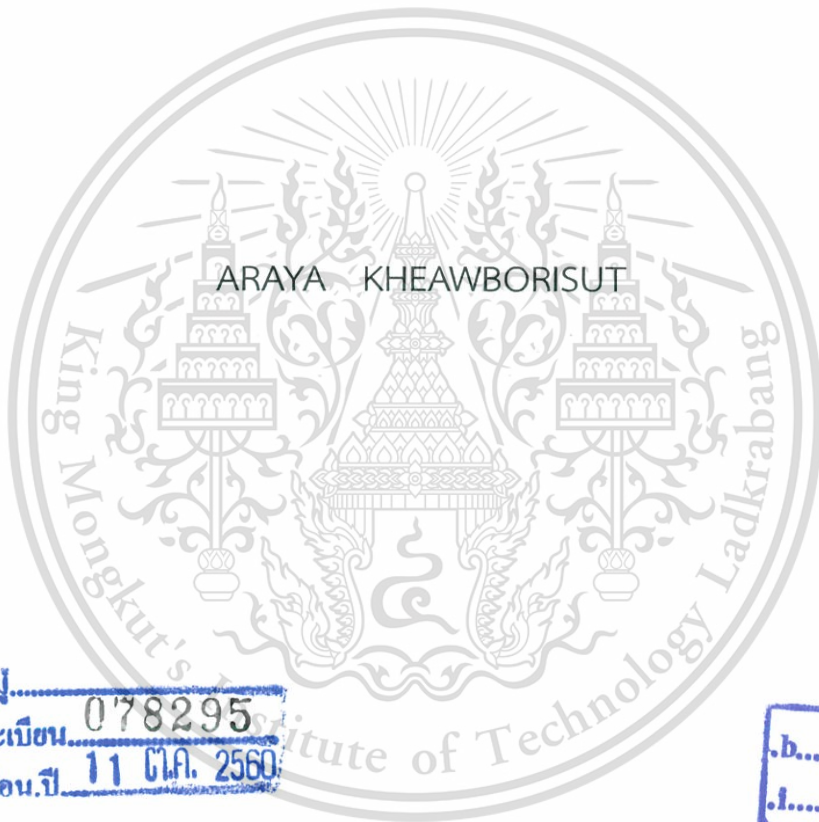


THE METHODS FOR EXISTENCE THEOREM ON CONVEX  
METRIC SPACE ENDOWED WITH GRAPH AND  
CONVERGENCE THEOREM IN HILBERT SPACE



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### Abstract

The purpose of this thesis, we first introduce a new type of multi-valued G-contraction mapping on a metric space endowed with a directed graph  $G$  and prove an existence theorem for fixed point problems in metric space endowed with a graph. Moreover, we prove fixed point theorems in partially ordered metric spaces by our main result. Secondly, we prove the strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed point of a finite family of  $\kappa_i$ -strictly pseudocontractive mapping (i.e, there exist  $\kappa_i \in [0, 1)$  such that  $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa_i \|(I - T)x - (I - T)y\|^2$ ,  $\forall x, y \in \mathbb{C}$  and  $i = 1, 2, \dots, N$ ) in framework of Hilbert spaces. The results obtained in this thesis improve and modify well-known results in the previous literature.

**Keywords :** fixed point theorems, multi-valued mapping, metric spaces, graph, Strictly pseudocontractive mapping, Equilibrium problem.

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### บทคัดย่อ

จุดประสงค์ของงานวิจัยฉบับนี้คือข้อแรกเราแนะนำการส่งแบบหดตัวหลายค่าจีชนิดใหม่บนปริภูมิเมตริกที่กำกับด้วยกราฟระบุทิศทางและพิสูจน์ทฤษฎีบทการมีอยู่จริงของปัญหาจุดตรึงในปริภูมิเมตริกที่มีกราฟ นอกจากนี้เรายังได้พิสูจน์ทฤษฎีบทจุดตรึงในปริภูมิเมตริกอันดับบางส่วนโดยใช้ทฤษฎีบทหลักดังกล่าว ข้อสองเราพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มสำหรับการหาสมาชิกร่วมของเซตของผลเฉลยของปัญหาเชิงดุลยภาพและเซตของจุดตรึงของการส่งวงค์จำกัดของการส่ง  $\kappa_i$  หดเทียมโดยแท้ (กล่าวคือ มี  $\kappa_i \in [0, 1)$  ซึ่งทำให้  $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa_i \|(I - T)x - (I - T)y\|^2$  , สำหรับทุกๆ  $x, y \in C$  และ  $i = 1, 2, \dots, N$ ) ในปริภูมิฮิลเบิร์ต โดยใช้วิธีที่แตกต่างจากงานวิจัยอื่นๆ ซึ่งผลการศึกษาที่ได้จากงานวิจัยนี้จะพัฒนาและปรับปรุงผลงานต่างๆของงานวิจัยในสาขา

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ARAYA KHEAWBORISUT

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# Chapter 1

## Introduction

### 1.1 Introduction

Fixed point problem can be reduced to the problem of economic, engineering, radiotherapy etc. So, we can use the process for fixed point solution to resolve such problem.

Many authors tried to use new methods to solve fixed point problem. The solution for fixed point problem in Hilbert and Banach spaces has been popular for a long time see, [1], [2], [3].

This research is to studied the fixed point problem on complete metric space endowed with graph.

Let  $(X, d)$  be a metric space and we let  $CB(X)$  and  $Comp(X)$  be the set of all closed bounded subsets of  $X$  and the set of all nonempty compact subsets of  $X$ , respectively. A mapping  $T : X \rightarrow X$  is called contraction if there exists  $\alpha \in [0, 1)$  such that  $d(Tx, Ty) \leq \alpha d(x, y)$  for all  $x, y \in X$ . A point  $x \in X$  is a fixed point of a multi-valued mapping  $T : X \rightarrow 2^X$  if  $x \in Tx$ . A point  $x \in X$  is a coincidence point of  $g : X \rightarrow X$  and  $T : X \rightarrow 2^X$  if  $g(x) \in T(x)$ . If  $g$  is the identity map on  $X$ , then  $x$  is a fixed point of  $T$ .

The Banach contraction principle is a very important tool for solving existence problem in many branches of mathematics and physics. It guarantees the existence and uniqueness of fixed points.

The Banach contraction principle was firstly given by as follows;

**Theorem 1.1.** Let  $(X, d)$  be complete metric space and  $T : X \rightarrow X$  be  $\alpha$ -contraction. Then  $T$  has a unique fixed point.

In 1969, Nadler [1] extended the Banach contraction principle to multi-valued contraction mapping in complete metric space as follows;

**Theorem 1.2.** [1] Let  $(X, d)$  be complete metric space and let  $T : X \rightarrow CB(X)$ . Assume that there exists  $k \in [0, 1)$  such that

$$H(Tx, Ty) \leq kd(x, y). \quad \forall x, y \in X.$$

Then there exists  $z \in Tz$ .

Reich [2] extended Nadler's fixed point theorem by replacing  $k$  with  $d(x, y)$  and  $CB(X)$  with  $Comp(X)$  as follows.

**Theorem 1.3.** [2] Let  $(X, d)$  be complete metric space and let  $T : X \rightarrow Comp(X)$ . Assume that there exists a function  $\alpha : [0, \infty) \rightarrow [0, 1)$  such that  $\lim_{r \rightarrow t^+} \sup \alpha(r) < 1$  for

each  $t \in (0, \infty)$  and

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y), \quad \forall x, y \in X.$$

Then there exists  $z \in X$  such that  $z \in Tz$ .

In 1989, Mizoguchi and Takahachi [3] relaxed the compactness assumption on the mapping to closed and bounded subset of  $X$  as follows;

**Theorem 1.4.** [3] Let  $(X, d)$  be complete metric space and let  $T : X \rightarrow CB(X)$ . Assume that there exists a function  $\alpha : [0, \infty) \rightarrow [0, 1)$  such that  $\lim_{r \rightarrow t+} \sup \alpha(r) < 1$  for each  $t \in (0, \infty)$  and

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y), \quad \forall x, y \in X.$$

Then there exists  $z \in X$  such that  $z \in Tz$ .

In 2007, Berinde and Berinde [4] introduce the following definition to prove an existence theory of fixed point problem in convex metric space;

**Definition 1.1.** [4] Let  $(X, d)$  be a metric space and  $T : X \rightarrow CB(X)$  is called a multi-valued weak contraction or a multi-valued  $(\theta, L)$ -weak contraction if there exists two constants  $\theta \in (0, 1)$  and  $L \geq 0$  such that

$$H(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx), \quad \forall x, y \in X.$$

For the last ten years, many research related to the existence theorem of fixed point multi-valued mapping have been extensively studied and used different techniques to prove the such theorem, see for example [5],[6].

Let a graph  $G$  be an ordered pair  $(V(G), E(G))$  consisting  $V(G)$  is a set of vertices of the graph and  $E(G)$  be a set of its edges.

In 2008, Jachymski [5] introduced a concept of  $G$ -contraction and generalized the Banach contraction principle in a metric space endowed with a directed graph in a definition below;

**Definition 1.2.** [5] Let  $(X, d)$  be a metric space and let  $G = (V(G), E(G))$  be a directed graph such that  $V(G) = X$  and  $E(G)$  contains all loop i.e,  $\Delta = \{(x, x) : x \in X\} \subseteq E(G)$ . The mapping  $f : X \rightarrow X$  is a  $G$ -contraction if  $f$  preserves edges of  $G$ , i.e.,

$$x, y \in X, (x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G), \quad (1.1)$$

and there exists  $\alpha \in (0, 1)$  such that

$$x, y \in X, (x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leq \alpha d(x, y).$$

The mapping  $f : X \rightarrow X$  satisfying condition (1.1) is also called a graph-preserving mapping. He proved some certain properties on  $X$ , a  $G$ -contraction  $f : X \rightarrow X$  has a fixed point if and only if there exists  $x \in X$  such that  $(x, f(x)) \in E(G)$ .

Recently, Beg and Butt [6] introduced the concept of  $G$ -contraction for multi-valued mapping as follows.

**Definition 1.3.** [6] Let  $T : X \rightarrow CB(X)$  be a multi-valued mapping. The mapping  $T$  is called  $G$ -contraction if there exists  $k \in (0, 1)$  such that

$$H(Tx, Ty) \leq kd(x, y) \text{ for all } (x, y) \in E(G),$$

and if  $u \in Tx$  and  $v \in Ty$  are such that

$$d(u, v) \leq kd(x, y) + \alpha \text{ for each } \alpha > 0,$$

then  $(u, v) \in E(G)$

**Property A** [5] For any sequence  $\{x_n\}$  in  $X$ , if  $x_n \rightarrow x$  and  $(x_n, x_{n+1}) \in E(G)$  for  $n \in \mathbb{N}$ , then there is a subsequence  $\{x_{n_k}\}$  such that  $(x_{n_k}, x) \in E(G)$  for  $k \in \mathbb{N}$ .

Beg and Butt [6] proved that if  $(X, d)$  is complete metric space and  $(X, d)$  has property A, then  $G$ -contraction mapping  $T : X \rightarrow CB(X)$  has a fixed point if and only if there exists  $x \in X$  and  $y \in Tx$  such that  $(x, y) \in E(G)$ .

Tiammee and Suatai [7] introduced concept of graph-preserving multi-valued mapping and a type of multi-valued weak  $G$ -contraction on a metric space endowed with graph as follows;

**Definition 1.4.** Let  $X$  be a nonempty set,  $G = (V(G), E(G))$  a directed graph such that  $V(G) = X$ . The multi-valued mapping  $T : X \rightarrow CB(X)$  is said to be graph preserving if

$$(x, y) \in E(G) \implies (u, v) \in E(G)$$

for all  $u \in Tx$  and  $v \in Ty$ .

**Definition 1.5.** Let  $X$  be a nonempty set,  $G = (V(G), E(G))$  a directed graph such that  $V(G) = X$ ,  $g : X \rightarrow X$ . The multi-valued mapping  $T : X \rightarrow CB(X)$  is said to be  $g$ -graph preserving if for any  $x, y \in X$ , such that

$$(g(x), g(y)) \in E(G) \implies (u, v) \in E(G)$$

for all  $u \in Tx$  and  $v \in Ty$ .

**Definition 1.6.** Let  $(X, d)$  be a metric space,  $G = (V(G), E(G))$  be a directed graph such that  $V(G) = X$ ,  $g : X \rightarrow X$ . The multi-valued mapping  $T : X \rightarrow CB(X)$  is said to be a multi-valued weak  $G$ -contraction with respect to  $g$  or  $(g, \alpha, L)$ - $G$ -contraction if there exists a function  $\alpha : [0, \infty) \rightarrow [0, 1)$  satisfying  $\limsup_{r \rightarrow t^+} \alpha(r) < 1$  for every  $t \in [0, \infty)$  and  $L \geq 0$  with

$$H(Tx, Ty) \leq \alpha(d(g(x), g(y)))d(g(x), g(y)) + LD(g(y), Tx),$$

for all  $x, y \in X$  such that  $(g(x), g(y)) \in E(G)$ .

They prove the following theorem in complete metric space endowed with graph.

**Theorem 1.5.** [7] Let  $(X, d)$  be a complete metric space and  $G = (V(G), E(G))$  be a directed graph such that  $V(G) = X$ , and let  $g : X \rightarrow X$  be a surjective mapping. If  $T : X \rightarrow CB(X)$  is a multi-valued mapping satisfying the following properties:

- (1)  $T$  is a  $g$ -graph-preserving mapping;
- (2) there exists  $x_0 \in X$  such that  $(g(x_0), y) \in E(G)$  for some  $y \in Tx_0$ ;
- (3)  $X$  has Property A;
- (4)  $T$  is a  $(g, \alpha, L)$ - $G$ -contraction;

then there exists  $u \in X$  such that  $g(u) \in Tu$ .

Many authors tried to introduce a new techniques to prove existence of fixed point theorem such as Phon-on et al [8] introduced a new concept of weak graph preserving and proved their fixed point theorem in a complete metric space endowed with a graph, see more example [6], [7], [8].

Many authors proved their results by using the following lemma 1.6, see for more detial [6], [7], [8].

**Lemma 1.6.** [3] Let  $(X, d)$  be metric space in  $CB(X)$ ,  $\{x_k\}$  be in  $X$  such that  $x_k \in A_{k-1}$ . Let  $\alpha : [0, \infty) \rightarrow [0, 1)$  be a function satisfying  $\limsup_{t \rightarrow t^+} \alpha(t) < 1$  for every  $t \in [0, \infty)$ . Suppose that  $d(x_{k-1}, x_k)$  is a non-increasing sequence such that

$$\begin{aligned} H(A_{k-1}, A_k) &\leq \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k) \\ d(x_{k-1}, x_k) &\leq H(A_{k-1}, A_k) + \alpha^{n_k}(d(x_{k-1}, x_k)), \end{aligned}$$

where  $n_1 < n_2 < \dots$  and  $k, n_k \in \mathbb{N}$ . Then  $\{x_k\}$  is a Cuachy sequence in  $X$ .

In this paper, we introduced a new type of multi-valued  $G$ -contraction on a metric space endowed with a directed graph  $G$  and new techniques to simplify the proof of existence theorem for coincidence point of such mapping, that is, we do not use lemma 1.6 to prove our main theorem.

The next results are equilibrium problems and fixed point problems in a real Hilbert space. Let  $H$  be a real Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . A mapping  $T : C \rightarrow C$  is called

- nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ .
- $\kappa$ -strictly pseudocontractive mapping, if there exist  $\kappa \in [0, 1)$  such that  $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2$ ,  $\forall x, y \in C$ .

The set of fixed points of  $T$  denote by  $F(T)$  (i.e.  $F(T) = \{x \in H : Tx = x\}$ ).

Let  $F : C \times C \rightarrow \mathbb{R}$  be a bifunction. The equilibrium problem for  $F$  is to determine its equilibrium points, i.e. the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}. \quad (1.2)$$

The problem is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others, see [9],[10] [14].

In 2007, Takahashi, Takahashi [14] introduced viscosity approximation method for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping in a Hilbert space. They defined the sequences  $\{x_n\}$  and  $\{u_n\}$  as follows: Let  $x_1 \in H$  and  $f : H \rightarrow H$  be contraction mapping with  $\alpha \in (0, 1)$ .  $\{x_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C. \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T u_n, & \forall n \in \mathbb{N}, \end{cases} \quad (1.3)$$

They proved strong convergence theorem of sequence  $\{x_n\}$  generated by (1.3) to  $z \in F(T) \cap EP(F)$ , where  $z = P_{F(T) \cap EP(F)} f(z)$  under certain appropriate conditions imposed on the sequences  $\{\alpha_n\}, \{r_n\}$  and bifunction  $F$ .

In this paper, motivated by (1.3), we introduce a new algorithm as follows: for  $u \in C$  and the sequence  $\{x_n\}$  generated by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - (\beta_n S x_n + (1 - \beta_n) x_n) \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) u_n, & \forall n \geq 1, \end{cases} \quad (1.4)$$

where  $F : C \times C \rightarrow \mathbb{R}$  is a bifunction and  $S$  is  $S$ -mapping generated by a finite family of  $\kappa_i$ -strictly pseudo-contractive mapping and finite real numbers. Under suitable conditions of parameters  $\{\alpha_n\}, \{\beta_n\}, \{r_n\}$ , we prove strong convergence theorem for finding a common element of the set of solutions of equilibrium problem and the set of solutions of the set of fixed point problems of a finite family of  $\kappa_i$ -strictly pseudocontractive mapping in Hilbert spaces.

## 1.2 Objectives of the study

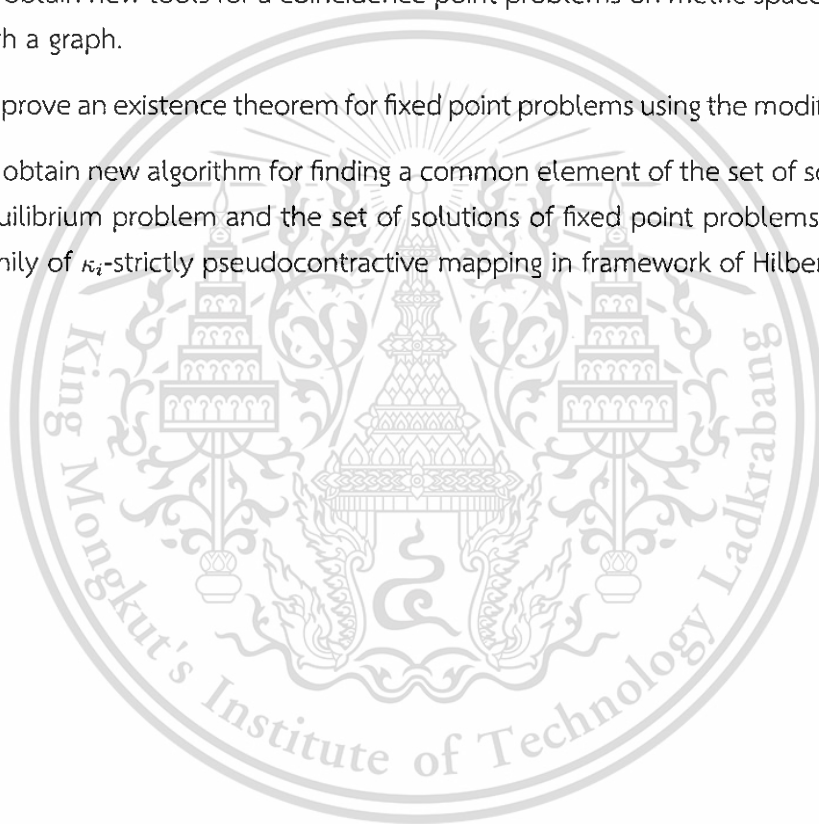
- 1) To introduce a new type of multi-valued  $G$ -contraction on a metric space endowed with a directed graph  $G$ .
- 2) To prove an existence theorem for a coincidence point problems in convex metric space endowed with graph.
- 3) To introduce a new algorithm for finding a common element of the set of solutions of equilibrium problem and the set of solutions of fixed point problems of a finite family of  $\kappa_i$ -strictly pseudocontractive mapping in framework of Hilbert spaces.

### 1.3 Scope(s) of the study

- 1) We following to prove an existence theorem for a coincidence point problems in metric space endowed with a graph.
- 2) we prove strong convergence theorem for finding a common element of the set of solutions of equilibrium problem and the set of solutions of fixed point problems of a finite family of  $\kappa_i$ -strictly pseudocontractive mapping in framework of Hilbert spaces.

### 1.4 Benefits of the study

- 1) To obtain new tools for a coincidence point problems on metric space endowed with a graph.
- 2) To prove an existence theorem for fixed point problems using the modify method.
- 3) To obtain new algorithm for finding a common element of the set of solutions of equilibrium problem and the set of solutions of fixed point problems of a finite family of  $\kappa_i$ -strictly pseudocontractive mapping in framework of Hilbert spaces.



## Chapter 2

### Theory and Literature Reviews

The purpose of this chapter is to explain fundamental concepts and definitions used throughout this thesis. Moreover, we give some lemmas, remarks and useful results used in the later chapters.

#### 2.1 Basic concepts

**Definition 2.1.** ([15]). Let  $X$  be a non-empty set. A mapping  $d : X \times X \rightarrow \mathbb{R}$  is called a metric or a distance on  $X$  provide that

- 1)  $d(x, y) \geq 0$ ;
- 2)  $d(x, y) = 0 \Leftrightarrow x = y$ ; (separation axiom)
- 3)  $d(x, y) = d(y, x)$ , for all  $x, y \in X$ ; (symmetry)
- 4)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ ; (the triangle inequality).

A set  $X$  endowed with a metric  $d$  is called metric space and is denoted by  $(X, d)$ .

**Definition 2.2.** ([15]). Let  $\{x_n\}_{n=0}^{\infty}$  be a sequence in metric space  $(X, d)$ . We say that the sequence  $\{x_n\}_{n=0}^{\infty}$

- a) is convergent to  $a \in X$  if, for any  $\epsilon > 0$ , there exists  $n_0 = n_0(\epsilon)$  such that

$$d(x_n, a) < \epsilon, \text{ for all } n, p \in \mathbb{N}, n \geq n_0$$

- b) is fundamental or Cuachy sequence if, for any  $\epsilon > 0$ , there exists  $n_0 = n_0(\epsilon)$  such that

$$d(x_n, x_{n+p}) < \epsilon, \text{ for all } n, p \in \mathbb{N}, n \geq n_0$$

**Definition 2.3.** ([15]). Let  $X$  be a metric space with metric  $d$ , and let  $\{x_n\}$  be a sequence in  $X$ . We say that  $\{x_n\}$  is convergent if there exists a point  $x$  in  $X$  such that  $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ . We usually symbolize this by writing  $x_n \rightarrow x$ . The point  $x$  is called the limit of sequence.

**Theorem 2.1.** ([15]). If  $x_n \rightarrow x$ , then  $x$  is the limit of any subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$ .

**Theorem 2.2.** ([15]). Let  $X$  be a metric space with metric  $d$ . A subset  $F$  of  $X$  is a closed set if and only if whenever  $\{x_n\} \subset F$  and  $x_n \rightarrow x$ , imply  $x \in F$ .

**Definition 2.4.** ([15]). A metric space  $(X, d)$  is called complete if any Cuachy sequence in  $X$  is convergent.

**Definition 2.5.** ([7]) A partial order is a binary relation  $\leq$  over the set  $X$  which the followings conditions:

1.  $x \leq x$  (reflexivity);
2. If  $x \leq y$  and  $y \leq x$ , then  $x = y$  (antisymmetry);
3. If  $x \leq y$  and  $y \leq z$ , then  $x \leq z$  (transitivity);

for all  $x, y, z \in X$ . A set with a partial order  $\leq$  is called a partially ordered set. We write  $x < y$  if  $x \leq y$  and  $x \neq y$ .

**Definition 2.6.** ([7]). Let  $(X, \prec)$  be a partially ordered set. For each  $A, B \subset X$ .

$$A \prec B \text{ if } a < b \text{ for all } a \in A, b \in B.$$

**Definition 2.7.** ([7]). Let  $(X, d)$  be metric space endowed with a partial order  $\leq$ . Let  $g : X \rightarrow X$  be surjective. A multi-valued mapping  $T : X \rightarrow CB(X)$  is said to be  $g$ -increasing if for any  $x, y \in X$ ,

$$g(x) < g(y) \implies Tx \prec Ty.$$

in the case  $g = I_X$ , the identity map, the mapping  $T$  is called an increasing mapping.

## 2.2 Some useful lemmas and theorems for fixed point with graph

**Definition 2.8.** ([15]). Let  $X$  be a nonempty set and  $T : X \rightarrow X$  be a self-mapping. We say that  $x \in X$  is a fixed point of  $T$  if

$$T(x) = x$$

and denote by  $F_T$  or  $Fix(X)$  the set of all fixed points of  $T$ .

**Definition 2.9.** ([7]). Let  $X$  be a nonempty set. A point  $x \in X$  is a fixed point of a multi-valued mapping  $T : X \rightarrow 2^X$  if  $x \in Tx$ .

**Definition 2.10.** ([7]). Let  $g : X \rightarrow X$  be a self-mapping and  $T : X \rightarrow 2^X$  be a multivalued map. A point  $x \in X$  is a coincidence point of  $g$  and  $T$  if  $g(x) \in T(x)$ .

If  $g$  is the identity map on  $X$ , then  $x$  is a fixed point of  $T$ .

Let  $(X, d)$  be a metric space and let  $CB(X)$  be the set of all closed bounded subsets of  $X$ . For  $x \in X$  and  $A, B \in CB(X)$ , define

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

$$\delta(A, B) = \sup\{d(x, B) : x \in A\}.$$

$$D(A, B) = \inf\{d(x, B) : x \in A\}$$

The symbol  $H(A, B)$  is the Pompeiu-Hausdorff metric [4], that is,

$$H(A, B) = \max\{\sup_{u \in A} d(u, B), \sup_{v \in B} d(v, A)\}.$$

**Lemma 2.3.** Let  $(X, d)$  be metric space, and let  $A \subset X$  be a non-empty set. For a point  $x \in X$  with  $d(x, A) = 0$ . Then  $x \in \bar{A}$ .

The following lemmas is useful for our main results.

**Lemma 2.4.** ([1]). Let  $(X, d)$  be metric space. If  $A, B \in CB(X)$  and  $a \in A$ , then, for each  $\varepsilon > 0$ , there exists  $b \in B$  such that

$$d(a, b) \leq H(A, B) + \varepsilon.$$

### 2.3 Fixed point sets of nonexpansive mappings and strictly pseudo-contractive mappings

**Definition 2.11** ([4]). Let  $H$  be a Hilbert space, let  $C$  be a nonempty closed convex subset of  $H$ . The metric projection or nearest point projection of  $H$  on to  $C$ , denoted by  $P_C$ , is defined, for any  $x \in H$ , as the only point in  $C$  with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

**Definition 2.12.** ([20]). Let  $C$  be a nonempty convex subset of real Banach space. Let  $\{T_i\}_{i=1}^N$  be a finite family of mapping of  $C$  into  $C$ . For each  $j = 1, 2, \dots, N$ , let  $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$  where  $I \in [0, 1]$  and  $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ . We define the mapping  $S : C \rightarrow C$  as follows:

$$\begin{aligned} U_0 &= I \\ U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\ U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\ U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \end{aligned}$$

$$\begin{aligned} U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\ S &= U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I. \end{aligned} \quad (2.1)$$

This mapping is called *S-mapping* generated by  $T_1, T_2, \dots, T_N$  and  $\alpha_1, \alpha_2, \dots, \alpha_N$ .

**Lemma 2.5.** ([22]) Assume that  $T$  is a nonexpansive self-mapping of closed convex subset  $C$  of a Hilbert space  $H$ . If  $T$  has a fixed point, then  $I - T$  is demiclosed. That is, whenever  $\{x_n\}$  is a sequence in  $C$  weakly converging to some  $x \in C$  and the sequence  $\{(I - T)x_n\}$  strongly converges to some  $y$  it follows that  $(I - T)x = y$ . Here,  $I$  is the identity mapping of  $H$ .

Lemma 2.6. ([15]). Given  $x \in H$  and  $y \in C$ . Then  $P_C x = y$  if and only if there holds the inequality

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.7. ([18]). Let  $E$  be a uniformly convex Banach space,  $C$  be a nonempty closed convex subset of  $E$  and  $S : C \rightarrow C$  be a nonexpansive mapping. Then  $I - S$  is demi-closed at zero.

Lemma 2.8. ([19]). Let  $\{s_n\}$  be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0$$

where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  and  $\{\delta_n\}$  is a sequence such that

$$(1) \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$(2) \quad \limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then  $\lim_{n \rightarrow \infty} s_n = 0$ .

For solving the equilibrium problem for a bifunction  $F : C \times C \rightarrow \mathbb{R}$ , let us assume that  $F$  satisfies the following conditions:

$$(A1) \quad F(x, x) = 0 \quad \forall x \in C;$$

$$(A2) \quad F \text{ is monotone, i.e. } F(x, y) + F(y, x) \leq 0, \quad \forall x, y \in C;$$

$$(A3) \quad \forall x, y, z \in C,$$

$$\lim_{t \rightarrow 0^+} F(tz + (1-t)x, y) \leq F(x, y);$$

$$(A4) \quad \forall x \in C, \quad y \mapsto F(x, y) \text{ is convex and lower semicontinuous.}$$

The following lemma appears implicitly in [9].

Lemma 2.9. ([9]). Let  $C$  be a nonempty closed convex subset of  $H$  and let  $F$  be a bifunction of  $C \times C$  into  $\mathbb{R}$  satisfying (A1) – (A4). Let  $r > 0$  and  $x \in H$ . Then, there exists  $z \in C$  such that

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad (2.2)$$

for all  $x \in C$ .

Lemma 2.10. ([10]). Assume that  $F : C \times C \rightarrow \mathbb{R}$  satisfies (A1) – (A4). For  $r > 0$  and  $x \in H$ , define a mapping  $S_r : H \rightarrow C$  as follows:

$$S_r(x) = \{z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C\}. \quad (2.3)$$

for all  $z \in H$ . Then, the following hold:

(1)  $S_r$  is single-valued;

(2)  $S_r$  is firmly nonexpansive i.e.

$$\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y \rangle \quad \forall x, y \in H;$$

(3)  $F(S_r) = EP(F)$ ;

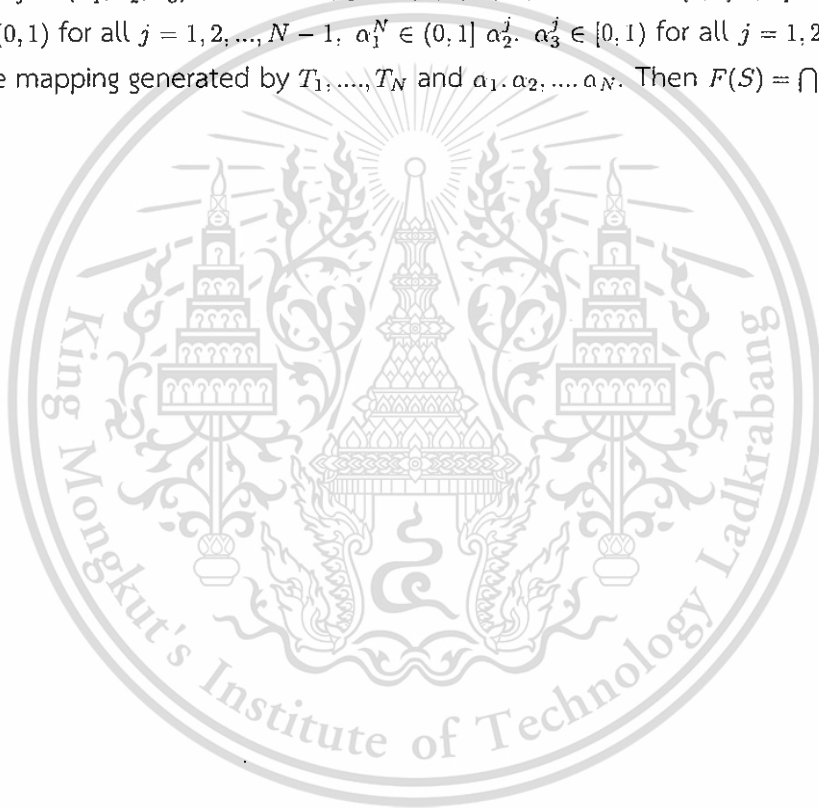
(4)  $EP(F)$  is closed and convex.

**Lemma 2.11.** ([23]). In a real Hilbert spaces  $H$ , there holds the following inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$$

for all  $x, y \in H$ .

**Lemma 2.12.** ([20]). Let  $C$  be a nonempty closed convex subset of strictly convex. Let  $\{T_i\}_{i=1}^N$  be a finite family of nonexpansive mappings of  $C$  into itself with  $\bigcap_{i=1}^N F(T_i) \neq \emptyset$  and let  $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ ,  $j = 1, 2, 3, \dots, N$ , where  $I = [0, 1]$ ,  $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ ,  $\alpha_1^j \in (0, 1)$  for all  $j = 1, 2, \dots, N - 1$ ,  $\alpha_1^N \in (0, 1]$ ,  $\alpha_2^j, \alpha_3^j \in [0, 1)$  for all  $j = 1, 2, \dots, N$ . Let  $S$  be the mapping generated by  $T_1, \dots, T_N$  and  $\alpha_1, \alpha_2, \dots, \alpha_N$ . Then  $F(S) = \bigcap_{i=1}^N F(T_i)$ .



## Chapter 3

### Main Results and Discussion

#### 3.1 Fixed point with graph

We define a new type of multi-valued mapping on a metric space endowed with a directed graph  $G$  as follow:

**Definition 3.1** Let  $(X, d)$  be a metric space and  $G = (V(G), E(G))$  be directed graph such that  $V(G) = X$  and let  $g : X \rightarrow X$  be a mapping. The multi-valued mapping  $T : X \rightarrow CB(X)$  is said to be  $(\alpha, \beta, L, g)$ - $G$ -contraction if there exists  $\alpha, \beta \in (0, \frac{1}{3})$  with  $\alpha < \beta$  and  $L \geq 0$  such that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta D(g(x), Ty) + LD(g(y), Tx) \quad (3.1)$$

for all  $x, y \in X$  and  $(g(x), g(y)) \in E(G)$ .

**Example 3.2** Let  $X = \{0, 1, 2, 3, 4, 5\}$  and  $d(x, y) = |x - y|$ . Let  $G = (V(G), E(G))$  be a directed graph defined by  $V(G) = X$  and

$$E(G) = \{(0, 2), (0, 4), (1, 3), (1, 5), (2, 5), (0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$$

Let  $T : X \rightarrow CB(X)$  and  $g : X \rightarrow X$  be defined by

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \\ 3 & \text{if } x = 2 \\ 2 & \text{if } x = 3 \\ 5 & \text{if } x = 4 \\ 4 & \text{if } x = 5 \end{cases}$$

and

$$T(x) = \begin{cases} \{1\} & \text{if } x = 0, 2, 4 \\ \{0, 2, 4\} & \text{if } x = 1, 3 \\ \{3, 5\} & \text{if } x = 5 \end{cases}$$

Then  $T$  is  $(\frac{1}{6}, \frac{1}{4}, 3, g)$ - $G$ -contraction mapping.

**Solution** Let  $(g(x), g(y)) \in E(G)$ .

If  $(g(x), g(y)) = (0, 2)$ , then  $x = 0, y = 3$ . It follows that  $\sup_{u \in T0} d(u, T3) = 1$  and  $\sup_{v \in T3} d(v, T0) = 3$ . From the definition of Pompeiu-Hausdorff, we have  $H(Tx, Ty) = 3$ , It is clear that  $D(g(0), T3) = 0, D(g(3), T0) = 1$ .

Put  $\alpha = \frac{1}{6}, \beta = \frac{1}{4}$  and  $L = 3$ .

Then  $H(Tx, Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}D(g(x), Ty) + (3)D(g(y), Tx)$ .

If  $(g(x), g(y)) = (0, 4)$ , then  $x = 0, y = 5$ . It follows that  $\sup_{u \in T_0} d(u, T_5) = 2$  and  $\sup_{v \in T_5} d(v, T_0) = 4$ . From the definition of Pompeiu-Hausdorff, we have  $H(Tx, Ty) = H(T_0, T_5) = 4$ , It is clear that  $D(g(0), T_5) = 3, D(g(5), T_0) = 3$ .

Put  $\alpha = \frac{1}{6}, \beta = \frac{1}{4}$  and  $L = 3$ .

Then  $H(Tx, Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}D(g(x), Ty) + (3)D(g(y), Tx)$ .

If  $(g(x), g(y)) = (1, 3)$ , then  $x = 1, y = 2$ . It follows that  $\sup_{u \in T_1} d(u, T_2) = 3$  and  $\sup_{v \in T_2} d(v, T_1) = 1$ . From the definition of Pompeiu-Hausdorff, we have  $H(Tx, Ty) = H(T_1, T_2) = 3$ , It is clear that  $D(g(1), T_2) = 0, D(g(2), T_1) = 1$ .

Put  $\alpha = \frac{1}{6}, \beta = \frac{1}{4}$  and  $L = 3$ .

Then  $H(Tx, Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}D(g(x), Ty) + (3)D(g(y), Tx)$ .

If  $(g(x), g(y)) = (1, 5)$ , then  $x = 1, y = 4$ . It follows that  $\sup_{u \in T_1} d(u, T_4) = 3$  and  $\sup_{v \in T_4} d(v, T_1) = 1$ . From the definition of Pompeiu-Hausdorff, we have  $H(Tx, Ty) = H(T_1, T_4) = 3$ , It is clear that  $D(g(1), T_4) = 0, D(g(4), T_1) = 1$ .

Put  $\alpha = \frac{1}{6}, \beta = \frac{1}{4}$  and  $L = 3$ .

Then  $H(Tx, Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}D(g(x), Ty) + (3)D(g(y), Tx)$ .

If  $(g(x), g(y)) = (2, 5)$ , then  $x = 3, y = 4$ . It follows that  $\sup_{u \in T_3} d(u, T_4) = 3$  and  $\sup_{v \in T_4} d(v, T_3) = 1$ . From the definition of Pompeiu-Hausdorff, we have  $H(Tx, Ty) = H(T_3, T_4) = 3$ , It is clear that  $D(g(3), T_4) = 1, D(g(4), T_3) = 1$ .

Put  $\alpha = \frac{1}{6}, \beta = \frac{1}{4}$  and  $L = 3$ .

Then  $H(Tx, Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}D(g(x), Ty) + (3)D(g(y), Tx)$ .

If  $(g(x), g(y)) \in \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$ . It obvious that  $H(Tx, Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}D(g(x), Ty) + (3)D(g(y), Tx)$ . Hence  $T$  is  $(\frac{1}{6}, \frac{1}{4}, 3, g)$ -G-contraction mapping.

**Theorem 3.1.** Let  $(X, d)$  be a complete metric space and  $G = (V(G), E(G))$  be directed graph such that  $V(G) = X$ , and let  $g : X \rightarrow X$  be a surjective mapping. If  $T : X \rightarrow CB(X)$  is a multi-valued mapping satisfying the following properties :

- 1)  $T$  is a  $g$ -graph-preserving mapping ;
- 2) there exists  $x_0 \in X$  such that  $(g(x_0), y) \in E(G)$  for some  $y \in Tx_0$ ;
- 3)  $X$  has Property A;
- 4)  $T$  is a  $(\alpha, \beta, L, g)$ -G-contraction;

Then there exists  $u \in X$  such that  $g(u) \in Tu$ .

**Proof.** Let  $x_0 \in X$ . Since  $g$  is surjective, there exists  $x_1 \in X$  such that  $g(x_1) \in Tx_0$ .

By 2), we have  $(g(x_0), g(x_1)) \in E(G)$ . Since  $(\beta - \alpha)d(g(x_0), g(x_1)) > 0$  and Lemma 2.4, there exists  $g(x_2) \in Tx_1$  such that

$$d(g(x_1), g(x_2)) \leq H(Tx_0, Tx_1) + (\beta - \alpha)d(g(x_0), g(x_1)). \quad (3.2)$$

Since  $(g(x_0), g(x_1)) \in E(G), g(x_1) \in Tx_0, g(x_2) \in Tx_1$  and  $T$  is  $g$ -graph-preserving mapping,

then  $(g(x_1), g(x_2)) \in E(G)$ . From (3.2) and  $T$  is  $(\alpha, \beta, L, g)$ -G-contraction, we have

$$\begin{aligned}
 d(g(x_1), g(x_2)) &\leq H(Tx_0, Tx_1) + (\beta - \alpha)d(g(x_0), g(x_1)) \\
 &\leq \alpha d(g(x_0), g(x_1)) + \beta D(g(x_0), Tx_1) + LD(g(x_1), Tx_0) + \\
 &\quad (\beta - \alpha)d(g(x_0), g(x_1)) \\
 &= \alpha d(g(x_0), g(x_1)) + \beta d(g(x_0), Tx_1) + (\beta - \alpha)d(g(x_0), g(x_1)) \\
 &\leq \beta d(g(x_0), g(x_1)) + \beta d(g(x_0), g(x_2)) \\
 &\leq \beta [d(g(x_0), g(x_1)) + d(g(x_1), g(x_2))] + \beta d(g(x_0), g(x_1)) \\
 &= 2\beta d(g(x_0), g(x_1)) + \beta d(g(x_1), g(x_2)).
 \end{aligned}$$

It implies that  $d(g(x_1), g(x_2)) \leq \frac{2\beta}{1-\beta} d(g(x_0), g(x_1))$ .

Since  $(\beta - \alpha)d(g(x_1), g(x_2)) > 0$  and lemma 2.4, there exists  $g(x_3) \in Tx_2$  such that

$$d(g(x_2), g(x_3)) \leq H(Tx_1, Tx_2) + (\beta - \alpha)d(g(x_1), g(x_2)). \quad (3.3)$$

Since  $(g(x_1), g(x_2)) \in E(G)$ ,  $g(x_2) \in Tx_1$ ,  $g(x_3) \in Tx_2$  and  $T$  is  $g$ -graph-preserving mapping, we have  $(g(x_2), g(x_3)) \in E(G)$ . From (3.3) and  $T$  is  $(\alpha, \beta, L, g)$ -G-contraction, we have

$$\begin{aligned}
 d(g(x_2), g(x_3)) &\leq H(Tx_1, Tx_2) + (\beta - \alpha)d(g(x_1), g(x_2)) \\
 &\leq \alpha d(g(x_1), g(x_2)) + \beta D(g(x_1), Tx_2) + LD(g(x_2), Tx_1) + \\
 &\quad (\beta - \alpha)d(g(x_1), g(x_2)) \\
 &= \alpha d(g(x_1), g(x_2)) + \beta d(g(x_1), Tx_2) + (\beta - \alpha)d(g(x_1), g(x_2)) \\
 &\leq \beta d(g(x_1), g(x_2)) + \beta d(g(x_1), g(x_3)) \\
 &\leq \beta [d(g(x_1), g(x_2)) + d(g(x_2), g(x_3))] + \beta d(g(x_1), g(x_2)) \\
 &= 2\beta d(g(x_1), g(x_2)) + \beta d(g(x_2), g(x_3)).
 \end{aligned}$$

It implies that  $d(g(x_2), g(x_3)) \leq \frac{2\beta}{1-\beta} d(g(x_1), g(x_2)) \leq (\frac{2\beta}{1-\beta})^2 d(g(x_0), g(x_1))$ .

By the same way, we have

$$d(g(x_k), g(x_{k+1})) \leq (\frac{2\beta}{1-\beta})^k d(g(x_0), g(x_1)),$$

$g(x_{k+1}) \in Tx_k$  and  $(g(x_k), g(x_{k+1})) \in E(G)$ , for all  $k \in \mathbb{N}$ .

For any numbers  $p \in \mathbb{N}$ , we have

$$\begin{aligned}
 d(g(x_{k+p}), g(x_k)) &\leq \sum_{n=k}^{k+p-1} d(g(x_{n+1}), g(x_n)) \\
 &\leq \sum_{n=k}^{k+p-1} (\frac{2\beta}{1-\beta})^n d(g(x_1), g(x_0)) \\
 &\leq \frac{(\frac{2\beta}{1-\beta})^k}{1 - (\frac{2\beta}{1-\beta})} d(g(x_1), g(x_0)). \quad (3.4)
 \end{aligned}$$

From  $(\frac{2\beta}{1-\beta})^k \rightarrow 0$  as  $k \rightarrow \infty$ , and (3.4), we can conclude that  $\{g(x_k)\}_{k=0}^{\infty}$  is a Cauchy sequence. Since  $X$  is complete, there exists  $u \in X$  such that  $\lim_{k \rightarrow \infty} g(x_k) = g(u)$ .

From  $(g(x_k), g(x_{k+1})) \in E(G)$  for all  $k \in \mathbb{N}$  and assumption 3), there is a subsequence  $g(x_{k_n})$  of  $g(x_k)$  such that  $(g(x_{k_n}), g(u)) \in E(G)$ .

From 4) and  $g(x_{k+1}) \in Tx_k$ , we have

$$\begin{aligned} D(g(u), Tu) &\leq d(g(u), g(x_{k_n+1})) + D(g(x_{k_n+1}), Tu) \\ &\leq d(g(u), g(x_{k_n+1})) + H(Tx_{k_n}, Tu) \\ &\leq d(g(u), g(x_{k_n+1})) + \alpha d(g(x_{k_n}), g(u)) + \beta D(g(x_{k_n}), Tu) + LD(g(u), Tx_{k_n}) \\ &\leq d(g(u), g(x_{k_n+1})) + \alpha d(g(x_{k_n}), g(u)) + \beta (d(g(x_{k_n}), g(u)) + D(g(u), Tu)) + \\ &\quad + Ld(g(u), Tx_{k_n}). \end{aligned}$$

It follows that

$$\begin{aligned} D(g(u), Tu) &\leq \left(\frac{1}{1-\beta}\right)(d(g(u), g(x_{k_n+1})) + (\alpha + \beta)d(g(x_{k_n}), g(u)) + Ld(g(u), Tx_{k_n})) \\ &\leq \left(\frac{1}{1-\beta}\right)(d(g(u), g(x_{k_n+1})) + (\alpha + \beta)d(g(x_{k_n}), g(u)) + Ld(g(u), g(x_{k_n+1}))). \end{aligned}$$

Since  $g(x_{k_n})$  converges to  $g(u)$  as  $n \rightarrow \infty$ , thus  $D(g(u), Tu) \leq 0$ . Then  $D(g(u), Tu) = 0$ . By lemma 2.3 and  $Tu$  is closed, we conclude that  $g(u) \in Tu$ .  $\square$

**Corollary 3.2.** Let  $(X, d)$  be a metric space endowed with a partial order  $\leq$ ,  $g : X \rightarrow X$  be surjective and  $T : X \rightarrow CB(X)$  be a multivalued mapping. Suppose that

- 1)  $T$  is a  $g$ -increasing ;
- 2) there exists  $x_0 \in X$  and  $u \in Tx_0$  such that  $g(x_0) < u$  ;
- 3) for each sequence  $\{x_k\}$  such that  $g(x_k) < g(x_{k+1})$  for all  $k \in \mathbb{N}$  and  $g(x_k)$  converge to  $g(x)$ , for some  $x \in X$ , then  $g(x_k) < g(x)$  for all  $k \in \mathbb{N}$ ;
- 4) there exists  $\alpha, \beta \in (0, 1)$  with  $0 < \alpha < \beta < \frac{1}{3}$  and  $L \geq 0$  such that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta D(g(x), Ty) + LD(g(y), Tx)$$

for any  $x, y \in X$  with  $g(x) < g(y)$ ;

- 5) the metric  $d$  is complete.

Then there exists  $u \in X$  such that  $g(u) \in Tu$ .

**Proof.** Put  $G = (V(G), E(G))$  by  $V(G) = X$  and  $E(G) = \{(x, y) : x < y\}$ . Let  $x, y \in X$  such that  $(g(x), g(y)) \in E(G)$ . Then  $g(x) < g(y)$  so  $Tx \prec Ty$ . For any  $u \in Tx$  and  $v \in Ty$ , we have  $u < v$ , i.e.,  $(u, v) \in E(G)$ . So  $T$  is graph-preserving. By assumption 2), there exist  $x_0$  and  $u \in Tx_0$  such that  $g(x_0) < u$ , thus  $(g(x_0), u) \in E(G)$ . Hence 2) of theorem 3.1 is satisfied. It's obviously 3) and 4) of theorem 3.1 are also satisfied. Therefore Corollary is obtained directly by theorem 3.1.  $\square$

### 3.2 A Strong Convergence Theorem for a $\kappa_i$ -Strictly Pseudocontractive Mapping in Hilbert space

**Theorem 3.3.** Let  $C$  be a nonempty closed convex subset of a Hilbert space  $H$ . Let  $F$  be a bifunctions from  $C \times C$  into  $\mathbb{R}$  satisfying  $(A_1) - (A_4)$ . Let  $\{T_i\}_{i=1}^N$  be a finite

family of  $\kappa_i$ -pseudo contractions with  $\mathfrak{F} = \bigcap_{i=1}^N F(T_i) \cap EP(F) \neq \emptyset$ . Define a mapping  $T_{\kappa_i}$  by  $T_{\kappa_i} = \kappa_i x + (1 - \kappa_i)T_i x, \forall x \in C, i \in \{1, 2, \dots, N\}$ . Let  $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, j = 1, 2, 3, \dots, N$ , where  $I = [0, 1], \alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j \in (0, 1)$  for all  $j = 1, 2, \dots, N-1, \alpha_1^N \in (0, 1), \alpha_2^j, \alpha_3^j \in [0, 1)$  for all  $j = 1, 2, \dots, N$ .

Let  $S$  be the mapping generated by  $T_{\kappa_1}, T_{\kappa_2}, \dots, T_{\kappa_N}$  and  $\alpha_1, \alpha_2, \dots, \alpha_N$ . Let  $\{x_n\}$  be sequences generated by  $x_1, u \in C$ :

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - (\beta_n S x_n + (1 - \beta_n)x_n) \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)u_n, \quad \forall n \geq 1, \end{cases} \quad (3.5)$$

where  $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$  such that  $r_n \in (a, b)$  and  $\beta_n \in (c, d] \subset (0, 1)$

Assume that

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,
- (ii)  $\sum_{n=1}^{\infty} |r_{n+1} - r_n|, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ .

Then the sequence  $\{x_n\}$  converge strongly to  $z = P_{\mathfrak{F}}u$ .

**Proof.** We shall divide our proof into 5 steps.

**Step 1.** We shall show that the sequence  $\{x_n\}$  is bounded. Let  $z \in \mathfrak{F}$ , since

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - (\beta_n S x_n + (1 - \beta_n)x_n) \rangle \geq 0, \forall y \in C,$$

by Lemma 2.10, it implies that  $u_n = S_{r_n}(\beta_n S x_n + (1 - \beta_n)x_n)$  and  $z \in F(S_{r_n})$ . Then, we have

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n \|u - z\| + (1 - \alpha_n) \|u_n - z\| \\ &= \alpha_n \|u - z\| + (1 - \alpha_n) \|S_{r_n}(\beta_n S x_n + (1 - \beta_n)x_n) - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n) \|\beta_n S x_n + (1 - \beta_n)x_n - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n) (\beta_n \|S x_n - z\| + (1 - \beta_n) \|x_n - z\|) \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n) (\beta_n \|x_n - z\| + (1 - \beta_n) \|x_n - z\|) \\ &= \alpha_n \|u - z\| + (1 - \alpha_n) \|x_n - z\| \\ &\leq \max \{ \|u - z\|, \|x_n - z\| \}. \end{aligned}$$

By induction we can prove that  $\{x_n\}$  is bounded, so are  $\{S x_n\}, \{u_n\}$ .

**Step 2.** We will show that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ .

Putting  $y_n = \beta_n S x_n + (1 - \beta_n)x_n$ . Since

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n u + (1 - \alpha_n)u_n - \alpha_{n-1}u - (1 - \alpha_{n-1})u_{n-1}\| \\ &= \|\alpha_n u + (1 - \alpha_n)u_n - (1 - \alpha_n)u_{n-1} + (1 - \alpha_n)u_{n-1} \\ &\quad - \alpha_{n-1}u - (1 - \alpha_{n-1})u_{n-1}\| \\ &= \|(\alpha_n - \alpha_{n-1})u + (1 - \alpha_n)(u_n - u_{n-1}) + (\alpha_{n-1} - \alpha_n)u_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|u_n - u_{n-1}\| \\ &\quad + |\alpha_{n-1} - \alpha_n| \|u_{n-1}\|. \end{aligned} \quad (3.6)$$

Since  $u_n = S_{r_n} y_n$  and  $u_{n-1} = S_{r_{n-1}} y_{n-1}$ , we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C, \quad (3.7)$$

and

$$F(u_{n-1}, y) + \frac{1}{r_{n-1}} \langle y - u_{n-1}, u_{n-1} - y_{n-1} \rangle \geq 0, \quad \forall y \in C. \quad (3.8)$$

Since  $u_n, u_{n-1} \in C$ , by (3.7) and (3.8), we have

$$F(u_n, u_{n-1}) + \frac{1}{r_n} \langle u_{n-1} - u_n, u_n - y_n \rangle \geq 0,$$

and

$$F(u_{n-1}, u_n) + \frac{1}{r_{n-1}} \langle u_n - u_{n-1}, u_{n-1} - y_{n-1} \rangle \geq 0,$$

from (A2), we have

$$\frac{1}{r_n} \langle u_{n-1} - u_n, u_n - y_n \rangle + \frac{1}{r_{n-1}} \langle u_n - u_{n-1}, u_{n-1} - y_{n-1} \rangle \geq 0,$$

it follows that

$$\langle u_n - u_{n-1}, u_{n-1} - y_{n-1} - \frac{r_{n-1}}{r_n} (u_n - y_n) \rangle \geq 0,$$

then, we have

$$\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - y_{n-1} - \frac{r_{n-1}}{r_n} (u_n - y_n) \rangle \geq 0,$$

it implies that

$$\begin{aligned} \|u_{n-1} - u_n\|^2 &\leq \langle u_n - u_{n-1}, u_n - y_{n-1} - \frac{r_{n-1}}{r_n} (u_n - y_n) \rangle \\ &= \langle u_n - u_{n-1}, u_n - y_n + y_n - y_{n-1} - \frac{r_{n-1}}{r_n} (u_n - y_n) \rangle \\ &= \langle u_n - u_{n-1}, y_n - y_{n-1} + (1 - \frac{r_{n-1}}{r_n})(u_n - y_n) \rangle \\ &\leq \|u_n - u_{n-1}\| \{ \|y_n - y_{n-1}\| + |1 - \frac{r_{n-1}}{r_n}| \|u_n - y_n\| \}. \end{aligned} \quad (3.9)$$

By (3.9), we have

$$\begin{aligned} \|u_{n-1} - u_n\| &\leq \|y_n - y_{n-1}\| + |1 - \frac{r_{n-1}}{r_n}| \|u_n - y_n\| \\ &= \|\beta_n S x_n + (1 - \beta_n) x_n - \beta_{n-1} S x_{n-1} - (1 - \beta_{n-1}) x_{n-1}\| \\ &\quad + |1 - \frac{r_{n-1}}{r_n}| \|u_n - y_n\| \\ &= \|\beta_n S x_n + (1 - \beta_n) x_n - (1 - \beta_n) x_{n-1} + (1 - \beta_n) x_{n-1} - \beta_n S x_{n-1} + \beta_n S x_{n-1} \\ &\quad - \beta_{n-1} S x_{n-1} - (1 - \beta_{n-1}) x_{n-1}\| + |1 - \frac{r_{n-1}}{r_n}| \|u_n - y_n\| \\ &\leq (1 - \beta_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \beta_n \|S x_n - S x_{n-1}\| \\ &\quad + |\beta_{n-1} - \beta_n| \|S x_{n-1}\| + |1 - \frac{r_{n-1}}{r_n}| \|u_n - y_n\| \\ &\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + |\beta_{n-1} - \beta_n| \|S x_{n-1}\| \\ &\quad + |1 - \frac{r_{n-1}}{r_n}| \|u_n - y_n\| \\ &\leq \|x_n - x_{n-1}\| + 2M |\beta_{n-1} - \beta_n| + \frac{1}{a} |r_n - r_{n-1}| M. \end{aligned} \quad (3.10)$$

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where  $M = \max_{n \in \mathbb{N}} \{\|x_n\|, \|Sx_n\|, \|u_n - y_n\|\}$ . Substitute (3.10) into (3.6)

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|u_n - u_{n-1}\| \\
&\quad + |\alpha_{n-1} - \alpha_n| \|u_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) (\|x_n - x_{n-1}\| + 2M |\beta_{n-1} - \beta_n|) \\
&\quad + \frac{1}{a} |r_n - r_{n-1}| M + |\alpha_{n-1} - \alpha_n| \|u_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|x_n - x_{n-1}\| + 2M |\beta_{n-1} - \beta_n| \\
&\quad + \frac{1}{a} |r_n - r_{n-1}| M + |\alpha_{n-1} - \alpha_n| \|u_{n-1}\|
\end{aligned} \tag{3.11}$$

By (3.11), condition  $i$  and Lemma 2.8, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

Step 3. We will show that

$$\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0. \tag{3.13}$$

Since  $y_n = \beta_n Sx_n + (1 - \beta_n)x_n$ , we have

$$\beta_n (Sx_n - x_n) = y_n - x_n. \tag{3.14}$$

Claim that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0. \tag{3.15}$$

Since

$$\|x_{n+1} - y_n\| \leq \alpha_n \|u - y_n\| + (1 - \alpha_n) \|u_n - y_n\|. \tag{3.16}$$

For  $z \in EP(F) \cap F(T)$ , by firmly nonexpansiveness, we have

$$\begin{aligned}
\|u_n - z\|^2 &= \|S_{r_n} y_n - S_{r_n} z\|^2 \\
&\leq \langle u_n - z, y_n - z \rangle \\
&= \frac{1}{2} (\|u_n - z\|^2 + \|y_n - z\|^2 - \|u_n - y_n\|^2).
\end{aligned} \tag{3.17}$$

From (3.17)

$$\|u_n - z\|^2 \leq \|y_n - z\|^2 - \|u_n - y_n\|^2. \tag{3.18}$$

By definition of  $\{x_n\}$  and (3.18), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|u_n - z\|^2 \\
&\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) (\|y_n - z\|^2 - \|u_n - y_n\|^2) \\
&= \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|y_n - z\|^2 - (1 - \alpha_n) \|u_n - y_n\|^2 \\
&\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) ((1 - \beta_n) \|x_n - z\|^2 + \beta_n \|Sx_n - z\|^2) \\
&\quad - (1 - \alpha_n) \|u_n - y_n\|^2 \\
&\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|x_n - z\|^2 - (1 - \alpha_n) \|u_n - y_n\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned} (1 - \alpha_n)\|u_n - y_n\|^2 &\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\|, \end{aligned}$$

by condition  $i$  and (3.12), we have

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0. \quad (3.19)$$

By (3.19), condition  $i$  and (3.16), we obtain (3.15). Again by (3.15) and (3.12), we have

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \quad (3.20)$$

By (3.20), (3.14) and  $\beta_n \in (c, d]$ , we have (3.13)

Step 4. Putting  $z_0 \in \mathfrak{F}$ . Show that

$$\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle \leq 0.$$

To show this inequality, choose subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that

$$\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{k \rightarrow \infty} \langle u - z_0, x_{n_k} - z_0 \rangle. \quad (3.21)$$

Without loss of generality, we may assume that  $x_{n_k} \rightarrow \omega$  as  $k \rightarrow \infty$  where  $\omega \in C$ . We first show  $\omega \in EP(F)$ . By (3.20), we have  $y_{n_k} \rightarrow \omega$  as  $k \rightarrow \infty$ . Since  $u_n = S_{r_n}y_n$ , for every  $y \in C$ , we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C,$$

from (A2), we have

$$\frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq F(y, u_n) \quad \forall y \in C.$$

In particular, we have

$$\langle y - u_{n_k}, \frac{u_{n_k} - y_{n_k}}{r_{n_k}} \rangle \geq F(y, u_{n_k}) \quad \forall y \in C. \quad (3.22)$$

By (3.19) and  $y_{n_k} \rightarrow \omega$  as  $k \rightarrow \infty$ , we have  $u_{n_k} \rightarrow \omega$  as  $k \rightarrow \infty$ . Again by (3.19), (3.22) and  $u_{n_k} \rightarrow \omega$  as  $k \rightarrow \infty$ , we have

$$0 \geq F(y, \omega), \quad \forall y \in C. \quad (3.23)$$

Let  $t \in (0, 1]$  and  $y \in C$ , setting  $y_t = ty + (1 - t)\omega$ , then  $y_t \in C$ . By (3.23)

$$0 \geq F(y_t, \omega). \quad \forall y \in C. \quad (3.24)$$

By (A1), (A4) and (3.24), we have

$$\begin{aligned} 0 &= F(y_t, y_t) \leq tF(y_t, y) + (1 - t)F(y_t, \omega) \\ &\leq tF(y_t, y). \end{aligned}$$

it follows that

$$0 \leq F(y_t, y). \quad (3.25)$$

From (A3) and (3.25), we have

$$0 \leq F(\omega, y),$$

then  $\omega \in EP(F)$ . By  $x_{n_k} \rightarrow \omega$  as  $k \rightarrow \infty$ , (3.13) and Lemma 2.7, we have  $\omega \in F(S)$ .

Hence  $\omega \in \mathfrak{F}$ . By (3.21)

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle &= \lim_{k \rightarrow \infty} \langle u - z_0, x_{n_k} - z_0 \rangle \\ &= \langle u - z_0, \omega - z_0 \rangle \leq 0. \end{aligned}$$

Step 5. Finally, we show that  $x_n \rightarrow z_0$  as  $n \rightarrow \infty$ , where  $z_0 = P_{\mathfrak{F}}u$ . By nonexpansiveness of  $T$  and  $S_{r_n}$ , we have

$$\begin{aligned} \|x_{n+1} - z_0\|^2 &= \|\alpha_n(u - z_0) + (1 - \alpha_n)(u_n - z_0)\|^2 \\ &\leq (1 - \alpha_n)^2 \|u_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\leq (1 - \alpha_n) \|S_{r_n} y_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\leq (1 - \alpha_n) \|y_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &= (1 - \alpha_n) \|\beta_n(Sx_n - z_0) + (1 - \beta_n)(x_n - z_0)\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\leq (1 - \alpha_n) (\beta_n \|Sx_n - z_0\|^2 + (1 - \beta_n) \|x_n - z_0\|^2) + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\leq (1 - \alpha_n) \|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle. \end{aligned}$$

from step 4, condition  $i$  and Lemma 2.8, we can conclude  $\{x_n\}$  converge strongly to  $z_0 = P_{\mathfrak{F}}u$ . Complete the proof.  $\square$

## Chapter 4

### Conclusions

In this section, we conclude all main results obtain in the thesis

#### 4.1 Fixed point with graph

(1) Let  $(X, d)$  be a complete metric space and  $G = (V(G), E(G))$  be directed graph such that  $V(G) = X$ , and let  $g : X \rightarrow X$  be a surjective mapping. If  $T : X \rightarrow CB(X)$  is a multi-valued mapping satisfying the following properties :

- 1)  $T$  is a  $g$ -graph-preserving mapping ;
- 2) there exists  $x_0 \in X$  such that  $(g(x_0), y) \in E(G)$  for some  $y \in Tx_0$ ;
- 3)  $X$  has Property A;
- 4)  $T$  is a  $(\alpha, \beta, L, g)$ - $G$ -contraction;

Then there exists  $u \in X$  such that  $g(u) \in Tu$ .

(2) Let  $(X, d)$  be a metric space endowed with a partial order  $\leq$ ,  $g : X \rightarrow X$  be surjective and  $T : X \rightarrow CB(X)$  be a multivalued mapping. Suppose that

- 1)  $T$  is a  $g$ -increasing ;
- 2) there exists  $x_0 \in X$  and  $u \in Tx_0$  such that  $g(x_0) < u$  ;
- 3) for each sequence  $\{x_k\}$  such that  $g(x_k) < g(x_{k+1})$  for all  $k \in \mathbb{N}$  and  $g(x_k)$  converge to  $g(x)$ , for some  $x \in X$  , then  $g(x_k) < g(x)$  for all  $k \in \mathbb{N}$ ;
- 4) there exists  $\alpha, \beta \in (0, 1)$  with  $0 < \alpha < \beta < \frac{1}{3}$  and  $L \geq 0$  such that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta D(g(x), Ty) + LD(g(y), Tx)$$

for any  $x, y \in X$  with  $g(x) < g(y)$ ;

5) the metric  $d$  is complete.

Then there exists  $u \in X$  such that  $g(u) \in Tu$ .

#### 4.2 A Strong Convergence Theorem for a $\kappa_i$ -Strictly Pseudocontractive Mapping in Hilbert space

(1) Let  $C$  be a nonempty closed convex subset of a Hilbert space  $H$ . Let  $F$  be a bifunctions from  $C \times C$  into  $\mathbb{R}$  satisfying  $(A_1) - (A_4)$ . Let  $\{T_i\}_{i=1}^N$  be a finite family of  $\kappa_i$ -psuedo contractions with  $\mathfrak{F} = \bigcap_{i=1}^N F(T_i) \cap EP(F) \neq \emptyset$ . Define a mapping  $T_{\kappa_i}$  by  $T_{\kappa_i} = \kappa_i x + (1 - \kappa_i)T_i x, \forall x \in C, i \in \{1, 2, \dots, N\}$ . Let  $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, j = 1, 2, 3, \dots, N$ , where  $I = [0, 1]$  ,  $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j \in (0, 1)$  for all  $j = 1, 2, \dots, N - 1, \alpha_1^N \in (0, 1], \alpha_2^j, \alpha_3^j \in [0, 1)$  for all  $j = 1, 2, \dots, N$ . Let  $S$  be the mapping generated by

$T_{\kappa_1}, T_{\kappa_2}, \dots, T_{\kappa_N}$  and  $\alpha_1, \alpha_2, \dots, \alpha_N$ . Let  $\{x_n\}$  be sequences generated by  $x_1, u \in C$ :

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - (\beta_n Sx_n + (1 - \beta_n)x_n) \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)u_n, \quad \forall n \geq 1, \end{cases} \quad (4.1)$$

where  $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$  such that  $r_n \in (a, b)$  and  $\beta_n \in (c, d) \subset (0, 1]$

Assume that

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ .
- (ii)  $\sum_{n=1}^{\infty} |r_{n+1} - r_n|, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ .

Then the sequence  $\{x_n\}$  converge strongly to  $z = P_{\mathfrak{F}}u$ .



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# Appendix A

## The Basic Definitions



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**Definition 4.1.** Let  $\{x_n\}$  be a sequence in  $\mathbb{R}$ .  $\{x_n\}$  is called bounded sequence, if there exists  $M > 0$  such that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ .

**Definition 4.2.** Let  $\{a_n\}$  be a sequence of real numbers and let  $a$  be a real number. We say that  $\{a_n\}$  converges to  $a$  if  $|a_n - a| \rightarrow 0$ . We usually symbolize this by writing  $a_n \rightarrow a (n \rightarrow \infty)$ ,  $\lim_{n \rightarrow \infty} a_n = a$ , or  $a_n \rightarrow a$ . The point  $a$  is called the limit of the sequence  $\{a_n\}$ . We also say that  $\{a_n\}$  is convergent, or  $\{a_n\}$  converges.

**Theorem 4.1.** Let  $\{a_n\}$  be a bounded sequence of real numbers. Then, there exists a convergent subsequence of  $\{a_n\}$ .

**Definition 4.3.** Let  $\mathbb{R}$  be the set of real number and  $A \subset \mathbb{R}$ .  $c$  is called limit point of  $A$  if every  $\epsilon > 0$ , there is  $x \in A$  such that  $|x - c| < \epsilon$ .

**Definition 4.4.** Let  $X$  be a metric space and  $F \subset X$  is called closed set if and only if for every limits point of  $F$  is in  $F$ . i.e. for every  $X$  is limit point of  $F$  imply  $x \in F$ .

**Definition 4.5.** Let  $X$  be a metric space and  $A \subset X$ . Then  $\bar{A} = A' \cup A$  is called closure set of  $A$  where  $A'$  is the set of all limit point of  $A$ .

**Theorem 4.2.** Let  $X$  be a metric space. If  $A$  is closed set then  $A = \bar{A}$

**Definition 4.6.** A vector space over  $\mathbb{F}$  is a set  $V$  together with the operations of addition  $V \times V \rightarrow V$  and scalar multiplication  $\mathbb{F} \times V \rightarrow V$  satisfying the following properties:

- 1) Commutativity:  $u + v = v + u$  for all  $u, v \in V$ ;
- 2) Associativity:  $(u + v) + w = u + (v + w)$  and  $(ab)v = a(bv)$  for all  $u, v, w \in V$  and  $a, b \in \mathbb{F}$ ;
- 3) Additive identity: There exists an element  $0 \in V$  such that  $0 + v = v$  for all  $v \in V$ ;
- 4) Additive inverse: For every  $v \in V$ , there exists an element  $w \in V$  such that  $v + w = 0$ ;
- 5) Multiplicative identity:  $1v = v$  for all  $v \in V$ ;
- 6) Distributivity:  $a(u + v) = au + av$  and  $(a + b)u = au + bu$  for all  $u, v \in V$  and  $a, b \in \mathbb{F}$ .

Usually, a vector space over  $\mathbb{R}$  is called a real vector space and a vector space over  $\mathbb{C}$  is called a complex vector space. The element  $v \in V$  of a vector space are called vectors.

**Definition 4.7.** A set  $C$  in a vector space is called *convex* if for any  $x, y \in C$  and  $\alpha \in (0, 1)$  we have

$$\alpha x + (1 - \alpha)y \in C.$$

**Definition 4.8.** Let  $X$  be a vector space and the function  $\|\cdot\| : X \rightarrow \mathbb{R}$  is norm if it satisfies the following conditions:

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- 1)  $\|x\| \geq 0$ ;
- 2)  $\|x\| = 0 \Leftrightarrow x = 0$ ;
- 3)  $\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in \mathbb{F}$ ;
- 4)  $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$  (triangle inequality).

Then  $(X, \|\cdot\|)$  or  $X$  is called normed space.

**Definition 4.9.** Let  $(E, \|\cdot\|)$  be a normed space. We say that a sequence  $\{x_n\}$  of elements of  $E$  converges to some  $x \in E$  if for every  $\epsilon > 0$  there exists a number  $M$  such that for every  $n \geq M$  we have  $\|x_n - x\| < \epsilon$ . In such a case we write  $\lim_{n \rightarrow \infty} x_n = x$  or simply  $x_n \rightarrow x$ .

**Definition 4.10.** A sequence of vectors  $\{x_n\}$  in a normed space is called a *Cauchy sequence* if for every  $\epsilon > 0$  there exists a number  $M$  such that  $\|x_m - x_n\| < \epsilon$  for all  $m, n > M$ .

**Definition 4.11.** A normed space  $E$  is called *complete* if every Cauchy sequence in  $E$  converges to an element of  $E$ . A complete normed space is called a *Banach space*.

**Definition 4.12.** An inner product space is a vector space  $X$  with an inner product defined on  $X$ . Here, an inner product on  $X$  is a mapping of  $X \times X$  into the scalar field  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ; that is, with every pair of vector  $x$  and  $y$  there is associated a scalar which is written and is called the inner product of  $x$  and  $y$ , such that for all vectors  $x, y, z$  and scalar  $\alpha \in \mathbb{F}$  we have:

- 1)  $\langle x, x \rangle \geq 0$  and  $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ ;
- 2)  $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ ;
- 3)  $\langle x, y \rangle = \overline{\langle y, x \rangle}$ ;
- 4)  $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ .

An inner product on  $X$  defines a norm on  $X$  given by  $\|x\| = \sqrt{\langle x, x \rangle}$ .

**Remark 4.3.** 1) An inner product space is called a *real inner product space* for the case when the scalars are the real numbers and  $\langle x, y \rangle$  is a real number. For the case, 3) mean

$$\langle x, y \rangle = \langle y, x \rangle.$$

2) Using 2), 3) and 4) we obtain that for  $x, y \in X$  and  $\alpha, \beta \in \mathbb{C}$ ,

$$\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle.$$

**Theorem 4.4.** If  $x$  and  $y$  are any two vectors in an inner product space  $X$ , then

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

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**Theorem 4.5.** For any two elements  $x$  and  $y$  of inner product space  $X$  we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

**Definition 4.13.** Let  $X$  be an inner product space and  $X$  is called Hilbert space if  $X$  is complete inner product space.

**Definition 4.14.** A sequence  $\{x_n\}$  of vectors in an inner product space  $K$  is called strongly convergent to a vector  $x$  in  $K$  if

$$\|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Definition 4.15.** A sequence  $\{x_n\}$  of vectors in an inner product space  $K$  is called weakly convergent to a vector  $x$  in  $K$  if

$$\langle x_n, y \rangle \rightarrow \langle x, y \rangle \text{ as } n \rightarrow \infty \text{ for every } y \in K.$$

**Theorem 4.6.** A strongly convergence sequence is weakly convergence (to the same limit), that is,  $x_n \rightarrow x$  implies  $x_n \rightarrow x$ .

**Remark 4.7.** If  $x_n \rightarrow x$  and  $x_n \rightarrow y$ , then  $x = y$ .

**Lemma 4.8.** Let  $\{x_n\}$  be a Cauchy sequence of an inner product space  $C$  such that  $x_n \rightarrow x$ . Then  $x_n \rightarrow x$ .

**Theorem 4.9.** Let  $H$  be a Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . Suppose that  $\{x_n\} \subset C$  and  $x_n \rightarrow x$ . Then  $x \in C$ .

**Definition 4.16.** Let  $H$  be a Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . Let  $f$  be a function of  $C$  into  $(-\infty, \infty]$ , where  $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$ . Then,  $f$  is called lower semicontinuous if for any  $a \in \mathbb{R}$ , the set

$$\{x \in C : f(x) \leq a\} \text{ is closed.}$$

Moreover,  $f$  is called convex if for any  $x_1, x_2 \in C$  and  $t \in (0, 1)$ ,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Similarly,  $f$  is said to be concave if for any  $x_1, x_2 \in C$  and  $t \in (0, 1)$ ,

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2).$$

**Theorem 4.10.** Let  $\{a_n\}$  be a bounded of real numbers. Then, there exists subsequence  $\{a_{n_i}\}$  of  $\{a_n\}$  such that

$$\alpha = \limsup_{n \rightarrow \infty} a_n = \lim_{i \rightarrow \infty} a_{n_i}.$$

Similarly, there exists a subsequence  $\{a_{n_j}\}$  of  $\{a_n\}$  such that

$$\beta = \liminf_{n \rightarrow \infty} a_n = \lim_{j \rightarrow \infty} a_{n_j}.$$

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Remark 4.11. Let  $H$  be an inner product space. Then we know that the following (1) and (2) are equivalent:

- 1)  $H$  is complete,
- 2) each bounded sequence  $\{x_n\}$  of  $H$  has a weakly convergence subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$ .

Definition 4.17. Let  $E$  and  $F$  be linear spaces with the same scalars, and let  $T$  be a mapping of  $E$  into  $F$ . Then  $T$  is called *linear* if for any  $x, y \in E$  and any scalar  $\alpha \in \mathbb{R}$ ,

$$T(x + y) = T(x) + T(y) \text{ and } T(\alpha x) = \alpha T(x).$$

In particular, for the case of  $F = \mathbb{R}$ ,  $T$  is called a *linear functional*.

Note that if  $T : E \rightarrow F$  is a linear mapping, then

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \forall x, y \in E \text{ and } \alpha, \beta \in \mathbb{R}.$$

Definition 4.18. Let  $E$  and  $F$  be normed linear spaces with the same scalars, and let  $T$  be a linear mapping of  $E$  into  $F$ . Then  $T$  is called *bounded* if there exists  $K \geq 0$  such that

$$\|T(x)\| \leq K\|x\| \text{ for all } x \in E.$$

Let  $T$  be a bounded linear mapping of  $E$  into  $F$ . So, we have that for  $x \in E$  with  $\|x\| \leq 1$ ,

$$\|T(x)\| \leq K, \tag{4.2}$$

where  $T(x)$  is often denoted by  $Tx$ .

Definition 4.19. A graph is a mathematical structure consisting of a set of vertices and a set of edges connecting the vertices.

Formally:  $G = (V(G), E(G))$ , where  $V(G)$  is a set and  $E(G) = V \times V$ .

$G = (V(G), E(G))$  undirected if for all  $v, w \in V$ ,  $(v, w) \in E$  if and only if  $(w, v) \in E$ .

Otherwise directed.

Example Let  $G = (V(G), E(G))$  with vertex set  $V(G) = \{0, 1, 2, 3, 4, 5, 6\}$  and  $E(G) = \{(0,2),(0,4),(0,5),(1,0),(2,1),(2,5),(3,1),(3,6),(4,0),(4,5),(6,3),(6,5)\}$ .

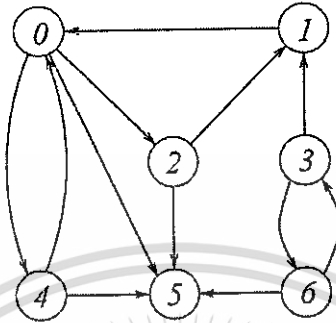


Figure 4.1: A directed graph  $G$ .

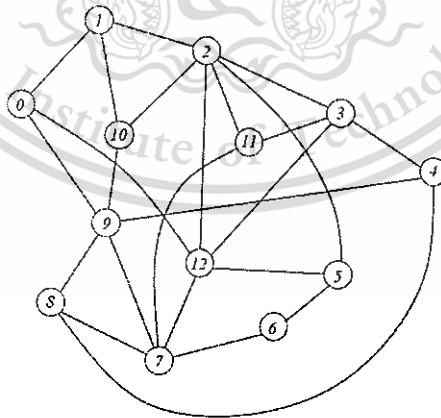


Figure 4.2: An undirected graph  $G$ .

# Appendix B

## The Research papers



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การประชมวิชาการเสนอผลงานวิจัยบัณฑิตศึกษา ระดับชาติและนานาชาติ 2560  
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PMP4

**A new method for equilibrium problems and a finite family of pseudocontractive mapping**  
**วิธีการใหม่สำหรับปัญหาเชิงดุลยภาพและการส่งของวงที่จำกัดแบบหดเทียม**

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**ABSTRACT**

The purpose of this paper is to prove the strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed point of a finite family of  $K_i$ -strictly pseudocontractive mapping (i.e. there exist  $K_i \in [0,1)$  such that  $\|Tx - Ty\|^2 \leq \|x - y\|^2 + K_i \|(I - T)x - (I - T)y\|^2$   $\forall x, y \in C$  and  $i = 1, 2, \dots, N$ ) in framework of Hilbert spaces by using different methods. Our main theorem improve and modify some previously proposed results.

**บทคัดย่อ**

วัตถุประสงค์ของงานวิจัยฉบับนี้คือพิสูจน์ทฤษฎีบทการส่งเวียนแบบเข้มสำหรับการหาสมาชิกร่วมของเซตของผลเฉลยของปัญหาเชิงดุลยภาพและเซตของจุดตรึงของการส่งวงที่จำกัดของทรงส่ง  $K_i$  หดเทียม โดยแท้ (กล่าวคือ  $K_i \in [0,1)$  ซึ่งทำให้  $\|Tx - Ty\|^2 \leq \|x - y\|^2 + K_i \|(I - T)x - (I - T)y\|^2$  สำหรับทุก  $x, y \in C$  และ  $i = 1, 2, \dots, N$ ) ในปริภูมิฮิลเบิร์ต โดยใช้วิธีการที่แตกต่างกันจากงานวิจัยอื่นๆ ซึ่งผลการศึกษางานวิจัยฉบับนี้จะพัฒนาและปรับปรุงผลการศึกษางานวิจัยก่อนหน้าได้

**Keywords:** Strictly pseudocontractive mapping, Equilibrium problem, Fixed point

**คำสำคัญ:** การส่งหดเทียมโดยแท้ ปัญหาดุลยภาพ จุดตรึง

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### Introduction

Let  $H$  be a real Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . Let  $P_C$  be the projection of  $H$  onto  $C$ . A mapping  $T$  of  $H$  into itself is called nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in H$ . The set of fixed points of  $T$  is denoted by  $F(T)$  (i.e.,  $F(T) = \{x \in H : Tx = x\}$ ). A mapping  $T : C \rightarrow C$  is said to be a  $\kappa$ -strictly pseudo contraction mapping, if there exist  $\kappa \in [0, 1)$  such that  $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2$ ,  $\forall x, y \in C$ .

Let  $F : C \times C \rightarrow R$  be a bifunction. The equilibrium problem for  $F$  is to determine its equilibrium points, i.e. the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}. \quad (1.1)$$

The problem is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others, see (Blum, Oettli, 1994; Combettes, Hirstoaga, 2005; Takahashi, Takahashi, 2007).

Takahashi, Takahashi (2007) introduced viscosity approximation method for finding a common element of the set of solutions of problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. They defined the sequences  $\{x_n\}$  and  $\{u_n\}$  as follows: Let  $x_1 \in H$  and  $f : H \rightarrow H$  be a contraction mapping with  $\alpha \in (0, 1)$ ,  $\{r_n\} \subset [0, 1]$  and  $\{r'_n\} \subset (0, \infty)$

$$\begin{cases} F(u_n, y) + \frac{1}{r'_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T u_n, & \forall n \in N, \end{cases} \quad (1.2)$$

They proved strong convergence theorem of sequence  $\{x_n\}$  generated by (1.2) to  $z \in F(T) \cap EP(F)$  where  $z = P_{F(T) \cap EP(F)} f(z)$  under certain appropriate conditions imposed on the sequences  $\{\alpha_n\}$ ,  $\{r_n\}$  and bifunction  $F$ .

In this paper, motivated by (1.2), we introduce a new algorithm as follows: for  $u \in C$  and the sequence  $\{x_n\}$  generated by

$$\begin{cases} F(u_n, y) + \frac{1}{r'_n} \langle y - u_n, u_n - (\beta_n T x_n + (1 - \beta_n) x_n) \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) u_n, & \forall n \geq 1, \end{cases} \quad (1.3)$$

where  $F : C \times C \rightarrow R$  is a bifunction and  $T : C \rightarrow C$  is  $\kappa_1$ -strictly pseudocontractive mapping.

Under suitable conditions of parameters  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{r'_n\}$ , we prove strong convergence theorem for finding a common element of the set of equilibrium problem of finite family of  $\kappa_1$ -strictly pseudocontractive mapping in Hilbert spaces.

### Objective of the study

To introduce a new algorithm for finding a common element of the set of solutions of equilibrium problem and the set of fixed point of a finite family of  $\kappa_1$ -strictly pseudocontractive mapping in framework of Hilbert spaces.



### Preliminaries

In this section, we give some useful lemmas and definitions that will be needed for our main result.

**Definition 2.1.**[5] Let  $E$  be a real Banach space and  $D$  be a closed subset of  $E$ . A mapping  $T : D \rightarrow D$  is said to be demi-closed at the origin if, for any sequence  $\{x_n\}$  in  $D$ , the conditions  $x_n \rightarrow x_0$  weakly and  $Tx_n \rightarrow 0$  strongly imply  $Tx_0 = 0$ .

**Lemma 2.1.**[7] Let  $\{s_n\}$  be a sequence of nonnegative real numbers satisfying  $s_{n+1} = (1 - \alpha_n)s_n + \beta_n$ ,  $\forall n \geq 0$  where  $\{\alpha_n\}, \{\beta_n\}$  satisfy conditions

$$(1) \{\alpha_n\} \subset (0, 1), \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$(2) \limsup_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} \leq 0.$$

$$\text{Then } \lim_{n \rightarrow \infty} s_n = 0.$$

Let  $C$  be closed convex subset of real Hilbert space  $H$  and  $P_C$  be the metric projection of  $H$  onto  $C$  i.e, for  $x \in H, P_C x$  satisfies the property  $\|x - P_C x\| = \min_{y \in C} \|x - y\|$ .

The following lemma 2.2. characterizes the projection  $P_C$ .

**Lemma 2.2.**[5] Given  $x \in H$  and  $y \in C$ . Then  $P_C x = y$  if and only if there holds the inequality  $\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C$ .

**Lemma 2.3.**[2] Let  $E$  be a uniformly convex Banach space,  $C$  be a nonempty closed convex subset of  $E$  and  $S : C \rightarrow C$  be a nonexpansive mapping. Then  $I - S$  is demi-closed at zero.

**Lemma 2.4.**[8] Let  $\{s_n\}$  be a sequence of nonnegative real numbers satisfying

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0, \text{ where } \{\alpha_n\} \text{ is a sequence in } (0, 1) \text{ and } \{\delta_n\} \text{ is a sequence such that}$$

$$(1) \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$(2) \limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

$$\text{Then } \lim_{n \rightarrow \infty} s_n = 0.$$

For solving the equilibrium problem for a bifunction  $F : C \times C \rightarrow R$ , let us assume that  $F$  satisfy the following conditions:

$$(A1) F(x, x) = 0 \quad \forall x \in C;$$

$$(A2) F \text{ is monotone, } F(x, y) + F(y, x) \leq 0, \quad \forall x, y \in C;$$



$$(A3) \forall x, y, z \in C, \lim_{t \rightarrow 0^+} F(tx - (1-t)x, y) \leq F(x, y);$$

$$(A4) \forall x \in C, y \mapsto F(x, y) \text{ is convex and lower semi continuous.}$$

**Lemma 2.5.[1]** Let  $C$  be a nonempty closed convex subset of  $H$  and let  $F$  be a bifunction of  $C \times C$  into  $R$  satisfying (A1) - (A4). Let  $r > 0$  and  $x \in H$ . Then, there exists  $z \in C$  such that

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle > 0 \quad (2.1)$$

for all  $y \in C$ .

**Lemma 2.6.[3]** Assume that  $F : C \times C \rightarrow R$  satisfies (A1) - (A4). For  $r > 0$  and  $x \in H$ , define a mapping  $S_r : H \rightarrow C$  as follows:

$$S_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}. \quad (2.2)$$

for all  $z \in H$ . Then, the following hold:

(1)  $S_r$  is single-valued;

(2)  $S_r$  is firmly nonexpansive i.e.  $\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y \rangle \forall x, y \in H$ ;

(3)  $F(S_r) = EP(F)$ ;

(4)  $EP(F)$  is closed and convex.

**Lemma 2.7.[9]** In a real Hilbert spaces  $H$ , there holds the following inequality  $\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle$  for all  $x, y \in H$ .

**Lemma 2.8.[9]** Let  $C$  be a nonempty closed convex subset of real Hilbert space  $H$  and  $T : C \rightarrow C$  a  $\kappa$ -strict pseudo contraction. Define  $S : C \rightarrow C$  by  $Sx = \alpha x + (1 - \alpha)Tx$  for each  $x \in C$ . Then  $\alpha \in [\kappa, 1)$ ,  $S$  is nonexpansive such that  $F(S) = F(T)$ .

**Lemma 2.9.[4]** Let  $C$  be a nonempty closed convex subset of strictly convex. Let  $\{T_i\}_{i=1}^N$  be a finite family of nonexpansive mapping of  $C$  into itself with  $\bigcap_{i=1}^N F(T_i) \neq \emptyset$  and let  $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ ,  $j = 1, 2, 3, \dots, N$ , where  $I = [0, 1]$ ,  $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ ,  $\alpha_1^j \in (0, 1)$  for all  $j = 1, 2, 3, \dots, N - 1$ ,  $\alpha_1^N \in (0, 1]$ ,  $\alpha_2^j, \alpha_3^j \in [0, 1)$  for all  $j = 1, 2, 3, \dots, N$ . Let  $S$  be the mapping generated by  $T_1, T_2, \dots, T_N$  and  $\alpha_1, \alpha_2, \dots, \alpha_N$ . Then  $F(S) = \bigcap_{i=1}^N F(T_i)$ .

#### Main result

**Theorem 3.1** Let  $C$  be a nonempty closed convex subset of a Hilbert space  $H$ . Let  $F$  be a bifunctions from  $C \times C$  into  $R$  satisfying (A1) - (A4). Let  $\{T_i\}_{i=1}^N$  be a finite family of  $\kappa_i$ -strictly pseudocontractions with

$F = \bigcap_{i=1}^N F(T_i) \cap EP(F) \neq \emptyset$ . Define a mapping  $T_{\kappa_i}$  by  $T_{\kappa_i} = \kappa_i x + (1 - \kappa_i)T_i x$ ,  $\forall x \in C$ ,

$i \in \{1, 2, \dots, N\}$ . Let  $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ ,  $j = 1, 2, 3, \dots, N$ , where  $I = [0, 1]$ ,

$\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ ,  $\alpha_1^j \in (0, 1)$  for all  $j = 1, 2, 3, \dots, N - 1$ ,  $\alpha_1^N \in (0, 1]$ ,  $\alpha_2^j, \alpha_3^j \in [0, 1)$

for all  $j = 1, 2, 3, \dots, N$ . Let  $S$  be the mapping generated by  $T_{\kappa_1}, T_{\kappa_2}, \dots, T_{\kappa_N}$  and  $\alpha_1, \alpha_2, \dots, \alpha_N$ . Let  $\{x_n\}$  be sequence generated by  $x_1, u \in C$ :



$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - (\beta_n Sx_n + (1 - \beta_n)x_n) \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)u_n, & \forall n \geq 1, \end{cases} \quad (3.1)$$

where  $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$  such that  $r_n \in (a, b)$  and  $\beta_n \in (c, d] \subset (0, 1]$ .

Assume that

$$(i) \lim_{n \rightarrow \infty} \alpha_n = 0 \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty,$$

$$(ii) \sum_{n=0}^{\infty} |r_{n+1} - r_n|, \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n|, \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

Then the sequence  $\{x_n\}$  converge strongly to  $z = P_F u$ .

*Proof.* Step 1. We shall show that the sequence  $\{x_n\}$  is bounded.

Let  $z \in F$ . Since

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - (\beta_n Sx_n + (1 - \beta_n)x_n) \rangle \geq 0, \quad \forall y \in C, \quad (3.2)$$

by Lemma 2.6, it implies that  $u_n = S_{r_n}(\beta_n Sx_n + (1 - \beta_n)x_n)$  and  $z \in F(S_{r_n})$ .

Then, we have

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n \|u - z\| + (1 - \alpha_n) \|u_n - z\| \\ &= \alpha_n \|u - z\| + (1 - \alpha_n) \|S_{r_n}(\beta_n Sx_n + (1 - \beta_n)x_n) - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n) \|(\beta_n Sx_n + (1 - \beta_n)x_n) - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n) (\beta_n \|Sx_n - z\| + (1 - \beta_n) \|x_n - z\|) \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n) (\beta_n \|x_n - z\| + (1 - \beta_n) \|x_n - z\|) \\ &= \alpha_n \|u - z\| + (1 - \alpha_n) \|x_n - z\| \\ &\leq \max\{\|u - z\|, \|x_n - z\|\}. \end{aligned} \quad (3.3)$$

By induction we can prove that  $\{x_n\}$  is bounded, so are  $\{Sx_n\}$  and  $\{u_n\}$ .

Step 2. We will show that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ .

Put  $y_n = \beta_n Sx_n + (1 - \beta_n)x_n$ .

Consider

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n u + (1 - \alpha_n)u_n - \alpha_{n-1}u - (1 - \alpha_{n-1})u_{n-1}\| \\ &= \|\alpha_n u + (1 - \alpha_n)u_n + (1 - \alpha_n)u_{n-1} + (1 - \alpha_n)u_{n-1} - \alpha_{n-1}u - (1 - \alpha_{n-1})u_{n-1}\| \\ &= \|(\alpha_n - \alpha_{n-1})u + (1 - \alpha_n)(u_n - u_{n-1}) + (\alpha_n - \alpha_{n-1})u_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|u_{n-1}\|. \end{aligned} \quad (3.4)$$

and  $u_n = S_{r_n} y_n$  and  $u_{n-1} = S_{r_{n-1}} y_{n-1}$ , we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C, \quad (3.5)$$

and



$$F(u_{n-1}, y) + \frac{1}{r_{n-1}} \langle y - u_{n-1}, u_{n-1} - y_{n-1} \rangle \geq 0, \quad \forall y \in C, \quad (3.6)$$

Since  $u_n, u_{n-1} \in C$ , by (3.5) and (3.6), we have

$$F(u_n, u_{n-1}) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0. \quad (3.7)$$

and

$$F(u_{n-1}, u_n) + \frac{1}{r_{n-1}} \langle y - u_{n-1}, u_{n-1} - y_{n-1} \rangle \geq 0. \quad (3.8)$$

From (A2), we have

$$\frac{1}{r_n} \langle u_{n-1} - u_n, u_n - y_n \rangle + \frac{1}{r_{n-1}} \langle u_n - u_{n-1}, u_{n-1} - y_{n-1} \rangle \geq 0. \quad (3.9)$$

It follows that

$$\left\langle u_n - u_{n-1}, u_{n-1} - y_{n-1} - \frac{r_{n-1}}{r_n} (u_n - y_n) \right\rangle \geq 0. \quad (3.10)$$

Then, we have

$$\left\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - y_{n-1} - \frac{r_{n-1}}{r_n} (u_n - y_n) \right\rangle \geq 0. \quad (3.11)$$

It implies that

$$\begin{aligned} \|u_{n-1} - u_n\|^2 &\leq \left\langle u_n - u_{n-1}, u_{n-1} - y_{n-1} - \frac{r_{n-1}}{r_n} (u_n - y_n) \right\rangle \\ &= \left\langle u_n - u_{n-1}, u_n + y_n - y_{n-1} - \frac{r_{n-1}}{r_n} (u_n - y_n) \right\rangle \\ &= \left\langle u_n - u_{n-1}, y_n - y_{n-1} \left( 1 - \frac{r_{n-1}}{r_n} \right) + \frac{r_{n-1}}{r_n} (u_n - y_n) \right\rangle \\ &\leq \|u_n - u_{n-1}\| \left\{ \|y_n - y_{n-1}\| + \left| 1 - \frac{r_{n-1}}{r_n} \right| \|u_n - y_n\| \right\}. \end{aligned} \quad (3.12)$$

By (3.12), we have



$$\begin{aligned}
 \|u_{n+1} - u_n\| &\leq \|y_n - y_{n-1}\| + \frac{r_n}{r_{n-1}} \|u_n - y_n\| \\
 &= \|\beta_n Sx_n - (1 - \beta_n)x_n - \beta_{n-1} Sx_{n-1} + (1 - \beta_{n-1})x_{n-1}\| + \frac{r_n}{r_{n-1}} \|u_n - y_n\| \\
 &= \|\beta_n Sx_n - (1 - \beta_n)x_n - (1 - \beta_n)x_{n-1} + (1 - \beta_n)x_{n-1} - \beta_{n-1} Sx_{n-1} - \beta_n Sx_{n-1} + \beta_n Sx_{n-1} \\
 &\quad + (1 - \beta_{n-1})x_{n-1}\| + \frac{r_n}{r_{n-1}} \|u_n - y_n\| \\
 &\leq (1 - \beta_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \beta_n \|Sx_n - Sx_{n-1}\| + |\beta_{n-1} - \beta_n| \|Sx_{n-1}\| \\
 &\quad + \left|1 - \frac{r_n}{r_{n-1}}\right| \|u_n - y_n\| \\
 &\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + |\beta_{n-1} - \beta_n| \|Sx_{n-1}\| + \left|1 - \frac{r_n}{r_{n-1}}\right| \|u_n - y_n\| \\
 &\leq \|x_n - x_{n-1}\| + 2M |\beta_{n-1} - \beta_n| + \frac{1}{\alpha} |r_n - r_{n-1}| M, \tag{3.13}
 \end{aligned}$$

where  $M = \max_{n \in \mathbb{N}} \{\|x_n\|, \|Sx_n\|, \|u_n - y_n\|\}$ . Substitute (3.13) into (3.4), we obtain

$$\begin{aligned}
 \|x_{n+1} - x_n\| &\leq |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|u_n - u_{n-1}\| + |\alpha_{n-1} - \alpha_n| \|u_{n-1}\| \\
 &\leq |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \left( \|x_n - x_{n-1}\| + 2M |\beta_{n-1} - \beta_n| + \frac{1}{\alpha} |r_n - r_{n-1}| M \right) \\
 &\quad + |\alpha_n - \alpha_{n-1}| \|u_{n-1}\| \\
 &\leq |\alpha_n - \alpha_{n-1}| \|u\| + (1 - \alpha_n) \|x_n - x_{n-1}\| + 2M |\beta_{n-1} - \beta_n| + \frac{1}{\alpha} |r_n - r_{n-1}| M \\
 &\quad + |\alpha_n - \alpha_{n-1}| \|u_{n-1}\| \tag{3.14}
 \end{aligned}$$

By (3.14), condition (i), (ii) and Lemma 2.4, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.15}$$

Step 3. We will show that

$$\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0. \tag{3.16}$$

Since  $y_n = \beta_n Sx_n + (1 - \beta_n)x_n$ , we have

$$\beta_n (Sx_n - x_n) = y_n - x_n. \tag{3.17}$$

Claim that

$$\lim_{n \rightarrow \infty} \|x_{n-1} - y_n\| = 0. \tag{3.18}$$

$$\text{Since } \|x_{n+1} - y_n\| \leq \alpha_n \|u - y_n\| + (1 - \alpha_n) \|u_n - y_n\| \tag{3.19}$$

and for all  $z \in EP(F) \cap F(S)$ , by firmly nonexpansiveness, we have

$$\begin{aligned}
 \|u_n - z\|^2 &= \|S_n y_n - S_n z\|^2 \\
 &\leq \langle u_n - z, y_n - z \rangle \\
 &= \frac{1}{2} (\|u_n - z\|^2 + \|y_n - z\|^2 - \|u_n - y_n\|^2).
 \end{aligned}$$



$$= \frac{1}{2} (\|u_n - z\|^2 + \|y_n - z\|^2 - \|u_n - y_n\|^2). \quad (3.20)$$

From (3.20),  $\|u_n - z\|^2 \leq \|y_n - z\|^2 - \|u_n - y_n\|^2$ . (3.21)

By definition of  $\{x_n\}$  and (3.21), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|u_n - z\|^2 \\ &\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) (\|y_n - z\|^2 - \|u_n - y_n\|^2) \\ &= \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|y_n - z\|^2 - (1 - \alpha_n) \|u_n - y_n\|^2 \\ &\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) ((1 - \beta_n) \|x_n - z\|^2 + \beta_n \|Sx_n - z\|^2) \\ &\quad - (1 - \alpha_n) \|u_n - y_n\|^2 \\ &\leq \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|x_n - z\|^2 - (1 - \alpha_n) \|u_n - y_n\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} (1 - \alpha_n) \|u_n - y_n\|^2 &\leq \alpha_n \|u - z\|^2 + \|x_n - z\|^2 + \|x_{n+1} - z\|^2 \\ &\leq \alpha_n \|u - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\|. \end{aligned}$$

By condition (i) and (3.15), we have

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0. \quad (3.22)$$

By (3.22), condition (i) and (3.19), we obtain (3.18). Again by (3.18) and (3.15), we have

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \quad (3.23)$$

By (3.23), (3.17) and  $\beta_n \in (c, d]$ , we have (3.16).

Step 4. Put  $z_0 \in F$ , we must show that

$$\lim_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle \leq 0. \quad (3.24)$$

To show this inequality, choose subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that

$$\limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{n \rightarrow \infty} \langle u - z_0, x_{n_k} - z_0 \rangle \quad (3.25)$$

Without loss of generality, we may assume that  $x_{n_k} \rightarrow \omega$  as  $k \rightarrow \infty$  where  $\omega \in C$ . We first show

$\omega \in EP(F)$ . By (3.23), we have  $y_{n_k} \rightarrow \omega$  as  $k \rightarrow \infty$ . Since  $u_n = S_{r_n} y_{n_k}$  for every  $y \in C$ ,

we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C,$$

From (A2), we have

$$\frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq F(y, u_n), \quad \forall y \in C.$$



In particular, we have

$$\left\langle y - u_n, \frac{u_n - y_n}{r_n} \right\rangle \geq F(y, u_n), \quad \forall y \in C. \quad (3.26)$$

By (3.22) and  $y_n \rightarrow \omega$  as  $k \rightarrow \infty$ , we have  $u_n \rightarrow \omega$  as  $k \rightarrow \infty$ . Again by (3.22), (3.26) and  $u_n \rightarrow \omega$  as  $k \rightarrow \infty$ , we have

$$0 \geq F(y, \omega), \quad \forall y \in C \quad (3.27)$$

Let  $t \in (0, 1]$  and  $y \in C$ , put  $y_t = ty + (1-t)\omega$ , we have  $y_t \in C$ . By (3.27)

$$0 \geq F(y_t, \omega), \quad \forall y \in C \quad (3.28)$$

By (A1), (A4) and (3.28), we have

$$\begin{aligned} 0 &= F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, \omega) \\ &\leq tF(y_t, y). \end{aligned}$$

It follows that

$$0 \leq F(y_t, y). \quad (3.29)$$

From (A3) and (3.29), we have

$$0 \leq F(\omega, y),$$

Then  $\omega \in EP(F)$ . By  $x_n \rightarrow \omega$  as  $k \rightarrow \infty$ , (3.16) and Lemma 2.3, we have  $\omega \in F(T)$ . Hence  $\omega \in F$ .

By (3.25), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle &= \lim_{n \rightarrow \infty} \langle u - z_0, x_n - z_0 \rangle \\ &= \langle u - z_0, \omega - z_0 \rangle \leq 0 \end{aligned}$$

Step 5. Finally, we show that  $x_n \rightarrow \omega$  as  $n \rightarrow \infty$ , where  $z_0 = P_T u$ . By nonexpansiveness of  $S$  and  $S'_{x_n}$ , we have

$$\begin{aligned} \|x_{n+1} - z_0\|^2 &= \|\alpha_n(u - z_0) + (1 - \alpha_n)(u_n - z_0)\|^2 \\ &\leq (1 - \alpha_n)^2 \|u_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\leq (1 - \alpha_n) \|S_n x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\leq (1 - \alpha_n) \|y_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &= (1 - \alpha_n) \|\beta_n(Sx_n - z_0) + (1 - \beta_n)(x_n - z_0)\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\leq (1 - \alpha_n) (\beta_n \|Sx_n - z_0\|^2 + (1 - \beta_n) \|x_n - z_0\|^2) + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\leq (1 - \alpha_n) \|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \end{aligned}$$

From step 4, condition (i) and lemma 2.1, we can conclude that  $\{x_n\}$  converges strongly to  $z_0 = P_T u$ . This completes the proof.



#### Conclusion

We define a new iterative scheme to generate the sequence  $\{x_n\}$  in theorem 3.1 and prove that the sequence  $\{x_n\}$  converges strongly to  $z = P_{F,H}u$ . This paper is to solve the equilibrium problem and fixed point problem of strictly pseudocontractive mapping.

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## Academic Publications

1. Kheawborisut, A., Kangtunyakarn, A. and Suantai, S. 2018.  
"The existence theorem of a new multi-valued mapping in metric space endowed with graph" *Carpathian Journal of Mathematics*. 2018:1-7.
2. Kheawborisut, A. and Kangtunyakarn, A. "A new method for equilibrium problems and a finite family of pseudocontractive mapping." *The National and International Graduate Research Conference 2017* March 10,2017 at Pote Sarasin Building, Khon Kaen University.