KRONECKER PRODUCT OF MATRICES OVER A COMMUTATIVE SEMIRING

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บทคัดย่อ

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Abstract

In this research, we investigate algebraic properties of the Kronecker product of matrices over a commutative semiring. It turns out that this matrix product has a rich and pleasing algebra. These algebraic properties include the associativity, the distribution over the addition and the compatability with other familar matrix operations. We also discuss the relationship between the Kronecker product and a kind of vector operation, namely, the vec operator.

Keywords : commutative semiring, vec opertator, Kronecker product, vec-permutation matrix

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Chapter 1

Introduction

1.1 Research Motivation

Motivated from computer science and statistics, a rectangular matrix is a two-dimensional array for stacking data. To produce a new data, we can use a variety of matrix products. Each type of product is suitable for particular problems occurred in science, engineering, economics, etc. One of the most important matrix products is the Kronecker product, which has a rich and very pleasing algebra that supports a wide range of applications. The theory of Kronecker product of real/complex matrices was established by many authors; see e.g. [11, 17]. Kronecker products are applied widely in matrix/operator theory, numerical linear algebra,statistics,economics, etc.; see [14, 16].

On the other hand, the theory of matrices whose entries come from a suitable algebraic structure such as a semiring or a commuative semiring are practically useful, especially in information sciences and fuzzy systems. Earlier results in this theory concerning the inversion of matrices over a commutative semiring were obtained in [15]. Since then a theory parallel to that of linear algebra to systems of linear equations, linear independence, rank and eigenvalue problems over various kinds of semirings has been obtained by many authors; see e.g.[4, 6, 7, 18]. Examples of applications of matrices over a semiring can be seen in [7].

In this research, we introduce the notion of Kronecker products for matrices over a commutative semiring. This kind of matrix product is not commutative in general. This product satisfies much nice algebraic properties, e.g. the associativity, the distributivity over the addition and the compatability with the scalar multiplication, the transpose and the usual multiplication. Moreover, we introduce a linear transformation that associates each matrix with a vector by stacking each sequential column of the matrix. This operation, called the vec operator, turns usual matrix products to the Kronecker product. It follows that any linear matrix equation of the form

$$\sum_{i=1}^{k} A_i X B_i = C$$

in an unknown matrix X can be reduced to the vector-matrix equation $Ax = B$ Finally, we introduce and investigate basic properties of the vec-permutation matrix which is a kind of permutation matrix that involves the Kronecker product. It follows that $A \otimes B$ and $B \otimes A$ are permutatively similar. Our results extend the results for real/complex matrices in the literature. However, some of the algebraic properties for Kronecker products of real/complex matrices do not hold in this setting.

1.2 Objectives of the study

1) To investigate the notion of Kronecker product of matrices over a commutative semiring
2) To investigate the relationship between the Kronecker product and other matrix operations, such as the addition, the scalar multiplication, the multiplication and the transposition
3) To investigate algebraic properties of the vec operator and the vec-permutation matrix.

1.3 Scope of the study

All matrices considered here are matrices over an arbitrary commutative semiring.

1.4 Benefits of the Study

1) To provide further mathematical theory for matrices.
2) To obtain mathematical tools for scientific computing, information sciences, fuzzy systems and related fields

1.5 Research methodology

1) Study related topics in matrix theory.
2) Study related topics in multilinear algebra.
3) Study basic properties of commutative semiring.
4) Collect and study research papers and textbooks concerned with the Kronecker product of matrices.
5) Determine the objectives and scope of the research.
6) Define the Kronecker product of matrices over a commutative semiring and investigate its basic properties

7) Define the vec operator and investigate its basic properties.
8) Investigate the relationship between the vec operator and the Kronecker product.
9) Define the vec permutation and investigate its relationship with the Kronecker product and the vec operator..
10) Conclude the results, make suggestions for further works and write.

Table 1.1: (Time table)

Activity	Time frame			
	2014	2015		2016
	Aug. - Dec.	Jan. - May.	Jun. - Dec.	Jan. - May.
1	⟺			
2		⟺		
3		⟺		
4		⟺		
5			⟺	
6			⟺	
7			⟺	
8			⟺	
9			⟺	
10				⟺

Chapter 2

Preliminaries

We first recall the definition and examples of a commutative semiring. Then we investigate basic properties for operations on matrices over this algebraic structure.

2.1 Commutative semiring

Definition 2.1. *Let M be a set, and $\cdot$ a binary operation on M. We say that $(M,\cdot)$ is a monoid if it satisfies the following properties:*

1. *for any $x,y \in M$, we have $x \cdot y \in M$*
2. *for any $x,y,z \in M$, we have $(x \cdot y)\cdot z = x \cdot (y \cdot z)$*
3. *there exists $e \in M$ such that for any $x \in M$, we have $e \cdot x = x \cdot e = x$.*

We may also say that M is a monoid under the operation $\cdot$.

The following definition of a semiring was due to Zimmermann [20] and Golan [6].

Definition 2.2. *A semiring is a set L together with*

- *a binary operation $+$ on L, called the addition*
- *a binary operation $\cdot$ on L, called the multiplication*
- *two distinguish elements 0 and 1 in L*

satisfying the following properties:

1. *$(L,+,0)$ is a commutative monoid,*
2. *$(L,\cdot,1)$ is a monoid,*
3. *the multiplication is left and right distributive over the addition,*
4. *$0 \cdot r = 0 = r \cdot 0$ for all $r \in L$.*

This algebraic system can be written as $(L,+,\cdot,0,1)$ or L for short.

Definition 2.3. *A semiring $(L,+,\cdot,1,0)$ is said to be commutative if $r \cdot s = s \cdot r$ for all $r,s \in L$.*

It is clear that a commutative semiring is a generalization of a field. We then provide examples of a commutative semiring which is not a field. These examples arise naturally in applications.

Example 2.1. The unit interval [0,1] under the operations

$$a+b=\sup\{a,b\} \text{ and } a\cdot b=\inf\{a,b\} \qquad \text{for } a,b\in L.$$

This semiring is known as the fuzzy algebra[9].

Example 2.2. The set $[0,\infty)$ of nonnegative real numbers with the usual operations of addition and multiplication.

Example 2.3. The set $\mathbb{N}\cup\{0\}$ of nonnegative integers under the operations

$$a+b=\begin{cases}0, & a=b=0\\ \gcd(a,b), & \text{otherwise}\end{cases}$$

$$a\cdot b=\begin{cases}\operatorname{lcm}(a,b), & a,b\in\mathbb{N}\\ 0, & \text{otherwise.}\end{cases}$$

Definition 2.4. *Let $a,b\in\mathbb{Z}$, where not both of a,b are not zero. A greatest common divisor d of a and b is a positive integer d such that*

1. *$d|a$ and $d|b$*
2. *if d' is an integers such that $d'|a$ and $d'|b$, then $d'|d$.*

Definition 2.5. *Let $a,b\in\mathbb{Z}-\{0\}$ .A least common multiple m of a and b is a positive integer m such that*

1. *$a|m$ and $b|m$*
2. *if m' is an integers such that $a|m'$ and $b|m'$ then $m|m'$.*

Example 2.4. For each natural number N the set $\mathbb{Z}_n$ of integers modulo n together with its usual operations.

Example 2.5. Consider the extended real number system $\mathbb{R}\cup\{-\infty,\infty\}$. Then $\mathbb{R}\cup\{-\infty\}$ is a commutative semiring under the operations

$$a+b=\max\{a,b\} \text{ and } a\cdot b=a+b \qquad \text{for } a,b\in\mathbb{R}\cup\{-\infty\}$$

This semiring is usually called a max-plus algebra or a schedule algebra[2].

2.2 Matrices over a commutative semiring

Form now on, let L be a commutative semiring. For each $m, n \in \mathbb{N}$, let $M_{m,n}(L)$ be the set of all m-by-n matrices whose entries come from L. If $A \in M_{m,n}(L)$ has a_{ij} as its $(i,j)th$ entry for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$, we write $A = [a_{ij}]_{i,j=1}^{m,n}$ or $A = [a_{ij}]$ for short. We abbreviate $M_n(L) = M_{n,n}(L)$ and $V_n(L) = M_{n,1}(L)$. For each matrix A, we denote its $(i,j)th$ block by A_{ij}.

Definition 2.6. *For each $A = [a_{ij}], B = [b_{ij}] \in M_{m,n}(L)$ and $k \in L$, define the addition and the scalar mutiplication as follows:*

$$A + B = [a_{ij} + b_{ij}] \in M_{m,n}(L),$$
$$kA = [ka_{ij}] \in M_{m,n}(L).$$

Definition 2.7. *If $A = [a_{ij}] \in M_{m,n}(L)$ and $B = [b_{jk}] \in M_{n,p}(L)$, we define the multiplication of A and B by*

$$AB = \left[\sum_{j=1}^{n} a_{ij}b_{jk}\right] \in M_{m,p}(L).$$

Definition 2.8. *A matrix in which each entry is zero is called a zero-matrix, denoted by*

$$0 = [0] \in M_n(L). \tag{2.1}$$

Definition 2.9. *The identity matrix defined by*

$$I_n = [\delta_{ij}] \in M_n(L) \tag{2.2}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Theorem 2.6. *The set $M_n(L)$ is a semiring with respect to the addition and the scalar multiplication defined as above. More precisely,*

1. *$(A + B) + C = A + (B + C)$ for all $A, B, C \in M_n(L)$*
2. *there is $0 \in M_n(L)$ such that $A + 0 = A = 0 + A$ for all $A \in M_n(L)$*
3. *$A + B = B + A$ for all $A, B \in M_n(L)$*
4. *$(AB)C = A(BC)$ for all $A, B, C \in M_n(L)$*
5. *there is $I \in M_n(L)$ such that $AI = A = IA$ for all $A \in M_n(L)$*
6. *$A(B + C) = AB + AC$ for all $A, B, C \in M_n(L)$*

7. $(B+C)A = BA + CA$ *for all* $A, B, C \in M_n(L)$

8. $0A = 0 = A0$ *for all* $A \in M_n(L)$

9. $0 \neq I$

This semiring is not commutative unless $n = 1$.

Proposition 2.7. *Let* A,B *be matrices over a commutative semiring* L *and let* $\alpha, \beta \in L$. *If the following matrix operations exist, we have*

1. $\alpha(A+B) = \alpha A + \alpha B$

2. $(\alpha+\beta)A = \alpha A + \beta A$

3. $(\alpha A)B = \alpha(AB) = A(\alpha B)$.

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$.Then

$$\alpha(A+B) = \alpha[a_{ij} + b_{ij}] = [\alpha(a_{ij} + b_{ij})] = [\alpha a_{ij} + \alpha b_{ij}] = [\alpha a_{ij}] + [\alpha b_{ij}] = \alpha A + \alpha B,$$

$$(\alpha+\beta)A = [(\alpha+\beta)a_{ij}] = [\alpha a_{ij} + \beta a_{ij}] = [\alpha a_{ij}] + [\beta b_{ij}] = \alpha A + \beta A,$$

For the last property Let $A = [a_{ij}]$ and $B = [b_{jk}]$

$$\begin{aligned}
\alpha(AB) &= \alpha\left[\sum_k a_{ik}b_{kj}\right] \\
&= \left[\alpha\sum_k a_{ik}b_{kj}\right] \\
&= \left[\sum_k \alpha(a_{ik}b_{kj})\right] \\
&= \left[\sum_k a_{ik}(\alpha b_{kj})\right] \\
&= A(\alpha B),
\end{aligned}$$

$$\begin{aligned}
\alpha(AB) &= \alpha\left[\sum_k a_{ik}b_{kj}\right] \\
&= \left[\alpha\sum_k a_{ik}b_{kj}\right] \\
&= \left[\sum_k \alpha(a_{ik}b_{kj})\right] \\
&= \left[\sum_k (\alpha a_{ik})b_{kj}\right] \\
&= (\alpha A)B.
\end{aligned}$$

□

Definition 2.10. *The transpose of a matrix* $A = [a_{ij}] \in M_{m,n}(L)$, *denoted by* A^T, *is defined to be the matrix in* $M_{n,m}(L)$ *whose* $(i,j)th$ *entry is given by* a_{ji} *for each* $i = 1,\dots,n$ *and* $j = 1,\dots,m$.

Basic properties of the transpose of matrices over a commutative semiring in connections with the addition, the scalar multiplication and the multiplication are given here.

Proposition 2.8. *Let* $A, B \in M_{m,n}(L), C \in M_{n,p}(L)$ *and* $k \in L$. *Then*

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AC)^T = C^T A^T$

Proof. Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{jk}]$. Then

$$(A^T)^T = ([a_{ij}]^T)^T = [a_{ji}]^T = [a_{ij}] = A,$$

$$(A+B)^T = [a_{ij} + b_{ij}]^T = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = A^T + B^T,$$

$$(kA)^T = [ka_{ij}]^T = [ka_{ji}] = k[a_{ji}] = kA^T.$$

For the last property, observe that the $(k,i) - th$ entry of $(AC)^T$ is given by

$$[AC]^T_{ki} = [AC]_{ik} = \sum_{j=1}^{n} a_{ij}c_{jk} = \sum_{j=1} c_{jk}a_{ij} = \sum_{j=1}^{n} [C^T]_{kj}[A^T]_{ji} = [C^T A^T]_{ki}.$$

This means that $(AC)^T = C^T A^T$. ☐

Definition 2.11. *A matrix* $A \in M_n(L)$ *is said to be invertible if there is a matrix* $B \in M_n(L)$ *such that* $AB = I_n = BA$.

Definition 2.12. *The direct sum of* $A \in M_n(L)$ *and* $B \in M_m(L)$ *is defined by*

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in M_{n+m}(L).$$

Definition 2.13. *The trace of a matrix* $A = (a_{ij}) \in M_n(L)$ *is defined by*

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

The following linearity properties of the trace of a matrix over a commutative semiring will be used in later discussions.

Proposition 2.9. *For each $A, B \in M_n(L)$ and $k \in L$, we have*

1. $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
2. $\mathrm{tr}(kA) = k\,\mathrm{tr}(A)$.

Proof. Let $A = [a_{ii}]$ and $B = [b_{ii}]$

$$\mathrm{tr}(A+B) = \sum_{i=1}^{n} a_{ii} + b_{ii} = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \mathrm{tr}(A) + \mathrm{tr}(B).$$

$$\mathrm{tr}(kA) = \sum_{i=1}^{n} ka_{ii} = k\sum_{i=1}^{n} a_{ii} = k\,\mathrm{tr}(A).$$

□

2.3 Kronecker product for complex matrices

In this section, we review fundamental properties of the Kronecker product for complex matrices. See more information in [1, 8, 17].

Definition 2.14. *Let $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$. The Kronecker product of A and B is defined to be*

$$A \otimes B = [a_{ij}B]_{ij} \in M_{mp,nq}(\mathbb{C}) \tag{2.3}$$

That is each $(i,j)-th$ block of $A \otimes B$ is given by $a_{ij}B$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

Theorem 2.10. *Let A, B and C be matrices over $\mathbb{C}$ If the following matrix operations exist, then one has*

1. $(kA) \otimes B = k(A \otimes B) = A \otimes (kB)$
2. $(A+B) \otimes C = (A \otimes C) + (B \otimes C)$
3. $A \otimes (B+C) = (A \otimes B) + (A \otimes C)$
4. $(A \otimes B)^T = A^T \otimes B^T$
5. $A \otimes 0 = 0 \otimes A = 0$.

Proposition 2.11. *If $x \in V_m(\mathbb{C})$ and $y \in V_n(\mathbb{C})$ then $xy^T = x \otimes y^T = y^T \otimes x$.*

Proposition 2.12. *The Kronecker product of two identity matrices $I_n \in M_n(\mathbb{C})$ and $I_m \in M_m(\mathbb{C})$ results in the identity matrix I_{mn}, that is*

$$I_m \otimes I_n = I_{mn}. \tag{2.4}$$

Proposition 2.13. *The k-copies direct sum of a matrix $A \in M_{m,n}(\mathbb{C})$ can be written*

$$I_k \otimes A = A \oplus A \oplus \ldots \oplus A. \tag{2.5}$$

Proposition 2.14. *Let $A \in M_{m,n}(\mathbb{C})$, $B \in M_{p,q}(\mathbb{C})$ and $C \in M_{r,s}(\mathbb{C})$ then*

$$(A \otimes B) \otimes C = A \otimes (B \otimes C). \tag{2.6}$$

Proposition 2.15. *If $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ then,*

$$\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B). \tag{2.7}$$

Proposition 2.16. *Let $A \in M_n(\mathbb{C})$,$B \in M_m(\mathbb{C})$ and $C \in M_p(\mathbb{C})$.Then*

$$(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C). \tag{2.8}$$

Corollary 2.17. *Let $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$.Then, $A \otimes B = 0$ if and only if either $A = 0$ or $B = 0$.*

Theorem 2.18. *Let $A \in M_{m,n}(\mathbb{C})$, $B \in M_{p,q}(\mathbb{C})$, $C \in M_{n,k}(\mathbb{C})$ and $D \in M_{q,r}(\mathbb{C})$. Then,*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD). \tag{2.9}$$

Corollary 2.19. *Let $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$. If both A and B are invertible, then $A \otimes B$ is invertible and*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{2.10}$$

Chapter 3

Basic properties of the Kronecker product of matrices over a commutative semiring

In this chapter, we define the Kronecker product of matrices over a commutative semiring by means of block matrices and establish its basic algebraic properties associated to the matrix operations introduced in chapter 2.

3.1 Definition and example of the Kronecker product of matrices over a commutative semiring

Definition 3.1. *Let* $A \in M_{m,n}(L)$ *and* $B \in M_{p,q}(L)$*. The Kronecker product of* A *and* B *is defined to be*

$$A \otimes B = [a_{ij}B]_{ij} \in M_{mp,nq}(L) \tag{3.1}$$

That is each $(i,j)-th$ *block of* $A \otimes B$ *is given by* $a_{ij}B$ *for* $i = 1, 2, ..., m$ *and* $j = 1, 2, ..., n$*.*

Example 3.1. Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. For instance, consider the following matrices over the set $\mathbb{N} \cup \{0\}$ of nonnegative integers:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & 5 \\ 6 & 3 \end{bmatrix}$$

$$\begin{aligned}
A \otimes B &= \begin{bmatrix} 2\begin{bmatrix} 1 & 5 \\ 6 & 3 \end{bmatrix} & 3\begin{bmatrix} 1 & 5 \\ 6 & 3 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 5 \\ 6 & 3 \end{bmatrix} & 4\begin{bmatrix} 1 & 5 \\ 6 & 3 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 2\cdot 1 & 2\cdot 5 & 3\cdot 1 & 3\cdot 5 \\ 2\cdot 6 & 2\cdot 3 & 3\cdot 6 & 3\cdot 3 \\ 1\cdot 1 & 1\cdot 5 & 4\cdot 1 & 4\cdot 5 \\ 1\cdot 6 & 1\cdot 3 & 4\cdot 6 & 4\cdot 3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{lcm}(2,1) & \mathrm{lcm}(2,5) & \mathrm{lcm}(3,1) & \mathrm{lcm}(3,5) \\ \mathrm{lcm}(2,6) & \mathrm{lcm}(2,3) & \mathrm{lcm}(3,6) & \mathrm{lcm}(3,3) \\ \mathrm{lcm}(1,1) & \mathrm{lcm}(1,5) & \mathrm{lcm}(4,1) & \mathrm{lcm}(4,5) \\ \mathrm{lcm}(1,6) & \mathrm{lcm}(1,3) & \mathrm{lcm}(4,6) & \mathrm{lcm}(4,3) \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} 2 & 10 & 3 & 15 \\ 6 & 6 & 6 & 3 \\ 1 & 5 & 4 & 20 \\ 6 & 3 & 12 & 12 \end{bmatrix}.$$

Example 3.2. Let $A \in M_{m,n}(L)$and$B \in M_{p,q}(L)$. For instance, consider the following matrices over The fuzzy algebra [0,1]:

$$A = \begin{bmatrix} 0.2 & 0.3 \\ 1 & 0.7 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 0.5 & 0.1 \\ 0.8 & 0.4 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} 0.2\begin{bmatrix} 0.5 & 0.1 \\ 0.8 & 0.4 \end{bmatrix} & 0.3\begin{bmatrix} 0.5 & 0.1 \\ 0.8 & 0.4 \end{bmatrix} \\ 1\begin{bmatrix} 0.5 & 0.1 \\ 0.8 & 0.4 \end{bmatrix} & 0.7\begin{bmatrix} 0.5 & 0.1 \\ 0.8 & 0.4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 \cdot 0.5 & 0.2 \cdot 0.1 & 0.3 \cdot 0.5 & 0.3 \cdot 0.1 \\ 0.2 \cdot 0.8 & 0.2 \cdot 0.4 & 0.3 \cdot 0.8 & 0.3 \cdot 0.4 \\ 1 \cdot 0.5 & 1 \cdot 0.1 & 0.7 \cdot 0.5 & 0.7 \cdot 0.1 \\ 1 \cdot 0.8 & 1 \cdot 0.4 & 0.7 \cdot 0.8 & 0.7 \cdot 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} \inf(0.2, 0.5) & \inf(0.2, 0.1) & \inf(0.3, 0.5) & \inf(0.3, 0.1) \\ \inf(0.2, 0.8) & \inf(0.2, 0.4) & \inf(0.3, 0.8) & \inf(0.3, 0.4) \\ \inf(1, 0.5) & \inf(1, 0.1) & \inf(0.7, 0.5) & \inf(0.7, 0.1) \\ \inf(1, 0.8) & \inf(1, 0.4) & \inf(0.7, 0.8) & \inf(0.7, 0.4) \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0.1 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.3 & 0.3 \\ 0.5 & 0.1 & 0.5 & 0.1 \\ 0.8 & 0.4 & 0.7 & 0.4 \end{bmatrix}$$

Example 3.3. Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. For instance, consider the following matrices over $\mathbb{R} \cup \{-\infty\}$:

$$A = \begin{bmatrix} -\infty & 2 \\ 3 & -\infty \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & -\infty \\ -4 & 2 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} -\infty \begin{bmatrix} 1 & -\infty \\ -4 & 2 \end{bmatrix} & 2 \begin{bmatrix} 1 & -\infty \\ -4 & 2 \end{bmatrix} \\ 3 \begin{bmatrix} 1 & -\infty \\ -4 & 2 \end{bmatrix} & -\infty \begin{bmatrix} 1 & -\infty \\ -4 & 2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -\infty \cdot 1 & -\infty \cdot -\infty & 2 \cdot 1 & 2 \cdot -\infty \\ -\infty \cdot -4 & -\infty \cdot 2 & 2 \cdot -4 & 2 \cdot 2 \\ 3 \cdot 1 & 3 \cdot -\infty & -\infty \cdot 1 & -\infty \cdot -\infty \\ 3 \cdot -4 & 3 \cdot 2 & -\infty \cdot -4 & -\infty \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} -\infty + 1 & -\infty + -\infty & 2 + 1 & 2 + -\infty \\ -\infty + -4 & -\infty + 2 & 2 + -4 & 2 + 2 \\ 3 + 1 & 3 + -\infty & -\infty + 1 & -\infty + -\infty \\ 3 + -4 & 3 + 2 & -\infty + -4 & -\infty + 2 \end{bmatrix}$$

$$= \begin{bmatrix} -\infty & -\infty & 3 & -\infty \\ -\infty & -\infty & -2 & 4 \\ 4 & -\infty & -\infty & -\infty \\ -1 & 5 & -\infty & -\infty \end{bmatrix}$$

Proposition 3.4. *If $x \in V_m(L)$ and $y \in V_n(L)$ then $xy^T = x \otimes y^T = y^T \otimes x$.*

Proof. Write $x = [x_1, x_2, \cdots, x_m]^T$ and $y = [y_1, y_2, \cdots, y_n]^T$

$$y^T \otimes x = [y_1, \ldots, y_n] \otimes \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

$$= \begin{bmatrix} y_1x_1 & \ldots & y_nx_1 \\ \vdots & \ddots & \vdots \\ y_1x_m & \ldots & y_nx_m \end{bmatrix}$$

$$= \begin{bmatrix} x_1(y_1, \ldots, y_n) \\ \vdots \\ x_m(y_1, \ldots, y_n) \end{bmatrix}$$

$$= \begin{bmatrix} x_1y^T \\ \vdots \\ x_my^T \end{bmatrix}$$

$$= x \otimes y^T.$$

□

Proposition 3.5. *The Kronecker product of two identity matrices* $I_n \in M_n(L)$ *and* $I_m \in M_m(L)$ *results in the identity matrix* I_{mn}*, that is*

$$I_m \otimes I_n = I_{mn}. \tag{3.2}$$

Proof. Let $I_n \in M_n(L)$ and $I_m \in M_m(L)$

$$\begin{aligned} I_m \otimes I_n &= \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix} \otimes I_n \\ &= \begin{bmatrix} 1I_n & \dots & 0I_n \\ \vdots & \ddots & \vdots \\ 0I_n & \dots & 1I_n \end{bmatrix} \\ &= \begin{bmatrix} I_n & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_n \end{bmatrix} \\ &= I_{mn} \end{aligned}$$

$\square$

Proposition 3.6. *The* k*-copies direct sum of a matrix* $A \in M_{m,n}(L)$ *can be written*

$$I_k \otimes A = A \oplus A \oplus \dots \oplus A. \tag{3.3}$$

Proof. Let $A \in M_{m,n}(L)$

$$\begin{aligned} I_k \otimes A &= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \otimes A \\ &= \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{bmatrix} \\ &= A \oplus A \oplus \dots \oplus A. \end{aligned}$$

$\square$

3.2 The compatability of the Kronecker product and matrix operations

Proposition 3.7. *Let A, B and C be matrices over L. If the following matrix operations exist, then one has*

1. $(kA) \otimes B = k(A \otimes B) = A \otimes (kB)$
2. $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$
3. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$
4. $(A \otimes B)^T = A^T \otimes B^T$.

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. It follows from Proposition 2.7 that

$$(kA) \otimes B = [(ka_{ij})B]_{ij} = k[a_{ij}B]_{ij} = k(A \otimes B)$$

$$A \otimes (kB) = [a_{ij}(kB)]_{ij} = [(ka_{ij})B]_{ij} = [k(a_{ij}B)]_{ij} = k[a_{ij}B]_{ij} = k(A \otimes B)$$

To prove the second property, it suffices to show that each $(i, j)th$ block of $(A+B) \otimes C$ and $(A \otimes C) + (B \otimes C)$ coincides. Indeed, we have

$$[(A + B) \otimes C]_{ij} = (a_{ij} + b_{ij})C = a_{ij}C + b_{ij}C = [A \otimes C]_{ij} + [B \otimes C]_{ij} = [(A \otimes C) + (B \otimes C)]_{ij}.$$

Similary, we have $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$. For the last property, we have by Proposition 2.8 that

$$(A \otimes B)^T = [a_{ij}B]^T_{ij} = [(a_{ji}B)^T]_{ij} = [a_{ji}B^T]_{ij} = A^T \otimes B^T$$

The following observation is very useful in dealing with Kronecker products by means of block matrices. □

Lemma 3.8. *For any matrix A and B, each $(i, j) - th$ block of $A \otimes B$ is given by $A_{ij} \otimes B$.*

Proof. Suppose that we partition into $s \times r$ block matrix with $A_{ij} \in M_{m_i, n_j}(L)$ for each $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, r$. Then we have

$$\mathrm{A} \otimes \mathrm{B} = \begin{bmatrix} a_{11}B & \ldots & a_{1n_1}B & & a_{11}B & \ldots & a_{1n_r}B \\ \vdots & \ddots & \vdots & \ldots & \vdots & \ddots & \vdots \\ a_{m_1 1}B & \ldots & a_{m_1 n_1}B & & a_{m_1 1}B & \ldots & a_{m_1 n_r}B \\ & \vdots & & \ddots & & \vdots & \\ a_{11}B & \ldots & a_{1n_1}B & & a_{11}B & \ldots & a_{1n_r}B \\ \vdots & \ddots & \vdots & \ldots & \vdots & \ddots & \vdots \\ a_{m_s 1}B & \ldots & a_{m_s n_s}B & & a_{m_s 1}B & \ldots & a_{m_s n_r}B \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} \otimes B & \dots & A_{1r} \otimes B \\ \vdots & \ddots & \vdots \\ A_{s1} \otimes B & \dots & A_{sr} \otimes B \end{bmatrix}$$

$$= \Big[A_{ij} \otimes B\Big]_{ij}.$$

$\square$

Remark 3.9. In contrast to Lemma 3.8, it is not true that $A \otimes B = (A \otimes B_{ij})_{ij}$. To see this, consider the following matrices over the fuzzy algebra $[0,1]$

$$A = \begin{bmatrix} 0.5 & 0.3 & 0.5 \\ 0.2 & 0.4 & 0.6 \\ 0.1 & 0.7 & 0.3 \end{bmatrix}, B_{11} = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, B_{12} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, B_{21} = \begin{bmatrix} 0.4 \end{bmatrix} B_{22} = \begin{bmatrix} 0.8 \end{bmatrix}$$

Then

$$[A \otimes B_{ij}]_{ij} = \begin{bmatrix} A \otimes B_{11} & A \otimes B_{12} \\ A \otimes B_{21} & A \otimes B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.3 & 0.5 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.5 & 0.3 & 0.5 \\ 0.2 & 0.4 & 0.6 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.2 & 0.4 & 0.5 \\ 0.1 & 0.7 & 0.3 & 0.1 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.3 \\ 0.4 & 0.3 & 0.4 & 0.5 & 0.3 & 0.5 \\ 0.2 & 0.4 & 0.4 & 0.2 & 0.4 & 0.1 \\ 0.1 & 0.4 & 0.3 & 0.1 & 0.7 & 0.3 \end{bmatrix}$$

The next result shows the associativity of the Kronecker product

Proposition 3.10. *Let $A \in M_{m,n}(L)$, $B \in M_{p,q}(L)$ and $C \in M_{r,s}(L)$ then*

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \tag{3.4}$$

Proof. It follows from Lemma 3.8 that

$$A \otimes (B \otimes C) = \begin{bmatrix} a_{11}(B \otimes C) & \dots & a_{1n}(B \otimes C) \\ \vdots & \ddots & \vdots \\ a_{m1}(B \otimes C) & \dots & a_{mn}(B \otimes C) \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}B) \otimes C & \dots & (a_{1n}B) \otimes C \\ \vdots & \ddots & \vdots \\ (a_{m1}B) \otimes C & \dots & (a_{mn}B) \otimes C \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \otimes C$$

$$= (A \otimes B) \otimes C$$

□

Remark 3.11. $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. Then the property that $A \otimes B = 0$ if and only if $A = 0$ or $B = 0$ is not true in general. For instance, let $L = (\mathbb{Z}_6, +, \cdot)$ and consider the following matrices

$$A = \begin{bmatrix} \overline{4} & \overline{0} \\ \overline{0} & \overline{2} \end{bmatrix}, B = \begin{bmatrix} \overline{3} & \overline{3} \\ \overline{0} & \overline{3} \end{bmatrix}$$

Then

$$A \otimes B = \begin{bmatrix} \overline{4}\begin{bmatrix} \overline{3} & \overline{3} \\ \overline{0} & \overline{3} \end{bmatrix} & \overline{0}\begin{bmatrix} \overline{3} & \overline{3} \\ \overline{0} & \overline{3} \end{bmatrix} \\ \overline{0}\begin{bmatrix} \overline{3} & \overline{3} \\ \overline{0} & \overline{3} \end{bmatrix} & \overline{2}\begin{bmatrix} \overline{3} & \overline{3} \\ \overline{0} & \overline{3} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \overline{4}\cdot\overline{3} & \overline{4}\cdot\overline{3} & \overline{0}\cdot\overline{3} & \overline{0}\cdot\overline{3} \\ \overline{4}\cdot\overline{0} & \overline{4}\cdot\overline{3} & \overline{0}\cdot\overline{0} & \overline{0}\cdot\overline{3} \\ \overline{0}\cdot\overline{3} & \overline{0}\cdot\overline{3} & \overline{2}\cdot\overline{3} & \overline{2}\cdot\overline{3} \\ \overline{0}\cdot\overline{0} & \overline{0}\cdot\overline{3} & \overline{2}\cdot\overline{0} & \overline{2}\cdot\overline{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The next result shows that the Kronecker product is compatible with the matrix trace.

Proposition 3.12. *If* $A \in M_n(L)$ *and* $B \in M_m(L)$ *then,*

$$\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B) \tag{3.5}$$

Proof. Observe that the sum of the main diagonal entries of $A \otimes B$ is $\sum_{i=1}^{n}(a_{ii}B)$.So, taking the trace of $A \otimes B$ gives.

$$\begin{aligned}\operatorname{tr}(A \otimes B) &= \sum_{i=1}^{a}\left(a_{ii}\sum_{j=1}^{b}(b_{jj})\right) = \sum_{i=1}^{a}(a_{ii}\operatorname{tr}(B)) = \operatorname{tr}(B)\sum_{i=1}^{a}(a_{ii}) \\ &= \operatorname{tr}(B)\operatorname{tr}(A) = \operatorname{tr}(A)\operatorname{tr}(B).\end{aligned}$$

Here, we use the linearity of the trace in Proposition2.9 ☐

Definition 3.2. *The k^{th} Kronecker power of $A^{\otimes k}$ is $A \in M_{m,n}(L)$. defined inductively for all positive integers k by*

$$A^{\otimes 1} \equiv A \text{ and } A^{\otimes k} \equiv A \otimes A^{\otimes (k-1)} \text{ for } k = 2, 3, \cdots \tag{3.6}$$

This definition implies that for $A \in M_{m,n}(L)$,the matrix $A^{\otimes k} \in M_{m^k,n^k}(L)$

Example 3.13. Let $A \in M_{m,n}(L)$. For instance, consider the following matrices over the set $\mathbb{N} \cup \{0\}$ of nonnegative integers

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 18 & 6 \end{bmatrix} \text{ and } \quad k = 2.$$

Then

$$A^{\otimes 2} = \begin{bmatrix} 4 & 12 & 4 & 12 & 3 & 6 & 4 & 6 & 2 \\ 4 & 8 & 12 & 3 & 24 & 6 & 2 & 8 & 6 \\ 4 & 3 & 2 & 8 & 24 & 8 & 12 & 6 & 6 \\ 1 & 8 & 6 & 8 & 8 & 24 & 6 & 24 & 6 \end{bmatrix}.$$

Proposition 3.14. *Let $A \in M_{m,n}(L)$. Then$A^{\otimes r} \otimes A^{\otimes s} = A^{\otimes (r+s)}$ and $(A^{\otimes r})^{\otimes s} = A^{\otimes (rs)}$ for all the positive integer r and s.*

Proof. Induction on r, $r = 1$, $A^{\otimes 1} \otimes A^{\otimes S} = A^{\otimes (1+S)}$

Suppose the property hold for r, i.e.,

$$A^{\otimes r} \otimes A^{\otimes s} = A^{\otimes (r+s)}$$

Then

$$\begin{aligned}A^{\otimes (r+1)} \otimes A^{\otimes s} &= A \otimes A^{\otimes r} \otimes A^{\otimes s} \\ &= A \otimes A^{\otimes (r+s)} \\ &= A^{\otimes (r+1)+s},\end{aligned}$$

Finally, Induction on s, $s = 1$, $(A^{\otimes r})^{\otimes 1} = A^{\otimes r \cdot 1}$

Suppose the property hold for s, i.e.,

$$(A^{\otimes r})^{\otimes s} = A^{\otimes rs}$$

Then

$$\begin{aligned}(A^{\otimes r})^{\otimes (s+1)} &= (A^{\otimes r})^{1} \otimes (A^{\otimes r})^{s} \\ &= A^{\otimes r} \otimes A^{\otimes rs} \\ &= A^{\otimes (r+rs)} \\ &= A^{\otimes r(s+1)}.\end{aligned}$$

☐

The next proposition asserts the compatibility between the Kronecker product and the direct sum of matrices.

Proposition 3.15. *Let* $A \in M_n(L), B \in M_m(L)$ *and* $C \in M_p(L)$. *Then*

$$(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C). \tag{3.7}$$

Proof. It follows from Lemma 3.8 that

$$\begin{aligned}(A \oplus B) \otimes C &= \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \otimes C \\ &= \begin{bmatrix} A \otimes C & 0 \otimes C \\ 0 \otimes C & B \otimes C \end{bmatrix} \\ &= \begin{bmatrix} A \otimes C & 0 \\ 0 & B \otimes C \end{bmatrix} \\ &= (A \otimes C) \oplus (B \otimes C).\end{aligned}$$

☐

Remark 3.16. In general, it does not hold that $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$ To see this, consider the following matrices over the fuzzy algebra $[0,1]$

$$A = \begin{bmatrix} 0.2 & 0.3 \\ 0.7 & 0.1 \end{bmatrix}, B = \begin{bmatrix} 0.1 & 0.8 \\ 0.4 & 0.6 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

Then

$$A \otimes (B \oplus C) = \begin{bmatrix} 0.2 & 0.3 \\ 0.7 & 0.1 \end{bmatrix} \otimes \begin{bmatrix} 0.1 & 0.8 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2\cdot0.1 & 0.2\cdot0.8 & 0.2\cdot0 & 0.2\cdot0 & 0.3\cdot0.1 & 0.3\cdot0.8 & 0.3\cdot0 & 0.3\cdot0 \\ 0.2\cdot0.4 & 0.2\cdot0.6 & 0.2\cdot0 & 0.2\cdot0 & 0.3\cdot0.4 & 0.3\cdot0.6 & 0.3\cdot0 & 0.3\cdot0 \\ 0.2\cdot0 & 0.2\cdot0 & 0.2\cdot0 & 0.2\cdot0 & 0.3\cdot0 & 0.3\cdot0 & 0.3\cdot0 & 0.3\cdot0 \\ 0.2\cdot0 & 0.2\cdot0 & 0.2\cdot0 & 0.2\cdot0.3 & 0.3\cdot0 & 0.3\cdot0 & 0.3\cdot0 & 0.3\cdot0.3 \\ 0.7\cdot0.1 & 0.7\cdot0.8 & 0.7\cdot0 & 0.7\cdot0 & 0.1\cdot0.1 & 0.1\cdot0.8 & 0.1\cdot0 & 0.1\cdot0 \\ 0.7\cdot0.4 & 0.7\cdot0.6 & 0.7\cdot0 & 0.7\cdot0 & 0.1\cdot0.4 & 0.1\cdot0.6 & 0.1\cdot0 & 0.1\cdot0 \\ 0.7\cdot0 & 0.7\cdot0 & 0.7\cdot0 & 0.7\cdot0 & 0.1\cdot0 & 0.1\cdot0 & 0.1\cdot0 & 0.1\cdot0 \\ 0.7\cdot0 & 0.7\cdot0 & 0.7\cdot0 & 0.7\cdot0.3 & 0.1\cdot0 & 0.1\cdot0 & 0.1\cdot0 & 0.1\cdot0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.2 & 0 & 0 & 0.1 & 0.3 & 0 & 0 \\ 0.2 & 0.2 & 0 & 0 & 0.3 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.3 \\ 0.1 & 0.7 & 0 & 0 & 0.1 & 0.1 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0.1 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$$(A \otimes B) \oplus (A \otimes C) = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.3 \\ 0.2 & 0.2 & 0.3 & 0.3 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.4 & 0.6 & 0.1 & 0.1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.3 \\ 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.3 & 0 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0 & 0 & 0 & 0 \\ 0.1 & 0.7 & 0.1 & 0.1 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0.1 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 03 & 0 & 0.1 \end{bmatrix}.$$

3.3 The mixed product property and its consequences

In this section, we prove that the Kronecker product is compatible with the usual matrix product. This property is called the mixed product property. Then some consequences of this result are given.

Lemma 3.17. *For each* $A \in M_{m,n}(L)$ *and* $B \in M_{p,q}(L)$ *we have*

$$A \otimes B = (A \otimes I_p)(I_n \otimes B) = (I_m \otimes B)(A \otimes I_q). \tag{3.8}$$

Proof. A direct computation shows that

$$\begin{aligned} A \otimes B &= \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \\ &= \begin{bmatrix} a_{11}I_p & \dots & a_{1n}I_p \\ \vdots & \ddots & \vdots \\ a_{m1}I_p & \dots & a_{mn}I_p \end{bmatrix} \begin{bmatrix} B & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B \end{bmatrix} \\ &= (A \otimes I_p)(I_n \otimes B), \end{aligned}$$

$$\begin{aligned} A \otimes B &= \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \\ &= \begin{bmatrix} B & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B \end{bmatrix} \begin{bmatrix} a_{11}I_p & \dots & a_{1n}I_p \\ \vdots & \ddots & \vdots \\ a_{m1}I_p & \dots & a_{mn}I_p \end{bmatrix} \\ &= (I_m \otimes B)(A \otimes I_q). \end{aligned}$$

□

Theorem 3.18. *Let* $A \in M_{m,n}(L), C \in M_{n,p}(L), B \in M_{q,r}(L)$ *and* $D \in M_{r,s}(L)$*. Then*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD). \tag{3.9}$$

Proof. By making use of Lemma 3.17, we obtain

$$\begin{aligned} (A \otimes B)(C \otimes D) &= (A \otimes I_q)(I_n \otimes B)(C \otimes I_r)(I_p \otimes D) \\ &= (A \otimes I_q)[(I_n \otimes B)(C \otimes I_r)(I_p \otimes D)] \\ &= (A \otimes I_q)(C \otimes B)(I_p \otimes D) \\ &= (A \otimes I_q)[(C \otimes I_q)(I_p \otimes B)](I_p \otimes D) \end{aligned}$$

$$= [(A \otimes I_q)(C \otimes I_q)][(I_p \otimes B)(I_p \otimes D)]$$
$$= [(AC) \otimes I_q][I_p \otimes (BD)]$$
$$= (AC) \otimes (BD)$$

which is the desired result. ☐

Corollary 3.19. *If the following matrix products exist, then one has*

1. $(A_1 \otimes B_1)(A_2 \otimes B_2) \cdots (A_p \otimes B_p) = (A_1 A_2 \cdots A_p) \otimes (B_1 B_2 \cdots B_p)$,
2. $(A_1 \otimes A_2 \otimes \cdots \otimes A_p)(B_1 \otimes B_2 \otimes \cdots \otimes B_p) = (A_1 B_1) \otimes (A_2 B_2) \otimes \cdots \otimes (A_p B_p)$,
3. $(AB)^{\otimes k} = A^{\otimes k} B^{\otimes k}$ *for any* $k \in \mathbb{N}$,
4. $(A^{\otimes r})^s = (A^s)^{\otimes r}$ *for any* $r, s \in \mathbb{N}$.

Proof. It follows immediately from the mixed product property (3.9) via mathematical inductions.

☐

Corollary 3.20. *Let* $A \in M_n(L)$ *and* $B \in M_m(L)$. *If both* A *and* B *are invertible, then* $A \otimes B$ *is invertible and*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{3.10}$$

Proof. Using the mixed product property (3.9), we have

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = (AA^{-1}) \otimes (BB^{-1}) = I \otimes I = I.$$

and, similarly $(A^{-1} \otimes B^{-1})(A \otimes B) = I$. This means that $A \otimes B$ is invertible with $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ ☐

Chapter 4

Kronecker Product, The Vec Operator and Vec-permutation matrices

4.1 Kronecker Product and The Vec Operator

In this chapter, we introduce a column-stacking operator that associates a matrix to a column vector, called the vec operator. This operator is linear and bijective. Moreover, we discuss the usage of the vec operator to the area of linear matrix equations.

For each matrix A, we write A_i for the $i-th$ column of A The precise definition of the vec operator is given here.

Definition 4.1. *For each $A \in M_{m,n}(L)$, we define*

$$\text{vec}(A) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \in L^{mn} \tag{4.1}$$

That is, vec(A) *is a column vector of each sequential column of A stacked on top of one another.*

Example 4.1. Let $A \in M_{m,n}(L)$. For instance, consider the following matrices over The fuzzy algebra [0,1]

$$A = \begin{bmatrix} 0.2 & 0.5 & 0.35 & 0.9 \\ 0.45 & 0.7 & 0.6 & 0.1 \end{bmatrix}$$

Then

$$\text{vec}(A) = \begin{bmatrix} 0.2 \\ 0.45 \\ 0.5 \\ 0.7 \\ 0.35 \\ 0.6 \\ 0.9 \\ 0.1 \end{bmatrix}$$

Proposition 4.2. *The vec operator* $\operatorname{vec} : M_{m,n}(L) \to L^{mn}$ *is a bijection satisfying*

$$\operatorname{vec}(A+B) = \operatorname{vec}(A) + \operatorname{vec}(B)$$
$$\operatorname{vec}(kA) = k\operatorname{vec}(A)$$

for any $A, B \in M_{m,n}(L)$ *and* $k \in L$.

Proof.

$$\begin{aligned}\operatorname{vec}(A+B) &= \begin{bmatrix} A_1 + B_1 \\ A_2 + B_2 \\ \vdots \\ A_n + B_n \end{bmatrix} \\ &= \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} \\ &= \operatorname{vec}(A) + \operatorname{vec}(B), \\ \operatorname{vec}(kA) &= \begin{bmatrix} kA_1 \\ kA_2 \\ \vdots \\ kA_n \end{bmatrix} \\ &= k\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \\ &= k\operatorname{vec}(A).\end{aligned}$$

☐

Lemma 4.3. *Let* $A \in M_{m,n}(L)$*,* $B \in M_{n,p}(L)$ *and* $C \in M_{p,n}(L)$*. Then*

1. $\operatorname{vec}(AB) = (I_p \otimes A)\operatorname{vec}(B)$
2. $\operatorname{vec}(CA^T) = (A \otimes I_p)\operatorname{vec}(C)$.

Proof. Observe that the i-th column of AB is given by AB_i for each $i = 1, 2, \dots, n$.

It follows that

$$\operatorname{vec}(AB) = \begin{bmatrix} (AB)_1 \\ (AB)_2 \\ \vdots \\ (AB)_p \end{bmatrix}$$

$$= \begin{bmatrix} AB_1 \\ AB_2 \\ \vdots \\ AB_p \end{bmatrix}$$

$$= \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \dots & A \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_p \end{bmatrix}$$

$$= (I_p \otimes A)\operatorname{vec}(B).$$

To prove another assertion, note that for each $T \in M_{m,n}(L)$ and $x \in V_n(L)$, the product Tx can be written as a linear combination of the columns of A where the coefficients are given by the coordinates of x. Now, we have

$$\operatorname{vec}(CA^T) = \begin{bmatrix} (CA^T)_1 \\ \vdots \\ (CA^T)_m \end{bmatrix}$$

$$= \begin{bmatrix} C(A^T)_1 \\ \vdots \\ C(A^T)_m \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}C_1 + & \dots & +a_{1n}C_n \\ \vdots & \ddots & \vdots \\ a_{m1}C_1 + & \dots & +a_{mn}C_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}I_p & \dots & a_{1n}I_p \\ \vdots & \ddots & \vdots \\ a_{m1}I_p & \dots & a_{mn}I_p \end{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$$

$$= (A \otimes I_p)\operatorname{vec}(C).$$

□

Theorem 4.4. *Let* $A \in M_{m,n}(L)$,$B \in M_{n,p}(L)$ *and* $C \in M_{p,q}(L)$. *Then*

$$\operatorname{vec}(ABC) = (C^T \otimes A) \operatorname{vec}(B). \tag{4.2}$$

Proof. It follows from Theorem 3.18 and Lemma 4.3 that

$$\begin{aligned} \operatorname{vec}(ABC) &= \operatorname{vec}((AB)C) \\ &= (C^T \otimes I_m) \operatorname{vec}(AB) \\ &= (C^T \otimes I_m)(I_p \otimes A) \operatorname{vec}(B) \\ &= [(C^T \otimes I_m)(I_p \otimes A)] \operatorname{vec}(B) \\ &= (C^T \otimes A) \operatorname{vec}(B). \end{aligned}$$

☐

Given $A \in M_{m,n}(L), B \in M_{p,q}(L)$ and $C \in M_{m,q}(L)$, consider the linear matrix equation

$$AXB = C$$

where $X \in M_{n,p}(L)$ is an unknown matrix. Proposition 4.2 and Theorem 4.4 assert that this matrix equation is equivalent to the system of mq equations in np unknowns given by

$$(B^T \otimes A) \operatorname{vec}(X) = \operatorname{vec}(C).$$

More generally, the linear matrix equation

$$\sum_{i=1}^{k} A_i X B_i = C$$

is equivalent to the vector-matrix equation

$$\left[\sum_{i=1}^{k} (B_i^T \otimes A_i)\right] \operatorname{vec}(X) = \operatorname{vec}(C)$$

which is easier to solve.

4.2 Vec-permutation matrices

For this chapter, we introduce and investigate basic properties of vecpermutation matrices. It follows that $A \otimes B$ and $B \otimes A$ are permutatively similar via a vec-permutation matrix.

Definition 4.2. *Let* e_{in} *denote the* n*-dimensional column vector which has 1 in the* i*th position and* $0's$ *elsewhere; that is,*

$$e_{in} := [0, 0, \ldots, 0, 1, 0, \ldots, 0]^T \in L^n. \tag{4.3}$$

Define the vec -permutation matrix

$$P_{mn} := \begin{bmatrix} I_m \otimes e_{1n}^T \\ I_m \otimes e_{2n}^T \\ \vdots \\ I_m \otimes e_{nn}^T \end{bmatrix} \in M_{mn}(L) \qquad (4.4)$$

The vec-permutation is a permutation matrix, that is, its rows are obtained from a permutation of the rows of identity matrix.

Example 4.5. Let $P_{mn} \in M_{mn}(L)$.Consider P_{32} and P_{23}

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Lemma 4.6. *The following properties for vec-permutation matrices hold:*

1. $P_{mn}^T = P_{nm}$
2. $P_{mn}^T P_{mn} = P_{mn} P_{mn}^T = I_{mn}$

Proof. It follows from Proposition 2.8 that

$$
\begin{aligned}
P_{mn}^{T} &= \begin{bmatrix} I_m^T \otimes e_{1n}, \dots, I_m^T \otimes e_{nn} \end{bmatrix} \\
&= \begin{bmatrix} e_{1m}^T \otimes e_{1n} & \dots & e_{1m}^T \otimes e_{nn} \\ \vdots & \ddots & \vdots \\ e_{mm}^T \otimes e_{1n} & \dots & e_{mm}^T \otimes e_{nn} \end{bmatrix} \\
&= \begin{bmatrix} e_{1n} \otimes e_{1m}^T & \dots & e_{nn} \otimes e_{1m}^T \\ \vdots & \ddots & \vdots \\ e_{1n} \otimes e_{mm}^T & \dots & e_{nn} \otimes e_{mm}^T \end{bmatrix} \\
&= \begin{bmatrix} I_n \otimes e_{1m}^T \\ \vdots \\ I_n \otimes e_{mm}^T \end{bmatrix} \\
&= P_{nm}
\end{aligned}
$$

For the second assertion, by using Theorem3.18 , we have

$$
\begin{aligned}
P_{mn}^{T} P_{mn} &= \begin{bmatrix} I_m \otimes e_{1n}, \dots, I_m \otimes e_{nn} \end{bmatrix} \begin{bmatrix} I_m \otimes e_{1n}^T \\ \vdots \\ I_m \otimes e_{nn}^T \end{bmatrix} \\
&= (I_m \otimes e_{1n})(I_m \otimes e_{1n}^T) + \dots + (I_m \otimes e_{nn})(I_m \otimes e_{nn}^T) \\
&= (I_m \otimes e_{1n} e_{1n}^T) + \dots + (I_m \otimes e_{nn} e_{nn}^T) \\
&= I_m \otimes \left[\textstyle\sum_{i=1}^{n} e_{in} e_{in}^T \right] \\
&= I_m \otimes I_n \\
&= I_{mn}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
P_{mn} P_{mn}^{T} &= \begin{bmatrix} I_m \otimes e_{1n}^T \\ \vdots \\ I_m \otimes e_{nn}^T \end{bmatrix} \begin{bmatrix} I_m \otimes e_{1n}, \dots, I_m \otimes e_{nn} \end{bmatrix} \\
&= \begin{bmatrix} I_m \otimes e_{1n}^T e_{1n} & \dots & I_m \otimes e_{1n}^T e_{nn} \\ \vdots & \ddots & \vdots \\ I_m \otimes e_{nn}^T e_{1n} & \dots & I_m \otimes e_{nn}^T e_{nn} \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} I_m & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_m \end{bmatrix}$$

$$= I_{mn}.$$

$\square$

Theorem 4.7. *If $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$, then one has*

$$P_{mp}(A \otimes B)P_{nq}^T = B \otimes A \tag{4.5}$$

Proof. For each $i = 1, 2, \dots, p$, we write B^i for the $i-th$ row of B. According to the definition of P_{mn} and Lemma 3.8 , we have

$$\begin{aligned} P_{mp}(A \otimes B)P_{nq}^T &= \begin{bmatrix} I_m \otimes e_{1p}^T \\ \vdots \\ I_m \otimes e_{pp}^T \end{bmatrix} \begin{bmatrix} A_1 \otimes B, \dots, A_n \otimes B \end{bmatrix} P_{nq}^T \\ &= \begin{bmatrix} A_1 \otimes B^1 & \dots & A_n \otimes B^1 \\ \vdots & \vdots & \vdots \\ A_1 \otimes B^p & \dots & A_n \otimes B^p \end{bmatrix} P_{nq}^T \\ &= \begin{bmatrix} A \otimes B^1 \\ \vdots \\ A \otimes B^p \end{bmatrix} \begin{bmatrix} I_n \otimes e_{1q}, \dots, I_n \otimes e_{qq} \end{bmatrix} \\ &= \begin{bmatrix} Ab_{11} & \dots & Ab_{1q} \\ \vdots & \vdots & \vdots \\ Ab_{p1} & \dots & Ab_{pq} \end{bmatrix} \\ &= B \otimes A. \end{aligned}$$

$\square$

Definition 4.3. *A permutation matrix is a matrix obtained by permuting the rows of an $n \times n$ identity matrix according to some permutation of the numbers 1 to n. Every row and column therefore contains precisely a single 1 with $0s$ everywhere else, and every permutation corresponds to a unique permutation matrix. There are therefore $n!$ permutation matrices of size n, where $n!$ is a factorial.*

A matrix $A \in M_n(L)$ is said to be permutatively similar to a matrix $B \in M_n(L)$ if there is a permutation matrix P such that $PAP^T = B$.

Corollary 4.8. *If* $A \in M_m(L)$ *and* $B \in M_n(L)$, *then* $A \otimes B$ *is permutatively similar to* $B \otimes A$.

Proof. According to Theorem 4.7 , for $n = m$ and $q = p$,one has $P_{mp}(A \otimes B)P_{mp}^T = B \otimes A$. Since $P_{mp}^T = P_{mp}^{-1}$. is a permutation matrix, we conclude that $A \otimes B$ is permutatively imilar to $B \otimes A$. □

Chapter 5

Conclusions and Suggestions

5.1 Conclusions

We extend the notion of Kronecker products for complex matrices to matrices over an arbitrary commutative semiring. This product turns out to be compatible with the following algebraic operations:

- the addition
- the scalar multiplication
- the usual matrix multiplication
- the direct sum
- the transpose

Moreover, we introduce the vector operator and the vec-permutation matrices. We investigate certain relations between the Kronecker product and these notions.

5.2 Suggestion for Further Works

- In this research, we consider Kronecker product over a commutative semiring. What happen when we extend the study to certain generalizations of Kronecker product over a commutative semiring?
- In this research, we investigate vec operator over a commutative semiring. What happen when we extend the study to certain generalizations of vec operator over a commutative semiring?

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Appendix

Appendix A

The research paper

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Kronecker Product of Matrices over a Commutative Semiring

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Abstract : In this paper, we investigate algebraic properties of the Kronecker product of matrices over a commutative semiring. It turns out that this matrix product has a rich and pleasing algebra. These algebraic properties include the associativity, the distribution over the addition, and the compatibility with other matrix operations. We also discuss the relationship between the Kronecker product and a kind of vector operation, namely, the vector operator.

Keywords : : commutative semiring, Kronecker product, vec operator.
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1 Introduction

Motivated from computer science and statistics, a rectangular matrix is a two-dimensional array for stacking data. To produce a new data, we can use a variety of matrix products. Each type of product is suitable for particular problems occurred in science, engineering, economics, etc. One of the most important matrix products is the Kronecker product, which has a rich and very pleasing algebra that supports a wide range of applications. The theory of Kronecker product of real/complex

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matrices was established by many authors; see e.g. [13, 17, 23, 25, 26, 27]. Kronecker products are applied widely in matrix/operator theory, numerical linear algebra, physics, statistics, computer science, signal processing, control theory, etc.; see [14, 19, 21, 23, 24].

On the other hand, the theory of matrices whose entries come from a suitable algebraic structure such as a semiring or a commutative semiring are practically useful, especially in information sciences and fuzzy systems. Earlier results in this theory concerning the inversion of matrices over a commutative semiring were obtained in [22]. Since then a theory parallel to that of linear algebra to systems of linear equations, linear independence, rank and eigenvalue problems over various kinds of semirings has been obtained by many authors; see e.g. [4, 8, 9, 11, 12, 28]. Examples of applications of matrices over a semiring in optimization theory, operation research, automatic control, and network theory can be seen in [2, 4, 7, 12].

In this paper, we introduce the notion of Kronecker products for matrices over a commutative semiring. This kind of matrix product is not commutative in general. This product satisfies much nice algebraic properties, e.g. the associativity, the distributivity over the addition, and the compatibility with the scalar multiplication, the transpose and the usual multiplication. Moreover, we introduce a linear transformation that associates each matrix with a vector by stacking each sequential column of the matrix. This operation, called the vec operator, turns usual matrix products to the Kronecker product. It follows that any linear matrix equation of the form

$$\sum_{i=1}^{k} A_i X B_i = C$$

in an unknown matrix X can be reduced to the vector-matrix equation $Ax = b$. Finally, we introduce and investigate basic properties of the vec-permutation matrix which is a kind of permutation matrix that involves the Kronecker product. It follows that $A \otimes B$ and $B \otimes A$ are permutatively similar. Our results extend the results for real/complex matrices in the literature. However, some of the algebraic properties for Kronecker products of real/complex matrices do not hold in this setting.

This paper is organized as follows. The next section provides some preliminaries about matrices whose entries come from a commutative semiring. In Section 3, we introduce the Kronecker product of matrices over a commutative semiring in terms of block matrices and investigate its basic algebraic properties. Section 4 deals with the property that associates the usual product and the Kronecker product, namely, the mixed product property. Section 5 discusses the vec operator and its applications to linear matrix equations. In the final section, we define the vec-permutation matrix and use it to show that $A \otimes B$ and $B \otimes A$ are permutatively similar.

2 Matrices over a commutative semiring

In this section, we first recall the definition and examples of a commutative semiring. Then we investigate basic properties for operations on matrices over this algebraic structure.

The following definition of a semiring was due to Zimmermann [30] and Golan [11].

Definition 2.1. A semiring is a set L together with

- a binary operation $+$ on L called the addition,
- a binary operation $\cdot$ on L called the multiplication,
- two distinguish elements 0 and 1 in L

satisfying the following properties:

1. $(L,+,0)$ is a commutative monoid,
2. $(L,\cdot,1)$ is a monoid,
3. the multiplication is left and right distributive over the addition,
4. $0 \cdot r = 0 = r \cdot 0$ for all $r \in L$.

This algebraic system can be written as $(L,+,\cdot,0,1)$ or L for short.

Definition 2.2. A semiring $(L,+,\cdot,0,1)$ is said to be commutative if $r \cdot s = s \cdot r$ for all $r, s \in L$.

It is clear that a commutative semiring is a generalization of a field. We then provide examples of a commutative semiring which is not a field. These examples arise naturally in applications.

Example 2.3. The following structures are commutative semirings.

1. The unit interval $[0,1]$ under the operations

$$a + b = \sup\{a,b\} \text{ and } a \cdot b = \inf\{a,b\} \text{ for } a, b \in L.$$

This semiring is known as the fuzzy algebra (see e.g. [16]).

2. The set $[0,\infty)$ of nonnegative real numbers with the usual operations of addition and multiplication.

3. The set $\mathbb{N}\cup\{0\}$ of nonnegative integers under the operations

$$a+b=\begin{cases}0, & a=b=0\\ \gcd(a,b), & otherwise\end{cases}$$

$$a\cdot b=\begin{cases}\operatorname{lcm}(a,b), & a,b\in\mathbb{N}\\ 0, & otherwise.\end{cases}$$

4. For each natural number n, the set $\mathbb{Z}_n$ of integers modulo n together with its usual operations.

Example 2.4. Consider the extended real number system $\mathbb{R}\cup\{-\infty,\infty\}$. Then $\mathbb{R}\cup\{-\infty\}$ is a commutative semiring under the operations

$$a\oplus b=\max\{a,b\} \text{ and } a\odot b=a+b \text{ for } a,b\in\mathbb{R}\cup\{-\infty\}.$$

This semiring is usually called a max-plus algebra or a schedule algebra (see more information in [1, 3, 8]).

Example 2.5. Recall that an MV-algebra (see e.g. [5, 6]) is a set L together with

- two binary operations $\oplus$ and $\odot$ on L
- a unary operation $\neg$ on L
- two distinguish elements 0 and 1 in L

such that the following properties hold for all $x,y,z\in L$

(i) $(x\oplus y)\oplus z=x\oplus(y\oplus z)$

(ii) $x\oplus y=y\oplus x$

(iii) $x\oplus 0=x$

(iv) $\neg(\neg x)=x$

(v) $x\oplus 1=1$

(vi) $\neg 0=1$

(vii) $x\odot y=\neg(\neg x\oplus\neg y)$

(viii) $\neg(\neg x\oplus y)\oplus y=\neg(\neg y\oplus x)\oplus x.$

For each $x,y\in L$, define

$$x \vee y = (x \odot \neg y) \oplus y$$
$$x \wedge y = (x \oplus \neg y) \odot y.$$

Then $(L, \vee, \odot, 0, 1)$ and $(L, \wedge, \oplus, 0, 1)$ are commutative semirings.

From now on, let L be a commutative semiring. For each $m, n \in \mathbb{N}$, let $M_{m,n}(L)$ be the set of all m-by-n matrices whose entries come from L. If $A \in M_{m,n}(L)$ has a_{ij} as its (i, j)th entry for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$, we write $A = \left[a_{ij}\right]_{i,j=1}^{m,n}$ or $A = \left[a_{ij}\right]$ for short. We abbreviate $M_n(L) = M_{n,n}(L)$ and $V_n(L) = M_{n,1}(L)$. For each matrix A, the notation $A = \left[A_{ij}\right]_{ij}$ means that A is a block matrix whose (i, j)th block is given by A_{ij}.

For each $A = \left[a_{ij}\right], B = \left[b_{ij}\right] \in M_{m,n}(L)$ and $k \in L$, define the addition and the scalar multiplication as follows:

$$A + B = \left[a_{ij} + b_{ij}\right] \in M_{m,n}(L),$$
$$kA = \left[ka_{ij}\right] \in M_{m,n}(L).$$

If $A = \left[a_{ij}\right] \in M_{m,n}(L)$ and $B = \left[b_{ij}\right] \in M_{n,p}(L)$, we define the multiplication of A and B by

$$AB = \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right] \in M_{m,p}(L).$$

It is straightforward to the show that $M_n(L)$ is a semiring with respect to the addition and the multiplication defined above. Here, the zero element is the zero matrix $0 = [0] \in M_n(L)$ and the multiplicative identity is the identity matrix $I_n = \left[\delta_{ij}\right] \in M_n(L)$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. This semiring is not commutative unless $n = 1$.

Proposition 2.6. Let A, B and C be matrices over a commutative semiring L. If the following matrix operations exist, we have

1. $\alpha(A + B) = \alpha A + \alpha B$
2. $(\alpha + \beta)A = \alpha A + \beta A$
3. $(\alpha A)C = \alpha(AC) = A(\alpha C)$.

Proof. It follows straightforward from the fact that L is a commutative semiring.

The transpose of a matrix $A = [a_{ij}] \in M_{m,n}(L)$, denoted by A^T, is defined to be the matrix in $M_{n,m}(L)$ whose (i,j)th entry is given by a_{ji} for each $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Basic properties of the transpose of matrices over a commutative semiring in connections with the addition, the scalar multiplication and the multiplication are given here.

Proposition 2.7. Let $A, B \in M_{m,n}(L)$, $C \in M_{n,p}(L)$ and $k \in L$. We have

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AC)^T = C^T A^T$
5. if $A = [A_{ij}]_{ij}$ then $A^T = [A_{ji}{}^T]_{ij}$.

Proof. The proof is straightforward.

A matrix $A \in M_n(L)$ is said to be invertible if there is a matrix $B \in M_n(L)$ such that $AB = I_n = BA$. The direct sum of $A \in M_n(L)$ and $B \in M_m(L)$ is defined by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in M_{n+m}(L).$$

The trace of a matrix $A = (a_{ij}) \in M_n(L)$ is defined by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

The following linearity properties of the trace of a matrix over a commutative semiring will be used in later discussions. The proofs of them are straightforward.

Proposition 2.8. For each $A, B \in M_n(L)$ and $k \in L$, we have

1. $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$
2. $\operatorname{tr}(kA) = k\operatorname{tr}(A)$.

3 Basic properties of the Kronecker product of matrices over a commutative semiring

In this section, we define the Kronecker product of matrices over a commutative semiring by means of block matrices and establish its basic algebraic properties associated to the matrix operations introduced in Section 2.

Definition 3.1. Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. The Kronecker product of A and B is defined to be

$$A \otimes B = \left[a_{ij}B\right]_{ij} \in M_{mp,nq}(L). \tag{3.1}$$

That is, each (i,j)th block of $A \otimes B$ is given by $a_{ij}B$ for $i = 1,\dots,m$ and $j = 1,\dots,n$.

It is interesting to note that $xy^T = x \otimes y^T = y^T \otimes x$ for any $x \in V_m(L)$ and $y \in V_n(L)$. That is, in special cases, the Kronecker product coincides with the usual matrix product. The Kronecker product of two identity matrices $I_n \in M_n(L)$ and $I_m \in M_m(L)$ results in the identity matrix $I_{mn} \in M_{mn}(L)$. Similar result holds for zero matrices. Note also that the n-copies direct sum of a matrix can be written in the form involving the Kronecker product as follows:

$$A \oplus A \oplus \cdots \oplus A = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{bmatrix} = I_n \otimes A.$$

The next proposition shows that the Kronecker product is compatible with the addition, the scalar multiplication and the transpose.

Proposition 3.2. Let A, B and C be matrices over L. If the following matrix operations exist, then one has

1. $(kA) \otimes B = k(A \otimes B) = A \otimes (kB)$
2. $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$
3. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$
4. $(A \otimes B)^T = A^T \otimes B^T$.

Proof. Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$. It follows from Proposition 2.6 that

$$(kA) \otimes B = [(ka_{ij})B]_{ij} = \left[k(a_{ij}B)\right]_{ij} = k\left[a_{ij}B\right]_{ij} = k(A \otimes B),$$

$$A \otimes (kB) = [a_{ij}(kB)]_{ij} = \left[(ka_{ij})B\right]_{ij} = \left[k(a_{ij}B)\right]_{ij} = k\left[a_{ij}B\right]_{ij} = k(A \otimes B).$$

To prove the second property, it suffices to show that each (i,j)th block of $(A+B) \otimes C$ and $(A \otimes C) + (B \otimes C)$ coincides. Indeed, we have

$$\begin{aligned}\left[(A+B) \otimes C\right]_{ij} &= (a_{ij} + b_{ij})C = a_{ij}C + b_{ij}C \\ &= [A \otimes C]_{ij} + \left[B \otimes C\right]_{ij} = [(A \otimes C) + (B \otimes C)]_{ij}.\end{aligned}$$

Similarly, we have $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$. For the last property, we have by Proposition 2.7 that

$$(A \otimes B)^T = [a_{ij}B]_{ij}^{\ T} = [(a_{ji}B)^T]_{ij} = [a_{ji}B^T]_{ij} = A^T \otimes B^T.$$

The following observation is very useful in dealing with Kronecker products by means of block matrices.

Lemma 3.3. For any matrix A and B, each $(i,j)th$ block of $A \otimes B$ is given by $A_{ij} \otimes B$.

Proof. Suppose that we partition into $s \times r$ block matrix with $A_{ij} \in M_{m_i,n_j}(L)$ for each $i = 1,2,\ldots,s$ and $j = 1,2,\ldots,r$. Then we have

$$\begin{aligned} A \otimes B &= \begin{bmatrix} a_{11}B & \cdots & a_{1n_1}B & & a_{11}B & \cdots & a_{1n_r}B \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{m_1 1}B & \cdots & a_{m_1 n_1}B & & a_{m_1 1}B & \cdots & a_{m_1 n_r}B \\ & \vdots & & \ddots & & \vdots & \\ a_{11}B & \cdots & a_{1n_1}B & & a_{11}B & \cdots & a_{1n_r}B \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{m_s 1}B & \cdots & a_{m_s n_1}B & & a_{m_s 1}B & \cdots & a_{m_s n_r}B \end{bmatrix} \\ &= \begin{bmatrix} A_{11} \otimes B & \cdots & A_{1r} \otimes B \\ \vdots & \ddots & \vdots \\ A_{s1} \otimes B & \cdots & A_{sr} \otimes B \end{bmatrix} \\ &= [A_{ij} \otimes B]_{ij}. \end{aligned}$$

Remark. In contrast to Lemma 3.3, it is not true that $A \otimes B = \left[A \otimes B_{ij}\right]_{ij}$. To see this, consider the following matrices over the fuzzy algebra [0,1] (see Example 2.3):

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The next result shows the associativity of the Kronecker product.

Proposition 3.4. Let $A \in M_{m,n}(L)$, $B \in M_{p,q}(L)$ and $C \in M_{r,s}(L)$. Then

$$(A \otimes B) \otimes C = A \otimes (B \otimes C). \tag{3.2}$$

Proof. It follows from Lemma 3.3 that

$$A \otimes (B \otimes C) = \begin{bmatrix} a_{11}(B \otimes C) & \dots & a_{1n}(B \otimes C) \\ \vdots & \ddots & \vdots \\ a_{m1}(B \otimes C) & \cdots & a_{mn}(B \otimes C) \end{bmatrix} = \begin{bmatrix} (a_{11}B) \otimes C & \dots & (a_{1n}B) \otimes C \\ \vdots & \ddots & \vdots \\ (a_{m1}B) \otimes C & \cdots & (a_{mn}B) \otimes C \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}C \end{bmatrix} \otimes C = (A \otimes B) \otimes C.$$

Remark. The property that $A \otimes B = 0$ if and only if $A = 0$ or $B = 0$ is not true in general. For instance, let $L = (\mathbb{Z}_6, +, \cdot)$ and consider the following matrices

$$A = \begin{bmatrix} \overline{4} & \overline{0} \\ \overline{0} & \overline{2} \end{bmatrix}, B = \begin{bmatrix} \overline{3} & \overline{3} \\ \overline{0} & \overline{3} \end{bmatrix}.$$

The next result shows that the Kronecker product is compatible with the matrix trace.

Proposition 3.5. If $A \in M_n(L)$ and $B \in M_m(L)$, then

$$\operatorname{tr}(A \otimes B) = (\operatorname{tr} A)(\operatorname{tr} B). \tag{3.2}$$

Proof. Observe that the sum of the main diagonal entries of $A \otimes B$ is $\sum_{i=1}^{n} a_{ii}B$. So, taking the trace of $A \otimes B$ gives

$$\operatorname{tr}(A \otimes B) = \sum_{i=1}^{n}\left(a_{ii}\sum_{j=1}^{m} b_{jj}\right) = \sum_{i=1}^{n} a_{ii}\operatorname{tr}(B) = \operatorname{tr}(B)\sum_{i=1}^{n} a_{ii} = \operatorname{tr}(A)\operatorname{tr}(B).$$

Here, we use the linearity of the trace in Proposition 2.8.

Definition 3.6. The Kronecker power of $A \in M_{m,n}(L)$ is defined inductively for all positive integers k by

$$A^{\otimes 1} = A \quad \text{and} \quad A^{\otimes k} = A \otimes A^{\otimes(k-1)} \quad \text{for } k = 2, 3, \ldots \tag{3.4}$$

This definition implies that for $A^{\otimes k} \in M_{m^k, n^k}(L)$.

It is easy to see that $(A^{\otimes r})^{\otimes s} = A^{\otimes(rs)}$ and $A^{\otimes r} \otimes A^{\otimes s} = A^{\otimes(r+s)}$ for all positive integers r and s. The next proposition asserts the compatibility between the Kronecker product and the direct sum of matrices.

Proposition 3.7. Let $A \in M_n(L)$ and $B \in M_m(L)$ and $C \in M_p(L)$. Then

$$(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C). \tag{3.5}$$

Proof. It follows from Lemma 3.3 that

$$\begin{aligned}(A \oplus B) \otimes C &= \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \otimes C = \begin{bmatrix} A \otimes C & 0 \otimes C \\ 0 \otimes C & B \otimes C \end{bmatrix} = \begin{bmatrix} A \otimes C & 0 \\ 0 & B \otimes C \end{bmatrix} \\ &= (A \otimes C) \oplus (B \otimes C).\end{aligned}$$

Remark. In general, it does not hold that $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$. To see this, consider the following matrices over the fuzzy algebra $[0,1]$ (see Example 2.8):

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

4 The mixed product property and its consequences

In this section, we prove that the Kronecker product is compatible with the usual matrix product. This property is called the mixed product property. Then some consequences of this result are given.

Lemma 4.1. For each $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$ we have

$$A \otimes B = (A \otimes I_p)(I_n \otimes B) = (I_m \otimes B)(A \otimes I_q) \tag{4.1}$$

Proof. A direct computation shows that

$$A \oslash B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} = \begin{bmatrix} a_{11}I_p & \dots & a_{1n}I_p \\ \vdots & \ddots & \vdots \\ a_{m1}I_p & \cdots & a_{mn}I_p \end{bmatrix} \begin{bmatrix} B & \dots & B \\ \vdots & \ddots & \vdots \\ B & \cdots & B \end{bmatrix} = (A \oslash I_p)(I_n \oslash B)$$

$$A \oslash B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} = \begin{bmatrix} B & \dots & B \\ \vdots & \ddots & \vdots \\ B & \cdots & B \end{bmatrix} \begin{bmatrix} a_{11}I_p & \dots & a_{1n}I_p \\ \vdots & \ddots & \vdots \\ a_{m1}I_p & \cdots & a_{mn}I_p \end{bmatrix} = (I_m \oslash B)(A \oslash I_q).$$

Theorem 4.2. Let $A \in M_{m,n}(L)$, $C \in M_{n,p}(L)$, $B \in M_{q,r}(L)$ and $D \in M_{r,s}(L)$. Then

$$(A \oslash B)(C \oslash D) = (AC) \oslash (BD). \tag{4.2}$$

Proof. By making use of Lemma 4.1, we obtain

$$\begin{aligned} (A \oslash B)(C \oslash D) &= (A \oslash I_q)(I_n \oslash B)(C \oslash I_r)(I_p \oslash D) \\ &= (A \oslash I_q)(C \oslash B)(I_p \oslash D) \\ &= (A \oslash I_q)(C \oslash I_q)(I_q \oslash B)(I_p \oslash D) \\ &= \left[(AC) \oslash I_q\right]\left[I_p \oslash (BD)\right] \\ &= (AC) \oslash (BD) \end{aligned}$$

which is the desired result.

Corollary 4.3. If the following matrix products exist, then one has

1. $(A_1 \oslash B_1)(A_2 \oslash B_2)\cdots(A_p \oslash B_p) = (A_1A_2\cdots A_p) \oslash (B_1B_2\cdots B_p)$
2. $(A_1 \oslash A_2 \oslash \cdots \oslash A_p)(B_1 \oslash B_2 \oslash \cdots \oslash B_p) = (A_1B_1) \oslash (A_2B_2) \oslash \cdots \oslash (A_pB_p)$
3. $(AB)^{\oslash k} = A^{\oslash k}B^{\oslash k}$ for any $k \in \mathbb{N}$
4. $(A^{\oslash r})^s = (A^s)^{\oslash r}$ for any $r, s \in \mathbb{N}$.

Proof. It follows immediately from the mixed product property (4.3) via mathematical inductions.

Corollary 4.4. Let $A \in M_n(L)$ and $B \in M_m(L)$. If both A and B are invertible, then $A \oslash B$ is invertible and

$$(A \oslash B)^{-1} = A^{-1} \oslash B^{-1}. \tag{4.3}$$

Proof. Using the mixed product property (4.3), we have

$$(A \oslash B)(A^{-1} \oslash B^{-1}) = (AA^{-1}) \oslash (BB^{-1}) = I \oslash I = I$$

and, similarly $(A^{-1} \otimes B^{-1})(A \otimes B) = I$. This means that $A \otimes B$ is invertible with $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

5 The vec operator and linear matrix equations

In this section, we introduce a column-stacking operator that associates a matrix to a column vector, called the vec operator. This operator is linear and bijective. Moreover, we discuss the usage of the vec operator to the area of linear matrix equations.

For each A, matrix we write A_i for the i th column of A. The precise definition of the vec operator is given here.

Definition 5.1. For each $A \in M_{m,n}(L)$, we define

$$\mathrm{vec}(A) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \in L^{mn}. \tag{5.1}$$

That is, $\mathrm{vec}(A)$ is a column vector of each sequential column of A stacked on top of one another.

Proposition 5.2. The vec operator $\mathrm{vec}: M_{m,n}(L) \to L^{mn}$ is a bijection satisfying

$$\mathrm{vec}(A+B) = \mathrm{vec(A)} + \mathrm{vec(B)}$$
$$\mathrm{vec}(kA) = k\mathrm{vec(A)}$$

for any $A, B \in M_{m,n}(L)$ and $k \in L$.

Proof. It is straightforward from the definition of vec operator.

Lemma 5.3. Let $A \in M_{m,n}(L), B \in M_{n,p}(L)$ and $C \in M_{p,n}(L)$. Then

1. $\mathrm{vec}(AB) = (I_p \otimes A)\mathrm{vec}(B)$
2. $\mathrm{vec}(CA^T) = (A \otimes I_p)\mathrm{vec}(C)$.

Proof. Observe that the i th column of AB is given by AB_i for each $i = 1, 2, \ldots, n$. It follows that

$$\operatorname{vec}(AB) = \begin{bmatrix} (AB)_1 \\ (AB)_2 \\ \vdots \\ (AB)_p \end{bmatrix} = \begin{bmatrix} AB_1 \\ AB_2 \\ \vdots \\ AB_p \end{bmatrix} = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_p \end{bmatrix}$$
$$= (I_p \otimes A)\operatorname{vec}(B).$$

To prove another assertion, note that for each $T \in M_{m,n}(L)$ and $x \in V_n(L)$, the product Tx can be written as a linear combination of the columns of A where the coefficients are given by the coordinates of x. Now, we have

$$\operatorname{vec}(CA^T) = \begin{bmatrix} (CA^T)_1 \\ \vdots \\ (CA^T)_m \end{bmatrix} = \begin{bmatrix} C(A^T)_1 \\ \vdots \\ C(A^T)_m \end{bmatrix} = \begin{bmatrix} a_{11}C_1 + & \cdots & +a_{1n}C_n \\ \vdots & \ddots & \vdots \\ a_{m1}C +_1 & \cdots & +a_{mn}C_n \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}I_p & \cdots & a_{1n}I_p \\ \vdots & \ddots & \vdots \\ a_{m1}I_p & \cdots & a_{mn}I_p \end{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} = (A \otimes I_p)\operatorname{vec}(C).$$

Theorem 5.4. Let $A \in M_{m,n}(L)$, $B \in M_{n,p}(L)$ and $sC \in M_{p,q}(L)$. Then

$$\operatorname{vec}(ABC) = (C^T \otimes A)\operatorname{vec}(C). \tag{5.2}$$

Proof. It follows from Theorem 4.2 and 5.3 that

$$\begin{aligned} \operatorname{vec}(ABC) &= \operatorname{vec}((AB)C) \\ &= (C^T \otimes I_m)\operatorname{vec}(AB) \\ &= (C^T \otimes I_m)(I_p \otimes A)\operatorname{vec}(B) \\ &= \left[(C^T \otimes I_m)(I_p \otimes A)\right]\operatorname{vec}(B) \\ &= (C^T \otimes A)\operatorname{vec}(B). \end{aligned}$$

Given $A \in M_{m,n}(L)$, $B \in M_{p,q}(L)$ and $C \in M_{m,q}(L)$. Consider the linear matrix equation

$$AXB = C$$

where $X \in M_{n,p}(L)$ is an unknown matrix. Proposition 5.2 and Theorem 5.4 assert that this matrix equation is equivalent to the system of mq equations in np unknowns given by

$$(B^T \otimes A)\operatorname{vec}(X) = \operatorname{vec}(C).$$

More generally, the linear matrix equation

$$\sum_{i=1}^{k} A_i X B_i = C$$

is equivalent to the vector-matrix equation

$$\left(\sum_{i=1}^{k} (B_i^T \otimes A_i) \right) \mathrm{vec}(X) = \mathrm{vec}(C)$$

which is easier to solve.

6 Vec-permutation matrices

For this section, we introduce and investigate basic properties of vec-permutation matrices. It follows that $A \otimes B$ and $B \otimes A$ are permutatively similar via a vec-permutation matrix.

Definition 6.1. Let c_{in} denote the n-dimensional column vector which has 1 in the ith position and $0's$ elsewhere; that is,

$$c_{in} = \left[0\,0 \cdots 0\,1\,0 \cdots 0 \right]^T \in L^n. \tag{6.1}$$

Define the vec-permutation matrix

$$P_{mn} = \begin{bmatrix} I_m \otimes c_{1n}^T \\ I_m \otimes c_{2n}^T \\ \vdots \\ I_m \otimes c_{nn}^T \end{bmatrix} \in M_{mn}(L). \tag{6.2}$$

The vec-permutation is a permutation matrix, that is, its rows are obtained from a permutation of the rows of identity matrix.

Lemma 6.2. The following properties for vec-permutation matrices hold:

1. $P_{mn}^T = P_{nm}$
2. $P_{mn}^T P_{mn} = P_{mn} P_{mn}^T = I_{mn}$.

Proof. It follows from Proposition 2.7 that

$$
\begin{aligned}
P^T_{mn} = \quad & \left[I^T_m \otimes e_{1n} \cdots I^T_m \otimes e_{nn}\right] \quad = \begin{bmatrix} e^T_{1m} \otimes e_{1n} & \cdots & e^T_{1m} \otimes e_{nn} \\ \vdots & \ddots & \vdots \\ e^T_{mm} \otimes e_{1n} & \cdots & e^T_{mm} \otimes e_{nn} \end{bmatrix} \\
= \quad & \begin{bmatrix} e_{1n} \otimes e^T_{1m} & \cdots & e_{nn} \otimes e^T_{1m} \\ \vdots & \ddots & \vdots \\ e_{1n} \otimes e^T_{mm} & \cdots & e_{nn} \otimes e^T_{mm} \end{bmatrix} = \begin{bmatrix} I_n \otimes e^T_{1m} \\ \vdots \\ I_n \otimes e^T_{mm} \end{bmatrix} \\
= \quad & P_{nm}
\end{aligned}
$$

For the second assertion, by using Theorem 4.2, we have

$$
\begin{aligned}
P^T_{mn} P_{mn} &= \left[I_m \otimes e_{1n} \cdots I_m \otimes e_{nn}\right] \begin{bmatrix} I_m \otimes e^T_{1n} \\ \vdots \\ I_m \otimes e^T_{nn} \end{bmatrix} \\
&= (I_m \otimes e_{1n})(I_m \otimes e^T_{1m}) + \ldots + (I_m \otimes e_{nn})(I_m \otimes e^T_{nn}) \\
&= (I_m \otimes e_{1n} e^T_{1n}) + \ldots + (I_m \otimes e_{nn} e^T_{nn}) \\
&= I_m \otimes \left(\textstyle\sum_{i=1}^{n} e_{in} e^T_{in}\right) \\
&= I_m \otimes I_n \\
&= I_{mn}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
P_{mn} P^T_{mn} &= \begin{bmatrix} I_m \otimes e^T_{1n} \\ \vdots \\ I_m \otimes e^T_{nn} \end{bmatrix} \left[I_m \otimes e_{1n} \cdots I_m \otimes e_{nn}\right] = \begin{bmatrix} I_m \otimes e^T_{1n} e_{1n} & \cdots & I_m \otimes e^T_{nn} e_{nn} \\ \vdots & \ddots & \vdots \\ I_m \otimes e^T_{nn} e_{1n} & \cdots & e^T_{nn} \otimes e_{nn} \end{bmatrix} \\
&= \begin{bmatrix} I_m & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_m \end{bmatrix} = I_{mn}.
\end{aligned}
$$

Theorem 6.3. Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$, then one has

$$P_{mp}(A \otimes B)P^T_{nq} = B \otimes A. \tag{6.3}$$

Proof. For each $i = 1, 2, \ldots, p$, we write B_i for the i th row of B. According to the definition of P_{mn} and Lemma 3.3, we have

$$
\begin{aligned}
P_{mp}(A \otimes B)P_{nq}^T &= \begin{bmatrix} I_m \otimes c_{1p}^T \\ \vdots \\ I_m \otimes c_{pp}^T \end{bmatrix} \begin{bmatrix} A_1 \otimes B \cdots A_n \otimes B \end{bmatrix} P_{nq}^T = \begin{bmatrix} A_1 \otimes B^1 & \ldots & A_n \otimes B^1 \\ \vdots & \ddots & \vdots \\ A_1 \otimes B^p & \cdots & A_n \otimes B^p \end{bmatrix} . \\
&= \begin{bmatrix} A \otimes B^1 \\ \vdots \\ A \otimes B^p \end{bmatrix} \begin{bmatrix} I_n \otimes c_{1q} \cdots I_n \otimes c_{qq} \end{bmatrix} \quad = \begin{bmatrix} b_{11}A & \ldots & b_{1q}A \\ \vdots & \ddots & \vdots \\ b_{p1}A & \cdots & b_{pq}A \end{bmatrix} \\
&= B \otimes A.
\end{aligned}
$$

A matrix $A \in M_n(L)$ is said to be permutatively similar to a matrix $B \in M_n(L)$ if there is a permutation matrix P such that $PAP^T = B$.

Corollary 6.4. If $A \in M_m(L)$ and $B \in M_p(L)$, then $A \otimes B$ is permutatively similar to $B \otimes A$.

Proof. According to Theorem 6.3, for $n = m$ and $q = p$, one has $P_{mp}(A \otimes B)P_{mp}^T = B \otimes A$. Since P_{mp} is a permutation matrix, we conclude that $A \otimes B$ is permutatively imilar to $B \otimes A$.

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1. Stangam, R. and Chansangiam, P. 2016. Kronecker Product of Matrices over a Commutative Semiring . The Thai Journal of Mathematics, Special Issue(The 11th IMT-GT International Conference on Mathematics, Statistics and Its Application 2015-ICMSA2015). Page 20-37.