

POLYNOMIALS WITH GENERALIZED FIBONACCI NUMBER  
COEFFICIENTS



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พหุนามที่มีสัมประสิทธิ์เป็นจำนวนพีโบนัชชีทั่วไป



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### Abstract

Garth, Mills and Mitchell in 2007 started the investigation on the nature of real zeros, and the Mahler measures of Fibonacci-coefficient polynomials, which are polynomials whose coefficients are Fibonacci numbers. Their results have later been extended in various directions. In this work, we complement these earlier works by investigating generalized Fibonacci-coefficient polynomials, which are polynomials whose coefficients are generalized Fibonacci numbers. Our main results give precise information about the location of the real roots of generalized Fibonacci-coefficient polynomials, and the computation of their Mahler measures.

Keywords : Generalized Fibonacci number, Mahler measure, polynomial, zero.

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### บทคัดย่อ

ในปีค.ศ. 2007 การ์ต, มิลส์ และมิทเชล ได้เริ่มวิเคราะห์คุณสมบัติของรากที่เป็นจำนวนจริงและมาตรา Mahler ของพหุนามที่มีสัมประสิทธิ์เป็นจำนวนฟีโบนัชชี โดยผลลัพธ์เหล่านั้นได้รับการขยายไปหลายทิศทาง โดยในงานวิจัยนี้เราจะเติมเต็มงานวิจัยก่อนหน้านี้โดยการวิเคราะห์พหุนามที่มีสัมประสิทธิ์เป็นจำนวนฟีโบนัชชีทั่วไป โดยผลลัพธ์หลักของเราจะให้ข้อมูลเกี่ยวกับตำแหน่งของรากที่เป็นจำนวนจริงของพหุนามที่มีสัมประสิทธิ์เป็นจำนวนฟีโบนัชชีทั่วไปและคำนวณมาตราวัด Mahler ของพหุนามนั้นด้วย

คำสำคัญ : จำนวนฟีโบนัชชีทั่วไป, มาตราวัด Mahler, พหุนาม, ค่าราก

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# Chapter 1

## Introduction

### 1.1 Research motivation

A Fibonacci-coefficient polynomial (FCP) is a polynomials of the form

$$p_n(x) = \sum_{k=0}^n F_{k+1} x^{n-k} \quad (n \geq 1),$$

where the  $F_i$ 's are Fibonacci numbers defined by

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$

In 2007, Garth et al. [4], investigated the zeros and the Mahler measures of the FCP's.

Their main results are summarized as follows:

- A. A1) For even  $n \geq 2$ , each polynomial  $p_n(x)$  has no real zeros.
- A2) For odd  $n \geq 3$ , each polynomial  $p_n(x)$  has exactly one real zero lying in the interval  $[-\varphi, -1]$ , where  $\varphi = (1 + \sqrt{5})/2$ .
- A3) The sequence of real zeros of  $p_n(x)$  with  $n$  odd decreases monotonically to zero.
- B. The zeros of  $p_n(x)$  approach  $\varphi$  in modulus as  $n \rightarrow \infty$ .
- C. For  $m \geq 2$ , let  $\{p_n^{(m)}(x)\}$  be the sequence obtained from the FCP's by reducing the coefficients modulo  $m$ .

Suppose  $p_n^{(m)}(x) = \sum_{k=0}^n \bar{F}_{k+1} x^{n-k}$ .

Defined the coefficients of  $p_n^{(m)}(x)$  by  $\bar{F}_{k+1} = F_{k+1} \pmod{m}$  if  $F_{k+1} \pmod{m} \leq \frac{m}{2}$  and  $\bar{F}_{k+1} = F_{k+1} \pmod{m} - m$  if  $F_{k+1} \pmod{m} > \frac{m}{2}$ .

C1) Let  $t$  be the number of terms in one period of the Fibonacci sequence reduced modulo  $m$ . If  $k \equiv -2$  or  $-1 \pmod{t}$ , then  $M(p_k^{(m)}(x)) = M(p_{t-2}^{(m)}(x))$ , where  $M(p)$  denotes the Mahler measure of the polynomial  $p$ .

C2) If  $k \equiv -2$  or  $-1 \pmod{2n}$ , then  $M(p_k^{(L_n)}(x)) = \varphi^{n-1}$ , where  $(L_n)$  is the sequence of Lucas numbers defined by  $L_0 = 2, L_1 = 1$ , and  $L_n = L_{n-1} + L_{n-2}$  ( $n \geq 2$ ).

Our objectives are to extend the results A), B) and C) to those of a generalized Fibonacci-coefficient polynomial (GFCP)

$$P_n(x) = \sum_{k=0}^n \mathcal{F}_{k+1} x^{n-k}, \quad (1.1)$$

where the coefficients are generalized Fibonacci numbers defined, for fixed positive integers  $a$  and  $b$ , by

$$\mathcal{F}_0 = 0, \mathcal{F}_1 = 1, \mathcal{F}_{n+1} = a\mathcal{F}_n + b\mathcal{F}_{n-1} \quad (n \geq 1). \quad (1.2)$$

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There have already appeared some extensions of the results A), B) and C), namely, Mátyás, [5], showed that the results A1) and A2) hold in this generalized setting. Mátyás [6], further extended A1) and A2) to sequence of GFCP's with shifted indices. Mátyás et al, [7], refined the results A1) and A2) further by considering polynomials whose coefficients satisfy a linear recursive sequence with constant coefficients. Mátyás and Szalay, [8], proved the validity of A1) and A2) for Tribonacci-coefficient polynomials. Mansour and Shattuck, [9], considered polynomials whose coefficients are  $k$ -Fibonacci numbers, i.e. numbers satisfying the recurrence  $a_n = a_{n-1} + a_{n-2} + \dots + a_{n-k}$ , and proved the results A1), A2) and B) in this setting as well as refining the results in [8]. Here, we complement all the above-mentioned works by establishing the results B) and C) for GFCP's.

## 1.2 Objectives of the study

The objectives of this research are as follows :

- 1) To find the precise information about the location of the real zeros of generalized Fibonacci-coefficient polynomials.
- 2) To compute the Mahler measures of generalized Fibonacci-coefficient polynomials.

## 1.3 Scope of the study

We consider a generalized Fibonacci-coefficient polynomial(GFCP) of the form

$$F_n(x) = \sum_{k=0}^n F_{k+1} x^{n-k}, \quad (1.3)$$

where the coefficients are generalized Fibonacci numbers defined, for fixed positive integers  $a$  and  $b$ , by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = aF_n + bF_{n-1} \quad (n \geq 1). \quad (1.4)$$

## 1.4 Benefits of the study

We receive the results, information and values of the Mahler measure of

- 1)  $p_k^{(m)}(x)$  if  $k \equiv -2$  or  $-1 \pmod{t}$  where  $t$  is the number of terms in one period of the generalized Fibonacci sequence modulo  $m \geq 2$ ,
- 2)  $p_k^{(m)}(x)$  if  $k \equiv -2$  or  $-1 \pmod{2n}$  where  $(\mathcal{L}_n)$  is the sequence of Lucas numbers defined by  $\mathcal{L}_{n+1} = a\mathcal{L}_n + b\mathcal{L}_{n-1}$  ( $n \geq 2$ ) for fixed positive integers  $a$  and  $b$ .

## 1.5 Research methodology

- 1) Study background in generalized Fibonacci sequences.
- 2) Study background in generalized Fibonacci-coefficient polynomials.
- 3) Study research papers about sequences and generalized Fibonacci-coefficient polynomials.
- 4) Investigate the location of the real zeros of generalized Fibonacci-coefficient polynomials.
- 5) Compute the Mahler measures of generalized Fibonacci-coefficient polynomials.
- 6) Summarize obtained results and write the thesis.



Table 1.1: The research schedule

Activity	Time frame						
	2015	2016			2017		
	Sep.-Dec.	Jan.-Apr.	May.-Aug.	Sep.-Dec.	Jan.-Apr.	May.-Aug.	Sep.-Dec.
Step 1	←→						
Step 2		←→					
Step 3			←→				
Step 4				←→			
Step 5					←→		
Step 6							←→



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## Chapter 2

### Preliminaries

The Fibonacci sequence  $(F_n)$  is defined by the recurrence relation  $F_{n+1} = F_n + F_{n-1}$  for  $n \geq 1$  with  $F_0 = 0$  and  $F_1 = 1$  and the Lucas sequence  $(L_n)$  is defined by  $L_{n+1} = L_n + L_{n-1}$  for  $n \geq 1$  with  $L_0 = 2$  and  $L_1 = 1$ .

The characteristic equation of the recurrence relation of Fibonacci numbers is  $x^2 - x - 1 = 0$  and its zeros are  $\varphi = \frac{1+\sqrt{5}}{2}$  and  $\tau = \frac{1-\sqrt{5}}{2}$ . Note that  $\varphi + \tau = 1$  and  $\varphi\tau = -1$ .

The well known identities, Binet's formulas, for  $F_n$  and  $L_n$  are shown in the following Lemma :

**Lemma 2.1.** [T., Koshy, p.145], for  $n \geq 1$ , we have

$$F_n = \frac{\varphi^n - \tau^n}{\varphi - \tau},$$
$$L_n = \varphi^n + \tau^n.$$

**Definition 2.1.** (Descartes' rule of signs) Let  $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n = 0$  has real coefficient. Let  $r$  be the number of positive real roots of  $f(x)$  (counted with multiplicity) and let  $V$  be the number of variations in the sequence of its coefficients  $a_0, a_1, \dots, a_n$  then  $r \leq V$  and if  $r < V$  then  $V - r$  always by an even number.

**Definition 2.2.** (Cyclotomic Polynomials) [2, p.81]. If  $n \in \mathbb{N}$ , then the  $n^{\text{th}}$  cyclotomic polynomial is given by

$$\Phi_n(x) = \prod_{\substack{j=1 \\ \gcd(n,j)=1}}^n (x - \zeta_n^j), \quad \zeta_n = \exp(2\pi i/n).$$

The degree of  $\Phi_n(x)$  is  $\phi(n)$  where  $\phi(n)$  is the Euler totient.

**Definition 2.3.** (Mahler measures) Let  $G(x) \in \mathbb{R}[X]$ ,  $\deg G = n$ , with leading coefficient  $a_n$ , and complex zeros  $\alpha_1, \dots, \alpha_n$ . It's Mahler measure is defined by

$$M(G(x)) = |a_n| \prod_{j=1}^n \max\{1, |\alpha_j|\}.$$

**Theorem 2.2.** (Rouché's Theorem) [3, p.2]. If  $P(z)$  and  $Q(z)$  are analytic interior in a simple closed Jordan curve  $C$ , and if they are continuous on  $C$  and

$$|P(z)| < |Q(z)|, z \in C,$$

then the function  $H(z) = P(z) + Q(z)$  has the same number of zero interior to  $C$  as does  $Q(z)$ .

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## 2.1 Literature reviews

### 2.1.1 The work of D. Garth, D. Mills and P. Mitchell ([4])

In this work, they considered the zeros and the Mahler measures of the Fibonacci-coefficient polynomial (FCP) of order  $n$  which is a polynomial of the form

$$p_n(x) = \sum_{k=0}^n F_{k+1} x^{n-k} \quad (2.1)$$

where  $F_0 = 0$ ,  $F_1 = 1$  and  $F_k = F_{k-1} + F_{k-2}$  ( $k \geq 2$ ).

**Lemma 2.3** ([4]). *If  $n$  is odd, then the period of the Fibonacci sequence modulo  $L_n$ , with residue classes adjusted to the range between  $-\frac{L_n-1}{2}$  to  $\frac{L_n-1}{2}$  for odd  $L_n$  and  $-\frac{L_n}{2} + 1$  to  $\frac{L_n}{2}$  for even  $L_n$ , is*

$$F_0, F_1, F_2, F_3, \dots, F_n, (-F_{n-1}), F_{n-2}, (-F_{n-3}), F_{n-4}, \dots, (-F_n), F_1.$$

*Example 2.4.* For  $n = 3$ ,  $L_3 = 4$ , the period of  $F_n \pmod{L_3}$  is

$$0, 1, 1, 2, -1, 1.$$

For  $n = 5$ ,  $L_5 = 11$ . The period of  $F_n \pmod{L_5}$  is

$$0, 1, 1, 2, 3, 5, -3, 2, -1, 1.$$

**Lemma 2.5** ([4]). *If  $n$  is even, then the period of the Fibonacci sequence modulo  $L_n$ , with residue classes adjusted as in Lemma 2.3 is*

$$F_0, F_1, F_2, F_3, \dots, F_n, (-F_{n+1}), F_{n+2}, (-F_{n+3}), \dots, F_2, (-F_1), F_0, (-F_1), (-F_2), (-F_3), \dots, (-F_n), F_{n-1}, (-F_{n-2}), F_{n-3}, \dots, (-F_2), F_1.$$

*Example 2.6.* For  $n = 2$ ,  $L_2 = 3$ , the period of  $F_n \pmod{L_2}$  is

$$0, 1, 1, -1, 0, -1, -1, 1.$$

For  $n = 6$ ,  $L_6 = 18$ , the period of  $F_n \pmod{L_6}$  is

$$0, 1, 1, 2, 3, 5, 8, -5, 3, -2, 1, -1, 0, -1, -1, -2, -3, -5, -8, 5, -3, 2, -1, 1.$$

**Theorem 2.7** ([4]). *For even  $n$ ,  $p_n(x)$  has no real zeros. For odd  $n$ ,  $p_n(x)$  has exactly one real zero, which lies in the interval  $(-\varphi, -1]$ . Moreover, the sequence of real zeros of the  $p_n(x)$ , with  $n$  odd, decreases monotonically to zero.*

**Theorem 2.8** ([4]). *As  $n$  increases without bound, the zeros of  $p_n(x)$  approach  $\varphi$  in modulus.*

Let  $G(x) \in \mathbb{R}[X]$ ,  $\deg G = n$ , with leading coefficient  $a_n$ , and complex zeros  $\alpha_1, \dots, \alpha_n$ . It's Mahler measure is defined by

$$M(G(x)) = |a_n| \prod_{j=1}^n \max\{1, |\alpha_j|\}.$$

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*Example 2.9.* Given  $p(x) = 3x^3 - 8x^2 + 10x - 4$ ,

it's zeros are  $\frac{2}{3}$ ,  $1 + i$  and  $1 - i$ . The Mahler measure of  $p(x)$  is  $(3)(1 \cdot \sqrt{2} \cdot \sqrt{2}) = 6$ .

Let  $\{p_n^{(m)}(x)\}$  ( $m \geq 2$ ) be the sequence obtained from FCP,  $p_n(x) = \sum_{k=0}^n F_{k+1}x^{n-k}$ , by reducing the coefficients  $F_{k+1}$  ( $0 \leq k \leq n$ ) modulo  $m$ .

Suppose  $p_n^{(m)}(x) = \sum_{k=0}^n \bar{F}_{k+1}x^{n-k}$ .

The coefficients of  $p_n^{(m)}(x)$  is defined by

$$\begin{aligned} \bar{F}_{k+1} &= F_{k+1} \pmod{m} \text{ if } F_{k+1} \pmod{m} \leq \frac{m}{2} \text{ and} \\ \bar{F}_{k+1} &= F_{k+1} \pmod{m} - m \text{ if } F_{k+1} \pmod{m} > \frac{m}{2}. \end{aligned}$$

*Example 2.10.* We will find  $P_6^{(4)}(x)$ .

Since  $P_6(x) = x^6 + x^5 + 2x^4 + 3x^3 + 5x^2 + 8x + 13$ , thus  $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8$ , and  $F_7 = 13$ .

Since  $m = 4$ , we get  $F_1 \pmod{4} = 1, F_2 \pmod{4} = 1, F_3 \pmod{4} = 2, F_5 \pmod{4} = 1, F_6 \pmod{4} = 0$  and  $F_7 \pmod{4} = 1$  which they are less than or equal to  $\frac{m}{2} = 2$ .

Thus  $\bar{F}_1 = \bar{F}_2 = \bar{F}_5 = \bar{F}_7 = 1, \bar{F}_3 = 2$  and  $\bar{F}_6 = 0$ . Since  $F_4 \pmod{4} = 3 > \frac{m}{2} = 2$ , thus  $\bar{F}_4 = 3 - 4 = -1$ . Hence we get  $P_6^{(4)}(x) = x^6 + x^5 + 2x^4 - x^3 + x^2 + 1$ .

**Theorem 2.11** ([4]). *Let  $m \geq 2$ , and let  $t$  be the number of terms in one period of the Fibonacci sequence reduced modulo  $m$ .*

*If  $k \equiv -2$  or  $-1 \pmod{t}$ , then*

$$M(p_k^{(m)}(x)) = M(p_{t-2}^{(m)}(x)).$$

*Example 2.12.* Let  $m = 4$ . From Example 2.4,  $t = 6$ , given  $k = 5$ , we have

$P_5^{(4)}(x) = x^5 + x^4 + 2x^3 - x^2 + x$  and its zeros are

$$0, -0.8090 + 1.4013i, -0.8090 - 1.4013i, 0.3090 + 0.5352i \text{ and } 0.3090 - 0.5352i.$$

$P_4^{(4)}(x) = x^4 + x^3 + 2x^2 - x + 1$  and its zeros are

$$-0.8090 + 1.4013i, -0.8090 - 1.4013i, 0.3090 + 0.5352i \text{ and } 0.3090 - 0.5352i.$$

We find that

$$M(P_5^{(4)}(x)) = 2.6182,$$

$$M(P_4^{(4)}(x)) = 2.6182.$$

**Theorem 2.13** ([4]). *If  $k \equiv -2$  or  $-1 \pmod{2n}$ , then  $M(P_k^{(\mathcal{L}_n)}(x)) = \varphi^{n-1}$ .*

*Example 2.14.* Let  $n = 2$ , then  $L_2 = 3$ . Choose  $k = 2$ . Then

$P_2^{(3)}(x) = x^2 + x - 1$ , and we get

$$M(P_2^{(3)}(x)) = (1)\left(\frac{1 + \sqrt{5}}{2}\right)(1) = \frac{1 + \sqrt{5}}{2} = \varphi.$$

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### 2.1.2 The work of F. Mátyás ([5])

Mátyás considered the generalized Fibonacci-coefficient polynomial (GFCP)

$$P_n(x) = \sum_{k=0}^n \mathcal{F}_{k+1} x^{n-k} \quad (2.2)$$

where, for fixed positive integers  $a$  and  $b$ , the coefficient are generalized Fibonacci numbers defined by

$$\mathcal{F}_0 = 0, \mathcal{F}_1 = 1, \mathcal{F}_{n+1} = a\mathcal{F}_n + b\mathcal{F}_{n-1} \quad (n \geq 1).$$

He found the number and the location of the real zeros of the polynomials  $P_n(x)$  as follows :

**Theorem 2.15 ([5]).** *If  $n \geq 2$  and even, then the polynomial  $P_n(x)$  has no real zero, that is, every zeros are non-real complex numbers. If  $n \geq 3$  and odd, then the polynomial  $P_n(x)$  has only one real zero and this is negative. That is, every but one zero is a non-real complex numbers.*

**Theorem 2.16 ([5]).** *Let  $z \in \mathbb{C}$  denote an arbitrary zero of the polynomial  $P_n(x)$ . For  $n \geq 1$ , we have*

$$a \leq |z| \leq a + \frac{b}{a}$$

where  $a$  and  $b$  are positive integers from the definition of generalized Fibonacci numbers.

## Chapter 3

### Preliminary lemmas

Our first lemma gives two recurrence relations for GFCP's.

**Lemma 3.1.** *For natural numbers  $n$ , we have  $P_0(x) = 1, P_1(x) = x + a$*

$$P_n(x) = x^n + aP_{n-1}(x) + bP_{n-2}(x) \quad \text{for } n \geq 2 \quad (3.1)$$

$$P_n(x) = xP_{n-1}(x) + \mathcal{F}_{n+1} \quad \text{for } n \geq 1. \quad (3.2)$$

*Proof.* By the definition of the GFCP, we have

$$\begin{aligned} P_n(x) &= \sum_{k=0}^n \mathcal{F}_{k+1} x^{n-k} \\ &= \mathcal{F}_1 x^n + \mathcal{F}_2 x^{n-1} + \mathcal{F}_3 x^{n-2} + \mathcal{F}_4 x^{n-3} + \cdots + \mathcal{F}_{n-1} x^2 + \mathcal{F}_n x + \mathcal{F}_{n+1} \\ &= \mathcal{F}_1 x^n + (a\mathcal{F}_1 + b\mathcal{F}_0) x^{n-1} + (a\mathcal{F}_2 + b\mathcal{F}_1) x^{n-2} + (a\mathcal{F}_3 + b\mathcal{F}_2) x^{n-3} \\ &\quad + \cdots + (a\mathcal{F}_{n-2} + b\mathcal{F}_{n-3}) x^2 + (a\mathcal{F}_{n-1} + b\mathcal{F}_{n-2}) x + (a\mathcal{F}_n + b\mathcal{F}_{n-1}) \\ &= \mathcal{F}_1 x^n + a\mathcal{F}_1 x^{n-1} + a\mathcal{F}_2 x^{n-2} + a\mathcal{F}_3 x^{n-3} + \cdots + a\mathcal{F}_{n-2} x^2 + a\mathcal{F}_{n-1} x + a\mathcal{F}_n \\ &\quad + b\mathcal{F}_0 x^{n-1} + b\mathcal{F}_1 x^{n-2} + b\mathcal{F}_2 x^{n-3} + \cdots + b\mathcal{F}_{n-3} x^2 + b\mathcal{F}_{n-2} x + b\mathcal{F}_{n-1} \\ &= \mathcal{F}_1 x^n + aP_{n-1}(x) + b\mathcal{F}_0 x^{n-1} + bP_{n-2}(x) \\ &= x^n + aP_{n-1}(x) + bP_{n-2}(x), \end{aligned}$$

and

$$\begin{aligned} P_n(x) &= \sum_{k=0}^n \mathcal{F}_{k+1} x^{n-k} \\ &= \mathcal{F}_1 x^n + \mathcal{F}_2 x^{n-1} + \mathcal{F}_3 x^{n-2} + \mathcal{F}_4 x^{n-3} + \cdots + \mathcal{F}_{n-1} x^2 + \mathcal{F}_n x + \mathcal{F}_{n+1} \\ &= x(\mathcal{F}_1 x^{n-1} + \mathcal{F}_2 x^{n-2} + \mathcal{F}_3 x^{n-3} + \mathcal{F}_4 x^{n-4} + \cdots + \mathcal{F}_{n-1} x + \mathcal{F}_n) + \mathcal{F}_{n+1} \\ &= xP_{n-1}(x) + \mathcal{F}_{n+1}. \end{aligned}$$

□

From Lemma 3.1, the characteristic equation of  $P_n(x)$  is  $x^2 - ax - b = 0$  and its characteristic zeros are  $\frac{a \pm \sqrt{a^2 + 4b}}{2}$ . For convenience, we let

$$\varphi_{a,b} = \frac{a + \sqrt{a^2 + 4b}}{2} > 0, \quad \tau_{a,b} = \frac{a - \sqrt{a^2 + 4b}}{2} < 0$$

and notice that  $\varphi_{a,b} + \tau_{a,b} = a$ ,  $\varphi_{a,b} - \tau_{a,b} = \sqrt{a^2 + 4b}$  and  $\varphi_{a,b}\tau_{a,b} = -b$ .

Recall that generalized Lucas numbers are defined by  $\mathcal{L}_0 = 2$ ,  $\mathcal{L}_1 = a$ ,  $\mathcal{L}_{n+2} = a\mathcal{L}_{n+1} + b\mathcal{L}_n$  ( $n \geq 0$ ). The next lemma, gives some relations among generalized Fibonacci numbers and generalized Lucas numbers.

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**Theorem 3.2.** [3, p.143] Let  $\varphi_{a,b}$  and  $\tau_{a,b}$  be the distinct solutions of the equation  $x^2 - ax - b = 0$  where  $a$  and  $b$  are non-zero constants. Then every solution of a recurrence relation of the form  $a_n = aa_{n-1} + ba_{n-2}$ , where  $a_0 = C_0$  and  $a_1 = C_1$ , is of the form  $a_n = A\varphi_{a,b}^n + B\tau_{a,b}^n$  for some constants  $A$  and  $B$ .

*Remark :* In our work,  $a$  and  $b$  are natural numbers.

**Lemma 3.3.** For  $n \in \mathbb{Z}$ ,  $n \geq 1$ , we have

$$i) \mathcal{F}_n = \frac{\varphi_{a,b}^n - \tau_{a,b}^n}{\varphi_{a,b} - \tau_{a,b}}, \quad \mathcal{L}_n = \varphi_{a,b}^n + \tau_{a,b}^n, \quad \mathcal{L}_n = \mathcal{F}_{n+1} + b\mathcal{F}_{n-1} \quad (n \geq 1);$$

$$ii) \mathcal{F}_{n+k} = \mathcal{F}_k \mathcal{L}_n + (-b)^k \mathcal{F}_{n-k} \quad (k \in \mathbb{N}, 1 \leq k \leq n).$$

*Proof.* i) Consider the recurrence relation  $\mathcal{F}_{n+1} = a\mathcal{F}_n + b\mathcal{F}_{n-1}$ . From Theorem 3.2, the general solution is  $\mathcal{F}_n = A\varphi_{a,b}^n + B\tau_{a,b}^n$  for some constants  $A$  and  $B$ . To find the values of  $A$  and  $B$ , the initial values  $\mathcal{F}_0 = 0$  and  $\mathcal{F}_1 = 1$  yield

$$\begin{aligned} A + B &= 0 \\ A\varphi_{a,b} + B\tau_{a,b} &= 1. \end{aligned}$$

Solving these system, we get  $A = \frac{1}{\sqrt{a^2 + 4b}}$  and  $B = \frac{-1}{\sqrt{a^2 + 4b}}$ . Since  $\varphi_{a,b} - \tau_{a,b} = \sqrt{a^2 + 4b}$ , we get  $\mathcal{F}_n = \frac{\varphi_{a,b}^n - \tau_{a,b}^n}{\varphi_{a,b} - \tau_{a,b}}$  as desired.

Since the generalized Lucas numbers are defined by a recurrence relation  $\mathcal{L}_{n+2} = a\mathcal{L}_{n+1} + b\mathcal{L}_n$  with  $\mathcal{L}_0 = 2$  and  $\mathcal{L}_1 = a$ . By the same argument above, we have

$\mathcal{L}_n = \varphi_{a,b}^n + \tau_{a,b}^n$ . Finally, we are going to show that  $\mathcal{L}_n = \mathcal{F}_{n+1} + b\mathcal{F}_{n-1}$

Since  $\mathcal{F}_n = \frac{\varphi_{a,b}^n - \tau_{a,b}^n}{\sqrt{a^2 + 4b}}$ , then

$$\begin{aligned} \mathcal{F}_{n+1} + b\mathcal{F}_{n-1} &= \left( \frac{1}{\sqrt{a^2 + 4b}} \varphi_{a,b}^{n+1} - \frac{1}{\sqrt{a^2 + 4b}} \tau_{a,b}^{n+1} \right) + b \left( \frac{1}{\sqrt{a^2 + 4b}} \varphi_{a,b}^{n-1} - \frac{1}{\sqrt{a^2 + 4b}} \tau_{a,b}^{n-1} \right) \\ &= \frac{\varphi_{a,b}^{n+1} - \tau_{a,b}^{n+1} + b\varphi_{a,b}^{n-1} - b\tau_{a,b}^{n-1}}{\sqrt{a^2 + 4b}} \\ &= \frac{\varphi_{a,b}^n \left( \varphi_{a,b} + \frac{b}{\varphi_{a,b}} \right) - \tau_{a,b}^n \left( \tau_{a,b} + \frac{b}{\tau_{a,b}} \right)}{\sqrt{a^2 + 4b}} \\ &= \frac{\varphi_{a,b}^n (\sqrt{a^2 + 4b}) - \tau_{a,b}^n (-\sqrt{a^2 + 4b})}{\sqrt{a^2 + 4b}} \\ &= \frac{(\sqrt{a^2 + 4b})(\varphi_{a,b}^n + \tau_{a,b}^n)}{\sqrt{a^2 + 4b}} \\ &= \varphi_{a,b}^n + \tau_{a,b}^n \\ &= \mathcal{L}_n. \end{aligned}$$

ii) Let  $k \in \mathbb{N}$  and  $1 \leq k \leq n$ , by first two formulas in i), we have

$$\begin{aligned}
 \mathcal{F}_{n+k} - \mathcal{F}_k \mathcal{L}_n &= \frac{\varphi_{a,b}^{n+k} - \tau_{a,b}^{n+k}}{\varphi_{a,b} - \tau_{a,b}} - \left( \frac{\varphi_{a,b}^k - \tau_{a,b}^k}{\varphi_{a,b} - \tau_{a,b}} \right) (\varphi_{a,b}^n + \tau_{a,b}^n) \\
 &= \frac{\varphi_{a,b}^{n+k} - \tau_{a,b}^{n+k} - (\varphi_{a,b}^{n+k} + \varphi_{a,b}^k \tau_{a,b}^n - \varphi_{a,b}^n \tau_{a,b}^k - \tau_{a,b}^{n+k})}{\varphi_{a,b} - \tau_{a,b}} \\
 &= \frac{\varphi_{a,b}^{n+k} - \tau_{a,b}^{n+k} - \varphi_{a,b}^{n+k} - \varphi_{a,b}^k \tau_{a,b}^n + \varphi_{a,b}^n \tau_{a,b}^k + \tau_{a,b}^{n+k}}{\varphi_{a,b} - \tau_{a,b}} \\
 &= \frac{\varphi_{a,b}^n \tau_{a,b}^k - \varphi_{a,b}^k \tau_{a,b}^n}{\varphi_{a,b} - \tau_{a,b}} \\
 &= \frac{\varphi_{a,b}^k \tau_{a,b}^k (\varphi_{a,b}^{n-k} - \tau_{a,b}^{n-k})}{\varphi_{a,b} - \tau_{a,b}} \\
 &= \frac{(\varphi_{a,b} \tau_{a,b})^k (\varphi_{a,b}^{n-k} - \tau_{a,b}^{n-k})}{\varphi_{a,b} - \tau_{a,b}} \\
 &= \frac{(-b)^k (\varphi_{a,b}^{n-k} - \tau_{a,b}^{n-k})}{\varphi_{a,b} - \tau_{a,b}} \\
 &= (-b)^k \mathcal{F}_{n-k}.
 \end{aligned}$$

□

For  $n \in \mathbb{N}$ , let

$$G_n(x) = P_n(x)(x^2 - ax - b). \quad (3.3)$$

Two useful relations are listed in the next lemma.

**Lemma 3.4.** For  $n \in \mathbb{N}$ , we have

$$G_n(x) = x^{n+2} - \mathcal{F}_{n+2}x - b\mathcal{F}_{n+1}. \quad (3.4)$$

$$x^n + a\mathcal{F}_n = P_n(x) - (ax + b)P_{n-2}(x). \quad (3.5)$$

*Proof.* Using (3.3), (2.2) and  $\mathcal{F}_n = a\mathcal{F}_{n-1} + b\mathcal{F}_{n-2}$ , we get

$$\begin{aligned}
 G_n(x) &= P_n(x)(x^2 - ax - b) \\
 &= \left( \sum_{k=0}^n \mathcal{F}_{k+1} x^{n-k} \right) (x^2 - ax - b) \\
 &= (\mathcal{F}_1 x^n + \mathcal{F}_2 x^{n-1} + \mathcal{F}_3 x^{n-2} + \dots + \mathcal{F}_n x + \mathcal{F}_{n+1})(x^2 - ax - b) \\
 &= \mathcal{F}_1 x^{n+2} + \mathcal{F}_2 x^{n+1} + \mathcal{F}_3 x^n + \mathcal{F}_4 x^{n-1} + \dots + \mathcal{F}_n x^3 + \mathcal{F}_{n+1} x^2 \\
 &\quad - a\mathcal{F}_1 x^{n+1} - a\mathcal{F}_2 x^n - a\mathcal{F}_3 x^{n-1} - \dots - a\mathcal{F}_{n-1} x^3 - a\mathcal{F}_n x^2 - a\mathcal{F}_{n+1} x \\
 &\quad - b\mathcal{F}_1 x^n - b\mathcal{F}_2 x^{n-1} - \dots - b\mathcal{F}_{n-2} x^3 - b\mathcal{F}_{n-1} x^2 - b\mathcal{F}_n x - b\mathcal{F}_{n+1} \\
 &= \mathcal{F}_1 x^{n+2} - (a\mathcal{F}_{n+1} + b\mathcal{F}_n)x - b\mathcal{F}_{n+1} \\
 &= \mathcal{F}_1 x^{n+2} - \mathcal{F}_{n+2}x - b\mathcal{F}_{n+1} \\
 &= x^{n+2} - \mathcal{F}_{n+2}x - b\mathcal{F}_{n+1}.
 \end{aligned}$$

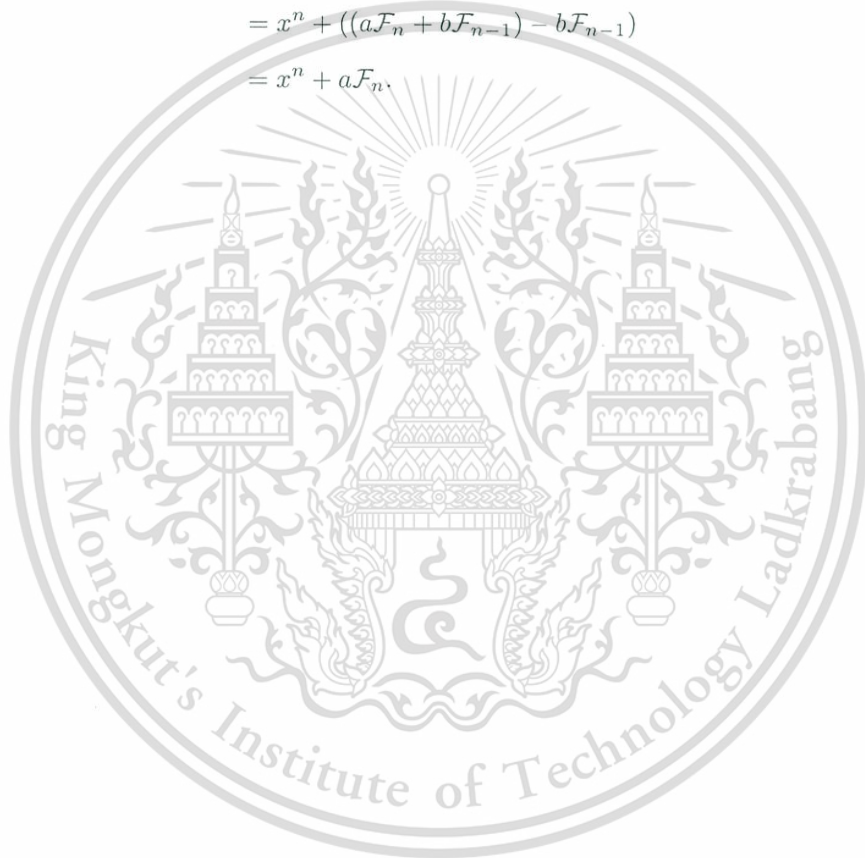
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and

$$\begin{aligned}
P_n(x) - (ax + b)P_{n-2}(x) &= \left(\sum_{k=0}^n \mathcal{F}_{k+1}x^{n-k}\right) - (ax + b)\left(\sum_{k=0}^{n-2} \mathcal{F}_{k+1}x^{n-2-k}\right) \\
&= \sum_{k=0}^n \mathcal{F}_{k+1}x^{n-k} - a\sum_{k=0}^{n-2} \mathcal{F}_{k+1}x^{n-1-k} - b\sum_{k=0}^{n-2} \mathcal{F}_{k+1}x^{n-2-k} \\
&= \mathcal{F}_1x^n + \mathcal{F}_2x^{n-1} + \mathcal{F}_3x^{n-2} + \mathcal{F}_4x^{n-3} + \cdots + \mathcal{F}_n x + \mathcal{F}_{n+1} \\
&\quad - a\mathcal{F}_1x^{n-1} - a\mathcal{F}_2x^{n-2} - a\mathcal{F}_3x^{n-3} - \cdots - a\mathcal{F}_{n-1}x \\
&\quad - b\mathcal{F}_1x^{n-2} - b\mathcal{F}_2x^{n-3} - \cdots - b\mathcal{F}_{n-2}x - b\mathcal{F}_{n-1} \\
&= \mathcal{F}_1x^n + \mathcal{F}_{n+1} - b\mathcal{F}_{n-1} \\
&= x^n + (\mathcal{F}_{n+1} - b\mathcal{F}_{n-1}) \\
&= x^n + ((a\mathcal{F}_n + b\mathcal{F}_{n-1}) - b\mathcal{F}_{n-1}) \\
&= x^n + a\mathcal{F}_n.
\end{aligned}$$

□



## Chapter 4

### Real zeros

In this chapter, we investigate the real zeros of GFCP's.

**Theorem 4.1.** For even  $n \geq 2$ , each polynomial  $P_n(x)$  has no real zeros. For odd  $n \geq 3$ , each polynomial  $P_n(x)$  has exactly one real zero which is negative; if  $a^2 \geq b$ , then this zero lies in the interval  $(-\varphi_{a,b}, -a]$ . Moreover, the sequence of real zeros of the polynomials  $P_n(x)$  with odd  $n$  converges to  $-\varphi_{a,b}$ .

*Proof.* For even  $n \geq 2$ , by Descart's rule of signs,  $G_n(x) = x^{n+2} - \mathcal{F}_{n+2}x - b\mathcal{F}_{n+1}$  has exactly one positive real zero and one negative real zero.

Since  $G_n(x) = P_n(x)(x^2 - ax - b)$  and the zeros of  $x^2 - ax - b$  are  $\varphi_{a,b} > 0$  and  $\tau_{a,b} < 0$ , then  $P_n(x)$  has no real zeros.

For odd  $n \geq 3$ , by Descart's rule of signs,  $G_n(x) = x^{n+2} - \mathcal{F}_{n+2}x - b\mathcal{F}_{n+1}$  has one positive real zero and two negative real zeros. By the same argument above, thus  $P_n(x)$  has exactly one negative real zero.

For odd  $n \geq 3$ , assuming  $a^2 \geq b$ , we now show that the negative real zero of  $P_n(x)$  lies in  $(-\varphi_{a,b}, -a]$ . Note first that

$$G_n(-\varphi_{a,b}) = -\varphi_{a,b}(\varphi_{a,b}^{n+1} - \mathcal{F}_{n+2}) + b\mathcal{F}_{n+1}. \quad (4.1)$$

By Lemma 3.3 i), for odd  $n \geq 3$  we have

$$\begin{aligned} \varphi_{a,b}^{n+1} - \mathcal{F}_{n+2} &= \varphi_{a,b}^{n+1} - \frac{\varphi_{a,b}^{n+2}}{\sqrt{a^2 + 4b}} + \frac{1}{\sqrt{a^2 + 4b}\varphi_{a,b}^{n+2}} \\ &= \varphi_{a,b}^{n+2} \left( \frac{1}{\varphi_{a,b}} - \frac{1}{\sqrt{a^2 + 4b}} + \frac{1}{\sqrt{a^2 + 4b}\varphi_{a,b}^{2n+4}} \right) > 0, \end{aligned}$$

since

$$\begin{aligned} \frac{1}{\varphi_{a,b}} - \frac{1}{\sqrt{a^2 + 4b}} &= \frac{2}{a + \sqrt{a^2 + 4b}} - \frac{1}{\sqrt{a^2 + 4b}} \\ &= \frac{2\sqrt{a^2 + 4b} - (a + \sqrt{a^2 + 4b})}{(a + \sqrt{a^2 + 4b})\sqrt{a^2 + 4b}} \\ &= \frac{\sqrt{a^2 + 4b} - a}{a\sqrt{a^2 + 4b} + a^2 + 4b} > 0, \end{aligned}$$

we get  $G_n(-\varphi_{a,b}) < 0$ .

Next, using  $\mathcal{F}_n > a^{n-1}$  ( $n \geq 3$ ) and  $a^2 \geq b$ , we get

$$\begin{aligned} G_n(-a) &= -a^{n+2} + a\mathcal{F}_{n+2} - b\mathcal{F}_{n+1} \\ &= -a^{n+2} + (a^2 - b)\mathcal{F}_{n+1} + ab\mathcal{F}_n \\ &> -a^{n+2} + (a^2 - b)a^n + ab \cdot a^{n-1} \geq 0, \end{aligned}$$

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which shows that the only real zero of  $G_n(x)$  lies in  $(-\varphi_{a,b}, -a]$ .

Finally, we show that for odd  $n \geq 3$ , the sequence of negative real zeros of  $P_n(x)$  converges, as  $n \rightarrow \infty$ , to  $-\varphi_{a,b}$ .

Let  $\alpha_n$  be the negative real zero of  $P_n(x)$ , for odd  $n \geq 3$ .

From (3.1), it is easily checked that  $G_n(x)$  has two critical points at  $\pm(\mathcal{F}_{n+2}/(n+2))^{1/(n+1)}$ .

Since  $\alpha_n$ ,  $\varphi_{a,b}$  and  $\tau_{a,b}$  are all the zeros of  $G_n(x)$ , with  $\alpha_n \in (-\varphi_{a,b}, -a]$ , it follows that  $G_n(x)$  has a local maximum at  $-(\mathcal{F}_{n+2}/(n+2))^{1/(n+1)}$ , with

$$-\varphi_{a,b} < \alpha_n \leq -(\mathcal{F}_{n+2}/(n+2))^{1/(n+1)}.$$

We show that  $\lim_{n \rightarrow \infty} (\mathcal{F}_{n+2})^{1/(n+1)} = \varphi_{a,b}$ . From Lemma 3.3 i),

$$\begin{aligned} \lim_{n \rightarrow \infty} (\mathcal{F}_{n+2})^{1/(n+1)} &= \lim_{n \rightarrow \infty} \varphi_{a,b}^{n+2} \left[ \left( \frac{1}{\sqrt{a^2+4b}} \right) - \left( \frac{1}{\sqrt{a^2+4b}} \right) \left( \frac{-1}{\varphi_{a,b}^{2n+4}} \right) \right]^{1/(n+1)} \\ &= \lim_{n \rightarrow \infty} (\varphi_{a,b}^{(n+2)/(n+1)}) \lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{a^2+4b}} - \left( \frac{1}{\sqrt{a^2+4b}} \right) \left( \frac{-1}{\varphi_{a,b}^{2n+4}} \right) \right)^{1/(n+1)} \\ &= \varphi_{a,b}^{\lim_{n \rightarrow \infty} ((n+2)/(n+1))} \lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{a^2+4b}} - \left( \frac{1}{\sqrt{a^2+4b}} \right) \left( \frac{-1}{\varphi_{a,b}^{2n+4}} \right) \right)^{1/(n+1)} \\ &= \varphi_{a,b} \lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{a^2+4b}} - \left( \frac{1}{\sqrt{a^2+4b}} \right) \left( \frac{-1}{\varphi_{a,b}^{2n+4}} \right) \right)^{1/(n+1)}. \end{aligned}$$

For convenience, we denote  $\frac{1}{\sqrt{a^2+4b}} + \left( \frac{1}{\sqrt{a^2+4b}} \right) \left( \frac{1}{\varphi_{a,b}^{2n+4}} \right)$  by  $X_n$ .

Note that  $\frac{1}{\sqrt{a^2+4b}} < X_n < \frac{2}{\sqrt{a^2+4b}}$ . By the squeeze theorem,

$$\lim_{n \rightarrow \infty} X_n^{1/(n+1)} = 1.$$

Therefore,  $\lim_{n \rightarrow \infty} (\mathcal{F}_{n+2})^{1/(n+1)} = \varphi_{a,b}$ , the last assertion follows. □

We proceed next to analyze the nature of all other zeros of  $P_n(x)$ .

**Lemma 4.2.** For  $n \in \mathbb{N}$ , we have

- i)  $\varphi_{a,b}^n = \varphi_{a,b} \mathcal{F}_n + b \mathcal{F}_{n-1}$  ;
- ii) if  $c > \varphi_{a,b}$ , then  $c^{n+2} > \mathcal{F}_{n+2}c + b \mathcal{F}_{n+1}$  ;
- iii) if  $0 < c < \varphi_{a,b}$ , then  $c^{n+2} < \mathcal{F}_{n+2}c + b \mathcal{F}_{n+1}$ .

*Proof.* i) When  $n = 1$ ,  $\varphi_{a,b} = \varphi_{a,b} \mathcal{F}_1 + b \mathcal{F}_0$ . So the result is true when  $n = 1$ .

For  $n \geq 2$ , from (3.3) and Lemma 3.4, we get

$$\begin{aligned} P_{n-2}(x)(x^2 - ax - b) &= x^n - \mathcal{F}_n x - b \mathcal{F}_{n-1} \\ \left( \sum_{k=0}^{n-2} \mathcal{F}_{k+1} x^{n-2-k} \right) (x^2 - ax - b) &= x^n - \mathcal{F}_n x - b \mathcal{F}_{n-1}. \end{aligned}$$

Since  $\varphi_{a,b}$  is one of the zeros of  $x^2 - ax - b$ . then

$$\varphi_{a,b}^n - \mathcal{F}_n \varphi_{a,b} - b \mathcal{F}_{n-1} = 0.$$

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Therefore, the formula holds for  $n \geq 2$ .

From (3.3) and Lemma 3.4, we have

$$P_n(x)(x^2 - ax - b) = x^{n+2} - \mathcal{F}_{n+2}x - b\mathcal{F}_{n+1}.$$

Since  $P_n(x) > 0$  for all  $x > 0$  and  $x^2 - ax - b < 0$  for  $0 < x < \varphi_{a,b}$ , and  $x^2 - ax - b > 0$  for  $x > \varphi_{a,b}$ , then  $c^{n+2} - \mathcal{F}_{n+2}c - b\mathcal{F}_{n+1} > 0$  for  $c > \varphi_{a,b}$  and  $c^{n+2} - \mathcal{F}_{n+2}c - b\mathcal{F}_{n+1} < 0$  for  $0 < c < \varphi_{a,b}$ . Yielding ii) and iii).  $\square$

**Theorem 4.3.** *The zeros of  $P_n(x)$  approach  $\varphi_{a,b}$  in modulus as  $n \rightarrow \infty$ .*

*Proof.* Consider the circle  $|z| = c$  in the complex plane, where  $c > \varphi_{a,b}$ . By Lemma 4.2 we have

$$|\mathcal{F}_{n+2}z + b\mathcal{F}_{n+1}| \leq \mathcal{F}_{n+2}|z| + b\mathcal{F}_{n+1} = \mathcal{F}_{n+2}c + b\mathcal{F}_{n+1} < c^{n+2} = |z^{n+2}|.$$

It then follows from Theorem 2.15 that  $P_n(z)(z^2 - az - b) = z^{n+2} - \mathcal{F}_{n+2}z - b\mathcal{F}_{n+1}$  has  $n+2$  zeros inside  $|z| = c$ . Since  $c > \varphi_{a,b}$  is arbitrary, it follows that  $P_n(z)(z^2 - az - b)$  has  $n+2$  zeros inside or on  $|z| = \varphi_{a,b}$ .

Now consider the circle  $|z| = c$  with  $\frac{b}{\varphi_{a,b}} < c < \varphi_{a,b}$ . We have

$$\begin{aligned} |\mathcal{F}_{n+2}z - b\mathcal{F}_{n+1}|^2 &= \mathcal{F}_{n+2}^2 c^2 + 2b\mathcal{F}_{n+2}\mathcal{F}_{n+1} \operatorname{Re}(z) + b^2 \mathcal{F}_{n+1}^2 \\ &\geq \mathcal{F}_{n+2}^2 c^2 - 2b\mathcal{F}_{n+2}\mathcal{F}_{n+1}c + b^2 \mathcal{F}_{n+1}^2 \\ &= (\mathcal{F}_{n+2}c - b\mathcal{F}_{n+1})^2, \end{aligned}$$

yielding

$$|\mathcal{F}_{n+2}z + b\mathcal{F}_{n+1}| \geq |\mathcal{F}_{n+2}c - b\mathcal{F}_{n+1}|. \quad (4.2)$$

Since

$$\mathcal{F}_{n+2}c - b\mathcal{F}_{n+1} = \varphi_{a,b}^{n+2} \left( \frac{c}{\sqrt{a^2 + 4b}} - \frac{b}{\varphi_{a,b} \sqrt{a^2 + 4b}} \right) + \frac{(-1)^{n+1}}{\sqrt{a^2 + 4b} \varphi_{a,b}^{n+1}} \left( \frac{c}{\varphi_{a,b}} + 1 \right),$$

it follows that for  $n$  large enough, we have

$$\mathcal{F}_{n+2}c - b\mathcal{F}_{n+1} > c^{n+2}. \quad (4.3)$$

Using (4.2) and (4.3), for  $|z| = c$ , with  $\frac{b}{\varphi_{a,b}} < c < \varphi_{a,b}$ , we have, for large  $n$ ,

$$|\mathcal{F}_{n+2}z + b\mathcal{F}_{n+1}| \geq |\mathcal{F}_{n+2}c - b\mathcal{F}_{n+1}| > c^{n+2}.$$

By Theorem 2.15 again, it follows that if  $\frac{b}{\varphi_{a,b}} < c < \varphi_{a,b}$ , then, for  $n$  large enough,  $P_n(z)(z^2 - az - b)$  has one zero in  $|z| = c$ . Since  $c$  is arbitrary, as  $n \rightarrow \infty$ , the zeros of  $P_n(z)$  approach  $\varphi_{a,b}$  in modulus.  $\square$

## Chapter 5

### Mahler measures

Let  $\{P_n^{(m)}(x)\}$  be the sequence obtained from the GFCP's by reducing the coefficients modulo  $m$  and adjusting the residue class in the same way as mentioned in chapter 2 section 2.1.1. It is well known that generalized Fibonacci sequence modulo  $m$  is periodic, and so let  $t$  to be the number of terms in one period.

**Theorem 5.1.** *Let  $m \geq 2$  and  $t$  be as above. If  $k \equiv -2$  or  $-1 \pmod{t}$ , then*

$$M(P_k^{(m)}(x)) = M(P_{t-2}^{(m)}(x)).$$

*Proof.* Let  $a_i \equiv F_{i-1} \pmod{m}$  ( $i \geq 1$ ) be the terms of the generalized Fibonacci sequence modulo  $m$ . Since  $a_1 = 0$  and  $a_2 = 1$ , we have

$$P_{t-2}^{(m)}(x) = x^{t-2} + a_3x^{t-3} + a_4x^{t-4} + \cdots + a_{t-1}x + a_t.$$

Let  $C_j(x) = x^j + x^{j-1} + \cdots + x + 1$ , where  $j \geq 1$ .

Setting  $P_k^{(m)}(x) = P_{t-2}^{(m)}(x)C_l(x^t)$ , and equating the degree of  $P_k^{(m)}(x)$  and  $P_{t-2}^{(m)}(x)C_l(x^t)$ , we have  $k = (t-2) + lt$ . From  $l = (k+2-t) \pmod{t}$ , we get  $k \equiv -2 \pmod{t}$ .

Setting  $P_k^{(m)}(x) = xP_{t-2}^{(m)}(x)C_l(x^t)$ , and equating the degree, we get  $k = (t-1) + lt$ , i.e.,  $k \equiv -1 \pmod{t}$ .

Since  $(C_j(x))(x-1) = x^{j+1} - 1$ , all the zeros of  $C_j(x^t)$  have modulus 1. Since the Mahler measure is multiplicative, the result follows. □

From the identity in Lemma 3.3, we have

$$\mathcal{L}_n = \mathcal{F}_{n+1} + b\mathcal{F}_{n-1} = 2\mathcal{F}_{n+1} - a\mathcal{F}_n > 2\mathcal{F}_{n+1} > 2\mathcal{F}_i \quad (0 \leq i \leq n).$$

Thus the first  $n+1$  elements of  $\mathcal{F}_i \pmod{\mathcal{L}_n}$  are  $\mathcal{F}_0 = 0, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n$ . From Lemma 3.3 ii), we have  $\mathcal{F}_{n+k} = \mathcal{F}_n \mathcal{L}_n + (-b)^k \mathcal{F}_{n-k}$  ( $1 \leq k \leq n$ ). Thus, the next  $n$  element of  $\mathcal{F}_i \pmod{\mathcal{L}_n}$ , i.e., for  $n+1 \leq i \leq 2n$ , are  $-b\mathcal{F}_{n-1}, b^2\mathcal{F}_{n-2}, -b^3\mathcal{F}_{n-3}, \dots, (-b)^{n-1}, 0$ , (noting that  $\mathcal{F}_0 = 0, \mathcal{F}_1 = 1$ ). This shows that the period of the generalized Fibonacci sequence modulo  $\mathcal{L}_n$  is  $2n$ .

**Theorem 5.2.** For  $\frac{b}{\varphi_{a,b}} < 1$ . If  $k \equiv -2$  or  $-1 \pmod{2n}$ , then  $M(P_k^{(\mathcal{L}_n)}(x)) = \varphi_{a,b}^{n-1}$ .

*Proof.* By Theorem 5.1, if  $k \equiv -2$  or  $-1 \pmod{2n}$ , then  $M(P_k^{(\mathcal{L}_n)}(x)) = M(P_{2n-2}^{(\mathcal{L}_n)}(x))$ .

Since

$$P_{2n-2}^{(\mathcal{L}_n)}(x) = x^{2n-2} + \mathcal{F}_2 x^{2n-3} + \mathcal{F}_3 x^{2n-4} + \dots + \mathcal{F}_n x^{n-1} + \mathcal{F}_{n+1} x^{n-2} + \mathcal{F}_{n+2} x^{n-3} + \dots + \mathcal{F}_{2n-2} x + \mathcal{F}_{2n-1},$$

we get

$$\begin{aligned} P_{2n-2}^{(\mathcal{L}_n)}(x) &= x^{2n-2} + \mathcal{F}_2 x^{2n-3} + \mathcal{F}_3 x^{2n-4} + \dots + \mathcal{F}_n x^{n-1} + (-b)\mathcal{F}_{n-1} x^{n-2} + (-b)^2 \mathcal{F}_{n-2} x^{n-3} \\ &\quad + \dots + (-b)^{n-2} \mathcal{F}_2 x + (-b)^{n-1} \mathcal{F}_1 \\ &= x^{n-1} P_{n-1}(x) - b x^{n-2} P_{n-2} \left( \frac{-b}{x} \right). \end{aligned} \quad (5.1)$$

To prove theorem, we have to show that

$$M \left( P_{2n-2}^{(\mathcal{L}_n)}(x) \right) = \varphi_{a,b}^{n-1}. \quad (5.2)$$

For odd  $n$ , using (5.1), we have

$$P_{2n-2}^{(\mathcal{L}_n)}(x)(x^2 - ax - b) = x^{2n} - \mathcal{L}_n x^n - b. \quad (5.3)$$

Using the  $k^{\text{th}}$  cyclotomic polynomial [section 2.2 p.81 [2]]  $\Phi_k(x)$ , it is easily checked that

$$b^{n-1} \prod_{d|n, d \neq 1} \Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right) \Phi_d \left( \frac{x}{\varphi_{a,b}} \right) = \frac{x^{2n} - \mathcal{L}_n x^n - b^n}{x^2 - ax - b},$$

which in turn gives

$$P_{2n-2}^{(\mathcal{L}_n)}(x) = b^{n-1} \prod_{d|n, d \neq 1} \Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right) \Phi_d \left( \frac{x}{\varphi_{a,b}} \right).$$

Since all the zeros of  $\Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right)$  have modulus  $\frac{b}{\varphi_{a,b}}$ , and all the zeros of  $\Phi_d \left( \frac{x}{\varphi_{a,b}} \right)$  have modulus  $\varphi_{a,b}$ , it follows that

$$M(P_{2n-2}^{(\mathcal{L}_n)}(x)) = M(b^{n-1}) M \left( \prod_{d|n, d \neq 1} \Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right) \right) M \left( \prod_{d|n, d \neq 1} \Phi_d \left( \frac{x}{\varphi_{a,b}} \right) \right)$$

From Definition 2.2, we see that

$$\Phi_d(x) = \prod_{j=1}^{\phi(d)} (x - \zeta_d^j).$$

Thus

$$\begin{aligned} \Phi_d \left( \frac{x}{\varphi_{a,b}} \right) &= \prod_{j=1}^{\phi(d)} \left( \frac{x}{\varphi_{a,b}} - \zeta_d^j \right) \\ &= \prod_{j=1}^{\phi(d)} \frac{1}{\varphi_{a,b}} (x - \zeta_d^j \varphi_{a,b}) \\ &= \frac{1}{\varphi_{a,b}^{\phi(d)}} \prod_{j=1}^{\phi(d)} (x - \zeta_d^j \varphi_{a,b}). \end{aligned}$$

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We see that all the zeros of  $\Phi_d(\frac{x}{\varphi_{a,b}})$  have modulus  $\varphi_{a,b} > 1$ .

It follows that  $M(\Phi_d(\frac{x}{\varphi_{a,b}})) = (\frac{1}{\varphi_{a,b}})^{\phi(d)} \varphi_{a,b}^{\phi(d)} = 1$ . And

$$\begin{aligned}\Phi_d(\frac{-\varphi_{a,b}x}{b}) &= \prod_{j=1}^{\phi(d)} (\frac{-\varphi_{a,b}x}{b} - \zeta_d^j) \\ &= \prod_{j=1}^{\phi(d)} (\frac{-\varphi_{a,b}}{b})(x + \zeta_d^j \frac{b}{\varphi_{a,b}}).\end{aligned}$$

Thus all the zeros of  $\Phi_d(\frac{-\varphi_{a,b}x}{b})$  have modulus  $\frac{b}{\varphi_{a,b}} < 1$ .

It follows that  $M(\Phi_d(\frac{-\varphi_{a,b}x}{b})) = (\frac{\varphi_{a,b}}{b})^{\phi(d)}$ .

We get

$$\begin{aligned}M(\Phi_{2n-2}^{(\mathcal{L}_n)}(x)) &= b^{n-1} M(\prod_{d|n, d \neq 1} \Phi_d(\frac{-\varphi_{a,b}x}{b})) M(\prod_{d|n, d \neq 1} \Phi_d(\frac{x}{\varphi_{a,b}})) \\ &= b^{n-1} \prod_{d|n, d \neq 1} M(\Phi_d(\frac{-\varphi_{a,b}x}{b})) \prod_{d|n, d \neq 1} M(\Phi_d(\frac{x}{\varphi_{a,b}})) \\ &= b^{n-1} (\prod_{d|n, d \neq 1} (\frac{\varphi_{a,b}}{b})^{\phi(d)}) (\prod_{d|n, d \neq 1} 1) \\ &= b^{n-1} [(\frac{\varphi_{a,b}}{b})^{\sum_{d|n, d \neq 1} \phi(d)}] \\ &= b^{n-1} (\frac{\varphi_{a,b}}{b})^{n-1} \\ &= \varphi_{a,b}^{n-1}.\end{aligned}$$

Since the zeros of  $x^2 + ax - b$  are  $-\tau_{a,b}$  and  $-\varphi_{a,b}$ , then  $M(x^2 + ax - b) = \varphi_{a,b}$  and by the same arguments as in the case  $n$  is odd, we get  $M(\Phi_d(\frac{x}{\varphi_{a,b}})) = 1$  and

$M(\Phi_d(\frac{-\varphi_{a,b}x}{b})) = (\frac{\varphi_{a,b}}{b})^{\phi(d)}$ . It follows that

$$\begin{aligned}M(P_{2n-2}^{(\mathcal{L}_n)}(x)) &= b^{n-2} M(x^2 + ax - b) M(\prod_{d|n, d \neq 1, 2} \Phi_d(\frac{-\varphi_{a,b}x}{b})) M(\prod_{d|n, d \neq 1, 2} \Phi_d(\frac{x}{\varphi_{a,b}})) \\ &= b^{n-2} \varphi_{a,b} (\prod_{d|n, d \neq 1, 2} (\frac{\varphi_{a,b}}{b})^{\phi(d)}) (\prod_{d|n, d \neq 1, 2} 1) \\ &= b^{n-2} \varphi_{a,b} [(\frac{\varphi_{a,b}}{b})^{\sum_{d|n, d \neq 1, 2} \phi(d)}] \\ &= b^{n-2} \varphi_{a,b} (\frac{\varphi_{a,b}}{b})^{n-2} \\ &= \varphi_{a,b}^{n-1}.\end{aligned}$$

For even  $n \geq 2$ , let

$$\begin{aligned}R_n(x) &= x^{2n-4} + \frac{\mathcal{F}_4}{a} x^{2n-6} + \frac{\mathcal{F}_6}{a} x^{2n-8} + \frac{\mathcal{F}_8}{a} x^{2n-10} + \dots + \frac{\mathcal{F}_{n-4}}{a} x^{n+2} + \frac{\mathcal{F}_{n-2}}{a} x^n + \frac{\mathcal{F}_n}{a} x^{n-2} \\ &\quad + \frac{(-b)^2}{a} \mathcal{F}_{n-2} x^{n-4} + \frac{(-b)^4}{a} \mathcal{F}_{n-4} x^{n-6} + \dots + \frac{(-b)^{n-4}}{a} \mathcal{F}_4 x^2 + \frac{(-b)^{n-2}}{a} \mathcal{F}_2.\end{aligned}$$

Using (5.1), we have

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$$P_{2n-2}^{(\mathcal{L}_n)}(x) = (x^2 + ax - b)R_n(x).$$

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It is also easy to show that

$$R_n(x)(x^4 - (a^2 + 2b)x^2 + b^2) = x^{2n} - \mathcal{L}_n x^n + b^n. \quad (5.5)$$

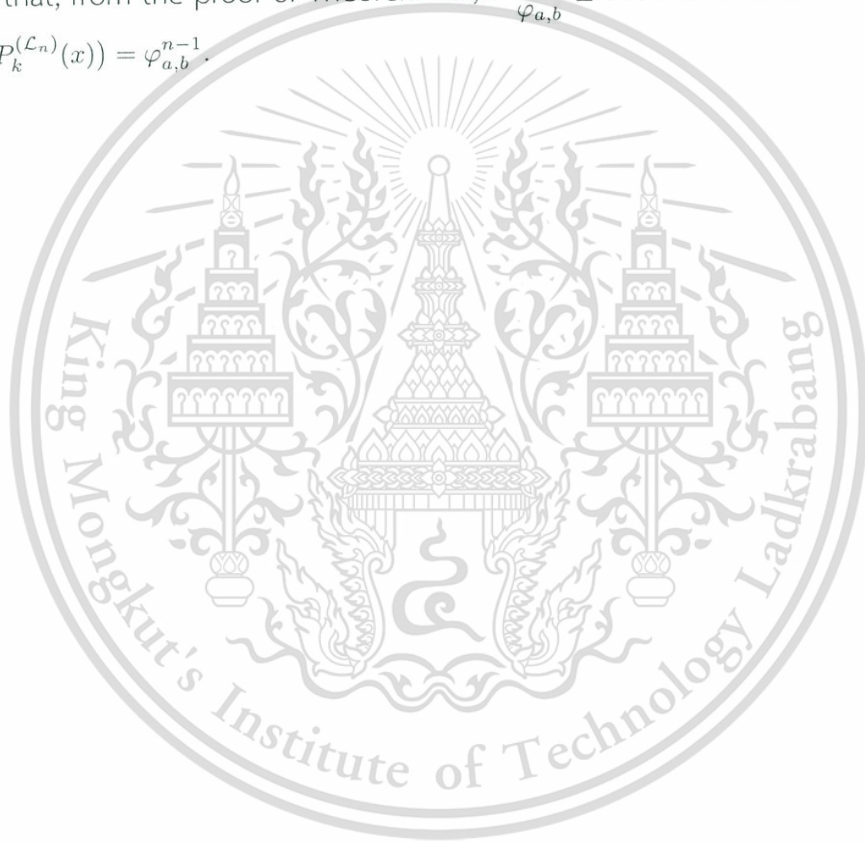
Similar to the case when  $n$  is odd, we have

$$b^{n-2} \prod_{d|n, d \neq 1, 2} \Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right) \Phi_d \left( \frac{x}{\varphi_{a,b}} \right) = \frac{x^{2n} - \mathcal{L}_n x^n + b^n}{x^4 - (a^2 + 2b)x^2 + b^2},$$

and so

$$P_{2n-2}^{(\mathcal{L}_n)}(x) = (x^2 + ax - b) b^{n-2} \prod_{d|n, d \neq 1, 2} \Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right) \Phi_d \left( \frac{x}{\varphi_{a,b}} \right).$$

We note that, from the proof of Theorem 5.2, if  $\frac{b}{\varphi_{a,b}} \geq 1$  and  $k \equiv -2$  or  $-1 \pmod{2n}$ , then  $M(P_k^{(\mathcal{L}_n)}(x)) = \varphi_{a,b}^{n-1}$ . □



## Chapter 6

### Conclusions

In our work, we consider a generalized Fibonacci-coefficient polynomial (GFCP) of the form

$$P_n(x) = \sum_{k=0}^n \mathcal{F}_{k+1} x^{n-k}$$

where the coefficients are generalized Fibonacci numbers defined, for fixed positive integers  $a$  and  $b$ , by

$$\mathcal{F}_0 = 0, \mathcal{F}_1 = 1, \mathcal{F}_{n+1} = a\mathcal{F}_n + b\mathcal{F}_{n-1} \quad (n \geq 1).$$

We investigated the zeros and the Mahler measures of the GFCP's. The main results are summarized as follows :

For  $m \geq 2$ , let  $\{p_n^{(m)}(x)\}$  ( $m \geq 2$ ) be the sequence obtained from FCP,  $p_n(x) = \sum_{k=0}^n \mathcal{F}_{k+1} x^{n-k}$ , by reducing the coefficients  $\mathcal{F}_{k+1}$  ( $0 \leq k \leq n$ ) modulo  $m$ .

Suppose  $p_n^{(m)}(x) = \sum_{k=0}^n \overline{\mathcal{F}}_{k+1} x^{n-k}$ .

The coefficients of  $p_n^{(m)}(x)$  is defined by

$$\overline{\mathcal{F}}_{k+1} = \mathcal{F}_{k+1} \pmod{m} \text{ if } \mathcal{F}_{k+1} \pmod{m} \leq \frac{m}{2} \text{ and}$$

$$\overline{\mathcal{F}}_{k+1} = \mathcal{F}_{k+1} \pmod{m} - m \text{ if } \mathcal{F}_{k+1} \pmod{m} > \frac{m}{2}.$$

1) For  $\frac{b}{\varphi_{a,b}} < 1$ ;  $\varphi_{a,b} = \frac{a + \sqrt{a^2 + 4b}}{2}$ . If  $k \equiv -2$  or  $-1 \pmod{t}$ , then

$M(P_k^{(m)}(x)) = M(P_{t-2}^{(m)}(x))$ , where  $M(P)$  denotes the Mahler measure of the polynomial  $P$ ,  $t$  is the number of terms in one period of the generalized Fibonacci sequence modulo  $m$ .

2) For  $\frac{b}{\varphi_{a,b}} < 1$ ;  $\varphi_{a,b} = \frac{a + \sqrt{a^2 + 4b}}{2}$ . If  $k \equiv -2$  or  $-1 \pmod{2n}$ , then

$M(P_k^{(\mathcal{L}_n)}(x)) = \varphi_{a,b}^{n-1}$ , where  $(\mathcal{L}_n)$  is the sequence of Lucas numbers defined by

$$\mathcal{L}_0 = 2, \mathcal{L}_1 = a, \text{ and } \mathcal{L}_{n+1} = a\mathcal{L}_n + b\mathcal{L}_{n-1} \quad (n \geq 2).$$

Remark : If  $a = b = 1$ , the solution will be the same as work of Garth, D., Mills, D., Mitchell, P. ([4]).

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# Appendix A

The first 20 Fibonacci and Lucas numbers

Table 6.1: The first 20 Fibonacci and Lucas numbers

$n$	$F_n$	$L_n$
0	0	2
1	1	1
2	1	3
3	2	4
4	3	7
5	5	11
6	8	18
7	13	29
8	21	47
9	34	76
10	55	123
11	89	199
12	144	322
13	233	521
14	377	843
15	610	1,364
16	987	2,207
17	1,597	3,571
18	2,584	5,778
19	4,181	9,349

## Appendix B

The first 20 Generalized Fibonacci numbers

Table 6.2: The first 20 Generalized Fibonacci numbers

$n$	$\mathcal{F}_n = a\mathcal{F}_{n-1} + b\mathcal{F}_{n-2}$
0	0
1	1
2	$a$
3	$a^2 + b$
4	$a^3 + 2ab$
5	$a^4 + 3a^2b + b^2$
6	$a^5 + 4a^3b + 3ab^2$
7	$a^6 + 5a^4b + 6a^2b^2 + b^3$
8	$a^7 + 6a^5b + 10a^3b^2 + 4ab^3$
9	$a^8 + 7a^6b + 15a^4b^2 + 10a^2b^3 + b^4$
10	$a^9 + 8a^7b + 21a^5b^2 + 20a^3b^3 + 5ab^4$
11	$a^{10} + 9a^8b + 28a^6b^2 + 35a^4b^3 + 15a^2b^4 + b^5$
12	$a^{11} + 10a^9b + 36a^7b^2 + 56a^5b^3 + 35a^3b^4 + 6ab^5$
13	$a^{12} + 11a^{10}b + 45a^8b^2 + 84a^6b^3 + 70a^4b^4 + 21a^2b^5 + b^6$
14	$a^{13} + 12a^{11}b + 55a^9b^2 + 120a^7b^3 + 126a^5b^4 + 56a^3b^5 + 7ab^6$
15	$a^{14} + 13a^{12}b + 66a^{10}b^2 + 165a^8b^3 + 210a^6b^4 + 126a^4b^5 + 28a^2b^6 + b^7$
16	$a^{15} + 14a^{13}b + 73a^{11}b^2 + 220a^9b^3 + 330a^7b^4 + 252a^5b^5 + 84a^3b^6 + 8ab^7$
17	$a^{16} + 15a^{14}b + 91a^{12}b^2 + 286a^{10}b^3 + 495a^8b^4 + 462a^6b^5 + 210a^4b^6 + 36a^2b^7 + b^8$
18	$a^{17} + 16a^{15}b + 105a^{13}b^2 + 106a^{11}b^3 + 715a^9b^4 + 792a^7b^5 + 462a^5b^6 + 120a^3b^7 + 9ab^8$
19	$a^{18} + 17a^{16}b + 120a^{14}b^2 + 197a^{12}b^3 + 1.001a^{10}b^4 + 1.287a^8b^5 + 924a^6b^6 + 330a^4b^7 + 45a^2b^8 + b^9$

## Appendix C

The first 20 Generalized Lucas numbers

Table 6.3: The first 20 Generalized Lucas numbers

$n$	$\mathcal{L}_n = a\mathcal{L}_{n-1} + b\mathcal{L}_{n-2}$
0	2
1	$a$
2	$a^2 + 2b$
3	$a^3 + 3ab$
4	$a^4 + 4a^2b + 2b^2$
5	$a^5 + 5a^3b + 5ab^2$
6	$a^6 + 6a^4b + 9a^2b^2 + 2b^3$
7	$a^7 + 6a^5b + 14a^3b^2 + 7ab^3$
8	$a^8 + 7a^6b + 20a^4b^2 + 16a^2b^3 + 2b^4$
9	$a^9 + 8a^7b + 26a^5b^2 + 30a^3b^3 + 9ab^4$
10	$a^{10} + 9a^8b + 33a^6b^2 + 50a^4b^3 + 25a^2b^4 + 2b^5$
11	$a^{11} + 10a^9b + 41a^7b^2 + 56a^5b^3 + 55a^3b^4 + 11ab^5$
12	$a^{12} + 11a^{10}b + 50a^8b^2 + 89a^6b^3 + 105a^4b^4 + 36a^2b^5 + 2b^6$
13	$a^{13} + 12a^{11}b + 60a^9b^2 + 130a^7b^3 + 161a^5b^4 + 91a^3b^5 + 13ab^6$
14	$a^{14} + 13a^{12}b + 71a^{10}b^2 + 180a^8b^3 + 250a^6b^4 + 196a^4b^5 + 49a^2b^6 + 2b^7$
15	$a^{15} + 14a^{13}b + 83a^{11}b^2 + 240a^9b^3 + 380a^7b^4 + 357a^5b^5 + 140a^3b^6 + 15ab^7$
16	$a^{16} + 15a^{14}b + 96a^{12}b^2 + 311a^{10}b^3 + 560a^8b^4 + 607a^6b^5 + 336a^4b^6 + 64a^2b^7 + 2b^8$
17	$a^{17} + 16a^{15}b + 110a^{13}b^2 + 394a^{11}b^3 + 800a^9b^4 + 987a^7b^5 + 693a^5b^6 + 204a^3b^7 + 17ab^8$
18	$a^{18} + 17a^{16}b + 125a^{14}b^2 + 490a^{12}b^3 + 1,111a^{10}b^4 + 1,547a^8b^5 + 1,300a^6b^6 + 540a^4b^7 + 81a^2b^8 + 2b^9$
19	$a^{19} + 18a^{17}b + 141a^{15}b^2 + 600a^{13}b^3 + 1,505a^{11}b^4 + 2,347a^9b^5 + 2,487a^7b^6 + 1,233a^5b^7 + 285a^3b^8 + 199ab^9$

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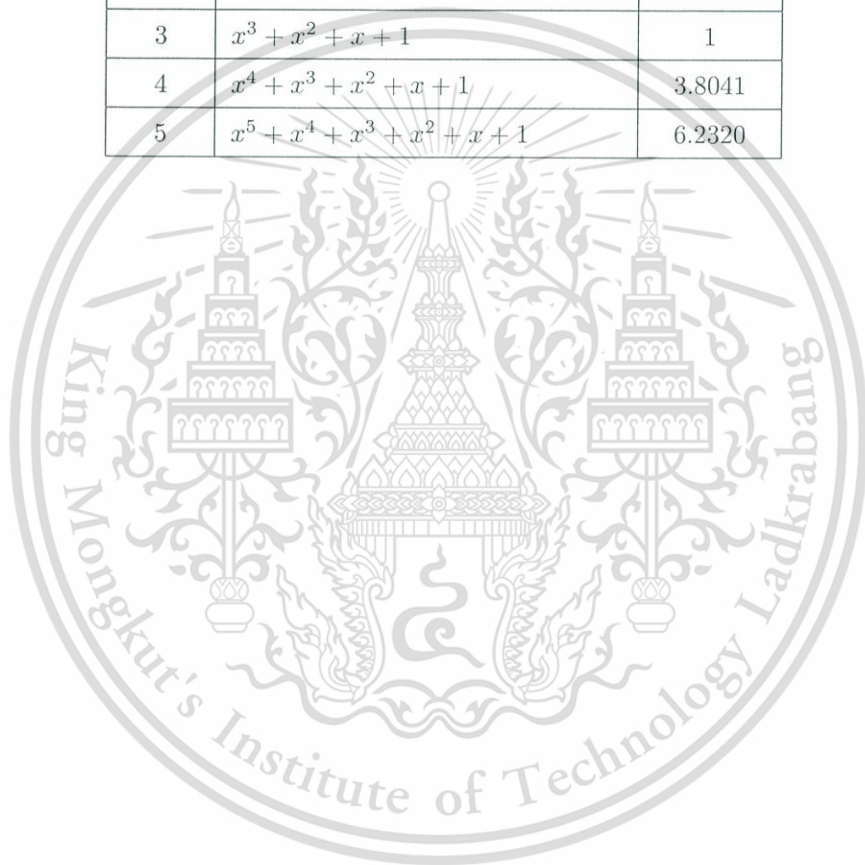
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## Appendix D

Sample of Mahler measures for polynomials of low degrees

**Table 6.4:** Sample of Mahler measures for polynomials of low degrees

<i>degree</i>	Sample of polynomials ( $p_n(x)$ )	$M(p_n(x))$
0	1	1
1	$x + 1$	1
2	$x^2 + x + 1$	2.4964
3	$x^3 + x^2 + x + 1$	1
4	$x^4 + x^3 + x^2 + x + 1$	3.8041
5	$x^5 + x^4 + x^3 + x^2 + x + 1$	6.2320



# Appendix E

The research paper



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# Polynomials with Generalized Fibonacci Number Coefficients

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**Abstract.** Garth, Mills and Mitchell in 2007 started the investigation on the nature of real zeros, and the Mahler measures of Fibonacci-coefficient polynomials, which are polynomials whose coefficients are Fibonacci numbers. Their results have later been extended in various directions. We complement these earlier works by investigating generalized Fibonacci-coefficient polynomials, which are polynomials whose coefficients are generalized Fibonacci numbers. Our main results give precise information about the location of the real roots of generalized Fibonacci-coefficient polynomials, and the computation of their Mahler measures.

## INTRODUCTION

A Fibonacci-coefficient polynomial (FCP) is a polynomial of the form

$$p_n(x) = \sum_{k=0}^n F_{k+1} x^{n-k} \quad (n \geq 1),$$

where the  $F_i$ 's are Fibonacci numbers defined by  $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$  ( $n \geq 2$ ). In 2007, Garth et al. [4], investigated the zeros and the Mahler measures of the FCP's. Their main results are summarized as follows:

- A. A1) For even  $n \geq 2$ , each polynomial  $p_n(x)$  has no real zeros.
- A2) For odd  $n \geq 3$  each polynomial  $p_n(x)$  has exactly one real zero lying in the interval  $[-\varphi, -1]$ , where  $\varphi = (1 + \sqrt{5})/2$ .
- A3) The sequence of real zeros of  $p_n(x)$  with  $n$  odd decreases monotonically to zero.
- B. The roots of  $p_n(x)$  approach  $\varphi$  in modulus as  $n \rightarrow \infty$ .
- C. For  $m \geq 2$ , let  $\{p_n^{(m)}(x)\}$  be the sequence obtained from the FCP's by reducing the coefficients modulo  $m$  and adjusting the residue class using  $F_k \pmod{m} - m$  if  $F_k \pmod{m} > m/2$ . Let  $t$  be the number of terms in one period of the Fibonacci sequence reduced modulo  $m$ .
- C1) If  $k \equiv -2$  or  $-1 \pmod{t}$ , then  $M(p_k^{(m)}(x)) = M(p_{t-2}^{(m)}(x))$ , where  $M(p)$  denotes the Mahler measure of the polynomial  $p$ .
- C2) If  $k \equiv -2$  or  $-1 \pmod{t}$ , then  $M(p_k^{(L_n)}(x)) = \varphi^{n-1}$ , where  $(L_n)$  is the sequence of Lucas numbers defined by  $L_0 = 2, L_1 = 1$ , and  $L_n = L_{n-1} + L_{n-2}$  ( $n \geq 2$ ).

Our objective is to extend the results A), B) and C) to those of a generalized Fibonacci-coefficient polynomial (GFCP)

$$P_n(x) = \sum_{k=0}^n \mathcal{F}_{k+1} x^{n-k}, \quad (1)$$

where the coefficients are generalized Fibonacci numbers defined, for fixed positive integers  $a$  and  $b$ , by

$$\mathcal{F}_0 = 0, \mathcal{F}_1 = 1, \mathcal{F}_{n+1} = a\mathcal{F}_n + b\mathcal{F}_{n-1} \quad (n \geq 1). \quad (2)$$

There have already appeared some extensions of the results A), B) and C), namely, Matyas, [5], showed that the results A1) and A2) hold in this generalized setting. Matyas [6], further extended A1) and A2) to sequence of GFCEP's with shifted indices. Matyas et al. [7], refined the results A1) and A2) further by considering polynomials whose coefficients satisfy a linear recursive sequence with constant coefficients. Matyas and Szalay, [8], proved the validity of A1) and A2) for Tribonacci-coefficient polynomials. Mansour and Shattuck, [9], considered polynomials whose coefficients are  $k$ -Fibonacci numbers, i.e. numbers satisfying the recurrence  $a_n = a_{n-1} + a_{n-2} + \dots + a_{n-k}$ , and proved the results A1), A2) and B) in this setting as well as refining the results in [8]. Here, we complement all the above-mentioned works by establishing the results B) and C) for GFCEP's.

### PRELIMINARY LEMMAS

Our first lemma, whose easy proof is omitted, gives two recurrence relations for GFCEP's.

**Lemma 1.** For all natural numbers  $n$ , we have

$$P_n(x) = x^n + aP_{n-1}(x) + bP_{n-2}(x), \quad (3)$$

$$P_n(x) = xP_{n-1}(x) + \mathcal{F}_{n+1}. \quad (4)$$

From Lemma 1., the characteristic equation of  $P_n(x)$  is  $x^2 - ax - b = 0$  and its characteristic roots are  $\frac{a \pm \sqrt{a^2 + 4b}}{2}$ .

For convenience, we let  $\varphi_{a,b} = \frac{a + \sqrt{a^2 + 4b}}{2}$ ,  $\tau_{a,b} = \frac{a - \sqrt{a^2 + 4b}}{2}$  and notice that  $\varphi_{a,b} + \tau_{a,b} = a$ ,  $\varphi_{a,b} - \tau_{a,b} = \sqrt{a^2 + 4b}$  and  $\varphi_{a,b}\tau_{a,b} = -b$ . Recall that generalized Lucas numbers are defined by  $\mathcal{L}_0 = 2$ ,  $\mathcal{L}_1 = a$ ,  $\mathcal{L}_{n+2} = a\mathcal{L}_{n+1} + b\mathcal{L}_n$  ( $n \geq 0$ ). The next lemma, whose straightforward proof is omitted, gives some relations between generalized Fibonacci numbers and generalized Lucas numbers.

**Lemma 2.** For  $n \in \mathbb{Z}$ ,  $n \geq 0$ , we have

$$\text{i) } \mathcal{F}_n = \frac{\varphi_{a,b}^n - \tau_{a,b}^n}{\varphi_{a,b} - \tau_{a,b}}, \quad \mathcal{L}_n = \varphi_{a,b}^n + \tau_{a,b}^n, \quad \mathcal{L}_n = \mathcal{F}_{n+1} + b\mathcal{F}_{n-1};$$

$$\text{ii) } \mathcal{F}_{n+k} = \mathcal{F}_k \mathcal{L}_n + (-b)^k \mathcal{F}_{n-k} \quad (k \in \mathbb{N}, 1 \leq k \leq n).$$

For  $n \in \mathbb{N}$ , let  $G_n(x) = P_n(x)(x^2 - ax - b)$ . Two useful relations, whose easy proofs are omitted, are listed in the next lemma.

**Lemma 3.** For  $n \in \mathbb{Z}$ , we have

$$G_n(x) = x^{n+2} - \mathcal{F}_{n+2}x - b\mathcal{F}_{n+1}, \quad (5)$$

$$x^n + a\mathcal{F}_n = P_n(x) - (ax + b)P_{n-2}(x). \quad (6)$$

### REAL ZEROS

In this section, we investigate the real zeros of the GFCEP's.

**Theorem 1.** For even  $n \geq 2$ , each polynomial  $P_n(x)$  has no real zeros, For odd  $n \geq 3$ , each polynomial  $P_n(x)$  has exactly one real zero which is negative; if  $a^2 \geq b$ , then this zero lies in the interval  $(-\varphi_{a,b}, -a]$ . Moreover, the sequence of real zeros of the polynomials  $P_n(x)$  with odd  $n$  converges to  $-\varphi_{a,b}$ .

**Proof.** The first two assertions have already been proved in [5, Theorem 3.1]. For odd  $n \geq 3$ , assuming  $a^2 \geq b$ , we now show that the negative real zero of  $P_n(x)$  lies in  $(-\varphi_{a,b}, -a]$ . Notes first that

$$G_n(-\varphi_{a,b}) = -\varphi_{a,b}(\varphi_{a,b}^{n+1} - \mathcal{F}_{n+2}) - b\mathcal{F}_{n+1}.$$

By Lemma 2., for odd  $n \geq 3$  we have

$$\varphi_{a,b}^{n+1} - \mathcal{F}_{n+2} = \varphi_{a,b}^{n+2} \left\{ \frac{1}{\varphi_{a,b}} - \frac{1}{\sqrt{a^2 + 4b}} + \frac{1}{\sqrt{a^2 + 4b\varphi_{a,b}^{2n+4}}} \right\} > 0,$$

yielding  $G_n(-\varphi_{a,b}) < 0$ . Next, using  $\mathcal{F}_n > a^{n-1}$  ( $n \geq 3$ ) and  $a^2 \geq b$ , we get

$$G_n(-a) = -a^{n+2} + a\mathcal{F}_{n+2} - b\mathcal{F}_{n+1} = -a^{n+2} + (a^2 - b)\mathcal{F}_{n+1} + ab\mathcal{F}_n > -a^{n+2} + (a^2 - b)a^n + ab \cdot a^{n-1} \geq 0,$$

which shows that the only real zero of  $G_n(x)$  lies in  $(-\varphi_{a,b}, -a]$ .

Finally, we show that for odd  $n \geq 3$ , the sequence of negative real zeros of  $P_n(x)$  converges, as  $n \rightarrow \infty$ , to  $-\varphi_{a,b}$ . Let  $\alpha_n$  be the negative real zero of  $P_n(x)$ . From (3), it is easily checked that  $G_n(x)$  has two critical points at  $\pm(\mathcal{F}_{n+2}/(n+2))^{1/(n+1)}$ . Since  $\alpha_n, \varphi_{a,b}, \tau_{a,b}$  are all the real zeros of  $G_n(x)$ , with  $\alpha_n \in (-\varphi_{a,b}, -a]$ , it follows that  $G_n(x)$  has a local maximum at  $-(\mathcal{F}_{n+2}/(n+2))^{1/(n+1)}$ , with  $-\varphi_{a,b} < \alpha_n \leq -(\mathcal{F}_{n+2}/(n+2))^{1/(n+1)}$ .

Since  $\lim_{n \rightarrow \infty} \mathcal{F}_{n+2}^{1/(n+1)} = \varphi_{a,b}$ , the last assertion follows.

We proceed next to analyze the nature of all other roots of  $P_n(x)$ . The next lemma is easily checked by induction.

**Lemma 4.** For  $n \in \mathbb{Z}$ , we have

- (i)  $\varphi_{a,b}^n = \varphi_{a,b} \mathcal{F}_n + b\mathcal{F}_{n-1}$ ;
- (ii) If  $c > \varphi_{a,b}$ , then  $c^{n+2} > \mathcal{F}_{n+2}c + b\mathcal{F}_{n+1}$ ;
- (iii) If  $0 < c < \varphi_{a,b}$ , then  $c^{n+2} < \mathcal{F}_{n+2}c + b\mathcal{F}_{n+1}$ .

We now state and prove our second main result.

**Theorem 2.** The roots of  $P_n(x)$  approach  $\varphi_{a,b}$  in modulus as  $n \rightarrow \infty$ .

**Proof.** Consider the circle  $|z| = c$  in the complex plane, where  $c > \varphi_{a,b}$ . By Lemma 4. We have

$$|\mathcal{F}_{n+2}z + b\mathcal{F}_{n+1}| \leq \mathcal{F}_{n+2}|z| + b\mathcal{F}_{n+1} = \mathcal{F}_{n+2}c + b\mathcal{F}_{n+1} < c^{n+2} = |z^{n+2}|.$$

It then follows from Roche's Theorem [3, p.2] that  $P_n(z)(z^2 - az - b) = z^{n+2} - \mathcal{F}_{n+2}z - b\mathcal{F}_{n+1}$  has  $n+2$  roots inside  $|z| = c$ . Since  $c > \varphi_{a,b}$  is arbitrary, it follows that  $P_n(z)(z^2 - az - b)$  has  $n+2$  roots inside or on

$|z| = \varphi_{a,b}$ .

Now consider the circle  $|z| = c$  with  $b/\varphi_{a,b} < c < \varphi_{a,b}$ . We have  $|\mathcal{F}_{n+2}z - b\mathcal{F}_{n+1}|^2 = \mathcal{F}_{n+2}^2 c^2 + 2b\mathcal{F}_{n+2}\mathcal{F}_{n+1} \operatorname{Re}(z) + b^2\mathcal{F}_{n+1}^2 \geq \mathcal{F}_{n+2}^2 c^2 - 2b\mathcal{F}_{n+2}\mathcal{F}_{n+1}c + b^2\mathcal{F}_{n+1}^2 = (\mathcal{F}_{n+2}c - b\mathcal{F}_{n+1})^2$ ,

yielding

$$|\mathcal{F}_{n+2}z + b\mathcal{F}_{n+1}| \geq |\mathcal{F}_{n+2}c - b\mathcal{F}_{n+1}|. \quad (7)$$

Since

$$\mathcal{F}_{n+2}c - b\mathcal{F}_{n+1} = \varphi_{a,b}^{n+2} \left( \frac{c}{\sqrt{a^2 + 4b}} - \frac{b}{\varphi_{a,b} \sqrt{a^2 + 4b}} \right) + \frac{(-1)^{n+1}}{\sqrt{a^2 + 4b} \varphi_{a,b}^{n+1}} \left( \frac{c}{\varphi_{a,b}} + 1 \right),$$

it follows that for  $n$  large enough,

$$\mathcal{F}_{n+2}c - b\mathcal{F}_{n+1} > c^{n+2}. \quad (8)$$

Using (7) and (8), for  $|z| = c$ , with  $\frac{b}{\varphi_{a,b}} < c < \varphi_{a,b}$ , we have for large  $n$

$$|\mathcal{F}_{n+2}z + b\mathcal{F}_{n+1}| \geq |\mathcal{F}_{n+2}c - b\mathcal{F}_{n+1}| > c^{n+2}.$$

By Roche's theorem again, it follows that if  $\frac{b}{\varphi_{a,b}} < c < \varphi_{a,b}$ , then, for  $n$  large enough,  $P_n(z)(z^2 - az - b)$  has one

zero in  $|z| = c$ . Since  $c$  is arbitrary, as  $n \rightarrow \infty$ , the zeros of  $P_n(x)$  approach  $\varphi_{a,b}$  in modulus.  $\square$

## MAHLER MEASURES

Let  $G(x) \in \mathbb{R}[X]$ ,  $\deg G = n$ , with leading coefficient  $a_n$ , and complex roots  $\alpha_1, \dots, \alpha_n$ . Its Mahler measure is defined by

$$M(G(x)) = |a_n| \prod_{j=1}^n \max\{1, |\alpha_j|\}.$$

Let  $\{P_n^{(m)}(x)\}$  be the sequence obtained from the GFCP's by reducing the coefficients modulo  $m$  and adjusting the residue class in the same way as mentioned in the introduction. It is well known that generalized Fibonacci sequence modulo  $m$  is periodic, and so let  $t$  to be the number of terms in one period.

**Theorem 3.** Let  $m \geq 2$  and  $t$  be as above. If  $k \equiv -2$  or  $-1 \pmod{t}$ , then

$$M(P_k^{(m)}(x)) = M(P_{t-2}^{(m)}(x)).$$

**Proof.** Let  $a_i \equiv \bar{F}_{i-1} \pmod{m}$  ( $i \geq 1$ ) be the terms of the generalized Fibonacci sequence modulo  $m$ . Since  $a_1 = 0$  and  $a_2 = 1$ , we have

$$P_{t-2}^{(m)}(x) = x^{t-2} + a_3x^{t-3} + a_4x^{t-4} + \dots + a_{t-1}x + a_t.$$

Let  $C_j(x) = x^j + x^{j-1} + \dots + x + 1$  ( $j \geq 1$ ). Setting  $P_k^{(m)}(x) = P_{t-2}^{(m)}(x)C_l(x')$ , and equating the degree of

$P_k^{(m)}(x)$  and  $P_{t-2}^{(m)}(x)C_l(x')$ , we have  $k = (t-2) + lt$ . From  $l \equiv (k+2-t) \pmod{t}$ , we get

$$k \equiv -2 \pmod{t}.$$

Setting  $P_k^{(m)}(x) = xP_{t-2}^{(m)}(x)C_l(x')$ , and equating the degrees, we get  $k = (t-1) + lt$ , i.e.,

$$k \equiv -1 \pmod{t}.$$

Since all the roots of  $C_j(x')$  have modulus 1 and the Mahler measure is multiplicative, the result follows.  $\square$

From the identity in Lemma 2., we have

$$\mathcal{L}_n = \mathcal{F}_{n+1} + b\mathcal{F}_{n-1} = 2\mathcal{F}_{n+1} - a\mathcal{F}_n > 2\mathcal{F}_{n+1} > 2\mathcal{F}_i \quad (0 \leq i \leq n).$$

Thus, the first  $n+1$  elements of  $\mathcal{F}_i \bmod \mathcal{L}_n$  are  $\mathcal{F}_0 = 0, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n$ . From Lemma 2. iii), we have  $\mathcal{F}_{n+k} = \mathcal{F}_n \mathcal{L}_n + (-b)^k \mathcal{F}_{n-k}$  ( $1 \leq k \leq n$ ). Thus, the next  $n$  element of  $\mathcal{F}_i \bmod \mathcal{L}_n$ , i.e., for  $n+1 \leq i \leq 2n$ , are  $-b\mathcal{F}_{n-1}, b^2\mathcal{F}_{n-2}, -b^3\mathcal{F}_{n-3}, \dots, (-b)^{n-1}, 0$  (noting that  $\mathcal{F}_0 = 0, \mathcal{F}_1 = 1$ ). This shows that the period of the generalized Fibonacci sequence modulo  $\mathcal{L}_n$  is  $2n$ .

We are now ready to prove our last result, keeping in mind that  $a_1 = 0$  and  $a_2 = 1$ .

**Theorem 4.** If  $k \equiv -2$  or  $-1 \pmod{2n}$ , then  $M(P_k^{(\mathcal{L}_n)}(x)) = \bar{\varphi}_{a,b}^{n-1}$ .

**Proof.** By Theorem 3., If  $k \equiv -2$  or  $-1 \pmod{2n}$ , then  $M(P_k^{(\mathcal{L}_n)}(x)) = M(P_{2n-2}^{(\mathcal{L}_n)}(x))$ . Since

$$P_{2n-2}(x) = x^{2n-2} + \mathcal{F}_2 x^{2n-3} + \mathcal{F}_3 x^{2n-4} + \dots + \mathcal{F}_n x^{n-1} + \mathcal{F}_{n+1} x^{n-2} + \mathcal{F}_{n+2} x^{n-3} + \dots + \mathcal{F}_{2n-2} x + \mathcal{F}_{2n-1},$$

we get

$$\begin{aligned} P_{2n-2}^{(\mathcal{L}_n)}(x) &= x^{2n-2} + \mathcal{F}_2 x^{2n-3} + \mathcal{F}_3 x^{2n-4} + \dots + \mathcal{F}_n x^{n-1} + (-b)\mathcal{F}_{n-1} x^{n-2} + (-b)^2 \mathcal{F}_{n-2} x^{n-3} + \dots + (-b)^{n-2} \mathcal{F}_2 x + (-b)^{n-1} \mathcal{F}_1 \\ &= x^{n-1} P_{n-1}(x) - b x^{n-2} P_{n-2} \left( \frac{-b}{x} \right). \end{aligned} \quad (9)$$

To prove theorem, we have to show that

$$M(P_{2n-2}^{(\mathcal{L}_n)}(x)) = \varphi_{a,b}^{n-1}. \quad (10)$$

For odd  $n$ , using (9), we have

$$P_{2n-2}^{(\mathcal{L}_n)}(x)(x^2 - ax - b) = x^{2n} - \mathcal{L}_n x^n - b. \quad (11)$$

Using the  $k^{\text{th}}$  cyclotomic polynomial, [2, section 2.2 p.81],  $\Phi_k(x)$ , it is easily checked that

$$b^{n-1} \prod_{d|n, d \neq 1} \Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right) \Phi_d \left( \frac{x}{\varphi_{a,b}} \right) = \frac{x^{2n} - \mathcal{L}_n x^n - b^n}{x^2 - ax - b},$$

which in turn gives

$$P_{2n-2}^{(\mathcal{L}_n)}(x) = b^{n-1} \prod_{d|n, d \neq 1} \Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right) \Phi_d \left( \frac{x}{\varphi_{a,b}} \right)$$

Since all the roots of  $\Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right)$  have modulus  $b/\varphi_{a,b}$ , and all the roots of  $\Phi_d \left( \frac{x}{\varphi_{a,b}} \right)$  have modulus  $\varphi_{a,b}$ , it

follows that

$$M(P_{2n-2}^{(\mathcal{L}_n)}(x)) = M(b^{n-1}) M \left( \prod_{d|n, d \neq 1} \Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right) \right) M \left( \prod_{d|n, d \neq 1} \Phi_d \left( \frac{x}{\varphi_{a,b}} \right) \right) = \varphi_{a,b}^{n-1}.$$

For even  $n \geq 2$ , let

$$\begin{aligned} R_n(x) &= x^{2n-4} + \frac{\mathcal{F}_4}{a} x^{2n-6} + \frac{\mathcal{F}_6}{a} x^{2n-8} + \frac{\mathcal{F}_8}{a} x^{2n-10} + \dots + \frac{\mathcal{F}_{n-4}}{a} x^{n+2} + \frac{\mathcal{F}_{n-2}}{a} x^n + \frac{\mathcal{F}_n}{a} x^{n-2} \\ &\quad + \frac{(-b)^2}{a} \mathcal{F}_{n-2} x^{n-4} + \frac{(-b)^4}{a} \mathcal{F}_{n-4} x^{n-6} + \dots + \frac{(-b)^{n-4}}{a} \mathcal{F}_4 x^2 + \frac{(-b)^{n-2}}{a} \mathcal{F}_2. \end{aligned}$$

Using (9), we have

$$P_{2n-2}^{(\mathcal{L}_n)}(x) = (x^2 + ax - b)R_n(x). \quad (12)$$

It is easy to show that

$$R_n(x)(x^4 - (a^2 + 2b)x^2 + b^2) = x^{2n} - \mathcal{L}_n x^n + b^n. \quad (13)$$

Similar to the case when  $n$  is odd, we have

$$b^{n-2} \prod_{d|n, d \neq 1, 2} \Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right) \Phi_d \left( \frac{x}{\varphi_{a,b}} \right) = \frac{x^{2n} - \mathcal{L}_n x^n + b^n}{x^4 - (a^2 + 2b)x^2 + b^2},$$

and so

$$P_{2n-2}^{(\mathcal{L}_n)}(x) = (x^2 + ax - b) b^{n-2} \prod_{d|n, d \neq 1, 2} \Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right) \Phi_d \left( \frac{x}{\varphi_{a,b}} \right),$$

yielding

$$M \left( P_{2n-2}^{(\mathcal{L}_n)}(x) \right) = M(x^2 + ax - b) M(b^{n-2}) M \left( \prod_{d|n, d \neq 1, 2} \Phi_d \left( \frac{-\varphi_{a,b} x}{b} \right) \right) M \left( \prod_{d|n, d \neq 1, 2} \Phi_d \left( \frac{x}{\varphi_{a,b}} \right) \right) = \varphi_{a,b}^{n-1}.$$

□

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