SOLVING A GENERAL SYSTEM OF NONHOMOGENEOUS COUPLED LINEAR MATRIX DIFFERENTIAL EQUATIONS IN TERMS OF MITTAG-LEFFLER MATRIX FUNCTIONS

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หัวข้อวิทยานิพนธ์	การแก้ระบบทั่วไปของสมการเชิงอนุพันธ์เมทริกซ์เชิงเส้นควบคู่ไม่เอกพันธ์ในพจน์ของฟังก์ชันเมทริกซ์มิทแท็ก-เลฟเฟอร์
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บทคัดย่อ

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Abstract

In this research, we investigate a system of nonhomogeneous coupled linear matrix differential equations. Applying Kronecker products, the vector operator, and matrix convolution product, we obtain an explicit formula of the general solution to this system in terms of matrix series concerning exponentials and Mittag-Leffler functions. Several special cases of this system are also discussed.

Keywords : vector operator, Kronecker product, matrix convolution product, Mittag-Leffler function, linear matrix differential equation.

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Introduction

1.1 Research Motivation

Theory of linear matrix differential equations can be applied in a broad range of scientific fields, e.g. statistics [3, 9, 11], game theory [5], ecometrics and Leondief model [9, 11, 15], control and system theory [4, 10]. The simplest first-order homogeneous linear matrix differential equation with time-invariant coefficient is given by

$$X'(t) = AX(t). \tag{1.1}$$

Here, A is a given square matrix, and $X(t)$ is an unknown matrix-valued function to be solved. The system (1.1) has been widely studied, and the solution relies on the computation of e^{tA}; see more information in [16, 17, 18, 19]. The nonhomogeneous case appears in the form

$$X'(t) = AX(t) + U(t), \tag{1.2}$$

here $U(t)$ is a given matrix-valued function. In fact, the equation (1.2) has a general solution given by a one-parameter matrix-valued function

$$X(t) = e^{(t-t_0)A}X(t_0) + e^{tA} * U(t), \tag{1.3}$$

where $*$ denotes the matrix convolution product.

A general system of nonhomogeneous coupled linear matrix differential equations with time-invariant coefficient takes the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t). \end{aligned} \tag{1.4}$$

Here, $A,B,C,D,E,F,G,H \in M_n(\mathbb{C})$ are given constant matrices, $U(t), V(t)$ are given matrix-valued functions and $X(t), Y(t)$ are unknown matrix-valued functions to be solved. Coupled matrix differential equation have numerous application in pure and applied mathematics. For example, to obtain the solution of an optimal control problem with performance index we need to solve the following system [12]:

$$\begin{aligned} X'(t) &= AX(t) + BY(t), \\ Y'(t) &= CX(t) - A^TY(t), \end{aligned}$$

Kiliman and Al Zhour [7] investigated a special case of (1.4) when $E = C$, $F = D$, $G = A$, $H = B$, $U(t) = V(t) = 0$ under the assumptions that $AC = CA$ and $BD = DB$. In this case, the solution is given in terms of Kronecker products, the vector operator, and matrix power series concerning exponentials and hyperbolic functions.

Al Zhour [1] discussed the system (1.4) when $E = C$, $F = D$, $G = A$, $H = B$, in which $X(t)$ and $Y(t)$ are diagonal matrix-valued functions. In this case, the solution is obtained in terms of Hadamard products, the diagonal extraction operator, matrix convolution product, and matrix series.

In this work, we investigate a general system of nonhomogeneous coupled linear matrix differential equations. We apply Kronecker products and the vector operator to reduce our complex system to the simplest form Thus, an explicit formula of the general solution to this system is obtained in terms of Mittag-Leffler matrix functions. In particular, we obtain general solution of several special cases of the main system. When initial conditions are imposed to these problems, the solution of each case is uniquely determined.

This thesis is structured as follows. In Chapter 2, we supply useful facts for solving linear matrix differential equations, including matrix functions defined by power series, Kronecker product, vector operator, and matrix convolution product. Chapter 3 is the main part of the research. We solve a general system of nonhomogeneous coupled linear matrix differential equations. We also investigate certain special cases of the system (1.4).

1.2 Objectives of the study

1) To solve a system of nonhomogeneous linear matrix differential equations.

2) To investigate initial value problems for the system of nonhomogeneous linear matrix differential equations when initial conditions are imposed.

1.3 Scopes of the study

We solve the following nonhomogeneous coupled linear matrix differential equations:

$$X'(t) = AX(t)B + CY(t)D + U(t),$$
$$Y'(t) = EX(t)F + GY(t)H + V(t).$$

Here A, B, C, D, E, F, G, H are given square matrices satisfying certain assumptions, $U(t)$, $V(t)$ are given matrix-valued functions, and $X(t)$, $Y(t)$ are unknown matrix-valued functions. All matrices considered here are real.

1.4 Benefits of the Study

To provide further mathematical theory for nonhomogeneous linear matrix differential equations.

1.5 Research methodology

1) Study advanced topics in matrix analysis.
2) Study research papers and textbooks concerning Kronecker products and the vector operator, and matrix convolution product.
3) Study background in Mittag-Leffler matrix functions.
4) Study research papers about linear matrix differential equations.
5) Determine the objectives and scope of the research.
6) Solve a system of nonhomogeneous linear matrix differential equations by using Kronecker products, the vector operator, and matrix convolution product.
7) Investigate initial value problems for systems of nonhomogeneous linear matrix differential equations when initial conditions are imposed.
8) Conclude the results, make suggestions for further works, and write the thesis.

Table 1.1: The research schedule

Activity	Time frame							
	2016		2017				2018	
	Aug.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.	Aug.-Sep.	Oct.-Dec.	Jan.-Mar.	Apr.-Jun.
Step 1	⟷							
Step 2		⟵	⟶					
Step 3			⟵	⟶				
Step 4					⟷			
Step 5						⟷		
Step 6							⟷	
Step 7							⟷	
Step 8								⟷

Chapter 2

Preliminaries and Literature review

In this chapter, we provide adequate tools for solving system of linear matrix differential equations. We shall denote the set of all m-by-n complex matrices by $M_{m,n}$ and we set $M_n = M_{n,n}$. We shall denote the set of all m-by-n real matrices by $M_{m,n}(\mathbb{R})$ and $M_n(\mathbb{R})$.

2.1 Kronecker product

Definition 2.1. Given two matrices $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{p,q}$ the Kronecker product of A and B is defined by

$$A \otimes B = [a_{ij}B]_{ij} \in M_{mp,nq}.$$

Example 2.2. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 3 & 5 \end{bmatrix}$. Find $A \otimes B$.

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 1B & 0B \\ 2B & 3B \end{bmatrix} \\ &= \begin{bmatrix} 1\begin{bmatrix} 0 & 1 \\ 3 & 5 \end{bmatrix} & 0\begin{bmatrix} 0 & 1 \\ 3 & 5 \end{bmatrix} \\ 2\begin{bmatrix} 0 & 1 \\ 3 & 5 \end{bmatrix} & 3\begin{bmatrix} 0 & 1 \\ 3 & 5 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 2 & 0 & 3 \\ 6 & 10 & 9 & 15 \end{bmatrix}. \end{aligned}$$

Lemma 2.3. The following properties hold for matrices of appropriate sizes:

1. $I_m \otimes I_n = I_{mn}$
2. $(\alpha A) \otimes B = \alpha(A \otimes B) = A \otimes (\alpha B)$ for all $\alpha \in \mathbb{F}$.
3. The map $(A, B) \mapsto A \otimes B$ is bilinear.
4. $(A \otimes B)^T = A^T \otimes B^T$.
5. $(A \otimes B)(C \otimes D) = AC \otimes BD$

2.2 The vector operator

Let us introduce a linear operator that converts a matrix to a vector. The vector operator Vec : $M_{m,n} \to \mathbb{C}^{mn}$ defined for each $A = [a_{ij}]$ by

$$\text{Vec}\, A = [a_{11} \dots a_{m1} \dots a_{12} \dots a_{m2} \dots a_{1m} \dots a_{mn}]^T.$$

It is clear that Vec is a bijection (injection and surjection).

Example 2.4. Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 7 & 1 & 2 \\ 4 & 8 & 0 \end{bmatrix}$. Then

$$\text{Vec}(A) = \begin{bmatrix} 1 \\ 7 \\ 4 \\ 3 \\ 1 \\ 8 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

Theorem 2.5 (see e.g. [10]). The Kronecker product and the vector operator are related by

$$\text{Vec}(AXB) = (B^T \otimes A)\,\text{Vec}\, X$$

for any matrices $A \in M_{m,n}$, $B \in M_{p,q}$ and $X \in M_{n,p}$.

2.3 Complex power series

Definition 2.6. Let $U \subseteq \mathbb{C}$ be an open set. A function $f : U \to \mathbb{C}$ is said to be analytic at $z_0 \in U$ if and only if there is a neighbourhood V of z_0 such that f has a Taylor expansion around z_0. That is,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \quad \forall z \in V$$

1. If $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ converges for all $z \in \mathbb{C}$ such that $|z - z_0| < R$ and diverges for all $z \in \mathbb{C}$ such that $|z - z_0| > R$ then we call R the radius of convergence for this series.
2. If $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges only for $z = z_0$, then we say that the radius of converges for this series is 0

3. If $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges for all $z \in \mathbb{C}$, then we say that the radius of converges for the series is ∞

Consider the following complex-valued functions represented by power series.

1. $e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$, $z \in \mathbb{C}$.
2. $\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$, $z \in \mathbb{C}$.
3. $\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$, $z \in \mathbb{C}$.
4. $\sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$, $z \in \mathbb{C}$.
5. $\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$, $z \in \mathbb{C}$.

2.4 Functions of a matrix defined by power series

(see e.g. [20]). Consider $A \in M_n$ and an analytic function f defined on a region in a complex plane containing the origin and the spectrum of A. Then there is a positive constant R such that f admits the Taylor series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{for } |z| < R,$$

where $a_0 = f(0)$ and $a_k = f^{(k)}(0)/k!$ for any $k \in \mathbb{N}$. If the spectral radius of A is less than R, then the matrix power series $\sum_{k=0}^{\infty} a_k A^k$ converges, denoted by $f(A)$. Hence if f is an entire function then $f(A)$ is a well-defined matrix for any $A \in M_n$. In particular, the following matrix series converge for any $A \in M_n$:

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

$$\sinh(A) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1}$$

$$\cosh(A) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k}$$

Lemma 2.7 (see e.g. [13]). If (A, B) is a pair of commuting complex matrices, then

$$e^{A+B} = e^A e^B.$$

Theorem 2.8 (see e.g. [21]). Let $A \in M_n(\mathbb{C})$. Then

1. $(e^A)^{-1} = e^{-A}$,
2. $(e^A)^T = e^{A^T}$.

2.5 Mittage-Leffler functions

Definition 2.9. The two-parameter Mittage-Leffler functions with parameters $\alpha > 0$ and $\beta > 0$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{2.1}$$

where the Gamma function Γ is defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \qquad \text{Re}(z) > 0.$$

The power series (2.1) converges for all complex numbers z.

Definition 2.10. The Mittage-Leffler function of a matrix $A \in M_n$ with parameters $\alpha > 0$ and $\beta > 0$ is defined by

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} A^k = I_n + \frac{1}{\Gamma(\alpha + \beta)} A + \frac{1}{\Gamma(2\alpha + \beta)} A^2 + \cdots .$$

The class of matrix Mittage-Leffler functions include the following functions:

$$E_{1,1}(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = e^A$$

$$E_{2,1}(A^2) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k} = \cosh(A).$$

We have

$$(E_{2,1}(A^2))A = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1} = \sinh(A).$$

The next lemmas is useful for deriving explicit formulas of solutions for system of linear matrix differential equations in Chapter 3.

Lemma 2.11. For any $A \in M_n(\mathbb{C})$ and $B \in M_n(\mathbb{C})$

$$e^{\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}} = \begin{bmatrix} e^A & 0 \\ 0 & e^B \end{bmatrix}.$$

Proof.

$$e^{\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^k$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} A^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} B^k \end{bmatrix} = \begin{bmatrix} e^A & 0 \\ 0 & e^B \end{bmatrix}$$

□

Lemma 2.12. For any $A \in M_n(\mathbb{C})$ and $B \in M_n(\mathbb{C})$

$$e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} = \begin{bmatrix} E_{2,1}(AB) & (E_{2,2}(AB))A \\ (E_{2,2}(BA))B & E_{2,1}(BA) \end{bmatrix}.$$

Proof.

$$e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \begin{bmatrix} (AB)^k & 0 \\ 0 & (BA)^k \end{bmatrix} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \begin{bmatrix} 0 & (AB)^k A \\ (BA)^k B & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{(2k)!}(AB)^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{(2k)!}(BA)^k \end{bmatrix} + \begin{bmatrix} 0 & \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(AB)^k A \\ \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(BA)^k B & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{(2k)!}(AB)^k & \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(AB)^k A \\ \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(BA)^k B & \sum_{k=0}^{\infty} \frac{1}{(2k)!}(BA)^k \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+1)}(AB)^k & \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+2)}(AB)^k A \\ \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+2)}(BA)^k B & \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+1)!}(BA)^k \end{bmatrix}$$

$$= \begin{bmatrix} E_{2,1}(AB) & (E_{2,2}(AB))A \\ (E_{2,2}(BA))B & E_{2,1}(BA) \end{bmatrix}.$$

□

Lemma 2.13 (see e.g. [13]). Let f be an analytic function defined on a region including the origin and the spectrum of A. Then

$$f(I \otimes A) = I \otimes f(A) \quad \text{and} \quad f(A \otimes I) = f(A) \otimes I.$$

In particular, the following relations hold for any complex square matrix A :

$$E_{\alpha,\beta}(A \otimes I) = E_{\alpha,\beta}(A) \otimes I \quad \text{and} \quad E_{\alpha,\beta}(I \otimes A) = I \otimes E_{\alpha,\beta}(A),$$
$$\sinh(A \otimes I) = \sinh(A) \otimes I \quad \text{and} \quad \sinh(I \otimes A) = I \otimes \sinh(A),$$
$$\cosh(A \otimes I) = \cosh(A) \otimes I \quad \text{and} \quad \cosh(I \otimes A) = I \otimes \cosh(A).$$

2.6 Matrix convolution product

Definition 2.14. Let $\Omega = [0, \infty)$ or $\Omega = [0, b]$ for some $b > 0$. The convolution is a binary operator assinged to each pair of integrable function f and g defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \qquad t \in \Omega.$$

The convolution is bilinear and commutative. Convolutions of elementary functions can be obtained via integration by parts and changes of variables. The following table illustrates the resulting convolutions.

Table 2.1: The table of convolution product

$f(t)$	$g(t)$	$(f*g)(t)$
t^a	t^b	$\sum_{i=0}^{b}(-1)^i\binom{b}{i}\frac{t^{b+a+1}}{i+a+1}$ $a, b \in \mathbb{N}$
$\sin at$	$\sin bt$	$\frac{-a\sin bt + b\sin bt}{b^2-a^2}$ $a \neq b$ $\frac{-bt\cos bt + b\sin at}{2b}$ $a = b \neq 0$
$\sin at$	$\cos bt$	$\frac{-b(\cos bt - \cos at)}{b^2-a^2}$ $a \neq b$ $\frac{t\sin bt}{2}$ $a = b \neq 0$
$\sin at$	e^{bt}	$\frac{e^{bt}b - b\cos at + a\sin at}{b^2+a^2}$ $a^2+b^2 \neq 0$
$\cos at$	$\cos bt$	$\frac{b\sin bt - a\sin at}{a^2-b^2}$ $a \neq b$ $\frac{bt\cos bt + \sin bt}{2b}$ $a = b \neq 0$
$\cos at$	e^{bt}	$\frac{e^{bt}b - b\cos at + a\sin at}{a^2+b^2}$ $a^2+b^2 \neq 0$
e^{at}	e^{bt}	$\frac{e^{bt}-e^{at}}{a-b}$ $a \neq b$ $e^{bt}t$ $a = b$

Definition 2.15. Given two integrable matrix-valued functions $A:\Omega \to M_{m,n}(\mathbb{R})$, $A(t) = [a_{ij}(t)]$ and $B:\Omega \to M_{n,p}(\mathbb{R})$, $B(t) = [b_{ij}(t)]$ the matrix convolution product of A and B is defined by

$$(A * B)(t) = \left[\sum_{k=1}^{n} a_{ik}(t) * b_{kj}(t)\right] \in M_{m,p}(\mathbb{R}), \qquad t \in \Omega.$$

We may write $A(t) * B(t)$ for $(A * B)(t)$.

The matrix convolution product is bilinear, but not commutative in general.

Example 2.16. Consider

$$A(t) = \begin{bmatrix} \sin t & t^2 \\ -1 & t-1 \end{bmatrix}, \qquad B(t) = \begin{bmatrix} e^t & \sin t \\ 1 & t \end{bmatrix}.$$

The convolution product of A and B is

$$\begin{aligned}(A * B)(t) &= \begin{bmatrix} \sin t * e^t + t^2 * 1 & \sin t * \sin t + t^2 * t \\ -1 * e^t + (t-1) * 1 & -1 * \sin t + (t-1) * t \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}(e^t - \cos t - \sin t + \frac{t^3}{3}) & \frac{1}{2}(-t\cos t + \sin t + \frac{t^4}{12}) \\ 1 - e^t - t + \frac{t^2}{2} & -1 + \cos t - \frac{t^2}{2} + \frac{t^6}{6} \end{bmatrix}.\end{aligned}$$

2.7 Literature review

2.7.1 System of linear ordinary differential equations

We explain here fundamental knowledge of systems of linear differential equation; see e.g. [22] for more details.

First-order linear differential equation

Definition 2.17. The system of ordinary differential equations in the form

$$\begin{aligned} \frac{dx_1(t)}{dt} &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t) \\ \frac{dx_2(t)}{dt} &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t) \\ &\vdots \\ \frac{dx_n(t)}{dt} &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + b_n(t), \end{aligned} \tag{2.2}$$

where the $a_{ij}(t)$ and $b_i(t)$ are specified functions on interval I, is called a first-order linear system. If $b_1 = b_2 = \cdots b_n = 0$, then the system is said to be homogeneous. Otherwise, it is nonhomogeneous.

Solving the system (2.2) subject to n auxiliary conditions imposed at the same value of the independent variable is called an initial-value problem (IVP). Thus, the general form of the auxiliary conditions for an IVP is:

$$x_1(t_0) = \alpha_1, x_2(t_0) = \alpha_2, \ldots, x_n(t_0) = \alpha_n,$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are constants.

Vector Formulation

The first step in developing the general theory for first-order linear systems is to formulate the problem of solving such a system as an appropriate vector space problem. The key to this formulation is the realization that the scalar system

$$\begin{aligned} x_1' &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t), \\ x_2' &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t), \\ &\vdots \\ x_n' &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + b_n(t), \end{aligned} \tag{2.3}$$

can be written as the equivalent vector equation

$$x'(t) = A(t)x(t) + b(t), \tag{2.4}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \; x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

and

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \; b(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}.$$

We see from (2.4) that matrices provide a natural framework for studying first order linear systems. Notice that in this formulation, we are dealing with matrix functions, that is, matrices whose elements are themselves functions. The algebra of matrix functions is the same as that for matrices of constants. The derivative of a matrix function is obtained by differentiating every element of the matrix. Thus, if $A(t) = [a_{ik}(t)]$, then $\frac{dA}{dt} = \left[\frac{da_{ik}(t)}{dt}\right]$, provided that each of the $a_{ik}(t)$ are differentiable. If A and B are both differentiable and the product AB is defined, then

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B.$$

Vector differential equations

An $n \times 1$ matrix function is called a column n-vector function. We let $V_n(I)$ denote the set of all column n-vector functions defined on an interval I, and $V_n(I)$ is a vector space.

A system of linear differential equation written in the vector form

$$x'(t) = A(t)x(t) + b(t)$$

will be called a vector differential equation (VDE).The problem of determining all solutions to the general first-order linear system of (2.3) can now be formulated as the vector space problem.

The vector space $V_n(I)$ is not finite-dimensional, since there is no finite set of linearly independent vectors that spans $V_n(I)$. The key to solving linear differential systems comes from the realization that, if $A(t)$ is an $n \times n$ matrix function, then the set of all solutions to the homogeneous VDE

$$x'(t) = A(t)x(t)$$

is an n-dimensional subspace of $V_n(I)$.

General results for first order linear differential systems

Theorem 2.18. The IVP

$$x'(t) = A(t)x + b(t), x(t_0) = x_0,$$

where $A(t)$ and $b(t)$ are continuous on an interval I, has a unique solution on I.

Theorem 2.19. The set of all solutions to $x'(t) = A(t)x + b(t)$, where $A(t)$ is an $n \times n$ matrix function that is continuous on an interval I, is a vector space of dimension n.

Definition 2.20. Let $A(t)$ be an $n \times n$ matrix function that is continuous on an interval I. Any set of n solutions, $x_1(t), x_2(t), ..., x_n(t)$, to $x' = Ax$ that is linearly independent on I is called a fundamental solution set on I. The corresponding matrix $X(t)$ defined by

$$X(t) = [x_1(t), x_2(t), ..., x_n(t)]$$

is called a fundamental matrix for the VDE $x' = Ax$.

Nonhomogeneous VDE

Theorem 2.21. Let $A(t)$ be a matrix function that is continuous on the interval I, and let $x_1, x_2, ..., x_n$ be a fundamental solution set for the VDE $x'(t) = A(t)x(t)$ on I. If $x_p(t)$ is any particular solution to the nonhomogeneous VDE

$$x'(t) = A(t)x(t) + b(t) \tag{2.5}$$

on I, then every solution to (2.5) on I is of the form

$$x(t) = c_1x_1 + c_2x_2 + \cdots + c_nx_nn + x_p.$$

Homogeneous constant coefficient VDE

Consider the vector function

$$x(t) = e^{\lambda t} v \tag{2.6}$$

where λ is a scalar and v is a constant vector. Differentiating (2.6) with respect to t yields

$$x' = \lambda e^{\lambda t} v.$$

Thus, $x(t) = e^{\lambda t} v$ is a solution to $x' = Ax$ if and only if

$$\lambda e^{\lambda t} v = e^{\lambda t} A v,$$

that is, if and only if λ and v satisfy

$$Av = \lambda v.$$

But this is the statement that λ and v must be an eigenvalue/eigenvector pair for A. Consequently, we have established the following fundamental result:

Theorem 2.22. Let A be an $n \times n$ matrix of real constants, and let λ be an eigenvalue of A with corresponding eigenvector v. Then

$$x(t) = e^{\lambda t} v$$

is a solution to the constant coefficient VDE $x' = Ax$ on any interval.

Theorem 2.23. Let A be an $n \times n$ matrix of real constants. If A has n real linearly independent eigenvectors $v_1, v_2, \cdots, v_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, (not necessarily distinct), then the vector functions $x_1, x_2, \cdots, x_n$ defined by

$$x_k(t) = e^{\lambda_k t} v_k, k = 1, 2, \cdots, n,$$

for all t, are linearly independent solutions to $x' = Ax$ on any interval. The general solution to this VDE is

$$x(t) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n.$$

2.7.2 Linear matrix differential equations

The simplest forms of linear matrix differential equations:

$$X'(t) = AX(t),\ X(t_0) = C. \tag{2.7}$$

was investigated in [7]. Here, $A, C \in M_n$ are given constant matrices, $X(t)$ is the unknown $n \times n$ matrix function to be solved. The solution of (2.7) is given by

$$X(t) = e^{A(t-t_0)} C$$

and

$$X'(t) = AX(t) + X(t)B,\ X(t_0) = C. \tag{2.8}$$

Here, $A, B \in M_n$ and $C \in M_{n,m}$ are given constant matrices. The solution of (2.8) is given by

$$X(t) = e^{A(t-t_0)}Ce^{B(t-t_0)}.$$

The nonhomogeneous:

$$X'(t) = AX(t) + U(t),\ X(t_0) = C, \tag{2.9}$$

was discussed in [2] Here, $A \in M_m$ and $C \in M_{m,n}$ are given constant matrices, $U(t)$ is a given $m \times n$ matrix function and $X(t)$ is the unknown $m \times n$ matrix function. The general solution of (2.9) is given by

$$X(t) = e^{A(t-t_0)}C + e^{A(t-t_0)} * U(t),$$

where $*$ is the matrix convolution product and

$$X'(t) = AX(t) + X(t)B + U(t),\ X(t_0) = C, \tag{2.10}$$

Here, $A, B, C \in M_m$ are given constant matrices, $U(t) \in M_n$ is a given matrix function and $X(t) \in M_n$ is the unknown diagonal matrix function. The general solution of (2.10) is given by

$$\begin{aligned}\text{Vecd}\,(X(t)) &= diag(e^{(a_{11}+b_{11})t}, \cdots, e^{(a_{nn}+b_{nn})t})\,\text{Vecd}\,(C)\\ &\quad + diag(e^{(a_{11}+b_{11})t}, \cdots, e^{(a_{nn}+b_{nn})t}) * \text{Vecd}\,(U(t)),\end{aligned}$$

where Vecd is the diagonal extraction operator.

2.7.3 System of linear matrix differential equations

In [7], the authors apply the Kronecker product of matrices and the vector operator to solve the following coupled linear matrix differential equations:

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D,\\ Y'(t) &= CX(t)D + AY(t)B. \end{aligned} \tag{2.11}$$

Here, $AC = CA, BD = DB$, $X(t_0) = E$, $Y(t_0) = F$, $A, B, C, D, E, F \in M_n$ are given constant matrices and $X(t), Y(t) \in M_n$ are the unknown matrices to be solved.

The general exact solution of coupled linear matrix differential equations defined in (2.11) is given by

$$\begin{aligned}\text{Vec}\,X(t) &= e^{(t-t_0)(B^T\otimes A)}\{\cosh((t-t_0)(D^T\otimes C))\,\text{Vec}\,E) + \sinh((t-t_0)(D^T\otimes C))\,\text{Vec}\,F\},\\ \text{Vec}\,Y(t) &= e^{(t-t_0)(B^T\otimes A)}\{\sinh((t-t_0)(D^T\otimes C))\,\text{Vec}\,E + \cosh((t-t_0)((D^T\otimes C))\,\text{Vec}\,F\}.\end{aligned}$$

In [2], the authors use the Kronecker product of matrices and the vector operator to solve the nonhomogeneous coupled linear matrix differential equations:

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= CX(t)D + AY(t)B + V(t). \end{aligned} \tag{2.12}$$

Here, $A, B, C, D, E, F \in M_n$ are given constant matrices such that $AC = CA, BD = DB$, $U(t)$, $V(t) \in M_n$ are given matrix functions and subject to the initial condition $X(0) = E$, $Y(0) = F$, $X(t), Y(t) \in M_n$ are the unknown matrices. Then the general solution of nonhomogeneous coupled linear matrix differential equations defined in (2.12) is given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{t(B^T \otimes A)}\{\cosh(t(D^T \otimes C)) \operatorname{Vec} E + \sinh(t(D^T \otimes C)) \operatorname{Vec} F \\ &\quad + \cosh(t(D^T \otimes C)) * \operatorname{Vec} U(t) + \sinh(t(D^T \otimes C)) * \operatorname{Vec} V(t)\}, \\ \operatorname{Vec} Y(t) &= e^{t(B^T \otimes A)}\{\sinh(t(D^T \otimes C)) \operatorname{Vec} E + \cosh(t((D^T \otimes C)) \operatorname{Vec} F \\ &\quad + \sinh(t(D^T \otimes C)) * \operatorname{Vec} U(t) + \cosh(t((D^T \otimes C)) * \operatorname{Vec} V(t)\}. \end{aligned}$$

and

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= CX(t)D + AY(t)B + V(t). \end{aligned} \tag{2.13}$$

Here, $A, B, C, D, E, F \in M_n$ are given constant matrices such that $(D^T \circ C)(B^T \circ A) = (B^T \circ A)(D^T \circ C)$, $U(t), V(t) \in M_n$ are given matrix functions and subject to the initial condition $X(0) = E$, $Y(0) = F$, $X(t), Y(t) \in M_n$ are the unknown diagonal matrices. Then the general solution of nonhomogeneous coupled linear matrix differential equations defined in (2.13) is given by

$$\begin{aligned} \operatorname{Vecd} X(t) &= e^{t(B^T \circ A)}\{\cosh(t(D^T \circ C)) \operatorname{Vecd} E + \sinh(t(D^T \circ C)) \operatorname{Vecd} F \\ &\quad + \cosh(t(D^T \circ C)) * \operatorname{Vecd} U(t) + \sinh(t(D^T \circ C)) * \operatorname{Vecd} V(t)\}, \\ \operatorname{Vecd} Y(t) &= e^{t(B^T \circ A)}\{\sinh(t(D^T \circ C)) \operatorname{Vecd} E + \cosh(t((D^T \circ C)) \operatorname{Vecd} F \\ &\quad + \sinh(t(D^T \circ C)) * \operatorname{Vecd} U(t) + \cosh(t((D^T \circ C)) * \operatorname{Vecd} V(t)\}, \end{aligned}$$

where $\circ$ is the Hadamard product.

Chapter 3

Solving a general system of nonhomogeneous coupled linear matrix differential equations in terms of Mittag-Leffler matrix functions

In this chapter, we solve a general system of nonhomogeneous the coupled linear matrix differential equations:

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t). \end{aligned}$$

Here, $A, B, C, D, E, F, G, H \in M_n(\mathbb{R})$ be given constant matrices and let $U, V : \Omega \to M_n(\mathbb{R})$ be given matrix-valued functions. We wish to solve certain systems of linear matrix differential equations in unknown matrix-valued functions $X, Y : \Omega \to M_n(\mathbb{R})$.

3.1 The main result

Theorem 3.1. The general solution of the system of nonhomogeneous coupled linear matrix differential equations:

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t) \end{aligned} \tag{3.1}$$

under the assumption that $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$ is given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)}\big\{\big(E_{2,1}((t-t_0)^2((FD)^T \otimes CE))\big) \operatorname{Vec} X(t_0) \\ &\quad + (t-t_0)\big(E_{2,2}((t-t_0)^2((FD)^T \otimes CE))\big)(D^T \otimes C) \operatorname{Vec} Y(t_0) \\ &\quad + \big(E_{2,1}((t-t_0)^2(((FD)^T \otimes CE))\big) * \operatorname{Vec} U(t) \\ &\quad + (t-t_0)\big(E_{2,2}((t-t_0)^2((FD)^T \otimes CE))\big)(D^T \otimes C) * \operatorname{Vec} V(t)\big\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T \otimes G)}\big\{(t-t_0)\big(E_{2,2}((t-t_0)^2((DF)^T \otimes EC)\big)(F^T \otimes E) \operatorname{Vec} X(t_0) \\ &\quad + \big(E_{2,1}((t-t_0)^2((DF)^T \otimes EC))\big) \operatorname{Vec} Y(t_0) \\ &\quad + (t-t_0)\big(E_{2,2}((t-t_0)^2((DF)^T \otimes EC))\big)(F^T \otimes E) * \operatorname{Vec} U(t) \\ &\quad + \big(E_{2,1}((t-t_0)^2((DF)^T \otimes EC))\big) * \operatorname{Vec} V(t)\big\}. \end{aligned} \tag{3.2}$$

Proof. Using Theorem 2.5, we can transform the system (3.1) into the vector form:

$$\begin{aligned}
\operatorname{Vec} X'(t) &= \operatorname{Vec}(AX(t)B + CY(t)D + U(t)) \\
&= \operatorname{Vec}(AX(t)B) + \operatorname{Vec}(CY(t)D) + \operatorname{Vec} U(t) \\
&= (B^T \otimes A)\operatorname{Vec} X(t) + (D^T \otimes C)\operatorname{Vec} Y(t) + \operatorname{Vec} U(t), \\
\operatorname{Vec} Y'(t) &= \operatorname{Vec}(EX(t)F + GY(t)H + V(t)) \\
&= \operatorname{Vec}(EX(t)F) + \operatorname{Vec}(GY(t)H) + \operatorname{Vec} V(t) \\
&= (F^T \otimes E)\operatorname{Vec} X(t) + (H^T \otimes G)\operatorname{Vec} Y(t) + \operatorname{Vec} V(t).
\end{aligned}$$

We have the following system:

$$\begin{bmatrix} \operatorname{Vec} X'(t) \\ \operatorname{Vec} Y'(t) \end{bmatrix} = \begin{bmatrix} B^T \otimes A & D^T \otimes C \\ F^T \otimes E & H^T \otimes G \end{bmatrix} \begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix} + \begin{bmatrix} \operatorname{Vec} U(t) \\ \operatorname{Vec} V(t) \end{bmatrix}$$

.

denote $P = \begin{bmatrix} B^T \otimes A & 0 \\ 0 & H^T \otimes G \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & D^T \otimes C \\ F^T \otimes E & 0 \end{bmatrix}$.

From (1.3), this system has the following solution:

$$\begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix} = e^{(t-t_0)S} \begin{bmatrix} \operatorname{Vec} X(t_0) \\ \operatorname{Vec} Y(t_0) \end{bmatrix} + e^{(t-t_0)S} * \begin{bmatrix} \operatorname{Vec} U(t) \\ \operatorname{Vec} V(t) \end{bmatrix},$$

where $S = P + Q$.

Now we will compute e^S since $DB = HD, AC = CG, FH = BF, GE = EA$, then we have

$$\begin{aligned}
PQ &= \begin{bmatrix} B^T \otimes A & 0 \\ 0 & H^T \otimes G \end{bmatrix} \begin{bmatrix} 0 & D^T \otimes C \\ F^T \otimes E & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & (B^T \otimes A)(D^T \otimes C) \\ (H^T \otimes G)(F^T \otimes E) & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & (DB)^T \otimes AC \\ (FH)^T \otimes GE & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & (HD)^T \otimes CG \\ (BF)^T \otimes EA & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & (D^T \otimes C)(H^T \otimes G) \\ (F^T \otimes E)(B^T \otimes A) & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & D^T \otimes C \\ F^T \otimes E & 0 \end{bmatrix} \begin{bmatrix} B^T \otimes A & 0 \\ 0 & H^T \otimes G \end{bmatrix} \\
&= QP.
\end{aligned}$$

From which it follows from Lemma 2.7 that $e^S = e^{P+Q} = e^P e^Q$.

By Lemma 2.11, we have

$$e^P = \begin{bmatrix} e^{B^T \otimes A} & 0 \\ 0 & e^{H^T \otimes G} \end{bmatrix}.$$

By Lemma 2.12, we have

$$e^{Q} = \begin{bmatrix} E_{2,1}((FD)^T \otimes CE) & (E_{2,2}((FD)^T \otimes CE))(D^T \otimes C) \\ (E_{2,2}((DF)^T \otimes EC))(F^T \otimes E) & E_{2,1}((DF)^T \otimes EC) \end{bmatrix}.$$

Thus

$$\begin{aligned} e^{S} &= \begin{bmatrix} e^{B^T \otimes A} & 0 \\ 0 & e^{H^T \otimes G} \end{bmatrix} \begin{bmatrix} E_{2,1}((FD)^T \otimes CE) & (E_{2,2}((FD)^T \otimes CE))(D^T \otimes C) \\ (E_{2,2}((DF)^T \otimes EC))(F^T \otimes E) & E_{2,1}((DF)^T \otimes EC) \end{bmatrix} \\ &= \begin{bmatrix} e^{B^T \otimes A} E_{2,1}((FD)^T \otimes CE) & e^{B^T \otimes A}(E_{2,2}((FD)^T \otimes CE))(D^T \otimes C) \\ e^{H^T \otimes G}(E_{2,2}((DF)^T \otimes EC))(F^T \otimes E) & e^{H^T \otimes G} E_{2,1}((DF)^T \otimes EC) \end{bmatrix}. \end{aligned}$$

Denoting

$$\begin{aligned} R_1 &= e^{(t-t_0)(B^T \otimes A)} E_{2,1}((t-t_0)^2((FD)^T \otimes CE)), \\ R_2 &= e^{(t-t_0)(B^T \otimes A)} (t-t_0)(E_{2,2}((t-t_0)^2((FD)^T \otimes CE)))(D^T \otimes C), \\ R_3 &= e^{(t-t_0)(H^T \otimes G)} (t-t_0)(E_{2,2}((t-t_0)^2((DF)^T \otimes EC)))(F^T \otimes E), \\ R_4 &= e^{(t-t_0)(H^T \otimes G)} E_{2,1}((t-t_0)^2((DF)^T \otimes EC)), \end{aligned}$$

we obtain

$$e^{(t-t_0)S} \begin{bmatrix} \operatorname{Vec} X(t_0) \\ \operatorname{Vec} Y(t_0) \end{bmatrix} = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} \begin{bmatrix} \operatorname{Vec} X(t_0) \\ \operatorname{Vec} Y(t_0) \end{bmatrix} = \begin{bmatrix} R_1 \operatorname{Vec} X(t_0) + R_2 \operatorname{Vec} Y(t_0) \\ R_3 \operatorname{Vec} X(t_0) + R_4 \operatorname{Vec} Y(t_0) \end{bmatrix}.$$

We also have

$$e^{(t-t_0)S} * \begin{bmatrix} \operatorname{Vec} U(t) \\ \operatorname{Vec} V(t) \end{bmatrix} = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} * \begin{bmatrix} \operatorname{Vec} U(t) \\ \operatorname{Vec} V(t) \end{bmatrix} = \begin{bmatrix} R_1 * \operatorname{Vec} U(t) + R_2 * \operatorname{Vec} V(t) \\ R_3 * \operatorname{Vec} U(t) + R_4 * \operatorname{Vec} V(t) \end{bmatrix}.$$

Therefore, the general solution of (3.1) is given by (3.2). □

3.2 Special cases

Corollary 3.2. The general solution of the system

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D, \\ Y'(t) &= EX(t)F + GY(t)H \end{aligned}$$

under the assumption that $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$ is given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)} \{ (E_{2,1}((t-t_0)^2((FD)^T \otimes CE))) \operatorname{Vec} X(t_0) \\ &\quad + (t-t_0)(E_{2,2}((t-t_0)^2((FD)^T \otimes CE)))(D^T \otimes C) \operatorname{Vec} Y(t_0), \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T \otimes G)} \{ (t-t_0)(E_{2,2}((t-t_0)^2((DF)^T \otimes EC))(F^T \otimes E) \operatorname{Vec} X(t_0) \\ &\quad + (E_{2,1}((t-t_0)^2((DF)^T \otimes EC))) \operatorname{Vec} Y(t_0). \end{aligned} \tag{3.3}$$

Proof. Putting $U(t) = V(t) = 0$ in (3.2) we obtain

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)}\{(E_{2,1}((t-t_0)^2((FD)^T \otimes CE))) \operatorname{Vec} X(t_0) \\
&\quad + (t-t_0)(E_{2,2}((t-t_0)^2((FD)^T \otimes CE)))(D^T \otimes C) \operatorname{Vec} Y(t_0) \\
&\quad + (E_{2,1}((t-t_0)^2((FD)^T \otimes CE))) * \operatorname{Vec}(0) \\
&\quad + (t-t_0)(E_{2,2}((t-t_0)^2((FD)^T \otimes CE)))(D^T \otimes C) * \operatorname{Vec}(0)\} \\
&= e^{(t-t_0)(B^T \otimes A)}\{(E_{2,1}((t-t_0)^2((FD)^T \otimes CE))) \operatorname{Vec} X(t_0) \\
&\quad + (t-t_0)(E_{2,2}((t-t_0)^2((FD)^T \otimes CE)))(D^T \otimes C) \operatorname{Vec} Y(t_0). \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T \otimes G)}\{(t-t_0)(E_{2,2}((t-t_0)^2((DF)^T \otimes EC))(F^T \otimes E) \operatorname{Vec} X(t_0) \\
&\quad + (E_{2,1}((t-t_0)^2((DF)^T \otimes EC))) \operatorname{Vec} Y(t_0) \\
&\quad + (t-t_0)(E_{2,2}((t-t_0)^2((DF)^T \otimes EC)))(F^T \otimes E) * \operatorname{Vec}(0) \\
&\quad + (E_{2,1}((t-t_0)^2((DF)^T \otimes EC))) * \operatorname{Vec}(0)\} \\
&= e^{(t-t_0)(H^T \otimes G)}\{(t-t_0)(E_{2,2}((t-t_0)^2((DF)^T \otimes EC))(F^T \otimes E) \operatorname{Vec} X(t_0) \\
&\quad + (E_{2,1}((t-t_0)^2((DF)^T \otimes EC))) \operatorname{Vec} Y(t_0).
\end{aligned}$$

□

Corollary 3.3. The general solution of the system

$$\begin{aligned}
X'(t) &= AX(t)B + CY(t)D + U(t), \\
Y'(t) &= CX(t)D + AY(t)B + V(t)
\end{aligned} \tag{3.4}$$

under the assumption that $AC = CA$ and $BD = DB$, is given by

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)}\{\cosh((t-t_0)(D^T \otimes C)) \operatorname{Vec} X(t_0) \\
&\quad + \sinh((t-t_0)(D^T \otimes C)) \operatorname{Vec} Y(t_0) + \cosh((t-t_0)(D^T \otimes C)) * \operatorname{Vec} U(t) \\
&\quad + \sinh((t-t_0)(D^T \otimes C)) * \operatorname{Vec} V(t)\}, \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T \otimes A)}\{\sinh((t-t_0)(D^T \otimes C)) \operatorname{Vec} X(t_0) \\
&\quad + \cosh((t-t_0)((D^T \otimes C)) \operatorname{Vec} Y(t_0) + \sinh((t-t_0)(D^T \otimes C)) * \operatorname{Vec} U(t) \\
&\quad + \cosh((t-t_0)((D^T \otimes C)) * \operatorname{Vec} V(t)\}.
\end{aligned} \tag{3.5}$$

Proof. Putting $E = C$, $F = D$, $G = A$ and $H = B$ in (3.2) we obtain

$$
\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,1}((t-t_0)^2((D^2)^T \otimes C^2)))VecX(t_0) \\
&\quad + (E_{2,2}((t-t_0)^2((D^2)^T \otimes C^2)))(t-t_0)(D^T \otimes C)\operatorname{Vec} Y(t_0) \\
&\quad + (E_{2,1}((t-t_0)^2((D^2)^T \otimes C^2))) * \operatorname{Vec} U(t) \\
&\quad + (E_{2,2}((t-t_0)^2((D^2)^T \otimes C^2)))(t-t_0)(D^T \otimes C) * \operatorname{Vec} V(t)\} \\
&= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,1}((t-t_0)^2(D^T \otimes C)^2))\operatorname{Vec} X(t_0) \\
&\quad + (E_{2,2}((t-t_0)^2(D^T \otimes C)^2))(t-t_0)(D^T \otimes C)\operatorname{Vec} Y(t_0) \\
&\quad + (E_{2,1}((t-t_0)^2(D^T \otimes C)^2)) * \operatorname{Vec} U(t) \\
&\quad + (E_{2,2}((t-t_0)^2(D^T \otimes C)^2))(t-t_0)(D^T \otimes C) * \operatorname{Vec} V(t)\} \\
&= e^{(t-t_0)(B^T\otimes A)}\{\cosh((t-t_0)(D^T \otimes C))\operatorname{Vec} X(t_0) \\
&\quad + \sinh((t-t_0)(D^T \otimes C))\operatorname{Vec} Y(t_0) + \cosh((t-t_0)(D^T \otimes C)) * \operatorname{Vec} U(t) \\
&\quad + \sinh((t-t_0)(D^T \otimes C)) * \operatorname{Vec} V(t)\}. \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,2}((t-t_0)^2((D^2)^T \otimes C^2)))(t-t_0)(D^T \otimes C)\operatorname{Vec} X(t_0) \\
&\quad + (E_{2,1}((t-t_0)^2((D^2)^T \otimes C^2)))\operatorname{Vec} Y(t_0) \\
&\quad + (E_{2,2}((t-t_0)^2((D^2)^T \otimes C^2)))(t-t_0)(D^T \otimes C) * \operatorname{Vec} U(t) \\
&\quad + (E_{2,1}((t-t_0)^2((D^2)^T \otimes C^2))) * \operatorname{Vec} V(t)\} \\
&= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,2}((t-t_0)^2(D^T \otimes C)^2))(t-t_0)(D^T \otimes C)\operatorname{Vec} X(t_0) \\
&\quad + (E_{2,1}((t-t_0)^2(D^T \otimes C)^2))\operatorname{Vec} Y(t_0) \\
&\quad + (E_{2,2}((t-t_0)^2(D^T \otimes C)^2))(t-t_0)(D^T \otimes C) * \operatorname{Vec} U(t) \\
&\quad + (E_{2,1}((t-t_0)^2(D^T \otimes C)^2) * \operatorname{Vec} V(t)\} \\
&= e^{(t-t_0)(B^T\otimes A)}\{\sinh((t-t_0)(D^T \otimes C))\operatorname{Vec} X(t_0) \\
&\quad + \cosh((t-t_0)(D^T \otimes C))\operatorname{Vec} Y(t_0) + \sinh((t-t_0)(D^T \otimes C)) * \operatorname{Vec} U(t) \\
&\quad + \cosh((t-t_0)(D^T \otimes C)) * \operatorname{Vec} V(t)\}.
\end{aligned}
$$

□

Corollary 3.4. The general solution of the system

$$
\begin{aligned}
X'(t) &= AX(t)B + CY(t)D, \\
Y'(t) &= CX(t)D + AY(t)B
\end{aligned}
\tag{3.6}
$$

under the assumption that $AC = CA$ and $BD = DB$, is given by

$$
\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\{\cosh((t-t_0)(D^T \otimes C))\operatorname{Vec} X(t_0) \\
&\quad + \sinh((t-t_0)(D^T \otimes C))\operatorname{Vec} Y(t_0)\}, \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T\otimes A)}\{\sinh((t-t_0)(D^T \otimes C))\operatorname{Vec} X(t_0) \\
&\quad + \cosh((t-t_0)((D^T \otimes C))\operatorname{Vec} Y(t_0)\}.
\end{aligned}
\tag{3.7}
$$

Proof. Putting $U(t) = V(t) = 0$ in (3.5) we obtain

$$
\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)}\{(E_{2,1}((t-t_0)^2((D^2)^T \otimes C^2)))VecX(t_0)\\
&\quad + (E_{2,2}((t-t_0)^2((D^2)^T \otimes C^2)))(t-t_0)(D^T \otimes C)\operatorname{Vec} Y(t_0)\\
&\quad + (E_{2,1}((t-t_0)^2((D^2)^T \otimes C^2))) * \operatorname{Vec}(0)\\
&\quad + (E_{2,2}((t-t_0)^2((D^2)^T \otimes C^2)))(t-t_0)(D^T \otimes C) * \operatorname{Vec}(0)\}\\
&= e^{(t-t_0)(B^T \otimes A)}\{(E_{2,1}((t-t_0)^2(D^T \otimes C)^2))\operatorname{Vec} X(t_0)\\
&\quad + (E_{2,2}((t-t_0)^2(D^T \otimes C)^2))(t-t_0)(D^T \otimes C)\operatorname{Vec} Y(t_0)\}\\
&= e^{(t-t_0)(B^T \otimes A)}\{\cosh((t-t_0)(D^T \otimes C))\operatorname{Vec} X(t_0)\\
&\quad + \sinh((t-t_0)(D^T \otimes C))\operatorname{Vec} Y(t_0)\}.\\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T \otimes A)}\{(E_{2,2}((t-t_0)^2((D^2)^T \otimes C^2)))(t-t_0)(D^T \otimes C)\operatorname{Vec} X(t_0)\\
&\quad + (E_{2,1}((t-t_0)^2((D^2)^T \otimes C^2)))\operatorname{Vec} Y(t_0)\\
&\quad + (E_{2,2}((t-t_0)^2((D^2)^T \otimes C^2)))(t-t_0)(D^T \otimes C) * \operatorname{Vec}(0)\\
&\quad + (E_{2,1}((t-t_0)^2((D^2)^T \otimes C^2))) * \operatorname{Vec}(0)\}\\
&= e^{(t-t_0)(B^T \otimes A)}\{(E_{2,2}((t-t_0)^2(D^T \otimes C)^2))(t-t_0)(D^T \otimes C)\operatorname{Vec} X(t_0)\\
&\quad + (E_{2,1}((t-t_0)^2(D^T \otimes C)^2))\operatorname{Vec} Y(t_0)\}\\
&= e^{(t-t_0)(B^T \otimes A)}\{\sinh((t-t_0)(D^T \otimes C))\operatorname{Vec} X(t_0)\\
&\quad + \cosh((t-t_0)(D^T \otimes C))\operatorname{Vec} Y(t_0)\}.
\end{aligned}
$$

□

Corollary 3.5. The general solution of the system

$$
\begin{aligned}
X'(t) &= AX(t)B + Y(t) + U(t),\\
Y'(t) &= X(t) + AY(t)B + V(t)
\end{aligned}
$$

is given by

$$
\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)}\{\cosh(t-t_0)\operatorname{Vec} X(t_0) + \sinh(t-t_0)\operatorname{Vec} Y(t_0)\\
&\quad + \cosh(t-t_0)(I_{n^2} * \operatorname{Vec} U(t)) + \sinh(t-t_0)(I_{n^2} * \operatorname{Vec} V(t))\},\\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T \otimes A)}\{\sinh(t-t_0)\operatorname{Vec} X(t_0) + \cosh(t-t_0)\operatorname{Vec} Y(t_0)\\
&\quad + \sinh(t-t_0)(I_{n^2} * \operatorname{Vec} U(t)) + \cosh(t-t_0)(I_{n^2} * \operatorname{Vec} V(t))\}.
\end{aligned} \tag{3.8}
$$

Proof. Putting $C = D = I_n$ in (3.5) we obtain

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,1}((t-t_0)^2(I_n\otimes I_n)))\operatorname{Vec} X(t_0)\\
&\quad + (E_{2,2}((t-t_0)^2(I_n\otimes I_n)))(t-t_0)(I_n\otimes I_n)\operatorname{Vec} Y(t_0)\\
&\quad + (E_{2,1}((t-t_0)^2(I_n\otimes I_n))) * \operatorname{Vec} U(t)\\
&\quad + (E_{2,2}((t-t_0)^2(I_n\otimes I_n)))(t-t_0)(I_n\otimes I_n) * \operatorname{Vec} V(t)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,1}((t-t_0)^2(I_{n^2})))\operatorname{Vec} X(t_0)\\
&\quad + (E_{2,2}((t-t_0)^2(I_{n^2})))(t-t_0)(I_{n^2})\operatorname{Vec} Y(t_0)\\
&\quad + (E_{2,1}((t-t_0)^2(I_{n^2}))) * \operatorname{Vec} U(t)\\
&\quad + (E_{2,2}((t-t_0)^2(I_{n^2})))(t-t_0)(I_{n^2}) * \operatorname{Vec} V(t)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{\cosh((t-t_0)(I_{n^2}))\operatorname{Vec} X(t_0)\\
&\quad + \sinh((t-t_0)(I_{n^2}))\operatorname{Vec} Y(t_0)\\
&\quad + \cosh((t-t_0)(I_{n^2})) * \operatorname{Vec} U(t) + \sinh((t-t_0)(I_{n^2})) * \operatorname{Vec} V(t)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{\cosh(t-t_0)\operatorname{Vec} X(t_0) + \sinh(t-t_0)\operatorname{Vec} Y(t_0)\\
&\quad + \cosh(t-t_0)(I_n^2 * \operatorname{Vec} U(t)) + \sinh(t-t_0)(I_n^2 * \operatorname{Vec} V(t))\}.\\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,2}((t-t_0)^2(I_n\otimes I_n)))(t-t_0)(I_n\otimes I_n)\operatorname{Vec} X(t_0)\\
&\quad + (E_{2,1}((t-t_0)^2(I_n\otimes I_n))\operatorname{Vec} Y(t_0)\\
&\quad + (E_{2,2}((t-t_0)^2(I_n\otimes I_n)))(t-t_0)(I_n\otimes I_n) * \operatorname{Vec} U(t)\\
&\quad + (E_{2,1}((t-t_0)^2(I_n\otimes I_n)) * \operatorname{Vec} V(t)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,2}((t-t_0)^2(I_{n^2})))(t-t_0)(I_{n^2})\operatorname{Vec} X(t_0)\\
&\quad + (E_{2,1}((t-t_0)^2(I_{n^2}))\operatorname{Vec} Y(t_0)\\
&\quad + (E_{2,2}((t-t_0)^2(I_{n^2})))(t-t_0)(I_{n^2}) * \operatorname{Vec} U(t)\\
&\quad + (E_{2,1}((t-t_0)^2(I_{n^2})) * \operatorname{Vec} V(t)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{\sinh((t-t_0)(I_{n^2}))\operatorname{Vec} X(t_0)\\
&\quad + \cosh((t-t_0)(I_{n^2}))\operatorname{Vec} Y(t_0)\\
&\quad + \sinh((t-t_0)(I_{n^2})) * \operatorname{Vec} U(t) + \cosh((t-t_0)(I_{n^2})) * \operatorname{Vec} V(t)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{\sinh(t-t_0)\operatorname{Vec} X(t_0) + \cosh(t-t_0)\operatorname{Vec} Y(t_0)\\
&\quad + \sinh(t-t_0)(I_n^2 * \operatorname{Vec} U(t)) + \cosh(t-t_0)(I_n^2 * \operatorname{Vec} V(t))\}.
\end{aligned}$$

□

Corollary 3.6. The general solution of the system

$$\begin{aligned}
X'(t) &= AX(t)B + Y(t),\\
Y'(t) &= X(t) + AY(t)B
\end{aligned}$$

is given by

$$\begin{aligned}\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\{\cosh(t-t_0)\operatorname{Vec} X(t_0)+\sinh(t-t_0)\operatorname{Vec} Y(t_0)\},\\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T\otimes A)}\{\sinh(t-t_0)\operatorname{Vec} X(t_0)+\cosh(t-t_0)\operatorname{Vec} Y(t_0)\}.\end{aligned}$$

Proof. Putting $U(t)=V(t)=0$ in (3.8) we obtain

$$\begin{aligned}\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,1}((t-t_0)^2(I_n\otimes I_n)))\operatorname{Vec} X(t_0)\\
&\quad+(E_{2,2}((t-t_0)^2(I_n\otimes I_n)))(t-t_0)(I_n\otimes I_n)\operatorname{Vec} Y(t_0)\\
&\quad+(E_{2,1}((t-t_0)^2(I_n\otimes I_n)))*\operatorname{Vec}(0)\\
&\quad+(E_{2,2}((t-t_0)^2(I_n\otimes I_n)))(t-t_0)(I_n\otimes I_n)*\operatorname{Vec}(0)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,1}((t-t_0)^2(I_{n^2})))\operatorname{Vec} X(t_0)\\
&\quad+(E_{2,2}((t-t_0)^2(I_{n^2})))(t-t_0)(I_{n^2})\operatorname{Vec} Y(t_0)\\
&\quad+(E_{2,1}((t-t_0)^2(I_{n^2})))*\operatorname{Vec}(0)\\
&\quad+(E_{2,2}((t-t_0)^2(I_{n^2})))(t-t_0)(I_{n^2})*\operatorname{Vec}(0)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{\cosh((t-t_0)(I_{n^2}))\operatorname{Vec} X(t_0)+\sinh((t-t_0)(I_{n^2}))\operatorname{Vec} Y(t_0)\\
&\quad+\cosh((t-t_0)(I_{n^2}))*\operatorname{Vec}(0)+\sinh((t-t_0)(I_{n^2}))*\operatorname{Vec}(0)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{\cosh(t-t_0)\operatorname{Vec} X(t_0)+\sinh(t-t_0)\operatorname{Vec} Y(t_0)\}.\\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,2}((t-t_0)^2(I_n\otimes I_n)))(t-t_0)(I_n\otimes I_n)\operatorname{Vec} X(t_0)\\
&\quad+(E_{2,1}((t-t_0)^2(I_n\otimes I_n))\operatorname{Vec} Y(t_0)\\
&\quad+(E_{2,2}((t-t_0)^2(I_n\otimes I_n)))(t-t_0)(I_n\otimes I_n)*\operatorname{Vec}(0)\\
&\quad+(E_{2,1}((t-t_0)^2(I_n\otimes I_n))*\operatorname{Vec}(0)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,2}((t-t_0)^2(I_{n^2})))(t-t_0)(I_{n^2})\operatorname{Vec} X(t_0)\\
&\quad+(E_{2,1}((t-t_0)^2(I_{n^2}))\operatorname{Vec} Y(t_0)\\
&\quad+(E_{2,2}((t-t_0)^2(I_{n^2})))(t-t_0)(I_{n^2})*\operatorname{Vec}(0)\\
&\quad+(E_{2,1}((t-t_0)^2(I_{n^2}))*\operatorname{Vec}(0)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{\sinh((t-t_0)(I_{n^2}))\operatorname{Vec} X(t_0)+\cosh((t-t_0)(I_{n^2}))\operatorname{Vec} Y(t_0)\\
&\quad+\sinh((t-t_0)(I_{n^2}))*\operatorname{Vec}(0)+\cosh((t-t_0)(I_{n^2}))*\operatorname{Vec}(0)\}\\
&= e^{(t-t_0)(B^T\otimes A)}\{\sinh(t-t_0)\operatorname{Vec} X(t_0)+\cosh(t-t_0)\operatorname{Vec} Y(t_0)\}.\end{aligned}$$

□

Corollary 3.7. The general solution of the system

$$\begin{aligned}X'(t) &= AX(t)B+CY(t)\quad+U(t),\\ Y'(t) &= EX(t)\quad+GY(t)B+V(t)\end{aligned}$$

under the condition $AC = CG, GE = EA$, is given by

$$
\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)} \operatorname{Vec}\Big\{\big(E_{2,1}((t-t_0)^2 CE)\big)X(t_0) + (t-t_0)\big(E_{2,2}((t-t_0)^2 CE)\big)CY(t_0)\Big\} \\
&\quad + e^{(t-t_0)(B^T \otimes A)}\Big\{\big(I_n \otimes E_{2,1}((t-t_0)^2 CE)\big) * \operatorname{Vec} U(t) \\
&\quad + (t-t_0)\big(I_n \otimes \big(E_{2,2}((t-t_0)^2 CE)\big)C\big) * \operatorname{Vec} V(t)\Big\}, \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T \otimes G)} \operatorname{Vec}\Big\{(t-t_0)\big(E_{2,2}((t-t_0)^2 EC)\big)EX(t_0) + \big(E_{2,1}((t-t_0)^2 EC)\big)Y(t_0)\Big\} \\
&\quad + e^{(t-t_0)(B^T \otimes G)}\Big\{(t-t_0)\big(I_n \otimes \big(E_{2,2}((t-t_0)^2 EC)\big)E\big) * \operatorname{Vec} U(t) \\
&\quad + \big(I_n \otimes E_{2,1}((t-t_0)^2 EC)\big) * \operatorname{Vec} V(t)\Big\}.
\end{aligned} \tag{3.9}
$$

Proof. Putting $H = B,\ D = F = I_n$ in (3.2) we obtain

$$
\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)}\{\big(E_{2,1}((t-t_0)^2(I_n \otimes CE))\big) \operatorname{Vec} X(t_0) \\
&\quad + (t-t_0)\big(E_{2,2}((t-t_0)^2(I_n \otimes CE))\big)(I_n \otimes C) \operatorname{Vec} Y(t_0) \\
&\quad + \big(E_{2,1}((t-t_0)^2(I_n \otimes CE))\big) * \operatorname{Vec} U(t) \\
&\quad + (t-t_0)\big(E_{2,2}((t-t_0)^2(I_n \otimes CE))\big)(I_n \otimes C) * \operatorname{Vec} V(t)\} \\
&= e^{(t-t_0)(B^T \otimes A)}\{\big(I_n \otimes E_{2,1}((t-t_0)^2(CE))\big) \operatorname{Vec} X(t_0) \\
&\quad + (t-t_0)\big(I_n \otimes E_{2,2}((t-t_0)^2(CE))\big)(I_n \otimes C) \operatorname{Vec} Y(t_0) \\
&\quad + \big(I_n \otimes E_{2,1}((t-t_0)^2(CE))\big) * \operatorname{Vec} U(t) \\
&\quad + (t-t_0)\big(I_n \otimes E_{2,2}((t-t_0)^2(CE))\big)(I_n \otimes C) * \operatorname{Vec} V(t)\} \\
&= e^{(t-t_0)(B^T \otimes A)}\{\big(I_n \otimes E_{2,1}((t-t_0)^2(CE))\big) \operatorname{Vec} X(t_0) \\
&\quad + (t-t_0)\big(I_n \otimes (E_{2,2}((t-t_0)^2(CE))C)\big) \operatorname{Vec} Y(t_0) \\
&\quad + \big(I_n \otimes E_{2,1}((t-t_0)^2(CE))\big) * \operatorname{Vec} U(t) \\
&\quad + (t-t_0)\big(I_n \otimes (E_{2,2}((t-t_0)^2(CE))C)\big) * \operatorname{Vec} V(t)\} \\
&= e^{(t-t_0)(B^T \otimes A)}\{\operatorname{Vec}\big(E_{2,1}((t-t_0)^2(CE))\big)X(t_0)I_n\big) \\
&\quad + (t-t_0)\operatorname{Vec}\big((E_{2,2}((t-t_0)^2(CE)))CY(t_0)I_n\big) \\
&\quad + \big(I_n \otimes E_{2,1}((t-t_0)^2(CE))\big) * \operatorname{Vec} U(t) \\
&\quad + (t-t_0)\big(I_n \otimes (E_{2,2}((t-t_0)^2(CE))C)\big) * \operatorname{Vec} V(t)\} \\
&= e^{(t-t_0)(B^T \otimes A)} \operatorname{Vec}\{\big(E_{2,1}((t-t_0)^2(CE))\big)X(t_0)\big) \\
&\quad + (t-t_0)\big((E_{2,2}((t-t_0)^2(CE)))CY(t_0)\big)\} \\
&\quad + e^{(t-t_0)(B^T \otimes A)}\{\big(I_n \otimes E_{2,1}((t-t_0)^2(CE))\big) * \operatorname{Vec} U(t) \\
&\quad + (t-t_0)\big(I_n \otimes (E_{2,2}((t-t_0)^2(CE))C)\big) * \operatorname{Vec} V(t)\}.
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T\otimes G)}\big\{(t-t_0)\big(E_{2,2}((t-t_0)^2(I_n\otimes EC))\big)(I_n\otimes E)\operatorname{Vec} X(t_0)\\
&\quad+\big(E_{2,1}((t-t_0)^2(I_n\otimes EC))\big)\operatorname{Vec} Y(t_0)\\
&\quad+(t-t_0)\big(E_{2,2}((t-t_0)^2(I_n\otimes EC))\big)(I_n\otimes E)*\operatorname{Vec} U(t)\\
&\quad+\big(E_{2,1}((t-t_0)^2(I_n\otimes EC))\big)*\operatorname{Vec} V(t)\big\}\\
&= e^{(t-t_0)(B^T\otimes G)}\big\{(t-t_0)\big(I_n\otimes E_{2,2}((t-t_0)^2(EC))\big)(I_n\otimes E)\operatorname{Vec} X(t_0)\\
&\quad+\big(I_n\otimes E_{2,1}((t-t_0)^2(EC))\big)\operatorname{Vec} Y(t_0)\\
&\quad+(t-t_0)\big(I_n\otimes E_{2,2}((t-t_0)^2(EC))\big)(I_n\otimes E)*\operatorname{Vec} U(t)\\
&\quad+\big(I_n\otimes E_{2,1}((t-t_0)^2(EC))\big)*\operatorname{Vec} V(t)\big\}\\
&= e^{(t-t_0)(B^T\otimes G)}\big\{(t-t_0)\big(I_n\otimes (E_{2,2}((t-t_0)^2(EC))E)\big)\operatorname{Vec} X(t_0)\\
&\quad+\big(I_n\otimes E_{2,1}((t-t_0)^2(EC))\big)\operatorname{Vec} Y(t_0)\\
&\quad+(t-t_0)\big(I_n\otimes (E_{2,2}((t-t_0)^2(EC))E)\big)*\operatorname{Vec} U(t)\\
&\quad+\big(I_n\otimes E_{2,1}((t-t_0)^2(EC))\big)*\operatorname{Vec} V(t)\big\}\\
&= e^{(t-t_0)(B^T\otimes G)}\big\{(t-t_0)\operatorname{Vec}\big((E_{2,2}((t-t_0)^2(EC)))\big)EX(t_0)I_n\\
&\quad+\operatorname{Vec}\big(E_{2,1}((t-t_0)^2(EC))\big)Y(t_0)I_n\big)\\
&\quad+(t-t_0)\big(I_n\otimes (E_{2,2}((t-t_0)^2(EC))E)\big)*\operatorname{Vec} U(t)\\
&\quad+\big(I_n\otimes E_{2,1}((t-t_0)^2(EC))\big)*\operatorname{Vec} V(t)\big\}\\
&= e^{(t-t_0)(B^T\otimes G)}\operatorname{Vec}\big\{(t-t_0)\big((E_{2,2}((t-t_0)^2(EC)))EX(t_0)\big)\\
&\quad+\big(E_{2,1}((t-t_0)^2(EC)))Y(t_0)\big)\big\}\\
&\quad+e^{(t-t_0)(B^T\otimes G)}\big\{(t-t_0)\big(I_n\otimes (E_{2,2}((t-t_0)^2(EC))E)\big)*\operatorname{Vec} U(t)\\
&\quad+\big(I_n\otimes E_{2,1}((t-t_0)^2(EC))\big)*\operatorname{Vec} V(t)\big\}.
\end{aligned}
$$

□

Corollary 3.8. The general solution of the system

$$
\begin{aligned}
X'(t) &= AX(t)B + CY(t),\\
Y'(t) &= EX(t) \quad + GY(t)B
\end{aligned}
$$

under the condition $AC = CG, GE = EA$, is given by

$$
\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\operatorname{Vec}\Big\{\big(E_{2,1}((t-t_0)^2 CE)\big)X(t_0) + (t-t_0)\big(E_{2,2}((t-t_0)^2 CE)\big)CY(t_0)\Big\},\\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T\otimes G)}\operatorname{Vec}\Big\{(t-t_0)\big(E_{2,2}((t-t_0)^2 EC)\big)EX(t_0) + \big(E_{2,1}((t-t_0)^2 EC)\big)Y(t_0)\Big\}.
\end{aligned} \tag{3.10}
$$

Proof. Putting $U(t) = V(t) = 0$ in (3.9) we obtain

$$
\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)} \{ (E_{2,1}((t-t_0)^2 (I_n \otimes CE))) \operatorname{Vec} X(t_0) \\
&\quad + (t-t_0) (E_{2,2}((t-t_0)^2 (I_n \otimes CE))) (I_n \otimes C) \operatorname{Vec} Y(t_0) \\
&= e^{(t-t_0)(B^T \otimes A)} \{ (I_n \otimes E_{2,1}((t-t_0)^2 (CE))) \operatorname{Vec} X(t_0) \\
&\quad + (t-t_0) (I_n \otimes E_{2,2}((t-t_0)^2 (CE))) (I_n \otimes C) \operatorname{Vec} Y(t_0) \\
&= e^{(t-t_0)(B^T \otimes A)} \{ (I_n \otimes E_{2,1}((t-t_0)^2 (CE))) \operatorname{Vec} X(t_0) \\
&\quad + (t-t_0) (I_n \otimes (E_{2,2}((t-t_0)^2 (CE)) C)) \operatorname{Vec} Y(t_0) \\
&= e^{(t-t_0)(B^T \otimes A)} \{ \operatorname{Vec} (E_{2,1}((t-t_0)^2 (CE))) X(t_0) I_n) \\
&\quad + (t-t_0) \operatorname{Vec} ((E_{2,2}((t-t_0)^2 (CE))) C Y(t_0) I_n) \\
&= e^{(t-t_0)(B^T \otimes A)} \operatorname{Vec} \{ (E_{2,1}((t-t_0)^2 (CE))) X(t_0)) \\
&\quad + (t-t_0) ((E_{2,2}((t-t_0)^2 (CE))) C Y(t_0)) \}. \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T \otimes G)} \{ (t-t_0) (E_{2,2}((t-t_0)^2 (I_n \otimes EC))) (I_n \otimes E) \operatorname{Vec} X(t_0) \\
&\quad + (E_{2,1}((t-t_0)^2 (I_n \otimes EC))) \operatorname{Vec} Y(t_0) \\
&= e^{(t-t_0)(B^T \otimes G)} \{ (t-t_0) (I_n \otimes E_{2,2}((t-t_0)^2 (EC))) (I_n \otimes E) \operatorname{Vec} X(t_0) \\
&\quad + (I_n \otimes E_{2,1}((t-t_0)^2 (EC))) \operatorname{Vec} Y(t_0) \\
&= e^{(t-t_0)(B^T \otimes G)} \{ (t-t_0) (I_n \otimes (E_{2,2}((t-t_0)^2 (EC)) E)) \operatorname{Vec} X(t_0) \\
&\quad + (I_n \otimes E_{2,1}((t-t_0)^2 (EC))) \operatorname{Vec} Y(t_0) \\
&= e^{(t-t_0)(B^T \otimes G)} \{ (t-t_0) \operatorname{Vec} ((E_{2,2}((t-t_0)^2 (EC)))) E X(t_0) I_n \\
&\quad + \operatorname{Vec} (E_{2,1}((t-t_0)^2 (EC))) Y(t_0) I_n) \\
&= e^{(t-t_0)(B^T \otimes G)} \operatorname{Vec} \{ (t-t_0) ((E_{2,2}((t-t_0)^2 (EC))) E X(t_0)) \\
&\quad + (E_{2,1}((t-t_0)^2 (EC))) Y(t_0)) \}.
\end{aligned}
$$

□

Corollary 3.9. The general solution of the system

$$
\begin{aligned}
X'(t) &= AX(t)B + U(t), \\
Y'(t) &= EX(t)F + GY(t)H + V(t)
\end{aligned}
$$

under the condition $FH = BF$ and $GE = EA$, is given by

$$
\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)} \{ \operatorname{Vec} X(t_0) + I * \operatorname{Vec} U(t) \}, \\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T \otimes G)} \operatorname{Vec} \{ (t-t_0) E X(t_0) F + Y(t_0) \} \\
&\quad + e^{(t-t_0)(H^T \otimes G)} \{ (t-t_0)(F^T \otimes E) * \operatorname{Vec} U(t) + I * \operatorname{Vec} V(t) \}.
\end{aligned}
\tag{3.11}
$$

Proof. Putting $C = D = 0$ in (3.2) we obtain

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{\big(E_{2,1}((t-t_0)^2(0))\big)\operatorname{Vec} X(t_0)\\
&\quad + \big(E_{2,2}((t-t_0)^2(0))\big)(t-t_0)(D^T\otimes 0)\operatorname{Vec} Y(t_0)\\
&\quad + \big(E_{2,1}((t-t_0)^2(0))\big) * \operatorname{Vec} U(t)\\
&\quad + \big(E_{2,2}((t-t_0)^2(0))\big)(t-t_0)(D^T\otimes 0) * \operatorname{Vec} V(t)\big\}\\
&= e^{(t-t_0)(B^T\otimes A)}\big\{\operatorname{Vec} X(t_0) + I * \operatorname{Vec} U(t)\big\}.\\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T\otimes G)}\big\{\big(E_{2,2}((t-t_0)^2(0))\big)(t-t_0)(F^T\otimes E)\operatorname{Vec} X(t_0)\\
&\quad + \big(E_{2,1}((t-t_0)^2(0))\big)\operatorname{Vec} Y(t_0)\\
&\quad + \big(E_{2,2}((t-t_0)^2(0))\big)(t-t_0)(F^T\otimes E) * \operatorname{Vec} U(t)\\
&\quad + \big(E_{2,1}((t-t_0)^2(0))\big) * \operatorname{Vec} V(t)\big\}\\
&= e^{(t-t_0)(H^T\otimes G)}\big\{(t-t_0)(F^T\otimes E)\operatorname{Vec} X(t_0) + \operatorname{Vec} Y(t_0)\\
&\quad + (t-t_0)(F^T\otimes E) * \operatorname{Vec} U(t) + I * \operatorname{Vec} V(t)\big\}\\
&= e^{(t-t_0)(H^T\otimes G)}\big\{\operatorname{Vec}(t-t_0)EX(t_0)F + \operatorname{Vec} Y(t_0)\\
&\quad + (t-t_0)(F^T\otimes E) * \operatorname{Vec} U(t) + I * \operatorname{Vec} V(t)\big\}\\
&= e^{(t-t_0)(H^T\otimes G)}\operatorname{Vec}\big\{(t-t_0)EX(t_0)F + Y(t_0)\big\}\\
&\quad + e^{(t-t_0)(H^T\otimes G)}\big\{(t-t_0)(F^T\otimes E) * \operatorname{Vec} U(t) + I * \operatorname{Vec} V(t)\big\}
\end{aligned}$$

□

Corollary 3.10. The general solution of the system

$$\begin{aligned}
X'(t) &= AX(t)B,\\
Y'(t) &= EX(t)F + GY(t)H
\end{aligned}$$

under the condition $FH = BF$ and $GE = EA$, is given by

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{\operatorname{Vec} X(t_0)\big\},\\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T\otimes G)}\operatorname{Vec}\big\{(t-t_0)EX(t_0)F + Y(t_0)\big\}.
\end{aligned}$$

Proof. Putting $U(t) = V(t) = 0$ in (3.11) we obtain

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{\big(E_{2,1}((t-t_0)^2(0))\big)\operatorname{Vec} X(t_0)\\
&\quad + \big(E_{2,2}((t-t_0)^2(0))\big)(t-t_0)(D^T\otimes 0)\operatorname{Vec} Y(t_0)\big\}\\
&= e^{(t-t_0)(B^T\otimes A)}\big\{\operatorname{Vec} X(t_0)\big\}.\\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T\otimes G)}\big\{\big(E_{2,2}((t-t_0)^2(0))\big)(t-t_0)(F^T\otimes E)\operatorname{Vec} X(t_0)\\
&\quad + \big(E_{2,1}((t-t_0)^2(0))\big)\operatorname{Vec} Y(t_0)\big\}\\
&= e^{(t-t_0)(H^T\otimes G)}\big\{(t-t_0)(F^T\otimes E)\operatorname{Vec} X(t_0) + \operatorname{Vec} Y(t_0)\big\}\\
&= e^{(t-t_0)(H^T\otimes G)}\big\{\operatorname{Vec}\big(E(t-t_0)X(t_0)F\big) + \operatorname{Vec} Y(t_0)\big\}\\
&= e^{(t-t_0)(H^T\otimes G)}\operatorname{Vec}\big\{(t-t_0)EX(t_0)F + Y(t_0)\big\}
\end{aligned}$$

□

Corollary 3.11. The general solution of equation

$$X'(t) = AX(t)B + U(t)$$

is given by

$$\text{Vec}\, X(t) = e^{(t-t_0)(B^T \otimes A)}\{\text{Vec}\, X(t_0) + I * \text{Vec}\, U(t)\}. \tag{3.12}$$

Proof. Put $E = F = 0$ in Corollary 3.10. □

Corollary 3.12. The general solution of equation

$$X'(t) = AX(t)B$$

is given by

$$\text{Vec}\, X(t) = e^{(t-t_0)(B^T \otimes A)}\{\text{Vec}\, X(t_0)\}. \tag{3.13}$$

Proof. Put $U(t) = 0$ in Corollary 3.11. □

Chapter 4

Unique solution of initial value problem and numerical examples

4.1 Unique solution of initial value problem associated with the main system

Let $A, B, C, D, E, F, G, H, J, K \in M_n(\mathbb{R})$ be given constant matrices and $U, V : \Omega \to M_n(\mathbb{R})$ be given matrix-valued functions.

Consider the following initial value problem associated with the system (3.1):

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t) \end{aligned}$$

subject to initial conditions $X(0) = J$ and $Y(0) = K$. Suppose $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$. In this case, the solution of this problem is unique and given by

$$\begin{aligned} \operatorname{Vec} X(t) = e^{t(B^T \otimes A)}\{ & (E_{2,1}(t^2((FD)^T \otimes CE))) \operatorname{Vec} J \\ & + t(E_{2,2}(t^2((FD)^T \otimes CE)))(D^T \otimes C) \operatorname{Vec} K \\ & + (E_{2,1}(t^2((FD)^T \otimes CE))) * \operatorname{Vec} U(t) \\ & + t(E_{2,2}(t^2((FD)^T \otimes CE)))(D^T \otimes C) * \operatorname{Vec} V(t)\}, \\ \operatorname{Vec} Y(t) = e^{t(H^T \otimes G)}\{ & t(E_{2,2}(t^2((DF)^T \otimes EC))(F^T \otimes E) \operatorname{Vec} J \\ & + (E_{2,1}(t^2((DF)^T \otimes EC))) \operatorname{Vec} K \\ & + t(E_{2,2}(t^2((DF)^T \otimes EC)))(F^T \otimes E) * \operatorname{Vec} U(t) \\ & + (E_{2,1}(t^2((DF)^T \otimes EC))) * \operatorname{Vec} V(t)\}. \end{aligned}$$

4.2 Numerical examples

Example 4.1. Consider the initial value problem

$$\begin{aligned} X'(t) &= AX(t)B + \quad Y(t) \quad + U(t), \\ Y'(t) &= \quad X(t) \quad + AY(t)B + V(t) \\ X(0) &= J \quad \text{and} \quad Y(0) = K \end{aligned}$$

with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$, $U(t) = \begin{bmatrix} -e^{2t} & 1 \\ 1 & \sin t \end{bmatrix}$, $V(t) = \begin{bmatrix} 1 & e^{2t} \\ \cos t & \sin 2t \end{bmatrix}$.

We have $B^T \otimes A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ -1 & -2 & 1 & 2 \\ -3 & -4 & 3 & 4 \end{bmatrix}$.

By using Corollary 3.5, the above initial value problem has a unique solution given by

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{t(B^T \otimes A)} \left\{ \cosh(t) \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \sinh(t) \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \cosh(t) \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -e^{2t} \\ 1 \\ 1 \\ \sin(t) \end{bmatrix} \right) + \sinh(t) \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ \cos(t) \\ e^{2t} \\ \sin(2t) \end{bmatrix} \right) \right\} \\
&= e^{t(B^T \otimes A)} \left\{ \begin{bmatrix} 2\cosh(t) \\ \cosh(t) \\ -\cosh(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 3\sinh(t) \\ \sinh(t) \\ \sinh(t) \\ -\sinh(t) \end{bmatrix} + \cosh(t) \begin{bmatrix} 1 * -e^{2t} \\ 1 * 1 \\ 1 * 1 \\ 1 * \sin(t) \end{bmatrix} + \sinh(t) \begin{bmatrix} 1 * 1 \\ 1 * \cos(t) \\ 1 * e^{2t} \\ 1 * \sin(2t) \end{bmatrix} \right\} \\
&= e^{t(B^T \otimes A)} \left\{ \begin{bmatrix} 2\cosh(t) \\ \cosh(t) \\ -\cosh(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 3\sinh(t) \\ \sinh(t) \\ \sinh(t) \\ -\sinh(t) \end{bmatrix} + \begin{bmatrix} (1 * -e^{2t})\cosh(t) \\ (1 * 1)\cosh(t) \\ (1 * 1)\cosh(t) \\ (1 * \sin(t))\cosh(t) \end{bmatrix} + \begin{bmatrix} (1 * 1)\sinh(t) \\ (1 * \cos(t))\sinh(t) \\ (1 * e^{2t})\sinh(t) \\ (1 * \sin(2t))\sinh(t) \end{bmatrix} \right\} \\
&= e^{t(B^T \otimes A)} \left\{ \begin{bmatrix} 2\cosh(t) \\ \cosh(t) \\ -\cosh(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 3\sinh(t) \\ \sinh(t) \\ \sinh(t) \\ -\sinh(t) \end{bmatrix} + \begin{bmatrix} (\frac{-e^{2t}-1}{2})\cosh(t) \\ (t)\cosh(t) \\ (t)\cosh(t) \\ (1-\cos(t))\cosh(t) \end{bmatrix} + \begin{bmatrix} (t)\sinh(t) \\ (\sin(t))\sinh(t) \\ (\frac{e^{2t}-1}{2})\sinh(t) \\ (\frac{1-\cos(2t)}{2})\sinh(t) \end{bmatrix} \right\} \\
&= e^{t(B^T \otimes A)} \begin{bmatrix} (\frac{3-e^{2t}}{2})\cosh(t) + (3+t)\sinh(t) \\ (1+t)\cosh(t) + (1+\sin(t))\sinh(t) \\ (-1+t)\cosh(t) + (\frac{1+e^{2t}}{2})\sinh(t) \\ (1-\cos(t))\cosh(t) + (\frac{-1-\cos(2t)}{2})\sinh(t) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\operatorname{Vec} Y(t) &= e^{t(B^T \otimes A)} \left\{ \sinh(t) \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \cosh(t) \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \sinh(t) \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -e^{2t} \\ 1 \\ 1 \\ \sin(t) \end{bmatrix} \right) + \cosh(t) \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ \cos(t) \\ e^{2t} \\ \sin(2t) \end{bmatrix} \right) \right\} \\
&= e^{t(B^T \otimes A)} \left\{ \begin{bmatrix} 2\sinh(t) \\ \sinh(t) \\ -\sinh(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 3\cosh(t) \\ \cosh(t) \\ \cosh(t) \\ -\cosh(t) \end{bmatrix} + \sinh(t) \begin{bmatrix} 1 * -e^{2t} \\ 1 * 1 \\ 1 * 1 \\ 1 * \sin(t) \end{bmatrix} + \cosh(t) \begin{bmatrix} 1 * 1 \\ 1 * \cos(t) \\ 1 * e^{2t} \\ 1 * \sin(2t) \end{bmatrix} \right\} \\
&= e^{t(B^T \otimes A)} \left\{ \begin{bmatrix} 2\sinh(t) \\ \sinh(t) \\ -\sinh(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 3\cosh(t) \\ \cosh(t) \\ \cosh(t) \\ -\cosh(t) \end{bmatrix} + \begin{bmatrix} (1 * -e^{2t})\sinh(t) \\ (1 * 1)\sinh(t) \\ (1 * 1)\sinh(t) \\ (1 * \sin(t))\sinh(t) \end{bmatrix} + \begin{bmatrix} (1 * 1)\cosh(t) \\ (1 * \cos(t))\cosh(t) \\ (1 * e^{2t})\cosh(t) \\ (1 * \sin(2t))\cosh(t) \end{bmatrix} \right\} \\
&= e^{t(B^T \otimes A)} \left\{ \begin{bmatrix} 2\sinh(t) \\ \sinh(t) \\ -\sinh(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 3\cosh(t) \\ \cosh(t) \\ \cosh(t) \\ -\cosh(t) \end{bmatrix} + \begin{bmatrix} (\frac{-e^{2t}-1}{2})\sinh(t) \\ (t)\sinh(t) \\ (t)\sinh \\ (1-\cos(t))\sinh(t) \end{bmatrix} + \begin{bmatrix} (t)\cosh(t) \\ (\sin(t))\cosh(t) \\ (\frac{e^{2t}-1}{2})\cosh(t) \\ (\frac{1-\cos(2t)}{2})\cosh(t) \end{bmatrix} \right\} \\
&= e^{t(B^T \otimes A)} \begin{bmatrix} (\frac{3-e^{2t}}{2})\sinh(t) + (3+t)\cosh(t) \\ (1+t)\sinh(t) + (1+\sin(t))\cosh(t) \\ (-1+t)\sinh(t) + (\frac{1+e^{2t}}{2})\cosh(t) \\ (1-\cos(t))\sinh(t) + (\frac{-1-\cos(2t)}{2})\cosh(t) \end{bmatrix}.
\end{aligned}$$

Example 4.2. Consider the initial value problem

$$\begin{aligned} X'(t) &= AX(t)B \\ Y'(t) &= EX(t)F + GY(t)H, \end{aligned}$$

with $A = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ -6 & 3 \end{bmatrix}$, $E = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$,
$H = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $J = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, and $K = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$.

We have

$$B^T \otimes A = \begin{bmatrix} -6 & 0 & -12 & 0 \\ 9 & -3 & 18 & -6 \\ 2 & 0 & 6 & 0 \\ -3 & 1 & -9 & 3 \end{bmatrix} \text{ and } H^T \otimes G = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & -2 & 4 \\ 1 & 0 & -1 & 0 \\ -1 & 2 & 1 & -2 \end{bmatrix}.$$

By using Corollary 3.10, the above initial value problem has a unique solution given by

$$\text{Vec}\, X(t) = e^{t(B^T \otimes A)} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

$$\begin{aligned} \text{Vec}\, Y(t) &= e^{t(H^T \otimes G)}\, \text{Vec} \left\{ t \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \right\} \\ &= e^{t(H^T \otimes G)}\, \text{Vec} \left\{ t \begin{bmatrix} -7 & 5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \right\} \\ &= e^{t(H^T \otimes G)}\, \text{Vec} \left\{ \begin{bmatrix} -7t+3 & 5t+1 \\ 1-t & 2t-1 \end{bmatrix} \right\} \\ &= e^{t(H^T \otimes G)} \begin{bmatrix} -7t+3 \\ 1-t \\ 5t+1 \\ 2t-1 \end{bmatrix}. \end{aligned}$$

Example 4.3. Consider the initial value problem

$$\begin{aligned} X'(t) &= AX(t)B \\ Y'(t) &= EX(t)F + GY(t)H, \end{aligned}$$

with $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $E = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, $F = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$,
$H = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$, and $K = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.

We have

$$B^T \otimes A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \text{ and } H^T \otimes G = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

By using Corollary 3.10, the above initial value problem has a unique solution given by

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{t(B^T \otimes A)} \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -2e^{-t} \\ 0 \\ 0 \\ e^{-2t} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\operatorname{Vec} Y(t) &= e^{t(H^T \otimes G)} \operatorname{Vec} \left\{ t \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \right\} \\
&= e^{t(H^T \otimes G)} \operatorname{Vec} \left\{ t \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \right\} \\
&= e^{t(H^T \otimes G)} \operatorname{Vec} \left\{ \begin{bmatrix} 4t+3 & 0 \\ 0 & 3t-1 \end{bmatrix} \right\} \\
&= \begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 4t+3 \\ 0 \\ 0 \\ 3t-1 \end{bmatrix} \\
&= \begin{bmatrix} (4t+3)e^{-t} \\ 0 \\ 0 \\ (3t-1)e^{-2t} \end{bmatrix}.
\end{aligned}$$

Thus

$$X(t) = \begin{bmatrix} -2e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix},$$

$$Y(t) = \begin{bmatrix} (4t+3)e^{-t} & 0 \\ 0 & (3t-1)e^{-2t} \end{bmatrix}.$$

Let us verify that $X(t)$ and $Y(t)$ are solutions of the above problem. Indeed, we have

$$X'(t) = \begin{bmatrix} 2e^{-t} & 0 \\ 0 & -2e^{-2t} \end{bmatrix},$$

$$AX(t)B = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -2e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} & 0 \\ 0 & -2e^{-2t} \end{bmatrix}$$

and

$$Y'(t) = \begin{bmatrix} (1-4t)e^{-t} & 0 \\ 0 & (5-6t)e^{-2t} \end{bmatrix},$$

$$\begin{aligned} EX(t)F + GY(t)H &= \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -2e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} (4t+3)e^{-t} & 0 \\ 0 & (3t-1)e^{-2t} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4e^{-t} & 0 \\ 0 & 3e^{-2t} \end{bmatrix} + \begin{bmatrix} -(4t+3)e^{-t} & 0 \\ 0 & -2(3t-1)e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} (1-4t)e^{-t} & 0 \\ 0 & (5-6t)e^{-2t} \end{bmatrix}. \end{aligned}$$

Thus $X(t)$ and $Y(t)$ are solution of the above problem.

Chapter 5

Conclusions

We solve a general system of nonhomogeneous coupled linear matrix differential equations. We apply Kronecker products and the vector operator to reduce our complex system to the simplest form we have an explicit formula of the general solution to this system is obtained in terms of Mittag-Leffler matrix functions.

Table 5.1: A General solution of coupled linear matrix differential equations

System of nonhomogeneous coupled linear matrix	A general solution of the system
$X'(t) = AX(t)B + CY(t)D + U(t)$, $Y'(t) = EX(t)F + GY(t)H + V(t)$ Assumption $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$.	$\text{Vec}\,X(t) = e^{(t-t_0)(B^T \otimes A)}\{(E_{2,1}((t-t_0)^2M))\,\text{Vec}\,X(t_0)$ $+(t-t_0)(E_{2,2}((t-t_0)^2M))(D^T \otimes C)\,\text{Vec}\,Y(t_0)$ $+(E_{2,1}((t-t_0)^2M)) * \text{Vec}\,U(t)$ $+(t-t_0)(E_{2,2}((t-t_0)^2M))(D^T \otimes C) * \text{Vec}\,V(t)\}$, $\text{Vec}\,Y(t) = e^{(t-t_0)(H^T \otimes G)}$ $\{(t-t_0)(E_{2,2}((t-t_0)^2N)(F^T \otimes E)\,\text{Vec}\,X(t_0)$ $+(E_{2,1}((t-t_0)^2N))\,\text{Vec}\,Y(t_0)$ $+(t-t_0)(E_{2,2}((t-t_0)^2N))(F^T \otimes E) * \text{Vec}\,U(t)$ $+(E_{2,1}((t-t_0)^2N)) * \text{Vec}\,V(t)\}$, where $M = (FD)^T \otimes CE$ and $N = (DF)^T \otimes EC$.
$X'(t) = AX(t)B + CY(t)D$, $Y'(t) = EX(t)F + GY(t)H$ Assumption $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$.	$\text{Vec}\,X(t) = e^{(t-t_0)(B^T \otimes A)}\{(E_{2,1}((t-t_0)^2M))\,\text{Vec}\,X(t_0)$ $+(t-t_0)(E_{2,2}((t-t_0)^2M))(D^T \otimes C)\,\text{Vec}\,Y(t_0)$ $\text{Vec}\,Y(t) = e^{(t-t_0)(H^T \otimes G)}$ $\{(t-t_0)(E_{2,2}((t-t_0)^2N)(F^T \otimes E)\,\text{Vec}\,X(t_0)$ $+(E_{2,1}((t-t_0)^2N))\,\text{Vec}\,Y(t_0)$ where $M = (FD)^T \otimes CE$ and $N = (DF)^T \otimes EC$.
$X'(t) = AX(t)B + CY(t)D + U(t)$, $Y'(t) = CX(t)D + AY(t)B + V(t)$ Assumption $AC = CA$ and $BD = DB$.	$\text{Vec}\,X(t) = e^{(t-t_0)(B^T \otimes A)}\{\cosh L\,\text{Vec}\,X(t_0)$ $+\sinh L\,\text{Vec}\,Y(t_0) + \cosh L * \text{Vec}\,U(t) + \sinh L * \text{Vec}\,V(t)\}$, $\text{Vec}\,Y(t) = e^{(t-t_0)(B^T \otimes A)}\{\sinh L\,\text{Vec}\,X(t_0)$ $+\cosh L\,\text{Vec}\,Y(t_0) + \sinh L * \text{Vec}\,U(t) + \cosh L * \text{Vec}\,V(t)\}$, where $L = (t-t_0)(D^T \otimes C)$.

System of nonhomogeneous coupled linear matrix	A general solution of the system
$X'(t) = AX(t)B + CY(t)D,$ $Y'(t) = CX(t)D + AY(t)B$ Assumption $AC = CA$ and $BD = DB$.	$\mathrm{Vec}\,X(t) = e^{(t-t_0)(B^T \otimes A)}\{ \cosh L\,\mathrm{Vec}\,X(t_0),$ $+ \sinh L\,\mathrm{Vec}\,Y(t_0)\}$ $\mathrm{Vec}\,Y(t) = e^{(t-t_0)(B^T \otimes A)}\{ \sinh L\,\mathrm{Vec}\,X(t_0).$ $+ \cosh L\,\mathrm{Vec}\,Y(t_0)\}$ where $L = (t - t_0)(D^T \otimes C)$
$X'(t) = AX(t)B + Y(t) + U(t),$ $Y'(t) = X(t) + AY(t)B + V(t)$	$\mathrm{Vec}\,X(t) = e^{(t-t_0)(B^T \otimes A)}\{ \cosh(t-t_0)\,\mathrm{Vec}\,X(t_0)$ $+ \sinh(t-t_0)\,\mathrm{Vec}\,Y(t_0) + \cosh(t-t_0)(I_{n^2} * \mathrm{Vec}\,U(t))$ $+ \sinh(t-t_0)(I_{n^2} * \mathrm{Vec}\,V(t))\},$ $\mathrm{Vec}\,Y(t) = e^{(t-t_0)(B^T \otimes A)}\{ \sinh(t-t_0)\,\mathrm{Vec}\,X(t_0)$ $+ \cosh(t-t_0)\,\mathrm{Vec}\,Y(t_0) + \sinh(t-t_0)(I_{n^2} * \mathrm{Vec}\,U(t))$ $+ \cosh(t-t_0)(I_{n^2} * \mathrm{Vec}\,V(t))\}.$
$X'(t) = AX(t)B + Y(t),$ $Y'(t) = X(t) + AY(t)B$	$\mathrm{Vec}\,X(t) = e^{(t-t_0)(B^T \otimes A)}\{ \cosh(t-t_0)\,\mathrm{Vec}\,X(t_0)$ $+ \sinh(t-t_0)\,\mathrm{Vec}\,Y(t_0)\}$ $\mathrm{Vec}\,Y(t) = e^{(t-t_0)(B^T \otimes A)}\{ \sinh(t-t_0)\,\mathrm{Vec}\,X(t_0)$ $+ \cosh(t-t_0)\,\mathrm{Vec}\,Y(t_0)\}.$
$X'(t) = AX(t)B + CY(t) + U(t),$ $Y'(t) = EX(t) + GY(t)B + V(t)$ Assumption $AC = CG, GE = EA$	$\mathrm{Vec}\,X(t) = e^{(t-t_0)(B^T \otimes A)}\,\mathrm{Vec}\Big\{(E_{2,1}(K_1))X(t_0)$ $+(t-t_0)(E_{2,2}(K_1))CY(t_0)\Big\}$ $+e^{(t-t_0)(B^T \otimes A)}\Big\{(I_n \otimes E_{2,1}(K_1)) * \mathrm{Vec}\,U(t)$ $+(I_n \otimes (t-t_0)(E_{2,2}(K_1))C) * \mathrm{Vec}\,V(t)\Big\},$ $\mathrm{Vec}\,Y(t) = e^{(t-t_0)(B^T \otimes G)}\,\mathrm{Vec}\Big\{(t-t_0)(E_{2,2}(K_2))EX(t_0)$ $+(E_{2,1}(K_2))Y(t_0)\Big\}$ $+e^{(t-t_0)(B^T \otimes G)}\Big\{(I_n \otimes (t-t_0)(E_{2,2}(K_2))E) * \mathrm{Vec}\,U(t)$ $+(I_n \otimes E_{2,1}(K_2)) * \mathrm{Vec}\,V(t)\Big\},$ where $K_1 = (t-t_0)^2CE$ and $K_2 = (t-t_0)^2EC$.
$X'(t) = AX(t)B + CY(t),$ $Y'(t) = EX(t) + GY(t)B$ Assumption $AC = CG, GE = EA$	$\mathrm{Vec}\,X(t) = e^{(t-t_0)(B^T \otimes A)}\,\mathrm{Vec}\Big\{(E_{2,1}(K_1))X(t_0)$ $+(t-t_0)(E_{2,2}(K_1))CY(t_0)\Big\},$ $\mathrm{Vec}\,Y(t) = e^{(t-t_0)(B^T \otimes G)}\,\mathrm{Vec}\Big\{(t-t_0)(E_{2,2}(K_2))EX(t_0)$ $+(E_{2,1}(K_2))Y(t_0)\Big\}$ where $K_1 = (t-t_0)^2CE$ and $K_2 = (t-t_0)^2EC$.

System of nonhomogeneous coupled linear matrix	A general solution of the system
$X'(t) = AX(t)B + U(t)$, $Y'(t) = EX(t)F + GY(t)H + V(t)$ Assumption $FH = BF$ and $GE = EA$.	$\text{Vec}\, X(t) = e^{(t-t_0)(B^T \otimes A)}\{\text{Vec}\, X(t_0) + I * \text{Vec}\, U(t)\}$, $\text{Vec}\, Y(t) = e^{(t-t_0)(H^T \otimes G)}\, \text{Vec}\,\{(t-t_0)EX(t_0)F + Y(t_0)\}$ $+e^{(t-t_0)(H^T \otimes G)}\{(t-t_0)(F^T \otimes E) * \text{Vec}\, U(t) + I * \text{Vec}\, V(t)\}$.
$X'(t) = AX(t)B$, $Y'(t) = EX(t)F + GY(t)H$ Assumption $FH = BF$ and $GE = EA$.	$\text{Vec}\, X(t) = e^{(t-t_0)(B^T \otimes A)}\{\text{Vec}\, X(t_0)\}$, $\text{Vec}\, Y(t) = e^{(t-t_0)(H^T \otimes G)}\, \text{Vec}\,\{(t-t_0)EX(t_0)F + Y(t_0)\}$.
$X'(t) = AX(t)B + U(t)$	$\text{Vec}\, X(t) = e^{(t-t_0)(B^T \otimes A)}\{\text{Vec}\, X(t_0) + I * \text{Vec}\, U(t)\}$.
$X'(t) = AX(t)B$	$\text{Vec}\, X(t) = e^{(t-t_0)(B^T \otimes A)}\{\text{Vec}\, X(t_0)\}$.

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Appendix

Appendix A

The research paper

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Solving Systems of Nonhomogeneous Coupled Linear Matrix Differential Equations in Terms of Mittag-Leffler Matrix Functions

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Abstract

In this paper, we investigate systems of nonhomogeneous coupled linear matrix differential equations. Applying Kronecker products, the vector operator, and matrix convolution product, we obtain explicit formula of the general solution to this system in terms of matrix series concerning exponentials and Mittag-Leffler functions.

Keywords: linear matrix differential equation, Kronecker product, vector operator, matrix convolution product, Mittag-Leffler function.
Mathematics Subject Classifications 2010: 15A16, 15A69, 33E12, 34A30, 44A35.

1 Introduction

Theory of linear matrix differential equations can be applied in a broad range of scientific fields, e.g. statistics [2, 6, 8], game theory [4], ecometrics and Leondief model [6, 8, 11], control and system theory [3, 7]. The simplest first-order homogeneous linear matrix differential equation with time-invariant coefficient is given by

$$X'(t) = AX(t). \tag{1.1}$$

Here, A is a given square matrix and $X(t)$ is an unknown matrix-valued function to be solved. The system (1.1) has been widely studied, and the solution relies on the computation of e^{tA}; see more information in [12, 13]. The nonhomogeneous case appears in the form

$$X'(t) = AX(t) + U(t), \tag{1.2}$$

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here $U(t)$ is a given matrix-valued function. In fact, the equation (1.2) has a general solution given by a one-parameter matrix-valued function

$$X(t) = e^{(t-t_0)A}X(t_0) + e^{tA} * U(t), \tag{1.3}$$

where $*$ denotes the matrix convolution product. See related works on nonhomogeneous case in [10, 15] and references therein.

Coupled matrix differential equations have numerous applications in pure and applied mathematics. For example, to obtain the solution of an optimal control problem with performance index we need to solve the system [7]

$$\begin{aligned} X'(t) &= AX(t) + BY(t), \\ Y'(t) &= CX(t) - A^T Y(t). \end{aligned}$$

A general system of nonhomogeneous coupled linear matrix differential equations with time-invariant coefficient takes the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t). \end{aligned} \tag{1.4}$$

In [5], a homogeneous case of (1.4) when $E = C$, $F = D$, $G = A$, $H = B$ was investigated under the assumption that $AC = CA$ and $BD = DB$. In this case, the solution is given in terms of Kronecker products, the vector operator, and matrix series concerning exponentials and hyperbolic functions. A nonhomogeneous case of (1.4) was discussed in [1].

In this work, we investigate the system (1.4) under the assumption that $AC = CG$, $GE = EA$, $DB = HD$, $FH = BF$. We apply Kronecker products and the vector operator to reduce our complex system to the simplest form. Thus, an explicit formula of the general solution to this system is obtained in terms of Mittag-Leffler matrix functions. In particular, we obtain general solution of several special cases of the main system. When initial conditions are imposed to these problems, its solution is uniquely determined. Our results also include the previous works [1, 5].

This paper is structured as follows. In Section 2, we supply useful facts for solving linear matrix differential equations, including matrix functions defined by power series, Kronecker product, vector operator, and matrix convolution product. The main part of the paper, Section 3, deals with solving the system (1.4) and its interesting special cases. In Sections 4, we treat an initial value problem related to (1.4) and illustrate it with a numerical example.

2 Preliminaries

In this section, we provide adequate tools for solving system of linear matrix differential equations. We shall denote the set of all m-by-n complex matrices by $M_{m,n}$, and we set $M_n = M_{n,n}$.

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2.1 Functions of a matrix defined by power series

Consider $A \in M_n$ and a holomorphic function f defined on a region in the complex plane containing the origin and the spectrum of A. Let $R > 0$ be such that f admits the Taylor series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{for } |z| < R,$$

where $a_0 = f(0)$ and $a_k = f^{(k)}(0)/k!$ for any $k \in \mathbb{N}$. If the spectral radius of A is less than R, then the matrix power series $\sum_{k=0}^{\infty} a_k A^k$ converges, denoted by $f(A)$. Hence if f is an entire function then $f(A)$ is a well-defined matrix for any $A \in M_n$. In particular, the following matrix series converge for any $A \in M_n$:

$$\sinh(A) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1}, \quad \cosh(A) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k}.$$

Recall that the two-parameter Mittag-Leffler functions (e.g. [14]) is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{2.1}$$

where Γ is the Gamma function. The power series (2.1) converges for all complex numbers z.

The Mittag-Leffler function of a matrix $A \in M_n$ with parameters $\alpha > 0$ and $\beta > 0$ is defined by

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} A^k = I_n + \frac{1}{\Gamma(\alpha + \beta)} A + \frac{1}{\Gamma(2\alpha + \beta)} A^2 + \cdots .$$

The class of matrix Mittag-Leffler functions include the following functions:

$$E_{1,1}(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = e^A, \quad E_{2,1}(A^2) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k} = \cosh(A).$$

An expansion shows that $(E_{2,2}(A^2))A = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1} = \sinh(A)$.

Lemma 2.1 (see e.g. [9]). *If (A, B) is a pair of commuting complex matrices, then $e^{A+B} = e^A e^B$.*

The next lemma is useful for deriving explicit formulas of solutions for system of linear matrix differential equations in Section 3.

Lemma 2.2. *For any $A \in M_n(\mathbb{C})$ and $B \in M_n(\mathbb{C})$, we have*

$$e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} = \begin{bmatrix} E_{2,1}(AB) & (E_{2,2}(AB))A \\ (E_{2,2}(BA))B & E_{2,1}(BA) \end{bmatrix}.$$

3

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Proof. A computation using matrix analysis reveals that

$$
\begin{aligned}
e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^k \\
&= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \begin{bmatrix} (AB)^k & 0 \\ 0 & (BA)^k \end{bmatrix} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \begin{bmatrix} 0 & (AB)^k A \\ (BA)^k B & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{(2k)!} (AB)^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{(2k)!} (BA)^k \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (AB)^k A \\ \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (BA)^k B & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+1)} (AB)^k & \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+2)} (AB)^k A \\ \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+2)} (BA)^k B & \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+1)} (BA)^k \end{bmatrix} \\
&= \begin{bmatrix} E_{2,1}(AB) & (E_{2,2}(AB))A \\ (E_{2,2}(BA))B & E_{2,1}(BA) \end{bmatrix}.
\end{aligned}
$$

□

2.2 Kronecker product and vector operator

Given two matrices $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{p,q}$ the Kronecker product of A and B is defined by

$$A \otimes B = [a_{ij}B]_{ij} \in M_{mp,nq}.$$

The the vector operator $\operatorname{Vec} : M_{m,n} \to \mathbb{C}^{mn}$ is defined for each $A = [a_{ij}]$ by

$$\operatorname{Vec} A = [a_{11} \dots a_{m1} \dots a_{12} \dots a_{m2} \dots a_{1m} \dots a_{mn}]^T.$$

It is clear that Vec is a linear isomorphism. Algebraic properties of the Kronecker product and the vector operator used in this paper are as follows:

Lemma 2.3 (see e.g. [9]). *The map $(A, B) \mapsto A \otimes B$ is bilinear. The following properties hold for matrices of appropriate sizes:*

1. $I_m \otimes I_n = I_{mn}$,
2. $(A \otimes B)(C \otimes D) = AC \otimes BD$,
3. $\operatorname{Vec}(AXB) = (B^T \otimes A) \operatorname{Vec} X$.

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The Kronecker product is compatible with holomorphic functions in the following sense.

Lemma 2.4 (see e.g.[9]). *Let f be a holomorphic function defined on a region including the origin and the spectrum of $A \in M_n$. Then $f(I \otimes A) = I \otimes f(A)$ and $f(A \otimes I) = f(A) \otimes I$. In particular, the following relations hold for any $A \in M_n$:*

$$\begin{array}{lcl} E_{\alpha,\beta}(A \otimes I) = E_{\alpha,\beta}(A) \otimes I & \text{and} & E_{\alpha,\beta}(I \otimes A) = I \otimes E_{\alpha,\beta}(A), \\ \sinh(A \otimes I) = \sinh(A) \otimes I & \text{and} & \sinh(I \otimes A) = I \otimes \sinh(A), \\ \cosh(A \otimes I) = \cosh(A) \otimes I & \text{and} & \cosh(I \otimes A) = I \otimes \cosh(A). \end{array}$$

2.3 Matrix convolution product

Let $\Omega = [0, \infty)$ or $\Omega = [0, b]$ for some $b > 0$. The convolution is a binary operation assigned to each pair of integrable function f and g defined by

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau, \quad t \in \Omega.$$

The convolution is bilinear and commutative. Given two integrable matrix-valued functions $A : \Omega \to M_{m,n}(\mathbb{R})$, $A(t) = [a_{ij}(t)]$ and $B : \Omega \to M_{n,p}(\mathbb{R})$, $B(t) = [b_{ij}(t)]$, we define the matrix convolution product of A and B by

$$(A * B)(t) = \left[\sum_{k=1}^{n} a_{ik}(t) * b_{kj}(t) \right] \in M_{m,p}(\mathbb{R}), \quad t \in \Omega.$$

We may write $A(t) * B(t)$ for $(A * B)(t)$. The matrix convolution product is bilinear, but not commutative in general.

3 General solutions of systems of nonhomogeneous coupled linear matrix differential equations

From now on, let $A, B, C, D, E, F, G, H, J, K \in M_n(\mathbb{C})$ be given constant matrices and let $U, V : \Omega \to M_n(\mathbb{C})$ be given matrix-valued functions. We wish to solve certain systems of linear matrix differential equations in unknown matrix-valued functions $X, Y : \Omega \to M_n(\mathbb{C})$.

Theorem 3.1. *Assume that $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$. Then the general solution of the system of nonhomogeneous coupled linear matrix differential equations:*

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t) \end{aligned} \qquad (3.1)$$

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is given by

$$\begin{aligned}\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,1}((t-t_0)^2M))\operatorname{Vec} X(t_0)\\&+(t-t_0)(E_{2,2}((t-t_0)^2M))(D^T\otimes C)\operatorname{Vec} Y(t_0)\\&+(E_{2,1}((t-t_0)^2M)) * \operatorname{Vec} U(t)\\&+(t-t_0)(E_{2,2}((t-t_0)^2M))(D^T\otimes C) * \operatorname{Vec} V(t)\},\\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T\otimes G)}\{(t-t_0)(E_{2,2}((t-t_0)^2N)(F^T\otimes E)\operatorname{Vec} X(t_0)\\&+(E_{2,1}((t-t_0)^2N))\operatorname{Vec} Y(t_0)\\&+(t-t_0)(E_{2,2}((t-t_0)^2N))(F^T\otimes E) * \operatorname{Vec} U(t)\\&+(E_{2,1}((t-t_0)^2N)) * \operatorname{Vec} V(t)\},\end{aligned} \tag{3.2}$$

where $M=(FD)^T\otimes CE$ *and* $N=(DF)^T\otimes EC$.

Proof. Using Lemma 2.3, we can transform the system (3.1) into the vector form:

$$\begin{bmatrix}\operatorname{Vec} X'(t)\\ \operatorname{Vec} Y'(t)\end{bmatrix}=\begin{bmatrix}B^T\otimes A & D^T\otimes C\\ F^T\otimes E & H^T\otimes G\end{bmatrix}\begin{bmatrix}\operatorname{Vec} X(t)\\ \operatorname{Vec} Y(t)\end{bmatrix}+\begin{bmatrix}\operatorname{Vec} U(t)\\ \operatorname{Vec} V(t)\end{bmatrix}.$$

Let us denote $P=\begin{bmatrix}B^T\otimes A & 0\\ 0 & H^T\otimes G\end{bmatrix}$ and $Q=\begin{bmatrix}0 & D^T\otimes C\\ F^T\otimes E & 0\end{bmatrix}$. From (1.3), this system has the following solution:

$$\begin{bmatrix}\operatorname{Vec} X(t)\\ \operatorname{Vec} Y(t)\end{bmatrix}=e^{(t-t_0)S}\begin{bmatrix}\operatorname{Vec} X(t_0)\\ \operatorname{Vec} Y(t_0)\end{bmatrix}+e^{(t-t_0)S} * \begin{bmatrix}\operatorname{Vec} U(t)\\ \operatorname{Vec} V(t)\end{bmatrix},$$

where $S=P+Q$. Now, we will compute e^S. Since $DB=HD$, $AC=CG$, $FH=BF$ and $GE=EA$, by Lemma 2.3 we have $PQ=QP$. From which it follows from Lemma 2.1 that $e^S=e^{P+Q}=e^Pe^Q$. By expanding the power series of matrix exponential, we have

$$e^P=\begin{bmatrix}e^{B^T\otimes A} & 0\\ 0 & e^{H^T\otimes G}\end{bmatrix}.$$

By Lemma 2.2, we have

$$e^Q=\begin{bmatrix}E_{2,1}(M) & (E_{2,2}(M))(D^T\otimes C)\\ (E_{2,2}(N))(F^T\otimes E) & E_{2,1}(N)\end{bmatrix}.$$

Thus

$$\begin{aligned}e^S &= \begin{bmatrix}e^{B^T\otimes A} & 0\\ 0 & e^{H^T\otimes G}\end{bmatrix}\begin{bmatrix}E_{2,1}(M) & (E_{2,2}(M))(D^T\otimes C)\\ (E_{2,2}(N))(F^T\otimes E) & E_{2,1}(N)\end{bmatrix}\\ &= \begin{bmatrix}e^{B^T\otimes A}E_{2,1}(M) & e^{B^T\otimes A}(E_{2,2}(M))(D^T\otimes C)\\ e^{H^T\otimes G}(E_{2,2}(N))(F^T\otimes E) & e^{H^T\otimes G}E_{2,1}(N)\end{bmatrix}.\end{aligned}$$

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Denoting

$$\begin{aligned} R_1 &= e^{(t-t_0)(B^T\otimes A)}E_{2,1}((t-t_0)^2M), \\ R_2 &= e^{(t-t_0)(B^T\otimes A)}(t-t_0)(E_{2,2}((t-t_0)^2M))(D^T\otimes C), \\ R_3 &= e^{(t-t_0)(H^T\otimes G)}(t-t_0)(E_{2,2}((t-t_0)^2N))(F^T\otimes E), \\ R_4 &= e^{(t-t_0)(H^T\otimes G)}E_{2,1}((t-t_0)^2N), \end{aligned}$$

we obtain

$$e^{(t-t_0)S}\begin{bmatrix}\operatorname{Vec}X(t_0)\\ \operatorname{Vec}Y(t_0)\end{bmatrix} = \begin{bmatrix}R_1 & R_2\\ R_3 & R_4\end{bmatrix}\begin{bmatrix}\operatorname{Vec}X(t_0)\\ \operatorname{Vec}Y(t_0)\end{bmatrix} = \begin{bmatrix}R_1\operatorname{Vec}X(t_0)+R_2\operatorname{Vec}Y(t_0)\\ R_3\operatorname{Vec}X(t_0)+R_4\operatorname{Vec}Y(t_0)\end{bmatrix}.$$

We also have

$$e^{(t-t_0)S}*\begin{bmatrix}\operatorname{Vec}U(t)\\ \operatorname{Vec}V(t)\end{bmatrix} = \begin{bmatrix}R_1 & R_2\\ R_3 & R_4\end{bmatrix}*\begin{bmatrix}\operatorname{Vec}U(t)\\ \operatorname{Vec}V(t)\end{bmatrix} = \begin{bmatrix}R_1*\operatorname{Vec}U(t)+R_2*\operatorname{Vec}V(t)\\ R_3*\operatorname{Vec}U(t)+R_4*\operatorname{Vec}V(t)\end{bmatrix}.$$

Therefore, the general solution of (3.1) is given by (3.2). □

Corollary 3.2. *Assume that $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$. Then the general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D, \\ Y'(t) &= EX(t)F + GY(t)H \end{aligned}$$

is given by

$$\begin{aligned} \operatorname{Vec}X(t) &= e^{(t-t_0)(B^T\otimes A)}\{(E_{2,1}((t-t_0)^2M))\operatorname{Vec}X(t_0) \\ &\quad + (t-t_0)(E_{2,2}((t-t_0)^2M))(D^T\otimes C)\operatorname{Vec}Y(t_0), \\ \operatorname{Vec}Y(t) &= e^{(t-t_0)(H^T\otimes G)}\{(t-t_0)(E_{2,2}((t-t_0)^2N))(F^T\otimes E)\operatorname{Vec}X(t_0) \\ &\quad + (E_{2,1}((t-t_0)^2N))\operatorname{Vec}Y(t_0)\} \end{aligned} \tag{3.3}$$

where $M = (FD)^T\otimes CE$ and $N = (DF)^T\otimes EC$.

Proof. Put $U(t) = V(t) = 0$ in (3.2) and then use Lemma 2.3. □

The next result was firstly established in [1].

Corollary 3.3. *The general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= CX(t)D + AY(t)B + V(t) \end{aligned} \tag{3.4}$$

under the assumption that $AC = CA$ and $BD = DB$, is given by

$$\begin{aligned} \operatorname{Vec}X(t) =& e^{(t-t_0)(B^T\otimes A)}\{\cosh L\operatorname{Vec}X(t_0) + \sinh L\operatorname{Vec}Y(t_0) \\ &+ \cosh L*\operatorname{Vec}U(t) + \sinh L*\operatorname{Vec}V(t)\}, \\ \operatorname{Vec}Y(t) =& e^{(t-t_0)(B^T\otimes A)}\{\sinh L\operatorname{Vec}X(t_0) + \cosh L\operatorname{Vec}Y(t_0) \\ &+ \sinh L*\operatorname{Vec}U(t) + \cosh L*\operatorname{Vec}V(t)\}, \end{aligned} \tag{3.5}$$

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where $L = (t - t_0)(D^T \otimes C)$.

Proof. Put $E = C$, $F = D$, $G = A$ and $H = B$ in (3.2), and use Lemma 2.3. □

The corresponding homogeneous system of (3.4) is given by

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D, \\ Y'(t) &= CX(t)D + AY(t)B. \end{aligned} \tag{3.6}$$

If $AC = CA$ and $BD = DB$, then the general solution of (3.6) is reduced to

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)}\{\cosh L \operatorname{Vec} X(t_0) + \sinh L \operatorname{Vec} Y(t_0)\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T \otimes A)}\{\sinh L \operatorname{Vec} X(t_0) + \cosh L \operatorname{Vec} Y(t_0)\}. \end{aligned}$$

This result was firstly obtained in [5].

Corollary 3.4. *The general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B + CY(t) \quad + U(t), \\ Y'(t) &= EX(t) \quad + GY(t)B + V(t) \end{aligned}$$

under the condition $AC = CG, GE = EA$, *is given by*

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)} \operatorname{Vec}\Big\{(E_{2,1}(K_1))X(t_0) + (t-t_0)(E_{2,2}(K_1))CY(t_0)\Big\} \\ &\quad + e^{(t-t_0)(B^T \otimes A)}\Big\{(I_n \otimes E_{2,1}(K_1)) * \operatorname{Vec} U(t) \\ &\quad + (I_n \otimes (t-t_0)(E_{2,2}(K_1))C) * \operatorname{Vec} V(t)\Big\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T \otimes G)} \operatorname{Vec}\Big\{(t-t_0)(E_{2,2}(K_2))EX(t_0) + (E_{2,1}(K_2))Y(t_0)\Big\} \\ &\quad + e^{(t-t_0)(B^T \otimes G)}\Big\{(I_n \otimes (t-t_0)(E_{2,2}(K_2))E) * \operatorname{Vec} U(t) \\ &\quad + (I_n \otimes E_{2,1}(K_2)) * \operatorname{Vec} V(t)\Big\}, \end{aligned}$$

where $K_1 = (t-t_0)^2 CE$ *and* $K_2 = (t-t_0)^2 EC$.

Proof. Put $H = B$, $D = F = I_n$ in (3.2) and then use Lemmas 2.3 and 2.4. □

Corollary 3.5. *The general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B + \quad Y(t) \quad + U(t), \\ Y'(t) &= \quad X(t) \quad + AY(t)B + V(t) \end{aligned}$$

is given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)}\{\cosh(t-t_0) \operatorname{Vec} X(t_0) + \sinh(t-t_0) \operatorname{Vec} Y(t_0) \\ &\quad + \cosh(t-t_0)(I_{n^2} * \operatorname{Vec} U(t)) + \sinh(t-t_0)(I_{n^2} * \operatorname{Vec} V(t))\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T \otimes A)}\{\sinh(t-t_0) \operatorname{Vec} X(t_0) + \cosh(t-t_0) \operatorname{Vec} Y(t_0) \\ &\quad + \sinh(t-t_0)(I_{n^2} * \operatorname{Vec} U(t)) + \cosh(t-t_0)(I_{n^2} * \operatorname{Vec} V(t))\}. \end{aligned}$$

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Proof. Put $C = D = I_n$ in (3.5) and then use Lemma 2.3. □

Corollary 3.6. *The general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t) \end{aligned}$$

under the condition $FH = BF$ and $GE = EA$, is given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\{\operatorname{Vec} X(t_0) + I * \operatorname{Vec} U(t)\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T\otimes G)}\operatorname{Vec}\{(t-t_0)EX(t_0)F + Y(t_0)\} \\ &\quad + e^{(t-t_0)(H^T\otimes G)}\{(t-t_0)(F^T\otimes E) * \operatorname{Vec} U(t) + I * \operatorname{Vec} V(t)\}. \end{aligned}$$

Proof. Put $C = D = 0$ in (3.2) and then use Lemma 2.3. □

Corollary 3.7. *The general solution of equation $X'(t) = AX(t)B + U(t)$ is given by $\operatorname{Vec} X(t) = e^{(t-t_0)(B^T\otimes A)}\{\operatorname{Vec} X(t_0) + I * \operatorname{Vec} U(t)\}$.*

Proof. Put $E = F = 0$ in Corollary 3.6. □

4 Unique solution of initial value problem and a numerical example

Consider the following initial value problem associated with the system (3.1):

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t) \end{aligned}$$

subject to initial conditions $X(0) = J$ and $Y(0) = K$. Suppose $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$. In this case, the solution of this problem is unique and given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{t(B^T\otimes A)}\{(E_{2,1}(t^2M))\operatorname{Vec} J + t(E_{2,2}(t^2M))(D^T\otimes C)\operatorname{Vec} K \\ &\quad + (E_{2,1}(t^2M)) * \operatorname{Vec} U(t) + t(E_{2,2}(t^2M))(D^T\otimes C) * \operatorname{Vec} V(t)\}, \\ \operatorname{Vec} Y(t) &= e^{t(H^T\otimes G)}\{t(E_{2,2}(t^2N)(F^T\otimes E)\operatorname{Vec} J + (E_{2,1}(t^2N))\operatorname{Vec} K \\ &\quad + t(E_{2,2}(t^2N))(F^T\otimes E) * \operatorname{Vec} U(t) + (E_{2,1}(t^2N)) * \operatorname{Vec} V(t)\}, \end{aligned}$$

where $M = (FD)^T \otimes CE$ and $N = (DF)^T \otimes EC$.

Let us see a numerical example.

Example 4.1. *The initial value problem*

$$\begin{aligned} X'(t) &= AX(t)B + \ \ Y(t) \quad + U(t), \\ Y'(t) &= \ \ X(t) \quad + AY(t)B + V(t) \\ X(0) &= J \quad and \quad Y(0) = K \end{aligned}$$

with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, J = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, K = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$,
$U(t) = \begin{bmatrix} -e^{2t} & 1 \\ 1 & \sin t \end{bmatrix}, V(t) = \begin{bmatrix} 1 & e^{2t} \\ \cos t & \sin 2t \end{bmatrix}$ *has a unique solution given by*

$$\operatorname{Vec} X(t) = e^{tW} \operatorname{Vec} \begin{bmatrix} w_1(t)\cosh t + w_2(t)\sinh t & w_3(t)\cosh t + w_4(t)\sinh t \\ w_5(t)\cosh t + w_6(t)\sinh t & w_7(t)\cosh t + w_8(t)\sinh t \end{bmatrix},$$

$$\operatorname{Vec} Y(t) = e^{tW} \operatorname{Vec} \begin{bmatrix} w_2(t)\cosh t + w_1(t)\sinh t & w_4(t)\cosh t + w_3(t)\sinh t \\ w_6(t)\cosh t + w_5(t)\sinh t & w_8(t)\cosh t + w_7(t)\sinh t \end{bmatrix}.$$

Here, $W = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ -1 & -2 & 1 & 2 \\ -3 & -4 & 3 & 4 \end{bmatrix}$,

$w_1(t) = \frac{1}{2}(5 - e^{2t}),\ w_2(t) = 3 + t,\ w_3(t) = -1 + t,\ w_4(t) = \frac{1}{2}(1 + e^{2t}),$
$w_5(t) = 1 + t,\ w_6(t) = 1 + \sin t,\ w_7(t) = 1 - \cos t,\ w_8(t) = -\frac{1}{2}(1 + \cos 2t).$

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