THE NUMBER OF LABELLED TREES IN COMPLETE BIPARTITE GRAPH

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หัวข้อวิทยานิพนธ์	จำนวนของกราฟต้นไม้แบบติดป้ายในกราฟสองส่วนบริบูรณ์
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บทคัดย่อ

จำนวนของกราฟต้นไม้แบบติดป้ายที่มีจุดปลาย r_1, r_2 ในกราฟสองส่วนบริบูรณ์ $K_{m,n}$ แสดงโดย $L(m,n,r_1,r_2)$ ในวิทยานิพนธ์นี้เราใช้ฟังก์ชันก่อกำเนิดเลขชี้กำลังในการหาจำนวนของ $L(m,n,r_1,r_2)$ ซึ่งสามารถหาได้จากสูตร

$$L(m,n,r_1,r_2)=\binom{m}{r_1}\binom{n}{r_2}A_{r_1}A_{r_2}$$

โดยที่ $A_{r_1}=\sum_{i=0}^{m-r_1-1}(-1)^i\binom{m-r_1}{i}(m-r_1-i)^{n-1}$ และ $A_{r_2}=\sum_{j=0}^{n-r_2-1}(-1)^j\binom{n-r_2}{j}(n-r_2-j)^{m-1}$

นอกจากนั้นเรายังได้จำนวนของกราฟต้นไม้แบบติดป้ายที่มีจุดปลาย r_1, r_2 ในกราฟสองส่วนบริบูรณ์ $K_{n,n}$

$$L(n,n,r_1,r_2)=\begin{cases} B_k^2 & ;\ r_1=r_2=k, \\ 2B_{r_1}B_{r_2} & ;\ r_1\neq r_2, \end{cases}$$

โดยที่ $2\le k\le n-1$ และ $B_j=\binom{n}{j}\sum_{i=0}^{n-j-1}(-1)^i\binom{n-j}{i}(n-j-i)^{n-1}$

คำสำคัญ : กราฟสองส่วนบริบูรณ์ แบบติดป้าย กราฟต้นไม้ จุดปลาย การนับ

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ABSTRACT

The number of labelled trees with r_1, r_2 end-vertices in complete bipartite graph $K_{m,n}$ denoted by $L(m,n,r_1,r_2)$. In this thesis, we use the exponential generating function to find the number $L(m,n,r_1,r_2)$ can be obtained from the formula

$$L(m,n,r_1,r_2)=\binom{m}{r_1}\binom{n}{r_2}A_{r_1}A_{r_2},$$

where $A_{r_1}=\sum_{i=0}^{m-r_1-1}(-1)^i\binom{m-r_1}{i}(m-r_1-i)^{n-1}$ and $A_{r_2}=\sum_{j=0}^{n-r_2-1}(-1)^j\binom{n-r_2}{j}(n-r_2-j)^{m-1}$.

Moreover, we obtained the number of labelled trees with r_1, r_2 end-vertices in $K_{n,n}$.

$$L(n,n,r_1,r_2)=\begin{cases} B_k^2 & ;\ r_1=r_2=k, \\ 2B_{r_1}B_{r_2} & ;\ r_1\neq r_2, \end{cases}$$

where $2\le k\le n-1$ and $B_j=\binom{n}{j}\sum_{i=0}^{n-j-1}(-1)^i\binom{n-j}{i}(n-j-i)^{n-1}$.

Keywords : Complete bipartite graph, Labelled, Trees, End-vertices, Counting.

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Chapter 1
Introduction

Graph Theory is a branch of mathematics that can be applied to many different fields of science such as physics, chemistry, communication science, computer technology, electrical and civil engineering, architecture, operational research, genetics, phychology, sociology, economics, anthropology, linguistics, etc. For example of some application problems, the Königsberg bridge problem, the electric networks, the chemical isomers, the four color conjecture, etc.

A labelled tree is a topic in graph theory that it can be applied to many applications such as In 1998, John, Mallion and Gutman [27] calculated the number of spanning trees in the (labelled) molecular graphs of cata-condensed systems containing rings of only one size. In 2015, Chin, Gordon, MacPhee et. al. [12] used a labelled tree to apply in topic "Pick a Tree-Any Tree".

This thesis is application of Cayley's Theorem that is the part of the number of labelled trees in a graph G, denoted by $T(G)$, is the total number of distinct spanning subgraphs that are trees.

In 1889, Cayley [8] proved the number of labelled trees with n vertices is

$$T(K_n) = n^{n-2}. \tag{1.1}$$

That is the number of labelled trees in complete graph K_n.

Figure 1.1 Arthur Cayley (1889)

Example 1.1 When $n = 4$, we have the labelled graph K_4 and the spanning trees in K_4 as following in Figure 1.2 and 1.3, respectively. So $T(K_4) = 4^{4-2} = 16$ as following in Figure 1.4.

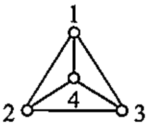


Figure 1.2 A labelled graph K_4

Figure 1.3 The spanning trees in K_4

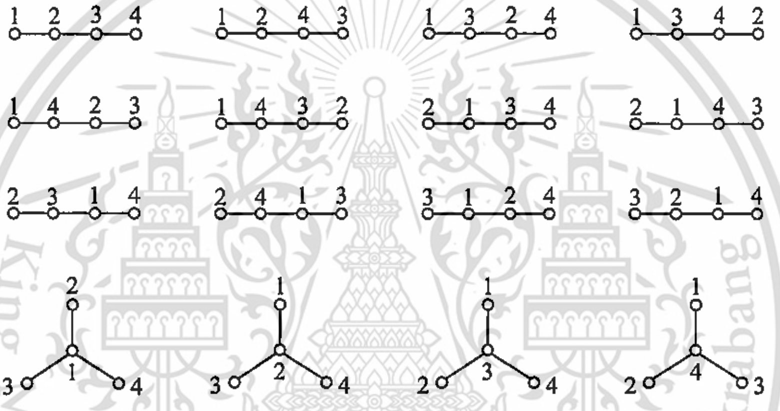


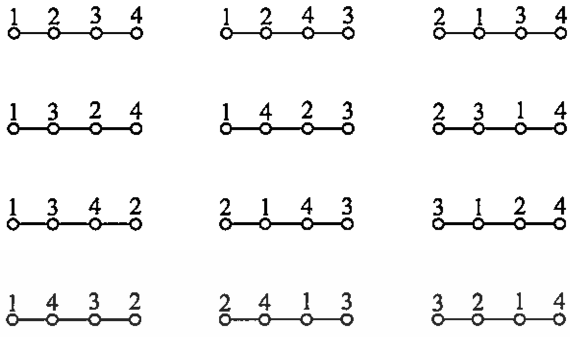
Figure 1.4 The labelled trees in K_4

Next, in 2008, Longani [29] has developed Cayley's Theorem to be the formula of the number of labelled trees with r end-vertices in K_n for $n \geq 3$,

$$L(n,r) = \binom{n}{n-r} \sum_{i=0}^{n-r-1} (-1)^i \binom{n-r}{i} (n-r-i)^{n-2}. \tag{1.2}$$

Example 1.2 From example 1.1, there are 2 trees but if the trees are labelled, then there are 4 possible ways to label the left tree, and 12 possible ways to label the right tree of Figure 1.3. Then in total, there are 16 labelled trees. So $L(4,2) = 12$ and $L(4,3) = 4$ that see in Figure 1.5.

$L(4,2) = 12$



(a) Labelled trees with 2 end-vertices in K_4.

$L(4,3) = 4$

(b) Labelled trees with 3 end-vertices in K_4.

Figure 1.5 Labelled trees in K_4.

From our intensive study, we have found many research about the number of labelled trees of some graphs such as in 2003, Jin and Liu [26] showed the formula of the number of labelled trees in complete bipartite graph $K_{m,n}$ that used exponential generating function as follows

$$T(K_{m,n}) = m^{n-1}n^{m-1}, \tag{1.3}$$

where $m \geq n$ and $m, n \geq 1$. So when $m = n$, we have

$$T(K_{n,n}) = n^{2n-2}. \tag{1.4}$$

Thus, we interested in the case of complete bipartite graph $K_{m,n}$.

Chapter 2

Preliminary and Basic Concept

In this chapter, we begin by introduce some basic definitions and theorems in graph theory and combinatorics that is necessary tools to study and research our work. First, we show knowledge and give examples about graph theory and combinatorics, respectively.

2.1 Graph Theory

2.1.1 The basic concepts of graph theory

In this section, we use the notation and definitions of graph theory that follow [9], [16], [20] and [46].

Definition 2.1 A ***graph*** $G=(V,E)$ consists of two sets V and E.

- The elements of V are called ***vertices*** (or ***nodes***). In this thesis, we called vertices.
- The elements of E are called ***edges***.
- Each edge has a set of one or two vertices associated to it, which are called ***end-vertices*** of edge e. An edge is said to ***join*** its end-vertices.

Example 2.1 The following figure, every figure is graphs.

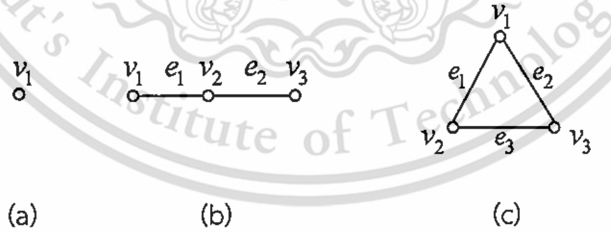
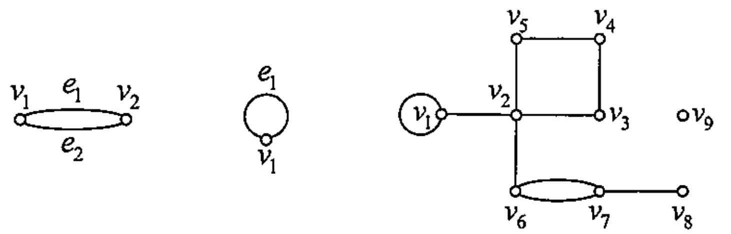


Figure 2.1 Graphs



(d) (e) (f)

Figure 2.2 Graphs (Cont.)

Terminology: The ***order*** of a graph is the cardinality of its vertex set, and the ***size*** of a graph is the cardinality of its edge set.

Definition 2.2 If vertex v is an end-vertex of edge e, then v is said to be ***incident*** on e, and e is incident on v.

Example 2.2 From Figure 2.1 (b), we see that v_1 and v_2 are incident on e_1. Similarly, v_2 and v_3 are incident on e_2.

Definition 2.3 A vertex u is ***adjacent*** to vertex v if they are joined by an edge.

Example 2.3 From Figure 2.1 (b), a vertex v_1 is adjacent to vertex v_2 and a vertex v_2 is adjacent to vertex v_3.

Definition 2.4 Two adjacent vertices may be called ***neighbors***.

Example 2.4 From Figure 2.1 (b), we see that a vertex v_1 is a neighbor of a vertex v_2 and a vertex v_2 is a neighbor of a vertex v_3.

Definition 2.5 A ***multi-edge*** is a collection of two or more edges having identical end-vertices.

Example 2.5 The graph in Figure 2.2 (d) and (f) have a multi-edge but the other figure have not a multi-edge.

Definition 2.6 A ***self-loop*** is an edge that joins a single end-vertex to itself.

Example 2.6 We see that the graph in Figure 2.2 (e) and (f) have a self-loop but the other figure have not a self-loop.

Definition 2.7 A ***simple graph*** is a graph that has no self-loops or multi-edges.

Example 2.7 From Figure 2.1, we see that Figure 2.1 (a), (b) and (c) are simple graphs.

Definition 2.8 A ***trivial graph*** is a graph consisting of one vertex and no-edge.

Example 2.8 From Figure 2.1 (a), it is a trivial graph.

Definition 2.9 The ***degree*** (or ***valence***) of a vertex v in a graph G, denoted by $\deg_G(v)$, is the number of proper edges incident on v plus twice the number of self-loops. (For simple graph graphs, of course, the degree is simply the number of neighbors.)

Example 2.9 From Figure 2.2 (f), we see that the degree of all vertices $v_1, v_2, v_3, \ldots, v_9$ are $3,4,2,2,2,4,4,1$ and 0, respectively.

Definition 2.10 The vertex v in graph G is an ***end-vertex*** of G if $\deg_G(v) = 1$.

Example 2.10 From Figure 2.1 (b), we see that the vertices v_1 and v_2 are end-vertices.

Definition 2.11 A ***walk*** in a graph G is an alternating sequence of vertices and edges,

$$W = v_0, e_1, v_1, e_2, \ldots, e_n, v_n$$

such that for $j = 1,2,3,\ldots,n$, the vertices v_{j-1} and v_j are the end-vertices of the edge e_j. If, moreover, the edge e_j is directed from v_{j-1} to v_j, then W is a ***directed walk***.

- In a simple graph, a walk may be represented simply by listed a sequence of vertices: $W = v_0, v_1, \ldots, v_n$ such that for $j = 1,2,3,\ldots,n$, the vertices v_{j-1} and v_j are adjacent.
- The ***initial vertex*** is v_0. In this thesis, let $v_0 = v_1$.

Example 2.11 Let a graph G as follows

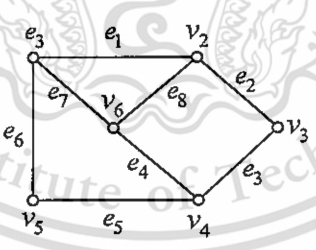


Figure 2.3 A graph G

From this figure, we see that $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_5, v_5, e_5, v_4, e_4, v_6$ is $v_1 - v_6$ walk W in G that we may be represented simply by $v_1, v_2, v_3, v_4, v_5, v_4, v_6$.

Definition 2.12 The ***length of a walk*** is the number of edges in walk W.

Example 2.12 From example 2.11, we see that the length of this walk W is 6.

Definition 2.13 A walk is ***closed*** if the initial vertex is also the final vertex.

Definition 2.14 A ***trail*** in a graph is a walk such that no edge occurs more than once.

Definition 2.15 A ***path*** in a graph is a trail such that no internal vertex is repeated.

Example 2.13 From Figure 2.3, we see that $W = v_1, v_2, v_3, v_4, v_5, v_4, v_6, v_1$ is a closed walk, $W = v_1, v_2, v_3, v_4, v_6, v_1, v_5$ is a trail and $W = v_1, v_6, v_2, v_3, v_4, v_5$ is a path.

Definition 2.16 A ***cycle*** is a closed path of length at least 1. A graph is ***acyclic*** if it has no cycle.

Example 2.14 From Figure 2.1 (b) and (c), Figure 2.1 (c) is a cycle but Figure 2.1 (b) is an acycle.

Definition 2.17 A graph $G' = (V', E')$ is called a ***subgraph*** of graph $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$ and V' contains all the end-vertices of all the edges in E'.

Example 2.15 From Figure 2.4, we see that G_1, G_2 and G_3 are subgraphs of graph G.

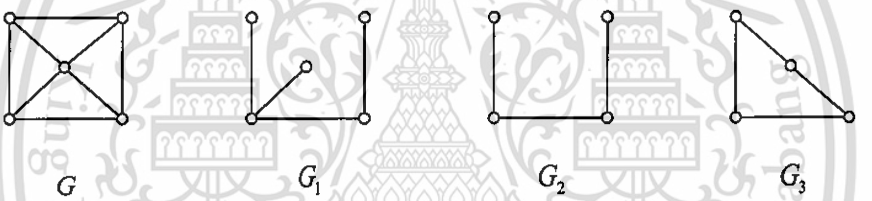


Figure 2.4 A graph G and its subgraph

Definition 2.18 A graph G is ***connected*** if there is a path in G between any given pair of vertices, and ***disconnected*** otherwise. Every disconnected graph can be split up into a number of connected subgraphs, called ***components***.

Example 2.16 From Figure 2.1, 2.2 and 2.3, we see that Figure 2.1 (a), (b), (c), 2.2 (d), (e) and Figure 2.3 are the connected graphs but Figure 2.2 (f) is disconnected graph which has two components.

Definition 2.19 An edges in connected graph is a ***bridge*** if its removal leaves a disconnected graph.

Example 2.17 From Figure 2.1 (b), we see that e_1 is a bridge.

Definition 2.20 A ***tree*** T is a connected graph with no cycles.

Example 2.18 Let a tree T have 4 vertices as following in Figure 2.5.

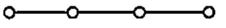


Figure 2.5 A tree T

Theorem 2.1 Let G be a graph with p vertices and q edges, the following statements are equivalent for a graph G.

1) G is a tree.
2) Every two vertices of G are joined by a unique path.
3) G is connected and $p = q + 1$.
4) G is acyclic and $p = q + 1$.
5) G is acyclic and if any two nonadjacent vertices of G are joined by an edge e, then $G + e$ has exactly one cycle.
6) G is connected, is not K_p for $p \geq 3$, and if any two adjacent vertices of G are joined by an edge e, then $G + e$ has exactly one cycle.
7) G is not $K_3 \cup K_1$, $K_3 \cup K_2$, $p = q + 1$, and if any two nonadjacent vertices of G are joined by an edge e, then $G + e$ has exactly one cycle.

Definition 2.21 A graph G is ***isomorphic*** to a graph G_2 $(G_1 \cong G_2)$ if there exists a one-to-one mapping θ, called an ***isomorphism***, from $V(G_1)$ onto $V(G_2)$ such that θ preserves adjacency and nonadjacency; that is, $uv \in E(G_1)$ if and only if $\theta(u)\theta(v) \in E(G_2)$.

Example 2.19 Consider G_1 and G_2 in Figure 2.6.

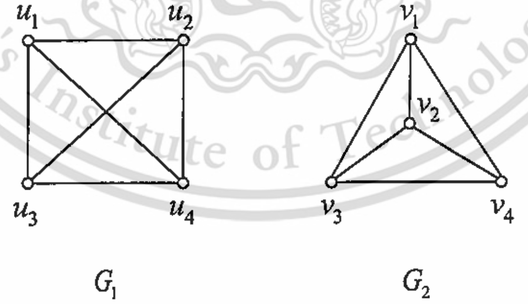


Figure 2.6 $G_1 \cong G_2$

From this figure, let $\theta : V(G_1) \to V(G_2)$ that $\theta(u_1) = v_1$, $\theta(u_2) = v_2$, $\theta(u_3) = v_3$, $\theta(u_4) = v_4$. We see that θ is one-to-one from G_1 onto G_2 and

$$u_1u_2 \in E(G_1) \Leftrightarrow \theta(u_1)\theta(u_2) = v_1v_2 \in E(G_2)$$
$$u_1u_3 \in E(G_1) \Leftrightarrow \theta(u_1)\theta(u_3) = v_1v_3 \in E(G_2)$$

$$u_1u_4 \in E(G_1) \Leftrightarrow \theta(u_1)\theta(u_4) = v_1v_4 \in E(G_2)$$
$$u_2u_3 \in E(G_1) \Leftrightarrow \theta(u_2)\theta(u_3) = v_2v_3 \in E(G_2)$$
$$u_2u_4 \in E(G_1) \Leftrightarrow \theta(u_2)\theta(u_4) = v_2v_4 \in E(G_2)$$
$$u_3u_4 \in E(G_1) \Leftrightarrow \theta(u_3)\theta(u_4) = v_3v_4 \in E(G_2)$$

So θ is isomorphism from G_1 to G_2. Thus $G_1 \cong G_2$.

Definition 2.22 A graph G is ***labelled*** when the n vertices are distinguished from one another by names such as $v_1, v_2, \ldots, v_n$.

Example 2.20 From Figure 2.5, let this tree labelled with $v_1, v_2, \ldots, v_4$ as following in Figure 2.7.

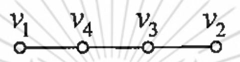


Figure 2.7 A tree that labelled with $v_1, v_2, \ldots, v_4$

Definition 2.23 A ***labelled tree*** is a tree in which labels, typically $v_1, v_2, \ldots, v_n$, have been assigned to the vertices.

Example 2.21 From Figure 2.7, let labelled trees that labelled with the simple integers $1, 2, 3$ and 4 as following in Figure 2.8.

Figure 2.8 Labelled trees in K_4 with 2 end-vertices

The two labelled trees with the same set of labels are considered the same only if there is an isomorphism from one to the other that preserves the labels.

Example 2.22 From Figure 2.8, it is the same labelled trees in Figure 2.9 (a) but different labelled trees in Figure 2.9 (b).

(a) (b)

Figure 2.9 Labelled trees in K_4 with 2 end-vertices (comparision)

Definition 2.24 A subgraph H of a graph G is a ***spanning subgraph*** if $V(H) = V(G)$. (Also, if H is isomorphic to a spanning subgraph of G, we may say that H spans G.)

Definition 2.25 A ***spanning tree*** of a graph G is a spanning subgraph of G that is a tree.

Example 2.23 From Figure 2.10, we see that G_1 and G_2 are spanning subgraphs and spanning trees of G. While G_3 is a spanning subgraph but is not spanning tree of G. So G_4 and G_5 are not spanning subgraph and spanning tree of G.

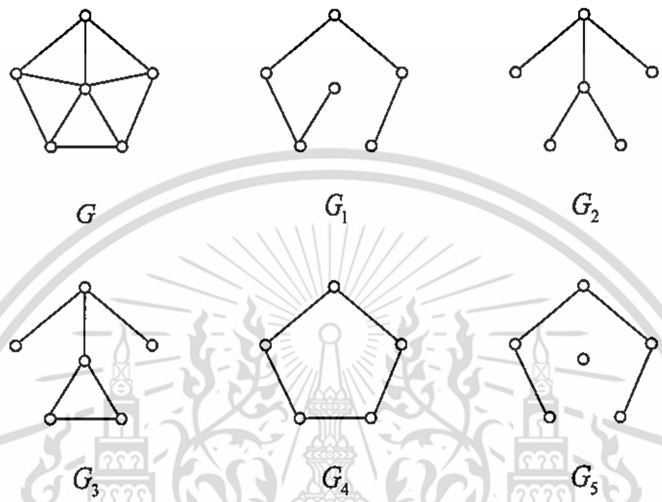


Figure 2.10 Spanning subgraphs and spanning trees

Definition 2.26 A ***spanning labelled tree*** in a graph G is a spanning subgraph of G that is a labelled tree.

Example 2.24 From Figure 1.3 and 1.4, we see that Figure 1.3 is the spanning labelled trees of Figure 1.4.

Example 2.25 From Figure 2.11, the spanning trees and spanning labelled trees of this figure is a figure as Figure 2.12 and 2.13, respectively.

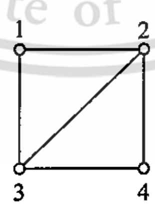


Figure 2.11 A labelled graph G

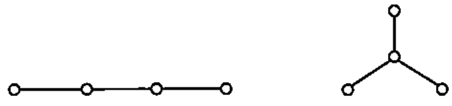


Figure 2.12 The spanning trees of a graph G

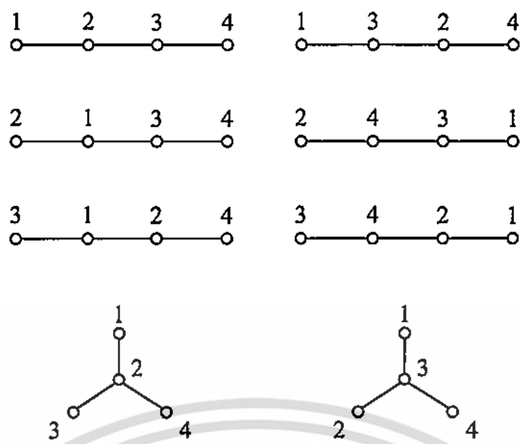


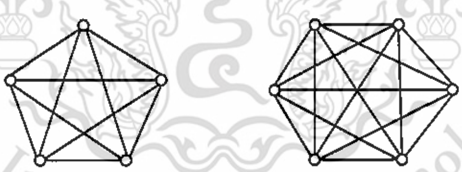
Figure 2.13 The spanning labelled trees of a graph G

For convenience, we use the number of labelled trees in G instead the number of spanning labelled trees in G.

Definition 2.27 A simple graph is a ***complete graph*** if every pair of vertices is joined by an edge. The complete graph with n vertices is denoted K_n.

Example 2.26 From Figure 2.6, we see that a graph G_1 is a complete graph K_4.

Example 2.27 Graphs in Figure 2.14 (a) and (b) are the complete graphs K_5 and K_6, respectively.



(a) K_5 (b) K_6

Figure 2.14 Complete graphs K_n with $n = 5, 6$

Definition 2.28 A ***complete bipartite graph*** is a simple bipartite graph in which each vertex in one partite set is adjacent to all vertices in the other partite set. If the two partite sets have cardinalities m and n, then this graph is denoted $K_{m,n}$.

Example 2.28 Let $m = 3$ and $n = 4$, we have a complete bipartite graph $K_{3,4}$ as following in Figure 2.15.

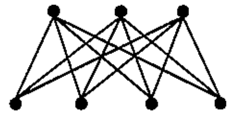


Figure 2.15 A complete bipartite graph $K_{3,4}$

When $m = n = 4$, we have a complete bipartite graph $K_{4,4}$ as following in Figure 2.16.

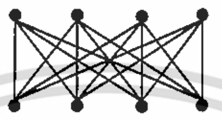
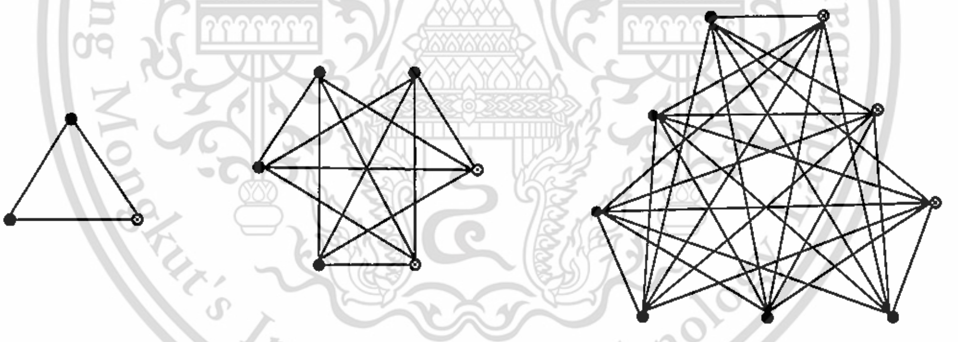


Figure 2.16 A complete bipartite graph $K_{4,4}$

Definition 2.29 A graph is k*-partite* if its vertices can be partitioned into k sets (called ***partite sets***) in such a way that no edge joins two vertices in the same set.

Example 2.29 Let $k = 3$, we have the example of the 3-partite graphs as following in Figure 2.17



(a) (b) (c)

Figure 2.17 k-partite graphs for $k = 3$

Definition 2.30 A ***complete*** k***-partite graph*** is a simple k-partite graph in which two vertices are adjacent if and only if they are in different partite sets. All such graphs are called ***complete multipartite graphs.*** If the k partite sets have order $n_1, n_2, \dots, n_k$, then the graph is denoted $K_{n_1,n_2,\dots,n_k}$, and if each partite set has order r, then $K_{k(r)}$.

Example 2.30 From Figure 2.17, we see that every graph is the complete 3-partite graphs or complete tripartite graph $K_{n,n,n}$ with $n = 1, 2$ and 3, respectively.

Example 2.31 Let $k=5$, $n=2$, we have a complete multipartite graph $K_{2,2,2,2,2}$ or $K_{5(2)}$ as following in Figure 2.18.

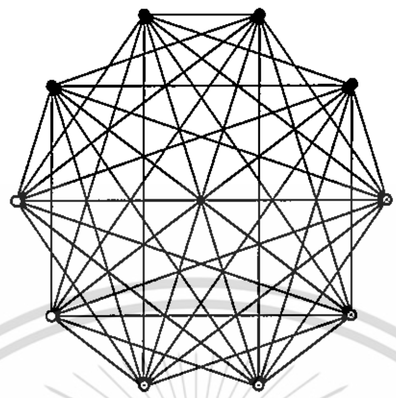


Figure 2.18 A complete multipartite graph $K_{2,2,2,2,2}$ or $K_{5(2)}$

2.1.2 Prüfer's construction [11]

We assume than $n \geq 3$, since the result is clearly true if $n=1$ or 2. We construct a one-to-one correspondence between the set of labelled trees with n vertices and the set of all sequence of the form $(a_1, a_2, a_3, \ldots, a_{n-2})$, is called **Prüfer sequence**, where each a_i is one of the integers $1, 2, 3, \ldots, n$ (allowing repetition). In order to obtain the required one-to-one correspondence, we take a labelled tree with n and apply three steps.

Step 1 Look at the vertices of degree 1 and choose the one with the smallest label.

Step 2 Look at the vertex adjacent to the one just chosen and place its label in the first available position in the sequence.

Step 3 Remove the vertex chosen in **Step 1** and its incident edge, leaving a smaller tree.

Repeat **Step 1-3** for the remaining tree, continuing until there are only two vertices left. By the time this happens, the required Prüfer sequence will have been constructed.

Example 2.32 Consider the labelled tree

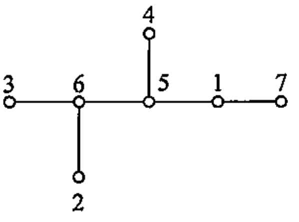


Figure 2.19 A labelled tree with 7 vertices

Step 1 The vertices of degree 1 are vertices 3,2,4 and 7; the one with the smallest label is vertex 2.

Step 2 The vertex adjacent to vertex 2 is vertex 6, so the sequence starts with 6.

Step 3 Removal of the vertex 2 and the edge 26 leaves.

Figure 2.20 The result of labelled tree in the first step

Step 1 The vertices of degree 1 are vertices 3,4 and 7; the one with the smallest label is vertex 3.

Step 2 The vertex adjacent to vertex 3 is vertex 6, so the next term in the sequence is 6.

Step 3 Removal of the vertex 3 and the edge 36 leaves.

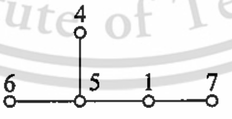


Figure 2.21 The result of labelled tree in the second step

Continuing in this way, we successively remove the edges 45, 65 and 51, and obtain the Prüfer sequence $(6,6,5,5,1)$.

In order to obtain the reverse correspondence, we take a Prüfer sequence and apply three steps.

Step 1 Draw the n vertices, labeling them from 1 to n and make a list of the numbers from 1 to n.

Step 2 Find the smallest number that is the list but not in the prüfer sequence, and also find the first number in the sequence; then add an edge joining the vertices with these labels.

Step 3 Remove the first number of **Step 2** from the list and the other number of **Step 2** from the sequence, leaving a smaller list and sequence.

Repeat **Step 2 and 3** for the remaining list and sequence, continuing until there are only two labels left in the list. Finally, join the vertices with these labels.

Example 2.33 Consider the Prüfer sequence $(6,6,5,5,1)$.

Step 1 Since the sequence contains $7-2=5$ numbers, we start with the list $(1,2,3,4,5,6,7)$, and draw the vertices 1 to 7 as follows

Figure 2.22 A graph with 7 vertices

Step 2 The smallest number in the list but not in the sequence is 2, and the first number in the sequence is 6, so we add an edge joining the vertices 2 and 6.

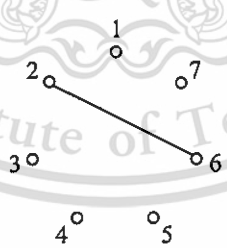


Figure 2.23 A graph in first step

Step 3 Removal of the number 2 from the list, and the number 6 from the sequence, leaves the list $(1,3,4,5,6,7)$ and the sequence $(6,5,5,1)$.

Step 2 The smallest number in the new list which is not in the new sequence is 3, and the first number in the new sequence is 6, so we add an edge joining the vertices 3 and 6.

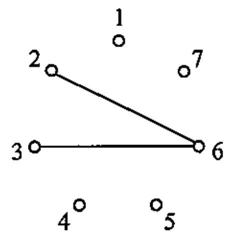


Figure 2.24 A graph in second step

Step 3 Removal the number 3 from the list, and the number 6 from the sequence, leaves the list $(1,4,5,6,7)$, and the sequence $(5,5,1)$.

Continuing in this way, we successively add edges joining the vertices 4 and 5, 6 and 5, and 5 and 1. The list is now $(1,7)$, and we join the vertices with these labels. This gives the labelled tree shown as following in Figure 2.25.

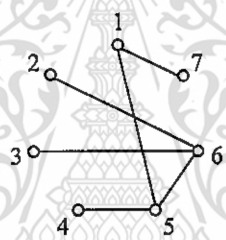


Figure 2.25 A labelled tree of sequence $(6,6,5,5,1)$ in K_7

Note that this labelled tree obtained from the Prüfer sequence $(6,6,5,5,1)$ is the same as the labelled tree which earlier gave rise to this sequence. This happens in general. If you start with any labelled tree corresponding to this sequence, you should get back to the original tree. This gives the required one-to-one correspondence:

Labelled tree $\longleftrightarrow$ Prüfer sequence

According to Prüfer's construction, we have it is the construction for complete graph K_n. We can extend to the complete bipartite graph that use the same algorithm in K_n. We show the algorithm in case $K_{m,n}$ as the following example 2.34 and 2.35.

Example 2.34 Consider the labelled tree in $K_{3,4}$ that $1,2,3$ are the labels in the first partition and $4,5,6,7$ are the labels in the second partition as follow

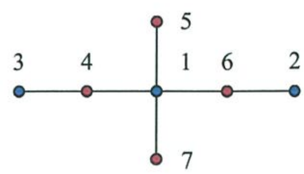


Figure 2.26 A labelled tree with $r_1 = r_2 = 2$ end-vertices in $K_{3,4}$

Step 1 The vertices of degree 6 are vertices $3,5,7$ and 2; the one with the smallest label is vertex 2.

Step 2 The vertex adjacent to vertex 2 is vertex 6, so the sequence starts with 6.

Step 3 Removal of the vertex 2 and the edge 26 leaves.

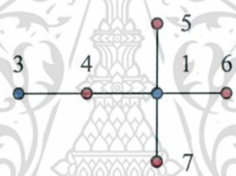


Figure 2.27 The first result of labelled tree in $K_{3,4}$

Step 1 The vertices of degree 1 are vertices $3,5$ and 7; the one with the smallest label is vertex 3.

Step 2 The vertex adjacent to vertex 3 is vertex 4, so the next term in the sequence is 4.

Step 3 Removal of the vertex 3 and the edge 34 leaves.

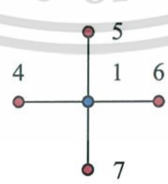


Figure 2.28 The second result of labelled tree in $K_{3,4}$

Continuing in this way, we successively remove the edges 14, 15 and 16, and obtain the Prüfer sequence $(6,4,1,1,1)$.

In order to obtain the reverse correspondence in complete bipartite graph, we take a Prüfer sequence and apply three steps as in complete graph.

Example 2.35 Consider the Prüfer sequence $(6,4,1,1,1)$ and initial graph in Figure 2.29

Step 1 Since the sequence contains $7-2=5$ numbers, we start with the list $(1,2,3,4,5,6,7)$, and draw the vertices 1 to 7 as follows

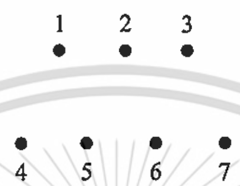


Figure 2.29 A graph of sequence $(6,4,1,1,1)$ in $K_{3,4}$

Step 2 The smallest number in the list but not in the sequence is 2, and the first number in the sequence is 6, so we add an edge joining the vertices 2 and 6.

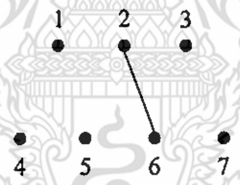


Figure 2.30 A graph in first step

Step 3 Removal of the number 2 from the list, and the number 6 from the sequence, leaves the list $(1,3,4,5,6,7)$ and the sequence $(4,1,1,1)$.

Step 2 The smallest number in the new list which is not in the new sequence is 3, and the first number in the new sequence is 4, so we add an edge joining the vertices 3 and 4.

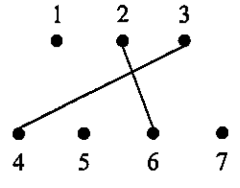


Figure 2.31 A graph in second step

Step 3 Removal the number 3 from the list, and the number 4 from the sequence, leaves the list $(1,4,5,6,7)$, and the sequence $(1,1,1)$.

Continuing in this way, we successively add edges joining the vertices 1 and 4, 1 and 5, and 1 and 6. The list is now $(1,7)$, and we join the vertices with these labels. This gives the labelled tree shown as following in Figure 2.32.

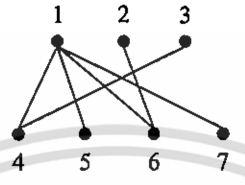


Figure 2.32 A labelled tree of sequence $(6,4,1,1,1)$ in $K_{3,4}$

Note that this labelled tree obtained from the Prüfer sequence $(6,4,1,1,1)$ is the same as the labelled tree which earlier gave rise to this sequence. This happens in general. Similary, We have the required one-to-one correspondence for complete bipartite graph as follow:

Labelled tree $\longleftrightarrow$ Prüfer sequence

2.1.3 The number of labelled trees

This fact is known as Cayley's Theorem. We outline a proof of this result, which is due to H. Prüfer and involves Prüfer's construction.

Theorem 2.2 (Cayley's Theorem) The number of labelled trees with n vertices is n^{n-2}. See in Table 2.1.

Table 2.1 The number of labelled trees with $1 \le n \le 15$ vertices

n	The number of labelled trees n
1	1
2	1
3	3
4	16
5	125

(Gross, JL. and Yellen, J. 2000. **Handbook of Graph Theory.** Florida : CRC Press.)

Table 2.2 The number of labelled trees with $1 \leq n \leq 15$ vertices (Cont.)

n	The number of labelled trees n
6	1,296
7	16,807
8	262,144
9	4,782,969
10	100,000,000
11	2,357,947,691
12	61,917,364,224
13	1,792,160,394,037
14	56,693,912,375,296
15	1,946,195,068,359,375

(Gross, JL. and Yellen, J. 2000. **Handbook of Graph Theory.** Florida : CRC Press.)

In 1985, Baron, Prodinger, Tichy, et. al. [4] showed that the formula of the number of labelled trees in a square of cycle; they have

$$T(C_n^2) = nF_n^2, \tag{2.1}$$

where F_n is Fibonacci number.

In 1990, Abu-Sbieh [1] used a new technique for proving the formula of the number of labelled trees in a complete graph K_n and a complete bipartite graph $K_{n,n}$.

In 1999, Lewis [30] showed the number of labelled trees in a complete multipartite graph that used Prüfer sequence.

$$T(K_{n_0,n_1,\ldots,n_{k-1}}) = n^{k-2}\prod_{0 \leq i < k}(n-n_i)^{n_i-i}, \tag{2.2}$$

where $n = n_0 + n_1 + \ldots + n_{k-1}$.

The following table are the closed formula and the number of spanning trees of other graphs.

Table 2.3 The closed formula of labelled trees of some graphs

Class	Abbreviations	The closed formula of labelled trees
Complete tripartite graph	$K_{n,n,n}$	$3 \cdot 8^{n-1} n^{3n-2}$
Cycle graph	C_n	n
Pan graph		n
Path graph	P_n	1
Star graph	S_n	1
Sunlet graph		n

(Weisstein, E. 2015. **Spanning tree.** [Online].
Available : http://mathworld.wolfram.com/SpanningTree.html.)

Table 2.4 The number of labelled trees of some graphs

Class	The number of labelled trees
Barbell graph	X, X, 9, 256, 15625, 1679616, 282475249, ...
Complete tripartite graph	3, 384, 419904, 1610612736, 15000000000000, ...
Cycle graph C_n	X, X, 3, 4, 5, 6, 7, 8, 9, 10, ...
Fan graph $F_{m,n}$	X, 8, 216, 13056, 1409375, ...
Pan graph	X, X, 3, 4, 5, 6, 7, 8, 9, 10, ...
Path graph P_n	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...
Square graph Sq_n	1, 4, 129600, 17210368000000, ...
Star graph S_n	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...
Sunlet graph	X, X, 3, 4, 5, 6, 7, 8, 9, 10
Wheel graph W_n	X, X, X, 16, 45, 121, 320, 841, 2205, ...

(Weisstein, E. 2015. **Spanning tree.** [Online].
Available : http://mathworld.wolfram.com/SpanningTree.html.)

2.2 Combinatorics

In this section, we use the notation and theorems of combinatorics that follows in [10] and [40].

2.2.1 Two basic counting combinations

In our lives, we often need to enumerate "events" such as the arrangement of objects in a certain way, the partition of things under a certain condition, the distribution of items according to a certain specification, and so on. For instance, we may come across counting problems of the following types :

"How many ways are there to arrange 5 boys and 3 girls in a row so that no two girls are adjacent?"

"How many ways are there to divide a group of 10 people into three groups consisting of 4, 3 and 2 people, respectively, with 1 person rejected?"

These are two very simple examples of counting problems related to what we call ***Permutations*** and ***combinations***. Before we introduce in the next three sections what permutations and combinations are, we state in this section two principles that are fundamental in all kinds of counting problems.

The addition principle (AP) Assume that there are

n_1 ways for the event E_1 to occur,

n_2 ways for the event E_2 to occur,

$\vdots$

n_k ways for the event E_k to occur,

where $k \geq 1$. If these ways for the different events to occur are pairwise disjoint, then the number of ways for at least one of the events $E_1, E_2, \ldots,$ or E_k to occur is $n_1 + n_2 + \ldots + n_k = \sum_{i=0}^{k} n_i$.

Let $A_1, A_2, \ldots, A_k$ be any k finite sets, where $k \geq 1$. If the given sets are pairwise disjoint, i.e., $A_i \cap A_j \neq \phi$ for $i, j = 1, 2, \ldots, k$, $i \neq j$, then

$$\left|\bigcup_{i=1}^{k} A_i\right| = |A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{i=1}^{k} |A_i| . \tag{2.3}$$

Example 2.36 A man wants to travel from city P to city Q. He has six ways to get there such as bicycle, car, taxi, bus, train and airplane. Suppose that the number of each vehicle to choose are $5, 2, 4, 4, 2$ and 3, respectively as following in Figure 2.33.

Figure 2.33 The vehicle from city P to city Q

Then by (AP), the total number of vehicles from P to Q by bicycle, car, taxi, bus, train and airplane is $5+2+4+4+2=20$. □

The multiciplication principle (MP) Assume that an event E can be decomposed into r ordered events $E_1, E_2, \ldots, E_r$, and that there are

n_1 ways for the event E_1 to occur,

n_2 ways for the event E_2 to occur,

$\vdots$

n_r ways for the event E_r to occur,

Then the total number of ways for the event E to occur is given by $n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i$.

Let $\prod_{i=1}^{r} A_i = A_1 \times A_2 \times \cdots \times A_r = \{(a_1, a_2, \ldots, a_r) \mid a_i \in A_i, i = 1, 2, \ldots, r\}$ denote the certesian product of the finite sets $A_1, A_2, \ldots, A_r$. Then

$$\left|\prod_{i=1}^{r} A_i\right| = |A_1| \times |A_2| \times \cdots \times |A_r| = \prod_{i=1}^{r} |A_i|. \tag{2.4}$$

Example 2.37 From example 2.36, A man wants to travel from city P to city Q via city W,X,Y and Z, respectively and the color of vehicles of each city are blue from city P to city W, red from city W to city X, green from city X to city Y, orange from city Y to city Z and purple from city Z to city Q, respectively. So there are 5 ways from city P to city W, 5 ways from city W to city X, 4 ways from city X to city Y, 3 ways from city Y to city Z and 1 way from city Z to city Q, respectively as following in Figure 2.34.

Figure 2.34 The way from city P to city Q via city W,X,Y and Z

Then by (MP), the total number of ways from city P to city Q via city W,X,Y and Z is given by $5\times5\times4\times3\times1=300$. □

2.2.2 The binomial theorem

In 1676, Issac Newton discovered the following simplest form of the binomial theorem.

Figure 2.35 Issac Newton (1676)

Theorem 2.3 (Binomial Theorem) For any integer $n\geq 0$,

$$(x+y)^n=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\cdots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r . \quad (2.5)$$

Proof In proving, we can proof by two ways that there are mathematical induction and combinatorial method. In this thesis, we prove by combinatorial method.

It suffices to prove that the coefficient of $x^{n-r}y^r$ in the expansion of $(x+y)^n$ is $\binom{n}{r}$. To expand the product $(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n}$, we choose either x or y from each factor $(x+y)$ and then multiply them together. Thus to form a term $x^{n-r}y^r$, we first select r of the n factors $(x+y)$ and then pick " y " from the r factors choosen (and of course pick " x " from the remaining $(n-r)$ factors). The first step can be done $\binom{n}{r}$ ways while the second in 1 way. Thus, the number of ways to form the term $x^{n-r}y^r$ is $\binom{n}{r}$ as required. □

2.2.3 Ordinary generating functions

This notion has its roots in the work of de Moivre around 1720 and was developed by Euler in 1748 in connection with his study on the partitions of integers. It was later on extensively and systematically treated by Laplace in the late 18^{th} century. In fact, the name "generating functions" owes its origin to Laplace in his great work "Théorie Analytique des Probabilities"

Suppose a_r is the number of ways to select r objects in the certain procedure. The $g(x)$ is a ***generating function*** for a_r if $g(x)$ has the polynomial expansion

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots + a_nx^n. \tag{2.6}$$

If the function has an infinite number of items, it is called a ***power series***.

We verified the well-known binomial expansion

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n. \tag{2.7}$$

Then $g(x) = (1+x)^n$ is the generating function for $a_r = C(n,r)$, the number of ways to select an r-subset from an n-set.

Example 2.38 For each $n \in \mathbb{N}$, let (a_r) be the sequence where

$$a_r = \begin{cases} 1 & ; r = n, \\ 0 & ; \textit{otherwise}. \end{cases} \tag{2.8}$$

That is $$(a_r) = (\underset{\substack{\uparrow \\ 0}}{0}, \underset{\substack{\uparrow \\ 1}}{0}, \ldots, 0, \underset{\substack{\uparrow \\ n}}{1}, 0, 0, \ldots). \tag{2.9}$$

Then then generating function for (a_r) is x^n. □

Example 2.39 The generating function for the sequence $(1,1,1,\ldots)$ is

$$1 + x + x^2 + \cdots = \frac{1}{1-x}. \tag{2.10}$$

More generally, the generating function for the sequence $(1, k, k^2, \ldots)$, where k is an arbitrary constant, is

$$1 + kx + k^3x^2 + k^3x^3 + \cdots = \frac{1}{1-kx}. \tag{2.11}$$

Example 2.40 The generating function for the sequence $(1,2,3,\ldots)$ is

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}. \tag{2.12}$$

Some modeling problems

In this section, we shall discuss how to notation of generating functions, as introduced in the preceding section, can be used to solve some combinatorial problems. Through the examples provided, the reader will be able to see the applicability of the technique studied here.

To begin with, let $S = \{a, b, c\}$. Consider the various ways of selecting objects from S.

To select one object from S, we have:

$\{a\}$ or $\{b\}$ or $\{c\}$ (denoted by $a+b+c$).

To select two objects from S, we have:

$\{a,b\}$ or $\{b,c\}$ or $\{c,a\}$ (denoted by $ab+bc+ca$).

To select three objects from S, we have:

$\{a,b,c\}$ (denoted by abc).

These symbols can be found in the following expression:

$$(1+ax)(1+bx)(1+cx) = 1x^0 + (a+b+c)x^1 + (ab+bc+ca)x^2 + (abc)x^3. \tag{2.13}$$

We may write $1+ax=x^0+ax^1$, which may be interpreted as "a is not selected or a is selected or a is selected once".

Similarly, $1+bx$ and $1+cx$ may be interpreted likewise. Now, expanding the product on the LHS of the equality (2.13), we obtain the expression on RHS, from which we see that the exponent of x in a term indicates the number of objects in a selection and the corresponding coefficient shows all the possible ways of selections.

Since we are only interested in the number of ways of selection, we may simply let $a=b=c=1$ and obtain the following:

$$(1+x)(1+x)(1+x)=1+3x+3x^2+1x^3, \tag{2.14}$$

which is the generating function for the sequence $(1,3,3,1,0,0,\ldots)$ (or simply $(1,3,3,1)$ after truncating the 0's at the end of the sequence). Hence the generating function for the number of ways to select r objects from 3 distinct objects is $(1+x)^3$.

Example 2.41 Let $S=\{s_1,s_2,\ldots,s_n\}$, and let a_r denote the number of ways of selecting r elements from S. Then the generating function for the sequence (a_r) is given by

$$\underset{(s_1)}{(1+x)}\underset{(s_2)}{(1+x)}\cdots\underset{(s_n)}{(1+x)}=(1+x)^n=\sum_{r=0}^{n}\binom{n}{r}x^r. \tag{2.15}$$

Thus $\sum_{r=0}^{\infty}a_rx^r=\sum_{r=0}^{n}\binom{n}{r}x^r$, which implies that

$$a_r=\begin{cases}\binom{n}{r} & ;\ 0\le r\le n,\\ 0 & ;\ r\ge n+1.\end{cases} \tag{2.16}$$

2.2.4 Exponential generating functions

We see that (ordinary) generating function are applicable in distribution problems or arrangement problems, in which the ordering of the objects involved is immaterial. In this section, we shall study the so-called exponential generating functions that will be useful in the counting of arrangements of objects where the ordering is taken into consideration.

The *exponential generating function* for the sequence of the numbers (a_r) is defined to be the power series

$$A(x) = a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots + a_r\frac{x^r}{r!} + \cdots = \sum_{r=0}^{\infty} a_r\frac{x^r}{r!}. \tag{2.17}$$

Example 2.42 The generating function for the sequence $(1,1,1,\ldots,1,\ldots)$ is

$$\sum_{r=0}^{\infty}\frac{x^r}{r!} = e^x. \tag{2.18}$$

Example 2.43 The generating function for the sequence $(0!,1!,2!,\ldots,r!,\ldots)$ is

$$\sum_{r=0}^{\infty} r!\frac{x^r}{r!} = 1 + x + x^2 + \cdots = \frac{1}{1-x}. \tag{2.19}$$

Example 2.44 The generating function for the sequence $(1,k,k^2,\ldots,k^r,\ldots)$, where k is a nonzero constant, is

$$1 + \frac{kx}{1!} + \frac{k^2x^2}{2!} + \cdots = \sum_{r=0}^{\infty}\frac{(kx)^r}{r!} = e^{kx}. \tag{2.20}$$

Exponential generating functions for permutation

Recall that P_r^n denotes the number of r-permutations of n distinct objects, and

$$P_r^n = \binom{n}{r}\cdot r!. \tag{2.21}$$

Then

$$\sum_{r=0}^{n} P_r^n\frac{x^r}{r!} = \sum_{r=0}^{n}\binom{n}{r}x^r = (1+x)^n. \tag{2.22}$$

Thus, by definition, the exponential generating function for the sequence $(P_r^n)_{r=0,1,2,\ldots}$ is $(1+x)^n$.

Note that

$$(1+x)^n = \underset{(1)}{(1+\frac{x^1}{1!})}\underset{(2)}{(1+\frac{x^1}{1!})}\cdots\underset{(n)}{(1+\frac{x^1}{1!})}, \tag{2.23}$$

where, as before, each bracket on the RHS corresponds to a distinct object in the arrangement.

Example 2.45 Let a_r denote the number of r-permutations of p identical objects. The exponential generating function for (a_r) is

$$1+2x+3x^2+\cdots=\frac{1}{(1-x)^2}\ . \tag{2.24}$$

Since $a_r=1$ for each $r=0,1,2,...,p$ and $a_r=0$ for each $r>p$. □

Example 2.46 Let a_r denote the number of r-permutations of p identical blue balls and q identical red balls. The exponential generating function for (a_r) is

$$\underset{(B)}{(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^p}{p!})}\underset{(R)}{(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^q}{q!})}. \tag{2.25}$$

Chapter 3

Main Results

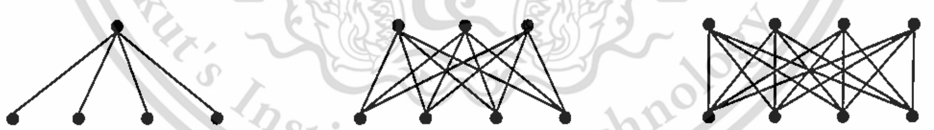
In this chapter, we show the process of formula of the number of labelled trees with r_1, r_2 end-vertices in complete bipartite graph and show some examples of our results.

3.1 The number of labelled trees with r_1, r_2 end-vertices in complete bipartite graph

In this section, we consider the labelled trees in complete bipartite graph and let the blue and red vertices be the vertices in the first and second partition, respectively. We divide three cases of complete bipartite graph such as $K_{1,n}$, $K_{n,n}$ and $K_{m,n}$ because spanning trees in $K_{1,n}$ have end-vertices only in the second partition, spanning trees in $K_{n,n}$ have end-vertices both partitions, and spanning trees in $K_{m,n}$ must have end-vertices in the second partition for $2 \le m < n$ as Example 3.2, 3.3 and 3.4.

Moreover, let $L(m,n,r_1,r_2)$ be the number of labelled trees with r_1, r_2 end-vertices in complete bipartite graph $K_{m,n}$.

Example 3.1 The complete bipartite graphs $K_{1,4}$, $K_{3,4}$ and $K_{4,4}$ as following that



(a) $K_{1,4}$ (b) $K_{3,4}$ (c) $K_{4,4}$

Figure 3.1 The complete bipartite graphs $K_{1,4}$, $K_{3,4}$ and $K_{4,4}$

For all labelled trees in each graph in Figure 3.1(a) and (b), see in Appendix C, have the pattern as the following examples.

Example 3.2 The pattern of labelled trees in $K_{1,4}$ has only one pattern as following that

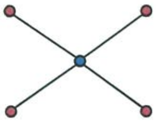


Figure 3.2 The pattern of labelled tree in $K_{1,4}$

Example 3.3 The pattern of labelled trees in $K_{3,4}$ as following that

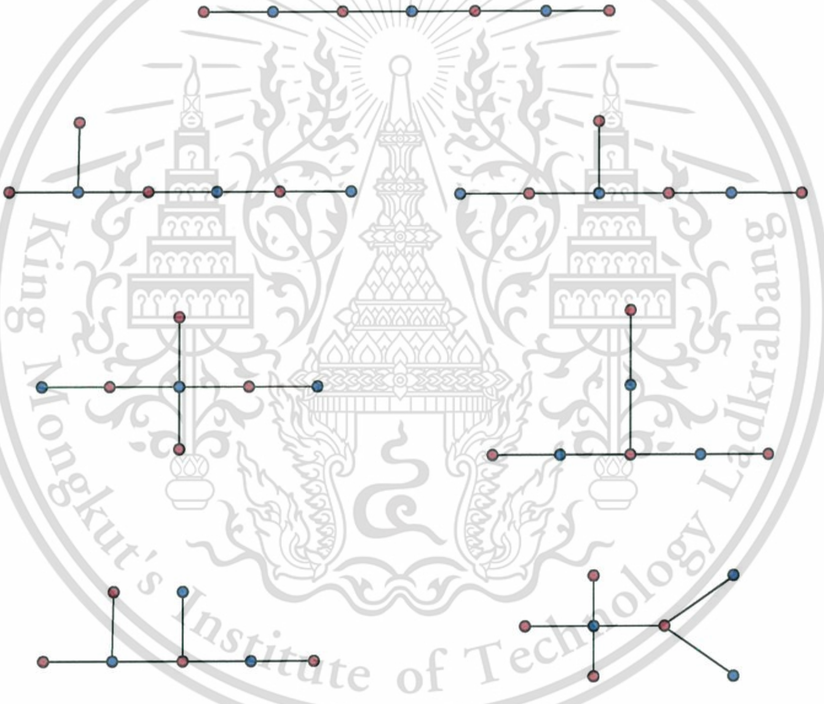


Figure 3.3 The pattern of labelled trees in $K_{3,4}$

Example 3.4 The pattern of labelled trees in $K_{4,4}$ as following that

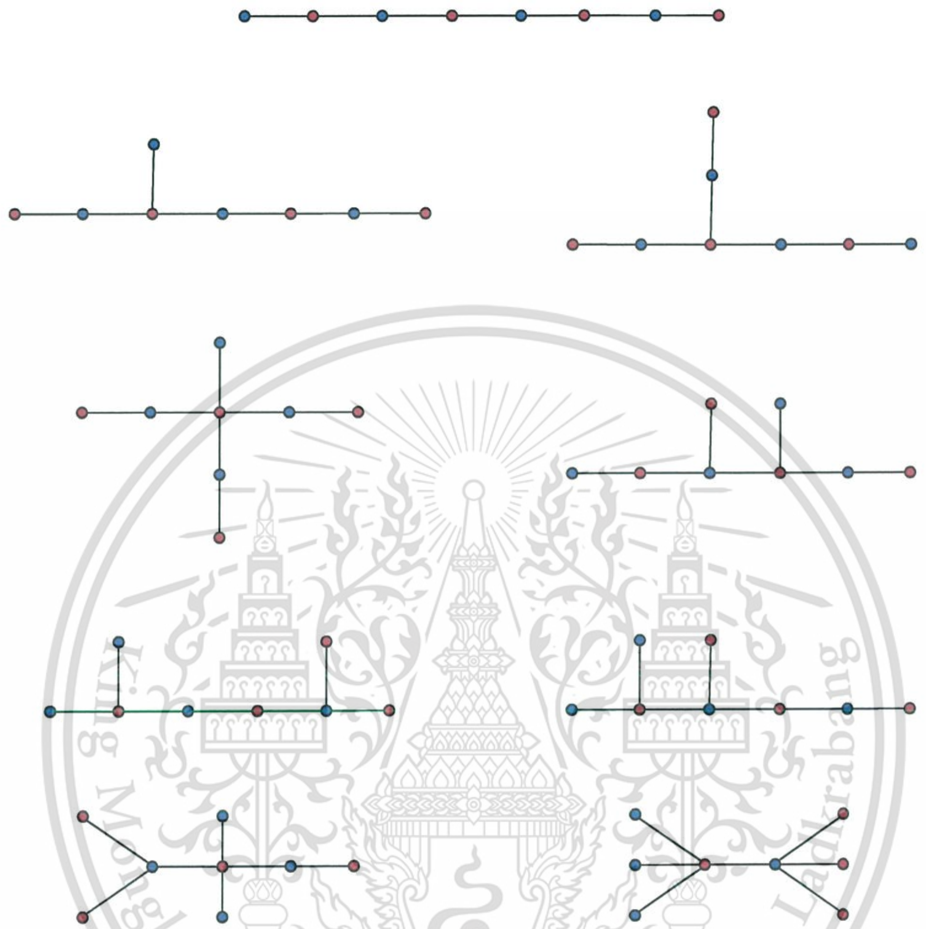


Figure 3.4 The pattern of labelled trees in $K_{4,4}$

Next, let r be the number of end-vertices of spanning trees in complete bipartite graph such that $r = r_1 + r_2$ where r_1 and r_2 are the number of end-vertices in the first (blue vertices) and second (red vertices) partition, respectively.

From example 3.2, 3.3 and 3.4, we show subcase of pattern of labelled trees in complete bipartite graph depend on r_1, r_2 end-vertices as following example 3.5, 3.6 and 3.7

Example 3.5 All case of labelled trees in $K_{1,4}$ as following that

- For $r_1 = 0$, $r_2 = 4$

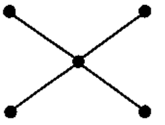


Figure 3.5 The labelled tree with $r_1 = 0$, $r_2 = 4$ in $K_{1,4}$

Example 3.6 All case of labelled trees in $K_{3,4}$ as following that

- For $r_1 = 0$, $r_2 = 2$

Figure 3.6 The labelled tree with $r_1 = 0$, $r_2 = 2$ in $K_{3,4}$

- For $r_1 = 0$, $r_2 = 3$

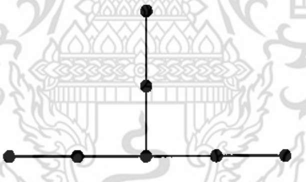
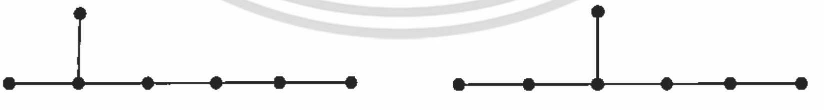


Figure 3.7 The labelled tree with $r_1 = 0$, $r_2 = 3$ in $K_{3,4}$

- For $r_1 = 1$, $r_2 = 2$



(a) (b)

Figure 3.8 The labelled trees with $r_1 = 1$, $r_2 = 2$ in $K_{3,4}$

- For $r_1 = 1$, $r_2 = 3$

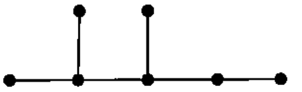


Figure 3.9 The labelled tree with $r_1 = 1$, $r_2 = 3$ in $K_{3,4}$

- For $r_1 = r_2 = 2$

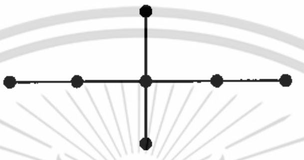


Figure 3.10 The labelled tree with $r_1 = r_2 = 2$ in $K_{3,4}$

- For $r_1 = 2$, $r_2 = 3$

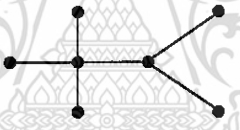


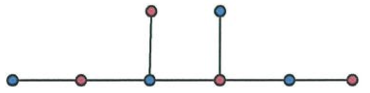
Figure 3.11 The labelled tree with $r_1 = 2$, $r_2 = 3$ in $K_{3,4}$

Example 3.7 All case of labelled trees in $K_{4,4}$ as following that

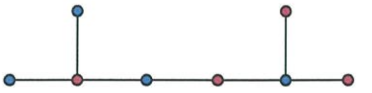
- For $r_1 = r_2 = 1$

Figure 3.12 The labelled tree with $r_1 = r_2 = 1$ in $K_{4,4}$

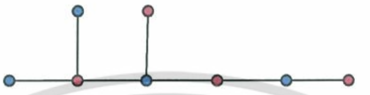
- For $r_1 = r_2 = 2$



(a)



(b)



(c)

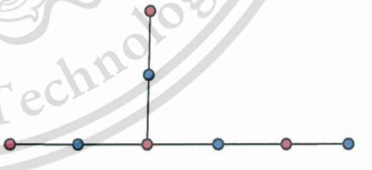
Figure 3.13 The labelled trees with $r_1 = r_2 = 2$ in $K_{4,4}$

- For $r_1 = r_2 = 3$

Figure 3.14 The labelled tree with $r_1 = r_2 = 3$ in $K_{4,4}$

- For $r_1 = 1$, $r_2 = 2$

(a)



(b)

Figure 3.15 The labelled trees with $r_1 = 1$, $r_2 = 2$ in $K_{4,4}$

- For $r_1 = 1$, $r_2 = 3$

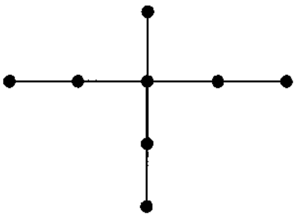


Figure 3.16 The labelled tree with $r_1 = 1$, $r_2 = 3$ in $K_{4,4}$

- For $r_1 = 2$, $r_2 = 3$

Figure 3.17 The labelled tree with $r_1 = 2$, $r_2 = 3$ in $K_{4,4}$

Therefore we will find the number of labelled trees in complete bipartite graph depend on end-vertices. According to finding the case of spanning trees in each graph, for $K_{1,n}$, is a star graph, we have only 1 case that is $r_1 = 0$, $r_2 = n$. So we have $L(1,n,0,n) = 1$, But, for $K_{m,n}$ and $K_{n,n}$, we necessary to consider all case of these cases as following that:

- For $K_{m,n}$, $2 \le r \le m+n-2$, $0 \le r_1 \le m-1$ and $2 \le r_2 \le n-1$.
- For $K_{n,n}$, $2 \le r \le 2n-2$, $1 \le r_1 \le n-1$ and $1 \le r_2 \le n-1$.

Lemma 3.1 Let $1,2,3,\ldots,m$ and $m+1, m+2, \ldots, m+n$ be the labels of a labelled tree in the first and second partition of complete bipartite graph $K_{m,n}$, respectively, with $2 \le m \le n$ that correspond to $(a_1, a_2, \ldots, a_{m+n-2})$. Then $b_1, b_2, \ldots, b_q$ are the labels of the end-vertices if and only if none of $b_1, b_2, \ldots, b_q$ appear in $(m+n-2)$-tuple $(a_1, a_2, \ldots, a_{m+n-2})$.

Proof We note that each of any two end-vertices cannot be adjacent to the other both on the same and other partition. First, we show that each $a_1, a_2, \ldots, a_{m+n-2}$ in the $(m+n-2)$-tuple cannot be the label of end-vertex. From Prufer's construction, since a_1 is adjacent to the deleted end-vertex that a partition of a_1 is opposite of a partition of end-vertex; so a_1 cannot be the label of an end-vertex. And by these construction, a_2 is adjacent to the deleted end-vertex; so a_2 cannot be the label of an end-vertex. Similarly, each of $a_3, a_4, ..., a_{m+n-2}$ cannot be the labels of an end-vertices. Next, we show that label b_j is a label of end-vertex and does not appear in the $(m+n-2)$-tuple. Suppose there is a vertex whose b_j does not be an end-vertex but does not appear in the $(m+n-2)$-tuple. From Prufer's construction, the label b_j must be adjacent to one of the deleting vertices, so b_j must be in the $(m+n-2)$-tuple which this is contradiction. Therefore, label b_j must be the label of end-vertex and does not appear in the $(m+n-2)$-tuple. This proof is completed. □

In fact, this lemma can be use in every pattern of labeling. But In this thesis, we will study in only complete bipartite graph which labelled with $1,2,3,...,m$ in the first partition and $m+1, m+2, ..., m+n$ in the second partition. Next, we shall find the formula of the number of labelled trees with r_1, r_2 end-vertices in complete bipartite graph $K_{m,n}$ and $K_{n,n}$.

Theorem 3.1 Let $L(m,n,r_1,r_2)$ be the number of labelled trees in complete bipartite graph $K_{m,n}$ with m,n,r_2 are positive numbers and r_1 is non-negative number. Then

$$L(m,n,r_1,r_2) = \binom{m}{r_1}\binom{n}{r_2} A_{r_1} A_{r_2}, \tag{3.1}$$

where $2 \le m < n$, $0 \le r_1 \le m-1$, $2 \le r_2 \le n-1$,

$$A_{r_1} = \sum_{i=0}^{m-r_1-1} (-1)^i \binom{m-r_1}{i} (m-r_1-i)^{n-1},$$

and $$A_{r_2} = \sum_{j=0}^{n-r_2-1} (-1)^j \binom{n-r_2}{j} (n-r_2-j)^{m-1}.$$

Proof For $2 \le m < n$, r_1 is non-negative number and r_2 is positive numbers, we consider $(m+n-2)$-tuple $(a_1, a_2, ..., a_{m+n-2})$ of the complete bipartite graph $K_{m,n}$ which the first and second partition have m and n vertices, respectively, and it consists $q = q_1 + q_2$ different labels that q_1 is the label in the first partition and q_2 is the label in the second partition. Let $r = r_1 + r_2$ be the number of labels that are not in $(m+n-2)$-tuple and r_1, r_2 be the number of end-vertices in each partitions. Thus $q = m+n-r$, $q_1 = m - r_1$ and $q_2 = n - r_2$. According to lemma 3.1, we use exponential generating function in counting all of these $(m+n-2)$-tuple. Since the q_1,q_2 labels must appear at least once in $(m+n-2)$-tuple, the related generating function would be

$$
\begin{aligned}
f(x)g(x) &= \left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^{q_1}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^{q_2} \\
&= \left(e^x-1\right)^{q_1}\left(e^x-1\right)^{q_2} \\
&= \left[\binom{q_1}{0}e^{q_1x}-\binom{q_1}{1}e^{(q_1-1)x}+\cdots+(-1)^{q_1-1}\binom{q_1}{q_1-1}e^x+(-1)^{q_1}\binom{q_1}{q_1}\right] \\
&\quad\times\left[\binom{q_2}{0}e^{q_2x}-\binom{q_2}{1}e^{(q_2-1)x}+\cdots+(-1)^{q_2-1}\binom{q_2}{q_2-1}e^x+(-1)^{q_2}\binom{q_2}{q_2}\right] \\
&= \left[\left[(-1)^0\binom{q_1}{0}\left[1+q_1x+\frac{q_1^2x^2}{2!}+\cdots+\frac{q_1^{n-1}x^{n-1}}{(n-1)!}+\cdots\right]\right]\right. \\
&\quad+\left[(-1)^1\binom{q_1}{1}\left[1+(q_1-1)x+\frac{(q_1-1)^2x^2}{2!}+\cdots+\frac{(q_1-1)^{n-1}x^{n-1}}{(n-1)!}+\cdots\right]\right] \\
&\quad+\left[(-1)^2\binom{q_1}{2}\left[1+(q_1-2)x+\frac{(q_1-2)^2x^2}{2!}+\cdots+\frac{(q_1-2)^{n-1}x^{n-1}}{(n-1)!}+\cdots\right]\right] \\
&\quad\left.+\cdots+\left[(-1)^{q_1-1}\binom{q_1}{q_1-1}\left[1+x+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\cdots\right]\right]+(-1)^{q_1}\right] \\
&\quad\times\left[\left[(-1)^0\binom{q_2}{0}\left[1+q_2x+\frac{q_2^2x^2}{2!}+\cdots+\frac{q_2^{m-1}x^{m-1}}{(m-1)!}+\cdots\right]\right]\right. \\
&\quad+\left[(-1)^1\binom{q_2}{1}\left[1+(q_2-1)x+\frac{(q_2-1)^2x^2}{2!}+\cdots+\frac{(q_2-1)^{m-1}x^{m-1}}{(m-1)!}+\cdots\right]\right] \\
&\quad+\left[(-1)^2\binom{q_2}{2}\left[1+(q_2-2)x+\frac{(q_2-2)^2x^2}{2!}+\cdots+\frac{(q_2-2)^{m-1}x^{m-1}}{(m-1)!}+\cdots\right]\right] \\
&\quad\left.+\cdots+\left[(-1)^{q_2-1}\binom{q_2}{q_2-1}\left[1+x+\frac{x^2}{2!}+\cdots+\frac{x^{m-1}}{(m-1)!}+\cdots\right]\right]+(-1)^{q_2}\right]. \qquad (3.2)
\end{aligned}
$$

From equation (3.2), the coefficient of $\frac{x^{n-1}}{(n-1)!}$ and $\frac{x^{m-1}}{(m-1)!}$ in expand $f(x)g(x)$ are

$$\left[\sum_{i=0}^{q_1-1}(-1)^i\binom{q_1}{i}(q_1-i)^{n-1}\right]\left[\sum_{j=0}^{q_2-1}(-1)^j\binom{q_2}{j}(q_2-j)^{m-1}\right].$$

This coefficient is the number of permutation of $a_1,a_2,\dots,a_{m+n-2}$ in $(a_1,a_2,\dots,a_{m+n-2})$ that consists of q_1,q_2 labels and equal to the number of labelled trees in $K_{m,n}$ with r_1,r_2 end-vertices and q_1,q_2 particular non-end-vertices.

If $r_2=0$ and $m<n$, then every end-vertices of spanning subgraph must be in the first partition and spanning subgraph contain cycle. So it has not a spanning tree in $K_{m,n}$.

If $r_2\neq 0$, the labeling of non-end-vertices can be done in $\binom{m}{q_1}\binom{n}{q_2}$ possible ways which $q_1=m-r_1$ and $q_2=n-r_2$. Thus, we have

$$L(m,n,r_1,r_2)=\binom{m}{r_1}\binom{n}{r_2}A_{r_1}A_{r_2}, \tag{3.3}$$

where $2\le m<n,\ 0\le r_1\le m-1,\ 2\le r_2\le n-1,$

$$A_{r_1}=\sum_{i=0}^{m-r_1-1}(-1)^i\binom{m-r_1}{i}(m-r_1-i)^{n-1},$$

and $$A_{r_2}=\sum_{j=0}^{n-r_2-1}(-1)^j\binom{n-r_2}{j}(n-r_2-j)^{m-1}.$$ □

Theorem 3.2 Let $L(n,n,r_1,r_2)$ be the number of labelled trees in complete bipartite graph $K_{n,n}$ with n,r_1 and r_2 are positive numbers. Then

$$L(n,n,r_1,r_2)=\begin{cases} B_k^2 & ;\ r_1=r_2=k,\\ 2B_{r_1}B_{r_2} & ;\ r_1\neq r_2,\end{cases} \tag{3.4}$$

where $n\ge 2,\ 2\le k\le n-1,\ 1\le r_1\le n-1,\ 1\le r_2\le n-1,$

and $B_j=\binom{n}{j}\sum_{i=0}^{n-j-1}(-1)^i\binom{n-j}{i}(n-j-i)^{n-1}$.

Proof For $n\ge 2$, we consider $(2n-2)$-tuple $(a_1,a_2,\dots,a_{2n-2})$ of the complete bipartite graph $K_{n,n}$ which it consists $q=q_1+q_2$ different labels that q_1 is the label in the first partition and q_2 is the label in the second partition. Let $r=r_1+r_2$ be the number of labels that are not in $(2n-2)$-tuple and r_1,r_2 be the number of end-vertices in each partitions. Thus $q=2n-r$, $q_1=n-r_1$ and $q_2=n-r_2$. Moreover, we see that $n-r$ possible ways of end-vertices in each partition. According to lemma 3.1, we use exponential generating function in counting all of these $(2n-2)$-tuple. Since the q_1, q_2 labels must appear at least once in $(2n-2)$-tuple, the related generating function would be

$$
\begin{aligned}
f(x)g(x) &= \left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^{q_1}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^{q_2} \\
&= \left(e^x-1\right)^{q_1}\left(e^x-1\right)^{q_2} \\
&= \left[\binom{q_1}{0}e^{q_1x}-\binom{q_1}{1}e^{(q_1-1)x}+\cdots+(-1)^{q_1-1}\binom{q_1}{q_1-1}e^x+(-1)^{q_1}\binom{q_1}{q_1}\right] \\
&\quad\times\left[\binom{q_2}{0}e^{q_2x}-\binom{q_2}{1}e^{(q_2-1)x}+\cdots+(-1)^{q_2-1}\binom{q_2}{q_2-1}e^x+(-1)^{q_2}\binom{q_2}{q_2}\right] \\
&= \left[\left[(-1)^0\binom{q_1}{0}\left[1+q_1x+\frac{q_1^2x^2}{2!}+\cdots+\frac{q_1^{n-1}x^{n-1}}{(n-1)!}+\cdots\right]\right]\right. \\
&\quad+\left[(-1)^1\binom{q_1}{1}\left[1+(q_1-1)x+\frac{(q_1-1)^2x^2}{2!}+\cdots+\frac{(q_1-1)^{n-1}x^{n-1}}{(n-1)!}+\cdots\right]\right] \\
&\quad+\left[(-1)^2\binom{q_1}{2}\left[1+(q_1-2)x+\frac{(q_1-2)^2x^2}{2!}+\cdots+\frac{(q_1-2)^{n-1}x^{n-1}}{(n-1)!}+\cdots\right]\right] \\
&\quad\left.+\cdots+\left[(-1)^{q_1-1}\binom{q_1}{q_1-1}\left[1+x+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\cdots\right]\right]+(-1)^{q_1}\right] \\
&\quad\times\left[\left[(-1)^0\binom{q_2}{0}\left[1+q_2x+\frac{q_2^2x^2}{2!}+\cdots+\frac{q_2^{n-1}x^{n-1}}{(n-1)!}+\cdots\right]\right]\right. \\
&\quad+\left[(-1)^1\binom{q_2}{1}\left[1+(q_2-1)x+\frac{(q_2-1)^2x^2}{2!}+\cdots+\frac{(q_2-1)^{n-1}x^{n-1}}{(n-1)!}+\cdots\right]\right] \\
&\quad+\left[(-1)^2\binom{q_2}{2}\left[1+(q_2-2)x+\frac{(q_2-2)^2x^2}{2!}+\cdots+\frac{(q_2-2)^{n-1}x^{n-1}}{(n-1)!}+\cdots\right]\right] \\
&\quad\left.+\cdots+\left[(-1)^{q_2-1}\binom{q_2}{q_2-1}\left[1+x+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\cdots\right]\right]+(-1)^{q_2}\right].
\end{aligned}
\tag{3.5}
$$

From equation (3.5), the coefficient of $\frac{x^{n-1}}{(n-1)!}$ in expand $f(x)g(x)$ are

$$\left[\sum_{i=0}^{q_1-1}(-1)^i\binom{q_1}{i}(q_1-i)^{n-1}\right]\left[\sum_{i=0}^{q_2-1}(-1)^i\binom{q_2}{i}(q_2-i)^{n-1}\right].$$

This coefficient is the number of permutation of $a_1, a_2, \dots, a_{2n-2}$ in $(a_1, a_2, \dots, a_{2n-2})$ that consists of q_1, q_2 labels and equal to the number of labelled trees in $K_{n,n}$ with r_1, r_2 end-vertices and q_1, q_2 particular non-end-vertices. We have 2 cases that are $r_1 = r_2$ and $r_1 \neq r_2$.

Case I $r_1 = r_2 = k$.

This q labeling of non-end-vertices can be done in $\binom{n}{k}\binom{n}{k}$ possible ways and $q_1 = q_2 = n-k$. Thus, for $r_1 = r_2 = k$, we have

$$L(n,n,r_1,r_2) = \left[\binom{n}{k}\sum_{i=0}^{n-k-1}(-1)^i\binom{n-k}{i}(n-k-i)^{n-1}\right]^2. \tag{3.6}$$

Case II $r_1 \neq r_2$.

Each tree is the non-symmetrical tree. Then we must look both partitions. Since $r_1 \neq r_2$, q_1, q_2 labeling of non-end-vertices can be done in $2\binom{n}{k}\binom{n}{k}$ possible ways and $q_1 = n - r_1$, $q_2 = n - r_2$. Hence, for $r_1 \neq r_2$, we get

$$\begin{aligned} L(n,n,r_1,r_2) = 2&\left[\binom{n}{r_1}\sum_{i=0}^{n-r_1-1}(-1)^i\binom{n-r_1}{i}(n-r_1-i)^{n-1}\right] \\ &\left[\binom{n}{r_2}\sum_{i=0}^{n-r_2-1}(-1)^i\binom{n-r_2}{i}(n-r_2-i)^{n-1}\right]. \end{aligned} \tag{3.7}$$

Therefore we have proved the following theorem. □

According to above theorems, some number of labelled trees in $K_{m,n}$ and $K_{n,n}$ are

- $L(m,n,1,1) = 0$,
- $L(n,n,1,1) = (n!)^2$,
- $L(m,n,m-1,n-1) = mn$ and
- $L(n,n,n-1,n-1) = n^2$.

Next, we show some examples from above theorem.

Note that the sum of number of labelled trees with r_1, r_2 end-vertices in complete bipartite graph is the number of labelled trees in complete bipartite graph.

3.2 Examples of the number of labelled trees with r_1, r_2 end-vertices in complete bipartite graph

In this section, we show the number of labelled trees with r_1, r_2 end-vertices in some complete bipartite graphs.

Example 3.8 We have the number of labelled trees with r_1, r_2 end-vertices in $K_{1,4}$ is equal to 1.

$$L(1,4,0,4) = 1.$$

Hence, $T(K_{1,4}) = 1$. □

Example 3.9 The number of labelled trees with r_1, r_2 end-vertices in $K_{3,4}$. We have

$$\begin{aligned}
L(3,4,0,2) &= \binom{3}{0}\left[\sum_{i=0}^{3-0-1}(-1)^i\binom{3-0}{i}(3-0-i)^{4-1}\right]\binom{4}{2}\left[\sum_{j=0}^{4-2-1}(-1)^j\binom{4-2}{j}(4-2-j)^{3-1}\right] \\
&= (1)[27-24+3](6)[4-2] \\
&= (1)[6](6)[2] \qquad = 72; \\
L(3,4,0,3) &= \binom{3}{0}\left[\sum_{i=0}^{3-0-1}(-1)^i\binom{3-0}{i}(3-0-i)^{4-1}\right]\binom{4}{3}\left[\sum_{j=0}^{4-3-1}(-1)^j\binom{4-3}{j}(4-3-j)^{3-1}\right] \\
&= (1)[27-24+3](4)[1] \\
&= (1)[6](4)[1] \qquad = 24; \\
L(3,4,1,2) &= \binom{3}{1}\left[\sum_{i=0}^{3-1-1}(-1)^i\binom{3-1}{i}(3-1-i)^{4-1}\right]\binom{4}{2}\left[\sum_{j=0}^{4-2-1}(-1)^j\binom{4-2}{j}(4-2-j)^{3-1}\right] \\
&= (3)[8-2](6)[4-2] \\
&= (3)[6](6)[2] \qquad = 216;
\end{aligned}$$

$$L(3,4,1,3) = \binom{3}{1}\left[\sum_{i=0}^{3-1-1}(-1)^i\binom{3-1}{i}(3-1-i)^{4-1}\right]\binom{4}{3}\left[\sum_{j=0}^{4-3-1}(-1)^j\binom{4-3}{j}(4-3-j)^{3-1}\right]$$

$$= (3)[8-2](4)[1]$$

$$= (3)[6](4)[1] \qquad = 72;$$

$$L(3,4,2,2) = \binom{3}{2}\left[\sum_{i=0}^{3-2-1}(-1)^i\binom{3-2}{i}(3-2-i)^{4-1}\right]\binom{4}{2}\left[\sum_{j=0}^{4-2-1}(-1)^j\binom{4-2}{j}(4-2-j)^{3-1}\right]$$

$$= (3)[1](6)[4-2]$$

$$= (3)[1](6)[2] \qquad = 36;$$

$$L(3,4,2,3) = \binom{3}{2}\left[\sum_{i=0}^{3-2-1}(-1)^i\binom{3-2}{i}(3-2-i)^{4-1}\right]\binom{4}{3}\left[\sum_{j=0}^{4-3-1}(-1)^j\binom{4-3}{j}(4-3-j)^{3-1}\right]$$

$$= (3)[1](4)[1] \qquad = 12.$$

Hence, $T(K_{3,4}) = 72+24+216+72+36+12 = 432 = (3^{4-1})(4^{3-1})$. □

Example 3.10 The number of labelled trees with r_1, r_2 end-vertices in $K_{4,4}$. We have

$$L(4,4,1,1) = B_1^2$$

$$= \left[\binom{4}{1}\sum_{i=0}^{4-1-1}(-1)^i\binom{4-1}{i}(4-1-i)^{4-1}\right]^2$$

$$= \left[4(27-24+3)\right]^2$$

$$= 24^2 \qquad = 576;$$

$$L(4,4,1,2) = 2B_1B_2$$

$$= 2\left[\binom{4}{1}\sum_{i=0}^{4-1-1}(-1)^i\binom{4-1}{i}(4-1-i)^{4-1}\right]\left[\binom{4}{2}\sum_{i=0}^{4-2-1}(-1)^i\binom{4-2}{i}(4-2-i)^{4-1}\right]$$

$$= 2\left[(27-24+3)\right]\left[6(8-2)\right]$$

$$= 2[24][36] \qquad = 1728;$$

$$L(4,4,1,3) = 2B_1B_3$$

$$= 2\left[\binom{4}{1}\sum_{i=0}^{4-1-1}(-1)^i\binom{4-1}{i}(4-1-i)^{4-1}\right]\left[\binom{4}{3}\sum_{i=0}^{4-3-1}(-1)^i\binom{4-3}{i}(4-3-i)^{4-1}\right]$$

$$= 2\left[(27-24+3)\right]\left[4(1)\right]$$

$$= 2[24][4] \qquad = 192;$$

$$
\begin{aligned}
L(4,4,2,2) &= B_2^2 \\
&= \left[\binom{4}{2}\sum_{i=0}^{4-2-1}(-1)^i\binom{4-2}{i}(4-2-i)^{4-1}\right]^2 \\
&= \left[6(8-2)\right]^2 \\
&= 36^2 \qquad\qquad = 1296;
\end{aligned}
$$

$$
\begin{aligned}
L(4,4,2,3) &= 2B_2B_3 \\
&= 2\left[\binom{4}{2}\sum_{i=0}^{4-2-1}(-1)^i\binom{4-2}{i}(4-2-i)^{4-1}\right]\left[\binom{4}{3}\sum_{i=0}^{4-3-1}(-1)^i\binom{4-3}{i}(4-3-i)^{4-1}\right] \\
&= 2\left[6(8-2)\right]\left[4(1)\right] \\
&= 2[36][4] \qquad\qquad = 288;
\end{aligned}
$$

$$
\begin{aligned}
L(4,4,3,3) &= B_3^2 \\
&= \left[\binom{4}{3}\sum_{i=0}^{4-3-1}(-1)^i\binom{4-3}{i}(4-3-i)^{4-1}\right]^2 \\
&= \left[4(1)\right]^2 \\
&= 4^2 \qquad\qquad = 16.
\end{aligned}
$$

Hence, $T(K_{4,4}) = 576+1728+192+1296+288+16 = 4096 = 4^{2(4)-2}$. □

Example 3.11 The number of labelled trees with r_1, r_2 end-vertices in $K_{5,7}$. The results as following in Table 3.1.

Table 3.1 The number of labelled trees in $K_{5,7}$

$L(m,n,r_1,r_2)$	$n = 7$								$T(K_{m,n}) = m^{n-1}n^{m-1}$
	r_2 \ r_1	0	1	2	3	4	5	6	
$m = 5$	0	-	-	-	1512000	2268000	529200	12600	
	1	-	-	-	6552000	9828000	2293200	54600	
	2	-	-	-	4536000	6804000	1587600	37800	
	3	-	-	-	520800	781200	182280	4340	
	4	-	-	-	4200	6300	1470	35	
	5	-	-	-	-	-	-	-	
	6	-	-	-	-	-	-	-	
	Total	-	-	-	13125000	19687500	4593750	109375	37515625

Example 3.12 The number of labelled trees with r_1, r_2 end-vertices in $K_{7,7}$. The results as following in Table 3.2.

Table 3.2 The number of labelled trees in $K_{7,7}$

$L(m,n,r_1,r_2)$		$n=7$						
	r_1 \ r_2	1	2	3	4	5	6	$T(K_{n,n}) = n^{2n-2}$
$n=7$	1	25401600	-	-	-	-	-	
	2	381024000	1428840000	-	-	-	-	
	3	550368000	4127760000	2981160000	-	-	-	
	4	190512000	1428840000	2063880000	357210000	-	-	
	5	13124160	98431200	142178400	49215600	1695204	-	
	6	70560	529200	764400	264600	18228	49	
	Total	1160500320	7084400400	5187982800	406690200	1713432	49	13841287201

Chapter 4
Conclusions

The purpose of this chapter is conclude from previous chapter which is the results of our research. We assemble all results in this chapter.

4.1 The number of labelled trees with r_1, r_2 end-vertices in complete bipartite graph

According to the previous chapter and consider the complete bipartite graph, we have three cases such as $K_{1,n}$, $K_{m,n}$ and $K_{n,n}$ depend on r_1, r_2 end-vertices.

For finding the number of labelled trees with r_1, r_2 end-vertices in complete bipartite graph that used Prufer's construction and exponential generating function as following that:

Lemma 4.1 Let $1,2,3,\ldots,m$ and $m+1, m+2, \ldots, m+n$ be the labels of a labelled tree in the first and second partition of complete bipartite graph $K_{m,n}$, respectively, with $2 \le m \le n$ that correspond to $(a_1, a_2, \ldots, a_{m+n-2})$. Then $b_1, b_2, \ldots, b_q$ are the labels of the end-vertices if and only if none of $b_1, b_2, \ldots, b_q$ appear in $(m+n-2)$-tuple $(a_1, a_2, \ldots, a_{m+n-2})$.

Theorem 4.1 Let $L(m,n,r_1,r_2)$ be the number of labelled trees in complete bipartite graph $K_{m,n}$ with m, n, r_2 are positive numbers and r_1 is non-negative number. Then

$$L(m,n,r_1,r_2) = \binom{m}{r_1}\binom{n}{r_2} A_{r_1} A_{r_2}, \tag{4.1}$$

where $2 \le m < n$, $0 \le r_1 \le m-1$, $2 \le r_2 \le n-1$,

$$A_{r_1} = \sum_{i=0}^{m-r_1-1} (-1)^i \binom{m-r_1}{i} (m-r_1-i)^{n-1},$$

and $$A_{r_2} = \sum_{j=0}^{n-r_2-1} (-1)^j \binom{n-r_2}{j} (n-r_2-j)^{m-1}.$$

Theorem 4.2 Let $L(n,n,r_1,r_2)$ be the number of labelled trees in complete bipartite graph $K_{n,n}$ with n, r_1 and r_2 are positive numbers. Then

$$L(n,n,r_1,r_2)=\begin{cases} B_k^2 & ; r_1=r_2=k, \\ 2B_{r_1}B_{r_2} & ; r_1 \neq r_2, \end{cases} \tag{4.2}$$

where $n \geq 2$, $2 \leq k \leq n-1$, $1 \leq r_1 \leq n-1$, $1 \leq r_2 \leq n-1$,

and $B_j = \binom{n}{j}\sum_{i=0}^{n-j-1}(-1)^i\binom{n-j}{i}(n-j-i)^{n-1}$.

Moreover, we have

- For $K_{1,n}$, that is a star graph S_n, we have $L(1,n,0,n)=1$.
- For $K_{m,n}$, if $r_1=r_2=1$, then $L(m,n,1,1)=0$,
  if $r_1=m-1$ and $r_2=n-1$, then $L(m,n,m-1,n-1)=mn$.
- For $K_{n,n}$, if $r_1=r_2=1$, then $L(n,n,1,1)=(n!)^2$,
  if $r_1=r_2=n-1$, then $L(n,n,n-1,n-1)=n^2$.

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Appendices

Appendix A
Research paper

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The Number of Labelled Trees with r_1, r_2 End-Vertices in $K_{n,n}$

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Abstract : Let $L(n,r_1,r_2)$ be the number of labelled trees with r_1, r_2 end-vertices in $K_{n,n}$. In this paper, we use the exponential generating function to find the formula of $L(n,r_1,r_2)$.

Keywords : Complete bipartite graph, Labelled, Trees, End-vertices, Counting.
2010 Mathematics Subject Classification : 05A15; 05C05; 05C78.

1 Introduction

We shall follow the terminology and notation of the book by Gary Chartrand, Linda Lesniak [1] and Jonathan L. Gross, Jay Yellen [2]. A labelled tree is a tree in which labels, typically $v_1, v_2, \ldots, v_n$, have been assigned to the vertices. Two labelled trees with the same set of labels are considered the same only if there is an isomorphism from one to the other that preserves the labels.

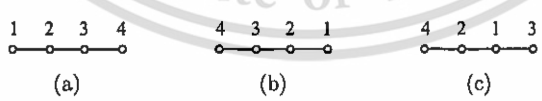


Fig.1 Labelled trees in K_4 with 2 end-vertices

[1]Corresponding author.

In Figure 1, (a) and (b) are the same labelled trees but (a) and (c) are the different labelled trees.

The number of labelled trees in a graph G, denoted by $T(G)$, is the total number of distinct spanning subgraphs that are trees. According to Cayley's Theorem, the number of of labelled trees with n vertices is $T(K_n) = n^{n-2}$. That is the number of labelled trees in complete graph K_n.

For example, when $n = 4$, $T(K_4) = 4^{4-2} = 16$ as following in Figure 4.

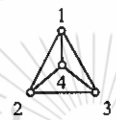
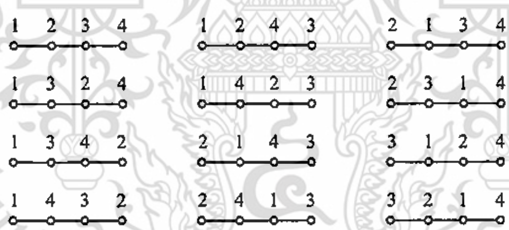


Fig.2 Labelled graph K_4

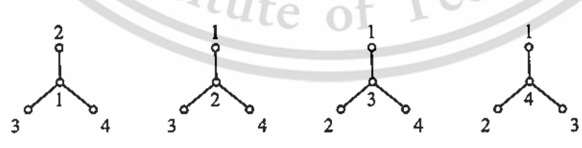
Fig.3 The spanning trees in K_4

$L(4,2) = 12$



(a) Labelled trees with 2 end-vertices in K_4.

$L(4,3) = 4$



(b) Labelled trees with 3 end-vertices in K_4.

Fig.4 Labelled trees in K_4.

Next, Vites Longani [3] has developed Cayley's Theorem to be the formula of the number of labelled trees with r end-vertices in K_n.

$$L(n,r)=\binom{n}{n-r}\sum_{i=0}^{n-r-1}(-1)^i\binom{n-r}{i}(n-r-i)^{n-2}. \tag{1.1}$$

For example, when $n=4$, there are 2 trees but if the trees are labelled, then there are 4 possible ways to label the left tree, and 12 possible ways to label the right tree of Figure 3. Then, in total, there are 16 labelled trees. So, $L(4,2)=12$ and $L(4,3)=4$ that see in Figure 4.

In 1990, Mohammad Z. Abu-Sbieh [4] used a new technique for proving the formula of labelled trees in complete graph and complete bipartite graph. But Yinglie Jin, Chunlin Liu [5] showed the formula of labelled trees in complete bipartite graph using exponential generating function $T(K_{m,n})=m^{n-1}n^{m-1}$ where $m\geq n$ and $m,n\geq 1$. So, when $m=n$, we have $T(K_{n,n})=n^{2n-2}$.

Consider $K_{4,4}$, the pattern of labelled trees in $K_{4,4}$ as follows

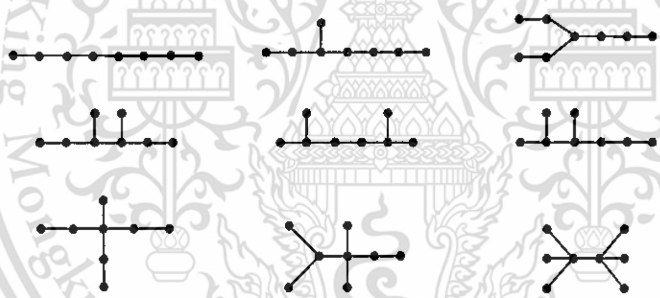


Fig.5 The pattern of labelled trees in $K_{4,4}$

Thus, the number of labeled trees in $K_{4,4}$ depends on the number of end-vertices of labelled trees. In this paper, we define $L(n,r_1,r_2)$ to be the number of labelled trees with r_1,r_2 end-vertices in $K_{n,n}$. Observe that $T(K_{n,n})=\sum_{r_1,r_2=1}^{n-1}L(n,r_1,r_2)$.

Moreover, Richard P. Lewis [6] showed the number of labelled trees in complete multipartite graph that used Prufer's sequence. He has

$$T\left(K_{n_0,n_1,\ldots,n_{k-1}}\right) = n^{k-2}\prod_{0\le i<k}\left(n-n_i\right)^{n_i-i},$$

where $n = n_0 + n_1 + \ldots + n_{k-1}$.

Next, G. Baron et. al. [7] showed that the formula of the number of labelled trees in square of cycle using Matrix Tree Theorem; they have

$$T(C_n^2) = nF_n^2,$$

where F_n is Fibonacci number.

2 The number of labelled trees in $K_{n,n}$

Let r be the number of end-vertices of spanning trees in $K_{n,n}$ such that $r = r_1 + r_2$ and r_1, r_2 are the number of end-vertices in each partition. For example, the pattern of spanning trees in $K_{4,4}$ as following that

- For $r_1 = r_2 = 1$

Fig.6 The spanning tree of $K_{4,4}$, $r_1 = r_2 = 1$

- For $r_1 = 1$, $r_2 = 2$

Fig.7 The spanning trees of $K_{4,4}$, $r_1 = 1$ and $r_2 = 2$

- For $r_1 = 1$, $r_2 = 3$ and $r_1 = r_2 = 2$ see in Fig.8,9 respectively.

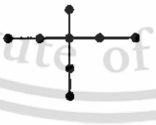


Fig.8 The spanning tree of $K_{4,4}$, $r_1 = 1$ and $r_2 = 3$

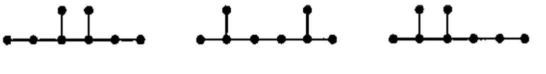


Fig.9 The spanning trees of $K_{4,4}$, $r_1 = r_2 = 2$

- For $r_1 = 2,\ r_2 = 3$

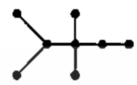


Fig.10 The spanning tree of $K_{4,4}$, $r_1 = 2$ and $r_2 = 3$

- For $r_1 = r_2 = 3$

Fig.11 The spanning tree of $K_{4,4}$ with $r_1 = r_2 = 3$

Therefore we will find the number of labelled trees in $K_{n,n}$ depend on end-vertices.

Lemma 2.1. Let $1,2,3,\dots,2n$ be the labels of a labelled tree in the complete bipartite graph $K_{n,n}$ with $n \geq 2$ that correspond to $(a_1, a_2, \dots, a_{2n-2})$. Then $b_1, b_2, \dots, b_q$ are the labels of the end-vertices if and only if none of $b_1, b_2, \dots, b_q$ appear in $(2n-2)$-tuple $(a_1, a_2, \dots, a_{2n-2})$.

Proof We note that each of any two-vertices cannot be adjacent to the other both on the same and other partition. First, we show that each $a_1, a_2, \dots, a_{2n-2}$ in the $(2n-2)$-tuple cannot be the label of end-vertex. From Prufer's construction, since a_1 is adjacent to the deleted end-vertex that a partition of a_1 is opposite of a partition of end-vertex; so a_1 cannot be the label of an end-vertex. And by these construction, a_2 is adjacent to the deleted end-vertex; so a_2 cannot be the label of an end-vertex. Similarly, each of $a_3, a_4, \dots, a_{2n-2}$ cannot be the labels of an end-vertices. Next, we show that label b_j is a label of end-vertex and does not appear in the $(2n-2)$-tuple. Suppose there is a vertex whose b_j does not be an end-vertex but does not appear in the $(2n-2)$-tuple. From Prufer's construction, the label b_j must be adjacent to one of the deleting vertices, so b_j must be in the $(2n-2)$-tuple which this is contradiction. Therefore, label b_j must be the a label of end-vertex and does not appear in the $(2n-2)$-tuple. This proof is completed. □

By definition of labelled trees in $K_{n,n}$. It easy to see that

$$L(n,1,1) = (n!)^2 \text{ and } L(n, n-1, n-1) = n^2.$$

Next, we shall find the formula of the number of labelled trees $L(n,r_1,r_2)$.

Theorem 2.2. Let $L(n,r_1,r_2)$ be the number of labelled trees in $K_{n,n}$ with $n \geq 2$ and r_1, r_2 end-vertices. Then,

$$L(n,r_1,r_2) = \begin{cases} A_k^2 & ;\text{if } r_1 = r_2 = k \\ 2A_{r_1}A_{r_2} & ;\text{if } r_1 \neq r_2 \end{cases}, \tag{2.1}$$

where $2 \leq k \leq n-1$ and $A_j = \binom{n}{j}\sum_{i=0}^{n-j-1}(-1)^i\binom{n-j}{i}(n-j-i)^{n-1}$.

Proof For $n \geq 2$, we consider $(2n-2)$-tuple $(a_1,a_2,\ldots,a_{2n-2})$ of the complete bipartite graph $K_{n,n}$ which it consists $q = q_1 + q_2$ different labels that q_1 is the label in the first partition and q_2 is the label in the second partition. Let $r = r_1 + r_2$ be the number of labels that are not in $(2n-2)$-tuple and r_1, r_2 be the number of end-vertices in each partitions. Thus $q = 2n - r$, $q_1 = n - r_1$ and $q_2 = n - r_2$. Moreover, we see that $n-1$ possible ways of end-vertices in each partition. According to lemma 1, we use exponential generating function in counting all of these $(2n-2)$-tuple. Since the q_1, q_2 labels must appear at least once in $(2n-2)$-tuple, the related generating function would be

$$\begin{aligned}
f(x)g(x) &= \left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)^{q_1}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)^{q_2} \\
&= (e^x-1)^{q_1}(e^x-1)^{q_2} \\
&= \left[\binom{q_1}{0}e^{q_1x}-\binom{q_1}{1}e^{(q_1-1)x}+\ldots+(-1)^{q_1-1}\binom{q_1}{q_1-1}e^x+(-1)^{q_1}\binom{q_1}{q_1}\right] \\
&\quad\times\left[\binom{q_2}{0}e^{q_2x}-\binom{q_2}{1}e^{(q_2-1)x}+\ldots+(-1)^{q_2-1}\binom{q_2}{q_2-1}e^x+(-1)^{q_2}\binom{q_2}{q_2}\right] \\
&= \Bigg[\left[(-1)^0\binom{q_1}{0}\left[1+q_1x+\frac{{q_1}^2x^2}{2!}+\ldots+\frac{{q_1}^{n-1}x^{n-1}}{(n-1)!}+\ldots\right]\right] \\
&\quad+\left[(-1)^1\binom{q_1}{1}\left[1+(q_1-1)x+\frac{(q_1-1)^2x^2}{2!}+\ldots+\frac{(q_1-1)^{n-1}x^{n-1}}{(n-1)!}+\ldots\right]\right]
\end{aligned}$$

$$+\left[(-1)^2\binom{q_1}{2}\left[1+(q_1-2)x+\frac{(q_1-2)^2x^2}{2!}+\ldots+\frac{(q_1-2)^{n-1}x^{n-1}}{(n-1)!}+\ldots\right]\right]$$

$$+\ldots+\left[(-1)^{q_1-1}\binom{q_1}{q_1-1}\left[1+x+\frac{x^2}{2!}+\ldots+\frac{x^{n-1}}{(n-1)!}+\ldots\right]\right]+(-1)^{q_1}\Bigg]$$

$$\times\Bigg[\left[(-1)^0\binom{q_2}{0}\left[1+q_2x+\frac{{q_2}^2x^2}{2!}+\ldots+\frac{{q_2}^{n-1}x^{n-1}}{(n-1)!}+\ldots\right]\right]$$

$$+\left[(-1)^1\binom{q_2}{1}\left[1+(q_2-1)x+\frac{(q_2-1)^2x^2}{2!}+\ldots+\frac{(q_2-1)^{n-1}x^{n-1}}{(n-1)!}+\ldots\right]\right]$$

$$+\left[(-1)^2\binom{q_2}{2}\left[1+(q_2-2)x+\frac{(q_2-2)^2x^2}{2!}+\ldots+\frac{(q_2-2)^{n-1}x^{n-1}}{(n-1)!}+\ldots\right]\right]$$

$$+\ldots+\left[(-1)^{q_2-1}\binom{q_2}{q_2-1}\left[1+x+\ldots+\frac{x^{n-1}}{(n-1)!}+\ldots\right]\right]+(-1)^{q_2}\Bigg]. \qquad (2.2)$$

From equation (2.2), the coefficient of $\frac{x^{n-1}}{(n-1)!}$ in expand $f(x)g(x)$ are

$$\left[\sum_{i=0}^{q_1-1}(-1)^i\binom{q_1}{i}(q_1-i)^{n-1}\right]\left[\sum_{i=0}^{q_2-1}(-1)^i\binom{q_2}{i}(q_2-i)^{n-1}\right].$$

This coefficient is the number of permutation of $a_1,a_2,\ldots,a_{2n-2}$ in $(a_1,a_2,\ldots,a_{2n-2})$ that consists of q_1,q_2 labels and equal to the number of label trees in $K_{n,n}$ with r_1,r_2 end-vertices and q_1,q_2 particular non-end-vertices. Consider the following 2 cases.

Case I $\quad r_1=r_2=k$.

This q labeling of non-end-vertices can be done in $\binom{n}{k}\binom{n}{k}$ possible ways and $q_1=q_2=n-k$. Thus, we have

$$L(n,r_1,r_2)=\left[\binom{n}{k}\sum_{i=0}^{n-k-1}(-1)^i\binom{n-k}{i}(n-k-i)^{n-1}\right]^2,\ \text{if}\ \ r_1=r_2=k. \qquad (2.3)$$

Case II $r_1 \neq r_2$.

Each tree is the non-symmetrical tree. Then we must look both partitions. Since $r_1 \neq r_2$, q_1, q_2 labeling of non-end-vertices can be done in $2\binom{n}{r_1}\binom{n}{r_2}$ possible ways and $q_1 = n - r_1$, $q_2 = n - r_2$. Hence, we get

$$L(n,r_1,r_2) = 2\left[\binom{n}{r_1}\sum_{i=0}^{n-r_1-1}(-1)^i\binom{n-r_1}{i}(n-r_1-i)^{n-1}\right]$$
$$\times\left[\binom{n}{r_2}\sum_{i=0}^{n-r_2-1}(-1)^i\binom{n-r_2}{i}(n-r_2-i)^{n-1}\right], \text{ if } r_1 \neq r_2. \tag{2.4}$$

Therefore we have proved the following theorem. □

Example 2.3. The number of labelled trees with r_1, r_2 end-vertices in $K_{4,4}$. We have

$$\begin{aligned}
L(4,1,1) &= A_1^2 \\
&= \left[\binom{4}{1}\sum_{i=0}^{4-1-1}(-1)^i\binom{4-1}{i}(4-1-i)^{4-1}\right]^2 \\
&= \left[4(27-24+3)\right]^2 \\
&= 24^2 \qquad = 576;
\end{aligned}$$

$$\begin{aligned}
L(4,1,2) &= 2A_1A_2 \\
&= 2\left[\binom{4}{1}\sum_{i=0}^{4-1-1}(-1)^i\binom{4-1}{i}(4-1-i)^{4-1}\right]\left[\binom{4}{2}\sum_{i=0}^{4-2-1}(-1)^i\binom{4-2}{i}(4-2-i)^{4-1}\right] \\
&= 2\left[(27-24+3)\right]\left[6(8-2)\right] \\
&= 2\left[24\right]\left[36\right] \qquad = 1728;
\end{aligned}$$

$$L(4,1,3) = 2A_1A_3$$

$$= 2\left[\binom{4}{1}\sum_{i=0}^{4-1-1}(-1)^i\binom{4-1}{i}(4-1-i)^{4-1}\right]\left[\binom{4}{3}\sum_{i=0}^{4-3-1}(-1)^i\binom{4-3}{i}(4-3-i)^{4-1}\right]$$

$$= 2\left[(27-24+3)\right]\left[4(1)\right]$$

$$= 2\left[24\right]\left[4\right] \qquad = 192\,;$$

$$L(4,2,2) = A_2^{\,2}$$

$$= \left[\binom{4}{2}\sum_{i=0}^{4-2-1}(-1)^i\binom{4-2}{i}(4-2-i)^{4-1}\right]^2$$

$$= \left[6(8-2)\right]^2$$

$$= 24^2 \qquad = 576\,;$$

$$L(4,2,3) = 2A_2A_3$$

$$= 2\left[\binom{4}{2}\sum_{i=0}^{4-2-1}(-1)^i\binom{4-2}{i}(4-2-i)^{4-1}\right]\left[\binom{4}{3}\sum_{i=0}^{4-3-1}(-1)^i\binom{4-3}{i}(4-3-i)^{4-1}\right]$$

$$= 2\left[6(8-2)\right]\left[4(1)\right]$$

$$= 2\left[36\right]\left[4\right] \qquad = 288\,;$$

$$L(4,3,3) = A_3^{\,2}$$

$$= \left[\binom{4}{3}\sum_{i=0}^{4-3-1}(-1)^i\binom{4-3}{i}(4-3-i)^{4-1}\right]^2$$

$$= \left[4(1)\right]^2$$

$$= \ 4^2 \qquad\qquad = \ 16 \,.$$

Hence, $T\left(K_{4,4}\right) = 576 + 1728 + 192 + 576 + 288 + 16 = 4096 = 4^{2(4)-2}$. □

Example 2.4. The number of labelled trees with r_1, r_2 end-vertices in $K_{7,7}$. The results as follow in Table 1.

Table 1. The number of labelled trees in $K_{7,7}$.

$L(n,r_1,r_2)$	$n = 7$							
	$r_2 \backslash r_1$	1	2	3	4	5	6	$T(K_{n,n}) = n^{2n-2}$
$n = 7$	1	25401600						
	2	381024000	1428840000					
	3	550368000	4127760000	2981160000				
	4	190512000	1428840000	2063880000	357210000			
	5	13124160	98431200	142178400	49215600	1695204		
	6	70560	529200	764400	264600	18228	49	
	Total	1160500320	7084400400	5187982800	406690200	1713432	49	13841287201

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Appendix B
Other graph

Type of graphs

Definition B1 A ***cycle graph*** C_n is the n-vertex graph with n edges, all on a single cycle.

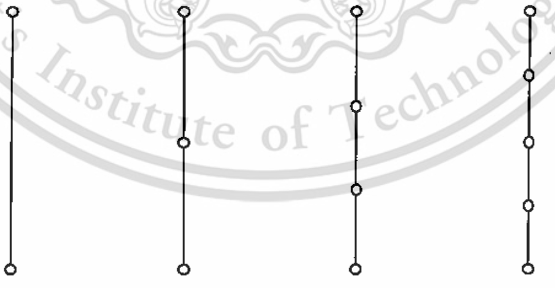
Example B1 For $n=3,4$ and 5, we have cycle graphs C_n as following Figure B1.

(a) $n=3$ (b) $n=4$ (c) $n=5$

Figure B1 Cycle graphs C_n, for $n=3,4$ and 5

Definition B2 The ***path graph*** P_n is the n-vertices graph with $n-1$ edges, all on a single open path.

Example B2 For $n=2,3,4$ and 5, we have path graphs P_n as following in Figure B2.



(a) $n=2$ (b) $n=3$ (c) $n=4$ (d) $n=5$

Figure B2 Path graphs P_n for $n=2,3,4$ and 5

The **singleton graph** is the graph consisting of a single isolated vertex with no edges. It is therefore the empty graph on one vertex. It is commonly denoted K_1 (i.e., the complete graph on one vertex).

Example B3 From Figure 2.1 (a), we have it is singleton graph.

The ***n*-barbell graph** is the simple graph obtained by connecting two copies of a complete graph K_n by a bridge.

Example B4 For $n=3,4,5$ and 6, we have n-barbell graphs as following in Figure B3.

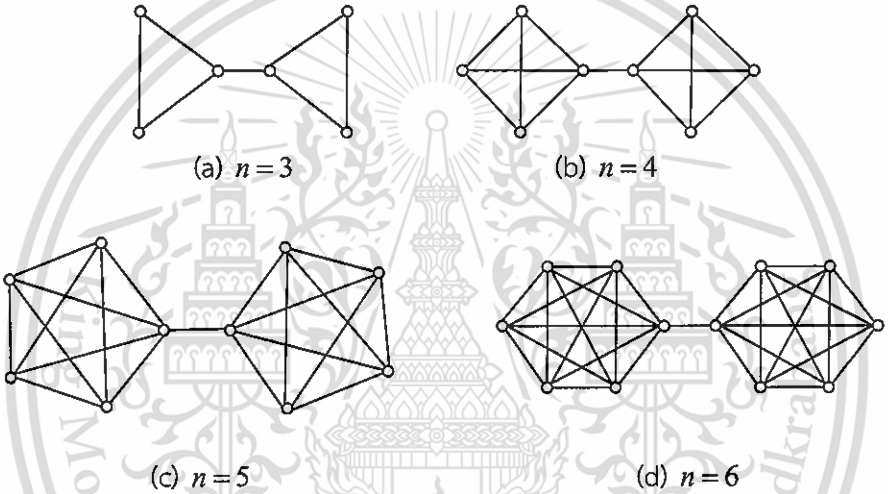


Figure B3 n-Barbell graphs for $n=3,4,5$ and 6

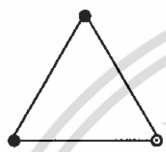
A ***complete tripartite graph*** is the $k=3$ case of a complete k-partite graph. In other words, it is a tripartite graph (i.e., a set of graph vertices decomposed into three disjoint sets such that no two graph vertices within the same set are adjacent) such that every vertex of each set graph vertices is adjacent to every vertex in the other two sets. If there are m,n and r graph vertices in the three sets, the complete tripartite graph (sometimes also called a ***complete trigraph***) is denoted $K_{m,n,r}$.

Example B5 The following table and figure summarizes the first few complete tripartite graphs.

Table B1 Complete tripartite graphs $K_{m,n,r}$

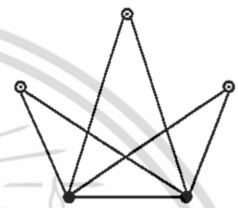
n	Name	$K_{m,n,r}$
$r+2$	$(r,2)$-fan graph	$K_{1,1,r}$
$r+3$	$(r,3)$-fan graph	$K_{1,2,r}$
5	5-wheel graph	$K_{1,2,2}$

(Weisstein E. 2016. **CompleteTripartiteGraph.** [Online]. Available : http://mathworld.wolfram.com/CompleteTripartiteGraph.html.)

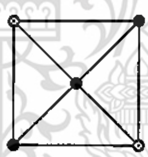


(a) $K_{1,1,1}$

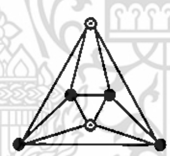
(b) $K_{1,1,2}$
(2,2)-fan graph



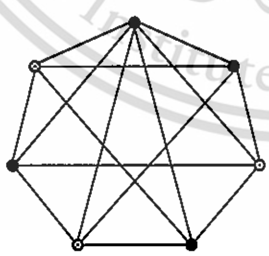
(c) $K_{1,1,3}$
(3,2)-fan graph



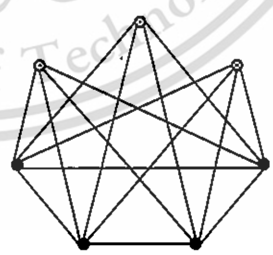
(d) $K_{1,2,2}$
5-wheel graph



(e) $K_{2,2,2}$



(f) $K_{1,3,3}$



(g) $K_{2,2,3}$

Figure B4 Complete tripartite graphs $K_{m,n,r}$

A *fan graph* $F_{m,n}$ is defined as the graph join $\overline{K_m}+P_n$, where $\overline{K_m}$ is the empty graph on m vertices and P_n is the path graph on n vertices. The case $m=1$ corresponds to the usual fan graphs, while $m=2$ corresponds to the double fan, etc.

The $(r,2)$- fan graph is isomorphic to the complete tripartite graph $K_{1,1,r}$, and the $(r,3)$-fan graph to $K_{1,2,r}$.

Example B6 For $m=1,2$ and $n=2,3,4$, we have fan graphs $F_{m,n}$ as following Figure B5.

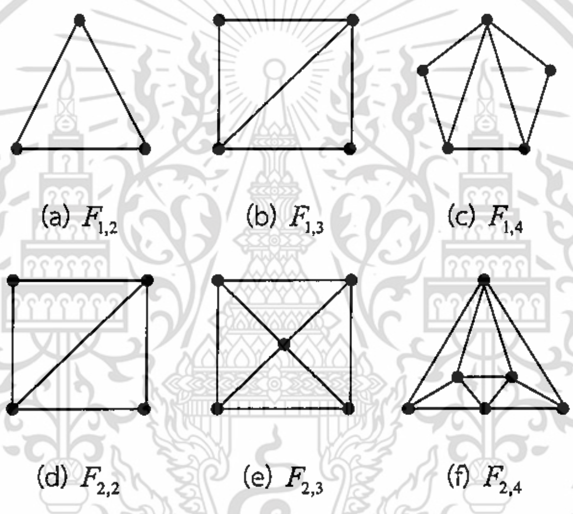


Figure B5 Fan graphs $F_{m,n}$ for $m=1,2$ and $n=2,3,4$

Definition B4 For integers $n\geq 4$, the ***wheel graph*** W_n in the n-vertex graph obtained by joining a vertex to each of the $n-1$ vertices of the cycle graph C_{n-1}.

Example B7 For $n=4,5$ and 6, we have wheel graphs W_n as following in Figure B6.

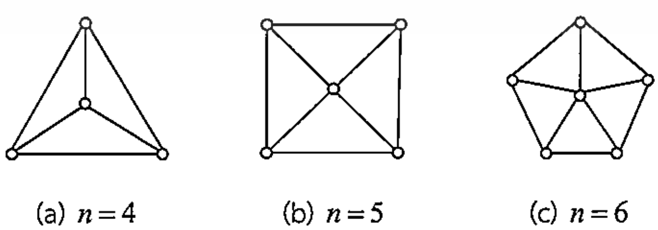
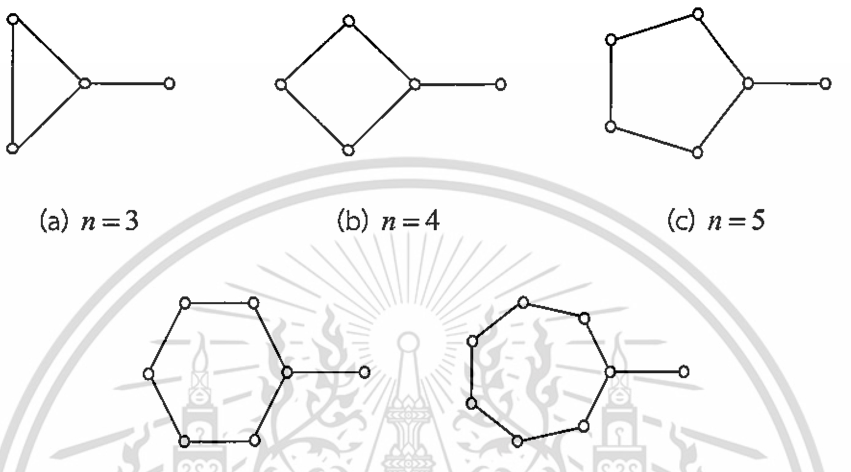


Figure B6 Wheel graphs W_n for $n=4,5$ and 6

The *n-pan graph* is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge.

Example B8 For $n=3,4,5,6$ and 7, we have n-pan graphs as following in Figure B7.



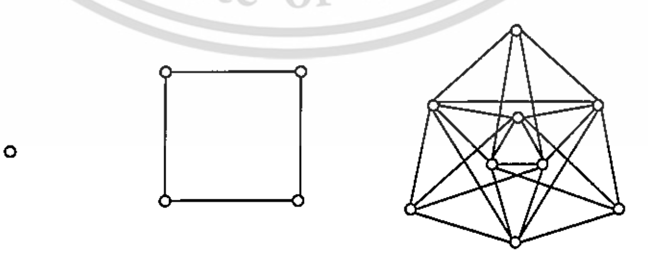
(a) $n=3$ (b) $n=4$ (c) $n=5$

(d) $n=6$ (e) $n=7$

Figure B7 n-pan graphs for $n=3,4,5,6$ and 7

A generalization of the *square graph* Sq_n is the graph obtained by taking the n^2 ordered pairs of the first n positive integers as vertices and drawing an edge between all pairs having exactly one number in common.

Example B9 For $n=1,2$ and 3, we have square graphs Sq_n as following in Figure B8.

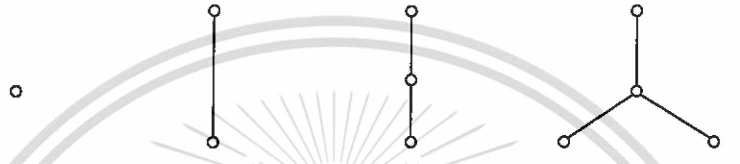


(a) $n=1$ (b) $n=2$ (c) $n=3$

Figure B8 Square graphs Sq_n for $n=1,2$ and 3

The ***star graph*** S_n of order n , sometimes simply known as an "***n*-star**", is a tree on n vertices with one vertex having vertex degree $n-1$ and the other $n-1$ having vertex degree 1. The star graph S_n is therefore isomorphic to the complete bipartite graph $K_{1,n-1}$.

Example B10 For $n=1,2,3$ and 4, we have star graphs S_n as following in Figure B9.



(a) $n=1$ (b) $n=2$ (c) $n=3$ (d) $n=4$

Figure B9 Star graphs S_n for $n=1,2,3$ and 4

The ***n*-sunlet graph** is the graph on $2n$ vertices obtained by attaching n pendant edges to a cycle graph C_n.

Example B11 For $n=3$ and 4, we have n-Sunlet graphs as following in Figure B10.

(a) $n=3$ (b) $n=4$

Figure B10 n-Sunlet graphs for $n=3$ and 4

Appendix C
All labelled trees in complete bipartite graph

The labelled trees in complete bipartite graph $K_{1,4}$

- $L(1,4,0,4)=1$.

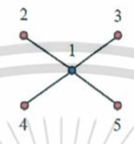


Figure C1 A labelled tree with $r_1=0$ and $r_2=4$ in $K_{1,4}$

The labelled trees in complete bipartite graph $K_{3,4}$

- $L(3,4,0,2)=72$.

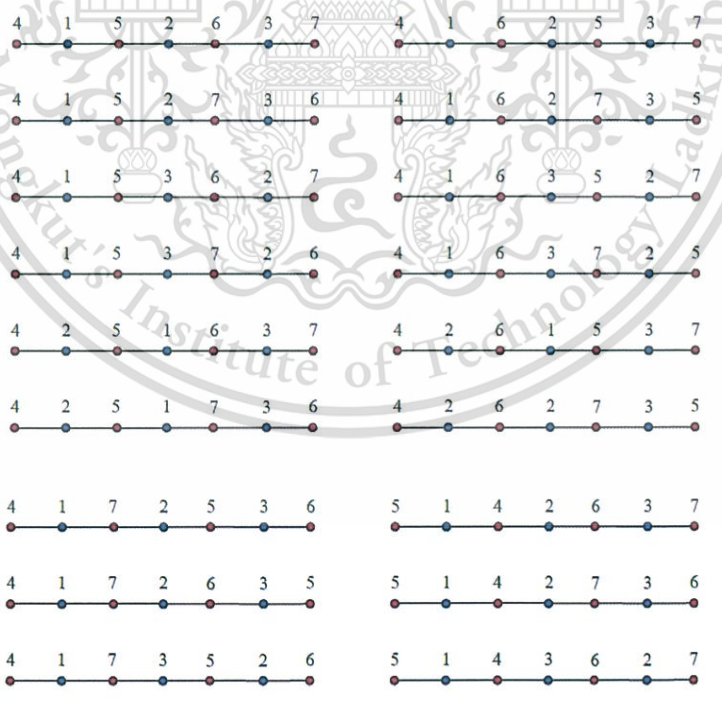


Figure C2 The labelled trees with $r_1=0$ and $r_2=2$ in $K_{3,4}$

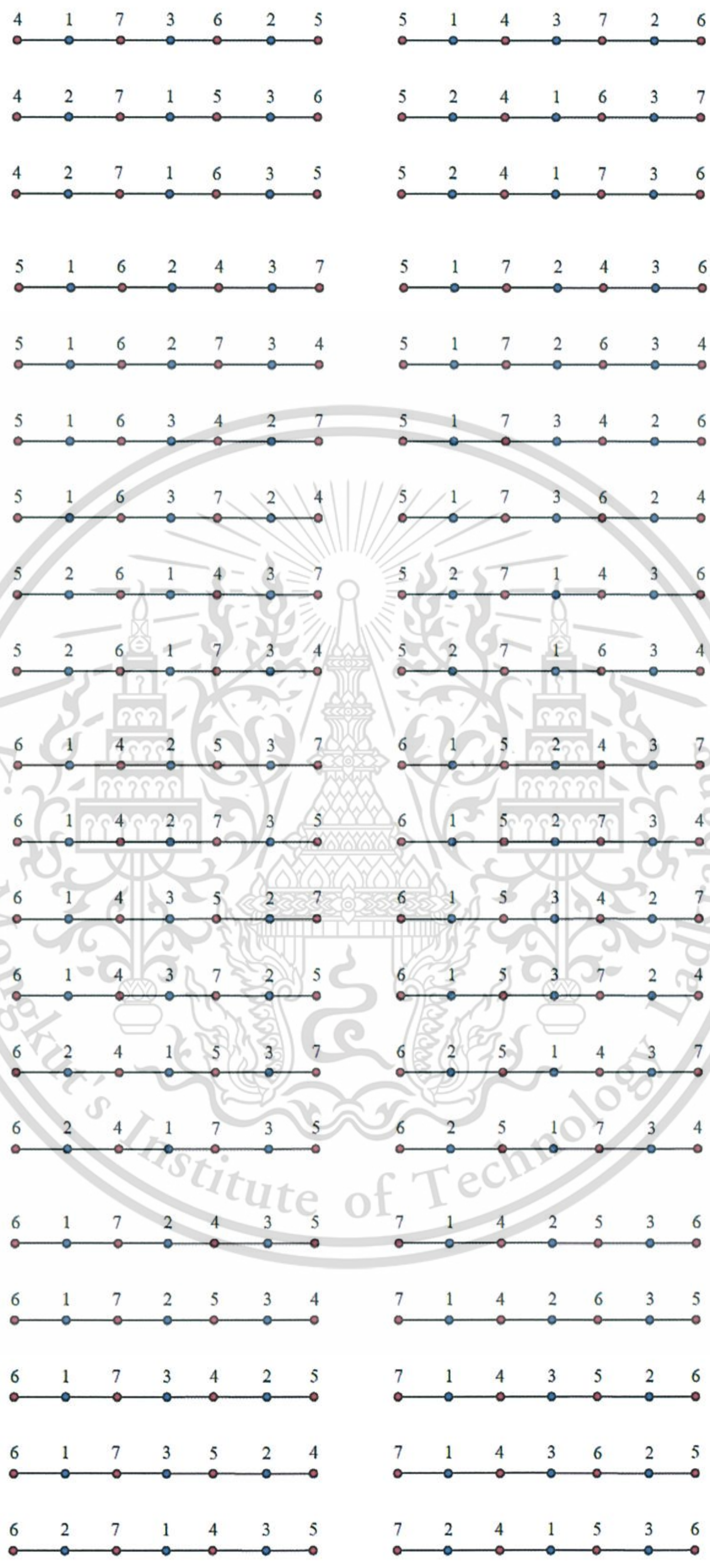


Figure C3 The labelled trees with $r_1 = 0$ and $r_2 = 2$ in $K_{3,4}$ (Cont.1)

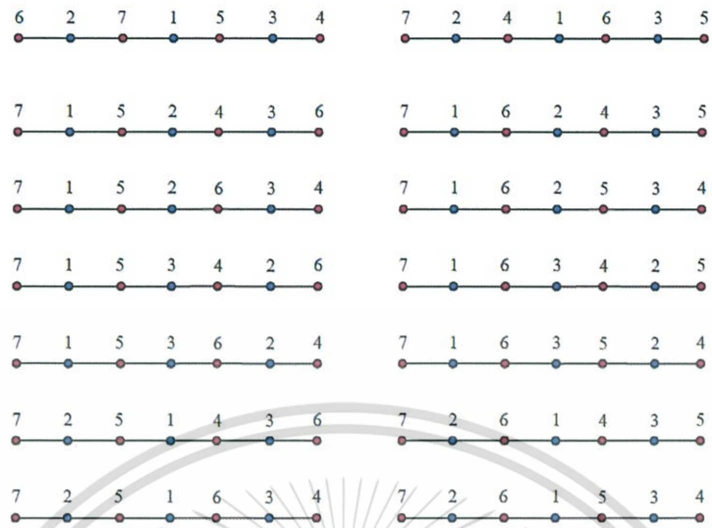


Figure C4 The labelled trees with $r_1 = 0$ and $r_2 = 2$ in $K_{3,4}$ (Cont.2)

- $L(3,4,0,3) = 24$.

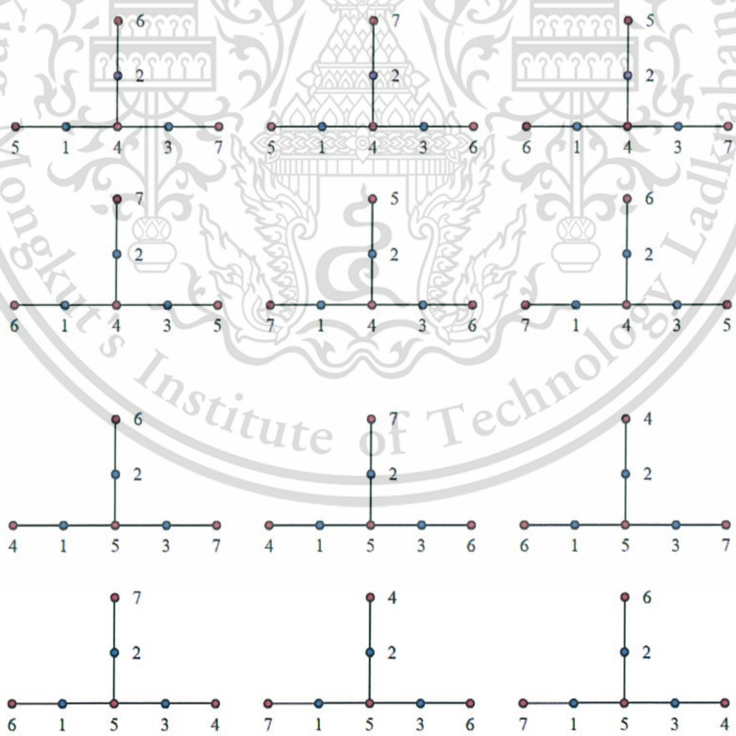


Figure C5 The labelled trees with $r_1 = 0$ and $r_2 = 3$ in $K_{3,4}$

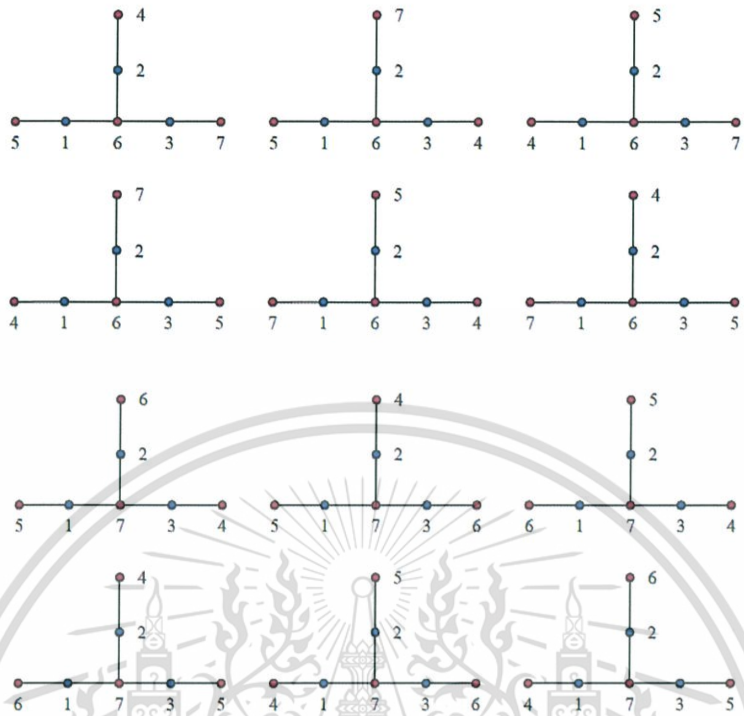
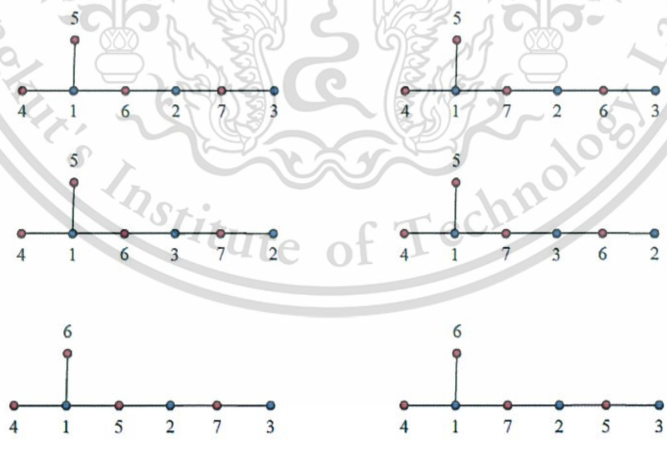


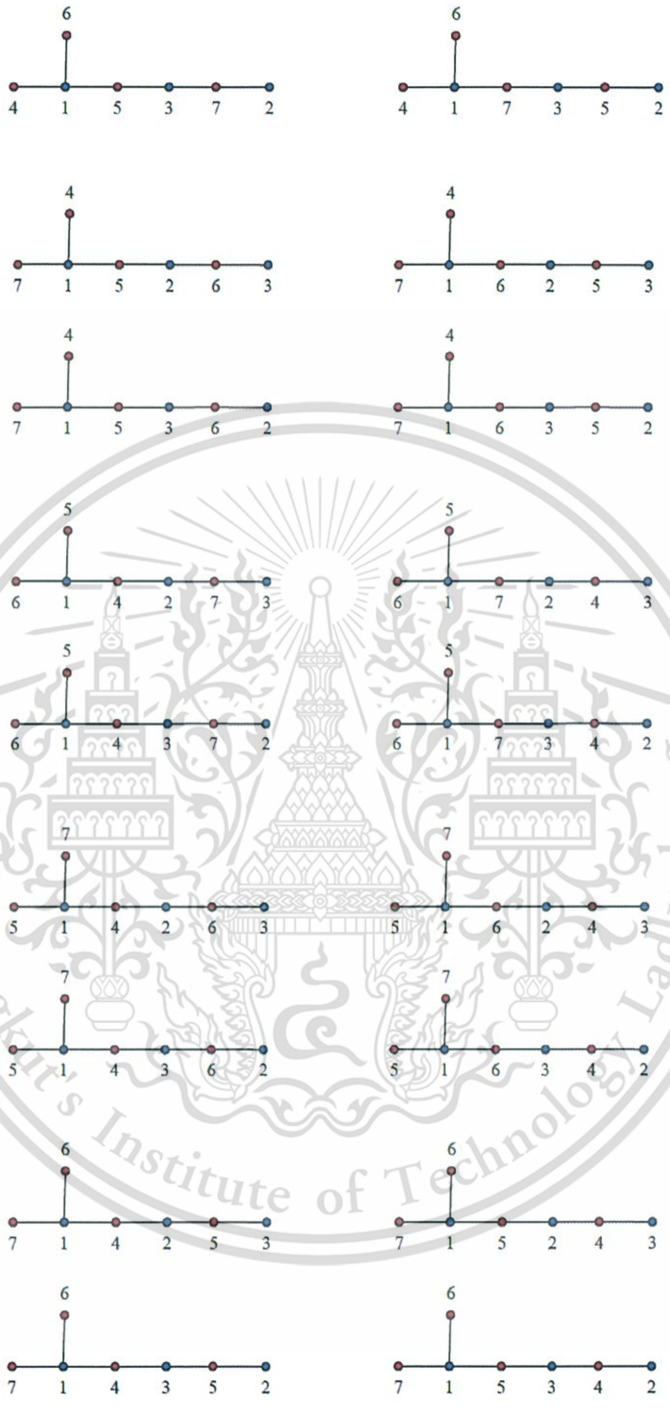
Figure C6 The labelled trees with $r_1 = 0$ and $r_2 = 3$ in $K_{3,4}$ (Cont.)

- $L(3,4,1,2) = 216$.



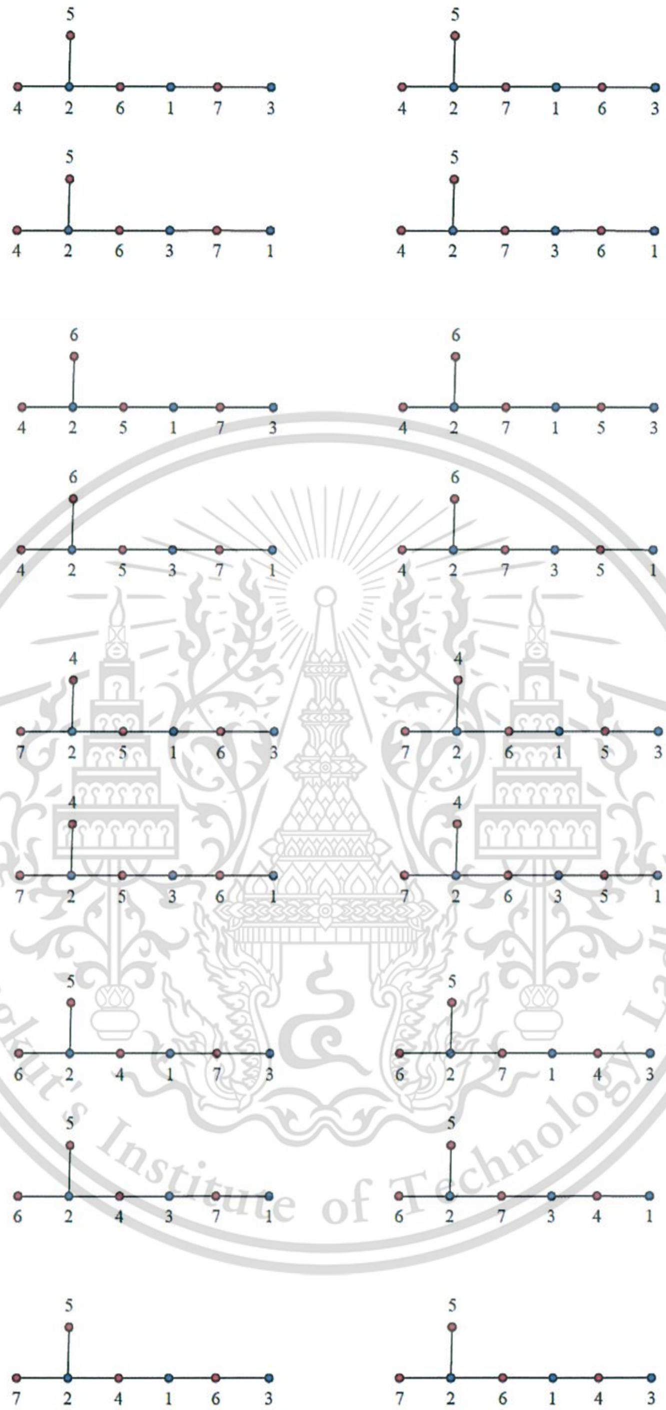
(a) $r_1 = 1$ and $r_2 = 2$

Figure C7 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$



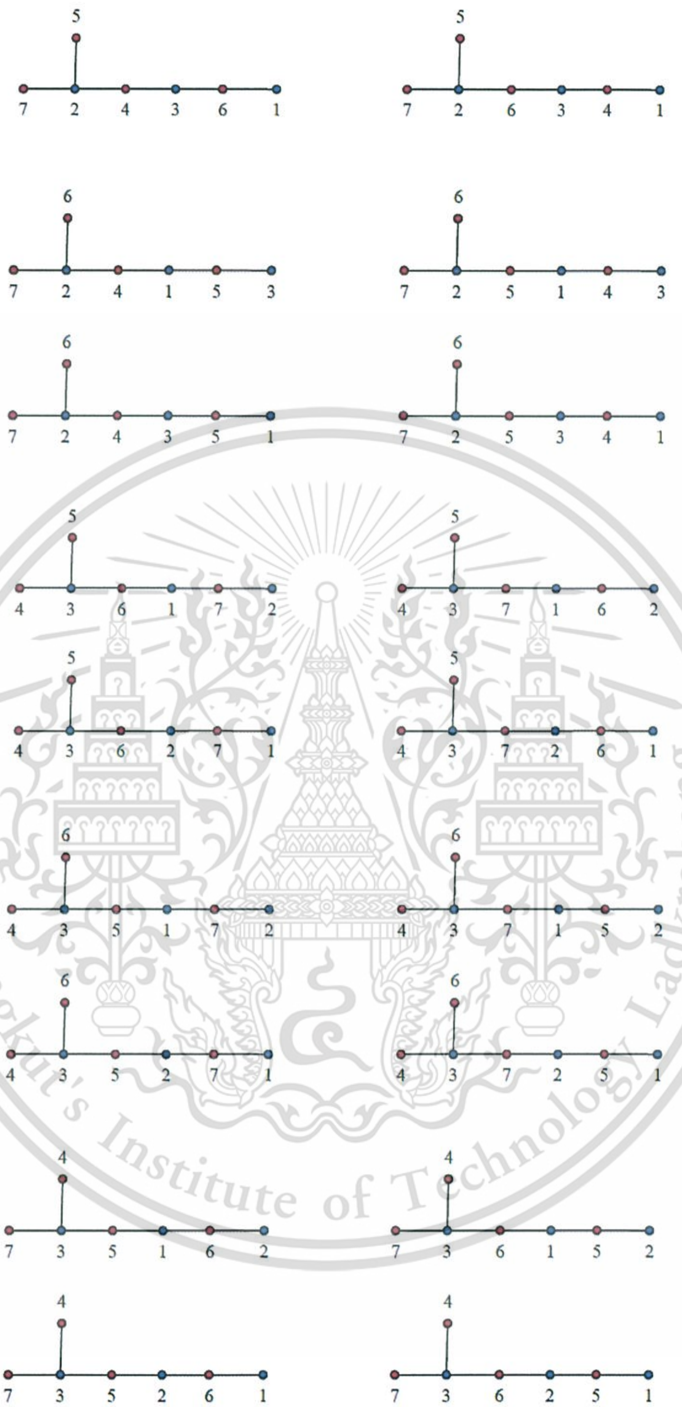
(a) $r_1 = 1$ and $r_2 = 2$ (Cont.)

Figure C8 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.1)



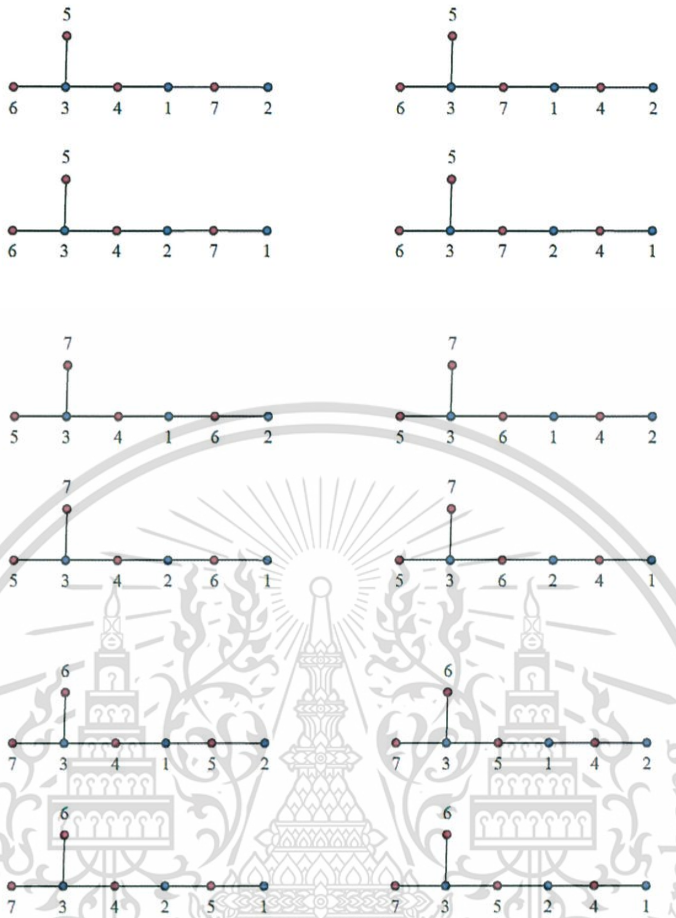
(a) $r_1 = 1$ and $r_2 = 2$ (Cont.)

Figure C9 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.2)

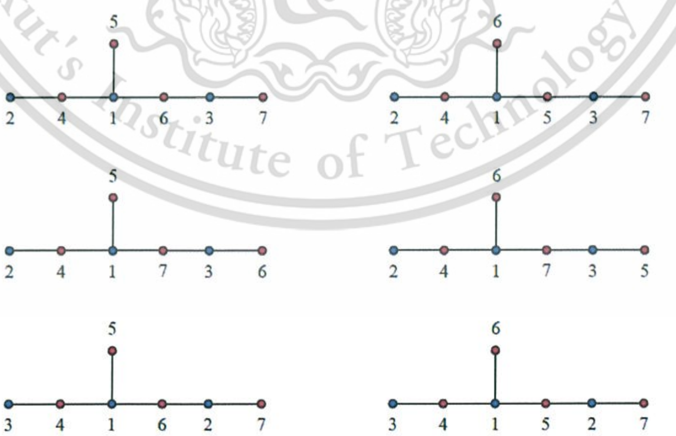


(a) $r_1 = 1$ and $r_2 = 2$ (Cont.)

Figure C10 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.3)

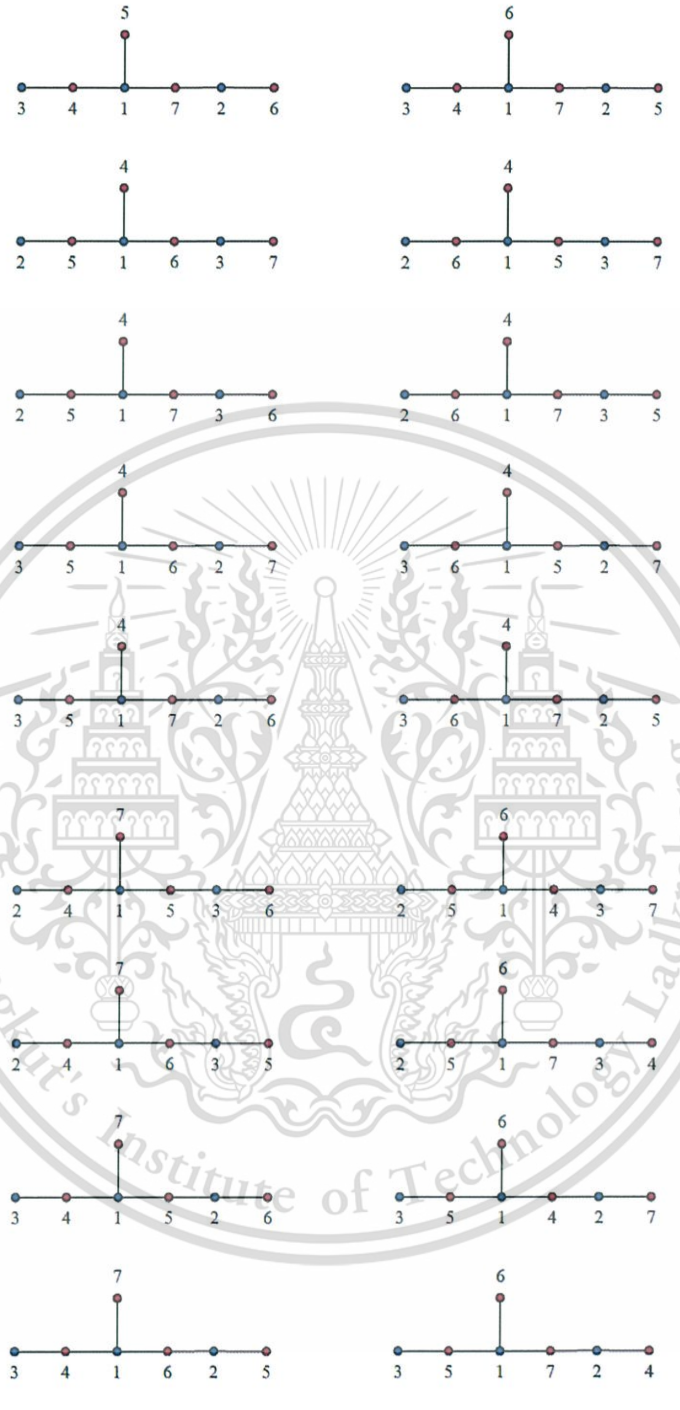


(a) $r_1 = 1$ and $r_2 = 2$ (Cont.)



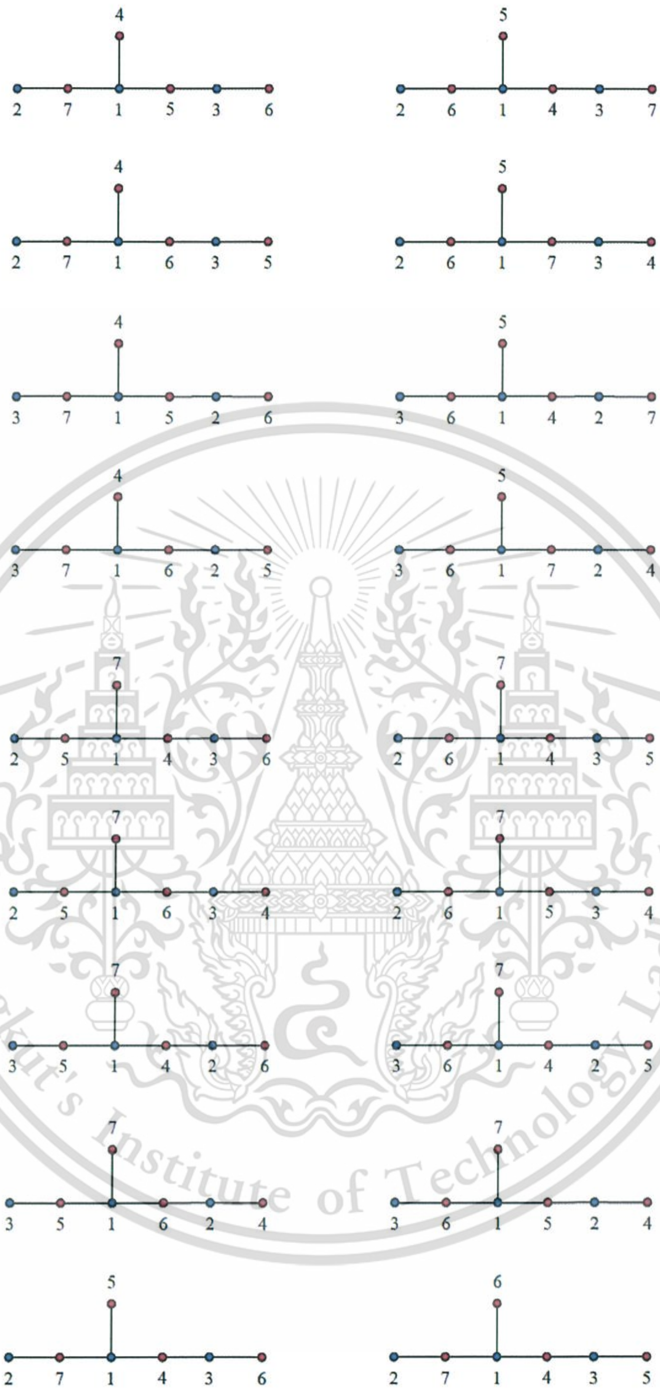
(b) $r_1 = 1$ and $r_2 = 2$

Figure C11 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.4)



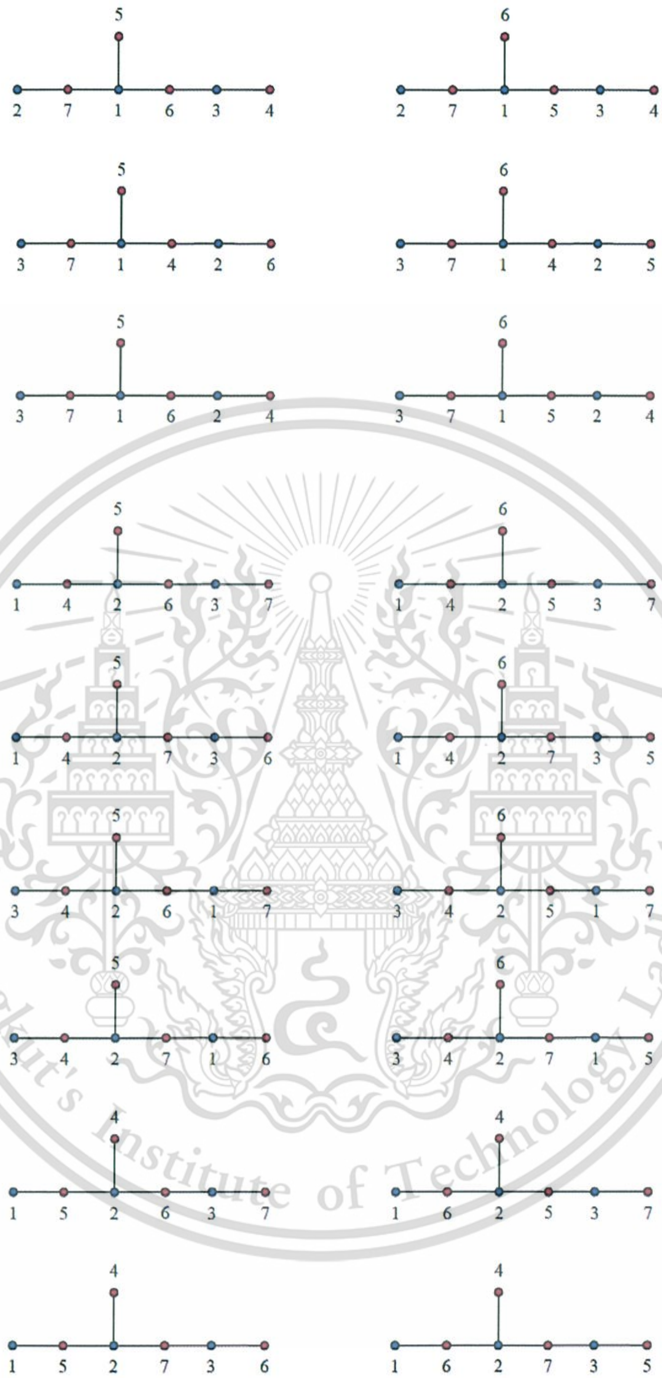
(b) $r_1 = 1$ and $r_2 = 2$ (Cont.)

Figure C12 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.5)



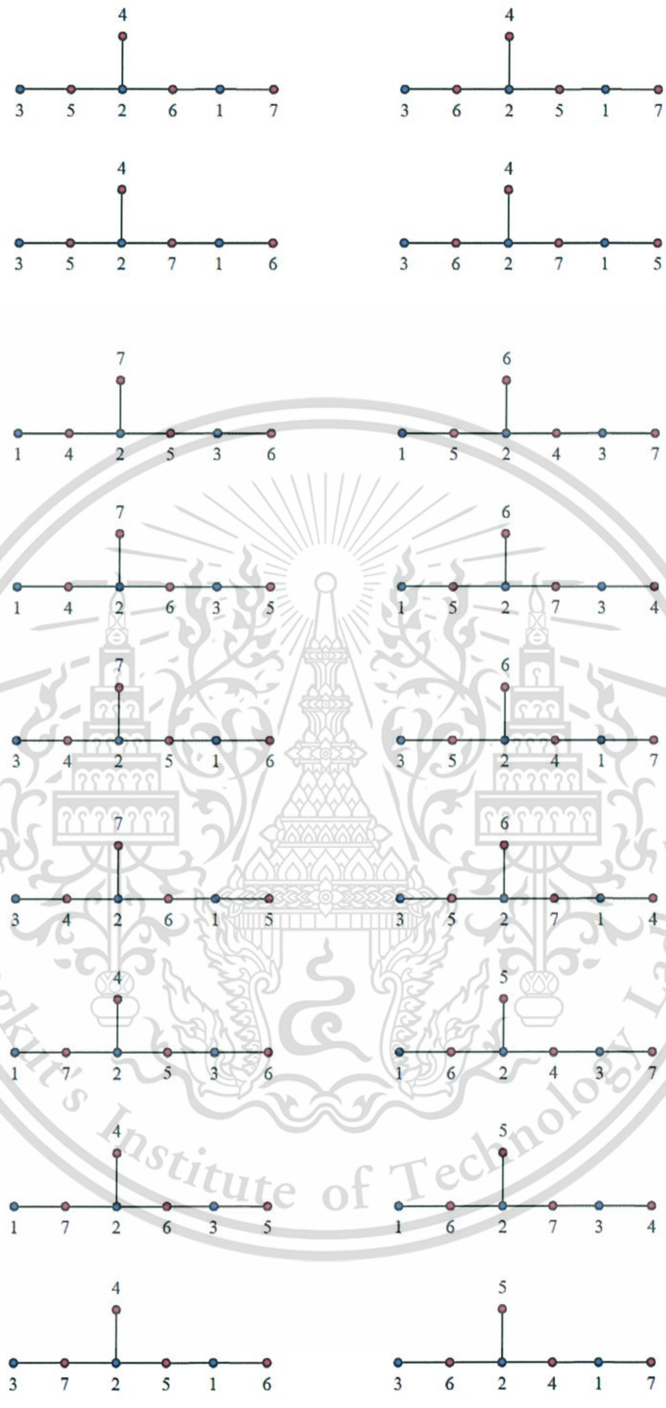
(b) $r_1 = 1$ and $r_2 = 2$ (Cont.)

Figure C13 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.6)



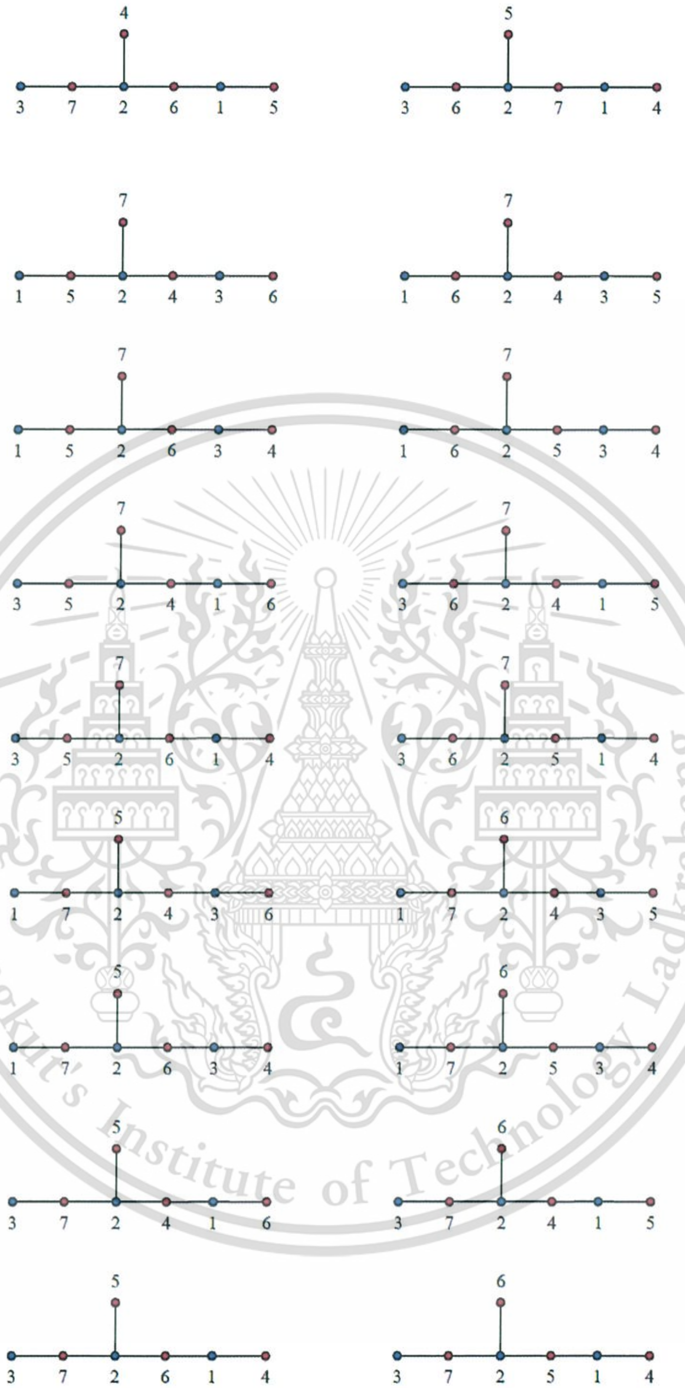
(b) $r_1 = 1$ and $r_2 = 2$ (Cont.)

Figure C14 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.7)



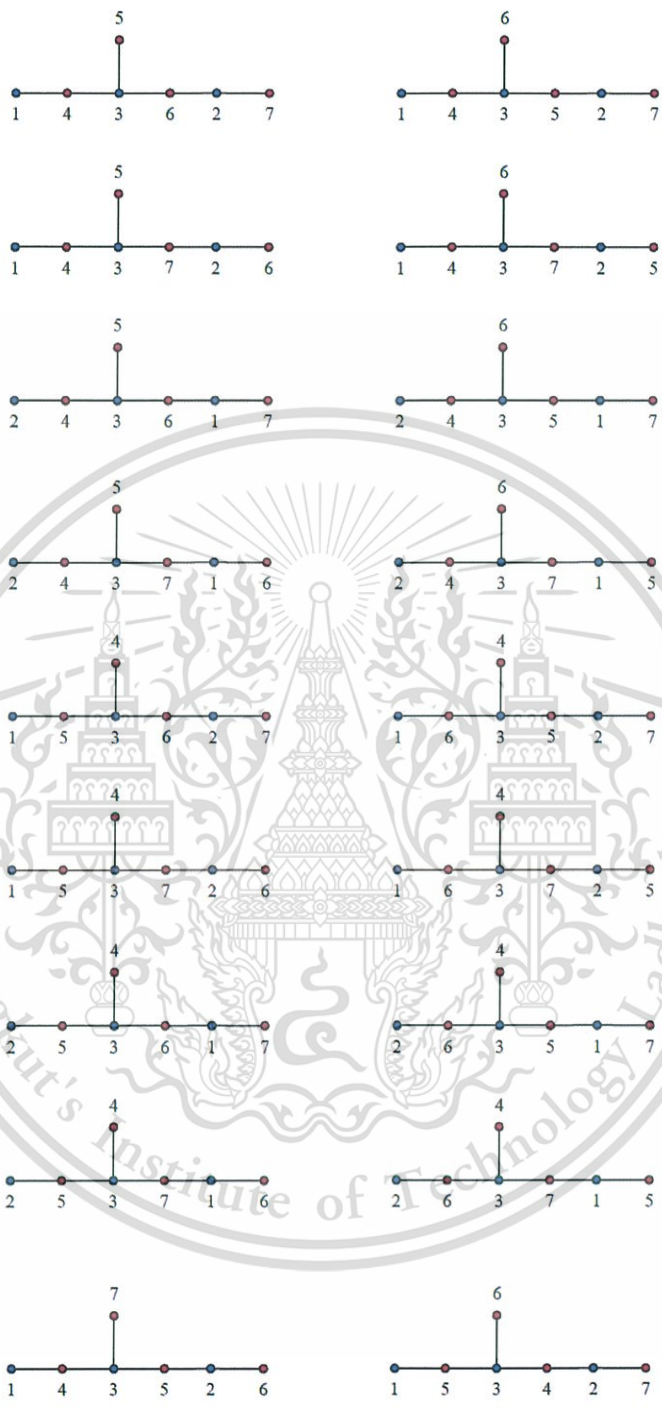
(b) $r_1 = 1$ and $r_2 = 2$ (Cont.)

Figure C15 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.8)



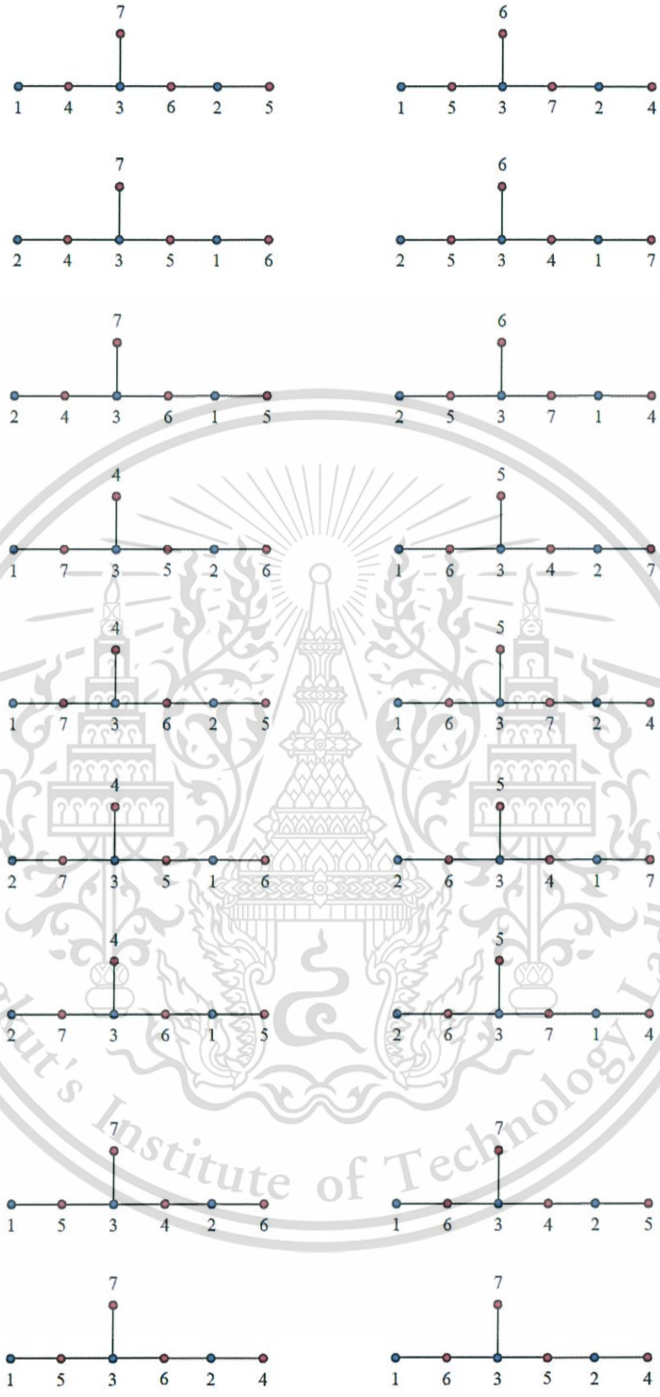
(b) $r_1 = 1$ and $r_2 = 2$ (Cont.)

Figure C16 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.9)



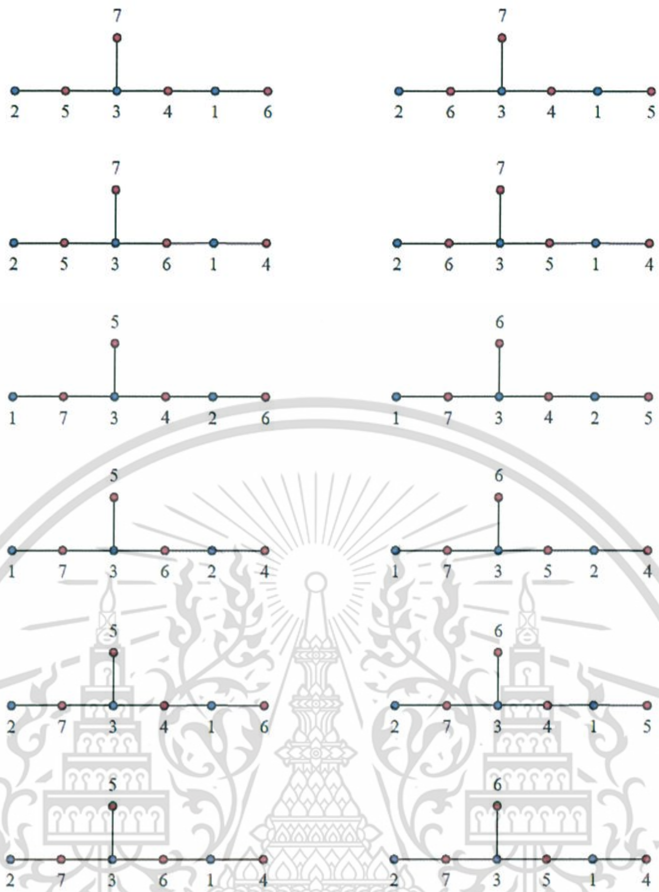
(b) $r_1 = 1$ and $r_2 = 2$ (Cont.)

Figure C17 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.10)



(b) $r_1 = 1$ and $r_2 = 2$ (Cont.)

Figure C18 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.11)



(b) $r_1 = 1$ and $r_2 = 2$ (Cont.)

Figure C19 The labelled trees with $r_1 = 1$ and $r_2 = 2$ in $K_{3,4}$ (Cont.12)

- $L(3,4,1,3) = 72$.

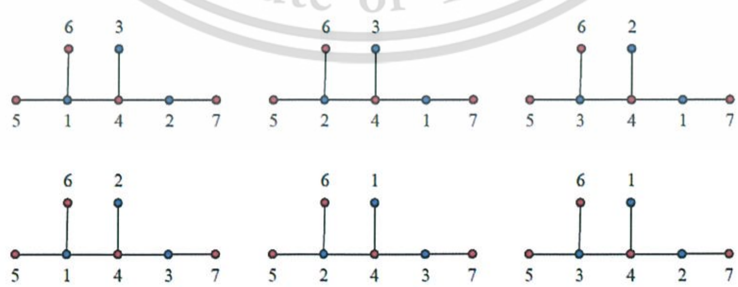


Figure C20 The labelled trees with $r_1 = 1$ and $r_2 = 3$ in $K_{3,4}$

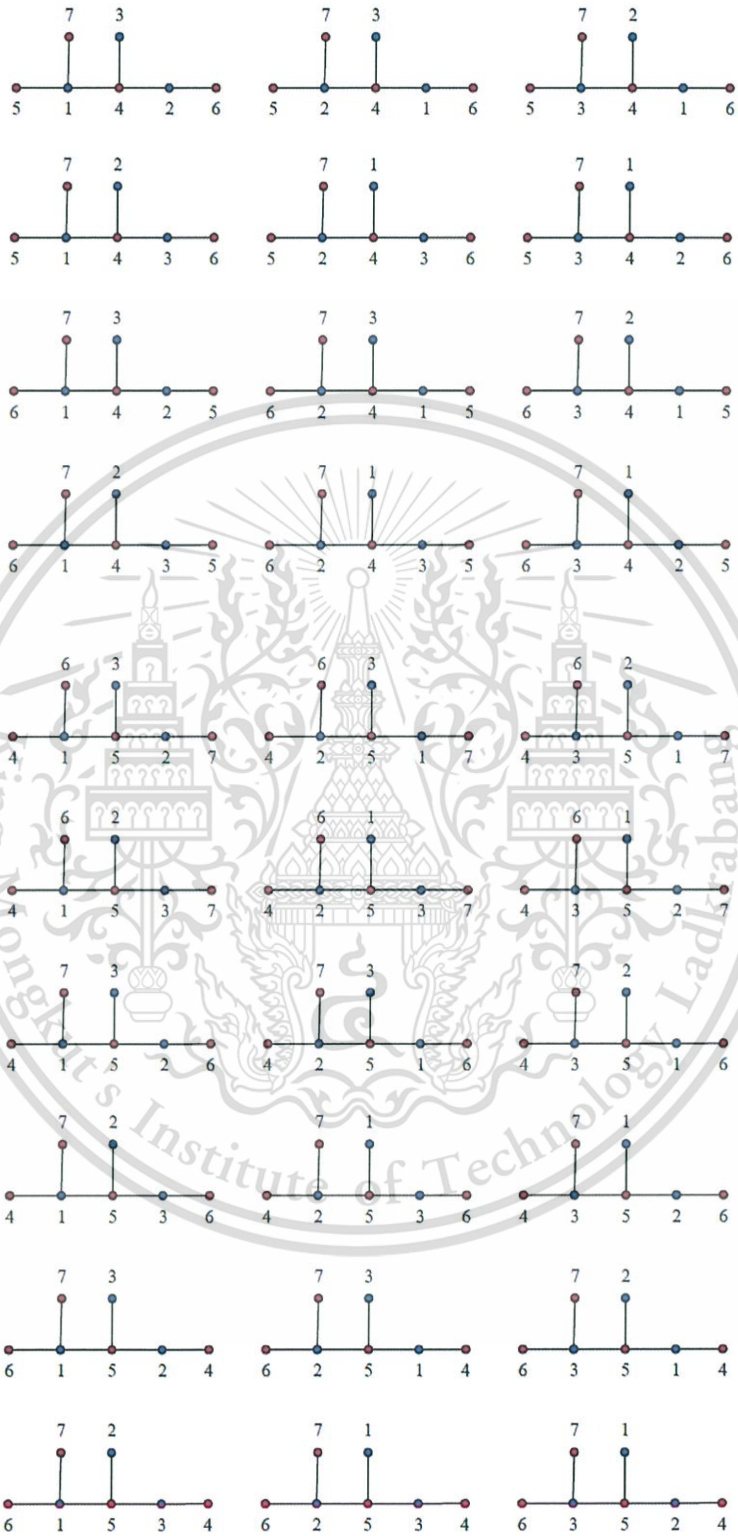


Figure C21 The labelled trees with $r_1 = 1$ and $r_2 = 3$ in $K_{3,4}$ (Cont.1)

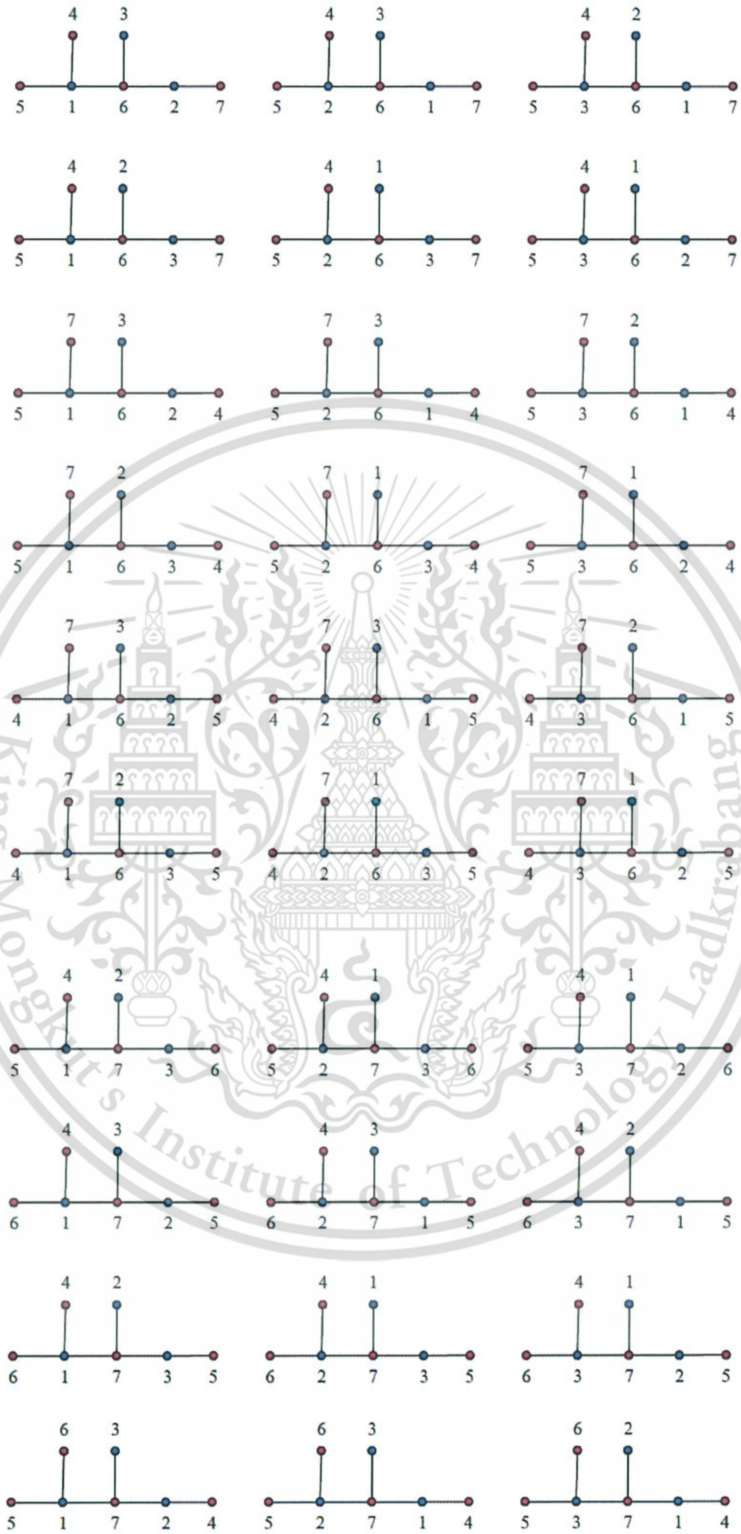


Figure C22 The labelled trees with $r_1 = 1$ and $r_2 = 3$ in $K_{3,4}$ (Cont.2)

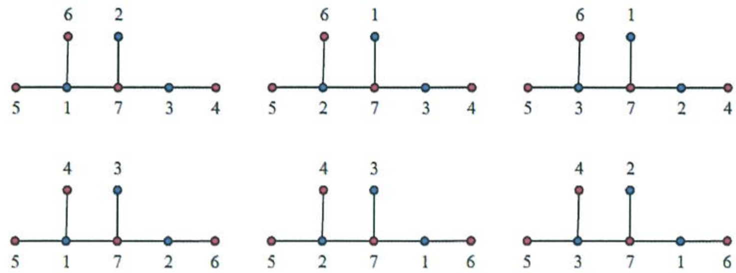


Figure C23 The labelled trees with $r_1 = 1$ and $r_2 = 3$ in $K_{3,4}$ (Cont.3)

- $L(3,4,2,2) = 36$.

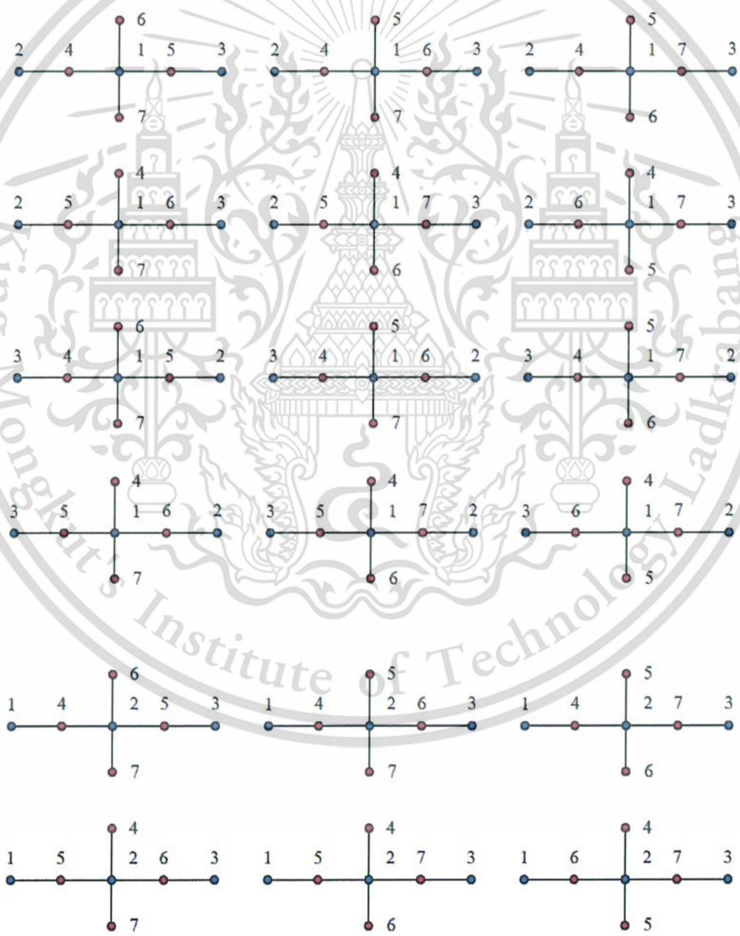


Figure C24 The labelled trees with $r_1 = r_2 = 2$ in $K_{3,4}$

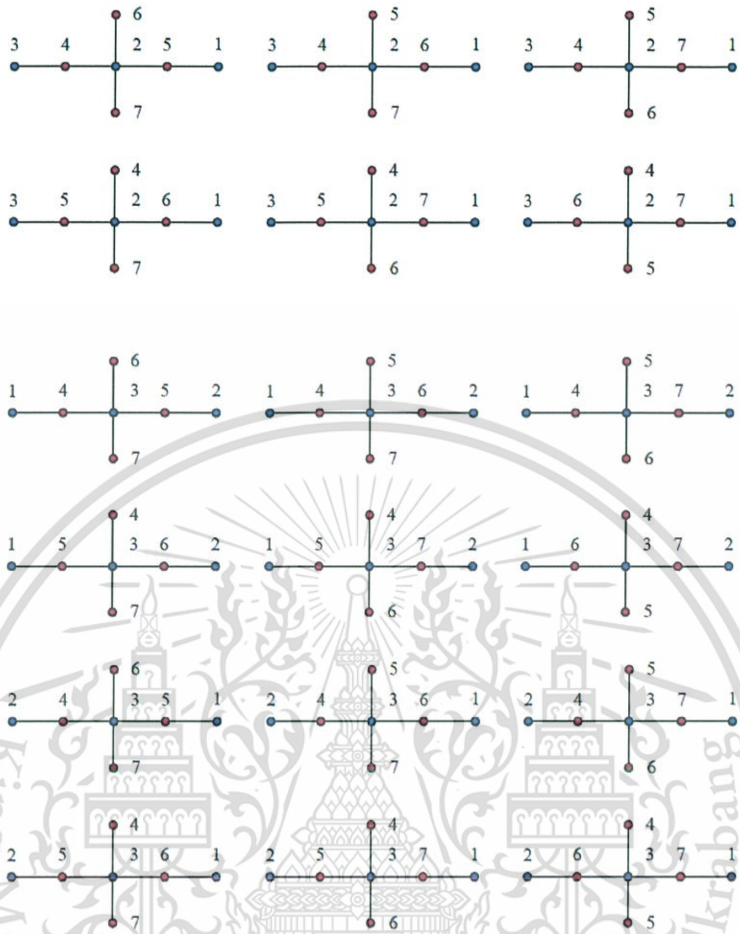


Figure C25 The labelled trees with $r_1 = r_2 = 2$ in $K_{3,4}$ (Cont.)

- $L(3,4,2,3) = 12$.

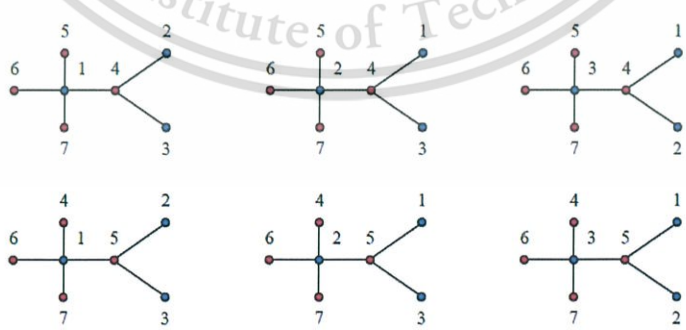


Figure C26 The labelled trees with $r_1 = 2$ and $r_2 = 3$ in $K_{3,4}$

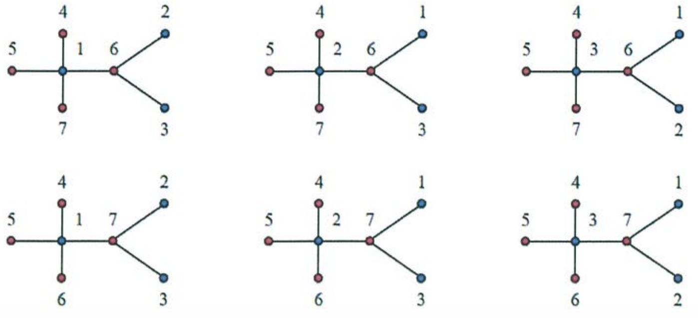


Figure C27 The labelled trees with $r_1 = 2$ and $r_2 = 3$ in $K_{3,4}$ (Cont.)

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