lemma 2.4 (see, e.g., [10]). *Let $A \in M_{m,n}(\mathbb{C})$ and $b \in \mathbb{C}^m$. The following statements are equivalent:*

(i) The vector-matrix equation $Ax = b$ has a solution x.

(ii) $\operatorname{rank}[A \mid b] = \operatorname{rank} A$.

(iii) $AA^\dagger b = b$.

In the above case, the general solution is given by $x = A^\dagger b + (I - A^\dagger A)q$ where $q \in \mathbb{C}^m$ is arbitrary.

3 Criterions for existence and uniqueness of solutions for systems of coupled linear matrix equations

In this section, we investigate existence and uniqueness of solution for the system (1.1) and its interesting special cases. Exact formula of the unique solution is explicitly presented.

Theorem 3.1. *Let $A, C \in M_{m,n}(\mathbb{C})$, $B, D \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{m,q}(\mathbb{C})$. Denote*

$$H = \begin{bmatrix} B^T \otimes A & D^T \otimes C \\ D^T \otimes C & B^T \otimes A \end{bmatrix}, \quad P = B^T \otimes A + D^T \otimes C \text{ and } Q = B^T \otimes A - D^T \otimes C.$$

Then, the following statements are equivalent:

(i) The system (1.1) *has a solution.*

(ii) $\operatorname{rank} P + \operatorname{rank} Q = \operatorname{rank} \left[\begin{array}{cc|c} B^T \otimes A & D^T \otimes C & \operatorname{Vec} E \\ D^T \otimes C & B^T \otimes A & \operatorname{Vec} F \end{array}\right]$.

(iii) $(2I_{mq} - (PP^\dagger + QQ^\dagger)) \operatorname{Vec} E = (PP^\dagger - QQ^\dagger) \operatorname{Vec} F$ *and*
$(2I_{mq} - (PP^\dagger + QQ^\dagger)) \operatorname{Vec} F = (PP^\dagger - QQ^\dagger) \operatorname{Vec} E$.

In the above case, the general solution is given by

$$\operatorname{Vec} X = \frac{1}{2}\left[(P^\dagger + Q^\dagger)\operatorname{Vec} E + (P^\dagger - Q^\dagger)\operatorname{Vec} F + (2I_{np} - (P^\dagger P + Q^\dagger Q))q_1 - (P^\dagger P - Q^\dagger Q)q_2\right],$$
$$\operatorname{Vec} Y = \frac{1}{2}\left[(P^\dagger - Q^\dagger)\operatorname{Vec} E + (P^\dagger + Q^\dagger)\operatorname{Vec} F - (P^\dagger P - Q^\dagger Q)q_1 + (2I_{np} - (P^\dagger P + Q^\dagger Q))q_2\right].$$

where $q_1, q_2 \in \mathbb{C}^{np}$ *are arbitrary.*

Proof. Taking the vector operator to (1.1) and then using Lemma 2.2, we get

$$(B^T \otimes A)\operatorname{Vec} X + (D^T \otimes C)\operatorname{Vec} Y = \operatorname{Vec} E,$$
$$(D^T \otimes C)\operatorname{Vec} X + (B^T \otimes A)\operatorname{Vec} Y = \operatorname{Vec} F.$$

For convenience, let us denote

$$x = \begin{bmatrix}\operatorname{Vec} X\\ \operatorname{Vec} Y\end{bmatrix},\quad b = \begin{bmatrix}\operatorname{Vec} E\\ \operatorname{Vec} F\end{bmatrix},\quad U = \frac{1}{\sqrt{2}}\begin{bmatrix} I_{mq} & -I_{mq}\\ I_{mq} & I_{mq}\end{bmatrix},\quad V = \frac{1}{\sqrt{2}}\begin{bmatrix} I_{np} & I_{np}\\ -I_{np} & I_{np}\end{bmatrix}.$$

The system (1.1) is equivalent to the vector-matrix equation $Hx = b$ due to the injectivity of the vector operator. One can decompose H as follows:

$$\begin{aligned} H &= U\begin{bmatrix} B^T \otimes A + D^T \otimes C & 0\\ 0 & B^T \otimes A - D^T \otimes C\end{bmatrix} V\\ &= U\begin{bmatrix} P & 0\\ 0 & Q\end{bmatrix} V. \end{aligned} \tag{3.1}$$

Since U and V are invertible, we have

$$\operatorname{rank} H = \operatorname{rank}\begin{bmatrix} P & 0\\ 0 & Q\end{bmatrix} = \operatorname{rank} P + \operatorname{rank} Q.$$

Hence, the conditions *(i)* and *(ii)* are equivalent. On the other hand, according to Lemma 2.4, the system (1.1) is consistent if and only if $HH^\dagger b = b$. One can compute

$$\begin{aligned} HH^\dagger b &= U \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} VV^* \begin{bmatrix} P^\dagger & 0 \\ 0 & Q^\dagger \end{bmatrix} U^* \begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} PP^\dagger + QQ^\dagger & PP^\dagger - QQ^\dagger \\ PP^\dagger - QQ^\dagger & PP^\dagger + QQ^\dagger \end{bmatrix} \begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} (PP^\dagger + QQ^\dagger) \operatorname{Vec} E + (PP^\dagger - QQ^\dagger) \operatorname{Vec} F \\ (PP^\dagger - QQ^\dagger) \operatorname{Vec} E + (PP^\dagger + QQ^\dagger) \operatorname{Vec} F \end{bmatrix}. \end{aligned}$$

Hence, the conditions *(i)* and *(iii)* are equivalent.

If the solution x exists, Lemma 2.4 ensure that it must be in the form

$$x = H^\dagger b + (I_{2np} - H^\dagger H) q \ ,$$

where $q \in \mathbb{C}^{2np}$ is arbitrary. Setting $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ when $q_1, q_2 \in \mathbb{C}^{np}$, we have

$$\begin{aligned} x &= V^* \begin{bmatrix} P^\dagger & 0 \\ 0 & Q^\dagger \end{bmatrix} U^* \begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix} + \left(\begin{bmatrix} I_{np} & 0 \\ 0 & I_{np} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} P^\dagger P + Q^\dagger Q & P^\dagger P - Q^\dagger Q \\ P^\dagger P - Q^\dagger Q & P^\dagger P + Q^\dagger Q \end{bmatrix} \right) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} (P^\dagger + Q^\dagger) \operatorname{Vec} E + (P^\dagger - Q^\dagger) \operatorname{Vec} F \\ (P^\dagger - Q^\dagger) \operatorname{Vec} E + (P^\dagger + Q^\dagger) \operatorname{Vec} F \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (2I_{np} - (P^\dagger P + Q^\dagger Q))\, q_1 - (P^\dagger P - Q^\dagger Q) q_2 \\ -(P^\dagger P - Q^\dagger Q) q_1 + (2I_{np} - (P^\dagger P + Q^\dagger Q))\, q_2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} (P^\dagger + Q^\dagger) \operatorname{Vec} E + (P^\dagger - Q^\dagger) \operatorname{Vec} F + (2I_{np} - (P^\dagger P + Q^\dagger Q))\, q_1 - (P^\dagger P - Q^\dagger Q) q_2 \\ (P^\dagger - Q^\dagger) \operatorname{Vec} E + (P^\dagger + Q^\dagger) \operatorname{Vec} F - (P^\dagger P - Q^\dagger Q) q_1 + (2I_{np} - (P^\dagger P + Q^\dagger Q))\, q_2 \end{bmatrix}. \end{aligned}$$

Therefore, the general (vector) solution of system (1.1) is given by

$$\begin{aligned} \operatorname{Vec} X &= \frac{1}{2} \left[(P^\dagger + Q^\dagger) \operatorname{Vec} E + (P^\dagger - Q^\dagger) \operatorname{Vec} F + (2I_{np} - (P^\dagger P + Q^\dagger Q)) q_1 - (P^\dagger P - Q^\dagger Q) q_2 \right], \\ \operatorname{Vec} Y &= \frac{1}{2} \left[(P^\dagger - Q^\dagger) \operatorname{Vec} E + (P^\dagger + Q^\dagger) \operatorname{Vec} F - (P^\dagger P - Q^\dagger Q) q_1 + (2I_{np} - (P^\dagger P + Q^\dagger Q)) q_2 \right]. \end{aligned}$$

□

Theorem 3.2. *Let $A, B, C, D, E, F \in M_n(\mathbb{C})$. Denote $P = B^T \otimes A + D^T \otimes C$ and $Q = B^T \otimes A - D^T \otimes C$. Then the system* (1.1) *has a unique solution if and only if P and Q* *are invertible. In this case, the unique solution is given by*

$$\begin{aligned}\operatorname{Vec} X &= \frac{1}{2}\Big[P^{-1}(\operatorname{Vec} E + \operatorname{Vec} F) + Q^{-1}(\operatorname{Vec} E - \operatorname{Vec} F)\Big],\\ \operatorname{Vec} Y &= \frac{1}{2}\Big[P^{-1}(\operatorname{Vec} E + \operatorname{Vec} F) - Q^{-1}(\operatorname{Vec} E - \operatorname{Vec} F)\Big].\end{aligned}$$

Proof. The system (1.1) is equivalent to the vector-matrix equation $Hx = b$. The decomposition (3.1) says that the system (1.1) has a unique solution if and only if both P and Q are invertible. To obtain the unique solution of (1.1), we substitute $P^{\dagger} = P^{-1}$ and $Q^{\dagger} = Q^{-1}$ into the general solution of Theorem 3.1, and we thus obtain

$$\begin{aligned}\operatorname{Vec} X &= \frac{1}{2}\Big[(P^{-1} + Q^{-1})\operatorname{Vec} E + (P^{-1} - Q^{-1})\operatorname{Vec} F\\ &\quad + (2I_{n^2} - (P^{-1}P + Q^{-1}Q))q_1 - (P^{-1}P - Q^{-1}Q)q_2\Big]\\ &= \frac{1}{2}\Big[P^{-1}\operatorname{Vec} E + Q^{-1}\operatorname{Vec} E + P^{-1}\operatorname{Vec} F - Q^{-1}\operatorname{Vec} F\Big]\\ &= \frac{1}{2}\Big[P^{-1}(\operatorname{Vec} E + \operatorname{Vec} F) + Q^{-1}(\operatorname{Vec} E - \operatorname{Vec} F)\Big].\end{aligned}$$

Similarly, we get the above formula of $\operatorname{Vec} Y$. □

Theorem 3.3. *Let $A \in M_{m,n}(\mathbb{C})$, $B, D \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{m,q}(\mathbb{C})$. Consider the following coupled linear matrix equations*

$$\begin{aligned}AXB + AYD &= E,\\ AXD + AYB &= F.\end{aligned} \tag{3.2}$$

Then, the following statements are equivalent:

(i) The system (3.2) *has a solution.*

(ii) $(\operatorname{rank} A)\,(\operatorname{rank}(B + D) + \operatorname{rank}(B - D)) = \operatorname{rank}\begin{bmatrix} B^T \otimes A & D^T \otimes A & \operatorname{Vec} E\\ D^T \otimes A & B^T \otimes A & \operatorname{Vec} F\end{bmatrix}.$

(iii) The following two conditions hold:

$$AA^{\dagger}\left[(E + F)(B + D)^{\dagger}(B + D) + (E - F)(B - D)^{\dagger}(B - D)\right] = 2E, \tag{3.3}$$

$$AA^{\dagger}\left[(E + F)(B + D)^{\dagger}(B + D) - (E - F)(B - D)^{\dagger}(B - D)\right] = 2F. \tag{3.4}$$

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In the above case, the general solution is given by

$$X = Q_1 + \frac{1}{2}\Big[A^\dagger\{(E+F)(B+D)^\dagger + (E-F)(B-D)^\dagger\} - A^\dagger A\{(Q_1+Q_2)(B+D)(B+D)^\dagger + (Q_1-Q_2)(B-D)(B-D)^\dagger\}\Big], \quad (3.5)$$

$$Y = Q_2 + \frac{1}{2}\Big[A^\dagger\{(E+F)(B+D)^\dagger - (E-F)(B-D)^\dagger\} - A^\dagger A\{(Q_1+Q_2)(B+D)(B+D)^\dagger - (Q_1-Q_2)(B-D)(B-D)^\dagger\}\Big], \quad (3.6)$$

where $Q_1, Q_2 \in M_{n,p}(\mathbb{C})$ *are arbitrary.*

Proof. Denote $R = (B+D)^T \otimes A$ and $S = (B-D)^T \otimes A$. In the viewpoint of Theorem 3.1 when $A = C$, the existence of a solution of the system (3.2) is equivalent to any of the following conditions:

(ii)' $\operatorname{rank} R + \operatorname{rank} S = \operatorname{rank}\left[\begin{array}{cc|c} B^T\otimes A & D^T\otimes A & \operatorname{Vec} E \\ D^T\otimes A & B^T\otimes A & \operatorname{Vec} F\end{array}\right]$,

(iii)' The following two conditions hold:

$$(2I_{mq} - (RR^\dagger + SS^\dagger))\operatorname{Vec} E = (RR^\dagger - SS^\dagger)\operatorname{Vec} F, \quad (3.7)$$

$$(2I_{mq} - (RR^\dagger + SS^\dagger))\operatorname{Vec} F = (RR^\dagger - SS^\dagger)\operatorname{Vec} E. \quad (3.8)$$

By Lemma 2.1, we have

$$\operatorname{rank} R = (\operatorname{rank} A)(\operatorname{rank}(B+D)), \quad \operatorname{rank} S = (\operatorname{rank} A)(\operatorname{rank}(B-D)).$$

Hence, the condition *(ii)'* becomes the condition *(ii)*. Now, we shall show that the equation (3.7) is reduced to (3.3). Indeed, note that by Lemma 2.3, we have

$$R^\dagger = ((B+D)^\dagger)^T \otimes A^\dagger, \quad S^\dagger = ((B-D)^\dagger)^T \otimes A^\dagger.$$

It follows that

$$RR^\dagger = \left((B+D)^\dagger(B+D)\right)^T \otimes AA^\dagger,$$
$$SS^\dagger = \left((B-D)^\dagger(B-D)\right)^T \otimes AA^\dagger.$$

Hence,

$$(RR^\dagger + SS^\dagger)\operatorname{Vec} E = \operatorname{Vec}\left[AA^\dagger E\left\{(B+D)^\dagger(B+D) + (B-D)^\dagger(B-D)\right\}\right],$$
$$(RR^\dagger - SS^\dagger)\operatorname{Vec} F = \operatorname{Vec}\left[AA^\dagger F\left\{(B+D)^\dagger(B+D) - (B-D)^\dagger(B-D)\right\}\right].$$

Now, (3.7) becomes

$$2E - AA^\dagger E\Big[(B+D)^\dagger(B+D) + (B-D)^\dagger(B-D)\Big] = AA^\dagger F\left[(B+D)^\dagger(B+D) - (B-D)^\dagger(B-D)\right].$$

which is equivalent to (3.3). Similarly, the conditions (3.8) and (3.4) are equivalent.

To obtain a formula of the general solution, note that Theorem 3.1, Lemmas 2.1 and 2.3 together imply that

$$\operatorname{Vec} X = \frac{1}{2}\Big[K_1 \operatorname{Vec} E + K_2 \operatorname{Vec} F + K_3 q_1 + K_4 q_2\Big],$$
$$\operatorname{Vec} Y = \frac{1}{2}\Big[K_2 \operatorname{Vec} E + K_1 \operatorname{Vec} F + K_4 q_1 + K_3 q_2\Big],$$

where $q_1, q_2 \in \mathbb{C}^{np}$ are arbitrary and

$$K_1 = \left((B+D)^\dagger + (B-D)^\dagger\right)^T \otimes A^\dagger,$$
$$K_2 = \left((B+D)^\dagger - (B-D)^\dagger\right)^T \otimes A^\dagger,$$
$$K_3 = 2I_{np} - \Big[\left((B+D)(B+D)^\dagger + (B-D)(B-D)^\dagger\right)^T \otimes A^\dagger A\Big],$$
$$K_4 = -\left((B+D)(B+D)^\dagger - (B-D)(B-D)^\dagger\right)^T \otimes A^\dagger A.$$

Since the vector operator is bijective, we can write $q_1 = \operatorname{Vec} Q_1$ and $q_2 = \operatorname{Vec} Q_2$ for some unique $Q_1, Q_2 \in M_{n,p}(\mathbb{C})$, respectively. Then, by Lemmas 2.2 and 2.3, we have

$$K_1 \operatorname{Vec} E = \operatorname{Vec}\left[A^\dagger E\left((B+D)^\dagger + (B-D)^\dagger\right)\right],$$
$$K_2 \operatorname{Vec} F = \operatorname{Vec}\left[A^\dagger F\left((B+D)^\dagger - (B-D)^\dagger\right)\right],$$
$$K_3 q_1 = \operatorname{Vec}\left[2Q_1 - A^\dagger A Q_1\left((B+D)(B+D)^\dagger + (B-D)(B-D)^\dagger\right)\right],$$
$$K_4 q_2 = \operatorname{Vec}\left[-A^\dagger A Q_2\left((B+D)(B+D)^\dagger - (B-D)(B-D)^\dagger\right)\right].$$

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Since the vector operator is linear and bijective, we obtain

$$\begin{aligned} X =& \frac{1}{2}\Big[A^{\dagger}E\left((B+D)^{\dagger}+(B-D)^{\dagger}\right)+A^{\dagger}F\left(B+D)^{\dagger}-(B-D)^{\dagger}\right) \\ &+2Q_1-A^{\dagger}AQ_1\left((B+D)(B+D)^{\dagger}+(B-D)(B-D)^{\dagger}\right) \\ &-A^{\dagger}AQ_2\left((B+D)(B+D)^{\dagger}-(B-D)(B-D)^{\dagger}\right)\Big], \end{aligned}$$

which can be reformed to the desired formula (3.5). Similarly, we get the formula of Y as (3.6). □

Corollary 3.4. *Let $A, B, D, E, F \in M_n(\mathbb{C})$. The system* (3.2) *has a unique solution if and only if A, $B+D$ and $B-D$ are invertible. In this case, the unique solution is given by*

$$\begin{aligned} X &= \frac{1}{2}A^{-1}\Big[(E+F)(B+D)^{-1}+(E-F)(B-D)^{-1}\Big], \\ Y &= \frac{1}{2}A^{-1}\Big[(E+F)(B+D)^{-1}-(E-F)(B-D)^{-1}\Big]. \end{aligned}$$

Proof. In the viewpoint of Theorem 3.2 when $A = C$, the system (3.2) has a unique solution if and only if $(B+D)^T \otimes A$ and $(B-D)^T \otimes A$ are invertible. By Lemma 2.1, this condition is equivalent to the invertibility of A, $B+D$ and $B-D$. The formula of the unique solution can be obtained by substituting $A^{\dagger} = A^{-1}$, $(B+D)^{\dagger} = (B+D)^{-1}$ and $(B-D)^{\dagger} = (B-D)^{-1}$ in (3.5) and (3.6). □

Corollary 3.4 was obtained in [8, Theorem 4.11] under the restrict condition $AB = BA$.

Corollary 3.5. *Let $B, D \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{n,q}(\mathbb{C})$. Consider the following system*

$$\begin{aligned} XB+YD &= E, \\ XD+YB &= F. \end{aligned} \tag{3.9}$$

Then, the following statements are equivalent:

(i) The system (3.9) *has a solution.*

(ii) $n\left(\operatorname{rank}(B+D)+\operatorname{rank}(B-D)\right) = \operatorname{rank}\left[\begin{array}{cc|c} B^T\otimes I_n & D^T\otimes I_n & \operatorname{Vec} E \\ D^T\otimes I_n & B^T\otimes I_n & \operatorname{Vec} F \end{array}\right].$

(iii) The following two conditions hold:

$$\begin{aligned} \left[(E+F)(B+D)^{\dagger}(B+D)+(E-F)(B-D)^{\dagger}(B-D)\right] &= 2E, \\ \left[(E+F)(B+D)^{\dagger}(B+D)-(E-F)(B-D)^{\dagger}(B-D)\right] &= 2F. \end{aligned}$$

In the above case, the general solution is given by

$$\begin{aligned}
X &= Q_1 + \frac{1}{2}\Big[\big\{(E+F)(B+D)^\dagger + (E-F)(B-D)^\dagger\big\} \\
&\quad - \big\{(Q_1+Q_2)(B+D)(B+D)^\dagger + (Q_1-Q_2)(B-D)(B-D)^\dagger\big\}\Big], \\
Y &= Q_2 + \frac{1}{2}\Big[\big\{(E+F)(B+D)^\dagger - (E-F)(B-D)^\dagger\big\} \\
&\quad - \big\{(Q_1+Q_2)(B+D)(B+D)^\dagger - (Q_1-Q_2)(B-D)(B-D)^\dagger\big\}\Big],
\end{aligned}$$

where Q_1, $Q_2 \in M_{n,p}(\mathbb{C})$ are arbitrary.

Proof. This is a special case of Theorem 3.3 when $A = I_n$. □

Corollary 3.6. *Let $B, D, E, F \in M_n(\mathbb{C})$. The system* (3.9) *has a unique solution if and only if $B+D$ and $B-D$ are invertible. In this case, the unique solution is given by*

$$\begin{aligned}
X &= \frac{1}{2}\Big[(E+F)(B+D)^{-1} + (E-F)(B-D)^{-1}\Big], \\
Y &= \frac{1}{2}\Big[(E+F)(B+D)^{-1} - (E-F)(B-D)^{-1}\Big].
\end{aligned}$$

Proof. The proof is similar to the Corollary 3.4 by substituting $A = I_n$. □

Theorem 3.7. *Let $A, C \in M_{m,n}(\mathbb{C})$, $B \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{m,q}(\mathbb{C})$. Consider the following coupled linear matrix equations*

$$\begin{aligned}
AXB + CYB &= E, \\
CXB + AYB &= F.
\end{aligned} \tag{3.10}$$

Then, the following statements are equivalent:

(i) The system (3.10) *has a solution.*

(ii) $(\operatorname{rank} B)\,(\operatorname{rank}(A+C) + \operatorname{rank}(A-C)) = \operatorname{rank}\left[\begin{array}{cc|c} B^T\otimes A & B^T\otimes C & \operatorname{Vec} E \\ B^T\otimes C & B^T\otimes A & \operatorname{Vec} F \end{array}\right].$

(iii) The following two conditions hold:

$$\big[(A+C)(A+C)^\dagger(E+F) + (A-C)(A-C)^\dagger(E+F)\big]\,B^\dagger B = 2E, \tag{3.11}$$

$$\big[(A+C)(A+C)^\dagger(E+F) - (A-C)(A-C)^\dagger(E+F)\big]\,B^\dagger B = 2F. \tag{3.12}$$

In the above case, the general solution is given by

$$X = Q_1 + \frac{1}{2}\Big[\left\{(A+C)^\dagger(E+F) + (A-C)^\dagger(E-F)\right\}B^\dagger - \left\{(A+C)^\dagger(A+C)(Q_1+Q_2) + (A-C)^\dagger(A-C)(Q_1-Q_2)\right\}BB^\dagger\Big], \tag{3.13}$$

$$Y = Q_2 + \frac{1}{2}\Big[\left\{(A+C)^\dagger(E+F) - (A-C)^\dagger(E-F)\right\}B^\dagger - \left\{(A+C)^\dagger(A+C)(Q_1+Q_2) - (A-C)^\dagger(A-C)(Q_1-Q_2)\right\}BB^\dagger\Big], \tag{3.14}$$

where $Q_1, Q_2 \in M_{n,p}(\mathbb{C})$ are arbitrary.

Proof. The idea of proof is similar to that of Theorem 3.3. In this case, the general solution is given by

$$\operatorname{Vec} X = \frac{1}{2}\Big[J_1 \operatorname{Vec} E + J_2 \operatorname{Vec} F + J_3 q_1 + J_4 q_2\Big],$$
$$\operatorname{Vec} Y = \frac{1}{2}\Big[J_2 \operatorname{Vec} E + J_1 \operatorname{Vec} F + J_4 q_1 + J_3 q_2\Big],$$

where $q_1, q_2 \in \mathbb{C}^{np}$ are arbitrary and

$$J_1 = (B^\dagger)^T \otimes \left\{(A+C)^\dagger + (A-C)^\dagger\right\},$$
$$J_2 = (B^\dagger)^T \otimes \left\{(A+C)^\dagger - (A-C)^\dagger\right\},$$
$$J_3 = 2I_{np} - \Big[(BB^\dagger)^T \otimes \left\{(A+C)^\dagger(A+C) + (A-C)^\dagger(A-C)\right\}\Big],$$
$$J_4 = -(BB^\dagger)^T \otimes \left\{(A+C)^\dagger(A+C) - (A-C)^\dagger(A-C)\right\}.$$

We arrive at (3.13) and (3.14) by using properties of the vector operator. □

Corollary 3.8. *Let $A, B, C, E, F \in M_n(\mathbb{C})$. The system* (3.10) *has a unique solution if and only if $B, A+C$ and $A-C$ are invertible. In this case, the unique solution is given by*

$$X = \frac{1}{2}\Big[(A+C)^{-1}(E+F) + (A-C)^{-1}(E-F)\Big]B^{-1},$$
$$Y = \frac{1}{2}\Big[(A+C)^{-1}(E+F) - (A-C)^{-1}(E-F)\Big]B^{-1}.$$

Proof. The criterion for uniqueness of solution follows from Theorem 3.2 by setting $B = D$. The formula of the solutions X and Y can be derived from (3.13) and (3.14) by substituting $B^\dagger = B^{-1}$, $(A+C)^\dagger = (A+C)^{-1}$ and $(A-C)^\dagger = (A-C)^{-1}$. □

Corollary 3.8, under the restrict condition $AB = BA$ was obtained in [8, Theorem 4.10].

Corollary 3.9. *Let* $A, E, F \in M_n(\mathbb{C})$. *Consider the following coupled linear matrix equations*

$$\begin{aligned} AX + Y &= E, \\ X + AY &= F. \end{aligned} \tag{3.15}$$

The following conditions are equivalent:

(i) The system (3.15) *has a unique solution.*

(ii) $A^2 - I_n$ *is invertible.*

(iii) 1 *and* -1 *are not eigenvalues of* A.

In this case, the unique solution is given by

$$\begin{aligned} X &= \frac{1}{2}\Big[(A+I_n)^{-1}(E+F) + (A-I_n)^{-1}(E-F)\Big], \\ Y &= \frac{1}{2}\Big[(A+I_n)^{-1}(E+F) - (A-I_n)^{-1}(E-F)\Big]. \end{aligned}$$

Proof. It follows directly from Corollary 3.8 by setting $B = C = D = I_n$. Note that the conditions *(ii)* and *(iii)* are equivalent since $A^2 - I_n = (A - I_n)(A + I_n)$. □

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1. T. Nuchniyom and P. Changsangiam, Existence and Uniqueness of Solutions for Systems of Coupled Linear Matrix Equations, Annual Pure and Applied Mathematics Conference 2017, Chulalongkorn University, Bankkok, Thailand, 28–30 June. 62–75, (2017).