รายงานการวิจัยฉบับสมบูรณ์

ทฤษฎีบทสำหรับการแก้ปัญหาระบบอสมการการแปรผันทั่วไปแบบแยก และการประยุกต์

A Theory for Solving the Split General System of Variational Inequalities Problem and Applications

รศ.ดร.อาทิตย์ แข็งธัญการ

ได้รับทุนสนับสนุนงานวิจัยจากเงินงบประมาณเงินรายได้

ประจำปีงบประมาณ 2562

คณะวิทยาศาสตร์

สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง

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บทคัดย่อ

ในงานวิจัยนี้ เราเสนอขั้นตอนวิธีแบบใหม่สำหรับแก้ปัญหาความเป็นไปได้แบบแยกทั่วไป และพิสูจน์บทแทรกซึ่งเป็นเครื่องมือที่สำคัญสำหรับการพิสูจน์ทฤษฎีบทสำหรับแก้ปัญหาความเป็นไปได้แบบแยกทั่วไป นอกจากนี้ได้นำทฤษฎีบทหลักไปประยุกต์ใช้กับทฤษฎีที่เกี่ยวข้องกับปัญหาค่าต่ำสุดจำกัด ซึ่งผลลัพธ์ที่ได้จะขยายงานของ Ceng, Ansari และ Yao และปรับปรุงงานวิจัยของ Xu

คำสำคัญ : ปัญหาความเป็นไปได้แบบแยกทั่วไป ปัญหาค่าต่ำสุดจำกัด ปัญหาจุดตรึง

Research Title: A theory for solving the split general system of variational inequalities problem and applications

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ABSTRACT

The purpose of this research , we introduce a new method for solving the general split feasibility problem and we establish the important lemma as a tool for proving the theorem that solves the general split feasibility problem. Applying our main theorem to prove the theorem related to the general constrained minimization problem in the last section. Our results expand some results of Ceng, Ansari and Yao and modify the results of Xu.

Keywords : General split feasibility problem Constrained minimization problem Fixed point problem

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Chapter 1

Introduction

1.1 The importance of the research

Currently, there are new technologies such as computer technology, engineering, transportation, economics and statistic etc., but these technologies still have problems that need to adjust. Mathematical models are an important tool to solve problems as mention previously, which one of an important mathematical tool is the fixed point theory. This can be used to solve computer problems, physic, economics and etc. The fixed point theory is a study about the existence solution and uniqueness solution.

There are many mathematicians that established the strong convergence theorem for finding the solution of a fixed point problem. This makes the fixed point theory develop extensively and can apply in various disciplines. The equilibrium problem, variational problem and optimization problem are the important problem in economics, engineering and physic which it can be converted to the fixed point problem in order to solve the problem easier. So, the study about the theorems related to fixed point of all mapping and establish an iterative scheme is the one tool that many mathematicians pay attention to this method. The split variational inequality is a popular problem in the sense that many mathematicians establish the model for solving this problem. In addition, the split variational inequality is also used to solve problems in medical such as Radiation therapy planning for cancer and repairing damaged pictures

Let H_1, H_2 be two real Hilbert spaces. Let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. The split feasibility problem (SFP) in the finite-dimensional Hilbert spaces is introduced by Censor and Elfving, which is to find a point x such that $x \in C$ and $Dx \in Q$, where $D : H_1 \to H_2$ is a bounded linear operator. The SFP is useful in various disciplines such as signal processing, image reconstruction, and computer tomography, seother works. The SFP has been studied by many researchers.

This researches use the concept of the solving of fixed point and the split variational inequality for create a new split variational inequality and for finding the solution of this problem. It is a better method than the original method. Furthermore, we introduce a new iterative for solving the new split variational inequality. The study about the split variational inequality requires knowledge about the functional analysis and fixed point theory to bring it together for solving the problem.

1.2 Objective of the research

1) To establish a fixed point theorem for solving the problem in science and technology such as transportation, computer, engineering and physic.

2) To establish a new iterative for solving fixed point problem and a new split variational inequality.

3) To publish this research in the international journal in ISI database with the impact factor.

1.3 Scope of the research

1) Study variational inequality problems ,constrained convex minimization problem and equilibrium problems in Hilbert space.

2) We establish a fixed point theorem for solving the mathematic problem.

3) We create mathematical tools for applying to solve the computer problem, engineering and transportation by convert the problem into an iterative scheme.

4) All strong convergence theorems are considered and proved in Hilbert spaces.

1.4 Method

1) In first – third month, we will study about the basic knowledge of fixed point theory from text book, such as Fixed Point Theory for Lipschitziantype Mappings with Applications, Nonlinear and Convex Analysis, Contemporary Radiation Therapy etc.

2) In third – fifth month, we will study related researches about fixed point, which all research is published in international journals such as fixed point theory and applications, nonlinear analysis, journal of optimization and etc.

3) In fifth – seventh month, we will establish a new iterative with higher quality than previous research

4) In seventh – tenth month, we will analyze the result from 3.) by using the mathematical proof.

5) In tenth – eleventh month, we will establish and prove a new theorem by using the information from 1.) – 4.) and write program for testing the speed of the rate of convergence of a sequence generated by the proposed iterative.

6) In eleventh – twelfth month, we will write the research for publish this research in the international journal in ISI database with the impact factor.

7) In twelfth – sixteenth month, we will establish another fixed point theorems which modified by Halpern, Ishikawa, Mann iterative and etc.

8) In sixteenth – nineteenth month, we will study another related researches for establish a new theorem.

9) In nineteenth – twenty second month, we will analyze the information from 8.) and study about the suitable conditions.

10) In twenty second – twenty fourth month, we will establish a theorem and write a paper for publish this research in the international journal in ISI database with the impact factor.

1.5 Expected benefits

1) Obtain new mathematical tools for the properties of variational inequality problems, equilibrium problems and fixed problems of nonlinear mappings in Hilbert spaces.

2) To obtain the new knowledge for solving split problem.

3) To obtain mathematical tools for split problem.

Chapter 2

Theory and related research

The purpose of this chapter is to explain fundamental concepts and definitions used throughout this thesis. Moreover, we give some lemmas, remarks and useful results used in the later chapters. Throughout this chapter, we use the letter $\mathbb{R}$ for the set of all real numbers, $\mathbb{C}$ for the set of all complex numbers and $\mathbb{F}$ for the set of all real or complex numbers.

The fixed point theory is one of the important mathematical tools applied to solve problems in many branches of science and technology. In the past few years, many mathematicians have been developed and widely studied fixed point theory. Finding the answer of the equation by using the fixed point theory may have the answer or no answer. Therefore, fixed point theory is involved with finding conditions on the set X and the mapping $T : X \to X$ to guarantee the existence and uniqueness of fixed points. Moreover, researchers have been studying about the structure of fixed point set and the approximation of fixed points. Iterative schemes for finding the solution set of nonlinear mappings such as nonexpansive mappings, quasi-nonexpansive mappings, nonspreading mappings have been increasingly studied by many mathematicians. They have introduced various types of iterative methods to approximate fixed points.

Throughout this paper, let $H_1,\ H_2$ be real Hilbert spaces and let $C,\ Q$ be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator.

For a mapping T of C into itself, we denote $F(T)$ by the set of all *fixed points* of T i.e.,

$$F(T) = \{x \in C : Tx = x\}.$$

Example 2.1.

1. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = \frac{x+1}{2}$, then $F(T) = \{1\}$.
2. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = x^2$, then $F(T) = \{0, 1\}$.
3. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = x + 5$, then $F(T) = \emptyset$.
4. If $T : \mathbb{R} \to \mathbb{R}$ and $Tx = x$, then $F(T) = \mathbb{R}$.

The split feasibility problem (SFP) is to find a point $x \in C$ and $Ax \in Q$. This problem was introduced by Censor and Elfving [5].

Such models were successfully developed for instance in radiation therapy treatment planning, sensor networks, resolution enhancement.

In 2012, Ceng, Ansari and Yao [2] introduced the following lemma to solve SFP;

Lemma 2.2. Given $x^* \in H_1$, the following statements are equivalent.

i) x^* solves the SFP;

ii) $x^* = P_C\left(I - \lambda A^*\left(I - P_Q\right)A\right)x^*$, where A^* is adjoint of A;

iii) x^* solves the variational inequality problem (VIP) of finding $x^* \in C$ such that $\langle y - x^*, \nabla g(x^*)\rangle \geq 0$, for all $y \in C$ and $\nabla g = A^*(I - P_Q)A$.

Many authors use this lemma to prove their results, see for example, [3], [8].

Let $p, q \in \mathbb{N}$. For each $1 \leq i \leq p$, let C_i be a nonempty closed convex subset of a real Hilbert space H_1. For each $1 \leq j \leq q$, let Q_j be a nonempty closed convex subset of another real Hilbert space H_2 and let $A_j : H_1 \to H_2$ be a bounded linear operator. Suppose that K is another nonempty closed convex subset of H_1. *The constrained multiple-set split convex feasibility problem (MSCFP)* raised by Masad and Reich [11] is finding a point $x^* \in K$ such that

$$x^* \in \bigcap_{i=1}^{p} C_i \text{ and } A_j x^* \in Q_i,\ 1 \leq j \leq q. \tag{2.1}$$

The MSCFP introduced by Censor et al. [6] and Xu [14] is a special case of (2.1), which is formulated as finding $x^* \in H_1$ such that

$$x^* \in \bigcap_{i=1}^{p} C_i \text{ and } Ax^* \in \bigcap_{j=1}^{q} Q_i, \tag{2.2}$$

where A is a bounded linear operator from H_1 to H_2. If $p = q = 1$, (2.2) is reduced to SFP. Let $A, B : H_1 \to H_2$ be bounded linear operators. Inspired by (2.1), (2.2) and SFP, we introduce *the general split feasibility problem* which is to find a point $x^* \in C$ and $Ax^*, Bx^* \in Q$. The set of this solution is denoted by $\Gamma = \{x \in C : Ax, Bx \in Q\}$.

By applying Mann 's iterative algorithm with SFP, Xu [17] proved the best following result;

Theorem 2.3. Assume that SFP is consistent and $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$. Let $\{x_n\}$ be defined by the following averaged CQ algorithm:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C(I - \gamma A^*(I - P_Q)A)x_n,$$

for all $n \geq 0$ where $\{\alpha_n\}$ is a sequence in a interval $\left[0, \frac{4}{2 + \gamma\|A\|^2}\right]$ satisfying the condition

$$\sum_{n=1}^{\infty}\left(\frac{4}{2 + \gamma\|A\|^2} - \alpha_n\right) = \infty.$$

Then $\{x_n\}$ converges weakly to a solution of SFP.

The such theorem is used as a model for proving some result to solve the split feasibility problem, see for example, [2, 3, 7].

symbol " $\longrightarrow$ " and " $\rightharpoonup$ " represent strong and weak convergence, respectively. Let C be a subset of a real Hilbert space H. A mapping $T : C \to C$ is called *α-contractive* if there exists $\alpha \in [0, 1]$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in C$. A mapping T is calll *nonexpansive* if $\alpha = 1$. The fixed point problem of T is to find a point $x^* \in C$ such that $Tx^* = x^*$. The set of all fixed point of T is denoted by $F(T)$. A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists $\alpha > 0$ such that $\alpha\|Ax - Ay\|^2 \leq \langle Ax - Ay, x - y\rangle$ for all $x, y \in C$.

The variational inequality problem (VIP) is a well known problem. That is to find a point $\varpi_* \in C$ such that

$$\langle y - \varpi_*, G\varpi_*\rangle \geq 0, \text{ for all } y \in C, \tag{2.3}$$

where $G : C \longrightarrow H$ is a mapping. The set of all solutions of (2.3) is denoted by $VI(C, G)$.

The variational inequality problem has been applied in various fields such as industry, finance, economics, social, ecology, regional, pure and applied sciences; see, [9],[10].

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following lemma is a property of P_C.

Lemma 2.4. (See [13]) Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.5. (See [12]) Let H be a Hilbert space, let C be nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,

$$u \in VI(C, A) \Leftrightarrow u = P_C(I - \lambda A)u$$

where P_C is the metric projection of H onto C.

Lemma 2.6. (See [16]) Let $\{s_n\}$ be a sequence of nonnegative real number satisfying

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions

$$(1) \quad \{\alpha_n\} \subset [0, 1], \ \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(2) \quad \limsup_{n\to\infty} \beta_n \leq 0 \text{ or } \sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7. (See [15].) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

$$(1) \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$(2) \quad \limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.8. Let H_1 and H_2 be real Hilbert spaces and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* are adjoint of A and B, respectively with $\Gamma \neq \emptyset$. Then the followings are equivalent.

$$i)\ x^* \in \Gamma,$$
$$ii)\ P_C\left(I - a\left(\frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}\right)\right)x^* = x^*,\ \forall\, a > 0,$$

where L_A, L_B are spectal redius of A^*A and B^*B, respectively with $a \in (0, \frac{2}{L})$ and $L = \max\{L_A, L_B\}$.

Proof. Let the conditions holds

$i) \Rightarrow ii)$ Let $x^* \in \Gamma$, we have $x^* \in C$ and $Ax^*, Bx^* \in Q$. It implies that

$$(I-P_Q)Ax^* = 0 = (I-P_Q)Bx^*.$$

Then
$$\frac{A^*(I-P_Q)Ax^*}{2} = \frac{B^*(I-P_Q)Bx^*}{2} = 0.$$

It follow that
$$P_C\left(I - a\left(\frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}\right)\right)x^* = x^*.$$

$ii) \Rightarrow i)$ Let $P_C\left(I - a\left(\frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}\right)\right)x^* = x^*$ and let $w \in \Gamma$, we have $w \in C$ and $Aw, Bw \in Q$. From $i) \Rightarrow ii)$, we have
$$P_C\left(I - a\left(\frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}\right)\right)w = w.$$

Then, we have

$$\begin{aligned}
\|x^*-w\|^2 &\leq \left\|x^* - w - a\left(\frac{A^*(I-P_Q)Ax^*}{2} + \frac{B^*(I-P_Q)Bx^*}{2}\right)\right\|^2 \\
&= \|x^*-w\|^2 - 2a\langle x^*-w, \frac{A^*(I-P_Q)Ax^*}{2} + \frac{B^*(I-P_Q)Bx^*}{2}\rangle \\
&\quad + a^2\left\|\frac{A^*(I-P_Q)Ax^*}{2} + \frac{B^*(I-P_Q)Bx^*}{2}\right\|^2 \\
&\leq \|x^*-w\|^2 - a\langle Ax^*-Aw, (I-P_Q)Ax^*\rangle - a\langle Bx^*-Bw, (I-P_Q)Bx^*\rangle \\
&\quad + \frac{a^2}{2}\|A^*(I-P_Q)Ax^*\|^2 + \frac{a^2}{2}\|B^*(I-P_Q)Bx^*\|^2 \\
&\leq \|x^*-w\|^2 - a\langle Ax^*-P_QAx^*, (I-P_Q)Ax^*\rangle - a\langle P_QAx^*-Aw, (I-P_Q)Ax^*\rangle \\
&\quad - a\langle Bx^*-P_QBx^*, (I-P_Q)Bx^*\rangle - a\langle P_QBx^*-Bw, (I-P_Q)Bx^*\rangle \\
&\quad + \frac{a^2L}{2}\|(I-P_Q)Ax^*\|^2 + \frac{a^2L}{2}\|(I-P_Q)Bx^*\|^2 \\
&\leq \|x^*-w\|^2 - a(1-\frac{aL}{2})\|(I-P_Q)Ax^*\|^2 - (1-\frac{aL}{2})\|(I-P_Q)Bx^*\|^2.
\end{aligned}$$

It implies that $Ax^* = P_QAx^*, Bx^* = P_QBx^* \in Q$.

It follows that
$$x^* = P_C\left(I - a\left(\frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}\right)\right)x^* = P_Cx^* \in C.$$

Hence $x^* \in \Gamma$. □

Let $H_1 = \mathbb{R}^2, H_2 = \mathbb{R}$ and let $C = \overline{H}(a, a_1 - a_2) = \{x = (x_1, x_2) \in H_1 : a_1x_1 + a_2x_2 = a_1 - a_2$ for all $a = (a_1, a_2) \in H_2$ and $Q = [-2, 3] \subseteq H_1$. Defined mappings $A, B : H_1 \to H_2$ by $Ax = x_1, Bx = x_2$ for all $x = (x_1, x_2) \in H_1$. It is obvious that $(1, -1) \in \Gamma$.

Next, we will show that $P_C\bigg(I - \lambda\Big(\frac{A^*(I - P_Q)A}{2} + \frac{B^*(I - P_Q)B}{2}\Big)\bigg)(1, -1) = (1, -1)$.

From the definition of A, B, we can defined adjoint operators $A^*, B^* : H_2 \to H_1$ of A, B by $A^*z = (z, 0), B^*z = (0, z)$ for all $z \in H_2$. From the definition of C, We can defined metric projection $P_C : H_1 \to C$ by

$$P_Cz = (z_1, z_2) - \bigg(\frac{a_1z_1 + a_2z_2 - (a_1 - a_2)}{\sqrt{a_1^2 + a_2^2}}\bigg)(a_1, a_2), \tag{2.4}$$

for all $z = (z_1, z_2) \in H_1$.

From A, A^*, B and B^*, we have

$$\frac{A^*(I - P_Q)A(1, -1)}{2} = \frac{A^*0}{2} = (0, 0), \tag{2.5}$$

and

$$\frac{B^*(I - P_Q)B(1, -1)}{2} = \frac{B^*0}{2} = (0, 0). \tag{2.6}$$

From (2.4) (2.5) and (2.6), we have

$$\begin{aligned} P_C\bigg(I - \lambda\Big(\frac{A^*(I - P_Q)A}{2} + \frac{B^*(I - P_Q)B}{2}\Big)\bigg) &= P_C(1, -1) \\ &= (1, -1) - \bigg(\frac{a_1 - a_2 - (a_1 - a_2)}{\sqrt{a_1^2 + a_2^2}}\bigg)(a_1, a_2) \\ &= (1, -1). \end{aligned}$$

Remark 2.9. The result of this example is guaranteed by Lemma 2.8.

Chapter 3

The methods of research

In this chapter, we introduce methods for solving the following problems;

Inspired by the works of [11], [6], and [14], we introduce a method to solve solution of the general split feasibility problem. We give the general constrained minimization problem and a lemma to show the relationship between these problems. The method utilized to solve this problem is presented.

Chapter 4

The results of research

This chapter, we solve problems in Chapter 3.

Theorem 4.1. Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* are adjoint of A and B, respectively and $L = \max\{L_A, L_B\}$, where L_A and L_B are special radius of A^*A and B^*B and let $D : C \to H_1$ be d-inverse strongly monotone. Assume that $\Gamma \cap VI(C, D) \neq \emptyset$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C (I - \lambda D) x_n + \gamma_n P_C \left(aI - a\left(\frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}\right)\right) x_n, \tag{4.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $f : C \to C$ is α-contractive mapping with $\alpha \in (0,1)$. Suppose that the following conditions hold;

$$\begin{aligned}
&i) && \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,\\
&ii) && c \le \beta_n, \gamma_n \le d, \text{ for some } c, d > 0,\\
&iii) && \lambda \in (0, 2d), a \in \left(0, \frac{2}{L}\right),\\
&iv) && \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty.
\end{aligned}$$

Then the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Gamma\cap VI(C,D)} f(x_0)$.

Proof. Putting $\nabla g = \frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}$. First, we show that ∇g is $\frac{1}{L}$-inverse strongly monotone. Let $x, y \in C$. Since $\nabla g = \frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}$, we have

$$\begin{aligned}
\|\nabla g(x) - \nabla g(y)\|^2 &= \|\frac{A^*(I-P_Q)Ax}{2} + \frac{B^*(I-P_Q)Bx}{2} - \frac{A^*(I-P_Q)Ay}{2} - \frac{B^*(I-P_Q)By}{2}\|^2\\
&= \|\frac{A^*(I-P_Q)Ax}{2} - \frac{A^*(I-P_Q)Ay}{2} + \frac{B^*(I-P_Q)Bx}{2} - \frac{B^*(I-P_Q)By}{2}\|^2\\
&\le \frac{1}{2}\|A^*(I-P_Q)Ax - A^*(I-P_Q)Ay\|^2 + \frac{1}{2}\|B^*(I-P_Q)Bx - B^*(I-P_Q)By\|^2\\
&\le \frac{L}{2}\|(I-P_Q)Ax - (I-P_Q)Ay\|^2 + \frac{L}{2}\|(I-P_Q)Bx - (I-P_Q)By\|^2. \qquad (4.2)
\end{aligned}$$

From property of P_C, we have

$$\begin{aligned}\|(I-P_Q)Ax-(I-P_Q)Ay\|^2 &= \langle (I-P_Q)Ax-(I-P_Q)Ay,(I-P_Q)Ax-(I-P_Q)Ay\rangle\\ &= \langle (I-P_Q)Ax-(I-P_Q)Ay, Ax-Ay-(P_QAx-P_QAy)\rangle\\ &= \langle A^*(I-P_Q)Ax-A^*(I-P_Q)Ay,x-y\rangle\\ &\quad -\langle (I-P_Q)Ax-(I-P_Q)Ay,P_QAx-P_QAy\rangle\\ &= \langle A^*(I-P_Q)Ax-A^*(I-P_Q)Ay,x-y\rangle\\ &\quad -\langle (I-P_Q)Ax,P_QAx-P_QAy\rangle\\ &\quad +\langle (I-P_Q)Ay,P_QAx-P_QAy\rangle\\ &\le \langle A^*(I-P_Q)Ax-A^*(I-P_Q)Ay,x-y\rangle. \end{aligned} \tag{4.3}$$

By using the same method as (4.3), we have

$$\|(I-P_Q)Bx-(I-P_Q)By\|^2 \le \langle B^*(I-P_Q)Bx-B^*(I-P_Q)By,x-y\rangle. \tag{4.4}$$

Substitute (4.3), (4.4) into (4.2), we have

$$\begin{aligned}\|\nabla g(x)-\nabla g(y)\|^2 &\le \frac{L}{2}\langle A^*(I-P_Q)Ax-A^*(I-P_Q)Ay,x-y\rangle\\ &\quad +\frac{L}{2}\langle B^*(I-P_Q)Bx-B^*(I-P_Q)By,x-y\rangle\\ &= L\langle \frac{A^*(I-P_Q)Ax}{2}+\frac{B^*(I-P_Q)Bx}{2}-\left(\frac{A^*(I-P_Q)Ay}{2}+\frac{B^*(I-P_Q)By}{2}\right),x-y\rangle\\ &= L\langle \nabla g(x)-\nabla g(y),x-y\rangle.\end{aligned}$$

So, we have ∇g is $\frac{1}{L}$-inverse strongly monotone. From the definition of ∇g , we have

$$\begin{aligned}\|P_C(I-a\nabla g)x-P_C(I-a\nabla g)y\|^2 &\le \|x-y-a(\nabla g(x)-\nabla g(y))\|^2\\ &= \|x-y\|^2-2a\langle x-y,\nabla g(x)-\nabla g(y)\rangle+a^2\|\nabla g(x)-\nabla g(y)\|^2\\ &\le \|x-y\|^2-\frac{2a}{L}\|\nabla g(x)-\nabla g(y)\|^2\rangle+a^2\|\nabla g(x)-\nabla g(y)\|^2\\ &= \|x-y\|^2-a\left(\frac{2}{L}-a\right)\|\nabla g(x)-\nabla g(y)\|^2\\ &\le \|x-y\|^2,\end{aligned} \tag{4.5}$$

for all $x,y\in C$. By using the same method as (4.5), we have

$$\|P_C(I-\lambda D)x-P_C(I-\lambda D)y\| \le \|x-y\|, \tag{4.6}$$

for all $x,y\in C$.

From the definition of x_n, (4.5) and (4.6), we have

$$\begin{aligned}\|x_{n+1}-z\| &\le \alpha_n\|f(x_n)-z\|+\beta_n\|P_C(I-\lambda D)x_n-z\|+\gamma_n\|P_C(I-a\nabla g)x_n-z\|\\ &\le \alpha_n\Big(\alpha\|x_n-z\|+\|f(z)-z\|\Big)+\beta_n\|P_C(I-\lambda D)x_n-z\|+\gamma_n\|P_C(I-a\nabla g)x_n-z\|\\ &\le (1-\alpha_n(1-\alpha))\|x_n-z\|+\alpha_n\|f(z)-z\|\\ &\le \max\left\{\|x_1-z\|,\frac{\|f(z)-z\|}{1-\alpha}\right\},\end{aligned}$$

for all $n \in \mathbb{N}$ and $z \in \Gamma \cap VI(C, D)$. By induction, we conclude that the sequence $\{x_n\}$ is bounded.
From (4.1), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \alpha_n \alpha \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|P_C (I - \lambda D) x_{n-1}\| \\ &\quad + \beta_n \|P_C (I - \lambda D) x_n - P_C (I - \lambda D) x_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|P_C (1 - a\nabla g) x_{n-1}\| \\ &\quad + \gamma_n \|P_C (1 - a\nabla g) x_n - P_C (1 - a\nabla g) x_{n-1}\| \\ &\leq (1 - \alpha_n (1 - \alpha)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + |\beta_n - \beta_{n-1}| \|P_C (I - \lambda D) x_{n-1}\| \\ &\quad + |\gamma_n - \gamma_{n-1}| \|P_C (1 - a\nabla g) x_{n-1}\|. \end{aligned}$$

From the conditions $i), iv)$ and Lemma 2.7, we have

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{4.7}$$

We can rewrite (4.1) by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) E_n x_n, \tag{4.8}$$

where $E_n = \frac{\beta_n}{1 - \alpha_n} P_C (I - \lambda D) + \frac{\gamma_n}{1 - \alpha_n} P_C(I - a\nabla g)$ for all $n \in \mathbb{N}$.
Since $P_C (I - \lambda D)$ and $P_C(I - a\nabla g)$ are nonexpansive mappings, we have E_n is a nonexpansive mappings, for all $n \in \mathbb{N}$.
It is easy to see that

$$F (P_C (I - \lambda D)) \cap F (P_C(I - a\nabla g) \subseteq F (E_n), \tag{4.9}$$

for all $n \in \mathbb{N}$.
From Lemma 2.5 and 2.8, we have

$$F (P_C (I - \lambda D)) \cap F (P_C(I - a\nabla g) = \Gamma \cap VI (C, D) \neq \emptyset.$$

Let $z_0 \in F (E_n)$, for all $n \in \mathbb{N}$ and $z \in \Gamma \cap VI (C, D)$, we have

$$\begin{aligned} \|z_0 - z\|^2 &\leq \frac{\beta_n}{1 - \alpha_n} \|P_C (1 - \lambda D) z_0 - z\|^2 + \frac{\gamma_n}{1 - \alpha_n} \|P_C (I - a\nabla g) z_0 - z\|^2 \\ &\quad - \frac{\beta_n \gamma_n}{(1 - \alpha_n)^2} \|P_C (1 - \lambda D) z_0 - P_C (I - a\nabla g) z_0\|^2 \\ &\leq \|z_0 - z\|^2 - \frac{\beta_n \gamma_n}{(1 - \alpha_n)^2} \|P_C (1 - \lambda D) z_0 - P_C (I - a\nabla g) z_0\|^2. \end{aligned}$$

From condition $iii)$, we can conclude that $P_C (1 - \lambda D) z_0 = P_C (I - a\nabla g) z_0$.
Since $z_0 \in F (E_n)$, for all $n \in \mathbb{N}$, we have

$$z_0 = \frac{\beta_n}{1 - \alpha_n} P_C (I - \lambda D) z_0 + \frac{\gamma_n}{1 - \alpha_n} P_C(I - a\nabla g) z_0 = P_C (I - \lambda D) z_0 = P_C(I - a\nabla g) z_0.$$

So, we get

$$z_0 \in F (P_C (I - \lambda D)) \cap F (P_C(I - a\nabla g) = \Gamma \cap VI (C, D).$$

It follows that

$$F (E_n) \subseteq F (P_C (I - \lambda D)) \cap F (P_C(I - a\nabla g).$$

Then

$$F(E_n) = F(P_C(I - \lambda D)) \cap F(P_C(I - a\nabla g), \tag{4.10}$$

for all $n \in \mathbb{N}$. From (4.8), we have

$$x_{n+1} - x_n = \alpha_n(f(x_n) - x_n) + (1 - \alpha_n)(E_n x_n - x_n). \tag{4.11}$$

From (4.7) and condition $i)$, we have

$$\lim_{n\to\infty} \|E_n x_n - x_n\| = 0. \tag{4.12}$$

Since the sequence $\{x_n\}$ is bounded in a real Hilbert space H_1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to w, where $w \in C$.
From the condition $ii)$ we may assume that $\beta_{n_k} \to \beta$ and $\gamma_{n_k} \to \gamma$ as $k \to \infty$ with $\beta, \gamma \in [c, d]$.
It follows that

$$1 = \lim_{k\to\infty} \left(\frac{\beta_{n_k}}{1 - \alpha_{n_k}} + \frac{\gamma_{n_k}}{1 - \alpha_{n_k}} \right) = \beta + \gamma.$$

Putting $E = \beta P_C(I - \lambda D) + \gamma P_C(I - a\nabla g)$. It is easy to see that E is a nonexpansive mapping. By using method as $F(E_n) = F(P_C(I - \lambda D)) \cap F(P_C(I - a\nabla g)$, we have

$$F(E) = F(P_C(I - \lambda D)) \cap F(P_C(I - a\nabla g). \tag{4.13}$$

From the definition of E_n and E, we have

$$E_{n_k} x_{n_k} - E x_{n_k} = \left(\frac{\beta_{n_k}}{1 - \alpha_{n_k}} - \beta \right) P_C(I - \lambda D) x_{n_k} + \left(\frac{\gamma_{n_k}}{1 - \alpha_{n_k}} - \gamma \right) P_C(I - a\nabla g) x_{n_k}.$$

From $\lim_{k\to\infty} \beta_{n_k} = \beta$, $\lim_{k\to\infty} \gamma_{n_k} = \gamma$ and condition $i)$, we have

$$\lim_{k\to\infty} \|E_{n_k} x_{n_k} - E x_{n_k}\| = 0. \tag{4.14}$$

From (4.12) and (4.14), we have

$$\lim_{k\to\infty} \|x_{n_k} - E x_{n_k}\| = 0. \tag{4.15}$$

Assume that $w \notin \Gamma \cap VI(C, D)$. From (4.13), Lemma 2.5 and 2.8, we have $w \notin F(E)$. From Opial's conditions and (4.15), we have

$$\begin{aligned}
\lim_{k\to\infty} \|x_{n_k} - w\| &< \lim_{k\to\infty} \|x_{n_k} - Ew\| \\
&\le \lim_{k\to\infty} \Big(\|x_{n_k} - E x_{n_k}\| + \|E x_{n_k} - Ew\| \Big) \\
&\le \lim_{k\to\infty} \|x_{n_k} - w\|.
\end{aligned}$$

This is a contradiction. Then $w \in \Gamma \cap VI(C, D)$.
Since the sequence $\{x_n\}$ is bounded, we have

$$\limsup_{n\to\infty} \langle f(x_0) - x_0, x_n - x_0 \rangle = \lim_{k\to\infty} \langle f(x_0) - x_0, x_{n_k} - x_0 \rangle = \langle f(x_0) - x_0, w - x_0 \rangle \le 0, \tag{4.16}$$

where $x_0 = P_{\Gamma \cap VI(C,D)} f(x_0)$.
From (4.1), we have

$$
\begin{aligned}
\|x_{n+1} - x_0\|^2 &= \left\|\alpha_n f(x_n) + \beta_n P_C (I - \lambda D) x_n + \gamma_n P_C \left(I - a\left(\frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}\right)\right) x_n - x_0\right\|^2 \\
&\leq \left\|\beta_n (P_C (I - \lambda D) x_n - x_0) + \gamma_n \left(P_C \left(I - a\left(\frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}\right)\right) x_n - x_0\right)\right\|^2 \\
&\quad + 2\alpha_n \langle f(x_n) - x_0, x_{n+1} - x_0\rangle \\
&\leq (1-\alpha_n)^2 \|x_n - x_0\|^2 + 2\alpha_n \langle f(x_0) - x_0, x_{n+1} - x_0\rangle + 2\alpha_n \alpha \|x_n - x_0\| \|x_{n+1} - x_0\| \\
&\leq (1-\alpha_n)^2 \|x_n - x_0\|^2 + 2\alpha_n \langle f(x_0) - x_0, x_{n+1} - x_0\rangle + \alpha_n \alpha \|x_n - x_0\|^2 + \alpha_n \alpha \|x_{n+1} - x_0\|^2.
\end{aligned}
$$

It implies that

$$
\|x_{n+1} - x_0\|^2 \leq \left(1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha_n \alpha}\right) \|x_n - x_0\|^2 + \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\left(\frac{\alpha_n}{2(1-\alpha)}\|x_n - x_0\|^2 + \frac{1}{1-\alpha}\langle f(x_0) - x_0, x_{n+1} - x_0\rangle\right).
$$

From Lemma 2.6, condition $i)$ and (4.16), we obtain that the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Gamma \cap VI(C,D)} f(x_0)$. This complete the proof. □

Using Theorem 4.1 , we can solve split feasibility problem.

Theorem 4.2. Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be bounded linear operator with A^* is adjoint of A where L is special radius of A^*A and let $D : C \to H_1$ be d-inverse strongly monotone. Assume that $\Gamma_A \cap VI(C,D) \neq \emptyset$, where $\Gamma_A = \{x \in C : Ax \in Q\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$
x_{n+1} = \alpha_n f(x_n) + \beta_n P_C (I - \lambda D) x_n + \gamma_n P_C (I - a(A^*(I - P_Q)A)) x_n,
$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$ with $a_n + \beta_n + \gamma_n = 1$ and $f : C \to C$ is α-contractive mapping with $\alpha \in (0,1)$. Suppose that the following conditions hold;

$$
\begin{aligned}
&i) && \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \\
&ii) && c \leq \beta_n, \gamma_n \leq d, \text{ for some } c, d > 0, \\
&iii) && \lambda \in (0, 2d), a \in \left(0, \frac{2}{L}\right), \\
&iv) && \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty.
\end{aligned}
$$

Then the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Gamma_A \cap VI(C,D)} f(x_0)$.

Remark 4.3. If we take $D \equiv 0$ in Theorem 4.2, we have

$$
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n P_C (I - a(A^*(I - P_Q)A)) x_n, \tag{4.17}
$$

for all $n \in \mathbb{N}$, which is modification iterative scheme $\{x_n\}$ in Theorem 2.3 and by Theorem 4.2, we have the sequence $\{x_n\}$ generated by (4.17) converges strongly to a solution of SFP under the sufficient conditions of Theorem 4.2.

Let $C \subseteq H_1, Q \subseteq H_2$ of Hilbert space H_1, H_2 and let $A : H_1 \to H_2$ be a bounded linear operator.

Let $g : H_1 \to \mathbb{R}$ be a continuous differentiable function. The minimization problem;

$$\min_{x \in C} g(x) := \frac{1}{2}\|(I - P_Q)Ax\|^2, \tag{4.18}$$

is to find a point $x^* \in C$ such that $g(x^*) \leq g(x)$ for all $x \in C$.

From studying the minimization problem, we introduce *the general constrained minimization problem* as follows,

$$\min_{x \in C} g(x) := \frac{\|(I - P_Q)Ax\|^2}{4} + \frac{\|(I - P_Q)Bx\|^2}{4}. \tag{4.19}$$

The set of all solution of (4.19) is denoted by $\Gamma_g = \{x^* \in C : g(x^*) \leq g(x), \ \forall x \in C\}$.

The following results show the relationship between the general split feasibility problem and the general constrained minimization problem.

Lemma 4.4. Let H_1 and H_2 be real Hilbert space and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* are adjoint of A and B, respectively and let $g : H_1 \to \mathbb{R}$ be a continuous differentiable function defined by $g(x) = \frac{\|(I - P_Q)Ax\|^2}{4} + \frac{\|(I - P_Q)Bx\|^2}{4}$ for all $x \in H_1$. Assume that $\Gamma \neq \emptyset$. Then the followings are equivalent.

$i)$ $x^* \in \Gamma$,

$ii)$ $x^* \in \Gamma_g$.

Proof. $ii) \Rightarrow i)$ Let $x^* \in \Gamma_g$ and let $\overline{x} \in \Gamma$, we get $\overline{x} \in C$ and $A\overline{x}, B\overline{x} \in Q$.

Since $x^* \in \Gamma_g$, we have

$$\frac{\|Ax^* - P_Q Ax^*\|^2}{4} + \frac{\|Bx^* - P_Q Bx^*\|^2}{4} \leq \frac{\|Ay - P_Q Ay\|^2}{4} + \frac{\|By - P_Q By\|^2}{4}, \tag{4.20}$$

for all $y \in C$.

Since $\overline{x} \in C$, we have

$$\frac{\|Ax^* - P_Q Ax^*\|^2}{4} + \frac{\|Bx^* - P_Q Bx^*\|^2}{4} \leq \frac{\|A\overline{x} - P_Q A\overline{x}\|^2}{4} + \frac{\|B\overline{x} - P_Q B\overline{x}\|^2}{4}. \tag{4.21}$$

Since $A\overline{x}, B\overline{x} \in Q$, we have $A\overline{x} = P_Q A\overline{x}$ and $B\overline{x} = P_Q B\overline{x}$.

From (4.21), we have

$$\frac{\|Ax^* - P_Q Ax^*\|^2}{4} + \frac{\|Bx^* - P_Q Bx^*\|^2}{4} = 0.$$

It implies that $Ax^* = P_Q Ax^* \in Q$ and $Bx^* = P_Q Bx^* \in Q$.

Since $x^* \in \Gamma_g$, we have $x^* \in C$.

Hence $x^* \in \Gamma$.

$i) \Rightarrow ii)$ Let $x^* \in \Gamma$, we have $x^* \in C$ and $Ax^*, Bx^* \in Q$. Then, we have

$$\frac{\|Ax^* - P_Q Ax^*\|^2}{4} + \frac{\|Bx^* - P_Q Bx^*\|^2}{4} = 0 \leq \frac{\|Ay - P_Q Ay\|^2}{4} + \frac{\|By - P_Q By\|^2}{4}$$

for all $y \in C$. It implies that $x^* \in \Gamma_g$. □

Remark 4.5. We observe that $\nabla g = \frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}$, where A^* and B^* are adjoint of A and B, respectively and ∇g is a gradient of g. From Lemma 2.8 and 4.4, we have $\Gamma_g = \Gamma = VI(C, \nabla g)$, where $\Gamma \neq \emptyset$.

Theorem 4.6. Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* are adjoint of A and B, respectively and $L = \max\{L_A, L_B\}$, where L_A and L_B are special radius of A^*A and B^*B. Let the function $g : H_1 \to \mathbb{R}$ be differentiable continuous function defined by $g(x) = \frac{\|(I-P_Q)Ax\|}{4} + \frac{\|(I-P_Q)Bx\|^2}{4}$ and let $D : C \to H_1$ be d-inverse strongly monotone. Assume that $\Gamma \cap VI(C, D) \neq \emptyset$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda D)x_n + \gamma_n P_C(I - a\nabla g)x_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$ with $a_n + \beta_n + \gamma_n = 1$ and $f : C \to C$ is α-contractive mapping with $\alpha \in (0,1)$. Suppose that the following conditions hold;

$$\begin{aligned}
&i) && \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \\
&ii) && c \leq \beta_n, \gamma_n \leq d, \text{ for some } c, d > 0, \\
&iii) && \lambda \in (0, 2d), a \in \left(0, \frac{2}{L}\right), \\
&iv) && \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty.
\end{aligned}$$

Then the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Gamma_g \cap VI(C,D)} f(x_0)$.

Proof. From Theorem 4.1 and Lemma 4.4, we can conclude Theorem 4.6. □

Chapter 5

Conclusions of research

In this research, we establish new mathematical theorems for solving problems as variational inequality problems, split combination variational inequalities problem and the split general system of variational inequalities problem which these theorems have more potential than the previous method. We give the general constrained minimization problem and a lemma to show the relation ship between these problems. Our results expand some results of Ceng, Ansari and Yao [2] and modify the results of Xu [17].

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Appendix

แบบรายงานผลผลิตจากโครงการวิจัยที่ได้รับการจัดสรรเงินวิจัย

ปี 2562

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Iterative Scheme for Finding Solutions of the General Split Feasibility Problem and the General Constrained Minimization Problems

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Abstract. Inspired by the works of [11], [6], and [14], we introduce a method to solve solution of the general split feasibility problem. In the last section, we give the general constrained minimization problem and a lemma to show the relationship between these problems. The method utilized to solve this problem is presented. Our results expand some results of Ceng, Ansari and Yao [2] and modify the results of Xu [17].

1. Introduction

Given closed convex subset $C \subseteq H_1, Q \subseteq H_2$ of Hilbert space H_1, H_2 and let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (*SFP*) is to find a point $x \in C$ and $Ax \in Q$. This problem was introduced by Censor and Elfving [5].

Such models were successfully developed for instance in radiation therapy treatment planning, sensor networks, resolution enhancement.

In 2012, Ceng, Ansari and Yao [2] introduced the following lemma to solve *SFP*;

Lemma 1.1. *Given $x^* \in H_1$, the following statements are equivalent.*
i) x^* solves the SFP;
ii) $x^* = P_C\left(I - \lambda A^*\left(I - P_Q\right)A\right)x^*$, where A^* is adjoint of A;
iii) x^* solves the variational inequality problem (VIP) of finding $x^* \in C$ such that $\langle y - x^*, \nabla g(x^*)\rangle \geq 0$, for all $y \in C$ and $\nabla g = A^*\left(I - P_Q\right)A$.

Many authors use this lemma to prove their results, see for example, [3], [8].

Let $p, q \in \mathbb{N}$. For each $1 \leq i \leq p$, let C_i be a nonempty closed convex subset of a real Hilbert space H_1. For each $1 \leq j \leq q$, let Q_j be a nonempty closed convex subset of another real Hilbert space H_2 and let $A_j : H_1 \to H_2$ be a bounded linear operator. Suppose that K is another nonempty closed convex subset of H_1. *The constrained multiple-set split convex feasibility problem (MSCFP)* raised by Masad and Reich [11] is finding a point $x^* \in K$ such that

$$x^* \in \bigcap_{i=1}^{p} C_i \text{ and } A_j x^* \in Q_i,\ 1 \leq j \leq q. \tag{1}$$

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The MSCFP introduced by Censor et al. [6] and Xu [14] is a special case of (1), which is formulated as finding $x^* \in H_1$ such that

$$x^* \in \bigcap_{i=1}^{p} C_i \text{ and } Ax^* \in \bigcap_{j=1}^{q} Q_i, \tag{2}$$

where A is a bounded linear operator from H_1 to H_2. If $p = q = 1$, (2) is reduced to SFP. Let $A, B : H_1 \to H_2$ be bounded linear operators. Inspired by (1), (2) and SFP, we introduce *the general split feasibility problem* which is to find a point $x^* \in C$ and $Ax^*, Bx^* \in Q$. The set of this solution is denoted by $\Gamma = \{x \in C : Ax, Bx \in Q\}$.

By applying Mann 's iterative algorithm with *SFP*, Xu [17] proved the best following result;

Theorem 1.2. *Assume that SFP is consistent and* $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$. *Let* $\{x_n\}$ *be defined by the following averaged CQ algorithm:*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_C \left(I - \gamma A^* \left(I - P_Q\right) A\right) x_n,$$

for all $n \geq 0$ *where* $\{\alpha_n\}$ *is a sequence in a interval* $\left[0, \frac{4}{2 + \gamma \|A\|^2}\right]$ *satisfying the condition*

$$\sum_{n=1}^{\infty} \left(\frac{4}{2 + \gamma \|A\|^2} - \alpha_n\right) = \infty.$$

Then $\{x_n\}$ *converges weakly to a solution of SFP.*

The such theorem is used as a model for proving some result to solve the split feasibility problem, see for example, [2, 3, 7].

In the next section, we prove the important lemma as a tool for proving the theorem that solves the general split feasibility problem.

The purpose of this research, we introduce a new method for solving the general split feasibility problem and apply our main theorem to prove the theorem related to the general constrained minimization problem in the last section. Our results expand some results of Ceng, Ansari and Yao [2] and modify the results of Xu [17].

2. Preliminaries

In order to prove our main theorem. Therefore, these tools are needed.

Throughout this research, we uses the symbol " $\longrightarrow$ " and " $\rightharpoonup$ " represent strong and weak convergence, respectively. Let C be a subset of a real Hilbert space H. A mapping $T : C \to C$ is called *α-contractive* if there exists $\alpha \in [0, 1]$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in C$. A mapping T is calll *nonexpansive* if $\alpha = 1$. The fixed point problem of T is to find a point $x^* \in C$ such that $Tx^* = x^*$. The set of all fixed point of T is denoted by $F(T)$. A mapping $A : C \to H$ is called *α*-inverse strongly monotone if there exists $\alpha > 0$ such that $\alpha\|Ax - Ay\|^2 \leq \langle Ax - Ay, x - y\rangle$ for all $x, y \in C$.

The variational inequality problem (VIP) is a well known problem. That is to find a point $\varpi_* \in C$ such that

$$\langle y - \varpi_*, G\varpi_*\rangle \geq 0, \text{ for all } y \in C, \tag{3}$$

where $G : C \longrightarrow H$ is a mapping. The set of all solutions of (3) is denoted by $VI(C, G)$.

The variational inequality problem has been applied in various fields such as industry, finance, economics, social, ecology, regional, pure and applied sciences; see, [9],[10].

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following lemma is a property of P_C.

Lemma 2.1. *(See [13]) Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality* $\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C$.

Lemma 2.2. *(See [12]) Let H be a Hilbert space, let C be nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u \in VI(C, A) \Leftrightarrow u = P_C(I - \lambda A)u$$

where P_C is the metric projection of H onto C.

Lemma 2.3. *(See [16]) Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions

(1) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. *(See [15].) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5. *Let H_1 and H_2 be real Hilbert spaces and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* are adjoint of A and B, respectively with $\Gamma \neq \emptyset$. Then the followings are equivalent.*

i) $x^* \in \Gamma$,

ii) $P_C\left(I - a\left(\frac{A^*\left(I - P_Q\right)A}{2} + \frac{B^*\left(I - P_Q\right)B}{2}\right)\right)x^* = x^*, \ \forall\, a > 0,$

where L_A, L_B are spectal redius of A^*A and B^*B, respectively with $a \in (0, \frac{2}{L})$ and $L = \max\{L_A, L_B\}$.

Proof. Let the conditions holds

i) $\Rightarrow$ *ii)* Let $x^* \in \Gamma$, we have $x^* \in C$ and $Ax^*, Bx^* \in Q$. It implies that

$$\left(I - P_Q\right)Ax^* = 0 = \left(I - P_Q\right)Bx^*.$$

Then

$$\frac{A^*\left(I-P_Q\right)Ax^*}{2}=\frac{B^*\left(I-P_Q\right)Bx^*}{2}=0.$$

It follow that

$$P_C\left(I-a\left(\frac{A^*\left(I-P_Q\right)A}{2}+\frac{B^*\left(I-P_Q\right)B}{2}\right)\right)x^*=x^*.$$

ii) $\Rightarrow$ *i)* Let $P_C\left(I-a\left(\frac{A^*\left(I-P_Q\right)A}{2}+\frac{B^*\left(I-P_Q\right)B}{2}\right)\right)x^*=x^*$ and let $w\in\Gamma$, we have $w\in C$ and $Aw,Bw\in Q$.
From *i)* $\Rightarrow$ *ii)*, we have

$$P_C\left(I-a\left(\frac{A^*\left(I-P_Q\right)A}{2}+\frac{B^*\left(I-P_Q\right)B}{2}\right)\right)w=w.$$

Then, we have

$$\begin{aligned}
\|x^*-w\|^2 &\le \left\|x^*-w-a\Big(\frac{A^*\left(I-P_Q\right)Ax^*}{2}+\frac{B^*\left(I-P_Q\right)Bx^*}{2}\Big)\right\|^2\\
&= \|x^*-w\|^2-2a\langle x^*-w,\frac{A^*\left(I-P_Q\right)Ax^*}{2}+\frac{B^*\left(I-P_Q\right)Bx^*}{2}\rangle\\
&\quad +a^2\left\|\frac{A^*\left(I-P_Q\right)Ax^*}{2}+\frac{B^*\left(I-P_Q\right)Bx^*}{2}\right\|^2\\
&\le \|x^*-w\|^2-a\langle Ax^*-Aw,\left(I-P_Q\right)Ax^*\rangle-a\langle Bx^*-Bw,\left(I-P_Q\right)Bx^*\rangle\\
&\quad +\frac{a^2}{2}\|A^*(I-P_Q)Ax^*\|^2+\frac{a^2}{2}\|B^*(I-P_Q)Bx^*\|^2\\
&\le \|x^*-w\|^2-a\langle Ax^*-P_QAx^*,\left(I-P_Q\right)Ax^*\rangle-a\langle P_QAx^*-Aw,\left(I-P_Q\right)Ax^*\rangle\\
&\quad -a\langle Bx^*-P_QBx^*,\left(I-P_Q\right)Bx^*\rangle-a\langle P_QBx^*-Bw,\left(I-P_Q\right)Bx^*\rangle\\
&\quad +\frac{a^2L}{2}\|(I-P_Q)Ax^*\|^2+\frac{a^2L}{2}\|(I-P_Q)Bx^*\|^2\\
&\le \|x^*-w\|^2-a(1-\frac{aL}{2})\|(I-P_Q)Ax^*\|^2-(1-\frac{aL}{2})\|(I-P_Q)Bx^*\|^2.
\end{aligned}$$

It implies that $Ax^*=P_QAx^*,Bx^*=P_QBx^*\in Q$.
It follows that

$$x^*=P_C\left(I-a\left(\frac{A^*\left(I-P_Q\right)A}{2}+\frac{B^*\left(I-P_Q\right)B}{2}\right)\right)x^*=P_Cx^*\in C.$$

Hence $x^*\in\Gamma$. □

Example 2.6. *Let $H_1=\mathbb{R}^2,H_2=\mathbb{R}$ and let $C=\overline{H}(a,a_1-a_2)=\{x=(x_1,x_2)\in H_1:a_1x_1+a_2x_2=a_1-a_2$ for all $a=(a_1,a_2)\in H_2$ and $Q=[-2,3]\subseteq H_1$. Defined mappings $A,B:H_1\to H_2$ by $Ax=x_1,Bx=x_2$ for all $x=(x_1,x_2)\in H_1$. It is obvious that $(1,-1)\in\Gamma$.*

Next, we will show that $P_C\Big(I-\lambda\Big(\frac{A^*(I-P_Q)A}{2}+\frac{B^*(I-P_Q)B}{2}\Big)\Big)(1,-1)=(1,-1)$.

From the definition of A, B, we can defined adjoint operators $A^*, B^*: H_2 \to H_1$ of A, B by $A^*z = (z, 0), B^*z = (0, z)$ for all $z \in H_2$. From the definition of C, We can defined metric projection $P_C : H_1 \to C$ by

$$P_C z = (z_1, z_2) - \left(\frac{a_1 z_1 + a_2 z_2 - (a_1 - a_2)}{\sqrt{a_1^2 + a_2^2}}\right)(a_1, a_2), \tag{4}$$

for all $z = (z_1, z_2) \in H_1$.

From A, A^*, B and B^*, we have

$$\frac{A^*(I-P_Q)A(1,-1)}{2} = \frac{A^*0}{2} = (0,0), \tag{5}$$

and

$$\frac{B^*(I-P_Q)B(1,-1)}{2} = \frac{B^*0}{2} = (0,0). \tag{6}$$

From (4) (5) and (6), we have

$$\begin{aligned} P_C\Big(I-\lambda\Big(\frac{A^*(I-P_Q)A}{2}+\frac{B^*(I-P_Q)B}{2}\Big)\Big) &= P_C(1,-1) \\ &= (1,-1) - \left(\frac{a_1 - a_2 - (a_1 - a_2)}{\sqrt{a_1^2 + a_2^2}}\right)(a_1, a_2) \\ &= (1,-1). \end{aligned}$$

Remark 2.7. *The result of this example is guaranteed by Lemma 2.5.*

3. Main results

Theorem 3.1. *Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* are adjoint of A and B, respectively and $L = \max\{L_A, L_B\}$, where L_A and L_B are special radius of A^*A and B^*B and let $D : C \to H_1$ be d-inverse strongly monotone. Assume that $\Gamma \cap VI(C, D) \neq \emptyset$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C (I - \lambda D) x_n + \gamma_n P_C \left(aI - a \left(\frac{A^*\left(I-P_Q\right)A}{2} + \frac{B^*\left(I-P_Q\right)B}{2} \right) \right) x_n, \tag{7}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $f : C \to C$ is α-contractive mapping with $\alpha \in (0,1)$. Suppose that the following conditions hold;

i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$

ii) $c \le \beta_n, \gamma_n \le d$, *for some* $c, d > 0$,

iii) $\lambda \in (0, 2d), a \in \left(0, \frac{2}{L}\right),$

iv) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty.$

Then the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Gamma\cap VI(C,D)} f(x_0)$.

Proof. Putting $\nabla g = \frac{A^*\left(I-P_Q\right)A}{2} + \frac{B^*\left(I-P_Q\right)B}{2}$. First, we show that ∇g is $\frac{1}{L}$-inverse strongly monotone. Let $x, y \in C$. Since $\nabla g = \frac{A^*\left(I-P_Q\right)A}{2} + \frac{B^*\left(I-P_Q\right)B}{2}$, we have

$$\begin{aligned} \|\nabla g(x) - \nabla g(y)\|^2 &= \|\frac{A^*\left(I-P_Q\right)Ax}{2} + \frac{B^*\left(I-P_Q\right)Bx}{2} - \frac{A^*\left(I-P_Q\right)Ay}{2} - \frac{B^*\left(I-P_Q\right)By}{2}\|^2 \\ &= \|\frac{A^*\left(I-P_Q\right)Ax}{2} - \frac{A^*\left(I-P_Q\right)Ay}{2} + \frac{B^*\left(I-P_Q\right)Bx}{2} - \frac{B^*\left(I-P_Q\right)By}{2}\|^2 \\ &\le \frac{1}{2}\|A^*\left(I-P_Q\right)Ax - A^*\left(I-P_Q\right)Ay\|^2 + \frac{1}{2}\|B^*\left(I-P_Q\right)Bx - B^*\left(I-P_Q\right)By\|^2 \\ &\le \frac{L}{2}\|\left(I-P_Q\right)Ax - \left(I-P_Q\right)Ay\|^2 + \frac{L}{2}\|\left(I-P_Q\right)Bx - \left(I-P_Q\right)By\|^2. \end{aligned} \tag{8}$$

From property of P_C, we have

$$\begin{aligned} \|(I-P_Q)Ax - (I-P_Q)Ay\|^2 &= \langle (I-P_Q)Ax - (I-P_Q)Ay, (I-P_Q)Ax - (I-P_Q)Ay\rangle \\ &= \langle (I-P_Q)Ax - (I-P_Q)Ay, Ax - Ay - (P_QAx - P_QAy)\rangle \\ &= \langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x-y\rangle \\ &\quad -\langle (I-P_Q)Ax - (I-P_Q)Ay, P_QAx - P_QAy\rangle \\ &= \langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x-y\rangle \\ &\quad -\langle (I-P_Q)Ax, P_QAx - P_QAy\rangle \\ &\quad +\langle (I-P_Q)Ay, P_QAx - P_QAy\rangle \\ &\le \langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x-y\rangle. \end{aligned} \tag{9}$$

By using the same method as (9), we have

$$\|(I-P_Q)Bx - (I-P_Q)By\|^2 \le \langle B^*(I-P_Q)Bx - B^*(I-P_Q)By, x-y\rangle. \tag{10}$$

Substitute (9), (10) into (8), we have

$$\begin{aligned} \|\nabla g(x) - \nabla g(y)\|^2 &\le \frac{L}{2}\langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x-y\rangle \\ &\quad +\frac{L}{2}\langle B^*(I-P_Q)Bx - B^*(I-P_Q)By, x-y\rangle \\ &= L\langle \frac{A^*(I-P_Q)Ax}{2} + \frac{B^*(I-P_Q)Bx}{2} - \left(\frac{A^*(I-P_Q)Ay}{2} + \frac{B^*(I-P_Q)By}{2}\right), x-y\rangle \\ &= L\langle \nabla g(x) - \nabla g(y), x-y\rangle. \end{aligned}$$

So, we have ∇g is $\frac{1}{L}$-inverse strongly monotone. From the definition of ∇g , we have

$$\begin{aligned} \|P_C(I-a\nabla g)x - P_C(I-a\nabla g)y\|^2 &\le \|x-y-a(\nabla g(x) - \nabla g(y))\|^2 \\ &= \|x-y\|^2 - 2a\langle x-y, \nabla g(x) - \nabla g(y)\rangle + a^2\|\nabla g(x) - \nabla g(y)\|^2 \\ &\le \|x-y\|^2 - \frac{2a}{L}\|\nabla g(x) - \nabla g(y)\|^2\rangle + a^2\|\nabla g(x) - \nabla g(y)\|^2 \\ &= \|x-y\|^2 - a\left(\frac{2}{L} - a\right)\|\nabla g(x) - \nabla g(y)\|^2 \\ &\le \|x-y\|^2, \end{aligned} \tag{11}$$

for all $x, y \in C$. By using the same method as (11), we have

$$\|P_C(I-\lambda D)x - P_C(I-\lambda D)y\| \le \|x-y\|, \tag{12}$$

for all $x, y \in C$.
From the definition of x_n, (11) and (12), we have

$$\begin{aligned}\|x_{n+1}-z\| &\leq \alpha_n \left\|f(x_n)-z\right\| + \beta_n \|P_C(I-\lambda D)x_n - z\| + \gamma_n \left\|P_C(I-a\nabla g)x_n - z\right\| \\ &\leq \alpha_n\Big(\alpha\|x_n-z\| + \|f(z)-z\|\Big) + \beta_n \|P_C(I-\lambda D)x_n - z\| + \gamma_n \left\|P_C(I-a\nabla g)x_n - z\right\| \\ &\leq (1-\alpha_n(1-\alpha))\|x_n-z\| + \alpha_n\|f(z)-z\| \\ &\leq \max\left\{\|x_1-z\|, \frac{\left\|f(z)-z\right\|}{1-\alpha}\right\},\end{aligned}$$

for all $n \in \mathbb{N}$ and $z \in \Gamma \cap VI(C,D)$. By induction, we conclude that the sequence $\{x_n\}$ is bounded.
From (7), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq |\alpha_n-\alpha_{n-1}|\left\|f(x_{n-1})\right\| + \alpha_n\alpha\|x_n-x_{n-1}\| + \left|\beta_n-\beta_{n-1}\right|\|P_C(I-\lambda D)x_{n-1}\| \\ &\quad +\beta_n\|P_C(I-\lambda D)x_n - P_C(I-\lambda D)x_{n-1}\| + \left|\gamma_n-\gamma_{n-1}\right|\|P_C(1-a\nabla g)x_{n-1}\| \\ &\quad +\gamma_n\|P_C(1-a\nabla g)x_n - P_C(1-a\nabla g)x_{n-1}\| \\ &\leq (1-\alpha_n(1-\alpha))\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|\left\|f(x_{n-1})\right\| + \left|\beta_n-\beta_{n-1}\right|\|P_C(I-\lambda D)x_{n-1}\| \\ &\quad +\left|\gamma_n-\gamma_{n-1}\right|\|P_C(1-a\nabla g)x_{n-1}\|.\end{aligned}$$

From the conditions $i), iv)$ and Lemma 2.4, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{13}$$

We can rewrite (7) by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)E_n x_n, \tag{14}$$

where $E_n = \frac{\beta_n}{1-\alpha_n}P_C(I-\lambda D) + \frac{\gamma_n}{1-\alpha_n}P_C(I-a\nabla g)$ for all $n \in \mathbb{N}$.
Since $P_C(I-\lambda D)$ and $P_C(I-a\nabla g)$ are nonexpansive mappings, we have E_n is a nonexpansive mappings, for all $n \in \mathbb{N}$.
It is easy to see that

$$F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g) \subseteq F(E_n), \tag{15}$$

for all $n \in \mathbb{N}$.
From Lemma 2.2 and 2.5, we have

$$F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g) = \Gamma \cap VI(C,D) \neq \emptyset.$$

Let $z_0 \in F(E_n)$, for all $n \in \mathbb{N}$ and $z \in \Gamma \cap VI(C,D)$, we have

$$\begin{aligned}\|z_0-z\|^2 &\leq \frac{\beta_n}{1-\alpha_n}\|P_C(1-\lambda D)z_0 - z\|^2 + \frac{\gamma_n}{1-\alpha_n}\|P_C(I-a\nabla g)z_0 - z\|^2 \\ &\quad -\frac{\beta_n\gamma_n}{(1-\alpha_n)^2}\|P_C(1-\lambda D)z_0 - P_C(I-a\nabla g)z_0\|^2 \\ &\leq \|z_0-z\|^2 - \frac{\beta_n\gamma_n}{(1-\alpha_n)^2}\|P_C(1-\lambda D)z_0 - P_C(I-a\nabla g)z_0\|^2.\end{aligned}$$

From condition $iii)$, we can conclude that $P_C(1-\lambda D)z_0 = P_C(I-a\nabla g)z_0$.
Since $z_0 \in F(E_n)$, for all $n \in \mathbb{N}$, we have

$$z_0 = \frac{\beta_n}{1-\alpha_n}P_C(I-\lambda D)z_0 + \frac{\gamma_n}{1-\alpha_n}P_C(I-a\nabla g)z_0 = P_C(I-\lambda D)z_0 = P_C(I-a\nabla g)z_0.$$

So, we get

$$z_0 \in F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g) = \Gamma \cap VI(C,D).$$

It follows that

$$F(E_n) \subseteq F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g).$$

Then

$$F(E_n) = F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g), \tag{16}$$

for all $n \in \mathbb{N}$. From (14), we have

$$x_{n+1} - x_n = \alpha_n(f(x_n) - x_n) + (1-\alpha_n)(E_n x_n - x_n). \tag{17}$$

From (13) and condition i), we have

$$\lim_{n\to\infty} \|E_n x_n - x_n\| = 0. \tag{18}$$

Since the sequence $\{x_n\}$ is bounded in a real Hilbert space H_1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to w, where $w \in C$.
From the condition ii) we may assume that $\beta_{n_k} \to \beta$ and $\gamma_{n_k} \to \gamma$ as $k \to \infty$ with $\beta, \gamma \in [c,d]$.
It follows that

$$1 = \lim_{k\to\infty}\left(\frac{\beta_{n_k}}{1-\alpha_{n_k}} + \frac{\gamma_{n_k}}{1-\alpha_{n_k}}\right) = \beta + \gamma.$$

Putting $E = \beta P_C(I-\lambda D) + \gamma P_C(I-a\nabla g)$. It is easy to see that E is a nonexpansive mapping. By using method as $F(E_n) = F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g)$, we have

$$F(E) = F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g). \tag{19}$$

From the definition of E_n and E, we have

$$E_{n_k}x_{n_k} - Ex_{n_k} = \left(\frac{\beta_{n_k}}{1-\alpha_{n_k}} - \beta\right)P_C(I-\lambda D)x_{n_k} + \left(\frac{\gamma_{n_k}}{1-\alpha_{n_k}} - \gamma\right)P_C(I-a\nabla g)x_{n_k}.$$

From $\lim_{k\to\infty}\beta_{n_k} = \beta$, $\lim_{k\to\infty}\gamma_{n_k} = \gamma$ and condition i), we have

$$\lim_{k\to\infty}\|E_{n_k}x_{n_k} - Ex_{n_k}\| = 0. \tag{20}$$

From (18) and (20), we have

$$\lim_{k\to\infty}\|x_{n_k} - Ex_{n_k}\| = 0. \tag{21}$$

Assume that $w \notin \Gamma \cap VI(C,D)$. From (19), Lemma 2.2 and 2.5, we have $w \notin F(E)$. From Opial's conditions and (21), we have

$$\begin{aligned}
\lim_{k\to\infty}\|x_{n_k} - w\| &< \lim_{k\to\infty}\|x_{n_k} - Ew\| \\
&\leq \lim_{k\to\infty}\left(\|x_{n_k} - Ex_{n_k}\| + \|Ex_{n_k} - Ew\|\right) \\
&\leq \lim_{k\to\infty}\|x_{n_k} - w\|.
\end{aligned}$$

This is a contradiction. Then $w \in \Gamma \cap VI(C, D)$.
Since the sequence $\{x_n\}$ is bounded, we have

$$\limsup_{n\to\infty}\langle f(x_0) - x_0, x_n - x_0\rangle = \lim_{k\to\infty}\langle f(x_0) - x_0, x_{n_k} - x_0\rangle = \langle f(x_0) - x_0, w - x_0\rangle \leq 0, \tag{22}$$

where $x_0 = P_{\Gamma\cap VI(C,D)}f(x_0)$.
From (7), we have

$$\begin{aligned}
\|x_{n+1} - x_0\|^2 &= \left\|\alpha_n f(x_n) + \beta_n P_C(I - \lambda D)x_n + \gamma_n P_C\left(I - a\left(\frac{A^*\left(I - P_Q\right)A}{2} + \frac{B^*\left(I - P_Q\right)B}{2}\right)\right)x_n - x_0\right\|^2 \\
&\leq \left\|\beta_n(P_C(I - \lambda D)x_n - x_0) + \gamma_n\left(P_C\left(I - a\left(\frac{A^*\left(I - P_Q\right)A}{2} + \frac{B^*\left(I - P_Q\right)B}{2}\right)\right)x_n - x_0\right)\right\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - x_0, x_{n+1} - x_0\rangle \\
&\leq (1 - \alpha_n)^2\|x_n - x_0\|^2 + 2\alpha_n\langle f(x_0) - x_0, x_{n+1} - x_0\rangle + 2\alpha_n\alpha\|x_n - x_0\|\|x_{n+1} - x_0\| \\
&\leq (1 - \alpha_n)^2\|x_n - x_0\|^2 + 2\alpha_n\langle f(x_0) - x_0, x_{n+1} - x_0\rangle + \alpha_n\alpha\|x_n - x_0\|^2 + \alpha_n\alpha\|x_{n+1} - x_0\|^2.
\end{aligned}$$

It implies that

$$\|x_{n+1} - x_0\|^2 \leq \left(1 - \frac{2\alpha_n(1 - \alpha)}{1 - \alpha_n\alpha}\right)\|x_n - x_0\|^2 + \frac{2\alpha_n(1 - \alpha)}{1 - \alpha_n\alpha}\left(\frac{\alpha_n}{2(1 - \alpha)}\|x_n - x_0\|^2 + \frac{1}{1 - \alpha}\langle f(x_0) - x_0, x_{n+1} - x_0\rangle\right).$$

From Lemma 2.3, condition *i*) and (22), we obtain that the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Gamma\cap VI(C,D)}f(x_0)$. This complete the proof. □

Using Theorem 3.1 , we can solve split feasibility problem.

Theorem 3.2. *Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be bounded linear operator with A^* is adjoint of A where L is special radius of A^*A and let $D : C \to H_1$ be d-inverse strongly monotone. Assume that $\Gamma_A \cap VI(C, D) \neq \emptyset$, where $\Gamma_A = \{x \in C : Ax \in Q\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda D)x_n + \gamma_n P_C\left(I - a\left(A^*\left(I - P_Q\right)A\right)\right)x_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, 1)$ with $a_n + \beta_n + \gamma_n = 1$ and $f : C \to C$ is α-contractive mapping with $\alpha \in (0, 1)$. Suppose that the following conditions hold;

i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty,$

ii) $c \leq \beta_n, \gamma_n \leq d$, *for some* $c, d > 0,$

iii) $\lambda \in (0, 2d), a \in \left(0, \frac{2}{L}\right),$

iv) $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}|, \sum_{n=1}^{\infty}|\beta_n - \beta_{n-1}| < \infty.$

Then the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Gamma_A\cap VI(C,D)}f(x_0)$.

Remark 3.3. *If we take $D \equiv 0$ in Theorem 3.2, we have*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n P_C\left(I - a\left(A^*\left(I - P_Q\right)A\right)\right)x_n, \tag{23}$$

for all $n \in \mathbb{N}$, which is modification iterative scheme $\{x_n\}$ in Theorem 1.2 and by Theorem 3.2, we have the sequence $\{x_n\}$ generated by (23) converges strongly to a solution of SFP under the sufficient conditions of Theorem 3.2.

4. Application

Let $C \subseteq H_1, Q \subseteq H_2$ of Hilbert space H_1, H_2 and let $A : H_1 \to H_2$ be a bounded linear operator.

Let $g : H_1 \to \mathbb{R}$ be a continuous differentiable function. The minimization problem;

$$\min_{x \in C} g(x) := \frac{1}{2}\|(I - P_Q)Ax\|^2, \tag{24}$$

is to find a point $x^* \in C$ such that $g(x^*) \leq g(x)$ for all $x \in C$.

From studying the minimization problem, we introduce *the general constrained minimization problem* as follows,

$$\min_{x \in C} g(x) := \frac{\left\|\left(I - P_Q\right)Ax\right\|^2}{4} + \frac{\left\|\left(I - P_Q\right)Bx\right\|^2}{4}. \tag{25}$$

The set of all solution of (25) is denoted by $\Gamma_g = \{x^* \in C : g(x^*) \leq g(x),\ \forall x \in C\}$.

The following results show the relationship between the general split feasibility problem and the general constrained minimization problem.

Lemma 4.1. *Let H_1 and H_2 be real Hilbert space and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* are adjoint of A and B, respectively and let $g : H_1 \to \mathbb{R}$ be a continuous differentiable function defined by $g(x) = \frac{\left\|\left(I - P_Q\right)Ax\right\|^2}{4} + \frac{\left\|\left(I - P_Q\right)Bx\right\|^2}{4}$ for all $x \in H_1$. Assume that $\Gamma \neq \emptyset$. Then the followings are equivalent.*

i) $x^* \in \Gamma$,

ii) $x^* \in \Gamma_g$.

Proof. *ii)* ⇒ *i)* Let $x^* \in \Gamma_g$ and let $\bar{x} \in \Gamma$, we get $\bar{x} \in C$ and $A\bar{x}, B\bar{x} \in Q$.

Since $x^* \in \Gamma_g$, we have

$$\frac{\|Ax^* - P_QAx^*\|^2}{4} + \frac{\|Bx^* - P_QBx^*\|^2}{4} \leq \frac{\|Ay - P_QAy\|^2}{4} + \frac{\|By - P_QBy\|^2}{4}, \tag{26}$$

for all $y \in C$.

Since $\bar{x} \in C$, we have

$$\frac{\|Ax^* - P_QAx^*\|^2}{4} + \frac{\|Bx^* - P_QBx^*\|^2}{4} \leq \frac{\|A\bar{x} - P_QA\bar{x}\|^2}{4} + \frac{\|B\bar{x} - P_QB\bar{x}\|^2}{4}. \tag{27}$$

Since $A\bar{x}, B\bar{x} \in Q$, we have $A\bar{x} = P_QA\bar{x}$ and $B\bar{x} = P_QB\bar{x}$.

From (27), we have

$$\frac{\|Ax^* - P_QAx^*\|^2}{4} + \frac{\|Bx^* - P_QBx^*\|^2}{4} = 0.$$

It implies that $Ax^* = P_QAx^* \in Q$ and $Bx^* = P_QBx^* \in Q$.

Since $x^* \in \Gamma_g$, we have $x^* \in C$.

Hence $x^* \in \Gamma$.

i) ⇒ *ii)* Let $x^* \in \Gamma$, we have $x^* \in C$ and $Ax^*, Bx^* \in Q$. Then, we have

$$\frac{\left\|Ax^* - P_QAx^*\right\|^2}{4} + \frac{\left\|Bx^* - P_QBx^*\right\|^2}{4} = 0 \leq \frac{\left\|Ay - P_QAy\right\|^2}{4} + \frac{\left\|By - P_QBy\right\|^2}{4}$$

for all $y \in C$. It implies that $x^* \in \Gamma_g$. □

Remark 4.2. *We observe that* $\nabla g = \frac{A^*\left(I-P_Q\right)A}{2} + \frac{B^*\left(I-P_Q\right)B}{2}$, *where* A^* *and* B^* *are adjoint of* A *and* B, *respectively and* ∇g *is a gradient of* g. *From Lemma 2.5 and 4.1, we have* $\Gamma_g = \Gamma = VI(C, \nabla g)$, *where* $\Gamma \neq \emptyset$.

Theorem 4.3. *Let* H_1 *and* H_2 *be real Hilbert spaces and let* C, Q *be nonempty closed convex subsets of* H_1 *and* H_2, *respectively. Let* $A, B : H_1 \to H_2$ *be bounded linear operators with* A^*, B^* *are adjoint of* A *and* B, *respectively and* $L = \max\{L_A, L_B\}$, *where* L_A *and* L_B *are special radius of* A^*A *and* B^*B. *Let the function* $g : H_1 \to \mathbb{R}$ *be differentiable continuous function defined by* $g(x) = \frac{\left\|\left(I-P_Q\right)Ax\right\|}{4} + \frac{\left\|\left(I-P_Q\right)Bx\right\|^2}{4}$ *and let* $D : C \to H_1$ *be d-inverse strongly monotone. Assume that* $\Gamma \cap VI(C, D) \neq \emptyset$. *Let the sequence* $\{x_n\}$ *generated by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C (I - \lambda D) x_n + \gamma_n P_C (I - a\nabla g) x_n,$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, 1)$ *with* $a_n + \beta_n + \gamma_n = 1$ *and* $f : C \to C$ *is* α*-contractive mapping with* $\alpha \in (0, 1)$. *Suppose that the following conditions hold;*

i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$

ii) $c \leq \beta_n, \gamma_n \leq d$, *for some* $c, d > 0$,

iii) $\lambda \in (0, 2d), a \in \left(0, \frac{2}{L}\right),$

iv) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty.$

Then the sequence $\{x_n\}$ *converges strongly to* $x_0 = P_{\Gamma_g \cap VI(C,D)} f(x_0)$.

Proof. From Theorem 3.1 and Lemma 4.1, we can conclude Theorem 4.3. □

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สรุปค่าใช้จ่ายการดำเนินงานโครงการวิจัย

รายการ	จำนวนเงิน
1. งบบุคลากร	
1.1 ค่าจ้างวิเคราะห์ทฤษฎีบททางคณิตศาสตร์ที่คิดขึ้นมาใหม่ คาดว่าจะตีพิมพ์ในฐานข้อมูล ISI (JCR) Q3 แบ่งออกเป็น	50,000
1) การวิเคราะห์ทฤษฎีบทสำหรับนำไปใช้กับปัญหาดุลยภาพ ปัญหาค่าเหมาะสม ปัญหาอสมการการแปรผันแบบแบ่งแยก	50,000
2) การวิเคราะห์ทฤษฎีบทเพื่อไปแก้ปัญหา เช่น การหาระยะทางในการขนส่งสินค้าที่สั้นที่สุด การวางแผนรักษามะเร็ง	50,000
1.2 ค่าจ้างตรวจสอบขั้นตอนการพิสูจน์ทฤษฎีบท	
1) ทฤษฎีบทอันที่ 1	
2) ทฤษฎีบทอันที่ 2	**150,000**
รวม	
2. งบดำเนินงาน (ค่าตอบแทน ค่าใช้สอย ค่าวัสดุ ค่าสาธารณูปโภค)	
2.2 ค่าใช้สอย	
1) ค่าจ้างเหมาเขียนโปรแกรม	100,000
2) ค่าจ้างพิมพ์รายงาน	3,000
3) ค่าจัดทำรูปเล่มรายงาน	1,000
รวม	**104,000**
2.3 ค่าวัสดุ	
1) ค่าวัสดุสำนักงาน	1,000
2) ค่าถ่ายเอกสาร	1,000
3) ค่าหนังสือ วารสารและตำราต่างๆ	4,000
4) ค่าวัสดุคอมพิวเตอร์ เช่น ค่าหมึกพิมพ์ ค่าปากกา stylus ยูเอสบีแฟลชไดรฟ์	10,000
รวม	**16,000**
รวมงบดำเนินงาน(ค่าตอบแทน+ค่าใช้สอย+ค่าวัสดุ+ค่าสาธาณูปโภค)	**120,000**
รวมงบประมาณที่เสนอขอ	**270,000**

ประวัติคณะผู้ดำเนินการวิจัย

หัวหน้าโครงการวิจัย

1. ชื่อ - นามสกุล (ภาษาไทย) ดร. อาทิตย์ แข็งธัญการ
   ชื่อ - นามสกุล (ภาษาอังกฤษ) Dr. Atid Kangtunyakarn
2. ตำแหน่งปัจจุบัน รองศาสตราจารย์
3. หน่วยงานและสถานที่อยู่ที่ติดต่อได้สะดวก พร้อมหมายเลขโทรศัพท์ โทรสาร และไปรษณีย์อิเล็กทรอนิกส์ (e-mail)

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   เลขที่ 1 ซอย ฉลองกรุง 1 ถนน ฉลองกรุง แขวง ลาดกระบัง
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4. ประวัติการศึกษา

คุณวุฒิ	ปี พ.ศ. ที่จบ	ชื่อสถานศึกษา
วิทยาศาสตรบัณฑิต	2543	มหาวิทยาลัยนเรศวร
วิทยาศาสตรมหาบัณฑิต	2548	มหาวิทยาลัยนเรศวร
วิทยาศาสตรดุษฎีบัณฑิต	2553	มหาวิทยาลัยเชียงใหม่

5. สาขาวิชาการที่มีความชำนาญพิเศษ ระบุสาขาวิชาการ
   สาขาวิทยาศาสตร์กายภาพและคณิตศาสตร์
6. ประสบการณ์ที่เกี่ยวข้องกับการบริหารงานวิจัยทั้งภายในและภายนอกประเทศ โดยระบุสถานภาพในการทำการวิจัยว่าเป็นผู้อำนวยการแผนงานวิจัย หัวหน้าโครงการวิจัย หรือผู้ร่วมวิจัยในแต่ละผลงานวิจัย

   ผู้อำนวยการแผนงานวิจัย : -

   หัวหน้าโครงการวิจัย :

   1. ชื่อโครงการวิจัย ปัญหาการแบ่งแยกการปรับปรุงเชิงดุลยภาพสำหรับปัญหาจุดตรึงของการส่งไม่เชิงเส้น

แหล่งทุน โครงการปริญญาเอกกาญจนาภิเษก (คปก.)

2. ชื่อโครงการวิจัย ทฤษฎีจุดตรึงสำหรับการแก้ปัญหาโครงข่ายประสาทเทียม

แหล่งทุน กองทุนวิจัย สจล.

3. ชื่อโครงการวิจัย ทฤษฎีจุดตรึงสำหรับปัญหาการแบ่งแยกอสมการแปรผันแบบใหม่และวิทยาการของรังสีรักษา

แหล่งทุน กองทุนวิจัย สจล.

4. ชื่อโครงการวิจัย วิธีการค่าเหมาะสมแบบใหม่สำหรับแบบจำลองเส้นทางการขนส่ง

แหล่งทุน ทุนวิจัย วช.

งานวิจัยที่ทำเสร็จแล้ว : ชื่อผลงานวิจัย ปีที่พิมพ์ การเผยแพร่ และแหล่งทุน (อาจมากกว่า 1 เรื่อง)

1. ชื่อผลงานวิจัย A new approximation method for finding common elements of equilibrium problems, variational inequality problems and fixed point problems of nonspreading mappings

ปีที่พิมพ์ 2017

การเผยแพร่ Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas

แหล่งทุน โครงการปริญญาเอกกาญจนาภิเษก (คปก.) และกองทุนวิจัย สจล.

2. ชื่อผลงานวิจัย Iterative algorithms for finding a common solution of system of the set of variational inclusion problems and the set of fixed point problems

ปีที่พิมพ์ 2011

การเผยแพร่ Fixed Point Theory and Applications

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