FIXED POINT THEOREM OF GENERALIZED EQUILIBRIUM PROBLEM WITHOUT SOME CONDITIONS ON A QUASI-NONEXPANSIVE MAPPING

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หัวข้อวิทยานิพนธ์	ทฤษฎีบทจุดตรึงของปัญหาดุลยภาพทั่วไปโดยปราศจากบางเงื่อนไขสำหรับการส่งแบบกึ่งไม่ขยาย
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บทคัดย่อ

จุดประสงค์ของวิทยานิพนธ์นี้คือการแก้ปัญหาดุลยภาพทั่วไปซึ่งแนะนำโดย Ceng Ansari และ Yao[2] และแนะนำวิธีการใหม่สำหรับการหาสมาชิกร่วมของเซตของผลเฉลยของปัญหาเชิงดุลยภาพทั่วไปและเซตของจุดตรึงสำหรับการส่งแบบกึ่งไม่ขยาย โดยปราศจากเงื่อนไขครึ่งปิด และ $T_\omega := (1-\omega)I + \omega T$, เมื่อ T เป็นการส่งแบบกึ่งไม่ขยาย และ $\omega \in (0, \frac{1}{2})$ ในปริภูมิฮิลเบิร์ต โดยประยุกต์ทฤษฎีบทหลัก ผู้วิจัยได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มเกี่ยวกับวงศ์จำกัดของการส่งแบบไม่กระจายและ κ-การส่งแบบกึ่งหดเทียม

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Abstract

The purpose of this thesis is to solve the generalized equilibrium problem introduced by Ceng et al.[2] and introduce the new method for finding a common element of the set of solutions of such a problem and the set of fixed points of a quasi-nonexpansive mapping without demiclose condition and $T_\omega := (1-\omega)I + \omega T$, when T is a quasi-nonexpansive mapping and $\omega \in (0, \frac{1}{2})$ in a framework of a real Hilbert space. Moreover, using our main result, we prove strong convergence theorems involving a finite family of nonspreading mapping and κ-demicontractive mapping.

Keywords : quasi-nonexpansive mapping, equilibrium-like function, fixed point problem, generalized equilibrium problem.

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Anchalee Sripattanet

Table of Contents

Chapter 1

Introduction

1.1 Research Motivation

The fixed point theorem based on the contraction principle is applied usually in proving the existence and uniqueness of the solutions of some scalar and mainly functional equations. In the past few decades, fixed point theorem has been applied in many branches such as mathematics, engineering, economics, optimization, physics, and computer sciences, etc. For an interesting example, equilibriam or stability are fundamental concepts that can be described in terms of fixed points. For example, in economics, a Nash equilibrium of a game is a fixed point of the game's best response correspondence. However, in physics, more precisely in the theory of phase transitions, linearization near an unstable. Furthermore, many mathematicians have been studying about the structure of fixed point set.

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that the mapping $T : C \to C$ is called quasi-nonexpansive if $\|Tx - y\| \leq \|x - y\|$, for all $x \in C$ and $y \in F(T)$. We denote by $F(T)$ the set of fixed points of T. We now recall some well-known concepts and results as follows.

Definition 1.1.1. Let $f : C \to C$ be contractive if there exists a constant $\xi \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \xi\|x - y\|,$$

for all $x, y \in C$.

Let $CB(H)$ be the family of all nonempty closed bounded subsets of H and $\mathcal{H}(\cdot, \cdot)$ be the Hausdorff metric on $CB(H)$ defined as

$$\mathcal{H}(U, V) = max\left\{\sup_{u \in U} d(u, V), \sup_{v \in V} d(U, v)\right\},$$

$\forall U, V \in CB(H)$, where $d(u, V) = inf_{v \in V} d(u, v), d(U, v) = inf_{u \in U} d(u, v)$ and $d(u, v) = \|u - v\|$.

Let C be nonempty closed convex subset of H. Let $\varphi : C \to \mathbb{R}$ be real-valued function, $T : C \to CB(H)$ a multivalued mapping and $\Phi : H \times C \times C \to \mathbb{R}$ be equilibrium-like function, that is, $\Phi(w, u, v) + \Phi(w, v, u) = 0$ for all $(w, v, u) \in H \times C \times C$ which satisfies the following conditions with respect to the multivalued mapping $T : C \to CB(H)$;

$(H1)$ For each fixed $v \in C, (w, u) \mapsto \Phi(w, u, v)$ is an upper semicontinuous function from $H \times C$ into $\mathbb{R}$, that is, for $(w, u) \in H \times C$, whenever $w_n \to w$ and $u_n \to u$ as $n \to \infty$,

$$\limsup_{n \to \infty} \Phi(w_n, u_n, v) \leq \Phi(w, u, v).$$

$(H2)$ For each fixed $(w,v) \in H \times C,\ \ u \mapsto \Phi(w,u,v)$ is a concave function.

$(H3)$ For each fixed $(w,u) \in H \times C,\ \ v \mapsto \Phi(w,u,v)$ is a convex function.

Fixed point problems has been widely studied and developed in the literature.

In 2009, Ceng et al.[2] introduced the generlized equilibrium problem(GEP) as follows:

$$(GEP)\begin{cases} Find\ \ u \in C\ \ and\ \ w \in T(u)\ \ \text{such that} \\ \Phi(w,u,v) + \varphi(v) - \varphi(u) \geq 0, \forall v \in C. \end{cases}$$

The set of such solutions $u \in C$ of (GEP) is denoted by $(GEP)_s(\Phi,\varphi)$.

By using Nadler's theorem [10], they introduced the following algorithm:

Let $x_1 \in C$ and $w_1 \in T(x_1)$, there exist sequences $w_n \subseteq H$ and $\{x_n\},\{u_n\} \subseteq C$ such that

$$\begin{cases} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq \left(1+\frac{1}{n}\right)\mathcal{H}(T(x_n),T(x_{n+1})), \\ \Phi(w_n,u_n,v) + \varphi(v) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, v - u_n\rangle \geq 0, \forall v \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, n = 1,2,... \end{cases} \tag{1.1.1}$$

and proved the strong convergence theorem of the sequence $\{x_n\}$ generated by (1.1.1) as follows.

Theorem 1.1.1. [2] *Let C be a nonempty, bounded, closed and convex subset of a real Hilbert space H and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with coefficients μ, $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1)$–$H3)$ and S be a nonexpansive mapping of C into itself such that $F(T) \cap (GEP)_s(\Phi,\varphi) \neq \emptyset$. Let f be a contraction of C into itself and let $\{x_n\},\{w_n\}$ and $\{u_n\}$ be a sequence generated by (1.1.1), where $\{\alpha_n\} \subseteq [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy*
$\lim_{n\to\infty} \alpha_n = 0, \sum_{i=1}^{\infty} \alpha_n = \infty \sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \liminf_{n\to\infty} r_n > 0$ *and* $\sum_{i=1}^{\infty} |r_{n+1} - r_n| < \infty.$
If there exists a constant $\lambda > 0$ such that

$$\Phi(w_1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi(w_2, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\lambda\|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2,$$

for all $(r_1,r_2) \in \Xi \times \Xi, (x_1,x_2) \in C \times C$ and $w_i \in T(x_i),\ i = 1,2,$ where $\Xi = r_n : n \geq 1,$ then for $\hat{x} = P_{F(s)\cap(GEP)_s(\Phi,\varphi)}f(\hat{x})$, there exists $\hat{w} \in T(\hat{x})$ such that $(\hat{x},\hat{w})$ is a solution of (GEP) and $x_n \to \hat{x}$, $w_n \to \hat{w}$ and $u_n \to \hat{x}$ as $n \to \infty$.

In 2011, Tian and Jin[16] proved the following strong convergence theorem of iterative scheme $\{x_n\}$ generated by (1.1.2)

Theorem 1.1.2. *Let H be a real Hilbert space, let F be a κ-Lipschitzian and η-strongly monotone operator on H with $k > 0, \eta > 0$, and let T be a quasi-nonexpansive mapping on H, and f is a L-Lipschitzian mapping with coefficient $L > 0$. Assume the set Fix(T) of fixed points of T is nonempty. Let $0 < \mu < 2\eta/k^2, 0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/L =$*

τ/L, and start with an arbitrary chosen $x_0 \in H$, let the sequence$\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F)T_\omega x_n, \tag{1.1.2}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Also $\omega \in (0, \frac{1}{2})$, $T_\omega := (1-\omega)I + \omega I$ with two conditions on T:

$(C1)$ $\|Tx - q\| \leq \|x - q\|$ for any $x \in H$, and $q \in Fix(T)$; this means that T is a quasi-nonexpansive mapping;

$(C2)$ T is demi-closed on H; that is: if $\{y_k\} \in H$, $y_k \rightharpoonup z$, and $(I-T)y_k \to 0$, then $z \in Fix(T)$.
Then, $\{x_n\}$ converges strongly to the $x^* \in Fix(T)$ which is the unique solution of the VIP:

$$\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in Fix(T).$$

Remark 1.1.3. From theorem 1.1.2, they proved strong convergence theorem for a quasi-nonexpansive mapping by using conditions:

(i) $T_\omega := (1-\omega)I + \omega T$,

(ii) T is demiclosed on H. where T is a quasi-nonexpansive mapping and for all $\omega \in (0, \frac{1}{2})$.

Many authors proved strong convergence theorem involving a quasi-nonexpansive mapping T by using conditions (i) and (ii). See for example [9],[12] and [15].

In 2015, Cheawchan and Kangtunyakarn[3] introduced the new method for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a variational inequalities without the conditions (i) and (ii) in a framework of a real Hilbert space as follows:

Theorem 1.1.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be $\alpha, \beta-$inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $Gx = P_C(I - \lambda_1 A)(ax + (1-a)P_C(I - \lambda_2 B)x)$ for all $x \in C$. Assume $\mathcal{F} = VI(C, A) \cap VI(C, B) \cap F(T) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n G x_n, \forall n \geq 1,$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ and $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions holds:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{i=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < a \leq \beta_n \leq c < 1$ *for all* $n \geq 1$;

(iv) $\sum\limits_{i=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$;

(v) $\sum_{i=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{i=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{i=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Then $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}}u$.

From motivated of Theorem1.1.1 and 1.1.4, we proved the strong convergence theorem for finding a common element of the set of solution of a finite family of the generalized equilibrium problem and the set of fixed point of a quasi-nonexpansive mapping without the conditions (i) and (ii). Moreover, using our main result, we obtain the additional results involving a finite family of a nonspreading mapping and κ-demicontractive mapping.

1.2 Objectives of the study

1) To prove a strong convergence theorems for finding a common element of the set of solution of a finite family of the generalized equilibrium problem and the set of fixed point of a quasi-nonexpansive mapping without some conditions.
2) To prove a strong convergence theorems for equilibrium problem and the set of fixed point of a nonspreading mapping and κ-demicontractive mapping
3) To give some method for fixed point theory of a finite family of nonspreading mappings

1.3 Scopes of the study

1) Study equilibrium problems in a real Hilbert space.
2) Investigate the fixed point problems of nonlinear mappings including a quasi-nonexpansive mapping, nonspreading mapping and κ-demicontractive mapping in a real Hilbert space.
3) All strong convergence theorems are considered and proved in a real Hilbert space.

1.4 Benefits of the Study

1) Obtain the strong convergence theorem for finding a common element of the set of solution of a finite family of the generalized equilibrium problem and the set of fixed point of a quasi-nonexpansive mapping.
2) Obtain a strong convergence theorem for finding a common element of the set of solution of a finite family of nonspreading mapping.
3) Obtain a strong convergence theorem for finding a common element of the set of solution of a finite family of κ-demicontractive mapping.

1.5 Research methodology

1) Study advanced topics in fixed point theory for a quasi-nonexpansive mapping.
2) Study background in a real Hilbert space.
3) Collect and study research papers concerning fixed point theorem.
4) Determine the objecttives and scope of the research.
5) Produce tools for theorem of fixed point theorem.
6) Prove the strong convergence theorem for finding a common element of the set of solution of a finite family of the generalized equilibrium problem and the set of fixed point of a quasi-nonexpansive mapping.
7) Provide applications.
8) Conclude the result, make suggestions for further works and write the thesis.

Chapter 2

Preliminaries

2.1 Fundamental properties

Lemma 2.1.1. Let H be a real Hilbert space. Then the following identities hold:

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H;$

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H.$

Definition 2.1.1 (Strong convergence [8]). A sequence $\{x_n\}$ of vectors in an inner product space K is called strongly convergent to a vector x in K if

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

Definition 2.1.2 (Weak convergence [8]). A sequence $\{x_n\}$ of vectors in an inner product space K is called weakly convergent to a vector x in K if

$$\langle x_n, y\rangle \to \langle x, y\rangle \text{ as } n \to \infty \text{ for every } y \in K.$$

Theorem 2.1.2 ([8]). *A strongly convergence sequence is weakly convergence (to the same limit), that is,* $x_n \to x$ *implies* $x_n \rightharpoonup x$.

Remark 2.1.3 ([14]). If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Lemma 2.1.4 ([14]). Let $\{x_n\}$ be a Cauchy sequence of an inner product space C such that $x_n \rightharpoonup x$. Then $x_n \to x$.

Theorem 2.1.5 ([14]). *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H*. Suppose that* $\{x_n\} \subset C$ *and* $x_n \rightharpoonup x$*. Then* $x \in C$.

Lemma 2.1.6 ([11]). Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{u_n\} \subset H$ with $u_n \rightharpoonup u$, the inequality

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

holds for every $v \in H$ with $v \neq u$.

Theorem 2.1.7 ([14]). *Let* $\{a_n\}$ *be a bounded of real numbers. Then, there exists subsequence* $\{a_{n_i}\}$ *of* $\{a_n\}$ *such that*

$$\alpha = \limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{n_i}.$$

Similarly, there exists a subsequence $\{a_{n_j}\}$ *of* $\{a_n\}$ *such that*

$$\beta = \liminf_{n\to\infty} a_n = \lim_{j\to\infty} a_{n_j}.$$

Definition 2.1.3 ([14]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called lower semicontinuous if for any $a \in \mathbb{R}$, the set

$$\{x \in C : f(x) \leq a\} \text{ is closed.}$$

Moreover, f is called convex if for any $x_1, x_2 \in C$ and $t \in (0, 1)$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Similarly, f is said to be concave if for any $x_1, x_2 \in C$ and $t \in (0, 1)$,

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2).$$

Definition 2.1.4 (Metric projection [14]). The (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.1.8 ([13]). For a given $z \in H$ and $u \in C$,

$$u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \forall v \in C.$$

It is well-known that P_C is a firmly nonexpansive mapping of H onto C, that is

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \forall x, y \in H.$$

Lemma 2.1.9 ([17]). Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that
(i) $\sum_{i=1}^{\infty} \alpha_n = \infty$;
(ii) lim $sup_{n \to \infty} \dfrac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} \delta_n < \infty$.
Then $\lim_{n \to \infty} s_n = 0$.

2.2 Fixed point sets of quasi-nonexpansive mapping, nonspreading mapping and κ-demicontractive mapping with some properties

Let X be a nonempty set and $T : X \to X$. We say that $x \in X$ is a fixed point of T if and only if $Tx = x$ and $Fix(T)$ represents the set of all fixed points of T, i.e.,

$$F(T) = \{x \in C : Tx = x\}.$$

Example 2.2.1 ([1]). Let $X = \mathbb{R}$.

1) If $T(x) = x$, then $Fix(T) = \mathbb{R}$;

2) If $T(x) = x + 2$, then $Fix(T) = \emptyset$;

3) If $T(x) = x^2 + 5x + 4$, then $Fix(T) = \{-2\}$.

Definition 2.2.1. [13] Let C be a nonempty closed convex subset of H and let A be an operator of C into H. Consider the following problem: Find $x \in C$ such that

$$\langle Ax, u - x\rangle \geq 0, \quad for \ \ all \ \ u \in C.$$

Such an $x \in C$ is called a solution of the variational inequality of A. We denote $VI(C, A)$ the set of all solutions of the variational inequality of A.

Lemma 2.2.2. [13] Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let D_1 be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,

$$u = P_C(I - \lambda D_1)u \Leftrightarrow u \in VI(C, D_1),$$

where P_C is the metric projection of H onto C.

Lemma 2.2.3. [3] Let H be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then $VI(C, I-T) = F(T)$.

Remark 2.2.4. [3] From Lemmas 2.2.2 and 2.2.3, we have

$$F(T) = VI(C, I - T) = F(P_C(I - \lambda(I - T))),$$

for all $\lambda > 0$.

Definition 2.2.2. Let C be a nonempty subset of H, a mapping $T : C \to C$ is called

1) κ-demicontractive if there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Tx^*\|^2 \leq \|x - x^*\|^2 + k\|(I - T)x\|^2,$$

2) nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2,$$

for all $x, y \in C$ and $x^* \in F(T)$.

A nonspreading mapping T is introduced by Kohsaka and Takahashi[7]. In 2009, Iemoto and Takahashi[4] proved that 2) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \forall x, y \in C. \qquad (2.2.1)$$

Theorem 2.2.5 ([7]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a nonspreading mapping of C into itself. Then $Fix(T)$ is closed and convex.*

Remark 2.2.6. From (2.2.1), if T is a nonspreading mapping with $Fix(T) \neq \emptyset$, then T is quasi-nonexpansive.

In 2009, Kangtunyakarn and Suantai[6] introduced the K-mapping generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$ as follows;

Definition 2.2.3. Let C be a nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of mappings of C into itself, and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, ..., N$. Define a mapping $K : C \to C$ as follows:

$$\begin{aligned}
U_1 &= \lambda_1 T_1 + (1-\lambda_1)I, \\
U_2 &= \lambda_2 T_2 U_1 + (1-\lambda_2)U_1, \\
U_3 &= \lambda_3 T_3 U_2 + (1-\lambda_3)U_2, \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1})U_{N-2}, \\
K = U_N &= \lambda_N T_N U_{N-1} + (1-\lambda_N)U_{N-1}. \qquad (2.2.2)
\end{aligned}$$

Such a mapping K is called the K-mapping generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$.

Lemma 2.2.7. [5] Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, ..., N-1$. and $0 \leq \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$. Then $F(K) = \bigcap_{i=1}^{N} F(T_i)$ and K is a quasi-nonexpansive mapping.

2.3 Generalized equilibrium problem in a real Hilbert spaces

In this section, we give some useful lemma and theorem for equilibrium problems to prove the main results.

Definition 2.3.1. A mulitivalued mapping $T : C \to CB(H)$ is said to be $\mathcal{H}$-Lipschitz continuous if there exists a constant $\mu > 0$ such that

$$\mathcal{H}(T(u), T(v)) \leq \mu \|u - v\|, \quad \forall u, v \in C,$$

where $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$.

Lemma 2.3.1. [10] Let $(X, \|\cdot\|)$ be a normed vector space and $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$. If $U, V \in CB(H)$, then for every $\varepsilon > 0$ and $u \in U$, there exists $v \in V$ such that

$$\|u - v\| \leq (1 + \varepsilon)\mathcal{H}(U, V).$$

Theorem 2.3.2. *[2] Let C be a nonempty, bounded, closed, and convex subset of a real Hilbert space H, and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with coefficients μ, and $\Phi_i : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1) - H3)$. Let $r > 0$ be a constant. For each $x \in C$, take $w_x \in T(x)$ arbitrarily and define a mapping $T_r : C \to C$ as follow:*

$$T_r(x) = \left\{ u \in C : \Phi(w_x, u, v) + \varphi(v) - \varphi(u) + \frac{1}{r}\langle u - x, v - u\rangle \geq 0, \forall v \in C \right\}.$$

Then we have the following:

(a) T_r is single-valued;

(b) T_r is firmly nonexpansive(that is, for any $u, v \in C, \|T_r u - T_r v\|^2 \leq \langle T_r u - T_r v, u - v\rangle$) if

$$\Phi(\bar{w}_1, T_r(x_1), T_r(x_2)) + \Phi(\bar{w}_2, T_r(x_2), T_r(x_1)) \leq 0,$$

for all $(x_1, x_2) \in C \times C$ and all $\bar{w}_i \in T(x_i)$, $i = 1, 2$;

(c) $F(T_r) = (GEP)_s(\Phi, \varphi)$;

(d) $(GEP)_s(\Phi, \varphi)$ is closed and convex.

Chapter 3

Main result

In this chapter, we introduce and study the new iterative methods for fixed point theorem of nonlinear mappings in Hilbert space.

3.1 Convergence theorem for finding a common element of the set of solution of a finite family of the generalized equilibrium problem and the set of fixed point of a quasi-nonexpansive mapping

In this section, we introduce a strong convergence theorem for finding a common element of the set of solution of a finite family of the generalized equilibrium problem and the set of fixed point of a quasi-nonexpansive mapping without demiclose condition and $T_\omega := (1-\omega)I + \omega T$, when T is a quasi-nonexpansive mapping and $\omega \in (0, \frac{1}{2})$.

Theorem 3.1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $S_i : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with coefficients μ_i and let $\Phi_i : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1) - H3)$. Let $\varphi : C \to R$ be lower semicontinuous and convex function. Let $T : C \to C$ be a quasi-nonexpansive mapping with $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} (GEP)_s(\Phi_i, \varphi) \neq \emptyset$. Let $\{x_n\}$ be sequence generated by $x_1 \in C$, $w_1^i \in S_i x_1$ and there exists sequence $\{w_n^i\} \subseteq H$ such that*

$$\begin{cases} \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n})\mathcal{H}(S_i x_n, S_i x_{n+1}), w_n^i \in S_i x_n, \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r}\langle u_n^i - x_n, y - u_n^i \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i, \\ \forall n \geq 1 \;\; \text{and} \;\; i = 1, 2, 3, ..., N, \end{cases} \tag{3.1.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $f : C \to C$ be ξ-contraction mapping. Suppose the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{i=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ *for all* $n \geq 1$ *and* $\sum_{i=1}^{N} a_i = 1$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$ *for all* $n \geq 1$;

(iv) $r \in (0, 2\alpha)$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(vi) for each $i = 1, 2, ..., N$, *there exists* $\rho_i > 0$ *such that*

$$\Phi_i(\bar{w}_1^i, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_i(\bar{w}_2^i, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\rho_i \|T_r(x_1) - T_r(x_2)\|^2;$$

for all $(r_1, r_2) \in E \times E$, $(x_1, x_2) \in C \times C$ *and* $\bar{w}_j^i \in S_i(x_j)$, *for all* $j = 1, 2$, *where* $E = \{r\}$.
Then $\{x_n\}$ *converges strongly to* $z = P_{\mathcal{F}} f(z)$.

Proof. It follows from condition(vi) that for each $r_1, r_2 \in E$, we have

$$\Phi_i(\bar{w}_1^i, T_r(x_1), T_r(x_2)) + \Phi_i(\bar{w}_2^i, T_r(x_2), T_r(x_1)) \leq -\rho_i \|T_r(x_1) - T_r(x_2)\|^2 \leq 0,$$

for all $(x_1, x_2) \in C \times C$ and $\bar{w}_j^i \in S_i(x_j)$, $j = 1, 2$.
Hence all conclusions$(a) - (d)$ of Theorem 2.3.2 hold.
We divide the remainder of the proof into six steps.

Step 1. We will show that $\{x_n\}$ is bounded.
Let $z \in \mathcal{F}$, we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n f(x_n) + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i - z\| \\
&\leq \alpha_n \|f(x_n) - z\| + \beta_n \|P_C(I - \lambda_n(I - T))x_n - z\| + \gamma_n \|\sum_{i=1}^{N} a_i u_n^i - z\| \\
&\leq \alpha_n (\|f(x_n) - f(z)\| + \|f(z) - z\|) + \beta_n \|P_C(I - \lambda_n(I - T))x_n - z\| \\
&\quad + \gamma_n \sum_{i=1}^{N} a_i \|u_n^i - z\|. \qquad (3.1.2)
\end{aligned}$$

Consider:

$$\begin{aligned}
\|P_C(I - \lambda_n(I - T))x_n - z\|^2 &= \|P_C(I - \lambda_n(I - T))x_n - P_C z\|^2 \\
&\leq \|(I - \lambda_n(I - T))x_n - z\|^2 \\
&= \|x_n - z - \lambda_n(I - T)x_n\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n \langle x_n - z, (I - T)x_n \rangle \\
&\quad + \lambda_n^2 \|(I - T)x_n\|^2. \qquad (3.1.3)
\end{aligned}$$

Consider, the result below,

$$\begin{aligned}
\|Tx - y\|^2 &= \|x - y - (I - T)x\|^2 \\
&= \|x - y\|^2 - 2\langle x - y, (I - T)x \rangle + \|(I - T)x\|^2 \\
&\leq \|x - y\|^2,
\end{aligned}$$

for all $x \in C$ and $y \in F(T)$. It implies that

$$\|(I - T)x\|^2 \leq 2\langle x - y, (I - T)x \rangle, \qquad (3.1.4)$$

From (3.1.3) and (3.1.4), we have

$$\begin{aligned}
\|P_C(I - \lambda_n(I - T))x_n - z\|^2 &\leq \|x_n - z\|^2 - \lambda_n \|(I - T)x_n\|^2 + \lambda_n^2 \|(I - T)x_n\|^2 \\
&= \|x_n - z\|^2 - \lambda_n(1 - \lambda_n)\|(I - T)x_n\|^2 \\
&\leq \|x_n - z\|^2. \qquad (3.1.5)
\end{aligned}$$

From (3.1.2) and (3.1.5) and $u_n^i = T_r^i x_n$, we have

$$\begin{aligned}
\|x_{n+1} - z\| &\leq \alpha_n(\|f(x_n) - f(z)\| + \|f(z) - z\|) + \beta_n\|x_n - z\| + \gamma_n \sum_{i=1}^{N} a_i\|u_n^i - z\| \\
&\leq \alpha_n(\xi\|x_n - z\| + \|f(z) - z\|) + \beta_n\|x_n - z\| + \gamma_n \sum_{i=1}^{N} a_i\|T_r^i x_n - z\| \\
&\leq \alpha_n(\xi\|x_n - z\| + \|f(z) - z\|) + \beta_n\|x_n - z\| + \gamma_n\|x_n - z\| \\
&= (1 - \alpha_n(1 - \xi))\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\leq max\left\{\|x_1 - z\|, \frac{\|f(z) - z\|}{1 - \xi}\right\}.
\end{aligned}$$

By induction, we have $\|x_n - z\| \leq max\left\{\|x_1 - z\|, \dfrac{\|f(z) - z\|}{1 - \xi}\right\}$, $\forall n \in \mathbb{N}$.
It follows that $\{x_n\}$ is bounded and so is $\{u_n^i\}, \forall i = 1, 2, ..., N$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. By the definition of x_n, we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i \\
&\qquad - \alpha_{n-1}f(x_{n-1}) - \beta_{n-1}P_C(I - \lambda_{n-1}(I - T))x_{n-1} \\
&\qquad - \gamma_{n-1}\sum_{i=1}^{N} a_i u_{n-1}^i\| \\
&\leq \alpha_n\|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\qquad + \gamma_n\|\sum_{i=1}^{N} a_i u_n^i - \sum_{i=1}^{N} a_i u_{n-1}^i\| + |\gamma_n - \gamma_{n-1}|\|\sum_{i=1}^{N} a_i u_{n-1}^i\| \\
&\qquad + \beta_n\|P_C(I - \lambda_n(I - T))x_n - P_C(I - \lambda_{n-1}(I - T))x_{n-1}\| \\
&\qquad + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda_{n-1}(I - T))x_{n-1}\| \\
&\leq \alpha_n\xi\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\qquad + \gamma_n \sum_{i=1}^{N} a_i\|u_n^i - u_{n-1}^i\| + |\gamma_n - \gamma_{n-1}|\sum_{i=1}^{N} a_i\|u_{n-1}^i\| \\
&\qquad + \beta_n\|x_n - x_{n-1}\| + \lambda_n\|(I - T)x_n\| + \lambda_{n-1}\|(I - T)x_{n-1}\| \\
&\qquad + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda_{n-1}(I - T))x_{n-1}\| \\
&\leq \alpha_n\xi\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\qquad + \gamma_n\|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\sum_{i=1}^{N} a_i\|u_{n-1}^i\| \\
&\qquad + \beta_n\|x_n - x_{n-1}\| + \lambda_n\|(I - T)x_n\| + \lambda_{n-1}\|(I - T)x_{n-1}\| \\
&\qquad + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda_{n-1}(I - T))x_{n-1}\| \\
&= (1 - \alpha_n(1 - \xi))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\qquad + |\gamma_n - \gamma_{n-1}|\sum_{i=1}^{N} a_i\|u_{n-1}^i\| \\
&\qquad + \lambda_n\|(I - T)x_n\| + \lambda_{n-1}\|(I - T)x_{n-1}\|
\end{aligned}$$

$$+ |\beta_n - \beta_{n-1}| \|P_C(I - \lambda_{n-1}(I - T))x_{n-1}\|$$

Applying the conditions $(i), (iii), (v)$ and Lemma 2.1.9, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.1.6}$$

Step 3. We will show that

$$\lim_{n\to\infty} \|u_n^i - x_n\| = \lim_{n\to\infty} \|P_C(I - \lambda_n(I - T))x_n - x_n\| = 0, \forall i = 1, 2, 3, ..., N.$$

Since T_r^i is a firmly nonexpansive mapping, for all $i = 1, 2, ..., N$, we obtain

$$\begin{aligned} \|T_r^i x_n - z\|^2 &= \|T_r^i x_n - T_r^i z\|^2 \\ &\leq \langle x_n - z, u_n^i - z\rangle \\ &= \frac{1}{2}(\|x_n - z\|^2 + \|u_n^i - z\|^2 - \|x_n - u_n^i\|^2), \end{aligned}$$

which implies that

$$\|u_n^i - z\|^2 \leq \|x_n - z\|^2 - \|x_n - u_n^i\|^2. \tag{3.1.7}$$

From the definition of x_n and (3.1.7), we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \alpha_n \|f(x_n) - z\|^2 + \beta_n \|P_C(I - \lambda_n(I - T))x_n - z\|^2 \\ &\quad + \gamma_n \sum_{i=1}^N a_i \|u_n^i - z\|^2 \\ &\leq \alpha_n \|f(x_n) - z\|^2 + \beta_n \|P_C(I - \lambda_n(I - T))x_n - z\|^2 \\ &\quad + \gamma_n \sum_{i=1}^N a_i (\|x_n - z\|^2 - \|u_n^i - x_n\|^2) \\ &\leq \alpha_n \|f(x_n) - z\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \sum_{i=1}^N a_i \|x_n - z\|^2 \\ &\quad - \gamma_n \sum_{i=1}^N a_i \|u_n^i - x_n\|^2 \\ &\leq \alpha_n \|f(x_n) - z\|^2 + \|x_n - z\|^2 - \gamma_n \sum_{i=1}^N a_i \|u_n^i - x_n\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \gamma_n \sum_{i=1}^N a_i \|u_n^i - x_n\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n \|f(x_n) - z\|^2 \\ &\leq (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - z\|^2. \end{aligned} \tag{3.1.8}$$

From (3.1.6), (3.1.8) and the condition (i) and (ii), we obtain

$$\lim_{n\to\infty} \|u_n^i - x_n\| = 0. \tag{3.1.9}$$

for all $i = 1, 2, 3, ..., N$.

By the definition of x_n, we obtain

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n f(x_n) + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i - x_n \\ &= \alpha_n(f(x_n) - x_n) + \beta_n(P_C(I - \lambda_n(I - T))x_n - x_n) \\ &\quad + \gamma_n \sum_{i=1}^{N} a_i(u_n^i - x_n). \end{aligned}$$

From (3.1.6), (3.1.9) and the condition (i) and (ii), we get

$$\lim_{n\to\infty} \|P_C(I - \lambda_n(I - T))x_n - x_n\| = 0.$$

Step 4. We will show that $\{x_n\}$, $\{w_n^i\}$ are Cauchy sequences, for every $i = 1, 2, 3, ..., N$.

Let $a \in (0, 1)$, by (3.1.6), there exists $N \in \mathbb{N}$ such that

$$\|x_{n+1} - x_n\| < a^n, \forall n \geq N. \tag{3.1.10}$$

Thus, for any $n \geq N$ and $p \in \mathbb{N}$, we have

$$\|x_{n+p} - x_n\| \leq \sum_{k=n}^{n+p-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{n+p-1} a^k < \frac{a^n}{1-a}. \tag{3.1.11}$$

Since $a \in (0, 1)$, we get $\lim_{n\to\infty} a^n = 0$. From (3.1.11), we obtain $\{x_n\}$ is a Cauchy sequence in a Hilbert space H. Then there is $x^* \in C$ such that $\lim_{n\to\infty} x_n = x^*$.

Since $S_i : C \to CB(H)$ be $\mathcal{H}$-*Lipschitz* continuous on H with coefficients μ_i, for every $i = 1, 2, 3, ..., N$ and (3.1.1), we have

$$\begin{aligned} \|w_n^i - w_{n+1}^i\| &\leq \left(1 + \frac{1}{n}\right) \mathcal{H}(S_i x_n, S_i x_{n+1}) \\ &\leq \left(1 + \frac{1}{n}\right) \mu_i \|x_n - x_{n+1}\|. \end{aligned} \tag{3.1.12}$$

From (3.1.6) and (3.1.12), we obtain

$$\lim_{n\to\infty} \|w_n^i - w_{n+1}^i\| = 0,$$

for every $i = 1, 2, 3, ..., N$.

By continuing the same argument as (3.1.10) and (3.1.11), we have $\{w_n^i\}$ is a Cauchy sequence in a Hilbert H. Then there is $w_i^* \in H$, for all $i = 1, 2, ..., N$ such that $\lim_{n\to\infty} w_n^i = w_i^* \in C$, for every $i = 1, 2, 3, ..., N$.

Next, we will prove that $w_i^* \in S_i x^*$, for all $i = 1, 2, 3, ..., N$. Since $w_n^i \in S_i x_n$, we obtain

$$\begin{aligned} d(w_n^i, S_i x^*) &\leq max\left\{d(w_n^i, S_i x^*), \sup_{\tilde{w}_i \in S_i x^*} d(S_i x_n, \tilde{w}_i)\right\} \\ &\leq max\left\{\sup_{\hat{w}_i \in S_i x_n} d(\hat{w}_i, S_i x^*), \sup_{\tilde{w}_i \in S_i x^*} d(S_i x_n, \tilde{w}_i)\right\} \end{aligned}$$

$$= \mathcal{H}(S_i x_n, S_i x^*), \quad \text{for every} \quad i = 1, 2, 3, ..., N. \tag{3.1.13}$$

From (3.1.13), we have

$$\begin{aligned} d(w_i^*, S_i x^*) &\leq \|w_i^* - w_n^i\| + d(w_n^i, S_i x^*) \\ &\leq \|w_i^* - w_n^i\| + \mathcal{H}(S_i x_n, S_i x^*) \\ &= \|w_i^* - w_n^i\| + \mu_i \|x_n - x^*\|, \end{aligned}$$

for all $i = 1, 2, ..., N$.

Taking $n \to \infty$, we have

$$d(w_i^*, S_i x^*) = 0,$$

which implies that

$$w_i^* \in S_i x^*, \quad \text{for all} \quad i = 1, 2, 3, ..., N. \tag{3.1.14}$$

Step 5. We will show that $\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle \leq 0$, where $z = P_{\mathcal{F}} f(z)$. To show this, choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle = \lim_{j\to\infty} \langle f(z) - z, x_{n_j} - z\rangle.$$

Without loss of generality we can assume that $x_{n_j} \rightharpoonup x^*$ as $j \to \infty$.

Claim that $x^* \in \mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} (GEP)_s(\Phi_i, \varphi)$.

From Remark 2.2.4, we have

$$F(T) = F(P_C(I - \lambda_{n_j}(I - T))).$$

Assume that $x^* \notin F(T)$, we have $x^* \neq P_C(I - \lambda_{n_j}(I - T))x^*$. By Opial's property and the condition (iii), we have

$$\begin{aligned} \liminf_{j\to\infty} \|x_{n_j} - x^*\| &< \liminf_{j\to\infty} \|x_{n_j} - P_C(I - \lambda_{n_j}(I - T))x^*\| \\ &= \liminf_{j\to\infty} \|x_{n_j} + P_C(I - \lambda_{n_j}(I - T))x_{n_j} \\ &\qquad - P_C(I - \lambda_{n_j}(I - T))x_{n_j} - P_C(I - \lambda_{n_j}(I - T))x^*\| \\ &\leq \liminf_{j\to\infty} (\|x_{n_j} - x^*\| + \lambda_{n_j} \|(I - T)x^* - (I - T)x_{n_j}\| \\ &\qquad + \|x_{n_j} - P_C(I - \lambda_{n_j}(I - T)x_{n_j}\|) \\ &= \liminf_{j\to\infty} \|x_{n_j} - x^*\|. \end{aligned}$$

This is a contradiction. So we have

$$x^* \in F(T). \tag{3.1.15}$$

Next, we show that $x^* \in \bigcap_{i=1}^{N}(GEP)_s(\Phi_i,\varphi)$.
Since $x_{n_j} \to x^*$ as $j \to \infty$ and (3.1.9), we have

$$u^i_{n_j} \to x^* as \;\; j \to \infty, \quad \text{for every} \quad i = 1,2,3,...,N. \tag{3.1.16}$$

From (3.1.1), we obtain

$$\Phi_i(w^i_{n_j}, u^i_{n_j}, y) + \varphi(y) - \varphi(u^i_{n_j}) + \frac{1}{r}\langle u^i_{n_j} - x_{n_j}, y - u^i_{n_j}\rangle \geq 0,$$

for every $y \in C$ and $i = 1,2,3,...,N$. From (3.1.9), (3.1.16), the condition (H1), and the lower semicontinuity of φ, we get

$$\Phi_i(w^*_i, x^*, y) + \varphi(y) - \varphi(x^*) \geq 0,$$

for every $y \in C$ and $i = 1,2,3,...,N$.
From (3.1.14) that

$$x^* \in (GEP)_s(\Phi_i,\varphi),$$

for every $i = 1,2,3,...,N$.
It implies that

$$x^* \in \bigcap_{i=1}^{N}(GEP)_s(\Phi_i,\varphi). \tag{3.1.17}$$

From (3.1.15) and (3.1.17), we have

$$x^* \in \mathcal{F}.$$

Then, we have

$$\lim_{n\to\infty} sup\langle f(z) - z, x_n - z\rangle = \lim_{j\to\infty}\langle f(z) - z, x_{n_j} - z\rangle = \langle f(z) - z, x^* - z\rangle \leq 0, \tag{3.1.18}$$

where $z = P_{\mathcal{F}}f(z)$.

Step 6. Finally, we will prove that $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}f(z)$, for every $i = 1,2,3,...,N$. By Lemma 2.1.1, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \Big\|\alpha_n(f(x_n) - z) + \beta_n(P_C(I - \lambda_n(I-T))x_n - z) \\
&\qquad + \gamma_n \sum_{i=1}^{N} a_i(u^i_n - z)\Big\|^2 \\
&\leq \Big\|\beta_n(P_C(I - \lambda_n(I-T))x_n - z) + \gamma_n \sum_{i=1}^{N} a_i(u^i_n - z)\Big\|^2 \\
&\qquad + 2\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle \\
&\leq \left(\beta_n\|(P_C(I - \lambda_n(I-T))x_n - z)\| + \gamma_n\Big\|\sum_{i=1}^{N} a_i(u^i_n - z)\Big\|\right)^2 \\
&\qquad + 2\alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle
\end{aligned}$$

$$
\begin{aligned}
&\le ((1-\alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\xi\|x_n - z\|\|x_{n+1} - z\| \\
&\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\le (1-\alpha_n)^2\|x_n - z\|^2 + \alpha_n\xi(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\le \frac{(1-\alpha_n)^2 + \alpha_n\xi}{1-\alpha_n\xi}\|x_n - z\|^2 + \frac{2\alpha_n}{1-\alpha_n\xi}\langle f(z) - z, x_{n+1} - z\rangle \\
&= \frac{1-\alpha_n\xi - 2\alpha_n(1-\xi)}{1-\alpha_n\xi}\|x_n - z\|^2 \\
&\quad + \frac{\alpha_n^2}{1-\alpha_n\xi}\|x_n - z\|^2 + \frac{2\alpha_n}{1-\alpha_n\xi}\langle f(z) - z, x_{n+1} - z\rangle \\
&= \left(1 - \frac{2\alpha_n(1-\xi)}{1-\alpha_n\xi}\right)\|x_n - z\|^2 \\
&\qquad + \frac{2\alpha_n(1-\xi)}{1-\alpha_n\xi}\left(\frac{\alpha_n}{2(1-\xi)}\|x_n - z\|^2 + \frac{1}{1-\xi}\langle f(z) - z, x_{n+1} - z\rangle\right).
\end{aligned}
$$

Applying the condition (i), (3.1.18) and Lemma 2.1.9, we have the sequence $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}f(z)$. This completes the proof. $\square$

The following corollary is directed results form Theorem 3.1.1.

Corollary 3.1.2. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $S_i : C \to CB(H)$ be $\mathcal{H}$-*Lipschitz* continuous with coefficients μ_i and let $\Phi_i : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1) - H3)$. Let $\varphi : C \to R$ be lower semicontinuous and convex function. Let $T : C \to C$ be a quasi-nonexpansive mapping with $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N}(GEP)_s(\Phi_i, \varphi) \neq \emptyset$. Let $\{x_n\}$ be sequence generated by $u, x_1 \in C$, $w_1^i \in S_i x_1$ and there exists sequence $\{w_n^i\} \subseteq H$ such that

$$
\begin{cases}
\|w_n^i - w_{n+1}^i\| \le (1 + \frac{1}{n})\mathcal{H}(S_i x_n, S_i x_{n+1}), w_n^i \in S_i x_n, \\
\Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r}\langle u_n^i - x_n, y - u_n^i\rangle \ge 0, \forall y \in C, \\
x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i, \\
\forall n \ge 1 \text{ and } i = 1, 2, 3, ..., N,
\end{cases}
\tag{3.1.19}
$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \ge 1$. Suppose the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{i=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < c \le \beta_n, \gamma_n \le d < 1$ for all $n \ge 1$ and $\sum_{i=1}^{N} a_i = 1$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$ for all $n \ge 1$;

(iv) $r \in (0, 2\alpha)$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$; for each $i = 1, 2, ..., N$, there exists $\rho_i > 0$ such that

$$\Phi_i(\bar{w}_1^i, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_i(\bar{w}_2^i, T_{r_2}(x_2), T_{r_1}(x_1)) \le -\rho_i\|T_r(x_1) - T_r(x_2)\|^2;$$

for all $(r_1, r_2) \in E \times E$, $(x_1, x_2) \in C \times C$ and $\bar{w}_j^i \in S_i(x_j)$, for all j $= 1, 2,$

where E={r}.

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}u$.

3.2 Some method for fixed point theory of a finite family of nonspreading mappings

In this section, we prove the strong convergence theorem for finding a common element of the set of fixed points of a finite family of nonspreading mappings.

Theorem 3.2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. $T_i : C \to C$ be a nonspreading mapping, for all $i = 1, 2, ..., N$ with $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let K be K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $\{x_n\}$ be sequence generated by $x_1 \in C$ such that*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_C(I - \lambda_n(I - K))x_n, \quad \forall n \geq 1$$

where $\{\alpha_n\} \subseteq [0, 1]$ for all $n \geq 1$ and $f : C \to C$ be ξ-contraction mapping. Suppose the following conditions hold :

$\lim_{n\to\infty} \alpha_n = 0$, $\sum_{i=1}^{\infty} \alpha_n = \infty$;

$\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$ *for all* $n \geq 1$;

$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$.

Proof. We divide the remainder of the proof into five steps.

Step 1. We will show that $\{x_n\}$ is bounded.

Let $z \in \mathcal{F}$, we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n f(x_n) + (1 - \alpha_n) P_C(I - \lambda_n(I - K))x_n - z\| \\ &\leq \alpha_n \|f(x_n) - z\| + (1 - \alpha_n)\|P_C(I - \lambda_n(I - K))x_n - z\| \\ &\leq \alpha_n(\|f(x_n) - f(z)\| + \|f(z) - z\|) + (1 - \alpha_n)\|P_C(I - \lambda_n(I - K))x_n - z\|. \end{aligned} \tag{3.2.1}$$

Consider:

$$\begin{aligned} \|P_C(I - \lambda_n(I - K))x_n - z\|^2 &= \|P_C(I - \lambda_n(I - K))x_n - P_C z\|^2 \\ &\leq \|(I - \lambda_n(I - K))x_n - z\|^2 \\ &= \|x_n - z - \lambda_n(I - K)x_n\|^2 \\ &= \|x_n - z\|^2 - 2\lambda_n\langle x_n - z, (I - K)x_n\rangle + \lambda_n^2\|(I - K)x_n\|^2. \end{aligned} \tag{3.2.2}$$

Consider the result below,

$$\begin{aligned}\|Kx-y\|^2 &= \|x-y-(I-K)x\|^2\\ &= \|x-y\|^2-2\langle x-y,(I-K)x\rangle+\|(I-K)x\|^2\\ &\le \|x-y\|^2,\end{aligned}$$

for all $x \in C$ and $y \in \bigcap_{i=1}^{N} F(T_i)$. It implies that

$$\|(I-K)x\|^2 \le 2\langle x-y,(I-K)x\rangle. \tag{3.2.3}$$

From (3.2.2), (3.2.3) and (i), we have

$$\begin{aligned}\|P_C(I-\lambda_n(I-K))x_n-z\|^2 &\le \|x_n-z\|^2-\lambda_n\|(I-K)x_n\|^2+\lambda_n^2\|(I-K)x_n\|^2\\ &= \|x_n-z\|^2-\lambda_n(1-\lambda_n)\|(I-K)x_n\|^2\\ &\le \|x_n-z\|^2. \end{aligned}\tag{3.2.4}$$

From (3.2.1) and (3.2.4), we have

$$\begin{aligned}\|x_{n+1}-z\| &\le \alpha_n(\|f(x_n)-f(z)\|+\|f(z)-z\|)+(1-\alpha_n)\|x_n-z\|\\ &\le \alpha_n(\xi\|x_n-z\|+\|f(z)-z\|)+(1-\alpha_n)\|x_n-z\|\\ &= (1-\alpha_n(1-\xi))\|x_n-z\|+\alpha_n\|f(z)-z\|\\ &\le max\left\{\|x_1-z\|,\frac{\|f(z)-z\|}{1-\xi}\right\}.\end{aligned}$$

By induction, we have $\|x_n-z\| \le max\left\{\|x_1-z\|,\frac{\|f(z)-z\|}{1-\xi}\right\}$, $\forall n \in \mathbb{N}$.
It follows that $\{x_n\}$ is bounded.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. By the definition of x_n, we obtain

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n f(x_n)+(1-\alpha_n)P_C(I-\lambda_n(I-K))x_n\\ &\qquad -\alpha_{n-1}f(x_{n-1})-(1-\alpha_{n-1})P_C(I-\lambda_{n-1}(I-K))x_{n-1}\|\\ &\le \alpha_n\|f(x_n)-f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|\\ &\qquad +(1-\alpha_n)\|P_C(I-\lambda_n(I-K))x_n-P_C(I-\lambda_{n-1}(I-K))x_{n-1}\|\\ &\qquad +|(1-\alpha_n)-(1-\alpha_{n-1})|\|P_C(I-\lambda_{n-1}(I-K))x_{n-1}\|\\ &\le \alpha_n\xi\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|\\ &\qquad +(1-\alpha_n)\|x_n-x_{n-1}\|+\lambda_n\|(I-K)x_n\|+\lambda_{n-1}\|(I-K)x_{n-1}\|\\ &\qquad +|\alpha_{n-1}-\alpha_n|\|P_C(I-\lambda_{n-1}(I-K))x_{n-1}\|\\ &= (1-\alpha_n(1-\xi))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|\\ &\qquad +\lambda_n\|(I-K)x_n\|+\lambda_{n-1}\|(I-K)x_{n-1}\|\\ &\qquad +|\alpha_{n-1}-\alpha_n|\|P_C(I-\lambda_{n-1}(I-K))x_{n-1}\|.\end{aligned}$$

Applying the conditions $(i), (ii), (iii)$ and Lemma 2.1.9, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.2.5}$$

Step 3. We will show that

$$\lim_{n\to\infty} \|P_C(I - \lambda_n(I-K))x_n - x_n\| = 0, \forall i = 1, 2, 3, ..., N.$$

By the definition of x_n, we obtain

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n f(x_n) + (1-\alpha_n)P_C(I - \lambda_n(I-K))x_n - x_n \\ &= \alpha_n(f(x_n) - x_n) + (1-\alpha_n)(P_C(I - \lambda_n(I-K))x_n - x_n). \end{aligned}$$

From (3.2.5) and the condition (i), we get

$$\lim_{n\to\infty} \|P_C(I - \lambda_n(I-K))x_n - x_n\| = 0.$$

Step 4. We will show that $\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle \leq 0$,
where $z_0 = P_{\mathcal{F}} f(z_0)$.
To show this, choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle = \lim_{j\to\infty} \langle f(z_0) - z_0, x_{n_j} - z_0\rangle.$$

Without loss of generality, we can assume that $x_{n_j} \rightharpoonup x^*$ as $j \to \infty$.
Assume that $x^* \notin \bigcap_{i=1}^{N} F(T_i)$.
From Remark 2.2.4 and Lemma 2.2.7, we have $x^* \neq P_C(I - \lambda_{n_j}(I-K))x^*$.
By Opial's property and the condition (ii), we have

$$\begin{aligned} \liminf_{j\to\infty} \|x_{n_j} - x^*\| &< \liminf_{j\to\infty} \|x_{n_j} - P_C(I - \lambda_{n_j}(I-K))x^*\| \\ &= \liminf_{j\to\infty} \|x_{n_j} + P_C(I - \lambda_{n_j}(I-K))x_{n_j} \\ &\qquad - P_C(I - \lambda_{n_j}(I-K))x_{n_j} - P_C(I - \lambda_{n_j}(I-K))x^*\| \\ &\leq \liminf_{j\to\infty} (\|x_{n_j} - x^*\| + \lambda_{n_j}\|(I-K)x^* - (I-K)x_{n_j}\| \\ &\qquad + \|x_{n_j} - P_C(I - \lambda_{n_j}(I-K)x_{n_j}\|) \\ &= \liminf_{j\to\infty} \|x_{n_j} - x^*\|. \end{aligned}$$

This is a contradiction. So we have

$$x^* \in \bigcap_{i=1}^{N} F(T_i) = \mathcal{F}.$$

Then, we have

$$\lim_{n\to\infty} sup\langle f(z_0) - z_0, x_n - z_0\rangle = \lim_{j\to\infty} \langle f(z_0) - z_0, x_{n_j} - z_0\rangle = \langle f(z_0) - z_0, x^* - z_0\rangle \leq 0, \tag{3.2.6}$$

Step 5. Finally, we will prove that $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$, for every $i = 1, 2, ..., N$. By Lemma 2.1.1, we have

$$
\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n(f(x_n) - z_0) + (1-\alpha_n)(P_C(I - \lambda_n(I-K))x_n - z_0)\|^2 \\
&\leq \|(1-\alpha_n)(P_C(I - \lambda_n(I-K))x_n - z_0)\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\leq ((1-\alpha_n)\|x_n - z_0\|)^2 + 2\alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq ((1-\alpha_n)\|x_n - z_0\|)^2 + 2\alpha_n\xi\|x_n - z_0\|\|x_{n+1} - z_0\| + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + \alpha_n\xi(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \frac{(1-\alpha_n)^2 + \alpha_n\xi}{1-\alpha_n\xi}\|x_n - z_0\|^2 + \frac{2\alpha_n}{1-\alpha_n\xi}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= \frac{1-\alpha_n\xi - 2\alpha_n(1-\xi)}{1-\alpha_n\xi}\|x_n - z_0\|^2 + \frac{\alpha_n^2}{1-\alpha_n\xi}\|x_n - z_0\|^2 + \frac{2\alpha_n}{1-\alpha_n\xi}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= \left(1 - \frac{2\alpha_n(1-\xi)}{1-\alpha_n\xi}\right)\|x_n - z_0\|^2 \\
&\qquad + \frac{2\alpha_n(1-\xi)}{1-\alpha_n\xi}\left(\frac{\alpha_n}{2(1-\xi)}\|x_n - z_0\|^2 + \frac{1}{1-\xi}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle\right).
\end{aligned}
$$

Applying the condition(i), (3.2.6) and Lemma 2.1.9, we have the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$. This completes the proof. □

Chapter 4

Applications

4.1 Strong convergence theorems for equilibrium problem and the set of fixed point of a nonspreading mapping and κ-demicontractive mapping

In this section, by using our main result, we obtain Theorem 4.1.2 and 4.1.4. Next we prove a result involving demicontractive mapping as follows;

Lemma 4.1.1. Let C be a nonempty closed convex subset of H and let $T : C \to C$ be κ-demicontractive mapping with $F(T) \neq \emptyset$. Definded $Sx = ax + (1-a)Tx \ \forall x \in C$ and $a \in (k, 1)$. Then the following properties holds;

(i) $F(S) = F(T)$,

(ii) S is a quasi-nonexpansive mapping.

Proof. It obvious that $F(T) \subseteq F(S)$.
Let $x_0 \in F(S)$ and $x^* \in F(T)$, we have

$$\begin{aligned}
\|x_0 - x^*\|^2 &= \|Sx_0 - x^*\|^2 \\
&= \|a(x_0 - x^*) + (1-a)(Tx_0 - x^*)\|^2 \\
&= a\|x_0 - x^*\|^2 + (1-a)\|Tx_0 - x^*\|^2 - a(1-a)\|x_0 - Tx_0\|^2 \\
&\leq a\|x_0 - x^*\|^2 + (1-a)(\|x_0 - x^*\|^2 + k\|(I-T)x_0\|^2) \\
&\quad - a(1-a)\|x_0 - Tx_0\|^2 \\
&= \|x_0 - x^*\|^2 - (1-a)(a-k)\|x_0 - Tx_0\|^2. \qquad (4.1.1)
\end{aligned}$$

It implies that

$$(1-a)(a-k)\|x_0 - Tx_0\| = 0.$$

Then,

$$x_0 \in F(T).$$

Hence $F(S) \subseteq F(T)$.
Applying (4.1.1), we have S is a quasi-nonexpansive mapping. ☐

By using these results, we obtain the following theorems.

Theorem 4.1.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $S_i : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with coefficients μ_i and let $\Phi_i : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1) - H3)$. Let $\varphi : C \to R$ be lower semicontinuous and convex function. Let $T_i : C \to C$ be a* *nonspreading mapping with* $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} (GEP)_s(\Phi_i, \varphi) \neq \emptyset$. *Let* K *be* K*-mapping generated by* $T_1, ..., T_N$ *and* $\lambda_1, ..., \lambda_N$ *and let* $\{x_n\}$ *be sequence generated by* $x_1 \in C$, $w_1^i \in S_i x_1$ *and there exists sequence* $\{w_n^i\} \subseteq H$ *such that*

$$\begin{cases} \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n})\mathcal{H}(S_i x_n, S_i x_{n+1}), w_n^i \in S_i x_n, \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r}\langle u_n^i - x_n, y - u_n^i \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda_n(I - K))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i, \\ \forall n \geq 1 \;\; \textit{and} \;\; i = 1, 2, 3, ..., N, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$ *and* $f : C \to C$ *be* ξ*-contraction mapping. Suppose the following conditions hold :*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{i=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ *for all* $n \geq 1$ *and* $\sum_{i=1}^{N} a_i = 1$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$ *for all* $n \geq 1$;

(iv) $r \in (0, 2\alpha)$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(vi) for each $i = 1, 2, ..., N$, *there exists* $\rho_i > 0$ *such that*

$$\Phi_i(\bar{w}_1^i, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_i(\bar{w}_2^i, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\rho_i \|T_r(x_1) - T_r(x_2)\|^2;$$

for all $(r_1, r_2) \in E \times E$, $(x_1, x_2) \in C \times C$ *and* $\bar{w}_j^i \in S_i(x_j)$, *for all* $j = 1, 2$, *where* $E = \{r\}$.

Then $\{x_n\}$ *converges strongly to* $z = P_{\mathcal{F}} f(z)$.

Proof. By using Theorem 3.1.1 and Lemma 2.2.7 , we obtain the conclusion. □

Corollary 4.1.3. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to CB(H)$ be $\mathcal{H}$*-Lipschitz* continuous with coefficients μ and let $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1) - H3)$. Let $\varphi : C \to \mathbb{R}$ be lower semicontinuous and convex function. Let $T_i : C \to C$ be a nonspreading mapping, for all $i = 1, 2, ..., N$ with $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap (GEP)_s(\Phi, \varphi) \neq \emptyset$. Let K be K-mapping generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$ and let $\{x_n\}$ be sequence generated by $x_1 \in C$, $w_1 \in S x_1$ and there exists sequence $\{w_n\} \subseteq H$ such that

$$\begin{cases} \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n})\mathcal{H}(S x_n, S x_{n+1}), w_n \in S x_n, \\ \Phi(w_n, u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle u_n - x_n, y - u_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda_n(I - K))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n, \forall n \geq 1 \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $f : C \to C$ be ξ-contraction mapping. Suppose the following conditions hold :

(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{i=1}^{\infty} \alpha_n = \infty;$

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for all $n \geq 1$ and $\sum_{i=1}^{N} a_i = 1;$

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$ for all $n \geq 1;$

(iv) $r \in (0, 2\alpha);$

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty;$

(vi) there exists $\rho > 0$ such that

$$\Phi(\bar{w}_1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi(\bar{w}_2, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\rho \|T_r(x_1) - T_r(x_2)\|^2;$$

for all $(r_1, r_2) \in E \times E$, $(x_1, x_2) \in C \times C$ and $\bar{w}_j^i \in S_i(x_j)$, for all $j = 1, 2$, where $E = \{r\}$

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.

Theorem 4.1.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $S_i : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with coefficients μ_i and let $\Phi_i : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1) - H3)$. Let $\varphi : C \to R$ be lower semicontinuous and convex function. Let $T : C \to C$ be κ-demicontractive mapping. Defined $S : C \to C$ by $ax + (1-a)Tx = Sx$, for all $a \in (k, 1)$ and $x \in C$ with $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} (GEP)_s(\Phi_i, \varphi) \neq \emptyset$. Let $\{x_n\}$ be sequence generated by $x_1 \in C$, $w_1^i \in S_i x_1$ and there exists sequence $\{w_n^i\} \subseteq H$ such that*

$$\begin{cases} \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n}) \mathcal{H}(S_i x_n, S_i x_{n+1}), w_n^i \in S_i x_n, \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r} \langle u_n^i - x_n, y - u_n^i \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda_n(I - S)) x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i, \\ \forall n \geq 1 \text{ and } i = 1, 2, 3, ..., N, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $f : C \to C$ be ξ-contraction mapping. Suppose the following conditions hold :

(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{i=1}^{\infty} \alpha_n = \infty;$

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ *for all* $n \geq 1$ *and* $\sum_{i=1}^{N} a_i = 1;$

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$ *for all* $n \geq 1;$

(iv) $r \in (0, 2\alpha);$

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty;$

(vi) for each $i = 1, 2, ..., N$, *there exists* $\rho_i > 0$ *such that*

$$\Phi_i(\bar{w}_1^i, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_i(\bar{w}_2^i, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\rho_i \|T_r(x_1) - T_r(x_2)\|^2;$$

for all $(r_1, r_2) \in E \times E$, $(x_1, x_2) \in C \times C$ *and* $\bar{w}_j^i \in S_i(x_j)$, *for all j* $= 1, 2$, *where* $E = \{r\}$. *Then* $\{x_n\}$ *converses strongly to* $z = P_{\mathcal{F}} f(z)$.

Proof. By using Theorem 3.1.1 and Lemma 4.1.1, we obtain the conclusion. □

Chapter 5

Conclusions

In this chapter, we conclude all main results obtained in this thesis.

5.1 Convergence theorem for finding a common element of the set of solution of a finite family of the generalized equilibrium problem and the set of fixed point of a quasi-nonexpansive mapping

Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $S_i : C \to CB(H)$ be $\mathcal{H}$-*Lipschitz* continuous with coefficients μ_i and let $\Phi_i : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1) - H3)$. Let $\varphi : C \to R$ be lower semicontinuous and convex function. Let $T : C \to C$ be a quasi-nonexpansive mapping with $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} (GEP)_s(\Phi_i, \varphi) \neq \emptyset$. Let $\{x_n\}$ be sequence generated by $x_1 \in C$, $w_1^i \in S_i x_1$ and there exists sequence $\{w_n^i\} \subseteq H$ such that

$$\begin{cases} \|w_n^i - w_{n+1}^i\| \leq (1+\frac{1}{n})\mathcal{H}(S_i x_n, S_i x_{n+1}), w_n^i \in S_i x_n, \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r}\langle u_n^i - x_n, y - u_n^i\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i, \\ \forall n \geq 1 \text{ and } i = 1, 2, 3, ..., N, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $f : C \to C$ be ξ-contraction mapping. Suppose the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{i=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for all $n \geq 1$ and $\sum_{i=1}^{N} a_i = 1$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$ for all $n \geq 1$;

(iv) $r \in (0, 2\alpha)$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(vi) for each $i = 1, 2, ..., N$, there exists $\rho_i > 0$ such that

$$\Phi_i(\bar{w}_1^i, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_i(\bar{w}_2^i, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\rho_i \|T_r(x_1) - T_r(x_2)\|^2;$$

for all $(r_1, r_2) \in E \times E$, $(x_1, x_2) \in C \times C$ and $\bar{w}_j^i \in S_i(x_j)$, for all j $= 1, 2,$ where $E = \{r\}$.

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.

5.2 Modified Halpern Iteration for finding a common element of the set of solution of a finite family of the generalized equilibrium problem and the set of fixed point of a quasi-nonexpansive mapping

Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $S_i : C \to CB(H)$ be $\mathcal{H}$-*Lipschitz* continuous with coefficients μ_i and let $\Phi_i : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1) - H3)$. Let $\varphi : C \to R$ be lower semicontinuous and convex function. Let $T : C \to C$ be a quasi-nonexpansive mapping with $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} (GEP)_s(\Phi_i, \varphi) \neq \emptyset$. Let $\{x_n\}$ be sequence generated by $u, x_1 \in C$, $w_1^i \in S_i x_1$ and there exists sequence $\{w_n^i\} \subseteq H$ such that

$$\begin{cases} \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n})\mathcal{H}(S_i x_n, S_i x_{n+1}), w_n^i \in S_i x_n, \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r}\langle u_n^i - x_n, y - u_n^i\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i, \\ \forall n \geq 1 \text{ and } i = 1, 2, 3, ..., N, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$. Suppose the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{i=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for all $n \geq 1$ and $\sum_{i=1}^{N} a_i = 1$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$ for all $n \geq 1$;

(iv) $r \in (0, 2\alpha)$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$; for each $i = 1, 2, ..., N$, there exists $\rho_i > 0$ such that

$$\Phi_i(\bar{w}_1^i, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_i(\bar{w}_2^i, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\rho_i \|T_r(x_1) - T_r(x_2)\|^2;$$

for all $(r_1, r_2) \in E \times E$, $(x_1, x_2) \in C \times C$ and $\bar{w}_j^i \in S_i(x_j)$, for all j $= 1, 2$,

where E={r}.

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} u$.

5.3 Some method for fixed point theory of a finite family of nonspreading mappings

Let C be a nonempty closed convex subset of a real Hilbert space H. $T_i : C \to C$ be a nonspreading mapping, for all $i = 1, 2, ..., N$ with $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let K be K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ and let $\{x_n\}$ be sequence generated by $x_1 \in C$ such that

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)P_C(I - \lambda_n(I-K))x_n, \quad \forall n \geq 1$$

where $\{\alpha_n\} \subseteq [0,1]$ for all $n \geq 1$ and $f : C \to C$ be ξ-contraction mapping. Suppose the following conditions hold :

(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{i=1}^{\infty} \alpha_n = \infty;$

(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$ for all $n \geq 1;$

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} f(z_0)$.

5.4 Strong convergence theorems for equilibrium problem and the set of fixed point of a nonspreading mapping and κ-demicontractive mapping

(1) Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $S_i : C \to CB(H)$ be $\mathcal{H}$-$Lipschitz$ continuous with coefficients μ_i and let $\Phi_i : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1) - H3)$. Let $\varphi : C \to R$ be lower semicontinuous and convex function. Let $T_i : C \to C$ be a nonspreading mapping with $\mathcal{F} := \bigcap\limits_{i=1}^{N} F(T_i) \cap \bigcap\limits_{i=1}^{N} (GEP)_s(\Phi_i, \varphi) \neq \emptyset$. Let K be K-mapping generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$ and let $\{x_n\}$ be sequence generated by $x_1 \in C$, $w_1^i \in S_i x_1$, and there exists sequence $\{w_n^i\} \subseteq H$ such that

$$\begin{cases} \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n})\mathcal{H}(S_i x_n, S_i x_{n+1}), w_n^i \in S_i x_n, \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r}\langle u_n^i - x_n, y - u_n^i \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda_n(I-K))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i, \\ \forall n \geq 1 \text{ and } i = 1, 2, 3, ..., N, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $f : C \to C$ be ξ-contraction mapping. Suppose the following conditions hold :

(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{i=1}^{\infty} \alpha_n = \infty;$

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for all $n \geq 1$ and $\sum_{i=1}^{N} a_i = 1;$

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$ for all $n \geq 1;$

(iv) $r \in (0, 2\alpha);$

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty;$

(vi) for each $i = 1, 2, ..., N$, there exists $\rho_i > 0$ such that

$$\Phi_i(\bar{w}_1^i, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_i(\bar{w}_2^i, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\rho_i \|T_r(x_1) - T_r(x_2)\|^2;$$

for all $(r_1, r_2) \in E \times E$, $(x_1, x_2) \in C \times C$ and $\bar{w}_j^i \in S_i(x_j)$, for all j $= 1, 2$, where $E = \{r\}$.

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.

(2) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to CB(H)$ be $\mathcal{H}$-*Lipschitz* continuous with coefficients μ and let $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1) - H3)$. Let $\varphi : C \to \mathbb{R}$ be lower semicontinuous and convex function. Let $T_i : C \to C$ be a nonspreading mapping, for all $i = 1, 2, ..., N$ with $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap (GEP)_s(\Phi, \varphi) \neq \emptyset$. Let K be K-mapping generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$ and let $\{x_n\}$ be sequence generated by $x_1 \in C$, $w_1 \in Sx_1$ and there exists sequence $\{w_n\} \subseteq H$ such that

$$\begin{cases} \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n})\mathcal{H}(Sx_n, Sx_{n+1}), w_n \in Sx_n, \\ \Phi(w_n, u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle u_n - x_n, y - u_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda_n(I - K))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n, \forall n \geq 1 \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $f : C \to C$ be ξ-contraction mapping. Suppose the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{i=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for all $n \geq 1$ and $\sum_{i=1}^{N} a_i = 1$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$ for all $n \geq 1$;

(iv) $r \in (0, 2\alpha)$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(vi) there exists $\rho > 0$ such that

$$\Phi(\bar{w}_1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi(\bar{w}_2, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\rho \|T_r(x_1) - T_r(x_2)\|^2;$$

for all $(r_1, r_2) \in E \times E$, $(x_1, x_2) \in C \times C$ and $\bar{w}_j^i \in S_i(x_j)$, for all j $= 1, 2$, where $E = \{r\}$.

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$.

(3) Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let $S_i : C \to CB(H)$ be $\mathcal{H}$-*Lipschitz* continuous with coefficients μ_i and let $\Phi_i : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying $H1) - H3)$. Let $\varphi : C \to R$ be lower semicontinuous and convex function. Let $T : C \to C$ be κ-demicontractive mapping. Defined $S : C \to C$ by $ax + (1 - a)Tx = Sx$, for all $a \in (k, 1)$ and $x \in C$ with

$\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} (GEP)_s(\Phi_i, \varphi) \neq \emptyset$. Let $\{x_n\}$ be sequence generated by $x_1 \in C$, $w_1^i \in S_i x_1$ and there exists sequence $\{w_n^i\} \subseteq H$ such that

$$\begin{cases} \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n})\mathcal{H}(S_i x_n, S_i x_{n+1}), w_n^i \in S_i x_n, \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r}\langle u_n^i - x_n, y - u_n^i \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda_n(I - S))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i, \\ \forall n \geq 1 \text{ and } i = 1, 2, 3, ..., N, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $f : C \to C$ be ξ-contraction mapping. Suppose the following conditions hold :

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{i=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < c \leq \beta_n, \gamma_n \leq d < 1$ for all $n \geq 1$ and $\sum_{i=1}^{N} a_i = 1$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$ for all $n \geq 1$;

(iv) $r \in (0, 2\alpha)$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(vi) for each $i = 1, 2, ..., N$, there exists $\rho_i > 0$ such that

$$\Phi_i(\bar{w}_1^i, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_i(\bar{w}_2^i, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\rho_i \|T_r(x_1) - T_r(x_2)\|^2;$$

for all $(r_1, r_2) \in E \times E$, $(x_1, x_2) \in C \times C$ and $\bar{w}_j^i \in S_i(x_j)$, for all j $= 1, 2$, where $E = \{r\}$.

Then $\{x_n\}$ converses strongly to $z = P_{\mathcal{F}} f(z)$.

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