SEMI-TENSOR PRODUCT OF MATRICES OVER A COMMUTATIVE SEMIRING

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บทคัดย่อ

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Abstract

In this research, we introduce and investigate the notion of semi-tensor product of matrices over an arbitrary commutative semiring. This kind of matrix product is a generalization of the usual matrix product for the case when the two factor matrices do not meet the dimension matching condition. It turns out that the semi-tensor product has nice algebraic properties in such the way that it is compatible with other matrix operations.

Keywords : commutative semiring, semi-tensor product, Kronecker product

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Sowaros Toim

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Chapter 1

Introduction

1.1 Inception and importance

Motivated from scientific computing, a rectangular matrix is a two-dimensional array for stacking data. To produce a new data, we can use a variety of matrix products. Each matrix product is suitable for particular problems occurred in science, engineering, economics, etc.

Consider two real/complex matrices A and B of dimension $m \times n$ and $p \times q$, respectively. In the case $n = p$, the matrices A and B are said to satisfy matching dimension condition. As n is a factor of p or p is a factor of n, they are said to satisfy factor dimension condition. From Linear Algebra, it is well known that as A and B have matching dimension, the conventional matrix product AB is well defined. In the literature, there is a kind of matrix product that generalizes the conventional matrix product in such the way that the two factor matrices do not meet the dimension matching condition, namely, the semi-tensor product (Cheng [2]). The theory of semi-tensor products for real/complex matrices has been developed by many authors; see e.g. [3, 4, 6]. The concept of semi-tensor product can be applied widely in science (see e.g. [4, 5, 7, 8])

In the viewpoint of scientific computing, the theory of matrices whose entries come from a suitable algebraic structure such as a ring or a (commutative) semiring are practically useful. Such theory was investigated by many mathematicians; see e.g. ([1, 12, 13, 19]). Applications of this theory can be seen in [9, 11, 17].

In this research, we introduce the notion of semi-tensor product of matrices over a commutative semiring. We investigate properties of semi-tensor product related to other matrix operations such as the addition, the scalar multiplication, the usual matrix multiplication, the transposition and traces. Our results extend the results known so far for real or complex matrices in the literature.

1.2 Objectives

1) To define the notion of semi-tensor product for matrices over a commutative semiring.

2) To investigate properties of semi-tensor product related to other matrix operations such as the addition, the scalar multiplication, the usual matrix multiplication, the Kronecker product, the transposition and traces.

1.3 Scope of the study

We consider the semi-tensor product of two matrices over an arbitrary commutative semiring.

1.4 Utilization of the study

1) To develop further mathematical theory for matrices over a commutative semiring.

2) To obtain mathematical tools for analysis and control of Boolean networks, mathematical logic, scientific computing and related fields.

1.5 Research methodology

1) Study related topics in matrix theory.

2) Study related topics in multilinear algebra.

3) Study basic properties of commutative semiring.

4) Collect and study research papers and textbooks concerned with the semi-tensor product of matrices.

5) Determine the objectives and scope of the research.

6) Define the semi-tensor product of matrices over a commutative semiring.

7) Investigate the relationship between semi-tensor products and other matrix operations.

8) Conclude the results, make suggestions for further works and write the thesis.

Table 1.1: The research schedule

Activity	Time frame						
	2014	2015				2016	
	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.	Jul.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.
Step 1	⟷						
Step 2		⟷	⟷				
Step 3			⟷				
Step 4			⟷				
Step 5				⟷			
Step 6				⟷			
Step 7				⟷	⟷	⟷	
Step 8							⟷

Chapter 2

Preliminaries

2.1 Commutative semiring

Definition 2.1. *Let M be a set, and $\circ$ an operation on M. We say that $(M, \circ)$ is a monoid if it satisfies the following properties:*

(i) (closure) for any $x, y \in M$, we have $x \circ y \in M$,

(ii) (associativity) for any $x, y, z \in M$, we have $(x \circ y) \circ z = x \circ (y \circ z)$,

(iii) (identity) there exists $e \in M$ such that for any $x \in M$, we have $e \circ x = x \circ e = x$.

We may also say that M is a monoid under $\circ$.

Example 2.1. $(\mathbb{R}, +), (\mathbb{R}, \cdot), (\mathbb{N}, +), (\mathbb{N}, \cdot), (\mathbb{Q}, +), (\mathbb{R} \setminus \{0\}, \cdot), (\mathbb{Z}, +), (\mathbb{Z}, \cdot)$ are monoids.

Definition 2.2. *(Zimmerman [20] and Golan [9]). A commutative semiring $L = \langle L, +, \cdot, 0, 1\rangle$ is an algebraic structure with the following properties:*

(i) $(L, +, 0)$ is a commutative monoid,

(ii) $(L, \cdot, 1)$ is a monoid,

(iii) $a \cdot b = b \cdot a$ for all $a, b \in L$,

(iv) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in L$,

(v) $0 \cdot a = a \cdot 0 = 0$ for all $a \in L$,

(vi) $0 \neq 1$.

Definition 2.3. *A field is a set F with two binary operations, + and ·, and two distinct elements 0 and 1, satisfying the following properties:*

(i) $\forall x, y, z \in F$, $(x + y) + z = x + (y + z)$,

(ii) $\forall x \in F$, $x + 0 = 0 + x = x$,

(iii) $\forall x \in F\ \exists y \in F$, $x + y = y + x = 0$,

(iv) $\forall x, y \in F$, $x + y = y + x$,

(v) $\forall x, y, z \in F$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(vi) $\forall x \in F$, $x \cdot 1 = 1 \cdot x = x$,

(vii) $\forall x \in F - \{0\}\ \exists y \in F$, $x \cdot y = y \cdot x = 1$,

(viii) $\forall x, y \in F$, $x \cdot y = y \cdot x$,

(ix) $\forall x, y, z \in F$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

Example 2.2. $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_p$, where p is a prime number, are fields. We have every field with its operations is a commutative semiring.

Example 2.3. The nonnegative real numbers with the usual operations of addition and multiplication is a commutative semiring.

Definition 2.4. *Let $a, b \in \mathbb{Z}$ be such that $a, b \neq 0$. Any positive integer d is called the greatest common divisor between a and b if and only if*

(i) $d \mid a$ and $d \mid b$,

(ii) $\forall d' \in \mathbb{N}$, $(d' \mid a) \wedge (d' \mid b) \rightarrow (d' \mid d)$.

Write gcd(a, b) for the greatest common divisor between a and b.

Definition 2.5. *Let $a, b \in \mathbb{Z} - \{0\}$. Any positive integer m is called the smallest common multiple between a and b if and only if*

(i) $a \mid m$ and $b \mid m$,

(ii) $\forall m' \in \mathbb{N}$, $(a \mid m') \wedge (b \mid m') \rightarrow (m \mid m')$.

Write lcm(a, b) for the smallest common multiple between a and b.

Example 2.4. The nonnegative integers is a commutative semiring under the operations

$$a + b = \begin{cases} 0, & a = b = 0 \\ \text{gcd}(a, b), & \text{otherwise} \end{cases}$$

$$a \cdot b = \begin{cases} \text{lcm}(a, b), & a, b \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.5. The fuzzy algebra $[0, 1]$ (see e.g. [10]) is a commutative semiring under the operations

$$a + b = \max\{a, b\} \text{ and } a \cdot b = \min\{a, b\} \qquad \text{for all } a, b \in [0, 1].$$

Example 2.6. $[0,\infty]$ is a commutative semiring under the operations

$$a+b=\max\{a,b\} \text{ and } a\cdot b=\min\{a,b\} \quad \text{for all } a,b\in[0,\infty].$$

That is $([0,\infty],\max,\min,0,\infty)$ is a commutative semiring.

Example 2.7. Let $n\in\mathbb{N}$ be such that $n>1$. The set $\mathbb{Z}_n$ of integer modulo n with its usual operations is a commutative semiring.

Example 2.8. The max-plus algebra (or schedule algebra) $\mathbb{R}\cup\{-\infty\}$ (see e.g. [14, 16]) is a commutative semiring under the operations

$$a\oplus b=\max\{a,b\} \text{ and } a\odot b=a+b \quad \text{for all } a,b\in\mathbb{R}\cup\{-\infty\},$$

i.e $(\mathbb{R}\cup\{-\infty\},\oplus,\odot,-\infty,0)$ is a commutative semiring.

2.2 Matrices over a commutative semiring

From now on, let L be a commutative semiring. Denote by $M_{m,n}(L)$ the set of all m-by-n matrices whose entries come from L. We abbreviate $M_{n,n}(L)$ to $M_n(L)$.

Definition 2.6. *Let $A=[a_{ij}], B=[b_{ij}]\in M_{m,n}(L)$ and $\alpha\in L$. Define the addition and the scalar multiplication as follows*

$$A+B=[a_{ij}+b_{ij}]\in M_{m,n}(L),$$
$$\alpha A=[\alpha a_{ij}]\in M_{m,n}(L).$$

Definition 2.7. *Let $A=[a_{ij}]\in M_{m,n}(L)$ and $B=[b_{ij}]\in M_{n,p}(L)$. We define the multiplication of A and B by*

$$AB=\left[\sum_{k=1}^{n}a_{ik}b_{kj}\right]\in M_{m,p}(L).$$

Definition 2.8. *The zero matrix and the identity matrix in $M_n(L)$ is defined as*

$$0_n=\begin{bmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{bmatrix} \text{ and } I_n=\begin{bmatrix}1&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&1\end{bmatrix}.$$

Proposition 2.9. *Let A be matrix over a commutative semiring L. If the following matrix operations exist, we have*

$$(\alpha+\beta)A=\alpha A+\beta A.$$

Definition 2.9. *The transpose of a matrix $A=[a_{ij}]\in M_{m,n}(L)$ is defined to be the matrix $A^T=[a_{ji}]\in M_{n,m}(L)$.*

Definition 2.10. *A matrix $A \in M_n(L)$ is said to be invertible if there is a matrix $B \in M_n(L)$ such that $AB = I_n = BA$.*

Theorem 2.10. *(Reurenauer [1]) Let $A, B \in M_n(L)$. If $AB = I_n$, then $BA = I_n$.*

Definition 2.11. *A matrix $A \in M_n(L)$ is similar to $B \in M_n(L)$ if and only if there is a invertible matrix S such that $S^{-1}AS = B$.*

Definition 2.12. *The trace of a matrix $A = [a_{ij}] \in M_n(L)$ is defined by*

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Definition 2.13. *(see [15]) Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. The Kronecker product of A and B is defined to be*

$$A \otimes B = [a_{ij}B]_{ij} \in M_{mp,nq}(L).$$

That is, each $(i,j)\,th$ block of $A \otimes B$ is given by $a_{ij}B$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Example 2.11. Let $L = \langle [0,1], \max, \min, 0, 1 \rangle$ be a commutative semiring with operations $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$ for all $a, b \in [0,1]$. Consider

$$A = \begin{bmatrix} 0.8 & 0.2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0.3 & 0.1 \\ 0.5 & 0.9 & 0.6 \end{bmatrix}.$$

We have

$$A \otimes B = \begin{bmatrix} 0.8 \begin{bmatrix} 0 & 0.3 & 0.1 \\ 0.5 & 0.9 & 0.6 \end{bmatrix} & 0.2 \begin{bmatrix} 0 & 0.3 & 0.1 \\ 0.5 & 0.9 & 0.6 \end{bmatrix} \\ 1 \cdot \begin{bmatrix} 0 & 0.3 & 0.1 \\ 0.5 & 0.9 & 0.6 \end{bmatrix} & 0 \cdot \begin{bmatrix} 0 & 0.3 & 0.1 \\ 0.5 & 0.9 & 0.6 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 & 0.3 & 0.1 \\ 0.5 & 0.8 & 0.6 \end{bmatrix} & \begin{bmatrix} 0 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.2 \end{bmatrix} \\ \begin{bmatrix} 0 & 0.3 & 0.1 \\ 0.5 & 0.9 & 0.6 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.3 & 0.1 & 0 & 0.2 & 0.1 \\ 0.5 & 0.8 & 0.6 & 0.2 & 0.2 & 0.2 \\ 0 & 0.3 & 0.1 & 0 & 0 & 0 \\ 0.5 & 0.9 & 0.6 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem 2.12. *(The mixed product property, see [15]) Let $A \in M_{m,n}(L)$, $B \in M_{p,q}(L)$, $C \in M_{n,k}(L)$ and $D \in M_{q,r}(L)$. Then*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

2.3 Semi-tensor products of real matrices

Definition 2.14. *(Cheng [2])*

(i) Let $X \in M_{1,mn}(\mathbb{R})$ and $Y \in M_{m,1}(\mathbb{R})$. Then we split X into m equal-size blocks as $(X^1, X^2, \dots, X^m)$, such that $X^i \in M_{1,n}(\mathbb{R})$, $i = 1, \dots, m$ and define the left Semi-tensor product (STP) of X and Y, denoted by $X \ltimes Y$, as

$$X \ltimes Y = \sum_{i=1}^{m} y_i X^i \in M_{1,n}(\mathbb{R}). \tag{2.1}$$

(ii) Let $X \in M_{1,m}(\mathbb{R})$ and $Y \in M_{mn,1}(\mathbb{R})$. Then we define the left STP of X and Y, as

$$X \ltimes Y = [Y^T \ltimes X^T]^T \in M_{n,1}(\mathbb{R}). \tag{2.2}$$

Example 2.13. Let $X = \begin{bmatrix} 1 & 3 & 2 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Then

$$\begin{aligned} X \ltimes Y &= 2 \cdot \begin{bmatrix} 1 & 3 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 6 \end{bmatrix} + \begin{bmatrix} (-2) & (-4) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \end{bmatrix}. \end{aligned}$$

Example 2.14. Let $X = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 & 0 & -2 & 1 \end{bmatrix}^T$. Then

$$\begin{aligned} X \ltimes Y &= \left(\begin{bmatrix} 2 & 1 & -1 & 0 & -2 & 1 \end{bmatrix} \ltimes \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right)^T \\ &= \left(1 \cdot \begin{bmatrix} 2 & 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} -1 & 0 \end{bmatrix} + (-1) \cdot \begin{bmatrix} -2 & 1 \end{bmatrix} \right)^T \\ &= \left(\begin{bmatrix} 2 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \end{bmatrix} \right)^T \\ &= \begin{bmatrix} 2 & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \end{aligned}$$

Definition 2.15. *(Cheng [2]) Let $A \in M_{m,n}(\mathbb{R})$ and $B \in M_{p,q}(\mathbb{R})$. If either n is a factor of p, say $nt = p$ and denote it as $A \prec_t B$, or p is a factor of n, say $n = pt$ and denote it as $A \succ_t B$, then we define the left STP of A and B, denoted by $C = A \ltimes B$, as the following: C consists of $m \times q$ blocks as $C = (C^{ij})$ and each block is*

$$C^{ij} = A^i \ltimes B_j, \qquad i = 1, 2, ..., m,\ j = 1, 2, ..., q,$$

where A^i is i-th row of A and B_j is j-th column of B.

Example 2.15. Let $A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 3 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Then

$$A \ltimes B = \begin{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 2 \end{bmatrix} \ltimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & -1 & 2 \end{bmatrix} \ltimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} \ltimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} \ltimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 3 & 3 & 1 & 1 \end{bmatrix} \ltimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \begin{bmatrix} 3 & 3 & 1 & 1 \end{bmatrix} \ltimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} + (-1) \cdot \begin{bmatrix} -1 & 2 \end{bmatrix} & 2 \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} + 3 \cdot \begin{bmatrix} -1 & 2 \end{bmatrix} \\ 1 \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 2 & 3 \end{bmatrix} & 2 \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 & 3 \end{bmatrix} \\ 1 \cdot \begin{bmatrix} 3 & 3 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 1 & 1 \end{bmatrix} & 2 \cdot \begin{bmatrix} 3 & 3 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -1 & 10 \\ -2 & -2 & 6 & 11 \\ 2 & 2 & 9 & 9 \end{bmatrix}.$$

Remark 2.16. Let $A \in M_{m,n}(\mathbb{R})$ and $B \in M_{p,q}(\mathbb{R})$. It follows from the definition that when $n = p$ we have

$$A \ltimes B = AB.$$

Theorem 2.17. *(Associative rule, Cheng [2]) Let A, B and C be matrices over $\mathbb{R}$. If the following matrix operations are well-defined, then*

$$(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C). \tag{2.3}$$

Theorem 2.18. *(Cheng [2]) Let* $A \in M_{m,n}(\mathbb{R})$ *and* $B \in M_{p,q}(\mathbb{R})$ *where* $p \mid n$ *or* $n \mid p$. *Then*

$$(A \ltimes B)^T = B^T \ltimes A^T. \tag{2.4}$$

Theorem 2.19. *(Bilinearity, Cheng [2]) Let* A, B *and* C *be matrices over* $\mathbb{R}$. *If* A, B *and* C *are of multiplier dimension, then*

$$A \ltimes (\alpha B + \beta C) = \alpha (A \ltimes B) + \beta (A \ltimes C), \tag{2.5}$$

$$(\alpha B + \beta C) \ltimes A = \alpha (B \ltimes A) + \beta (C \ltimes A). \tag{2.6}$$

Definition 2.16. *(Cheng [2]) If* X *is a row vector or a column vector, we define for each* $k \in \mathbb{N}$,

$$X^{\ltimes k} = \underbrace{X \ltimes \cdots \ltimes X}_{k}. \tag{2.7}$$

Definition 2.17. *(Cheng [2]) Assume* $A \in M_{m,n}(\mathbb{R})$ *where either* $m \mid n$ *or* $n \mid m$. *Then* A^n *is inductively defined as*

$$A^{\ltimes 1} = A \text{ and } A^{\ltimes (k+1)} = A^{\ltimes k} \ltimes A \quad \text{for } k = 1, 2, \ldots \tag{2.8}$$

It is easy to see that A^k *is well defined. Moreover, if* $m = nt$, *then* $A^{\ltimes k} \in M_{t^k n,n}(\mathbb{R})$ *and if* $mt = n$, *then* $A^{\ltimes k} \in M_{m,t^k m}(\mathbb{R})$.

Theorem 2.20. *(The mixed product property, Cheng [2])*

(i) Let $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^q$ be two columns and $A \in M_{m,n}(\mathbb{R})$, $B \in M_{p,q}(\mathbb{R})$ be two matrices. Then

$$(AX) \ltimes (BY) = (A \otimes B)(X \ltimes Y). \tag{2.9}$$

(ii) Let $W \in \mathbb{R}^m$, $Z \in \mathbb{R}^p$ be two rows and $A \in M_{m,n}(\mathbb{R})$, $B \in M_{p,q}(\mathbb{R})$ be two matrices. Then

$$(WA) \ltimes (ZB) = (W \ltimes Z)(B \otimes A). \tag{2.10}$$

Theorem 2.21. *(Cheng [2]) Let* A *and* B *be matrices over* $\mathbb{R}$.

(i) If $A \succ_t B$, then $A \ltimes B = A(B \otimes I_t)$.

(ii) If $A \prec_t B$, then $A \ltimes B = (A \otimes I_t) B$.

Theorem 2.22. *(Cheng [2]) Assume that* $A \in M_m(\mathbb{R})$ *and* $B \in M_n(\mathbb{R})$ *with proper dimensions such that* $A \ltimes B$ *and* $B \ltimes A$ *are well-defined. Then*

$$\operatorname{tr}(A \ltimes B) = \operatorname{tr}(B \ltimes A). \tag{2.11}$$

Theorem 2.23. *(Cheng [2]) Assume that A and B are square matrices and both $A \ltimes B$ and $B \ltimes A$ are well-defined.*

(i) If $A \succ_t B$ and A^{-1} exists, then $A \ltimes B \sim B \ltimes A$.

(ii) If $A \prec_t B$ and B^{-1} exists, then $A \ltimes B \sim B \ltimes A$.

Theorem 2.24. *(Cheng [2]) Assume $A \in M_m(\mathbb{R})$ and $B \in M_n(\mathbb{R})$ are invertible and $A \ltimes B$ is well defined, then*

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \tag{2.12}$$

Theorem 2.25. *(Cheng [2]) Assume A and B are square matrices and A and B are of multiplier dimension.*

(i) If A and B are orthogonal, then so is $A \ltimes B$.

(ii) If A and B are upper triangular, then so is $A \ltimes B$.

(iii) If A and B are lower triangular, then so is $A \ltimes B$.

(iv) If A and B are diagonal, then so is $A \ltimes B$.

Theorem 2.26. *(Cheng [2]) The semi-tensor product of a matrix with an identity matrix has the following properties*

(i) If $A \in M_{m,pn}(\mathbb{R})$, then $A \ltimes I_n = A$.

(ii) If $A \in M_{m,n}(\mathbb{R})$, then $A \ltimes I_{pn} = A \otimes I_p$.

(iii) If $A \in M_{pm,n}(\mathbb{R})$, then $I_p \ltimes A = A$.

(iv) If $A \in M_{m,n}(\mathbb{R})$, then $I_{pm} \ltimes A = A \otimes I_p$.

Chapter 3

Semi-tensor products of matrices over a commutative semiring

In this chapter, we introduce the semi-tensor product of matrices over a commutative semiring. First, we define the semi-tensor product between row vectors and column vectors. Then we extend it to the product between row vectors and matrices. Finally, we can define the semi-tensor products for matrices by partitioning the first factor matrix into row-block matrix.

3.1 Semi-tensor products between row vectors and column vectors

Definition 3.1. *Let $X \in M_{1,m}(L)$ and $Y \in M_{p,1}(L)$.*

(i) Consider the case $m = tp$, denoted by $X \succ_t Y$. We split X into p equal-size blocks as $(X^1, X^2, \ldots, X^p)$, such that $X^i \in M_{1,t}(L)$ for all $i = 1, \ldots, p$. Define the left Semi-tensor product (STP), denote by $\ltimes$, as

$$X \ltimes Y = \sum_{i=1}^{p} y_i X^i \in M_{1,t}(L). \tag{3.1}$$

(ii) Consider the case $mt = p$, denoted by $X \prec_t Y$. We split Y into m equal-size blocks as $(Y^1, Y^2, \ldots, Y^m)$, such that $Y^i \in M_{t,1}(L)$ for all $i = 1, \ldots, m$. Define the left STP, denote by $\ltimes$, as

$$X \ltimes Y = \sum_{i=1}^{m} x_i Y^i \in M_{t,1}(L). \tag{3.2}$$

Example 3.1. Let $L = \langle [0, \infty], \max, \min, 0, \infty \rangle$ be a commutative semiring with operations $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$ for all $a, b \in [0, \infty]$. Consider

$$X = \begin{bmatrix} 0 & 2 & 1 & \infty \end{bmatrix} \text{ and } Y = \begin{bmatrix} \infty \\ 5 \end{bmatrix}.$$

We have

$$X \ltimes Y = \infty \cdot \begin{bmatrix} 0 & 2 \end{bmatrix} + 5 \cdot \begin{bmatrix} 1 & \infty \end{bmatrix}$$
$$= \begin{bmatrix} \infty \cdot 0 & \infty \cdot 2 \end{bmatrix} + \begin{bmatrix} 5 \cdot 1 & 5 \cdot \infty \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 0+1 & 2+5 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 5 \end{bmatrix}.$$

Example 3.2. Let $L = \langle \mathbb{Z}_6, +, \cdot, \overline{0}, \overline{1} \rangle$ be a commutative semiring with operations $\overline{a} + \overline{b} = \overline{a+b}$ and $\overline{a} \cdot \overline{b} = \overline{a \cdot b}$ for all $a, b \in \mathbb{Z}_6$. Consider $X = \begin{bmatrix} \overline{0} & \overline{2} & \overline{5} \end{bmatrix}$ and $Y = \begin{bmatrix} \overline{3} & \overline{0} & \overline{1} & \overline{2} & \overline{4} & \overline{5} \end{bmatrix}^T$. We have

$$X \ltimes Y = \overline{0} \cdot \begin{bmatrix} \overline{3} \\ \overline{0} \end{bmatrix} + \overline{2} \cdot \begin{bmatrix} \overline{1} \\ \overline{2} \end{bmatrix} + \overline{5} \cdot \begin{bmatrix} \overline{4} \\ \overline{5} \end{bmatrix}$$
$$= \begin{bmatrix} \overline{0} \cdot \overline{3} \\ \overline{0} \cdot \overline{0} \end{bmatrix} + \begin{bmatrix} \overline{2} \cdot \overline{1} \\ \overline{2} \cdot \overline{2} \end{bmatrix} + \begin{bmatrix} \overline{5} \cdot \overline{4} \\ \overline{5} \cdot \overline{5} \end{bmatrix}$$
$$= \begin{bmatrix} \overline{0} \\ \overline{0} \end{bmatrix} + \begin{bmatrix} \overline{2} \\ \overline{4} \end{bmatrix} + \begin{bmatrix} \overline{2} \\ \overline{1} \end{bmatrix}$$
$$= \begin{bmatrix} \overline{0} + \overline{2} + \overline{2} \\ \overline{0} + \overline{4} + \overline{1} \end{bmatrix}$$
$$= \begin{bmatrix} \overline{4} \\ \overline{5} \end{bmatrix}.$$

Lemma 3.3. *Let $X \in M_{1,m}(L)$ and $Y \in M_{p,1}(L)$ be such that $p \mid m$ or $m \mid p$. Then*

$$(X \ltimes Y)^T = Y^T \ltimes X^T. \tag{3.3}$$

Proof. For the case $X \succ_t Y$, we have

$$(X \ltimes Y)^T = \left(\sum_{i=1}^{p} y_i X^i \right)^T = \sum_{i=1}^{p} y_i (X^T)^i = Y^T \ltimes X^T.$$

For the case $X \prec_t Y$, we get

$$(X \ltimes Y)^T = \left(\sum_{i=1}^{m} x_i Y^i \right)^T = \sum_{i=1}^{m} x_i (Y^T)^i = Y^T \ltimes X^T.$$

□

Lemma 3.4.

(i) Let $X \in M_{1,m}(L)$, $Y, Z \in M_{p,1}(L)$ and $\alpha, \beta \in L$ where $p \mid m$ or $m \mid p$. Then

$$X \ltimes (\alpha Y + \beta Z) = \alpha (X \ltimes Y) + \beta (X \ltimes Z). \tag{3.4}$$

(ii) Let $X, Y \in M_{1,m}(L)$, $Z \in M_{p,1}(L)$ and $\alpha, \beta \in L$ where $p \mid m$ or $m \mid p$. Then

$$(\alpha X + \beta Y) \ltimes Z = \alpha (X \ltimes Z) + \beta (Y \ltimes Z). \tag{3.5}$$

Proof. Consider $X \in M_{1,m}(L)$, $Y, Z \in M_{p,1}(L)$ and $\alpha, \beta \in L$. For the case $p \mid m$, we obtain

$$\begin{aligned} X \ltimes (\alpha Y + \beta Z) &= \sum_{i=1}^{p} (\alpha y_i + \beta z_i) X^i \\ &= \sum_{i=1}^{p} (\alpha y_i X^i + \beta z_i X^i) \\ &= \alpha \sum_{i=1}^{p} y_i X^i + \beta \sum_{i=1}^{p} z_i X^i \\ &= \alpha (X \ltimes Y) + \beta (X \ltimes Z). \end{aligned}$$

For the case $m \mid p$, we have

$$\begin{aligned} X \ltimes (\alpha Y + \beta Z) &= \sum_{i=1}^{m} x_i (\alpha Y^i + \beta Z^i) \\ &= \sum_{i=1}^{m} (\alpha x_i Y^i + \beta x_i Z^i) \\ &= \alpha \sum_{i=1}^{m} x_i Y^i + \beta \sum_{i=1}^{m} x_i Z^i \\ &= \alpha (X \ltimes Y) + \beta (X \ltimes Z). \end{aligned}$$

Similarly, we can prove the assertion (ii). □

Lemma 3.5. *Let $X \in M_{1,n}(L)$ and $Y \in M_{p,1}(L)$. Then*

(i) If $n = tp$, then $X \ltimes Y = X (Y \otimes I_t)$.

(ii) If $nt = p$, then $X \ltimes Y = (X \otimes I_t) Y$.

Proof. For the case $X \succ_t Y$, we write $X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \end{bmatrix}$ and $Y = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{p1} \end{bmatrix}$.

We have

$$X \ltimes Y$$

$$= \left[x_{11} \cdots x_{1t} \vdots x_{1,t+1} \cdots x_{1,2t} \vdots \cdots \vdots x_{1,(p-1)t+1} \cdots x_{pt}\right] \ltimes \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{p1} \end{bmatrix}$$

$$= y_{11}\left[x_{11} \cdots x_{1t}\right] + y_{21}\left[x_{1,t+1} \cdots x_{1,2t}\right] + \ldots + y_{p1}\left[x_{1,(p-1)t+1} \cdots x_{pt}\right]$$

$$= \left[y_{11}x_{11} + y_{21}x_{1,t+1} + \ldots + y_{p1}x_{1,(p-1)t+1} \cdots y_{11}x_{1t} + y_{21}x_{1,2t} + \ldots + y_{p1}x_{p,t}\right]$$

$$= \left[x_{11} \cdots x_{1t} \ \ x_{1,t+1} \cdots x_{1,2t} \cdots x_{1,(p-1)t+1} \cdots x_{pt}\right] \ltimes \begin{bmatrix} y_{11} & 0 & \cdots & 0 \\ 0 & y_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_{11} \\ y_{21} & 0 & \cdots & 0 \\ 0 & y_{21} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_{21} \\ \vdots & \vdots & \vdots & \vdots \\ y_{p1} & 0 & \cdots & 0 \\ 0 & y_{p1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_{p1} \end{bmatrix}$$

$$= X\left(\begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{p1} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}\right)$$

$$= X\left(Y \otimes I_t\right).$$

Similarly, we can prove the assertion (ii). □

3.2 Semi-tensor products between row vectors and matrices

We use $\text{Row}_i(A)$ for the ith row of A and $\text{Col}_i(A)$ for the ith column of A.

Definition 3.2. *Let $X \in M_{1,m}(L)$ and $A \in M_{p,q}(L)$ where $p \mid m$ or $m \mid p$. We define*

$$X \ltimes A = \left[X \ltimes \mathrm{Col}_1(A) \cdots X \ltimes \mathrm{Col}_q(A)\right]. \tag{3.6}$$

Example 3.6. Let $L = \langle [0,1], \max, \min, 0, 1\rangle$ be a commutative semiring with operations $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$ for all $a, b \in [0,1]$. Consider

$$X = \begin{bmatrix} 0.1 & 1 & 0.9 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0.2 & 0 \\ 0.4 & 1 \end{bmatrix}.$$

We obtain

$$\begin{aligned} X \ltimes A &= \left[\begin{bmatrix} 0.1 & 1 & 0.9 & 0 \end{bmatrix} \ltimes \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} \quad \begin{bmatrix} 0.1 & 1 & 0.9 & 0 \end{bmatrix} \ltimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] \\ &= \left[(0.2)\begin{bmatrix} 0.1 & 1 \end{bmatrix} + (0.4)\begin{bmatrix} 0.9 & 0 \end{bmatrix} \quad 0 \cdot \begin{bmatrix} 0.1 & 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0.9 & 0 \end{bmatrix}\right] \\ &= \left[\begin{bmatrix} 0.1 & 0.2 \end{bmatrix} + \begin{bmatrix} 0.4 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.9 & 0 \end{bmatrix}\right] \\ &= \left[\begin{bmatrix} 0.4 & 0.2 \end{bmatrix} \quad \begin{bmatrix} 0.9 & 0 \end{bmatrix}\right] \\ &= \begin{bmatrix} 0.4 & 0.2 & 0.9 & 0 \end{bmatrix}. \end{aligned}$$

Lemma 3.7. *Let $X, Y \in M_{1,m}(L)$ and $A, B \in M_{p,q}(L)$ and $\alpha, \beta \in L$ where $p \mid m$ or $m \mid p$. Then*

$$X \ltimes (\alpha A + \beta B) = \alpha (X \ltimes A) + \beta (X \ltimes B), \tag{3.7}$$

$$(\alpha X + \beta Y) \ltimes A = \alpha (X \ltimes A) + \beta (Y \ltimes A). \tag{3.8}$$

Proof. From Definition 3.2 and Lemma 3.4, we have

$$\begin{aligned} & X \ltimes (\alpha A + \beta B) \\ &= X \ltimes \left[\alpha \mathrm{Col}_1(A) + \beta \mathrm{Col}_1(B) \cdots \alpha \mathrm{Col}_q(A) + \beta \mathrm{Col}_q(B)\right] \\ &= \left[X \ltimes \left(\alpha \mathrm{Col}_1(A) + \beta \mathrm{Col}_1(B)\right) \cdots X \ltimes \left(\alpha \mathrm{Col}_q(A) + \beta \mathrm{Col}_q(B)\right)\right] \\ &= \left[\alpha \left(X \ltimes \mathrm{Col}_1(A)\right) + \beta \left(X \ltimes \mathrm{Col}_1(B)\right) \cdots \alpha \left(X \ltimes \mathrm{Col}_q(A)\right) + \beta \left(X \ltimes \mathrm{Col}_q(B)\right)\right] \\ &= \left[\alpha \left(X \ltimes \mathrm{Col}_1(A)\right) \cdots \alpha \left(X \ltimes \mathrm{Col}_q(A)\right)\right] + \left[\beta \left(X \ltimes \mathrm{Col}_1(B)\right) \cdots \beta \left(X \ltimes \mathrm{Col}_q(B)\right)\right] \\ &= \alpha \left[X \ltimes \mathrm{Col}_1(A) \cdots X \ltimes \mathrm{Col}_q(A)\right] + \beta \left[X \ltimes \mathrm{Col}_1(B) \cdots X \ltimes \mathrm{Col}_q(B)\right] \\ &= \alpha (X \ltimes A) + \beta (X \ltimes B). \end{aligned}$$

Similarly, we obtain $(\alpha X + \beta Y) \ltimes A = \alpha (X \ltimes A) + \beta (Y \ltimes A)$. ☐

3.3 Semi-tensor products of matrices

Definition 3.3. *Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$ where $p \mid n$ or $n \mid p$. The left semi-tensor product of A and B is defined as*

$$A \ltimes B = \begin{bmatrix} \text{Row}_1(A) \ltimes B \\ \vdots \\ \text{Row}_m(A) \ltimes B \end{bmatrix} \tag{3.9}$$

which is an $m \times q$ block matrix.

Remark 3.8. Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$.

(i) If $n = tp$, then $A \ltimes B$ is an $m \times tq$ matrix.

(ii) If $nt = p$, then $A \ltimes B$ is an $mt \times q$ matrix.

Example 3.9. Let $L = \langle [0,\infty), +, \cdot, 0, 1\rangle$ be a commutative semiring with the usual operations of addition and multiplication. Consider

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}.$$

We get

$$A \ltimes B = \begin{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} \ltimes B \\ \begin{bmatrix} 2 & 1 \end{bmatrix} \ltimes B \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} \ltimes \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix} \ltimes \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix} \ltimes \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \end{bmatrix} \ltimes \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix} & \begin{bmatrix} 2 & 1 \end{bmatrix} \ltimes \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 & 1 \end{bmatrix} \ltimes \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 3 & 6 \\ 3 & 11 & 7 \\ 4 & 1 & 7 \\ 6 & 7 & 4 \end{bmatrix}.$$

Example 3.10. Let $L = \langle \mathbb{N} \cup \{0\}, +, \cdot, 0, 1\rangle$ be a commutative semiring with operations

$$a + b = \begin{cases} 0, & a = b = 0 \\ \mathrm{gcd}(a,b), & \text{otherwise} \end{cases}$$

$$a \cdot b = \begin{cases} \mathrm{lcm}(a,b), & a, b \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Consider $A = \begin{bmatrix} 12 & 6 & 8 & 3 \\ 2 & 5 & 1 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$. We have

$$\begin{aligned} A \ltimes B &= \begin{bmatrix} \begin{bmatrix} 12 & 6 & 8 & 3 \end{bmatrix} \ltimes \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 5 & 1 & 10 \end{bmatrix} \ltimes \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 12 & 6 & 8 & 3 \end{bmatrix} \ltimes \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 12 & 6 & 8 & 3 \end{bmatrix} \ltimes \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 5 & 1 & 10 \end{bmatrix} \ltimes \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 5 & 1 & 10 \end{bmatrix} \ltimes \begin{bmatrix} 4 \\ 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 12 & 6 \end{bmatrix} + \begin{bmatrix} 8 & 6 \end{bmatrix} & \begin{bmatrix} 12 & 12 \end{bmatrix} + \begin{bmatrix} 8 & 3 \end{bmatrix} \\ \begin{bmatrix} 2 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 10 \end{bmatrix} & \begin{bmatrix} 4 & 20 \end{bmatrix} + \begin{bmatrix} 1 & 10 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 & 4 & 3 \\ 2 & 5 & 1 & 10 \end{bmatrix}. \end{aligned}$$

Proposition 3.11. *Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$ where $p \mid n$ or $n \mid p$. Then*

$$A \ltimes B = \left[A \ltimes \mathrm{Col}_1(B) \cdots A \ltimes \mathrm{Col}_q(B)\right]. \tag{3.10}$$

Proof. From Definitions 3.3 and 3.2, we have

$$\begin{aligned} A \ltimes B &= \begin{bmatrix} \mathrm{Row}_1(A) \ltimes \mathrm{Col}_1(B) & \cdots & \mathrm{Row}_1(A) \ltimes \mathrm{Col}_q(B) \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A) \ltimes \mathrm{Col}_1(B) & \cdots & \mathrm{Row}_m(A) \ltimes \mathrm{Col}_q(B) \end{bmatrix} \\ &= \left[A \ltimes \mathrm{Col}_1(B) \cdots A \ltimes \mathrm{Col}_q(B)\right]. \end{aligned}$$

□

Theorem 3.12. *Let $A \in M_{m,n}(L)$ and $B \in M_{n,q}(L)$. Then $A \ltimes B = AB$.*

Proof. From Proposition 3.11, we have

$$\begin{aligned}
A \ltimes B &= \left[A \ltimes \mathrm{Col}_1(B) \cdots A \ltimes \mathrm{Col}_q(B)\right] \\
&= \begin{bmatrix} \mathrm{Row}_1(A) \ltimes \mathrm{Col}_1(B) & \cdots & \mathrm{Row}_1(A) \ltimes \mathrm{Col}_q(B) \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A) \ltimes \mathrm{Col}_1(B) & \cdots & \mathrm{Row}_m(A) \ltimes \mathrm{Col}_q(B) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=1}^{n} a_{1k}b_{k1} & \cdots & \sum_{k=1}^{n} a_{1k}b_{kq} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk}b_{k1} & \cdots & \sum_{k=1}^{n} a_{mk}b_{kq} \end{bmatrix} \\
&= AB.
\end{aligned}$$

□

Theorem 3.13. *(Associative rule) Let A, B, C be matrices over L. If the following matrix operations are well-defined, then*

$$(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C). \tag{3.11}$$

Proof. Consider the dimension of A, B and C.

Case 1: $A \succ B$ and $B \succ C$. Assume $A \in M_{m,np}(L), B \in M_{p,qr}(L)$ and $C \in M_{r,s}(L)$.

Case 2: $A \prec B$ and $B \prec C$. Assume $A \in M_{m,n}(L), B \in M_{np,q}(L)$ and $C \in M_{rq,s}(L)$.

Case 3: $A \prec B$ and $B \succ C$. Assume $A \in M_{m,n}(L), B \in M_{np,qr}(L)$ and $C \in M_{r,s}(L)$.

Case 4: $A \succ B$ and $B \prec C$. Assume $A \in M_{m,np}(L), B \in M_{p,q}(L)$ and $C \in M_{rq,s}(L)$.

Now the dimension of $A \ltimes B$ is $m \times nq$. To make it feasible for $(A \ltimes B) \ltimes C$, we need

Case 4.1: $(A \ltimes B) \succ C$, i.e., $n = n'r$.

Case 4.2: $(A \ltimes B) \prec C$, i.e., $r = nr'$.

The dimension of $B \ltimes C$ is $pr \times s$. To make it feasible for $(A \ltimes B) \ltimes C$, we need

Case 4.3: $A \succ (B \ltimes C)$, i.e., $n = n'r$.

Case 4.4: $A \prec (B \ltimes C)$, i.e., $r = nr'$.

Next, we prove the associativity. We have to prove it case by case. But case 1-3 are similar, we prove only case 1.

Let $A \in M_{m,np}(L), B \in M_{p,qr}(L)$ and $C \in M_{r,s}(L)$. Assume $m = 1$ and $s = 1$. Split A as $A = [A_1 \cdots A_p]$, where A_i, $i = 1, \ldots, p$ are $1 \times n$ blocks. Then

LHS:

$$A \ltimes B = [A_1 \cdots A_p] \ltimes \begin{bmatrix} b_{11}^1 & \cdots & b_{1q}^1 & \cdots & b_{r1}^1 & \cdots & b_{rq}^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \\ b_{11}^p & \cdots & b_{1q}^p & \cdots & b_{r1}^p & \cdots & b_{rq}^p \end{bmatrix}$$

$$= \left[\sum_{i=1}^{p} A_i b_{11}^i \cdots \sum_{i=1}^{p} A_i b_{1q}^i \cdots \sum_{i=1}^{p} A_i b_{r1}^i \cdots \sum_{i=1}^{p} A_i b_{rq}^i \right].$$

Then we obtain

$$(A \ltimes B) \ltimes C = (A \ltimes B) \ltimes \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} = \left[\sum_{j=1}^{r} \sum_{i=1}^{p} A_i b_{j1}^i c_j \cdots \sum_{j=1}^{r} \sum_{i=1}^{p} A_i b_{jq}^i c_j \right].$$

RHS:

$$\begin{bmatrix} b_{11}^1 & \cdots & b_{1q}^1 & \cdots & b_{r1}^1 & \cdots & b_{rq}^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \\ b_{11}^p & \cdots & b_{1q}^p & \cdots & b_{r1}^p & \cdots & b_{rq}^p \end{bmatrix} \ltimes \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{r} b_{j1}^1 c_j & \cdots & \sum_{j=1}^{r} b_{jq}^1 c_j \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{r} b_{j1}^p c_j & \cdots & \sum_{j=1}^{r} b_{jq}^p c_j \end{bmatrix}.$$

Then

$$A \ltimes (B \ltimes C) = (A_1, \ldots, A_p) \ltimes (B \ltimes C)$$

$$= \left[\sum_{j=1}^{r} \sum_{i=1}^{p} A_i b_{j1}^i c_j \cdots \sum_{j=1}^{r} \sum_{i=1}^{p} A_i b_{jq}^i c_j \right].$$

We have LHS = RHS.

Since Cases 4.1-4.4 are similar, we prove Case 4.1 only.

Let $A \in M_{m,npr}(L)$, $B \in M_{p,q}(L)$ and $C \in M_{rq,s}(L)$ be given. We also assume $m = 1$ and $s = 1$. Split A as $A = [A_{11} \cdots A_{1r} \cdots A_{p1} \cdots A_{pr}]$, where each $A_{ij}, i = 1, \ldots, p, j = 1, \ldots, r$ are $1 \times n$ blocks.

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix}, C = [c_{11} \cdots c_{1r} \cdots c_{q1} \cdots c_{qr}]^T.$$

A direct computation shows that

$$(A \ltimes B) \ltimes C = \sum_{i=1}^{p} \sum_{j=1}^{r} \sum_{k=1}^{q} A_{ij} b_{ik} c_{kj} = A \ltimes (B \ltimes C).$$

□

Chapter 4

Semi-tensor products and familiar matrix operations

We shall show that the semi-tensor product of matrices is compatible with familiar matrix operations, such as, addition, scalar multiplication, transposition and Kronecker multiplication.

Theorem 4.1. *Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$ where $p \mid n$ or $n \mid p$. Then*

$$(A \ltimes B)^T = B^T \ltimes A^T. \tag{4.1}$$

Proof. From Lemma 3.3, we have

$$\begin{aligned}
(A \ltimes B)^T &= \begin{bmatrix} [\mathrm{Row}_1(A) \ltimes \mathrm{Col}_1(B)]^T & \cdots & [\mathrm{Row}_m(A) \ltimes \mathrm{Col}_1(B)]^T \\ \vdots & \ddots & \vdots \\ [\mathrm{Row}_1(A) \ltimes \mathrm{Col}_q(B)]^T & \cdots & [\mathrm{Row}_m(A) \ltimes \mathrm{Col}_q(B)]^T \end{bmatrix} \\
&= \begin{bmatrix} [\mathrm{Col}_1(B)]^T \ltimes [\mathrm{Row}_1(A)]^T & \cdots & [\mathrm{Col}_1(B)]^T \ltimes [\mathrm{Row}_m(A)]^T \\ \vdots & \ddots & \vdots \\ [\mathrm{Col}_q(B)]^T \ltimes [\mathrm{Row}_1(A)]^T & \cdots & [\mathrm{Col}_q(B)]^T \ltimes [\mathrm{Row}_m(A)]^T \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Row}_1(B^T) \ltimes \mathrm{Col}_1(A^T) & \cdots & \mathrm{Row}_1(B^T) \ltimes \mathrm{Col}_m(A^T) \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_q(B^T) \ltimes \mathrm{Col}_1(A^T) & \cdots & \mathrm{Row}_q(B^T) \ltimes \mathrm{Col}_m(A^T) \end{bmatrix} \\
&= B^T \ltimes A^T.
\end{aligned}$$

□

The next theorem shows that the semi-tensor product is a bilinear map.

Theorem 4.2. *(Bilinearity) Let A, B, C be matrices over L and $\alpha, \beta \in L$. If the following matrix operations are well-defined, then*

$$A \ltimes (\alpha B + \beta C) = \alpha (A \ltimes B) + \beta (A \ltimes C), \tag{4.2}$$

$$(\alpha B + \beta C) \ltimes A = \alpha (B \ltimes A) + \beta (C \ltimes A). \tag{4.3}$$

Proof. From Definition 3.3 and Lemma 3.7, we have

$$
\begin{aligned}
A \ltimes (\alpha B + \beta C) &= \begin{bmatrix} \mathrm{Row}_1(A) \ltimes (\alpha B + \beta C) \\ \vdots \\ \mathrm{Row}_m(A) \ltimes (\alpha B + \beta C) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Row}_1(A) \ltimes (\alpha B) + \mathrm{Row}_1(A) \ltimes (\beta C) \\ \vdots \\ \mathrm{Row}_m(A) \ltimes (\alpha B) + \mathrm{Row}_m(A) \ltimes (\beta C) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Row}_1(A) \ltimes (\alpha B) \\ \vdots \\ \mathrm{Row}_m(A) \ltimes (\alpha B) \end{bmatrix} + \begin{bmatrix} \mathrm{Row}_1(A) \ltimes (\beta C) \\ \vdots \\ \mathrm{Row}_m(A) \ltimes (\beta C) \end{bmatrix} \\
&= \alpha \begin{bmatrix} \mathrm{Row}_1(A) \ltimes (B) \\ \vdots \\ \mathrm{Row}_m(A) \ltimes (B) \end{bmatrix} + \beta \begin{bmatrix} \mathrm{Row}_1(A) \ltimes (C) \\ \vdots \\ \mathrm{Row}_m(A) \ltimes (C) \end{bmatrix} \\
&= \alpha (A \ltimes B) + \beta (A \ltimes C).
\end{aligned}
$$

Similarly, we obtain $(\alpha B + \beta C) \ltimes A = \alpha (B \ltimes A) + \beta (C \ltimes A)$. □

Definition 4.1. *(i) If X is a row vector or a column vector, we define for each $k \in \mathbb{N}$,*

$$X^{\ltimes k} = \underbrace{X \ltimes \cdots \ltimes X}_{k}. \tag{4.4}$$

(ii) Let $A \in M_{m,n}(L)$ and $n \mid m$ or $m \mid n$. The k^{th} semi-tensor power $A^{\ltimes k}$ is defined inductively for all positive integers k by

$$A^{\ltimes 1} = A \text{ and } A^{\ltimes (k+1)} = A^{\ltimes k} \ltimes A \text{ for } k = 1, 2, \ldots \tag{4.5}$$

Moreover, if $m = nt$, then $A^{\ltimes k} \in M_{t^k n,n}(L)$ and if $mt = n$, then $A^{\ltimes k} \in M_{m,t^k m}(L)$.

Example 4.3. Let $L = \langle \mathbb{Z}_4, +, \cdot, \overline{0}, \overline{1} \rangle$ be a commutative semiring with operations $\overline{a} + \overline{b} = \overline{a+b}$ and $\overline{a} \cdot \overline{b} = \overline{a \cdot b}$ for all $a, b \in \mathbb{Z}_4$. Consider $A = \begin{bmatrix} \overline{2} & \overline{0} \\ \overline{1} & \overline{3} \end{bmatrix}$. Find $A^{\ltimes 3}$. We have

$$A^{\ltimes 2} = \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{bmatrix}.$$

From Definition 4.1, we obtain

$$A^{\ltimes 3} = A^{\ltimes 2} \ltimes A = \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{3} & \overline{3} \end{bmatrix}.$$

Lemma 4.4.

(i) For $X \in M_{m,1}(L)$ and $Y \in M_{n,1}(L)$, we have $X \ltimes Y = X \otimes Y$.

(ii) For $W \in M_{1,m}(L)$ and $Z \in M_{1,n}(L)$, we have $W \ltimes Z = Z \otimes W$.

Proof. For the case $X = [x_1 \cdots x_m]^T$ and $Y = [y_1 \cdots y_n]^T$, we have

$$
\begin{aligned}
X \ltimes Y &= \begin{bmatrix} x_1 \ltimes [y_1 \cdots y_n]^T \\ \vdots \\ x_m \ltimes [y_1 \cdots y_n]^T \end{bmatrix} \\
&= \begin{bmatrix} x_1 y_1 \\ \vdots \\ x_1 y_n \\ \vdots \\ x_m y_1 \\ \vdots \\ x_m y_n \end{bmatrix} \\
&= X \otimes Y.
\end{aligned}
$$

For the case $W = [w_1 \cdots w_m]$ and $Z = [z_1 \cdots z_n]$, we have

$$
\begin{aligned}
W \ltimes Z &= [[w_1 \cdots w_m] \ltimes [z_1] \cdots [w_1 \cdots w_m] \ltimes [z_n]] \\
&= [z_1 w_1 \cdots z_1 w_m \cdots z_n w_1 \cdots z_n w_m] \\
&= Z \otimes W.
\end{aligned}
$$

□

Theorem 4.5. *(The mixed product property)*

(i) Let $X \in M_{n,1}(L)$, $Y \in M_{q,1}(L)$, $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. Then

$$(AX) \ltimes (BY) = (A \otimes B)(X \ltimes Y). \tag{4.6}$$

(ii) Let $W \in M_{1,m}(L)$, $Z \in M_{1,p}(L)$, $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. Then

$$(WA) \ltimes (ZB) = (W \ltimes Z)(B \otimes A). \tag{4.7}$$

Proof. Consider the case $X \in M_{n,1}(L)$, $Y \in M_{q,1}(L)$, $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. Let $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix}$ and $A = [a_{ij}]$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $B = [b_{ij}]$, $i = 1, \ldots, p$,

$j = 1, \ldots, q$. From Lemma 4.4, we have

$$
\begin{aligned}
(AX) \ltimes (BY) &= \begin{bmatrix} a_{11}x_1 + \ldots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \end{bmatrix} \ltimes \begin{bmatrix} b_{11}y_1 + \ldots + b_{1q}y_q \\ \vdots \\ b_{q1}y_1 + \ldots + b_{pq}y_q \end{bmatrix} \\
&= \begin{bmatrix} [a_{11}x_1 + \ldots + a_{1n}x_n]\,[b_{11}y_1 + \ldots + b_{1q}y_q] \\ \vdots \\ [a_{11}x_1 + \ldots + a_{1n}x_n]\,[b_{q1}y_1 + \ldots + b_{pq}y_q] \\ \vdots \\ [a_{m1}x_1 + \ldots + a_{mn}x_n]\,[b_{11}y_1 + \ldots + b_{1q}y_q] \\ \vdots \\ [a_{m1}x_1 + \ldots + a_{mn}x_n]\,[b_{q1}y_1 + \ldots + b_{pq}y_q] \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11}x_1y_1 + \ldots + a_{11}b_{1q}x_1y_q + \ldots + a_{1n}b_{11}x_ny_1 + \ldots + a_{1n}b_{1q}x_ny_q \\ \vdots \\ a_{11}b_{q1}x_1y_1 + \ldots + a_{11}b_{pq}x_1y_q + \ldots + a_{1n}b_{q1}x_ny_1 + \ldots + a_{1n}b_{pq}x_ny_q \\ \vdots \\ a_{m1}b_{11}x_1y_1 + \ldots + a_{m1}b_{1q}x_1y_q + \ldots + a_{mn}b_{11}x_ny_1 + \ldots + a_{mn}b_{1q}x_ny_q \\ \vdots \\ a_{m1}b_{q1}x_1y_1 + \ldots + a_{m1}b_{pq}x_1y_q + \ldots + a_{mn}b_{q1}x_ny_1 + \ldots + a_{mn}b_{pq}x_ny_q \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} & \cdots & a_{11}b_{1q} & \cdots & a_{1n}b_{11} & \cdots & a_{1n}b_{1q} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{11}b_{q1} & \cdots & a_{11}b_{pq} & \cdots & a_{1n}b_{q1} & \cdots & a_{1n}b_{pq} \\ & \vdots & & & & \vdots & \\ a_{m1}b_{11} & \cdots & a_{m1}b_{1q} & \cdots & a_{mn}b_{11} & \cdots & a_{mn}b_{1q} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1}b_{q1} & \cdots & a_{m1}b_{pq} & \cdots & a_{mn}b_{q1} & \cdots & a_{mn}b_{pq} \end{bmatrix} \begin{bmatrix} x_1y_1 \\ \vdots \\ x_1y_q \\ \vdots \\ x_ny_1 \\ \vdots \\ x_ny_q \end{bmatrix} \\
&= (A \otimes B)\,(X \ltimes Y).
\end{aligned}
$$

The proof of (ii) is similar to that of (i). □

According to Theorem 4.5, we have the next corollary.

Corollary 4.6. *(i) Let $X \in M_{n,1}(L)$ and $A \in M_{m,n}(L)$. Then*

$$(AX)^{\ltimes k} = \Big(\underbrace{A \otimes \cdots \otimes A}_{k}\Big) X^{\ltimes k}. \tag{4.8}$$

(ii) Let $W \in M_{1,m}(L)$ and $A \in M_{m,n}(L)$. Then

$$(WA)^{\ltimes k} = W^{\ltimes k}\left(\underbrace{A \otimes \cdots \otimes A}_{k}\right). \tag{4.9}$$

Theorem 4.7. *Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$.*

(i) If $n = tp$, then $A \ltimes B = A(B \otimes I_t)$.

(ii) If $nt = p$, then $A \ltimes B = (A \otimes I_t)B$.

Proof. To prove (i), write

$$A = \begin{bmatrix} \mathrm{Row}_1(A) \\ \vdots \\ \mathrm{Row}_m(A) \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathrm{Col}_1(B) & \cdots & \mathrm{Col}_q(B) \end{bmatrix}.$$

From Lemma 3.5, we obtain

$$\begin{aligned}
A \ltimes B &= \begin{bmatrix} \mathrm{Row}_1(A) \ltimes \mathrm{Col}_1(B) & \cdots & \mathrm{Row}_1(A) \ltimes \mathrm{Col}_q(B) \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A) \ltimes \mathrm{Col}_1(B) & \cdots & \mathrm{Row}_m(A) \ltimes \mathrm{Col}_q(B) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Row}_1(A)(\mathrm{Col}_1(B) \otimes I_t) & \cdots & \mathrm{Row}_1(A)(\mathrm{Col}_q(B) \otimes I_t) \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A)(\mathrm{Col}_1(B) \otimes I_t) & \cdots & \mathrm{Row}_m(A)(\mathrm{Col}_q(B) \otimes I_t) \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \mathrm{Row}_1(A) \\ \vdots \\ \mathrm{Row}_m(A) \end{bmatrix} (\mathrm{Col}_1(B) \otimes I_t) & \cdots & \begin{bmatrix} \mathrm{Row}_1(A) \\ \vdots \\ \mathrm{Row}_m(A) \end{bmatrix} (\mathrm{Col}_q(B) \otimes I_t) \end{bmatrix} \\
&= \begin{bmatrix} A(\mathrm{Col}_1(B) \otimes I_t) & \cdots & A(\mathrm{Col}_q(B) \otimes I_t) \end{bmatrix} \\
&= \begin{bmatrix} A(\mathrm{Col}_1(B \otimes I_t)) & \cdots & A(\mathrm{Col}_q(B \otimes I_t)) \end{bmatrix} \\
&= A \begin{bmatrix} \mathrm{Col}_1(B \otimes I_t) & \cdots & \mathrm{Col}_q(B \otimes I_t) \end{bmatrix} \\
&= A(B \otimes I_t).
\end{aligned}$$

Similarly, we can prove (ii). ☐

Corollary 4.8. *Assume that $A \in M_m(L)$ and $B \in M_n(L)$ with proper dimensions such that $A \ltimes B$ and $B \ltimes A$ are well-defined. Then*

$$\mathrm{tr}(A \ltimes B) = \mathrm{tr}(B \ltimes A). \tag{4.10}$$

Proof. For the case $A \succ_t B$, we have

$$\begin{aligned}\operatorname{tr}(A \ltimes B) &= \operatorname{tr}(A(B \otimes I_t)) \\ &= \operatorname{tr}((B \otimes I_t)A) \\ &= \operatorname{tr}(B \ltimes A)..\end{aligned}$$

For the case $A \prec_t B$, we obtain

$$\begin{aligned}\operatorname{tr}(A \ltimes B) &= \operatorname{tr}((A \otimes I_t)B) \\ &= \operatorname{tr}(B(A \otimes I_t)) \\ &= \operatorname{tr}(B \ltimes A).\end{aligned}$$

□

Corollary 4.9. *Assume that $A \in M_m(L)$, $B \in M_n(L)$ and both $A \ltimes B$ and $B \ltimes A$ are well-defined.*

(i) If $A \succ_t B$ and A^{-1} exists, then $A \ltimes B \sim B \ltimes A$.

(ii) If $A \prec_t B$ and B^{-1} exists, then $A \ltimes B \sim B \ltimes A$.

Proof. First, consider the case $A \succ_t B$ and A^{-1} exists. By Theorem 4.7, we get

$$\begin{aligned}A \ltimes B &= A(B \otimes I_t) \\ &= A(B \otimes I_t)AA^{-1} \\ &= A(B \ltimes A)A^{-1}.\end{aligned}$$

It means that $A \ltimes B \sim B \ltimes A$.
Next, consider the case $A \prec_t B$ and B^{-1} exists. By Theorem 4.7, we have

$$\begin{aligned}A \ltimes B &= (A \otimes I_t)B \\ &= B^{-1}B(A \otimes I_t)B \\ &= B^{-1}(B \ltimes A)B.\end{aligned}$$

That is, $A \ltimes B \sim B \ltimes A$. □

Corollary 4.10. *The semi-tensor product of a matrix with an identity matrix has the following properties.*

(i) If $A \in M_{m,pn}(L)$, then $A \ltimes I_n = A$.

(ii) If $A \in M_{m,n}(L)$, then $A \ltimes I_{pn} = A \otimes I_p$.

(iii) If $A \in M_{pm,n}(L)$, then $I_p \ltimes A = A$.

(iv) If $A \in M_{m,n}(L)$, then $I_{pm} \ltimes A = A \otimes I_p$.

Proof. All assertions follow from Theorem 4.7.

(i) Since $A \succ_p I$, we have

$$A \ltimes I_n = A(I_n \otimes I_p) = A(I_{np}) = A.$$

(ii) Since $A \prec_p I$, we get

$$A \ltimes I_{pn} = (A \otimes I_p) I_{pn} = A \otimes I_p.$$

(iii) Since $I \prec_m A$, we obtain

$$I_p \ltimes A = (I_p \otimes I_m) A = (I_{pm}) A = A.$$

(iv) Since $I \succ_p A$, we have

$$I_{pm} \ltimes A = I_{pm}(A \otimes I_p) = A \otimes I_p.$$

□

Chapter 5

Structure properties of the semi-tensor product

In this chapter that the semi-tensor product preserves invertibility and orthogonality of matrices. Moreover, special kind of matrices, namely, triangular and diagonal matrices, are preserved under this matrix product.

Theorem 5.1. *If $A \in M_m(L)$ and $B \in M_n(L)$ are invertible where $n \mid m$ or $m \mid n$, then $A \ltimes B$ is invertible with*

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \tag{5.1}$$

Proof. For the case $A \succ_t B$, we have

$$\begin{aligned}(A \ltimes B)^{-1} &= (A(B \otimes I_t))^{-1} \\ &= (B \otimes I_t)^{-1} A^{-1} \\ &= B^{-1} \ltimes A^{-1}.\end{aligned}$$

For the case $A \prec_t B$, it follows that

$$\begin{aligned}(A \ltimes B)^{-1} &= ((A \otimes I_t) B)^{-1} \\ &= B^{-1} (A \otimes I_t)^{-1} \\ &= B^{-1} \ltimes A^{-1}.\end{aligned}$$

□

Definition 5.1. *A matrix $A \in M_n(L)$ is said to be orthogonal if $A^T A = I_n$.*

Corollary 5.2. *Let $A \in M_m(L)$ and $B \in M_n(L)$ be such that $n \mid m$ or $m \mid n$. If A and B are orthogonal, then so is $A \ltimes B$.*

Proof. The orthogonality of A and B means that $A^T A = I_m$ and $B^T B = I_n$. Consider the case $A \succ_t B$. From Theorems 4.7 and 2.12, we have

$$\begin{aligned}(A \ltimes B)^T (A \ltimes B) &= \left(B^T \ltimes A^T\right)(A \ltimes B) \\ &= \left(B^T \otimes I_t\right) A^T A (B \otimes I_t) \\ &= \left(B^T B \otimes I_t I_t\right) \\ &= I_n \otimes I_t \\ &= I_{nt}.\end{aligned}$$

It means that $A \ltimes B$ is orthogonal. For the case $A \prec_t B$, we get

$$\begin{aligned}(A \ltimes B)^T (A \ltimes B) &= \left(B^T \ltimes A^T\right)(A \ltimes B) \\ &= B^T \left(A^T \otimes I_t\right)(A \otimes I_t) B \\ &= B^T \left(A^T A \otimes I_t I_t\right) B \\ &= B^T (I_{mt}) B \\ &= B^T B \\ &= I_{mt}.\end{aligned}$$

Therefore $A \ltimes B$ is orthogonal. □

Corollary 5.3. *Let $A \in M_m(L)$ and $B \in M_n(L)$ be such that $n \mid m$ or $m \mid n$. If A and B are upper triangular, then so is $A \ltimes B$.*

Proof. Assume that A and B are upper triangular. For the case $A \succ_t B$, we obtain

$$A \ltimes B = A(B \otimes I_t).$$

Since A and $B \otimes I_t$ are upper triangular, it follows that $A \ltimes B$ is upper triangular. For the case $A \prec_t B$, we get

$$A \ltimes B = (A \otimes I_t)B.$$

Since $A \otimes I_t$ and B are upper triangular, so is $A \ltimes B$. □

Corollary 5.4. *Let $A \in M_m(L)$ and $B \in M_n(L)$ be such that $n \mid m$ or $m \mid n$. If A and B are lower triangular, then so is $A \ltimes B$.*

Proof. It follows from Corollary 5.3 and the fact that the transpose of a upper triangular matrix is a lower one. □

Corollary 5.5. *Let $A \in M_m(L)$ and $B \in M_n(L)$ be such that $n \mid m$ or $m \mid n$. If A and B are diagonal, then so is $A \ltimes B$.*

Proof. From Corollary 5.3 and 5.4 , we obtain $A \ltimes B$ is diagonal. □

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Appendix

Appendix A

The research paper

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Department of Mathematics , Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Thailand

Type: Oral

Semi-tensor product of matrices over a commutative semiring

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Abstract

We introduce the notion of semi-tensor product of matrices over an arbitrary commutative semiring and investigate its algebraic properties. This matrix product is a generalization of usual matrix product for the case when the two factor matrices do not meet the dimension-matching condition. We show that the semi-tensor product is associative and compatible with other matrix operations. It turns out that some attractive properties of matrices are preserved under semi-tensor products.

Mathematics Subject Classification: 15A69, 15B33, 16Y60

Keywords: commutative semiring, semi-tensor product, Kronecker product

1 Introduction

Motivated from scientific computing, a rectangular matrix is a two-dimensional array for stacking data. To produce a new data, we can use a variety of matrix products. Each matrix product is suitable for particular problems occurred in science, engineering, economics, etc.

Consider two real/complex matrices A and B of dimension $m \times n$ and $p \times q$, respectively. In the case $n = p$, the matrices A and B are said to satisfy matching dimension condition. As n is a factor of p or p is a factor of n, they are said to satisfy factor dimension condition. From Linear Algebra, it is well known that as A and B have matching dimension, the conventional matrix product AB is well defined. In the literature, there is a kind of matrix product that generalizes the conventional matrix product in such the way that the two factor matrices do not meet the dimension matching condition, namely, the semi-tensor product. The theory of semi-tensor products for real/complex matrices has been developed by many authors; see e.g. [3, 4, 7, 28]. The concept of semi-tensor product can be applied widely in mathematical logic (see e.g. [4, 5, 9]), differential geometry ([31]), abstract algebra ([13]), dynamic systems ([6, 12, 14]), analysis and control of Boolean networks ([8, 28]).

In the viewpoint of scientific computing, the theory of matrices whose entries come from a suitable algebraic structure such as a ring or a (commutative) semiring are practically useful. Such theory was investigated by many mathematicians; see e.g. ([2, 20, 21, 29]). Applications of this theory go to the areas of operation research, optimization theory, automatic control, graph and network theory (see e.g. [14, 15, 17, 18, 19, 26, 32]).

In this paper, we introduce the notion of semi-tensor product of matrices over a commutative semiring. We investigate properties of semi-tensor product related to other matrix operations such as the addition, the scalar multiplication, the usual matrix multiplication, the transposition and traces. Our results extend the results known so far for real or complex matrices in the literature.

2 Matrices over a commutative semiring

Definition 2.1. (Zimmermann [30] and Golan [15]). A commutative semiring $L = \langle L, +, \cdot, 0, 1\rangle$ is an algebraic structure with the following properties:

* *Corresponding author*

(i) $(L, +, 0)$ is a commutative monoid,

(ii) $(L, \cdot, 1)$ is a monoid,

(iii) $a \cdot b = b \cdot a$ for all $a, b \in L$,

(iv) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in L$,

(v) $0 \cdot a = a \cdot 0 = 0$ for all $a \in L$,

(vi) $0 \neq 1$.

Example 2.2. The following structures are commutative semirings.

(i) Every field with its operations.

(ii) The nonnegative real numbers with the usual operations of addition and multiplication.

(iii) The nonnegative integers under the operations $a + b = \text{g.c.d}\,\{a, b\}$ and $a \cdot b = \text{l.c.m}\,\{a, b\}$, where g.c.d stands for the greatest common divisor and l.c.m stands for the smallest common multiple between a and b.

(iv) The fuzzy algebra $[0, 1]$ (see e.g. [16]) under the operations $a + b = \sup\{a, b\}$ and $a \cdot b = \inf\{a, b\}$.

(v) $[0, \infty]$ under the operations $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$.

(vi) The integers modulo n $(\mathbb{Z}_n)$ with its usual operations.

(vii) The max-plus algebra (or schedule algebra) $\mathbb{R} \cup \{\infty\}$ under the operations $a \vee b = \max\{a, b\}$ and $a \cdot b = a + b$ (see e.g. [11, 22, 24]).

Example 2.3. An MV-algebra (see e.g. [1, 25]) is an algebraic structure of the type $L = \langle L, \oplus, \odot, \neg, 0, 1\rangle$ satisfying the following axioms hold for all $x, y, z \in L$

(i) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,

(ii) $x \oplus y = y \oplus x$,

(iii) $x \oplus 0 = x$,

(iv) $\neg\neg x = x$,

(vi) $\neg 0 = 1$,

(vii) $x \odot y = \neg(\neg x \oplus \neg y)$,

(viii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

We put $x \vee y = (x \odot \neg y) \oplus y$ and $x \wedge y = (x \oplus \neg y) \odot y$ for $x, y \in L$. Then $(L, \vee, \odot, 0, 1)$ and $(L, \wedge, \oplus, 1, 0)$ are commutative semirings.

We denote $M_n(L) = M_{n,n}(L)$. Let $A = [a_{ij}], B = [b_{ij}] \in M_{m,n}(L), C = [c_{ij}] \in M_{n,p}(L)$ and $\alpha \in L$. Define the addition and the scalar multiplication as follows

$$A + B = [a_{ij} + b_{ij}] \in M_{m,n}(L),$$
$$\alpha A = [\alpha a_{ij}] \in M_{m,n}(L).$$

We define the multiplication of A and C by

$$AC = \left[\sum_{k=1}^{n} a_{ik} c_{kj}\right] \in M_{m,p}(L).$$

Denote the zero matrix $0 = [0] \in M_n(L)$ and identity matrix $I_n = [\delta_{ij}] \in M_n(L)$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

The transpose of a matrix $A = [a_{ij}] \in M_{m,n}(L)$ is defined to be the matrix $A^T = [a_{ji}] \in M_{n,m}(L)$. It follows that the operations of addition, scalar multiplication, multiplication and transposition satisfy usual properties for those of real/complex matrices.

A matrix $A \in M_n(L)$ is said to be invertible if there is a matrix $B \in M_n(L)$ such that $AB = I_n = BA$, equivalently, $AB = I_n$ or $BA = I_n$ (see). A matrix $A \in M_n(L)$ is similar to $B \in M_n(L)$ if and only if there is a invertible matrix S such that $S^{-1}AS = B$.

The trace of a matrix $A = [a_{ij}] \in M_n(L)$ is defined by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

It holds that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any $A, B \in M_n(L)$.

Definition 2.4. (see [23]) Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. The Kronecker product of A and B is defined to be

$$A \otimes B = [a_{ij}B]_{ij} \in M_{mp,nq}(L).$$

That is, each $(i,j)th$ block of $A \otimes B$ is given by $a_{ij}B$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Theorem 2.5. *(see [23]) (The mixed product property) Let $A \in M_{m,n}(L)$, $B \in M_{p,q}(L)$, $C \in M_{n,k}(L)$ and $D \in M_{q,r}(L)$. Then*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

3 Semi-tensor products of matrices over a commutative semiring

In this section, we introduce the semi-tensor products of matrices and show that this product is associative. To do this job, first we define the semi-tensor product between row vectors and column vectors. Then we extend it to the product between row vectors and matrices. Finally, we can define the semi-tensor products for matrices by partitioning the first factor matrix into row-block matrix.

3.1 Semi-tensor products between row vectors and column vectors

Definition 3.1. Let $X \in M_{1,m}(L)$ and $Y \in M_{p,1}(L)$.

(i) Consider the case $m = tp$, denoted by $X \succ_t Y$. We split X into p equal-size blocks as $(X^1, X^2, \ldots, X^p)$, such that $X^i \in M_{1,t}(L)$ for all $i = 1, \ldots, p$. Define the left Semi-tensor product (STP), denote by $\ltimes$, as

$$X \ltimes Y = \sum_{i=1}^{p} y_i X^i \in M_{1,t}(L). \tag{3.1}$$

(ii) Consider the case $mt = p$, denoted by $X \prec_t Y$. We split Y into m equal-size blocks as $(Y^1, Y^2, \ldots, Y^m)$, such that $Y^i \in M_{t,1}(L)$ for all $i = 1, \ldots, m$. Define the left STP, denote by $\ltimes$, as

$$X \ltimes Y = \sum_{i=1}^{m} x_i Y^i \in M_{t,1}(L). \tag{3.2}$$

Lemma 3.2. *Let $X \in M_{1,m}(L)$ and $Y \in M_{p,1}(L)$ be such that $p \mid m$ or $p \mid m$. Then*

$$(X \ltimes Y)^T = Y^T \ltimes X^T. \tag{3.3}$$

Proof. For the case $X \succ_t Y$, we have $(X \ltimes Y)^T = \sum_{i=1}^{p} y_i X^{i^T} = Y^T \ltimes X^T$. For the case $X \prec_t Y$, we get $(X \ltimes Y)^T = \sum_{i=1}^{m} x_i Y^{i^T} = Y^T \ltimes X^T$. □

Lemma 3.3. *(i) Let $X \in M_{1,m}(L)$, $Y, Z \in M_{p,1}(L)$ and $\alpha, \beta \in L$ where $p \mid m$ or $m \mid p$. Then*

$$X \ltimes (\alpha Y + \beta Z) = \alpha (X \ltimes Y) + \beta (X \ltimes Z). \tag{3.4}$$

(ii) Let $X, Y \in M_{1,m}(L)$, $Z \in M_{p,1}(L)$ and $\alpha, \beta \in L$ where $p \mid m$ or $m \mid p$. Then

$$(\alpha X + \beta Y) \ltimes Z = \alpha (X \ltimes Z) + \beta (Y \ltimes Z). \tag{3.5}$$

Proof. Consider $X \in M_{1,m}(L)$, $Y, Z \in M_{p,1}(L)$ and $\alpha, \beta \in L$. For the case $p \mid m$, we obtain

$$\begin{aligned} X \ltimes (\alpha Y + \beta Z) &= \sum_{i=1}^{p} (\alpha y_i + \beta z_i) X^i \\ &= \sum_{i=1}^{p} \left(\alpha y_i X^i + \beta z_i X^i\right) \\ &= \alpha \sum_{i=1}^{p} y_i X^i + \beta \sum_{i=1}^{p} z_i X^i \\ &= \alpha (X \ltimes Y) + \beta (X \ltimes Z). \end{aligned}$$

For the case $m \mid p$, we have

$$\begin{aligned} X \ltimes (\alpha Y + \beta Z) &= \sum_{i=1}^{m} x_i (\alpha Y_i + \beta Z_i) \\ &= \sum_{i=1}^{m} (\alpha x_i Y_i + \beta x_i Z_i) \\ &= \alpha \sum_{i=1}^{m} x_i Y^i + \beta \sum_{i=1}^{m} x_i Z^i \\ &= \alpha (X \ltimes Y) + \beta (X \ltimes Z). \end{aligned}$$

□

Similarly, we can prove the assertion (ii).

Lemma 3.4. *Let $X \in M_{1,n}(L)$ and $Y \in M_{p,1}(L)$. Then*

(i) If $n = tp$, then $X \ltimes Y = X(Y \otimes I_t)$.

(ii) If $nt = p$, then $X \ltimes Y = (X \otimes I_t)Y$.

Proof. For the case $X \succ_t Y$, we write $X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \end{bmatrix}$ and $Y = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{p1} \end{bmatrix}$. We have

$$
\begin{aligned}
X \ltimes Y
&= \begin{bmatrix} x_{11} \cdots x_{1t} \,\vdots\, x_{1,t+1} \cdots x_{1,2t} \,\vdots \cdots \vdots\, x_{1,(p-1)t+1} \cdots x_{pt} \end{bmatrix} \ltimes \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{p1} \end{bmatrix} \\
&= y_{11} \begin{bmatrix} x_{11} \cdots x_{1t} \end{bmatrix} + y_{21} \begin{bmatrix} x_{1,t+1} \cdots x_{1,2t} \end{bmatrix} + \ldots + y_{p1} \begin{bmatrix} x_{1,(p-1)t+1} \cdots x_{pt} \end{bmatrix} \\
&= \begin{bmatrix} y_{11}x_{11} + y_{21}x_{1,t+1} + \ldots + y_{p1}x_{1,(p-1)t+1} \cdots y_{11}x_{1t} + y_{21}x_{1,2t} + \ldots + y_{p1}x_{pt} \end{bmatrix} \\
&= \begin{bmatrix} x_{11} \cdots x_{1t} \;\; x_{1,t+1} \cdots x_{1,2t} \cdots x_{1,(p-1)t+1} \cdots x_{pt} \end{bmatrix} \ltimes
\begin{bmatrix}
y_{11} & 0 & \cdots & 0 \\
0 & y_{11} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & y_{11} \\
y_{21} & 0 & \cdots & 0 \\
0 & y_{21} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & y_{21} \\
\vdots & \vdots & \vdots & \vdots \\
y_{p1} & 0 & \cdots & 0 \\
0 & y_{p1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & y_{p1}
\end{bmatrix} \\
&= X \left(\begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{p1} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right) \\
&= X (Y \otimes I_t).
\end{aligned}
$$

□

Similarly, we can prove the assertion (ii).

3.2 Semi-tensor products between row vectors and matrices

We use $\mathrm{Row}_i(A)$ for the ith row of A and $\mathrm{Col}_i(A)$ for the ith column of A.

Definition 3.5. Let $X \in M_{1,m}(L)$ and $A \in M_{p,q}(L)$ where $p \mid m$ or $m \mid p$. We define

$$X \ltimes A = [X \ltimes \mathrm{Col}_1(A) \cdots X \ltimes \mathrm{Col}_q(A)]. \tag{3.6}$$

Lemma 3.6. *Let $X, Y \in M_{1,m}(L)$ and $A, B \in M_{p,q}(L)$ and $\alpha, \beta \in L$ where $p \mid m$ or $m \mid p$. Then*

$$X \ltimes (\alpha A + \beta B) = \alpha (X \ltimes A) + \beta (X \ltimes B), \tag{3.7}$$

$$(\alpha X + \beta Y) \ltimes A = \alpha (X \ltimes A) + \beta (Y \ltimes A). \tag{3.8}$$

Proof. From Definition 3.5 and Lemma 3.3, we have

$$
\begin{aligned}
&X \ltimes (\alpha A + \beta B) \\
&= X \ltimes [\alpha \operatorname{Col}_1(A) + \beta \operatorname{Col}_1(B) \cdots \alpha \operatorname{Col}_q(A) + \beta \operatorname{Col}_q(B)] \\
&= [X \ltimes (\alpha \operatorname{Col}_1(A) + \beta \operatorname{Col}_1(B)) \cdots X \ltimes (\alpha \operatorname{Col}_q(A) + \beta \operatorname{Col}_q(B))] \\
&= [\alpha (X \ltimes \operatorname{Col}_1(A)) + \beta (X \ltimes \operatorname{Col}_1(B)) \cdots \alpha (X \ltimes \operatorname{Col}_q(A)) + \beta (X \ltimes \operatorname{Col}_q(B))] \\
&= [\alpha (X \ltimes \operatorname{Col}_1(A)) \cdots \alpha (X \ltimes \operatorname{Col}_q(A))] + [\beta (X \ltimes \operatorname{Col}_1(B)) \cdots \beta (X \ltimes \operatorname{Col}_q(B))] \\
&= \alpha [X \ltimes \operatorname{Col}_1(A) \cdots X \ltimes \operatorname{Col}_q(A)] + \beta [X \ltimes \operatorname{Col}_1(B) \cdots X \ltimes \operatorname{Col}_q(B)] \\
&= \alpha (X \ltimes A) + \beta (X \ltimes B).
\end{aligned}
$$

Similarly, we obtain $(\alpha X + \beta Y) \ltimes A = \alpha (X \ltimes A) + \beta (Y \ltimes A)$. □

3.3 Semi-tensor products of matrices

Definition 3.7. Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$ where $p \mid n$ or $n \mid p$. The left semi-tensor product of A and B is defined as

$$
A \ltimes B = \begin{bmatrix} \operatorname{Row}_1(A) \ltimes B \\ \vdots \\ \operatorname{Row}_m(A) \ltimes B \end{bmatrix} \tag{3.9}
$$

which is an $m \times q$ block matrix.

Remark 3.8. (i) If $n = p$, then $A \ltimes B = AB$.

(ii) If $n = tp$, then $A \ltimes B$ is an $m \times tq$ matrix.

(iii) If $nt = p$, then $A \ltimes B$ is an $mt \times q$ matrix.

Proposition 3.9. *Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$ where $p \mid n$ or $n \mid p$. Then*

$$
A \ltimes B = [A \ltimes \operatorname{Col}_1(B) \cdots A \ltimes \operatorname{Col}_q(B)]. \tag{3.10}
$$

Proof. From Definitions 3.7 and 3.5, we have

$$
\begin{aligned}
A \ltimes B &= \begin{bmatrix} \operatorname{Row}_1(A) \ltimes \operatorname{Col}_1(B) & \cdots & \operatorname{Row}_1(A) \ltimes \operatorname{Col}_q(B) \\ \vdots & \ddots & \vdots \\ \operatorname{Row}_m(A) \ltimes \operatorname{Col}_1(B) & \cdots & \operatorname{Row}_m(A) \ltimes \operatorname{Col}_q(B) \end{bmatrix} \\
&= [A \ltimes \operatorname{Col}_1(B) \cdots A \ltimes \operatorname{Col}_q(B)].
\end{aligned}
$$

□

Theorem 3.10. *(Associative rule) If the following matrix operations are well-defined, then*

$$
(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C). \tag{3.11}
$$

Proof. Consider the dimension of A, B and C.

Case 1: $A \succ B$ and $B \succ C$. Assume $A \in M_{m,np}(L)$, $B \in M_{p,qr}(L)$ and $C \in M_{r,s}(L)$.
Case 2: $A \prec B$ and $B \prec C$. Assume $A \in M_{m,n}(L)$, $B \in M_{np,q}(L)$ and $C \in M_{rq,s}(L)$.
Case 3: $A \prec B$ and $B \succ C$. Assume $A \in M_{m,n}(L)$, $B \in M_{np,qr}(L)$ and $C \in M_{r,s}(L)$.
Case 4: $A \succ B$ and $B \prec C$. Assume $A \in M_{m,np}(L)$, $B \in M_{p,q}(L)$ and $C \in M_{rq,s}(L)$.
Now the dimension of $A \ltimes B$ is $m \times nq$. To make it feasible for $(A \ltimes B) \ltimes C$, we need
Case 4.1: $(A \ltimes B) \succ C$, i.e., $n = n'r$.
Case 4.2: $(A \ltimes B) \prec C$, i.e., $r = nr'$.
The dimension of $B \ltimes C$ is $pr \times s$. To make it feasible for $(A \ltimes B) \ltimes C$, we need
Case 4.3: $A \succ (B \ltimes C)$, i.e., $n = n'r$.
Case 4.4: $A \prec (B \ltimes C)$, i.e., $r = nr'$.

Next, we prove the associativity. We have to prove it case by case. But Case 1-3 are similar, we prove only Case 1.

Let $A \in M_{m,np}(L)$, $B \in M_{p,qr}(L)$ and $C \in M_{r,s}(L)$. Assume $m = 1$ and $s = 1$. Split A as $A = [A_1, \ldots, A_p]$, where A_i, $i = 1, \ldots, p$ are $1 \times n$ blocks. Then

LHS:

$$
\begin{aligned}
A \ltimes B &= [A_1, \ldots, A_p] \ltimes \begin{bmatrix} b_{11}^1 & \cdots & b_{1q}^1 & \cdots & b_{r1}^1 & \cdots & b_{rq}^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \\ b_{11}^p & \cdots & b_{1q}^p & \cdots & b_{r1}^p & \cdots & b_{rq}^p \end{bmatrix} \\
&= \left[\sum_{i=1}^{p} A_i b_{11}^i, \cdots, \sum_{i=1}^{p} A_i b_{1q}^i, \cdots, \sum_{i=1}^{p} A_i b_{r1}^i, \cdots, \sum_{i=1}^{p} A_i b_{rq}^i \right].
\end{aligned}
$$

Then we obtain

$$
(A \ltimes B) \ltimes C = (A \ltimes B) \ltimes \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} = \left[\sum_{j=1}^{r} \sum_{i=1}^{p} A_i b_{j1}^i c_j, \cdots, \sum_{j=1}^{r} \sum_{i=1}^{p} A_i b_{jq}^i c_j \right].
$$

RHS:

$$
\begin{aligned}
&\begin{bmatrix} b_{11}^1 & \cdots & b_{1q}^1 & \cdots & b_{r1}^1 & \cdots & b_{rq}^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \\ b_{11}^p & \cdots & b_{1q}^p & \cdots & b_{r1}^p & \cdots & b_{rq}^p \end{bmatrix} \ltimes \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^{r} b_{j1}^1 c_j & \cdots & \sum_{j=1}^{r} b_{jq}^1 c_j \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{r} b_{j1}^p c_j & \cdots & \sum_{j=1}^{r} b_{jq}^p c_j \end{bmatrix}.
\end{aligned}
$$

Then

$$
\begin{aligned}
A \ltimes (B \ltimes C) &= (A_1, \ldots, A_p) \ltimes (B \ltimes C) \\
&= \left[\sum_{j=1}^{r} \sum_{i=1}^{p} A_i b_{j1}^i c_j, \cdots, \sum_{j=1}^{r} \sum_{i=1}^{p} A_i b_{jq}^i c_j \right].
\end{aligned}
$$

We have LHS = RHS.

Since Cases 4.1-4.4 are similar, we prove Case 4.1 only.

Let $A \in M_{m,npr}(L)$, $B \in M_{p,q}(L)$ and $C \in M_{rq,s}(L)$ be given. We also assume $m = 1$ and $s = 1$. Split A as $A = [A_{11}, \cdots, A_{1r}, \cdots, A_{p1}, \cdots, A_{pr}]$, where each A_{ij}, $i = 1, \ldots, p$, $j = 1, \ldots, r$ are $1 \times n$ blocks.

$$
B = \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix}, \quad C = [c_{11}, \cdots, c_{1r}, \cdots, c_{q1}, \cdots, c_{qr}]^T.
$$

A direct computation shows that

$$
(A \ltimes B) \ltimes C = \sum_{i=1}^{p} \sum_{j=1}^{r} \sum_{k=1}^{q} A_{ij} b_{ik} c_{kj} = A \ltimes (B \ltimes C).
$$

□

4 The compatibility of Semi-tensor products and other matrix operations

We shall show that the semi-tensor product of matrices is compatible with familiar matrix operations, such as, addition, scalar multiplication, transposition and Kronecker multiplication. It follows that the semi-tensor product preserves invertibility and orthogonality of matrices. Moreover, special kind of matrices, namely, triangular and diagonal matrices, are preserved under this matrix product.

Theorem 4.1. *Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$ where $p \mid n$ or $n \mid p$. Then*

$$(A \ltimes B)^T = B^T \ltimes A^T. \tag{4.1}$$

Proof. Using Lemma 3.2, we have

$$\begin{aligned}
(A \ltimes B)^T &= \begin{bmatrix} [\mathrm{Row}_1(A) \ltimes \mathrm{Col}_1(B)]^T & \cdots & [\mathrm{Row}_m(A) \ltimes \mathrm{Col}_1(B)]^T \\ \vdots & \ddots & \vdots \\ [\mathrm{Row}_1(A) \ltimes \mathrm{Col}_q(B)]^T & \cdots & [\mathrm{Row}_m(A) \ltimes \mathrm{Col}_q(B)]^T \end{bmatrix} \\
&= \begin{bmatrix} [\mathrm{Col}_1(B)]^T \ltimes [\mathrm{Row}_1(A)]^T & \cdots & [\mathrm{Col}_1(B)]^T \ltimes [\mathrm{Row}_m(A)]^T \\ \vdots & \ddots & \vdots \\ [\mathrm{Col}_q(B)]^T \ltimes [\mathrm{Row}_1(A)]^T & \cdots & [\mathrm{Col}_q(B)]^T \ltimes [\mathrm{Row}_m(A)]^T \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Row}_1(B^T) \ltimes \mathrm{Col}_1(A^T) & \cdots & \mathrm{Row}_1(B^T) \ltimes \mathrm{Col}_m(A^T) \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_q(B^T) \ltimes \mathrm{Col}_1(A^T) & \cdots & \mathrm{Row}_q(B^T) \ltimes \mathrm{Col}_m(A^T) \end{bmatrix} \\
&= B^T \ltimes A^T.
\end{aligned}$$

□

The next theorem shows that the semi-tensor product is a bilinear map.

Theorem 4.2. *(Bilinearity) If the following matrix operations are well-defined, then*

$$A \ltimes (\alpha B + \beta C) = \alpha (A \ltimes B) + \beta (A \ltimes C), \tag{4.2}$$

$$(\alpha B + \beta C) \ltimes A = \alpha (B \ltimes A) + \beta (C \ltimes A). \tag{4.3}$$

Proof. From Definition 3.7 and Lemma 3.6, we have

$$\begin{aligned}
A \ltimes (\alpha B + \beta C) &= \begin{bmatrix} \mathrm{Row}_1(A) \ltimes (\alpha B + \beta C) \\ \vdots \\ \mathrm{Row}_m(A) \ltimes (\alpha B + \beta C) \end{bmatrix} \\
&= \begin{bmatrix} \alpha(\mathrm{Row}_1(A) \ltimes B) + \beta(\mathrm{Row}_1(A) \ltimes C) \\ \vdots \\ \alpha(\mathrm{Row}_m(A) \ltimes B) + \beta(\mathrm{Row}_m(A) \ltimes C) \end{bmatrix} \\
&= \alpha(A \ltimes B) + \beta(A \ltimes C).
\end{aligned}$$

Similarly, we obtain $(\alpha B + \beta C) \ltimes A = \alpha (B \ltimes A) + \beta (C \ltimes A)$. □

Definition 4.3. (i) If X is a row vector or a column vector, we define for each $k \in \mathbb{N}$,

$$X^{\ltimes k} = \underbrace{X \ltimes \cdots \ltimes X}_{k}. \tag{4.4}$$

(ii) Let $A \in M_{m,n}(L)$ and $n \mid m$ or $m \mid n$. The k^{th} semi-tensor power $A^{\ltimes k}$ is defined inductively for all positives integers k by

$$A^{\ltimes 1} = A \text{ and } A^{\ltimes (k+1)} = A^{\ltimes k} \ltimes A \text{ for } k = 2, 3, \ldots \tag{4.5}$$

Moreover, if $m = nt$, then $A^{\ltimes k} \in M_{t^k n,n}(L)$ and if $mt = n$, then $A^{\ltimes k} \in M_{m,t^k m}(L)$

Lemma 4.4. *(i) For $X \in M_{m,1}(L)$ and $Y \in M_{n,1}(L)$, we have $X \ltimes Y = X \otimes Y$.*

(ii) For $W \in M_{1,m}(L)$ and $Z \in M_{1,n}(L)$, we have $W \ltimes Z = Z \otimes W$.

Proof. For the case $X = [x_1 \cdots x_m]^T$ and $Y = [y_1 \cdots y_n]^T$, we have

$$\begin{aligned} X \ltimes Y &= \left[x_1 \ltimes [y_1 \cdots y_n]^T \cdots x_m \ltimes [y_1 \cdots y_n]^T\right]^T \\ &= [x_1 y_1 \cdots x_1 y_n \cdots x_m y_1 \cdots x_m y_n]^T \\ &= X \otimes Y. \end{aligned}$$

For the case $W = [w_1 \cdots w_m]$ and $Z = [z_1 \cdots z_n]$, we have

$$\begin{aligned} W \ltimes Z &= [[w_1 \cdots w_m] \ltimes [z_1] \cdots [w_1 \cdots w_m] \ltimes [z_n]] \\ &= [z_1 w_1 \cdots z_1 w_m \cdots z_n w_1 \cdots z_n w_m] \\ &= Z \otimes W. \end{aligned}$$

□

Theorem 4.5. *(The mixed product property)*

(i) Let $X \in M_{n,1}(L)$, $Y \in M_{q,1}(L)$, $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. Then

$$(AX) \ltimes (BY) = (A \otimes B)(X \ltimes Y). \tag{4.6}$$

(ii) Let $W \in M_{1,m}(L)$, $Z \in M_{1,p}(L)$, $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. Then

$$(WA) \ltimes (ZB) = (W \ltimes Z)(B \otimes A). \tag{4.7}$$

Proof. For the case $X \in M_{n,1}(L)$, $Y \in M_{q,1}(L)$, $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$. Using Lemma 4.4.

we have

$$
\begin{aligned}
(AX) \ltimes (BY) &= \begin{bmatrix} a_{11}x_1 + \ldots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \end{bmatrix} \ltimes \begin{bmatrix} b_{11}y_1 + \ldots + b_{1q}y_q \\ \vdots \\ b_{q1}y_1 + \ldots + b_{pq}y_q \end{bmatrix} \\
&= \begin{bmatrix} [a_{11}x_1 + \ldots + a_{1n}x_n][b_{11}y_1 + \ldots + b_{1q}y_q] \\ \vdots \\ [a_{11}x_1 + \ldots + a_{1n}x_n][b_{q1}y_1 + \ldots + b_{pq}y_q] \\ \vdots \\ [a_{m1}x_1 + \ldots + a_{mn}x_n][b_{11}y_1 + \ldots + b_{1q}y_q] \\ \vdots \\ [a_{m1}x_1 + \ldots + a_{mn}x_n][b_{q1}y_1 + \ldots + b_{pq}y_q] \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11}x_1y_1 + \ldots + a_{11}b_{1q}x_1y_q + \ldots + a_{1n}b_{11}x_ny_1 + \ldots + a_{1n}b_{1q}x_ny_q \\ \vdots \\ a_{11}b_{q1}x_1y_1 + \ldots + a_{11}b_{pq}x_1y_q + \ldots + a_{1n}b_{q1}x_ny_1 + \ldots + a_{1n}b_{pq}x_ny_q \\ \vdots \\ a_{m1}b_{11}x_1y_1 + \ldots + a_{m1}b_{1q}x_1y_q + \ldots + a_{mn}b_{11}x_ny_1 + \ldots + a_{mn}b_{1q}x_ny_q \\ \vdots \\ a_{m1}b_{q1}x_1y_1 + \ldots + a_{m1}b_{pq}x_1y_q + \ldots + a_{mn}b_{q1}x_ny_1 + \ldots + a_{mn}b_{pq}x_ny_q \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} & \cdots & a_{11}b_{1q} & \cdots & a_{1n}b_{11} & \cdots & a_{1n}b_{1q} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{11}b_{q1} & \cdots & a_{11}b_{pq} & \cdots & a_{1n}b_{q1} & \cdots & a_{1n}b_{pq} \\ & \vdots & & & & \vdots & \\ a_{m1}b_{11} & \cdots & a_{m1}b_{1q} & \cdots & a_{mn}b_{11} & \cdots & a_{mn}b_{1q} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1}b_{q1} & \cdots & a_{m1}b_{pq} & \cdots & a_{mn}b_{q1} & \cdots & a_{mn}b_{pq} \end{bmatrix} \begin{bmatrix} x_1y_1 \\ \vdots \\ x_1y_q \\ \vdots \\ x_ny_1 \\ \vdots \\ x_ny_q \end{bmatrix} \\
&= (A \otimes B)(X \ltimes Y).
\end{aligned}
$$

The proof of (ii) is similar to that of (i). □

According to Theorem 4.5, we have the next corollary.

Corollary 4.6. *(i) Let $X \in M_{n,1}(L)$ and $A \in M_{m,n}(L)$. Then*

$$(AX)^{\ltimes k} = \left(\underbrace{A \otimes \cdots \otimes A}_{k} \right) X^{\ltimes k}. \tag{4.8}$$

(ii) Let $W \in M_{1,m}(L)$ and $A \in M_{m,n}(L)$. Then

$$(WA)^{\ltimes k} = W^{\ltimes k} \left(\underbrace{A \otimes \cdots \otimes A}_{k} \right). \tag{4.9}$$

Theorem 4.7. *Let $A \in M_{m,n}(L)$ and $B \in M_{p,q}(L)$.*

(i) If $n = tp$, then $A \ltimes B = A(B \otimes I_t)$.

(ii) If $nt = p$, then $A \ltimes B = (A \otimes I_t)B$.

Proof. To prove (i), write

$$A = \begin{bmatrix} \mathrm{Row}_1(A) \\ \vdots \\ \mathrm{Row}_m(A) \end{bmatrix} \text{ and } B = \begin{bmatrix} \mathrm{Col}_1(B) & \cdots & \mathrm{Col}_q(B) \end{bmatrix}.$$ Using Lemma 3.4, we obtain

$$\begin{aligned} A \ltimes B &= \begin{bmatrix} \mathrm{Row}_1(A) \ltimes \mathrm{Col}_1(B) & \cdots & \mathrm{Row}_1(A) \ltimes \mathrm{Col}_q(B) \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A) \ltimes \mathrm{Col}_1(B) & \cdots & \mathrm{Row}_m(A) \ltimes \mathrm{Col}_q(B) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Row}_1(A)(\mathrm{Col}_1(B) \otimes I_t) & \cdots & \mathrm{Row}_1(A)(\mathrm{Col}_q(B) \otimes I_t) \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A)(\mathrm{Col}_1(B) \otimes I_t) & \cdots & \mathrm{Row}_m(A)(\mathrm{Col}_q(B) \otimes I_t) \end{bmatrix} \\ &= \begin{bmatrix} A(\mathrm{Col}_1(B \otimes I_t)) & \cdots & A(\mathrm{Col}_q(B \otimes I_t)) \end{bmatrix} \\ &= A \begin{bmatrix} \mathrm{Col}_1(B \otimes I_t) & \cdots & \mathrm{Col}_q(B \otimes I_t) \end{bmatrix} \\ &= A(B \otimes I_t). \end{aligned}$$

□

Similarly, we can prove (ii).

Corollary 4.8. *Assume that $A \in M_m(L)$ and $B \in M_n(L)$ with proper dimensions such that $A \ltimes B$ and $B \ltimes A$ are well-defined. Then*

$$\mathrm{tr}(A \ltimes B) = \mathrm{tr}(A(B \otimes I_t)) = \mathrm{tr}((B \otimes I_t)A) = \mathrm{tr}(B \ltimes A).$$

□

For the case $A \prec_t B$, we obtain $\mathrm{tr}(A \ltimes B) = \mathrm{tr}((A \otimes I_t)B) = \mathrm{tr}(B(A \otimes I_t)) = \mathrm{tr}(B \ltimes A)$.

Corollary 4.9. *Assume that $A \in M_m(L)$, $B \in M_n(L)$ and both $A \ltimes B$ and $B \ltimes A$ are well-defined. If either A or B is invertible, then*

$$A \ltimes B \sim B \ltimes A \tag{4.11}$$

Proof. Using Theorem 4.7. For the case $A \succ_t B$, since A^{-1} exists, we get

$$A \ltimes B = A(B \otimes I_t) = A(B \otimes I_t)AA^{-1} = A(B \ltimes A)A^{-1}.$$

It means that $A \ltimes B \sim B \ltimes A$. For the case $A \prec_t B$, since B^{-1} exists, we have

$$A \ltimes B = (A \otimes I_t)B = B^{-1}B(A \otimes I_t)B = B^{-1}(B \ltimes A)B.$$

That is, $A \ltimes B \sim B \ltimes A$. □

Theorem 4.10. *If $A \in M_m(L)$ and $B \in M_n(L)$ are invertible where $n \mid m$ or $m \mid n$, then*

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \tag{4.12}$$

Proof. Using Theorem 4.7. For the case $A \succ_t B$, we have

$$(A \ltimes B)^{-1} = (A(B \otimes I_t))^{-1} = (B \otimes I_t)^{-1}A^{-1} = B^{-1} \ltimes A^{-1}.$$

For the case $A \prec_t B$, it follows that $(A \ltimes B)^{-1} = ((A \otimes I_t)B)^{-1} = B^{-1}(A \otimes I_t)^{-1} = B^{-1} \ltimes A^{-1}$. □

A matrix $A \in M_n(L)$ is said to be orthogonal if $A^T A = I$.

Corollary 4.11. *Let $A \in M_m(L)$ and $B \in M_n(L)$ be such that $n \mid m$ or $m \mid n$.*

(i) If A and B are orthogonal, then so is $A \ltimes B$.

(ii) If A and B are upper triangular, then so is $A \ltimes B$.

(iii) If A and B are lower triangular, then so is $A \ltimes B$.

(iv) If A and B are diagonal, then so is $A \ltimes B$.

Proof. All assertions can be proved using Theorems 4.7 and 2.5 to convert the semi-tensor product into conventional product and Kronecker product.

(i) The orthogonality of A and B means that $A^T A = I$ and $B^T B = I$. Using Theorem 4.7. Then for the case $A \succ_t B$, we have

$$(A \ltimes B)^T (A \ltimes B) = (B^T \ltimes A^T)(A \ltimes B) = (B^T \otimes I_t) A^T A (B \otimes I_t) = (B^T B \otimes I_t I_t) = I.$$

It means that $A \ltimes B$ is orthogonal. For the case $A \prec_t B$, we get

$$(A \ltimes B)^T (A \ltimes B) = (B^T \ltimes A^T)(A \ltimes B) = B^T (A^T \otimes I_t)(A \otimes I_t) B = B^T B = I.$$

Therefore $A \ltimes B$ is orthogonal.

(ii) Assume that A and B are upper triangular. For the case $A \succ_t B$, we obtain

$$A \ltimes B = A (B \otimes I_t).$$

Since A and $(B \otimes I_t)$ are upper triangular. It follows that $A \ltimes B$ is upper triangular. For the case $A \prec_t B$, we get $A \ltimes B = (A \otimes I_t) B$. Since $(A \otimes I_t)$ and B are upper triangular. So is $A \ltimes B$.

(iii) It follows from (ii) and the fact that the transpose of a upper triangular matrix is a lower one.

(iv) From (ii) and (iii), we obtain $A \ltimes B$ is diagonal. □

Corollary 4.12. *The semi-tensor product of a matrix with an identity matrix has the following properties.*

(i) If $A \in M_{m,pn}(L)$, then $A \ltimes I_n = A$.

(ii) If $A \in M_{m,n}(L)$, then $A \ltimes I_{pn} = A \otimes I_p$.

(iii) If $A \in M_{pm,n}(L)$, then $I_p \ltimes A = A$.

(iv) If $A \in M_{m,n}(L)$, then $I_{pm} \ltimes A = A \otimes I_p$. □

Proof. All assertions follow from Theorem 4.7.

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