TRACY-SINGH PRODUCT FOR BLOCK OPERATOR MATRICES

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ในงานวิจัยนี้จะนิยามผลคูณเทรซี-ซิงห์สำหรับเมทริกซ์ตัวดำเนินแบบบล็อกบนปริภูมิฮิลเบิร์ต เราได้สมบัติต่างๆ ของผลคูณเทรซี-ซิงห์ซึ่งประกอบด้วยสมบัติเชิงพีชคณิต สมบัติเชิงอันดับ สมบัติเชิงโครงสร้าง และสมบัติเชิงการวิเคราะห์

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Abstract

In this research, we define the Tracy-Singh product for block operator matrices on a Hilbert space. We establish various properties of Tracy-Singh product involving algebraic properties, order properties, structure properties and analytic properties.

Keywords : Hilbert space, tensor product, Tracy-Singh product, block operator matrix

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Arnon Ploymukda

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Chapter 1

Introduction

1.1 Inception and importance

The Kronecker product (or tensor product) of two matrices, denoted by $A \odot B$, is very important in linear algebra and related fields. This product is named after Leopold Kronecker, even though there is little evidence that he was the first to define and use it. Indeed, in the past the Kronecker product was sometimes called the Zehfuss product, after Johann Georg Zehfuss who in 1858 described the matrix operation we now know as the Kronecker product. This kind of matrix product has wide applications in matrix theory, system theory, physics, statistics, computer science, signal processing and other special fields; see [1, 5, 14, 21].

The Tracy-Singh product of two partitioned matrices, denoted by $A \boxdot B$, is introduced and used in econometrics by Tracy and Singh [19]. Liu [13] shows that the Tracy-Singh product can be viewed as a generalized Kronecker product, i.e., for non-partitioned matrices A and B their Tracy-Singh product $A \boxdot B$ is the Kronecker product $A \odot B$. The Tracy-Singh product is studied and applied widely in matrix theory and statistics, see [9, 13, 18]. For example, Tracy and Jinadasa [18] derived the expectations and covariances of the Tracy–Singh product of random matrices with indicated statistical applications and Koning et al. [9] used the block Kronecker and Tracy–Singh products in the estimation of k-factorial covariance structures.

As a natural generalization of a complex matrix, a Hilbert-space operator is a bounded linear transformation of a Hilbert space into itself. The tensor product of two Hilbert-space operators can be viewed as an infinite-dimensional extension of the standard Kronecker product of complex matrices. Important results on tensor product of (bounded linear) operators have been continuously considered in the literature (see, e.g., [4, 11, 20]).

In this research, we define the Tracy-Singh product for block operator matrices. We then establish its algebraic properties, order properties, analytic properties and structure properties. Our results generalize the results known so far in the literature for both Tracy-Singh products of matrices and tensor products of operators.

1.2 Objectives

1) To propose a new product, namely, the Tracy-Singh product, for bounded linear operators.

2) To investigate algebraic properties, order properties, analytic properties and structure properties of Tracy-Singh product for bounded linear operators.

1.3 Scope of the study

We investigate algebraic, order, analytic and structure properties of Tracy-Singh products for bounded linear operators. All Hilbert spaces considered here are complex Hilbert spaces.

1.4 Benefits

1) To develop further theory for operators on a Hilbert space.

2) To provide mathematical tools for computer science, statistics, physics, engineering and related fields.

1.5 Research methodology

1) Study advanced topics in matrix theory.

2) Study background in functional analysis.

3) Study background in multilinear algebra.

4) Collect and study research papers and textbooks concerning Tracy-Singh products of matrices and tensor products of operators on a Hilbert space.

5) Define the Tracy-Singh product of block operator matrices.

6) Investigate algebraic, order, analytic and structure properties of the Tracy-Singh product of operators.

7) Conclude the results, make suggestions for further works and write the thesis.

Table 1.1: The research schedule

Activity	Time frame							
	2014		2015				2016	
	Jul.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.	Jul.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.
Step 1	↔	↔						
Step 2			↔	↔				
Step 3				↔	↔			
Step 4					↔			
Step 5						↔		
Step 6							↔	
Step 7								↔

Chapter 2

Preliminaries

The purpose of this chapter is to provide basic concept and tools in matrix theory, linear algebra and functional analysis used in the research.

The first and the second sections deal with preliminaries in matrix theory. We collect fundamental properties of the Kronecker product of complex matrices and the Tracy-Singh product, as a generalized Kronecker product, of complex partitioned matrices.

In the third section, we give definition and some properties about tensor product for vector spaces and linear maps.

The rest of this chapter consist of some background in functional analysis. We provide basic concept about Hilbert spaces and bounded linear operators on a Hilbert space, tensor product for Hilbert spaces and bounded linear operators on a Hilbert space, functions of operators and operator matrices.

2.1 Kronecker product of complex matrices

In this section, we recall some basic definitions and results about of the Kronecker product of complex matrices. More detailed description can be found in [7, 21].

Denote by $\mathbb{M}_{m,n}$ the set of all $m \times n$ complex matrices and $\mathbb{M}_{n,n}$ is abbreviated to $\mathbb{M}_n$. We write A^T to indicate the transpose of $A \in \mathbb{M}_{m,n}$. The conjugate transpose A^* of $A \in \mathbb{M}_{m,n}$ is defined by $A^* = \bar{A}^T$ where $\bar{A}$ is the componentwise conjugate of A.

Definition 2.1. Let $A = [a_{ij}]$ and $B = [b_{kl}]$ be matrices of order $m \times n$ and $p \times q$, respectively. The **Kronecker product** of A and B (or **tensor product**), denoted as $A \odot B$, is defined by

$$A \odot B \ = \ [a_{ij}B]_{ij}\,, \tag{2.1}$$

where $a_{ij}B$ is the (i,j)th submatrix of order $p \times q$ and $A \odot B$ of order $mp \times nq$.

Example 2.2. Consider

$$A \ = \ \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix}, \ B \ = \ \begin{bmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{bmatrix}.$$

The Kronecker product of A and B is

$$A \odot B = \begin{bmatrix} 0B & -2B \\ 3B & -1B \end{bmatrix}$$

$$= \begin{bmatrix} 0\begin{bmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{bmatrix} & -2\begin{bmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{bmatrix} \\ 3\begin{bmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{bmatrix} & -1\begin{bmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & -4 & -2 & -10 & 0 \\ 0 & 0 & 0 & 0 & 8 & 4 & -12 & -6 \\ 0 & 0 & 0 & 0 & 6 & -4 & 2 & -8 \\ 6 & 3 & 15 & 0 & -2 & -1 & -5 & 0 \\ -12 & -6 & 18 & 9 & 4 & 2 & -6 & -3 \\ -9 & 6 & -3 & 12 & 3 & -2 & 1 & -4 \end{bmatrix}.$$

Proposition 2.3 ([7]). *Let $A, C \in \mathbb{M}_{m,n}$, $B, D \in \mathbb{M}_{p,q}$, $E \in \mathbb{M}_{r,s}$ and $\alpha \in \mathbb{C}$. Then*

(i) $(\alpha A) \odot B = A \odot (\alpha B) = \alpha(A \odot B)$,

(ii) $(A + C) \odot B = A \odot B + C \odot B$,

(iii) $A \odot (B + D) = A \odot B + A \odot D$,

(iv) $A \odot (B \odot E) = (A \odot B) \odot E$,

(v) $(A \odot B)^* = A^* \odot B^*$.

Theorem 2.4 (Mixed product property [7]). *Let $A \in \mathbb{M}_{m,n}$, $B \in \mathbb{M}_{p,q}$, $C \in \mathbb{M}_{n,r}$ and $D \in \mathbb{M}_{q,s}$. Then*

$$(A \odot B)(C \odot D) = (AC) \odot (BD). \tag{2.2}$$

Definition 2.5. A matrix $A \in \mathbb{M}_n$ is called

- **Hermitian** if $A^* = A$,
- **normal** $A^*A = AA^*$,
- **unitary** if $A^*A = AA^* = I$.

Corollary 2.6 ([7]). *Let $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$.*

(i) *If A and B are invertible, then $A \odot B$ is also invertible with $(A \odot B)^{-1} = A^{-1} \odot B^{-1}$.*

(ii) *If A and B are Hermitain, then $A \odot B$ is also Hermitian.*

(iii) *If A and B are normal, then $A \odot B$ is also normal.*

(iv) *If A and B are unitary, then $A \odot B$ is also unitary.*

Definition 2.7. Let $A \in \mathbb{M}_n$ be Hermitian. Then A is called

- **positive semidefinite** if $x^* A x \geqslant 0$ for all $x \in \mathbb{C}^n$,
- **positive definite** if $x^* A x > 0$ for all $x \in \mathbb{C}^n - \{0\}$.

Proposition 2.8 ([7]). *Let $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$ be Hermitian.*

(i) *If A and B are positive semidefinite, then $A \odot B$ is also positive semidefinite.*

(i) *If A and B are positive definite, then $A \odot B$ is also positive definite.*

2.2 Tracy-Singh product of complex matrices

In this section, we recall basic definitions and properties related to the Tracy-Singh product. See [13, 22, 23] for more details.

Definition 2.9. Consider matrices A and B of order $m \times n$ and $p \times q$, respectively. Let $A = [A_{ij}]$ be partitioned with A_{ij} of order $m_i \times n_j$ as the (i, j)th submatrix, and let $B = [B_{kl}]$ be partitioned with B_{kl} of order $p_k \times q_l$ as the (k, l)th submatrix where

$$\sum_{i=1}^{r} m_i = m, \quad \sum_{j=1}^{s} n_j = n, \quad \sum_{k=1}^{t} p_k = p, \quad \sum_{l=1}^{u} q_l = q.$$

The **Tracy-Singh product** of A and B, denoted as $A \boxdot B$, is defined by

$$A \boxdot B = \left[[A_{ij} \odot B_{kl}]_{kl} \right]_{ij}, \tag{2.3}$$

where $A \boxdot B$ is of order $mp \times nq$.

Remark 2.10. For an element-wise partitioned matrix A (i.e., each submatrix of A is of order 1×1), their $A \boxdot B$ is $A \odot B$.

Example 2.11. Consider

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 7 & 8 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 9 \end{bmatrix},$$

and

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where

$$B_{11} = \begin{bmatrix} 1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 4 & 7 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 5 & 8 \\ 6 & 9 \end{bmatrix}.$$

The Tracy-Singh product of A and B is

$$A \boxdot B = \begin{bmatrix} A_{11} \odot B_{11} & A_{11} \odot B_{12} & A_{12} \odot B_{11} & A_{12} \odot B_{12} \\ A_{11} \odot B_{21} & A_{11} \odot B_{22} & A_{12} \odot B_{21} & A_{12} \odot B_{22} \\ A_{21} \odot B_{11} & A_{21} \odot B_{12} & A_{22} \odot B_{11} & A_{22} \odot B_{12} \\ A_{21} \odot B_{21} & A_{21} \odot B_{22} & A_{22} \odot B_{21} & A_{22} \odot B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} & \begin{bmatrix} 4 & 7 & 8 & 14 \\ 16 & 28 & 20 & 35 \end{bmatrix} & \begin{bmatrix} 3 \\ 6 \end{bmatrix} & \begin{bmatrix} 12 & 21 \\ 24 & 42 \end{bmatrix} \\ \begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 8 & 10 \\ 12 & 15 \end{bmatrix} & \begin{bmatrix} 5 & 8 & 10 & 16 \\ 6 & 9 & 12 & 18 \\ 20 & 32 & 25 & 40 \\ 24 & 36 & 30 & 45 \end{bmatrix} & \begin{bmatrix} 6 \\ 9 \\ 12 \\ 18 \end{bmatrix} & \begin{bmatrix} 15 & 24 \\ 18 & 27 \\ 30 & 48 \\ 36 & 54 \end{bmatrix} \\ \begin{bmatrix} 7 & 8 \end{bmatrix} & \begin{bmatrix} 28 & 49 & 32 & 56 \end{bmatrix} & \begin{bmatrix} 9 \end{bmatrix} & \begin{bmatrix} 36 & 63 \end{bmatrix} \\ \begin{bmatrix} 14 & 16 \\ 21 & 24 \end{bmatrix} & \begin{bmatrix} 35 & 56 & 40 & 64 \\ 42 & 63 & 48 & 72 \end{bmatrix} & \begin{bmatrix} 18 \\ 27 \end{bmatrix} & \begin{bmatrix} 45 & 72 \\ 54 & 81 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 & 7 & 8 & 14 & 3 & 12 & 21 \\ 4 & 5 & 16 & 28 & 20 & 25 & 6 & 24 & 42 \\ 2 & 4 & 5 & 8 & 10 & 16 & 6 & 15 & 24 \\ 3 & 6 & 6 & 9 & 12 & 18 & 9 & 18 & 27 \\ 8 & 10 & 21 & 32 & 25 & 40 & 12 & 30 & 48 \\ 12 & 15 & 24 & 36 & 30 & 45 & 18 & 36 & 54 \\ 7 & 8 & 28 & 49 & 32 & 56 & 9 & 36 & 63 \\ 14 & 16 & 35 & 56 & 40 & 64 & 18 & 45 & 72 \\ 21 & 24 & 42 & 63 & 48 & 72 & 27 & 54 & 81 \end{bmatrix}.$$

Proposition 2.12 ([13]). *Let A, B, C and D be compatibly partitioned matrices and $\alpha \in \mathbb{C}$. Then*

(i) $(\alpha A) \boxdot B = A \boxdot (\alpha B) = \alpha(A \boxdot B)$,

(ii) $(A + C) \boxdot B = A \boxdot B + C \boxdot B$,

(iii) $A \boxdot (B + D) = A \boxdot B + A \boxdot D$,

(iv) $(A \boxdot B)^* = A^* \boxdot B^*$.

Theorem 2.13 (Mixed product property [13]). *Let A, B, C and D be compatibly partitioned matrices. Then*

$$(A \boxdot B)(C \boxdot D) = (AC) \boxdot (BD). \tag{2.4}$$

Definition 2.14. Let $A \in \mathbb{M}_n$. A scalar $\lambda \in \mathbb{C}$ is called an **eigenvalue** of A if and only if there exists $x \in \mathbb{C}^n - \{0\}$ such that

$$Ax = \lambda x.$$

We call x an **eigenvector** of A associated with λ. The set of all eigenvalue of A is called the **spectrum** of A and denote by $\lambda(A)$.

Theorem 2.15 (Spectral theorem for normal matrices). *Let $A \in \mathbb{M}_n$ be such that $\lambda(A) = \{\lambda_1, \ldots, \lambda_n\}$. Then the followings are equivalent.*

(i) A is normal.

(ii) There is a unitary matrix U such that

$$A = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^*. \tag{2.5}$$

Definition 2.16. Let $A \in \mathbb{M}_n$ be normal matrix in the form (2.5) and $f : X \to \mathbb{C}$ be such that $\lambda(A) \subseteq X$ and f is continuous function on $\lambda(A)$. We define **functional calculus** of f at A to be

$$f(A) = U \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} U^*, \tag{2.6}$$

where $\lambda_i \in \lambda(A)$ for each $i = 1, \ldots, n$ and U is unitary matrix.

Theorem 2.17 ([23]). *Let A and B be compatibly partitioned matrices. If A and B are positive definite, then for any positive real α*

$$(A \boxdot B)^{\alpha} = A^{\alpha} \boxdot B^{\alpha}. \tag{2.7}$$

Here, $A^{\alpha} = f(A)$ where $f(z) = z^{\alpha}$.

Definition 2.18. For a complex matrix $A = [A_{ij}]$ of order $m \times n$, the **Tracy-Singh power** is defined by $A^{\boxdot 1} = A$ and

$$A^{\boxdot (r+1)} = A^{\boxdot r} \boxdot A \tag{2.8}$$

for $r \in \mathbb{N}$.

2.3 Tensor product of vector spaces

In this section, we review fundamental properties of tensor product of vector spaces. See [12] for more information.

Definition 2.19. A **vector space** (or **linear space**) over a field $\mathcal{F}$ is a set $\mathcal{V}$ together with an addition on $\mathcal{V} \times \mathcal{V}$ to $\mathcal{V}$ given by $(u, v) \mapsto u + v$ and a scalar multiplication on $\mathcal{F} \times \mathcal{V}$ to $\mathcal{V}$ given by $(\alpha, v) \mapsto \alpha v$ satisfy the following properties:

(i) $(u + v) + w = u + (v + w)$ for all $u, v, w \in \mathcal{V}$.

(ii) $u + v = v + u$ for all $u, v \in \mathcal{V}$.

(iii) There exists an element $0 \in \mathcal{V}$ such that $v + 0 = 0$ for all $v \in \mathcal{V}$.

(iv) For each $v \in \mathcal{V}$, there exists an element $-v \in \mathcal{V}$ such that $v + (-v) = 0$.

(v) $\alpha(u + v) = \alpha u + \alpha v$ for all $u, v \in \mathcal{V}$ and $\alpha \in \mathcal{F}$.

(vi) $(\alpha + \beta)v = \alpha v + \beta v$ for all $v \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{F}$.

(vii) $\alpha(\beta v) = (\alpha\beta)v$ for all $v \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{F}$.

(viii) $1v = v$ for all $v \in \mathcal{V}$.

Definition 2.20. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a field $\mathcal{F}$. A **linear operator** or **linear map** T from $\mathcal{V}$ to $\mathcal{W}$ is a mapping $T : \mathcal{V} \to \mathcal{W}$ such that

$$T(u + v) = T(u) + T(v),$$
$$T(\alpha v) = \alpha T(v)$$

for every $u, v \in \mathcal{V}$ and $\alpha \in \mathcal{F}$.

Denote by $\mathbb{L}(\mathcal{V},\mathcal{W})$, the set of all linear maps from $\mathcal{V}$ into $\mathcal{W}$ and abbreviate $\mathbb{L}(\mathcal{V},\mathcal{V})$ to $\mathbb{L}(\mathcal{V})$.

Definition 2.21. Let $T \in \mathbb{L}(\mathcal{V},\mathcal{W})$. Then T is **one-to-one** or **injective** if whenever $T(u) = T(v)$, then $u = v$.

Definition 2.22. Let $T \in \mathbb{L}(\mathcal{V},\mathcal{W})$. Then T is **onto** or **surjective** if for every $w \in \mathcal{W}$ there exists $v \in \mathcal{V}$ such that $T(v) = w$.

Definition 2.23. If $T \in \mathbb{L}(\mathcal{V},\mathcal{W})$ is a linear, one-to-one, and onto mapping, then T is called an **isomorphism**, and $\mathcal{V}$ and $\mathcal{W}$ are said to be **isomorphic**.

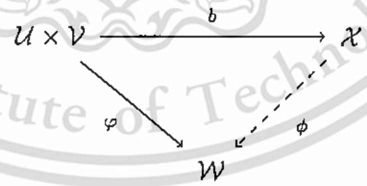
Definition 2.24. Let $\mathcal{V}_1,\ldots,\mathcal{V}_n$ and $\mathcal{W}$ be vector spaces over a field $\mathcal{F}$. A mapping $T : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \mathcal{W}$ is said to be **multilinear** if for each $i \in \{1,2,\ldots,n\}$,

$$\begin{aligned} T(v_1,\ldots,v_i+u_i,\ldots,v_n) &= T(v_1,\ldots,v_i,\ldots,v_n) + T(v_1,\ldots,u_i,\ldots,v_n), \\ T(v_1,\ldots,\alpha v_i,\ldots,v_n) &= \alpha T(v_1,\ldots,v_i,\ldots,v_n) \end{aligned}$$

for every $v_i, u_i \in \mathcal{V}_i$ and $\alpha \in \mathcal{F}$.

Remark 2.25. If $n = 1$, a multilinear map is simply a linear map. If $n = 2$, we call it a bilinear map.

Definition 2.26. Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over a field $\mathcal{F}$. Then a **tensor product** of $\mathcal{U}$ and $\mathcal{V}$ is a vector space $\mathcal{X}$ over $\mathcal{F}$ together with a bilinear map $b : \mathcal{U} \times \mathcal{V} \to \mathcal{X}$ such that for any vector space $\mathcal{W}$ and a bilinear map $\varphi : \mathcal{U} \times \mathcal{V} \to \mathcal{W}$, there exists a unique linear map $\phi : \mathcal{X} \to \mathcal{W}$ such that $\phi \circ b = \varphi$.



Theorem 2.27. *Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces. Then $\mathcal{U}$ and $\mathcal{V}$ has a tensor product.*

Theorem 2.28. *Let $(\mathcal{X}_1, b_1)$ and $(\mathcal{X}_2, b_2)$ be two tensor products of $\mathcal{U}$ and $\mathcal{V}$. Then there exists a unique isomorphism $T : \mathcal{X}_1 \to \mathcal{X}_2$ such that $T \circ b_1 = b_2$.*

Remark 2.29. Since the tensor product of $\mathcal{U}$ and $\mathcal{V}$ is unique up to isomorphism, we denote tensor products of $\mathcal{U}$ and $\mathcal{V}$ by $\mathcal{U} \otimes \mathcal{V}$ and write $u \otimes v$ for $b(u,v)$.

Proposition 2.30. *Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over a field $\mathcal{F}$. Then*

(i) For any $u_1, u_2 \in \mathcal{U}, v \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{F}$,

$$(\alpha u_1 + \beta u_2) \otimes v = \alpha (u_1 \otimes v) + \beta (u_2 \otimes v).$$

(ii) For any $u \in \mathcal{U}, v_1, v_2 \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{F}$,

$$u \otimes (\alpha v_1 + \beta v_2) = \alpha (u \otimes v_1) + \beta (u \otimes v_2).$$

Proposition 2.31. *Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces. Then any element in $\mathcal{U} \otimes \mathcal{V}$ can be written*

$$\sum_{i=1}^{n} u_i \otimes v_i,$$

where $n \in \mathbb{N}, u_i \in \mathcal{U}$ and $v_i \in \mathcal{V}$ for $i = 1, 2, \ldots, n$. In particular we have

$$\mathcal{U} \otimes \mathcal{V} = \left\{ \sum_{i=1}^{n} u_i \otimes v_i \mid n \in \mathbb{N}, u_i \in \mathcal{U}, v_i \in \mathcal{V} \right\}.$$

Theorem 2.32. *Let $S \in \mathbb{L}(\mathcal{U}, \mathcal{U}')$ and $T \in \mathbb{L}(\mathcal{V}, \mathcal{V}')$. Then there exists a unique linear map $\psi : \mathcal{U} \otimes \mathcal{V} \to \mathcal{U}' \otimes \mathcal{V}'$ such that*

$$\psi(u \otimes v) = S(u) \otimes T(v) \tag{2.9}$$

for any $u \in \mathcal{U}$ and $v \in \mathcal{V}$. The unique linear map ψ is said to be the tensor product of S and T, denoted by $S \otimes T$.

Proposition 2.33. *Let $S, S' \in \mathbb{L}(\mathcal{U}, \mathcal{U}')$ and $T, T' \in \mathbb{L}(\mathcal{V}, \mathcal{V}')$ be linear maps and $\alpha \in \mathcal{F}$. Then*

(i) $S \otimes (T + T') = S \otimes T + S \otimes T'$,

(ii) $(S + S') \otimes T = S \otimes T + S' \otimes T$,

(iii) $(\alpha S) \otimes T = S \otimes (\alpha T) = \alpha(S \otimes T)$.

Theorem 2.34. *Let $S, Q \in \mathbb{L}(\mathcal{U})$ and $T, R \in \mathbb{L}(\mathcal{V})$ be linear maps. Then*

$$(S \otimes T) \circ (Q \otimes R) = (S \circ Q) \otimes (T \circ R). \tag{2.10}$$

Here, $S \circ Q$ stands for the composition of S and Q.

2.4 Operators on a Hilbert space

In this section, we provide prerequisites about operators on a Hilbert space. More details can be found in [8, 10, 17].

From now on, $\mathbb{F}$ stands for the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For two operators S and T, we write ST for the composition $S \circ T$ (if it exists).

Definition 2.35. A **normed vector space** is a vector space $\mathcal{X}$ over a field $\mathbb{F}$ together with a mapping $\|\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, called a **norm**, satisfying the following conditions:

(NS1) $\|x\| \geqslant 0$ for all $x \in \mathcal{X}$.

(NS2) $\|x\| = 0$ if and only if $x = 0$.

(NS3) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in \mathcal{X}$ and $\alpha \in \mathbb{F}$.

(NS4) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$.

Definition 2.36. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces, and $T : \mathcal{X} \to \mathcal{Y}$ be a linear operator. Then T is said to be **bounded linear operator** if there exists a constant M such that

$$\|T(x)\| \leqslant M\|x\| \quad \text{for all } x \in \mathcal{X}. \tag{2.11}$$

From now on, we will write Tx instead of $T(x)$.

Definition 2.37. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces. Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator. Then the norm of T, called **operator norm**, is defined as

$$\|T\| = \inf\{M \geqslant 0 : \|Tx\| \leqslant M\|x\|,\ x \in \mathcal{X}\}. \tag{2.12}$$

Definition 2.38. An **inner product space** is a vector space $\mathcal{X}$ over a field $\mathbb{F}$ together with a mapping $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{F}$, called an **inner product**, satisfying the following conditions:

(IP1) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in \mathcal{X}$.

(IP2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in \mathcal{X}$ and $\alpha \in \mathbb{F}$.

(IP3) $\overline{\langle x, y \rangle} = \langle y, x \rangle$ for all $x, y \in \mathcal{X}$.

(IP4) $\langle x, x \rangle \geqslant 0$ for all $x \in \mathcal{X}$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

Remark 2.39. If $\mathbb{F} = \mathbb{R}$, we call $\mathcal{X}$ real inner product space. If $\mathbb{F} = \mathbb{C}$, we call $\mathcal{X}$ complex inner product space.

Definition 2.40. A real (respectively, complex) **Hilbert space** is a complete real (respectively, complex) inner product space.

When $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, denote by $\mathbb{B}(\mathcal{H},\mathcal{K})$ the set of all bounded linear operator from $\mathcal{H}$ into $\mathcal{K}$ and abbreviate $\mathbb{B}(\mathcal{H},\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$.

Example 2.41. Examples of Hilbert spaces.

(i) The complex vector space $\mathbb{C}^n$ with the inner product given by

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i\bar{y}_i.$$

(ii) The space of all $n\times n$ complex matrices $\mathbb{M}_n$ with the inner product given by

$$\langle A,B\rangle = \text{tr}(B^*A).$$

Here, $\text{tr}(A)=\sum_{i=1}^{n} a_{ii}$ for every $A=[a_{ij}]_{i,j=1}^{n,n}\in\mathbb{M}_n$.

(iii) Every finite dimensional inner product space.

(iv) The inner product space ℓ^2 of all square summable infinite sequences of complex numbers (the space of sequences of complex numbers $x=\{x_n\}_{n=1}^{\infty}$ such that $\sum_{i=1}^{\infty}|x_i|^2<\infty$) with the inner product given by

$$\langle x,y\rangle = \sum_{i=1}^{\infty} x_i\bar{y}_i.$$

(v) Any closed subspace of a Hilbert space.

Definition 2.42. Let $T\in\mathbb{B}(\mathcal{H})$. The **adjoint** of T is the unique linear operator T^* on $\mathcal{H}$ such that

$$\langle Tx,y\rangle = \langle x,T^*y\rangle \quad \text{for all } x,y\in\mathcal{H}.$$

Theorem 2.43. *Let T and S be bounded linear operators on a complex Hilbert space $\mathcal{H}$ and $\alpha\in\mathbb{C}$. Then*

(i) $(T+S)^* = T^*+S^*$,

(ii) $(\alpha T)^* = \overline{\alpha}T^*$,

(iii) $(TS)^* = S^*T^*$,

(iv) $T^{**} = T$,

(v) if T is invertible, then T^* is also invertible with $(T^*)^{-1}=(T^{-1})^*$.

Definition 2.44. Let $\mathcal{H}$ be a complex Hilbert space. Then $T \in \mathbb{B}(\mathcal{H})$ is called

- **self-adjoint** or **Hermitian** if $T^* = T$,
- **skew Hermitian** if $T^* = -T$,
- **normal** if $TT^* = T^*T$,
- **unitary** if $TT^* = T^*T = I$,
- **isometry** if $T^*T = I$,
- **co-isometry** if $TT^* = I$,
- **partial isometry** if $TT^*T = T$,
- **projection** if $T^2 = T$ and $T^* = T$,
- **idempotent** if $T^2 = T$,
- **involutory** if $T^2 = I$,
- **nilpotent** if $T^k = 0$ for some positive integer k.

Definition 2.45. Let T be a Hermitian operator on a complex Hilbert $\mathcal{H}$ into itself. Then T is said to be **positive**, denoted by $T \geqslant 0$, if

$$\langle Tx, x \rangle \geqslant 0 \text{ for all } x \in \mathcal{H}.$$

If a positive operator T is invertible, then we say that it is **strictly positive** and write $T > 0$.

Definition 2.46. Let S and T be two Hermitian operators on a complex Hilbert $\mathcal{H}$. We write $S \geqslant T$ if $S - T$ is positive and write $S > T$ if $S - T$ is strictly positive.

Proposition 2.47. *Let T be a bounded linear operator on a complex Hilbert $\mathcal{H}$. Then TT^* and T^*T are positive.*

Proposition 2.48. *Let S_1, S_2, T_1 and T_2 be bounded linear operators on a complex Hilbert $\mathcal{H}$.*

(i) If $S_1 \geqslant S_2$ and $T_1 \geqslant T_2$, then

$$S_1 + T_1 \geqslant S_2 + T_2.$$

(i2) If $S_1 > S_2$ and $T_1 > T_2$, then

$$S_1 + T_1 > S_2 + T_2.$$

Definition 2.49 (Positive square root). Let $T \in \mathbb{B}(\mathcal{H})$ be a positive operator on a complex Hilbert space $\mathcal{H}$. A positive operator $A \in \mathbb{B}(\mathcal{H})$ such that $A^2 = T$ is called a **positive square root** of T, denoted by $T^{1/2}$.

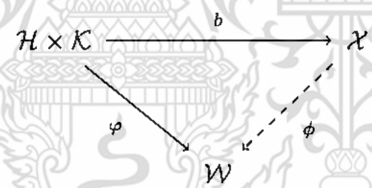
Definition 2.50 (Positive kth root). Let $T \in \mathbb{B}(\mathcal{H})$ be a positive operator on a complex Hilbert space $\mathcal{H}$ and $k \in \mathbb{N}$. A positive operator $A \in \mathbb{B}(\mathcal{H})$ such that $A^k = T$ is called a **positive kth root** of T, denoted by $T^{1/k}$.

Theorem 2.51. *Let $T \in \mathbb{B}(\mathcal{H})$ be a positive operator on a complex Hilbert space $\mathcal{H}$ and $k \in \mathbb{N}$. Then T has a unique positive kth root.*

2.5 Tensor product of Hilbert spaces

In this section, we review basic knowledge about tensor product of Hilbert spaces. See [2, 15] for more information.

Definition 2.52. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then a **tensor product** of $\mathcal{H}$ and $\mathcal{K}$ is a Hilbert space $\mathcal{X}$ together with a bounded bilinear map $b : \mathcal{H} \times \mathcal{K} \to \mathcal{X}$ such that for any Hilbert space $\mathcal{W}$ and a bounded bilinear map $\varphi : \mathcal{H} \times \mathcal{K} \to \mathcal{W}$, there exists a unique linear map $\phi : \mathcal{X} \to \mathcal{W}$ such that $\phi \circ b = \varphi$.



Theorem 2.53. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then $\mathcal{H}$ and $\mathcal{K}$ has a tensor product.*

If $(\mathcal{X}, b)$ is a tensor product of $\mathcal{H}$ and $\mathcal{K}$ it is customary to write $x \otimes y$ in place of $b(x, y)$ and $\mathcal{H} \otimes \mathcal{K}$ in place of $\mathcal{X}$.

Theorem 2.54. *Let $S \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $T \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. Then there exists a unique bounded linear operator $\psi : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H}' \otimes \mathcal{K}'$ such that*

$$\psi(x \otimes y) = S(x) \otimes T(y) \tag{2.13}$$

for any $x \in \mathcal{H}$ and $y \in \mathcal{K}$. The unique linear map ψ is said to be the tensor product of S and T denoted by $S \otimes T$.

Proposition 2.55. *Let $S, S' \in \mathbb{B}(\mathcal{H}, \mathcal{H}'), T, T' \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ and $\alpha \in \mathbb{F}$. Then*

(i) $S \otimes (T + T') = S \otimes T + S \otimes T'$,

(ii) $(S + S') \otimes T = S \otimes T + S' \otimes T$,

(iii) $(\alpha S) \otimes T = S \otimes (\alpha T) = \alpha(S \otimes T)$.

Proposition 2.56. *Let $S \in \mathbb{B}(\mathcal{H})$ and $T \in \mathbb{B}(\mathcal{K})$. Then*

$$(S \otimes T)^* = S^* \otimes T^*.$$

Theorem 2.57. *Let $S, Q \in \mathbb{B}(\mathcal{H})$ and $T, R \in \mathbb{B}(\mathcal{K})$. Then*

$$(S \otimes T)(Q \otimes R) = (SQ) \otimes (TR). \tag{2.14}$$

Theorem 2.58. *Let $S \in \mathbb{B}(\mathcal{H})$ and $T \in \mathbb{B}(\mathcal{K})$. Let $(S_n)_{n=1}^{\infty}$ and $(T_n)_{n=1}^{\infty}$ be sequences in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$, respectively. If $S_n \to S$ and $T_n \to T$, then $S_n \otimes T_n \to S \otimes T$.*

2.6 Functions of operators

In this section, we explain how to define functions of operators. A more detailed description can be found in [10, 16].

Throughout this section, a Hilbert space $\mathcal{H}$ means complex Hilbert space.

Definition 2.59. Let $T \in \mathbb{B}(\mathcal{H})$. The **spectrum** $\sigma(T)$ of T is the collection of complex numbers λ such that $T - \lambda I$ is not invertible.

Theorem 2.60 (Spectral family). *Let $T \in \mathbb{B}(\mathcal{H})$ be Hermitian. Set*

$$m = \inf_{\|x\|=1} \langle Tx, x\rangle, \qquad M = \sup_{\|x\|=1} \langle Tx, x\rangle. \tag{2.15}$$

Then there is a family $\{E(\lambda)\}$ of orthogonal projection operators on $\mathcal{H}$ depending on a real parameter λ and such that

(i) $E(\lambda_1) \leqslant E(\lambda_2)$ *for* $\lambda_1 \leqslant \lambda_2$,

(ii) $E(\lambda)x \to E(\lambda_0)x$ *as* $\lambda_0 < \lambda \to \lambda_0,\ x \in \mathcal{H}$,

(ii) $E(\lambda) = 0$ *for* $\lambda < m,\ E(\lambda) = I$ *for* $\lambda \geqslant M$,

(iv) $TE(\lambda) = E(\lambda)T$.

The family $\{E(\lambda)\}$ is called the **spectral family associated with the operator T**.

Theorem 2.61. *Let $T \in \mathbb{B}(\mathcal{H})$ be Hermitian. Let $a < m$ and $b \geqslant M$ where m and M be defined in (2.15). Then T has the spectral representation*

$$T = \int_a^b \lambda dE(\lambda), \tag{2.16}$$

where $\{E(\lambda)\}$ is the spectral family associated with T.
More generally, if p is polynomial, then

$$p(T) = \int_a^b p(\lambda) dE(\lambda). \tag{2.17}$$

Theorem 2.62 (Spectral Theorem). *Let $T \in \mathbb{B}(\mathcal{H})$ be Hermitian. Let $a < m$ and $b \geqslant M$ where m and M are defined in (2.15). If f be a continuous real-valued function on $[a, b]$, then*

$$f(T) = \int_a^b f(\lambda) dE(\lambda), \tag{2.18}$$

where $\{E(\lambda)\}$ is the spectral family associated with T.

2.7 Operator matrices

In this section, we recall standard results on operator matrices which can be found, e.g., in [6].

For Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ consider the direct sum Hilbert space

$$\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n := \{x_1 \oplus \ldots \oplus x_n \mid x_i \in \mathcal{H}_i, i = 1, \ldots, n\}$$

equipped with the inner product

$$\langle x_1 \oplus \ldots \oplus x_n, y_1 \oplus \ldots \oplus y_n \rangle = \langle x_1, y_1 \rangle + \ldots + \langle x_n, y_n \rangle, \; x_i, y_i \in \mathcal{H}_i \text{ for } i = 1, \ldots, n$$

with addition and multiplication defined (in usual way) as follows:

$$(x_1 \oplus \ldots \oplus x_n) + (y_1 \oplus \ldots \oplus y_n) = (x_1 + y_1) \oplus \ldots \oplus (x_n + y_n),$$
$$\alpha (x_1 \oplus \ldots \oplus x_n) = (\alpha x_1) \oplus \ldots \oplus (\alpha x_n).$$

Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ and $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be complex Hilbert spaces. Let $\mathcal{H}$ and $\mathcal{K}$ denote $\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$ and $\mathcal{K}_1 \oplus \ldots \oplus \mathcal{K}_m$, respectively. For each $i = 1, \ldots, m$, let P_i be the natural projection from $\mathcal{K}$ onto $\mathcal{K}_i$, defined by

$$P_i(x_1 \oplus \ldots \oplus x_i \oplus \ldots, x_m) = x_i.$$

For each $j = 1, \ldots, n$, let E_j be the canonical embedding from $\mathcal{H}_j$ into $\mathcal{H}$, defined by

$$E_j(x_j) = 0 \oplus \ldots \oplus 0 \oplus x_j \oplus 0 \oplus \ldots \oplus 0.$$

An operator A in $\mathbb{B}(\mathcal{H},\mathcal{K})$ can be represented uniquely as a block operator matrix

$$A = [A_{ij}]_{i,j=1}^{m,n} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}, \tag{2.19}$$

where $A_{ij} = P_i A E_j \in \mathbb{B}(\mathcal{H}_j,\mathcal{K}_i)$ for each $i = 1,\dots,m$ and $j = 1,\dots,n$. An operator $A \in \mathbb{B}(\mathcal{H},\mathcal{K})$ acts on $\mathcal{H}$ in the form

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + \ldots + A_{1n}x_n \\ \vdots \\ A_{m1}x_1 + \ldots + A_{mn}x_n \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n. \tag{2.20}$$

Proposition 2.63. *Let $A = [A_{ij}]_{i,j=1}^{m,n}$ and $B = [B_{ij}]_{i,j=1}^{m,n}$ be operator matrices in $\mathbb{B}(\mathcal{H},\mathcal{K})$. Then*

$$A + B = [A_{ij} + B_{ij}]_{ij}. \tag{2.21}$$

Proposition 2.64. *Let $A = [A_{ij}]_{i,j=1}^{m,n}$ be an operator matrix in $\mathbb{B}(\mathcal{H},\mathcal{K})$ and $\alpha \in \mathbb{C}$. Then*

$$\alpha A = [\alpha A_{ij}]_{ij}. \tag{2.22}$$

Proposition 2.65. *Let $\mathcal{G}_1,\dots,\mathcal{G}_r,\mathcal{H}_1,\dots,\mathcal{H}_n$ and $\mathcal{K}_1,\dots,\mathcal{K}_m$ be complex Hilbert spaces. Let $A = [A_{ij}]_{i,j=1}^{m,n}$ and $B = [B_{kl}]_{k,l=1}^{n,r}$ be operator matrices in $\mathbb{B}(\mathcal{H},\mathcal{K})$ and $\mathbb{B}(\mathcal{G},\mathcal{H})$, respectively, where $\mathcal{G} = \mathcal{G}_1 \oplus \ldots \oplus \mathcal{G}_r, \mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$ and $\mathcal{K} = \mathcal{K}_1 \oplus \ldots \oplus \mathcal{K}_m$. Then*

$$AB = \left[\sum_{t=1}^{n} A_{it}B_{tl}\right]_{il}, \tag{2.23}$$

where $AB \in \mathbb{B}(\mathcal{G},\mathcal{K})$.

Proposition 2.66. *Let $A = [A_{ij}]_{i,j=1}^{m,n}$ be an operator matrix in $\mathbb{B}(\mathcal{H})$. Then*

$$A^* = [A_{ji}^*]_{ij}. \tag{2.24}$$

Definition 2.67. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. For $A \in \mathbb{B}(\mathcal{H}_1,\mathcal{K}_1)$ and $B \in \mathbb{B}(\mathcal{H}_2,\mathcal{K}_2)$, the operator

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathbb{B}(\mathcal{H},\mathcal{K})$$

is said to be the **direct sum** of A and B, and denoted by $A \oplus B$.

Chapter 3

Algebraic and Order Properties of the Tracy-Singh Product

In this chapter, we present the definitions and some properties including algebraic and order properties of Tracy-Singh product for block operator matrices on a Hilbert space.

3.1 Algebraic properties of Tracy-Singh products

We decompose the complex Hilbert spaces $\mathcal{H}, \mathcal{H}', \mathcal{K}$ and $\mathcal{K}'$ as direct sums of certain Hilbert spaces as follows:

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{l=1}^{q} \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^{p} \mathcal{K}'_k.$$

Each operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ can be expressed uniquely as a block operator matrix

$$A = [A_{ij}]_{i,j=1}^{m,n} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}, \tag{3.1}$$

where $A_{ij} = P_i A E_j \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}'_i)$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$ where E_j is the canonical embedding of $\mathcal{H}_j$ in $\mathcal{H}$, defined by

$$x_j \mapsto 0 \oplus \ldots \oplus 0 \oplus x_j \oplus 0 \oplus \ldots \oplus 0,$$

and P_i is the projection from $\mathcal{H}'$ onto $\mathcal{H}'_i$, defined by

$$x_1 \oplus \ldots \oplus x_i \oplus \ldots \oplus x_m \mapsto x_i.$$

Similarly, an operator B in $\mathbb{B}(\mathcal{K}, \mathcal{K}')$ can be represented uniquely as a block operator matrix

$$B = [B_{kl}]_{k,l=1}^{p,q} = \begin{bmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & \ddots & \vdots \\ B_{p1} & \cdots & B_{pq} \end{bmatrix}, \tag{3.2}$$

where $B_{kl} = Q_k B F_l \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}'_k)$ for each $k = 1, \ldots, p$ and $l = 1, \ldots, q$ where F_l is the canonical embedding of $\mathcal{K}_l$ in $\mathcal{K}$, defined by

$$y_l \mapsto 0 \oplus \ldots \oplus 0 \oplus y_l \oplus 0 \oplus \ldots \oplus 0,$$

and Q_k is the natural projection from $\mathcal{K}'$ onto $\mathcal{K}'_k$, defined by

$$y_1 \oplus \ldots \oplus y_k \oplus \ldots \oplus y_p \mapsto y_k.$$

Definition 3.1. Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. The **Tracy-Singh product** of A and B, denoted as $A \boxtimes B$, is defined by

$$A \boxtimes B \;=\; \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}, \tag{3.3}$$

where $A_{ij} \otimes B_{kl}$ is a bounded linear operator from $\mathcal{H}_j \otimes \mathcal{K}_l$ into $\mathcal{H}'_i \otimes \mathcal{K}'_k$ and $A \boxtimes B$ is a bounded linear operator from $\bigoplus_{j,l=1}^{n,q} \mathcal{H}_j \otimes \mathcal{K}_l$ into $\bigoplus_{i,k=1}^{m,p} \mathcal{H}'_i \otimes \mathcal{K}'_k$.

Remark 3.2. If both A and B are 1×1 block operator matrices, their Tracy-Singh product $A \boxtimes B$ is the tensor product $A \otimes B$.

Example 3.3. Consider complex Hilbert spaces $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $\mathcal{H}' = \mathcal{H}'_1 \oplus \mathcal{H}'_2$, $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3$ and $\mathcal{K}' = \mathcal{K}'_1 \oplus \mathcal{K}'_2$. Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. Then A and B can be written as block operator matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix},$$

where $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}'_i)$ for each $i, j = 1, 2$ and $B_{kl} \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}'_k)$ for each $k = 1, 2$ and $l = 1, 2, 3$. The Tracy-Singh product of A and B is

$$A \boxtimes B = \begin{bmatrix} A_{11} \otimes B_{11} & A_{11} \otimes B_{12} & A_{11} \otimes B_{13} & A_{12} \otimes B_{11} & A_{12} \otimes B_{12} & A_{12} \otimes B_{13} \\ A_{11} \otimes B_{21} & A_{11} \otimes B_{22} & A_{11} \otimes B_{23} & A_{12} \otimes B_{21} & A_{12} \otimes B_{22} & A_{12} \otimes B_{23} \\ A_{21} \otimes B_{11} & A_{21} \otimes B_{12} & A_{21} \otimes B_{13} & A_{22} \otimes B_{11} & A_{22} \otimes B_{12} & A_{22} \otimes B_{13} \\ A_{21} \otimes B_{21} & A_{21} \otimes B_{22} & A_{21} \otimes B_{23} & A_{22} \otimes B_{21} & A_{22} \otimes B_{22} & A_{22} \otimes B_{23} \end{bmatrix},$$

where $A_{ij} \otimes B_{kl} \in \mathbb{B}(\mathcal{H}_j \otimes \mathcal{K}_l, \mathcal{H}'_i \otimes \mathcal{K}'_k)$.

Recall that for each $A \in \mathbb{M}_{m,n}$ and $B \in \mathbb{M}_{p,q}$, the induced maps

$$T_A : \mathbb{C}^n \to \mathbb{C}^m,\ x \mapsto Ax \quad \text{and} \quad T_B : \mathbb{C}^q \to \mathbb{C}^p,\ x \mapsto Bx$$

are bounded linear operators. We identify $\mathbb{C}^n \otimes \mathbb{C}^q$ with $\mathbb{C}^{nq}$ together with the canonical bilinear map $(x, y) \mapsto x \odot y$ for each $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^q$. It is similar for $\mathbb{C}^m \otimes \mathbb{C}^p$.

Lemma 3.4. *For each $A \in \mathbb{M}_{m,n}$ and $B \in \mathbb{M}_{p,q}$, we have*

$$T_A \otimes T_B \;=\; T_{A \odot B}. \tag{3.4}$$

Proof. By using the mixed product property of the Kronecker product (2.2), we have

$$\begin{aligned}(T_A \otimes T_B)(x \otimes y) &= T_A(x) \otimes T_B(y)\\ &= Ax \otimes By\\ &= Ax \odot By\\ &= (A \odot B)(x \odot y)\\ &= (A \odot B)(x \otimes y)\\ &= T_{A\odot B}(x \otimes y)\end{aligned}$$

for any $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^q$. Thus, by the uniqueness of tensor product, $T_A \otimes T_B = T_{A\odot B}$. □

Proposition 3.5. *The Tracy-Singh product of two linear maps induced by matrices is just the linear map induced by the Tracy-Singh product of these matrices. More precisely, for any complex matrices $A = [A_{ij}]$ and $B = [B_{kl}]$ partitioned in blockmatrix forms, we have*

$$T_A \boxtimes T_B = T_{A \boxdot B}. \tag{3.5}$$

Proof. Recall that the (i,j)th block of the matrix representation of T_A is the $T_{A_{ij}}$. By using Lemma 3.4, we obtain

$$\begin{aligned}T_A \boxtimes T_B &= [T_{A_{ij}}]_{ij} \boxtimes [T_{B_{kl}}]_{kl}\\ &= [[T_{A_{ij}} \otimes T_{B_{kl}}]_{kl}]_{ij}\\ &= [[T_{A_{ij}\odot B_{kl}}]_{kl}]_{ij}\\ &= T_{A\boxdot B}.\end{aligned}$$

□

Proposition 3.6. *The map* $\mathbb{B}(\mathcal{H},\mathcal{H}') \times \mathbb{B}(\mathcal{K},\mathcal{K}') \to \mathbb{B}\left(\bigoplus_{i,j=1}^{n,q} \mathcal{H}_i \otimes \mathcal{K}_j, \bigoplus_{i,j=1}^{m,p} \mathcal{H}'_i \otimes \mathcal{K}'_j\right)$ *determined by*

$$(A, B) \mapsto A \boxtimes B \tag{3.6}$$

is bilinear.

Proof. Let $A, B \in \mathbb{B}(\mathcal{H},\mathcal{H}'), C, D \in \mathbb{B}(\mathcal{K},\mathcal{K}')$ and $\alpha, \beta \in \mathbb{C}$. Write $A = [A_{ij}]\,, B = [B_{ij}]\,, C =$

$[C_{kl}]$ and $D = [D_{kl}]$. By using Propositions 2.55, 2.63 and 2.64, we have

$$\begin{aligned}
(\alpha A + \beta B) \boxtimes C &= [\alpha A_{ij} + \beta B_{ij}]_{ij} \boxtimes [C_{kl}]_{kl} \\
&= \left[[(\alpha A_{ij} + \beta B_{ij}) \otimes C_{kl}]_{kl}\right]_{ij} \\
&= \left[[\alpha A_{ij} \otimes C_{kl} + \beta B_{ij} \otimes C_{kl}]_{kl}\right]_{ij} \\
&= \left[[\alpha A_{ij} \otimes C_{kl}]_{kl}\right]_{ij} + \left[[\beta B_{ij} \otimes C_{kl}]_{kl}\right]_{ij} \\
&= \left[[\alpha (A_{ij} \otimes C_{kl})]_{kl}\right]_{ij} + \left[[\beta (B_{ij} \otimes C_{kl})]_{kl}\right]_{ij} \\
&= \alpha \left[[A_{ij} \otimes C_{kl}]_{kl}\right]_{ij} + \beta \left[[B_{ij} \otimes C_{kl}]_{kl}\right]_{ij} \\
&= \alpha(A \boxtimes C) + \beta(B \boxtimes C), \\
A \boxtimes (\alpha C + \beta D) &= [A_{ij}]_{ij} \boxtimes [\alpha C_{kl} + \beta D_{lk}]_{lk} \\
&= \left[[A_{ij} \otimes (\alpha C_{kl} + \beta D_{kl})]_{kl}\right]_{ij} \\
&= \left[[A_{ij} \otimes \alpha C_{kl} + A_{ij} \otimes \beta D_{kl}]_{kl}\right]_{ij} \\
&= \left[[A_{ij} \otimes \alpha C_{kl}]_{kl}\right]_{ij} + \left[[A_{ij} \otimes \beta D_{kl}]_{kl}\right]_{ij} \\
&= \left[[\alpha (A_{ij} \otimes C_{kl})]_{kl}\right]_{ij} + \left[[\beta (A_{ij} \otimes D_{kl})]_{kl}\right]_{ij} \\
&= \alpha \left[[A_{ij} \otimes C_{kl}]_{kl}\right]_{ij} + \beta \left[[A_{ij} \otimes D_{kl}]_{kl}\right]_{ij} \\
&= \alpha(A \boxtimes C) + \beta(A \boxtimes D).
\end{aligned}$$

Hence the Tracy-Singh product is a bilinear map. □

Proposition 3.7. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. Then*

$$(A \boxtimes B)^* = A^* \boxtimes B^*. \tag{3.7}$$

Proof. Write $A = [A_{ij}]$ and $B = [B_{kl}]$. By applying Propositions 2.56 and 2.66, we obtain

$$\begin{aligned}
(A \boxtimes B)^* &= \left([A_{ij}]_{ij} \boxtimes [B_{kl}]_{kl}\right)^* \\
&= \left[[A_{ij} \otimes B_{kl}]_{kl}\right]_{ij}^* \\
&= \left[[A_{ji} \otimes B_{kl}]_{kl}^*\right]_{ij} \\
&= \left[[(A_{ji} \otimes B_{lk})^*]_{kl}\right]_{ij} \\
&= \left[[A_{ji}^* \otimes B_{lk}^*]_{kl}\right]_{ij} \\
&= [A_{ji}^*]_{ij} \boxtimes [B_{lk}^*]_{kl} \\
&= A^* \boxtimes B^*.
\end{aligned}$$

□

Proposition 3.8. *Let* $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ *and* $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K},\mathcal{K}')$. *Then*

$$A \boxtimes B = [A_{ij} \boxtimes B]_{ij} = \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1n} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{m1} \boxtimes B & \cdots & A_{mn} \boxtimes B \end{bmatrix}.$$

That is, the (i,j)*th block of* $A \boxtimes B$ *is just* $A_{ij} \boxtimes B$.

Proof. By Definition of the Tracy-Singh product, we have

$$A \boxtimes B = \begin{bmatrix} A_{11} \otimes B_{11} & \cdots & A_{11} \otimes B_{1q} & & A_{1n} \otimes B_{11} & \cdots & A_{1n} \otimes B_{1q} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ A_{11} \otimes B_{p1} & \cdots & A_{11} \otimes B_{pq} & & A_{1n} \otimes B_{p1} & \cdots & A_{1n} \otimes B_{pq} \\ & \vdots & & \ddots & & \vdots & \\ A_{m1} \otimes B_{11} & \cdots & A_{m1} \otimes B_{1q} & & A_{mn} \otimes B_{11} & \cdots & A_{mn} \otimes B_{1q} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ A_{m1} \otimes B_{p1} & \cdots & A_{m1} \otimes B_{pq} & & A_{mn} \otimes B_{p1} & \cdots & A_{mn} \otimes B_{pq} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} \boxtimes \begin{bmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & \ddots & \vdots \\ B_{p1} & \cdots & B_{pq} \end{bmatrix} & \cdots & A_{1n} \boxtimes \begin{bmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & \ddots & \vdots \\ B_{p1} & \cdots & B_{pq} \end{bmatrix} \\ \vdots & \ddots & \vdots \\ A_{m1} \boxtimes \begin{bmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & \ddots & \vdots \\ B_{p1} & \cdots & B_{pq} \end{bmatrix} & \cdots & A_{mn} \boxtimes \begin{bmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & \ddots & \vdots \\ B_{p1} & \cdots & B_{pq} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1n} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{m1} \boxtimes B & \cdots & A_{mn} \boxtimes B \end{bmatrix}.$$

□

Remark 3.9. It is not true in general that the (k,l)th block of $A \boxtimes B$ is $A \boxtimes B_{kl}$.

Example 3.10. Consider

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 7 & 8 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 9 \end{bmatrix},$$

and

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where

$$B_{11} = \begin{bmatrix} 1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 4 & 7 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 5 & 8 \\ 6 & 9 \end{bmatrix}.$$

Let $T_A : \mathbb{C}^3 \to \mathbb{C}^3,\ x \mapsto Ax$ and $T_B : \mathbb{C}^3 \to \mathbb{C}^3,\ x \mapsto Bx$.

By Example 2.11 and Proposition 3.5, we have

$$T_A \boxtimes T_B = T_{A \boxdot B} : \mathbb{C}^9 \to \mathbb{C}^9,\ x \mapsto (A \boxdot B)x$$

where

$$A \boxdot B = \begin{bmatrix} A_{11} \boxdot B & A_{12} \boxdot B \\ A_{21} \boxdot B & A_{22} \boxdot B \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 & 4 & 7 & 8 & 14 & 3 & 12 & 21 \\ 4 & 5 & 16 & 28 & 20 & 25 & 6 & 24 & 42 \\ 2 & 4 & 5 & 8 & 10 & 16 & 6 & 15 & 24 \\ 3 & 6 & 6 & 9 & 12 & 18 & 9 & 18 & 27 \\ 8 & 10 & 21 & 32 & 25 & 40 & 12 & 30 & 48 \\ 12 & 15 & 24 & 36 & 30 & 45 & 18 & 36 & 54 \\ 7 & 8 & 28 & 49 & 32 & 56 & 9 & 36 & 63 \\ 14 & 16 & 35 & 56 & 40 & 64 & 18 & 45 & 72 \\ 21 & 24 & 42 & 63 & 48 & 72 & 27 & 54 & 81 \end{bmatrix}.$$

Consider

$$T_C : \mathbb{C}^9 \to \mathbb{C}^9,\ x \mapsto Cx$$

where

$$C = \begin{bmatrix} A \boxdot B_{11} & A \boxdot B_{12} \\ A \boxdot B_{21} & A \boxdot B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 & 7 & 8 & 14 & 12 & 21 \\ 4 & 5 & 6 & 16 & 28 & 20 & 35 & 24 & 42 \\ 7 & 8 & 9 & 28 & 49 & 32 & 56 & 36 & 63 \\ 2 & 4 & 6 & 5 & 8 & 10 & 16 & 15 & 24 \\ 3 & 6 & 9 & 6 & 9 & 12 & 18 & 18 & 27 \\ 8 & 10 & 12 & 20 & 32 & 25 & 40 & 30 & 48 \\ 12 & 15 & 18 & 24 & 36 & 30 & 45 & 36 & 54 \\ 14 & 16 & 18 & 35 & 56 & 40 & 64 & 45 & 72 \\ 21 & 24 & 27 & 42 & 63 & 48 & 72 & 54 & 81 \end{bmatrix}.$$

We have $T_A \boxtimes T_B \neq T_C$. This show that the (k,l)th block of $T_A \boxtimes T_B$ is not $T_{A \boxdot B_{kl}}$.

Proposition 3.11. *Let A, B and C be compatible operator matrices. Then*

$$(A \oplus B) \boxtimes C = (A \boxtimes C) \oplus (B \boxtimes C). \tag{3.8}$$

Proof. By applying Proposition 3.8, we obtain

$$(A \oplus B) \boxtimes C = \begin{bmatrix} A \boxtimes C & 0 \boxtimes C \\ 0 \boxtimes C & B \boxtimes C \end{bmatrix}.$$

Since $0 \boxtimes X = 0$ for every operator X, we have

$$(A \oplus B) \boxtimes C = \begin{bmatrix} A \boxtimes C & 0 \\ 0 & B \boxtimes C \end{bmatrix} = (A \boxtimes C) \oplus (B \boxtimes C).$$

□

This Proposition means that the Tracy-Singh product is right distributive over the direct sum of operators.

Remark 3.12. It is not true in general that the Tracy-Singh product is left distributive over the direct sum of operators.

Example 3.13. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \end{bmatrix} & \begin{bmatrix} 2 \end{bmatrix} \\ \begin{bmatrix} 3 \\ 5 \end{bmatrix} & \begin{bmatrix} 4 \\ 6 \end{bmatrix} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 4 & 2 \end{bmatrix} \end{bmatrix}$$

and

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} & \begin{bmatrix} 9 \\ 5 \end{bmatrix} \end{bmatrix}.$$

Then we have

$$A \boxdot (B \oplus C) = \left[\begin{array}{cccccccccc}
1 & 0 & & & & 2 & 0 & & & \\
4 & 2 & & & & 8 & 4 & & & \\
 & & 0 & 1 & 9 & & & 0 & 2 & 18 \\
 & & 2 & 0 & 5 & & & 4 & 0 & 10 \\
3 & 0 & & & & 4 & 0 & & & \\
5 & 0 & & & & 6 & 0 & & & \\
12 & 6 & & & & 16 & 8 & & & \\
20 & 10 & & & & 24 & 12 & & & \\
 & & 0 & 3 & 27 & & & 0 & 4 & 36 \\
 & & 6 & 0 & 15 & & & 8 & 0 & 20 \\
 & & 0 & 5 & 45 & & & 0 & 6 & 54 \\
 & & 10 & 0 & 25 & & & 12 & 0 & 30
\end{array}\right]$$

and

$$(A \boxdot B) \oplus (A \boxdot C) = \left[\begin{array}{cccccccccc}
1 & 0 & 2 & 0 & & & & & & \\
4 & 2 & 8 & 4 & & & & & & \\
3 & 0 & 4 & 0 & & & & & & \\
5 & 0 & 6 & 0 & & & & & & \\
12 & 6 & 16 & 8 & & & & & & \\
20 & 10 & 24 & 12 & & & & & & \\
 & & & & 0 & 1 & 9 & 0 & 2 & 18 \\
 & & & & 2 & 0 & 5 & 4 & 0 & 10 \\
 & & & & 0 & 3 & 27 & 0 & 4 & 36 \\
 & & & & 6 & 0 & 15 & 8 & 0 & 20 \\
 & & & & 0 & 5 & 45 & 0 & 6 & 54 \\
 & & & & 10 & 0 & 25 & 12 & 0 & 30
\end{array}\right].$$

Let T_A, T_B and T_C be linear maps induced by matrices A, B and C, respectively. By using the fact that $T_X \oplus T_Y = T_{X \oplus Y}$ for any matrices X and Y together with Proposition 3.5, we get

$$T_A \boxtimes (T_B \oplus T_C) = T_{A \boxdot (B \oplus C)} \neq T_{(A \boxdot B) \oplus (A \boxdot C)} = (T_A \boxtimes T_B) \oplus (T_A \boxtimes T_C).$$

Theorem 3.14 (Mixed product property). *Let $\mathcal{H},\mathcal{H}',\mathcal{H}'',\mathcal{K},\mathcal{K}'$ and $\mathcal{K}''$ be complex Hilbert spaces. Let $A \in \mathbb{B}(\mathcal{H}',\mathcal{H}'')$, $B \in \mathbb{B}(\mathcal{K}',\mathcal{K}'')$, $C \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ and $D \in \mathbb{B}(\mathcal{K},\mathcal{K}')$ be compatible block operator matrices. Then*

$$(A \boxtimes B)(C \boxtimes D) \;=\; (AC) \boxtimes (BD). \tag{3.9}$$

Proof. Write $A = [A_{ij}]_{i,j=1}^{m,n}$, $B = [B_{kl}]_{k,l=1}^{p,q}$, $C = [C_{ij}]_{i,j=1}^{n,r}$ and $D = [D_{kl}]_{k,l=1}^{q,s}$. Using Propositions 2.55 and 2.65, and Theorem 2.57, we get

$$\begin{aligned}
(A \boxtimes B)(C \boxtimes D) &= \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij}\,\big[[C_{ij} \otimes D_{kl}]_{kl}\big]_{ij} \\
&= \left[\left[\sum_{t=1}^{n}\sum_{u=1}^{q}(A_{it} \otimes B_{ku})(C_{tj} \otimes D_{ul})\right]_{kl}\right]_{ij} \\
&= \left[\left[\sum_{t=1}^{n}\sum_{u=1}^{q}(A_{it}C_{tj} \otimes B_{ku}D_{ul})\right]_{kl}\right]_{ij} \\
&= \left[\left[\sum_{t=1}^{n}\left(\sum_{u=1}^{q}A_{it}C_{tj} \otimes B_{ku}D_{ul}\right)\right]_{kl}\right]_{ij} \\
&= \left[\left[\left(\sum_{t=1}^{n}A_{it}C_{uj}\right) \otimes \left(\sum_{u=1}^{q}B_{ku}D_{ul}\right)\right]_{kl}\right]_{ij} \\
&= \left[\sum_{t=1}^{n}A_{it}C_{uj}\right]_{ij} \boxtimes \left[\sum_{u=1}^{q}B_{ku}D_{ul}\right]_{kl} \\
&= (AC) \boxtimes (BD).
\end{aligned}$$

□

For simplicity, we will write $AC \boxtimes BD$ for $(AC) \boxtimes (BD)$.

Corollary 3.15. *For any operator matrices $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$, we have*

$$(A \boxtimes B)^r \;=\; A^r \boxtimes B^r \tag{3.10}$$

for any $r \in \mathbb{N}$.

Proof. The proof is by induction on r. The claim (3.10) is true for $r = 1$ since

$$(A \boxtimes B)^1 \;=\; A \boxtimes B \;=\; A^1 \boxtimes B^1.$$

Suppose the property (3.10) holds for a positive integer r, that is

$$(A \boxtimes B)^r \;=\; A^r \boxtimes B^r.$$

By applying the mixed product property (Theorem 3.14), we have

$$\begin{aligned}(A \boxtimes B)^{r+1} &= (A \boxtimes B)^r (A \boxtimes B) \\ &= (A^r \boxtimes B^r)(A \boxtimes B) \\ &= A^r A \boxtimes B^r B \\ &= A^{r+1} \boxtimes B^{r+1}.\end{aligned}$$

This implies that (3.10) is true for $r+1$. □

Corollary 3.16. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If A and B are invertible, then $A \boxtimes B$ is invertible and*

$$(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}. \tag{3.11}$$

Proof. Applying the mixed product property in Theorem 3.14, we get

$$\begin{aligned}(A \boxtimes B)(A^{-1} \boxtimes B^{-1}) &= AA^{-1} \boxtimes BB^{-1} \\ &= I \boxtimes I \\ &= I,\end{aligned}$$

and

$$\begin{aligned}(A^{-1} \boxtimes B^{-1})(A \boxtimes B) &= A^{-1}A \boxtimes B^{-1}B \\ &= I \boxtimes I \\ &= I.\end{aligned}$$

This implies that $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$. □

Definition 3.17. For $A, B \in \mathbb{B}(\mathcal{H})$, the **commutator** and the **anticommutator** of A and B are respectively defined as follows:

$$\begin{aligned}[A, B] &= AB - BA, \\ [A, B]_+ &= AB + BA.\end{aligned}$$

Corollary 3.18. *Let $A, C \in \mathbb{B}(\mathcal{H})$ and $B, D \in \mathbb{B}(\mathcal{K})$. Then*

$$[A \boxtimes B, C \boxtimes D] = \frac{1}{2}\left([A, C] \boxtimes [B, D]_+ + [A, C]_+ \boxtimes [B, D]\right), \tag{3.12}$$

$$[A \boxtimes B, C \boxtimes D]_+ = \frac{1}{2}\left([A, C] \boxtimes [B, D] + [A, C]_+ \boxtimes [B, D]_+\right). \tag{3.13}$$

Proof. (i) By using Proposition 3.6 and Theorem 3.14, we obtain

$$\begin{aligned}
[A,C]\boxtimes[B,D]_+ + [A,C]_+\boxtimes[B,D] &= (AC-CA)\boxtimes(BD+DB)+(AC+CA)\boxtimes(BD-DB)\\
&= AC\boxtimes BD + AC\boxtimes DB - CA\boxtimes BD - CA\boxtimes DB\\
&\quad + AC\boxtimes BD - AC\boxtimes DB + CA\boxtimes BD - CA\boxtimes DB\\
&= 2(AC\boxtimes BD - CA\boxtimes DB)\\
&= 2\left((A\boxtimes B)(C\boxtimes D)-(C\boxtimes D)(A\boxtimes B)\right)\\
&= 2[A\boxtimes B, C\boxtimes D].
\end{aligned}$$

(ii) Applying Proposition 3.6 and Theorem 3.14, we get

$$\begin{aligned}
[A,C]\boxtimes[B,D] + [A,C]_+\boxtimes[B,D]_+ &= (AC-CA)\boxtimes(BD-DB)+(AC+CA)\boxtimes(BD+DB)\\
&= AC\boxtimes BD - AC\boxtimes DB - CA\boxtimes BD + CA\boxtimes DB\\
&\quad + AC\boxtimes BD + AC\boxtimes DB + CA\boxtimes BD + CA\boxtimes DB\\
&= 2(AC\boxtimes BD - CA\boxtimes DB)\\
&= 2\left((A\boxtimes B)(C\boxtimes D)+(C\boxtimes D)(A\boxtimes B)\right)\\
&= 2[A\boxtimes B, C\boxtimes D]_+.
\end{aligned}$$

□

Corollary 3.19. *Let $A,C\in\mathbb{B}(\mathcal{H})$ and $B,D\in\mathbb{B}(\mathcal{K})$. If*

$$[A,C] = 0 \quad \text{and} \quad [B,D] = 0,$$

then

$$[A\boxtimes B, C\boxtimes D] = 0.$$

Proof. Since $AC=CA$ and $BD=DB$, we have by using Theorem 3.14 that

$$\begin{aligned}
[A\boxtimes B, C\boxtimes D] &= (A\boxtimes B)(C\boxtimes D)-(C\boxtimes D)(A\boxtimes B)\\
&= AC\boxtimes BD - BD\boxtimes AC\\
&= AC\boxtimes BD - AC\boxtimes BD\\
&= 0.
\end{aligned}$$

□

Corollary 3.20. *Let* $A \in \mathbb{B}(\mathcal{H})$ *and* $B \in \mathbb{B}(\mathcal{K})$. *Then*

(i) $[A \boxtimes B, A \boxtimes I] = 0$,

(ii) $[A \boxtimes B, I \boxtimes B] = 0$,

(iii) $[A \boxtimes I, I \boxtimes B] = 0$.

Proof. (i) By using Theorem 3.14, we obtain

$$\begin{aligned} [A \boxtimes B, A \boxtimes I] &= (A \boxtimes B)(A \boxtimes I) - (A \boxtimes I)(A \boxtimes B) \\ &= AA \boxtimes BI - AA \boxtimes IB \\ &= A^2 \boxtimes B - A^2 \boxtimes B \\ &= 0. \end{aligned}$$

Similarly, we arrive at the properties (ii) and (iii). □

Definition 3.21. Let A and B be operators in $\mathbb{B}(\mathcal{H})$. Then we call A and B satisfy a **braid-like relation** if

$$ABA = BAB.$$

Corollary 3.22. *Let* $A, B \in \mathbb{B}(\mathcal{H})$ *be block operator matrices. If* A *and* B *satisfy a braid-like relation, then so are the following pairs of operators:*

(i) $A \boxtimes A$ *and* $B \boxtimes B$,

(ii) $A \boxtimes I$ *and* $B \boxtimes I$,

(iii) $I \boxtimes A$ *and* $I \boxtimes B$.

Proof. (i) By applying Theorem 3.14, we obtain

$$\begin{aligned} (A \boxtimes A)(B \boxtimes B)(A \boxtimes A) &= ABA \boxtimes ABA \\ &= BAB \boxtimes BAB \\ &= (B \boxtimes B)(A \boxtimes A)(B \boxtimes B). \end{aligned}$$

This implies that $A \boxtimes A$ and $B \boxtimes B$ satisfy a braid-like relation.

(ii) Making use of Theorem 3.14, we get

$$\begin{aligned} (A \boxtimes I)(B \boxtimes I)(A \boxtimes I) &= ABA \boxtimes I \\ &= BAB \boxtimes I \\ &= (B \boxtimes I)(A \boxtimes I)(B \boxtimes I). \end{aligned}$$

It follows that $A \boxtimes I$ and $B \boxtimes I$ satisfy a braid-like relation.

(iii) The proof is similar with the property (ii). □

3.2 Order properties of Tracy-Singh products

In this section, we focus on order properties of Tracy-Singh products related to algebraic properties.

Theorem 3.23. *Let* $A \in \mathbb{B}(\mathcal{H})$ *and* $B \in \mathbb{B}(\mathcal{K})$ *be block operator matrices.*

(i) If $A \geqslant 0$ *and* $B \geqslant 0$*, then*

$$A \boxtimes B \geqslant 0. \tag{3.14}$$

(ii) If $A > 0$ *and* $B > 0$*, then*

$$A \boxtimes B > 0. \tag{3.15}$$

Proof. (i) By using Theorems 2.51 and 3.14, we have

$$\begin{aligned} A \boxtimes B &= A^{1/2}A^{1/2} \boxtimes B^{1/2}B^{1/2} \\ &= \left(A^{1/2} \boxtimes B^{1/2}\right)\left(A^{1/2} \boxtimes B^{1/2}\right). \end{aligned}$$

We know that $A^{1/2}$ and $B^{1/2}$ are positive by Definition 2.49. By using Propositions 2.47 and 3.7, we obtain

$$\begin{aligned} A \boxtimes B &= \left(A^{1/2} \boxtimes B^{1/2}\right)\left(A^{1/2} \boxtimes B^{1/2}\right) \\ &= \left(\left(A^{1/2}\right)^* \boxtimes \left(B^{1/2}\right)^*\right)\left(A^{1/2} \boxtimes B^{1/2}\right) \\ &= \left(A^{1/2} \boxtimes B^{1/2}\right)^*\left(A^{1/2} \boxtimes B^{1/2}\right) \\ &\geqslant 0. \end{aligned}$$

(ii) We have by (i) that $A \boxtimes B \geqslant 0$. By using Corollary 3.16, we have $A \boxtimes B$ is invertible. This implies that $A \boxtimes B > 0$. □

Corollary 3.24. *Let* $A, C \in \mathbb{B}(\mathcal{H})$ *and* $B, D \in \mathbb{B}(\mathcal{K})$.

(i) If $A \geqslant C \geqslant 0$ *and* $B \geqslant D \geqslant 0$*, then*

$$A \boxtimes B \geqslant C \boxtimes D. \tag{3.16}$$

(ii) If $A > C > 0$ *and* $B > D > 0$*, then*

$$A \boxtimes B > C \boxtimes D. \tag{3.17}$$

Proof. (i) Applying Propositions 2.48 and 3.6, and Theorem 3.23, we get

$$\begin{aligned} A \boxtimes B - C \boxtimes D &= A \boxtimes B - A \boxtimes D + A \boxtimes D - C \boxtimes D \\ &= A \boxtimes (B - D) + (A - C) \boxtimes D \\ &\geqslant 0. \end{aligned}$$

This implies that $A \boxtimes B \geqslant C \boxtimes D$.

(ii) By using Proposition 2.48 and Theorem 3.23 together with the distributivity of the Tracy-Singh product over the addition (Proposition 3.6), we have

$$\begin{aligned} (A \boxtimes B) - (C \boxtimes D) &= A \boxtimes B - A \boxtimes D + A \boxtimes D - C \boxtimes D \\ &= A \boxtimes (B - D) + (A - C) \boxtimes D \\ &> 0. \end{aligned}$$

□

3.3 Tracy-Singh powers

In this section, we give the definition and some properties concern the exponents for Tracy-Singh products.

Definition 3.25. Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ be a block operator matrix. We define the **Tracy-Singh power** by $A^{\boxtimes 1} = A$ and

$$A^{\boxtimes (r+1)} = A^{\boxtimes r} \boxtimes A \tag{3.18}$$

for $r \in \mathbb{N}$.

Corollary 3.26. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be block operator matrices. Then*

$$\left(A^{\boxtimes r}\right)\left(B^{\boxtimes r}\right) = (AB)^{\boxtimes r} \tag{3.19}$$

for $r \in \mathbb{N}$.

Proof. We use induction on r. For $r = 1$ the property (3.19) is true since

$$\left(A^{\boxtimes 1}\right)\left(B^{\boxtimes 1}\right) = AB = (AB)^{\boxtimes 1}.$$

Suppose that the property (3.19) holds for integer $r \geqslant 1$. By Theorem 3.14,

$$\begin{aligned} \left(A^{\boxtimes (r+1)}\right)\left(B^{\boxtimes (r+1)}\right) &= \left(A^{\boxtimes r} \boxtimes A\right)\left(B^{\boxtimes r} \boxtimes B\right) \\ &= A^{\boxtimes r} B^{\boxtimes r} \boxtimes AB \\ &= (AB)^{\boxtimes r} \boxtimes AB \\ &= (AB)^{\boxtimes (r+1)}. \end{aligned}$$

This implies that the property (3.19) holds for $r+1$. $\square$

Corollary 3.27. *Let $A \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ be a block operator matrix and $r,s \in \mathbb{N}$. Then*

$$\left(A^{\boxtimes r}\right)^{s} = (A^{s})^{\boxtimes r}. \tag{3.20}$$

Proof. We use induction on s. If $s=1$, then the property (3.20) is true because

$$\left(A^{\boxtimes r}\right)^{1} = A^{\boxtimes r} = (A^{1})^{\boxtimes r},$$

where $r \in \mathbb{N}$. Suppose that the property (3.20) holds for integer $s \geqslant 1$. Then, By Proposition 3.26,

$$\begin{aligned}\left(A^{\boxtimes r}\right)^{s+1} &= \left(A^{\boxtimes r}\right)^{s}\left(A^{\boxtimes r}\right)\\ &= (A^{s})^{\boxtimes r}\left(A^{\boxtimes r}\right)\\ &= (A^{s}A)^{\boxtimes r}\\ &= \left(A^{s+1}\right)^{\boxtimes r}.\end{aligned}$$

where $r \in \mathbb{N}$. This shows that the property (3.20) holds for $s+1$. $\square$

Corollary 3.28. *The Tracy-Singh power of linear map induced by matrix is just the linear map induced by the Tracy-Singh power of this matrix. More precisely, for any complex matrices $A=[A_{ij}]$ partitioned in blockmatrix form, we have*

$$(T_A)^{\boxtimes r} = T_{A^{\boxtimes r}}. \tag{3.21}$$

Proof. We shall prove the property (3.21) by using induction on r. When $r=1$ the property (3.21) is true since

$$(T_A)^{\boxtimes 1} = T_A = T_{A^{\boxtimes 1}}.$$

Suppose the property (3.21) holds for $r \in \mathbb{N}$. By using the induction hypothesis and Proposition 3.5, we have

$$\begin{aligned}(T_A)^{\boxtimes(r+1)} &= (T_A)^{\boxtimes r} \boxtimes T_A\\ &= (T_{A^{\boxtimes r}}) \boxtimes T_A\\ &= T_{A^{\boxtimes r}\boxtimes A}\\ &= T_{A^{\boxtimes(r+1)}}.\end{aligned}$$

That is, the property (3.21) holds for $r+1$. $\square$

Corollary 3.29. *Let $A \in \mathbb{B}(\mathcal{H})$ be a block operator matrix. Then*

$$\left(A^{\boxtimes r}\right)^* = (A^*)^{\boxtimes r} \tag{3.22}$$

for $r \in \mathbb{N}$.

Proof. We use induction on r. It easy to see that

$$\left(A^{\boxtimes 1}\right)^* = A^* = (A^*)^{\boxtimes 1}.$$

This show that the property (3.22) is true for $r = 1$. Suppose that the property (3.22) holds for integer $r \geqslant 1$. By Proposition 3.7, we have

$$\begin{aligned}\left(A^{\boxtimes (r+1)}\right)^* &= \left(A^{\boxtimes r} \boxtimes A\right)^* \\ &= \left(A^{\boxtimes r}\right)^* \boxtimes A^* \\ &= (A^*)^{\boxtimes r} \boxtimes A^* \\ &= (A^*)^{\boxtimes (r+1)}.\end{aligned}$$

This implies that the property (3.19) holds for $r + 1$. □

Corollary 3.30. *Let $A \in \mathbb{B}(\mathcal{H})$ be a block operator matrix and $r \in \mathbb{N}$. If A is invertible, then $A^{\boxtimes r}$ is also invertible with*

$$\left(A^{\boxtimes r}\right)^{-1} = \left(A^{-1}\right)^{\boxtimes r}. \tag{3.23}$$

Proof. We proceed by induction on r. For $r = 1$,

$$\left(A^{\boxtimes 1}\right)^{-1} = A^{-1} = \left(A^{-1}\right)^{\boxtimes 1}.$$

This is true. Suppose the statement (3.23) is true for $r \in \mathbb{N}$. Then

$$\begin{aligned}\left(A^{\boxtimes (r+1)}\right)^{-1} &= \left(A^{\boxtimes r} \boxtimes A\right)^{-1} \\ &= \left(A^{\boxtimes r}\right)^{-1} \boxtimes A^{-1} \\ &= \left(A^{-1}\right)^{\boxtimes r} \boxtimes A^{-1} \\ &= \left(A^{-1}\right)^{\boxtimes (r+1)}.\end{aligned}$$

That is the statement (3.23) is true for $r + 1$. □

Corollary 3.31. *Let $A \in \mathbb{B}(\mathcal{H})$ be a block operator matrix and $r \in \mathbb{N}$.*

(i) If $A \geqslant 0$, then

$$A^{\boxtimes r} \geqslant 0. \tag{3.24}$$

(i) If $A > 0$, then

$$A^{\boxtimes r} > 0. \tag{3.25}$$

Proof. (i) We use induction on r. The property (3.24) holds for $r = 1$ since $A^{\boxtimes 1} = A \geqslant 0$. Suppose the property (3.24) holds for $r \in \mathbb{N}$, i.e. $A^{\boxtimes r} \geqslant 0$. Since $A \geqslant 0$ and $A^{\boxtimes r} \geqslant 0$, we have, by Corollary 3.23,

$$A^{\boxtimes (r+1)} = A^{\boxtimes r} \boxtimes A \geqslant 0.$$

(ii) By (i), we have $A^{\boxtimes r} \geqslant 0$. Since A is invertible, $A^{\boxtimes r}$ is invertible by Corollary 3.30. It follows that $A^{\boxtimes r} > 0$. □

Chapter 4

Tracy-Singh Products and Classes of Operators

In this chapter, we show that certain properties of operators are preserved under Tracy-Singh products and Tracy-Singh powers.

Theorem 4.1. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be block operator matrices. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$:*

(i) Hermitian,

(ii) normal,

(iii) isometry,

(iv) co-isometry,

(v) unitary,

(vi) partial isometry,

(vii) projection,

(viii) idempotent,

(ix) nilpotent,

(x) involutory.

Proof. (i) Suppose A and B are Hermitian. Applying Proposition 3.7, we get

$$\begin{aligned}(A \boxtimes B)^* &= A^* \boxtimes B^* \\ &= A \boxtimes B\end{aligned}$$

This implies that $A \boxtimes B$ is Hermitian.

(ii) Suppose A and B are normal. Applying Proposition 3.7 and Theorem 3.14, we get

$$\begin{aligned}(A \boxtimes B)(A \boxtimes B)^* &= (A \boxtimes B)(A^* \boxtimes B^*) \\ &= AA^* \boxtimes BB^* \\ &= A^*A \boxtimes B^*B \\ &= (A^* \boxtimes B^*)(A \boxtimes B) \\ &= (A \boxtimes B)^*(A \boxtimes B).\end{aligned}$$

Thus $A \boxtimes B$ is normal.

(iii) Suppose A and B are isometry. Then we have by using Proposition 3.7 and Theorem 3.14 that

$$\begin{aligned}(A \boxtimes B)^*(A \boxtimes B) &= (A^* \boxtimes B^*)(A \boxtimes B) \\ &= A^*A \boxtimes B^*B \\ &= I \boxtimes I \\ &= I.\end{aligned}$$

Thus $A \boxtimes B$ is isometry.

(iv) Suppose A and B are co-isometry. Applying Proposition 3.7 and Theorem 3.14, we get

$$\begin{aligned}(A \boxtimes B)(A \boxtimes B)^* &= (A \boxtimes B)(A^* \boxtimes B^*) \\ &= AA^* \boxtimes BB^* \\ &= I \boxtimes I \\ &= I.\end{aligned}$$

Hence, $A \boxtimes B$ is co-isometry.

(v) It follows that by (iii) and (iv).

(vi) If A and B are partial isometry, we have, by using Proposition 3.7 and Theorem 3.14, that

$$\begin{aligned}(A \boxtimes B)(A \boxtimes B)^*(A \boxtimes B) &= (A \boxtimes B)(A^* \boxtimes B^*)(A \boxtimes B) \\ &= (AA^*A) \boxtimes (BB^*B) \\ &= A \boxtimes B.\end{aligned}$$

This implies that $A \boxtimes B$ is partial isometry.

(vii) Suppose A and B are projection. By (i), we have $(A \boxtimes B)^* = A \boxtimes B$. Applying Corollary 3.15, we get

$$\begin{aligned}(A \boxtimes B)^2 &= (A \boxtimes B)(A \boxtimes B) \\ &= AA \boxtimes BB \\ &= A^2 \boxtimes B^2 \\ &= A \boxtimes B.\end{aligned}$$

Thus $A \boxtimes B$ is projection.

(viii) Suppose A and B are idempotent. By (vii), we have $(A \boxtimes B)^2 = A \boxtimes B$. Thus $A \boxtimes B$

is idempotent.

(ix) Suppose A and B are nilpotent. Then $A^r = 0$ and $B^s = 0$ for some positive integers r and s. Applying Corollary 3.15, we get

$$\begin{aligned}(A \boxtimes B)^{r+s} &= A^{r+s} \boxtimes B^{r+s} \\ &= A^r A^s \boxtimes B^r B^s \\ &= 0 \cdot A^s \boxtimes B^r \cdot 0 \\ &= 0 \boxtimes 0 \\ &= 0.\end{aligned}$$

It follows that $A \boxtimes B$ is nilpotent.

(x) Suppose A and B are involutory. Applying Theorem 3.14, we get

$$\begin{aligned}(A \boxtimes B)^2 &= (A \boxtimes B)(A \boxtimes B) \\ &= AA \boxtimes BB \\ &= A^2 \boxtimes B^2 \\ &= I \boxtimes I \\ &= I.\end{aligned}$$

This implies that $A \boxtimes B$ is involutory. □

Proposition 4.2. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be block operator matrices. If A and B are skew-Hermitian, then $A \boxtimes B$ is Hermitian.*

Proof. Applying Propositions 3.6 and 3.7, we obtain

$$\begin{aligned}(A \boxtimes B)^* &= A^* \boxtimes B^* \\ &= (-A) \boxtimes (-B) \\ &= A \boxtimes B.\end{aligned}$$

Hence, $A \boxtimes B$ is Hermitian. □

Proposition 4.3. *Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be block operator matrices. If A is Hermitian and B is skew-Hermitian, then $A \boxtimes B$ is skew-Hermitain.*

Proof. By Propositions 3.6 and 3.7, we get

$$\begin{aligned}(A \boxtimes B)^* &= A^* \boxtimes B^* \\ &= A \boxtimes (-B) \\ &= -(A \boxtimes B).\end{aligned}$$

Thus $A \boxtimes B$ is skew-Heimitain. □

Theorem 4.4. *Let $A \in \mathbb{B}(\mathcal{H})$ be a block operator matrix and $r \in \mathbb{N}$. If A satisfies one of the following properties, then the same property holds for $A^{\boxtimes r}$:*

(i) Hermitian,

(ii) normal,

(iii) isometry,

(iv) co-isometry,

(v) unitary,

(vi) partial isometry,

(vii) projection,

(viii) idempotent,

(ix) nilpotent,

(x) involutory.

Proof. (i) Suppose A is Hermitian. By Corollary 3.29, we obtain

$$\begin{aligned}\left(A^{\boxtimes r}\right)^* &= (A^*)^{\boxtimes r} \\ &= A^{\boxtimes r}.\end{aligned}$$

Thus $A^{\boxtimes r}$ is Hermitian.

(ii) Suppose A is normal. Using Corollaries 3.26 and 3.29, we get

$$\begin{aligned}\left(A^{\boxtimes r}\right)\left(A^{\boxtimes r}\right)^* &= A^{\boxtimes r}(A^*)^{\boxtimes r} \\ &= (AA^*)^{\boxtimes r} \\ &= (A^*A)^{\boxtimes r} \\ &= (A^*)^{\boxtimes r}A^{\boxtimes r} \\ &= \left(A^{\boxtimes r}\right)^*\left(A^{\boxtimes r}\right).\end{aligned}$$

Hence, $A^{\boxtimes r}$ is normal.

(iii) Let A be isometry. Then we have by using Corollaries 3.26 and 3.29 that

$$\begin{aligned}\left(A^{\boxtimes r}\right)^*\left(A^{\boxtimes r}\right) &= (A^*)^{\boxtimes r}A^{\boxtimes r}\\ &= (A^*A)^{\boxtimes r}\\ &= I^{\boxtimes r}\\ &= I.\end{aligned}$$

Thus $A^{\boxtimes r}$ is isometry.

(iv) Suppose A is co-isometry. Applying Corollaries 3.26 and 3.29, we get

$$\begin{aligned}\left(A^{\boxtimes r}\right)\left(A^{\boxtimes r}\right)^* &= A^{\boxtimes r}(A^*)^{\boxtimes r}\\ &= (AA^*)^{\boxtimes r}\\ &= I^{\boxtimes r}\\ &= I.\end{aligned}$$

Hence, $A^{\boxtimes r}$ is co-isometry.

(v) It follows that by (iii) and (iv).

(vi) If A is partial isometry, we have, by using Corollaries 3.26 and 3.29, that

$$\begin{aligned}\left(A^{\boxtimes r}\right)\left(A^{\boxtimes r}\right)^*\left(A^{\boxtimes r}\right) &= A^{\boxtimes r}(A^*)^{\boxtimes r}A^{\boxtimes r}\\ &= (AA^*A)^{\boxtimes r}\\ &= A^{\boxtimes r}.\end{aligned}$$

This implies that $A^{\boxtimes r}$ is partial isometry.

(vii) Suppose A is projection. By (i), we have $(A^{\boxtimes r})^* = A^{\boxtimes r}$. Applying Corollary 3.26, we get

$$\begin{aligned}\left(A^{\boxtimes r}\right)^2 &= A^{\boxtimes r}A^{\boxtimes r}\\ &= (AA)^{\boxtimes r}\\ &= (A^2)^{\boxtimes r}\\ &= A^{\boxtimes r}.\end{aligned}$$

Thus $A^{\boxtimes r}$ is projection.

(viii) Suppose A is idempotent. By (vii), we have $(A^{\boxtimes r})^2 = A^{\boxtimes r}$. Thus $A^{\boxtimes r}$ is idempotent.

(ix) Let A be nilpotent. Then $(A^{\boxtimes r})^k = 0$ for some positive integer k. Then we have by using Corollary 3.27 that

$$\begin{aligned}\left(A^{\boxtimes r}\right)^k &= \left(A^k\right)^{\boxtimes r}\\ &= 0^{\boxtimes r}\\ &= 0.\end{aligned}$$

It follows that $A^{\boxtimes r}$ is nilpotent.

(x) Suppose A is involutory. Applying Corollary 3.27, we get

$$\begin{aligned}\left(A^{\boxtimes r}\right)^2 &= A^{\boxtimes r}A^{\boxtimes r}\\ &= (AA)^{\boxtimes r}\\ &= \left(A^2\right)^{\boxtimes r}\\ &= I^{\boxtimes r}\\ &= I.\end{aligned}$$

This implies that $A^{\boxtimes r}$ is involutory. □

Proposition 4.5. *Let $A \in \mathbb{B}(\mathcal{H})$ be skew-Hermitian and $r \in \mathbb{N}$.*

(i) If r is even, then $A^{\boxtimes r}$ is Hermitian.

(ii) If r is odd, then $A^{\boxtimes r}$ is skew-Hermitian.

Proof. (i) If r is even, than $r = 2k$ for some $k \in \mathbb{N}$. Using Proposition 3.6 and Corollary 3.29, we have

$$\begin{aligned}\left(A^{\boxtimes r}\right)^* &= \left(A^{\boxtimes(2k)}\right)^*\\ &= (A^*)^{\boxtimes(2k)}\\ &= (-A)^{\boxtimes(2k)}\\ &= (-1)^{2k}A^{\boxtimes(2k)}\\ &= A^{\boxtimes(2k)}\\ &= A^{\boxtimes r}.\end{aligned}$$

This show that $A^{\boxtimes r}$ is Hermitian.

(ii) If r is odd, than $r = 2k+1$ for some $k \in \mathbb{N}$. Using Proposition 3.6 and Corollary 3.29, we have

$$\begin{aligned}\left(A^{\boxtimes r}\right)^* &= \left(A^{\boxtimes(2k+1)}\right)^* \\ &= (A^*)^{\boxtimes(2k+1)} \\ &= (-A)^{\boxtimes(2k+1)} \\ &= (-1)^{2k+1}A^{\boxtimes(2k+1)} \\ &= -A^{\boxtimes(2k+1)} \\ &= -A^{\boxtimes r}.\end{aligned}$$

This implies that $A^{\boxtimes r}$ is skew-Hermitian. □

Chapter 5

Analytic Properties of the Tracy-Singh Product

In this chapter, we establish some properties of the Tracy-Singh product from analytical point of view.

5.1 Continuity of Tracy-Singh products

Lemma 5.1 ([3]). *Let $A = [A_{ij}]_{i,j=1}^{n,n}$ be a block operator matrix in $\mathbb{B}(\mathcal{H})$. Then*

$$n^{-2} \sum_{i,j=1}^{n} \|A_{ij}\|^2 \leqslant \|A\|^2 \leqslant \sum_{i,j=1}^{n} \|A_{ij}\|^2. \tag{5.1}$$

Lemma 5.2. *Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ be a block operator matrix and let $(A_r)_{r=1}^{\infty}$ be a sequence in $\mathbb{B}(\mathcal{H})$ where $A_r = [A_{ij}^{(r)}]_{i,j=1}^{n,n}$ for each $r \in \mathbb{N}$. Then $A_r \to A$ if and only if $A_{ij}^{(r)} \to A_{ij}$ for all $i, j = 1, \ldots, n$.*

Proof. Suppose that the sequence $(A_r)_{r=1}^{\infty}$ converges to A. For any fixed $i, j \in \{1, ..., n\}$, we have by Lemma 5.1 that

$$\begin{aligned} \|A_{ij}^{(r)} - A_{ij}\|^2 &\leqslant \sum_{i,j=1}^{n} \|A_{ij}^{(r)} - A_{ij}\|^2 \\ &\leqslant n^2 \|A_r - A\|^2 \\ &\to 0 \text{ as } r \to \infty. \end{aligned}$$

Conversely, suppose $A_{ij}^{(r)} \to A_{ij}$ for each i, j. Then, by Lemma 5.1,

$$\begin{aligned} \|A_r - A\|_{\infty}^2 &\leqslant \sum_{i,j=1}^{n} \|A_{ij}^{(r)} - A_{ij}\|_{\infty}^2 \\ &\to 0 \text{ as } r \to \infty. \end{aligned}$$

□

Now, we consider the continuity of the Tracy-Singh product.

Theorem 5.3. *Let $A = [A_{ij}] \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}] \in \mathbb{B}(\mathcal{K})$ be block operator matrices, and let $(A_r)_{r=1}^{\infty}$ and $(B_r)_{r=1}^{\infty}$ be sequences in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$, respectively. If $A_r \to A$ and $B_r \to B$, then $A_r \boxtimes B_r \to A \boxtimes B$.*

Proof. Suppose that $A_r \to A$ and $B_r \to B$. By Lemma 5.2, we have $A_{ij}^{(r)} \to A_{ij}$ and $B_{kl}^{(r)} \to B_{kl}$. Since the tensor product is continuous (Theorem 2.58), we have $A_{ij}^{(r)} \otimes$

$B_{kl}^{(r)} \to A_{ij} \otimes B_{kl}$. It follows by Lemma 5.2 that

$$A_r \boxtimes B_r = \left[\left[A_{ij}^{(r)} \otimes B_{kl}^{(r)}\right]_{kl}\right]_{ij} \to \left[[A_{ij} \otimes B_{kl}]_{kl}\right]_{ij} = A \boxtimes B$$

□

5.2 Tracy-Singh products and certain continuous functions

Theorem 5.4. *Let A and B be positive operators in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$, respectively. Then*

$$(A \boxtimes B)^{\alpha} = A^{\alpha} \boxtimes B^{\alpha} \tag{5.2}$$

for any positive real α.

Proof. From Corollary 3.15, we have

$$\begin{aligned}\left(A^{r/s} \boxtimes B^{r/s}\right)^s &= \left(A^{r/s}\right)^s \boxtimes \left(B^{r/s}\right)^s \\ &= A^r \boxtimes B^r \\ &= (A \boxtimes B)^r,\end{aligned}$$

for any $r, s \in \mathbb{N}$. Hence, $(A \boxtimes B)^{r/s} = A^{r/s} \boxtimes B^{r/s}$. Let $\alpha \in \mathbb{R}^+$. Since the set of positive rational numbers is dense in the positive reals, there is a sequence $(q_n)_{n=1}^{\infty}$ in $\mathbb{Q}^+$ such that $q_n \to \alpha$. By using the continuity of the Tracy-Singh product (Theorem 5.3), we have

$$\begin{aligned}(A \boxtimes B)^{\alpha} &= \lim_{n\to\infty} (A \boxtimes B)^{q_n} \\ &= \lim_{n\to\infty} A^{q_n} \boxtimes B^{q_n} \\ &= \lim_{n\to\infty} A^{q_n} \boxtimes \lim_{n\to\infty} B^{q_n} \\ &= A^{\alpha} \boxtimes B^{\alpha}.\end{aligned}$$

□

Corollary 5.5. *Let A and B be strictly positive operators in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$, respectively. Then*

$$(A \boxtimes B)^{\alpha} = A^{\alpha} \boxtimes B^{\alpha} \tag{5.3}$$

for any real α.

Proof. It follows from Theorem 5.4 and Corollary 3.16. □

Corollary 5.6. *Let A and B be operators in $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $\mathbb{B}(\mathcal{K}, \mathcal{K}')$, respectively. Then*

$$|A \boxtimes B| = |A| \boxtimes |B|. \tag{5.4}$$

Here, $|A| = f(A)$ where $f(z) = |z|$.

Proof. Applying the properties (3.7), (3.9) and (5.2), we get

$$\begin{aligned} |A \boxtimes B| &= [(A \boxtimes B)^*(A \boxtimes B)]^{1/2} \\ &= [(A^* \boxtimes B^*)(A \boxtimes B)]^{1/2} \\ &= (A^*A \boxtimes B^*B)^{1/2} \\ &= (A^*A)^{1/2} \boxtimes (B^*B)^{1/2} \\ &= |A| \boxtimes |B|. \end{aligned}$$

□

5.3 Tracy-Singh products and polar decomposition

Recall the polar decomposition theorem: for any $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$, there exists a partial isometry P such that

$$A = P|A|.$$

The next result is a polar decomposition for the Tracy-Singh product of operators.

Corollary 5.7. *Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$. If $A = P|A|$ and $B = Q|B|$ are polar decompositions of A and B, respectively, then a polar decomposition of $A \boxtimes B$ is given by*

$$A \boxtimes B = (P \boxtimes Q)|A \boxtimes B|. \tag{5.5}$$

Proof. Let P and Q be partial isometries such that $A = P|A|$ and $B = Q|B|$. It follows from Theorem 3.14 and Corollary 5.6 that

$$\begin{aligned} A \boxtimes B &= P|A| \boxtimes Q|B| \\ &= (P \boxtimes Q)(|A| \boxtimes |B|) \\ &= (P \boxtimes Q)|A \boxtimes B|. \end{aligned}$$

By Theorem 4.1, $P \boxtimes Q$ is a partial isometry. Hence, the decomposition (5.5) is a polar one. □

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Appendix

Appendix A

The research paper

The 5th KMITL-TKU International Joint Symposium on Mathematics and Applied Mathematics (MAM2016)

Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Thailand

Type: Oral

Tracy-Singh Products for Bounded Linear Operators on a Hilbert Space

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Abstract

In this paper, we define the Tracy-Singh product for bounded linear operators on a Hilbert space. We establish various properties of Tracy-Singh product involving algebraic properties, structure properties, order properties, and analytic properties.

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1 Introduction

The Kronecker product (or tensor product) of two matrices is very important in linear algebra, scientific computing, and related fields. Recall that for $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ and $B = [b_{kl}] \in M_{p,q}(\mathbb{C})$, the Kronecker product

$$A \circledast B = [a_{ij}B]_{ij} \in M_{mp,nq}(\mathbb{C})$$

is an $m \times n$ block matrix whose (i,j)th block is the $p \times q$ matrix $a_{ij}B$. The Kronecker product has a lot of interesting properties. The most important one is the mixed product property:

$$(A \circledast B)(C \circledast D) = AC \circledast BD \tag{1.1}$$

for all $A \in M_{m,n}(\mathbb{C}), B \in M_{p,q}(\mathbb{C}), C \in M_{n,r}(\mathbb{C})$ and $D \in M_{q,s}(\mathbb{C})$. This kind of matrix product has wide applications in matrix theory, system theory, physics, statistics, computer science, signal processing and other special fields (see, e.g., [4, 5, 6, 9]). The Tracy-Singh product of partitioned matrices was introduced and used in econometrics by Tracy and Singh [3]. Let $A = [A_{ij}] \in M_{m,n}(\mathbb{C})$ be a partitioned matrix with A_{ij} of order $m_i \times n_j$ as the (i,j)th submatrix where $\sum_i m_i = m$ and $\sum_j n_j = n$. Let $B = [B_{kl}] \in M_{p,q}(\mathbb{C})$ be a partitioned matrix with B_{kl} of order $p_k \times q_l$ as the (k,l)th submatrix, where $\sum_k p_k = p$ and $\sum_l q_l = q$. The Tracy-Singh product of A and B is defined by

$$A \boxtimes B = \left[[A_{ij} \circledast B_{kl}]_{kl}\right]_{ij} \in M_{mp,nq}(\mathbb{C}),$$

where $A_{ij} \circledast B_{kl}$ is of order $m_i p_k \times n_j q_l$. In [11], Liu shows that the Tracy-Singh product can be viewed as a generalized Kronecker product. The Tracy-Singh product is studied and applied widely in matrix theory and statistics. See, e.g., [2, 11, 13, 14] for more information.

The tensor product of operators can be viewed as an extension of the Kronecker product to infinite-dimensional spaces. The tensor product of two operators $A \in \mathcal{B}(\mathbb{H}, \mathbb{H}')$ and $B \in \mathcal{B}(\mathbb{K}, \mathbb{K}')$ is the unique bounded linear operator $A \otimes B : \mathbb{H} \otimes \mathbb{K} \to \mathbb{H}' \otimes \mathbb{K}'$ such that

$$(A \otimes B)(x \otimes y) = Ax \otimes By \tag{1.2}$$

for any $x \in \mathbb{H}$ and $y \in \mathbb{K}$. Various attractive properties of tensor product of operators have been established in the literature (see, e.g., [1, 7, 8, 12]).

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In this paper, we extend the notion of tensor product for operators to the Tracy-Singh product of operators expressed in block-matrix form. We introduce the definition of the Tracy-Singh product and establish its algebraic and order properties in Section 2. Then in Section 3, we establish analytic properties of the Tracy-Singh product. Our results generalize the results known so far in the literature for both Tracy-Singh products of matrices and tensor products of operators.

Throughout this paper, let $\mathbb{H}$, $\mathbb{H}'$, $\mathbb{K}$ and $\mathbb{K}'$ be complex separable Hilbert spaces. When X and Y are Hilbert spaces, denote by $\mathcal{B}(X,Y)$ the space of bounded linear operator from X into Y and $\mathcal{B}(X)$ denote $\mathcal{B}(X,X)$. Recall that A is said to be positive, denoted by $A \geqslant 0$, if $\langle Ax, x\rangle \geqslant 0$ for any $x \in \mathbb{H}$. We write $A \geqslant B$ to mean $A - B \geqslant 0$. A positive invertible operator A is denoted by $A > 0$. Finally, we write A^* to indicate the adjoint of A.

2 Algebraic and Order Properties of the Tracy-Singh Product

In this section, we introduce the Tracy-Singh product for bounded linear operators on a Hilbert spaces. Then we investigate its algebraic and order properties. We decompose the Hilbert spaces $\mathbb{H}, \mathbb{H}', \mathbb{K}$ and $\mathbb{K}'$ as direct sums of certain Hilbert spaces as follows:

$$\mathbb{H} = \bigoplus_{j=1}^{n} \mathbb{H}_j, \quad \mathbb{H}' = \bigoplus_{i=1}^{m} \mathbb{H}'_i, \quad \mathbb{K} = \bigoplus_{l=1}^{q} \mathbb{K}_l, \quad \mathbb{K}' = \bigoplus_{k=1}^{p} \mathbb{K}'_k.$$

Each operator $A \in \mathcal{B}(\mathbb{H}, \mathbb{H}')$ can be expressed uniquely as an operator matrix

$$A = [A_{ij}]_{i,j=1}^{m,n}.$$

where $A_{ij} \in \mathcal{B}(\mathbb{H}_j, \mathbb{H}'_i)$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Similarly, an operator B in $\mathcal{B}(\mathbb{K}, \mathbb{K}')$ can be represented uniquely as an operator matrix

$$B = [B_{kl}]_{k,l=1}^{p,q}$$

where $B_{kl} \in \mathcal{B}(\mathbb{K}_l, \mathbb{K}'_k)$ for each $k = 1, \ldots, p$ and $l = 1, \ldots, q$. Such representations are user throughout this paper

Definition 2.1. Let $A = [A_{ij}]_{i,j=1}^{m,n}$ and $B = [B_{kl}]_{k,l=1}^{p,q}$ be operator matrices in $\mathcal{B}(\mathbb{H}, \mathbb{H}')$ and $\mathcal{B}(\mathbb{K}, \mathbb{K}')$, respectively. We define the *Tracy-Singh product* of A and B to be the operator matrix

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl}\right]_{ij} \tag{2.1}$$

where $A_{ij} \otimes B_{kl}$ is a bounded linear operator from $\mathbb{H}_j \otimes \mathbb{K}_l$ to $\mathbb{H}'_i \otimes \mathbb{K}_k$ and the resulting operator $A \boxtimes B$ is a bounded linear operator from $\bigoplus_{j,l=1}^{n,q} \mathbb{H}_j \otimes \mathbb{K}_l$ to $\bigoplus_{i,k=1}^{m,p} \mathbb{H}'_i \otimes \mathbb{K}'_k$.

Remark 2.2. If both A and B are 1×1 block operator matrices, their Tracy-Singh product $A \boxtimes B$ is the tensor product $A \otimes B$.

We shall show that the Tracy-Singh product of operator matrices coincides with the Tracy-Singh product of matrices. Recall that for each $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, the induced maps

$$T_A : \mathbb{C}^n \to \mathbb{C}^m,\ x \mapsto Ax \quad \text{and} \quad T_B : \mathbb{C}^q \to \mathbb{C}^p,\ x \mapsto Bx$$

are bounded linear operators. Using the universal mapping property, we identify $\mathbb{C}^n \otimes \mathbb{C}^q$ with $\mathbb{C}^{nq} \cong M_{n,q}(\mathbb{C})$ together with the canonical bilinear map $(x, y) \mapsto x \circledast y$ for each $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^q$. It is similar for $\mathbb{C}^m \otimes \mathbb{C}^p$.

Lemma 2.3. *For each $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, we have*

$$T_A \otimes T_B = T_{A \circledast B}. \tag{2.2}$$

Proof. By using the mixed product property of the Kronecker product (1.1), we have

$$\begin{aligned}(T_A \otimes T_B)(x \otimes y) &= T_A(x) \otimes T_B(y) = T_A(x) \circledast T_B(y) = Ax \circledast By \\ &= (A \otimes_K B)(x \circledast y) = (A \otimes_K B)(x \otimes y) = T_{A \circledast B}(x \otimes y).\end{aligned}$$

for any $x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^q$. Thus, by the uniqueness of tensor product, $T_A \otimes T_B = T_{A \circledast B}$. □

Proposition 2.4. *Let $A = [A_{ij}]$ and $B = [B_{kl}]$ be complex partitioned matrices of order $m \times n$ and $p \times q$, respectively. Then*

$$T_A \boxtimes T_B = T_{A \boxtimes B}. \tag{2.3}$$

Proof. Recall that the (i,j)th block of the matrix representation of T_A is the $T_{A_{ij}}$. By using Lemma 2.3, we obtain

$$T_A \boxtimes T_B = \left[\left[T_{A_{ij}} \otimes T_{B_{kl}} \right]_{kl} \right]_{ij} = \left[\left[T_{A_{ij} \otimes B_{kl}} \right]_{kl} \right]_{ij} = T_{A \boxtimes B}.$$

□

Proposition 2.4 means that the Tracy-Singh product of two linear maps induced by matrices is just the linear map induced by the Tracy-Singh product of these matrices.

Proposition 2.5. *Let $A \in \mathcal{B}(\mathbb{H}, \mathbb{H}')$, $B \in \mathcal{B}(\mathbb{K}, \mathbb{K}')$ and $\alpha \in \mathbb{C}$. Then*

$$(\alpha A) \boxtimes B = \alpha (A \boxtimes B) = A \boxtimes (\alpha B). \tag{2.4}$$

Proof. Since $(\alpha A)_{ij} = \alpha A_{ij}$ and $(\alpha B)_{kl} = \alpha B_{kl}$ for all i, j, k, l, we get

$$(\alpha A) \boxtimes B = \left[[\alpha A_{ij} \otimes B_{kl}]_{kl} \right]_{ij} = \left[[\alpha (A_{ij} \otimes B_{kl})]_{kl} \right]_{ij} = \alpha \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij} = \alpha (A \boxtimes B).$$

Similarly, $A \boxtimes (\alpha B) = \alpha (A \boxtimes B)$. □

Proposition 2.6. *Let $A \in \mathcal{B}(\mathbb{H}, \mathbb{H}')$ and $B, C \in \mathcal{B}(\mathbb{K}, \mathbb{K}')$ be operator matrices. Then*

$$(A \boxtimes B)^* = A^* \boxtimes B^* \tag{2.5}$$

$$A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C \tag{2.6}$$

$$(B + C) \boxtimes A = B \boxtimes A + C \boxtimes A. \tag{2.7}$$

Proof. Since $A^* = \left[A_{ji}^* \right]_{ij}$ and $B^* = [B_{lk}^*]_{kl}$ for all i, j, k, l, we obtain

$$(A \boxtimes B)^* = \left[[A_{ji} \otimes B_{kl}]_{kl}^* \right]_{ij} = \left[\left[A_{ji}^* \otimes B_{lk}^* \right]_{kl} \right]_{ij} = A^* \boxtimes B^*.$$

By using the fact that $(B + C)_{kl} = B_{kl} + C_{kl}$ for all k, l together with the left distributivity of the tensor product over the addition, we get

$$\begin{aligned} A \boxtimes (B + C) &= \left[[A_{ij} \otimes (B_{kl} + C_{kl})]_{kl} \right]_{ij} \\ &= \left[[(A_{ij} \otimes B_{kl}) + (A_{ij} \otimes C_{kl})]_{kl} \right]_{ij} \\ &= A \boxtimes B + A \boxtimes C. \end{aligned}$$

Similarly, we obtain the right distributivity (2.7). □

Proposition 2.7. *Let $A = [A_{ij}] \in \mathcal{B}(\mathbb{H}, \mathbb{H}')$ and $B = [B_{kl}] \in \mathcal{B}(\mathbb{K}, \mathbb{K}')$. Then*

$$A \boxtimes B = [A_{ij} \boxtimes B]_{ij} = \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1n} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{m1} \boxtimes B & \cdots & A_{mn} \boxtimes B \end{bmatrix}.$$

That is, the (i,j)th block of $A \boxtimes B$ is just $A_{ij} \boxtimes B$.

Proof. By Definition of the Tracy-Singh product, we have

$$A \boxtimes B = \begin{bmatrix} A_{11}\otimes B_{11} & \cdots & A_{11}\otimes B_{1q} & & A_{1n}\otimes B_{11} & \cdots & A_{1n}\otimes B_{1q} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ A_{11}\otimes B_{p1} & \cdots & A_{11}\otimes B_{pq} & & A_{1n}\otimes B_{p1} & \cdots & A_{1n}\otimes B_{pq} \\ & \vdots & & \ddots & & \vdots & \\ A_{m1}\otimes B_{11} & \cdots & A_{m1}\otimes B_{1q} & & A_{mn}\otimes B_{11} & \cdots & A_{mn}\otimes B_{1q} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ A_{m1}\otimes B_{p1} & \cdots & A_{m1}\otimes B_{pq} & & A_{mn}\otimes B_{p1} & \cdots & A_{mn}\otimes B_{pq} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}\boxtimes \begin{bmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & \ddots & \vdots \\ B_{p1} & \cdots & B_{pq} \end{bmatrix} & \cdots & A_{1n}\boxtimes \begin{bmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & \ddots & \vdots \\ B_{p1} & \cdots & B_{pq} \end{bmatrix} \\ \vdots & \ddots & \vdots \\ A_{m1}\boxtimes \begin{bmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & \ddots & \vdots \\ B_{p1} & \cdots & B_{pq} \end{bmatrix} & \cdots & A_{mn}\boxtimes \begin{bmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & \ddots & \vdots \\ B_{p1} & \cdots & B_{pq} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}\boxtimes B & \cdots & A_{1n}\boxtimes B \\ \vdots & \ddots & \vdots \\ A_{m1}\boxtimes B & \cdots & A_{mn}\boxtimes B \end{bmatrix}.$$

□

Remark 2.8. It is not true in general that the (k,l)th block of $A \boxtimes B$ is just $A \boxtimes B_{kl}$.

Let $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$ and $\mathbb{K} = \mathbb{K}_1 \oplus \mathbb{K}_2$. For $A \in \mathcal{B}(\mathbb{H}_1, \mathbb{K}_1)$ and $B \in \mathcal{B}(\mathbb{H}_2, \mathbb{K}_2)$, the operator

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathcal{B}(\mathbb{H}, \mathbb{K}),$$

is said to be the *direct sum* of A and B and denoted by $A \oplus B$.

The next result gives relation between the direct sum and the Tracy-Singh product.

Proposition 2.9. *Let A, B and C be compatible operator matrices. Then*

$$(A \oplus B) \boxtimes C = (A \boxtimes C) \oplus (B \boxtimes C). \tag{2.8}$$

Proof. We know that $0 \boxtimes X = 0$ for every operator X. By applying Proposition 2.7, we obtain

$$(A \oplus B) \boxtimes C = \begin{bmatrix} A\boxtimes C & 0\boxtimes C \\ 0\boxtimes C & B\boxtimes C \end{bmatrix} = \begin{bmatrix} A\boxtimes C & 0 \\ 0 & B\boxtimes C \end{bmatrix} = (A \boxtimes C) \oplus (B \boxtimes C).$$

□

This Proposition means that the Tracy-Singh product is right distributive over the direct sum of operators.

Remark 2.10. It is not true in general that the Tracy-Singh product is left distributive over the direct sum of operators.

The next theorem is called the *mixed product property* which involves both the ordinary operator product and the Tracy-Singh product.

Theorem 2.11. *Let $\mathbb{H}, \mathbb{H}', \mathbb{H}'', \mathbb{K}, \mathbb{K}'$ and $\mathbb{K}''$ be complex Hilbert spaces. Let $A = [A_{ij}]_{i,j=1}^{m,n} \in \mathcal{B}(\mathbb{H}', \mathbb{H}'')$, $C = [C_{ij}]_{i,j=1}^{n,r} \in \mathcal{B}(\mathbb{H}, \mathbb{H}')$, $B = [B_{kl}]_{k,l=1}^{p,q} \in \mathcal{B}(\mathbb{K}', \mathbb{K}'')$ and $D = [D_{kl}]_{k,l=1}^{q,s} \in \mathcal{B}(\mathbb{K}, \mathbb{K}')$ be compatible operator matrices. Then*

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD \tag{2.9}$$

Proof. Recall that for any $Q \in \mathcal{B}(\mathbb{H}', \mathbb{H}'')$, $S \in \mathcal{B}(\mathbb{H}, \mathbb{H}')$, $R \in \mathcal{B}(\mathbb{K}', \mathbb{K}'')$ and $T \in \mathcal{B}(\mathbb{K}, \mathbb{K}')$, we have

$$(Q \otimes R)(S \otimes T) = QS \otimes RT.$$

Using block multiplication of operators, we get

$$\begin{aligned}
(A \boxtimes B)(C \boxtimes D) &= \left[[A_{ij} \otimes B_{kl}]_{kl}\right]_{ij} \left[[C_{ij} \otimes D_{kl}]_{kl}\right]_{ij} \\
&= \left[\left[\sum_{r=1}^{n}\sum_{s=1}^{q}(A_{ir} \otimes B_{ks})(C_{rj} \otimes D_{sl})\right]_{kl}\right]_{ij} \\
&= \left[\left[\sum_{r=1}^{n}\sum_{s=1}^{q}(A_{ir}C_{rj} \otimes B_{ks}D_{sl})\right]_{kl}\right]_{ij} \\
&= \left[\sum_{r=1}^{n} A_{ir}C_{sj}\right]_{ij} \boxtimes \left[\sum_{s=1}^{q} B_{ks}D_{sl}\right]_{kl} \\
&= AC \boxtimes BD.
\end{aligned}$$

□

Corollary 2.12. *For any operator matrices $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$, we have*

$$(A \boxtimes B)^r = A^r \boxtimes B^r \tag{2.10}$$

for any $r \in \mathbb{N}$.

Proof. The proof is by induction on r. We have known from Theorem 2.11 that

$$(A \boxtimes B)^2 = A^2 \boxtimes B^2,$$

i.e., the claim (2.10) is true for $r = 2$. Now, assume that the property (2.10) holds for a positive integer r. By applying the mixed product property (Theorem 2.11), we have

$$(A \boxtimes B)^{r+1} = (A \boxtimes B)(A \boxtimes B)^r = (A \boxtimes B)(A^r \boxtimes B^r) = AA^r \boxtimes BB^r = A^{r+1} \boxtimes B^{r+1}.$$

This implies that (2.10) is true for $r + 1$. □

Corollary 2.13. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$. If A and B are invertible, then $A \boxtimes B$ is invertible and*

$$(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}. \tag{2.11}$$

Proof. By applying the mixed product property, we have

$$(A \boxtimes B)(A^{-1} \boxtimes B^{-1}) = AA^{-1} \boxtimes BB^{-1} = I \boxtimes I = A^{-1}A \boxtimes B^{-1}B = (A^{-1} \boxtimes B^{-1})(A \boxtimes B).$$

This implies that $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$. □

Theorem 2.14. *Let $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$. If both A and B satisfy one of the following properties, then the same property holds for $A \boxtimes B$: self-adjoint, normal, unitary, isometry, co-isometry, partial isometry, projection, idempotent, nilpotent, involutary.*

Proof. Suppose A and B are normal. Applying Theorem 2.11 and Proposition 2.6, we get

$$\begin{aligned}
(A \boxtimes B)(A \boxtimes B)^* &= (A \boxtimes B)(A^* \boxtimes B^*) = AA^* \boxtimes BB^* \\
&= A^*A \boxtimes B^*B = (A^* \boxtimes B^*)(A \boxtimes B) \\
&= (A \boxtimes B)^*(A \boxtimes B).
\end{aligned}$$

Thus $A \boxtimes B$ is normal. If A and B are partial isometry, we have, by using Theorem 2.11 and Proposition 2.6, that

$$(A \boxtimes B)(A \boxtimes B)^*(A \boxtimes B) = (A \boxtimes B)(A^* \boxtimes B^*)(A \boxtimes B) = (AA^*A) \boxtimes (BB^*B) = A \boxtimes B.$$

This implies that $A \boxtimes B$ is partial isometry. Suppose A and B are nilpotent. Then $A^r = 0$ and $B^s = 0$ for some integers r and s. Applying Corollary 2.12, we get

$$(A \boxtimes B)^{r+s} = A^{r+s} \boxtimes B^{r+s} = (A^r)^s \boxtimes (B^s)^r = 0^s \boxtimes 0^r = 0 \boxtimes 0.$$

It follows that $A \boxtimes B$ is nilpotent. Similarly, we have $A \boxtimes B$ is self-adjoint, unitary, isometry, co-isometry, projection, idempotent, and involutary. □

For $A, B \in \mathcal{B}(\mathbb{H})$, the *commutator* and the *anticommutator* of A and B are respectively defined as follows:

$$[A,B] = AB - BA,$$
$$[A,B]_+ = AB + BA.$$

Proposition 2.15. *Let $A, C \in \mathcal{B}(\mathbb{H})$ and $B, D \in \mathcal{B}(\mathbb{K})$. Then*

$$[A \boxtimes B, C \boxtimes D] = \frac{1}{2}\left([A,C] \boxtimes [B,D]_+ + [A,C]_+ \boxtimes [B,D]\right) \tag{2.12}$$

$$[A \boxtimes B, C \boxtimes D]_+ = \frac{1}{2}\left([A,C] \boxtimes [B,D] + [A,C]_+ \boxtimes [B,D]_+\right). \tag{2.13}$$

Proof. By using Proposition 2.6 and Theorem 2.11, we obtain

$$\begin{aligned}
&[A,C] \boxtimes [B,D]_+ + [A,C]_+ \boxtimes [B,D] \\
&\qquad = (AC - CA) \boxtimes (BD + DB) + (AC + CA) \boxtimes (BD - DB) \\
&\qquad = AC \boxtimes BD + AC \boxtimes DB - CA \boxtimes BD - CA \boxtimes DB \\
&\qquad \quad + AC \boxtimes BD - AC \boxtimes DB + CA \boxtimes BD - CA \boxtimes DB \\
&\qquad = 2(AC \boxtimes BD - CA \boxtimes DB) \\
&\qquad = 2\left((A \boxtimes B)(C \boxtimes D) - (C \boxtimes D)(A \boxtimes B)\right) \\
&\qquad = 2[A \boxtimes B, C \boxtimes D].
\end{aligned}$$

Similarly, we arrive at the property (2.13). □

For any nonzero operators A and B such that $A \neq B$, we say that A and B satisfy a *braid-like relation* if

$$ABA = BAB.$$

Corollary 2.16. *Let $A, B \in \mathcal{B}(\mathbb{H})$ be nonzero operator matrices such that $A \neq B$. If A and B satisfy a braid-like relation, then so are the following pairs of operators*

(i) $A \boxtimes A$ *and* $B \boxtimes B$

(ii) $A \boxtimes I$ *and* $B \boxtimes I$

(iii) $I \boxtimes A$ *and* $I \boxtimes B$.

Proof. By applying the mixed product property, we get

$$(A \boxtimes A)(B \boxtimes B)(A \boxtimes A) = ABA \boxtimes ABA = BAB \boxtimes BAB = (B \boxtimes B)(A \boxtimes A)(B \boxtimes B).$$

It follows that $A \boxtimes A$ and $B \boxtimes B$ satisfy a braid-like relation. Similarly, we have (ii) and (iii). □

Now, we focus on order properties of Tracy-Singh products related to algebraic properties.

Corollary 2.17. *Let $A, C \in \mathcal{B}(\mathbb{H})$ and $B, D \in \mathcal{B}(\mathbb{K})$.*

(i) If $A \geqslant C \geqslant 0$ and $B \geqslant D \geqslant 0$, then

$$A \boxtimes B \geqslant C \boxtimes D \geqslant 0. \tag{2.14}$$

(ii) If $A > C > 0$ and $B > D > 0$, then

$$A \boxtimes B > C \boxtimes D > 0. \tag{2.15}$$

Proof. (i) Using Theorem 2.11, we obtain

$$C \boxtimes D = \left(C^{1/2} \boxtimes D^{1/2}\right)\left(C^{1/2} \boxtimes D^{1/2}\right) = \left(C^{1/2} \boxtimes D^{1/2}\right)^{*}\left(C^{1/2} \boxtimes D^{1/2}\right) \geqslant 0.$$

By applying Proposition 2.6, we get

$$(A \boxtimes B) - (C \boxtimes D) = (A - C) \boxtimes B + C \boxtimes (B - D) \geqslant 0.$$

(ii) We have by (i) that $C \boxtimes D \geqslant 0$. By using Corollary 2.13, we have $C \boxtimes D$ is invertible. This implies that $C \boxtimes D > 0$. By applying Proposition 2.6, we get

$$(A \boxtimes B) - (C \boxtimes D) = (A - C) \boxtimes B + C \boxtimes (B - D) > 0.$$

□

The next definition concerns exponents for the Tracy-Singh product.

Definition 2.18. Let $A = [A_{ij}] \in \mathcal{B}(\mathbb{H}, \mathbb{H}')$. We define the *Tracy-Singh power* by $A^{\boxtimes 1} = A$ and

$$A^{\boxtimes(r+1)} = A^{\boxtimes r} \boxtimes A$$

for $r \in \mathbb{N}$.

Proposition 2.19. *Let $A = [A_{ij}] \in \mathcal{B}(\mathbb{H}, \mathbb{H}')$ and $r, s \in \mathbb{N}$. Then*

$$\left(A^{\boxtimes r}\right)^{s} = (A^{s})^{\boxtimes r}. \tag{2.16}$$

Proof. We use induction on s. In particular, when $s = 2$ we obtain, by the mixed product property and induction on r, that

$$\left(A^{\boxtimes r}\right)^{2} = (A^{2})^{\boxtimes r},$$

where $r \in \mathbb{N}$. Suppose that (2.16) holds for $s = k$. Then

$$\left(A^{\boxtimes r}\right)^{k+1} = \left(A^{\boxtimes r}\right)^{k}\left(A^{\boxtimes r}\right) = (A^{k})^{\boxtimes r}\left(A^{\boxtimes r}\right) = (A^{k+1})^{\boxtimes r}.$$

This shows that (2.16) holds for $s = k + 1$. □

For a complex matrix $A = [A_{ij}]$ of order $m \times n$, the *Tracy-Singh power* is defined by $A^{\boxdot 1} = A$ and

$$A^{\boxdot(r+1)} = A^{\boxdot r} \boxdot A$$

for any $r \in \mathbb{N}$. The following result provides that the Tracy-Singh power of two linear maps induced by matrices is just the linear map induced by the Tracy-Singh power of these matrices.

Corollary 2.20. *Let $A = [A_{ij}]$ be complex partitioned matrices and $r \in \mathbb{N}$. Then*

$$(T_A)^{\boxtimes r} = T_{A^{\boxdot r}}. \tag{2.17}$$

Proof. The proof is by applying Proposition 2.4 and induction on r. □

3 Analytic Properties of the Tracy-Singh Product

In this section, we establish some properties of the Tracy-Singh product from analytical point of view.

Recall the following bounds for the operator norm of operator matrices.

Lemma 3.1 ([10]). *Let $A = [A_{ij}]_{i,j=1}^{n,n}$ be an operator matrix in $\mathcal{B}(\mathbb{H})$. Then*

$$n^{-2} \sum_{i,j=1}^{n} \|A_{ij}\|_\infty^2 \leqslant \|A\|_\infty^2 \leqslant \sum_{i,j=1}^{n} \|A_{ij}\|_\infty^2. \tag{3.1}$$

Lemma 3.2. *Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathcal{B}(\mathbb{H})$ be an operator matrix and let $(A_r)_{r=1}^{\infty}$ be a sequence in $\mathcal{B}(\mathbb{H})$ where $A_r = [A_{ij}^{(r)}]_{i,j=1}^{n,n}$ for each $r \in \mathbb{N}$. Then $A_r \to A$ if and only if $A_{ij}^{(r)} \to A_{ij}$ for all $i,j = 1,\ldots,n$.*

Proof. Suppose that the sequence $(A_r)_{r=1}^{\infty}$ converges to A. For any fixed $i,j \in \{1,\ldots,n\}$, we have by Lemma 3.1 that

$$\begin{aligned} \|A_{ij}^{(r)} - A_{ij}\|_\infty^2 &\leqslant \sum_{i,j=1}^{n} \|A_{ij}^{(r)} - A_{ij}\|_\infty^2 \\ &\leqslant n^2 \|A^r - A\|_\infty^2 \\ &\to 0 \text{ as } r \to \infty. \end{aligned}$$

Conversely, suppose $A_{ij}^{(r)} \to A_{ij}$ for each i,j. Then, by Lemma 3.1,

$$\begin{aligned} \|A_r - A\|_\infty^2 &\leqslant \sum_{i,j=1}^{n} \|A_{ij}^{(r)} - A_{ij}\|_\infty^2 \\ &\to 0 \text{ as } r \to \infty. \end{aligned}$$

□

Now, we consider the continuity of the Tracy-Singh product.

Theorem 3.3. *Let $A = [A_{ij}] \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}] \in \mathcal{B}(\mathbb{K})$ be operator matrices, and let $(A_r)_{r=1}^{\infty}$ and $(B_r)_{r=1}^{\infty}$ be sequences in $\mathcal{B}(\mathbb{H})$ and $\mathcal{B}(\mathbb{K})$, respectively. If $A_r \to A$ and $B_r \to B$, then $A_r \boxtimes B_r \to A \boxtimes B$.*

Proof. Suppose that $A_r \to A$ and $B_r \to B$. By Lemma 3.2, we have $A_{ij}^{(r)} \to A_{ij}$ and $B_{kl}^{(r)} \to B_{kl}$. Since the tensor product is continuous, we have $A_{ij}^{(r)} \otimes B_{kl}^{(r)} \to A_{ij} \otimes B_{kl}$. It follows by Lemma 3.2 that

$$A_r \boxtimes B_r = \left[\left[A_{ij}^{(r)} \otimes B_{kl}^{(r)}\right]_{kl}\right]_{ij} \to \left[[A_{ij} \otimes B_{kl}]_{kl}\right]_{ij} = A \boxtimes B$$

□

Theorem 3.4. *Let A and B be positive operators in $\mathcal{B}(\mathbb{H})$ and $\mathcal{B}(\mathbb{K})$, respectively. Then*

$$(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha \tag{3.2}$$

for any positive real α.

Proof. From Corollary 2.12, we have

$$\left(A^{r/s} \boxtimes B^{r/s}\right)^s = \left(A^{r/s}\right)^s \boxtimes \left(B^{r/s}\right)^s = A^r \boxtimes B^r = (A \boxtimes B)^r,$$

for any $r,s \in \mathbb{N}$. Hence, $(A \boxtimes B)^{r/s} = A^{r/s} \boxtimes B^{r/s}$. Let $\alpha \in \mathbb{R}^+$. Since the set of positive rational numbers is dense in the positive reals, there is a sequence $(q_n)_{n=1}^{\infty}$ in $\mathbb{Q}^+$ such that $q_n \to \alpha$. By using the continuity of the Tracy-Singh product (Theorem 3.3), we have

$$(A \boxtimes B)^\alpha = \lim_{n\to\infty} (A \boxtimes B)^{q_n} = \lim_{n\to\infty} A^{q_n} \boxtimes B^{q_n} = \lim_{n\to\infty} A^{q_n} \boxtimes \lim_{n\to\infty} B^{q_n} = A^\alpha \boxtimes B^\alpha.$$

□

Corollary 3.5. *Let A and B be strictly positive operators in $\mathcal{B}(\mathbb{H})$ and $\mathcal{B}(\mathbb{K})$, respectively. Then*

$$(A \boxtimes B)^{\alpha} = A^{\alpha} \boxtimes B^{\alpha} \tag{3.3}$$

for any real α.

Proof. It follows from Theorem 3.2 and Corollary 2.13. □

Corollary 3.6. *Let A and B be operators in $\mathcal{B}(\mathbb{H}, \mathbb{H}')$ and $\mathcal{B}(\mathbb{K}, \mathbb{K}')$, respectively. Then*

$$|A \boxtimes B| = |A| \boxtimes |B|. \tag{3.4}$$

Here, $|A| = f(A)$ where $f(z) = |z|$.

Proof. Applying the properties (2.6), (2.9) and (3.2), we get

$$\begin{aligned}|A \boxtimes B| &= [(A \boxtimes B)^*(A \boxtimes B)]^{1/2} = [(A^* \boxtimes B^*)(A \boxtimes B)]^{1/2} \\ &= (A^*A \boxtimes B^*B)^{1/2} = (A^*A)^{1/2} \boxtimes (B^*B)^{1/2} = |A| \boxtimes |B|.\end{aligned}$$

□

Recall the polar decomposition theorem: for any $A \in \mathcal{B}(\mathbb{H}, \mathbb{H}')$, there exists a partial isometry P such that

$$A = P|A|.$$

The next result is a polar decomposition for the Tracy-Singh product of operators.

Corollary 3.7. *Let $A \in \mathcal{B}(\mathbb{H}, \mathbb{H}')$ and $B \in \mathcal{B}(\mathbb{K}, \mathbb{K}')$. If $A = P|A|$ and $B = Q|B|$ are polar decompositions of A and B, respectively, then a polar decomposition of $A \boxtimes B$ is given by*

$$A \boxtimes B = (P \boxtimes Q)|A \boxtimes B|. \tag{3.5}$$

Proof. Let P and Q be partial isometries such that $A = P|A|$ and $B = Q|B|$. It follows from Theorem 2.11 and Corollary 3.6 that

$$A \boxtimes B = P|A| \boxtimes Q|B| = (P \boxtimes V)(|A| \boxtimes |B|) = (P \boxtimes Q)|A \boxtimes B|.$$

Note that $P \boxtimes Q$ is also a partial isometry, according to Theorem 2.14. Hence, the decomposition (3.5) is a polar one. □

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