CQ METHOD TO SOLVE FIXED POINT PROBLEMS OF NONSPREADING MAPPINGS, GENERALIZED EQUILIBRIUM PROBLEMS AND VARIATIONAL INEQUALITY PROBLEMS WITH APPLICATIONS

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Abstract

In this research, by using hybrid method the a strong convergence theorem was proved to find a common element of the set of solutions of equilibrium problems, generalized equilibrium problems and fixed points problems by using the K-mapping generated by a finite family of nonspreading mappings and a finite real numbers. Moreover, we apply a strong convergence theorem involving minimization problems and fixed points of demicontractive mappings by using our main result.

Keywords : Generalized Equilibrium Problem, Variational Inequality Problem, Fixed Point Problem, Nonspreading Mapping

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Chapter 1

Introduction

1.1 Research Motivation

Fixed point theory is an important area of mathematical analysis. There are many inventions about the method for solving the solution in many problems in hilbert space such as nonlinear operator equations, equilibrium problems, variational inequality problems, optimization problems and minimal problems. The solution of some problem has many benefit in a variety of area such as physics, engineering, and economics.

Let $T : C \to C$ be mapping. A point $x \in C$ is called a fixed point of T if $Tx = x$. The set of fixed points of T is denoted $F(T) = \{x \in C : Tx = x\}$.

Let C be a nonempty closed convex subset of a real Hilbert spaces H. A mapping $T : C \to C$ is called

(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|,\ \forall x, y \in C,$$

(ii) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle,\ \forall x, y \in C,$$

(iii) quasi-nonexpansive if

$$\|Tx - z\| \leq \|x - z\|,\ \forall x \in C \text{ and } z \in F(T),$$

(iv) nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2,\ \forall x, y \in C,$$

A mapping $A : C \to H$ is called α-inverse strongly monotone, see [13] if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C.$$

Let $G : C \times C \to \mathbb{R}$ be a bifunction. The equilibriums problem for G is to determine a point $x^* \in C$ such that

$$G(x^*, y) \geq 0,\ \forall y \in C. \tag{1.1}$$

The set of all solution of (1.1) is denoted by

$$EP(G) = \{x^* \in C : G(x^*, y) \geq 0\}. \tag{1.2}$$

Numerous problems in physics, optimization, and economics reduce to find a solution of $EP(G)$, see, for instance [2]-[4]. In 2007, Takahashi and Takahashi [4] proved the following theorem:

Theorem 1.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(G) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, 1)$ satisfying
(C1) $\alpha_n \to 0$ as $n \to \infty$,
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(C3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$,
$\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(G)$, where $z = P_{F(S)\cap EP(G)} f(z)$.

Let $B : C \to H$ be a nonlinear mapping. The variational inequality problems is to find a point $u \in C$ such that

$$\langle v - u, Bu\rangle \geq 0, \text{ for all } v \in C. \tag{1.3}$$

The set of solutions of the variational inequality is denoted $VI(C, B)$. Numerous problems in physics, optimization, minimax problems are reduced to variational inequality problems.

The generalized equilibrium problem is to find $z \in C$ such that

$$G(x, y) + \langle Bz, y - z\rangle \geq 0, \ \forall y \in C. \tag{1.4}$$

The set of all solutions of generalized equilibrium problem is denoted by

$$EP(G, B) = \{z \in C : G(z, y) + \langle Bz, y - z\rangle \geq 0, \ \forall y \in C\}. \tag{1.5}$$

In the case of $B = 0, EP(G, B) = EP(G)$.

In 2010, Qin et al. [16] introduced an iterative algorithm as follows:

Algorithm 1.2. Let C be a nonempty closed and convex subset of a Hilbert space H. For every $i = 1, 2, ..., N$. Let T_i be nonexpansive mappings of C into itself. Let $\{x_n\}$ be a sequence generated in the following:

$$\begin{cases} x_1 \in C, \text{ arbitrarily}, \\ F_1(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r}\langle u - u_n, u_n - x_n\rangle \geq 0, \forall u \in C, \\ F_2(v_n, v) + \langle Bx_n, v - v_n\rangle + \frac{1}{s}\langle v - v_n, v_n - x_n\rangle \geq 0, \forall v \in C, \\ y_n = \delta_n u_n + (1 - \delta_n)v_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n x_n, \end{cases} \tag{1.6}$$

where $f : C \to C$ is a contraction mapping and $F_j : C \times C \to \mathbb{R}$ be a bifunction for all $j = 1, 2$ satisfying $(A1) - (A4)$, $A : C \to H$ is α-inverse strongly monotone mapping and $B : C \to H$ is β-inverse strongly monotone mapping. Let W_n is W-mapping generated by infinite family of nonexpansive mappings and infinite real number.

They proved under some control conditions on $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ that the sequence $\{x_n\}$ generated by (1.6) converge strongly to $z = P_{\mathbb{F}} f(z)$, where $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i)) \cap EP(F_1, A) \cap EP(F_2, B)$.

Let $CB(H)$ be a family of all nonempty closed bounded subsets of H and $\mathcal{H}(.,.)$ be the Hausdorff metric on $CB(H)$ defined as

$$\mathcal{H}(U, V) = \max\{\sup_{u \in U} d(u, V), \sup_{v \in V} d(U, v)\}, \forall U, V \in CB(H),$$

where $d(u, V) = \inf_{v \in V} d(u, v)$, $d(U, v) = \inf_{u \in U} d(u, v)$ and $d(u, v) = \|u - v\|$.

Let $\varphi : C \to H$ be a real-valued function, $T : C \to CB(H)$ be a multivalued mapping and $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function, that is, $\Phi(w, u, v) + \Phi(w, v, u) = 0$ for all $(w, u, v) \in H \times C \times C$ which satisfies the following conditions with respect to the multivalued $T : C \to CB(H)$;

(H1) For each fixed $v \in C$, $(w, u) \mapsto \Phi(w, u, v)$ is an upper semicontinuous function from $H \times C$ to $\mathbb{R}$, that is, for $(w, u) \in H \times C$, wherever $w_n \to w$ and $u_n \to u$ as $n \to \infty$,

$$\lim_{n \to \infty} sup \Phi(w_n, u_n, v) \to \Phi(w, u, v),$$

(H2) For each fixed $(w, v) \in H \times C \times C$, $u \mapsto \Phi(w, u, v)$ is a concave function,

(H3) For each fixed $(w, u) \in H \times C \times C$, $v \mapsto \Phi(w, u, v)$ is a convex function.

In 2009, Ceng et al.[5] introduced the following generalized equilibrium problem (GEP) as follows:

$$(GEP) \begin{cases} Find\, u \in C\, and\, w \in T(u)\, such\, that \\ \Phi(w, u, v) + \varphi(v) - \varphi(u) \geq 0, \forall v \in C. \end{cases} \tag{1.7}$$

The set of solutions of (GEP) is denoted by $(GEP)_s(\Phi, \varphi)$. In the case of $\varphi \equiv 0$ and $\Phi(w, u, v) \equiv G(u, v)$, then $(GEP)_s(\Phi, \varphi)$ is denoted by $EP(G)$.

In 2012, Kangtunyakarn [11] introduced an iterative algorithm as follows:

Algorithm 1.3. For every $i = 1, 2, ..., N$. Let T_i be κ_i–pseudo-contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1, 2, ..., N\}$ and let S_n be the S-mappings generated by $\{T_i\}_{i=1}^{N}$ and $\lambda_1^{(n)}, ..., \lambda_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{(n),j}, \alpha_2^{(n),j}, \alpha_3^{(n)}, j) \in I \times I \times I, I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1, 2, ..., N-1$, $\kappa \leq \alpha_1^{n,N} \leq 1$, $\kappa \leq \alpha_3^{n,N} \leq d < 1$, $\kappa \leq \alpha_2^{n,N} \leq e < 1$ for all $j = 1, 2, ..., N$. Let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exist sequences $\{w_n^1\}, \{w_n^2\} \subseteq H$ and $\{x_n\}, \{u_n\}, \{v_n\} \subseteq C$

such that

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C, s_n > 0, \\ z_n = \delta_n P_C(I - \lambda A)u_n + (1 - \delta_n)P_C(I - \eta B)v_n, \\ y_n = \alpha_n z_n + (1 - \alpha_n)S_n z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \forall n \geq 1, \end{cases} \tag{1.8}$$

where $D, T : C \to CB(H)$ are $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ are equilibrium-like function satisfying $(H1)-(H3)$, $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be a lower semicontinuous and convex function, $A : C \to H$ is α-inverse strongly monotone mapping and $B : C \to H$ is β-inverse strongly monotone mapping.

He proved under some control conditions on $\{\delta_n\}, \{\alpha_n\}, \{r_n\}$ and $\{s_n\}$ that the sequence $\{x_n\}$ generated by (1.8) converge strongly to $P_{\mathbb{F}}x_1$, where $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i)) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \cap F(G_1) \cap F(G_2)$, $G_1, G_2 : C \times C$ are defined by $G_1(x) = P_C(x - \lambda Ax)$, $G_2(x) = P_C(x - \eta Bx)$, $\forall x \in C$ and $P_{\mathbb{F}}x_1$ is a solution of the following system of variational inequalities:

$$\begin{cases} \langle Ax^*, x - x^*\rangle \geq 0, \\ \langle Bx^*, x - x^*\rangle \geq 0. \end{cases}$$

Motived by Algorithm 1.2 and Algorithm 1.3, we define the following algorithm as follows:

Algorithm 1.4. For every $i = 1, 2, ..., N$. Let T_i be nonspreading mappings of C into itself and let K be the $K-$mappings generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$. Let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exist sequences $\{x_n\}, \{u_n\}, \{v_n\}, \{\overline{u}_n\}, \{\overline{v}_n\} \subseteq C$

and $\{w_n^1.\}, \{w_n^2\} \subseteq H$ such that

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \Big(1 + \frac{1}{n}\Big)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \Big(1 + \frac{1}{n}\Big)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C, s_n > 0, \\ F_1(\overline{u}_n, \overline{u}) + \langle Ax_n, \overline{u} - \overline{u}_n\rangle + \frac{1}{r}\langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n\rangle \geq 0, \forall \overline{u} \in C, \\ F_2(\overline{v}_n, \overline{v}) + \langle Bx_n, \overline{v} - \overline{v}_n\rangle + \frac{1}{s}\langle \overline{v} - \overline{v}_n, \overline{v}_n - x_n\rangle \geq 0, \forall \overline{v} \in C, \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n \overline{u}_n + \delta_n \overline{v}_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases} \tag{1.9}$$

where $D, T : C \to CB(H)$ are $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ are equilibrium-like functions satisfying $(H1)-(H3)$, $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be a lower semicontinuous and convex functions, $F_j : C \times C \to \mathbb{R}$ be a bifunction for all $j = 1, 2$, $A : C \to H$ is α-inverse strongly monotone mapping and $B : C \to H$ is β-inverse strongly monotone mapping.

In this thesis, we prove a sequence $\{x_n\}$ generated by (1.9) converges strongly to an element of the set of solutions of equilibrium problems, generalized equilibrium problems and fixed points problems by using the K-mapping generated by a finite family of nonspreading mappings and finite real numbers introduced by Kangtunyakarn and Suantai [7]. Moreover, we apply a strong convergence theorem involving minimization problems and fixed points problems of nonlinear mappings by using our main result.

1.2 Objectives of the study

1) To prove a strong convergence theorem for finding a common element of the set of solutions of equilibrium problems, generalized equilibrium problems and fixed points problems by using the K-mapping generates by a finite family of nonspreading mappings and finite real numbers introduced by Kangtunyakarn and Suantai [7].

2) For apply our main result to obtain a strongly convergence theorem for finding a solutions of minimization problems, generalized equilibrium problems and fixed points problems of nonlinear mappings.

1.3 Scopes of the study

1) Study the fixed point problems of by using the K-mapping generates by a finite family of nonspreading mappings and finite real numbers introduced by Kangtunyakarn and Suantai [7] and a convergence theorem involving minimization problems.
2) Study generalized equilibrium problem in real Hilbert space.
3) Study variational inequality problem in real Hilbert space.
4) All strong convergence theorems are considered and proved in a real Hilbert space.

1.4 Benefits of the study

1) To obtain new tools for fixed point problem on real Hilbert space.
2) To obtain a strong convergence theorem for finding a common element of the set of solutions of equilibrium problems, generalized equilibrium problems and fixed points problems by using the K-mapping generates by a finite family of nonspreading mappings and finite real numbers introduced by Kangtunyakarn and Suantai [7].
3) To obtain a strongly convergence theorem for finding a solutions of minimization problems, generalized equilibrium problems and fixed points problems of nonlinear mappings.

1.5 Research methodology

1) Study advanced topics in fixed point theory for a nonspreading mapping.
2) Study background in a real Hilbert space.
3) Collect and study research papers and textbooks concerning fixed point theorem.
4) Determine the objectives and scope of the research.
5) Produce tools for a strong convergence theorem of fixed point problem.
6) Prove a strong convergence theorem for fixed point problem in a real Hilbert space.
7) Provide applications.
8) Conclude the results, make suggestions for further works and write the thesis.

Chapter 2

Preliminaries and Literature Reviews

The purpose of this chapter is to collect lemma, definition, theorem and terminology for used throughout of the thesis.

2.1 Fundamental properties in Hilbert spaces

Definition 2.1. (Cauchy sequence, See [14].) A sequence of vectors $\{x_n\}$ in a normed space is called a Cauchy sequence if for every $\epsilon > 0$ there exists a number M such that $\|x_m - x_n\| < \epsilon$ for all $m, n > M$.

Definition 2.2. (A complete normed space, See [14].) A normed space E is called complete if every Cauchy sequence in E converges to an element of E. A complete normed space is called a Banach space.

Definition 2.3. (Hilbert space, See [14].) Let X be an inner product space and X is called Hilbert space if X is complete inner product space.

Definition 2.4. (Strong convergence, See [14].) A sequence $\{x_n\}$ of vectors in an inner product space K is called strongly convergent to a vector x in K if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.5. (Weak convergence, See [14].) A sequence $\{x_n\}$ of vectors in an inner product space K is called weakly convergent to a vector x in K if

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ as } n \to \infty \text{ for every } y \in K.$$

Theorem 2.1. (Srong convergence, See [14].) A strongly convergence sequence is weakly convergence (to the same limit), that is, $x_n \to x$ implies $x_n \rightharpoonup x$.

Lemma 2.2. (See [17].) In a real Hilbert spaces H, the following results hold:
(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$ for all $x, y \in H$.
(ii) for all $x, y \in H$ and $\alpha \in [0, 1]$,

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2.$$

Definition 2.6. (See [15].) Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called lower semicontinuous if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \le a\} \text{ is closed.}$$

Moreover, f is called convex if for any $x_1, x_2 \in C$ and $t \in (0,1)$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Similarly, f is said to be concave if for any $x_1, x_2 \in C$ and $t \in (0,1)$,

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2).$$

Theorem 2.3. (See [15].) Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let f be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightharpoonup x_0$. Then

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

Theorem 2.4. (See [15].) Let $\{a_n\}$ be a bounded of real numbers. Then, there exists subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Remark 2.5. (See [15].) Let H be an inner product space. Then we know that the following (i) and (ii) are equivalent:
(i) H is complete,
(ii) each bounded sequence $\{x_n\}$ of H has a weakly convergence subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Definition 2.7. (Metric projection, See [15].) The (nearest point) projection P_C from H onto C assigns to each $x \in H$, the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

Lemma 2.6. (See [6].) Given $x \in H$ and $y \in C$. Then $P_C x = z$ if and only if there holds the inequality

$$\langle x - z, z - y\rangle \geq 0, \ \forall y \in C.$$

It is well-known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \forall x, y \in H.$$

Lemma 2.7. (See [6].) Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Let $\omega(x_n)$ be the set of all weakly ω-limit of $\{x_n\}$, i.e., $\omega(x_n) = \{c \,|x_{n_k} \rightharpoonup c \,\text{as}\, k \to \infty\}$ where $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Lemma 2.8. (See [9].) Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$, if $\{x_n\}$ is such that $\omega(x_n) \subset C$ and satisfies the condition

$$\|x_n - u\| \leq \|u - q\|, \forall n \in \mathbb{N}.$$

Then $x_n \to q$ as $n \to \infty$.

Definition 2.8. A multivalued mapping $T : C \to CB(H)$ is said to be $\mathcal{H}$-Lipschitz continuous if there exists a constant $\mu > 0$ such that

$$\mathcal{H}(T(u), T(v)) \leq \mu\|u - v\|, \forall u, v \in C,$$

where $\mathcal{H}(.,.)$ is the Hausdorff metric on $CB(H)$.

Lemma 2.9. (Nadler's theorem, See [10].) Let $(X, \|\cdot\|)$ be a normed vector space and $\mathcal{H}(.,.)$ be the Hausdorff metric on $CB(H)$. If $U, V \in CB(X)$, then for any given $\varepsilon > 0$ and $u \in U$, there exists $v \in V$ such that

$$\|u - v\| \leq (1 + \varepsilon)\mathcal{H}(U, V).$$

2.2 Fixed Point Theorems

Let X be a nonempty set and $T : X \to X$ a self-mapping. We say that $x \in X$ is a fixed point of T if and only if $Tx = x$ and $F(T)$ represents the set of all fixed points of T.

Example 2.10. (See [18].)

1) If $X = \mathbb{R}$ and $T(x) = x^2 - 7x + 15$, then $F(T) = \{3, 5\}$;

2) If $X = \mathbb{R}$ and $T(x) = x^2 - 3x$, then $F(T) = \{0, 4\}$;

3) If $X = \mathbb{R}$ and $T(x) = x + 6$, then $F(T) = \emptyset$;

4) If $X = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$;

Theorem 2.11. (See [15].) Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point in C.

Lemma 2.12. (See [1].)Let H be a Hilbert space and C a nonempty closed convex subset of H. Let T be a nonspreading mapping of C into itself. Then $F(T)$ is closed and convex.

Lemma 2.13. (See [8].)Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonspreading mapping with $F(T) \neq \emptyset$.Then T is demiclosed, that is, $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$ imply $u \in F(T)$.

In 2009, Kangtunyakarn and Suantai [7] introduced K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ as follows:

Definition 2.9. Let C be a nonempty convex subset of a real Banach space. For every $i = 1, 2, ..., N$. Let T_i be a finite family of mappings of C into itself, and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, ..., N$. Define a mapping $K : C \to C$ as follows:

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1 - \lambda_1) I \\ U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2) U_2 \\ U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3) U_2 \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1}) U_{N-2} \\ K = U_N &= \lambda_N T_N U_{N-1} + (1 - \lambda_N) U_{N-1}. \end{aligned}$$

Such a mapping K is called the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$.

Lemma 2.14. (See [7].)Let C be a nonempty closed convex subset of real Hilbert space. For every $i = 1, 2, ..., N$. Let T_i be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K-mapping generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$. Then $F(K) = \bigcap_{i=1}^{N} F(T_i)$ and K is a quasi-nonexpansive mapping.

2.3 Equilibrium problems and generalized equilibrium problem in Hilbert spaces

The equilibrium problem provides us a natural to study problems arising in economics, finance, minimization problems, Nash equilibria in noncooperative games and certain fixed point problems.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The classical equilibrium problem for F is to find $u \in C$ satisfying the following inequality

$$F(u, v) \geq 0, \forall v \in C. \tag{2.1}$$

We use $EP(F)$ to represent the set of solution of (2.1).

Theorem 2.15. (See [5].) Let C be a nonempty, bounded, closed and convex subset of a real Hilbert space H, and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with a constant μ, and $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying (H1)-(H3). Let $r > 0$ be a constant. For each $x \in C$, take $w_x \in T(x)$ arbitrarity and define a mapping $T_r : C \to C$ as follows:

$$T_r(x) = \{u \in C : \Phi(\omega_x, u, v) + \varphi(v) - \varphi(u) + \frac{1}{r}\langle u - x, v - u\rangle \geq 0, \forall v \in C\}.$$

Then, there hold the following:

(a) T_r is a single-valued;

(b) T_r is a firmly nonexpansive (that is, for any $u, v \in C$, $\|T_r u - T_r v\|^2 \leq \langle T_r u - T_r v, u - v\rangle$) if

$$\Phi(w_1, T_r(x_1), T_r(x_2)) + \Phi(w_2, T_r(x_2), T_r(x_1)) \leq 0,$$

for all $(x_1, x_2) \in C \times C$ and all $w_i \in T(x_i)$, $i = 1, 2$;

(c) $F(T_r) = (GEP)_s(\Phi, \varphi)$;

(d) $(GEP)_s(\Phi, \varphi)$ is closed and convex.

Let the bifunction F satisfy the following conditions for solving equilibrium problem.

(A1) $F(u, u) = 0$ for all $u \in C$;

(A2) F is monotone, i.e., $F(u, v) + F(v, u) \leq 0$ for all $u, v \in C$;

(A3) For each $u, v, w \in C$,

$$\lim_{t \to 0^+} F(tw + (1 - t)u, v) \leq F(u, v);$$

(A4) For each $u \in C, v \mapsto F(u, v)$ is convex and lower semicontinuous.

Example 2.16. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x, y) = 4x^2 + xy - 5y^2, \forall x, y \in \mathbb{R}.$$

Then a bifunction F satisfies the condition (A1)-(A4) and $0 \in EP(F)$.

Solution. Let $x, y, z \in \mathbb{R}$. Since

$$F(x, x) = 4x^2 + x^2 - 5x^2 = 0,$$

thus we obtain (A1) holds. Next, observe that

$$F(x, y) + F(y, x) = \left(4x^2 + xy - 5y^2\right) + \left(4y^2 + xy - 5x^2\right) = -x^2 + 4xy - y^2 = -(x - y)^2 \leq 0.$$

This implies that F satisfies (A2). Let $t \in [0, 1]$. Consider

$$\lim_{t \to 0^+} F(tz + (1 - t)x, y) = \lim_{t \to 0^+} [4(tz + (1 - t)x)^2 + (tz + (1 - t)x)y - 5y^2] = 4x^2 + xy - 5y^2 = F(x, y).$$

Therefore, (A3) is true. To show (A4), first let $\alpha \in (0,1)$. Then we derive that

$$\begin{aligned}
&F(x, \alpha z + (1-\alpha)y) \\
&= 4x^2 + x(\alpha z + (1-\alpha)y) - 5(\alpha z + (1-\alpha)y)^2 \\
&= 4x^2 + \alpha xz + (1-\alpha)xy - 5(\alpha^2 z^2 + 2\alpha(1-\alpha)zy + (1-\alpha)^2 y^2) \\
&\leq 4x^2 + \alpha xz + (1-\alpha)xy - 5(\alpha^2 z^2 + \alpha(1-\alpha)(z^2 + y^2) + (1-\alpha)^2 y^2) \\
&= \alpha\left(4x^2 + xz - 5(\alpha z^2 + (1-\alpha)z^2)\right) + (1-\alpha)\left(4x^2 + xy - 5(\alpha y^2 + (1-\alpha)y^2)\right) \\
&= \alpha\left(4x^2 + xz - 5z^2\right) + (1-\alpha)\left(4x^2 + xy - 5y^2\right) \\
&= \alpha F(x,z) + (1-\alpha)F(x,y).
\end{aligned}$$

Hence F is a convex function. Let $\{y_n\} \subset \mathbb{R}$ with $y_n \to y$ as $n \to \infty$. Thus we get

$$\lim_{n\to\infty} F(x, y_n) = \lim_{n\to\infty} 4x^2 + xy_n - 5y_n^2 = 4x^2 + xy - 5y^2. \tag{2.2}$$

This yields that F is lower semicontinuous and (A4) holds.

In 1994, Blum and Oettli [3] proved the following existence result:

Lemma 2.17. (See [3].) Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

Lemma 2.18. (See [2].) Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $\overline{T}_r : H \times C$ as follows:

$$\overline{T}_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\},$$

for all $z \in H$. Then the following hold:

(a) $\overline{T}_r$ is a single-valued;

(b) $\overline{T}_r$ is a firmly nonexpansive, that is,

$$\|\overline{T}_r u - \overline{T}_r v\|^2 \leq \langle \overline{T}_r u - \overline{T}_r v, u - v\rangle, \forall u, v \in H;$$

(c) $F(\overline{T}_r) = EP(F)$;

(d) $EP(F)$ is closed and convex.

Chapter 3

A Theorem for Solving Nonspreading Mapping, Generalized Equilibrium Problem and Variational Inequality Problem

The purpose of this chapter is a theorem for solving the set of fixed point of a nonspreading mapping, generalized equilibrium problem and variational inequality problem and corollaries.

Theorem 3.1. Let C be a nonempty bounded, closed, and convex subset of a real Hilbert space H and let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be lower semicontinuous and convex functions. Let $D, T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $(H1)-(H3)$. Let $A : C \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be an β-inverse strongly monotone mapping, let $F_j : C \times C \to \mathbb{R}$ satisfy $(A1)-(A4)$ for all $j = 1, 2$, let T_i be nonspreading mappings of C into itself for all $i = 1, 2, ..., N$ with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap EP(F_1, A) \cap EP(F_2, B) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mappings generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exist sequences $\{x_n\}, \{u_n\}, \{v_n\}, \{\overline{u}_n\}, \{\overline{v}_n\} \subseteq C$ and $\{w_n^1\}, \{w_n^2\} \subseteq H$ be sequence generated by (1.9) where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\eta_n\}$ are sequences in $[0, 1]$ for all $n \in \mathbb{N}$, $r_n, r \subset (0, 2\alpha)$ and $s_n, s \subset (0, 2\beta)$ and suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there are $\overline{\lambda}_1, \overline{\lambda}_2 > 0$ such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\overline{\lambda}_1 \|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \leq -\overline{\lambda}_2 \|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2, \end{cases} \tag{3.1}$$

for all $(r_1, r_2) \in \Theta \times \Theta$,$(s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$ and $\Xi = \{s_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}} x_1$.

Proof. From (3.1) for every $r \in \Theta$, we have

$$\Phi_1(w_1^1, T_r(x_1), T_r(x_2)) + \Phi_1(w_2^1, T_r(x_2), T_r(x_1)) \leq -\overline{\lambda}_1 \|T_r(x_1) - T_r(x_2)\|^2 \leq 0, \tag{3.2}$$

for all $(x_1, x_2) \in C \times C$ and $w_i^1 \in T(x_i), i = 1, 2$.
Similarly, for every $s \in \Xi$, we have

$$\Phi_2(w_1^2, T_s(x_1), T_s(x_2)) + \Phi_2(w_2^2, T_s(x_2), T_s(x_1)) \leq -\overline{\lambda}_2 \|T_s(x_1) - T_s(x_2)\|^2 \leq 0, \tag{3.3}$$

for all $(x_1, x_2) \in C \times C$ and $w_i^2 \in D(x_i), i = 1, 2.$ From (3.2) and (3.3), we have Theorem 2.15 holds.

It easy to see that $I - rA$ and $I - sB$ are nonexpansive mapping. Indeed, since A is an α-inverse strongly monotone mapping with $r \in (0, 2\alpha)$, we have

$$\begin{aligned}
\|(I - rA)x - (I - rA)y\|^2 &= \|x - y - r(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2r\langle x - y, Ax - Ay\rangle + r^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\alpha r\|Ax - Ay\|^2 + r^2\|Ax - Ay\|^2 \\
&= \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2. \qquad (3.4)
\end{aligned}$$

Thus $I - rA$ is a nonexpansive mapping. From (3.4), it obvious that $I - sB$ is a nonexpansive mapping.

From (1.9) and Theorem 2.15, we have $u_n = T_{r_n}x_n$ and $v_n = T_{s_n}x_n$.

From (1.9) and Lemma 2.18, we have $\bar{u}_n = \overline{T}_r(I - rA)x_n$ and $\bar{v}_n = \overline{T}_s(I - sB)x_n$.

Let $z \in \mathbb{F} = \bigcap_{i=1}^N F(T_i) \cap EP(F_1, A) \cap EP(F_2, B) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2)$.

From Theorem 2.15 and Lemma 2.18, we have $z = T_{r_n}z = T_{s_n}z = \overline{T}_r(I - rA)z = \overline{T}_s(I - sB)z$.

From nonexpansiveness of $\{T_{r_n}\}, \{T_{s_n}\}$, we have

$$\begin{aligned}
\|u_n - z\| &= \|T_{r_n}x_n - z\| \\
&= \|T_{r_n}x_n - T_{r_n}z\| \\
&\leq \|x_n - z\|,
\end{aligned}$$

and

$$\begin{aligned}
\|v_n - z\| &= \|T_{s_n}x_n - z\| \\
&= \|T_{s_n}x_n - T_{s_n}z\| \\
&\leq \|x_n - z\|.
\end{aligned}$$

From nonexpansiveness of $\{T_r\}, \{T_s\}, \{I - rA\}, \{I - sB\}$, we have

$$\begin{aligned}
\|\bar{u}_n - z\| &= \|\overline{T}_r(I - rA)x_n - z\| \\
&= \|\overline{T}_r(I - rA)x_n - \overline{T}_r(I - rA)z\| \\
&\leq \|x_n - z\|,
\end{aligned}$$

and

$$\begin{aligned}
\|\bar{v}_n - z\| &= \|\overline{T}_s(I - sB)x_n - z\| \\
&= \|\overline{T}_s(I - sB)x_n - \overline{T}_s(I - sB)z\| \\
&\leq \|x_n - z\|.
\end{aligned}$$

From the definition of y_n, we have

$$
\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n u_n + \beta_n v_n + \gamma_n \overline{u}_n + \delta_n \overline{v}_n + \eta_n K x_n - z\|^2 \\
&= \|\alpha_n u_n + \beta_n v_n + \gamma_n \overline{u}_n + \delta_n \overline{v}_n + \eta_n K x_n - (\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n) z\|^2 \\
&= \|\alpha_n (u_n - z) + \beta_n (v_n - z) + \gamma_n (\overline{u}_n - z) + \delta_n (\overline{v}_n - z) + \eta_n (K x_n - z)\|^2 \\
&\leq \alpha_n \|u_n - z\|^2 + \beta_n \|v_n - z\|^2 + \gamma_n \|\overline{u}_n - z\|^2 + \delta_n \|\overline{v}_n - z\|^2 \\
&\quad + \eta_n \|K x_n - z\|^2 \\
&\leq (\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n) \|x_n - z\|^2 \\
&= \|x_n - z\|^2. \qquad (3.5)
\end{aligned}
$$

Next, we show that C_n is closed and convex for every $n \in \mathbb{N}$. It is obvious that C_n is closed. In fact, we know that, for $z \in C_n$.

$$\|y_n - z\| \leq \|x_n - z\| \text{ is equivalent to } \|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \leq 0. \qquad (3.6)$$

Let $z_1, z_2 \in C_n$ and $t \in (0, 1)$, it follows that

$$
\begin{aligned}
&\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - (t z_1 + (1 - t) z_2)\rangle \\
&\quad = t(2\langle y_n - x_n, x_n - z_1\rangle + \|y_n - x_n\|^2) \\
&\quad\quad + (1 - t)(2\langle y_n - x_n, x_n - z_2\rangle + \|y_n - x_n\|^2) \\
&\quad \leq 0.
\end{aligned}
$$

From (3.6), we have $t z_1 + (1 - t) z_2 \in C_n$. Then, we have C_n is convex. By Theorem 2.15, Lemma 2.12 and Lemma 2.18, we conclude that $\mathbb{F}$ is closed and convex. Then $P_{\mathbb{F}}$ is well defined.

Next, we show that $\mathbb{F} \subset C_n$ for every $n \in \mathbb{N}$.

Putting $q \in \mathbb{F}$, by (3.5), then we have

$$\|y_n - q\| \leq \|x_n - q\|,$$

implies that $q \in C_n$ for all $n \in \mathbb{N}$. It implies that $\mathbb{F} \subset C_n$ for all $n \in \mathbb{N}$. Since $x_n = P_{C_n} x_1$, for every $w \in C_n$, we have

$$\|x_n - x_1\| \leq \|w - x_1\|, \forall n \in \mathbb{N}.$$

Since $P_{\mathbb{F}} x_1 \in \mathbb{F} \subset C_n$ and $x_n = P_{C_n} x_1$, we have

$$\|x_n - x_1\| \leq \|P_{\mathbb{F}} x_1 - x_1\|. \qquad (3.7)$$

We will show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$

Since C is bounded, we have $\{x_n\}$ is bounded, so are $\{u_n\}$,$\{v_n\}$,$\{\bar{u}_n\}$,$\{\bar{v}_n\}$,$\{y_n\}$.

Since $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$ and $x_n = P_{C_n} x_1$, we have

$$\begin{aligned} 0 &\le \langle x_1 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1} \rangle \\ &= -\langle x_n - x_1, x_n - x_1 \rangle + \langle x_1 - x_n, x_1 - x_{n+1} \rangle \\ &\le -\|x_n - x_1\|^2 + \|x_n - x_1\| \|x_1 - x_{n+1}\|. \end{aligned}$$

It implies that

$$\|x_n - x_1\| \le \|x_1 - x_{n+1}\|.$$

It follows that $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

Since

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1} \rangle \\ &\quad + \|x_1 - x_{n+1}\|^2 \\ &\le \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2, \end{aligned}$$

and $\lim_{n\to\infty} \|x_n - x_1\|$ exists, we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.8}$$

Since $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|.$$

From (3.8), we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.9}$$

Since

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

by (3.8) and (3.9), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.10}$$

Next, we show that

$$\lim_{n\to\infty} \|Kx_n - x_n\| = 0.$$

By definition y_n, we have

$$\begin{aligned} y_n - x_n &= \alpha_n u_n + \beta_n v_n + \gamma_n \overline{u}_n + \delta_n \overline{v}_n + \eta_n K x_n - x_n \\ &= \alpha_n (u_n - x_n) + \beta_n (v_n - x_n) + \gamma_n (\overline{u}_n - x_n) \\ &\quad + \delta_n (\overline{v}_n - x_n) + \eta_n (K x_n - x_n). \end{aligned}$$

It implies that

$$\begin{aligned}\eta_n(Kx_n - x_n) = (y_n - x_n) + \alpha_n(x_n - u_n) + \beta_n(x_n - v_n) \\ + \gamma_n(x_n - \overline{u}_n) + \delta_n(x_n - \overline{v}_n).\end{aligned} \tag{3.11}$$

Since T_{r_n} is a firmly nonexpansive mapping and $T_{r_n}x_n = u_n$, we have

$$\begin{aligned}\|u_n - z\|^2 &= \|T_{r_n}x_n - T_{r_n}z\|^2 \\ &\leq \langle T_{r_n}x_n - T_{r_n}z, x_n - z\rangle \\ &= \frac{1}{2}(\|u_n - z\|^2 + \|x_n - z\|^2 - \|u_n - x_n\|^2),\end{aligned} \tag{3.12}$$

it implies that

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{3.13}$$

Since T_{s_n} is a firmly nonexpansive mapping and $T_{s_n}x_n = v_n$. By using the same method as (3.12), we have

$$\|v_n - z\|^2 \leq \|x_n - z\|^2 - \|v_n - x_n\|^2. \tag{3.14}$$

From (3.5), (3.13) and (3.14), we have

$$\begin{aligned}\|y_n - z\|^2 &= \|\alpha_n u_n + \beta_n v_n + \gamma_n \overline{u}_n + \delta_n \overline{v}_n + \eta_n K x_n - z\|^2 \\ &= \|\alpha_n(u_n - z) + \beta_n(v_n - z) + \gamma_n(\overline{u}_n - z) + \delta_n(\overline{v}_n - z) + \eta_n(Kx_n - z)\|^2 \\ &\leq \alpha_n\|u_n - z\|^2 + \beta_n\|v_n - z\|^2 + \gamma_n\|\overline{u}_n - z\|^2 + \delta_n\|\overline{v}_n - z\|^2 \\ &\quad + \eta_n\|Kx_n - z\|^2 \\ &\leq \alpha_n(\|x_n - z\|^2 - \|u_n - x_n\|^2) + \beta_n(\|x_n - z\|^2 - \|v_n - x_n\|^2) \\ &\quad + \gamma_n\|\overline{u}_n - z\|^2 + \delta_n\|\overline{v}_n - z\|^2 + \eta_n\|Kx_n - z\|^2 \\ &\leq \|x_n - z\|^2 - \alpha_n\|u_n - x_n\|^2 - \beta_n\|v_n - x_n\|^2,\end{aligned} \tag{3.15}$$

it implies that

$$\begin{aligned}\alpha_n\|u_n - x_n\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 - \beta_n\|v_n - x_n\|^2 \\ &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\ &\leq (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|.\end{aligned} \tag{3.16}$$

By (3.10) and condition (ii), we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.17}$$

By using the same method as (3.16), we have

$$\lim_{n\to\infty}\|v_n - x_n\| = 0. \tag{3.18}$$

Since $\overline{T}_r$ is a nonexpansive mapping, $\overline{T}_r(I-rA)x_n=\bar{u}_n$ and A is an $\alpha-$inverse strongly monotone mapping with $r\in(0,2\alpha)$, we have

$$\begin{aligned}\|\bar{u}_n-z\|^2&=\|\overline{T}_r(I-rA)x_n-\overline{T}_r(I-rA)z\|^2\\&\le\|(x_n-z)-r(Ax_n-Az)\|^2\\&\le\|x_n-z\|^2-2r\langle x_n-z,Ax_n-Az\rangle+r^2\|Ax_n-Az\|^2\\&\le\|x_n-z\|^2-r(2\alpha-r)\|Ax_n-Az\|^2.\end{aligned}\tag{3.19}$$

From $\overline{T}_s$ is a nonexpansive mapping, $\overline{T}_s(I-sB)x_n=\bar{u}_n$ and B is an $\beta-$inverse strongly monotone mapping with $s\in(0,2\beta)$ and using the same method as (3.19), we have

$$\|\bar{v}_n-z\|^2\le\|x_n-z\|^2-s(2\beta-s)\|Bx_n-Bz\|^2.\tag{3.20}$$

By (3.15), (3.19) and (3.20), we have

$$\begin{aligned}\|y_n-z\|^2&=\|\alpha_nu_n+\beta_nv_n+\gamma_n\overline{u}_n+\delta_n\overline{v}_n+\eta_nKx_n-z\|^2\\&=\|\alpha_n(u_n-z)+\beta_n(v_n-z)+\gamma_n(\overline{u}_n-z)+\delta_n(\overline{v}_n-z)+\eta_n(Kx_n-z)\|^2\\&\le\alpha_n\|u_n-z\|^2+\beta_n\|v_n-z\|^2+\gamma_n\|\overline{u}_n-z\|^2\\&\quad+\delta_n\|\overline{v}_n-z\|^2+\eta_n\|Kx_n-z\|^2\\&\le\alpha_n\|u_n-z\|^2+\beta_n\|v_n-z\|^2+\gamma_n(\|x_n-z\|^2-r(2\alpha-r)\|Ax_n-Az\|^2)\\&\quad+\delta_n(\|x_n-z\|^2-s(2\beta-s)\|Bx_n-Bz\|^2)+\eta_n\|Kx_n-z\|^2\\&\le\|x_n-z\|^2-\gamma_nr(2\alpha-r)\|Ax_n-Az\|^2-\delta_ns(2\beta-s)\|Bx_n-Bz\|^2.\end{aligned}$$

It implies that

$$\begin{aligned}\gamma_nr(2\alpha-r)\|Ax_n-Az\|^2&\le\|x_n-z\|^2-\|y_n-z\|^2-\delta_ns(2\beta-s)\|Bx_n-Bz\|^2\\&\le\|x_n-z\|^2-\|y_n-z\|^2\\&\le(\|x_n-z\|+\|y_n-z\|)\|x_n-y_n\|.\end{aligned}\tag{3.21}$$

By condition (ii), $r\in(0,2\alpha)$ and (3.10), we have

$$\lim_{n\to\infty}\|Ax_n-Az\|=0.\tag{3.22}$$

By using the same method as (3.22), we have

$$\lim_{n\to\infty}\|Bx_n-Bz\|=0.\tag{3.23}$$

By $\overline{T}_r$ is a firmly nonexpansive mapping and $\overline{T}_r(I - rA)x_n = \bar{u}_n$, we have

$$\begin{aligned}
\|\bar{u}_n - z\|^2 &= \|\overline{T}_r(I - rA)x_n - \overline{T}_r(I - rA)z\|^2 \\
&\le \langle \bar{u}_n - z, (I - rA)x_n - (I - rA)z\rangle \\
&= \frac{1}{2}\Big(\|\bar{u}_n - z\|^2 + \|(I - rA)x_n - (I - rA)z\|^2 \\
&\quad - \|((I - rA)x_n - (I - rA)z) - (\bar{u}_n - z)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|\bar{u}_n - z\|^2 + \|x_n - z\|^2 - \|((x_n - \bar{u}_n) - r(Ax_n - Az)\|^2\Big) \\
&= \frac{1}{2}\Big(\|\bar{u}_n - z\|^2 + \|x_n - z\|^2 - \|x_n - \bar{u}_n\|^2 \\
&\quad + 2r\langle x_n - \bar{u}_n, Ax_n - Az\rangle - r^2\|Ax_n - Az\|^2\Big), \qquad (3.24)
\end{aligned}$$

it implies that

$$\|\bar{u}_n - z\| \le \|x_n - z\|^2 - \|x_n - \bar{u}_n\|^2 + 2r\langle x_n - \bar{u}_n, Ax_n - Az\rangle. \qquad (3.25)$$

By using the same method as (3.24), we have

$$\|\bar{v}_n - z\| \le \|x_n - z\|^2 - \|x_n - \bar{v}_n\|^2 + 2s\langle x_n - \bar{v}_n, Bx_n - Bz\rangle. \qquad (3.26)$$

From (3.15), (3.25) and (3.26), we have

$$\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n u_n + \beta_n v_n + \gamma_n \bar{u}_n + \delta_n \bar{v}_n + \eta_n Kx_n - z\|^2 \\
&= \|\alpha_n(u_n - z) + \beta_n(v_n - z) + \gamma_n(\bar{u}_n - z) + \delta_n(\bar{v}_n - z) + \eta_n(Kx_n - z)\|^2 \\
&\le \alpha_n\|u_n - z\|^2 + \beta_n\|v_n - z\|^2 + \gamma_n\|\bar{u}_n - z\|^2 \\
&\quad + \delta_n\|\bar{v}_n - z\|^2 + \eta_n\|Kx_n - z\|^2 \\
&\le \alpha_n\|x_n - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n(\|x_n - z\|^2 - \|x_n - \bar{u}_n\|^2 + 2r\langle x_n - \bar{u}_n, Ax_n - Az\rangle) \\
&\quad + \delta_n(\|x_n - z\|^2 - \|x_n - \bar{v}_n\|^2 + 2s\langle x_n - \bar{v}_n, Bx_n - Bz\rangle) + \eta_n\|x_n - z\|^2 \\
&\le \|x_n - z\|^2 - \gamma_n\|x_n - \bar{u}_n\|^2 - \delta_n\|x_n - \bar{v}_n\|^2 \\
&\quad + 2r\gamma_n\langle x_n - \bar{u}_n, Ax_n - Az\rangle + 2s\delta_n\langle x_n - \bar{v}_n, Bx_n - Bz\rangle.
\end{aligned}$$

It implies that

$$\begin{aligned}
\gamma_n\|x_n - \bar{u}_n\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 - \delta_n\|x_n - \bar{v}_n\|^2 \\
&\quad + 2r\gamma_n\langle x_n - \bar{u}_n, Ax_n - Az\rangle + 2s\delta_n\langle x_n - \bar{v}_n, Bx_n - Bz\rangle \\
&\le \|x_n - z\|^2 - \|y_n - z\|^2 + 2r\gamma_n\langle x_n - \bar{u}_n, Ax_n - Az\rangle \\
&\quad + 2s\delta_n\langle x_n - \bar{v}_n, Bx_n - Bz\rangle \\
&\le (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\| + 2r\gamma_n\langle x_n - \bar{u}_n, Ax_n - Az\rangle \\
&\quad + 2s\delta_n\langle x_n - \bar{v}_n, Bx_n - Bz\rangle \\
&\le (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\| + 2r\gamma_n\|x_n - \bar{u}_n\|\|Ax_n - Az\| \\
&\quad + 2s\delta_n\|x_n - \bar{v}_n\|\|Bx_n - Bz\|. \qquad (3.27)
\end{aligned}$$

From (3.10), (3.22) , (3.23), condition (ii), $r \in (0, 2\alpha)$ and $s \in (0, 2\beta)$, we have

$$\lim_{n\to\infty} \|x_n - \bar{u}_n\| = 0. \tag{3.28}$$

By using the same method as (3.28), we have

$$\lim_{n\to\infty} \|x_n - \bar{v}_n\| = 0. \tag{3.29}$$

Since

$$\begin{aligned}\eta_n\|Kx_n - x_n\| \leq \|y_n - x_n\| + \alpha_n\|x_n - u_n\| + \beta_n\|x_n - v_n\| \\ + \gamma_n\|x_n - \overline{u}_n\| + \delta_n\|x_n - \overline{v}_n\|.\end{aligned}$$

From (3.11), (3.17), (3.18), (3.28) and (3.29) and condition (ii), we have

$$\lim_{n\to\infty} \|Kx_n - x_n\| = 0.$$

Next, we will show that $\{x_n\}$, $\{w_n^1\}$ and $\{w_n^2\}$ are Cauchy sequences.
Let $a \in (0,1)$, by (3.8), there exists $N_0 \in \mathbb{N}$ such that

$$\|x_{n+1} - x_n\| < a^n, \forall n \geq N_0. \tag{3.30}$$

Thus, for any number n, $p \in \mathbb{N}$, we have

$$\|x_{n+p} - x_n\| \leq \sum_{k=n}^{n+p-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{n+p-1} a^k \leq \frac{a^n}{1-a}. \tag{3.31}$$

Since $a \in (0,1)$, we have $\lim_{n\to\infty} a^n = 0$. By (3.31), we have $\{x_n\}$ is a Cauchy sequence in Hilbert space. Then, there exists $x^* \in C$ such that $\lim_{n\to\infty} x_n = x^*$.

Since $T : C \to CB(H)$ is a $\mathcal{H}$-Lipschitz continuous with constant μ_1 and (1.9), we have

$$\|w_n^1 - w_{n+1}^1\| \leq \Big(1 + \frac{1}{n}\Big)\mathcal{H}(T(x_n), T(x_{n+1})) \leq \Big(1 + \frac{1}{n}\Big)\mu_1\|x_{n+1} - x_n\|. \tag{3.32}$$

By (3.30), (3.32) and for any number n, $p \in \mathbb{N}$, $p > 0$, we have

$$\begin{aligned}\|w_{n+p}^1 - w_n^1\| &\leq \Sigma_{k=n}^{n+p-1}\|w_{k+1}^1 - w_k^1\| \\ &\leq \Sigma_{k=n}^{n+p-1}\Big(1 + \frac{1}{k}\Big)\mu_1\|x_{k+1} - x_k\| \\ &\leq \Sigma_{k=n}^{n+p-1} 2\mu_1 a^k \\ &\leq 2\mu_1\frac{a^n}{1-a}.\end{aligned} \tag{3.33}$$

Since $a \in (0,1)$, we have $\lim_{n\to\infty} a^n = 0$. By (3.33), we have $\{w_n^1\}$ is a Cauchy sequence in Hilbert space. Then, there exists $w_1^* \in C$ such that $\lim_{n\to\infty} w_n^1 = w_1^*$.

Since $D : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_2 and (1.9), we have

$$\|w_n^2 - w_{n+1}^2\| \leq \Big(1 + \frac{1}{n}\Big)\mathcal{H}(D(x_n), D(x_{n+1})) \leq \Big(1 + \frac{1}{n}\Big)\mu_2\|x_{n+1} - x_n\|. \tag{3.34}$$

By using the same method as (3.33), there is $w_2^* \in H$ such that $\lim_{n\to\infty} w_n^2 = w_2^*$.
Next, we show that $w_1^* \in T(x^*)$ and $w_2^* \in D(x^*)$.
Since $w_n^1 \in T(x_n)$, we have

$$\begin{aligned} d(w_n^1, T(x^*)) &\leq \max\left\{ d(w_n^1, T(x^*)), \sup_{w_1 \in T(x^*)} d(T(x_n), w_1) \right\} \\ &\leq \max\left\{ \sup_{z \in T(x_n)} d(z, T(x^*)), \sup_{w_1 \in T(x^*)} d(T(x_n), w_1) \right\} \\ &= \mathcal{H}(T(x_n), T(x^*)). \end{aligned} \tag{3.35}$$

It implies that

$$\begin{aligned} d(w_1^*, T(x^*)) &\leq \|w_1^* - w_n^1\| + d(w_n^1, T(x^*)) \\ &\leq \|w_1^* - w_n^1\| + \mathcal{H}(T(x_n), T(x^*)) \\ &\leq \|w_1^* - w_n^1\| + \mu_1 \|x_n - x^*\| \end{aligned}$$

By $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} w_n^1 = w_1^*$, we have $d(w_1^*, T(x^*)) = 0$. Since $T(x^*)$ is a closed set, we have $w_1^* \in T(x^*)$. Since $\lim_{n\to\infty} w_n^2 = w_2^*$ and using the same method as above, we get $w_2^* \in D(x^*)$.

Next, we show that $\omega(x_n) \subset \mathbb{F}$.

Since $\{x_n\}$ is bounded, then $\omega(x_n) \neq \emptyset$. Let $q \in \omega(x_n)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converge weakly to q. Since $\{x_n\}$ is a Cauchy sequence in Hilbert space, we have $x_n \to q$ as $n \to \infty$. Since $\lim_{n\to\infty} x_n = x^*$, we have $x^* = q$, it follows that $w_1^* \in T(q)$ and $w_2^* \in D(q)$.

From (3.17) and $x_n \to q$ as $n \to \infty$, we have $u_n \to q$ as $n \to \infty$.
By $u_n = T_{r_n} x_n$, we have

$$\Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0.$$

By (3.17), $(H1)$ and lower semicontinuity of φ_1, we have

$$\Phi_1(w_1^*, q, u) + \varphi_1(u) - \varphi_1(q) \geq 0, \forall u \in C.$$

Then, we have

$$q \in (GEP)_s(\Phi_1, \varphi_1). \tag{3.36}$$

From (3.18) and $x_n \to q$ as $n \to \infty$, we have $v_n \to q$ as $n \to \infty$.
By $v_n = T_{s_n} x_n$, we have

$$\Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C.$$

By using the same method as (3.36), we have

$$q \in (GEP)_s(\Phi_2, \varphi_2). \tag{3.37}$$

From (3.28) and $x_n \to q$ as $n \to \infty$, we have $\bar{u}_n \to q$ as $n \to \infty$.
By $\bar{u}_n = \overline{T}_r(I - rA)x_n$, we have

$$F_1(\overline{u}_n, \overline{u}) + \langle Ax_n, \overline{u} - \overline{u}_n\rangle + \frac{1}{r}\langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n\rangle \geq 0, \forall \overline{u} \in C.$$

From $(A2)$, we have

$$\begin{aligned} F_1(\overline{u}, \overline{u}_n) &\leq F_1(\overline{u}_n, \overline{u}) + F_1(\overline{u}, \overline{u}_n) + \langle Ax_n, \overline{u} - \overline{u}_n\rangle + \frac{1}{r}\langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n\rangle \\ &\leq \langle Ax_n, \overline{u} - \overline{u}_n\rangle + \frac{1}{r}\langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n\rangle \\ &= \langle Ax_n, \overline{u} - \overline{u}_n\rangle + \langle \overline{u} - \overline{u}_n, \frac{\overline{u}_n - x_n}{r}\rangle, \end{aligned} \tag{3.38}$$

Put $z_t = ty + (1-t)q$ for all $t \in (0, 1]$ and $y \in C$. Then we have $z_t \in C$.
So, from (3.38), we have

$$F_1(z_t, \bar{u}_n) - \langle z_t - \bar{u}_n, Ax_n\rangle - \langle z_t - \bar{u}_n, \frac{\bar{u}_n - x_n}{r}\rangle \leq 0. \tag{3.39}$$

By (3.39), we have

$$\begin{aligned} \langle z_t - \bar{u}_n, Az_t\rangle &\geq \langle z_t - \bar{u}_n, Az_t\rangle - \langle z_t - \bar{u}_n, Ax_n\rangle - \langle z_t - \bar{u}_n, \frac{\bar{u}_n - x_n}{r}\rangle \\ &\quad + F_1(z_t, \bar{u}_n) \\ &= \langle z_t - \bar{u}_n, Az_t - A\bar{u}_n\rangle + \langle z_t - \bar{u}_n, A\bar{u}_n\rangle - \langle z_t - \bar{u}_n, Ax_n\rangle \\ &\quad - \langle z_t - \bar{u}_n, \frac{\bar{u}_n - x_n}{r}\rangle + F_1(z_t, \bar{u}_n) \\ &= \langle z_t - \bar{u}_n, Az_t - A\bar{u}_n\rangle + \langle z_t - \bar{u}_n, A\bar{u}_n - Ax_n\rangle \\ &\quad - \langle z_t - \bar{u}_n, \frac{\bar{u}_n - x_n}{r}\rangle + F_1(z_t, \bar{u}_n). \end{aligned} \tag{3.40}$$

Since $\lim_{n\to\infty} \|\bar{u}_n - x_n\| = 0$, we have $\lim_{n\to\infty} \|A\bar{u}_n - Ax_n\| = 0$.

From monotone of A, we have $\langle z_t - \bar{u}_n, Az_t - A\bar{u}_n\rangle \geq 0$. From $\bar{u}_n \to q$ as $n \to \infty$ and $(A4)$, we have

$$\langle z_t - q, Az_t\rangle \geq F_1(z_t, q). \tag{3.41}$$

From $(A1), (A4)$ and (3.41), we have

$$\begin{aligned} 0 = F_1(z_t, z_t) &= F_1(z_t, ty - (1-t)q) \\ &\leq tF_1(z_t, y) + (1-t)F_1(z_t, q) \\ &\leq tF_1(z_t, y) + (1-t)\langle z_t - q, Az_t\rangle \\ &= tF_1(z_t, y) + (1-t)t\langle y - q, Az_t\rangle. \end{aligned}$$

It implies that

$$F_1(z_t, y) + (1-t)\langle y - q, Az_t\rangle \geq 0. \tag{3.42}$$

Letting $t \to 0^+$ and (3.42), we have $0 \leq F_1(q, y) + \langle y - q, Aq\rangle$, for all $y \in C$. Then

$$q \in EP(F_1, A). \tag{3.43}$$

From (3.29) and $x_n \to q$ as $n \to \infty$, we have $\bar{v}_n \to q$ as $n \to \infty$.
By $\bar{v}_n = \overline{T}_s(I - sB)x_n$, we have

$$F_2(\overline{v}_n, \overline{v}) + \langle Bx_n, \overline{v} - \overline{v}_n \rangle + \frac{1}{s}\langle \overline{v} - \overline{v}_n, \overline{v}_n - x_n \rangle \geq 0, \forall \overline{v} \in C,$$

By using the same method as (3.43), we have

$$q \in EP(F_2, B). \tag{3.44}$$

By Lemma 2.14, we have K is quasi-noneapansive mapping and $F(K) = \bigcap_{i=1}^{N} F(T_i)$. Since $x_{n_i} \rightharpoonup q$ as $i \to \infty$ and $\lim_{n\to\infty} \|Kx_n - x_n\| = 0$ and Lemma 2.13, we have

$$q \in F(K) = \bigcap_{i=1}^{N} F(T_i) \tag{3.45}$$

From (3.36), (3.37), (3.43), (3.44) and (3.45), we have $q \in \mathbb{F}$.

Hence $\omega(x_n) \subset \mathbb{F}$. By Lemma 2.8 and (3.7), it implies that $\{x_n\}$ converges strongly to $P_{\mathbb{F}}x_1$. This completes the proof. ☐

The following corollary is consequences which are reduced iterative scheme of Theorem (3.1).

Corollary 3.2. Let C be a nonempty bounded, closed, and convex subset of a real Hilbert space H and let $\varphi : C \to \mathbb{R}$ be lower semicontinuous and convex functions. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ, respectively, $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $(H1) - (H3)$. Let $A : C \to H$ be an α-inverse strongly monotone mapping, let $F : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$, let T_i be nonspreading mappings of C into itself for all $i = 1, 2, ..., N$ with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap EP(F, A) \cap (GEP)_s(\Phi, \varphi) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1 \in T(x_1)$. Support that there exist sequences $\{x_n\}, \{u_n\}, \{\overline{u}_n\} \subseteq C$ and $\{w_n\} \subseteq H$ be sequences generated by

$$\begin{cases} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ \Phi(w_n, u_n, u) + \varphi(u) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n \rangle \geq 0, \forall u \in C, r_n > 0, \\ F(\overline{u}_n, \overline{u}) + \langle Ax_n, \overline{u} - \overline{u}_n \rangle + \frac{1}{r}\langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n \rangle \geq 0, \forall \overline{u} \in C, \\ y_n = \alpha_n u_n + \gamma_n \overline{u}_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases} \tag{3.46}$$

where $\{\alpha_n\}, \{\gamma_n\}$ and $\{\eta_n\}$ are sequences in $[0, 1]$ for all $n \in \mathbb{N}$, $r, r_n \subset (0, 2\alpha)$ and suppose the following conditions hold:

(i) $\alpha_n + \gamma_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \gamma_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there is $\overline{\lambda} > 0$ such that

$$\Phi(w_1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi(w_2, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\overline{\lambda}\|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \tag{3.47}$$

for all $(r_1, r_2) \in \Theta \times \Theta$, $w_i \in T(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}}x_1$.

The following corollary is consequences which are expand iterative scheme of Theorem 3.1. Therefore, we omit the proof.

Corollary 3.3. Let C be a nonempty bounded, closed and convex subset of a real Hilbert space H and let $\varphi_i : C \to \mathbb{R}$ be a lower semicontinuous and convex function, for all $i = 1, 2, ..., N$. Let $T^i : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_i, for all $i = 1, 2, ..., N$, respectively, $\Phi_i : H \times C \times C \to \mathbb{R}$ be equilibrium-like function satisfying $(H1) - (H3)$, for all $i = 1, 2, ..., N$. Let $A_i : C \to H$ be an α_i-inverse strongly monotone mapping, for all $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$ for all $i = 1, 2, ..., N$.For every $i = 1, 2, ..., N$. let T_i be a nonspreading mapping of C into itself with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} EP(F_i, A_i) \cap \bigcap_{i=1}^{N} (GEP)_s(\Phi_i, \varphi_i) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1^i \in T^i(x_1)$, for all $i = 1, 2, ..., N$, there exist sequences $\{x_n\}, \{u_n^i\}, \{\overline{u}_n^i\} \subseteq C$ and $\{w_n^i\} \subseteq H$, for all $i = 1, 2, ..., N$ generated by

$$\begin{cases} w_n^i \in T^i(x_n), \|w_n^i - w_{n+1}^i\| \leq \Big(1 + \frac{1}{n}\Big)\mathcal{H}(T(x_n), T(x_{n+1})), \\ \Phi_i(w_n^i, u_n^i, u_i) + \varphi_i(u_i) - \varphi_i(u_n^i) + \frac{1}{r_n^i}\langle u_n^i - x_n, u_i - u_n^i \rangle \geq 0, \forall u_i \in C, r_n^i > 0 \\ F_i(\overline{u}_n^i, \overline{u}_i) + \langle A_i x_n, \overline{u}_i - \overline{u}_n^i \rangle + \frac{1}{r^i}\langle \overline{u}_i - \overline{u}_n^i, \overline{u}_n^i - x_n \rangle \geq 0, \forall u_i \in C, \\ y_n = \Sigma_{i=1}^N \alpha_n^i u_n^i + \Sigma_{i=1}^N \gamma_n^i \overline{u}_n^i + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases} \tag{3.48}$$

where $\{\alpha_n^i\}, \{\gamma_n^i\}$ and $\{\eta_n\}$ are sequence in $[0, 1]$, $r_n^i, r^i \subset (0, 2\alpha)$, for every $i = 1, 2, ..., N$ and suppose the following conditions hold:

(i) $\sum_{i=1}^{N} \alpha_n^i + \sum_{i=1}^{N} \gamma_n^i + \eta_n = 1$,

(ii) $0 < b < \alpha_n^i, \gamma_n^i, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there exists $\overline{\lambda}^i > 0, \forall i = 1, 2, ..., N$ such that

$$\Phi_i(w_1^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)) + \Phi_i(w_2^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)) \leq -\overline{\lambda}^i \|T_{r_1^i}(x_1) - T_{r_2^i}(x_2)\|^2, \tag{3.49}$$

where $\Theta^i = \{r_n^i : n \geq 1\}$ with $(r_1^i, r_2^i) \in \Theta^i \times \Theta^i$, $w_k^i \in T^i(x_k)$, for $k = 1, 2$ and $i = 1, 2, ..., N$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}}x_1$.

The following corollaries are consequences which are applied by Theorem 3.1. Therefore, we omit the proof.

Corollary 3.4. Let C be a nonempty bounded, closed, and convex subset of a real Hilbert space H and let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be lower semicontinuous and convex function. Let $D, T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ be equilibrium-like function satisfying $(H1) - (H3)$. Let $A : C \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be an β-inverse strongly monotone mapping, let $F_j : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$ for all $j = 1, 2$. Let $\mathbb{F} = EP(F_1, A) \cap EP(F_2, B) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \neq \emptyset$ and let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exist sequence $\{x_n\}, \{u_n\}, \{v_n\}, \{\overline{u}_n\}, \{\overline{v}_n\} \subseteq C$ and $\{w_n^1\}, \{w_n^2\} \subseteq H$ generated by

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \Big(1 + \frac{1}{n}\Big)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \Big(1 + \frac{1}{n}\Big)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C, s_n > 0, \\ F_1(\overline{u}_n, \overline{u}) + \langle Ax_n, \overline{u} - \overline{u}_n\rangle + \frac{1}{r}\langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n\rangle \geq 0, \forall \overline{u} \in C, \\ F_2(\overline{v}_n, \overline{v}) + \langle Bx_n, \overline{v} - \overline{v}_n\rangle + \frac{1}{s}\langle \overline{v} - \overline{v}_n, \overline{v}_n - x_n\rangle \geq 0, \forall \overline{v} \in C, \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n \overline{u}_n + \delta_n \overline{v}_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases} \tag{3.50}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are sequence in $[0, 1]$ for all $n \in \mathbb{N}$, $r_n, r \subset (0, 2\alpha)$ and $s_n, s \subset (0, 2\beta)$ and suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$,

(ii) $0 < b < \alpha_n, \beta_n, \gamma_n, \delta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there are $\overline{\lambda}_1, \overline{\lambda}_2 > 0$ such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\overline{\lambda}_1\|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \leq -\overline{\lambda}_2\|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2, \end{cases} \tag{3.51}$$

for all $(r_1, r_2) \in \Theta \times \Theta$,$(s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$ and $\Xi = \{s_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}} x_1$.

In the case of $F_1, F_2 \equiv 0$, then $EP(F_1, A)$ is reduced to $VI(C, A)$ and $EP(F_2, B)$ is reduced to $VI(C, B)$. So, we prove the next result as follows:

Corollary 3.5. Let C be a nonempty bounded, closed and convex subset of a real Hilbert space H and let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be lower semicontinuous and convex functions. Let $D, T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ be equilibrium-like function satisfying $(H1) - (H3)$. Let $A : C \to H$ be an $\alpha-$inverse strongly monotone mapping and $B : C \to H$ be an β-inverse strongly monotone mapping. For every $i = 1, 2, ..., N$, let T_i be nonspreading mappings of C into itself with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A) \cap VI(C, B) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$ .Let K be the K-mappings generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exist sequences $\{x_n\}, \{u_n\}, \{v_n\}, \{\overline{u}_n\}, \{\overline{v}_n\} \subseteq C$ and $\{w_n^1\}, \{w_n^2\} \subseteq H$ generated by

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \Big(1 + \frac{1}{n}\Big)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \Big(1 + \frac{1}{n}\Big)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C, s_n > 0, \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n P_C(I - rA)x_n + \delta_n P_C(I - sB)x_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases} \tag{3.52}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\eta_n\}$ are sequence in $[0, 1]$, $r_n, r \subset (0, 2\alpha)$ and $s_n, s \subset (0, 2\beta)$, for every $n \in \mathbb{N}$ and suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there exists $\overline{\lambda}_1, \overline{\lambda}_2 > 0$ such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\overline{\lambda}_1 \|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \leq -\overline{\lambda}_2 \|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2, \end{cases} \tag{3.53}$$

for all $(r_1, r_2) \in \Theta \times \Theta$,$(s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$ and $\Xi = \{s_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}} x_1$.

Chapter 4

Application

4.1 Constrained convex optimization problems

In this section, by using our main result, we obtain Theorem 4.1. Before we prove strong convergence theorem in this section, we consider the following standard constrained convex optimization problem as follows:

$$\text{find } x^* \in C, \text{ such that } f(x^*) = \min_{x \in C} f(x), \tag{4.1}$$

where $f : C \to \mathbb{R}$ is a convex, Fréchet differentiable function, C is a closed convex subset of H.

It is known that the optimization problem (4.1) is equivalent to the following variational inequality problem

$$\text{find } x^* \in C, \text{ such that } \langle v - x^*, \nabla f(x^*) \rangle \geq 0, \forall v \in C, \tag{4.2}$$

where $\nabla f : C \to C$ is the gradient of f.

It is also known that the optimality condition (4.2) is equivalent to the following fixed point equation

$$x^* = P_C(x^* - \mu \nabla f(x^*)), \tag{4.3}$$

where P_C is the metric projection onto C and $\mu > 0$ is a positive constant. The set of all solution of (4.1) is denoted by Ω_f

Convex minimization problem has applications in a wide range of disciplines, such as automatic control systems, estimation and signal processing, communications and networks, electronic circuit design, finance and structural optimization.

Next, we prove a result involving optimization problem as follows:

Theorem 4.1. Let C be a nonempty bounded,closed and convex subset of a real Hilbert space H and let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be lower semicontinuous and convex functions. Let $D, T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ be equilibrium-like function satisfying $(H1) - (H3)$. Let $f : C \to \mathbb{R}$ be a convex function with ∇f be $\frac{1}{L_f}$-inverse strongly monotone mapping and $g : C \to \mathbb{R}$ be a convex function with ∇g be $\frac{1}{L_f}$-inverse strongly monotone mapping, where $L_f,\ L_g > 0$. For every $i = 1, 2, ..., N$, let T_i be nonspreading mappings of C into itself with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap \Omega_f \cap \Omega_g \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mappings generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exists sequence $\{x_n\}, \{u_n\}, \{v_n\}, \{\overline{u}_n\}, \{\overline{v}_n\} \subseteq C$ and

$\{w_n^1\}, \{w_n^2\} \subseteq H$ be sequence generated by

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \Big(1 + \frac{1}{n}\Big)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \Big(1 + \frac{1}{n}\Big)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C, s_n > 0, \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n P_C(I - r\nabla f)x_n + \delta_n P_C(I - s\nabla g)x_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases} \tag{4.4}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\eta_n\}$ are a sequence in $[0,1]$, $r \subset (0, \frac{2}{L_f})$ and $s \subset (0, \frac{2}{L_g})$, for every $n \in \mathbb{N}$ and suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there exists $\overline{\lambda}_1, \overline{\lambda}_2 > 0$ such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\overline{\lambda}_1\|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \leq -\overline{\lambda}_2\|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2, \end{cases} \tag{4.5}$$

for all $(r_1, r_2) \in \Theta \times \Theta$,$(s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$ and $\Xi = \{s_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}} x_1$.

Proof. Putting $A \equiv \nabla f$, $B \equiv \nabla g$ and Corollary 3.5, we can conclude the desired conclusion. $\square$

4.2 Quasi-nonexpansive mapping

Next we prove a result involving quasi-nonexpansive mapping as follows:

Corollary 4.2. Let C be a nonempty bounded, closed, and convex subset of a real Hilbert space H and let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be lower semicontinuous and convex function. Let $D, T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ be equilibrium-like function satisfying $(H1) - (H3)$. Let $A : C \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be ab β-inverse strongly monotone mapping, let $F_j : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$ for all $j = 1, 2$, let T be quasi-nonexpansive mappings of C into itself with $\mathbb{F} = F(T) \cap EP(F_1, A) \cap EP(F_2, B) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \neq \emptyset$. Let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exist sequences $\{x_n\}, \{u_n\}, \{v_n\}, \{\overline{u}_n\}, \{\overline{v}_n\} \subseteq C$ and $\{w_n^1\}, \{w_n^2\} \subseteq H$ generated by

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C, s_n > 0, \\ F_1(\overline{u}_n, \overline{u}) + \langle Ax_n, \overline{u} - \overline{u}_n\rangle + \frac{1}{r}\langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n\rangle \geq 0, \forall \overline{u} \in C, \\ F_2(\overline{v}_n, \overline{v}) + \langle Bx_n, \overline{v} - \overline{v}_n\rangle + \frac{1}{s}\langle \overline{v} - \overline{v}_n, \overline{v}_n - x_n\rangle \geq 0, \forall \overline{v} \in C, \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n \overline{u}_n + \delta_n \overline{v}_n + \eta_n T x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases} \tag{4.6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\eta_n\}$ are sequence in $[0, 1]$ for all $n \in \mathbb{N}$, $r_n, r \subset (0, 2\alpha)$ and $s_n s \subset (0, 2\beta)$ and suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there are $\overline{\lambda}_1, \overline{\lambda}_2 > 0$ such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\overline{\lambda}_1 \|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \leq -\overline{\lambda}_2 \|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2, \end{cases} \tag{4.7}$$

for all $(r_1, r_2) \in \Theta \times \Theta, (s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$ and $\Xi = \{s_n : n \geq 1\}$. Then $\{x_n\}$ converge strongly to $P_{\mathbb{F}} x_1$.

Corollary 4.3. Let C be a nonempty bounded, closed and convex subset of a real Hilbert space H and let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be lower semicontinuous and convex function. Let $D, T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ be equilibrium-like functions satisfying $(H1)-(H3)$. Let $A : C \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be an β-inverse strongly monotone mapping, let $F_j : C \times C \to \mathbb{R}$ satisty $(A1) - (A4)$ for all $j = 1, 2$, let $T_i : C \to C$ be quasi-nonexpansive mappings for all $i = 1, 2, ..., N$ with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap EP(F_1, A) \cap EP(F_2, B) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mappings generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exist sequences $\{x_n\}, \{u_n\}, \{v_n\}, \{\overline{u}_n\}, \{\overline{v}_n\} \subseteq C$ and $\{w_n^1\}, \{w_n^2\} \subseteq H$ be sequence generated by

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \le \left(1+\frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \le \left(1+\frac{1}{n}\right)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \ge 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \ge 0, \forall v \in C, s_n > 0, \\ F_1(\overline{u}_n, \overline{u}) + \langle Ax_n, \overline{u} - \overline{u}_n\rangle + \frac{1}{r}\langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n\rangle \ge 0, \forall \overline{u} \in C, \\ F_2(\overline{v}_n, \overline{v}) + \langle Bx_n, \overline{v} - \overline{v}_n\rangle + \frac{1}{s}\langle \overline{v} - \overline{v}_n, \overline{v}_n - x_n\rangle \ge 0, \forall \overline{v} \in C, \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n \overline{u}_n + \delta_n \overline{v}_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \ge 1, \end{cases} \tag{4.8}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\eta_n\}$ are sequence in $[0,1]$ for all $n \in \mathbb{N}$, $r_n, r \subset (0, 2\alpha)$ and $s_n, s \subset (0, 2\beta)$ and suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1,$

(ii) $0 < b < \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n \le c,$ for some $b, c \in \mathbb{R},$

(iii) there are $\overline{\lambda}_1, \overline{\lambda}_2 > 0$ such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \le -\overline{\lambda}_1\|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \le -\overline{\lambda}_2\|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2, \end{cases} \tag{4.9}$$

for all $(r_1, r_2) \in \Theta \times \Theta$,$(s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \ge 1\}$ and $\Xi = \{s_n : n \ge 1\}$. Then $\{x_n\}$ converge strongly to $P_{\mathbb{F}} x_1$.

4.3 Demicontractive mapping

Next we prove a result involving demicontractive mapping as follows:

Let C be a nonempty subset of H. A mapping $T : C \to C$ is called κ-demicontractive if there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Tx^*\|^2 \le \|x - x^*\|^2 + \kappa\|(I - T)x\|^2, \ \forall x \in C \text{ and } x^* \in F(T). \tag{4.10}$$

In 2009, Iemoto and Takahashi [8] proved that (4.10) is equivalent to

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \ \forall x, y \in C.$$

Lemma 4.4. Let C be a nonempty closed convex subset of H and let $T : C \to C$ be κ-demicontractive mapping with $F(T) \ne \emptyset$. Defined
$Sx = ax + (1 - a)Tx$, $\forall x \in C$ and $a \in (\kappa, 1)$. Then the following properties holds:

(i) $F(S) = F(T)$,

(ii) S is a quasi-nonexpansive mapping.

Proof. It obvious that $F(T) \subseteq F(S)$. Let $x_0 \in F(S)$ and $x^* \in F(T)$, we have

$$\begin{aligned}
\|x_0 - x^*\|^2 &= \|Sx_0 - x^*\|^2 \\
&= \|a(x_0 - x^*) + (1-a)(Tx_0 - x^*)\|^2 \\
&= a\|x_0 - x^*\|^2 + (1-a)\|Tx_0 - x^*\|^2 - a(1-a)\|x_0 - Tx_0\|^2 \\
&\le a\|x_0 - x^*\|^2 + (1-a)(\|x_0 - x^*\|^2 \\
&\quad + \kappa\|(I-T)x_0\|^2) - a(1-a)\|x_0 - Tx_0\|^2 \\
&= \|x_0 - x^*\|^2 - (1-a)(a-\kappa)\|(I-T)x_0\|^2. \qquad (4.11)
\end{aligned}$$

It implies that

$$(1-a)(a-\kappa)\|(I-T)x_0\|^2 = 0.$$

Then

$$x_0 \in F(T).$$

Therefore $F(S) \subseteq F(T)$. Hence $F(S) = F(T)$.

From $F(S) = F(T)$ and applying (4.11) we can conclude that S is a quasi-nonexpansive mapping. ☐

By using Lemma 4.4, we obtain the following theorem.

Theorem 4.5. Let C be a nonempty bounded, closed and convex subset of a real Hilbert space H and let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be lower semicontinuous and convex functions. Let $D, T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ be equilibrium-like functions satisfying $(H1)-(H3)$. Let $A : C \to H$ be α-inverse strongly monotone mapping and $B : C \to H$ be β-inverse strongly monotone mapping, let $F_j : C \times C \to \mathbb{R}$ satisty $(A1)-(A4)$ for all $j = 1, 2$, let T_i be κ_i-demicontractive mappings of C into itself for all $i = 1, 2, ..., N$. Defined $S_i x = ax + (1-a)T_i x$, $\forall x \in C$ and $a \in (\kappa_i, 1)$ for all $i = 1, 2, ..., N$ with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap EP(F_1, A) \cap EP(F_2, B) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \le 1$. Let K be the K-mappings generated by $S_1, S_2, ..., S_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exists sequence $\{x_n\}, \{u_n\}, \{v_n\}, \{\overline{u}_n\}, \{\overline{v}_n\} \subseteq C$ and $\{w_n^1\}, \{w_n^2\} \subseteq H$ be sequence generated by

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \left(1+\frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \left(1+\frac{1}{n}\right)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C, s_n > 0, \\ F_1(\overline{u}_n, \overline{u}) + \langle Ax_n, \overline{u} - \overline{u}_n\rangle + \frac{1}{r}\langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n\rangle \geq 0, \forall \overline{u} \in C, \\ F_2(\overline{v}_n, \overline{v}) + \langle Bx_n, \overline{v} - \overline{v}_n\rangle + \frac{1}{s}\langle \overline{v} - \overline{v}_n, \overline{v}_n - x_n\rangle \geq 0, \forall \overline{v} \in C, \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n \overline{u}_n + \delta_n \overline{v}_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases} \tag{4.12}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\eta_n\}$ are a sequence in $[0,1]$ for all $n \in \mathbb{N}$, $r \subset (0, 2\alpha)$ and $s \subset (0, 2\beta)$ and suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there are $\overline{\lambda}_1, \overline{\lambda}_2 > 0$ such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\overline{\lambda}_1\|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \leq -\overline{\lambda}_2\|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2, \end{cases} \tag{4.13}$$

for all $(r_1, r_2) \in \Theta \times \Theta$,$(s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$ and $\Xi = \{s_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}} x_1$.

Proof. By Lemma 4.4 and Corollary 4.3, we can conclude the desired result. ☐

Chapter 5

Conclusion

In this chapter, we summarize all main theorems and applications obtained in this thesis.

(1) Let C be a nonempty bounded, closed, and convex subset of a real Hilbert space H and let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be lower semicontinuous and convex functions. Let $D, T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ be equilibrium-like functions satisfying $(H1) - (H3)$. Let $A : C \to H$ be α-inverse strongly monotone mapping and $B : C \to H$ be β-inverse strongly monotone mapping, let $F_j : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$ for all $j = 1, 2$, let T_i be nonspreading mappings of C into itself for all $i = 1, 2, ..., N$ with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap EP(F_1, A) \cap EP(F_2, B) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mappings generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exists sequence $\{x_n\}, \{u_n\}, \{v_n\}, \{\overline{u}_n\}, \{\overline{v}_n\} \subseteq C$ and $\{w_n^1\}, \{w_n^2\} \subseteq H$ be sequence generated by

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C, s_n > 0, \\ F_1(\overline{u}_n, \overline{u}) + \langle Ax_n, \overline{u} - \overline{u}_n\rangle + \frac{1}{r}\langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n\rangle \geq 0, \forall \overline{u} \in C, \\ F_2(\overline{v}_n, \overline{v}) + \langle Bx_n, \overline{v} - \overline{v}_n\rangle + \frac{1}{s}\langle \overline{v} - \overline{v}_n, \overline{v}_n - x_n\rangle \geq 0, \forall \overline{v} \in C, \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n \overline{u}_n + \delta_n \overline{v}_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases} \tag{5.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\eta_n\}$ are a sequence in $[0,1]$ for all $n \in \mathbb{N}$, $r \subset (0, 2\alpha)$ and $s \subset (0, 2\beta)$ and suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there are $\overline{\lambda}_1, \overline{\lambda}_2 > 0$ such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\overline{\lambda}_1 \|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \leq -\overline{\lambda}_2 \|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2, \end{cases} \tag{5.2}$$

for all $(r_1, r_2) \in \Theta \times \Theta$,$(s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$ and $\Xi = \{s_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}} x_1$.

(2) Let C be a nonempty bounded, closed and convex subset of a real Hilbert space H and let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be lower semicontinuous and convex functions. Let $D, T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ be equilibrium-like functions satisfying $(H1)-(H3)$. Let $A : C \to H$ be $\alpha-$inverse strongly monotone mapping and $B : C \to H$ be β-inverse strongly monotone mapping. For every $i = 1, 2, ..., N$, let T_i be nonspreading mappings of C into itself with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A) \cap VI(C, B) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$ .Let K be the K-mappings generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exists sequence $\{x_n\}, \{u_n\}, \{v_n\}, \{\overline{u}_n\}, \{\overline{v}_n\} \subseteq C$ and $\{w_n^1\}, \{w_n^2\} \subseteq H$ be sequence generated by

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C, s_n > 0, \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n P_C(I - rA)x_n + \delta_n P_C(I - sB)x_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases} \tag{5.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\eta_n\}$ are a sequence in $[0, 1]$, $r \subset (0, 2\alpha)$ and $s \subset (0, 2\beta)$, for every $n \in \mathbb{N}$ and suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there exists $\overline{\lambda}_1, \overline{\lambda}_2 > 0$ such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\overline{\lambda}_1\|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \leq -\overline{\lambda}_2\|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2, \end{cases} \tag{5.4}$$

for all $(r_1, r_2) \in \Theta \times \Theta$,$(s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$ and $\Xi = \{s_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}} x_1$.

(3) Let C be a nonempty bounded,closed and convex subset of a real Hilbert space H and let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be lower semicontinuous and convex functions. Let $D, T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ be equilibrium-like functions satisfying $(H1)-(H3)$. Let $f : C \to \mathbb{R}$ be a convex function with ∇f be $\frac{1}{L_f}$-inverse strongly monotone mapping and $g : C \to \mathbb{R}$ be a convex function with ∇g be $\frac{1}{L_f}$-inverse strongly monotone mapping, where L_f, $L_g > 0$. For every $i = 1, 2, ..., N$, let T_i be nonspreading mappings of C into itself

with $\mathbb{F} = \bigcap_{i=1}^{N} F(T_i) \cap \Omega_f \cap \Omega_g \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \neq \emptyset$. Let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, ..., N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mappings generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exists sequence $\{x_n\}, \{u_n\}, \{v_n\}, \{\overline{u}_n\}, \{\overline{v}_n\} \subseteq C$ and $\{w_n^1\}, \{w_n^2\} \subseteq H$ be sequence generated by

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C, s_n > 0, \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n P_C(I - r\nabla f)x_n + \delta_n P_C(I - s\nabla g)x_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \forall n \geq 1, \end{cases} \tag{5.5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\eta_n\}$ are a sequence in $[0, 1]$, $r \subset (0, \frac{2}{L_f})$ and $s \subset (0, \frac{2}{L_g})$, for every $n \in \mathbb{N}$ and suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there exists $\overline{\lambda}_1, \overline{\lambda}_2 > 0$ such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\overline{\lambda}_1\|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \leq -\overline{\lambda}_2\|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2, \end{cases} \tag{5.6}$$

for all $(r_1, r_2) \in \Theta \times \Theta$,$(s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$ and $\Xi = \{s_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}}x_1$.

(4) Let C be a nonempty bounded, closed, and convex subset of a real Hilbert space H and let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be lower semicontinuous and convex function. Let $D, T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ be equilibrium-like function satisfying $(H1) - (H3)$. Let $A : C \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be ab β-inverse strongly monotone mapping, let $F_j : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$ for all $j = 1, 2$, let T be quasi-nonexpansive mappings of C into itself with $\mathbb{F} = F(T) \cap EP(F_1, A) \cap EP(F_2, B) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \neq \emptyset$. Let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there

exist sequences $\{x_n\},\{u_n\},\{v_n\},\{\overline{u}_n\},\{\overline{v}_n\}\subseteq C$ and $\{w_n^1\},\{w_n^2\}\subseteq H$ generated by

$$\begin{cases} w_n^1\in T(x_n), \|w_n^1-w_{n+1}^1\|\le\left(1+\frac{1}{n}\right)\mathcal{H}(T(x_n),T(x_{n+1})),\\ w_n^2\in D(x_n), \|w_n^2-w_{n+1}^2\|\le\left(1+\frac{1}{n}\right)\mathcal{H}(D(x_n),D(x_{n+1})),\\ \Phi_1(w_n^1,u_n,u)+\varphi_1(u)-\varphi_1(u_n)+\frac{1}{r_n}\langle u_n-x_n,u-u_n\rangle\ge 0,\forall u\in C, r_n>0,\\ \Phi_2(w_n^2,v_n,v)+\varphi_2(v)-\varphi_2(v_n)+\frac{1}{s_n}\langle v_n-x_n,v-v_n\rangle\ge 0,\forall v\in C, s_n>0,\\ F_1(\overline{u}_n,\overline{u})+\langle Ax_n,\overline{u}-\overline{u}_n\rangle+\frac{1}{r}\langle\overline{u}-\overline{u}_n,\overline{u}_n-x_n\rangle\ge 0,\forall\overline{u}\in C,\\ F_2(\overline{v}_n,\overline{v})+\langle Bx_n,\overline{v}-\overline{v}_n\rangle+\frac{1}{s}\langle\overline{v}-\overline{v}_n,\overline{v}_n-x_n\rangle\ge 0,\forall\overline{v}\in C,\\ y_n=\alpha_n u_n+\beta_n v_n+\gamma_n\overline{u}_n+\delta_n\overline{v}_n+\eta_n Tx_n,\\ C_{n+1}=\{z\in C_n:\|y_n-z\|\le\|x_n-z\|\},\\ x_{n+1}=P_{C_{n+1}}x_1,\forall n\ge 1, \end{cases}\tag{5.7}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}$ and $\{\eta_n\}$ are sequence in $[0,1]$ for all $n\in\mathbb{N}$, $r_n,r\subset(0,2\alpha)$ and $s_n s\subset(0,2\beta)$ and suppose the following conditions hold:

(i) $\alpha_n+\beta_n+\gamma_n+\delta_n+\eta_n=1$,

(ii) $0<b<\alpha_n,\beta_n,\gamma_n,\delta_n,\eta_n\le c$, for some $b,c\in\mathbb{R}$,

(iii) there are $\overline{\lambda}_1,\overline{\lambda}_2>0$ such that

$$\begin{cases} \Phi_1(w_1^1,T_{r_1}(x_1),T_{r_2}(x_2))+\Phi_1(w_2^1,T_{r_2}(x_2),T_{r_1}(x_1))\le-\overline{\lambda}_1\|T_{r_1}(x_1)-T_{r_2}(x_2)\|^2,\\ \Phi_2(w_1^2,T_{s_1}(x_1),T_{s_2}(x_2))+\Phi_2(w_2^2,T_{s_2}(x_2),T_{s_1}(x_1))\le-\overline{\lambda}_2\|T_{s_1}(x_1)-T_{s_2}(x_2)\|^2, \end{cases}\tag{5.8}$$

for all $(r_1,r_2)\in\Theta\times\Theta$,$(s_1,s_2)\in\Xi\times\Xi$, $w_i^1\in T(x_i)$ and $w_i^2\in D(x_i)$, for $i=1,2$ where $\Theta=\{r_n:n\ge 1\}$ and $\Xi=\{s_n:n\ge 1\}$. Then $\{x_n\}$ converge strongly to $P_{\mathbb{F}}x_1$.

(5) Let C be a nonempty bounded, closed and convex subset of a real Hilbert space H and let $\varphi_1,\varphi_2:C\to\mathbb{R}$ be lower semicontinuous and convex functions. Let $D,T:C\to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ_1,μ_2, respectively, $\Phi_1,\Phi_2:H\times C\times C\to\mathbb{R}$ be equilibrium-like functions satisfying $(H1)-(H3)$. Let $A:C\to H$ be α-inverse strongly monotone mapping and $B:C\to H$ be β-inverse strongly monotone mapping, let $F_j:C\times C\to\mathbb{R}$ satisty $(A1)-(A4)$ for all $j=1,2$, let T_i be κ_i-demicontractive mappings of C into itself for all $i=1,2,...,N$. Defined $S_i x=ax+(1-a)T_i x$, $\forall x\in C$ and $a\in(\kappa_i,1)$ for all $i=1,2,...,N$ with $\mathbb{F}=\bigcap_{i=1}^N F(T_i)\cap EP(F_1,A)\cap EP(F_2,B)\cap(GEP)_s(\Phi_1,\varphi_1)\cap(GEP)_s(\Phi_2,\varphi_2)\neq\emptyset$. Let $\lambda_1,...,\lambda_N$ be real numbers such that $0<\lambda_i<1$ for every $i=1,...,N-1$ and $0<\lambda_N\le 1$. Let K be the K-mappings generated by $S_1,S_2,...,S_N$ and $\lambda_1,\lambda_2,...,\lambda_N$ and let $x_1\in C=C_1$ and $w_1^1\in T(x_1),w_1^2\in D(x_1)$, there exists sequence $\{x_n\},\{u_n\},\{v_n\},\{\overline{u}_n\},\{\overline{v}_n\}\subseteq C$ and $\{w_n^1\},\{w_n^2\}\subseteq H$ be sequence generated by

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \left(1+\frac{1}{n}\right)\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq \left(1+\frac{1}{n}\right)\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \forall u \in C, r_n > 0, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \forall v \in C, s_n > 0, \\ F_1(\overline{u}_n, \overline{u}) + \langle Ax_n, \overline{u} - \overline{u}_n\rangle + \frac{1}{r}\langle \overline{u} - \overline{u}_n, \overline{u}_n - x_n\rangle \geq 0, \forall \overline{u} \in C, \\ F_2(\overline{v}_n, \overline{v}) + \langle Bx_n, \overline{v} - \overline{v}_n\rangle + \frac{1}{s}\langle \overline{v} - \overline{v}_n, \overline{v}_n - x_n\rangle \geq 0, \forall \overline{v} \in C, \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n \overline{u}_n + \delta_n \overline{v}_n + \eta_n K x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases} \tag{5.9}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\eta_n\}$ are a sequence in $[0,1]$ for all $n \in \mathbb{N}$, $r \subset (0, 2\alpha)$ and $s \subset (0, 2\beta)$ and suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$,

(ii) $0 < b < \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n \leq c$, for some $b, c \in \mathbb{R}$,

(iii) there are $\overline{\lambda}_1, \overline{\lambda}_2 > 0$ such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\overline{\lambda}_1\|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2, \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \leq -\overline{\lambda}_2\|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2, \end{cases} \tag{5.10}$$

for all $(r_1, r_2) \in \Theta \times \Theta$,$(s_1, s_2) \in \Xi \times \Xi$, $w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$ and $\Xi = \{s_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}} x_1$.

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